Inverse Problems for Fractional Operators Involving a Magnetic Potential

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Abstract

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In this thesis, we study forward and inverse problems for fractional operators involving a magnetic potential. We show that many properties of fractional operators are preserved under the perturbation by a magnetic potential. Besides, we carefully use Runge approximation properties to obtain strong results when we study inverse problems. More precisely, we determine both the magnetic potential and the electric potential from exterior partial measurements of the Dirichlet-to-Neumann map in the linear fractional inverse problem by using the Runge approximation property and an integral identity; we also determine both the magnetic potential and the non-linearity in the semi-linear fractional inverse problem by using the Runge approximation property and a first order linearization.

TABLE OF CONTENTS

Page

GLOSSARY

DIMENSION: $n \geq 2$ denotes the space dimension.

POWER: $0 < s < 1$ denotes the fractional power.

DOMAIN: Ω denotes a bounded domain with smooth boundary; $\Omega_e := \mathbb{R}^n \setminus \bar{\Omega}$.

BALL: $B_r(0)$ denotes the open ball centered at the origin with radius $r > 0$.

CONSTANTS: $c, C, C', C_1, \cdots$ denote positive constants.

INTEGRATION: $\int \cdots \int = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n}$.

DUAL SPACE: X^* denotes the continuous dual space of X.

PAIRING: $\langle \cdot, \cdot \rangle$ denotes the distributional pairing.

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DEDICATION

to my parents and grandparents

Chapter 1

INTRODUCTION

In the context of partial differential equations, the study of inverse problems concerns the recovery of internal properties of a medium (corresponding to certain terms in an equation or operator) from indirect external measurements (corresponding to information on solutions of equations at the boundary or in the exterior domain). In fact, such problems lie at the foundations of many scientific disciplines such as electrostatics, electromagnetism, geophysics, medical imaging and biology, and consequently the study of inverse problems associated with partial differential equations remains active and influential as a research area in mathematics.

1.1 Classical Inverse Problems

One of the most famous examples in this area is the Calderón problem arising in electrostatics. The classical Calderón problem is whether one can determine the electrical conductivity of a medium by making voltage and current measurements at its boundary. Now we describe this problem in the language of mathematics.

Suppose we have a conductor filling Ω and we apply a voltage f at its boundary $\partial\Omega$. We denote the electrical conductivity at x by $\gamma(x)$ and the induced voltage at x by $u(x)$. By Ohm's Law, the current at x is $-\gamma(x)\nabla u(x)$ and in the absence of sinks or sources of current, u solves the Dirichlet problem

$$\operatorname{div}(\gamma\nabla u) = 0 \text{ in } \Omega, \qquad u|_{\partial\Omega} = f.$$

The inverse problem is whether one can determine γ from the knowledge of the Dirichlet-to-

Neumann (voltage-to-current) map defined by

$$\Lambda_\gamma : f \to (\gamma \frac{\partial u}{\partial \nu})|_{\partial\Omega}.$$

Based on the construction of complex geometrical optics (CGO) solutions, Sylvester and Uhlmann proved the following fundamental interior uniqueness theorem in [43].

Theorem 1.1.1. *Suppose $n \geq 3$. Let $\gamma_{1,2} \in C^2(\bar{\Omega})$ be strictly positive. If*

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2},$$

then $\gamma_1 = \gamma_2$ in Ω.

As a variation of the classical Calderón problem, the Calderón problem for the magnetic Schrödinger operator has been studied later. The magnetic operator has the form

$$(-i\nabla + A(x))^2 + q(x)$$

where $A(x)$ is a (vector-valued) magnetic potential. It can be viewed as a generalization of the Schrödinger operator $(-i\nabla)^2 + q = -\Delta + q$.

In [41, 34, 11, 20] the authors considered the Dirichlet problem

$$(-i\nabla + A)^2 u + qu = 0 \text{ in } \Omega, \qquad u|_{\partial\Omega} = f$$

and define the Dirichlet-to-Neumann map by

$$\Lambda_{A,q} : f \to (\partial_\nu + iA \cdot \nu)u|_{\partial\Omega}.$$

They proved that if $\Lambda_{A_1,q_1} = \Lambda_{A_2,q_2}$, then

$$\operatorname{rot} A_1 = \operatorname{rot} A_2, \qquad q_1 = q_2$$

in Ω when $n \geq 3$ and $A_{1,2}, q_{1,2}$ satisfy certain a priori assumptions. Here

$$\operatorname{rot} A := (\partial_j A^{(l)} - \partial_l A^{(j)})_{l,j}$$

where $A^{(j)}$ is the j-component of A and we call $\operatorname{rot} A$ the magnetic field induced by A.

We remark that it is impossible to completely determine A from the Dirichlet-to-Neumann map. In fact, if A_1 and A_2 are gauge equivalent, i.e.

$$A_1 - A_2 = \nabla\phi$$

for some smooth ϕ in $\bar{\Omega}$ with $\phi|_{\partial\Omega} = 0$, then by using the operator identity

$$e^{-i\phi}((-i\nabla + A)^2 + q)e^{i\phi} = (-i\nabla + (A + \nabla\phi))^2 + q$$

we can verify that $\Lambda_{A_1,q} = \Lambda_{A_2,q}$.

1.2 The Fractional Calderón Problem

Recently, equations involving non-local operators have attracted much attention. A typical non-local operator is the fractional Laplacian $(-\Delta)^s$ given by the Fourier transform definition

$$(-\Delta)^s u(x) := \mathcal{F}^{-1}(|\xi|^{2s}\mathcal{F}u(\xi))(x),$$

as well as the equivalent singular integral definition

$$(-\Delta)^s u(x) := c_{n,s} \lim_{\epsilon\to 0^+} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{u(x) - u(y)}{|x - y|^{n+2s}}\, dy.$$

See [23] for more equivalent definitions of the fractional Laplacian.

These kinds of equations have been introduced to describe anomalous diffusion and random processes with jumps in physics and probability theory. See [44] for a continuous limit of discrete, long jump random walks described by the fractional diffusion equation

$$\partial_t u + (-\Delta)^s u = 0.$$

Correspondingly, inverse problems associated with fractional operators involving $(-\Delta)^s$ have been studied. The study in this direction was initiated in [16]. Instead of the (boundary value) Dirichlet problem associated with the classical Calderón problem, the exterior Dirichlet problem

$$((-\Delta)^s + q)u = 0 \text{ in } \Omega, \qquad u|_{\Omega_e} = f \tag{1.1}$$

was considered in [16]. The inverse problem is whether one can determine the potential q in Ω from the exterior partial measurements of the Dirichlet-to-Neumann map

$$\Lambda_q : f \to (-\Delta)^s u|_{\Omega_e}.$$

See Appendix in [16] for a probabilistic interpretation of Λ_q.

The following theorem is the main result in [16].

Theorem 1.2.1. *Let* $0 \leq q_{1,2} \in L^\infty(\Omega)$ *and let* $W_{1,2} \subset \Omega_e$ *be open. If*

$$\Lambda_{q_1} g|_{W_2} = \Lambda_{q_2} g|_{W_2}, \qquad g \in C_c^\infty(W_1),$$

then $q_1 = q_2$ *in* Ω.

The main ingredient of the proof of the theorem above is the use of the following unique continuation property of $(-\Delta)^s$. See Theorem 1.2 in [16].

Proposition 1.2.2. *Let* $0 < s < 1$ *and* $u \in H^s(\mathbb{R}^n)$. *Let* W *be open and non-empty. If*

$$(-\Delta)^s u = u = 0 \quad \text{in } W,$$

then $u = 0$ *in* $\mathbb{R}^n$.

The proof of the unique continuation property above is based on the Caffarelli-Silvestre definition of the fractional Laplacian

$$(-\Delta)^s f(x) = c_s \lim_{y \to 0^+} y^{1-2s} \partial_y u(x, y)$$

where u is the solution of the extension problem

$$\begin{cases} \operatorname{div}(y^{1-2s} \nabla u) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \qquad u(x, 0) = f(x) & \text{on } \mathbb{R}^n \times \{0\}. \end{cases}$$

This definition enables us to derive properties of $(-\Delta)^s$ from local arguments in the extension problem. See [2, 37] for more details.

Based on the unique continuation property above, the following Runge approximation property was proved in [16].

Proposition 1.2.3. *Let* $0 \leq q \in L^\infty(\Omega)$ *and let* $W \subset \Omega_e$ *be open. Then*

$$S := \{u_g|_\Omega : g \in C_c^\infty(W)\}$$

is dense in $L^2(\Omega)$*. Here* u_g *is the solution of (1.1) corresponding to the exterior data* g*.*

Now Theorem 1.2.1 immediately follows from the Runge approximation property above and an integral identity for Dirichlet-to-Neumann maps. See Section 5 in [16] for details.

We remark that there are distinct differences between the classical and fractional Calderón problems. No construction of CGO solutions is required in dealing with the fractional problem. Both the unique continuation property of $(-\Delta)^s$ and the associated Runge approximation property are typical non-local phenomenons, which make the inverse problem more manageable and help us obtain strong results.

The fractional Calderón problem has been further studied in many settings. Low regularity and stability results have been obtained in [38]. Reconstruction and single measurement results have been obtained in [15]. Runge type approximation results for a large class of operators involving Fourier multipliers have been obtained in [39]. Inverse problems for variable coefficients fractional elliptic operators have been studied in [14, 3]. Inverse problems for fractional Schrödinger operators with local and non-local perturbations have been studied in [4, 1, 6].

1.3 Motivations and Goals

We will study fractional magnetic operators related with the fractional magnetic Laplacian

$$(-\Delta)^s_A u(x) := c_{n,s} \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{u(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})} u(y)}{|x-y|^{n+2s}} \, dy$$

where the vector-valued function A is a magnetic potential. This definition was first introduced in [10]. It was proved in [40] that $(-\Delta)^s_A$ converge to the magnetic Laplacian $(\nabla - iA(x))^2$ as $s \to 1^-$ in an appropriate sense.

A formal computation shows that

$$(-\Delta)^s_A u(x) = (2\pi)^{-n} \iint e^{i(x-y)\cdot\xi} |\xi - A(\frac{x+y}{2})|^{2s} u(y) \, dy d\xi$$

so $(-\Delta)^s_A$ has the form of a Weyl pseudo-differential operator. See more details in [28]. In particular, when $s = \frac{1}{2}$, $(-\Delta)^{1/2}_A$ is one of the quantized kinetic energy operators corresponding to the symbol $|\xi - A(x)|$ for a spinless particle of zero mass under the influence of the magnetic potential A. See for instance, [33, 18].

We will study forward and inverse problems associated with our fractional magnetic operators. Our problems can be viewed as variants of the fractional Calderón problem studied in [16] as well as a non-local analogues of the Calderón problem for the magnetic Schrödinger operator studied in [41, 34, 11, 20].

The rest of this thesis is organized in the following way. In Chapter 2, we focus on the linear fractional magnetic operator $\mathcal{R}^s_A$ involving a time-independent magnetic potential; we study the associated exterior Dirichlet problem and the Dirichlet-to-Neumann map; then we formulate and solve the associated inverse problem. In Chapter 3, we focus on the linear fractional magnetic operator involving a time-dependent magnetic potential; we study parabolic analogues of elliptic problems studied in Chapter 2. In Chapter 4, we study semilinear analogues of problems studied in Chapter 2.

Chapter 2

THE FRACTIONAL MAGNETIC ELLIPTIC PROBLEM

In this chapter, we study linear problems associated with the fractional elliptic operator involving a magnetic potential. Semi-linear analogues will be studied in Chapter 4.

2.1 Preliminaries

We introduce some basic notations and results in this section.

2.1.1 Function Spaces

Throughout this thesis we refer all function spaces to real-valued function spaces.

For $\alpha \in \mathbb{R}$, we have Sobolev spaces

$$H^\alpha(\mathbb{R}^n) := \{u \in \mathcal{S}'(\mathbb{R}^n) : \int (1+|\xi|^2)^\alpha |\mathcal{F}u(\xi)|^2 d\xi < \infty\}$$

where $\mathcal{F}$ is the Fourier transform and $\mathcal{S}'(\mathbb{R}^n)$ is the space of temperate distributions.

We have the natural identification

$$H^{-\alpha}(\mathbb{R}^n) = H^\alpha(\mathbb{R}^n)^*.$$

Let U be an open set in $\mathbb{R}^n$. Let F be a closed set in $\mathbb{R}^n$. Then

$$H^\alpha(U) := \{u|_U : u \in H^\alpha(\mathbb{R}^n)\}, \qquad H^\alpha_F(\mathbb{R}^n) := \{u \in H^\alpha(\mathbb{R}^n) : \operatorname{supp} u \subset F\},$$

$$\tilde{H}^\alpha(U) := \text{the closure of } C^\infty_c(U) \text{ in } H^\alpha(\mathbb{R}^n).$$

Ω is a bounded domain with smooth boundary, which implies

$$\tilde{H}^\alpha(\Omega) = H^\alpha_{\bar{\Omega}}(\mathbb{R}^n).$$

2.1.2 The Operator $\mathcal{R}^s_A$

We first give the formal pointwise definition of our fractional magnetic operator

$$\mathcal{R}^s_A u(x) := 2 \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} (u(x) - R_A(x,y)u(y))K(x,y)\, dy. \tag{2.1}$$

Here K is a function associated with a heat kernel (see subsection 2.2 in [28]) satisfying

$$K(x,y) = K(y,x), \qquad c/|x-y|^{n+2s} \le K(x,y) \le C/|x-y|^{n+2s},$$

A is a time-independent real vector-valued magnetic potential, and

$$R_A(x,y) := \cos((x-y)\cdot A(\frac{x+y}{2})). \tag{2.2}$$

Clearly $\mathcal{R}^s_A$ generalizes the fractional elliptic operator studied in [14], which coincides with $\mathcal{R}^s_0$. $\mathcal{R}^s_A$ also coincides with the real part of the fractional magnetic Laplacian $(-\Delta)^s_A$ when $K(x,y) = c_{n,s}/|x-y|^{n+2s}$ for real-valued u.

It is clear from (2.2) that

$$R_A(x,y) = R_A(y,x).$$

Hence for real-valued u, v, we can formally compute that

$$\langle \mathcal{R}^s_A u, v\rangle = 2 \iint (u(x) - R_A(x,y)u(y))v(x)K(x,y)\, dydx$$

$$= \iint [(u(x) - R_A(x,y)u(y))v(x)K(x,y) + (u(y) - R_A(x,y)u(x))v(y)K(x,y)]\, dydx$$

$$= \mathrm{Re} \iint (u(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}u(y))(v(x) - e^{-i(x-y)\cdot A(\frac{x+y}{2})}v(y))K(x,y)\, dxdy$$

and it is easy to verify that

$$\langle \mathcal{R}^s_A u, v\rangle = \langle \mathcal{R}^s_A v, u\rangle.$$

We remark that a different fractional magnetic operator can be found in [5] where the fractional gradient $\nabla^s : H^s(\mathbb{R}^n) \to L^2(\mathbb{R}^n \times \mathbb{R}^n)$ defined by

$$\nabla^s u(x,y) := c_{n,s}(u(x) - u(y))\frac{y-x}{|y-x|^{n/2+s+1}}$$

was considered. Based on the identity

$$\langle (-\Delta)^s u, v\rangle = \langle \nabla^s u, \nabla^s v\rangle,$$

the author defined the operator $(-\Delta)^s_{\mathcal{A}}$ by

$$\langle (-\Delta)^s_{\mathcal{A}} u, v\rangle := \langle (\nabla^s + \mathcal{A}(x,y))u, (\nabla^s + \mathcal{A}(x,y))v\rangle$$

for a bivariate vector-valued function $\mathcal{A}(x,y)$ and studied the associated inverse problem.

2.1.3 The Norm Equivalence

The following norm $||\cdot||_{H^s_A}$ was introduced in [10, 40].

Definition 2.1.1. *The magnetic Sobolev norm* $||\cdot||_{H^s_A}$ *is defined by*

$$||u||_{H^s_A} := (||u||^2_{L^2} + [u]^2_{H^s_A})^{1/2}$$

where the semi-norm part is given by

$$[u]_{H^s_A} := (\iint \frac{|u(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}u(y)|^2}{|x-y|^{n+2s}}\,dxdy)^{1/2}. \tag{2.3}$$

For $0 < s < 1$, it is well-known that one of the equivalent forms of the Sobolev norm $||\cdot||_{H^s}$ is given by

$$||u||_{H^s} := (||u||^2_{L^2} + \iint \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dxdy)^{1/2}.$$

In fact, we can show the equivalence between $||\cdot||_{H^s_A}$ and $||\cdot||_{H^s}$ (for complex-valued u) when A is bounded. The key estimate we will use is the following elementary inequality

$$|e^{i(x-y)\cdot A(\frac{x+y}{2})} - 1| \le C\min\{1, |x-y|\}.$$

Here the constant C depends on $||A||_{L^\infty}$.

Lemma 2.1.2. *Suppose* $0 < s < 1$ *and* $A \in L^\infty(\mathbb{R}^n)$*. Then* $||\cdot||_{H^s_A} \sim ||\cdot||_{H^s}$*.*

Proof. We only need to show that

$$|[u]_{H^s} - [u]_{H^s_A}| \leq C'||u||_{L^2}. \tag{2.4}$$

In fact, by using the identity

$$|a|^2 - |b|^2 = (a-b)\bar{a} + (\bar{a} - \bar{b})b,$$

we can write

$$|[u]^2_{H^s} - [u]^2_{H^s_A}| = |I_1 + I_2|$$

where

$$I_1 := \iint \frac{(E_A(x,y)-1)u(y)\overline{u(x)-u(y)}}{|x-y|^{n+2s}}\,dxdy,$$
$$I_2 := \iint \frac{\overline{(E_A(x,y)-1)u(y)}(u(x)-E_A(x,y)u(y))}{|x-y|^{n+2s}}\,dxdy,$$
$$E_A(x,y) := e^{i(x-y)\cdot A(\frac{x+y}{2})}. \tag{2.5}$$

By the Cauchy-Schwarz inequality we have

$$\begin{aligned}
|I_1| &\leq (\iint \frac{|(E_A(x,y)-1)u(y)|^2}{|x-y|^{n+2s}}\,dxdy)^{1/2}(\iint \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dxdy)^{1/2} \\
&= ((\iint_{\{|x-y|\leq 1\}} + \iint_{\{|x-y|\geq 1\}})\frac{|(E_A(x,y)-1)u(y)|^2}{|x-y|^{n+2s}}\,dxdy)^{1/2}[u]_{H^s} \\
&\leq (\iint_{\{|x-y|\leq 1\}} \frac{C^2|x-y|^2|u(y)|^2}{|x-y|^{n+2s}}\,dxdy + \iint_{\{|x-y|\geq 1\}} \frac{C^2|u(y)|^2}{|x-y|^{n+2s}}\,dxdy)^{1/2}[u]_{H^s} \\
&= C(\int |u(y)|^2 \int_{\{|x-y|\leq 1\}} \frac{dx}{|x-y|^{n+2s-2}}\,dy + \int |u(y)|^2 \int_{\{|x-y|\geq 1\}} \frac{dx}{|x-y|^{n+2s}}\,dy)^{1/2}[u]_{H^s} \\
&\leq C'||u||_{L^2}[u]_{H^s}.
\end{aligned}$$

Similarly we can show

$$|I_2| \leq C'||u||_{L^2}[u]_{H^s_A}.$$

Hence we have

$$|[u]^2_{H^s} - [u]^2_{H^s_A}| \leq C'||u||_{L^2}([u]_{H^s} + [u]_{H^s_A}),$$

which implies (2.4). □

The boundness of $\mathcal{R}_A^s$ follows from the Cauchy-Schwarz inequality and the lemma above.

Proposition 2.1.3. *Suppose* $0 < s < 1$ *and* $A \in L^\infty(\mathbb{R}^n)$. *Then*

$$\mathcal{R}_A^s : H^s(\mathbb{R}^n) \to H^{-s}(\mathbb{R}^n)$$

is linear and bounded.

2.2 The Forward Problem

In the rest of this chapter we always assume $A \in L^\infty(\mathbb{R}^n)$ and $q \in L^\infty(\Omega)$.

The bilinear form associated with $\mathcal{R}_A^s + q$ is given by

$$B_{A,q}[u, v] := \langle \mathcal{R}_A^s u, v\rangle + \int_\Omega quv, \qquad u, v \in H^s(\mathbb{R}^n). \tag{2.6}$$

Proposition 2.2.1. *Suppose* $c \leq q \in L^\infty(\Omega)$ *for some* $c > 0$. *Then the problem*

$$\begin{cases} (\mathcal{R}_A^s + q)u = 0 & \textit{in } \Omega \\ u = g & \textit{in } \Omega_e \end{cases} \tag{2.7}$$

has a unique (weak) solution $u_g \in H^s(\mathbb{R}^n)$ *for each* $g \in H^s(\mathbb{R}^n)$ *and the solution operator*

$$P_{A,q} : g \to u_g$$

is bounded on $H^s(\mathbb{R}^n)$.

Proof. The boundness of $\mathcal{R}_A^s$ implies the boundedness of $B_{A,q}$ and the norm equivalence implies that $B_{A,q}$ is coercive on $\tilde{H}^s(\Omega) \times \tilde{H}^s(\Omega)$.

By Lax-Milgram Theorem, there exists an invertible bounded linear map $f \to w_f$ from $(\tilde{H}^s(\Omega))^*$ to $\tilde{H}^s(\Omega)$ s.t. w_f satisfies

$$B_{A,q}[w, \phi] = f(\phi), \qquad \phi \in \tilde{H}^s(\Omega).$$

For each $g \in H^s(\mathbb{R}^n)$, let $f = -(\mathcal{R}_A^s + q)g$, then $u_g := w_f + g$ is the unique solution of (2.7) and the boundness of $P_{A,q}$ on $H^s(\mathbb{R}^n)$ is clear. $\square$

2.3 The Inverse Problem

In the rest of this chapter we always assume $c \leq q \in L^\infty(\Omega)$ for some $c > 0$.

2.3.1 The Dirichlet-to-Neumann Map

Let $X := H^s(\mathbb{R}^n)/\tilde{H}^s(\Omega) = H^s(\Omega_e)$ and $\tilde{g} :=$ the natural image of $g \in H^s(\mathbb{R}^n)$ in X.

Definition 2.3.1. *We define the Dirichlet-to-Neumann map* $\Lambda_{A,q}$ *by*

$$\langle \Lambda_{A,q}\tilde{g}, \tilde{h} \rangle := B_{A,q}[P_{A,q}g, h], \qquad g, h \in H^s(\mathbb{R}^n). \tag{2.8}$$

Note that if $g_2 - g_1 \in \tilde{H}^s(\Omega)$ and $h_2 - h_1 \in \tilde{H}^s(\Omega)$, then $u_{g_1} = u_{g_2}$ and

$$B_{A,q}[u_{g_2}, h_2] - B_{A,q}[u_{g_1}, h_1] = B_{A,q}[u_{g_2} - u_{g_1}, h_2] + B_{A,q}[u_{g_1}, h_2 - h_1] = 0$$

so $\Lambda_{A,q}$ is well-defined.

If g and h belong to the orthogonal complement of $\tilde{H}^s(\Omega)$ in $H^s(\mathbb{R}^n)$, then we have

$$|\langle \Lambda_{A,q}\tilde{g}, \tilde{h} \rangle| \leq C||u_g||_{H^s}||h||_{H^s} \leq C'||g||_{H^s}||h||_{H^s} = C'||\tilde{g}||_X||\tilde{h}||_X$$

so $\Lambda_{A,q} : X \to X^*$ is bounded.

For convenience, we will write $\Lambda_{A,q}g$ and $\langle \Lambda_{A,q}g, h\rangle$ instead of $\Lambda_{A,q}\tilde{g}$ and $\langle \Lambda_{A,q}\tilde{g}, \tilde{h}\rangle$.

We remark that the formal computation

$$\langle \Lambda_{A,q}g, h\rangle = \int (\mathcal{R}^s_A u_g)h + \int_\Omega q u_g h$$
$$= (\int_{\Omega_e} + \int_\Omega)(\mathcal{R}^s_A u_g)h + \int_\Omega q u_g h = \int_{\Omega_e} (\mathcal{R}^s_A u_g)h.$$

shows that

$$\Lambda_{A,q}g = \mathcal{R}^s_A u_g|_{\Omega_e}.$$

We prove an integral identity which will be used later.

Note that by the symmetry of $B_{A,q}$, we have

$$\langle \Lambda_{A,q}g, h\rangle = B_{A,q}[P_{A,q}g, P_{A,q}h] = B_{A,q}[P_{A,q}h, P_{A,q}g] = \langle \Lambda_{A,q}h, g\rangle.$$

Hence for $g_j \in H^s(\mathbb{R}^n)$ and $u_j = P_{A_j,q_j} g_j$ $(j = 1, 2)$, we have

$$\begin{aligned}
&\langle(\Lambda_{A_1,q_1} - \Lambda_{A_2,q_2})g_1, g_2\rangle = \langle\Lambda_{A_1,q_1} g_1, g_2\rangle - \langle\Lambda_{A_2,q_2} g_2, g_1\rangle \\
&= \langle\mathcal{R}^s_{A_1} u_1, u_2\rangle + \int_\Omega q_1 u_1 u_2 - \langle\mathcal{R}^s_{A_2} u_2, u_1\rangle - \int_\Omega q_2 u_2 u_1 \\
&= \langle\mathcal{R}^s_{A_1} u_1, u_2\rangle - \langle\mathcal{R}^s_{A_2} u_1, u_2\rangle - \int_\Omega (q_2 - q_1) u_1 u_2 \\
&= 2\iint (R_{A_2}(x,y) - R_{A_1}(x,y)) u_1(y) u_2(x) K(x,y)\, dxdy - \int_\Omega (q_2 - q_1) u_1 u_2. \qquad (2.9)
\end{aligned}$$

2.3.2 The Runge Approximation

The following proposition was proved in [14]. It generalizes Proposition 1.2.2.

Proposition 2.3.2. *Let* $0 < s < 1$ *and* $u \in H^s(\mathbb{R}^n)$. *Let* W *be open and non-empty. If*

$$\mathcal{R}^s_0 u = u = 0 \quad \text{in } W,$$

then $u = 0$ *in* $\mathbb{R}^n$.

It is not clear whether the unique continuation property above holds for general $\mathcal{R}^s_A$. Fortunately, we have the following observation which enables us to show the Runge approximation property of $\mathcal{R}^s_A + q$ by using the unique continuation property of $\mathcal{R}^s_0$.

Lemma 2.3.3. *Suppose* $\operatorname{supp} A \subset \Omega \subset B_r(0)$ *for some* $r > 0$, W *is a non-empty open set s.t.* $W \cap B_{3r}(0) = \emptyset$. *Then we have*

$$\mathcal{R}^s_0 u|_W = \mathcal{R}^s_A u|_W, \qquad u \in \tilde{H}^s(\Omega).$$

Proof. Let $u \in C^\infty_c(\Omega)$ and $v \in C^\infty_c(W)$. Note that

$$\langle(\mathcal{R}^s_0 - \mathcal{R}^s_A)u, v\rangle = 2\iint (R_A(x,y) - 1) u(y) v(x) K(x,y)\, dxdy. \qquad (2.10)$$

If $x \notin W$, then $v(x) = 0$; If $y \notin \Omega$, then $u(y) = 0$; If $x \in W$ and $y \in \Omega$, then

$$\left|\frac{x+y}{2}\right| \geq \frac{|x| - |y|}{2} \geq \frac{3r - r}{2} = r,$$

which implies $R_A(x,y) = 1$ in this case. Hence the integrand in (2.10) is always zero. □

We prove the following Runge approximation property which will be used later.

Proposition 2.3.4. *Suppose* $\operatorname{supp} A \subset \Omega \subset B_r(0)$ *for some* $r > 0$*,* W *is a non-empty open set s.t.* $W \subset \Omega_e$ *and* $W \cap B_{3r}(0) = \emptyset$*. Then*

$$S := \{P_{A,q}f|_\Omega : f \in C_c^\infty(W)\}$$

is dense in $L^2(\Omega)$*.*

Proof. By the Hahn-Banach Theorem, it suffices to show that:

If $v \in L^2(\Omega)$ and $\int_\Omega vw = 0$ for all $w \in S$, then $v = 0$ in Ω.

In fact, for any given $v \in L^2(\Omega)$, let ϕ be the unique weak solution of

$$\begin{cases} (\mathcal{R}_A^s + q)\phi = v & \text{in } \Omega \\ \phi = 0 & \text{in } \Omega_e, \end{cases} \tag{2.11}$$

then for any $f \in C_c^\infty(W)$, we have

$$\int_\Omega vP_{A,q}f = \langle v, P_{A,q}f - f\rangle = B_{A,q}[\phi, P_{A,q}f - f] = B_{A,q}[(P_{A,q}f - f), \phi]$$

since $P_{A,q}f - f \in \tilde{H}^s(\Omega)$. Also note that

$$B_{A,q}[P_{A,q}f, \phi] = 0$$

since $P_{A,q}$ is the solution operator and $\phi \in \tilde{H}^s(\Omega)$, so we have

$$\int_\Omega vP_{A,q}f = -\langle \mathcal{R}_A^s f, \phi\rangle = -\langle \mathcal{R}_A^s \phi, f\rangle.$$

Hence, if $v \in L^2(\Omega)$ and $\int_\Omega vw = 0$ for all $w \in S$, then the corresponding ϕ satisfies

$$\phi \in \tilde{H}^s(\Omega), \qquad \mathcal{R}_A^s \phi|_W = 0.$$

This implies $\phi = 0$ in $\mathbb{R}^n$ by Proposition 2.3.2 and Lemma 2.3.3 and thus $v = 0$ in Ω. □

2.3.3 The Main Result

The inverse problem associated with (2.7) is whether both the magnetic potential A and the electric potential q can be determined from partial exterior measurements of the Dirichlet-to-Neumann map $\Lambda_{A,q}$. The following theorem (Theorem 1.1 in [29]) is our main result in this chapter.

Theorem 2.3.5. *Suppose* $\operatorname{supp} A_j \subset \Omega \subset B_r(0)$ *for some constant* $r > 0$, $c \leq q_j \in L^\infty(\Omega)$ *for some* $c > 0$, $A_j \in L^\infty(\mathbb{R}^n)$, W_j *are open sets s.t.* $W_j \cap B_{3r}(0) = \emptyset$ *(*$j = 1, 2$*). Let*

$$W^{(1,2)} := \{\frac{x+y}{2} : x \in W_1, y \in W_2\}.$$

Also assume $W^{(1,2)} \setminus (\operatorname{supp} A_1 \cup \operatorname{supp} A_2) \neq \emptyset$*. If*

$$\Lambda_{A_1,q_1} g|_{W_2} = \Lambda_{A_2,q_2} g|_{W_2} \tag{2.12}$$

for any $g \in C_c^\infty(W_1)$*, then* $A_1 = \pm A_2$ *and* $q_1 = q_2$*.*

It is remarkable that A can only be determined up to a gauge equivalence in the classical magnetic Calderón problem while it can be totally determined (up to the sign) in this fractional inverse problem.

The assumption on the set $W^{(1,2)}$ seems unnatural but it is essential when we deal with the integral identity (2.9) to prove our main theorem.

In fact, for $g_j \in C_c^\infty(W_j)$ and $u_j := P_{A_j,q_j} g_j$ $(j = 1, 2)$, by (2.12) and (2.9) we have

$$\iint G(x,y) u_1(y) u_2(x)\, dxdy = \int_\Omega (q_2 - q_1) u_1 u_2$$

where we write

$$G(x,y) := 2(R_{A_2}(x,y) - R_{A_1}(x,y)) K(x,y).$$

Note that $\operatorname{supp} u_j \subset \Omega \cup W_j$ so the double integral on the left hand side is

$$\int_{\Omega \cup W_2} \int_{\Omega \cup W_1} G(x,y) u_1(y) u_2(x)\, dxdy = I_1 + I_2 + I_3 + I_4$$

where we write

$$I_1 := \int_\Omega \int_\Omega, \quad I_2 := \int_\Omega \int_{W_1}, \quad I_3 := \int_{W_2} \int_\Omega, \quad I_4 := \int_{W_2} \int_{W_1}.$$

Note that $(x, y) \in W_2 \times \Omega$ (or $(x, y) \in \Omega \times W_1$) implies $(x + y)/2 \geq r$. In this case $R_{A_1}(x, y) = R_{A_2}(x, y) = 1$, $G(x, y) = 0$ so $I_2 = I_3 = 0$.

Also note that by the assumption on $W^{(1,2)}$, we can choose $x_0 \in W_2, y_0 \in W_1$ s.t. $\frac{x_0+y_0}{2} \notin \operatorname{supp} A_j$ so $(x, y) \in W_2 \times W_1$ implies $(x + y)/2 \notin \operatorname{supp} A_j$ if we replace W_2, W_1 by small open balls centered at x_0 and y_0 when necessary. Hence $I_4 = 0$ so we have

$$\int_\Omega \int_\Omega G(x, y) u_1(y) u_2(x)\, dxdy = \int_\Omega (q_2 - q_1) u_1 u_2. \tag{2.13}$$

Based on the framework given in [16], we will make a more careful use of the Runge approximation property to finish our proof. We will first determine A and then determine q.

Proof. **Determine** A: We fix open sets $\Omega_{1,2} \subset \Omega$ s.t. $\Omega_1 \cap \Omega_2 = \emptyset$. We also fix $\phi_j \in C_c^\infty(\Omega_j)$ $(j = 1, 2)$ and a small constant $\epsilon > 0$. By Proposition 2.3.4, we can choose $g_1 \in C_c^\infty(W_1)$ s.t.

$$||u_1 - \phi_1||_{L^2(\Omega)} \leq \epsilon$$

and for this chosen g_1, we can choose $g_2 \in C_c^\infty(W_2)$ s.t.

$$||u_1||_{L^2(\Omega)} ||u_2 - \phi_2||_{L^2(\Omega)} \leq \epsilon.$$

Note that $\phi_1(x)\phi_2(x) = 0$ for $x \in \Omega$ so

$$|\int_\Omega (q_2 - q_1) u_1 u_2| = |\int_\Omega (q_2 - q_1)(u_1 - \phi_1)\phi_2 + \int_\Omega (q_2 - q_1) u_1 (u_2 - \phi_2)|$$
$$\leq ||(q_2 - q_1)||_{L^\infty} ||\phi_2||_{L^2} ||u_1 - \phi_1||_{L^2} + ||(q_2 - q_1)||_{L^\infty} ||u_1||_{L^2} ||u_2 - \phi_2||_{L^2} \leq C\epsilon. \tag{2.14}$$

Also note that

$$|G(x, y)| \leq 4|\sin(\frac{x - y}{2} \cdot (A_1 - A_2)(\frac{x + y}{2})) \sin(\frac{x - y}{2} \cdot (A_1 + A_2)(\frac{x + y}{2}))| K(x, y)$$
$$\leq C_A |x - y|^2 K(x, y) \leq \frac{C}{|x - y|^{n+2s-2}},$$

which implies

$$\int_{\Omega} |G(x,y)|dy \le C_0,\ x \in \Omega; \qquad \int_{\Omega} |G(x,y)|dx \le C_0,\ y \in \Omega.$$

By the generalized Young's Inequality (see Proposition 0.10 on page 9 in [12]),

$$||Tf||_{L^2(\Omega)} \le C_0 ||f||_{L^2(\Omega)}, \qquad (Tf)(x) := \int_{\Omega} |G(x,y)f(y)|\, dy$$

so we have

$$\begin{aligned} &|\int_{\Omega}\int_{\Omega} G(x,y)u_1(y)u_2(x)\, dxdy - \int_{\Omega_1}\int_{\Omega_2} G(x,y)\phi_1(y)\phi_2(x)\, dxdy| \\ = {} & |\int_{\Omega}\int_{\Omega} [G(x,y)(u_1(y)-\phi_1(y))\phi_2(x) + G(x,y)u_1(y)(u_2(x)-\phi_2(x))]\, dxdy| \\ & \le \int_{\Omega}\int_{\Omega} |G(x,y)\phi_2(x)|\, dx |u_1(y)-\phi_1(y)|\, dy \\ & + \int_{\Omega}\int_{\Omega} |G(x,y)u_1(y)|\, dy |u_2(x)-\phi_2(x)|\, dx \\ & \le C_0 ||\phi_2||_{L^2} ||u_1-\phi_1||_{L^2} + C_0 ||u_1||_{L^2} ||u_2-\phi_2||_{L^2} \le C'\epsilon. \end{aligned} \tag{2.15}$$

We combine (2.14), (2.15) with (2.13). ϵ is arbitrary, implying that

$$\int_{\Omega_1}\int_{\Omega_2} G(x,y)\phi_1(y)\phi_2(x)\, dxdy = 0.$$

Note that the set

$$\{\phi_1 \otimes \phi_2 :\ \phi_j \in C_c^{\infty}(\Omega_j),\ j = 1,2\}$$

generates a space dense in $C_c^{\infty}(\Omega_1 \times \Omega_2)$ so $G(x,y) = 0$ in $\Omega_1 \times \Omega_2$. Ω_1, Ω_2 are arbitrary, implying that $G(x,y) = 0$ for $x, y \in \Omega$ whenever $x \neq y$ so

$$R_{A_1}(x,y) = R_{A_2}(x,y), \quad x, y \in \Omega.$$

Now we fix $x_0 \in \Omega$. Let $A^{(k)}$ denote the k^{th} component of A and let $\{e_k\}_{k=1}^n$ denote the standard basis of the vector space $\mathbb{R}^n$. Consider $x = x_0 + \epsilon e_k$ and $y = x_0 - \epsilon e_k$ for small $\epsilon > 0$. Since $|2\epsilon A_j^{(k)}(x_0)| < \frac{\pi}{2}$, $R_{A_1}(x,y) = R_{A_2}(x,y)$ implies that $|A_1^{(k)}(x_0)| = |A_2^{(k)}(x_0)|$.

Suppose there exist $l \neq k$ s.t. $A_1^{(k)}(x_0) = A_2^{(k)}(x_0) \neq 0$ and $A_1^{(l)}(x_0) = -A_2^{(l)}(x_0) \neq 0$. Consider $x = x_0 + \epsilon(e_k + e_l)$ and $y = x_0 - \epsilon(e_k + e_l)$. Then we have

$$(x-y) \cdot A_j(\frac{x+y}{2}) = 2\epsilon(A_j^{(k)}(x_0) + A_j^{(l)}(x_0)), \quad j = 1, 2.$$

This contradicts with $R_{A_1}(x,y) = R_{A_2}(x,y)$. Hence the only possibility is $A_1(x_0) = \pm A_2(x_0)$. x_0 is arbitrary, implying that $A_1 = \pm A_2$.

Determine q: Now (2.13) has become

$$\int_\Omega (q_2 - q_1)u_1u_2 = 0.$$

Fixing $\epsilon > 0$ and $f \in L^2(\Omega)$, by Proposition 2.3.4 we can choose $g_1 \in C_c^\infty(W_1)$ s.t.

$$||u_1 - f||_{L^2(\Omega)} \leq \epsilon$$

and for this chosen u_1, we can choose $g_2 \in C_c^\infty(W_2)$ s.t.

$$||u_1||_{L^2(\Omega)}||u_2 - 1||_{L^2(\Omega)} \leq \epsilon.$$

Now we have

$$|\int_\Omega (q_1 - q_2)f| = |\int_\Omega (q_1 - q_2)(f - u_1) + \int_\Omega (q_1 - q_2)u_1(1 - u_2)| \leq C\epsilon.$$

We conclude that $q_1 = q_2$ since ϵ, f are arbitrary. $\square$

Chapter 3

THE FRACTIONAL MAGNETIC PARABOLIC PROBLEM

Throughout this chapter, we fix the constant $T > 0$ and we use t to denote the time variable. For an $(n+1)$-variable function u, $u(t)$ denotes the n-variable function $u(\cdot, t)$.

3.1 Preliminaries

This section is a supplement to its counterpart in the previous chapter.

3.1.1 Function Spaces

Let X be a Banach space. For $m \in \mathbb{N}$, we use $C^m([-T,T];X)$ (resp. $AC([-T,T];X)$) to denote the space consisting of the corresponding Banach space-valued continuously differentiable (resp. absolutely continuous) functions on $[-T,T]$.

$L^p(-T,T;X)$ denotes the space consisting of the corresponding Banach space-valued L^p functions, equipped with the standard norm

$$||u||_{L^p(-T,T;X)} := (\int_{-T}^{T} ||u(t)||_X^p \, dt)^{1/p}.$$

3.1.2 The Time-dependent Operator $\mathcal{R}^s_{A(t)}$

We still focus on the fractional magnetic operator (2.1) but now we allow the magnetic potential A to be time-dependent.

The following lemma is a time-dependent version of Lemma 2.1.2.

Lemma 3.1.1. *Suppose $A \in C([-T,T], L^\infty(\mathbb{R}^n))$. Then for $u \in H^s(\mathbb{R}^n)$, we have*

$$c||u||_{H^s} \le ||u||_{H^s_{A(t)}} \le C||u||_{H^s}$$

where the constants c, C depend on $\sup_{t\in[-T,T]} ||A(t)||_{L^\infty}$ but do not depend on t.

We also allow the electric potential q to be time-dependent and we define the time-dependent bilinear form associated with A, q by

$$B_t[u,v] := \langle \mathcal{R}^s_{A(t)}u, v\rangle + \int_\Omega q(t)uv, \qquad t \in [-T,T]. \tag{3.1}$$

The following estimates will be used in the next section.

Lemma 3.1.2. *Suppose $A \in C^2([-T,T]; L^\infty(\mathbb{R}^n))$, $q \in C^2([-T,T]; L^\infty(\Omega))$ and $q \geq c'$ in $\Omega \times [-T,T]$ for some constant $c' > 0$. Then*

$$|B_t[u,v]| \leq C_0 ||u||_{H^s}||v||_{H^s}, \qquad u, v \in H^s(\mathbb{R}^n) \tag{3.2}$$

and for $u, v \in \tilde{H}^s(\Omega)$, we have

$$|B_t[u,u]| \geq c_0 ||u||^2_{H^s}, \tag{3.3}$$

$$|B_{t+h}[u,v] - B_t[u,v]| \leq C_1 h ||u||_{H^s}||v||_{H^s}, \tag{3.4}$$

$$|B_{t+h}[u,v] + B_{t-h}[u,v] - 2B_t[u,v]| \leq C_2 h^2 ||u||_{H^s}||v||_{H^s} \tag{3.5}$$

where the constants c_0, C_0, C_1, C_2 do not depend on u, v, t and $h > 0$.

Proof. (3.2) and (3.3) follow from Lemma 3.1.1 immediately.

Note that for $u, v \in \tilde{H}^s(\Omega)$, we have

$$\frac{d}{dt}\langle \mathcal{R}^s_{A(t)}u, v\rangle = -2\iint \frac{d}{dt}R_{A(t)}(x,y)u(y)v(x)K(x,y)\,dxdy$$

$$= -2\int_\Omega\int_\Omega \frac{d}{dt}R_{A(t)}(x,y)u(y)v(x)K(x,y)\,dxdy$$

$$= 2\int_\Omega\int_\Omega \sin((x-y)\cdot A(\frac{x+y}{2},t))(x-y)\cdot\partial_t A(\frac{x+y}{2},t)u(y)v(x)K(x,y)\,dxdy$$

so by the Cauchy-Schwarz inequality we have

$$|\frac{d}{dt}\langle \mathcal{R}^s_{A(t)}u, v\rangle| \leq C\int_\Omega\int_\Omega \frac{|u(y)v(x)|}{|x-y|^{n+2s-2}}\,dxdy$$

$$\leq C(\int_\Omega\int_\Omega \frac{|u(y)|^2}{|x-y|^{n+2s-2}}\,dxdy)^{\frac{1}{2}}(\int_\Omega\int_\Omega \frac{|v(x)|^2}{|x-y|^{n+2s-2}}\,dxdy)^{\frac{1}{2}}.$$

Note that

$$\int_\Omega \int_\Omega \frac{|u(y)|^2}{|x-y|^{n+2s-2}}\, dxdy = \int_\Omega |u(y)|^2 (\int_\Omega \frac{dx}{|x-y|^{n+2s-2}})\, dy \leq C' ||u||^2_{L^2}$$

and similarly

$$\int_\Omega \int_\Omega \frac{|v(x)|^2}{|x-y|^{n+2s-2}}\, dxdy \leq C' ||v||^2_{L^2}$$

so we have

$$|\frac{d}{dt} \langle \mathcal{R}^s_{A(t)} u, v\rangle| \leq C'' ||u||_{L^2} ||v||_{L^2} \leq C'' ||u||_{H^s} ||v||_{H^s},$$

which implies that (3.4) holds.

Also note that

$$\begin{aligned} \frac{d^2}{dt^2} R_{A(t)}(x,y) = & -\sin((x-y)\cdot A(\frac{x+y}{2},t))(x-y)\cdot \partial_{tt} A(\frac{x+y}{2},t) \\ & -\cos((x-y)\cdot A(\frac{x+y}{2},t))((x-y)\cdot \partial_t A(\frac{x+y}{2},t))^2 \end{aligned}$$

so we have

$$|\frac{d^2}{dt^2} R_{A(t)}(x,y)| \leq C''' |x-y|^2$$

and we can similarly show that

$$|\frac{d^2}{dt^2} \langle \mathcal{R}^s_{A(t)} u, v\rangle| \leq C'''' ||u||_{H^s} ||v||_{H^s}, \qquad u, v \in \tilde{H}^s(\Omega),$$

which implies that (3.5) holds. □

3.2 The Forward Problem

In the rest of this chapter, we always assume $q \in C^2([-T,T]; L^\infty(\Omega))$ and $q \geq c'$ in $\Omega \times [-T,T]$ for some constant $c' > 0$. We also always assume $\operatorname{supp} A(t) \subset \Omega$ for $t \in [-T,T]$, $A \in C^2([-T,T]; L^\infty(\mathbb{R}^n))$.

3.2.1 Discretization in Time

We first study the initial value problem

$$\begin{cases} \partial_t u + \mathcal{R}^s_{A(t)} u + q(t) u = f & \text{in } \Omega \times (-T,T) \\ u = 0 & \text{in } \Omega \times \{-T\}. \end{cases} \tag{3.6}$$

Proposition 3.2.1. *Suppose* $f \in L^2(\Omega \times (-T,T))$ *and*

$$||f(t+h) - f(t)||_{L^2(\Omega)} \leq Ch \tag{3.7}$$

for some C *independent of* t, h. *Then (3.6) has a unique (weak) solution satisfying*

$$u \in L^2(-T,T;\tilde{H}^s(\Omega)) \cap AC([-T,T];L^2(\Omega)), \quad \partial_t u \in L^2(\Omega \times (-T,T)).$$

We will use the method of discretization in time to show the existence of solutions, which enables us to relate the parabolic problem with the elliptic one. This method can be viewed as a nonlocal analogue of the Rothe's method for classical parabolic problems (see for instance, Chapter 15 in [35]).

Proof. **Existence**: We first divide $[-T,T]$ into p subintervals of length $h = 2T/p < \frac{1}{2}$. Let $t_j = -T + jh$.

Consider the discretization in t

$$\begin{cases} \dfrac{z_j - z_{j-1}}{h} + \mathcal{R}^s_{A(t_j)} z_j + q(t_j) z_j = f(t_j) \quad j = 1, \cdots, p \\ z_0 = 0. \end{cases} \tag{3.8}$$

We can iteratively determine $z_j \in \tilde{H}^s(\Omega)$, which solves the elliptic equation

$$\mathcal{R}^s_{A(t_j)} z_j + (q(t_j) + \frac{1}{h}) z_j = f(t_j) + \frac{z_{j-1}}{h}.$$

((3.2) and (3.3) ensure the existence and uniqueness of z_j by the Lax-Milgram Theorem.) We define $Z_j := (z_j - z_{j-1})/h$ and $u^{(1)} : [-T,T] \to \tilde{H}^s(\Omega)$ given by

$$u^{(1)}(t) := z_{j-1} + Z_j(t - t_{j-1}), \qquad t \in [t_{j-1}, t_j].$$

In general, we divide $[-T,T]$ into $2^{m-1}p$ subintervals of length $h_m = 2T/(2^{m-1}p)$. Let $t_j^{(m)} = -T + jh_m$. We similarly consider the discretization and obtain a sequence

$$\{z_0^{(m)} = 0, \cdots, z^{(m)}_{2^{m-1}p}\}$$

in $\tilde{H}^s(\Omega)$. We define $Z_j^{(m)} := (z_j^{(m)} - z_{j-1}^{(m)})/h_m$ and

$$u^{(m)}(t) := z_{j-1}^{(m)} + Z_j^{(m)}(t - t_{j-1}^{(m)}), \qquad t \in [t_{j-1}^{(m)}, t_j^{(m)}].$$

We also define the step functions

$$\tilde{u}^{(m)}(t) := z_j^{(m)}, \quad \tilde{U}^{(m)}(t) := Z_j^{(m)}, \qquad t \in (t_{j-1}^{(m)}, t_j^{(m)}).$$

In fact, we can show that

$$||z_j^{(m)}||_{H^s} \leq c_2, \qquad ||Z_j^{(m)}||_{L^2} \leq c_3$$

where the constants c_2, c_3 do not depend on the small interval length h_m. See Lemma A.2 in [30] for more details. This implies the boundedness of $\{u^{(m)}\}$ in $L^2(-T, T; \tilde{H}^s(\Omega))$ and the boundedness of $\{\tilde{U}^{(m)}\}$ in $L^2(\Omega \times (-T, T))$. Hence we can choose weakly convergent sequences s.t.

$$u^{(m_k)} \rightharpoonup u \quad \text{in } L^2(-T, T; \tilde{H}^s(\Omega)), \quad \tilde{U}^{(m_k)} \rightharpoonup \tilde{U} \quad \text{in } L^2(\Omega \times (-T, T)).$$

Note that $\tilde{U}^{(m)}$ is the weak derivative as well as the pointwise derivative of $u^{(m)}$ so we can let $k \to \infty$ in

$$u^{(m_k)}(t) = \int_{-T}^{t} \tilde{U}^{(m_k)}(\tau)\, d\tau$$

to obtain

$$u(t) = \int_{-T}^{t} \tilde{U}(\tau)\, d\tau.$$

Hence u is absolutely continuous in t, $u(-T) = 0$ and $\partial_t u(t) = \tilde{U}(t)$.

Now we show that this u satisfies the equation in (3.6).

We define the step function

$$\tilde{f}^{(m)}(t) := f(t_j^{(m)}), \qquad t \in (t_{j-1}^{(m)}, t_j^{(m)})$$

and the step bilinear form

$$B_t^{(m)}[\cdot, \cdot] := B_{t_j^{(m)}}[\cdot, \cdot], \qquad t \in (t_{j-1}^{(m)}, t_j^{(m)}).$$

Fixing $v \in L^2(-T,T;\tilde{H}^s(\Omega))$, we let both sides of the discretized equation

$$Z_j^{(m)} + \mathcal{R}^s_{A(t_j^{(m)})} z_j^{(m)} + q(t_j^{(m)}) z_j^{(m)} = f(t_j^{(m)})$$

act on $v(t)$ for $t \in (t_{j-1}^{(m)}, t_j^{(m)})$ and integrate from $-T$ to T. Then we have

$$\int_{-T}^{T} \langle \tilde{U}^{(m)}(t), v(t)\rangle\, dt + \int_{-T}^{T} B_t^{(m)}[\tilde{u}^{(m)}(t), v(t)]\, dt = \int_{-T}^{T} \langle \tilde{f}^{(m)}(t), v(t)\rangle\, dt. \tag{3.9}$$

(3.7) ensures that

$$\int_{-T}^{T} \langle \tilde{f}^{(m)}(t), v(t)\rangle\, dt \to \int_{-T}^{T} \langle f(t), v(t)\rangle\, dt. \tag{3.10}$$

The weak convergence of $\tilde{U}^{(m_k)}$ implies that

$$\int_{-T}^{T} \langle \tilde{U}^{(m_k)}(t), v(t)\rangle\, dt \to \int_{-T}^{T} \langle \partial_t u(t), v(t)\rangle\, dt. \tag{3.11}$$

Note that (3.2) ensures that

$$\int_{-T}^{T} B_t[\cdot, v(t)]\, dt$$

is a bounded linear functional on $L^2(-T,T;\tilde{H}^s(\Omega))$ and we can show that

$$\tilde{u}^{(m_k)} \rightharpoonup u \quad \text{in } L^2(-T,T;\tilde{H}^s(\Omega)).$$

See Lemma A.3 in [30] for more details. Hence we have

$$\int_{-T}^{T} B_t[\tilde{u}^{(m_k)}(t), v(t)]\, dt \to \int_{-T}^{T} B_t[u(t), v(t)]\, dt. \tag{3.12}$$

Also we can show that

$$\int_{-T}^{T} B_t^{(m_k)}[\tilde{u}^{(m_k)}(t), v(t)]\, dt - \int_{-T}^{T} B_t[\tilde{u}^{(m_k)}(t), v(t)]\, dt \to 0. \tag{3.13}$$

In fact, we first assume $v(t) = 1_{[\alpha,\beta]}(t) v_0$ where $v_0 \in \tilde{H}^s(\Omega)$ and α, β are endpoints of subintervals in the m_k-division for some k. For each large m_k, we write $\alpha = t_{j_1}^{(m_k)}, \beta = t_{j_2}^{(m_k)}$ for some j_1, j_2. Then we have

$$|\int_{-T}^{T} B_t^{(m_k)}[\tilde{u}^{(m_k)}(t), v(t)]\, dt - \int_{-T}^{T} B_t[\tilde{u}^{(m_k)}(t), v(t)]\, dt|$$

$$
\begin{aligned}
&= \Big| \sum_{j=j_1+1}^{j_2} \int_{t_{j-1}^{(m_k)}}^{t_j^{(m_k)}} B_{t_j^{(m_k)}}[z_j^{(m_k)}, v_0] - B_t[z_j^{(m_k)}, v_0]\, dt \Big| \\
&\leq \sum_{j=j_1+1}^{j_2} \int_{t_{j-1}^{(m_k)}}^{t_j^{(m_k)}} C_1 ||z_j^{(m_k)}||_{H^s} ||v_0||_{H^s} h_{m_k}\, dt \\
&\leq 2TC_1 c_2 h_{m_k} ||v_0||_{H^s}
\end{aligned}
$$

by using (3.4) and the boundedness of $\{z_j^{(m)}\}$.

Since the set consisting of such v spans a space dense in $L^2(-T, T; \tilde{H}^s(\Omega))$, we know that (3.13) holds for all $v \in L^2(-T, T; \tilde{H}^s(\Omega))$.

Combining (3.10), (3.11), (3.12), (3.13) with (3.9), we conclude that

$$
\int_{-T}^{T} \langle \partial_t u(t), v(t) \rangle\, dt + \int_{-T}^{T} B_t[u(t), v(t)]\, dt = \int_{-T}^{T} \langle f(t), v(t) \rangle\, dt. \tag{3.14}
$$

Uniqueness: Let v be the solution u constructed in the existence part.

Then (3.14) becomes

$$
\frac{1}{2} ||u(T)||^2_{L^2(\Omega)} + \int_{-T}^{T} B_t[u(t), u(t)]\, dt = \int_{-T}^{T} \langle f(t), u(t) \rangle\, dt,
$$

which implies

$$
||u||_{L^2(-T,T;\tilde{H}^s(\Omega))} \leq C' ||f||_{L^2(-T,T;H^{-s}(\Omega))}.
$$

By (3.6) and (3.2),

$$
\begin{aligned}
||\partial_t u||_{L^2(-T,T;H^{-s}(\Omega))} &\leq ||f||_{L^2(-T,T;H^{-s}(\Omega))} + ||\mathcal{R}_A^s u + qu||_{L^2(-T,T;H^{-s}(\Omega))} \\
&\leq ||f||_{L^2(-T,T;H^{-s}(\Omega))} + C'' ||u||_{L^2(-T,T;\tilde{H}^s(\Omega))}.
\end{aligned}
$$

Hence we have

$$
||u||_{L^2(-T,T;\tilde{H}^s(\Omega))} + ||\partial_t u||_{L^2(-T,T;H^{-s}(\Omega))} \leq C ||f||_{L^2(-T,T;H^{-s}(\Omega))}. \tag{3.15}
$$

The uniqueness is clear when we let $f = 0$. □

3.2.2 The Initial Exterior Problem

Now we consider (3.6) for general $f \in L^2(\Omega \times (-T, T))$.

In fact, we can choose f_m satisfying (3.7) s.t. $f_m \to f$ in $L^2(\Omega \times (-T, T))$. For each f_m, let u_m be the corresponding solution of (3.6). Then

$$||u_m - u_l||_{L^2(-T,T;\tilde{H}^s(\Omega))} + ||\partial_t(u_m - u_l)||_{L^2(-T,T;H^{-s}(\Omega))} \leq ||f_m - f_l||_{L^2(-T,T;H^{-s}(\Omega))}$$

so we have

$$u_m \to u \quad \text{in } L^2(-T, T; \tilde{H}^s(\Omega)), \quad \partial_t u_m \to v \quad \text{in } L^2(-T, T; H^{-s}(\Omega))$$

for some u, v and $\partial_t u = v$. This implies the convergence of $\{u_m\}$ in $C([-T, T]; L^2(\Omega))$ (see, for instance, Theorem 1 in Section 1.2 in Chapter 18 in [7]) and u satisfies (3.15).

Hence we reach the following conclusion.

Corollary 3.2.2. *Suppose $f \in L^2(\Omega \times (-T, T))$. Then (3.6) has a unique solution satisfying*

$$u \in L^2(-T, T; \tilde{H}^s(\Omega)) \cap C([-T, T]; L^2(\Omega)), \quad \partial_t u \in L^2(-T, T; H^{-s}(\Omega)).$$

In the rest of this chapter we always assume $\Omega \subset B_r(0)$ for some constant $r > 0$ and W is an open set in $\mathbb{R}^n$ s.t. $W \cap B_{3r}(0) = \emptyset$.

Proposition 3.2.3. *Suppose $g \in C_c^\infty(W \times (-T, T))$. Then the initial exterior problem*

$$\begin{cases} \partial_t u + \mathcal{R}^s_{A(t)} u + q(t)u = 0 & \text{in } \Omega \times (-T, T) \\ u = g & \text{in } \Omega_e \times (-T, T) \\ u = 0 & \text{in } \mathbb{R}^n \times \{-T\} \end{cases} \tag{3.16}$$

has a unique solution u satisfying

$$w \in L^2(-T, T; \tilde{H}^s(\Omega)) \cap C([-T, T]; L^2(\Omega)), \quad \partial_t w \in L^2(-T, T; H^{-s}(\Omega))$$

where $w := u - g$. We denote the associated solution operator $g \to u_g$ by $P_{A,q}$.

Proof. By using arguments similar to the ones in Lemma 2.3.3, we can show that

$$\mathcal{R}^s_A g|_{\Omega\times(-T,T)} = \mathcal{R}^s_0 g|_{\Omega\times(-T,T)}.$$

This ensures that $-\mathcal{R}^s_A g|_{\Omega\times(-T,T)} =: f \in L^2(\Omega\times(-T,T))$. Then we consider the problem

$$\begin{cases} \partial_t w + \mathcal{R}^s_{A(t)} w + q(t)w = f & \text{in } \Omega\times(-T,T) \\ w = 0 & \text{in } \Omega\times\{-T\} \end{cases} \tag{3.17}$$

and apply the corollary above to complete the proof. ☐

By a change of variables we know that the proposition above holds for the dual problem

$$\begin{cases} -\partial_t u + \mathcal{R}^s_{A(t)} u + q(t)u = 0 & \text{in } \Omega\times(-T,T) \\ u = g & \text{in } \Omega_e\times(-T,T) \\ u = 0 & \text{in } \mathbb{R}^n\times\{T\} \end{cases} \tag{3.18}$$

and we denote the associated solution operator by $P^*_{A,q}$.

3.3 The Inverse Problem

We will see that the inverse problem for our fractional parabolic operator shares many common features with its elliptic counterpart. One noticeable difference is that we need to consider (3.16) together with its dual problem (3.18) in dealing with the fractional parabolic inverse problem.

We remark that a different fractional parabolic operator

$$(\partial_t - \Delta)^s + q$$

and its associated inverse problem have been studied in [25].

3.3.1 The Dirichlet-to-Neumann Map

Proposition 3.2.3 ensures that the Dirichlet-to-Neumann map $\Lambda_{A,q}$ given by

$$\Lambda_{A,q} g := \mathcal{R}^s_A(P_{A,q} g)|_{\Omega_e\times(-T,T)} \tag{3.19}$$

is well-defined at least for $g \in C_c^\infty(W \times (-T,T))$.

Let W_j be open in $\mathbb{R}^n$ and assume $W_j \cap B_{3r}(0) = \emptyset$ $(j = 1, 2)$. Let $g \in C_c^\infty(W_1 \times (-T,T))$ and $h \in C_c^\infty(W_2 \times (-T,T))$. By the definition of $P_{A,q}$ we have

$$\int_{-T}^{T} \langle \Lambda_{A,q} g(t), h(t)\rangle \, dt = \int_{-T}^{T} B_t[u(t), \tilde{h}(t)] \, dt + \int_{-T}^{T} \langle \partial_t u(t), \tilde{h}(t)\rangle_\Omega \, dt \tag{3.20}$$

for any $\tilde{h}$ satisfying $\tilde{h} - h \in L^2(-T,T;\tilde{H}^s(\Omega))$. Here $u := P_{A,q} g$, $w := u - g$ and

$$\langle \partial_t u(t), \tilde{h}(t)\rangle_\Omega := \langle \partial_t w(t), \tilde{h}(t) - h(t)\rangle.$$

Similarly we can define

$$\Lambda^*_{A,q} h := \mathcal{R}^s_A u^*|_{\Omega_e \times (-T,T)}$$

where $u^* := P^*_{A,q} h$ and we have

$$\int_{-T}^{T} \langle \Lambda^*_{A,q} h(t), g(t)\rangle \, dt = \int_{-T}^{T} B_t[u^*(t), \tilde{g}(t)] \, dt + \int_{-T}^{T} \langle -\partial_t u^*(t), \tilde{g}(t)\rangle_\Omega \, dt \tag{3.21}$$

for any $\tilde{g}$ satisfying $\tilde{g} - g \in L^2(-T,T;\tilde{H}^s(\Omega))$.

Proposition 3.3.1. *For $g \in C_c^\infty(W_1 \times (-T,T))$ and $h \in C_c^\infty(W_2 \times (-T,T))$,*

$$\int_{-T}^{T} \langle \Lambda_{A,q} g(t), h(t)\rangle \, dt = \int_{-T}^{T} \langle \Lambda^*_{A,q} h(t), g(t)\rangle \, dt. \tag{3.22}$$

Proof. Let $\tilde{h} = u^*$ in (3.20) and $\tilde{g} = u$ in (3.21). Since $u(-T) = u^*(T) = 0$, we have

$$\begin{aligned} &\int_{-T}^{T} \langle \Lambda_{A,q} g(t), h(t)\rangle \, dt - \int_{-T}^{T} \langle \Lambda^*_{A,q} h(t), g(t)\rangle \, dt \\ &= \int_{-T}^{T} \langle \partial_t u(t), u^*(t)\rangle_\Omega + \langle \partial_t u^*(t), u(t)\rangle_\Omega \, dt \\ &= \langle u(t), u^*(t)\rangle_\Omega |_{t=-T}^{t=T} = 0. \end{aligned}$$

by the symmetry of B_t. □

Now we prove an integral identity which will be used later.

For $g_j \in C_c^\infty(W_j \times (-T,T))$ $(j=1,2)$, let $u_1 = P_{A_1,q_1}(g_1)$ and $u_2^* = P^*_{A_2,q_2}(g_2)$, i.e. u_1 is the unique weak solution of

$$\begin{cases} \partial_t u + \mathcal{R}^s_{A_1(t)} u + q_1(t)u = 0 & \text{in } \Omega \times (-T,T) \\ u = g_1 & \text{in } \Omega_e \times (-T,T) \\ u = 0 & \text{in } \mathbb{R}^n \times \{-T\} \end{cases} \tag{3.23}$$

and u_2^* is the unique weak solution of

$$\begin{cases} -\partial_t u + \mathcal{R}^s_{A_2(t)} u + q_2(t)u = 0 & \text{in } \Omega \times (-T,T) \\ u = g_2 & \text{in } \Omega_e \times (-T,T) \\ u = 0 & \text{in } \mathbb{R}^n \times \{T\}. \end{cases} \tag{3.24}$$

Then we have

$$\begin{aligned} &\int_{-T}^{T} \langle \Lambda_{A_1,q_1} g_1(t), g_2(t)\rangle - \langle \Lambda_{A_2,q_2} g_1(t), g_2(t)\rangle \, dt \\ &= \int_{-T}^{T} \langle \Lambda_{A_1,q_1} g_1(t), g_2(t)\rangle \, dt - \int_{-T}^{T} \langle \Lambda^*_{A_2,q_2} g_2(t), g_1(t)\rangle \, dt \\ &= \int_{-T}^{T} B_t^{(1)}[u_1(t), u_2^*(t)] + \langle \partial_t u_1(t), u_2^*(t)\rangle_\Omega \, dt \\ &- \int_{-T}^{T} B_t^{(2)}[u_2^*(t), u_1(t)] + \langle -\partial_t u_2^*(t), u_1(t)\rangle_\Omega \, dt \end{aligned}$$

Now we use $u(-T) = u^*(T) = 0$ and the symmetry of B_t to get

$$\begin{aligned} &\int_{-T}^{T} \langle \Lambda_{A_1,q_1} g_1(t), g_2(t)\rangle - \langle \Lambda_{A_2,q_2} g_1(t), g_2(t)\rangle \, dt \\ &= \int_{-T}^{T} B_t^{(1)}[u_1(t), u_2^*(t)] \, dt - \int_{-T}^{T} B_t^{(2)}[u_1(t), u_2^*(t)] \, dt \\ &= \int_{-T}^{T} \iint G(x,y,t) u_1(y,t) u_2^*(x,t) - \int_{-T}^{T} \int_\Omega (q_2 - q_1) u_1 u_2^* \end{aligned} \tag{3.25}$$

where we write

$$G(x,y,t) := 2(R_{A_2(t)}(x,y) - R_{A_1(t)}(x,y))K(x,y).$$

3.3.2 The Runge Approximation

We will use the following the Runge approximation property later, which can viewed as a parabolic analogue of Proposition 2.3.4. We refer readers to [9, 39] for more approximation results for non-local evolution problems.

Proposition 3.3.2. *Suppose* $\Omega \subset B_r(0)$ *for some constant* $r > 0$ *and* W *is an open set in* $\mathbb{R}^n$ *s.t.* $W \cap B_{3r}(0) = \emptyset$. *Then*

$$S := \{P_{A,q}g|_{\Omega\times(-T,T)} : g \in C_c^\infty(W \times (-T,T))\},$$

$$S^* := \{P^*_{A,q}g|_{\Omega\times(-T,T)} : g \in C_c^\infty(W \times (-T,T))\}$$

are dense in $L^2(\Omega \times (-T,T))$.

Proof. By the Hahn-Banach Theorem, it suffices to show that:

If $v \in L^2(\Omega \times (-T,T))$ and $\int_{-T}^T \int_\Omega vw = 0$ for all $w \in S$, then $v = 0$ in $\Omega \times (-T,T)$.

In fact, for a given $v \in L^2(\Omega \times (-T,T))$, let $\phi \in L^2(-T,T;\tilde{H}^s(\Omega))$ be the solution of

$$\begin{cases} -\partial_t\phi + \mathcal{R}^s_{A(t)}\phi + q(t)\phi = v & \text{in } \Omega \times (-T,T) \\ \phi = 0 & \text{in } \Omega \times \{T\}. \end{cases} \tag{3.26}$$

For $g \in C_c^\infty(W \times (-T,T))$, Let $u_g := P_{A,q}g$. Then we have

$$\int_{-T}^T \int_\Omega vu_g = \int_{-T}^T \langle -\partial_t\phi(t) + \mathcal{R}^s_{A(t)}\phi(t) + q(t)\phi(t), u_g(t) - g(t)\rangle\, dt$$

$$= \int_{-T}^T \langle \partial_t u_g(t), \phi(t)\rangle + B_t[u_g(t), \phi(t)]\, dt - \int_{-T}^T \langle \mathcal{R}^s_{A(t)}g(t), \phi(t)\rangle\, dt$$

$$= -\int_{-T}^T \langle \mathcal{R}^s_{A(t)}\phi(t), g(t)\rangle\, dt. \tag{3.27}$$

The first equality holds since $u_g - g \in L^2(-T,T;\tilde{H}^s(\Omega))$. The second equality holds since $u_g(-T) = \phi(T) = 0$. The last equality holds since $\phi \in L^2(-T,T;\tilde{H}^s(\Omega))$ and u_g is the solution of (3.16).

Hence, if $\int_{-T}^{T}\int_{\Omega} vw = 0$ for all $w \in S$, then (3.27) yields

$$\int_{-T}^{T} \langle \mathcal{R}^s_{A(t)}\phi(t), g(t)\rangle \, dt = 0, \qquad g \in C_c^\infty(W \times (-T,T))$$

so for each t we have

$$\phi(t) \in \tilde{H}^s(\Omega), \qquad \mathcal{R}^s_{A(t)}\phi(t)|_W = 0,$$

which implies $\phi(t) = 0$ in $\mathbb{R}^n$ by Proposition 2.3.2 and Lemma 2.3.3, and thus $v = 0$ in $\Omega \times (-T,T)$. Similarly we can show that S^* is dense in $L^2(\Omega \times (-T,T))$. □

3.3.3 The Main Result

The following theorem (Theorem 1.1 in [30]) is our main result in this chapter, which can be viewed as a parabolic version of Theorem 2.3.5.

Theorem 3.3.3. *Suppose $\Omega \subset B_r(0)$ for some constant $r > 0$, $\operatorname{supp} A_j(t) \subset \Omega$ for $t \in [-T,T]$, $A_j \in C^2([-T,T]; L^\infty(\mathbb{R}^n))$, $q_j \in C^2([-T,T]; L^\infty(\Omega))$ and $q_j \geq c$ for some constant $c > 0$, W_j are open sets s.t. $W_j \cap B_{3r}(0) = \emptyset$ $(j = 1,2)$. Let*

$$W^{(1,2)} := \{\frac{x+y}{2} : x \in W_1, y \in W_2\}.$$

Also assume $W^{(1,2)} \setminus \Omega \neq \emptyset$. If

$$\Lambda_{A_1,q_1} g|_{W_2\times(-T,T)} = \Lambda_{A_2,q_2} g|_{W_2\times(-T,T)} \tag{3.28}$$

for any $g \in C_c^\infty(W_1 \times (-T,T))$, then $A_1(t) = \pm A_2(t)$ and $q_1 = q_2$ in $\Omega \times (-T,T)$.

As in the proof of Theorem 2.3.5, we exploit the integral identity and the Runge approximation property to determine magnetic and electric potentials.

Proof. Write $u_1 = P_{A_1,q_1}(g_1)$ and $u_2^* = P^*_{A_2,q_2}(g_2)$ for $g_j \in C_c^\infty(W_j \times (-T,T))$ $(j = 1,2)$. As in the proof of Theorem 2.3.5, the assumptions on $W_1, W_2, W^{(1,2)}$ ensure that

$$\iint G(x,y,t)u_1(y,t)u_2^*(x,t)\, dxdy = \int_\Omega\int_\Omega G(x,y,t)u_1(y,t)u_2^*(x,t)\, dxdy$$

for each t (if we shrink W_1, W_2 when necessary).

By the integral identity (3.25), (3.28) implies that

$$\int_{-T}^{T}\int_{\Omega}\int_{\Omega} G(x,y,t)u_1(y,t)u_2^*(x,t) = \int_{-T}^{T}\int_{\Omega}(q_2-q_1)u_1u_2^*. \tag{3.29}$$

Determine A: We fix open sets $\Omega_{1,2} \subset \Omega$ s.t. $\Omega_1 \cap \Omega_2 = \emptyset$. We also fix $\phi_j \in C_c^\infty(\Omega_j)$ $(j=1,2)$ and the constants $a, b \in (-T,T)$ and $\epsilon > 0$. Write

$$\tilde{\phi}_j(x,t) := 1_{[a,b]}(t)\phi_j(x).$$

By Proposition 3.3.2, we can choose $g_1 \in C_c^\infty(W_1 \times (-T,T))$ s.t.

$$||u_1 - \tilde{\phi}_1||_{L^2(\Omega\times(-T,T))} \leq \epsilon$$

and for this chosen g_1, we can choose $g_2 \in C_c^\infty(W_2 \times (-T,T))$ s.t.

$$||u_1||_{L^2(\Omega\times(-T,T))}||u_2^* - \tilde{\phi}_2||_{L^2(\Omega\times(-T,T))} \leq \epsilon.$$

Note that $\phi_1(x)\phi_2(x) = 0$ for $x \in \Omega$ so

$$\begin{aligned} &|\int_{-T}^{T}\int_{\Omega}(q_2-q_1)u_1u_2^*| \\ &= |\int_{-T}^{T}\int_{\Omega}(q_2-q_1)(u_1-\tilde{\phi}_1)\tilde{\phi}_2 + \int_{-T}^{T}\int_{\Omega}(q_2-q_1)u_1(u_2^*-\tilde{\phi}_2)| \\ &\leq ||(q_2-q_1)||_{L^\infty}||\tilde{\phi}_2||_{L^2}||u_1-\tilde{\phi}_1||_{L^2} + ||(q_2-q_1)||_{L^\infty}||u_1||_{L^2}||u_2^*-\tilde{\phi}_2||_{L^2} \leq C\epsilon. \end{aligned} \tag{3.30}$$

Also note that

$$|G(x,y,t)| \leq \frac{C'}{|x-y|^{n+2s-2}},$$

which implies that

$$\int_{\Omega}|G(x,y,t)|dy \leq C'',\ x \in \Omega; \qquad \int_{\Omega}|G(x,y,t)|dx \leq C'',\ y \in \Omega$$

where C', C'' do not depend on t. By the generalized Young's Inequality,

$$||T_t f||_{L^2(\Omega)} \leq C''||f||_{L^2(\Omega)}, \qquad (T_t f)(x) := \int_{\Omega}|G(x,y,t)f(y)|\,dy.$$

Now note that

$$\begin{aligned} &\int_{-T}^{T}\int_{\Omega}\int_{\Omega} G(x,y,t)u_1(y,t)u_2^*(x,t)\,dxdydt \\ &-\int_{a}^{b}\int_{\Omega_1}\int_{\Omega_2} G(x,y,t)\phi_1(y)\phi_2(x)\,dxdydt \\ =&\int_{-T}^{T}\int_{\Omega}\int_{\Omega} G(x,y,t)(u_1(y,t)-\tilde{\phi}_1(y,t))\tilde{\phi}_2(x,t)\,dxdydt \\ +&\int_{-T}^{T}\int_{\Omega}\int_{\Omega} G(x,y,t)u_1(y,t)(u_2^*(x,t)-\tilde{\phi}_2(x,t))\,dxdydt. \end{aligned}$$

By Cauchy-Schwarz inequality, we have the estimate

$$\begin{aligned} &\int_{-T}^{T}\int_{\Omega}(\int_{\Omega}|G(x,y,t)u_1(y,t)|\,dy)|u_2^*(x,t)-\tilde{\phi}_2(x,t)|\,dxdt \\ &\leq (\int_{-T}^{T}\int_{\Omega}|(T_t u_1(t))(x)|^2\,dxdt)^{\frac{1}{2}}||u_2^*-\tilde{\phi}_2||_{L^2(\Omega\times(-T,T))} \\ &\leq C''(\int_{-T}^{T}||u_1(t)||^2_{L^2(\Omega)}\,dt)^{\frac{1}{2}}||u_2^*-\tilde{\phi}_2||_{L^2(\Omega\times(-T,T))} \\ &= C''||u_1||_{L^2(\Omega\times(-T,T))}||u_2^*-\tilde{\phi}_2||_{L^2(\Omega\times(-T,T))}. \end{aligned}$$

Similarly, we have the estimate

$$\begin{aligned} &\int_{-T}^{T}\int_{\Omega}(\int_{\Omega}|G(x,y,t)\tilde{\phi}_2(x,t)|\,dx)|(u_1(y,t)-\tilde{\phi}_1(y,t))|\,dydt \\ &\leq C''||\tilde{\phi}_2||_{L^2(\Omega\times(-T,T))}||u_1-\tilde{\phi}_1||_{L^2(\Omega\times(-T,T))}. \end{aligned}$$

Hence we have

$$|\int_{-T}^{T}\int_{\Omega}\int_{\Omega} G(x,y,t)u_1(y,t)u_2^*(x,t)-\int_{a}^{b}\int_{\Omega_1}\int_{\Omega_2} G(x,y,t)\phi_1(y)\phi_2(x)|\leq C'''\epsilon. \tag{3.31}$$

We combine (3.30), (3.31) with (3.29). ϵ is arbitrary, implying that

$$\int_{a}^{b}\int_{\Omega_1}\int_{\Omega_2} G(x,y,t)\phi_1(y)\phi_2(x)\,dxdydt=0.$$

Then $[a,b]$ is arbitrary, implying that

$$\int_{\Omega_1}\int_{\Omega_2} G(x,y,t)\phi_1(y)\phi_2(x)\,dxdy=0$$

for each t and thus $G(x,y,t)=0$ in $\Omega_1\times\Omega_2$ for each t since ϕ_1,ϕ_2 are arbitrary. We conclude that $G(x,y,t)=0$ for $x,y\in\Omega$ whenever $x\neq y$ since Ω_1,Ω_2 are arbitrary. Hence

$$R_{A_1(t)}(x,y)=R_{A_2(t)}(x,y),\quad x,y\in\Omega \tag{3.32}$$

for each t, which implies $A_1(t)=\pm A_2(t)$ as in the proof of Theorem 2.3.5.

Determine q: Now (3.29) have become

$$\int_{-T}^{T}\int_{\Omega}(q_2-q_1)u_1u_2^*=0.$$

Fixing $\epsilon>0$ and $f\in L^2(\Omega\times(-T,T))$, by the Runge approximation property we can choose $g_1\in C_c^\infty(W_1\times(-T,T))$ s.t.

$$||u_1-f||_{L^2(\Omega\times(-T,T))}\leq\epsilon$$

and for this chosen u_1, we can choose $g_2\in C_c^\infty(W_2)$ s.t.

$$||u_1||_{L^2(\Omega\times(-T,T))}||u_2^*-1||_{L^2(\Omega\times(-T,T))}\leq\epsilon.$$

Now we have

$$|\int_{-T}^{T}\int_{\Omega}(q_1-q_2)f|=|\int_{-T}^{T}\int_{\Omega}(q_1-q_2)(f-u_1)+\int_{-T}^{T}\int_{\Omega}(q_1-q_2)u_1(1-u_2^*)|\leq C\epsilon.$$

We conclude that $q_1=q_2$ since ϵ,f are arbitrary. □

Chapter 4

THE SEMI-LINEAR FRACTIONAL MAGNETIC PROBLEM

In this chapter, we study a semi-linear analogue of the linear fractional elliptic problem studied in Chapter 2. We return to the time-independent magnetic potential A and we further study the fractional magnetic operator $\mathcal{R}^s_A$ when it coincides with the real part of the fractional magnetic Laplacian $(-\Delta)^s_A$. To formulate the semi-linear problem, we introduce a power type non-linearity instead of the electric potential q.

4.1 Preliminaries

Let $U \subset \mathbb{R}^n$ be open. We use $C^s(U)$ to denote the Hölder space equipped with the norm

$$||f||_{C^s(U)} := ||f||_{L^\infty(U)} + \sup_{x\neq y, x,y\in U} \frac{|f(x)-f(y)|}{|x-y|^s}.$$

For $k \in \mathbb{N}$, the norm $||\cdot||_{C^k(\mathbb{R}^n)}$ is defined by

$$||f||_{C^k(\mathbb{R}^n)} := \sum_{|\alpha|\leq k} ||\partial^\alpha f||_{L^\infty(\mathbb{R}^n)}.$$

4.1.1 The Non-linearity

We consider the non-linearity $a(x,z) : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying

(i) $z \to a(\cdot, z)$ is analytic with values in $C^s(\Omega)$;

(ii) $a(x,0) = 0$ and $\partial_z a(x,0) \geq c$ for some constant $c > 0$

so we have the Taylor expansion

$$a(x,z) = \sum_{k=1}^{\infty} a_k(x) \frac{z^k}{k!}, \quad a_k(x) = \partial_z^k a(x,0) \in C^s(\Omega) \tag{4.1}$$

where the series converges in $C^s(\Omega)$ topology. We write

$$R_m(x,z) := \sum_{k=m+1}^{\infty} \frac{a_k(x)}{k!} z^k.$$

Note that $C^s(\Omega)$ is an algebra. In fact, for $u_1, u_2 \in C^s(\Omega)$, we have

$$||u_1 u_2||_{C^s(\Omega)} \leq C_0(||u_1||_{C^s(\Omega)}||u_2||_{L^\infty(\Omega)} + ||u_1||_{L^\infty(\Omega)}||u_2||_{C^s(\Omega)})$$

(see Theorem A.7 in [17]) implying that

$$||u_1 u_2||_{C^s(\Omega)} \leq 2C_0||u_1||_{C^s(\Omega)}||u_2||_{C^s(\Omega)}.$$

Also note that by Cauchy's estimate, we have

$$||a_k||_{C^s(\Omega)} \leq \frac{k!}{R^k} \sup_{z \in \mathbb{C}, |z|=R} ||a(\cdot, z)||_{C^s(\Omega)}, \quad R > 0.$$

Based on the estimates above, we have the following estimates if we let $R := \max\{4C_0, 1\}$.

Proposition 4.1.1. *Suppose* $||u||_{C^s(\Omega)} \leq 1$. *Then we have*

$$\sum_{k=m+1}^{\infty} ||\frac{a_k(x)}{k!}u^k||_{C^s(\Omega)} \leq (\sum_{k=m+1}^{\infty} \frac{1}{2^k}) \sup_{z \in \mathbb{C}, |z|=R} ||a(\cdot, z)||_{C^s(\Omega)}||u||^{m+1}_{C^s(\Omega)},$$

$$\sum_{k=m+1}^{\infty} ||\frac{a_k(x)}{(k-1)!}u^{k-1}||_{C^s(\Omega)} \leq (\sum_{k=m+1}^{\infty} \frac{k}{2^{k-1}}) \sup_{z \in \mathbb{C}, |z|=R} ||a(\cdot, z)||_{C^s(\Omega)}||u||^{m}_{C^s(\Omega)}.$$

4.1.2 L^∞ Estimates

The following estimates will be used later.

Lemma 4.1.2. *Suppose* $g \in C_c^\infty(\mathbb{R}^n)$. *Then we have*

$$||(-\Delta)^s g||_{L^\infty(\mathbb{R}^n)} \leq C||g||_{C^2(\mathbb{R}^n)}.$$

Proof. For $g \in C_c^\infty(\mathbb{R}^n)$, we have

$$(-\Delta)^s g(x) = c_{n,s} \int \frac{2g(x) - g(x+y) - g(x-y)}{|y|^{n+2s}} dy$$

(see for instance, Lemma 3.2 in [8]) so by using the Taylor expansion, we have

$$|(-\Delta)^s g(x)| \leq c_{n,s}(\int_{|y|\leq 1} + \int_{|y|>1}) \frac{|2g(x) - g(x+y) - g(x-y)|}{|y|^{n+2s}} dy \leq C||g||_{C^2(\mathbb{R}^n)}.$$

□

Lemma 4.1.3. *Suppose* $A, g \in L^\infty(\mathbb{R}^n)$. *Then we have*

$$||((-\Delta)^s - \mathcal{R}_A^s)g||_{L^\infty(\mathbb{R}^n)} \leq C||g||_{L^\infty(\mathbb{R}^n)}.$$

Proof. Note that

$$0 \leq 1 - R_A(x,y) = 2\sin^2(\frac{1}{2}(x-y)\cdot A(\frac{x+y}{2})) \leq C_A \min\{1, |x-y|^2\}$$

so we have

$$|((-\Delta)^s - \mathcal{R}_A^s)g(x)| \leq \int (1 - R_A(x,y))K(x,y)|g(y)|dy$$

$$= (\int_{|y-x|\leq 1} + \int_{|y-x|>1})(1 - R_A(x,y))K(x,y)|g(y)|dy \leq C||g||_{L^\infty(\mathbb{R}^n)}.$$

□

4.2 More Results for the Linear Problem

In the rest of this chapter we always assume $\operatorname{supp} A \subset \Omega \subset B_r(0)$ for some constant $r > 0$ and $||A||_{L^\infty(\mathbb{R}^n)} \leq \pi/(8\sqrt{n}r)$. Under these assumptions, we have

$$0 \leq R_A(x,y) \leq 1, \qquad (x,y) \in (\mathbb{R}^n \times \mathbb{R}^n) \setminus (\Omega_e \times \Omega_e).$$

In fact, if $(x,y) \in \Omega \times \mathbb{R}^n$, then

(i) $y \in B_{3r}(0)$, which implies $|x-y| \leq 4r$, $|(x-y)\cdot A(\frac{x+y}{2})| \leq \frac{\pi}{2}$;

(ii) $y \notin B_{3r}(0)$, which implies $|\frac{x+y}{2}| \geq r$, $R_A = 1$.

By the symmetry of R_A, we know the claim also holds for $(x,y) \in \mathbb{R}^n \times \Omega$.

4.2.1 The Maximum Principle

The following proposition generalizes the maximum principle for the fractional Laplacian. The non-negativity of R_A is essential in the proof.

Proposition 4.2.1. *Suppose* $0 < c \leq q(x) \in L^\infty(\Omega)$. *If* $u \in H^s(\mathbb{R}^n)$ *solves the problem*

$$\begin{cases} (\mathcal{R}_A^s + q)u = f & in\ \Omega \\ u = g & in\ \Omega_e \end{cases} \tag{4.2}$$

for $0 \leq f \in H^{-s}(\Omega)$ *and* $0 \leq g|_{\Omega_e} \in L^\infty(\Omega_e)$, *then* $u \geq 0$.

Proof. Write $u = u^+ - u^-$ where $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$. Note that

$$|u^+(x) - u^+(y)| + |u^-(x) - u^-(y)| = |u(x) - u(y)|$$

so $u^+, u^- \in H^s(\mathbb{R}^n)$ and $g|_{\Omega_e} \geq 0$ implies $u^- \in \tilde{H}^s(\Omega)$. Hence we have

$$\langle \mathcal{R}^s_A u, u^- \rangle + \int_\Omega q u u^- = f(u^-).$$

Now write

$$\langle \mathcal{R}^s_A u, u^- \rangle = 2 \iint (u(x) - R_A(x,y)u(y))u^-(x)K(x,y)\, dxdy$$

$$= 2\Big(\int_\Omega \int_\Omega + \int_{\Omega_e} \int_\Omega\Big)(u(x) - R_A(x,y)u(y))u^-(x)K(x,y)\, dxdy =: I_1 + I_2.$$

Since $u^- u \leq 0$, we have

$$I_2 = 2\int_{\Omega_e}\int_\Omega (u(x) - R_A(x,y)g(y))u^-(x)K(x,y)\, dxdy \leq 0, \qquad \int_\Omega q u u^- \leq 0.$$

Note that

$$I_1 = 2\int_\Omega\int_\Omega (u(x) - R_A(x,y)u(y))u^-(x)K(x,y)\, dxdy$$

$$= 2\int_\Omega\int_\Omega [(u^+(x) - R_A(x,y)u^+(y))u^-(x) - (u^-(x) - R_A(x,y)u^-(y))u^-(x)]K(x,y)\, dxdy$$

$$= -2\int_\Omega\int_\Omega [R_A(x,y)u^+(y)u^-(x) + (u^-(x) - R_A(x,y)u^-(y))u^-(x)]K(x,y)\, dxdy$$

$$\leq -2\int_\Omega\int_\Omega (u^-(x) - R_A(x,y)u^-(y))u^-(x)K(x,y)\, dxdy$$

$$= -\int_\Omega\int_\Omega [(u^-(x) - R_A(x,y)u^-(y))u^-(x) + (u^-(y) - R_A(x,y)u^-(x))u^-(y)]K(x,y)\, dxdy$$

$$= -\int_\Omega\int_\Omega (|u^-(x)|^2 + |u^-(y)|^2 - 2R_A(x,y)u^-(x)u^-(y))K(x,y)\, dxdy$$

$$\leq -\int_\Omega\int_\Omega |u^-(x) - u^-(y)|^2 K(x,y)\, dxdy.$$

Since $f(u^-) \geq 0$, the only possibility is

$$\int_\Omega\int_\Omega |u^-(x) - u^-(y)|^2 K(x,y)\, dxdy = 0,$$

implying that u^- is a non-negative constant c_0 in Ω. Now we show that c_0 has to be 0.

Otherwise $u = -c_0 < 0$ in Ω. In this case, for $x \in \Omega$, we have

$$\mathcal{R}^s_A u(x) = 2 \lim_{\epsilon \to 0^+} (\int_{\Omega \setminus B_\epsilon(x)} + \int_{\Omega_e})(u(x) - R_A(x,y)u(y))K(x,y)\, dy$$

$$= 2\int_\Omega (1 - R_A(x,y))(-c_0)K(x,y)dy + 2\int_{\Omega_e}(-c_0 - R_A(x,y)g(y))K(x,y)dy \leq 0.$$

Both integrals converge since $g|_{\Omega_e} \in L^\infty(\Omega_e)$ and

$$0 \leq 1 - R_A(x,y) \leq C_A|x-y|^2.$$

Now we have reached the contradiction

$$f = \mathcal{R}^s_A u + qu < 0 \quad \text{in } \Omega.$$

□

The following L^∞ estimate follows from the maximum principle above.

Proposition 4.2.2. *Suppose* $0 < c \leq q(x) \in L^\infty(\Omega)$. *If* $u \in H^s(\mathbb{R}^n)$ *solves the problem 4.2 for* $f \in L^\infty(\Omega)$ *and* $g \in C^\infty_c(\Omega_e)$, *then*

$$||u||_{L^\infty} \leq C||f||_{L^\infty(\Omega)} + ||g||_{L^\infty(\Omega_e)}.$$

Proof. We fix a function $\phi \in C^\infty_c(\mathbb{R}^n)$ s.t. $0 \leq \phi \leq 1$ and $\phi = 1$ on $\bar{\Omega} \cup \operatorname{supp} g$.

It is clear from the pointwise definition of $\mathcal{R}^s_A$ that $\mathcal{R}^s_A\phi \geq 0$ in Ω so

$$(\mathcal{R}^s_A + q)\phi \geq c \quad \text{in } \Omega.$$

Now we consider

$$\tilde{\phi} := (\frac{1}{c}||f||_{L^\infty(\Omega)} + ||g||_{L^\infty(\Omega_e)})\phi.$$

Then $\tilde{\phi} \pm u \geq 0$ in Ω_e and

$$(\mathcal{R}^s_A + q)(\tilde{\phi} \pm u) \geq 0 \quad \text{in } \Omega$$

so we have $|u| \leq \tilde{\phi}$ by Proposition 4.2.1. □

4.2.2 C^s Regularity

The following regularity theorem was proved in [36].

Proposition 4.2.3. *Suppose $u \in \tilde{H}^s(\Omega)$ and $(-\Delta)^s u = f \in L^\infty(\Omega)$. Then $u \in C^s(\mathbb{R}^n)$ and*

$$||u||_{C^s(\mathbb{R}^n)} \leq C||f||_{L^\infty(\Omega)}.$$

We remark that u is not in general $C^{s'}$ for any $s' > s$ so this C^s regularity is optimal. See Theorem 5.2 in [13] for a concrete example.

Now we prove the Hölder regularity theorem for the linear exterior problem (2.7).

Proposition 4.2.4. *Suppose* $\operatorname{supp} A \subset \Omega \subset B_r(0)$ *for some* $r > 0$ *and* $||A||_{L^\infty(\mathbb{R}^n)} \leq \pi/(8\sqrt{n}r)$, $0 < c \leq q(x) \in L^\infty(\Omega)$ *for some* $c > 0$, $W \cap B_{3r}(0) = \emptyset$. *If* $g \in C_c^\infty(W)$, *then* $u = P_{A,q}g \in C^s(\mathbb{R}^n)$ *and*

$$||u||_{C^s(\mathbb{R}^n)} \leq C||g||_{C^2(\mathbb{R}^n)}.$$

Proof. Note that $v := u - g \in \tilde{H}^s(\Omega)$ and

$$(-\Delta)^s v = ((-\Delta)^s - \mathcal{R}_A^s)v - \mathcal{R}_A^s g - qu \quad \text{in } \Omega.$$

By Proposition 4.2.2, $||v||_{L^\infty} \leq C_1||g||_{L^\infty}$. By Lemma 4.1.3,

$$||((-\Delta)^s - \mathcal{R}_A^s)v||_{L^\infty(\mathbb{R}^n)} \leq C_2||v||_{L^\infty} \leq C_3||g||_{L^\infty}.$$

Using arguments similar to the ones in the proof of Lemma 2.3.3, we can show that

$$\mathcal{R}_A^s g = (-\Delta)^s g \quad \text{in } \Omega.$$

By Lemma 4.1.2, $||\mathcal{R}_A^s g||_{L^\infty(\Omega)} \leq C'||g||_{C^2(\mathbb{R}^n)}$. Hence

$$||((-\Delta)^s - \mathcal{R}_A^s)v - \mathcal{R}_A^s g - qu||_{L^\infty(\Omega)} \leq C''||g||_{C^2(\mathbb{R}^n)}$$

Now by Proposition 4.2.3, we have $||v||_{C^s(\mathbb{R}^n)} \leq C||g||_{C^2(\mathbb{R}^n)}$. □

4.3 The Semi-linear Exterior Problem

Now we study the well-posedness of the semi-linear exterior problem

$$\begin{cases} \mathcal{R}^s_A u + a(x,u) = 0 & \text{in } \Omega \\ u = g & \text{in } \Omega_e. \end{cases} \tag{4.3}$$

The following proposition generalizes Theorem 2.1 in [24].

Proposition 4.3.1. *Suppose* $\operatorname{supp} A \subset \Omega \subset B_r(0)$ *for some* $r > 0$ *and* $||A||_{L^\infty(\mathbb{R}^n)} \leq \pi/(8\sqrt{n}r)$*,* $W \cap B_{3r}(0) = \emptyset$ *and* $g \in C^\infty_c(W)$*. There exists a small constant* $\rho > 0$ *s.t. if* $||g||_{C^2(\mathbb{R}^n)} \leq \rho$*, then (4.3) has a unique solution* $u \in H^s(\mathbb{R}^n) \cap C^s(\mathbb{R}^n)$ *satisfying*

$$(u - P_{A,a_1} g) \in M := \{v \in C^s(\mathbb{R}^n) : v|_{\Omega_e} = 0, ||v||_{C^s(\mathbb{R}^n)} \leq \rho\}.$$

Moreover, we have

$$||u||_{C^s(\mathbb{R}^n)} \leq C||g||_{C^2(\mathbb{R}^n)}.$$

Proof. Let $u_0 := P_{A,a_1} g$. By Proposition 4.2.4 we have

$$||u_0||_{C^s(\mathbb{R}^n)} \leq C_1||g||_{C^2(\mathbb{R}^n)}.$$

The semi-linear exterior problem (4.3) can be written as

$$\begin{cases} \mathcal{R}^s_A(u-u_0) + a_1(x)(u-u_0) = -R_1(x,u) & \text{in } \Omega \\ u - u_0 = 0 & \text{in } \Omega_e. \end{cases} \tag{4.4}$$

For $f \in L^\infty(\Omega)$, we consider the solution operator $J : f \to u_f \in \tilde{H}^s(\Omega)$ associated with

$$\mathcal{R}^s_A u + a_1(x)u = f \quad \text{in } \Omega.$$

We write

$$(-\Delta)^s u = ((-\Delta)^s - \mathcal{R}^s_A)u - a_1(x)u + f \quad \text{in } \Omega.$$

By Lemma 4.1.3, Proposition 4.2.2 and 4.2.3, we have $J(f) \in C^s(\mathbb{R}^n)$ and

$$||J(f)||_{C^s(\mathbb{R}^n)} \leq C_2||f||_{L^\infty(\Omega)}.$$

We define maps G, F by

$$G(v) := R_1(x, u_0 + v), \qquad F := J \circ G.$$

We will show that F is a contraction map on the complete metric space M for small ρ, which will be chosen later. In fact, for small ρ and $v \in M$, we have

$$||F(v)||_{C^s(\mathbb{R}^n)} \leq C_2||G(v)||_{L^\infty(\Omega)} = C_2||R_1(x, u_0 + v)||_{L^\infty(\Omega)}$$

$$\leq C_2'||u_0 + v||^2_{C^s(\Omega)} \leq C_2''\rho^2.$$

Here we use the first estimate in Proposition 4.1.1 and the constant C_2'' is independent of ρ. Also for small ρ and $v_1, v_2 \in M$, we have

$$||F(v_1) - F(v_2)||_{C^s(\mathbb{R}^n)} \leq C_2||G(v_1) - G(v_2)||_{L^\infty(\Omega)}$$

$$= C_2||R_1(x, u_0 + v_1) - R_1(x, u_0 + v_2)||_{L^\infty(\Omega)}$$

$$\leq ||v_1 - v_2||_{L^\infty(\Omega)} \sum_{k=2}^{\infty} ||\frac{a_k(x)}{(k-1)!}(|u_0 + v_1|^{k-1} + |u_0 + v_1|^{k-1})||_{L^\infty(\Omega)}$$

$$\leq C_3||v_1 - v_2||_{L^\infty(\Omega)}(||u_0 + v_1||_{C^s(\Omega)} + ||u_0 + v_2||_{C^s(\Omega)})$$

$$\leq C_4\rho||v_1 - v_2||_{L^\infty(\Omega)}.$$

Here we use the inequality

$$|a^m - b^m| \leq m|a - b|(|a|^{m-1} + |b|^{m-1})$$

and the second estimate in Proposition 4.1.1. The constant C_4 is independent of ρ.

Hence, for small $\rho < 1/(C_2'' + C_4)$, F is a contraction map on M so by the Banach fixed-point theorem, there exists a unique $v_0 \in M$ s.t. $F(v_0) = v_0$.

Now note that $v_0 = F(v_0) \in \tilde{H}^s(\Omega)$ and

$$||v_0||_{C^s(\mathbb{R}^n)} = ||F(v_0)||_{C^s(\mathbb{R}^n)} \leq C_2'||u_0 + v_0||^2_{C^s(\Omega)}$$

$$\leq C_3'\rho(||u_0||_{C^s(\Omega)} + ||v_0||_{C^s(\Omega)})$$

where the constant C_3' is independent of ρ. Hence, for small $\rho < 1/(2C_3')$, we have

$$||v_0||_{C^s(\mathbb{R}^n)} \leq 2C_3'\rho||u_0||_{C^s(\mathbb{R}^n)},$$

implying that $u := u_0 + v_0$ satisfies

$$||u||_{C^s(\mathbb{R}^n)} \leq C||g||_{C^2(\mathbb{R}^n)}.$$

$\square$

4.4 The Inverse Problem

Proposition 4.3.1 ensures that the solution operator $Q_{A,a} : g \to u_g$ associated with (4.3) is well-defined for g satisfying the conditions assumed in the statement of the proposition. Hence the Dirichlet-to-Neumann map $\Lambda_{A,a}$ given by

$$\Lambda_{A,a}g := \mathcal{R}_A^s(Q_{A,a}g)|_{\Omega_e} \tag{4.5}$$

is well-defined for such g.

The following first order linearization in $H^s(\mathbb{R}^n)$ will be used later.

Proposition 4.4.1. *Suppose* $\operatorname{supp} A \subset \Omega \subset B_r(0)$ *for some* $r > 0$ *and* $||A||_{L^\infty(\mathbb{R}^n)} \leq \pi/(8\sqrt{n}r)$, $W \cap B_{3r}(0) = \emptyset$ *and* $g \in C_c^\infty(W)$. *Then*

$$Q_{A,a}(\epsilon g)/\epsilon \to P_{A,a_1}g$$

in $H^s(\mathbb{R}^n)$ *as* $\epsilon \to 0$.

Proof. Let $u_{\epsilon,g} := Q_{A,a}(\epsilon g)$ for sufficiently small ϵ. Let $u_g := P_{A,a_1}g$.

Note that $v_{\epsilon,g} := u_g - \frac{u_{\epsilon,g}}{\epsilon} \in \tilde{H}^s(\Omega)$ and we have

$$\mathcal{R}_A^s v_{\epsilon,g} + a_1(x)v_{\epsilon,g} = \frac{1}{\epsilon}R_1(x, u_{\epsilon,g}) \quad \text{in } \Omega.$$

Now choose $v_{\epsilon,g}$ as a test function. By the norm equivalence

$$\langle \mathcal{R}_A^s v_{\epsilon,g} + a_1 v_{\epsilon,g}, v_{\epsilon,g}\rangle \geq [v_{\epsilon,g}]^2_{H_A^s} + c||v_{\epsilon,g}||^2_{L^2(\Omega)} \geq c'||v_{\epsilon,g}||^2_{H^s}.$$

By Proposition 4.1.1 and 4.3.1, we have

$$|\langle\frac{1}{\epsilon}R_1(x,u_{\epsilon,g}),v_{\epsilon,g}\rangle|\leq\frac{C}{\epsilon}||R_1(x,u_{\epsilon,g})||_{L^\infty(\Omega)}||v_{\epsilon,g}||_{L^2(\Omega)}$$

$$\leq\frac{C'}{\epsilon}||u_{\epsilon,g}||^2_{C^s(\mathbb{R}^n)}||v_{\epsilon,g}||_{L^2(\Omega)}\leq C''\epsilon||g||^2_{C^2(\mathbb{R}^n)}||v_{\epsilon,g}||_{L^2(\Omega)}.$$

Hence we have

$$||v_{\epsilon,g}||_{H^s}\leq C'''\epsilon||g||^2_{C^2(\mathbb{R}^n)}.$$

Now it is clear that $v_{\epsilon,g}\to 0$ in $H^s(\mathbb{R}^n)$ as $\epsilon\to 0$. □

The inverse problem associated with (4.3) is whether both the magnetic potential A and the non-linearity can be determined from partial exterior measurements of the Dirichlet-to-Neumann map $\Lambda_{A,a}$.

We mention that inverse problems associated with non-linear equations have been extensively studied. See [42] for inverse problems associated with quasi-linear elliptic equations. See [19] for inverse problems associated with parabolic equations involving general semi-linear terms. See [22] for inverse problems associated with non-linear hyperbolic equations.

We remark that the higher order linearization technique has been commonly applied to determine the full nonlinearity in dealing with inverse problems associated with power type nonlinear equations. See [21, 27, 32] for this approach for classical elliptic problems. See [24, 26] for the higher order linearization approach for fractional elliptic problems. Here we will only use the first order linearization (Proposition 4.4.1) to determine all the coefficients a_k based on the Runge approximation property.

The following theorem (Theorem 1.2 in [29]) is our main result in this chapter.

Theorem 4.4.2. *Suppose* $\operatorname{supp}A_j\subset\Omega\subset B_r(0)$ *for some constant* $r>0$ *and* $||A_j||_{L^\infty(\mathbb{R}^n)}\leq\pi/(8\sqrt{n}r)$, $a^{(j)}$ *satisfy the conditions (i) and (ii) (see Subsection 4.1),* W_j *are open sets s.t.* $W_j\cap B_{3r}(0)=\emptyset$ $(j=1,2)$*. Let*

$$W^{(1,2)}:=\{\frac{x+y}{2}:x\in W_1,y\in W_2\}.$$

Also assume $W^{(1,2)} \setminus (\operatorname{supp} A_1 \cup \operatorname{supp} A_2) \neq \emptyset$. *If*

$$\Lambda_{A_1,a^{(1)}} g|_{W_2} = \Lambda_{A_2,a^{(2)}} g|_{W_2}, \qquad g \in C_c^\infty(W_1) \tag{4.6}$$

whenever $||g||_{C^2(\mathbb{R}^n)}$ *is sufficiently small, then* $A_1 = \pm A_2$ *in* Ω *and* $a^{(1)} = a^{(2)}$ *in* $\Omega \times \mathbb{R}$.

We first use the first order linearization to determine A based on Theorem 2.3.5. Then we use the Runge approximation property to determine the non-linearity.

Proof. **Determine** A: For $g \in C_c^\infty(W_1)$ and small $\epsilon > 0$, let $u_{\epsilon,g}^{(j)} := Q_{A_j,a^{(j)}}(\epsilon g)$ satisfying

$$\begin{cases} \mathcal{R}_{A_j}^s u + a^{(j)}(x,u) = 0 & \text{in } \Omega \\ u = \epsilon g & \text{in } \Omega_e \end{cases}$$

and let $u_g^{(j)} := P_{A_j,a_1^{(j)}} g$ satisfying

$$\begin{cases} \mathcal{R}_{A_j}^s u + a_1^{(j)}(x)u = 0 & \text{in } \Omega \\ u = g & \text{in } \Omega_e. \end{cases}$$

By Proposition 4.4.1, we have

$$u_{\epsilon,g}^{(j)}/\epsilon \to u_g^{(j)} \quad \text{in } H^s(\mathbb{R}^n),$$

which implies

$$\frac{1}{\epsilon}\mathcal{R}_{A_j}^s u_{\epsilon,g}^{(j)}|_{W_2} \to \mathcal{R}_{A_j}^s u_g^{(j)}|_{W_2} \quad \text{in } H^{-s}(W_2).$$

Note that (4.6) implies

$$\mathcal{R}_{A_1}^s u_{\epsilon,g}^{(1)}|_{W_2} = \mathcal{R}_{A_2}^s u_{\epsilon,g}^{(2)}|_{W_2}.$$

Let $\epsilon \to 0$. Then we have

$$\mathcal{R}_{A_1}^s u_g^{(1)}|_{W_2} = \mathcal{R}_{A_2}^s u_g^{(2)}|_{W_2},$$

i.e.

$$\Lambda_{A_1,a_1^{(1)}} g|_{W_2} = \Lambda_{A_2,a_1^{(2)}} g|_{W_2}, \qquad g \in C_c^\infty(W_1).$$

By Theorem 2.3.5, $\pm A_1 = A_2 =: A$ and $a_1^{(1)} = a_1^{(2)} =: a_1$.

Determine a: Now we have

$$\mathcal{R}_A^s u_{\epsilon,g}^{(1)} = \mathcal{R}_A^s u_{\epsilon,g}^{(2)} \quad \text{in } W_2.$$

Since $u_{\epsilon,g}^{(1)} = u_{\epsilon,g}^{(2)} = \epsilon g$ in Ω_e, by the unique continuation property (Proposition 1.2.2) and Lemma 2.3.3 we have $u_{\epsilon,g}^{(1)} = u_{\epsilon,g}^{(2)} =: u_{\epsilon,g}$ in $\mathbb{R}^n$ so

$$\mathcal{R}_A^s u_{\epsilon,g} + a^{(j)}(x, u_{\epsilon,g}) = 0 \quad \text{in } \Omega$$

$(j = 1, 2)$, implying that

$$(a_1^{(1)}(x) - a_1^{(2)}(x))u_{\epsilon,g} = R_1^{(2)}(x, u_{\epsilon,g}) - R_1^{(1)}(x, u_{\epsilon,g}) \quad \text{in } \Omega.$$

We will inductively show that $a_k^{(1)} = a_k^{(2)}$ for all k.

Suppose we have shown $a_j^{(1)} = a_j^{(2)}$ $(1 \leq j \leq l-1)$. Then we have

$$\frac{1}{l!}(a_l^{(1)}(x) - a_l^{(2)}(x))u_{\epsilon,g}^l = R_l^{(2)}(x, u_{\epsilon,g}) - R_l^{(1)}(x, u_{\epsilon,g}) \quad \text{in } \Omega.$$

Now note that

$$\begin{aligned} &||\,|a_l^{(1)}(x) - a_l^{(2)}(x)|^{\frac{1}{l}}\,||_{L^2(\Omega)} \\ \leq\ &||\,|a_l^{(1)}(x) - a_l^{(2)}(x)|^{\frac{1}{l}}\,||_{L^\infty(\Omega)}||1 - \frac{u_{\epsilon,g}}{\epsilon}||_{L^2(\Omega)} \\ &+\frac{1}{\epsilon}||\,|a_l^{(1)}(x) - a_l^{(2)}(x)|^{\frac{1}{l}} u_{\epsilon,g}||_{L^2(\Omega)}. \end{aligned}$$

For given $\delta > 0$, by Proposition 2.3.4 we can choose $g \in C_c^\infty(W_1)$ s.t.

$$||1 - u_g||_{L^2(\Omega)} \leq \delta.$$

For this chosen g, we have

$$\begin{aligned} \frac{1}{\epsilon}||\,|a_l^{(1)}(x) - a_l^{(2)}(x)|^{\frac{1}{l}} u_{\epsilon,g}||_{L^2(\Omega)} &\leq \frac{C}{\epsilon}||R_l^{(2)}(x, u_{\epsilon,g}) - R_l^{(1)}(x, u_{\epsilon,g})||_{L^\infty(\Omega)}^{\frac{1}{l}} \\ &\leq \frac{C'}{\epsilon}||u_{\epsilon,g}||_{C^s(\Omega)}^{\frac{l+1}{l}} \leq C''\epsilon^{\frac{1}{l}}||g||_{C^2(\mathbb{R}^n)}^{\frac{l+1}{l}} \end{aligned}$$

for small ϵ by Proposition 4.1.1 and 4.3.1.

Now let $\epsilon \to 0$. Then we have

$$|||a_l^{(1)}(x) - a_l^{(2)}(x)|^{\frac{1}{l}}||_{L^2(\Omega)} \leq 2\delta|||a_l^{(1)}(x) - a_l^{(2)}(x)|^{\frac{1}{l}}||_{L^\infty(\Omega)}.$$

δ is arbitrary, implying that $a_l^{(1)} = a_l^{(2)}$. □

We remark that the method used for determining the non-linearity in the proof above also works for non-local operators involving fractional power type nonlinearities. See a related work in [31].

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