A Tropical Framework for Using Porteous Formula

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Abstract

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Given a rational polyhedral space X (a tropical cycle with boundary, in the sense of Mikhalkin–Rau), one can define tropical vector bundles on X having real or tropical fibers. By restricting attention to bounded rational sections of these bundles, one obtains characteristic classes that behave as expected classically. We develop further properties of these classes and use them to prove a tropical analogue of the splitting principle, which allows us to establish the foundations for Porteous' formula in this setting: a determinantal expression for the fundamental class of degeneracy loci in terms of Chern classes. The boundary framework is essential, as it allows the rank of a bundle morphism to drop at sedentary strata, giving degeneracy loci their expected codimension. We also discuss the foundational considerations that necessitated the adoption of the Mikhalkin–Rau framework, explaining why the standard $\mathbb{R}^n$-based framework of Allermann–Rau is insufficient for a meaningful theory of degeneracy loci.

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DEDICATION

For Alexis Trilby Torres

Chapter 1
INTRODUCTION

Classically, Porteous' formula enters Brill–Noether theory through description of loci where an evaluation map of vector bundles drops rank. It turns a geometric condition on linear series into an intersection-theoretic calculation involving only the Chern classes of the bundles. This work began the quest for a tropical analogue of this statement: answering the question of whether the same consequences can hold when the calculations become entirely polyhedral and combinatorial.

1.1 The Classical Porteous Formula

In classical algebraic geometry, Porteous' formula describes the *fundamental class* of degeneracy loci: the locus of points where the rank of a bundle morphism $\varphi : \mathcal{E} \to \mathcal{F}$ over X is at most an integer k.

Theorem 1.1.1 (Porteous Formula)**.** *Let $\varphi : \mathcal{E} \to \mathcal{F}$ be a morphism of vector bundles of ranks e and f on a smooth variety X. If the scheme $D_k(\varphi) \subseteq X$ has codimension $(e-k)(f-k)$, then its class is given by*

$$[D_k(\varphi)] = \Delta^{e-k}_{f-k}\left[\frac{c(\mathcal{F})}{c(\mathcal{E})}\right].$$

The right-hand side is a Sylvester determinant whose entries depend only on the Chern classes of $\mathcal{F}$ and $\mathcal{E}$. By expressing a subspace of interest as a degeneracy locus of a bundle morphism, one deduces its class from the Chern classes of the bundles alone.

1.2 A Tropical Analogue

We adapt this statement to the tropical setting using the language of tropical vector bundles and Chern classes, building largely on the work of [1]. Our main result is a tropical analogue of Porteous' formula in the rank-0 case (Theorem 7.4.1), with the resulting fundamental class residing in the tropical Chow ring:

Theorem 1.2.1 (Tropical Porteous Formula in Rank 0). *Let $\varphi : \mathcal{E} \to \mathcal{F}$ be a morphism of tropical vector bundles of ranks e and f over a rational polyhedral space X, with bounded matrix entries. If $D_0(\varphi)$ has the expected codimension ef, then*

$$[D_0(\varphi)] = \Delta_f^e\left(\frac{c_{[t]}(\mathcal{F})}{c_{[t]}(\mathcal{E})}\right),$$

where $c_{[t]}(\mathcal{E}) := \sum_{p\geq 0} c_p(\mathcal{E})\, t^p$ denotes the Chern polynomial.

Along the way, we develop the constructions that make this formula possible: tropical vector bundles and their operations, Chern classes via bounded rational sections, tropical projectivization, a tropical splitting principle, and the Hom bundle correspondence that identifies rank-0 degeneracy loci with zero loci of sections.

The long-term motivation for this tropical Porteous theory is the Brill–Noether conjecture of Gross–Shokrieh [7]; the connection is expanded on in Chapter 8.

1.3 Contributions

The main contributions of this thesis, in order of appearance, are the following.

(i) We adopt the Mikhalkin–Rau framework of rational polyhedral spaces modelled on $\mathbb{T}^n$ as the correct foundation for a Porteous-type formula, and explain in Chapter 2 why the standard $\mathbb{R}^n$-based framework of Allermann–Rau is insufficient.

(ii) We develop the basic theory of tropical vector bundles needed here, including Chern classes defined via bounded rational sections and a tropical Whitney formula (Chapters 4–5).

(iii) We construct the tropical projectivization $\mathbb{TP}(\mathcal{E})$ and prove a tropical splitting principle (Chapter 6), which allows computations with virtual Chern roots.

(iv) We establish the Hom bundle correspondence identifying the rank-0 degeneracy locus $D_0(\varphi)$ with the zero locus of a canonical bounded rational section of $\mathcal{E}^\vee \otimes \mathcal{F}$ (Section 7.3).

(v) We prove the rank-0 tropical Porteous formula (Theorem 7.4.1), the main theorem of the thesis, and outline paths towards the full generalization in Chapter 8.

1.4 Outline of the Thesis

Chapter 2 explains why the theory of degeneracy loci cannot be developed over $\mathbb{R}^n$ alone and why the passage to $\mathbb{T}^n$ resolves the obstruction, utilizing the framework developed by Mikhalkin–Rau in [15].

Chapter 3 recalls the basic notions of rational polyhedral spaces in $\mathbb{T}^n$ and tropical intersection theory, following [15] and [2].

Chapter 4 develops the theory of tropical vector bundles: tropical linear algebra, bundle definitions and operations, morphisms of bundles, and upper semicontinuity of the rank function.

Chapter 5 defines Chern classes of tropical vector bundles via bounded rational sections and establishes their fundamental properties, including a tropical Whitney formula.

Chapter 6 constructs the tropical projectivization of a vector bundle and proves the tropical splitting principle.

Chapter 7 contains the main result: the tropical Porteous formula in rank 0 (Theorem 7.4.1), proved via the Hom bundle correspondence.

Chapter 8 discusses extensions to higher rank via a prospective tropical Grassmann bundle and the connection to the tropical Brill–Noether conjecture.

Chapter 2

THE QUESTION OF FIBERS: FROM $\mathbb{R}^N$ TO $\mathbb{T}^N$

The choice of ambient space directly determines whether tropical degeneracy loci carry the expected structure. This chapter describes the limitation of the standard $\mathbb{R}^n$-based framework, the resolution offered by the Mikhalkin–Rau theory of rational polyhedral spaces in $\mathbb{T}^n$, and the consequences of this choice for the rest of the thesis.

2.1 The Allermann–Rau Framework

The standard framework for tropical intersection theory was developed by Allermann and Rau [2], building on foundational work of Mikhalkin and others. In this framework, the ambient space is $\mathbb{R}^n$, and the primary objects are *tropical cycles*: equivalence classes of weighted, pure-dimensional, rational polyhedral complexes in $\mathbb{R}^n$ satisfying the balancing condition. That is, for each codimension-one face τ, the weighted sum of primitive normal vectors around τ vanishes.

On this foundation, Allermann and Rau constructed the full machinery of tropical intersection theory: rational and regular invertible functions, Cartier and Weil divisors, the affine intersection product, and the Chow ring $A(X)$. Utilizing this, Allermann [1] developed the theory of tropical vector bundles over tropical cycles: rank-r bundles with fibers $\mathbb{R}^r$, transition maps valued in the group $G(r)$ of invertible tropical matrices, Chern classes defined via bounded rational sections, and a tropical analogue of Whitney's formula. This framework provided the initial foundation for the results of this thesis.

2.2 The Boundary Problem

Classically, the rank of a bundle morphism $\varphi : \mathcal{E} \to \mathcal{F}$ drops on a proper closed subvariety, so degeneracy loci acquire their expected codimension. Over $\mathbb{R}^n$ the analogous statement was found to fail – precisely because of the lack of boundary strata.

More precisely, let U be a connected open subset of the interior of a top-dimensional face of X, and let $A(p) \in G(f \times e)$ be a local matrix for φ on U. Each entry of $A(p)$ is either a regular invertible function on U or the constant function $-\infty$; a regular invertible function is $\mathbb{R}$-valued and never takes the value $-\infty$ on U. Consequently, for each position (i, j), either $a_{ij}(p)$ is finite for every $p \in U$ or $a_{ij}(p) = -\infty$ for every $p \in U$, and the pattern of finite entries of $A(p)$ is constant on U. By Lemma 4.1.8, the tropical rank of a matrix in $G(f \times e)$ is determined by this pattern of finite entries. Hence troprank$(A(p))$, and therefore rank(φ), is constant on U; equivalently, rank(φ) is locally constant on the interior of each top-dimensional face of X.

It follows that $D_k(\varphi) = \{p \in |X| \mid \text{troprank}(\varphi_p) \leq k\}$ is a union of connected components of X and thus has codimension zero. This created three immediate problems for the Porteous formula:

1. Since $D_k(\varphi)$ is a union of connected components, its class $[D_k(\varphi)]$ lies in codimension 0 rather than the expected codimension $(e - k)(f - k)$.

2. The proof of the main theorem requires the equality $[D_0(\varphi)] = c_{ef}(\mathcal{E}^\vee \otimes \mathcal{F}) \cdot [X]$. In the $\mathbb{R}^n$ framework, the left-hand side lies in codimension 0 while the right-hand side lies in codimension ef.

3. The proof that $\varphi^* : A(X) \to A(\mathbb{TP}(\mathcal{E}))$ is a split monomorphism depends on the intersection theory of the projective bundle $\mathbb{TP}(\mathcal{E})$, whose fibers $\mathbb{TP}^{n-1}$ already carry boundary strata. That boundary was not represented in the $\mathbb{R}^n$ framework.

2.3 The Mikhalkin–Rau Resolution

All three of these issues were resolved by re-framing the work and utilizing the Mikhalkin–Rau framework [15], which enlarges the ambient space so that $-\infty$ is a legitimate coordinate value. The Mikhalkin–Rau framework [15] does exactly this, replacing $\mathbb{R}^n$ with the *extended tropical affine space*

$$\mathbb{T}^n = (\mathbb{R} \cup \{-\infty\})^n.$$

We defer the definitions of sedentarity $I(p)$, the sedentarity order $\mathrm{sed}(p)$, and the stratification $\mathbb{T}^n = \bigsqcup_{I \subseteq [n]} \mathbb{T}^n_I$ to Chapter 3. For now, we only need the picture of $\mathbb{T}^n$ as a manifold with corners whose open stratum is the usual $\mathbb{R}^n$ and whose boundary strata are indexed by subsets $I \subseteq [n]$.

The benefit of working in $\mathbb{T}^n$ is that a continuous piecewise-linear function on $|X|$ can now genuinely approach $-\infty$ as one moves toward a sedentary stratum. For a morphism $\varphi : \mathcal{E} \to \mathcal{F}$ locally represented by a matrix $A_i(p) \in G(f \times e)$, the entries that are regular invertible on the interior can approach $-\infty$ at a sedentary face of X. Additional entries become $-\infty$ at such faces, the tropical rank drops, and $D_k(\varphi)$ acquires faces of positive codimension where rank drops. This makes the expected codimension $(e-k)(f-k)$ attainable. This framework naturally extends the Allermann–Rau theory: the balancing condition, intersection product, Chern class constructions, and the rest of the basic algebraic formalism carry over straightforwardly, while providing the crucial addition of sedentarity in order for us to gain access to the boundary.

2.4 Mathematical Consequences

Passing to the $\mathbb{T}^n$ framework has had four notable consequences.

Upper semicontinuity of rank. The rank function $\mathrm{rank}(\varphi) : |X| \to \mathbb{Z}$ becomes upper semicontinuous rather than merely locally constant (Corollary 4.3.4). On the interior of each top-dimensional face of X the rank is still locally constant, but at sedentary strata regular invertible functions may take the value $-\infty$, causing additional matrix entries to vanish and the rank to drop. The sublevel sets $D_k(\varphi)$ are therefore closed, with their interesting faces occurring at the boundary.

Expected codimension for degeneracy loci. The degeneracy locus $D_k(\varphi)$ is a rational polyhedral subspace of X (Lemma 4.3.5) whose expected codimension is $(e-k)(f-k)$, matching the classical situation (Proposition 7.2.2). On the interior, $D_k(\varphi) \cap X^\circ$ remains a union of connected components, but the sedentary faces where rank drops give $D_k(\varphi)$ additional faces of positive codimension.

The Hom bundle correspondence. The equality $[D_0(\varphi)] = c_{ef}(\mathcal{E}^\vee \otimes \mathcal{F}) \cdot [X]$ can be justified through a Hom bundle argument, which we provide in Section 7.3. A morphism $\varphi : \mathcal{E} \to \mathcal{F}$ corresponds to a canonical bounded rational section σ_φ of $\mathcal{E}^\vee \otimes \mathcal{F}$ (Proposition 7.3.1), and $D_0(\varphi) = Z(\sigma_\varphi)$ (Proposition 7.3.2). Under the expected-codimension hypothesis, $[D_0(\varphi)] = c_{ef}(\mathcal{E}^\vee \otimes \mathcal{F}) \cdot [X]$ follows from the independence of Chern classes from the choice of section (Proposition 7.3.3).

The splitting principle on spaces with boundary. The proof that $\varphi^* : A(X) \to A(\mathbb{TP}(\mathcal{E}))$ is a split monomorphism (Lemma 6.2.4) is carried out directly by fiberwise computation. The total space $\mathbb{TP}(\mathcal{E})$ is a rational polyhedral space whose boundary arises from both the base X and the fibers $\mathbb{TP}^{n-1}$, and the intersection theory of [15] applies to this space with boundary.

Chapter 3

TROPICAL PRELIMINARIES

We recall the foundational terminology of polyhedral complexes, following the framework of rational polyhedral spaces developed in [15]. This framework incorporates *boundary* (via the extended tropical affine space $\mathbb{T}^n$), which is essential for defining degeneracy loci of the expected codimension. The reader unfamiliar with these notions may treat them in analogy to their classical algebraic-geometric counterparts; the manner in which we interact with them — for instance in manipulations with Chern classes — is almost indistinguishable from the classical theory.

For an equivalent approach to tropical vector bundles using *integral affine manifolds*, or through sheaf-theoretic language, see [13] or [6], respectively.

3.1 Rational Polyhedral Spaces

We take the polytopal perspective following [15, 2]. The key departure from the setup of [2] is that we work in the *extended tropical affine space* $\mathbb{T}^n = (\mathbb{R} \cup \{-\infty\})^n$ rather than $\mathbb{R}^n$. This allows polyhedra to have faces meeting the boundary of $\mathbb{T}^n$, which is the mechanism by which morphism rank can drop.

Definition 3.1.1 (Sedentarity)**.** For a point $p \in \mathbb{T}^n$, the *sedentarity* of p is the set $I(p) = \{i \in [n] \mid p_i = -\infty\}$. The *sedentarity order* $\mathrm{sed}(p) = |I(p)|$ counts the number of coordinates equal to $-\infty$. For each subset $I \subseteq [n]$, the *sedentary stratum* $\mathbb{T}^n_I = \{p \in \mathbb{T}^n \mid I(p) = I\}$ is naturally identified with $\mathbb{R}^{n-|I|}$, and we have

$$\mathbb{T}^n = \bigsqcup_{I \subseteq [n]} \mathbb{T}^n_I.$$

The open stratum $\mathbb{T}^n_\emptyset = \mathbb{R}^n$ is called the *interior*.

Definition 3.1.2 (Rational polyhedral complex in $\mathbb{T}^n$)**.** A *rational polyhedral complex* X in $\mathbb{T}^n$ is a weighted, pure-dimensional, rational, finite polyhedral complex in $\mathbb{T}^n$ (with underlying lattice $\mathbb{Z}^n \subseteq \mathbb{R}^n$) that satisfies the *balancing condition* for every $\tau \in X^{(k-1)}$:

$$\sum_{\sigma:\tau<\sigma} \omega(\sigma)\, u_{\sigma/\tau} = 0 \in \Gamma_\tau^\vee,$$

where, for a face $\sigma \in X$, we write $\Gamma_\sigma \subseteq \mathbb{Z}^n$ for the lattice of primitive integer vectors parallel to σ, and $u_{\sigma/\tau} \in \Gamma_\sigma/\Gamma_\tau$ denotes the primitive integer generator of the ray obtained by projecting σ to the quotient V_σ/V_τ, with V_τ the real linear span of τ. Faces of X may be *sedentary*, meaning they are contained in a boundary stratum $\mathbb{T}^n_I$ for some $I \neq \emptyset$. When a ridge τ is sedentary with $\tau \subseteq \mathbb{T}^n_I$, the balancing condition above is imposed within the stratum $\mathbb{T}^n_I \cong \mathbb{R}^{n-|I|}$ in which τ has its relative interior, using the induced lattice on that stratum; only facets $\sigma \supseteq \tau$ contained in the same stratum contribute to the sum. See [15, Ch. 4] for details.

We call the top-dimensional polyhedra in a pure-dimensional X the *facets* and the codimension one polyhedra the *corners* (or *ridges*) of X. We denote by $|X|$ the *support*, which is the union of all facets of X having non-zero weight. Two rational polyhedral complexes are *equivalent* if they admit a common refinement with the same induced weights. A *rational polyhedral space* (or *tropical cycle*) X is an equivalence class of rational polyhedral complexes. When X has no sedentary faces (i.e., $|X| \subseteq \mathbb{R}^n$), this recovers the framework of [2].

Remark 1. The framework of rational polyhedral spaces in $\mathbb{T}^n$ was developed by Mikhalkin–Rau [15]. In this setting, the boundary strata of $\mathbb{T}^n$ give the ambient space the structure of a manifold with corners. When a polyhedral complex X has faces meeting these boundary strata, continuous functions on $|X|$ (including matrix entries of a bundle morphism) can degenerate to $-\infty$ as one approaches the boundary, enabling rank to drop. This is precisely the mechanism by which degeneracy loci acquire positive codimension.

3.2 Tropical Intersection Theory

The intersection-theoretic machinery of Allermann–Rau [2] extends to rational polyhedral spaces in $\mathbb{T}^n$; see [15] for the general development. We recall the key definitions below, noting that they are stated for rational polyhedral spaces (which may have sedentary faces) but specialize to the original formulations of [2] when restricted to $\mathbb{R}^n$.

Definition 3.2.1 (Abstract tropical cycles)**.** Let $((X, |X|), \omega_X)$ be an n-dimensional rational polyhedral complex. Its equivalence class $[((X, |X|), \omega_X)]$ is an *(abstract) tropical n-cycle.* The *set of n-cycles* is denoted Z_n, and we define $|((X, |X|), \omega_X)| := |X^*|$, where X^* denotes the union of the facets of X having non-zero weight. An n-cycle $((X, |X|), \omega_X)$ is an *(abstract) tropical variety* if $\omega_X(\sigma) \geq 0$ for all $\sigma \in X^{(n)}$.

As an illustration, consider the following figure depicting an abstract tropical polyhedral complex, which becomes a tropical variety when each cone σ is assigned weight one.

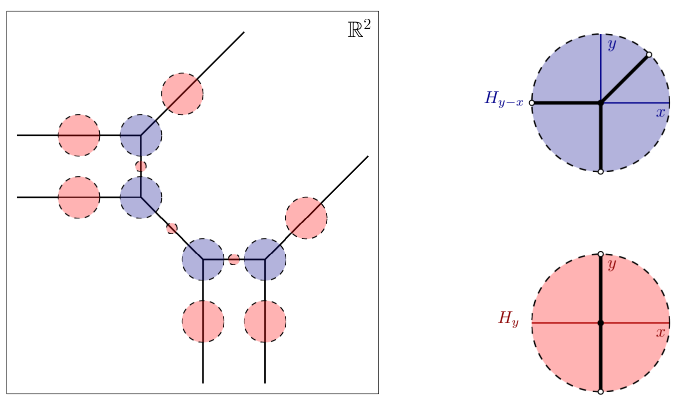

Figure 3.1: An abstract tropical polyhedral complex, as well as a covering with open fans. Here, $H_f \subseteq \mathbb{R}^2$ denotes the fan having cones $\{f(x) \leq 0\}$, $\{f(x) = 0\}$, and $\{f(x) \geq 0\}$.

A tropical subcycle is a tropical cycle living within a larger ambient cycle, inheriting the expected structure from the ambient space.

Definition 3.2.2 (Abstract tropical subcycles)**.** Let $C \in Z_n$ and $D \in Z_k$ be two tropical cycles. Then D is called an *(abstract) tropical cycle in* C or a *subcycle of* C if there exists a representative $(((Z,|Z|),\omega_Z),\{\Psi_\tau\})$ of D and a reduced representative $(((X,|X|),\omega_X),\{\Phi_\sigma\})$ of C such that

(a) $(Z,|Z|) \trianglelefteq (X,|X|)$,

(b) the tropical structures on Z and X agree, i.e. for every $\tau \in Z$ the maps $\Psi_\tau \circ \Phi^{-1}_{C_{Z,X}(\tau)}$ and $\Phi_{C_{Z,X}(\tau)} \circ \Psi_\tau^{-1}$ are integer affine linear where defined.

The *set of tropical* k*-cycles* in C is denoted by $Z_k(C)$.

In order to define Cartier divisors, we need the analogous notions of rational and regular invertible functions.

Definition 3.2.3 (Rational functions)**.** Let C be an abstract k-cycle and let U be an open set in $|C|$. A *(non-zero) rational function on* U is a continuous function $\varphi : U \to \mathbb{R}$ such that there exists a representative $(((X,|X|,\{m_\sigma\}_{\sigma\in X}),\omega_X),\{M_\sigma\}_{\sigma\in X})$ of C such that for each face $\sigma \in X$ the map $\varphi \circ m_\sigma^{-1}$ is locally integer affine linear (where defined). The *set of all non-zero rational functions on* U is denoted by $\mathcal{K}^*_C(U)$ (or $\mathcal{K}^*(U)$).

Definition 3.2.4 (Regular invertible functions)**.** With the same notation as above, if for each face $\sigma \in X$ the map $\varphi \circ M_\sigma^{-1}$ is locally integer affine linear (where defined), φ is called *regular invertible*. The *set of all regular invertible functions on* U is denoted by $\mathcal{O}^*_C(U)$ (or $\mathcal{O}^*(U)$).

Remark 2. For a k-cycle C, $(\mathcal{K}^*(C),\odot)$ is an abelian group and $(\mathcal{O}^*(C),\odot)$ is a subgroup.

With these ingredients in place, we can define divisors.

Definition 3.2.5 (Cartier divisors)**.** Let C be an abstract k-cycle. The set of *Cartier divisors on* C is denoted by $\mathrm{Div}(C)$.

- a *representative of a Cartier divisor on* C is a finite set

$$\{(U_1,\varphi_1),\ldots,(U_n,\varphi_n)\},$$

where $\{U_i\}$ is an open covering of $|C|$ and each $\varphi_i \in \mathcal{K}^*(U_i)$ is a rational function such that for $i \neq j$ we have[1]

$$\frac{\varphi_i|_{U_i \cap U_j}}{\varphi_j|_{U_i \cap U_j}} \in \mathcal{O}^*(U_i \cap U_j).$$

- We define the *product* of two representatives of Cartier divisors to be

$$\{(U_i, \varphi_i)\} \odot \{(V_j, \psi_j)\} := \{(U_i \cap V_j, \varphi_i \odot \psi_j)\}.$$

- We say two Cartier divisor representatives are *equivalent* if

$$\frac{\{(U_i, \varphi_i)\}}{\{(V_j, \psi_j)\}} = \{(W_k, \gamma_k)\} \quad \text{where } \gamma_k \in \mathcal{O}^*(W_k) \text{ for each } k.$$

It follows from the above constructions that the identity element of Div(C) is $[\{(|C|, 0)\}]$, where 0 denotes the constant zero function. We define the *group of Cartier divisors of* C to be the quotient group $\mathrm{Div}(C) := \mathcal{K}^*(C)/\mathcal{O}^*(C)$.

Lastly, the notion of a Weil divisor will be necessary for defining the intersection product, through which we later interact with higher Chern classes.

Definition 3.2.6 (Associated Weil divisors)**.** Let C be a k-cycle and $\varphi \in \mathcal{K}^*(C)$ a rational function on C. Let furthermore (X, ω) be a representative of C on whose cones φ is integer affine linear. We define the *Weil divisor* of φ, denoted by div(φ), to be

$$\mathrm{div}(\varphi) := \left[\left(\bigcup_{i=0}^{k-1} X^{(i)}, \omega_\varphi\right)\right] \in Z_{k-1}(C),$$

where the weight function $\omega_\varphi : X^{(k-1)} \to \mathbb{Z}$ is given by

$$\omega_\varphi(\tau) = \sum_{\substack{\sigma \in X^{(k)} \\ \tau < \sigma}} \varphi_\sigma(\omega(\sigma)\widetilde{u}_{\sigma/\tau}) - \varphi_\tau\left(\sum_{\substack{\sigma \in X^{(k)} \\ \tau < \sigma}} \omega(\sigma)\, \widetilde{u}_{\sigma/\tau}\right),$$

with $\widetilde{u}_{\sigma/\tau} \in \Gamma_\sigma$ an arbitrary integer lift of the normal vector $u_{\sigma/\tau} \in \Gamma_\sigma/\Gamma_\tau$ introduced above. (The difference is independent of the chosen lifts.) If D is an arbitrary subcycle of C, we set $\varphi \cdot D := \mathrm{div}(\varphi|_{|D|})$.

[1]Writing $U_{ij} = U_i \cap U_j$, this is equivalent to $\varphi_i|_{U_{ij}} - \varphi_j|_{U_{ij}} \in \mathcal{O}^*(U_{ij})$.

Definition 3.2.7 (Intersection product, [2])**.** Let C be a k-cycle and $\varphi \in \mathrm{Div}(\mathsf{C})$ a Cartier divisor. The natural bilinear mapping[2]

$$\begin{aligned} \mathrm{Div}(\mathsf{C}) \times \mathsf{Z}_{\mathsf{k}}(\mathsf{C}) &\longrightarrow \mathsf{Z}_{\mathsf{k}-1}(\mathsf{C}) \\ ([\varphi], \mathsf{D}) &\longmapsto \varphi \cdot \mathsf{D} := [\varphi] \cdot \mathsf{D} \end{aligned}$$

is called the *intersection product*.

[2]Where, recall, if D is a subcycle of C, then $\varphi \cdot \mathsf{D} = \mathrm{div}(\varphi|_{|\mathsf{D}|})$.

Chapter 4

TROPICAL VECTOR BUNDLES

4.1 Tropical Linear Algebra

Definition 4.1.1 (Tropical matrices)**.** A *tropical matrix* will be a matrix with entries in the tropical semi-ring $(\mathbb{T}, \oplus, \odot)$ where $\mathbb{T} := \mathbb{R} \cup \{-\infty\}$ and

$$a \oplus b := \max\{a, b\} \quad \text{and} \quad a \odot b := a + b.$$

We will be using the following notation:

- $\mathrm{Mat}(m \times n, \mathbb{T})$ for the set of tropical $m \times n$ matrices,
- $G(r \times s) \subseteq \mathrm{Mat}(r \times s, \mathbb{T})$ for the subset having at most one finite entry in each row,
- $G(r) \subseteq G(r \times r)$ for the subset with exactly one finite entry in every row and column.

Lastly, when multiplying matrices $A \in \mathrm{Mat}(m \times n, \mathbb{T})$ and $B \in \mathrm{Mat}(n \times p, \mathbb{T})$, we take their tropical product $A \odot B := (c_{ij}) \in \mathrm{Mat}(m \times p, \mathbb{T})$ where $c_{ij} = \bigoplus_{k=1}^{n} a_{ik} \odot b_{kj}$.

Remark 3. A matrix $A \in G(r \times s)$, for an $x \in \mathbb{R}^s$, may send x to $A \odot x \notin \mathbb{R}^r$ (i.e. contain entries that are $-\infty$). If we need to interpret such a matrix as a map $f_A : \mathbb{R}^s \to \mathbb{R}^r$, we redefine any entries that are $-\infty$ in $A \odot x$ to be zero.

For a permutation $\sigma \in S_n$, we denote by $E_\sigma \in \mathrm{Mat}(n \times n, \mathbb{T})$ the matrix

$$e_{ij} := \begin{cases} 0 & \text{if } j = \sigma(i) \\ -\infty & \text{otherwise,} \end{cases}$$

which plays a role analogous to the permutations matrices, but in the tropical setting. Furthermore, for choices of $a_i \in \mathbb{R}$, we denote by $D(a_1, \dots, a_n) \in G(n)$ the matrix with $a_1, a_2, \dots, a_n$ along the diagonal and $-\infty$ elsewhere.

Remark 4. Using this, we may then write any matrix $M \in G(n)$ as

$$M = E_\sigma \odot D(a_1, \dots, a_n)$$

for some $\sigma \in S_n$ and $a_i \in \mathbb{R}$. This makes it evident that $G(n)$ is a group under tropical matrix multiplication, having identity element $D(0, \dots, 0)$, which we denote by E_n to distinguish it from $I_n \in \mathrm{Mat}(n \times n, \mathbb{R})$.

Lemma 4.1.2. *The group $G(n) \subseteq Mat(n \times n, \mathbb{T})$ is precisely the collection of invertible tropical matrices:*

$$GL_n(\mathbb{T}) = G(n) = \{A \in Mat(n \times n, \mathbb{T}) \mid (\exists B \in Mat(n \times n, \mathbb{T})) \colon A \odot B = B \odot A = E_n\}.$$

The proof is a short computation (resembling those in the proof of Lemma 4.1.8 below), as these matrices behave similarly to permutation matrices. In fact, $\mathrm{GL}_n(\mathbb{T})$ is isomorphic to $S_n \ltimes \mathbb{R}^n$, so one may think of invertible tropical matrices as permutation matrices with 0's replaced by $-\infty$ and 1-entries replaced by elements of $\mathbb{R}$.

Definition 4.1.3 (Tropical determinant)**.** Consider a tropical matrix $A \in \mathrm{Mat}(n \times n, \mathbb{T})$. We define the *tropical determinant* to be

$$\mathrm{tropdet}(A) = \bigoplus_{\sigma \in S_n} (A_{1,\sigma(1)} \odot A_{2,\sigma(2)} \odot \cdots \odot A_{n,\sigma(n)})$$

Lemma 4.1.4. *Consider a tropical matrix $A \in G(n \times n)$. We then have that*

$$\mathit{tropdet}(A) \neq -\infty \iff A \in G(n).$$

That is to say, the only tropical matrices with "non-trivial" determinants are precisely those that are invertible.

Proof. The reverse direction is immediate: if $A \in G(n)$, then A has exactly one finite entry in each row and column, so there is a unique $\sigma \in S_n$ with $A_{i,\sigma(i)}$ finite for all i, and

$$\mathrm{tropdet}(A) = A_{1,\sigma(1)} \odot \cdots \odot A_{n,\sigma(n)} \in \mathbb{R}.$$

For the forward direction, assume that $\operatorname{tropdet}(A) \neq -\infty$. By definition

$$\operatorname{tropdet}(A) = \bigoplus_{\sigma \in S_n} A_{1,\sigma(1)} \odot A_{2,\sigma(2)} \odot \cdots \odot A_{n,\sigma(n)},$$

so there is some $\sigma \in S_n$ such that $A_{1,\sigma(1)} \odot \cdots \odot A_{n,\sigma(n)} \neq -\infty$. Hence each $A_{i,\sigma(i)}$ is finite. Since $A \in \mathrm{G}(n \times n)$ has at most one finite entry per row, $A_{i,\sigma(i)}$ is the unique finite entry of row i. It remains to check that every column also contains exactly one finite entry. Given $j \in [n]$, let $j = \sigma(i)$ for the corresponding unique $i \in [n]$; then it follows that $A_{i,j}$ is finite. If $A_{k,j}$ were also finite for some k, then the unique finite entry of row k would sit in column $\sigma(k)$, forcing $\sigma(k) = j = \sigma(i)$ and therefore $k = i$. Hence A has exactly one finite entry in every row and every column, so $A \in \mathrm{G}(n)$. $\square$

This is reminiscent of [10, Theorem 3.1]; the reader interested in pursuing tropical linear algebra further is directed there.

Remark 5. Observe that this does not hold for a general tropical matrix in $\mathrm{Mat}(n \times n, \mathbb{T})$:

$$\operatorname{tropdet}\begin{pmatrix} 0 & 0 \\ 0 & -\infty \end{pmatrix} = (0 \odot -\infty) \oplus (0 \odot 0) = -\infty \oplus 0 = 0,$$

but the above matrix has more than one finite entry in its first row, so it is not in $\mathrm{G}(2)$.

Definition 4.1.5 (Tropical rank)**.** For $A \in \mathrm{G}(n \times m)$, the *tropical rank* of A is defined in analogy with the classical case:

$$\operatorname{troprank}(A) = k \iff \begin{cases} \text{all } (k+1) \times (k+1) \text{ minors vanish, and} \\ \text{some } k \times k \text{ minor does not vanish,} \end{cases}$$

where a minor is said to *vanish* if it equals $-\infty$.

The "correct" notion of rank for tropical matrices is a matter of debate, but the definition above suffices for our purposes. Other variants, such as *Barvinok rank* and *Kapranov rank*, are related as follows:

Theorem 4.1.6 ([4])**.** *For every matrix* M *with entries in the tropical semiring,*

$$\textit{troprank}(M) \leq \textit{Kapranov rank}(M) \leq \textit{Barvinok rank}(M).$$

Both of these inequalities can be strict.

Geometrically, every $(d \times n)$-matrix has an image that is a tropical polytope in $\mathbb{R}^d/\mathbb{R}\mathbf{1}$ (see [14, §5.2]). The tropical rank of a matrix is the dimension of this tropical polytope *plus one.*

Example 4.1.7. Unlike working over $\mathbb{R}$, (tropical) rank is not generally additive over $\mathbb{T}$:

$$\text{troprank}\begin{pmatrix} 2 & -\infty & -\infty \\ -\infty & -\infty & 3 \end{pmatrix} = 2 \qquad \text{and} \qquad \text{troprank}\begin{pmatrix} 0 & -\infty & -\infty \\ -\infty & 0 & -\infty \\ -\infty & -\infty & -\infty \end{pmatrix} = 2$$

since $\left|\begin{smallmatrix} 2 & -\infty \\ -\infty & 3 \end{smallmatrix}\right| = 5 \neq -\infty$ and $\left|\begin{smallmatrix} 0 & -\infty \\ -\infty & 0 \end{smallmatrix}\right| = 0 \neq -\infty$, yet

$$\begin{pmatrix} 2 & -\infty & -\infty \\ -\infty & -\infty & 3 \end{pmatrix} \odot \begin{pmatrix} 0 & -\infty & -\infty \\ -\infty & 0 & -\infty \\ -\infty & -\infty & -\infty \end{pmatrix} = \begin{pmatrix} 2 & -\infty & -\infty \\ -\infty & -\infty & -\infty \end{pmatrix}$$

but the resulting matrix has tropical rank 1. Thus for matrices $A \in \mathrm{G}(n \times m)$ and $B \in \mathrm{G}(m \times s)$, generally

$$\text{troprank}(A \odot B) \neq \text{troprank}(A) + \text{troprank}(B).$$

However, repositioning the finite entries via a tropical change of basis restores equality. We make this precise below.

Lemma 4.1.8. *Let* $A \in G(n \times m)$ *and take* $N \in G(n)$ *and* $M \in G(m)$*. Then*

$$\textit{troprank}(A) = \textit{troprank}(N \odot A \odot M)$$

and troprank(A) *is the number of columns having at least one finite entry.*

Proof. First we prove that troprank(A) is the number of columns having at least one finite entry. The matrices in $G(n \times m)$ are only subject to having at most one finite entry in each row. As such, there may be empty (all $-\infty$) columns or rows, which do not contribute to the rank. Assume then without loss of generality that A has no such empty rows or columns. Then the dimension of A is $k \times \ell$ with $k \geq \ell$, and since a minor is a determinant of a square matrix, we must have that troprank$(A) = \ell$. But as ℓ was the number of columns having a finite entry, we are done.

To prove the second statement, by Remark 4, we may write

$$N = E_\sigma \odot D(\beta_1, \dots, \beta_n) \quad \text{and} \quad M = E_\tau \odot D(\gamma_1, \dots, \gamma_m)$$

where $\beta_i, \gamma_j \in \mathbb{R}$. This allows us to express the product as

$$N \odot A \odot M = E_\sigma \odot D(\beta_1, \dots, \beta_n) \odot A \odot E_\tau \odot D(\gamma_1, \dots, \gamma_m).$$

We may assume without loss of generality that β_i, γ_j are all zero, as the only aspect that would distinguish rank is whether entries are infinite. But as $D(0, \dots, 0)$ is the identity matrix, this simplifies this expression immediately:

$$E_\sigma \odot E_n \odot A \odot E_\tau \odot E_m = E_\sigma \odot A \odot E_\tau.$$

This situation is now analogous to multiplying with permutation matrices, where E_σ is permuting the rows of A and E_τ the columns. It follows that $\oplus_{\alpha \in S_k} \odot_{i=1}^{k} a_{i,\alpha(i)} \neq -\infty$ if and only if $\oplus_{\alpha \in S_k} \odot_{i=1}^{k} a_{\sigma(i),\tau(\alpha(i))} \neq -\infty$, hence troprank$(A) =$ troprank$(N \odot A \odot M)$. $\square$

Many common matrix operations carry over to the tropical setting, but multiplication and addition must be replaced by their tropical counterparts.

Definition 4.1.9 (Direct sum and tensor products of tropical matrices)**.** Consider two tropical matrices $A = (a_{ij}) \in \mathrm{Mat}(m \times n, \mathbb{T})$ and $B = (b_{st}) \in \mathrm{Mat}(k \times \ell, \mathbb{T})$, we then define

- the *direct sum* to be the $(m + k) \times (n + \ell)$ tropical matrix

$$A \oplus B = \begin{pmatrix} A & -\infty \\ -\infty & B \end{pmatrix},$$

- the *tensor product* to be the $mk \times n\ell$ tropical matrix

$$A \otimes B = \begin{pmatrix} a_{11} \odot B & \cdots & a_{1n} \odot B \\ \vdots & \ddots & \vdots \\ a_{m1} \odot B & \cdots & a_{mn} \odot B \end{pmatrix}.$$

It follows that the tensor product of tropical matrices expands over their direct sum, i.e.

$$\left(\bigoplus_{i=1}^{n} A_i \right) \otimes B = \bigoplus_{i=1}^{n} (A_i \otimes B),$$

which can be seen by expanding out the entries.

4.2 Definitions and Operations

We now introduce the primary objects of study. Most of these notions were introduced or formulated for our purposes in [1], to which the reader is directed for further details on tropical vector bundles.

Definition 4.2.1 (Tropical vector bundles)**.** Let X be a rational polyhedral space. A *tropical vector bundle over* X *of rank* r is a rational polyhedral space F together with a morphism $\pi : F \to X$ and a finite open covering $\{U_1, \ldots, U_s\}$ of X as well as a homeomorphism $\Phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^r$ for every $1 \leq i \leq s$ such that

(a) for all i we have that the diagram

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[r, "\Phi_i"] \arrow[dr, "\pi"'] & U_i \times \mathbb{R}^r \arrow[d, two heads, "p_1"] \\ & U_i \end{tikzcd}$$

commutes, where $p_1 : U_i \times \mathbb{R}^r \to U_i$ is the projection to the first factor,

(b) for all i, j the composition $p_j^{(i)} \circ \Phi_i : \pi^{-1}(U_i) \to \mathbb{R}$ is a regular invertible function, where $p_j^{(i)} : U_i \times \mathbb{R}^r \to \mathbb{R}$ is projection onto the jth component of $\mathbb{R}^r$, i.e. $(x, (a_1, \ldots, a_r)) \mapsto a_j$,

(c) for every $i, j \in \{1, \ldots, s\}$ there exists a *transition map* $M_{ij} : U_i \cap U_j \to G(r)$ such that

$$\Phi_j \circ \Phi_i^{-1} : (U_i \cap U_j) \times \mathbb{R}^r \to (U_i \cap U_j) \times \mathbb{R}^r$$

is given by $(x, a) \mapsto (x, f_{M_{ij}(x)}(a))$ and the entries of M_{ij} are regular invertible functions on $U_i \cap U_j$ or constantly $-\infty$,

(d) there exist representatives F_0 of F and X_0 of X such that

$$F_0 = \{\pi^{-1}(\tau) \mid \tau \in X_0\}$$

and $\omega_{F_0}(\pi^{-1}(\tau)) = \omega_{X_0}(\tau)$ for all maximal polyhedral $\tau \in X_0$.

An open set U_i with the map $\Phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^r$ is called a *local trivialization* of F.

Remark 6. In the original source [1], the fibers are all taken to be $\mathbb{R} = \mathbb{T}^{\times}$, but a reader familiar with other work in this area (e.g. [13] and [6]) will notice that at times the fibers are instead taken to be $\mathbb{T}$. This hinges on condition (c) above, where we funnel transition maps M_{ij} through $f_{M_{ij}}$. Throughout the work we will at times take fibers that are $\mathbb{T}$ instead, and this is meant to imply that our transition maps no longer go through this extra step to be re-interpreted as a map on $\mathbb{R}^n$.

There has been considerable work with the above definition of tropical vector bundles (introduced by Allermann [1]). In [9, 6], the authors studied the case where X is an *integral affine manifold* (X, Aff_X), with Aff_X a sheaf of affine functions.

Adopting this language, we may then interpret tropical vector bundles of rank n as being $S_n \ltimes \mathrm{Aff}_X^n$-torsors. This terminology continues in the expected way (e.g. morphisms of tropical vector bundles become morphisms of torsors, etc.). Most interestingly though is the following result they obtain, which applies just as well in our setting:

Proposition 4.2.2 ([6]). *The category of tropical vector bundles of rank $n \in \mathbb{Z}_{\geq 1}$ on an integral affine manifold X is equivalent to the category of pairs $(\widetilde{X} \xrightarrow{\pi} X, \mathcal{L})$ consisting of a degree n free cover $\pi : \widetilde{X} \to X$ and a tropical line bundle $\mathcal{L} \to X$.*

That is, a tropical vector bundle of rank n on X may be constructed by passing to a covering space $\widetilde{X}$ and assembling line bundle fibers over each point. The simplest example takes X to be a circle with a rank-2 bundle and $\widetilde{X}$ a double cover with a line bundle.

In perhaps a different flavor, [11] generalizes this definition of tropical vector bundles to apply more broadly to tropical/monoidal schemes, proving in Proposition 5.1 that for a pair $(X, \mathcal{O}_X)$ of a topological space with a sheaf (of semirings) of $\mathbb{T}$-valued functions, that there is an equivalence of categories between topological $\mathbb{T}$-vector bundles on X (as defined in Definition 4.2.1) and locally free $\mathcal{O}_X$-modules, given by sending a vector bundle to its sheaf of continuous sections. They later prove an analogue of the above equivalence of isomorphism classes of rank-n topological $\mathbb{T}$-vector bundles with line bundles on n-fold covering spaces for locally connected paracompact Hausdorff spaces in [11, Proposition 5.5].

Definition 4.2.3 (Pull-back of vector bundles)**.** Let $\pi : F \to X$ be a vector bundle of rank n with an open covering $U_1, \ldots, U_s$ and transition maps M_{ij}. Let $f : Y \to X$ be a morphism of rational polyhedral spaces. Then the *pullback bundle* $\pi' : f^*F \to Y$ is the vector bundle we obtain by gluing the patches $f^{-1}(U_1) \times \mathbb{R}^n, \ldots, f^{-1}(U_s) \times \mathbb{R}^n$ along the transition maps $M_{ij} \circ f$. Hence we have the following commutative diagram

$$\begin{tikzcd}
f^*F \arrow[r, dashed, "f'"] \arrow[d, dashed, "\pi'"'] & F \arrow[d, "\pi"] \\
Y \arrow[r, "f"] & X
\end{tikzcd}$$

where f' and π' are locally given by

$$\begin{aligned} f' \colon f^{-1}(U_i) \times \mathbb{R}^n &\longrightarrow U_i \times \mathbb{R}^n & \qquad \pi' \colon f^{-1}(U_i) \times \mathbb{R}^n &\longrightarrow f^{-1}(U_i) \\ (y, a) &\longmapsto (f(y), a) & (y, a) &\longmapsto y. \end{aligned}$$

Definition 4.2.4 (Subbundles)**.** Let $\pi : F \to X$ be a vector bundle with an open trivialization $U_1, \ldots, U_s$ with corresponding homeomorphisms Φ_i. A subcycle $E \in Z_\ell(F)$ is called a *subbundle of rank* r *of* F if $\pi|_E : E \to X$ is a bundle of rank r such that for $1 \leq i \leq s$ we have homeomorphisms

$$\Phi_i|_{(\pi|_E)^{-1}(U_i)} : (\pi|_E)^{-1}(U_i) \to U_i \times \langle e_{j_1}, \ldots, e_{j'_r} \rangle_{\mathbb{R}}$$

for some $1 \leq j_1 < \cdots < j_{r'} \leq r$, where e_j are the standard basis vectors in $\mathbb{R}^r$.

Allermann noted the following as Remark 1.13 in [1]; we record it as a proposition since we will invoke it repeatedly. Together with the definition of quotient bundles below, it implies that all short exact sequences of tropical vector bundles split.

Proposition 4.2.5 ([1, Remark 1.13])**.** *If $\pi : F \to X$ is a vector bundle of rank n with a subbundle E of rank r, then there exists a subbundle E' of F of rank $n-r$ such that $F \cong E \oplus E'$.*

Proof. Locally E is identified with the span of a standard subset $\{e_{j_1}, \ldots, e_{j_r}\}$ of basis vectors of $\mathbb{R}^n$; let E' be the subbundle given locally by the span of the complementary basis vectors. Transition maps of F lie in $G(n)$, so they act on $\mathbb{R}^n$ as generalized permutations (permutation matrices in which the 1's are replaced by finite scalars), preserving each standard coordinate span up to the induced permutation. Consequently E' inherits the structure of a subbundle of F, and $F \cong E \oplus E'$ as tropical bundles. See [1, Remark 1.13] for details. □

Definition 4.2.6. Let $\pi : F \to X$ be a vector bundle of rank n. We say F is *decomposable* if there exists a subbundle $\pi|_E : E \to X$ of F of rank $1 \leq r < n$. Otherwise, F is said to be *indecomposable.*

This motivates the following definition.

Definition 4.2.7 (Quotient bundles)**.** Let $\pi : F \to X$ be a vector bundle and $F \cong S \oplus Q$. Then we define the *quotient bundle* F/S to be Q.

These definitions are straightforwardly consistent with one another, i.e. every short exact sequence of tropical vector bundles is split.

Definition 4.2.8 (Direct Sum, Tensor, Dual)**.** Let E and F be two vector bundles of ranks e and f over X with local trivializations $U_1, \ldots, U_s$ and transition maps $\{M_{ij}^E\}$ and $\{M_{ij}^F\}$. Then we have the following operations:

- the *direct sum bundle* $E \oplus F \to X$ is the rank $e + f$ bundle having transition maps

$$\begin{aligned} M_{ij}^{E\oplus F}: \quad U_i \cap U_j &\longrightarrow G(e+f) \\ M_{ij}^{E\oplus F}(p) &\longmapsto M_{ij}^{E}(p) \oplus M_{ij}^{F}(p) \end{aligned}$$

- the *tensor product bundle* $E \otimes F \to X$ is the rank ef bundle having transition maps

$$\begin{aligned} M_{ij}^{E\otimes F}: \quad U_i \cap U_j &\longrightarrow G(ef) \\ M_{ij}^{E\otimes F}(p) &\longmapsto M_{ij}^{E}(p) \otimes M_{ij}^{F}(p) \end{aligned}$$

- the *dual bundle* $E^\vee \to X$ is the rank e bundle having transition maps

$$\begin{aligned} M_{ij}^{E^\vee}: \ U_i \cap U_j &\longrightarrow G(e) \\ M_{ij}^{\vee}(p) &\longmapsto \left(M_{ij}(p)^T\right)^{-1} \end{aligned}$$

  where the inverse is taken under the natural group structure of $G(e)$.

If $A \in G(n)$ and $B \in G(m)$, then $A \otimes B \in G(nm)$, so the transition maps land in the codomain $G(fe) \subseteq \mathrm{Mat}(fe \times fe, \mathbb{T})$ as required. Indeed, after a (tropical) change of basis we may assume that the finite entries of A lie along the diagonal entries a_{ii} for $1 \le i \le n$. We then have that

$$\begin{aligned} A \otimes B &= \begin{pmatrix} a_{11} \odot B & a_{12} \odot B & \cdots & a_{1n} \odot B \\ a_{21} \odot B & a_{22} \odot B & \cdots & a_{2n} \odot B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} \odot B & a_{n2} \odot B & \cdots & a_{nn} \odot B \end{pmatrix} \\ &= \begin{pmatrix} a_{11} \odot B & -\infty & \cdots & -\infty \\ -\infty & a_{22} \odot B & \cdots & -\infty \\ \vdots & \vdots & \ddots & \vdots \\ -\infty & -\infty & \cdots & a_{nn} \odot B \end{pmatrix} \end{aligned}$$

and as B has exactly one finite entry in each row and column, we conclude that $A \otimes B \in G(nm)$.

4.3 Morphisms and Rank

Definition 4.3.1 (Morphisms of vector bundles). Let $\pi_{\mathcal{E}} : \mathcal{E} \to X$ and $\pi_{\mathcal{F}} : \mathcal{F} \to X$ be tropical vector bundles of ranks e and f respectively. A *morphism of vector bundles* $\varphi : \mathcal{E} \to \mathcal{F}$ is a morphism of rational polyhedral spaces such that

(a) the diagram

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "\varphi"] \arrow[dr, "\pi_{\mathcal{E}}"'] & & \mathcal{F} \arrow[dl, "\pi_{\mathcal{F}}"] \\ & X & \end{tikzcd}$$

commutes, i.e. $\pi_{\mathcal{E}} = \pi_{\mathcal{F}} \circ \varphi$, and

(b) there exist local trivializations $U_1, \dots, U_s$ of $\mathcal{E}$ and $\mathcal{F}$ and associated maps $A_i : U_i \to G(f \times e)$ such that

$$\Phi_i^{\mathcal{F}} \circ \varphi \circ (\Phi_i^{\mathcal{E}})^{-1} : U_i \times \mathbb{R}^e \to U_i \times \mathbb{R}^f$$

is given by $(x, a) \mapsto (x, f_{A_i(x)}(a))$, and the entries of A_i are regular invertible functions on U_i or constantly $-\infty$.

Let $U \subseteq X$ be a local trivialization of both $\mathcal{E}$ and $\mathcal{F}$, and let $A : U \to G(f \times e)$ be the associated local matrix for φ. By Remark 3, A induces a map $f_A : U \to \mathrm{Hom}_{\mathsf{Set}}(\mathbb{R}^e, \mathbb{R}^f)$ with

$$\Phi^{\mathcal{F}} \circ \varphi \circ (\Phi^{\mathcal{E}})^{-1} : U \times \mathbb{R}^e \to U \times \mathbb{R}^f, \qquad (p, v) \mapsto (p, f_A(p)(v)).$$

We use this to define the pointwise behavior of φ.

Definition 4.3.2. With notation as above, we define the *induced stalk map at* p to be the tropical linear map $\varphi_p : \mathbb{T}^e \to \mathbb{T}^f$ given by $\varphi_p(v) = A(p)(v)$, i.e. having $A(p) \in G(f \times e)$.

Note that this definition of the induced stalk map is invariant of the choice of local trivialization, up to a change of basis. We prove this below.

Lemma 4.3.3. *Let* $U, V \subseteq X$ *be local trivializations of a tropical vector bundle morphism* $\varphi : \mathcal{E} \to \mathcal{F}$. *Write*

$$A_U : U \to G(f \times e) \quad \text{and} \quad A_V : V \to G(f \times e)$$

for the associated local matrices. Then for $p \in U \cap V$,

$$A_V(p) = M^{\mathcal{F}}_{U,V}(p) \odot A_U(p) \odot (M^{\mathcal{E}}_{U,V}(p))^{-1},$$

where

$$M^{\mathcal{F}}_{U,V} : U \cap V \to G(f) \quad \text{and} \quad M^{\mathcal{E}}_{U,V} : U \cap V \to G(e)$$

are the transition maps of $\mathcal{F}$ *and* $\mathcal{E}$.

Proof. This statement is equivalent to the content that the following diagram commutes:

$$\begin{tikzcd}
(U \cap V) \times \mathbb{T}^e \arrow[r, "\mathrm{id} \times M^{\mathcal{E}}_{U \cap V}"] \arrow[d, "\mathrm{id} \times A_U|_{U \cap V}"'] & (U \cap V) \times \mathbb{T}^e \arrow[d, "\mathrm{id} \times A_V|_{U \cap V}"] \\
(U \cap V) \times \mathbb{T}^f \arrow[r, "\mathrm{id} \times M^{\mathcal{F}}_{U \cap V}"] & (U \cap V) \times \mathbb{T}^f
\end{tikzcd}$$

which follows from the commutativity property of tropical vector bundle morphisms and that the above transitions are isomorphisms. □

The above allows us to establish what is one of the most important tools for our subsequent work.

When the base space X has sedentary strata (boundary), the entries of $A_i(p)$ that are regular invertible functions on the interior of a face can approach $-\infty$ as p approaches a sedentary face of X. At such boundary points, the induced stalk map φ_p has strictly lower rank than at interior points. This is the mechanism by which degeneracy loci acquire positive codimension.

Corollary 4.3.4. *Let $\varphi : \mathcal{E} \to \mathcal{F}$ be a morphism of tropical vector bundles over a rational polyhedral space X. The induced rank function*

$$\begin{aligned} \mathit{rank}(\varphi):\ |X| &\longrightarrow \mathbb{Z} \\ p &\longmapsto \mathit{troprank}(\varphi_p) \end{aligned}$$

is locally constant on the interior of each top-dimensional face of X and may drop at sedentary strata. In particular, rank(φ) is upper semicontinuous on $|X|$.

Proof. Well-definedness follows from Lemmas 4.3.3 and 4.1.8. On the interior of a top-dimensional face of X, each entry of the local matrix $A_i(p)$ is either a regular invertible (hence $\mathbb{R}$-valued) function or constantly $-\infty$, so the pattern of finite entries is constant on each connected component of the interior. By Lemma 4.1.8 the rank depends only on this pattern, and so $\mathrm{rank}(\varphi)$ is locally constant there.

At a sedentary face $\tau \subseteq |X|$, regular invertible functions on the interior may extend continuously to τ with value $-\infty$, causing additional entries of $A_i(p)$ to become $-\infty$; additional $-\infty$ entries can only decrease the tropical rank. Hence $\mathrm{rank}(\varphi)$ is upper semicontinuous. □

Remark 7. Upper semicontinuity of the rank function means that the sublevel sets $D_k(\varphi) = \{p \in |X| \mid \mathrm{troprank}(\varphi_p) \leq k\}$ are closed in $|X|$ for each k. On the interior, the locally constant behavior implies $D_k(\varphi) \cap X^\circ$ is a union of connected components of X°. The genuine rank drop occurs at sedentary strata, giving $D_k(\varphi)$ additional faces of positive codimension.

Lemma 4.3.5. *Let $\varphi : \mathcal{E} \to \mathcal{F}$ be a morphism of tropical vector bundles over a rational polyhedral space X, with rank $\mathcal{E} = e$ and rank $\mathcal{F} = f$. Then the sublevel sets*

$$D_k(\varphi) = \{p \in |X| \mid \mathit{troprank}(\varphi_p) \leq k\},$$

for $0 \leq k \leq \min(e, f)$, are closed subsets of $|X|$ that inherit the structure of rational polyhedral subspaces of X.

Proof. Closedness follows from upper semicontinuity of $p \mapsto \mathrm{troprank}(\varphi_p)$ (Corollary 4.3.4). For the polyhedral statement, pass to a rational polyhedral refinement of X on which

$\operatorname{troprank}(\varphi)$ is constant on the relative interior σ° of every facet σ; this is possible because tropical rank is locally constant on facet interiors. On such a refinement,

$$\mathsf{D}_k(\varphi) = \bigcup_{\operatorname{troprank}(\varphi)|_{\sigma^\circ} \leq k} \overline{\sigma^\circ},$$

together with the sedentary faces along which the rank may fall more. Hence $\mathsf{D}_k(\varphi)$ is a finite union of rational polyhedra in X, and inherits the structure of a rational polyhedral subspace. $\square$

Chapter 5

TROPICAL CHERN CLASSES

5.1 Rational Sections

Definition 5.1.1 (Rational sections of vector bundles)**.** Let $\pi : F \to X$ be a vector bundle of rank n. A *rational section* $s : X \to F$ of F is a continuous map $s : |X| \to |F|$ such that

(a) $\pi \circ s = \mathrm{id}_{|X|}$, i.e. s is a section of π, and

(b) there exists a cover of X by local trivializations and with associated homeomorphisms $\mathcal{U}$ such that for each $(U, \Phi) \in \mathcal{U}$ the components of $\Phi \circ s$ are rational functions on U, i.e.

$$\Phi \circ s = (\mathrm{id}_U, s_1, \ldots, s_n) : U \to U \times \mathbb{R}^n$$

and $s_j \in \mathcal{K}(U)$ for each $1 \leq j \leq n$.

We say a rational section $s : X \to F$ is *bounded* if the components are bounded over each local trivialization.

Remark 8. For the remainder of the thesis, unless explicitly stated otherwise, all the tropical vector bundles we discuss will be assumed to admit bounded rational sections.

An important fact we use routinely is the following: if $\mathcal{L} \to X$ is a line bundle, then a rational section $s : X \to \mathcal{L}$ naturally corresponds to a Cartier divisor $\{(U_i, \widehat{s}_i)\}$, which we denote by $\mathcal{D}(s)$. We state the following result, directing the reader to the source for the proof.

Lemma 5.1.2 ([1])**.** *Let* $\pi : L \to X$ *be a line bundle and let* $s_1, s_2 : X \to L$ *be two bounded rational sections. Then*

$$[\mathcal{D}(s_1)] = [\mathcal{D}(s_2)] \in \mathit{Div}(X).$$

Remark 9. Most importantly, the above lemma lets us now associate to any line bundle $\mathcal{L}$ that admits a bounded rational section $s : X \to \mathcal{L}$ a well-defined Cartier divisor class

$$\mathcal{D}(\mathcal{L}) := [\mathcal{D}(s)] \in \mathrm{Div}(X).$$

Theorem 5.1.3 ([1]). *Let $\pi : F \to X$ be a vector bundle of rank r on a simply connected rational polyhedral space X. Then F is a direct sum of line bundles, i.e. there exist line bundles $\mathcal{L}_1, \dots, \mathcal{L}_r$ on X such that*

$$F \cong \bigoplus_{i=1}^{r} \mathcal{L}_i.$$

Definition 5.1.4 (Global intersection cycles)**.** Let $\pi : F \to X$ be a vector bundle, $s : X \to F$ be a rational section, and $Y \in Z_\ell(X)$ a subcycle. Then for a positive integer $k \leq \operatorname{rank} F$, we define the *global intersection cycle* $s^{(k)} \cdot Y \in Z_{\ell-k}(X)$ on a local trivialization U as

$$(s^{(k)} \cdot Y) \cap U := \sum_{I \in \binom{[r]}{k}} s_{i_1} \cdot s_{i_2} \cdots s_{i_k} \cdot (Y \cap U),$$

where we are summing over all k-element subsets $I = \{i_1, \dots, i_k\} \subseteq [r]$.

Remark 10. Observe that if we let $U_1, \dots, U_s$ denote an open covering of X by local trivializations of the bundle $\pi : F \to X$, and we let $s_{ij} : U_i \to \mathbb{R}$ denote the jth component of s over U_i, then

$$(s^{(k)} \cdot Y) \cap U_i := \sum_{1 \leq j_1 < \cdots < j_k \leq n} s_{ij_1} \cdot s_{ij_2} \cdots s_{ij_k} \cdot (Y \cap U_i).$$

Since $s_{i'j} = s_{i\sigma(j)+\varphi_j}$ on $U_i \cap U_{i'}$ for some $\sigma \in S_n$ and some regular invertible map $\mathcal{O}^*(U_i \cap U_{i'})$, the intersection products $(s^{(k)} \cdot Y) \cap U_i$ and $(s^{(k)} \cdot Y) \cap U_{i'}$ coincide on $U_i \cap U_{i'}$ and we can glue them together, so this object is well-defined.

The Chow ring of a rational polyhedral space is $A(X) := \bigoplus_{i \geq 0} A_i(X)$; see [2] for details.

Lemma 5.1.5 ([1]). *With the same notation as above, let $Y \in Z_\ell(X)$ be a cycle and $\varphi \in \mathcal{K}^*(Y)$ a bounded rational function on Y. Then the following holds:*

$$s^{(k)} \cdot (\varphi \cdot Y) = \varphi \cdot (s^{(k)} \cdot Y).$$

Theorem 5.1.6 ([1]). *Let $\pi : F \to X$ be a vector bundle of rank r and $s_1, s_2 : X \to F$ two bounded rational sections. Then $s_1^{(k)} \cdot Y$ and $s_2^{(k)} \cdot Y$ are rationally equivalent, i.e.*

$$[s_1^{(k)} \cdot Y] = [s_2^{(k)} \cdot Y] \in A(X)$$

holds for all subcycles $Y \in Z_\ell(X)$.

5.2 Chern Classes

Definition 5.2.1 (Chern classes). Let $\pi : \mathcal{E} \to X$ be a vector bundle admitting *bounded* rational sections. The kth *Chern class* of $\mathcal{E}$ is the endomorphism

$$\begin{aligned} c_k(\mathcal{E}) :\ A(X) &\longrightarrow A(X) \\ [Y] &\longmapsto [s^{(k)} \cdot Y] \end{aligned}$$

where $s : X \to \mathcal{E}$ is any bounded rational section. It follows that $c_0(\mathcal{E}) = 1_{A(X)}$ and $c_k(\mathcal{E}) = 0$ for $k > \operatorname{rank}\mathcal{E}$. We often write $c_k(\mathcal{E}) \cdot Y$ for $c_k(\mathcal{E})(Y)$ to emphasize the reliance on the intersection product. The *full Chern class* of $\mathcal{E}$ is

$$c(\mathcal{E}) := \sum_{p \geq 0} c_p(\mathcal{E}) \in A(X),$$

and the *Chern polynomial* of $\mathcal{E}$ is the formal polynomial

$$c_{[t]}(\mathcal{E}) := \sum_{p \geq 0} c_p(\mathcal{E})\, t^p \in A(X)[t].$$

The class $c_k(\mathcal{E})$ is well-defined by Lemma 5.1.5 and independent of the choice of rational section by Theorem 5.1.6. Isomorphic vector bundles have the same Chern classes ([1, Theorem 2.11(d)]). We next record a tropical analogue of Whitney's formula.

Theorem 5.2.2 (Tropical Whitney's Formula). *If we have a short exact sequence of finite rank tropical vector bundles over X,*

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0,$$

then $c(\mathcal{F}) = c(\mathcal{E})c(\mathcal{G})$.

Proof. Every short exact sequence of finite-rank tropical vector bundles splits (Proposition 4.2.5), so $\mathcal{F} \cong \mathcal{E} \oplus \mathcal{G}$. By [1, Theorem 2.11(e)],

$$\begin{aligned} c(\mathcal{E})c(\mathcal{G}) &= \left(\sum_{p\geq 0} c_p(\mathcal{E})\right)\left(\sum_{p\geq 0} c_p(\mathcal{G})\right) \\ &= \sum_{p\geq 0}\left(\sum_{i+j=p} c_i(\mathcal{E})\, c_j(\mathcal{G})\right) = \sum_{p\geq 0} c_p(\mathcal{F}), \end{aligned}$$

as desired. □

An important consequence of Whitney's formula is that it expresses higher Chern classes of a splitting vector bundle in terms of first Chern classes.

Corollary 5.2.3. *If a finite-rank tropical vector bundle splits as a direct sum of tropical line bundles, $\mathcal{E} = \bigoplus_{j=1}^{e} \mathcal{L}_j$, then*

$$c(\mathcal{E}) = \prod_{j=1}^{e}(1 + c_1(\mathcal{L}_j)).$$

Proof. By definition, $c_0(\mathcal{L}_j) = 1$ for all j, and so we apply Whitney's formula. □

This decomposition of higher Chern classes into first Chern classes is a highly desirable property. Even when a vector bundle does not split, one can construct a space where its pullback does, perform computations with the resulting "virtual" Chern roots, and then push the result back down. This is the core principle behind the splitting construction, which we prove below in the tropical setting.

5.3 Properties of Chern Classes

Theorem 5.3.1 (Properties of Chern classes)**.** *Let $\mathcal{F} \to X$ and $\mathcal{E} \to X$ be tropical vector bundles admitting bounded rational sections. Furthermore, let $f : Y \to X$ be a morphism of rational polyhedral spaces. Then the following statements hold true:*

1. *classes commute under intersection product, i.e.*

$$c_i(\mathcal{F}) \cdot c_j(\mathcal{E}) = c_j(\mathcal{E}) \cdot c_i(\mathcal{F}),$$

2. *for all* $Z \in A(Y)$ *we have that*

$$f_*(c_i(f^*\mathcal{E}) \cdot Z) = c_i(\mathcal{E}) \cdot f_*(Z),$$

3. *functoriality, i.e. if* X *and* Y *are smooth varieties, then for all* $Z \in A(X)$

$$c_i(f^*\mathcal{F}) \cdot f^*(Y) = f^*(c_i(\mathcal{F}) \cdot Y),$$

4. *tropical Whitney's formula, i.e.*

$$c_k(\mathcal{F} \oplus \mathcal{E}) = \sum_{i+j=k} c_i(\mathcal{F}) \cdot c_j(\mathcal{E}),$$

5. *if* $\mathcal{L}$ *is a line bundle over* X*, then*

$$c_1(\mathcal{L}) = [\mathcal{D}(\mathcal{L})],$$

where $\mathcal{D}(\mathcal{L})$ *is the Cartier divisor class associated to the line bundle* $\mathcal{L}$*,*

6. *and lastly, for line bundles* $\mathcal{L}$ *and* $\mathcal{L}'$ *over a rational polyhedral space* X*, we have the following equalities*

$$c_1(\mathcal{L}^\vee) = -c_1(\mathcal{L}) \quad \text{and} \quad c_1(\mathcal{L} \otimes \mathcal{L}') = c_1(\mathcal{L}) + c_1(\mathcal{L}').$$

Proof. See [1] for (1), (2), (3), and (5). Statement (4) is Theorem 5.2.2 above. We now prove the two equalities of (6) hold true. We first prove the rational equivalence of sections of these bundles, in other words,

$$\mathcal{D}(\mathcal{L}^\vee) = -\mathcal{D}(\mathcal{L}) \quad \text{and} \quad \mathcal{D}(\mathcal{L} \otimes \mathcal{L}') = \mathcal{D}(\mathcal{L}) + \mathcal{D}(\mathcal{L}')$$

in $\text{Div}(X) = \mathcal{K}^*/\mathcal{O}^*$.

Let $\sigma : X \to \mathcal{L}$ be an arbitrary bounded section. We now define a section $\tau : X \to \mathcal{L}^\vee$. Let $\{U_1, \ldots, U_s\}$ be a cover of X by shared local trivializations of both $\mathcal{L}$ and $\mathcal{L}^\vee$. Define τ locally on the trivialization $U_i \subseteq |X|$ as

$$\widehat{\tau}_i := -\widehat{\sigma}_i$$

with $\widehat{\tau}_i$ denoting $p^{(i)} \circ \Phi_i \circ \tau$ (respectively, $\widehat{\sigma}_i := p^{(i)} \circ \Psi_i \circ \sigma$), where

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[r, "\Phi_i"] & U_i \times \mathbb{R} \arrow[d, "p^{(i)}"] \\ U_i \arrow[u, "\sigma"] \arrow[r, dashed, "\widehat{\sigma}_i"] & \mathbb{R} \end{tikzcd}$$

We now need only show that τ is well-behaved on intersections. Observe that on $U_i \cap U_j$ we have that

$$\begin{aligned} \frac{\widehat{\tau}_i|_{ij}}{\widehat{\tau}_j|_{ij}} &= \widehat{\tau}_i|_{ij} \odot (\widehat{\tau}_j|_{ij})^{-1} \\ &= \widehat{\tau}_i|_{ij} - \widehat{\tau}_j|_{ij} \\ &= (-\widehat{\sigma}_i|_{ij}) - (-\widehat{\sigma}_j|_{ij}) \\ &= -(\widehat{\sigma}_i|_{ij} - \widehat{\sigma}_j|_{ij}) = \frac{\widehat{\sigma}_j|_{ij}}{\widehat{\sigma}_i|_{ij}} \end{aligned}$$

and since σ is a bounded rational section of $\mathcal{L}$, we have that $\frac{\widehat{\sigma}_i|_{ij}}{\widehat{\sigma}_j|_{ij}} = \alpha \in \mathcal{O}^*(U_i \cap U_j)$, and so we may conclude that

$$\frac{\widehat{\tau}_i|_{ij}}{\widehat{\tau}_j|_{ij}} = -\alpha \in \mathcal{O}^*(U_i \cap U_j),$$

as desired. So indeed τ is a bounded rational section of $\mathcal{L}^\vee$, hence $\mathcal{D}(\tau)$ denotes the corresponding Cartier divisor, and we finally have

$$\begin{aligned} \mathcal{D}(\mathcal{L}) + \mathcal{D}(\mathcal{L}^\vee) &= [\mathcal{D}(\sigma)] + [\mathcal{D}(\tau)] \\ &= [\{(U_i, \widehat{\sigma}_i)\}] + [\{(U_i, \widehat{\tau}_i)\}] \\ &= [\{(U_i, \widehat{\sigma}_i + \widehat{\tau}_i)\}] \\ &= [\{(U_i, \widehat{\sigma}_i + (-\widehat{\sigma}_i))\}] = [\{(U_i, 0)\}]. \end{aligned}$$

As $[\{(U_i, 0)\}] \in \mathrm{Div}(X)$ is the identity, this successfully proves that $c_1(\mathcal{L}^\vee) = -c_1(\mathcal{L})$.

We now proceed to prove the second statement. Let $\sigma : X \to \mathcal{L}$ and $\tau : X \to \mathcal{L}'$ be two bounded rational sections of our bundles. We now construct a section $\sigma \otimes \tau : X \to \mathcal{L} \otimes \mathcal{L}'$ locally on a cover of X by local trivializations $\{U_1, \dots, U_s\}$ (shared by both bundles) and proceed as we had above. We define on the trivialization $U_i \subseteq |X|$

$$\widehat{(\sigma \otimes \tau)}_i := \widehat{\sigma}_i \otimes \widehat{\tau}_i.$$

Note as both $\widehat{\sigma}_i$ and $\widehat{\tau}_i$ are maps into $G(1)$, this is just equal to $\widehat{\sigma}_i \odot \widehat{\tau}_i$, i.e. $\widehat{\sigma}_i + \widehat{\tau}_i$. Checking transitions, we see that

$$\begin{aligned}
\frac{\widehat{(\sigma \otimes \tau)}_i|_{ij}}{\widehat{(\sigma \otimes \tau)}_j|_{ij}} &= \widehat{(\sigma \otimes \tau)}_i|_{ij} - \widehat{(\sigma \otimes \tau)}_j|_{ij} \\
&= (\widehat{\sigma}_i|_{ij} + \widehat{\tau}_i|_{ij}) - (\widehat{\sigma}_j|_{ij} + \widehat{\tau}_j|_{ij}) \\
&= \widehat{\sigma}_i|_{ij} + \widehat{\tau}_i|_{ij} - \widehat{\sigma}_j|_{ij} - \widehat{\tau}_j|_{ij} \\
&= (\widehat{\sigma}_i|_{ij} - \widehat{\sigma}_j|_{ij}) + (\widehat{\tau}_i|_{ij} - \widehat{\tau}_j|_{ij}) = \frac{\widehat{\sigma}_i|_{ij}}{\widehat{\sigma}_j|_{ij}} \odot \frac{\widehat{\tau}_i|_{ij}}{\widehat{\tau}_j|_{ij}}.
\end{aligned}$$

And as σ and τ are both bounded rational sections of line bundles, we have that

$$\frac{\widehat{\sigma}_i|_{ij}}{\widehat{\sigma}_j|_{ij}} = \alpha \in \mathcal{O}^*(U_i \cap U_j) \quad \text{and} \quad \frac{\widehat{\tau}_i|_{ij}}{\widehat{\tau}_j|_{ij}} = \beta \in \mathcal{O}^*(U_i \cap U_j).$$

Hence we can conclude that

$$\frac{\widehat{(\sigma \otimes \tau)}_i|_{ij}}{\widehat{(\sigma \otimes \tau)}_j|_{ij}} = \alpha \odot \beta \in \mathcal{O}^*(U_i \cap U_j),$$

and furthermore as both α and β are bounded, as must $\alpha \odot \beta = \alpha + \beta$ be. Therefore indeed $\sigma \otimes \tau$ is a section of $\mathcal{L} \otimes \mathcal{L}'$ and $\sigma \otimes \tau \in \mathrm{Div}(X)$. Finally, we can observe that

$$\begin{aligned}
\mathcal{D}(\mathcal{L}) + \mathcal{D}(\mathcal{L}') &= [\mathcal{D}(\sigma)] + [\mathcal{D}(\tau)] \\
&= [\{(U_i, \widehat{\sigma}_i)\}] + [\{(U_i, \widehat{\tau}_i)\}] \\
&= [\{(U_i, \widehat{\sigma}_i + \widehat{\tau}_i)\}] \\
&= [\{(U_i, \widehat{(\sigma \otimes \tau)}_i\}] = \mathcal{D}(\mathcal{L} \otimes \mathcal{L}').
\end{aligned}$$

So we may therefore conclude that $c_1(\mathcal{L} \otimes \mathcal{L}') = c_1(\mathcal{L}) + c_1(\mathcal{L}')$. □

By Theorem 2.11(f) of [1], we have that for a tropical line bundle $\mathcal{L}$ over a rational polyhedral space X that

$$c_1(\mathcal{L}) = \mathcal{D}(\mathcal{L}) = [\mathcal{D}(\sigma)]_{\mathrm{rat}}$$

where σ denotes a bounded rational section of $\mathcal{L}$ and $\mathcal{D}(\sigma)$ is the Cartier divisor class associated to σ. As all bounded rational sections of a line bundle are rationally equivalent, $\mathcal{D}(\sigma)$ uniquely determines a Cartier divisor class associated to $\mathcal{L}$ denoted $\mathcal{D}(\mathcal{L})$.

Chapter 6

TROPICAL SPLITTING PRINCIPLE

6.1 Tropical Projectivization

One construction we need is the tropical analogue of projective space; see Section 3 of [15] for further details.

Definition 6.1.1 (Tropical projective space)**.** The *tropical projective space* $\mathbb{TP}^n$ is defined to be the quotient $(\mathbb{T}^{n+1})^\times/\sim$, where the equivalence relation is given by

$$(x_0, \dots, x_n) \sim (y_0, \dots, y_n) \iff (\exists \lambda \in \mathbb{T}^\times): \ (x_0, \dots, x_n) = (\lambda \odot y_0, \dots, \lambda \odot y_n).$$

Note that $\mathbb{T}^\times = \mathbb{T} \setminus \{-\infty\}$ and the above condition asks that $x_i = \lambda + y_i$ for all $0 \leq i \leq n$. We denote elements (equivalence classes) in $\mathbb{TP}^n$ by $[x_0 : \dots : x_n]$.

Remark 11. We may cover $\mathbb{TP}^n$ with $(n+1)$ affine charts $\mathbb{T}^n$, in complete analogy with the classical setting: with $\odot$ playing the role of multiplication and 0 the multiplicative identity, if $x_i \neq -\infty$ we have

$$\begin{aligned} [x_0 : \dots : x_i : \dots : x_n] &= (-x_i) \odot [x_0 : \dots : x_{i-1} : x_i : x_{i+1} : \dots : x_n] \\ &= [\underbrace{x_0 - x_i}_{y_1} : \dots : \underbrace{x_{i-1} - x_i}_{y_i} : 0 : \underbrace{x_{i+1} - x_i}_{y_{i+1}} : \dots : \underbrace{x_n - x_i}_{y_n}] \end{aligned}$$

and so we have compatible canonical embeddings $\mathbb{T}^n \hookrightarrow \mathbb{TP}^n$ for each choice of $x_i \neq -\infty$ for $0 \leq i \leq n$, having an inverse given by

$$[x_0 : \dots : x_n] \mapsto (y_1, \dots, y_n)$$

on $U_i = \{x \in \mathbb{TP}^n \mid x_i \neq -\infty\}$, which cover $\mathbb{TP}^n$.

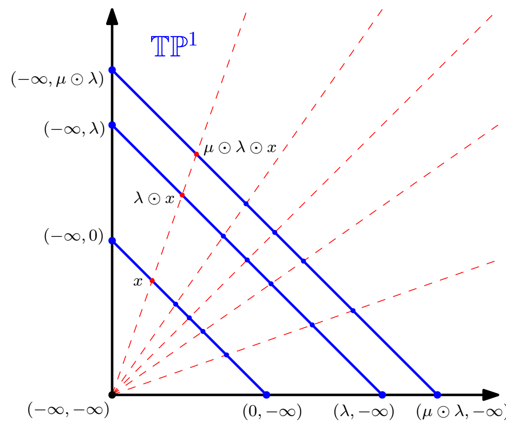

Figure 6.1: The tropical projective line $\mathbb{TP}^1$, with rays emanating from the origin analogous to the classical projective line.

This space has a canonical bundle on it that reflects the Serre twisting sheaf $\mathcal{O}_{\mathbb{P}^n}(1)$ one encounters in the classical setting.

Definition 6.1.2. We define the tropical line bundles $\mathcal{O}_{\mathbb{TP}^n}(d) \to \mathbb{TP}^n$, for $d \in \mathbb{Z}$, by the local data of

- local trivializations over the $(n+1)$-many affine charts U_i, $0 \le i \le n$, where

$$U_i = \{x \in \mathbb{TP}^n \mid x_i \neq -\infty\},$$

- transition functions $M_{ij} : U_i \cap U_j \to G(1) = \mathbb{R}$ given by[1]

$$M_{ij}(x) := \left(\frac{x_i}{x_j}\right)^{\odot d} = d(x_i - x_j),$$

  and hence it follows that $\Phi_j \circ \Phi_i^{-1} : (U_i \cap U_j) \times \mathbb{T} \to (U_i \cap U_j) \times \mathbb{T}$ is given by

$$(x, t) \mapsto (x, M_{ij}(x) \odot t) = \left(x, \left(\frac{x_i}{x_j}\right)^d \odot t\right) = (x, t + d(x_i - x_j)).$$

Note that $\mathcal{O}_{\mathbb{TP}^n}(0)$ is just the trivial line bundle $\mathbb{TP}^n \times \mathbb{T}$.

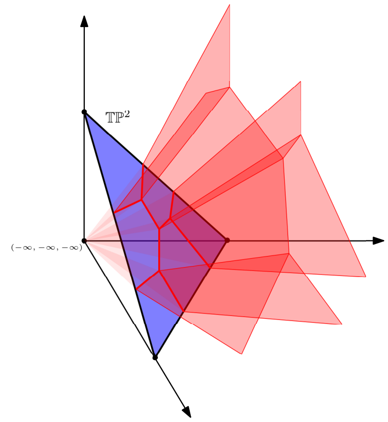

Figure 6.2: The pullback of $\mathcal{O}_{\mathbb{TP}^2}(1)$ along an embedding of a curve into $\mathbb{TP}^2$.

Theorem 6.1.3. *The tropical line bundle $\mathcal{O}_{\mathbb{TP}^n}(d) \to \mathbb{TP}^n$ has a first Chern class of*

$$c_1(\mathcal{O}_{\mathbb{TP}^n}(d)) = d.$$

Proof. Observe that

$$\mathcal{O}_{\mathbb{TP}^n}(1)^\vee \cong \mathcal{O}_{\mathbb{TP}^n}(-1) \quad \text{and} \quad \mathcal{O}_{\mathbb{TP}^n}(d) \cong \underbrace{\mathcal{O}_{\mathbb{TP}^n}(1) \otimes \cdots \otimes \mathcal{O}_{\mathbb{TP}^n}(1)}_{\text{d-many}}$$

for $d \geq 1$. It suffices to show that $c_1(\mathcal{O}_{\mathbb{TP}^n}(1)) = 1$, as the result then follows by properties of Chern classes of line bundles under dualizing and tensoring (see Theorem 5.3.1).

We claim that $f = 0 \oplus x_1 \oplus \cdots \oplus x_n = \max(0, x_1, \ldots, x_n)$ defines a bounded rational section of $\mathcal{O}_{\mathbb{TP}^n}(1)$. By Theorem 5.3.1(5), we then have $c_1(\mathcal{O}_{\mathbb{TP}^n}(1)) = [\mathcal{D}(f)]$, where $\mathcal{D}(f)$ denotes the associated Cartier divisor.

First observe that f is well-defined on projective equivalence classes, as

$$f(\lambda \odot x) = f([\lambda \odot 0 : \lambda \odot x_1 : \cdots : \lambda \odot x_n]) = \max\{\lambda, \lambda + x_1, \ldots, \lambda + x_n\} = \lambda \odot f(x)$$

for any $\lambda \in \mathbb{T}^\times = \mathbb{R}$. On the standard affine chart U_i (where $x_i = 0$), the local representative of f is

$$f_i = \max(x_0, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n),$$

[1]Importantly note that these are *tropical* operations!

which is a bounded rational function on U_i (as it is a tropical polynomial). We now verify the transition condition: on $U_i \cap U_j$, passing from the U_i-normalization (where $x_i = 0$) to the U_j-normalization (where $x_j = 0$) subtracts x_j from every coordinate, so

$$f_j = f_i - x_j = f_i + (x_i - x_j) = f_i + M_{ij}(x),$$

where $M_{ij}(x) = x_i - x_j$ is the transition function of $\mathcal{O}_{\mathbb{TP}^n}(1)$. Hence f is indeed a bounded rational section of $\mathcal{O}_{\mathbb{TP}^n}(1)$.

It remains to show that $[\mathcal{D}(f)] = 1$. On each chart $U_i \cong \mathbb{R}^n$, the Weil divisor $\mathrm{div}(f_i)$ is the locus where f_i fails to be linear, namely the standard tropical hyperplane — the $(n-1)$-dimensional balanced polyhedral complex where the maximum $\max(x_0, \dots, 0, \dots, x_n)$ is attained by at least two terms. This represents the degree-1 generator of $A_{n-1}(\mathbb{TP}^n)$, so $c_1(\mathcal{O}_{\mathbb{TP}^n}(1)) = [\mathcal{D}(f)] = 1$. □

Similarly, various bundles from the tropical toric setting can be reinterpreted in the context of [1]. For more on tropical toric varieties, see §5.3.3 of [15].

Proposition 6.1.4 (Birkhoff–Grothendieck Theorem). *Every rank n tropical vector bundle $E \to \mathbb{TP}^1$ splits into a direct sum of line bundles, i.e.*

$$E \cong \bigoplus_{i=1}^{n} \mathcal{O}(d_i).$$

Proof. Let $E \to \mathbb{TP}^1$ be a rank n tropical vector bundle with cover by local trivializations $\{U_1, \dots, U_s\}$. By Lemma 3.2 of [1], we may assume there is at most one non-trivial transition map $M_{ij} : U_i \cap U_j \to G(n)$. After a (tropical) change of basis, we may assume that this transition matrix is diagonal,

$$M_{ij} = \begin{pmatrix} g_1 & -\infty & \cdots & -\infty \\ -\infty & g_2 & \cdots & -\infty \\ \vdots & \vdots & \ddots & \vdots \\ -\infty & -\infty & \cdots & g_n \end{pmatrix},$$

where each $g_i \in \mathcal{O}^*(U_i \cap U_j) \cong \mathcal{O}^*(\mathbb{R})$ is a regular invertible function, hence of the form $g_i(x) = d_i \odot x$ for some integer d_i. Let $\mathcal{E}_i \subseteq E$ be the rank-1 subbundle whose local trivialization over U_j is the ith coordinate axis $\langle e_i \rangle_{\mathbb{R}} \subseteq \mathbb{R}^n$. Because M_{ij} is diagonal, each $\mathcal{E}_i$ is preserved by the transition data and is therefore a well-defined tropical line subbundle of E; the transition function it inherits is g_i, which defines the line bundle $\mathcal{O}(d_i)$ on $\mathbb{TP}^1$. Iterating Proposition 4.2.5 to split off each $\mathcal{E}_i$ in turn, we obtain $E \cong \bigoplus_{i=1}^n \mathcal{O}(d_i)$. □

Remark 12. Note this same result has been obtained independently in a different manner (namely by pushing forward line bundles on covers of metric graphs in the work of [9]).

6.2 The Splitting Principle

Definition 6.2.1 (Projectivization of a bundle)**.** Let $\pi : \mathcal{E} \to X$ be a tropical vector bundle of rank n. The *projectivization of* $\mathcal{E}$ is a pair $(\mathbb{TP}(\mathcal{E}), \varphi)$, where $\mathbb{TP}(\mathcal{E})$ is a rational polyhedral space (a "tropical fiber bundle") and $\varphi : \mathbb{TP}(\mathcal{E}) \to X$ is the projection, defined as follows:

- $\mathbb{TP}(\mathcal{E})$ is the fiber bundle obtained by projectivizing the stalks $\mathcal{E}_p \cong \mathbb{T}^n$ of $\mathcal{E}$, i.e.

$$\varphi^{-1}(p) := \mathbb{TP}(\mathcal{E}_p) = (\mathcal{E}_p \setminus \{-\infty\})/\sim\, \cong\ \mathbb{TP}^{n-1},$$

  where $v \sim v'$ in $\mathcal{E}_p \setminus \{-\infty\}$ if there exists some $\lambda \in \mathbb{T}^\times$ with $v = \lambda \odot v'$.

- On a trivialization U_i of $\mathcal{E}$, we have

$$\varphi^{-1}(U_i) \cong U_i \times \mathbb{TP}^{n-1},$$

  with transition maps on $U_i \cap U_j$ given by $[M_{ij}] \in \mathbb{P}G(n)$, where

$$\mathbb{P}G(n) := G(n)/(A \sim \lambda \odot A)$$

  for $\lambda \in \mathbb{T}^\times$. Locally on a trivialization U, the projection acts as $\varphi(p, [x]) = p$, where $p \in U$ and $[x] \in \mathbb{TP}(\mathcal{E}_p) \cong \mathbb{TP}^{n-1}$.

Remark 13. The total space $\mathbb{TP}(\mathcal{E})$ is a rational polyhedral space whose sedentary strata arise from two sources: the boundary of the base X and the boundary of the fibers $\mathbb{TP}^{n-1}$. The intersection theory on $\mathbb{TP}(\mathcal{E})$ is that of rational polyhedral spaces in the sense of [15].

We construct the projectivization so that the following property holds: pull back $\mathcal{E}$ along φ to obtain

$$\begin{tikzcd} \varphi^*(\mathcal{E}) \arrow[r, dashed, "\hat{\varphi}"] \arrow[d, dashed, "\hat{\pi}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E} \arrow[d, "\pi"] \\ \mathbb{TP}(\mathcal{E}) \arrow[r, "\varphi"] & X \end{tikzcd}$$

The pullback $\varphi^*(\mathcal{E})$ is also of rank n, and has a canonical line subbundle $\mathcal{L}_{\mathcal{E}}$: over the point $(p, [L]) \in \mathbb{TP}(\mathcal{E})$, the fiber of $\mathcal{L}_{\mathcal{E}}$ is the 1-dimensional subspace $L \subseteq \mathcal{E}_p \cong \mathbb{T}^n$. Writing $\mathcal{Q}_{\mathcal{E}}$ for the quotient $\varphi^*(\mathcal{E})/\mathcal{L}_{\mathcal{E}}$, we have a canonical short exact sequence as follows.

Lemma 6.2.2. *Let $\mathcal{E} \to X$ be a rank n tropical vector bundle. Then*

$$0 \longrightarrow \mathcal{L}_{\mathcal{E}} \longrightarrow \varphi^*(\mathcal{E}) \longrightarrow \mathcal{Q}_{\mathcal{E}} \longrightarrow 0$$

is a short exact sequence of tropical vector bundles over $\mathbb{TP}(\mathcal{E})$.

Proof. It suffices to show that $\mathcal{L}_{\mathcal{E}}$ is a rank-1 subbundle of $\varphi^*(\mathcal{E})$ in the tropical sense; then $\mathcal{Q}_{\mathcal{E}}$ is the corresponding quotient and the sequence is exact. Let $U_1, \dots, U_s$ be an open cover by trivializations of $\mathbb{TP}(\mathcal{E})$. Since $\mathcal{L}_{\mathcal{E}} \to \mathbb{TP}(\mathcal{E})$ is a line bundle, we need only show that the maps

$$\Phi_i|_{\hat{\pi}|^{-1}_{\mathcal{L}_{\mathcal{E}}}(U_i)} : (\hat{\pi}|^{-1}_{\mathcal{L}_{\mathcal{E}}})(U_i) \to U_i \times \langle e_j \rangle_{\mathbb{R}}$$

for $1 \le i \le s$ are isomorphisms, where $j \in [n]$ is fixed with $\mathcal{E}_p \cong \langle e_1, \dots, e_n \rangle_{\mathbb{R}}$ for $p \in U_i$. Since $\mathcal{L}_{\mathcal{E}}$ is a line bundle,

$$(\hat{\pi}|^{-1}_{\mathcal{L}_{\mathcal{E}}})(U_i) \cong U_i \times L^{\times} \cong U_i \times \mathbb{R},$$

where $L^{\times}$ denotes the units of $\mathbb{T} \cong L \subseteq \mathcal{E}_p$. Taking $\langle e_1 \rangle_{\mathbb{R}}$ as a basis vector of L, we obtain a natural isomorphism

$$U_i \times \langle e_1 \rangle_{\mathbb{R}} \to (\hat{\pi}|^{-1}_{\mathcal{L}_{\mathcal{E}}})(U_i) \cong U_i \times \mathbb{R},$$

as required, inside $\hat{\pi}^{-1}(U_i) \cong U_i \times \langle e_1, \dots, e_n \rangle_{\mathbb{R}}$. $\square$

Definition 6.2.3. We write $\mathcal{O}_{\mathcal{E}}(-1) := \mathcal{L}_{\mathcal{E}}$ for the canonical line subbundle of $\varphi^*(\mathcal{E})$ defined above, and call it the *tautological subbundle* on $\mathbb{TP}(\mathcal{E})$. Its dual is

$$\mathcal{O}_{\mathcal{E}}(1) := \mathcal{O}_{\mathcal{E}}(-1)^{\vee},$$

which we call the *tautological line bundle* (dually, the dual of the tautological subbundle) on $\mathbb{TP}(\mathcal{E})$.

Lemma 6.2.4. *Let $\mathcal{E} \to X$ be a rank-$(n+1)$ tropical vector bundle, and let $\varphi : \mathbb{TP}(\mathcal{E}) \to X$ be the projectivization projection. The induced pullback map*

$$\varphi^* : A_k(X) \to A_{k+n}(\mathbb{TP}(\mathcal{E}))$$

is a split monomorphism.

Proof. Define $L : \mathrm{im}\,\varphi^* \to A_k(X)$ by $L(\beta) = \varphi_*(c_1(\mathcal{O}_{\mathcal{E}}(1))^n \cdot \beta)$; we show $L \circ \varphi^* = \mathrm{id}$. Fix $\gamma = [Z] \in A_k(X)$ for an integral subcycle $Z \subseteq X$. Since φ is a tropical fiber bundle with transition maps in $\mathbb{P}\mathrm{G}(n+1)$, over each trivialization $\varphi^{-1}(U_i) \cong U_i \times \mathbb{TP}^n$ the pullback and pushforward are defined on trivialization via the products and projections on rational polyhedral spaces of [2], extended to $\mathbb{T}^n$ with sedentary boundary in [15], and $\mathcal{O}_{\mathcal{E}}(1)|_{U_i \times \mathbb{TP}^n} = \mathrm{pr}_2^* \mathcal{O}_{\mathbb{TP}^n}(1)$, giving $\varphi^*[Z \cap U_i] = [Z \cap U_i] \times [\mathbb{TP}^n]$. Theorem 6.1.3 then yields $c_1(\mathcal{O}_{\mathbb{TP}^n}(1))^n \cdot [\mathbb{TP}^n] = [\mathrm{pt}]$, so

$$\varphi_*\big(c_1(\mathcal{O}_{\mathcal{E}}(1))^n \cdot \varphi^*[Z \cap U_i]\big) \;=\; (\mathrm{pr}_1)_*\big([Z \cap U_i] \times [\mathrm{pt}]\big) \;=\; [Z \cap U_i].$$

These glue across $U_i \cap U_j$: the transition $g \in \mathbb{P}\mathrm{G}(n+1)$ pulls the bounded rational section $f = 0 \oplus x_1 \oplus \cdots \oplus x_n$ of Theorem 6.1.3 to another bounded rational section of the *same* bundle $\mathcal{O}_{\mathbb{TP}^n}(1)$, so $c_1(\mathcal{O}_{\mathbb{TP}^n}(1))$ — and hence $[\mathrm{pt}] = c_1(\mathcal{O}_{\mathbb{TP}^n}(1))^n \cdot [\mathbb{TP}^n]$ — is preserved. Assembling gives $\varphi_*(c_1(\mathcal{O}_{\mathcal{E}}(1))^n \cdot \varphi^*[Z]) = [Z]$, and linearity extends this to all of $A_k(X)$. □

Corollary 6.2.5 (Tropical Splitting Principle)**.** *For a tropical vector bundle $\pi : \mathcal{E} \to X$ of rank n over a rational polyhedral space X, there exist a rational polyhedral space Y and a morphism $f : Y \to X$ such that*

1. *the pullback bundle* $f^*\mathcal{E}$ *is a direct sum of tropical line bundles, and*

2. *the induced pullback map* $f^* : A(X) \to A(Y)$ *is injective.*

Proof. We adapt the classical *splitting construction*. At each step, $\mathbb{TP}(\mathcal{E}_i)$ is a rational polyhedral space with boundary coming from both the base and the fibers, and Lemma 6.2.4 supplies a left inverse to φ_i^*. We argue by induction on the rank of $\mathcal{E}$. Projectivization gives

$$\begin{tikzcd} \varphi^*(\mathcal{E}) \arrow[r, dashed] \arrow[d, dashed] & \mathcal{E} \arrow[d, "\pi"] \\ \mathbb{TP}(\mathcal{E}) \arrow[r, "\varphi"] & X \end{tikzcd}$$

in which the pullback $\varphi^*\mathcal{E}$ fits into the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{TP}(\mathcal{E})}(-1) \longrightarrow \varphi^*\mathcal{E} \longrightarrow \mathcal{Q} \longrightarrow 0.$$

If $\mathcal{E}$ has rank 2, then $\mathcal{Q} = \varphi^*\mathcal{E}/\mathcal{O}_{\mathbb{TP}(\mathcal{E})}(-1)$ is a line bundle; taking $Y = \mathbb{TP}(\mathcal{E})$, we have $\varphi^*\mathcal{E} \cong \mathcal{O}_{\mathbb{TP}(\mathcal{E})}(-1) \oplus \mathcal{Q}$, and φ^* is injective by Lemma 6.2.4. For higher rank we iterate this construction by repeated projectivization and quotienting:

$$\begin{tikzcd} \overbrace{\varphi_n^*(\mathcal{E}_{n-1})/\mathcal{O}_{\mathcal{E}_{n-1}}(-1)}^{\mathcal{E}_n} \arrow[d] & & \overbrace{\varphi_2^*(\mathcal{E}_1)/\mathcal{O}_{\mathcal{E}_1}(-1)}^{\mathcal{E}_2} \arrow[d] & \overbrace{\varphi_1^*(\mathcal{E}_0)/\mathcal{O}_{\mathcal{E}_0}(-1)}^{\mathcal{E}_1} \arrow[d] & \mathcal{E}_0 = \mathcal{E} \arrow[d] \\ \underbrace{\mathbb{TP}(\mathcal{E}_n)}_{Y_n} \arrow[r, "p_n"'] & \cdots \arrow[r, "p_3"'] & \underbrace{\mathbb{TP}(\mathcal{E}_1)}_{Y_2} \arrow[r, "p_2"'] & \underbrace{\mathbb{TP}(\mathcal{E}_0)}_{Y_1} \arrow[r, "p_1"'] & X \end{tikzcd}$$

Then taking $Y = Y_n$, we have that the pullback decomposes into a direct sum $f^*\mathcal{E} = \bigoplus_{j=1}^n \mathcal{L}_j$, and that $f^* : A(X) \to A(Y)$ is the desired injective map.

□

This allows us to write

$$c_{[t]}(\mathcal{E}) = \prod_{j=1}^{n} (1 + c_1(\mathcal{L}_j)t)$$

when computing with Chern classes. We call $\mathsf{c}_1(\mathcal{L}_1), \ldots, \mathsf{c}_1(\mathcal{L}_n)$ the *virtual Chern roots* of $\mathcal{E}$; they live in $\mathsf{A}(\mathsf{Y})$ rather than $\mathsf{A}(\mathsf{X})$, but the injectivity of f^* ensures that identities proved in $\mathsf{A}(\mathsf{Y})$ descend to $\mathsf{A}(\mathsf{X})$.

Chapter 7

PORTEOUS FORMULA

Throughout, $\varphi : \mathcal{E} \to \mathcal{F}$ denotes a morphism of tropical vector bundles of ranks e and f over a rational polyhedral space $X \subseteq \mathbb{T}^n$, whose local matrix representatives $A_i : U_i \to G(f \times e)$ have bounded regular-invertible entries. We write $D_k(\varphi)$ for the kth degeneracy locus, which we introduce in Section 7.2 below.

7.1 Sylvester Determinants

The goal of Porteous' formula is to express the class of the kth degeneracy locus of a morphism $\varphi : \mathcal{E} \to \mathcal{F}$ of tropical vector bundles over X, $[D_k(\varphi)] \in A_{(e-k)(f-k)}(X)$, in terms of the Chern classes of $\mathcal{E}$ and $\mathcal{F}$ alone. This expression involves the Sylvester determinant, which we now define.

Definition 7.1.1 (Sylvester determinant)**.** Let $\mathbf{c} = (c_0, c_1, c_2, \dots)$ be a sequence of elements in a commutative ring R. For integers e, f, we define the *Sylvester determinant* of $\mathbf{c}$ of order f and degree e to be the element

$$\Delta_f^e(\mathbf{c}) = \det \begin{bmatrix} c_f & c_{f+1} & \cdots & c_{f+e-1} \\ c_{f-1} & c_f & \cdots & c_{f+e-2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{f-e+1} & c_{f-e+2} & \cdots & c_f \end{bmatrix}$$

in the commutative ring R. We write $S_f^e(\mathbf{c})$ for the $e \times e$-matrix above.

Let $a(t) = \sum_{i=0}^{e} a_i t^i$ and $b(t) = \sum_{j=0}^{f} b_j t^j$ be two polynomials of degrees e and f over a polynomial ring $R[t]$. Assume that $a(0) = a_0 = 1$ and $b(0) = b_0 = 1$. We then have over the appropriate ring extension $R \subseteq S$ that we may factor these polynomials into linear terms

$$a(t) = \prod_{i=1}^{e}(1+\alpha_i t) \quad \text{and} \quad b(t) = \prod_{j=1}^{f}(1+\beta_j t)$$

for coefficients $\alpha_i, \beta_j \in S$.

Lemma 7.1.2. *In this setting,*

$$\Delta_f^e\left(\frac{b(t)}{a(t)}\right) = \prod_{\substack{1\le i\le e\\ 1\le j\le f}} (\beta_j - \alpha_i)$$

where $\frac{b(t)}{a(t)}$ *is identified with its sequence of coefficients in the ring of power series.*

Proof. This is proven as Proposition 12.2 of [5]. □

7.2 Degeneracy Loci

By Corollary 4.3.4, the rank function $\operatorname{rank}(\varphi) : |X| \to \mathbb{Z}$ is upper semicontinuous, and by Lemma 4.3.5 its sublevel sets inherit a rational polyhedral structure from X. We now name these sublevel sets and exhibit them as subcycles of X.

Definition 7.2.1. The k*th tropical degeneracy locus* of a morphism $\varphi : \mathcal{E} \to \mathcal{F}$ is

$$D_k(\varphi) := \{p \in X \mid \operatorname{troprank}(\varphi_p) \le k\}.$$

Proposition 7.2.2. *With notation as above,* $D_k(\varphi)$ *is a rational polyhedral subspace of* X*. On the interior* X°*,* $D_k(\varphi) \cap X^\circ$ *is a union of connected components. At sedentary strata of* X*, the rank can drop, giving* $D_k(\varphi)$ *additional faces of positive codimension.*

Proof. By Lemma 4.3.5, $D_k(\varphi)$ is a rational polyhedral subspace of X. On the interior, rank is locally constant (Corollary 4.3.4), so $D_k(\varphi) \cap X^\circ$ is a union of connected components. At a sedentary stratum, entries of the local matrix $A_i(p)$ may become $-\infty$, causing the tropical rank to drop below that of nearby interior points; these sedentary faces give $D_k(\varphi)$ faces of positive codimension. □

7.3 The Hom Bundle

A morphism $\varphi : \mathcal{E} \to \mathcal{F}$ determines a bounded rational section of the Hom bundle $\mathsf{Hom}(\mathcal{E}, \mathcal{F}) \cong \mathcal{E}^\vee \otimes \mathcal{F}$; this correspondence is what connects degeneracy loci to Chern classes via the zero-locus construction.

Proposition 7.3.1. *A morphism $\varphi : \mathcal{E} \to \mathcal{F}$ of tropical vector bundles over X determines a canonical bounded rational section σ_φ of $\mathcal{E}^\vee \otimes \mathcal{F}$.*

Remark 14. The bounded-entries hypothesis from the standing assumption holds automatically when $|X|$ is compact, since continuous functions on compact sets are bounded; in particular, it holds when the base space is a compact rational polyhedral space such as a tropical curve or a tropical toric variety.

Proof. On a local trivialization U_i, the morphism φ is represented by $A_i : U_i \to \mathrm{G}(f \times e)$ whose entries are regular invertible functions on U_i or constantly $-\infty$. We define σ_φ locally by viewing $A_i(p)$ as an element of $\mathbb{T}^{ef}$ (listing the ef entries of the matrix). The transition behavior of φ (Lemma 4.3.3),

$$A_j(p) = M_{ij}^{\mathcal{F}}(p) \odot A_i(p) \odot (M_{ij}^{\mathcal{E}}(p))^{-1},$$

is precisely the transition law for sections of $\mathcal{E}^\vee \otimes \mathcal{F}$, since the transition maps of $\mathcal{E}^\vee \otimes \mathcal{F}$ are $(M_{ij}^{\mathcal{E}})^{-T} \otimes M_{ij}^{\mathcal{F}}$, which acts on a matrix $A \in \mathbb{T}^{ef}$ as $M_{ij}^{\mathcal{F}} \odot A \odot (M_{ij}^{\mathcal{E}})^{-1}$. The entries of A_i that are not constantly $-\infty$ are bounded by hypothesis, so σ_φ is a bounded rational section. □

Proposition 7.3.2. $D_0(\varphi) = Z(\sigma_\varphi)$, *where $Z(\sigma_\varphi)$ denotes the zero locus of σ_φ (the locus where all ef components equal $-\infty$).*

Proof. By definition, $D_0(\varphi) = \{p \in X \mid \operatorname{troprank}(\varphi_p) = 0\}$, which is the set of points where $A_i(p)$ has all entries equal to $-\infty$. This is precisely the vanishing locus of σ_φ. □

Proposition 7.3.3. *If $D_0(\varphi)$ has the expected codimension ef in X, then*

$$[D_0(\varphi)] = c_{ef}(\mathcal{E}^\vee \otimes \mathcal{F}) \cdot [X].$$

Proof. By the definition of Chern classes (see Chapter 5), $c_{ef}(\mathcal{E}^\vee \otimes \mathcal{F}) \cdot [X] = [\sigma^{(ef)} \cdot X]$ for any bounded rational section σ of $\mathcal{E}^\vee \otimes \mathcal{F}$. The section σ_φ is bounded by Proposition 7.3.1 (using the standing bounded-entries hypothesis of the chapter), so by independence of Chern classes from the choice of section (Theorem 5.1.6),

$$c_{ef}(\mathcal{E}^\vee \otimes \mathcal{F}) \cdot [X] = [\sigma_\varphi^{(ef)} \cdot X].$$

We claim $[\sigma_\varphi^{(ef)} \cdot X] = [D_0(\varphi)]$. Observe that on the complement $|X| \setminus D_0(\varphi)$, at each point p at least one entry of $A_i(p)$ is finite, so the corresponding component $(\sigma_\varphi)_j$ is a regular invertible function in that local trivialization. The Weil divisor of a regular invertible function is the trivial divisor, so the iterated intersection $\sigma_\varphi^{(ef)} \cdot X$ receives no contribution on the complement of $D_0(\varphi)$, and is therefore supported on $Z(\sigma_\varphi) = D_0(\varphi)$. For the cycle structure: the expected-codimension hypothesis ensures $\dim D_0(\varphi) = \dim X - ef$, which matches the dimension of $\sigma_\varphi^{(ef)} \cdot X$, so the intersection is proper and the two classes agree in $A_{\dim X - ef}(X)$. $\square$

7.4 The Rank-0 Case

In the base case of Porteous' formula, we have

$$D_0(\varphi) = \{p \in X \mid \operatorname{troprank}(\varphi_p) = 0\}.$$

The 0th degeneracy locus consists of all points $p \in X$ where φ_p is an $(f \times e)$-matrix with all entries equal to $-\infty$ (the tropical analogue of a zero matrix). Equivalently,

$$D_0(\varphi) = \{p \in X \mid \varphi_p : \mathcal{E}_p \to \mathcal{F}_p \text{ is identically } -\infty\}.$$

This is *not* the same as asking that $f_{\varphi_p} : \mathbb{R}^e \to \mathbb{R}^f$ be identically zero; the statement refers specifically to the stalk map, so $\mathcal{E}_p = \mathbb{T}^e$ and $\mathcal{F}_p = \mathbb{T}^f$.

Theorem 7.4.1 (Tropical Porteous Formula in Rank 0). *Let $\varphi : \mathcal{E} \to \mathcal{F}$ be a morphism of tropical vector bundles with bounded matrix entries over a rational polyhedral space X, with*

$\mathcal{E}$ of rank e and $\mathcal{F}$ of rank f. If $D_0(\varphi)$ has the expected codimension ef, then

$$[D_0(\varphi)] = \Delta_f^e\left(\frac{c_{[t]}(\mathcal{F})}{c_{[t]}(\mathcal{E})}\right),$$

where $c_{[t]}(-)$ denotes the Chern polynomial of the vector bundle.

Proof. By Proposition 7.3.3, the expected-codimension hypothesis gives

$$[D_0(\varphi)] = c_{ef}(\mathcal{E}^\vee \otimes \mathcal{F}) \cdot [X].$$

It therefore suffices to show that $c_{ef}(\mathcal{E}^\vee \otimes \mathcal{F}) = \Delta_f^e\big(c_{[t]}(\mathcal{F})/c_{[t]}(\mathcal{E})\big)$.

By the splitting principle (Theorem 6.2.5), there exist a rational polyhedral space Y and a morphism $s : Y \to X$ for which $s^* : A(X) \to A(Y)$ is injective and both $s^*\mathcal{E}$ and $s^*\mathcal{F}$ split as direct sums of line bundles. Since $s^*c_i(\mathcal{E}) = c_i(s^*\mathcal{E})$ by pullback compatibility of Chern classes (Theorem 5.3.1(3)), we may identify $A(X)$ with its image in $A(Y)$. Write

$$s^*\mathcal{E} = \bigoplus_{i=1}^{e} \mathcal{L}_i, \qquad s^*\mathcal{F} = \bigoplus_{j=1}^{f} \mathcal{M}_j, \qquad \alpha_i := c_1(\mathcal{L}_i), \quad \beta_j := c_1(\mathcal{M}_j).$$

Then

$$s^*(\mathcal{E}^\vee \otimes \mathcal{F}) = \left(\bigoplus_{i=1}^{e} \mathcal{L}_i^\vee\right) \otimes \left(\bigoplus_{j=1}^{f} \mathcal{M}_j\right) = \bigoplus_{i,j} (\mathcal{L}_i^\vee \otimes \mathcal{M}_j),$$

and by Theorem 5.3.1(6), $c_1(\mathcal{L}_i^\vee \otimes \mathcal{M}_j) = -\alpha_i + \beta_j$. Applying Whitney's formula (Theorem 5.2.2),

$$c(s^*(\mathcal{E}^\vee \otimes \mathcal{F})) = \prod_{i,j}\big(1 + \underbrace{(-\alpha_i + \beta_j)}_{\gamma_{ij}}\big).$$

The top Chern class $c_{ef}(s^*(\mathcal{E}^\vee \otimes \mathcal{F}))$ is the highest-degree term, $\prod_{i,j} \gamma_{ij}$. Since the Chern polynomials factor as

$$c_{[t]}(s^*\mathcal{E}) = \prod_{i=1}^{e}(1 + \alpha_i t) \quad \text{and} \quad c_{[t]}(s^*\mathcal{F}) = \prod_{j=1}^{f}(1 + \beta_j t),$$

Lemma 7.1.2 yields

$$c_{ef}(s^*(\mathcal{E}^\vee \otimes \mathcal{F})) = \prod_{i,j}(\beta_j - \alpha_i) = \Delta_f^e\left(\frac{c_{[t]}(s^*\mathcal{F})}{c_{[t]}(s^*\mathcal{E})}\right).$$

Both sides of this equality pull back from $A(X)$, and s^* is injective, so the same identity holds downstairs:

$$c_{ef}(\mathcal{E}^\vee \otimes \mathcal{F}) = \Delta_f^e\left(\frac{c_{[t]}(\mathcal{F})}{c_{[t]}(\mathcal{E})}\right).$$

$\square$

Example 7.4.2. Consider a morphism of tropical vector bundles $\varphi : \mathcal{E} \to \mathcal{O}_{\mathbb{TP}^1}$ from a rank-e bundle to the trivial line bundle over the tropical projective line $\mathbb{TP}^1$. By Proposition 6.1.4, $\mathcal{E}$ splits as $\bigoplus_{i=1}^e \mathcal{O}_{\mathbb{TP}^1}(d_i)$ for some $(d_1, \ldots, d_e) \in \mathbb{Z}^e$. Applying Theorem 7.4.1,

$$[D_0(\varphi)] = \Delta_1^e\left(\frac{c_{[t]}(\mathcal{O}_{\mathbb{TP}^1})}{c_{[t]}(\bigoplus_{i=1}^e \mathcal{O}_{\mathbb{TP}^1}(d_i))}\right).$$

Expanding the Chern polynomials and using Corollary 5.2.3 for the denominator,

$$[D_0(\varphi)] = \Delta_1^e\left(\frac{c_0(\mathcal{O}_{\mathbb{TP}^1}) + c_1(\mathcal{O}_{\mathbb{TP}^1})t}{\prod_{i=1}^e(1 + c_1(\mathcal{O}_{\mathbb{TP}^1}(d_i))t)}\right).$$

We have $c_0(\mathcal{O}_{\mathbb{TP}^1}) = 1$, $c_1(\mathcal{O}_{\mathbb{TP}^1}) = c_1(\mathcal{O}_{\mathbb{TP}^1}(0)) = 0$, and $c_1(\mathcal{O}_{\mathbb{TP}^1}(d_i)) = d_i$ (Theorem 6.1.3). By Lemma 7.1.2,

$$[D_0(\varphi)] = \Delta_1^e\left(\frac{1}{\prod_{i=1}^e(1 + d_i t)}\right) = (-1)^e d_1 \cdots d_e.$$

Chapter 8

FUTURE DIRECTIONS

Summary of contributions

This thesis establishes: the foundational passage from the $\mathbb{R}^n$-based framework of Allermann–Rau to the Mikhalkin–Rau framework of rational polyhedral spaces in $\mathbb{T}^n$, which is what allows degeneracy loci to carry their expected codimension (Chapter 2); tropical Chern classes of vector bundles via bounded rational sections and a tropical Whitney formula (Chapter 5); the tropical projectivization and splitting principle (Chapter 6); the Hom bundle correspondence identifying $D_0(\varphi)$ with the zero locus of a section of $\mathcal{E}^\vee \otimes \mathcal{F}$ (Section 7.3); and the rank-0 tropical Porteous formula (Theorem 7.4.1), the main result of the thesis.

We conclude by addressing two questions: what is needed to extend the rank-0 result to the full Porteous formula for arbitrary $D_k(\varphi)$, and what motivates a tropical Porteous formula in the first place? The first leads to a discussion of the tropical Grassmannian and its role in reducing higher-rank degeneracy loci to the rank-0 case. The second returns to classical algebraic geometry and to the Brill–Noether conjecture of [7], which was the primary motivation for this work.

8.1 Higher Rank Porteous

In the classical setting, the full Porteous formula for $[D_k(\varphi)]$ is deduced from the rank-0 case by a *Grassmann bundle reduction*: one replaces the original morphism $\varphi : \mathcal{E} \to \mathcal{F}$ with a new morphism on a larger space whose rank-0 locus maps onto $D_k(\varphi)$. We outline how this strategy would proceed tropically, and identify the ingredients that remain to be constructed.

The classical strategy. Given $\varphi : \mathcal{E} \to \mathcal{F}$ over X with $\operatorname{rank}\mathcal{E} = e$ and $\operatorname{rank}\mathcal{F} = f$ and a fixed $0 \le k < \min(e, f)$, the classical proof proceeds as follows.

1. Form the Grassmann bundle $\pi : \mathrm{Gr}(e-k, \mathcal{E}) \to X$, whose fiber over $x \in X$ parametrizes $(e-k)$-dimensional subspaces of $\mathcal{E}_x$. Over $\mathrm{Gr}(e-k, \mathcal{E})$ there is a *universal exact sequence*

$$0 \to \mathcal{S} \to \pi^*\mathcal{E} \to \mathcal{Q} \to 0,$$

   where $\mathcal{S}$ is the tautological subbundle of rank $e-k$, with $\mathcal{S}_{(x,K_x)} = K_x$ and $\mathcal{Q}_{(x,K_x)} = \mathcal{E}_x/K_x$.

2. Restrict $\pi^*\varphi$ to $\mathcal{S}$ to obtain $\psi = \pi^*\varphi|_{\mathcal{S}} : \mathcal{S} \to \pi^*\mathcal{F}$, a morphism of ranks $e-k$ and f.

3. Observe that $D_0(\psi) = \{(x, K_x) \mid K_x \subseteq \ker \varphi_x\}$ maps onto $D_k(\varphi)$ under π, since $\mathrm{rank}(\varphi_x) \leq k$ if and only if $\ker \varphi_x$ contains an $(e-k)$-plane.

4. Apply the rank-0 formula: $[D_0(\psi)] = c_{(e-k)f}(\mathcal{S}^\vee \otimes \pi^*\mathcal{F}) \cdot [\mathrm{Gr}(e-k, \mathcal{E})]$.

5. Push forward via π_* and compute using the Giambelli formula to obtain

$$[D_k(\varphi)] = \pi_*[D_0(\psi)] = \Delta^{e-k}_{f-k}\left(\frac{c_{[t]}(\mathcal{F})}{c_{[t]}(\mathcal{E})}\right).$$

The projectivization $\mathbb{TP}(\mathcal{E})$ used in the splitting principle is the case $k = e-1$ of this construction, i.e. $\mathbb{TP}(\mathcal{E}) = \mathrm{Gr}(1, \mathcal{E})$.

Tropical ingredients needed. Writing $d = e-k$ throughout, to carry out this program tropically one would need:

(I) A *tropical Grassmann bundle* $\pi\colon \mathsf{trop}(\mathrm{Gr}(d, \mathcal{E})) \to X$ whose fiber over $x \in X$ is $\mathsf{trop}(\mathrm{Gr}(d, \mathcal{E}_x))$, realized as a rational polyhedral space. The Speyer–Sturmfels bijection between $\mathsf{trop}(G(d, n))$ and tropical d-planes [16] gives candidates for the fibers.

(II) A *universal exact sequence* $0 \to \mathcal{S} \to \pi^*\mathcal{E} \to \mathcal{Q} \to 0$ over $\mathsf{trop}(\mathrm{Gr}(d, \mathcal{E}))$, with $\mathcal{S}_{(x,K_x)} = K_x$ the tautological subbundle of rank d.

(III) A *left-inverse construction for* π^*, generalizing Lemma 6.2.4 from $\mathbb{TP}(\mathcal{E}) = \mathrm{Gr}(1, \mathcal{E})$ to the higher Grassmann bundle — presumably by the analogous fiberwise computation, with $c_1(\mathcal{O}(1))^n$ replaced by the top Chern class of the tautological quotient on $\mathrm{Gr}(d, \mathcal{E})$.

(IV) A *Giambelli-type computation* verifying

$$\pi_*\big(c_{df}(\mathcal{S}^\vee \otimes \pi^*\mathcal{F})\big) = \Delta^d_{f-k}\big(c_{[t]}(\mathcal{F})/c_{[t]}(\mathcal{E})\big),$$

which is formal given **(I)**–**(III)**.

Of these, **(I)** is the essential obstacle. The following steps follow the standard literature.

Question 8.1.1. Can one construct a tropical Grassmann bundle $\mathsf{trop}(\mathrm{Gr}(d, \mathcal{E})) \to X$ as a rational polyhedral space satisfying **(I)** and **(II)**?

8.2 Towards the Brill–Noether Conjecture

Let Γ be a smooth tropical curve of genus g, and for integers $0 \leq d \leq g$ and $r \geq 0$ let $W^r_d(\Gamma) \subseteq \mathrm{Pic}^d(\Gamma)$ denote the locus of divisor classes of degree d and (tropical) rank at least r (see, e.g., [3, 12]).

Conjecture 8.2.1 ([7])**.** Let Γ be a smooth tropical curve of genus g, and fix integers $0 \leq d \leq g$ and $r \geq 0$ with $\rho := g - (r+1)(g-d+r) \geq 0$. Then there exists a canonical tropical subvariety $Z^r_d \subseteq W^r_d(\Gamma)$ of pure dimension ρ such that

$$[Z^r_d] = \left(\prod_{i=0}^{r} \frac{i!}{(g-d+r+i)!}\right) [\Theta]^{g-\rho}$$

modulo tropical homological equivalence.

The conjecture is known for $r = 0$: then $W^0_d = W_d$ is pure-dimensional [8] and $Z^0_d = W_d$. This matches the classical picture for subvarieties $W_d \subset \mathrm{Pic}^d(C)$ that parametrize effective divisor classes of degree d on a curve C, i.e. images of the Abel–Jacobi maps $u_d : C_d \to \mathrm{Jac}(C) \cong \mathrm{Pic}^d(C)$.

Question 8.2.2. Do there exist tropical analogues $\mathcal{E}, \mathcal{F}$ of these bundles and an evaluation map φ on $\mathrm{Pic}^d(\Gamma)$ for which $W^r_d = D_{e-r-1}(\varphi)$, and to which one can apply the tropical Porteous formula (or a higher-rank extension thereof) to recover Conjecture 8.2.1?

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