Wild Automorphisms and Abelian Varieties

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A dissertation submitted in partial fulfillment of
the requirements for the degree of

Doctor of Philosophy

University of Washington

2010

Program Authorized to Offer Degree: Mathematics

University of Washington
Graduate School

This is to certify that I have examined this copy of a doctoral dissertation by

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Abstract

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An automorphism σ of a projective variety X is said to be *wild* if $\sigma(Y) \neq Y$ for every non-empty subvariety $Y \subsetneq X$. In [RRZ06] Z. Reichstein, D. Rogalski, and J.J. Zhang conjectured that if X is an irreducible projective variety admitting a wild automorphism then X is an abelian variety, and proved this conjecture for $\dim(X) \leq 2$. As a step toward answering this conjecture in higher dimensions we prove a structure theorem for projective varieties of Kodaira dimension 0 admitting wild automorphisms. This essentially reduces the Kodaira dimension 0 case to a study of Calabi-Yau varieties, which we also investigate. In support of this conjecture, we show that there are no wild automorphisms of certain Calabi-Yau varieties.

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ACKNOWLEDGMENTS

I thank Sándor Kovács for his constant guidance and support, without which this dissertation, and my graduate education would not have been possible. I also thank Julia Pevtsova and James Zhang for not only agreeing to serve on my committee, but for all they have taught me. I have been fortunate to take classes from both of them, and have benefited from the clarity with which they teach. In addition, I thank Peter Guttorp, who fit my defense into his schedule without even a meeting. Finally, my work has benefited from discussions with those in my cohort: Jeremy Berquist, Ariana Dundon, Dan Finkel, Jacob Lewis and Kiana Ross.

DEDICATION

To my family, Burt, Teresa, and Lia.

Chapter 1

INTRODUCTION

When studying mathematical objects, an important tool is studying or classifying their symmetries. In algebraic geometry, where the objects of study are algebraic varieties, the symmetries take the form of automorphisms. Given an automorphism, an important question to ask is whether or not it has a fixed point. Indeed, fixed point theorems are pervasive throughout mathematics. For example, we use the results of the Lefschetz Fixed Point Theorems, which is described in Section 2.3.

It is easy to see that any automorphism of $\mathbb{P}^N$ fixes a point. In Section 3.1 we show that the same is true for $\mathbb{P}^1 \times \mathbb{P}^1$. We will also show that there exist automorphisms of elliptic curves that do not fix any finite subset of points. This fact is true in general: if X is an abelian variety then there exists an automorphism of X that has no invariant subvarieties.

Given a variety X we define a *wild automorphism* of X to be an automorphism σ such that $\sigma(Y) \neq Y$ for every non-empty subvariety $Y \subsetneq X$. In [RRZ06] it is shown that any abelian variety admits a wild automorphism. Furthermore, the authors conjecture that these varieties are the only examples of such behavior. Namely:

Conjecture 1.0.1 ([RRZ06])**.** If an irreducible projective variety X admits a wild automorphism, then X is an abelian variety.

Note that in the language of schemes, subvarieties correspond to points, so the above conjecture can viewed as a type of fixed point theorem.

In [RRZ06] the authors consider this conjecture in the context of projectively simple rings. They give a construction of such a ring that requires the existence of a wild automorphism of a projective variety. Background for projectively simple rings and the details of this construction are addressed in Section 1.2. In addition, the authors give a proof of Conjecture 1.0.1 for $\dim(X) \leq 2$.

The existence of a wild automorphism imposes strong restrictions on two important numerical invariants of a projective variety, the Kodaira dimension and the Euler characteristic. The Albanese map must also have various special properties. All of these results are given in [RRZ06]; we review them in Chapter 4. In Chapter 5 we give a proof of Conjecture 1.0.1 for $\dim(X) \leq 2$.

In Chapter 6 we present original results. We prove a structure theorem for irreducible, projective varieties of Kodaira dimension zero admitting a wild automorphism, that essentially reduces Conjecture 1.0.1 to the study of Calabi-Yau varieties in this case.

Theorem 1.0.2. *Let X be an irreducible projective variety with $\kappa(X) = 0$, and σ a wild automorphism of X. Then $X \cong A \times (S/G)$ where A is an abelian variety, S is a Calabi-Yau variety, and $G \leq \operatorname{Aut}(S)$ is a finite subgroup acting freely on S.*

Lastly we give some partial results regarding the existence of wild automorphisms on Calabi-Yau varieties. We show that there are no wild automorphisms of Calabi-Yau varieties of Picard number one. Assuming a conjecture stated in [Ogu01], we show that there are no wild automorphisms of Calabi-Yau varieties of dimension three, and as a corollary we give a proof of Conjecture 1.0.1 for $\dim(X) \leq 4$ and Kodaira dimension zero.

1.1 Notation and Definitions

We will say that an automorphism σ of a variety X *fixes* or *preserves* a subvariety $Y \subset X$ if $\sigma(Y) = Y$. Note that σ does not have to fix the elements of Y pointwise. Unless otherwise noted, we will assume that all varieties are defined over the complex numbers $\mathbb{C}$, although not necessarily irreducible. There will be two exceptions to this: an *abelian variety* is by definition an irreducible, complex, projective algebraic group, and a *Calabi-Yau variety* is by definition an irreducible, simply connected, projective variety X, with trivial canonical bundle such that $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < \dim(X)$. ω_X will denote the canonical bundle on a variety X, and K_X will denote a correponding canonical divisor.

Given a Cartier divisor D on a variety X, we define $\kappa(X, D)$ to be the transcendence degree over $\mathbb{C}$ of the ring $R = \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nD))$ minus 1. The Kodaira dimension of X will be denoted $\kappa(X) = \kappa(X, K_X)$, and the anti-Kodaira dimension will be denoted

$\bar{\kappa}(X) = \kappa(X, -K_X)$. An equivalent definition of $\kappa(X)$ is that it is the largest dimension of the image of X in $\mathbb{P}^N$ by the rational map determined by the linear system $|nK_X|$ for some $n \geq 1$, or $\kappa(X) < 0$ if $|nK_X| = \emptyset$ for all $n \geq 1$. For any variety X, $\kappa(X) \leq \dim(X)$. When equality holds we say that X is a variety of *general type*.

The topological Euler characteristic of X will be denoted by $e(X)$, and the Euler characteristic of the structure sheaf $\mathcal{O}_X$ will be denoted by $\chi(\mathcal{O}_X)$. They are defined as the following numbers:

$$e(X) = \sum_{i=0}^{\infty}(-1)^i h^i(X, \mathbb{C})$$
$$\chi(\mathcal{O}_X) = \sum_{i=0}^{\infty}(-1)^i h^i(X, \mathcal{O}_X).$$

1.2 Motivation via Projectively Simple Rings

As we mentioned earlier, in [RRZ06] Conjecture 1.0.1 arises in an attempt to construct specific examples of projectively simple rings. Here we describe the background material necessary to state the main result that motivates their study of wild automorphisms.

Let k be a base field. An algebra A is $\mathbb{N}$-*graded*(or *graded*) if $A = \bigoplus_{i \geq 0} A_i$ with $1 \in A_0$ and $A_i A_j \subset A_{i+j}$ for all $i, j \geq 0$. The graded algebra A is *locally finite* if $\dim_k A_i < \infty$ for all i.

Definition 1.2.1. A locally finite graded algebra A is called *projectively simple* if $\dim_k A = \infty$ and $\dim_k A/I < \infty$ for every nonzero graded ideal I of A.

Note that since A is locally finite, the condition $\dim_k A/I < \infty$ for a graded ideal I is equivalent to the condition $A_{\geq n} \subset I$ for some n.

We now set out to produce some interesting examples of projectively simple rings. Assume throughout the remaining discussion that k is algebraically closed. Let X be a commutative projective scheme, σ an automorphism of X and $\mathcal{L}$ an invertible sheaf on X. For any sheaf $\mathscr{F}$ denote the pullback $\sigma^*(\mathscr{F})$ by $\mathscr{F}^\sigma$. Let $\mathcal{L}_n = \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \cdots \otimes \mathcal{L}^{\sigma^{n-1}}$.

Definition 1.2.2. The *twisted homogeneous coordinate ring* $B = B(X, \mathcal{L}, \sigma)$ is defined to be the graded vector space $\bigoplus_{n=0}^{\infty} H^0(X, \mathcal{L}_n)$ with the multiplication rule $fg = f \otimes g^{\sigma^m}$ for $f \in B_m$, $g \in B_n$.

Definition 1.2.3. The sheaf $\mathcal{L}$ is called *σ-ample* if for any coherent sheaf $\mathscr{F}$ on X, $H^i(X, \mathscr{F} \otimes \mathcal{L}_n) = 0$ for all $i > 0$ and $n \gg 0$.

Notice that when σ is the identity this is just the usual notion of ampleness.

With these definitions in hand we can now state a proposition that will lead to examples of projectively simple rings. These examples involve the following geometric notion.

Definition 1.2.4. Let σ be an automorphism of a projective scheme X. We call σ *wild* if $\sigma(Y) \neq Y$ for every nonempty reduced closed subscheme of $Y \subsetneq X$.

Proposition 1.2.5 ([RRZ06]). *Let $B = B(X, \mathcal{L}, \sigma)$, where $\mathcal{L}$ is σ-ample. Then B is projectively simple if and only if σ is a wild automorphism of X.*

□

Proposition 1.2.5 leads to two natural, purely geometric questions. Namely, we would like to classify the projective schemes that admit a wild automorphism σ, and for each such scheme, find a *σ-ample* line bundle. The first question is the focus of this paper.

To better understand which projective schemes admit wild automorphisms, the authors in [RRZ06] give several simple observations. These observations, along with their proofs are contained in the following lemma.

Lemma 1.2.6 ([RRZ06]). *Let X be a projective scheme. Let σ be an automorphism of X.*

1. *If σ is wild, then X is reduced.*
2. *If σ is wild, then X is smooth.*
3. *Assume that X is reduced with irreducible components $X_1, X_2, \ldots, X_m$. Then σ is wild if and only if the permutation of the X_i induced by σ is a single m-cycle, and σ^m restricts to a wild automorphism of each X_i. Moreover, if σ is wild then X is a disjoint union of $X_1, X_2, \ldots, X_m$.*
4. *If X is integral, then σ is wild if and only if σ^n is wild for every $n \geq 1$.*

Proof. 1. X_{red} is a nontrivial subscheme of X preserved by σ; hence, $X = X_{red}$.

2. By part (1), X is reduced. Let Y be the singular locus of X. Then Y is closed, σ-invariant and $Y \neq X$. Since σ is wild, Y is empty.

3. The orbit of each component is preserved by σ; hence, there can only be one such orbit. Since the subscheme

$$Y = \bigcup_{i \neq j} X_i \cap X_j \subsetneq X$$

is σ-invariant, it has to be empty, i.e., $X_1, X_2, \ldots, X_m$ are disjoint. If $\sigma^m(Z) = Z$ for some subscheme $Z \subsetneq X_i$, then $\sigma(Z') = Z'$ where $Z' = \bigcup_{j=0}^{m-1} \sigma^j(Z) \subsetneq X$; thus $Z = \emptyset$ and so $\sigma^m|_{X_i}$ is wild.

4. If σ preserves a subscheme $Y \subset X$, then so does σ^n. Conversely, if σ^n preserves $Y \subset X$ then σ preserves $\bigcup_{j=0}^{n-1} \sigma^j(Y) \subset X$.

$\square$

In view of 1.2.6 it is clear that we lose nothing essential in our understanding of wild automorphisms by restricting to the case where X is integral and smooth. In fact, for the remainder of the paper we will study the case where X is a smooth algebraic variety.

Chapter 2

BACKGROUND MATERIALS

This chapter reviews the background materials necessary to prove Conjecture 1.0.1 for $\dim(X) \leq 2$ as well as discuss it in higher dimensions. We will assume that all varieties in this chapter are irreducible.

2.1 Cones

To study the existence of wild automorphisms on Calabi-Yau varieties we will examine the cone of ample divisor classes.

Let V be a $\mathbb{R}$-vector space. A subset $N \subset V$ is called a *cone* if $0 \in N$ and N is closed under multiplication by positive scalars.

A subcone $M \subset N$ is called *extremal* if $u, v \in N, u + v \in M$ imply that $u, v \in M$. M is also called an *extremal face* of N. A 1-dimensional extremal subcone is called an *extremal* ray.

Definition 2.1.1. Let X be a proper variety. A *1-cycle* is a formal linear combination of irreducible, reduced and proper curves $C = \sum a_i C_i$. A 1-cycle is called *effective* if $a_i \geq 0$ for every i. Two 1-cycles C and C' are called *numerically equivalent* if $(C \cdot D) = (C' \cdot D)$ for any Cartier divisor D. 1-cycles with real coefficients modulo numerical equivalence form an $\mathbb{R}$-vector space; it is denoted by $N_1(X)$. The class of a 1-cycle C is denoted by $[C]$.

The Theorem of the Base of Néron-Severi asserts that $N_1(X)$ is finite dimensional. Its dimension is called the *Picard number* of X and is denoted by $\rho(X)$. If X is smooth over $\mathbb{C}$ there is an injection $N_1(X) \hookrightarrow H_2(X, \mathbb{R})$; in this case we will think of $N_1(X)$ as a subset of $H_2(X, \mathbb{R})$ via this inclusion.

Define $\mathrm{NE}(X) = \{\sum a_i [C_i] | C_i \subset X, a_i \geq 0\} \subset \mathrm{N}_1(X)$. The *cone of curves* of X is the closure of $\mathrm{NE}(X)$ in $\mathrm{N}_1(X)$; it will be denoted $\overline{\mathrm{NE}}(X)$.

On a smooth variety X over $\mathbb{C}$, the set of ample divisor classes form an open cone $A(X) \subset H^2(X, \mathbb{R})$. The closure, $\overline{A}(X)$, consists of divisor classes D such that $D \cdot C \geq 0$ for all curves C on X. By definition, such classes D are called *nef* classes, and $\overline{A}(X)$ is known as the nef cone.

2.2 Elementary Transformations

In this section we will assume C is a nonsingular curve, and $\pi : X \to C$ is a ruled surface. We will denote the fiber of π passing through a point $p \in X$ by l_p.

On the blowup of X at p, the strict transform of l_p is a (-1)-curve. Thus we can contract this curve to obtain a surface X' and a rational map $\mathrm{elm}_p : X \dashrightarrow X'$ called the *elementary transformation centered at* P. It is clear that X' is also a ruled surface over C. $\mathrm{elm}_{P_1,\ldots,P_n}(X)$ will denote the ruled surface that is obtained from a ruled surface X by a succession of elementary transformations centered at points $P_1, \ldots, P_n \in X$. Note that elm_{P_1,P_2} cannot be defined if P_1 and P_2 lie on the same fiber of X.

Theorem 2.2.1 ([Bea96]). *The ruled surface $\pi : X \to C$ can be obtained from the direct product $C \times \mathbb{P}^1$ by some successive elementary transformations.*

□

2.3 The Lefschetz Fixed Point Theorems

Let σ be an automorphism of a projective variety X over $\mathbb{C}$. We can think of σ as a holomorphic self map of a compact complex manifold X. Thus σ acts on the cohomology groups $H^q(X, \mathbb{C})$ and $H^q(X, \mathcal{O}_X)$ via the maps $\tilde{\sigma}_q : H^q(X, \mathbb{C}) \to H^q(X, \mathbb{C})$ and $\hat{\sigma}_q : H^q(X, \mathcal{O}_X) \to H^q(X, \mathcal{O}_X)$.

We say that a fixed point p of σ is nondegenerate if in terms of local coordinates $z_1, \ldots, z_n$ centered at p, the Jacobian matrix $\mathscr{J}_\sigma(p) : T_p(X) \to T_p(X)$ satisfies $\det(\mathscr{J}_\sigma(p) - \mathrm{Id}) \neq 0$. In this situation we define the *index* of σ at p to be $i_\sigma(p) = \mathrm{sgn}\ \det(\mathscr{J}_\sigma(p) - \mathrm{Id})$.

Denote the set of fixed points by $\{p_\alpha\}$. In terms of local coordinates $z_{\alpha i}$ for p_α we may write

$$\sigma(z_\alpha)_j = \sum b_{ij} z_{\alpha i} + \text{higher order terms},$$

or equivalently

$$\sigma(z_\alpha) = B_\alpha z_\alpha + \text{higher order terms}$$

where $B_\alpha = (b_{ij})$. By nondegeneracy, $I - B_\alpha$ is non singular.

Define the *Lefschetz number* of the map σ, denoted by $\mathcal{L}(\sigma)$, to be the number $\sum(-1)^q \text{trace}(\tilde{\sigma}_q)$. Similarly we define the *holomorphic Lefschetz number*, denoted $\mathcal{L}(\sigma, \mathcal{O}_X)$, to be the number $\sum(-1)^q \text{trace}(\hat{\sigma}_q)$. Note that $\mathcal{L}(\text{Id}) = e(X)$, the topological Euler characteristic of X, and $\mathcal{L}(\text{Id}, \mathcal{O}_X) = \chi(O_X)$, the Euler characteristic of $\mathcal{O}_X$.

In [GH94] the following two equalities are refered to as the *Lefschetz Fixed Point Theorem* and the *Holomorphic Lefschetz Fixed Point Theorem* respectively:

$$\mathcal{L}(\sigma) = \sum_{\sigma(p_\alpha)=p_\alpha} i_\sigma(p_\alpha) \tag{2.1}$$

$$\mathcal{L}(\sigma, \mathcal{O}_X) = \sum_{\sigma(p_\alpha)=p_\alpha} \frac{1}{\det(I - B_\alpha)} \tag{2.2}$$

Since both sums are taken over all fixed points of σ, if either $\mathcal{L}(\sigma)$ or $\mathcal{L}(\sigma, \mathcal{O}_X)$ is non-zero, the corresponding sum is also non-zero and hence there must be at least one fixed point.

2.4 Abelian Varieties and their Automorphisms

Let V be a complex vector space of dimension n. Let $L \subset V$ be a lattice, that is: $L \cong \mathbb{Z}^{2n}$ and the map $L \otimes_{\mathbb{Z}} \mathbb{R} \to V_{\mathbb{R}}$ is an isomorphism of $\mathbb{R}$-vector spaces. We say that $X := V/L$ is a complex torus. The variety X is abelian if it is projective.

Let $X = V/L$ be an abelian variety of dimension n. Since X is a complex Lie group its tangent bundle, T_X is trivial. The fiber of the bundle is $T_{X,0}$, the tangent space at the identity element of X. Thus $T_X = \mathcal{O}_X \otimes V \cong \mathcal{O}_X^{\oplus n}$. Similarly $\Omega_X = T_X^* = \mathcal{O}_X \otimes V^* \cong \mathcal{O}_X^{\oplus n}$, and $\omega_X = \bigwedge^n \Omega_X = \mathcal{O}_X$. Consequently, $K_X = 0$.

Let $f : X \to Y$ be a map of abelian varieties. Following the notation of [RRZ06], if f is a regular morphism that preserves the group structure we will say that f is a *homomorphism*, or an *endomorphism* if in addition $Y = X$. We reserve the words *automorphism* and *morphism* for maps that do not necessarily preserve the group structure. For example, for any abelian variety X and $b \in X$, the map $T_b(x) = x + b$ is an automorphism. To characterize the morphisms of abelian varieties we will need the following lemma.

Lemma 2.4.1 ([BL04] Rigidity Lemma)**.** *Let X be a complete variety, Y and Z any varieties, $f : X \times Y \to Z$ a morphism such that for some $y_0 \in Y$, $f(X \times \{y_0\})$ is a single point z_0 of Z. Then there is a morphism $g : Y \to Z$ such that if $p_2 : X \times Y \to Y$ is the projection, $f = g \circ p_2$.*

Proof. Choose any point $x_0 \in X$, and define $g : Y \to Z$ by $g(y) = f(x_0, y)$. Since $X \times Y$ is a variety, to show that $f = g \circ p_2$, it is sufficient to show that these morphisms agree on some open subset of $X \times Y$. Let U be an affine open neighborhood of z_0 in Z, $F = Z \setminus U$, and $G = p_2(f^{-1}(F))$; then G is closed in Y since X is complete and hence p_2 is a closed map. Further $y_0 \notin G$ since $f(X \times \{y_0\}) = \{z_0\}$. Therefore $V = Y \setminus G$ is a non empty open subset of Y. for each $y \in V$, the complete variety $X \times \{y\}$ gets mapped by f into the affine variety U, and hence to a single point of U. But this means that for any $x \in X$, $y \in V$, $f(x, y) = f(x_0, y) = g \circ p_2(x, y)$, and this completes the proof. □

Corollary 2.4.2 ([BL04])**.** *If X and Y are abelian varieties and $f : X \to Y$ is any morphism, $f(x) = \alpha(x) + b$ where α is a homomorphism of X into Y and $b \in Y$.*

Proof. Replacing f by $f - f(0)$, we may assume $f(0) = 0$ and we have to show under this assumption that f is a homomorphism.

Consider the morphism $\phi : X \times X \to Y$ defined by $\phi(x, y) = f(x + y) - f(y) - f(x)$. Then $\phi(X \times \{0\}) = \phi(\{0\} \times X) = 0$. By the above lemma $\phi \equiv 0$ on $X \times X$, in other words, f is a homomorphism. □

Thus any automorphism, σ, of an abelian variety X can be expressed as $\sigma = T_b \circ \alpha$ where α is an endomorphism (also an automorphism) of X and T_b is the translation automorphism by the element $b \in X$.

2.4.1 The Albanese Variety and Morphism

An important tool used to study a variety is its albanese morphism to its albanese variety. To briefly describe both of these we first need the following result which is proved using

Hodge Theory.

Lemma 2.4.3 ([Bea96])**.** *Let X be a smooth projective variety. The image of the map $i : H_1(X, \mathbb{Z}) \to \Gamma(X, \Omega_X)^*$ defined by $\gamma \mapsto (\omega \mapsto \int_\gamma \omega)$ is a lattice in $\Gamma(X, \Omega_X)^*$, and the quotient is an Abelian Variety.*

□

Thus we can make the following definition:

Definition 2.4.4. Let X be a smooth projective variety. The *Albanese Variety* Alb(X) is the complex torus of dimension $q = h^0(X, \Omega_X)$ given by

$$\mathrm{Alb}(X) := \Gamma(X, \Omega_X)^*/H_1(X, \mathbb{Z})$$

The *Albanese Morphism*, $\mathrm{alb}_X : X \to \mathrm{Alb}(X)$ is defined as follows: Fix a point p in X. Let c_x be a path joining p to a point $x \in X$, and let $a(c_x) \in \Omega_X^*$ be the linear form $\omega \mapsto \int_{c_x} \omega$. If we replace c_x by another path c'_x joining p to x, we change $a(c_x)$ by an element of $H_1(X, \mathbb{Z})$. Thus the class of $a(c_x)$ depends only on x. We define this class to be $\mathrm{alb}(x)$.

Properties of the Albanese Variety and Morphism(see [Bea96])

1. alb_X induces an isomorphism $\mathrm{alb}_X^* : H^0(\mathrm{Alb}(X), \Omega_{\mathrm{Alb}(X)}) \to H^0(X, \Omega_X)$

2. for any complex torus T and any morphism $f : X \to T$ there is a unique morphism $\tilde{f} : \mathrm{Alb}(X) \to T$ such that $\tilde{f} \circ \mathrm{alb}_X = f$.

3. $\mathrm{Alb}(X)$ is determined by Property 2 up to isomorphism.

4. from Property 2 it follows that the Albanese variety is functorial: if $f : X \to Y$ is a morphism of smooth projective varieties, there exists a unique morphism $F : \mathrm{Alb}(X) \to \mathrm{Alb}(Y)$ such that the following diagram is commutative.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle \mathrm{alb}_X} & & \downarrow{\scriptstyle \mathrm{alb}_Y} \\ \mathrm{Alb}(X) & \xrightarrow{\exists ! F} & \mathrm{Alb}(Y) \end{array}$$

5. from Property 2 it also follows that the Abelian Variety is generated by $\mathrm{alb}_X(X)$ (because the Abelian subvariety of $\mathrm{Alb}(X)$ generated by $\mathrm{alb}_X(X)$ satisfies this property). In particular if $q(X) \neq 0$ then $\mathrm{alb}_X(X)$ is not a point.

6. If X is an abelian variety then $\mathrm{Alb}(X) \cong X$.

Example 2.4.5. Let $\pi : X \to E$ be a ruled surface over an elliptic curve E. By [Har77, V.2.5] $q(X) = 1$ so $\mathrm{Alb}(X)$ is an elliptic curve. Using the Property 2 from above, π factors through the albanese map. In other words, we have: $\pi : X \xrightarrow{\mathrm{alb}_X} \mathrm{Alb}(X) \xrightarrow{\psi} E$. Since the image of alb_X is not a point, and alb_X is proper, it is a surjective morphism. Since π has connected fibers, ψ is injective. The map ψ is surjective since π is surjective. Thus ψ is a bijective morphism between two smooth curves, and hence is an isomorphism.

We have just shown that $\pi : X \to E$ is actually the albanese morphism. Using Property 4 from above we see that any automorphism of a ruled surface over an elliptic curve descends to an automorphism of the elliptic curve.

2.4.2 Isogenies

Next we define a special class of homomorphisms of abelian varieties, the isogenies. An *isogeny* of an abelian variety X to an abelian variety X' is by definition a surjective homomorphism $X \to X'$ with finite kernel. Obviously a homomorphism is an isogeny if an only if it is surjective and $\dim X = \dim X'$. If $\Gamma \subset X$ is a finite subgroup, the quotient X/Γ is an abelian variety and the natural projection $X \to X/\Gamma$ is an isogeny. To see this, let $X = V/L$ and note that $\pi^{-1}(\Gamma) \subset V$ is a lattice containing L. X/Γ is the abelian variety $V/\pi^{-1}(\Gamma)$. Up to isomorphism, every isogeny is of this type. Define the *exponent* $e = e(f)$ of an isogeny f to be the exponent of the finite group $\ker(f)$.

As an example, for any integer n define the homomorphism $n_X : X \to X$ by $x \mapsto nx$. The kernel of n_X is equal to $\frac{1}{n}L/L \cong L/nL$. Therefore for any $n \neq 0$ the homomorphism n_X is an isogeny.

Proposition 2.4.6. *For any isogeny $f : X \to X'$ of exponent e there exists an isogeny $g : X' \to X$, unique up to isomorphism, such that $gf = e_X$.*

Proof. As $\ker(f) \subset \ker(e_X)$, there is a unique map $g : X' \to X$ such that $gf = e_X$, with g and e_X an isogeny. □

Corollary 2.4.7. *Isogenies define an equivalence relation on the set of abelian varieties.*

Hence it makes sense to call two abelian varieties *isogenous*, if there is an isogeny between them.

2.5 Kähler Manifolds with $c_1^{\mathbb{R}} = 0$

One of our main results will use a structure theorem of Beauville given in [Bea83]. This section presents the relevant results of this paper.

Let X be a smooth variety. From the exponential sequence $0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$ we get a long exact sequence in cohomology, part of which is the morphism $d : H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z})$. Let X be a smooth projective variety, and $\mathscr{E}$ a vector bundle on X. The first chern class of $\mathscr{E}$, denoted $c_1(\mathscr{E})$, is defined as $d(\det \mathscr{E}) \in H^2(X, \mathbb{Z})$. The first chern class of X is denoted $c_1(X)$ and is defined as $c_1(\mathscr{T}_X) = d(-K_X)$. $c_1^{\mathbb{R}}(X)$ denotes the image of $c_1(X)$ in $H^2(X, \mathbb{Z}) \otimes \mathbb{R}$. Note that if ω_X is torsion, then $c_1^{\mathbb{R}}(X) = 0$

Theorem 2.5.1 ([Bea83])**.** *Let X be a compact Kähler Manifold with $c_1^{\mathbb{R}}(X) = 0$. Then there exists a finite étale cover $\tilde{X}$ of X which is isomorphic to a product $T \times \prod V_i \times \prod X_j$, where*

1. *T is a complex torus,*

2. *V_i is a simply connected projective manifold, of dimension ≥ 3, with trivial canonical bundle, such that $H^p(V_i, \mathcal{O}_{V_i}) = 0$ for $0 < p < \dim(V_i)$, and*

3. *X_j is a simply connected compact Kähler manifold of even dimension $n = 2r$ with $\chi(\mathcal{O}_{X_j}) = 2^r$.*

Moreover, this decomposition is unique up to the order of the V_i and X_j.

□

Lemma 2.5.2 ([Bea83]). *Let T be a complex torus and S a simply connected compact Kähler manifold. Then any automorphism of $T \times S$ is of the form (v, w), with $v \in \mathrm{Aut}(T)$ and $w \in \mathrm{Aut}(S)$.*

□

Definition 2.5.3. Let X be a compact manifold. A finite étale cover $\tilde{X} \to X$ is called a *split covering* if $\tilde{X}$ is isomorphic to the product of a torus and a simply connected compact manifold. A split covering $T \times S \to X$ is called *minimal* if it is Galois and if its Galois group does not contain any element of the form (τ, l_S) where τ is a translation of the torus T.

Proposition 2.5.4 ([Bea83]). *Let X be a compact complex manifold that admits a finite split covering. Then there exists a unique minimal split covering $\pi : T \times S \to X$*

□

From this, we deduce the following important remark:

Remark 2.5.5. Any compact Kähler manifold with $c_1^{\mathbb{R}}(X) = 0$ has a unique minimal split covering of the form $\tilde{X} = T \times \prod V_i \times \prod X_j$, where T, V_i, and X_j are as in Theorem 2.5.1.

With this structure in mind we can now describe the automorphism group of a Kähler manfiold X with $c_1^{\mathbb{R}}(X) = 0$ in terms of the automorphism group of $\tilde{X}$. Let us write $X = (T \times S)/G$, where the covering is minimal. Then any automorphism of X lifts to $T \times S$, so that the group $\mathrm{Aut}(X)$ is identified with the normalizer of G in $\mathrm{Aut}(T) \times \mathrm{Aut}(S)$. Write $S = \prod S_i^{n_i}$, where the S_i are non-isomorphic manifolds equal to some V_i or X_j. By the unicity property $\mathrm{Aut}(S) = \prod \mathrm{Aut}(S_i^{n_i})$.

Chapter 3

EXAMPLES

It is easy to see that any automorphism of $\mathbb{P}^N$ fixes a point: $\mathrm{Aut}(\mathbb{P}^N) = \mathrm{PGL}_{N+1}(\mathbb{C})$, so any automorphism of $\mathbb{P}^N$ can be represented by an $(N+1) \times (N+1)$ complex valued matrix. An eigenvector of this matrix will correpond to a fixed point in $\mathbb{P}^N$. However, determining the existence of wild automorphisms on other varieties quickly becomes non-trivial. In this section we will examine two examples that support Conjecture 1.0.1, and then give a classification of wild automorphisms of abelian varieties.

3.1 Automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1$

By examining the cone of curves, $\overline{\mathrm{NE}}(\mathbb{P}^1 \times \mathbb{P}^1)$, we will show that that are no wild automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1$. Note that $\mathbb{P}^1 \times \mathbb{P}^1$ is not an abelian variety so this is what we expect according to Conjecture 1.0.1.

Lemma 3.1.1. *Two Cartier divisors on $\mathbb{P}^1 \times \mathbb{P}^1$ are numerically equivalent if and only if they are linearly equivalent.*

Proof. First note that linear equivalence implies numerical equivalence in general: Assume D and F are linearly equivalent. By definition $D - F = 0 \in \mathrm{Cl}(X)$. Thus for any divisor C we have that $(D-F).C = 0$, in other words $D.C = F.C$.

Now let D and F be two numerically equivalent cartier divisors on $\mathbb{P}^1 \times \mathbb{P}^1$. We will show that their images in $\mathrm{Cl}(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathbb{Z} \times \mathbb{Z}$ are equal. Let the images of D and F be (a, b) and (a', b') respectively. Since D and F are numerically equivalent we know that $(a, b).(c, d) = (a', b').(c, d)$ for every divisor (c, d). Computing these intersection products, we get that $ad + bc = a'd + b'c$ for all $c, d \in \mathbb{Z}$. Thus $(a, b) = (a', b')$ so D and F are linearly equivalent. □

We are now ready to prove our claim for $\mathbb{P}^1 \times \mathbb{P}^1$.

Theorem 3.1.2. *There are no wild automorphisms of* $\mathbb{P}^1 \times \mathbb{P}^1$.

Proof. Let σ be an automorphism of $\mathbb{P}^1 \times \mathbb{P}^1$. Using the above lemma, we see that the cone of curves of $\mathbb{P}^1 \times \mathbb{P}^1$ is the first quadrant in $\mathbb{R}^2$. The automorphism σ preserves linear equivalence so by the above lemma, σ induces a linear automorphism of $\mathbb{R}^2$ preserving the cone of curves. Let $E_1 = \{\text{pt.}\} \times \mathbb{P}^1$ and $E_2 = \mathbb{P}^1 \times \{\text{pt.}\}$ generate the cone of curves. Since the boundary of $\overline{\text{NE}}(\mathbb{P}^1 \times \mathbb{P}^1)$ is preserved by σ we either have that $\sigma(E_1) \sim E_1$ or $\sigma(E_1) \sim E_2$

If $\sigma(E_1) \sim E_1$ then σ permutes the fibers of the projection morphism onto the first coordinate. Thus σ descends to an automorphism of $\mathbb{P}^1$; denote this $\bar{\sigma}$. As an automorphism of $\mathbb{P}^1$, $\bar{\sigma}$ will fix a point $x \in \mathbb{P}^1$. This implies that σ fixes the fiber $\{x\} \times \mathbb{P}^1$

If $\sigma(E_1) \sim E_2$ then $\sigma^2(E_1) \sim E_1$. Applying the above case to σ^2 we see that $\sigma^2(L) = L$ for some fiber L. This implies that the point $L \cap \sigma(L)$ is fixed by σ. □

3.2 A Wild Automorphism of an Elliptic Curve

Before we discuss wild automorphisms of abelian varities of arbitrary dimension, we consider the illuminating example of an elliptic curve. In [Har77, §IV.4] we see that elliptic curves are abelian varieties of dimension 1 and hence are isomorphic to $\mathbb{C}$ modulo a lattice. We first give a theorem describing the group structure of an elliptic curve E, and with this in hand we can then easily illustrate a wild automorphism σ of E.

Theorem 3.2.1 ([Har77])**.** *Let E be an elliptic curve over $\mathbb{C}$. Then as an abstract group, E is isomorphic to $\mathbb{R}/\mathbb{Z} \oplus \mathbb{R}/\mathbb{Z}$. In particular, for any n, the subgroup of points of order n is isomorphic to $\mathbb{Z}/n \oplus \mathbb{Z}/n$.*

Proof. As a group, E is isomorphic to $\mathbb{C}/\Lambda$ where Λ is the lattice $\{n + m\tau \in \mathbb{C} | n, m \in \mathbb{Z}\}$ for a fixed $\tau \in \mathbb{C} \setminus \mathbb{R}$. This in turn is isomorphic to $\mathbb{R}/\mathbb{Z} \oplus \mathbb{R}/\mathbb{Z}$. The points of order n are represented by $(a/n) + (b/n)\tau$, with $a, b = 0, 1, \ldots, n-1$. Thus the points whose coordinates are not rational combinations of $1, \tau$ are of infinite order. □

By the above theorem, we may choose an element $b \in E$ of infinite order. We will show that the automorphism T_b is wild. Suppose to the contrary that T_b has a fixed subvariety.

Such a subvariety would have to be dimension 0, in other words, some finite set of points Y. Since T_b is an automorphism, T_b permutes the points in Y. Thus T_b^N acts as the identity on Y for some N. This implies that for any $x \in Y$, $x = x + Nb$ which implies $Nb = 0$. This contradicts the fact that b is an element of infinite order. Thus T_b must be a wild automorphism.

3.3 Wild Automorphisms of Abelian Varieties

In [RRZ06], the authors give a very nice classification of the wild automorphisms of an abelian variety. From this it will be easy to see that every abelian variety admits a wild automorphism. Recall that any automorphism, σ, of an abelian variety X can be expressed as $\sigma = T_b \circ \alpha$ where α is an endomorphism (also an automorphism) of X and T_b is the translation automorphism by the element $b \in X$.

Definition 3.3.1. The algebraic subgroup of an abelian variety X *generated* by $S \subset X$ is the Zariski closure of the (abstract) subgroup $< S >$ of X generated by S. In particular, we say that S *generates* X if S is not contained in any proper closed subgroup of X. If $S = \{s_1, s_2, \ldots\}$, then we will say that $s_1, s_2, \ldots$ generate X.

Definition 3.3.2. If R is a ring with identity, we call an element $r \in R$ *unipotent* if $r - 1$ is nilpotent.

Theorem 3.3.3 ([RRZ06]). *Let $\sigma = T_b \circ \alpha$ be an automorphism of the abelian variety X. Let $\beta = \alpha - \mathrm{Id}$, and set $S = \{b, \beta(b), \beta^2(b), \ldots\} \subset X$. Then the following are equivalent:*

1. *σ is wild*

2. *α is unipotent and S generates X.*

3. *α is unipotent and the image $\bar{b}$ of b generates $X/\beta(X)$.*

□

In the remainder of this section we will state two propositions that study the above conditions for wildness. In light of part (3) of Theorem 3.3.3, we would like to understand under what conditions a single point of an abelian variety generates that variety. This is the content of the next proposition. The first two parts of the proposition show that "most" points will generate the variety.

Proposition 3.3.4 ([RRZ06]). *Let X be an abelian variety and $a \in X$ a point.*

1. *The element a generates X if and only if $f(a) \neq 0$ for every $0 \neq f \in End(X)$.*

2. *There is a countable set of closed subgroups $\{G_\alpha\}$ of X such that a generates X if and only if $a \notin \bigcup G_\alpha$.*

3. *If $f : X \to Y$ is an isogeny, then a generates X if and only if $f(a)$ generates Y.*

4. *If X is simple, then a generates X if and only if a is a point of infinite order on X.*

5. *Let $X = X_1 \times X_2 \times \cdots \times X_n$ where the X_i are abelian varieties such that $\mathrm{Hom}(X_i, X_j) = 0$ for all $i \neq j$. Then $a = (a_1, a_2, \ldots, a_n)$ generates X if and only if a_i generates X_i for every $i = 1, \ldots, n$.*

6. *$a = (a_1, a_2, \ldots, a_n) \in X^{\times n}$ generates $X^{\times n}$ if and only if the following condition holds: given endomorphisms $\theta_i \in End(X)$ with $\sum_{i=1}^{n} \theta_i(a_i) = 0$, one must have $\theta_i = 0$ for $i = 1, \ldots, n$.*

□

Remark 3.3.5. Statement Proposition 3.3.4(3) with an element a replaced with a subset S holds as well. The proof is identical.

Next we turn our attention to unipotent automorphisms of abelian varieties. Of course, the identity automorphism is unipotent for any abelian variety. The next proposition identifies those abelian varieties for which there exist nonidentity unipotent automorphisms, and shows how to construct some of them. Suppose that $X = Y^{\times n}$ where Y is an abelian variety. Then for any integer matrix $M \in M_n(\mathbb{Z})$ we may define an endomorphism $\alpha_M \in End(X)$, as follows. Write an arbitrary point in X as a column vector x with entries from Y. Then let α_M be defined by the formula $\alpha_M(x) = Mx$, where the right hand side is matrix multiplication performed using the $\mathbb{Z}$-module structure of Y.

Proposition 3.3.6 ([RRZ06]).

1. *Let $X = Y^{\times n}$ for some abelian variety Y. Then for $M \in M_n(\mathbb{Z})$, the automorphism $\alpha_M \in End(X)$ is unipotent if and only if M is a unipotent matrix in $GL_n(\mathbb{Z})$.*

2. *Let X be an abelian variety that is isogenous to a product $\Pi_i X_i$, where the X_i are simple abelian varieties. Then X has a unipotent automorphism $\mathrm{Id} \neq \alpha \in End(X)$ if and only if X_i and X_j are isogenous for some $i \neq j$.*

□

If one is only concerned about the existence of wild automorphisms of an abelian variety X then we can construct one as follows. By Proposition 3.3.4 (2), choose an element $b \in X$ that generates X. By Theorem 3.3.3 (2), the automorphism T_b is a wild automorphism of X.

Chapter 4

FIRST PROPERTIES

In [RRZ06] the authors prove many non trivial consequences of the existence of a wild automorphism. We will reproduce several of them here, not only for reference, but as an illustration of the methods used.

4.1 Wild Automorphisms and Euler Characteristics

In this section we will show that the existence of a wild automorphism imposes strong restrictions on both the topological Euler characteristic, $e(X)$, and the Euler characteristic of $\mathcal{O}_X$, $\chi(\mathcal{O}_X)$.

Let σ be an automorphism on a variety X. The automorphism σ induces maps on the cohomology groups $H^p(X, \mathbb{C})$ and $H^p(X, \mathcal{O}_X)$. In what follows, the distinction between the two is unimportant so we will denote both of these groups by H^p and the induced map by σ_p^*. We will also abuse notation by letting $\mathcal{L}(X, \sigma) = \sum_{p=0}^{\infty} (-1)^p \text{trace}(\sigma_p^*)$

Recall that by the Lefschetz Fixed Point Theorems if $\mathcal{L}(X, \sigma) \neq 0$ then σ has a fixed point.

Theorem 4.1.1. *Let X be a an irreducible projective variety that admits a wild automorphism σ. Then $\chi(\mathcal{O}_X) = e(X) = 0$.*

Proof. Let $\sigma^* : H^* \to H^*$ be the induced map on the total cohomology group. We can view σ^* as an endomorphism of a finite dimensional vector space, so σ^* satisfies its characteristic polynomial: $(\sigma^*)^n + c_{n-1}(\sigma^*)^{n-1} + \ldots + c_0(\text{Id}_{H^*}) = 0$ with $c_0 \neq 0$. This map preserves the grading of H^* so for each p we know $(\sigma_p^*)^n + c_{n-1}(\sigma_p^*)^{n-1} + \ldots + c_0(\text{Id}_{H^p}) = 0$. Using the linearity of trace we get $\mathcal{L}(X, \sigma^n) + c_{n-1}\mathcal{L}(X, \sigma^{n-1}) + \ldots + c_0\mathcal{L}(X, \text{Id}_X) = 0$. By Lemma 1.2.6 no power of σ fixes a point since σ is wild, and thus $\mathcal{L}(X, \sigma^i) = 0$ for $i > 0$. Hence $\mathcal{L}(X, \text{Id}_X) = 0$. This implies that $\chi(\mathcal{O}_X) = e(X) = 0$. $\square$

Corollary 4.1.2. *Let X be an irreducible projective variety that admits a wild automorphism σ. Then X is neither rationally connected, nor unirational, nor rational.*

Proof. If X is rationally connected then $H^i(X, \mathcal{O}_X) = 0$ for all $i > 0$ [Kol96], so that $\chi(\mathcal{O}_X) = 1$. A variety that is either unirational or rational is also rationally connected. □

4.2 Invariant Line Bundles

Given a wild automorphism of an irreducible projective variety X, we have already shown that X must be smooth. Thus we may consider the canonical bundle ω_X and the Kodaira dimension $\kappa(X)$ of the variety X. Note that $\sigma^*\omega_X \cong \omega_X$ for any automorphism σ of X, in particular for those σ that are wild. So the following question arises: what can we say about those line bundles that are invariant under the action of a wild automorphism? This section will give an answer to this question, and apply the results to the canonical bundle.

Proposition 4.2.1. *Suppose an irreducible projective variety X admits a wild automorphism σ. If $\mathcal{L}$ is a line bundle on X such that $\sigma^*\mathcal{L} \cong \mathcal{L}$ then either*

1. *$\mathcal{L}$ is isormophic to $\mathcal{O}_X$, or*

2. *$H^0(X, \mathcal{L}) = 0 = H^0(X, \mathcal{L}^{-1})$.*

Proof. Suppose that (2) fails, in other words that either $\mathcal{L}$ or $\mathcal{L}^{-1}$ has a nonzero global section. Reversing the roles of $\mathcal{L}$ and $\mathcal{L}^{-1}$, if necessary, we may assume that $V := H^0(X, \mathcal{L}) \neq 0$.

The pullback by σ induces an automorphism $\tilde{\sigma}$ of V. Since V is a vector space over the complex numbers, it must have an eigenvector f for the action of $\tilde{\sigma}$. The vanishing set $Z(f) \subsetneq X$ of the global section f is then fixed by σ. Since σ is wild, $Z(f) = \emptyset$. This implies that the map $i : \mathcal{O}_X \to \mathcal{L}$ sending 1 to f is an isomorphism. □

Corollary 4.2.2. *Suppose an irreducible projective variety X of dimension $d \geq 1$ admits a wild automorphism. Then either*

1. *$\kappa(X) = \bar{\kappa}(X) < 0$, or*

2. *$\omega_X^{\otimes n} \cong \mathcal{O}_X$ for some $n \geq 1$. (In this case $\kappa(X) = \bar{\kappa}(X) = 0$)*

In particular X cannot be a Fano variety or a variety of general type.

Proof. We apply Proposition 4.2.1 with $\mathcal{L} = \omega_X^{\otimes n}$. Either $\omega_X^{\otimes n} \cong \mathcal{O}_X$ for some n, or $H^0(X, \omega_X^{\otimes n}) = 0 = H^0(X, \omega_X^{\otimes n})$ for all n. In the latter case, $\kappa(X) = \bar{\kappa}(X) < 0$.

If X is Fano then $\bar{\kappa}(X) = d \geq 1$, and if X is of general type then $\kappa(X) = d \geq 1$. □

Corollary 4.2.3. *Let X be an abelian variety, and suppose that $Y \subset X$ is an irreducible subvariety of X such that Y admits a wild automorphism. Then Y is a translate of an abelian subvariety of X.*

Proof. By corollary 4.2.2, since Y has a wild automorphism, $\kappa(Y) \leq 0$. A theorem of Ueno, proved in [Uen75], states that Y must be a translate of an abelian subvariety of X. □

The following corollary is not explicitly stated in [RRZ06], but will be useful later.

Corollary 4.2.4. *Let σ be an automorphism of an irreducible projective variety X, and $\mathcal{L}$ a σ-invariant ample line bundle. Then σ is not wild.*

Proof. Choose an integer m such that $\mathcal{L}^{\otimes m}$ is very ample. The line bundle $\mathcal{L}^{\otimes m}$ is also a σ-invariant line bundle. Since $\mathcal{L}^{\otimes m}$ is very ample it is not isomorphic to $\mathcal{O}_X$ nor is $H^0(X, \mathcal{L}^{\otimes m}) = 0$. Thus σ is not wild. □

4.3 Wild Automorphisms and the Albanese Morphism

Recall that in Section 2.4.1 we reviewed some of the properties of the albanese variety and morphism associated to an irreducible variety X. Here we will see how these interact with a wild automorphism σ.

Let X be an irreducible projective variety of dimension n and irregularity q, and σ a wild automorphism of X. Let $\mathrm{alb}_X : X \to \mathrm{Alb}(X)$ denote the albanese morphism. By the universal property of the albanese morphism, σ induces an automorphism of $\mathrm{Alb}(X)$, which we will denote by $\bar{\sigma}$.

Lemma 4.3.1. *Let $\bar{X} = \mathrm{alb}_X(X)$. The automorphism $\bar{\sigma}$ restricts to a wild automorphism of $\bar{X}$.*

Proof. Assume the contrary. Let $\bar{\sigma}(\bar{Y}) \subset \bar{Y}$ for $\emptyset \subsetneq \bar{Y} \subsetneq \bar{X}$. Then setting $Y = \mathrm{alb}_X^{-1}(\bar{Y})$, we see that $\emptyset \subsetneq \sigma(Y) \subset Y$, a contradiction. □

Theorem 4.3.2. *With the notation as above. The following are true.*

1. alb_X *is surjective.*

2. $\bar{\sigma}$ *is a wild automorphism of* $\mathrm{Alb}(X)$*.*

3. alb_X *is smooth.*

4. *The fiber* $X_t = \mathrm{alb}_X^{-1}(t)$ *is a smooth irreducible variety of dimension* $n - q$ *for every* $t \in \mathrm{Alb}(X)$*.*

5. $q \leq n$ *and if* $q = n$ *then* X *is an abelian variety.*

Proof.

1. By Lemma 4.3.1 and Corollary 4.2.3 $\bar{X} = \mathrm{alb}_X(X)$ is a translate of an abelian subvariety in $\mathrm{Alb}(X)$. By the definition of the Albanese map, this implies that $\bar{X} = \mathrm{Alb}(X)$.

2. This follows from part (1) and Lemma 4.3.1

3. By generic smoothness there exists a non-empty Zariski open subset $U \subset \mathrm{Alb}(X)$ such that alb_X is smooth over U [Har77, Lemma III.10.5]. Then alb_X is smooth over the $\bar{\sigma}$-invariant open subset $W = \bigcup_{i\in\mathbb{Z}} \bar{\sigma}^i(U) \subset \mathrm{Alb}(X)$. Since $\bar{\sigma}$ is wild, $W = \mathrm{Alb}(X)$.

4. The fact that each X_t is smooth of dimension $n - q$ is immediate from part (3). To show that X_t is irreducible, consider the Stein factorization $\mathrm{alb}_X : X \xrightarrow{\alpha} X' \xrightarrow{\beta} \mathrm{Alb}(X)$, where α has connected fibers and β is finite. The variety X' is defined explicitly as $\mathbf{Spec}\ \pi_*(\mathcal{O}_X)$ [Har77, Corollary III.11.5], and is irreducible. We will show that β is an isomorphism. This will imply that each X_t is connected and smooth, so must be irreducible.

Since the automorphism $\bar{\sigma}$ acts on the sheaf of graded algebras $\pi_*(\mathcal{O}_X)$, we get an induced automorphism $\tilde{\sigma}$ of X'. Since β is a finite surjective morphism and $\bar{\sigma}$ is a wild automorphism of $\mathrm{Alb}(X)$, it follows that $\tilde{\sigma}$ is a wild automorphism of X'. As in the proof of part (3), β must be a smooth morphism of relative dimension 0,namely an étale map. By a theorem of Serre and Lang, see [Mum70, Section IV.18], X' has the structure of an abelian variety. By the universal property of the Albanese morphism, α factors through π. In other words, β has an inverse, so β is an isomorphism as desired.

5. The inequality $q \leq n$ is immediate from part (1). If $q = n$, then by (3) alb_X is a smooth morphism of relative dimension 0, so étale. Applying the theorem of Serre and Lang again, [Mum70, Section IV.18] , we conclude that X has the structure of an abelian variety.

□

Chapter 5

PROOF IN LOW DIMENSIONS

From the results we have given it is easy to see that if σ is a wild automorphism of an irreducible algebraic curve X, then X is an abelian variety. Theorem 4.1.1 implies that $\chi(\mathcal{O}_X) = 0$, which in turn implies that $h^1(\mathcal{O}_X) = 1$. Since the dimension of X is also one, Theorem 4.3.2 implies that X is an abelian variety.

The proof of Conjecture 1.0.1 in the case where $\dim(X) = 2$ relies on the Castelnuovo-Enriques classification of minimal algebraic surfaces. Before we give this proof we will first consider the automorphisms of two important examples, hyperelliptic surfaces and ruled surfaces. With the exception of Propositions 5.1.5 and 5.2.2, which give new proofs of known results, the proof we give of Conjecture 1.0.1 for $\dim(X) = 2$ is identical to that in [RRZ06].

5.1 Automorphisms of Hyperelliptic Surfaces

The automorphism groups of hyperelliptic surfaces were calculated in [BM90]. We will list the relevant facts below:

Lemma 5.1.1. *Every hyperelliptic surface is isomorphic to* $(E \times F)/G$*, the quotient of a product of elliptic curves by a finite group* G*.*

Remark 5.1.2. The action of G is given by $g \cdot (x, y) \mapsto (x + e, \alpha(g) \cdot y)$ for some $e \in E$ and injective homomorphism $\alpha : G \to \mathrm{Aut}(F)$. There are only seven possibilities for E, F, G, and α.

Lemma 5.1.3. *Let* $X = (E \times F)/G$ *be a hyperelliptic surface and let* M *denote the centralizer of* G *in* $\mathrm{Aut}(E) \times \mathrm{Aut}(F)$*. Then* $\mathrm{Aut}(X) \cong M/G$*.*

Lemma 5.1.4. *Let* $X = (E \times F)/G$ *be a hyperelliptic surface. The subgroup of* $\mathrm{Aut}(E \times F)$*, isomorphic to* E*, consisting of automorphisms of the form* (T_b, Id_F) *embeds into the quotient*

$M/G \cong \mathrm{Aut}(X)$ as a normal subgroup of finite index.

With these facts we can now prove the following:

Proposition 5.1.5. *A hyperelliptic surface cannot have a wild automorphism.*

Proof. Let σ be an automorphism of a hyperelliptic surface $X = (E \times F)/G$. Some power σ^N lifts to an automorphism of $E \times F$ of the form (T_b, Id_F). Thus any subvariety of the form $E \times \{\mathrm{pt.}\}/G$ is fixed by σ^N. We can now apply Lemma 1.2.6 to see that σ is not wild. □

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5.2 Automorphisms of Ruled Surfaces

Theorem 5.2.1. *A ruled surface cannot have a wild automorphism.*

Proof. Assume to the contrary that σ is a wild automorphism of a ruled surface $\pi : X \to C$. Since $0 = \chi(\mathcal{O}_X) = 1 - g(C)$, C must be an elliptic curve. Let C_0 be a section of π and f a fiber.

We first show that

$$\sigma(C_0) \equiv C_0. \tag{5.1}$$

We know from [Har77, V.2.3] that $\sigma(C_0) \equiv aC_0 + bf$. As we saw in Example 2.4.5, σ permutes the fibers of π. Thus $1 = C_0 \cdot f = \sigma(C_0) \cdot \sigma(f) = \sigma(C_0) \cdot f$, and hence $a = 1$. Since σ is an automorphism, $C_0^2 = \sigma(C_0)^2 = C_0^2 + 2b$, so $b = 0$. Thus $\sigma(C_0) \equiv C_0$.

As in [Har77] $\sigma(C_0) \cdot C_0 = C_0^2 = -e$. We now consider three cases.

1. $e > 0$. In view of (5.1), $\sigma(C_0) \cdot C_0 = C_0^2 < 0$; consequently $\sigma(C_0) = C_0$, contradicting the fact that σ is wild.

2. $e = 0$ The collection of sections of the form $\sigma^i(C_0)$ are pairwise disjoint since $\sigma^i(C_0) \cdot \sigma^j(C_0) = C_0^2 = 0$. We now appeal to the fact that since there are, in particular, three disjoint sections of the ruled surface X, $X \cong E \times \mathbb{P}^1$. In this case the anti canonical divisor $-K_X = 2(\{pt.\} \times C)$ is effective, contradicting Proposition 4.2.1.

3. $e < 0$ By [Har77, V.2.15] there is only one such surface in this category with $e = -1$. We will denote this surface $\mathbf{Q}_1$.

Proposition 5.2.2. *There are no wild automorphisms of $\boldsymbol{Q}_1$.*

Proof. In [Mar71, Lemma 6, Lemma 8] we see that there is a short exact sequence of groups $0 \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(\mathbf{Q}_1) \xrightarrow{f} \mathrm{Aut}(E) \to 0$. For a given automorphism of $\mathbf{Q}_1$, the map f outputs the induced automorphism of E. Thus $\sigma = \bar{\phi}$, the lift of some $\phi \in \mathrm{Aut}(E)$. We will examine the construction of $\bar{\phi}$ in [Mar71, Lemma 8] to show it fixes a subvariety of $\mathbf{Q}_1$.

Denote the natural projections of $E \times \mathbb{P}^1$ by π_E and $\pi_{\mathbb{P}^1}$. Let $P_1, P_2, P_3 \in E \times \mathbb{P}^1$ be three points such that $\pi_E(P_i)$ and $\pi_{\mathbb{P}^1}(P_i)$ are all distinct for $i = 1, 2, 3$, and let $g : \mathbf{Q}_1 \to \mathrm{elm}_{P_1P_2P_3}(E\times\mathbb{P}^1)$ be the isomorphism that identifies $\mathbf{Q}_1$ with $\mathrm{elm}_{P_1P_2P_3}(E\times\mathbb{P}^1)$. Given any $\phi \in \mathrm{Aut}(E)$, there is an induced automorphism $\tilde{\phi} : E \times \mathbb{P}^1 \to E \times \mathbb{P}^1$ that acts by ϕ on the first coordinate. Define $S_e = \mathrm{elm}_{P_1P_2P_3}(E \times \mathbb{P}^1)$ and $S_\phi = \mathrm{elm}_{\tilde{\phi}(P_1)\tilde{\phi}(P_2)\tilde{\phi}(P_3)}(E \times \mathbb{P}^1)$. There exist morphisms h_ϕ and T_ϕ such that $\bar{\phi} = g^{-1} \circ h_\phi \circ T_\phi \circ g$ and the following diagram commutes:

$$
\begin{array}{ccccc}
S_e & \xrightarrow{T_\phi} & S_\phi & \xrightarrow{h_\phi} & S_e \\
\uparrow {\scriptstyle \mathrm{elm}_{P_1P_2P_3}} & & \uparrow {\scriptstyle \mathrm{elm}_{\tilde{\phi}(P_1)\tilde{\phi}(P_2)\tilde{\phi}(P_3)}} & & \uparrow {\scriptstyle \mathrm{elm}_{P_1P_2P_3}} \\
E \times \mathbb{P}^1 & \xrightarrow{\tilde{\phi}} & E \times \mathbb{P}^1 & \xrightarrow{\mathrm{Id}} & E \times \mathbb{P}^1
\end{array}
$$

From the diagram, it is clear that $g^{-1}(E \times \{x\})$ is fixed by $\bar{\phi}$ for any $x \in \mathbb{P}^1$. □

Thus ruled surfaces do not admit wild automorphisms. □

5.3 Wild Automorphisms of Algebraic Surfaces

We are now ready to prove Conjecture 1.0.1 for $\dim X = 2$. We first show we can assume that X is a minimal surface.

Lemma 5.3.1 ([RRZ06])**.** *Let σ be a wild automorphism of an irreducible algebraic surface X. Then X is a minimal surface.*

Proof. By Theorem 4.1.1 we know that $0 = \chi(\mathcal{O}_X) = 1 - h^1(\mathcal{O}_X) + h^2(\mathcal{O}_X)$, so that $h^1(\mathcal{O}_X) = 1 + h^2(\mathcal{O}_X)$. By theorem 4.3.2, the irregularity $h^1(\mathcal{O}_X) \leq 2$. In addition, if $h^1(\mathcal{O}_X) = 2$, then X is an abelian variety. Thus we may assume $h^2(\mathcal{O}_X) = 0$ and $h^1(\mathcal{O}_X) = 1$.

Now consider the albanese morphism $\mathrm{alb}_X : X \to C$. Since $h^1(\mathcal{O}_X) = 1$, C is an elliptic curve. We now show that X is minimal, i.e., X does not contain (-1)-curves. If X contained a (-1)-curve D, then D must be contained in a fiber of alb_X. By Theorem 4.3.2 alb_X is a smooth map with irreducible fibers, so D must in fact be equal to a fiber. But then $D^2 = 0$ since the fibers of alb_X are numerically equivalent. This contradicts our assumption that $D^2 = -1$. □

Theorem 5.3.2 ([RRZ06])**.** *Let X be an irreducible projective variety with* $\dim X = 2$*. If X admits a wild automorphism then X is an abelian variety.*

Proof. Recall that if σ is a wild automorphism of an irreducible algebraic variety X, then $\kappa(X) < 0$ or $\kappa(X) = 0$. Using the Castelnuovo-Enriques classification of minimal algebraic surfaces, we will treat each of these cases separately.

If $\kappa(X) < 0$ then X is rational or ruled [Har77, Theorem V.6.1]. By Corollary 4.1.2 X cannot be rational, by Theorem 5.2.1 X cannot be ruled.

Now suppose $\kappa(X) = 0$. Here there are four possibilities: (1) a $K3$ surface, (2) an Enriques Surface, (3) an abelian surface, or (4) a hyperelliptic surface. As in Lemma 5.3.1 if X is not an abelian surface then $h^2(\mathcal{O}_X) = 0$ and $h^1(\mathcal{O}_X) = 1$. Of these four possibilities only hyperelliptic surfaces satify these equalities. By Proposition 5.1.5 X cannot be a hyperelliptic surface. Thus X is an abelian surface.

□

Chapter 6

PROGRESS IN HIGHER DIMENSIONS

Recall that the existence of a wild automorphism on an irreducible projective variety X implies that $\kappa(X) \leq 0$. In this chapter we will restrict our attention to the $\kappa(X) = 0$ case. A particularly important example are Calabi-Yau Varieties. First we prove a theorem that essentially reduces the study of Conjecture 1.0.1 to Calabi-Yau varieties in this case.

6.1 Wild Automorphisms of Irreducible Projective Varieties with $\kappa(X) = 0$

Before we give the main theorem we first state two technical lemmas that describe the implications of having a wild automorphism on an abelian variety A that centralizes a finite subgroup $G \leq \mathrm{Aut}(A)$.

Lemma 6.1.1. *Let $\Sigma = T_a \circ \alpha$ be an automorphism of an abelian variety A. Let $G \leq \mathrm{Aut}(A)$ be a finite subgroup such that for all $g = T_b \circ \beta$ in G, $\Sigma \circ g = g \circ \Sigma$. Then $\alpha \circ \beta = \beta \circ \alpha$ and $(\alpha - \mathrm{Id})(b) = (\beta - \mathrm{Id})(a)$.*

Proof. The assumption $\Sigma \circ g = g \circ \Sigma$ implies that for all $x \in A$

$$(\alpha \circ \beta)(x) + \alpha(b) + a = (\beta \circ \alpha)(x) + \beta(a) + b. \tag{6.1}$$

Setting $x = 0$ shows that $\alpha(b) + a = \beta(a) + b$, which is equivalent to $(\alpha - \mathrm{Id})(b) = (\beta - \mathrm{Id})(a)$. Referring back to Equation 6.1, it is clear that since $\alpha(b) + a = \beta(a) + b$, we must also have that $\alpha \circ \beta = \beta \circ \alpha$. □

Lemma 6.1.2. *Let Σ be a wild automorphism of an abelian variety A. Let $G \leq \mathrm{Aut}(A)$ be a finite subgroup such that for all g in G, $\Sigma \circ g = g \circ \Sigma$. Then every element of G is a translation.*

Proof. We will procede by induction on $n = \dim(A)$.

For $n = 1$, A is an elliptic curve, and some power of Σ is a translation [BPdV84, V.5]. By Lemma 1.2.6 we may replace Σ with an appropriate power, so we may assume without loss of generality that $\Sigma = T_a$ is a translation such that a generates A.

Let $g = T_b \circ \beta \in G$ be an arbitrary element of G. By Lemma 6.1.1, $\beta(a) = a$, which implies that $\beta = \mathrm{Id}$ since a generates A.

Now consider $\dim(A) = n$. Suppose $\Sigma = T_a \circ \alpha$. By Theorem 3.3.3, $(\alpha - \mathrm{Id})$ is nilpotent, so $\dim(\ker(\alpha - \mathrm{Id})) > 0$. Let $A' \subset \ker(\alpha - \mathrm{Id})$ be the connected component of the identity. Let $\pi : A \to A/A' = T$ denote the quotient map and note that the fibers are of the form $A' + t$ for $t \in A$.

We first show that Σ descends to a wild automorphism of T. Since $\Sigma(A' + t) = \alpha(A') + \alpha(t) + a = A' + \alpha(t) + a$, Σ takes fibers to fibers and hence it descends to an automorphism of $\overline{\Sigma}$ of T. The automorphism $\overline{\Sigma}$ must be wild or else the preimage, under π, of an invariant subvariety of $\overline{\Sigma}$ would be an invariant subvariety of Σ.

We next show that every $g = T_b \circ \beta$ in G descends to an automorphism of T. Note that $\beta(A') = \beta(\alpha(A')) = \alpha(\beta(A'))$, so that $\beta(A')$ is contained in $\ker(\alpha - \mathrm{Id})$. Since $\beta(0) = 0$, we must have that $\beta(A') = A'$. The same argument as above shows that g descends to an automorphism $\overline{g} = T_{\overline{b}} \circ \overline{\beta}$ of T. Let $\overline{G} \leq \mathrm{Aut}(T)$ denote the subgroup of all such $\overline{g}$.

Since $\Sigma \circ g = g \circ \Sigma$ for all g in G, we must have that $\overline{\Sigma} \circ \overline{g} = \overline{g} \circ \overline{\Sigma}$ for all $\overline{g}$ in $\overline{G}$. By the induction hypothesis, $\overline{G}$ contains only translations.

Let $g = T_b \circ \beta$ be arbitrary element of G. By above, $\overline{g}$ is a translation so $\overline{g} = T_{\overline{b}}$. Since g is of finite order $n\overline{b} = 0 \in T$ for some $n > 0$. Equivalently, $nb \in A' \subset \ker(\alpha - \mathrm{Id})$. Since $(\alpha - \mathrm{Id})(nb) = 0$, by Lemma 6.1.1, $(\beta - \mathrm{Id})(na) = 0$ as well. Thus $\beta(na) = na$. Furthermore since $\alpha \circ \beta = \beta \circ \alpha$, $\beta((\alpha - \mathrm{Id})^i(na)) = (\alpha - \mathrm{Id})^i(na)$ for all $i \geq 0$. Thus β fixes every element of the algebraic subgroup generated by $\{na, (\alpha - \mathrm{Id})(na), (\alpha - \mathrm{Id})^2(na), \ldots\}$. The map $n_A(x) = nx$ is an isogeny on A, so by Remark 3.3.5, this algebraic subgroup is all of A. Hence β fixes every element of A, and so $\beta = \mathrm{Id}$. □

To prove the main theorem we will use the decomposition theorem of Beauville described in Section 2.5. Since this decomposition theorem is stated for compact kähler manifolds we will first prove that we can use it and remain in the algebraic category.

Proposition 6.1.3. *Let X be an irreducible projective variety with $\kappa(X) = 0$, and σ a wild automorphism of X. Then $X \cong (A \times S)/G$ where A is an abelian variety, S is a Calabi-Yau variety, and $G \leq \mathrm{Aut}(A \times S)$ is a finite subgroup acting freely on $A \times S$.*

Proof. By Corollary 4.2.2, we know that $\omega_X^n \cong \mathcal{O}_X$ for some $n \geq 0$. Also, by Lemma 1.2.6, we know that X is smooth. Thus X is a compact kähler manifold with $c_1^{\mathbb{R}} = 0$. Using the results in Section 2.5 we can write $X = (T \times \prod V_i \times \prod X_j)/G$. The Generalized Riemann Existence Theorem [Har77, Theorem B.3.2] implies that T, V_i, and X_j are each irreducible projective algebraic varieties, and that the quotient map $T \times \prod V_i \times \prod X_j \to X$ is also algebraic.

Futhermore, we know that σ lifts to an automorphism Σ of $T \times \prod V_i \times \prod X_j$. Since σ is wild, Σ must also be wild. As in [Bea83] we know that some power of Σ splits as a product of automorphisms of T, each V_i, and each X_j. By Theorem 4.1.1, since $\chi(\mathcal{O}_{X_j}) > 0$ no X_j can appear in this decomposition. Thus $X \cong (A \times S)/G$ where A is an abelian variety, S is a Calabi-Yau variety, and $G \leq \mathrm{Aut}(A \times S)$ is a finite subgroup acting freely on $A \times S$. $\square$

Theorem 6.1.4. *Let X be an irreducible projective variety with $\kappa(X) = 0$, and σ a wild automorphism of X. Then $X \cong A \times (S/G)$ where A is an abelian variety, S is a Calabi-Yau variety, and $G \leq \mathrm{Aut}(S)$ is a finite subgroup acting freely on S.*

Proof. As in Proposition 6.1.3, we know $X \cong (A \times S)/G$ for a minimal cover $A \times S$, where A is an abelian variety and S is a Calabi-Yau variety. The automorphism σ lifts to an automorphism Σ of $A \times S$ that normalizes the subgroup G. By Lemma 1.2.6, by replacing σ with an appropriate power we may assume that Σ commutes with every g in G.

Let $g \in G$ be an arbitrary element of G. By Lemma 2.5.2 we can write $g = g_A \times g_S$ and $\Sigma = \Sigma_A \times \Sigma_S$. The set $\{g_A | g \in G\}$ is clearly a subgroup of $\mathrm{Aut}(A)$, and the automorphism Σ_A is wild since Σ is wild. The automorphism Σ_A commutes with every g_A which implies that every g_A is a translation by Lemma 6.1.2. However, $A \times S$ is a minimal cover, so $g_A = \mathrm{Id}$. Thus every element of G acts trivially on A, and so $X \cong A \times (S/G)$. $\square$

Thus in the case of $\kappa(X) = 0$, we have reduced Conjecture 1.0.1 to determining if Calabi-Yau varieties admit wild automorphisms. In fact the Calabi-Yau varieties V_i have

Euler characteristic $\chi(\mathcal{O}_{V_i}) = 1 + (-1)^{\dim(V_i)}$. Since an irreducible projective variety X admitting a wild automorphism must have $\chi(\mathcal{O}_X) = 0$, we may restrict our attention to Calabi-Yau varieties of odd dimension.

6.2 Calabi-Yau Varieties

As we have just seen the analysis of Conjecture 1.0.1 reduces to the study of wild automorphisms of Calabi-Yau varieties of the type V_i given in Beauville's decomposition theorem. In this section we will do just this. While a complete answer to this question is still unknown, we will give several partial results that lend support to Conjecture 1.0.1, namely that there do not exist wild automorphisms on Calabi-Yau varieties.

Theorem 6.2.1. *Let X be a Calabi-Yau variety with Picard number 1. Then X does not admit a wild automorphism.*

Proof. The exponential sequence $0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$ gives rise to a long exact sequence in cohomology, part of which is $H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X)$. We know that both $H^1(X, \mathcal{O}_X)$ and $H^2(X, \mathcal{O}_X)$ are 0. Hence $\operatorname{Pic}(X) \cong H^1(X, \mathcal{O}_X^*) \cong H^2(X, \mathbb{Z})$ is a finitely generated abelian group.

Since the Picard number of X is 1 we can write $\operatorname{Pic}(X) \cong \mathrm{T} \oplus \mathbb{Z}$, where T is the torsion subgroup of $\operatorname{Pic}(X)$. Let σ be an automorphism of X, and σ^* be the induced automorphism of $\operatorname{Pic}(X)$. Since $\sigma^*(T) = T$, σ^* descends to the an automorphism of the quotient $\operatorname{Pic}(X)/T \cong \mathbb{Z}$. Denote this automorphism by $\bar{\sigma}^*$. There are only two automorphisms of $\mathbb{Z}$, $\mathrm{Id}_{\mathbb{Z}}$ and $-\mathrm{Id}_{\mathbb{Z}}$, so in either case $(\bar{\sigma}^2)^* = \mathrm{Id}$.

Let $\mathcal{L}$ be an ample line bundle on X. Consider the line bundle $(\sigma^2)^*\mathcal{L} \otimes \mathcal{L}^{-1}$. By the above discussion, the image of this line bundle in the quotient $\operatorname{Pic}(X)/T$ is trivial, and so $(\sigma^2)^*\mathcal{L} \otimes \mathcal{L}^{-1}$ is torsion. Thus we can choose an integer n such that $(\sigma^2)^*\mathcal{L}^n \otimes \mathcal{L}^{-n} \cong \mathcal{O}_X$, or equivalently $(\sigma^2)^*\mathcal{L}^n \cong \mathcal{L}^n$. Using Corollary 4.2.4, since $\mathcal{L}$ is ample, σ^2, and equivalently σ, is not wild. □

Examining the Kähler cone of a Calabi-Yau variety seems to be promising. For Calabi-Yau threefolds much has been studied and many questions remain unanswered, see [Ogu01]

[Wil92] [Wil97]. Assuming a conjecture given in [Ogu01] we can show that Calabi-Yau threefolds do not admit wild automorphisms.

Definition 6.2.2. A surjective morphism $\phi : Y \to Z$ between irreducible projective varieties is called a contraction if ϕ has connected fibers. A contraction $\phi : Y \to Z$ is called a fiber space (resp. birational contraction) if $0 < \dim Z < \dim Y$ (resp. $\dim Z = \dim Y$ and ϕ is not an isomorphism). Two contractions $\phi_i : Y \to Z_i$, $i = 1, 2$, are regarded as the same if there exists an isomorphism $\tau_{12} : Z_1 \to Z_2$ such that $\phi_2 = \tau_{12} \circ \phi_1$.

The second chern class $c_2(X) \in H^4(X, \mathbb{Z})$ plays an important role. From this, we get a linear form $c_2 : H^2(X, \mathbb{Z}) \to \mathbb{Z}$ given by the cup product with $c_2(X)$. It is know that a minimal threefold with $c_2 = 0$ is the étale quotient of an abelian threefold. Since Calabi-Yau varieties of type V_i are simply connected this cannot happen. Thus, in what follows, we will assume that $c_2 \neq 0$.

The most important fact about c_2 is the following:

Theorem 6.2.3 ([Miy85])**.** *For every minimal threefold Y, $c_2 \geq 0$ on the nef cone $\bar{A}$ and $c_2 > 0$ on the ample cone A provided $c_2 \neq 0$.*

For a contraction $\phi : Y \to Z$ given by a linear system $\phi = \Phi_{|D|}$, it does not depend on the choice of representative D whether $c_2.D > 0$ or $c_2.D = 0$ [Ogu01]. Therefore we call a contraction $\Phi_{|D|} : Y \to Z$ with $c_2.D > 0$ (resp. $c_2.D = 0$) a $c_2 > 0$ (resp. $c_2 = 0$) contraction. The main theorem concerning these contractions is the following:

Theorem 6.2.4 ([Ogu01])**.** *Every Calabi-Yau threefold admits only finitely many $c_2 = 0$ contractions up to isomorphism.*

This result is especially useful for our purposes because we can reduce to the case where c_2 is not strictly positive on the nef cone. If c_2 were strictly positive we could construct an intrinsically defined ample class as follows. We merely look for non-zero elements of $\mathrm{Pic}(X) \cap A$ on which c_2 takes its minimum value. Our assumption on c_2 implies there are only finitely many such elements, and by adding these classes together, we form an intrinsically defined ample class. In light of Corollary 4.2.4, we may disregard this case.

Thus we may assume the existence of a nef divisor D such that $c_2.D = 0$. In [Ogu01] the following conjecture is given:

Conjecture 6.2.5. [Semi-ampleness conjecture] Every nef divisor (not necessarily effective) on a Calabi-Yau variety is semi-ample.

Note that if Conjecture 6.2.5 is true and the next Cone Conjecture is also true, then any Calabi-Yau variety admits only finitely many contractions, and not just $c_2 = 0$ contractions.

Conjecture 6.2.6 (Cone conjecture)**.** We set $\bar{A}^+(X) = \bar{A}(X) \cap \mathrm{Pic}(X) \otimes \mathbb{Q}$, the set of rational points of $\bar{A}(X)$. then there exists a finite rational polyhedral cone $\Delta \subset \bar{A}^+(X)$ such that $\bar{A}^+(X) = \mathrm{Aut}(X)\Delta$.

We can now state the following theorem:

Theorem 6.2.7. *Assume Conjecture 6.2.5 is true. Let X be a simply connected Calabi-Yau threefold. Then X does not admit a wild automorphism.*

Proof. Let σ be an automorphism of X. By the above discussion we may assume that there exists a nef divisor D that is not ample such that $c_2.D = 0$. Using Conjecture 6.2.5, some power nD defines a $c_2 = 0$ contraction $\phi : X \to Y$. Since $\sigma(nD)$ also defines a $c_2 = 0$ contraction, by Theorem 6.2.4 some power of σ descends to an automorphism $\bar{\sigma}$ of Y. If ϕ is a birational contraction, then the exceptional set is invariant under σ. Since $H^1(X, \mathcal{O}_X) = 0$, Y cannot be an abelian variety. Thus, in the case where ϕ is a fibration (and $\dim Y \leq 2$), it is true that $\bar{\sigma}$ is not wild. If $Z \subset Y$ is an invariant subvariety of $\bar{\sigma}$, then clearly $\phi^{-1}(Z)$ is an invariant subvariety of σ. □

Note that this, conjecturally, confirms Conjecture 1.0.1 in the case where $\dim X \leq 4$ and $\kappa(X) = 0$.

Corollary 6.2.8. *Assume Conjecture 6.2.5 is true. Let X be an irreducible projective variety, and σ a wild automorphism of X. Suppose $\kappa(X) = 0$ and $\dim X \leq 4$. Then X is an abelian variety*

Proof. We know X has a minimal split cover of the form $\tilde{X} = A \times \prod V_i$ where V_i is a Calabi-Yau variety with $\dim V_i \geq 3$ and $\dim V_i$ is odd. We only need to consider the cases where $\dim X = 3$ and $\dim X = 4$.

If $\dim X = 3$ and $\tilde{X}$ is not abelian then $\tilde{X} = S$, where S is a Calabi-Yau threefold. If $\dim X = 4$ and $\tilde{X}$ is not abelian then $\tilde{X} = E \times S$, where is a S Calabi-Yau threefold and E is an elliptic curve. The wild automorphism σ must lift to a wild automorphism of $\tilde{X}$. But, by Theorem 6.2.7, there are no wild automorphisms of Calabi-Yau threefolds.

Thus $\tilde{X}$ must be an abelian variety, and therefore X must be as well. □

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