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Improving Student Fluency in Applying Basic Skills when Solving Problems in Quantum Mechanics

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Abstract

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Research into student understanding of quantum mechanics has identified various conceptual and mathematical difficulties pervasive at all levels of education. While much work has been dedicated to the design and assessment of intervention to address student difficulties with conceptual components of the subject, little work has been dedicated to the design of intervention addressing mathematical student difficulties. To this end, this dissertation discusses the construction and implementation of practice question sequences designed to help students become more accurate and fluent in their use of basic mathematical and representational skills in quantum mechanics. After assessing the effect that the practice had on student learning, we found that the practice served as a beneficial instructional tool, where students not only saw improvements in performance on the practice itself but also exam question responses. In addition, we found that the practice served as a research tool, helping us identify new potential avenues of research into student ideas in quantum mechanics. These initial assessments are encouraging results on the effectiveness of the practice in helping improve student learning.

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Truly, thank you all for keeping me focused, impassioned, and human. Now, at the end of this chapter of my life, the endless farmland of our lives and this world stretch out before me, and everywhere I look I see everyone's efforts to cultivate this land. I promise to wield the kindness you all have given me and carve this kindness back into the world alongside all of you.

DEDICATION

To my parents and my brother. I am where I am now only from your constant support and unwavering belief in me.

Chapter 1

INTRODUCTION

The study of quantum mechanics represents an important stage for every student of physics. On one hand, the topic is the backbone of nearly every area of modern physics. From particle theory, electronics, and even the infant field of quantum information, quantum mechanics serves as the rule-book for constructing these physical theories. This is true even beyond physics and engineering: chemistry, computer science, mathematics, and even philosophy ride on the successes of quantum mechanics. On the other hand, quantum mechanics is a notoriously difficult subject for students and experts alike. It often operates orthogonal to, if not completely against, common intuitions we have about the world. Further, the language of quantum mechanics consists of a novel mathematical framework that students often have little practice working in before learning the subject. As such, students of quantum mechanics are required to build faculty with this mathematical framework while confronting the contradictions between the subject and their classical intuitions.

In quantum mechanics courses, much of student understanding is developed through solving problems. After all, quantum mechanics is famous for lacking the same experiential intuition that introductory classical mechanics has. As a result, understanding largely comes not from phenomenological priors but rather through homework assignments, the textbook, and lectures. Solving problems entails the use of the mathematical framework and the new quantities being used in quantum mechanics, such as probability, expectation values, and eigenvalue equations, etc.

At the University of Washington, our junior level courses on quantum mechanics use supplemental tutorials. These are based on growing research on student learning of quantum mechanics and help bolster student understanding of conceptual ideas in quantum mechanics [19, 131, 91]. Further, these tutorials have been shown to be an effective way of helping students learn these components of the subject. But tutorials do not address all of the difficulties with which students struggle. After analyzing student responses to exam questions, we found that while students acquire understanding of various conceptual components, they seem lack some facility with the mathematical aspects of the theory as well as the various representations that make up the subject. Students seem to struggle with quantum mechanics *as a mathematical framework*, as well as how to link this framework to the conceptual ideas that they refine in the tutorials. If we want to teach understanding in quantum mechanics effectively, we must also give students practice that familiarizes them with novel components.

In this dissertation, we will be discussing work done in the construction, implementation, and assessment of structured practice homework assignments designed to help improve how students utilize and apply common mathematical and conceptual skills when solving problems in upper-division quantum mechanics courses. Part I begins by discussing our motivation for the project, the framework we used to construct the practice, and the prior research on quantum mechanics education research and “deliberate practice” that influenced our work. Then, we will discuss our implementation of the aforementioned framework. In doing so, we combed through the research literature and synthesized the results to identify various skills students could practice and then wrote practice questions to help students improve their abilities with these skills. Part I concludes with a discussion of the resulting online homework system we used to administer the practice as well as how we organized the questions we developed for it within that system.

In Part II, we will outline various assessments of this practice. First, we discuss our implementation of the practice, the kinds of data used in this dissertation, and the various ways we analyzed this data in general. Then, we discuss brief example of how this data can be used to assess the effectiveness of the practice. We then outline a more in-depth assessment of the practice in the specific domain of translating between mathematical and verbal descriptions of common quantities in quantum mechanics. We then finish by describing other notable results from assessing the practice, such as student ability in using mathematical skills and applying more advanced quantum mechanical concepts.

Part I

OVERVIEW AND IMPLEMENTATION OF A FRAMEWORK TO CONSTRUCT AND ADMINISTER STRUCTURED PRACTICE

We begin Part I in Chapter 2 by discussing results from exam questions that motivated us to create practice question sequences helping students develop fluency in various mathematical and representational skills in quantum mechanics. We then outline the framework that we used to build this practice as well as the prior research on quantum mechanics education research and deliberate practice that motivated our work in constructing the practice. Chapter 3 then outlines various examples on how we used the research literature and student responses to exam questions to identify skills and design practice questions for these skills. Finally, Chapter 4 discusses the online homework system, STEM Fluency, that we used to administer the practice, as well as how we organized our practice questions into this system to administer them to students.

Chapter 2

MOTIVATION, THE ESSENTIAL SKILLS FRAMEWORK, AND PRIOR RESEARCH

Despite successful intervention targeting conceptual student difficulties in quantum mechanics, various issues still persist. After noticing these issues, we recognized the need for further, more mathematical intervention for our students. After looking closely at student responses to exam questions, which we discuss in detail below, we determined that deliberate practice would serve as a good guide for our design of the intervention.

In this chapter, we will discuss the motivations and framework for the structured homework practice that we incorporated into our junior level quantum mechanics courses. We also present some of the prior research that informed our choice in using this framework and guided the kind of practice we built. We begin by briefly discussing results from exam administrations prior to the implementation of the practice. It was our interpretations of the student responses that suggested that students may need to practice basic translational and mathematical skills related to quantum mechanics. Then, we will outline the Essential Skills Framework, which is the research-based framework we selected to help us build the practice given to students. Finally, we will discuss the prior research on quantum mechanics education research and directed practice to frame the considerations made when building practice.

2.1 Motivation for Practice: A Brief Look at Exam Question Results

In this section, we will briefly outline some results from an exam question administered to students. We discuss an exam question in detail and briefly give general trends that we noted in how students were answering it. We then use these observations to comment on the approach we took to the instructional intervention taken in this dissertation. The questions we asked ourselves were: what kind of intervention might be useful for students? what should this intervention generally target? and how could it be incorporated easily into the course?

2.1.1 Exam Question that Motivated our Intervention

In Fall 2018, the exam problem in Figure 2.1 was administered to $N = 92$ students on the second exam in the first course of a two quarter sequence on upper-division quantum mechanics. This course was largely a traditional lecture-based course, where students met 3 times a week for lecture. This course was a positions-first course using the textbook by Griffiths [27]. Students in the course met weekly to work on small-group tutorials that emphasized conceptual components of various aspects of quantum mechanics. The problem in Figure 2.1 was written to assess the effect of the tutorials on these students.

This problem gives students the state of a particle in a nontrivial superposition of eigenstates of the infinite square well (namely, the state is in a superposition, but it is not visually obvious what the superposition is). Students must produce expressions for two different probabilities. First, they must write an expression for the probability of measuring the energy to be the ground state energy. Then, students must write an expression for the probability of finding the particle to be between $x = 0$ and $x = a/3$. While there are many ways students may approach solving this problem, we will outline relevant correct solution strategies for each question below.

2.1.2 Outline of Correct Answers to Questions

Although there are many different ways to solve these questions, we will focus on common solutions and how the errors students made in trying to solve the questions informed our reasoning when deciding to use direct practice as an intervention.

Problem 1: The Probability of Measuring the Ground State Energy

Consider a particle in the quantum mechanical infinite square well of width a. At $t = 0$, the normalized wave function for the particle is given by

$$\Psi(x, t=0) = \begin{cases} \frac{\sqrt{15}}{a^{5/2}}\left(x^2 - ax\right) & 0 < x < a \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

You may receive full credit on this problem without evaluating any integrals; however, writing down an integral without explaining where it comes from will not result in credit.

A. Suppose you were to measure the energy of this particle at $t = 0$. Determine the probability that the energy is equal to E_1, the energy of the ground state. Explain your reasoning.

B. Suppose you were to measure the position of this particle at $t = 0$ (no other measurements have been made). Determine the probability that the particle is measured to be between $x = 0$ and $x = a/3$. Explain your reasoning.

Figure 2.1: An exam problem administered to $N = 92$ students in Fall 2018 in a first upper-division quantum mechanics course. The problem has two questions, tasking students to write down an expression for the probabilities of two particular measurements: 1) the probability of measuring the energy to be the ground state energy, and 2) the probability that the particle is located between $x = 0$ and $x = a/3$.

One way of solving this question is simply to write down the correct answer from memory, and many students took this approach. It is reasonable to do this since students often work with energy probabilities in upper-division quantum mechanics courses. However, many students tried to derive this result in various ways but did so incorrectly. One particular way - a way that students also commonly made mistakes in - involved students recognizing that the probability of measuring the ground state energy is the norm-square of the coefficient in front of the ground state eigenstate in the energy eigenbasis expansion of our state. A correct description of this approach goes as follows. We can expand our state $\Psi(x,0)$ as follows:

$$P(E_1) = |c_1|^2, \text{ where } \Psi(x,0) = c_1\psi_1(x) + c_2\psi_2(x) + ... \tag{2.2}$$

With this in mind, one can then solve for an expression for this coefficient. This can be done by taking the inner product of our state with $\psi_1(x)$:

$$\int_{-\infty}^{\infty} \psi_1^*(x)\Psi(x,0)\mathrm{d}x = c_1 \int_{-\infty}^{\infty} \psi_1^*(x)\psi_1(x)\mathrm{d}x + c_2 \int_{-\infty}^{\infty} \psi_1^*(x)\psi_2(x)\mathrm{d}x + ... \tag{2.3}$$

Finally, using orthonormality, we can simply this to obtain the following:

$$c_1 \int_{-\infty}^{\infty} \psi_1^*(x)\psi_1(x)\mathrm{d}x + c_2 \int_{-\infty}^{\infty} \psi_1^*(x)\psi_2(x)\mathrm{d}x + ... = c_1 \times 1 + c_2 \times 0 + ... = c_1 \tag{2.4}$$

Thus, we obtain an expression for the coefficient c_1 and, with it, an expression for the probability of measuring the ground state energy:

$$c_1 = \int_{-\infty}^{\infty} \psi_1^*(x)\Psi(x,0)\mathrm{d}x \longrightarrow P(E_1) = |c_1|^2 = \left| \int_{-\infty}^{\infty} \psi_1^*(x)\Psi(x,0)\mathrm{d}x \right|^2 \tag{2.5}$$

In this solution, students must recognize the desired probability is the norm-square of a *particular* coefficient. Further, they must identify the utility of executing various mathematical procedures that can be used to produce an expression for this probability and be able to execute them properly. Importantly, we found that students often struggled with the steps in this solution, as we will discuss below. Many of these procedures and identifications are new to students in this course, and thus there are a variety of ways for students to fail in constructing this solution.

Problem 2: The Probability of Measuring the Position to be between $x = 0$ and $x = a/3$:

Most students approached this problem with memorized responses. This is reasonable considering the textbook used by our students, the textbook by Griffiths, takes a position-first approach to quantum mechanics [27]. In this text, position probabilities are the first quantum mechanical quantities produced for students. Students spend ample time working with various scenarios in wave function notation answering questions related to position probabilities. As such, a correct solution will take the form of simply

$$P(x \in [0, a/3]) = \int_0^{a/3} |\Psi(x,0)|^2 \mathrm{d}x \tag{2.6}$$

Because students often memorized and attempted to reproduce this response, we cannot provide a detailed solution that students commonly attempted to demonstrate the various ways students struggled with this problem. Instead, we comment on a few characteristics of this expression that are important for students to answer with. First, students must integrate over the interval of interest. Second, students must know the integrand is the norm-square of the wave function for our particle (namely, the probability density). These aspects are important because we found students often missed on one or both of these aspects, as we will outline below.

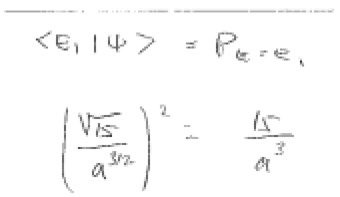
2.1.3 Trends in Student Responses

With these correct solutions and steps in mind, we identify some general trends that we noticed in these problems. We discuss first those trends specific to question 1 on the energy probability, then second those trends specific to question 2 on the position probability. Then we discuss the trends found in both questions.

Trends in Student Responses to Question 1

One of the first trends we noticed is that students seemed to recognize that the probability of measuring the ground state energy was the norm-square of *some* coefficient. However, many students did not seem to know which coefficient they needed.

As an example, many students took the expression for the wave function, $\frac{\sqrt{15}}{a^{5/2}}(x^2 - ax)$, and treated the coefficient in front of x, $-\frac{\sqrt{15}}{a^{5/2}}a = -\frac{\sqrt{15}}{a^{3/2}}$, as the coefficient to be norm-squared to find the desired probability. In fact, a few students claimed that x *was* the ground state eigenstate. Examples of such responses are in Figure 2.2. So in some sense students were correct in identifying the need for the coefficient in front of the ground state, but they misidentified this coefficient.

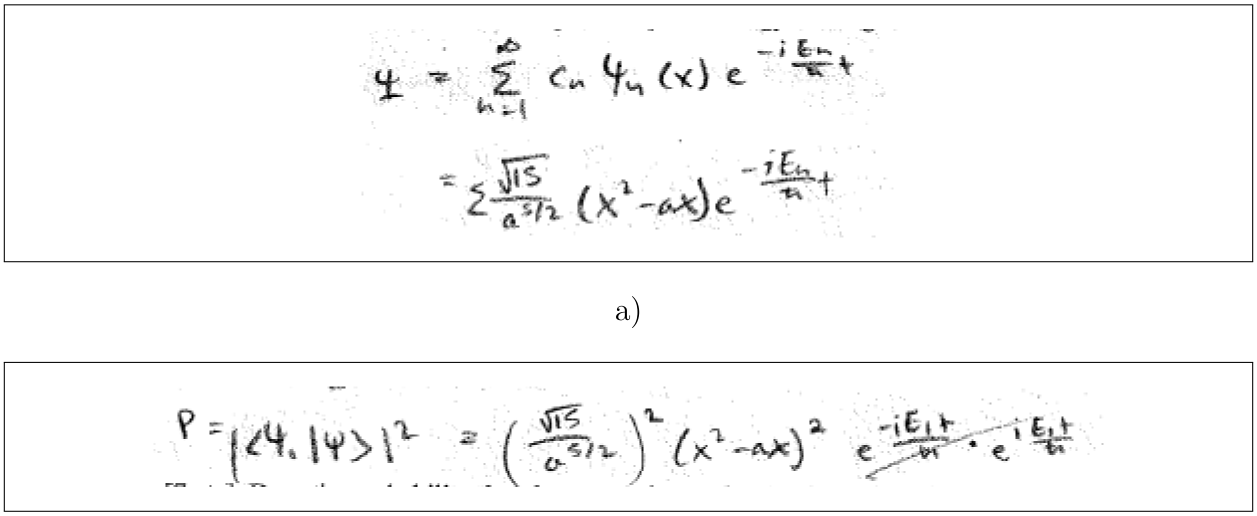


$$\langle E_1 | \Psi \rangle = P_{E=E_1}$$

$$\left(\frac{\sqrt{15}}{a^{3/2}}\right)^2 = \frac{15}{a^3}$$

Figure 2.2: In this response to the question posed in Figure 2.1, this student chose the coefficient in front of the x term in the wave function to be c_1, the coefficient in front of the ground state energy.

In addition, many students struggled with the mathematical steps required to isolate an expression for the desired coefficient. As suggested above, this included things like not expanding the state in the energy eigenbasis correctly or taking inner products correctly. Examples of these behaviors are presented in Figure 2.3.



$$\Psi = \sum_{n=1}^{\infty} c_n \psi_n(x)\, e^{-\frac{iE_n}{\hbar}t}$$

$$= \sum \frac{\sqrt{15}}{a^{5/2}}(x^2-ax)\, e^{-\frac{iE_n}{\hbar}t}$$

a)

$$P = |\langle \psi_1 | \Psi \rangle|^2 = \left(\frac{\sqrt{15}}{a^{5/2}}\right)^2 (x^2-ax)^2\ \cancel{e^{\frac{-iE_1t}{\hbar}} \cdot e^{\frac{iE_1t}{\hbar}}}$$

b)

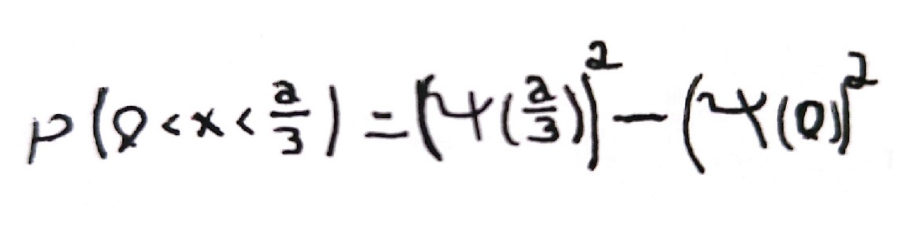
Figure 2.3: These student responses depict two examples of mathematical errors students made. a) shows a student attempting to expand the given state in the energy eigenbasis, but does so incorrectly (plugging in the state for each energy eigenstate). b) demonstrates a student taking an inner product incorrect, not utilizing the wave function form of the energy eigenstate nor an integral.

In summary, students seemed to struggle with finding a particular probability in a variety of ways. This commonly included misapplying a rule “probability-as-a-coefficient as well” as the mathematical procedures need to find an expression for the correct coefficient.

Trends in Student Responses to Question 2

As mentioned above, most students seemed to have the appropriate expression memorized. This may be because of how early position probabilities were introduced in the course. As such, most incorrect responses involved misremembering key aspects of the definition of a position probability.

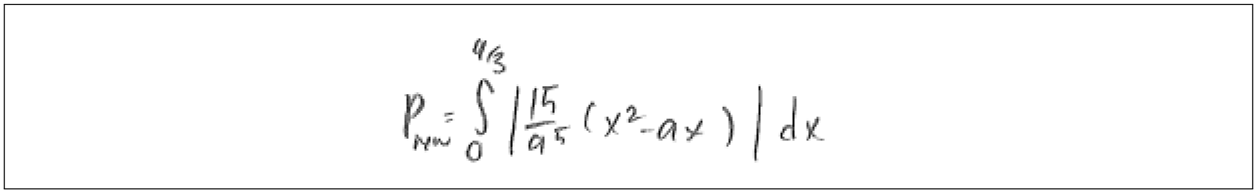
For example, many students failed to recognize that the probability density should be integrated in order to obtain the desired probability. Some students did not integrate and instead simply evaluated the probability density itself at the boundary points of the interval, as shown in Figure 2.4.



$$p(0<x<\tfrac{a}{3}) = (\Psi(\tfrac{a}{3}))^2 - (\Psi(0))^2$$

Figure 2.4: This student response involves finding a probability by evaluating the probability density at the bounds of the interval of interest, and taking the difference. Key here is that the students seems to have forgotten or did not think to perform an integral, or that the quantity has already been integrated.

As another example, some students merely integrated the wave function (a probability amplitude) and not the norm-square of the wave function (a probability density). Such a response is depicted in Figure 2.5.



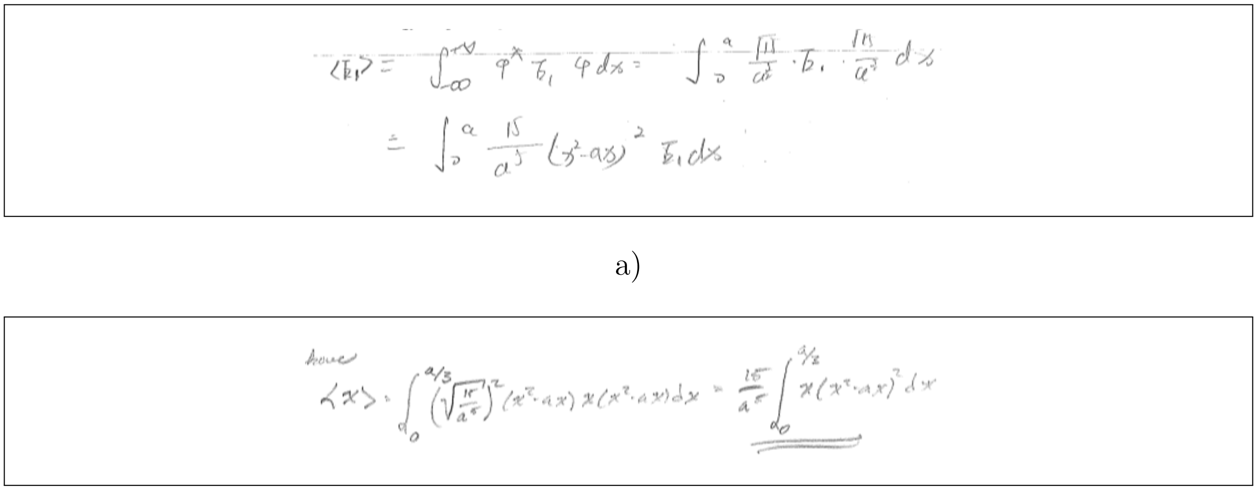
$$P_{new} = \int_0^{a/3} \left| \frac{15}{a^5}(x^2 - ax) \right| dx$$

Figure 2.5: This student response depicts a student integrating the (norm-square of) the wave function and not the probability density. Since the wave function is often called a probability amplitude, it is common for students to confuse these two quantities.

As discussed earlier, these errors seem to suggest that students sometimes employ memorized responses and often do so incorrectly.

Trend in Student Responses to Both Questions

Finally, we mention a common set of responses that appeared in both the energy and position probability questions. It was fairly common for students to respond to these questions using mathematical objects that were not probabilities. As an example, many students responded using expectation values, as shown in Figure 2.6. Recall that students were asked to produce expressions corresponding to the probability of measuring certain position and energy outcomes. The student responses depicted in this figure, though, correspond to the mathematical form of expectation values.



a)

b)

Figure 2.6: These responses depict students responding to a question asking to produce an expression for a particular probability with an expectation value. Both students even go as far as denote the common $\langle O \rangle$ symbol for the expectation value of an observable. As such, it seems students are conflating to some degree the concept and mathematical form of probabilities and expectation values.

As such, students sometimes seem to conflate the conceptual idea of probabilities and the mathematical form of the expression corresponding to expectation values. In addition, this was not the only mix-up that we noticed in student responses: many students responded using other common mathematical objects, including eigenvalue equations, probability densities, *etc.* At its core, these responses seem to suggest that students are having a hard time distinguishing between the various, new mathematical objects used in quantum mechanics, both mathematical *and* conceptual. This is somewhat to be expected in a course where students are seeing these mathematical objects for the first time. However, it is clearly desirable to help students learn these distinctions, especially between quantities that are commonly used in upper-division quantum mechanics.

2.1.4 Summary and Reflection: What Kind of Intervention Should we Use?

To summarize the observations above, students seemed to struggle with a variety of components of solutions when solving a typical exam problem administered to them. This included mathematical steps such as expanding states in a particular eigenbasis and orthonormality, as well as knowing the definition of the quantities that students must work with. Note that many of these steps did not have a clear conceptual component to them, or at least it was not evident that the questions require conceptual considerations when answering. This is important, because the intervention used in these courses were the UW tutorials, which focus on conceptual aspects of quantum mechanics. As such, we reasoned students may also benefit from activities emphasizing the use and production of mathematical objects in quantum mechanics as well as practice in translating between the various representations and notations that make up the theory. This led us to think about deliberate practice as a means for promoting fluency in these components. In the next section, we discuss the framework we chose to implement in our design of the practice that we believe best captured the perspectives on our desired outcomes and goals.

2.2 The Essential Skills Framework

As discussed above, our analysis of student responses to exam questions led us to believe that students may benefit from practicing the execution of various mathematical and translational skills often required in quantum mechanics problems. That being said, it would have been ill-informed to simply provide practice without drawing on a research basis. We wanted to design practice that was research-informed, utilizing various perspectives on effective practice as well as utilizing the existing research on how students struggle when learning quantum mechanics. As such, we needed a framework to help us construct this practice and frame what we want the practice to accomplish. After some considerations, we decided that the Essential Skills Framework designed by Brendon Mikula and Andrew Heckler at The Ohio State University best fit our goals and desires for this process.

In this section, we will discuss the Essential Skills Framework (ESF) in detail. We will focus on the aspects of the ESF that we believe make it suitable for our objectives in designing practice. We will first discuss the motivation for and language commonly used in the ESF, and then we will outline its components (namely, the steps it employs for the identification and design of practice). We will conclude by mentioning the research-validated considerations it suggests to take when designing the practice.

2.2.1 The Essential Skills Framework

Solving problems is an important way in which students both learn physics but also demonstrate their understanding to instructors. This is especially true in quantum mechanics, where there is little experiential basis for the phenomena they are learning. The ESF hypothesizes that solving problems requires accuracy and fluency in various relatively simple, yet "essential skills" common to a large array of problems [82]. Further, the ESF proposes that if students lack said accuracy and fluency, then solving problems becomes more difficult, which in turn can impede the learning process. Heckler *et al.* use vector math in the introductory setting as a prototypical example of a class of essential skills, where skills like adding and subtracting vectors, as well as solving for components, form essential steps in a wide array of introductory physics topics [82].

Although most instructors would consider skills like vector math "essential" in physics, many courses do not give ample time to systematically develop and practice these skills. This may be because instructors often see them as prerequisite skills, simple to learn, or potentially as something that students can develop as a natural component of the course. But, as the size of the corpus of physics education research may suggest, even after instruction, students still seem to struggle with important physics and reasoning skills. With this in mind, the ESF serves as a procedure one can use to identify skills deemed to be "essential", to design research-informed practice for it, and to refine this practice in order to help students develop the desired level of accuracy and fluency for these skills.

2.2.2 Accuracy, Fluency, and "Essential" Skills

Heckler *et al.* define an *essential skill* as a common skill for solving problems that is:

1) procedural and relatively simple, requiring at most a few steps,
2) necessary to solve commonly assigned problems, and
3) largely automated in experts [82].

This last characteristic is important, because it contextualizes what is meant by "fluency" in the context of the ESF. Students and experts interact with the steps in their solutions differently, in that many expert behaviors are automatic and performed in parallel fashion (as opposed to serial) [55]. In addition, this fluency is often developed as a result of practicing a given skill in a variety of contexts over several years. When students are solving problems, if they are not accurate nor fluent in these essential skills, this may induce a significant cognitive load that interferes with how students engage with other important aspects of the problem.

As an example, during a problem involving force-balancing, an expert may make the decision to break forces into their components subconsciously and fluidly, allowing for more mental resources to be dedicated to other aspects of the problem (say, a recognition of the use of Newton's second law). A student who is not fluent in breaking vectors into their components may spend considerable time and mental energy on this step. Even if they perform this step correctly, such a mental expenditure can make it harder for students to pay attention to other important steps and concepts. In other words, students may lack the necessary accuracy for skills when solving problems, and even when they have sufficient accuracy, they may not have the necessary fluency to utilize this accuracy in productive ways.

2.2.3 Components of the Essential Skills Framework

With accuracy, fluency, and essential skills so defined, we outline the components of the ESF. In short, the authors argue for the need to identify potential essential skills and to design research-informed practice for these skills. Per Heckler *et al.*, there are four steps, which we elaborate on some detail below [82].

1) First, one must identify potential essential skills. Candidates can come from published education research literature, prior experience in teaching courses, studying student work, and anecdotal comments by students. In all cases, it is key that each possible essential skill should have some sort of empirical basis.

2) Second, when constructing robust practice problem sets for students, one should identify common difficulties with these proposed essential skills. Such difficulties can again be obtained via the research literature, studying student work, prior experience, *etc.*. Again, the key factor is that such difficulties should have an empirical basis. Further, it is also important that such difficulties have a particular grain-size. In particular, some difficulties are complex in nature. Such difficulties may be related to other difficulties, not only in physics content, but representational and contextual content. These difficulties may not be suitable for essential skills practice due to their complex nature.

3) Third, one should employ research-validated methods in the design and implementation

of the practice. Examples of such methods will be elaborated on later when we discuss the STEM Fluency system. We wish merely to emphasize the importance of using methods that have research backing-up their utility.

4) Finally, the implementation of the essential skills practice should be tested and refined over time. One may refine the set of skills that seem to be essential, rewrite questions so that they provide the practice one intends, or rework the practice schedule so that it maximally benefits students. Ultimately, one should use past experience, the research literature, and past implementations of the practice to refine it.

By "difficulty" in this context, we use the definition that Heron gave as "the use of a specific idea or pattern of reasoning instead of those we consider correct and appropriate" [1]. It is not an idea itself, but rather the *use* or *misuse* of an idea, namely the decision-making process itself. As such, student difficulties can take many forms, including "*choosing* an inappropriate or incorrect rule, *ignoring* an important variable, *failing* to recognize a conflict, *blending* similar concepts, *making* an unwarranted assumption, [or] *drawing* an invalid inference" [30]. We wish to make clear that the term "difficulty" merely indicates that a student is performing actions that do not conform to objectives and goals the physics education research community has agreed to be reflective of understanding. In some sense, the existence of a difficulty may indicate the existence of precursor modes of reasoning and understanding that can be refined and made more to align with our goals. Historically, many approaches have been designed to help students address student difficulties, such as Elicit-Confront-Response models [23]. In this paper, we will outline the utility that deliberate practice can have in helping improve student understanding of concepts underlying common student difficulties.

In summary, the ESF outlines a procedure that one can use to identify potential essential skills and create effective practice for these skills for students to complete in order to improve their accuracy and fluency. It is important to recognize that the ESF is just a framework: it is not the platform ones uses to administer the practice, nor is it the actual questions one writes using the framework. Later, we will describe the work we did in identifying essential skills and designing practice for them in quantum mechanics for the first time. Before we do that, though, we will outline considerations the ESF describes as necessary when building practice for students.

2.2.4 Considerations taken when Constructing Practice using the Essential Skills Framework

Once one has built practice using the ESF, the practice should then be administered in a way that promotes student learning. Heckler *et al.* outline some principles to consider when designing and administering the practice so as to make it as effective as possible [82].

1) The first principle is the use of computer-based training with feedback mechanisms. These tools allow for students to complete the practice in their own time and at their own pace. In addition, it has been shown that providing students feedback can be beneficial to student learning when using deliberate practice [29]. As such, we wanted to administer practice that was online and incorporated various feedback mechanisms to optimize students' use of their time and learning.

2) A second principle is that the practice should be mastery-based. "Mastery" in this context involves providing students with practice until they achieve a desired standard of performance on each assignment. This allows students who may need more practice to get it. If a student who is starting the practice already has fluency in the relevant skills, they may finish the assignment quite quickly, whereas a student who is less fluent has the opportunity to get more practice.

3) Third, the practice should be distributed (students practice the skills several times across the term) and interleaved (students see a mix of skills, both within and between assignments). Thus, students get practice over an extended period of time and must learn to identify which skill(s) must be executed at a given point in time.

4) Fourth, students should be given practice using multiple representations. In quantum

mechanics, for example, students often encounter several different representations for various concepts, such as pictures, graphs, mathematical representations, *etc.* It is important that students practice skills in various representations so that they can attain fluency in each while also developing intuitions for how they are related to each other. As a result, accuracy and fluency in these skills are more likely to transfer in other representational contexts.

5) Fifth, students should be given practice in *simple* representations. While it's important for students to get practice in multiple representations, the representations shouldn't be so complicated that distracting features impede the students' ability to execute the skill.

Finally, there are a handful of desirable outcomes one might aim for in the implementation of the ESF. In particular, Heckler *et al.* comment that the practice designed using the ESF should do the following:

- The practice ideally maximizes the accuracy, fluency, and retention of the skills being practiced. This is sensible considering the motivation for the ESF, but this last aspect is important. Many skills are used in a variety of physics contexts and not just the course in which the practice is completed.
- The practice ideally does not add too much time to students' workload. Students are often already busy with other components of the course, and thus the practice should not add too much extra in terms of required workloads. In general, we wanted our practice to take at most 20-30 minutes a week. We also tried to remove other homework already in our courses so the homework total was unchanged.
- Thirdly, the practice should be a positive experience for the students. Ideally they should view the practice as unintrusive and enjoy the practice while they learn.
- Lastly, students should see the benefit that they are receiving from the practice and that they feel it is not a waste of time. To students, the practice should be clearly

productive. They should see that they are improving with continued practice, either by doing better on the practice or executing the skills correctly on problems.

2.2.5 Summary of the Essential Skills Framework

In summary, the Essential Skills Framework is a means by which instructors and researchers can identify simple, common skills that students are known to struggle with and then build practice for these skills so that students may become more accurate and fluent in their execution. The framework is not the practice itself, but a guide to its construction and implementation. The overall goal of the framework is to construct practice that is informed by research and administered using research-validated principles in order to make the practice as effective for student learning as possible.

2.3 Prior Education Research on Quantum Mechanics and Deliberate Practice

In this section, we will discuss 1) the prior literature on quantum mechanics education research and 2) deliberate practice as an instructional strategy. The focus here will to be to outline the research and not discuss detailed findings of this research. In chapter 4, we will discuss the research in more detail and illustrate how we used the findings of the prior research in detail.

2.3.1 Overview of Prior Research on Quantum Mechanics

Before the year 2000, work on quantum mechanics education research was largely anecdotal. Some studies identified common student misconceptions in university quantum mechanics courses [127]. Many investigated student understanding of various historical results of quantum mechanics, such as particle-wave duality, atomic orbitals, and atomic stability [38, 22, 86]. Gradually, the research began to engage directly with misconceptions and difficulties students have with the novel conceptual and mathematical aspects of quantum mechanics. Singh *et al.*'s literature review from 2015 summarizes the research in student understanding of quantum mechanics prior to that year [120]. We will briefly summarize the highlights of this review below.

In general, student difficulties in quantum mechanics tended to be present at both undergraduate and graduate levels [109, 114]. Some works noted that diversity in student preparation for quantum mechanics has drastically increased in recent years, meaning that more and more students are taking quantum mechanics with potentially insufficient backgrounds, *e.g.*, not having studied linear algebra [120]. Most research up to 2015, however, has been dedicated to student understanding of specific quantum concepts as discussed below.

A basic finding was that students tend to treat classical and quantum mechanical systems similarly and often confuse classical and quantum mechanical concepts [135, 84, 136, 110, 119, 69, 12]. This is understandable since their physics education prior to this point often has not had any quantum instruction. But such confusions often lead to various difficulties. For example, student understanding of energy in quantum tunneling scenarios has shown student retain classical intuitions about overcoming potential barriers [135, 84]. In addition, many students claim that quantities labeled with orthogonal spatial directions, such as S_x and S_z, are orthogonal as well [136, 110].

Next, it has been found that students tend to struggle with the mathematical framework and formalism of quantum mechanics [114, 136, 141, 118, 116, 109, 117]. This is to be expected as quantum mechanics is expressed in a novel mathematical language, namely that of linear algebra, but students tend to confuse even conceptual components of the mathematical framework. For example, it's been found that students seem to confuse the three-dimensional vector space associated with the real, physical space with the Hilbert space in which quantum mechanics is done [114, 136, 141]. It's also been documented that students have difficulties in contextualizing the role and function of eigenvalue equations [114]. Related to this is the observation that students view action by an operator as being conceptually equivalent to a measurement [116, 117, 111].

In addition to these findings, there have been a variety of works dedicated to probing student ability to translate between various representations. Quantum mechanics is often introduced using three representations: matrix, Dirac, and wave function notations. Singh found that students often know that $\psi(x) = \langle x|\psi\rangle$ but fail to apply this when solving problems [118]. At the core of this finding is that when students attempt to translate various expressions from Dirac notation into wave function notation, they do not do so correctly. In addition, it has been found that students struggle to relate and translate between expectation values and probabilities in various notations [109]. Finally, more generally, it's been found that students have trouble relating quantities in various notations (for example, the momentum operator acting on a state, $p|\psi\rangle$, vs its position-space representation $\langle x|p|\psi\rangle = -i\hbar\frac{d\psi}{dx}$) [118].

Students also often struggle with various aspects of wave functions [114, 2, 39, 140]. What Singh *et al.* found is that students tend to find the mathematical and physical structure of wave functions difficult to conceptualize. Such difficulties include knowing when $H\Psi = E\Psi$ is true [114], that states can be expanded in eigenbases (as well as what this looks like) [114, 140], as well as which wave functions are allowed to be considered as states for particles [2, 114, 39]. In particular, students often did not consider boundary conditions (correctly or at all), and many claimed that the only allowed wave functions for a given potential are its associated energy eigenstates [140, 2].

Students also seem to struggle with the concept of measurement in quantum mechanics [109, 2, 111, 114, 117, 140, 116]. This includes difficulty in assessing the probability that a particular measurement will occur [2, 140], which measurements are possible given a state [140], and the role that eigenstates play in the process of measurement [109, 111, 114, 140]. Students also tend to struggle with deciding what the result of a measurement is as well as how measurements of one observable affect the measurements of other observables [114, 140]. It has also been found that students will sometimes conflate action by an operator on a state with measurement of that observable [111, 117, 116]. In general, the findings related to measurement are quite diverse. This is especially relevant given that measurement forms a cornerstone of the novelty of quantum mechanics.

Somewhat relatedly, students often misunderstand the concept of expectation values [114]. This includes confusing the concepts of probability and expectation values [114], interpreting the expectation value as an ensemble average [114], and not understanding the difference between individual measurements and expectation values [114].

Students also struggle with the concept of time dependence in quantum mechanics [109, 114, 140, 110, 68, 6]. The difficulties they encounter encompass a large set of behaviors. For example, students often fail to identify the differences (conceptually and mathematically) between the time-dependent Schrodinger equation and the time-independent Schrodinger equation [114]. Students fail to identify how to make wave functions time dependent [140, 114, 110]. In addition, many students fail to assess the time dependence of various quantities, such as probabilities [140] and expectation values [140, 109, 68]. Further, students seem to struggle in identifying the conditions under which these quantities do and do not change in time [68].

In addition, student understanding of angular momentum has also been studied [141, 123, 113]. Difficulties with angular momentum include student understanding of the Hilbert space structure of particles with angular momentum [141, 123], uncertainty relationships between particles with angular momentum [113], and student conceptual understanding of particles with both single and multiple kinds of angular momentum [123].

The review by Singh also discusses a few studies that are not focused on student difficulties but that are related to problem solving and students' self-monitoring in quantum mechanics courses. For example, it has been found that students tend to classify problems in quantum mechanics differently than experts [79, 63]. It was also found that students fail to recognize how they can use their homework assignments and solutions to these assignments as well as solutions to exam questions for deepening their understanding of quantum mechanics [79, 78].

Since 2015 when this review by Singh was published, work focusing on student difficulties and various attempts to address these difficulties has continued. Some of this work has been dedicated to student understanding of perturbation theory [91, 45, 46, 53, 65, 121, 87] and multiple particle statistics [54, 44, 16, 52, 67, 48, 47]. There has even been continued study on previously examined topics such as time dependence, angular momentum, and student understanding of the mathematical structure of quantum mechanics [19, 37, 132, 105, 17, 89, 88, 72, 93, 74, 73, 99, 62, 130, 129, 18, 51].

Since 2015, there has also been a shift in focus on quantum mechanics education research. Prior to that point, most research was dedicated to student conceptual understanding. After 2015, though, some researchers began to focus on student ideas in general in quantum mechanics. In particular, there was a shift in focus from describing what students were doing incorrectly to examining what students were doing with an eye toward informing instructors of the ways in which students approach learning and the way in which they seem to have organized their quantum mechanical knowledge in their minds.

As an example, Gire *et al.* in 2015 published an article suggesting that students weigh the pros and cons of the various notations in quantum mechanics depending on the problem context, and that this affects the way they approach problem solving generally [25]. Using students' ideas on notation, the authors constructed a scheme that classified various characteristics of these notations that factor into students' decision-making processes with regards to notation. This scheme has been applied in various settings, such as Wawro's study of students' metarepresentational competence and Schermerhorn's study of notational preferences when solving expectation value problems [134, 103].

Further, research into student ideas has delved into various other aspects of student understanding of quantum mechanics. Wawro *et al.* studied students' perspectives on conceptual aspects of eigenvalue equations and their uses in various notations [133]. In addition, the symbolic forms framework of Sherin [107] has been applied to both student understanding of boundary conditions as well as the relationship between probabilities in both wave function and Dirac notation [97, 95]. There has also been work studying student ideas on how students reason about the different bases that can be used to represent a state and how the basis used can affect student reasoning about probabilities [10]. Finally, Corsiglia *et al.* has studied how students develop and characterize their intuition in quantum mechanics while learning the subject [8].

While there has been a large corpus of work dedicated to studying student understanding in quantum mechanics, there has also been ample work involving the construction and assessment of instructional materials geared towards improving student understanding of quantum mechanics [57, 19, 90, 77, 113, 100, 76, 75, 101, 15]. For example, broad tutorial sequences have been designed at various large universities. These include the Tutorials in Physics: Quantum Mechanics at the University of Washington [106], the tutorials designed at the University of Colorado: Boulder [94], and the Quantum Interactive Learning Tutorials (QuILT) from the University of Pittsburgh [115]. These tutorial sequences have each been found to be effective in developing student understanding of quantum mechanics [19, 102, 2, 18].

Materials focusing on more specific topics have also been developed. For example, the University of Pittsburgh has developed clicker question sequences on specific topics such as angular momentum, Dirac notation, and time evolution [12, 34, 35, 43, 36, 33, 42, 41]. In addition, various visualization tools have been developed to assist in aiding the development of conceptual understanding [60, 59, 61, 58, 92]. These include time dependence, measurements, and various features of wave functions and scattering in various potentials [13, 19].

In summary, quantum mechanics education research has probed a wide array of components of typical quantum mechanics course material. This ranges from student difficulties with essentially all topics in the course to student ideas about the material and meta-material. In addition, there has been much work dedicated to addressing these difficulties via curriculum and tools so that student understanding of quantum mechanics can improve.

That being said, to the best of our knowledge, there haven't been interventions designed that address student fluency in applying the tools of quantum mechanics. Those interventions that do cover diverse sets of topics tend to focus on conceptual understanding of quantum mechanical ideas instead of specifically improving student ability to apply the skills required to solve problems in and understand quantum mechanics. This dissertation aims to fill that gap in by supplementing our ability to improve student understanding of quantum mechanics by giving students directed practice on specific skills.

In summary, this literature review is intended to frame the breadth of research that one can take into account in trying to build essential skills practice for students. As mentioned before, the Essential Skills Framework outlines, as an essential step, using the research literature to inform the the identification of essential skills and to design practice for these skills. The breadth of quantum mechanics education research provides a large pool of information to consider when building practice for students. In Chapter 4, we will discuss in more detail how we used the results of individual studies to identify essential skills and build practice for these skills.

2.3.2 Prior Research on Practice as an Instructional Strategy

The Essential Skills Framework (ESF) suggests student fluency can be improved by developing targeted practice using research-validated methods. We have already outlined some of these practices above in Section 2.2.4. Here, we discuss some research that backs the effectiveness of these techniques as well as some prior research on the use of deliberate practice as relevant to the ESF.

Practice, as part of learning, is valued as a means for not only developing expertise in a particular skill but to also make such a skill second nature. But it is a common addage that "practice does not make perfect; perfect practice makes perfect". This suggests one needs a particular perspective on practice so as to develop student abilities more effectively. One such perspective is through "deliberate" practice. Ericsson *et al.* define deliberate practice as being focused and targeted practice on specific skills paired with expert feedback [21]. Such practice is highly structured and requires high effort, in contrast to what Ericsson calls “play”. The goal of such practice is not only just to improve performance in the skills themselves but to improve critical thinking in general.

The improvement of critical thinking skills is closely related to an important learning outcome of the ESF, namely fluency in the skills practiced. It is not enough to simply be able to correctly execute a skill: one must be able to do so fluidly and without the execution of the skill interfering with other aspects of problem solving. Fluency has a colloquial meaning associated with a kind of subconscious execution and understanding of a particular skill. This is particularly important considering that when students are learning, a lack of fluency in certain basic skills can induce significant cognitive load impeding their ability to solve other important aspects of a problem [128]. Mathematics education research regards fluency in a skill to mean that a task requires a relatively short completion time, minimal cognitive load, and the ability to process the skill in parallel with other skills when solving a problem [55].

Deliberate practice has been employed in Physics Education Research (PER) in a variety of contexts, including in homework assignments, clicker question sequences, and laboratory settings. It has been shown that deliberate practice in these contexts can yield improvements in student learning in the particular course component involved [83, 125, 32]. Deliberate practice has been been used to revamp entire coursework and has documented improvements in student learning without sacrificing course content [40].

Core to the philosophy of deliberate practice is the availability of feedback to the students. With the advent of computer-based systems, feedback can largely be automated and thus has been shown, in various circumstances, to be effective [11]. It has not been shown to be effective in all circumstances, though [108]. Heckler *et al.* have tried to demonstrate that feedback through computer-based feedback requires a sweet spot in the complexity of the feedback to make it beneficial for students [81]. This work compared two types of feedback to students: students are told correct answers after they have answered, *versus* simply being told their answer was incorrect. In this work, they found that immediately providing students with a correct answer and providing the option to view elaborated explanations on the topics being practiced had the most beneficial effect on students. The option of elaborated feedback is important, since students who need or want the elaborated explanations have the option to view them for their own benefit.

This last point here is important: students who need or want enhanced practice should be able to acquire such, even beyond the simple acquisition of feedback. With this in mind, another important component of deliberate practice is the usage of mastery-based training. Mastery-based training is practice that involves differing time-on-task for students [3]. The advantage of such an approach means that students who may need more practice (more time on individual practice questions, more questions to practice with, or both) [3]. Mastery-based training has been shown to be an effective means of furthering improvements on low performing students after practice [26, 104].

Another important aspect of the ESF is that it recommends the practice be distributed and interleaved. What is meant by this is that the practice for each skill should be spread out over the term (distributed), and that the practice should thread multiple skills within each other so that students must also recognize what skill they must apply before applying it (interleaved). Distributed practice has been shown to be an effective way for students to not only attain accuracy and fluency in skills but also retention [7, 96]. In addition, interleaving has been shown to help students learn to distinguish between the skills they are practicing [14].

Finally, the ESF promotes providing practice in skills using multiple yet simple representations for these skills. Once again, one objective of deliberate practice is the improvement of one's critical thinking skills generally, not just the improvement of specific skills in specific contexts. It has been shown that practicing a skill in multiple representations leads to transfer of the expertise in these skills to other concepts [5]. In addition, practicing skills with simple representations, where the skill in question is the main focus of the practice, has been shown to improve student learning by reducing any extra processing required for components of the practice irrelevant to the skills in question [124].

Deliberate practice has been shown to be an effective means of administering practice to students in improving their learning and fluency in skills. In addition, we have discussed various other perspectives on practice shown to be beneficial to student learning that are desirable to incorporate for the practice we built for students in upper-division quantum mechanics courses. These perspectives have been incorporated into the ESF in an attempt to guide the development of the most effective practice as possible.

2.4 Summary

In this chapter, we have discussed the motivation for selecting directed practice as a means for improving student understanding in quantum mechanics. We did so by discussing rough trends we have found in analyzing student responses to exam questions, where we found that students tend to make mistakes on a wide variety of small steps that are ubiquitous in quantum mechanics problems. This motivated our use of deliberate practice to improve student learning. We then discussed the framework we adopted for constructing this practice for students, the Essential Skills Framework. Finally, we discussed the prior research in quantum mechanics education research and directed practice that informed our building of the practice.

In the next chapter, we will discuss how we used the research literature and historical exam data to identify potential essential skills and construct practice for these skills. In doing so, we will discuss specific examples of the prior literature discussed above in more detail and how they guided the development of the practice questions. Using this procedure, we ended up with more than 1500 practice questions that we administered over a period of several years to students.

Chapter 3

IMPLEMENTATION OF THE ESSENTIAL SKILLS FRAMEWORK: WRITING THE PRACTICE

In this chapter, we discuss our design and implementation of the Essential Skills Framework (ESF). The process consisted of identifying a list of potential essential skills, creating a series of practice questions for them, and then organize them into STEM Fluency's topic-category-subcategory hierarchy. We will begin by outlining a few examples for the procedure we used to identify potential Essential Skills from both the research literature and student responses to exam questions. Then, utilizing these potential essential skills, we will discuss three examples for how the research literature and student responses to exam questions aided in our design of practice questions for these skills. Then, after briefly introducing the online practice homework system (STEM Fluency) we used for this project, we discuss how we organized these questions into a topical hierarchy for ease of use in the homework system. Finally, we give a few heuristics and considerations for how assignments were built for student practice. For a full list of, and descriptions of, the skills identified as well as the final hierarchy for their organization, see Appendix B (corresponding to the "subcategory" columns of the tables in this appendix). For a list of example questions in each skill, see Appendix C.

3.1 Illustrating the Process for Identifying Potential Essential Skills

First, we will discuss our work on how we identified the various skills that students seemed to lack facility with when solving problems in quantum mechanics. The purpose of this procedure is to identify the skills that students might benefit from practicing in a systematic way. Not all of the skills identified may be fruitful or skills that would benefit from practice, though. As such, any set of skills we identify may be seen as *potential* essential skills that we need to study in order to determine whether or not they are is “essential”. Although this process is asymptotic in nature, the ESF requires research on student difficulties as well as anecdotally observed behaviors students demonstrate to get good initial guesses. In this section, I will discuss four articles in the literature as well as two exam questions and illustrate how we used them when determining potential essential skills.

3.1.1 Examples of Essential Skills Identified from the Literature

In this section, we will discuss the results of four articles and how we used them to determine various potential essential skills for students to practice. In doing so, we will merely identify the broad difficulties themselves and not the *ways* students struggle with them. The latter will occupy section 3.2 when we discuss how we designed practice questions based off these skills.

Literature Example 1: Identifying a Broad Set of Essential Skills

As our first example, we discuss “Student Understanding of Quantum Mechanics“ (2001) by Chandralekha Singh [109]. This work discusses the results of a conceptual survey and interview data covering a variety of important ideas in quantum mechanics, such as eigenbases, stationary states, time dependence, and measurements. Two examples of observed difficulties are below.

- It was found that much of the formalism of quantum mechanics seemed difficult to students. For example, students were tasked with calculating a generic expression for the expectation value of an operator Q for a particle given in the state $|\psi\rangle$. Students were meant to expand $|\psi\rangle$ in the eigenbasis of Q, plug it into the expression $\langle\psi|Q|\psi\rangle$, and utilize orthonormality and eigenequation relationships to obtain the desired expression.

  In this problem, students seemed to struggle in various ways with correctly expanding states in Q's eigenbasis as well as applying orthonormality. These are key mathematical tools in quantum mechanics, appearing in many problems due to the linear-algebraic nature of the theory. In addition, many students did not recognize the utility and validity of eigenvalue equations in this problem.

All of these concepts and tools are fundamental to the mathematical components of quantum mechanics and appear in a wide variety of problems. As such, it seemed to us that these might be good skills for students to practice.

- Another important finding is that students seemed to struggle with assessing the time dependence of expectation values and in particular the role that eigenstates, stationary states, and commutativity of operators plays. One of the questions students were asked to this end involved giving them a particle in a harmonic oscillator potential with Hamiltonian H, and asking them to consider an arbitrary operator Q. Students were tasked with identifying whether or not a particle in 1) a momentum eigenstate, or 2) an energy eigenstate, will have an expectation value of Q that depends on time. To do so, students must consider the equation $\frac{d}{dt}\langle\psi|Q|\psi\rangle = \frac{i}{\hbar}\langle\psi|[H,Q]|\psi\rangle$ and identify the conditions for when the right hand side is 0. This includes cases where $[H,Q] = 0$, or that the expecatation value of this commutator is 0 (namely, if $|\psi\rangle$ is an eigenstate of Q or H).

  Many students did not identify one or the other of these conditions and appealed to other incorrect ideas. For example, many students seemed to believe that expectation values always depend on time unless $Q = H$. Most students did not appeal to what the state was (in this case, either a momentum eigenstate or an energy eigenstate). When they did, they often did so incorrectly (for example, if $|\psi\rangle$ is an eigenstate of Q, the expectation value will depend on time).

  At its core, these difficulties reflect that students often do not think of time dependence in a mathematical way, instead appealing to memorized rules or classical intuitions.

> Time dependence in quantum mechanics is important and is one of the four axioms of the theory. As such, helping students learn the mechanics of time dependence, both mathematically and conceptually, would be beneficial to them.

In summary: this article gives a broad picture for some of the ideas with which students struggle in quantum mechanics, presenting difficulties ranging from the formalism of the theory as well as the conceptual pieces. It suggests that students could benefit from practicing a wide variety of topics and gave us many ideas for where that practice can be done.

Example 2: Examples of Essential Skills Identified from an In-Depth Look at Time Dependence

As a second example of using the literature to identify potential essential skills, consider the work "Student Understanding of Time Dependence in Quantum Mechanics" (2015) by Paul J. Emigh, Gina Passante, and Peter S. Shaffer [17]. This work focuses specifically on various difficulties students have with: the time dependence of wave functions, probability densities, and probabilities; how to assess whether or not these quantities are time dependent in various cases; and the role that energy eigenstates play in determining the time dependence of these quantities.

These difficulties were identified and studied by administering four questions to students involving time dependence in various contexts, both in written and interview form, and analyzing the students' responses. The tasks involved assessing whether or not the state, the probability density, and various probabilities depend on time in various contexts (including ones involving degeneracy).

Students often confused the time dependence of various quantities. For example, some students would incorrectly claim that probability densities always change in time because their wave functions always changed in time, or that because a particular probability density depends on time, the energy probabilities depended on time. This was observed in a variety of contexts and thus seems to be a prevalent difficulty. A particularly prevalent difficulty involved students struggling to properly ascribe time dependence to wave functions/ states. In some sense, the skill associated with the difficulty of time evolving a state is the most important when considering time dependence since it is an axiom and can help determine the time dependence of other quantities of interest.

In summary, this article serves as an important basis for identifying various difficulties students have with time dependence. Overall, students seem to have difficulty identifying and assessing the relationship between various time dependencies as well as how to produce time dependent quantities. These skills, especially the latter, are extremely important in quantum mechanics. As such, this work can be used to give a good picture as to what students should practice in order to become more comfortable with analyzing time dependence.

Example 3: Example of Essential Skills Identified from a Specific Look at Student Uses of Mathematical Expressions

For a third example, consider "Surveying Students' Understanding of Quantum Mechanics in One Spatial Dimension" (2012) by G. Zhu and C. Singh [140]. This work discusses the administration of a broad survey on various quantum mechanical topics and ideas. It assesses student performance in order to identify various ideas that students struggle with. Many of the topics identified here match difficulties previously mentioned, so we will focus on some of the newer difficulties mentioned that we haven't discussed yet. We discuss them below.

- A common trend found among students who took the survey was that they seemed to interpret the expectation value in multiple ways. While not an error, such interpretations often led to students to attempt more complicated calculations than necessary, leading to incorrect answers. As an example, the expectation value of energy for a particle in the state $\Psi(x,0)$ is often given as $\int_{-\infty}^{\infty}\Psi^*(x,0)H\Psi(x,0)\mathrm{d}x$. But, it is also a weighted average of the energy eigenvalues for the energy eigenstates present in the state, which can be written as $\sum_{n=0}^{\infty}E_n|c_n|^2$, where c_n is the coefficient of $\psi_n(x)$ in the energy eigenbasis expansion of $\Psi(x,0)$. There are indeed various ways one can interpret or state the expectation value of an operator, but it was found that many students had tendencies one way or another.

It was found that many students did not recognize nor apply interpretations more amenable to problems at hand. This implies a disconnect between the various ways students engage with expectation values (and other quantum mechanical concepts in general): often, students miss the connections between verbal descriptions, mathematical representations, and physical interpretations. We decided that students may benefit from practicing recognizing the various interpretations of expectation values (and other quantum mechanical objects) and their uses.

- Elaborating on the last point above, another finding in this work involved the tendency for students to use and produce mathematical expressions corresponding to expectation values when asked about probabilities. More generally, students tended to blend aspects of expectation values and probabilities. For example, a common incorrect student response to being asked to write down an expression for the probability of measuring the energy to be E_n was $|\int_{-\infty}^{\infty}\psi_n^*(x)H\psi(x,0)\mathrm{d}x|^2$. This mathematical expression involves the correct components of a probability but includes the added operator H between the $\psi_n^*(x)$ and $\psi(x,0)$, which is characteristic of expectation values.

  This, again, reflects a tendency of students to incorrectly or incompletely connect various aspects of quantum mechanical concepts and quantities. This observation has already been made elsewhere, where it was found that students claimed that when measuring probabilities for quantities involving a particular operator, that operator must be present in the corresponding mathematical expression [109]. As such, students may benefit from having explicit practice asking them to produce, interpret, and recognize common representations (mathematical, verbal, conceptual) of the various quantities involved in quantum mechanics.

- As a final example, this work discusses the various methods students used when judging

> whether or not a particular wave function can be considered the state of a particle in a particular potential. In general, a wave function can be used to describe a state of a particle if it is square integrable and satisfies appropriate boundary conditions. Technically, it must also be in the domain of the Hamiltonian which is rarely the entire Hilbert space, but this is a technical point often ignored. It was found, though, that students tended to use various heuristics that they have invented when deciding if a given wave function could be the state of a particle.
>
> Some examples include the tendency for students, in some situations, to act as if they believe only eigenstates can be states for particles and that superpositions of eigenstates cannot. Others claim that superpositions are allowed, but only when they are explicitly written as a superposition (namely, the eigenstates are clearly identifiable in the mathematical form of the wave function in the form of, say, an explicit superposition). As an example, students may view the wave function $\Psi(x, 0) = N(x^2 - ax)$ (where N is a normalization constant) cannot be the state of a particle in an infinite square well, because it is not written in terms of sine functions (the eigenstates of the infinite square well Hamiltonian). These are only a few examples, but these observations reflect a potential need to gives students practice in judging what wave functions can be states of particles (or, more generally, helping students recognize how broad of a class of functions states can be).

Overall, this article serves as a broad picture of the various ways students struggle with common topics in quantum mechanics, but it also suggests some ideas for skills that may warrant practice for students. This latter observation on its own yielded us a large and diverse set of skills that we decided students would benefit from getting practice in. These involve students being led to recognize, produce, and interpret quantum mechanical quantities within these various mathematical and conceptual representations, as well as assess the suitability of wave functions as potentially allowed states for particles in various potentials.

Example 4: Examples of Essential Skills Regarding the Importance of Choices in Notation

Finally, we consider the work "Structural Features of Algebraic Quantum Notations" (2015) by E. Gire and E Price [25]. While this work does not explicitly identify student difficulties, we claim that it provides a valuable insights that we were able to use in identifying potential essential skills.

The article focuses on interview data. Students are given a state expanded in the energy eigenbasis *in words*. Namely, they are told the probabilities for measuring two energy eigenvalues that sum to 1, and they are asked to construct a candidate state. They are then tasked with calculating the expectation value of energy for this particle. The way that this question is framed allows room for students to choose a notation that they wish to work with. Afterward, students are asked about their choices in notation (Dirac, matrix, or wave function notation) when solving the problem. The authors found that students tended to prioritize various aspects of notations when solving a problem that depended on the problem context. For example, when solving position probability problems, students claimed that wave function notation was most preferred since position shows up most explicitly in this notation. Using students' thoughts about how various components of each notation factored into their decision to use it for a particular problem, the authors created a scheme of general characteristics that guided this decision making process.

This work shows that students have different ideas about the various notations. Students see these notations as useful in different contexts and that these notations communicate different information. Thus, they have preferences for which notation to use in a given problem context. We thought it might be useful to give students practice in certain skills in different notations. Namely, if students think about notations as being useful in different contexts, then students should get practice executing said skill in a variety of notational contexts. Further, it seems useful to give students practice in identifying advantages and disadvantages of these notations in specific cases. Overall, this work informs us that, when identifying essential skills, the skills could be notation dependent, and so students should have diverse notational practice as well as practice with skills identified above, matching a goal of the ESF mentioned above in Chapter 3.

3.1.2 Examples of Essential Skills Identified from Exam Questions

Here, we will discuss two questions in a problem on an exam administered to $N = 92$ students in Fall 2015. In particular, we will discuss the various ways students attempted to solve these problems in order to identify potential essential skills. We will do this by identifying common solution strategies used by students on these questions, summarize a correct version of these strategies, and break down these solutions into necessary steps that we believe could be practiced. Because there are many ways students can solve a given problem, one problem may contain potentially a large variety of skills needed to solve it.

The full problem is as follows:

> Consider a particle in the one-dimensional harmonic oscillator potential, $V(x) = \frac{1}{2}m\omega^2 x^2$. The initial wave function for this system is known to be $\Psi(x,0) = N\frac{\sin(2\pi x/b)}{x}$, where N is a normalization constant and b is a constant.
>
> 1) Write an expression for the probability of measuring the ground state energy, E_0.
> 2) State whether or not this quantity depends on time.
>
> Explain your reasoning in each case for full credit. If you do not have enough information to answer, state so explicitly. Note: you do not need to evaluate any expression.

Figure 3.1: Part of an exam problem administered to students in Fall 2015. The problem is abridged for convenience of presentation. In the first question, students must produce an expression for the probability of measuring the ground state energy. In the second question for this problem, students must assess the time dependence of this probability.

In particular, students are first tasked with producing an expression for the probability of measuring the energy to be the ground state. They are then asked to assess whether or not this probability depends on time. For each question, we will discuss the various ways students attempted to solve this question and outline complete, correct solutions for these methods, outlining the various steps needed in each case. The keyword to consider here is *attempted*: often students made errors when executing the steps discussed here, or they didn't recognize that the steps were necessary. At its core, what follows is not meant to demonstrate *required* steps students would need to perform solving these questions but instead to highlight the strategies that students employed (often incorrectly) when attempting to solve these exam problems. That students made errors in identifying or executing these steps indicates that students might benefit from practicing these steps.

Example 1: Producing the Expression for a Probability

The first part of the question asked students, given the state above (in wave function notation), to produce an expression for the probability of measuring the energy to be the ground state. Generally, students attempted to solve this problem using three main solution strategies, whose correct verstions are outlined in Table 3.1. We discuss in solution strategy below. Note that even though the state was given in wave function notation, students are not required to answer in wave function notation. In fact, many students responded using Dirac notation.

Approach 1:

Many students simply responded with what seemed to be a memorized response. This is not surprising, since probabilities are a common quantum mechanical quantity to work with. As such, simply responding with $|\int_{-\infty}^{\infty}\psi_0^*(x)\Psi(x,0)\mathrm{d}x|^2$ or $|\langle\psi_0|\psi\rangle|^2$ (after appropriately defining the ket and bra in question) was considered correct.

Outline of Correct Solutions Attempted by Students for Question 1

	Description	Result
Solution 1		
	Simply knowing the answer	$\lvert \int_{-\infty}^{\infty} \psi_0^*(x)\Psi(x,0)\mathrm{d}x \rvert^2$
Solution 2		
	Knowing the answer in Dirac notation	$\lvert\langle\psi_0\vert\Psi\rangle\rvert^2$
	Inserting a completeness relation	$\lvert \int_{-\infty}^{\infty} \langle\psi_0\vert x\rangle\langle x\vert\Psi\rangle\mathrm{d}x \rvert^2$
	Using the Dirac definition of the wave function	$\Psi(x,0) = \langle x\vert\Psi\rangle$
		$\rightarrow \lvert \int_{-\infty}^{\infty} \psi_0^*(x)\Psi(x,0)\mathrm{d}x \rvert^2$
Solution 3		
	Probability as a particular coefficient	$P(E_0) = \lvert c_0\rvert^2$
	Expand our state in the energy eigenbasis	$\Psi(x,0) = \sum_{n=0}^{\infty} c_n\psi_n(x)$
	Take a particular inner product	$\int_{\infty}^{\infty} \psi_0^*(x)\Psi(x,0)\mathrm{d}x = \sum_n c_n \int_{-\infty}^{\infty} \psi_0^*(x)\psi_n(x)\mathrm{d}x$
	Utilize orthonormality	$\int_{-\infty}^{\infty} \psi_0^*(x)\psi_n(x)\mathrm{d}x = \delta_{0n}$
		$\rightarrow c_0 = \int_{-\infty}^{\infty} \psi_0^*(x)\Psi(x,0)\mathrm{d}x$

Table 3.1: Outlines of three correct solution strategies employed by students when solving the first part of the exam question outlined in Figure 3.1. These are not all techniques used by students to solve this problem, and not all students using these strategies utilized them correctly. The full solution is provided for the purpose of outlining the various skills and steps one could use to solve a problem like this.

Approach 2:

Many students alternatively recognized the correct answer in Dirac notation and chose to express their answer in wave function notation. Although we accepted the initial step in this approach (namely, producing the correct answer in Dirac notation like the first approach above) as correct when grading, analyzing student responses when they tried to convert the Dirac expression into wave function notation served to be valuable.

Students began by writing the expression for the probability in Dirac notation: $|\langle\psi_0|\psi\rangle|^2$ (or something equivalent depending on how they chose to label their bra and ket). To mathematically convert this expression into wave function notation, students must insert the identity/ a completeness relation for position into this expression to obtain $|\int_{-\infty}^{\infty}\langle\psi_0|x\rangle\langle x|\psi\rangle \mathrm{d}x|^2$. Students must then recognize that $\langle x|\psi\rangle = \Psi(x,0)$ and $\langle\psi_0|x\rangle = \psi_0^*(x)$. After substituting these expressions into the above, we obtain the final answer: $|\int_{-\infty}^{\infty}\psi_0^*(x)\Psi(x,0)\mathrm{d}x|^2$.

In summary, for this second solution strategy, students needed to recall the expression for the desired probability in Dirac notation, insert the identity into this expression, and substitute expressions for the wave function's Dirac form in for the wave function. In addition to performing these steps, students would need to recognize their necessity. We believed that it may be beneficial for students to practice these skills, especially since they since unique to those skills identified using the literature.

Approach 3:

Many students seemed to recall that energy probabilities can be viewed as the norm-square of a particular coefficient. In doing so, students must recognize which coefficient encodes the probability (amplitude) of interest, and then devise a mathematical procedure to solve for an expression for this coefficient.

A correct solution in this vein would take the following form (potentially not in this order, but with these steps). Students first recognize that the probability of measuring the ground state energy takes the form of a particular coefficient, $P(E_0) = |c_0|^2$, where c_0 is the coefficient of $\psi_0(x)$ in the energy eigenbasis expansion of $\Psi(x,0)$. To find an expression for c_0, students then need to expand $\Psi(x,0)$ in the energy eigenbasis: $\Psi(x,0) = \sum_{n=0}^{\infty} c_n \psi_n(x)$, or $\Psi(x,0) = c_0\psi_0(x) + c_1\psi_1(x) + c_2\psi_2(x) + ...$ Students must then recognize that, to isolate the coefficient c_0, they must take an inner product of $\Psi(x,0)$ with $\psi_0(x)$, yielding $\int_{-\infty}^{\infty} \psi_0^*(x)\Psi(x,0)\mathrm{d}x = \sum_{n=0}^{\infty} c_n \int_{-\infty}^{\infty} \psi_0^*(x)\psi_n(x)\mathrm{d}x$. Then, students must recognize the utility of, and properly utilize, orthonormality of eigenstates, namely that $\int_{-\infty}^{\infty} \psi_0^*(x)\psi_n(x)\mathrm{d}x = \delta_{0n}$. This simplifies the above expression to $\int_{-\infty}^{\infty} \psi_0^*(x)\Psi(x,0)\mathrm{d}x = \sum_{n=0}^{\infty} c_n\delta_{0n} = c_0$. With this mathematical expression for the coefficient c_0, students can then write the full expression for the probability by taking its norm-square, yielding $P(E_0) = |c_0|^2 = |\int_{-\infty}^{\infty} \psi_0^*(x)\Psi(x,0)\mathrm{d}x|^2$.

As a summary of the steps required for this solutions, students need to recognize the probability as a particular coefficient, expand their state abstractly in the energy eigenbasis, recognize the utility of and execute an inner product, and utilize orthonormality to obtain an expression for the desired coefficient. Many of these skills are largely mathematical in nature, but certain skills also require students to recognize what certain mathematical procedures do so that they know to execute such a step at all. It was observed that students who tried this solution strategy failed to execute these steps correctly rather frequently, so we figured that giving students practice in these skills would be beneficial.

Summary of Approaches for Problem 1:

Each of the three solutions outlined above require a variety of steps, both mathematical and conceptual. In general, students must learn to recognize the mathematica form of what it is that they are solving for, execute a variety of mathematical procedures, and contextualize when these mathematical steps are useful. Further, these steps are ubiquitous in quantum mechanics.

In addition, notice that many skills students struggled with on these exam questions were also observed student difficulties in the literature. Steps like identifying proper mathematical expressions for the quantities that one solves for and expanding states in a basis correctly show up both here and in the literature. As a result, this suggests that these skills we identified are, indeed, likely candidates for being treated as essential skills that students could benefit from practicing.

Example 2: Examples of Essential Skills when Identifying whether or not a Probability Changes in Time

Now, we tackle the second portion of the question in Figure 3.1, where students are tasked to assess whether or not the quantity solved for in the first portion of the question depends on time. Below, I illustrate a correct solution strategy, followed by various incorrect solution strategies attempted by students. Note that many of these seem to be based on memorized responses. In the previous one, most solutions were based on strategies that were correct, but here many students didn't seem to have a correct solution strategy from the beginning. As before, an outline of these strategies is illustrated in Table 3.2.

First, we discuss a correct version of a formal argument students could make when assessing the time dependence of the probability of measuring the ground state energy. First, the wave function given is made fully time dependent by expanding the state in the energy eigenbasis and then acting on the state with $e^{-iHt/\hbar}$ (or, more commonly, attaching the phases $e^{-iE_nt/\hbar}$ to their corresponding eigenstates in the eigenbasis expansion of the state) to make a time evolved state:

$$\Psi(x,0) = c_0\psi_0(x) + c_1\psi_1(x) + c_2\psi_2(x) + ... \tag{3.1}$$

$$\Psi(x,t) = e^{-iHt/\hbar}\Psi(x,0) = c_0e^{-iE_0t/\hbar}\psi_0(x) + c_1e^{-iE_1t/\hbar}\psi_1(x) + c_2e^{-iE_2t/\hbar}\psi_2(x) + ... \tag{3.2}$$

Now, one must recall that the probability for measuring the ground state at time t is $|\int_{-\infty}^{\infty}\psi_0^*(x)\Psi(x,t)\mathrm{d}x|^2$. Executing this mathematical procedure to obtain an expression

Outline of Correct Solutions Attempted by Students for Question 2		
	Description	Result
Solution 1		
	Expand our state in the energy eigenbasis	$\Psi(x,0) = \sum_{n=0}^{\infty} c_n \psi_n(x)$
	Time evolve the state with complex phases	$\Psi(x,t) = \sum_{n=0}^{\infty} c_n e^{-iE_n t/\hbar} \psi_n(x)$
	Employ a probability-finding procedure (outlined above in Figure 3.1)	$P(E_0) = \lvert c_0 \rvert^2$ for all times t
Solution 2		
	Memorized response: energy probabilities never change in time	$P(E_0) = \lvert c_0 \rvert^2$ for all time t.

Table 3.2: Outlines of three correct solution strategies employed by students when solving the second part of the exam question outlined in Figure 3.1. These are not all techniques used by students to solve this problem, and not all students using these strategies utilized them correctly. The full solution is provided for the purpose of outlining the various skills and steps one could use to solve a problem like this.

for the integral, one must take an inner product and exploit orthonormality to obtain $\int_{-\infty}^{\infty} \psi_0^*(x)\Psi(x,t)\mathrm{d}x = c_0 e^{-iE_0 t/\hbar}$. The probability is obtained then by taking the norm-square of this quantity, which entails multiplying it with its complex conjugate, canceling out the complex (and time dependent!) exponential, yielding simply $|c_0|^2$. This quantity is not time dependent.

This procedure entails a variety of conceptual and mathematical steps. Many steps are the same as those mentioned above in solving the first part of the question, such as expanding a state in an eigenbasis or utilizing orthonormality. Some skills, though, are different, such as understanding how to obtain a time evolved wave function.

Many students answered this question by utilizing memorized responses as opposed to the more quantitative explanation provided above. For example, some students claimed (correctly) that energy probabilities never depend on time for time independent Hamiltonians. Others incorrectly claimed that because it wasn't in an eigenstate that the probability of measuring the ground state would in fact depend on time. At its core, identifying these answers as specifc "difficulties" in a way that can be classified for practice may not be productive, but it's clear students broadly tended to answer this question using memorized, qualitative arguments instead of quantitative, mathematical procedures. This potentially indicates a need for students to practice recognizing the conditions under which certain memorized responses apply.

3.1.3 Summary of Identifying Essential Skills

While the above section is not an exhaustive account of how we identified potential essential skills, it serves as a holistic means of demonstrating the process we took. Using the literature and student responses to exam questions, we identified ideas and procedures students struggled with. Some of these skills are more "fundamental" than others, but they all represent common, basic steps in larger physics problems that students complete in an undergraduate, junior-level quantum mechanics course.

We do not have room here to list all of the skills identified. Below in Table 3.3 we have listed a large number of skills identified. The purpose of the table to demonstrate the breadth and number of potential essential skills that we wrote practice questions for.

3.2 Writing Questions for Essential Skills Practice

The next step we undertook was to write practice problems for students in the skills identified above. While one can simply write a question that requires students to correctly apply the skill we want them to practice, the ESF makes an important note of writing questions that take into account the *ways* students struggle with the skills. In this way, students are forced

Partial List of Potential Essential Skills

Complex Conjugating Kets and Bras	Complex Conjugating Inner Products
Expanding a State in an Eigenbasis: WF	Expanding a State in an Eigenbasis: Dirac
Orthogonality of Kets	Orthonormality of Bases
Insert Discrete Completeness Relation	Insert Continuous Completeness Relation
Recognizing Probability Expressions	Producing Probability Expressions
Recognizing Expectation Value Expressions	Producing Expectation Value Expressions
Add Time Dependence to State: WF	Add Time Dependence to State: Dirac
Continuous Probability: Units	Recognize Time Independent Schrodinger Equation
Recognize Time Dependent Schrodinger Equation	Basic 3x3 Matrix Diagonalize
Basic 2x2 Matrix Diagonalize	Recognize First Order Energy Correction
Recognize First Order State Correction	Recognize Second Order Energy Correction
Translating Dirac Notation to Wave Function Notation: Discrete	Translating Dirac Notation to Wave Function Notation: Continuous
Translating Wave Function Notation to Dirac Notation: Discrete	Translating Wave Function Notation to Dirac Notation: Continuous
Produce Energy State for Bosonic Particles	Produce Spin State for Bosonic Particles
Produce Energy State for Fermionic Particles	Produce Spin State for Fermionic Particles
List States in Degenerate Energy Level	What is Degeneracy of an Energy Level?

Table 3.3: Partial list of potential Essential Skills identified using the research literature and student responses to exam questions. Note that there are more potential skills than those entered here (for space considerations). For a full list of skills, see Appendix B for a rough list of skills, which are roughly in correspondence to the "subcategory" columns.

to confront their incorrect tendencies. Above, we have listed the things students are known to struggle with. As such, integrating this component of the ESF into our practice design fit most naturally.

The homework system that we used to administer the practice (to be discussed below) allows for question-writers to utilize a variety of different answer types for each question they write. This includes multiple-choice, multiple-choice-multiple-response, short answer, and so on. When writing our questions for quantum mechanics, we primarily drew on multiple-choice and multiple-choice-multiple-response, in part to simplify our analysis of student performance on these questions.

In this section, we will walk through some examples of question writing in order to highlight some of the important considerations we took in utilizing the existing literature and our known experiences. Generally, the procedure looks as follows:

- We identified a skill from the list in Table 3.3 (or rather the full list that this table is meant to represent). We then wrote a question being mindful of the variety of ways such ideas have been identified in the existing literature and in typical quantum mechanics problems.
- Then, in addition to writing the correct answer choice (or multiple correct choices, depending on the objective of the question), we consulted the literature and student work to come up with with choices that reflect the *ways* students struggle with the skill/ difficulty in question. In general, physics education research articles that document student difficulties come with examples of particular patterns of incorrect applications of these skills. Our idea was to utilize these observable behaviors so that students not only need to correctly apply the skills, but also potentially recognize the incorrectness of other possibilities.

Below, we give a few examples on how questions for each skill were written. For the sake of space, we cannot do each of the skills. The purpose of this section is to provide the reader with wide perspective on the kinds of considerations taken into account when writing the questions.

3.2.1 Skill 1: Adding Time Dependence to Wave Functions

One of the essential skills mentioned above, and one that has had a large amount of research into student conceptual understanding, is that of time dependence. Specifically, we're concerned with how students add time dependence to wave functions and states.

First, we designed a question. Since we're concerned with targeting a single skill, namely adding time dependence, our question should contain only information related to this skill (and potentially any foundational information needed to set this up):

> Let H be an arbitrary Hamiltonian with energy eigenstates $\psi_n(x)$ with energies E_n. Let $\Psi(x,0) = \frac{1}{2}\psi_1(x) - \sqrt{\frac{3}{4}}\psi_5(x)$. What is $\Psi(x,t)$?

The state chosen for this problem was arbitrary, but has a few key characteristics: it is a superposition, so that each eigenstate needs its own individual phase (as was be mentioned above, a common difficulty for students to recognize). Further, the superposition is explicit, making the addition of the time dependent phases less involved than a nontrivial superposition. Next, we need to include an answer choice corresponding to the correct answer:

> Let H be an arbitrary Hamiltonian with energy eigenstates $\psi_n(x)$ with energies E_n. Let $\Psi(x,0) = \frac{1}{2}\psi_1(x) - \sqrt{\frac{3}{4}}\psi_5(x)$. What is $\Psi(x,t)$?
>
> ○ $\Psi(x,t) = \frac{1}{2}\psi_1(x)e^{-\frac{i}{\hbar}E_1 t} - \sqrt{\frac{3}{4}}\psi_5(x)e^{-\frac{i}{\hbar}E_5 t}$

For the remaining answer choices, we look to the literature and student responses and use the various observed ways students struggle with adding time dependence to wave functions.

For example, as mentioned above, [17] provides a detailed set of student behaviors associated with writing $\Psi(x,t)$ from $\Psi(x,0)$. We mention some below:

- Placing a single phase in front of $\Psi(x,0)$. Emigh *et al.* report that when asked to generate the time evolving wave function for the state described by $\frac{1}{\sqrt{2}}(\psi_1+\psi_2)$, students often wrote $\frac{1}{\sqrt{2}}e^{-iEt/\hbar}(\psi_1+\psi_2)$.

  In this work, they note that, for students, E has served the role as simply an arbitrary energy, or what is believed to be the "actual" energy of the system. This pattern has been observed in other works as well [112, 34].

- Instead of phases, using decaying exponentials. Emigh *et al.* also note that students will often, in words and in mathematics, utilize exponential decay instead of phases for time dependence. They provide an example from student interview data: "Since the wave equation will gain a $e^{E_2t/\hbar}$ term to represent its evolution as time goes on, the probability of finding the particle in the marked area will decreases [...] since the square of its wave equation will decreases as well".

These two observations suggest a form for two other answer choices: one where the time dependence is all in one overall phase (either taking the form of an arbitrary energy or the "real" energy), and one where the time dependence takes the form of decaying instead of complex exponentials. Accommodating these two possibilities, we obtain

Let H be an arbitrary Hamiltonian with energy eigenstates $\psi_n(x)$ with energies E_n. Let $\Psi(x,0)=\frac{1}{2}\psi_1(x)-\sqrt{\frac{3}{4}}\psi_5(x)$. What is $\Psi(x,t)$?

○ $\Psi(x,t)=\frac{1}{2}\psi_1(x)e^{-\frac{i}{\hbar}E_1t}-\sqrt{\frac{3}{4}}\psi_5(x)e^{-\frac{i}{\hbar}E_5t}$

○ $\Psi(x,t)=e^{-\frac{i}{\hbar}Et}(\frac{1}{2}\psi_1(x)-\sqrt{\frac{3}{4}}\psi_5(x))$ where E is the actual energy of our system.

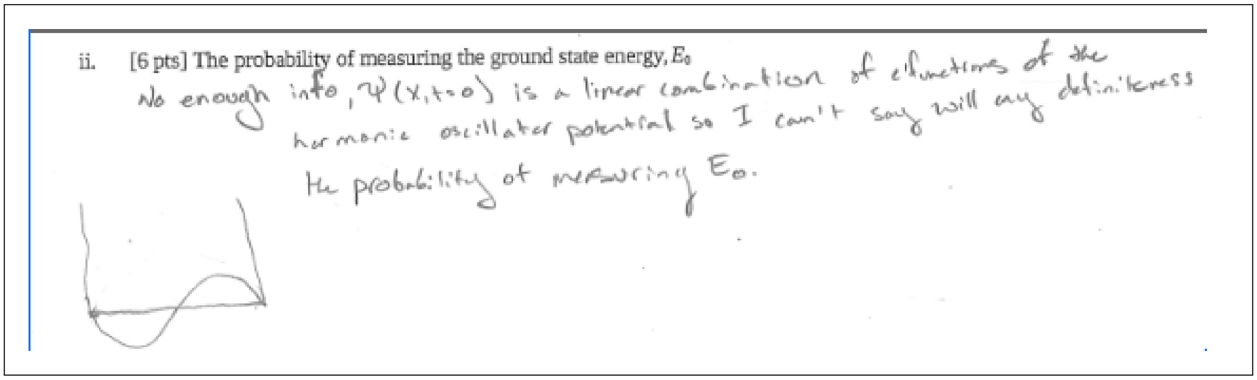
○ $\Psi(x,t)=\frac{1}{2}\psi_1(x)e^{-\frac{1}{\hbar}E_1t}-\sqrt{\frac{3}{4}}\psi_5(x)e^{-\frac{1}{\hbar}E_5t}$

In addition to these observed student behaviors documented in the literature (by no means exhaustive), we bring in anecdotal observations of student responses to our own exam questions. As an example, consider the second question asked above in Figure 3.1, discussed already in detail.

In this question, there were a variety of common incorrect answers, including ones already mentioned above. Other responses patterns were present though:

- A small handful of students answered the question claiming that you add time evolution to $\Psi(x, 0)$ by acting on it with the Hamiltonian H: $H\Psi(x, 0) = \Psi(x, t)$. An example response is given in Figure 3.3.

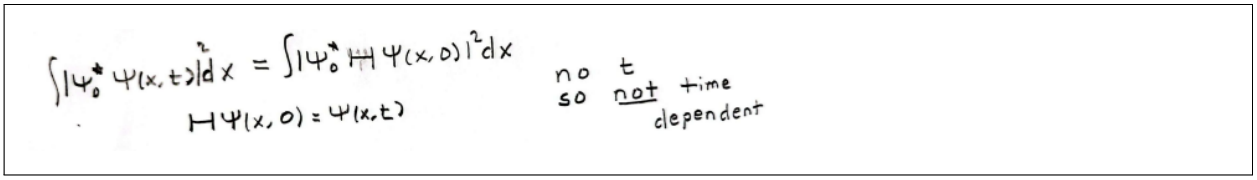
- A few students commented that they cannot say for various reasons, including "we can't be certain about its time evolution", and "it will be an unknown superposition of eigenstates". An example response is given in Figure 3.2



ii. [6 pts] The probability of measuring the ground state energy, E_0

No enough info, $\Psi(x, t=0)$ is a linear combination of e'functions of the harmonic oscillator potential so I can't say will any definiteness the probability of measuring E_0.

Figure 3.2: This student response comments on how the state provided to them is in a superposition of energy eigenstates, and thus they cannot say what the probability of measuring the energy to be the ground state is. Implicit in such a response is that they cannot assess the time dependence of this expression without knowing the state's energy eigenbasis expansion.



$\int |\psi_0^* \psi(x,t)|^2 dx = \int |\psi_0^* H \psi(x,0)|^2 dx$

$H\psi(x,0) = \psi(x,t)$

no t
so not time dependent

Figure 3.3: This student response involves assessing the time dependence of the probability of measuring the ground state. Although the expression they started with was incorrect (the norm-square is on the inside as opposed to the outside), the key feature of this question relevant to this situation is there incorrect assignation of time dependence by acting on the state with the Hamiltonian as opposed to the usual time evolution operator $e^{-iHt/\hbar}$.

While this is not exhaustive of all the ways students struggle with writing fully time evolving states, it gives us a sense as to what students may look for when doing so. As a result, one can create answer choices for this question accommodating for these patterns:

Let H be an arbitrary Hamiltonian with energy eigenstates $\psi_n(x)$ with energies E_n. Let $\Psi(x,0) = \frac{1}{2}\psi_1(x) - \sqrt{\frac{3}{4}}\psi_5(x)$. What is $\Psi(x,t)$?

- ○ $\Psi(x,t) = \frac{1}{2}\psi_1(x)e^{-\frac{i}{\hbar}E_1 t} - \sqrt{\frac{3}{4}}\psi_5(x)e^{-\frac{i}{\hbar}E_5 t}$
- ○ $\Psi(x,t) = \frac{1}{2}\psi_1(x)e^{-\frac{1}{\hbar}E_1 t} - \sqrt{\frac{3}{4}}\psi_5(x)e^{-\frac{1}{\hbar}E_5 t}$
- ○ $\Psi(x,t) = H\Psi(x,0)$
- ○ It is impossible to say: the state will be an arbitrary superposition of eigenfunctions of H.
- ○ $\Psi(x,t) = e^{-\frac{i}{\hbar}Et}(\frac{1}{2}\psi_1(x) - \sqrt{\frac{3}{4}}\psi_5(x))$ where E is the actual energy of our system.

To create more questions involving the practice of adding time dependence, now, we can utilize different student behaviors present in the literature, or simply change $\Psi(x,0)$ to a different initial state. This gives us a handful of questions that students can use to practice that target the skill they are practicing but not repeat the same exact question in doing this practice.

3.2.2 Skill 2: Identifying a Probability Expression

For a second example of how questions were constructed, we consider the essential skill of identifying expressions corresponding to probability given a verbal prompt to do so. In other words, students are asked to identify the general expression corresponding to a particular probability measurement: they are told, in words, what they are looking for, and then they must identify the corresponding mathematical expression. Being able to generate mathematical expressions corresponding to concepts is important in quantum mechanics, so we thought that practicing this skill may be beneficial for students.

As above, we start with a template question and its correct answers:

Consider a particle in a state $|\psi\rangle$ with Hamiltonian H, whose energies eigenvalues are E_n with corresponding energy eigenstates $|\phi_n\rangle$. Let $\psi(x) = \langle x|\psi\rangle$ be the position-space wave function for $|\psi\rangle$.

What expressions correspond to the probability of measuring the energy of our particle to be E_4? Select all that apply.

- ☐ $|\langle\phi_4|\psi\rangle|^2$
- ☐ $|\int_{-\infty}^{\infty} \phi_4^*(x)\psi(x)\mathrm{d}x|^2$

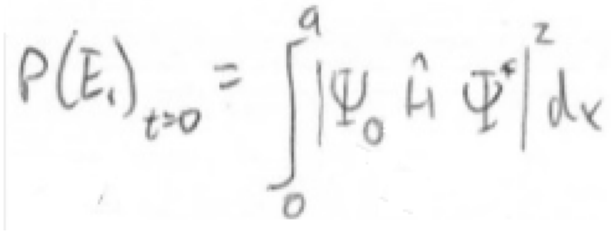
In this situation, we have opted to use both Dirac and wave function notation in the question so that students must practice connecting the notations alongside each other. Other questions could use just one notation, but, as [25] notes, students view different notations as serving different purposes. As such, helping students work with the different notations side by side could help build more robust understanding of the notations at play.

For the first possible answer choice, we deferred again to the literature. Zhu *et al.* notes that students often write expectation value expressions when solving for probabilities [140]. This matches what we observed in student responses as well. We had the option to use the Dirac notation expression or the wave function notation expression for this answer choice (or both). For this example, we simply chose the Dirac expression, yielding

Consider a particle in a state $|\psi\rangle$ with Hamiltonian H, whose energies eigenvalues are E_n with corresponding energy eigenstates $|\phi_n\rangle$. Let $\psi(x) = \langle x|\psi\rangle$ be the position-space wave function for $|\psi\rangle$.

What expressions correspond to the probability of measuring the energy of our particle to be E_4? Select all that apply.

- ☐ $|\langle\phi_4|\psi\rangle|^2$
- ☐ $|\int_{-\infty}^{\infty}\phi_4^*(x)\psi(x)\mathrm{d}x|^2$
- ☐ $\langle\psi|H|\psi\rangle$



$$P(E_1)_{t=0} = \int_0^a |\Psi_0 \hat{H} \Psi^*|^2 dx$$

Figure 3.4: This student response is simply an expression a student wrote down for the probability of measuring the ground state energy. It is all the student wrote for this response. It mixes some features of probability expressions (the presence of the $\psi_0(x)$ and the state), and expectation values H being between the $\psi_0(x)$ and the state).

For the other answer choices, we will rely on empirically observed student responses. Once again, consider the first question asked above in Figure 3.1. Students were tasked with producing an expression for a particular probability, much like the question asked of students that we are writing. While many students executed procedures to identify simplified expressions (namely, coefficients, or solving for the expression instead of producing it from memory), a considerable number of students responded by producing expressions for this probability from memory. An example of an incorrect response is depicted in Figure 3.4.

This answer choice seems to be a mixture of an expectation value and a probability: the Hamiltonian operator is between a bra and a ket, but the bra is the eigenstate corresponding to the energy we're interested in. Since this response was observed a nontrivial amount of times, it can be a good incorrect answer choice for the above question, yielding

> Consider a particle in a state $|\psi\rangle$ with Hamiltonian H, whose energies eigenvalues are E_n with corresponding energy eigenstates $|\phi_n\rangle$. Let $\psi(x) = \langle x|\psi\rangle$ be the position-space wave function for $|\psi\rangle$.
>
> What expressions correspond to the probability of measuring the energy of our particle to be E_4? Select all that apply.
>
> - ☐ $|\langle\phi_4|\psi\rangle|^2$
> - ☐ $|\int_{-\infty}^{\infty} \phi_4^*(x)\psi(x)\mathrm{d}x|^2$
> - ☐ $\langle\psi|H|\psi\rangle$
> - ☐ $\langle\phi_4|H|\psi\rangle$

It was also *very* common for students to respond by placing the norm-square on the integrand and not outside of the integral, as shown in Figure 3.6. When responding with wave function notation, this was one of the most common errors students made. It was so common that this answer became a common incorrect answer choice for questions over this skill. This gave a final question of

Consider a particle in a state $|\psi\rangle$ with Hamiltonian H, whose energies eigenvalues are E_n with corresponding energy eigenstates $|\phi_n\rangle$. Let $\psi(x) = \langle x|\psi\rangle$ be the position-space wave function for $|\psi\rangle$.

What expressions correspond to the probability of measuring the energy of our particle to be E_4? Select all that apply.

☐ $|\langle\phi_4|\psi\rangle|^2$
☐ $|\int_{-\infty}^{\infty} \phi_4^*(x)\psi(x)\mathrm{d}x|^2$
☐ $\langle\psi|H|\psi\rangle$
☐ $\langle\phi_4|H|\psi\rangle$
☐ $\int_{-\infty}^{\infty} |\psi_4^*(x)\psi(x)|^2\mathrm{d}x$

To create more questions of this form, one could focus on just using Dirac or wave function notation, have the correct answer involve only one of the notations, use various other common conceptual and mathematical errors students make, and use different operators such as position or momentum. Since probabilities are a very common quantity solved for in quantum mechanics, there was ample literature and empirical observations to utilize in designing questions for this skill.

3.2.3 Skill 3: Expanding a State in a Given Basis

As a final example of how questions were constructed, we consider the essential skill of "Expanding a State in an Eigenbasis". This skill is obviously important considering that state vectors are just that: vectors. Further, coefficients in eigenbasis expansions are associated with probabilities. As such, having students recognize how expanding a state in a basis works and what it says about a state is imperative.

The following Figure depicts one way one can write a question specifically asking students to expand an arbitrary state in a particular eigenbasis, as well as its associated correct answer.

Let $|\psi\rangle$ be the state of a particle. Let H be a Hamiltonian with energy eigenstates $|\psi\rangle$ with corresponding energy eigenvalues E_n. Finally, recall that the position-space wave functions for our state and the energy eigenstates are $\psi(x) = \langle x|\psi\rangle$ and $\psi_n(x) = \langle x|\psi_n\rangle$, respectively.

Which of the following corresponds to, generally, $|\psi\rangle$ and/or $\psi(x)$ expanded in the energy eigenbasis? Select all that apply.

○ $\psi(x) = c_1\psi_1(x) + c_2\psi_2(x) + ...$, where each c_n is complex.

Once again, we look to the literature and observed student responses to determine other answers. As a first example, we consider the work [109], which provides several examples of observed student responses and difficulties. One of the questions asked of students was to write for the expression for the expectation value of a generic operator Q in a general state $|\phi\rangle$. They are given the eigen-information of Q as $Q|\psi_i\rangle = \lambda_i|\psi_i\rangle$. Here are a few difficulties of note:

- A common mistake noted in this work is for students to expand states in the eigenbasis for an operator setting all of the coefficients to 1 (or not putting any coefficients). The article claims that 9% of the students began the problem by expanding the state as $|\phi\rangle = \sum_i |\psi_i\rangle$. They are either forgetting the coefficients in this expansion, or are setting them equal to 1 intentionally. This is incorrect, whereas the correct expansion takes the form $|\phi\rangle = \sum_i c_i|\psi_i\rangle$, where $c_i = \langle\psi_i|\phi\rangle$ is a complex number.

- In an interview, a student began a problem by expanding the state given to them in the eigenbasis for this operator using the eigenvalues for each eigenstate as the corresponding coefficient. To quote the article, "one student said that '*the eigenvalue gives the probability of getting a particular eigenstate*' and expanded the state as '$|\phi\rangle = \sum_i \lambda_i|\psi_i\rangle$.'". Here, $|\psi_i\rangle$ and λ_i are the eigenstates and eigenvalues for the operator Q. Again, the correct expansion takes the form $|\phi\rangle = \sum_i c_i|\psi_i\rangle$, where $c_i = \langle\psi_i|\phi\rangle$.

Using these, we can design two new incorrect answer choices:

Let $\psi(x)$ be the state of a particle. Let H be a Hamiltonian with energy eigenstates $\psi_n(x)$ with corresponding energy eigenvalues E_n.

Which of the following corresponds to, generally, $\psi(x)$ expanded in the energy eigenbasis? Select all that apply.

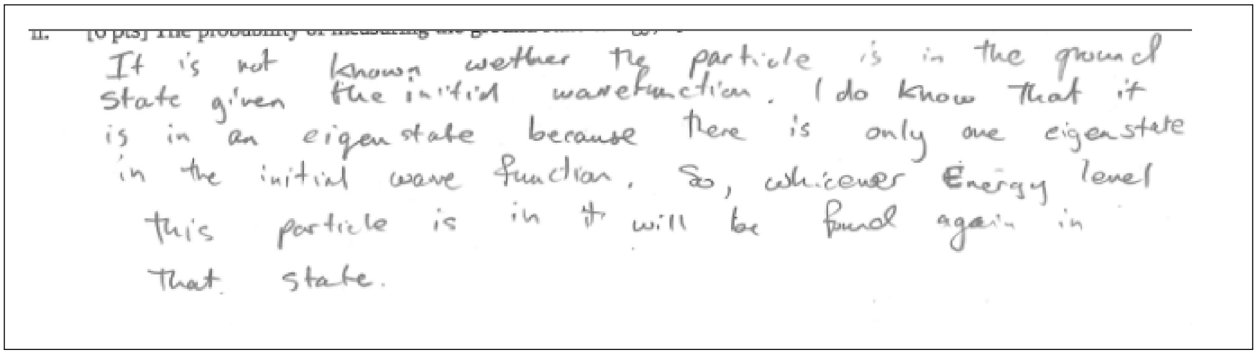
- ○ $\psi(x) = c_1\psi_1(x) + c_2\psi_2(x) + ...$, where each c_n is complex.
- ○ $\psi(x) = \psi_1(x) + \psi_2(x) + \psi_3(x) + ...$
- ○ $\psi(x) = E_1\psi_1(x) + E_2\psi_2(x) + ...$

As a final incorrect answer choice, we use an observed student response pattern from an exam question. Figure 3.1 mentioned that many students tried to use a mathematical procedure to identify a correct expression for the probability of a particular energy measurement. To do so, many students discussed the nontrivial superposition's energy eigenbasis expansion, and one student response is depicted in Figure 3.5. Some students said that the state *was* the energy we wished to measure with a particular coefficient (potentially the original coefficient) in front. Using this as a final incorrect answer choice, we obtain

Let $\psi(x)$ be the state of a particle. Let H be a Hamiltonian with energy eigenstates $\psi_n(x)$ with corresponding energy eigenvalues E_n.

Which of the following corresponds to, generally, $\psi(x)$ expanded in the energy eigenbasis? Select all that apply.

- ○ $\psi(x) = c_1\psi_1(x) + c_2\psi_2(x) + ...$, where each c_n is complex.
- ○ $\psi(x) = \psi_1(x) + \psi_2(x) + \psi_3(x) + ...$
- ○ $\psi(x) = E_1\psi_1(x) + E_2\psi_2(x) + ...$
- ○ $\psi(x) = \psi_n(x)$, where $\psi_n(x)$ is the actual state of the particle.



It is not known wether the particle is in the ground state given the initial wavefunction. I do know That it is in an eigenstate because there is only one eigenstate in the initial wave function. So, whichever Energy level this particle is in it will be found again in That state.

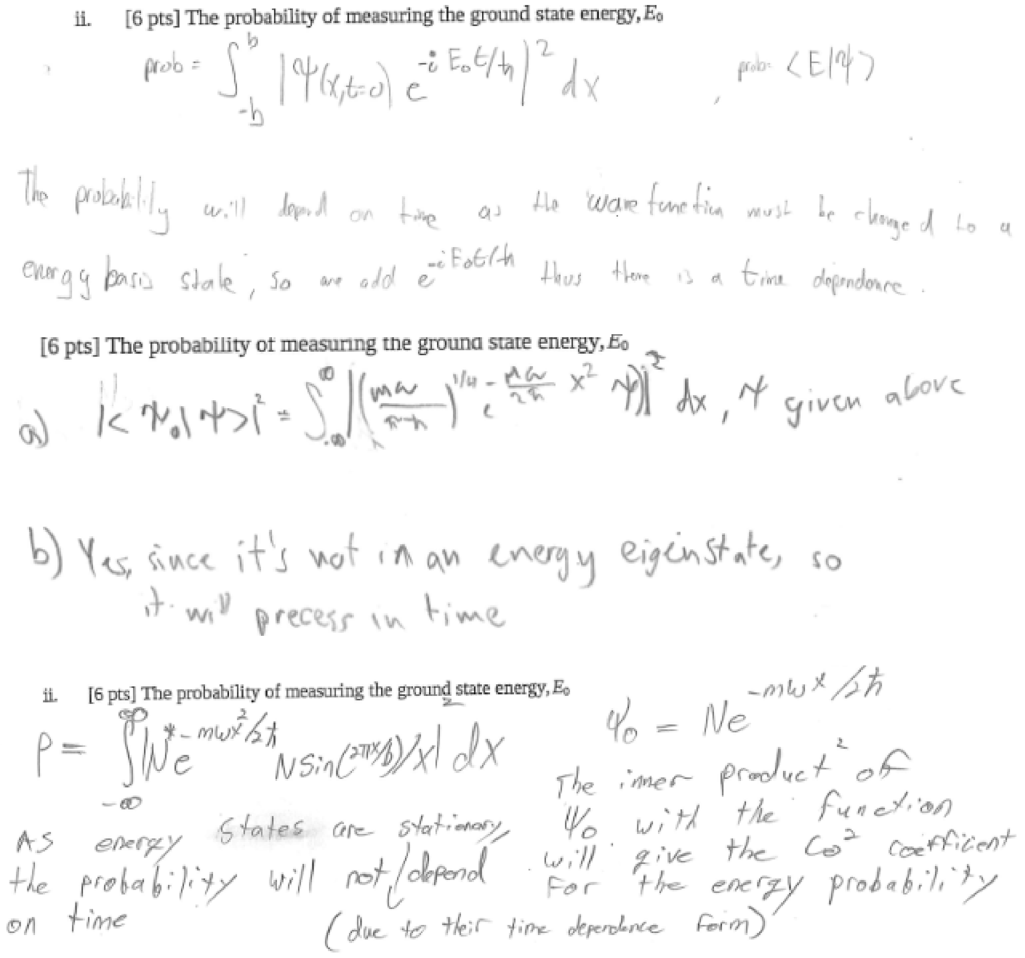
Figure 3.5: This student mentions that although the state is unknown, it *is* an eigenstate and thus whichever eigenstate the particle is in describes the state before and after a measurement is made. While the student does not use mathematics to describe this statement, it is equivalent to the statement that $\psi(x) = \psi_n(x)$, where $\psi_n(x)$ is the actual state of the particle.

To make questions similar to this, we can use arbitrary bases, different operators' eigenbases, and use Dirac notation instead of wave function notation (since, again, according to Gire *et al*, students think about the different notations in different ways, so giving students practice with multiple notations could likely be beneficial [25]).

3.3 Summary

To recap the procedure thus far: first, we identified potential skills using known student difficulties in the literature and our prior experiences as instructors. Second, we used our knowledge of the ways students struggled with these ideas, again from either the literature or our own experiences (say, with student responses on exams) to write questions designed to help students correctly practice skills they are known to struggle with and identify common incorrect applications of these skills. We created a set of over 1500 questions, many of them isomorphic to each other. The next step was to place these questions into a system that can be used to administer the questions in a sensible way to students. We used the STEM

Fluency system for this purpose. In the next chapter, we will discuss the STEM Fluency system we used to administer the practice, how we organized the questions we wrote into the system, and considerations we took when designing the homework assignments.



ii. [6 pts] The probability of measuring the ground state energy, E_0

prob = $\int_{-b}^{b} |\Psi(x,t=0)\, e^{-iE_0t/\hbar}|^2\, dx$, prob: $\langle E|\psi\rangle$

The probability will depend on time as the wave function must be changed to a energy basis state, so we add $e^{-iE_0t/\hbar}$ thus there is a time dependence.

[6 pts] The probability of measuring the ground state energy, E_0

a) $|\langle\psi_0|\psi\rangle|^2 = \int_{-\infty}^{\infty} \left|\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}\psi\right|^2 dx$, ψ given above

b) Yes, since it's not in an energy eigenstate, so it will precess in time

ii. [6 pts] The probability of measuring the ground state energy, E_0

$P = \int_{-\infty}^{\infty} |N^* e^{-m\omega x^2/2\hbar} N\sin(2\pi x/b)/x|\, dx$

$\psi_0 = Ne^{-m\omega x/2\hbar}$

The inner product^2 of ψ_0 with the function will give the c_0^2 coefficient for the energy probability

As energy states are stationary, the probability will not depend on time

(due to their time dependence form)

Figure 3.6: These student responses involve writing an expression for the probability of a particular energy measurement, and each response places the norm-square required by an energy probability expression on the inside of the integral as opposed to the outside. Three responses are included here to emphasize how common this response pattern is.

Chapter 4

STEM FLUENCY AND BUILDING THE PRACTICE

In the previous chapter, we outlined how we identified potential essential skills and how we wrote practice questions for them. Once we had the questions, we then wanted to administer these questions as practice to students. But we wanted to administer these questions in a way that strongly promoted student learning. Recall that the Essential Skills Framework (ESF) outlines a set of research-validated techniques that have been shown to be effective at helping students develop fluency. The online STEM Fluency platform is an homework system that has been shown to be consistent with the ESF, and we chose it for administering practice to students in our courses.

In this chapter we will introduce and discuss the intricacies of STEM Fluency and how we used it in order to organize the questions written for student practice. We begin by discussing STEM Fluency as an online homework-platform, presenting the features that it has that make it good software with the ESF in mind. Then, we will talk about how the structure of STEM Fluency led us to a particular organization for the questions written. Here, we present a partial overview. For a full outline of the organization of these questions, see Appendix B. Next, we will discuss the general rules of thumb we used to organize the questions we wrote into STEM Fluency in accordance to this structure.

4.1 Overview of STEM Fluency

STEM Fluency is an online, mastery-based system for organizing questions into structured homework assignments and administering them to students. Questions are incorporated into STEM Fluency's organizational hierarchy, which consists of subjects, topics, categories, and subcategories. Figure 4.1 contains a visual depiction of this organizational hierarchy for a single topic. Questions fall into various subcategories, and similar subcategories are organized into categories. Similar categories are then organized into topics, and topics into subjects.

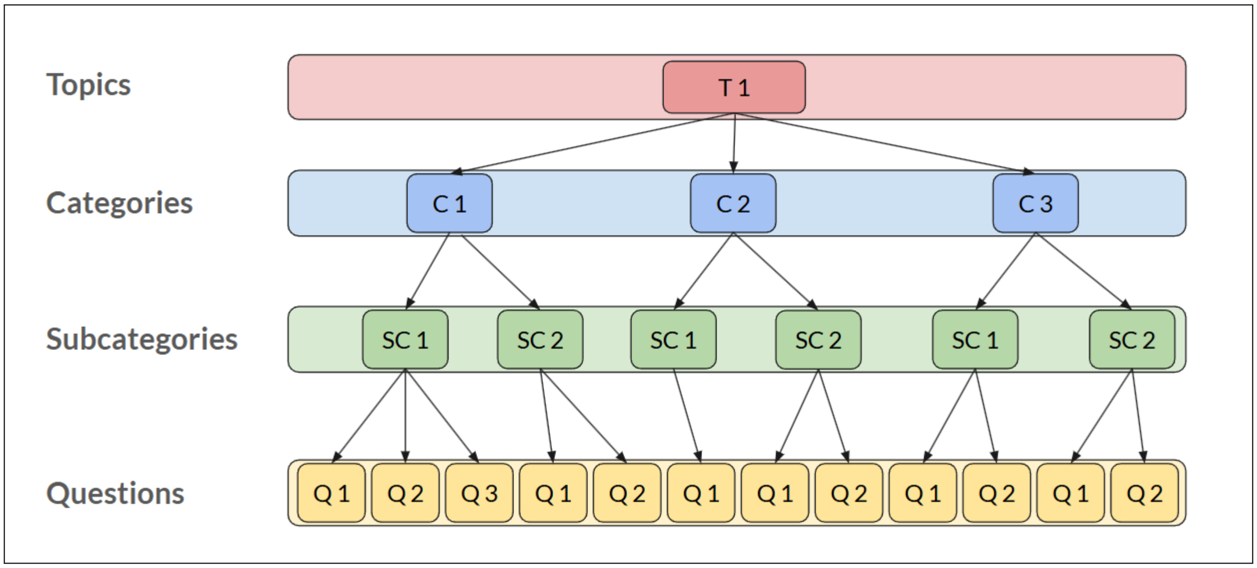


Figure 4.1: This figure gives an example of how the hierarchy is structured within STEM Fluency. For space constraints, Topics are abbreviated to "T", Categories to "C", Subcategories to "SC", and Questions to "Q". Each topic contains multiple categories. Each category in turn contains multiple subcategories. Finally, each subcategory contains each question corresponding to the skill associated with the respective subcategory. Roughly, each subcategory corresponds to a skill identified in Section 3.1 above.

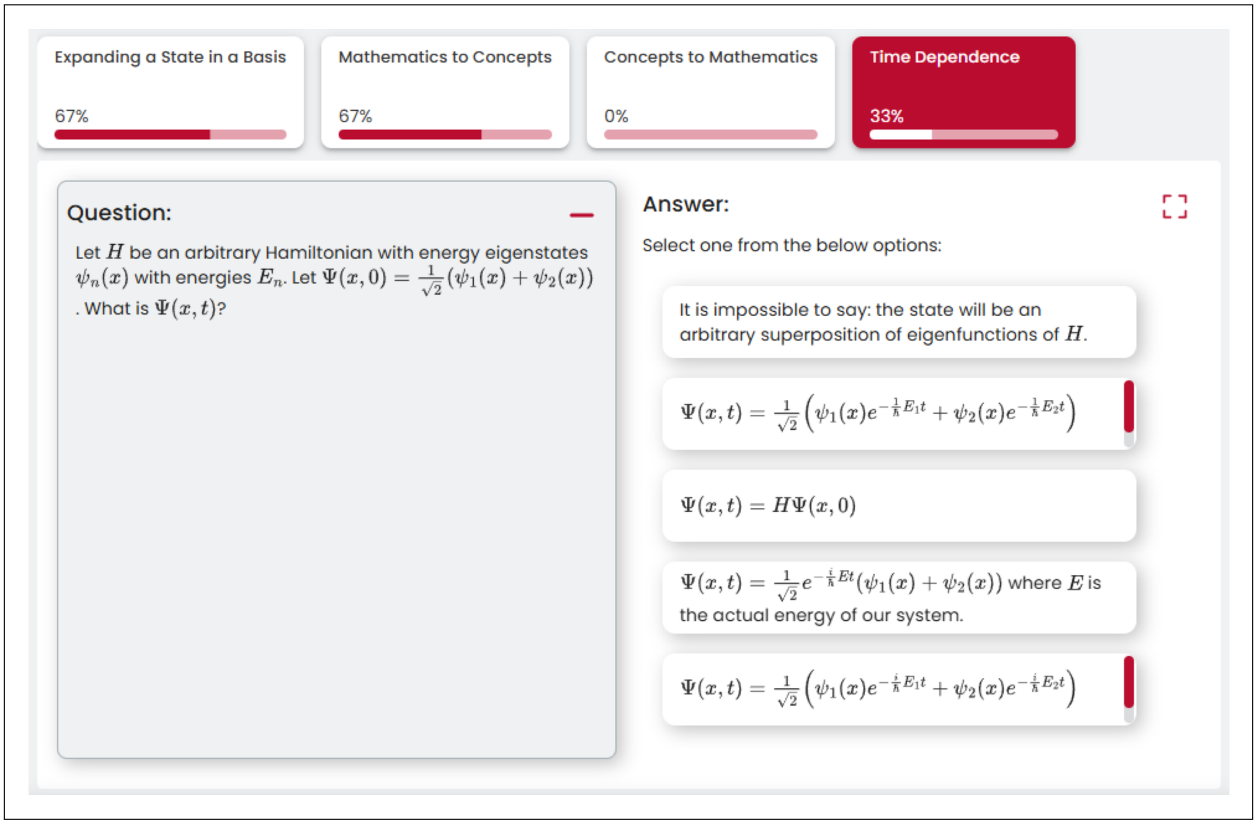
When constructing a homework assignment, instructors select a handful of categories (usually between 3 and 5) and then a set of subcategories within them. Then, a number is associated to each category called its "mastery" (also usually between 3 and 5). STEM Fluency then automatically assembles a sequences of questions belonging to the categories and associated subcategories selected into an assignment for students to complete.

See Figure 4.2 for an example of the student view of a particular question in an assignment. Note that Figure 4.2 contains the names of categories that we created that will be discussed in a later section. The body of the question and its associated answer choices take up the center of the figure. The answer choices can be multiple-choice-single-select, multiple-choice-multiple-select, short answer, or a variety of different answer styles. We only opted to use the first two so that analysis of the practice would be streamlined.

Above the question are the categories selected for the assignment together with meters showing progress for each of these categories. If a student shown the question in Figure 4.2 answers correctly, then the meter for the category to which the question belongs increments up by a percentage corresponding to $\frac{1}{\text{Mastery}}$ for that category. The student is then given a question from another (randomly determined) category in the list above, as depicted in Figure 4.3. If a student answers the question incorrectly, though, the meter for the category containing the question resets to 0.

Notice that in Figure 4.2, the question belongs to the category Time Dependence, involving adding time dependence to a state. Answering this question correctly would increment the Time Dependence meter, feeding students another question from one of the categories in this assignment. This is depicted in Figure 4.3. This next question belongs to the category Concepts to Mathematics, as it gives students a verbal description of a category and asks them to produce a mathematical expression for this quantity.

Once the meter for a particular category reaches 100%, the student achieves “mastery” in that category and will no longer see questions from that category for the rest of the assignment. An assignment is completed when all categories selected for that assignment have reached the mastery level. Since the meter for a category resets if a student answers a question from that category incorrectly, students must answer a certain number of questions from that category correctly in a row (not back to back, but as they are given to the students) in order to achieve the desired mastery for that category.



Question:

Let H be an arbitrary Hamiltonian with energy eigenstates $\psi_n(x)$ with energies E_n. Let $\Psi(x,0) = \frac{1}{\sqrt{2}}(\psi_1(x) + \psi_2(x))$. What is $\Psi(x,t)$?

Answer:

Select one from the below options:

It is impossible to say: the state will be an arbitrary superposition of eigenfunctions of H.

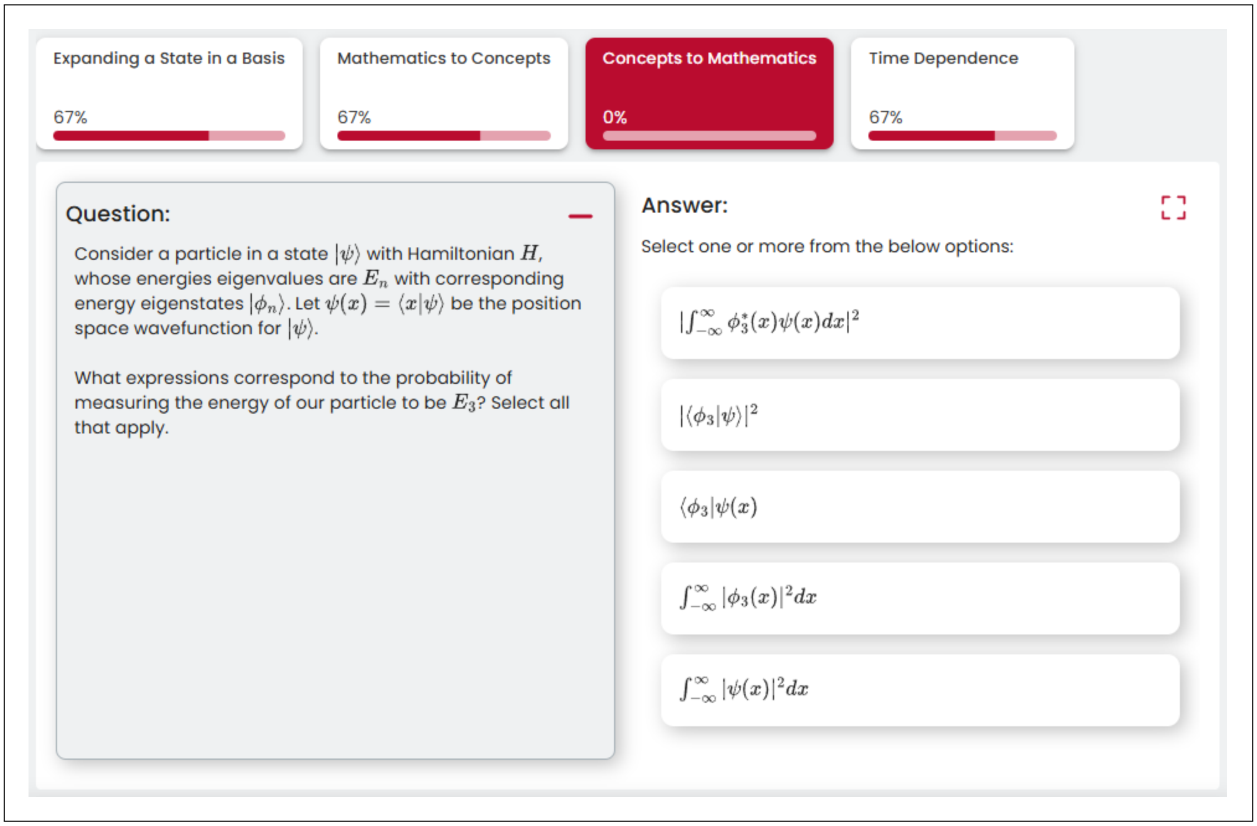
$\Psi(x,t) = \frac{1}{\sqrt{2}}\left(\psi_1(x)e^{-\frac{1}{\hbar}E_1 t} + \psi_2(x)e^{-\frac{1}{\hbar}E_2 t}\right)$

$\Psi(x,t) = H\Psi(x,0)$

$\Psi(x,t) = \frac{1}{\sqrt{2}}e^{-\frac{i}{\hbar}Et}(\psi_1(x) + \psi_2(x))$ where E is the actual energy of our system.

$\Psi(x,t) = \frac{1}{\sqrt{2}}\left(\psi_1(x)e^{-\frac{i}{\hbar}E_1 t} + \psi_2(x)e^{-\frac{i}{\hbar}E_2 t}\right)$

Figure 4.2: Example of the student view during a STEM Fluency assignment. Students are presented a question and answer choices for said question. The top outlines a list of categories that make up classes of questions that the assignment contains. A meter associated with each skill depicting their progress at “mastering” the skills for this assignment is also given. The current category of questions is highlighted.



Question:

Consider a particle in a state $|\psi\rangle$ with Hamiltonian H, whose energies eigenvalues are E_n with corresponding energy eigenstates $|\phi_n\rangle$. Let $\psi(x) = \langle x|\psi\rangle$ be the position space wavefunction for $|\psi\rangle$.

What expressions correspond to the probability of measuring the energy of our particle to be E_3? Select all that apply.

Answer:

Select one or more from the below options:

$|\int_{-\infty}^{\infty} \phi_3^*(x)\psi(x)dx|^2$

$|\langle\phi_3|\psi\rangle|^2$

$\langle\phi_3|\psi(x)$

$\int_{-\infty}^{\infty} |\phi_3(x)|^2 dx$

$\int_{-\infty}^{\infty} |\psi(x)|^2 dx$

Figure 4.3: After having answered the question in Figure 4.2 correctly, the meter for that category increases, and students are given questions from another category (highlighted). This category is selected randomly.

Notice how STEM Fluency's design encapsulates many of the important components of the design philosophy of the ESF. First, it is computer-based, which allows students to complete the practice on their own time and receive real-time feedback. In addition, STEM Fluency provides more practice to students who need it (namely, mastery-based practice) by resetting the mastery meters for each category when students get a question wrong. As a third example, the practice is distributed and interleaved: students practice multiple skills at once (evident by the multiple categories present in the assignment), and once a student answers a question, the next question is selected from a random category. As a result, students need to correctly execute the skills they're practicing, *and* identify which skill it is that they are practicing.

Here, it's important to emphasize the difference between the ESF and STEM Fluency. The ESF is a framework to follow in order to build practice, providing guidelines for things to consider when providing the practice. STEM Fluency, on the other hand, is a platform to implement this practice with features that satisfy the guidelines of the ESF. These are not the same things: not all practice put into STEM Fluency is necessarily built using the ESF, and practice built using the ESF need not be administered through STEM Fluency.

4.2 Organizing the Questions into STEM Fluency

Now that we've discussed how STEM Fluency works as an organizational structure and an online homework system, we continue by discussing our logic when organizing our questions into STEM Fluency's hierarchy. For the sake of simplicity, we will be working in a single **subject** which we will call "Junior-Level Quantum Mechanics". This is because when setting up the organization this was the space built for us. But, overall, this was arbitrary, and it simply matches the logic of other subjects already within the STEM Fluency system. Starting here, we will work from the bottom up. Looking at each of the skills (partially) outlined in Table 3.3, a few trends begin to emerge, as outlined below.

First, we chose to identify each skill with a subcategory of some kind. In particular, we collected each question associated with a particular skill into a subcategory corresponding to the skill in question. This is because at the finest grain we can chose subcategories for students to practice for each assignment. If we fit multiple skills into a subcategory, then students run the risk of not being able to practice a skill. Further, if we spread a skill across multiple subcategories, then we lose space at higher levels of the organizational structure (namely, the topics and categories). As such, it seemed natural to us to associated skills with subcategories, so that if we decided to give students practice in a particular skill, it amounts to selecting the associated subcategory in the assignment.

Our objective of organization the questions into STEM Fluency, then, amounts to grouping like subcategories into categories, and then like categories into topics. While there is no one definite way of doing this, we outline below three examples of our thinking that led us to a final organizational structure.

- First, notice that many skills we identified correspond to translations of some kind. This includes translating between different notations (wave function to Dirac, Dirac to wave function), but also translating between mathematical expressions and verbal descriptions or interpretations of these quantities (producing probability, expectation value, or eigenvalue equation expressions, or interpreting expectation values, for example).

  We decided that it may be fruitful to designate all skills involving translations into a topic called "Translating Representations". What we then needed to do was design a set of categories and fit our subcategories (namely, our skills) into these categories sensibly. One way of doing this (and the way we ultimately ended up using) is the following: we chose the categories to be the particular *kind* of translation occurring (*i.e.*, concept-to-mathematics, translating notations), and the subcategories to be specific instances of these translations corresponding to particular skills. As an example, in the Concept to Mathematics category, we can organize the subcategories Expectation Values: Concept to Mathematics, Probabilities: Concept to Mathematics, and so on. In addition, in a category titled Translating Dirac Notation to Wave Function Notation, we can place all of the subcategories involving translating moving from Dirac notation to wave function notation, and *vice versa*.

  What emerged was an organization of all skills involving translations of some kind. The final structure of this topic is depicted in Table B in Appendix B. We omitted discussion of other categories, but similar logic was used in constructing and assembling them. Once again, the skills here are all related by nature of the fact that they involve the same objective: *translating* various mathematical objects between different representations.

- Next, notice that many of these skills are mathematical. Mathematical skills involve things like expanding states in bases, using recognizing instances of orthonormality, and recognizing *when* to use orthonormality. There are a wide variety of skills involving the use and execution of mathematics. As such, we considered combining all mathematical skills into a topic. That being said, we agreed that there was a difference in how the skills are being *used*. For example, expanding a state in an eigenbasis when you are told to is different from recognizing when orthonormality can be applied: the first is a procedure to be executed, while the other relies on students recognizing whether or not a particular mathematical step can be executed (or if it would be useful). As such, it seems that grouping these skills together under the same topic may be too broad. We decided that they may best be split into two topics, one including mathematical procedures, and the other including practice priming various mathematical decision making strategies.

  We called the topic containing skills involving mathematical procedures "Mathematical Procedures and Linear Algebra" (not to suggest these are mutually exclusive but simply to emphasize the role that linear algebra plays in this skill set). From here, the classification was rather simple: many skills required practicing complex conjugation of various mathematical objects (complex numbers and variables, bras and kets, *etc.*), and so these fit into a category called Complex Conjugation. Many skills involve working with and expanding states in various bases, and so these classes of skills fit into a category Expanding a State in a Basis. Table B in Appendix B contains a full list of the categories and their subcategories.

  The topic containing skills related to mathematical decision making we decided to call "Mathematization". Various skills we identified, for example, choosing certain mathematical objects to use in probability calculations and thus we grouped into a category called Choosing a Bra in a Probability Calculation. Some skills involved choosing which notation one might use during a particular calculation, which we decided to group into the Choosing Which Notation category. All categories and their associated subcategories are outlined in Table B in Appendix B.

Notice that all together, both the “Mathematical Procedures and Linear Algebra” topic and the “Mathematization” topic make up a total of 10 categories. While there is nothing wrong with that many categories being a part of one topic, we found that dividing “procedural” mathematical skills and “decision-based” mathematical skills helped us think about these skills differently when analyzing data on student performance on these skills.

- Finally, we noticed that a lot of the skills we identified related to the particular physics concept of angular momentum. Further, many of these skills were related to each other in various ways. For example, many skills were related to the use and reading of Clebsch-Gordan tables. Many skills were also related to determining the probability of particular measurements in the context of multiple angular momenta. This motivated us to construct the topic “Angular Momentum”, and group these like skills into categories. The Clebsch Gordan skills were placed in a category called Addition of Angular Momentum - Change Bases, as an example. Note here that the skills for this topic were grouped on the basis of their similar physics content. This approach was also taken when constructing the topics “Perturbation Theory” and “Multiple Particle and Degeneracy”.

The examples presented above are not exhaustive, but consists of various considerations taken when organizing our practice questions into STEM Fluency. In summary, our scheme largely consisted of the following procedure: each skill corresponds precisely to a subcategory, and then groupings were established based on similarities therein (for example, mathematical procedures vs mathematical decision-making, the action of translation, or physics content).

The result of this procedure is a list of topics, categories, and subcategories organizing all 1500+ questions written for student practice into STEM Fluency. Assignments can be constructed using this hierarchy and the questions they contain, which will be described in the next section. A final list of all topics, categories, subcategories, and descriptions of each subcategory are contained in Appendix B.

Something important to note before closing out this section is that the organization described in Appendix B is not the initial organization determined, and it will not be the final organization. In the next section, we will discuss how edits are made, why edits are made, and examples of principles to consider when deciding to make edits.

4.3 Considerations when Making Changes to the Structure of STEM Fluency

The skill set outlined in Table 3.3 above and the organization of the practice questions into STEM Fluency represent the final stage of development that was used in this dissertation. This structure was not developed on our first attempt. It was developed over a period of several years as we tested the practice and modified it based on our findings and new insights from PER. This approach to curriculum development is consistent with one of the key components outlined in the ESF. In Part II, we indicate some of ways in the current structure seems to be productive, however, there is no questions that the structure can be improved with continued iterations.

In this section, we discuss a few considerations we made when making edits and revisions to the set of skills and organizational structure. This is by no means exhaustive, but serves to provide some examples of ways we decided to make changes.

First, we looked for at the effectiveness of the practice. This is discussed in later chapters in detail. For now, we will merely indicate that sometimes the practice designed was not as effective as intended. In these cases, one possibility is that a skill we had identified as "essential" may not actually be essential. In other cases, a skill we identified might be easy for students within the practice in isolation but is still difficult when combined with other steps to answer a problem. In such cases, it may imply more needs to be done beyond (or in contrast to) the practice.

In addition to analyzing the effectiveness of the practice, we took into account student input. As an example, the translation topic discussed above was initially organized differently: instead of categories being “concept to mathematics” and subcategories being various quantum mechanical objects, these levels were previously reversed. However, students could see the names of the categories in their assignments (the top portion of Figure 4.2). Comments from students suggested that students were using this as a hint for what quantities they were looking for in the case of the mathematics to concept questions. With this observation in mind, we decided to swap the levels to the current organization.

These are simple examples but provide various considerations we took into account when reorganizing and restructuring the questions within STEM Fluency. The approach took was consistent with the philosophy of essential skills practice. The structure is never final and should be constantly changing as a result of new research and insights from the results of the practice or other external sources. We will finish off this chapter in the next section by discussing considerations that were made when making the assignments administered to students.

4.4 Considerations when Making Assignments

As a final stage in the construction of the practice, we briefly outline considerations made when building assignments. As before, this discussion is not exhaustive but simply demonstrates various perspectives we took and that might be useful for others. Before discussing these considerations, though, we briefly outline how assignments are constructed within STEM Fluency.

4.4.1 Building Assignments in STEM Fluency

When making assignments in STEM Fluency, instructors select a handful of categories (and subcategories therein), and designate a mastery level for each category. For a graphic example of this process, see Figure 4.4. Note that when selecting a category, not all of the subcategories need to be added to the assignment: instructors can hand pick specific categories to be added to an assignment.

This last point leads us into the first consideration we made when making assignments: since mastery is determined at the category level, questions are randomly interleaved throughout an assignment. If too many subcategories are selected for a particular category, there's a chance that students may not be given practice a skill that is part of the assignment. As such, we needed to find a balance between the mastery level for a particular category and the number of subcategories selected within a category for a particular assignment. For us, the number of subcategories selected per category was never greater than 3, and the mastery for each category was between 3 and 5 (larger if more subcategories were selected, usually).

As a further consideration, recall that one of the goals of the ESF is to minimize any additions to student workload. The authors of the framework suggest building assignments that take between 10 and 20 minutes per week, and so instructors need to be cognizant about how much time their assignments will take. In principle, it's impossible to know this precisely ahead of time, but after a few administrations of the practice we were able to come up with a good heuristic about how much to include in an assignment. When we built assignments, we usually chose between 3 or 4 categories (with 1 to 3 subcategories in each), with mastery being between 3 and 5 for each category.

Another metric used to gauge assignment is mastery total, summing the masteries for each category for a particular assignment. Our total mastery was usually between 12 and 15, and this seemed to lead to a reasonable length assignment, although it did depend on the complexity of the skill being practiced and how much students had practiced the skills beforehand.

This led us to another consideration we made when selecting assignment length: have students practiced these skills before? If so (and if they did well on the previous administration of said skill), then we found that we could bump the mastery up by 1 level so students can

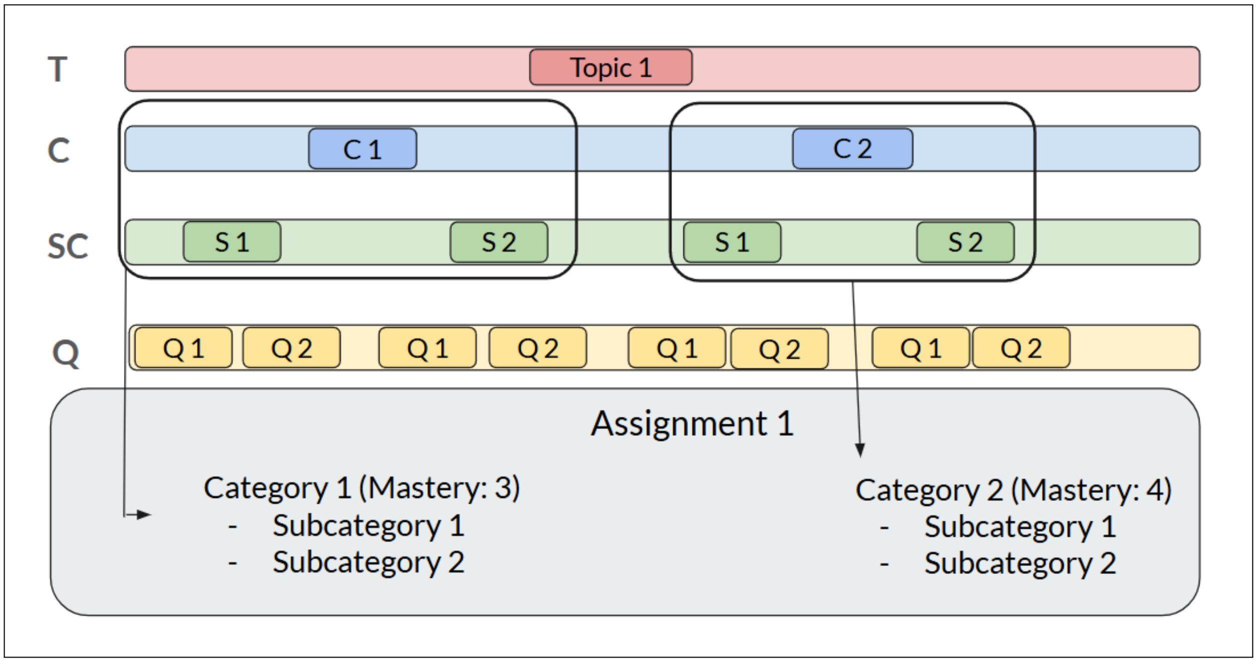


Figure 4.4: This figure gives an example of how homework assignments are constructed within STEM Fluency. For space constraints, Topics are abbreviated to "T", Categories to "C", Subcategories to "SC", and Questions to "Q". To construct assignments, categories (and subcategories within them) are selected, adding each question from each subcategory into the assignment. Not all subcategories for a particular category need to be added to the assignment. Each category is also assigned an "mastery" level, which the assignment builder can select (although it defaults to 3).

practice even more under the assumption it may take not much more time to do so. This may not hold in all cases, but empirically is seemed to be true in our experience).

4.4.2 What Skills Should be Practiced?

The previous portion of this section discussed quantitative considerations for how assignments were constructed. In this section, we discuss more qualitative considerations, such as what skills we decided students should practice, how much they should practice, *etc.* After all, the partial set of skills outline in Table 3.3 is quite large we did not believe students could practice every skill in our list. As a result, we made decisions as to what students would practice, when they practice particular skills, and how much they should practice a particular skill.

Immediate considerations we made concerned what students had seen in class at the time of practice. For example, a wave functions first quantum mechanics course practicing Dirac notation before they've seen Dirac notation seems anachronistic. As such, we realized we need to consider the material students had seen as well as the textbook being used as a guide for what to administer and when.

At UW, both junior-level quantum mechanics courses used Griffiths' "Introduction to Quantum Mechanics", which is known as a wave functions first approach to teaching quantum mechanics [27]. As such, for the first several weeks, we assigned practice based off the principles discussed early on in Griffiths, such as the differences between probabilities and probability densities, time dependence of wave functions, and various common difficulties associated with continuous probability. Once Dirac notation was introduced in lecture, we had the freedom to give students practice in skills related to the handling of Dirac notation as well as the more explicitly linear algebraic skills such as expanding states in eigenbases, orthonormality, *etc.*

In comparison, a spins first class (taught using, say, McIntyre's "Quantum Mechanics: A Paradigms Approach" [80]) might practice the mechanics of Dirac notation and matrix notation *first*, before discussions of continuous probability. Time dependence may come similarly early, but then questions associated with time dependence should be selected in the context of Dirac notation or matrix notation as opposed to wave function notation.

As such, even from the beginning, we needed to consider where students were *in the course* when making assignments. Students should practice topics they are currently learning, or at least had learned in weeks prior. This provides a rough outline for when certain topics fit into the course broadly.

That being said, this doesn't narrow down the set of skills we thought they *should* practice. The set of skills is still large, so we still needed to make decisions about which skills students should practice, and how many times they should practice the skills. To answer these questions, we tried to consider where our students were (both historically and currently). For example, we looked to see whether most of our students had taken a mathematical methods course in the past that taught complex numbers and complex variables. At the University of Washington they had, so many of the skills related to the basics of complex numbers we decided were not as necessary. In addition, we looked at questions given on past homework assignments and exams to see if students had performed well at questions involving adding time dependence to wave functions. Clearly if students had performed well we would have decided they may not need to practice that skill multiple times. In contrast, if students had had a hard time producing expressions for probabilities and expectation values from verbal descriptions of these quantities, then more practice would have been necessary for these students.

What "having an easy time" and "having a hard time" executing certain skills means is dependent on student performance on the STEM Fluency practice and their performance on other components of the course. In future chapters we will discuss various ways we did this, but for the time being it suffices to say that STEM Fluency provided us with ample data to analyze so that we could make well-informed decisions on what our students should practice more and what they needed less practice on.

For a final list of assignments for each quarter in which STEM Fluency was used, see Appendix D. Note that many of the assignments refer to categories and/or subcategories that do not appear in the final list referred to in Appendix B. This is because, as mentioned, many changes to the structure were made from quarter to quarter. As such, it suffices to look at the more recent quarters' assignment lists, but more historical lists were included for those curious as to seeing how assignments changed over time.

4.5 Summary

In this section, we discussed the work we did in organizing the questions we discussed in Chapter 3 into the online practice system STEM Fluency, and then detailed various considerations we made when making changes to the structure and assignments built for the practice. Of note was our attempt (consistent with the advice given by the authors of the ESF) in using the existing research literature and anecdotal, empirical observations at every step in this process. In addition, we make clear the reality that the process of identification of skills, to construction of questions, to organization and implementation, is never complete: as new research or new experiences informed us of new insights, we re-implemented the components of this procedure accordingly. As such, one is never truly finished implementing the ESF.

Now that we had a set of practice homework assignments in hand, the next set was to implement the practice and then assess its effectiveness. In the next several chapters, we discuss our efforts in assessing various components of the practice sets.

Part II

ASSESSING THE EFFECT OF THE PRACTICE

In Part I of this dissertation, we outlined the motivation, framework, and prior research relevant for the design of the practice question sequences we administered and assessed. Further, we discussed in detail: the process by which essential skills were identified; how practice questions for these questions were built; the online homework system (STEM Fluency) that we used to organize and administer the questions to students. In this part, we will outline our assessment of several administrations of the practice.

Beginning in Chapter 5, we will discuss our implementation of the practice as well as the various kinds of data analyzed and the analysis strategies used to make sense of this data. Chapter 6 will contain a small assessment on how the practice affected student understanding of probabilities. In Chapter 7 , we will discuss in more detail the effect of the practice on student ability to translate conceptual ideas into corresponding mathematical expressions. Chapter 8 will outline various other smaller assessments of the practice on student understanding of a wide variety of skills in quantum mechanics related to degenerate perturbation theory, angular momentum, and other mathematical ideas. Finally, we conclude with Chapter 9 summarizing the work in this dissertation as well as the educational and research takeaways we learned from providing essential skills practice to students.

Chapter 5

5. IMPLEMENTATION OF THE PRACTICE AND DISCUSSION OF THE DATA

In this chapter we will discuss the context in which the practice was administered and the data obtained. In Section 5.1, we discuss the details of the implementation, including the student populations and the terms during which the practice was administered. Then, in Section 5.2 we will outline the kinds of data relevant for assessing the effectiveness of the practice. In Section 5.3 we describe, generally, the various ways in which the data was analyzed to assess the practice's effectiveness in improving student understanding of quantum mechanics. Finally, in Section 5.4, we will address various potential sources of systematic error in our study.

5.1 Details of Implementation

The essential skills practice discussed in Part I was administered in various different courses across several years. At the University of Washington, the upper-division quantum mechanics sequence consists of two quarters (each around 11 weeks in duration). The first course, PHYS 324, traditionally covers standard topics such as the wave function in various potentials, the formalism of quantum mechanics, and the Hydrogen atom. The second course, PHYS 325, covers topics such as addition of angular momentum, time-independent perturbation theory, and time-dependent perturbation theory. Both courses historically have used the standard textbook by Griffiths [27]. We mention this specifically since there is another common, spins-first approach using other textbooks and proceeding in a different order material-wise [80]. Historically, PHYS 324 is offered in Fall quarters, whereas PHYS 325 is offered in Winter quarters.

Both PHYS 324 and PHYS 325 largely take the form of traditional university courses, utilizing lectures, problems sets, and exams. For each course discussed in this dissertation, there were roughly 7 problem set homework assignments and 3 exams. Each course often allowed students to drop one exam. In addition to the traditional course components, the courses had small-group, weekly, discussion-based tutorial sections using *Tutorials in Physics: Quantum Mechanics* [106]. These tutorials aided in student understanding of conceptual components of quantum mechanics, and students worked in small groups to discuss and refine their ideas related to a wide range of topics in quantum mechanics. The essential skills practice was administered on top of this workload (with some tutorial homework removed so as to not add more work for the students).

Table 8.2.1 outlines the various administrations of the essential skills practice. In this table, we also include the number of students enrolled (juniors and seniors) in the course as well as the number of practice assignments. For example, the first administration of the practice was given in Fall 2021 to 91 students in the form of 9 assignments. For a full outline of the content of the assignments, see Appendix D.

Table 5.1: A list of each administration of STEM Fluency administered to our upper-division quantum mechanics students. For each administration, we list the quarter of the administration, the course the administration occurred in, the number of students, N, in each course, and the total number of STEM Fluency homework assignments used.

List of Administrations of STEM Fluency Practice

Quarter of Administration	Course of Administration	N for each Course	Number of STEM Fluency Assignments
Fall 2021	PHYS 324	91	9
Fall 2022	PHYS 324	62	10

Continued on next page

Table 5.1: A list of each administration of STEM Fluency administered to our upper-division quantum mechanics students. For each administration, we list the quarter of the administration, the course the administration occurred in, the number of students, N, in each course, and the total number of STEM Fluency homework assignments used. (Continued)

Winter 2023	PHYS 325	43	10
Fall 2023	PHYS 324	71	10
Winter 2024	PHYS 325	45	9

All together, there were five administrations of the practice over the period of four years. Three of the administrations were in PHYS 324, and two of the administrations were in PHYS 325. The PHYS 324 administrations had between 70-90 students per course, and the PHYS 325 administrations around 40-50 students (since PHYS 325 is not a required course).

The essential skills practice was administered as weekly, online homework practice sets using the STEM Fluency system outlined in Chapter 4. The homework assignments were designed to take less than 30 minutes to complete. Each assignment contained between 3 and 5 categories, with a mastery level between 3 and 5 (larger mastery levels were usually associated with fewer categories, and *vice versa.*)

Before proceeding, it's important to note that the essential skills practice homework assignments varied from quarter to quarter. Further, many administrations of the practice used different organizations of the questions within STEM Fluency. As we mentioned in the previous chapter, modifications to the assignments were made on the basis of changes to the organization of the questions in STEM Fluency. As the study progressed, we adjusted the categories and assignments based on results that suggested that more or less practice in particular topics might be needed. For a full list of the assignments for each quarter, see Appendix D.

Fundamentally, though, there are a few common trends. Recall the skills that were described in Chapter 3. Most of these skills were practiced two or three times, and a few skills were practiced even more. Some skills were only practiced later in the quarter. For example, practicing the conjugation of complex numbers would always come before translating mathematical expressions between Dirac notation and wave function notation.

In addition, we note that not every student completed each assignment, and further, the students that did not complete a particular assignment were not the same from assignment to assignment. However, this only amounted to 2 to 3 students per assignment, so the data that we obtained through STEM Fluency is largely representative of the class itself.

Students received credit for an assignment if they completed the assignment or if the data STEM Fluency gave for a student for that particular assignment demonstrated that they put in effort to completing it. For example, a student who answered one question in STEM Fluency and then closed the assignment would not get credit, whereas a student who spent 30 minutes on an assignment but did not finish it *would* get credit. The STEM Fluency assignments made up 2% of the students' final grades, which previous studies suggest represents a sufficient incentive for students to do the practice [82].

5.2 Description of the Various Kinds of Data Obtained

Now that we've outlined the contexts in which the practice was administered, we will discuss the various data types collected to assess the effectiveness of the practice. There are two primary kinds of data obtained that we used for assessment: data obtained directly from STEM Fluency itself, and exam response data from student populations that either did completed the practice (the populations of which are mentioned in Table 8.2.1 above) or did not complete the practice. Both types of data are discussed below.

5.2.1 Description of STEM Fluency Data

Once a class finishes an assignment within STEM Fluency, the system generates data on the performance for each student who completed the assignment in the form of a .csv file. Each row in this file corresponds to a student, and each column gives student performance on particular aspects of the homework assignment. Examples of quantities that measure the performance include the following.

- Number of Correctly Answered Questions: There are columns that, for each student, give the number of correctly answered questions for each subcategory and category in the assignment, as well as for the whole assignment.
- Number of Attempted Questions: Similar to the above, the number of *attempted* questions for each student is reported for each subcategory, category, and the whole assignment.
- Total Time Taken: STEM Fluency also reports the total amount of time taken by each student to answer all of the questions they saw in each subcategory and category, as well as the whole assignment.

In Section 5.3, we discuss how we summarized the data above to assess the effectiveness of the practice.

There are other data that is reported by STEM Fluency, but these were not used in the analysis for this dissertation. For example, STEM Fluency reports the amount of time students spend on each hint associated with each category, as well as the number of skills mastered (*i.e.*, the number of categories whose metes maxed out), *etc.*

5.2.2 Description of Exam Data

To assess the impact of the essential skills practice on student fluency with certain skills, we looked at changes in performance on exam questions. To do this, we selected a handful of exam questions that had been administered before the design of the practice that we believed required students to use various skills that we had incorporated into the essential skills practice. We then readministered these exam questions to students who had completed the practice.

Table 5.2.2 contains a list of the exam questions selected. For each question, we report the quarters in which it was administered and the number of students who answered the question for each administration. Full statements of these questions are included in Appendix A and in the chapters in which the student responses to each question are discussed in detail.

As an example, consider question A.1. It was first administered in Fall 2018 in Exam 2 to a population of 98 students, and then readministered in Fall 2021 in Exam 2 to a population of 83 students. This question was discussed above in Chapter 2, but as a quick review, it asks students to produce a mathematical expression for the probability of two particular events given an initial state.

Table 5.2: A list of all of the exam questions used in assessing the effect of the STEM Fluency practice. In the table, each exam question is labeled according to its figure label in Appendix A. Each exam question was administered twice: once before STEM Fluency existed for that course, and once during an administration of STEM Fluency. The second and third columns contain the quarter, exam number, and number of students in the pre-STEM Fluency course for each exam, and the fourth and fifth columns contain similar information for the With-STEM Fluency course.

List of Exam Questions Used for Assessing the Essential Skills Practice

Exam Question	Pre-STEM Fluency Administration Quarter and N	With-STEM Fluency Administration Quarter and N	Chapter of Analysis
A.1	Fall 2018 (Exam 2) (98)	Fall 2021 (Exam 2) (83)	Chapter 6
A.2	Fall 2018 (Exam 1) (104)	Fall 2023 (Exam 1) (70)	Chapter 7
A.3	Fall 2015 (Exam 2) (76)	Fall 2023 (Exam 2) (63)	Chapter 7
A.4	Fall 2018 (Final Exam) (98)	Fall 2023 (Final Exam) (71)	Chapter 7
A.5	Winter 2019 (Exam 1) (81)	Winter 2024 (Exam 1) (42)	Chapter 8
A.6	Winter 2022 (Exam 2) (43)	Winter 2024 (Exam 2) (41)	Chapter 8

This question serves to assess the effect of the practice since, as we outlined in Chapter 2, solutions for this question require the execution of various skills administered through STEM Fluency. To solve this problem, students need to apply mathematical steps such as taking inner products, expanding the state in the energy eigenbasis, or alternatively knowing how to translate an answer in Dirac notation to wave function notation.

For each question and administration discussed in Table 5.2.2, we analyzed the responses from each student. As such, we can look at not only if students are getting a question correct but also the steps they use to answer the question. Thus, we can see in detail the effect of the practice.

5.2.3 Summary of Data Sets

The data from STEM Fluency itself, which consists of performance and time data for each student, and data for exam questions administered to students who did and did not complete the practice provide two independent ways to assess the practice. In the next section, we describe the ways in which we summarized and analyzed the data to make judgments on the effect of the practice.

5.3 Outline of the Various Methods of Analyzing our Data

In this section, we will discuss how we generally used the data described in Section 5.2 to assess the effectiveness of the practice. In Chapters 6-8, we will discuss the results of specific analyses. For now, we will simply discuss the general means by which we summarize and analyze the data.

5.3.1 Strategies for Analyzing STEM Fluency Data

As discussed in Section 5.2, STEM Fluency generates for each assignment a .csv file that contains various measures of student performance for each student who worked on the assignment. We looked for ways to summarize the data into numerical measures that capture notions of “accuracy” and “fluency” (see [82] on the Essential Skills Framework.)

Colloquially, “fluency” often encompasses “accuracy”. As such, we opted to find a measure of “accuracy” and “speed” that, taken together, can be used as a measure of fluency. We begin this subsection by discussing how we designed the measures of accuracy and speed using the data acquired through STEM Fluency.

Measures for Accuracy and Speed

STEM Fluency gives information for each student on the number of questions answered correctly, the number of questions attempted, and the total time spent on the questions seen by the student for each subcategory and category. Below we describe how we designed quantities that capture the accuracy and speed of students using these measures.

- Accuracy: For each subcategory and category in an assignment, we define the “accuracy” for a student in that particular (sub)category as the number of correctly answered questions in that (sub)category) divided by the number of questions attempted in that (sub)category. For example, consider a category in an assignment whose mastery was designated to be 3. Further, assume that a particular student answered 5 questions in that category correctly, but answered 4 of those questions correctly (say, by getting 1 question right, then missing 1 question, and then answering the final 3 correctly). Then this student will have an accuracy of 80%:

$$\frac{\text{4 questions answered correctly for a subcategory}}{\text{5 questions attempted for a subcategory}} = .8, \text{ or } 80\% \tag{5.1}$$

  Note that accuracies do not describe unique trajectories for individual students. For example, on a skill with mastery 3, an accuracy made up of 4 correct questions out of 7 attempts could describe 1) a correct answer, then 3 incorrect answers, and then 3 correct answers, 2) an incorrect answer, a correct answer, two incorrect answers, and then three correct answers, or 3) a handful of other possibilities. Our accuracy statistic treats students that perform in these differentiated ways “identical”. Thus, a student who takes a while to start getting correct answers is treated as equivalent to a student that gets one right, makes a mistake or two, and then achieves mastery. A more in depth analysis of the differences in these kinds of performances are beyond the scope of this dissertation.

- Speed: For each subcategory and category in an assignment, we define the “average time per question” (hereby simply called “time per question”) for a particular (sub)category as the total amount of time spent on questions in that (sub)category divided by the number of attempted questions in that (sub)category. We treat the time per question as a proxy for the speed of a student in a particular (sub)category.

  As an example, consider a student who attempted 6 questions in a particular category, and spent 300 total seconds on questions in that category. Then the average time per question for that category is 50 seconds per question:

$$\frac{\text{300 seconds spent on questions in a category}}{\text{6 questions attempted in that category}} = \text{50 seconds per question} \tag{5.2}$$

Note that each of the above quantities is defined for individual students.

Changes in Class-Wide Student Performance within STEM Fluency

Now that we have measures of performance for each individual student, we wish to be able to describe the performance of groups of students to be able to assess changes in these measures over time. For instance, one may be interested in the accuracy of a class for a particular skill as well as how this measure changes across various homework assignments in STEM Fluency that involve practicing this skill.

To motivate the way we formed these group-wide measures of performance, it is useful to visualize each individual student data point on a scatterplot as depicted in Figure 5.1.

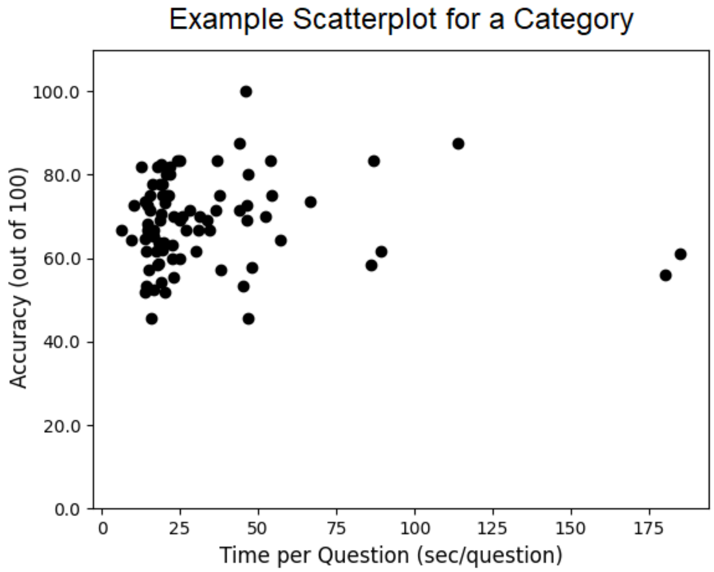


Figure 5.1: An example of the accuracy and time per question data for each student for a particular category on a particular assignment. Each dot represents a student, whose horizontal value represents their time per question and whose vertical value represents their accuracy. We can use a sequence of scatterplots to track the performance of individual students and the whole class over several iterations of practicing a skill. Ideally, students move up and to the left, getting more accurate and speeding up on average.

In this scatterplot, each point represents a student whose horizontal component represents their time per question on a particular (sub)category, and whose vertical component represents their accuracy for that same (sub)category. One can interpret this by observing that students who take longer per question appear further to the right, and students who have less accuracy appear farther down. The reader might imagine that, ideally, students shift upwards and to the left as they get more practice in a particular skill.

These scatterplots can be overwhelming to interpret, though. Further, the data seems to highly skewed in this particular instance (as is the case for most instances). As such, we must be careful as to how we describe group-wide data so as to not misrepresent it due to the highly skewed nature commonly encountered.

To mitigate the issues involved with skew, we opted to use the median as a central measure for describing group data (as opposed to the mean) for both accuracy and time per question. As a measure of dispersion, we opted to use the standard error of the median. The standard error of the median was found via a bootstrapping procedure on our data sets.

With class-wide measures of accuracy and time per question, we can now track changes in these quantities over multiples iterations of practice in these skills. As an example, consider the graph in Figure 5.2. Here, a particular skill is practiced over three homework assignments (in this case, in weeks 3, 5, and 8). We track how the class's median accuracy and median time per question change across each administration in order to assess how accuracies and fluencies change for this particular skill. Statistically significant changes are considered those changes in accuracy and time per question whose intervals for standard errors of the median in these quantities do not overlap. In this case, the median accuracies for the class increased significantly from week 3 to week 5 as well as week 5 to week 8. That being said, the median time per question for the class did not significantly change across the three assignments.

Note that sometimes particular students have anomalously large time per question measures. In every single instance that this was the case, it was the result of an anomalously large amount of time spent on a particular (sub)category. We use the word "anomalously" here to indicate that it was clear to us that students were not finding the practice questions in this particular skill time consuming, but rather that they likely left the system running while not focusing on it when on a question in that (sub)category. In fact, several students admitted that they were doing just this. Further, this phenomena was rare enough (around 1 to 2 students per assignment) that we decided to remove these students from the data set when calculating accuracies and time per questions for the cases where the time per question was too large.

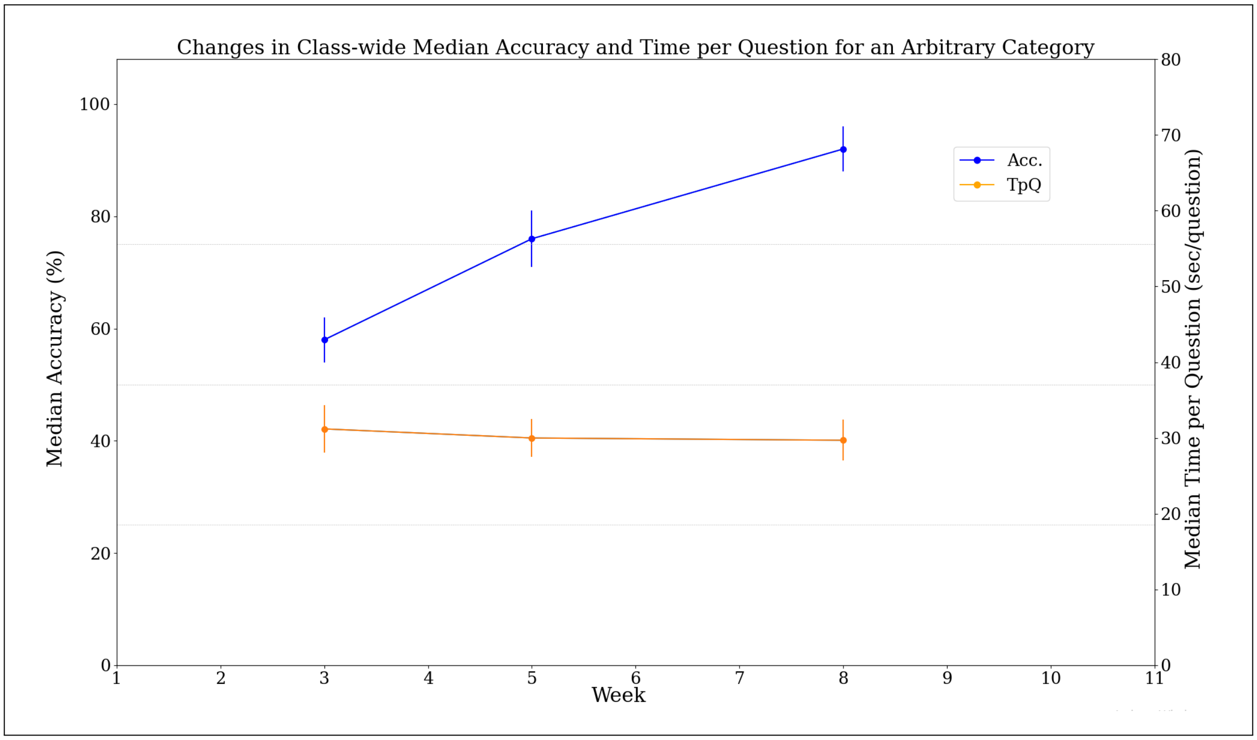


Figure 5.2: A graph depicting an example of changes in the classwide changes in median accuracy and median time per question. The blue line represents the median accuracy each week the skill was practiced (with error bars being the standard error of the median), and the orange line represents the median time per question each week (likewise with errors bars being the standard error of the median). As students practiced this skill, the median accuracy of the class increased significantly, whereas the time per question did not convey significant change. Such a graph is a useful way to read off the changes in median accuracy and time per question for a skill as students practice the skill.

Changes in Student Performance by Student Groups

In addition to summarizing class-wide data with a median accuracy and time per question, one might be interested in how various parts of the class compare to each other. For example, how do students who did not do well in the course improve in the practice compared to students who did do well in the course? The principles described above apply to arbitrary groups of students, and so we commonly split classes into groups to assess the effect of the practice on more fine-grained collections of students.

The most common grouping we opted for was breaking a class into a group consisting of students in the top quarter based on final course grades, students in the middle half based on final course grades, and students in the bottom quarter based on final course grades. In other words, we tracked the changes in performance among students who did well in the course, students who did average in the course, and students who (comparatively) did not do well in the course.

One way to visualize how to assess changes in student performance within and across these groups is through the use of box-and-whisker plots. An example of such is in Figure 5.3. In this example, we split the class into the scheme mentioned above: a group of the top quarter of performers in the course, a group of the middle two quarters of performers, and a group of the bottom performers. For each group, we display information on the accuracy on the first and second iterations of practice on a particular skill (colored blue and red, respectively). The solid black horizontal lines represent the medians in each distribution, the extent of the colored boxes show the middle two quartiles of the distributions, and the extent of the whiskers show the extent of the whole distribution.

From this, we are able to track how the median and interquartile ranges of these data sets change over multiple iterations. For example, in the middle group, the median for this skill did not change across both practice sessions, but the interquartile range grew, with the upward bound increasing. This suggests that while the median course performance did not change, the students doing better than the median got more accurate at this skill. Meanwhile, students in the top group vastly improved their median accuracies, with the median located at 100% (suggesting that over half of the students in the top group performed perfectly for this skill).

The reason we were interested in this more fine-grained tracing of performance is that we were interested in whether students in the middle and bottom of the course - whom we thought most needed the practice - were benefiting the most from it, and if students at the top might not be benefiting. Another possibility is that the practice was benefiting only the students who were already performing well. In either case, the practice would not be ideal as an instructional tool. As such, we found it helpful to break our data sets into groups according to final course performance to see how each groups' accuracies and times per question changed across several iterations of practice for a variety of skills. By using box and whisker plots, we obtained a fairly detailed picture of not only what was happening to each group but *within* each group as well.

Figure 5.3: Box and whisker plots for a particular skill across two assignments for three different groups of students: the top quartile of the class, the middle two quartiles of the class, and the bottom quartile of the class. The blue boxes represent the first iteration of the practice and the red boxes the second iteration for each group. The boxes enclose the interquartile range of the distribution for each group, the whiskers the full distribution, and the black line denotes the median.

Summary of Strategies for Analyzing STEM Fluency Data

To assess the effect of the practice on student understanding using data obtained from STEM Fluency, we designed two proxy quantities that apply to each category and subcategory: the accuracy (essentially the percent correct), and the time per question (the average time spent on questions). These quantities can be used to describe the performance of individual students but also arbitrary groups of students. To do the latter, we took the median accuracy and time per question instead of means in order to account for the highly skewed nature of the data acquired. Using these measures, we can track the accuracy and time per question for various groups of students as more and more practice is completed.

5.3.2 Strategies for Analyzing Exam Data

The second class of data that we used to assess the effect of the practice of student understanding was their responses to various exam questions. As discussed above, we have student response data to exam questions that were administered to populations who did and did not complete the practice. In this section, we will discuss how we used this data.

Exam Performance

The first way we used exam data is by simply comparing student performance on the questions between the populations who did complete the practice and who did not complete the practice. To do so, we graded the student response relative to a rubric we made that took into account both the correctness of their answer as well as the correctness of their explanation. Each response was simply graded as “correct” or “incorrect”.

By comparing the percentage of students who, without the practice, got the question correct to the percentage of students who, with the practice, got the question correct, we obtain a measure of the effect of the practice on student understanding more directly.

Exam Response Data

In addition to tracking student performance, we also analyzed student responses in detail and tracked whether or not the skills that they were practicing were being executed correctly or not, or if they were no being applied at all by the students. Just as with performance, we can track the extent that students are utilizing, correctly or not, the skills we identified in Chapter 3 with and without the practice. With this, we can determine whether or not the way students respond to commonly asked questions in quantum mechanics is affected by the practice. The hope is that, as students practice various skills, students are using these skills more consciously and do so correctly.

Summary of Strategies for Analyzing Exam Data

By utilizing exam data from various student populations, some of which completed the practice and some of which didn't, we are able to assess the effect of the practice on student understanding of quantum mechanics. To do so, we compared how well students did on the exam questions between the populations that did and did not complete the practice. We also analyzed the student responses themselves, tracking the ways they approached solving the problem as well as whether or not they were (correctly) executing the skills they practiced. The desired outcome is that, as students get practice, they respond to questions differently, utilizing the skills they practiced more correctly, ultimately leading to better overall performances on the exam questions.

5.4 Addressing Potential Sources of Systematic Error

There are many aspects of our study that could not be controlled for. In research of this type, it is difficult to create control groups. This could imply that changes in performance on exam questions could be attributable to sources other than the practicethat was provided to students. That being said, we have good reasons to suggest that these potential sources of error are not present in our work. In this section, I will address two potential concerns that might impact some of the conclusions found in this dissertation.

5.4.1 Exam Banks

As stated earlier, the conclusions made about the practice provided to students will be derived from differences in exam question performance. If the intervention group of students had access to these questions (since they were administered on exams prior to the intervention group's exams), then this could potentially produce improvements in performance that are not attributable solely to the practice.

At the University of Washington in our upper-division quantum mechanics courses, students are often given practice exams before their midterms and final exams so that they can review and prepare for their upcoming exams. These practice exams often have tutorial questions similar to questions we use to assess the effectiveness of the practice. To avoid giving students the exam questions being used for the assessments in this dissertation on the practice exams, we monitored which questions were being given as practice. In the cases where the practice question might have been given on a practice exam, we swapped the tutorial question with another question. This only needed to occur once (midterm 2 in Winter 2024). As such, to our knowledge, students had not seen the exam questions we used for assessing the practice prior to their respective exams.

That being said, this doesn't exclude the possibility that students have access to these question through other means, such as an exam bank shared from quarter to quarter. To our knowledge, students don't have access to such exam banks. Regardless, though, we have evidence that suggests that this would not affect the performance on the exam questions used for assessment. In our previous research in both the introductory and upper-division courses, we have not seen an impact on student performance when questions were repeated in subsequent exams. In addition, Mason and Singh showed in 2009 that students, upon being administered questions on exams that they had seen previously, did not yield improvements in performance [78]. As such, it seems that student performance on various exam questions likely don't change even if students had seen the questions being assessed prior to the exam. This matches what we see in our introductory sequence at the University of Washington, where students have decades worth of exam materials to study from, though we see little difference in performance on these questions.

In summary, we are fairly confident that students had not seen the exam questions that we used to assess the effectiveness of the practice and, even if they had, there is evidence that suggests there likely wouldn't have been a difference.

5.4.2 Differences in Student Populations

Over the years, improvements in pedagogy and instruction, both in the introductory level and upper-division level, may have resulted in changes to student populations. In particular, it might be argued that improvements in instruction at the introductory level may make our students more ready for upper-division physics courses, so that earlier quarters may have weaker performances compared to more contemporary quarters. In addition, better instruction in quantum mechanics courses themselves might result in improvements to student performance. If so, then these improvements could confound results suggesting any differences on exam question performance are due to the practice.

We have a few reasons that suggest that these possibilities do not apply to our courses. First, the courses themselves where assessment was being performed were essentially identical. Each course used Griffiths' *Introduction to Quantum Mechanics* (different editions, though the same material from both editions), our *Tutorials in Physics: Quantum Mechanics*, and the professors lectured over the same material in the same order on the same schedule. As such, there's nothing *structurally* different in the courses we're assessing.

It could be argued that different professors could potentially lead to different performances on the exam questions being assessed. While we do not wish to suggest that professors have *no* effect on student learning, we suspect that this does not apply to the exam questions used in this research. These were testing student understanding of tutorial material. While the tutorial material overlaps with lecture material, the kinds of responses and reasoning required for these questions is distinct enough that performance on these questions is largely decoupled from the lecture. As evidence for this, previous dissertations in the department have seen that performance on tutorial questions typically does not change from professor to professor or class to class until stable, effective tutorial material is administered to students [20, 2]. Again, this isn't to suggest that lecturers play no role in student learning, but rather that lecturers must explicitly address material and difficulties students experience for there to be differences (and this is often done through other course materials, like tutorials in our case).

As such, we suspect that the changes in the ways we are teaching our quantum mechanics courses over the years did not have an effect on the performance on the exam questions being assessed. It could be argued, however, that our students are coming into our courses more prepared from changes in instruction in prerequisite courses. We also, though, have reason to suspect that such improvements wouldn't have an effect on performance in the course for populations of students taking our quantum mechanics courses.

At the University of Washington, we assessed the effectiveness of the tutorials through the use of pretest-posttest performance on the material being tested. This material historically spans the whole of the topics typically presented in quantum mechanics courses. Generally, we observe that performance on pretests does not change from quarter to quarter [20, 2]. As such, students seem to be going into the classes with similar preparedness for quantum mechanics material. One could argue that our tutorials are designed to promote conceptual understanding which is not necessarily aligned with the objectives of the practice, meaning that pretests may not assess student preparedness in the material we give them practice in. Brahmia *et al.*, though, shows that students' quantitative literacy (more aligned with the goals of the practice) in the introductory sequence doesn't change as students progress through introductory material [4]. Further, recent work suggests that although we see changes in quantitative literacy as students take upper-division courses, not all students benefit equally, maintaining performances on the Physics Inventory for Quantitative Literacy (PIQL) across several years of physics instruction [28]. As such, there is a large pool of students who do not demonstrate changes in quantitative literacy, meaning that our students entering our upper-division quantum mechanics courses are largely constant on various components of the course between quarters.

In summary, we have evidence that suggests that students are roughly seeing similar instruction both during and prior to the courses we are using for our assessment. Not only are our courses structured the same, but students entering our quantum mechanics courses begin with roughly the same conceptual and quantitative understanding.

5.4.3 Summary of Potential Sources of Systematic Error

In this section, we have addressed some potential sources of systematic error involved with the assessment of the exam questions. In particular, we discussed the potential existence of exam banks and changes in student population between quarters of assessment. What we see is that even if these exam banks exist, they likely do not have an impact on student performance on exam questions. Further, we have evidence that our student populations have remained rather similar between quarters, suggesting that students roughly start off with the same tools (both conceptual and quantitative) needed to succeed in quantum mechanics. As such, it seems that any large differences on exam performance between intervention and non-intervention groups is likely due to the practice question sequences.

5.5 Summary of Implementation Details and Data Sets

In this chapter, we outlined the context in which the practice was administered. In addition, we discussed the various kinds of data used in this dissertation, as well as how we generally used that data to assess the effectiveness of the practice. This included data that we acquired directly from STEM Fluency itself on individual student performance on the categories and subcategories that made up the assignments, as well as exam question responses for student populations that did and did not complete the STEM Fluency practice. Finally, we address sources of systematic error present in the project. In the next chapter, we will discuss an example assessment of how we used this data in order to assess the effectiveness of the practice administered to students.

Chapter 6

6. EXAMPLE OF ASSESSMENT: CONCEPTUAL AND MATHEMATICAL SKILLS RELATED TO PROBABILITY

6.1 Introduction to the Chapter

The contents of this chapter consist of a paper written for the Physics Education Research Conference conference proceedings in the year 2023. The following sections are the sections of the paper themselves with no modifications. Note that the names of the categories and subcategories utilized in this paper are different from their modern names as outlined in Appendix B.

To summarize the contents of the paper, we use the abstract given to the paper, as follows. An important component of learning physics is being able to apply concepts and reasoning when solving problems. This can be especially challenging for students in quantum mechanics courses, in which the mathematical nature of the theory requires students adjust to new and unfamiliar ways of attaining "understanding". This paper describes the application of the Essential Skills Framework in an upper-division quantum mechanics course. A preliminary set of "essential skills" were proposed that underlie the solutions of common problems related to probabilities in quantum mechanics. Homework assignments were then developed that provided students with practice in applying these skills. The effect was assessed both by examining the accuracy and speed of students in using these skills over repeated homework assignments as well as through the impact on a standard course exam. Significant improvements were observed, although to a different extent on different types of questions. The results suggest that essential skills practice can be productively incorporated into courses on quantum mechanics, but certain skills are more difficult and may need special attention.

6.2 Introduction to the Paper

Quantum mechanics is recognized as a hard subject; not only are the concepts difficult and unintuitive, but it is expressed in abstract mathematical formalism, much of which is new to students. There has been increasing research examining the problems that students encounter in learning this topic, which ranges from probing their understanding of the many novel physical aspects of the theory [138, 139] to examining their ability to reason with and manipulate the mathematical objects in which the theory is encoded [118].

Ideally, when taking a course on quantum mechanics, students learn both the mathematics and the theory and become adept at translating between them. However, the unfamiliar language and the new ways in which concepts are expressed present a barrier [112, 109]. Even at the start of graduate instruction on quantum mechanics, many students have difficulty in relating mathematical quantities to the relevant concepts (*e.g.*, expectation values, probabilities, eigenvalues, *etc.*) [114].

One way of interpreting these results is that the combination of new tools and concepts induces considerable cognitive overhead for students [128]. When solving a problem or following an example presented in lecture, students may need to dedicate significant mental energy to simply recall which quantities are represented by different symbols and to distinguish between them. This can impede the ability of students to focus on and recognize the important takeaways of a lesson. It suggests that students need to develop skill in using the novel mathematics (*e.g.*, linear algebra) and learn how to quickly and coherently relate various mathematical representations to real-world phenomena.

In our courses we are using tutorials that are designed to address conceptual difficulties related to quantum mechanics [1]. However, we observe that students continue to struggle with applying the mathematical formalism. We decided to try to give students practice that

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[1] Tutorials in Quantum Mechanics, L.C. McDermott, Peter S. Shaffer, and Paula R.L. Heron, Physics Education Group, University of Washington.

could improve their fluency in applying the formalism and reduce this cognitive load.

To this end, we designed homework exercises using the online STEM Fluency system based on the Essential Skills Framework (ESF) by Andrew Heckler and Brendon Mikula at Ohio State University (OSU) [82]. It is intended to promote "fluency" by helping students develop both accuracy and speed in applying key concepts and skills. An example in the use of ESF is given by Heckler *et al.* in the context of vector algebra, which is ubiquitous in introductory sequences and presents difficulty for a wide variety of students [56].

This paper begins by discussing the STEM Fluency system used to provide students with focused practice and gives an overview of some of the skills we have targeted and the questions we have designed. It then presents preliminary results from student responses to a subset of these skills administered over the period of a quarter, as well as a comparison of results from an exam question given in courses with and without essential skills (ES) practice.

6.3 Implementation and Forms of Assessment

6.3.1 Implementation

Essential skills practice was administered in weekly, required homework assignments via the STEM Fluency system at Ohio State University. This mastery-based, online system allows instructors to create assignments that consist of sets of questions organized into "categories" and "subcategories". Each assignment includes a set of categories and subcategories chosen by the instructor. The relevant questions administered are then given randomly within the assignment, so sequential questions come from different subcategories. Each category has a "mastery" level, which indicates the number of questions in that category that students must answer correctly in a row before they stop seeing questions from that category in that assignment (thus, having acquired "mastery" in that category). The assignment is complete, and students receive credit, once they acquire mastery in all the categories assigned. Individual students thus saw different numbers of questions based on their performance on prior questions. The individual questions in a given category are all very similar and differ only in the details of the context.

Nine assignments were administered in a junior-level quantum mechanics course ($N = 93$) in Fall 2021. The course used a positions-first textbook (as opposed to a spins-first textbook). On average, each assignment had four categories, each with a mastery level of three. About half of the skills were repeated at least once in the quarter. The assignments were expected to take less than 30 minutes on average. To compensate for the extra work, some homework was removed. All assignments were completed by the majority of the class.

Discussed in this paper is the category we developed for "Probabilities" (offered in Weeks 6 and 7). It had two subcategories *Math to Idea* and *Idea to Math*. These were developed by taking into account known student difficulties in quantum mechanics as well as our personal experience in interacting with students [13, 98, 131]. It seemed to us that students have difficulty in expressing probabilistic ideas in a mathematical form (*Idea to Math*) and, conversely, interpreting the mathematical expressions in terms of the relevant concepts (*Math to Idea*). Note that these questions were designed, to the extent possible, to test single ideas and skills and to be possible to complete quickly.

Figure 6.1 contains an example question from each subcategory. A full list of the categories and subcategories that were developed, as well as how we decided on the particular question hierarchy, will be discussed in a future article.

6.3.2 Forms of Assessment

We have used two methods to assess the impact of the STEM Fluency system on student understanding. The first is by examining the progression of student performance on a given category/subcategory after repeated exposures within STEM Fluency. The second is comparing student performance on exam questions that rely on those skills and that have also been given in prior classes that did not use STEM Fluency.

Consider a spin$-\frac{1}{2}$ particle in the state $|\chi\rangle$. The eigenstates for spin in the z-direction are given by $|s_i\rangle$, where $s_i = \frac{1}{2}$ or $-\frac{1}{2}$. What do you call/how do you interpret the quantity given by $|\langle\frac{1}{2}|\chi\rangle|^2$?

(A) The probability of measuring the z-component of spin for our state to be $\frac{1}{2}$.
(B) The expectation value of the z-component of spin in the $|\chi\rangle$ state.
(C) The result of measuring the z-component of spin in our state $|\chi\rangle$.
(D) The wavefunction of $|\chi\rangle$ in the z-spin basis.
(E) The probability density for $|\chi\rangle$ in the z-spin basis.

(a)

Let H be a Hamiltonian with energy eigenstates $|n\rangle$ for $n = 1, 2, ...$ with associated energy eigenvalues E_n. In addition, consider a particle in the state $|\psi\rangle$. What is the probability of measuring E_2 for our particle?

(A) $|\langle 2|\psi\rangle|^2$
(B) $|\langle 2|H|\psi\rangle|^2$
(C) $\langle 2|\psi\rangle$
(D) $\langle 2|H|\psi\rangle$

(b)

Figure 6.1: An example question from a) the Math to Idea subcategory and b) the Idea to Math subcategory of the Probabilities category. Questions in each subcategory had similar structures, involving many different observables and many different representations when available.

Assessment within STEM Fluency

STEM Fluency records each student response and the time taken for each question. We have been using subcategory score (the fraction of correctly answered questions in a given subcategory) as a proxy for “accuracy”. A student who gets 100% accuracy in a given subcategory has not answered any of those questions incorrectly, whereas an accuracy of 50% indicates that the student had multiple incorrect answers. We use time per question (the total time spent on questions in a given subcategory divided by the number of attempted questions in that subcategory) as a proxy for “speed”. Fluency is then a combination of both measures.

For each STEM Fluency assignment, we have found that a few students took a very long time on the assignment. The assignments are designed to be done in less than 30 minutes. Most students take less time, but some take up to 2 hours. We have dropped these students from the analysis, since they likely took time away from the assignment before finishing and their time is not reflective of the time they spent on individual questions.

Assessment through Exam Questions

To assess the effect of the ES practice, we analyzed results from two exam questions that were given before (Fall 2018) and after (Fall 2021) ES practice was incorporated into the course. Figure 6.2 shows the two relevant questions. Both questions can be regarded as testing the *Idea to Math* subcategory. The content and structure of the course between the two quarters was identical except for the ES practice. Although the instructors differed, our past experience in this and other courses suggest that typically the effect of the instructor is small. Thus, we believe that differences in performance on the question can likely be attributed to the ES practice.

Consider a particle in the quantum mechanical infinite square well of width a. At $t = 0$, the normalized wavefunction for the particle is given by

$$\Psi(x, t = 0) = \begin{cases} \frac{\sqrt{15}}{a^{5/2}}(x^2 - ax) & 0 < x < a \\ 0 & \text{otherwise} \end{cases} \tag{6.1}$$

You may receive full credit on this problem without evaluating any integrals; however, writing down an integral without explaining where it comes from will not result in credit.

A. Suppose you were to measure the energy of this particle at $t = 0$. Determine the probability that the energy is equal to E_1, the energy of the ground state. Explain your reasoning.

C. Suppose you were to measure the position of this particle at $t = 0$ (no other measurements have been made). Determine the probability that the particle is measured to be between $x = 0$ and $x = a/3$. Explain your reasoning.

Figure 6.2: Portions of an exam question administered in classes before and after STEM Fluency was incorporated. The questions assess student ability to determine probabilities for various observables.

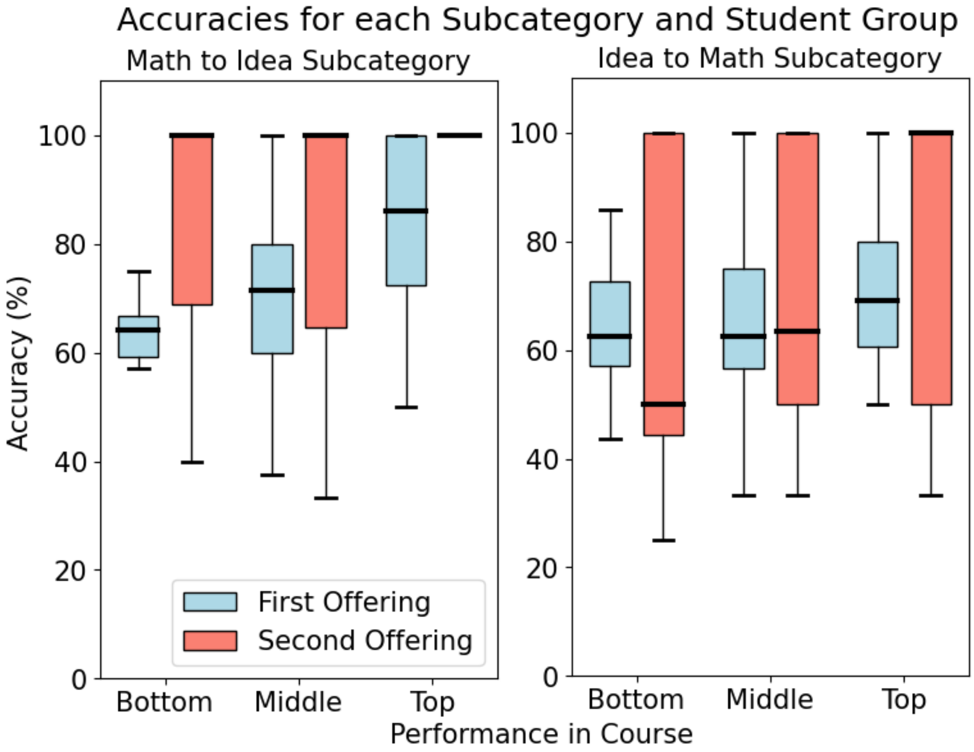

Figure 6.3: Box plots for both the *Math to Idea* and *Idea to Math* subcategories of the Probability category (N = 93). Data are shown for the Top quartile, Middle two quartiles, and Bottom quartile based on final course performance. (Note: the Top and Bottom quartiles are divided into four parts each with about 6 students, and the Middle two quartiles have four parts with about 12 students each.)

6.4 Results and Discussion

6.4.1 Assessment through STEM Fluency

The impact of the essential skills practice was assessed, in part, by analyzing changes in student accuracy and time per question (TpQ) from the first homework assignment (week 6) to the second (week 7). The results for the two probability subcategories *Math to Idea* and *Idea to Math* are shown through the box-plots in Figure 6.3. We were interested in determining the impact on all students, thus in each figure the data is divided according to class performance. The terms "Top", "Middle", and "Bottom" indicate whether the student's final course grade was in the top quartile, middle two quartiles, or in the bottom relative to final course grade. The large number of students with perfect accuracy on the second offering motivated the choice of using these three groups.

The "ideal" outcome from using STEM Fluency would be 1) accuracies would increase with a corresponding decrease in TpQ for each group, and 2) lower-performing students would have their final performance near that of the higher-performing students. Here, we focus on two aspects of the box-plots: the medians for each data set (represented by the solid horizontal black lines), and an overview of the distribution of student scores within the Top, Middle, and Bottom groups.

Math to Idea Subcategory Results

For the *Math to Idea* subcategory, the median accuracy increased substantially from the first to the second assignment. All three student groups (Top, Middle, and Bottom) started with a median accuracy between 65% and 85% on the first assignment, which increased to 100% on the second assignment. Thus, half or more of each group, and the class as a whole, performed perfectly with no errors before achieving mastery on the assignment. Interestingly, however, the median TpQ remained roughly the same across both assignments for all three groups. The medians started at 22, 22.4, and 22.9 seconds per question for the Top, Middle, and

Bottom groups, respectively, and ended up at 20.8, 21, and 22.1 seconds per question. Thus, after a single repetition, students were completing questions on the *Math to Idea* subcategory at roughly the same rate, but with much higher accuracy.

Note that in the Top group, essentially every student had 100% accuracy. The bottom halves of the Middle and Bottom groups, despite the large number of students who achieved 100% accuracy, ended with a distribution of accuracies that was broader than it was on the first assignment.

Idea to Math Subcategory Results

The *Idea to Math* subcategory had a different outcome than the *Math to Idea* subcategory. On the first assignment, performance for the Top, Middle, and Bottom groups was more compressed, ranging from 65% to 70%, so these questions appeared to be harder for the Top group. On the second assignment, only the median accuracy of the top group increased, reaching 100%. The Middle and Bottom groups had mixed improvement. Each had a quarter of their students achieve 100% accuracy and the next highest performing quarter had a higher and broader range than was true for the first administration. However, the median accuracy of the Middle group was unchanged and that of the Bottom group may have had a decrease in median accuracy.

In addition, we saw that the distributions for the lower half of each group broadened considerably. Although many students achieved 100% accuracy on the second assignment, those who did not obtained a wider range of scores compared to those who did not obtain 100% on the first assignment.

Meanwhile, the TpQ of all three groups significantly decreased from the first iteration to the second. The medians started at 20.6, 20.6, and 23.4 seconds per question for the Top, Middle, and Bottom groups, respectively. They ended up at 14, 14.3, and 19.4 seconds per question. Thus, it seems that while the Top group had improvements in both their accuracy and TpQ, the bottom three quartiles of students answered the questions more quickly but with no positive shifts in their median accuracy.

Summary and Interpretation

Student performance on the *Math to Idea* and *Idea to Math* subcategories demonstrated somewhat different behaviors. The increases in the *Math to Idea* median accuracies for all groups after only one repetition might suggest that not much practice is required for students to be able to identify the concepts associated with mathematical expressions. The lesser improvement in the *Idea to Math* subcategory may suggest that translating between probabilistic concepts and mathematics is an asymmetric process, with one direction being more difficult than the other. It suggests that the Middle and Bottom group of students (and perhaps all) may need more practice in the *Idea to Math* subcategory for them to reach the improvements observed in *Math to Idea* subcategory.

On the other hand, the *Math to Idea* subcategory had no significant shifts in the TpQs from the first to the second assignment. This was true for all groups despite significant improvements in accuracy. The converse was seen in the *Idea to Math* subcategory in which all of the groups had a decrease in TpQ despite the bottom two groups having little or no change in their median accuracies.

One might expect that ES practice would initially result in a simultaneous increase in accuracy and decrease in TpQ and that these would level off with repeated practice. These results suggest that these two aspects do not necessarily improve together; extra practice may be necessary to achieve the desired improvements for both measures.

6.4.2 Assessment Through Exam Question

The essential skills practice was also assessed by using two exam questions given in a class ($N = 98$) without ES practice and in a class with it ($N = 77$)[2]. The questions are shown in

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[2] We have evidence from prior quarters that the 2018 results are higher than they would have been, in part, as a result of the use of tutorials in quantum mechanics.

Figure 6.2. Table 6.4.2 shows the percentage of students from each class who answered each question correctly. An answer was regarded as "correct" if students had a correct procedure. Questions of this form (short-answer and compound in nature) are different from those used in the ES practice, which had questions that were more atomistic (for example, requiring students to select which bra to use in a probability calculation, or to associate a physics or math concept with a particular mathematical object). The exam question thus requires that students be able to use the skills developed in STEM Fluency when they are not explicitly prompted.

Table 6.1: Percentage of students who answered correctly the exam questions in Figure 6.2. These questions are of the type Idea to Math described previously.

Observable	No SF ($N = 98$)	SF ($N = 77$)
Energy	43%	66%
Position	80%	78%

On the energy probability question, about 43% of the students without STEM Fluency answered correctly. This increased to 66% for the class that used STEM Fluency. This 23% increase was statistically significant (with $\chi^2 = 8.547, df = 1$). On the position probability question, both classes did well (80% vs 78% correct), but there was no significant shift ($\chi^2 = .062, df = 1$). Given that the only difference was the introduction of STEM Fluency, it is possible that the ES practice may have played a role in improving student performance on the energy question. We believe that this question requires more steps in the solution than that for position and thus STEM Fluency may have played a larger role. However, additional research would be needed to draw conclusions.

6.5 Conclusion

Weekly essential skills practice was administered in an upper-division quantum mechanics course. At the start of this project, we identified some skills that we believed research had demonstrated are difficult for students and are essential for students to be able to apply the concepts they were learning. In this paper, we focus on skills related to the concept of probability.

As part of our assessment, we looked at changes in student accuracy and the time required to answer questions within the STEM Fluency system (*e.g.*, performance on repeated assignments). Preliminary results have been encouraging. The scores of students on repeated assignments suggest that students at all levels (Top, Middle, and Bottom, as defined relative to their final course grade) can benefit. There were significant increases in the median accuracies for questions involving translating mathematical expressions in terms of a relevant probability (*Math to Idea*) for each of these three student groups. The results were not as strong for the converse, identifying the mathematical expression corresponding to a given probability (*Idea to Math*). However, even for this type of question we saw considerable improvement, especially among students in the Top group. The disparity in improvement for these two lines of reasoning may not be surprising, since coming up with a mathematical expression requires identifying the component parts (*e.g.*, bras and kets or elements of an integral) and then assembling them. This asymmetry does not appear to be documented in the literature on quantum mechanics and we plan to examine the implications as part of our future work.

We also found that increases in accuracy were not necessarily correlated with the time taken per question by students. This finding suggests that accuracy alone is not sufficient as a measure of fluency. Indeed, learning is a complex process where different aspects may improve at different rates.

Finally, we examined results from exam questions that required students to produce the mathematical expression corresponding to a particular probability (*Idea to Math*). Although not originally designed with STEM Fluency in mind, we saw large improvements on student performance on some of these questions as well.

The preliminary results presented here suggest that incorporating ES practice into a junior level course on quantum mechanics can be productive. This type of practice is not intended to replace other course components that help students develop a conceptual framework. However, the mathematics and formalism in quantum mechanics is sufficiently difficult and new for students that we believe targeted practice in interpreting and generating the mathematical expressions can be useful. In developing our practice questions, we drew on existing research into student difficulties associated with quantum mechanics. However, additional work is needed to determine whether the skills we identified and the questions we designed effectively address the needs of students.

Chapter 7

7. ASSESSMENT OF STUDENT ABILITY TO TRANSLATE BETWEEN CONCEPTS AND MATHEMATICS

7.1 Introduction to the Chapter

The contents of this chapter consist of a paper written for Physical Review Physics Education Research in 2025. The following sections are the sections of the paper themselves with the only modifications being references to separate works being changed to parts of this dissertation.

To summarize the contents of the paper, we use the abstract given to the paper, as follows. Research on the learning and teaching of quantum mechanics has revealed many insights into the ways that students may fail to develop a functional understanding of the subject. Some of the difficulties they encounter appear to be conceptual in nature, while others seem to be due to a lack of fluency with the various representations that are used and the overarching mathematical formalism. There has been some development of curriculum to address the former, but less that specifically focuses on the latter. To address this gap, we created homework assignments that are designed to foster mathematical fluency. In this paper, we discuss the component of the homework that promotes skill in translating between conceptual quantities and mathematical expressions. The assignments were administered using an online homework system. The impact is illustrated through: (a) improvements in performance on the homework as the course progressed, (b) increases in the percentage of students who successfully translated concepts to mathematical expressions on exam questions, and (c) increases in performance on those questions.

7.2 *Introduction and Motivation for the Paper*

Quantum mechanics is widely acknowledged to be a particularly difficult subject to learn. Not only are there significant differences between the behaviors of quantum and classical systems, but the mathematical representations used are often unfamiliar to students (*e.g.*, Dirac, matrix, and wave function notations). There is by now ample evidence from physics education research demonstrating some of the ways in which students struggle with both aspects [120, 122, 114]. Research into the conceptual components [109, 43, 72, 17, 88, 10] has guided the development of curricula that have been shown to improve student learning [106, 94, 115]. Recent work has also focused on student ideas about the mathematical components [118, 71, 64, 24, 70]. Those results have also led to the design of instructional materials [103].

Ultimately, in order for students to construct a functional understanding of the subject, they must be able to move fluidly between the quantum mechanical concepts and their mathematical formulations. Research, however, suggests that students often fail in doing so [103, 70, 118, 99, 18, 91, 131, 19]. For example, many will write down incorrect expressions for key concepts, such as probabilities and expectation values.

In this article, we present results from an intervention to help students in a junior-level quantum mechanics course acquire accuracy and fluency in translating concepts into mathematical expressions. We begin by discussing relevant prior research and motivation. We then outline the framework and tools used to develop structured homework problems, as well as how they were implemented and how data were acquired. This is followed by a description and synthesis of the results that document the impact on student ability.

7.2.1 Prior Research

In constructing the online homework assignments assessed in this study, we drew on existing research on the learning and teaching of quantum mechanics. Our primary focus was on research that explicitly examined student understanding of mathematical representations. However, some studies that target conceptual ideas also provided insight into difficulties that students encounter with the mathematics.

Many investigations into student understanding of the mathematics used in quantum mechanics have examined student facility with specific notations. For example, Singh *et al.* identified difficulties in relating expressions in Dirac notation to the corresponding quantities [118]. Other studies detail how certain errors made by students are notation specific, with difficulties in one notation not necessarily present in others [71, 64, 24]. Finally, there is research on the ability of students to translate between various notations, for example between Dirac and wave function notation [70].

There is also research that probes student ideas about representations in general. Gire *et al.* found that students, when solving problems, view the use and function of the various notations differently depending on the problem context [25]. This led them to construct a framework for classifying various characteristics of the notations in quantum mechanics according to what students look for when choosing a notation to use in solving a problem. This scheme was applied in another study examining how notation affects student problem solving approaches, specifically for expectation value problems [103]. The results indicate that each notation seems to invoke different preferences on the part of the students due to the way various notations convey information. Wawro *et al.* examined how students interpret eigenvalue equations and found that, in some cases, students interpret them differently depending on whether or not the equations were presented in Dirac notation [133].

These studies focused mostly on student ideas and approaches related to particular mathematical notations. Additional insights have also come from investigations into student conceptual understanding of various quantum mechanical quantities such as expectation values, probability distributions, *etc.* [73, 72, 131]. One finding is that some of the difficulties documented in these studies involve errors related to translating from verbal descriptions of quantities into mathematical expressions. Marshman *et al.*, for example, found that some students fail to produce a correct expression for expectation values in both Dirac and wave function notation, with many of the incorrect expressions showing mixed aspects of other quantum mechanical quantities, such as probability [72].

Wawro *et al.*, using a framework on metarepresentational competence, found that success in quantum mechanics courses is often associated with a level of comfort and fluency with the various notations, especially matrix and Dirac notation [134]. They concluded that helping students develop fluidity in moving between representations, whether mathematical or conceptual, should thus be an important teaching objective.

There has been little work, however, in determining how to help students bridge the conceptual-mathematical gap. Students need help in relating the many concepts introduced in a course on quantum mechanics to their mathematical representations in each of the various notations used. The gap seems especially broad in quantum mechanics, in part, because the mathematical objects used to communicate it are new and involve a discontinuous jump in complexity for students. Below we provide some evidence from our work that it is not enough to help students understand the concepts alone.

7.2.2 Motivation for Study

For most of the development of *Tutorials in Physics: Quantum Mechanics* (the supplemental materials developed by the University of Washington Physics Education Group for junior-level quantum mechanics) the focus was on promoting student conceptual understanding. Significant improvements were documented in this area [19, 18, 89, 87]. However, when reviewing student responses on exam questions administered during the development period, one of the authors noticed the following pattern. On multi-step questions, students often failed on the first step, which typically required them to come up with a mathematical expression corresponding to the quantity of interest.

Three exam questions exhibiting this pattern are shown in Figure 7.1. As an example, we discuss the questions from Exam 2. The others are discussed in the later section *IV. Student*

Performance on STEM Fluency and on Exams.

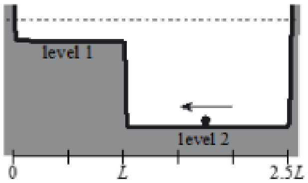
The questions from Exam 2 present students with a wave function for a particle in a one-dimensional simple harmonic oscillator, where the function is not an eigenstate of the potential. Students are asked to produce two expressions: (1) the probability of measuring the position of the particle to be in a particular interval, and (2) the probability of an energy measurement yielding the ground state energy. Students were told not to evaluate either expression, so our analysis is based on the form of the response, not a numerical calculation. The question was initially given to 104 students in a junior level quantum mechanics course.

A complete solution to the first question involves recognizing that the probability of measuring the particle to be in a specific region is given by the integral of the probability density over that region. The next step is to relate the probability density to the wave function to obtain $\int_{-b}^{b} |N|^2 \frac{\sin^2(2\pi x/b)}{x^2} \mathrm{d}x$. Answering the second question involves recognizing that the wave function can be expressed as a superposition of eigenstates, each with a complex coefficient. The norm-square of each coefficient is the probability of measuring the energy for that eigenstate. Using the orthonormality of the eigenstates, one can determine the probability for measuring the ground state energy by taking the inner product of the state with the ground state (*e.g.*, $\int_{-\infty}^{\infty} \psi_0^*(x) \left(N \frac{\sin(2\pi x/b)}{x} \right) \mathrm{d}x$), and then taking its norm-square.

Only 66% of the students answered question 1 correctly, and only 34% answered question 2 correctly. For both questions, many students gave an expression for an expectation value, not a probability. For question 1, it was also common for students to give a probability density. For question 2, many wrote a portion of the eigenvalue equation for the ground state energy, *e.g.*, $H\psi_0(x) = E_0\psi_0(x)$, or just the right side of this equation. Ultimately, they failed to recognize that these expressions do not represent a number between 0 and 1, which is required for a probability.

Performance on these and other questions demonstrate how difficult it can be for students to match verbal descriptions to mathematical expressions in quantum mechanics. This is

Exam 1 Questions: A (classical) ball moves back and forth in a track with steep sides. The track consists of two levels of unequal length joined by a steep ramp. Assume that the time spent in the steep portions is negligible, that there is no friction or energy loss in the system, and that the ball rolls smoothly, without bouncing, forever. Level 1 has length L, and level 2 has a length $1.5L$.



The ball has enough energy to reach both levels, and the speed of the ball on level 2 is 3 times the speed of the ball on level 1.

(1) Determine the probability density for both levels.
(2) What is the probability of finding the ball between $x = 3L/4$ and $x = 1.5L$?
(3) Determine the average position of the ball.

Exam 2 Questions: Consider a particle in the one-dimensional harmonic oscillator potential, $V(x) = \frac{1}{2}m\omega^2x^2$. The initial wave function for this system is known to be $\Psi(x,0) = N\frac{\sin(2\pi x/b)}{x}$, where N is a normalization constant and b is a constant.

In each part below, write an expression for the given quantity. Explain your reasoning in each case for full credit. If you do not have enough information to answer, state so explicitly. Note: you do not need to evaluate any expression.

(1) The probability of measuring the position to be $-b \leq x \leq b$.
(2) The probability of measuring the ground state energy, E_0.

Exam 3 Questions: The wave function for an election in a hydrogen atom is given by:

$$\Psi(\vec{r}, t=0) = \frac{1}{\sqrt{2}}\left(R_{21}(r)Y_1^{-1}(\theta,\varphi) - R_{21}(r)Y_1^{1}(\theta,\varphi)\right).$$

(1) Does the probability density in position space of this electron depend on time? Explain.
(2) Determine the expectation value of the L_z operator. Show your work.
(3) Is the given state an eigenstate of the Hamiltonian operator? Is the given state an eigenstate of the L^2 operator? Explain why or why not.

Figure 7.1: Examples of exam problems that motivated this study. These were also used to assess the impact of the practice problems developed as homework in the course.

especially the case on examinations where they need to do so quickly and efficiently. The tendency of students to provide incorrect mathematical expressions for the quantities of interest is consistent with findings by others [72, 131]. With this in mind, we decided to try to identify efficient and practical ways to provide students with practice in translating between different representations for various quantum mechanical concepts. A goal was to create supplemental homework that could be readily incorporated by instructors.

7.3 Framework for Adding Practice into the Course

In constructing the homework practice for students, we adopted the Essential Skills Framework (ESF) using the online STEM Fluency system developed by Andrew Heckler at The Ohio State University [82]. This framework provided a model for designing efficient, research-based homework that could be administered weekly to a large number of students and give them immediate feedback while also providing data that could be used to assess student progress.

7.3.1 The Essential Skills Framework

The Essential Skills Framework is predicated on the assumption that solving problems efficiently requires both accuracy and fluency in certain atomistic, "essential" skills common to large classes of problems. A prototypical example in the context of introductory physics is adding and subtracting vectors [82].

Heckler *et al.* hypothesize that if students lack accuracy and fluency in such skills, they must devote mental attention to them, which can induce significant cognitive load when solving problems, reading example problems, or following a derivation during lecture. Thus, it is desirable to find a way to identify these essential skills and then develop effective and efficient ways for students to practice them. Heckler *et al.* define an essential skill as one that:

- is procedural and relatively simple, requiring at most a few steps,
- constitutes steps necessary to solve commonly assigned problems, and

- is largely automated in experts [82].

For experts, essential skills can be executed nearly subconsciously. For example, an expert solving an introductory statics problem is likely to automatically decompose forces into their components giving minimal attention to the required steps.

To help students begin to develop a level of "mastery", Heckler *et al.* argue it is important not only to develop accuracy in performing these skills, but also to develop fluency, *i.e.*, to help students both recognize the steps that are needed and be able to perform them quickly.

The ESF outlines an iterative procedure that can help instructors identify skills that are likely essential. Often, the skills for which students need practice are not known *a priori*. One can make an educated guess, but research is needed to identify the ones that present difficulty to students and then to identify strategies that effectively address the issues that students have.

We describe the effect of the practice that we designed and provided to students through homework using the STEM Fluency system. The skills on which we focused were based on research by our group and others. The questions and their organization were developed as part of this study. Here, we describe only a few of the skills and the results of the practice.

7.3.2 The STEM Fluency System

A basic idea of the ESF is that a given practice session should focus on a small number of skills and that questions on those skills should be randomly assigned. Thus, students become fluent both in identifying which skill is relevant and in applying it correctly. STEM Fluency was developed with the ESF in mind. It allows instructors to construct questions and then organize them into structured homework assignments.

Within STEM Fluency, questions are grouped into *subcategories* consisting of nearly isomorphic questions that test students on a particular skill. Subcategories are grouped into *categories* that, in turn, are collected into *topics* and finally into *subjects*. As one moves up

Subject: Quantum Mechanics

Topic: Translating Representations

Category: Translating Concepts to Mathematics

Subcategories:

Expectation Values: Producing expressions for expectation values of observables in both Dirac notation and wave function notation.

Probabilities: Producing expressions for the probability of measurements (both continuous and discrete) in both Dirac and wave function notations.

Probability Density: Producing expressions for the probability density of a state both in Dirac and wave function notation.

Eigenvalue Equations: Producing eigenvalue equations for various observables.

Figure 7.2: The hierarchy in STEM Fluency for the subcategories discussed in this chapter.

the classification scheme, the groupings become more general: each *category* contains similar, related skills, while each *topic* is even more general. (See Figure 7.2 for the hierarchy relevant to this paper.)

When constructing STEM Fluency homework, instructors select 3 or 4 categories and a subset of their corresponding subcategories. To complete an assignment, students must correctly answer a certain number of questions for each category without having answered an intermediary question on that category incorrectly. The required number of questions for a given category that must be answered without making an error is called the *mastery level* for that category. The mastery level is selected by the instructor and is usually between 3 and 5.

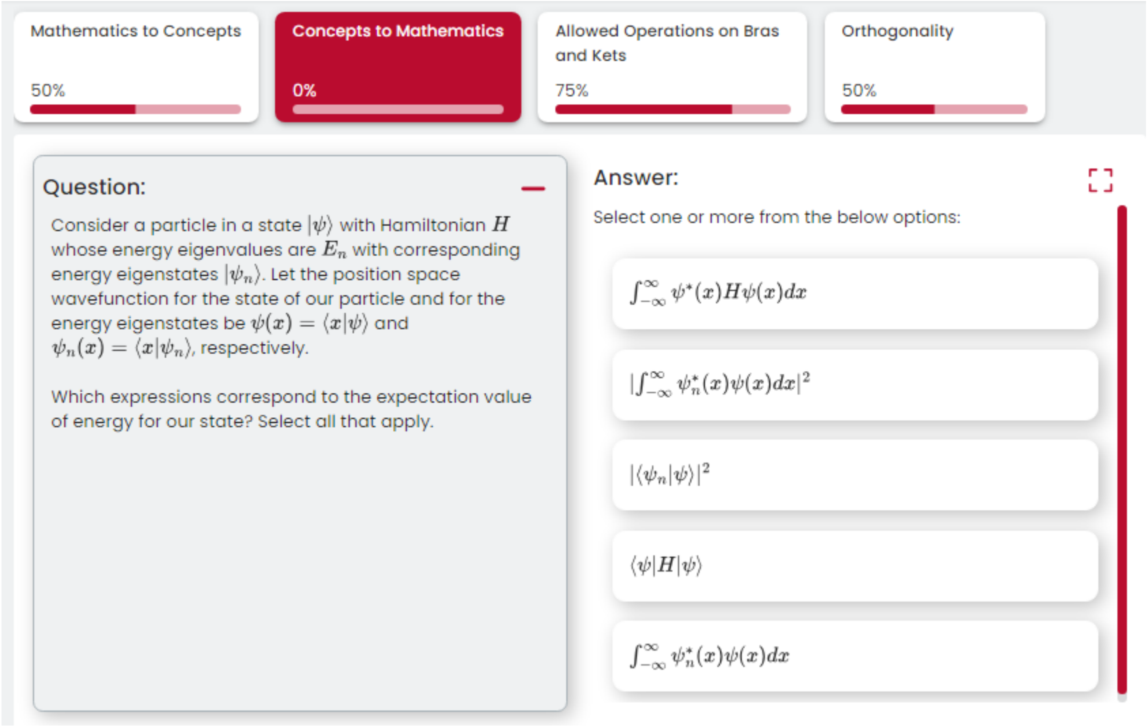
Figure 7.3 shows the student interface for one of the questions used in this study. The meters across the top indicate the percentage of the mastery level for each category completed thus far by the student. If the question is answered correctly, the meter increments for the corresponding category. If the answer is incorrect, the meter resets to 0%. The next question will come from a random category in the assignment. Once a meter reaches 100% (*i.e.*, the mastery level decided by the instructor), the student no longer sees questions from that category within that assignment. Each question can be either multiple-choice or multiple-choice, multiple-response. (Each question relevant to this work was multiple-choice, multiple-response.) The assignment is completed when *mastery* is achieved for all categories.

To use the STEM Fluency platform effectively, we needed to (1) decide on the essential skills, (2) design the questions, and (3) decide on ways to group them into subcategories, categories, and topics. Over several quarters, we reviewed the errors made by students on various exam questions and made reference to the research literature. The process of developing the questions and the full categorization scheme is discussed in Chapters 3 and 4. Here, we describe a few of the categories and subcategories that we developed and the impact of the practice on student learning.

7.4 Implementation of Essential Skills Practice

Essential Skills practice was implemented over several years in the first quarter of a junior-level quantum mechanics course. During this time, the questions and format of the practice were being developed and refined. The results from the last implementation (Autumn 2023) are discussed below.

The Autumn 2023 quantum mechanics course had 67 students enrolled. The textbook used a wave-functions-first (positions-first) approach [27]. Weekly small-group sections were held based on *Tutorials in Physics: Quantum Mechanics* developed by the University of Washington [106]. The standard course homework assigned by the instructor was supplemented with tutorial homework and STEM Fluency assignments. The latter made up 2% of the



Mathematics to Concepts
50%

Concepts to Mathematics
0%

Allowed Operations on Bras and Kets
75%

Orthogonality
50%

Question:

Consider a particle in a state $|\psi\rangle$ with Hamiltonian H whose energy eigenvalues are E_n with corresponding energy eigenstates $|\psi_n\rangle$. Let the position space wavefunction for the state of our particle and for the energy eigenstates be $\psi(x) = \langle x|\psi\rangle$ and $\psi_n(x) = \langle x|\psi_n\rangle$, respectively.

Which expressions correspond to the expectation value of energy for our state? Select all that apply.

Answer:

Select one or more from the below options:

$\int_{-\infty}^{\infty} \psi^*(x) H \psi(x) dx$

$|\int_{-\infty}^{\infty} \psi_n^*(x) \psi(x) dx|^2$

$|\langle \psi_n|\psi\rangle|^2$

$\langle \psi|H|\psi\rangle$

$\int_{-\infty}^{\infty} \psi_n^*(x) \psi(x) dx$

Figure 7.3: Example of STEM Fluency interface showing a question presented to a student for the category *Translating Concept to Mathematics*. At top are the names of the categories selected for the assignment being worked on. The associated meters depict how close students are to acquiring mastery in each category (*e.g.*, 50% for Orthogonality). In the administration of STEM Fluency discussed in this work, the category at top for the question being shown students was not highlighted.

students' final grade and were graded based on effort.

In designing the STEM Fluency homework, we focused on four areas that research suggests are especially difficult for students. These correspond to the following *topics* that we developed for STEM Fluency. The first (*Mathematical Procedures and Linear Algebra*) is on common mathematical procedures that students perform when solving quantum mechanics problems. The second (*Pre-Dirac Notation*) involves basic skills that often occur before Dirac notation in our courses, such as time-dependence or conceptual ideas related to probability densities. The third (*Mathematization*) involves decisions students must make when solving problems, such as deciding which notations to use or how to represent various quantities. The fourth (*Translating Representations*) involves translation, either between different mathematical notations or between these notations and their conceptual counterparts. Each topic contains multiple categories, and each category has a variety of subcategories.

The focus of this article is a particular category within the topic of *Translating Representations*: *Translating Concepts to Mathematics*. The four subcategories for this category are shown in Figure 7.2. This category was chosen since many exam questions require students to translate from a conceptual idea to a corresponding mathematical expression (*e.g.*, finding the probability density for a given situation). However, translating is a bi-directional process and students should also be able to associate mathematical expressions with the related concepts. Each homework that included the category *Translating Concepts to Mathematics* included the converse category, and the practice it provided likely played a role in any improvements in performance that we observed.

In the course, we administered weekly STEM Fluency assignments designed to be completed in less than 30 minutes. We repeated most categories across several assignments since spaced practice has been shown to be beneficial in helping students develop fluency [81]. Each STEM Fluency assignment contained between 3 and 4 categories, each of which had a subset of the associated subcategories. A typical category had its mastery level set at four. (Thus, students continued to see questions from that category until they had correctly answered four questions in a row for that category.)

7.5 Student Performance within STEM Fluency and on Exams

We present three related sets of data in our assessment of the homework. The first is (A) student performance on the STEM Fluency assignments relevant to this study. This includes both performance on the skills within an individual assignment and changes in performance on skills that appeared on repeated assignments. The second and third sets of data come from responses to exam questions administered to classes with and without Essential Skills practice. In particular we probed (B) the extent to which students were able to *identify* the correct concept that needed to be used to answer each question and (C) how successful students were in reaching a correct *final solution* to each exam question.

7.5.1 Student Performance within STEM Fluency

For each assignment and each student, STEM Fluency records the number of questions attempted, the number of questions answered correctly, and the total time taken on questions in a category or subcategory. Based on this data, we constructed two measures for student performance: *accuracy* and *average time per question* (TpQ). Since most categories and subcategories were included on multiple homework assignments, we also tracked changes in these measures.

Statistics used to Track Performance

Accuracy for an individual student: We define accuracy for a student within a category or subcategory in a particular assignment as the number of correctly answered questions in that category or subcategory divided by the number of attempted questions in that category or subcategory (*e.g.*, the percent correct). As an example, suppose that in a given homework, the mastery for a particular category was set at 3. If a student incorrectly answered the first question in that category and then correctly answered the next three questions in that category, their corresponding accuracy would be 3/4, or 75%.

Average Time per Question for an individual student: For a given category or subcategory, the average time per question (TpQ) for a student is defined as the total time spent on questions in that category or subcategory divided by the number of questions attempted for that category or subcategory. For example, if a student spent 300 seconds on 6 questions in a particular category or subcategory, then the average time per question is 300/6, or 50 seconds per question.

Class-wide Measures of Accuracy and TpQ: To summarize class-wide performance on a particular category or subcategory, we used the medians of the student accuracies and TpQ across the class. The standard error of the median was calculated using a bootstrapping procedure on the data set for each subcategory in each individual week. The median, not the average, was used as the measure of centrality, since the distributions were highly skewed for some subcategories. Hence, the mean did not seem to be a helpful measure for assessing how performance changed over time.

Although assignments were intended to be completed in less than 30 minutes, occasionally a student took much longer. In these cases, we believe a reasonable assumption is that the student left the assignment open while working on other tasks. This was corroborated by conversations with students. Thus, we opted to drop students from the analysis who had unusually large total times on a given administration. Ultimately, this amounted to only a few students across the entire course.

Performance

Figure 7.4 shows graphically the class performance on four subcategories within the category of translating from concepts to mathematics: (i) expectation values, (ii) probabilities, (iii) probability density, and (iv) eigenvalue equations. On each assignment, students were asked to choose the expression(s) denoting one of these quantities (see Figure 7.2). Both the median accuracy and median time per question are shown for each assignment that included that subcategory. Below is a discussion of how class performance in each subcategory changed

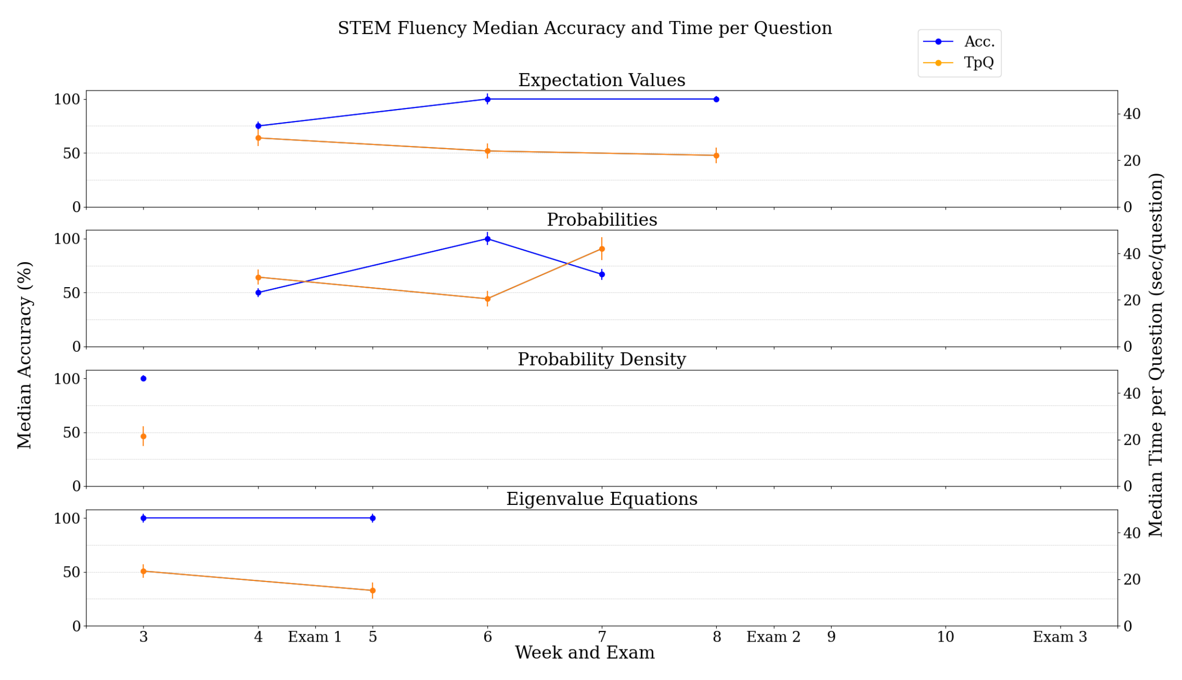


Figure 7.4: Class-wide median accuracies (blue), and median average time per question (orange) for each of the skills tested in the weekly STEM Fluency assignments. Weeks 1 and 2 are excluded since they did not include questions relevant to the ability to translate between physical concepts and their associated mathematical expressions.

during the quarter.

i. Expectation Values

Questions in the subcategory of *Expectation Values* appeared on the homework in weeks 4, 6, and 8. Initially the median class accuracy was 75%, so half of the students performed above that level, and half were below. Hence, many incorrectly answered one or more questions on expectation values. The week 6 and 8 homework assignments had a median of 100%, so more than half of the class had perfect accuracy. This improvement was significant relative to the standard error of the median.

The median *TpQ* in the subcategory for *Expectation Values* also improved, starting at about 30 seconds per question in week 4 and ending at 22 seconds per question in week 8. This change was also statistically significant.

Based on this data alone, it seems that the directed practice in identifying and producing expectation value expressions helped improve both student accuracy and speed in this subcategory.

ii. Probabilities

The *Probabilities* subcategory was administered on homework in weeks 4, 6, and 7. Questions in this category asked students to identify the mathematical expression(s) corresponding to the probability for various quantities. In week 4, the questions involved only wave function notation, which students had seen starting early in the course. In week 6, the questions were solely based on Dirac notation, which had recently been covered in lecture. Finally, the week 7 assignment had questions using both notations, mixed within each question.

Performance in week 4 (wave function notation) was relatively low, despite students having already worked through many problems involving probabilities in the course. The median accuracy was about 50%. In contrast, on the week 6 assignment on Dirac notation (which had just been introduced in lecture), students did very well, with more than half performing perfectly (a median score of 100%). In week 7 (mixed notation), the median accuracy went down to 67%.

Although the performance decreased between week 6 and 7, the latter assignment is arguably harder. Students needed to identify a mathematical expression for probability in both wave function and Dirac notation. It is promising however, that the accuracy in week 7 was statistically better than that in week 4.

The median time per question also showed improvement from week 4 (30 seconds/question) to week 6 (21 seconds/question). This suggests that students recognize probability expressions at different rates depending on the notational system used to express them. The median *TpQ* in week 7, however, increased significantly to 42 seconds/question. As might be expected, this suggests that students need more time to think about which expressions correspond to probabilities when having to consider two different notations.

iii. Probability Density

The subcategory of *Probability Density* was only administered once, in week 3 of the course. Even that early in the course, students seemed to perform well, with a median accuracy of 100% and a median *TpQ* of 22 seconds per question. Although one might infer from this result that students do not need practice with this subcategory, we found this is not necessarily the case when examining exam performance, as discussed later in this paper.

iv. Eigenvalue Equations

The subcategory of *Eigenvalue Equations* was administered in week 3 and week 5. The median accuracy was 100% for both assignments. Thus, in both cases, more than half of the students answered all questions correctly without making any errors. However, the median *TpQ* for this subcategory decreased from 24 to 15 seconds per question, a statistically significant decrease. Thus, although the accuracies were high on both assignments, the practice may have helped students identify and produce Eigenvalue Equation expressions more quickly.

Summary of STEM Fluency Data

Overall, the performance of students within STEM Fluency suggests that many benefited from practice on the subcategories that we identified. In two of the four cases (expectation values and probabilities), the median performance was not at 100% on the first administration but improved in either accuracy or time per question (or both) with repeated practice. In another case (probability density), students started out at 100% and, although they did not receive additional practice, we show later how this may not indicate that they had proficiency in this skill. In the final case (eigenvalue equations), the median accuracy started at 100%

but the time per question decreased significantly during the second administration.

The specific trends of the results for these four skills differ in some ways, but overall, the results suggest that the practice had a positive impact on the students. A more complete picture of the value of the homework, however, is obtained from comparing student performance on exam questions in quarters when Essential Skills practice was and was not available.

7.5.2 Impact on Student Ability to Identify the Correct Quantity on Exam Questions

The second method used to assess the Essential Skills practice was to examine the impact on student ability to apply the correct concept on exam questions. The questions came from three exams that had been administered in courses prior to the start of this study. (See Figure 7.1.) They were then re-administered on the corresponding exams in the course using STEM Fluency. These were selected because a correct answer required students apply the skills targeted in the STEM Fluency homework.

Nearly all aspects of all the courses with and without STEM Fluency were essentially the same. The same textbook and tutorials were used, and the courses had the same format. The primary difference was that the instructors differed, and that STEM Fluency was used. Our prior experience suggests that the exam results from the questions used to assess student learning were likely not significantly impacted by having had different instructors [31]. Thus, we believe that the STEM Fluency practice was responsible for any effect that we observed.

The essential skills practice was given in Autumn 2023 to a class of 67 students. The non-STEM Fluency courses on which the exam questions were asked were Autumn 2018 for exam 1 (N = 104) and exam 3 (N = 98) and Autumn 2015 for exam 2 (N = 76). (The numbers in Autumn 2018 on the two exams are not the same since not all students took both exams.)

Table 7.5.2 shows the concepts tested by each exam question and the concepts corresponding to the most common expressions written down by students. Note that not all students who wrote down the correct concept necessarily gave a correct solution to the problem.

Table 7.1: The correct (bold) and the most common incorrect concepts corresponding to the expressions given by students for each exam question. The percentage and numbers of students giving each response is shown for the classes without and with STEM Fluency (SF).

Exam/ Question	Mathematical Object	Percentage/Number of Responses	
Exam 1		w/o SF	w/ SF
Q1	**Probability Density**	(43%) 45	(52%) 36
	Probability	(40%) 42	(26%) 18
Q2	**Probability**	(85%) 88	(86%) 59
Q3	**Expectation Value**	(65%) 68	(87%) 60
	Probability	(15%) 16	(7%) 5
Exam 2			
Q1	**Probability**	(80%) 61	(95%) 62
	Expectation Value	(9%) 7	(2%) 1
	Probability Density	(5%) 4	(0%) 0
Q2	**Probability**	(75%) 57	(91%) 59
	Expectation Value	(12%) 9	(2%) 1
	Eigenvalue Equation	(5%) 4	(0%) 0
Exam 3			
Q1	**Probability Density**	(34%) 33	(67%) 43
	Wave Function/State	(56%) 55	(28%) 18
Q2	**Expectation Value**	(68%) 67	(86%) 55
	Eigenvalues	(9%) 9	(2%) 1
	Probabilities	(8%) 8	(0%) 0
Q3	**Eigenvalue Equations**	(15%) 15	(60%) 38
	Wave Function/State	(56%) 55	(31%) 20

Exam 1

For exam 1, question 1, students were asked to find the probability density for the position of a classical particle moving in a two-level well. Without Essential Skills practice, 40% of the students wrote down a probability, not a probability density. They did not seem to be distinguishing between the representations for these concepts. After Essential Skills practice was introduced, the percentage of students making this error was only 26%. (See Table 7.5.2.) The percentage correctly giving an expression for probability density increased from 43% to 52% - indicating an improved ability to identify the correct concept on this question.

For exam 1, question 2, students were asked to solve for the probability that the ball was in a particular region. Before and after STEM Fluency was introduced, the majority of students correctly wrote down an expression corresponding to a probability, 85% and 86%, respectively.

Exam 1 question 3 asked students about the average position of the ball, which corresponds to an expectation value. The percentage writing down a correct expression rose from 65% to 87% when Essential Skills practice was introduced. Prior to this practice, the most common error, given by 15% of the students, was to write down a quantity representing a probability. Only 7% did so in the class with this practice.

Exam 2

Exam 2 asked students for the probability of an energy measurement and a position measurement. Prior to the introduction of STEM Fluency, on the energy measurement question, 75% of the students responded correctly by giving an expression corresponding to a probability. About 12% gave an expression that could be interpreted as an expectation value and 5% wrote down some part of the eigenvalue equation for the energy. After STEM Fluency was introduced, 91% of the students correctly gave a probability expression and only one student (2%) gave an expectation value. No one gave an expression that appeared to be part of an eigenvalue equation.

A similar pattern was observed for the position measurement. As shown in Table 7.5.2, prior to Essential Skills practice, 80% of the students correctly responded with an expression for probability. About 9% gave an expectation value, and 5% gave a probability density.

After working through the STEM Fluency homework, 95% of the students wrote down an expression corresponding to a probability. Only one student (the same one as in the case of the energy measurement) gave an expectation value and none gave a probability density.

On both parts of Exam 2, the percentage of students who correctly gave an expression for probability was higher in the classes with STEM Fluency. However, the numbers of students were small, so it was not clear whether the changes were significant. It is notable, however, that the percentage of students giving a quantity that did not correspond to a probability decreased, often with no students making that error.

Exam 3

On exam 3, students were shown the wave function for an electron in the hydrogen atom and asked three questions. The first was about the time dependence of the probability density. The second was about the expectation value for the square of the orbital angular momentum. The third asked whether the state is an eigenstate of two observables (the Hamiltonian and the square of the orbital angular momentum).

For question 1, only about 34% of the students in the class without STEM Fluency correctly wrote down an expression for a probability density. The most common error was to reason based on the form of the wave function or the state itself. For example, 56% of the students claimed that because the state is a superposition of eigenstates, the probability density is time dependent. They failed to recognize the role of degeneracy. With STEM Fluency, 67% of students correctly responded by calculating a probability density. Only 28% referenced just the state itself.

For question 2, students were asked to calculate an expectation value of the z-component of the orbital angular momentum for the given state. Prior to STEM Fluency, about 68% of the students used the correct quantity in their response. All of these students either wrote down a weighted average or wrote down (and tried to simplify) the quantity $\langle\psi|L_z|\psi\rangle$. Other students gave other quantities such as a list of eigenvalues present in the wave function (9%), or the probabilities of the eigenvalues (8%). After STEM Fluency was used, about 86% of the students responded correctly with an expectation value. Only one student (2%) responded by writing down an eigenvalue, and no students responded with just probabilities.

Finally, for question 3, students needed to indicate whether the state was an eigenstate of the Hamiltonian or the square of the orbital angular momentum, or both. We treated an answer as correct only if both parts were correct. Prior to STEM Fluency, only 15% correctly responded by calculating or discussing whether the state obeys an eigenvalue equation. About 56% responded by only referencing the state, for example by saying that it was a superposition and therefore not an eigenstate. This changed after STEM Fluency, when about 60% correctly used an eigenvalue equation and only 31% of the students incorrectly referenced just the state.

Summary of Exam Response Data

For all questions but one, the STEM Fluency homework seemed to have had a positive impact on student responses. (On that question, the ability of students to write down an expression for a probability was essentially unchanged.) Overall, students were better able to translate the verbal description of the quantity into the appropriate mathematical expression. Since the practice was the only substantial difference between the courses, we are encouraged and believe that the changes are likely attributable to that component of the course.

Note that simply because students responded with the correct mathematical object, it does not follow that they then answered the question correctly. Most of the problems are multistep, sometimes requiring several mathematical steps and/or conceptual knowledge to frame the analysis. Although the impact documented above is promising, we were also interested in the extent to which the STEM Fluency practice helps students arrive at a correct solution. This analysis is discussed in the following section.

7.5.3 Impact on Exam Performance

Improving student performance on exam questions was a significant reason Essential Skills practice was incorporated into the quantum mechanics course. We had hypothesized that if students are more fluent in translating between concepts and the related mathematics, they would make fewer errors in choosing the appropriate quantity when solving a problem, and ultimately would be more successful in reaching a correct answer.

Table 7.5.3 shows the percentage of students in each class who correctly answered each question. In this analysis, we considered a response as "correct" if a student gave the correct answer with correct reasoning (for those questions that asked for an explanation). Since there were typically multiple ways a student could respond to each question, no single rubric was used.

Table 7.2: Exam performance data for both pre-STEM Fluency courses and the post-STEM Fluency course, as well as the primary skill tested for each question. Both N's for the corresponding pre-STEM Fluency course and the post-STEM Fluency course are provided. For a given course, the number of students taking each exam may not be the same because historically our quantum mechanics courses allow students to drop one exam so some students skip the second exam.

Exam	Question	Skill being Tested	w/o STEM Fluency	w/ STEM Fluency	Change in Performance	p-value
Exam 1	1	Probability Density	41	57	16	$< .05$
$N_{2018} = 104$, $N_{2023} = 69$	2	Probability	35	60	25	$< .001$
	3	Expectation Value	13	30	17	$< .01$
Exam 2	1	Probability	66	86	20	$< .01$

Continued on next page

Table 7.2: Exam performance data for both pre-STEM Fluency courses and the post-STEM Fluency course, as well as the primary skill tested for each question. Both N's for the corresponding pre-STEM Fluency course and the post-STEM Fluency course are provided. For a given course, the number of students taking each exam may not be the same because historically our quantum mechanics courses allow students to drop one exam so some students skip the second exam. (Continued)

$N_{2015} = 76$, $N_{2023} = 65$	2	Probability	34	62	28	$< .01$
Exam 3	1	Probability Density	26	70	44	$< .001$
(Final Exam)	2	Expectation Value	36	86	50	$< .001$
$N_{2018} = 98$, $N_{2023} = 64$	3	Eigenvalue Equations	26	67	41	$< .001$

The results demonstrate a statistically significant improvement on each exam question, based on a chi-square test of significance on the number of students. The p-values for each question are shown in Table 7.5.3. As an example, consider Exam 1 Question 1, on which students were asked to produce an expression for the probability density for a classical scenario of a ball rolling on a hill. Of the students who did not use STEM Fluency, 41% answered correctly, while this increased to 57% for students with STEM Fluency. The difference of 16% is statistically significant under a chi-square test of significance with $p < .05$. As a separate measure of significance, we calculated 95% confidence intervals on the effect size between the two groups of students for each exam question. We used the odds ratio as a measure of effect size. We found that each question had odds ratios > 1 with 95% confidence, indicating that the practice had a positive effect on student performance on these exam questions.

Another observation can be made: skills that were included in multiple STEM Fluency assignments saw larger gains after repeated practice. For example, on Exam 1, both probability density and expectation values were tested for and on the questions requiring the related translational skills, students demonstrated significant improvements compared to non-STEM Fluency. On Exam 3, after additional practice, both questions involving these skills saw even larger gains.

7.5.4 Summary of the Data Sets

Each of the three analyses discussed above tells a different, but consistent, story. First, we observed that for each of the four skills targeted in STEM Fluency, students generally became more accurate and faster. We also observed that, on exam questions, students who had STEM Fluency practice were more successful in translating verbal statements into correct mathematical expressions and ultimately were more able to answer the questions correctly than students without the practice.

7.6 Progression of Student Learning

The previous section discusses the impact of Essential Skills practice separately on individual course components (the STEM Fluency assignments themselves and the specific exam questions). This section draws on the same data presented above through a timeline to help illustrate *how* the practice improved student performance on the exam questions. Figure 7.5 shows the timing of the STEM Fluency assignments on each given topic together with the associated exam questions. The percentage of students answering each exam question correctly are given for the classes with and without STEM Fluency with arrows that indicate changes in performance.

7.6.1 Expectation Values

As shown in Figure 7.5, the STEM Fluency assignments that included questions on expectation values occurred in weeks 4, 6, and 8. Assessment on this topic was on Exam 1 in week 5 and again on Exam 3 in week 11.

On the initial STEM Fluency assignment, the class had a median accuracy of 75%. One

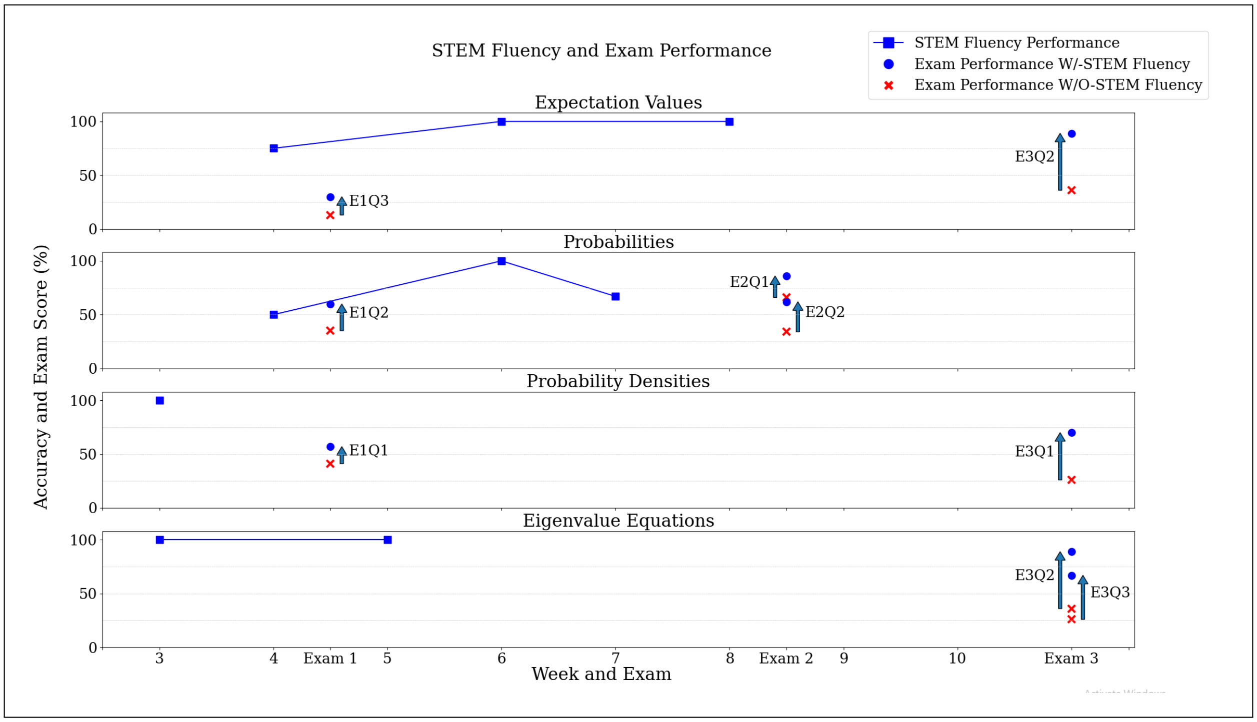


Figure 7.5: Timeline for each of the skills in STEM Fluency related to translating between concepts and mathematics. Each graph corresponds to a different subcategory. The lines on each graph indicate the weekly performance of students on that subcategory within STEM Fluency. The vertical arrows indicate shifts in exam performance between students who did and did not use STEM Fluency. Exam questions are coded EnQm for "Exam n, Question m."

week later, on Exam 1 Question 3, the correct answer was given by 30% of the students. In the class without STEM Fluency, only 13% had given a correct answer to the same question administered at the same time in the course. This is only a 17% increase but represents a doubling of correct responses. Moreover, there was a marked difference in the incorrect answers given by the two populations. As shown in Table 7.5.2, even after just the one practice assignment on this topic, 22% more of the students were able to give a quantity that represented an expectation value.

The class with STEM Fluency had two additional homework assignments on expectation values, and, as discussed earlier, the median accuracy jumped to 100% and the median TpQ decreased significantly. At the end of the course, on Exam 3 Question 2, the difference between the classes with and without STEM Fluency was even larger. On this question, students were asked to calculate the expectation value for the L_z operator. As shown in Figure 7.5, 50% more of the students with STEM Fluency answered correctly and 18% more correctly wrote down an expression for an expectation value (see Table 7.5.2).

On several of the exams, there was also a decrease in the percentage of students who wrote down an expectation value when asked for some other quantity. For example, on Exam 2 given in week 8, both questions 1 and 2 asked students to produce an expression for a *probability*. Without the practice, the most common incorrect mathematical object that students wrote down was an expectation value.

7.6.2 Probabilities

The sequence of STEM Fluency assignments and exam questions on probability is shown in the second row of Figure 7.5. This practice occurred in weeks 4, 6, and 7, with related assessment questions on Exams 1 and 2.

As mentioned earlier, the first assignment only involved wave function notation and proved difficult, yielding a class median accuracy of 50%. When tested on Exam 1 Question 2, about 60% of the students with the practice answered correctly, whereas only 35% did so in the class without the practice.

The results from the two additional practice assignments on probabilities in STEM Fluency involved Dirac notation (week 6), and a mixture of Dirac and wave function notations (week 7). This practice was followed up by Exam 2, on which the two questions asked students for the probability of a position measurement and the probability of an energy measurement. With the practice, the percentage of correct responses rose by 20% (from 66% to 86%) and 28% (from 34% to 62%), respectively.

There was also an increase in the percentage of students who answered the problem by writing down an expression for a probability and not some other quantity, as was the case for expectation values. As can be seen in Table 7.5.2, after the practice more students responded with probabilities, and almost no students wrote down an expression for an expectation value, a probability density, or a part of an eigenvalue equation.

7.6.3 Probability Densities

Practice on probability densities occurred only once - in week 3 of the quarter. (In previous quarters, students did well on the first homework involving this skill, and we had thus reduced the amount of practice.) As can be seen in Figure 7.5, the median accuracy of the class was 100%. There were two assessments through exams, one early in the quarter (Exam 1 Question 1) and one at the end (Exam 3 Question 1).

On Exam 1 Question 1, students were asked to find a probability density. In the class that had not used STEM Fluency, 43% of the students answered correctly. With STEM Fluency, this increased to 52%. In addition, the percentage of students who wrote down an expression for a different quantity decreased significantly (see Table 7.5.2). The most common error, giving a probability, decreased by 14%.

Exam 3 Question 1, given in week 11, asked students about the time dependence of the probability density for a particular state. Although the students had not received any additional practice in STEM Fluency on probability density there was a considerable gain, changing from 26% to 70% correct.

As we found with other exam questions, the gains in performance seemed to result at least partly from improvement in student ability to distinguish expressions for probability density from those for other quantities. On Exam 3 Question 1, 56% of the students without STEM Fluency commonly answered by referencing the state itself or some characteristic of the state (*i.e.*, the fact that it is a superposition). This dropped to 28% after STEM Fluency practice (see Table 7.5.2).

7.6.4 Eigenvalue Equations

Practice in STEM Fluency on eigenvalue equations occurred in weeks 3 and 5. On each, the class performed well, with median accuracies of 100%. Even though they did well on the first administration, it seemed to have an impact, since the time per question decreased on the second assignment (see Figure 7.4).

Questions examining the effect of the practice occurred only late in the course. These were Questions 2 and 3 given on Exam 3 in the final week.

For Question 2, students were asked for the expectation value of the observable, L_z, for a given state. Although this question did not ask directly about eigenvalue equations, we observed an effect that seemed to be related to student ability to identify and apply eigenvalue equations. Prior to the use of STEM Fluency, only 36% of the students answered correctly. Most of the successful students found the sum of the eigenvalues for the two eigenstates present in the state, weighted by their respective probabilities. Others gave the correct expression $\langle\psi|L_z|\psi\rangle$ but *none* solved it correctly by utilizing an eigenvalue equation. Some explicitly stated that the effect of L_z on the bra represented a measurement of L_z. After STEM Fluency, 89% of the students answered correctly with a significant increase in the students who were able to identify $L_z|\psi\rangle$ as an eigenvalue equation.

For Exam 3 Question 3, students were asked to determine whether the given state was an eigenstate of two quantities (the Hamiltonian and the square of the angular momentum). At its core, the answer depends on whether the state obeys the eigenvalue equation for each. The percentage of students giving a correct answer was 41% greater in the class with STEM Fluency (increasing from 26% to 67% correct).

On this question, in the class without STEM Fluency, 56% of the students answered by referencing only the state. For example, many claimed that since it is a superposition of eigenstates it is not an eigenstate. This argument would be true if the angular momentum was not degenerate, but it is not a general result. Only 15% of the students included an eigenvalue equation as a part of their answer. With STEM Fluency practice, many more students responded by giving eigenvalue equations (60%).

7.6.5 Summary

The timelines presented in this section provide additional insight into the impact of the STEM Fluency practice on student ability to translate between the targeted concepts and their mathematical formulations. For some skills, students worked through several STEM Fluency assignments and in those cases we generally observed increasingly large impacts as the course progressed. For other skills, there was only one (or possibly two homework assignments) given early in the course, but even in those cases, we observed an impact at the end of the course. We even observed large effects on exam performance for skills on which students seemed strong on early STEM Fluency assignments.

The results suggest that overall, the class benefited significantly from using STEM Fluency. In essentially all cases, we found that students who completed the practice identified the relevant concepts more frequently then those without the practice and were more able to arrive at a correct final answer.

7.7 Conclusion

In this paper, we document the impact of weekly homework assignments developed to give students targeted practice in the skill of translating between core concepts in quantum mechanics and their mathematical representations. The homework was administered using an online system, STEM Fluency, designed to help students develop speed and accuracy in applying various essential skills. The impact was assessed by examining performance both on STEM Fluency assignments throughout the course and on examinations that had been given to students with and without the practice.

There was improvement on nearly all the questions we examined. Some of the most significant findings include:

1. Repetition of a given skill within STEM Fluency typically resulted in improved accuracy and/or speed on later assignments. This was true even if students did well on their first trial.
2. In the class with STEM Fluency, student ability to translate from concepts to mathematics on exam questions was typically significantly better than for classes in which the STEM Fluency homework was not used.
3. Even if a given category was only covered in STEM Fluency homework early in the course, there was an impact on exam performance late in the course.
4. The tendency of students to write down incorrect concepts on exam questions was significantly lower in the class with STEM Fluency homework.
5. The students who used STEM Fluency were much more likely to arrive at a final, correct answer to the exam questions that were posed.

This paper assesses only the skill of translating from certain concepts to mathematics (expectation values, probabilities, probability densities, and eignvalue equations). The results, however, cannot be taken in isolation. The students who used STEM Fluency worked through questions addressing several other skills and that practice could have contributed to the improvement in student performance. In addition, all the classes in this study included small group sections based on *Tutorials in Physics: Quantum Mechanics*. These tutorials focus on strengthening student understanding of many of the concepts addressed in the STEM Fluency homework. Given that each class in the study used the *Tutorials*, the improvements seem to be due to the use of STEM Fluency, not the *Tutorials*. Nonetheless, if the *Tutorials* had not been included it is possible that the improvements would not have been as great.

A question one might ask is *which* students benefit from the practice. In an early, preliminary analysis, we found that, for most of the STEM Fluency categories, all students benefit to some extent, through improvement in accuracy, speed, or both (as seen in Chapter 6). Although students who received a grade in the upper third of the class typically had higher accuracy on the first administration of a given category or subcategory than the others, they still improved with practice. Students in the lower third benefited even more, resulting in a narrowing of the gap between the groups. These findings are consistent with those from the use of STEM Fluency in introductory courses [85].

Finally, we note that there are many ways in which the results of research into student learning of quantum mechanics might be synthesized and used to identify essential skills. There is more that can be done, both in terms of identifying and categorizing the difficulties that students encounter and in incorporating the results into an effective and efficient practice regime. Nonetheless, the results are promising and suggest that the use of homework that provides students with practice in performing certain critical skills can significantly benefit students in their learning.

Chapter 8

8. ADDITIONAL ASSESSMENTS OF ESSENTIAL SKILLS PRACTICE

In the previous two chapters, we demonstrated how we utilized STEM Fluency and student responses to exam questions to assess the effectiveness of the essential skills practice that we constructed. In those scenarios, the practice appeared to help students develop fluency as demonstrated by their performance within STEM Fluency and their ability to answer exam questions that tested the various skills that were required to answer them. As such, it seems that the practice benefited the students' ability to reason with and understand quantum mechanics.

In this chapter, we provide additional assessments of the effect of the practice by discussing smaller-scale analyses of other skills. We first begin in Section 8.1 by discussing how focused practice on the use of Clebsch-Gordan tables affected students' solution strategies on an exam problem involving multiple angular momenta. In Section 8.2, we discuss the effect of the practice on student performance on a problem involving degenerate perturbation theory. Section 8.3 will detail findings within STEM Fluency data associated with skills over expanding states in various bases and what we can learn from the atypical findings. Finally, Section 8.4 compares STEM Fluency results with exam performance, detailing an example where improvements in the practice can change student responses though not correctly.

We note from the start that the initial implementation of STEM Fluency discussed in this dissertation did not always lead to demonstrably beneficial outcomes. We will outline our thoughts on the implications of these observations and what we believe can be done to improve our practice regimen.

8.1 Assessing Student Reasoning on a Question Involving a Sum of Multiple Angular Momenta

Various research articles have documented observed student difficulties in reasoning about systems that involve a sum of angular momenta (*e.g.*, a particle with both orbital and spin angular momentum or a two particle system in which each particle has spin.) Some articles have described students difficulties in determining the dimension of the Hilbert space associated with particles with single and multiple angular momenta [141]. Others describe student difficulties in using Clebsch-Gordan tables to move between the two bases commonly used in scenarios with multiple angular momenta or distinguish between these two bases [141, 123]. Learning how to manage multiple angular momenta in quantum mechanics is important, since various applications of quantum mechanics rely on a proficiency in this management, such as the Stark effect, Zeeman effect, and fine structure applications [50, 121, 66].

In an attempt to address these difficulties, we combed the research literature and synthesized the results to identify eight total skills, We organized these into three categories. These categories encompass the following classes of skills.

- "Addition of Angular Momentum - Dimension of Hilbert Space": This category includes skills involving the identification of the dimension of the Hilbert space of particular with particular magnitude-square quantum numbers. There are two subcategories in this category:
    - "Dimension of Hilbert Space - Single": Students must identify the dimension of the Hilbert space for a particle with one angular momentum.
    - "Dimension of Hilbert Space - Multiple": Students must identify the dimension of the Hilbert space for a situation involving a sum of multiple angular momenta.
- "Addition of Angular Momentum - Change Bases": This category includes skills that require students to use Clebsch-Gordan tables to translate between the different bases

prevalent in problems with multiple angular momenta. There are two subcategories in this category:

  - “Change Bases - Separate to Total”: Students must translate a state expanded in the “separate” basis consisting of eigenstates of the z-components of both angular momentum operators, to the “total” basis consisting of eigenstates of the total angular momentum squared operator using a Clebsch-Gordan table.
  - “Change Bases - Total to Separate”: Students must translate a state expanded in the “total” basis consisting of eigenstates of the z-components of both angular momentum operators, to the “separate” basis consisting of eigenstates of the total angular momentum squared operator using a Clebsch-Gordan table.

- “Addition of Angular Momentum - Probabilities”: This category has skills involving calculating probabilities of measuring various outcomes from particles with multiple angular momenta. Students may need to translate from one basis to another using Clebsch-Gordan tables to do so. There are four subcategories in this category:
  - “Separate to Total - Given”: Students are given information needed to translate a state in the “separate” basis to the “total” basis in order to calculate the probability of a particular angular momentum operator measurement.
  - “Total to Separate - Given”: Students are given information needed to translate a state in the “total” basis to the “separate” basis in order to calculate the probability of a particular angular momentum operator measurement.
  - “Separate to Total - Build”: Students are presented a state in the “separate” basis and must use a Clebsch-Gordan table to translate it into the “total” basis in order to calculate the probability of a particular angular momentum measurement.
  - “Total to Separate - Build”: Students are presented a state in the “total” basis and must use a Clebsch-Gordan table to translate it into the “separate” basis in

order to calculate the probability of a particular angular momentum measurement.

In this section, we will discuss changes in student performance on the skill “Change Bases - Separate to Total” within the “Addition of Angular Momentum - Change Bases” category. The reason we only focus on this one skill is due to the fact that we had an exam question that had been given previously that required students to apply this skill to solve the problem. As such, we could use this skill to assess the effect of this practice on both student responses and student performance on this exam question.

8.1.1 Implementation Details for Addition of Angular Momenta and Clebsch-Gordan Tables

Skills associated with angular momentum in quantum mechanics were practiced by students in PHYS 325 in both the Winter 2023 and Winter 2024 quarters. See Table 8.2.1 for the details of both administrations. Before discussing the analysis of the impact of the practice in one of these skills, we outline the exam question used to assess the effectiveness of the practice in improving student understanding.

Figure 8.1 contains a problem that was asked of students in Winter 2019 and Winter 2024. The students in Winter 2019 did not complete essential skills practice, whereas those in Winter 2024 did. The problem presents students with a particle whose state in the $|l, m_l; s, m_s\rangle$ (henceforth referred to as the “separate” basis) is given. The question that we are concerned with is question a), which asks students to determine if j, the quantum number associated with the total angular momentum operator $\vec{J} = \vec{L} + \vec{S}$, is well-defined.

To answer question 1a) correctly, students must express the state of the particle in the basis $|j, m_j\rangle$ (henceforth referred to as the “total” basis). In order to do this, a Clebsch-Gordan table must be used. In this case, the state of the particle is a linear combination of states with different j values. As such, j is not well-defined.

The solution described above requires, among other things, that students be able to use a Clebsch-Gordan table to translate a state written in the separate basis to the total basis.

(1) An electron is in the state $|l, m_l\rangle|s, m_s\rangle = |2, -2\rangle|\frac{1}{2}, \frac{1}{2}\rangle$. Here $\hat{J} = \hat{L} + \hat{S}$.

  (a) Is the square of the total angular momentum ($\hat{J}^2$) of the electron well-defined? If yes, write down the eigenvalue. If no, write down the possible eigenvalues. In either case, explain how you arrive at your answer.

  (b) What are the possible outcomes if you measure the z-component of the total angular momentum, $\hat{J}_z$, of the electron? Explain.

  (c) What are the possible outcomes if you instead measure the x-component of the total angular momentum, $\hat{J}_x$, of the electron? Explain.

(2) Consider two non-interacting, identical spin-$\frac{1}{2}$ particles in the one-dimensional harmonic oscillator potential. The energy of each particle is independent of its spin state. The energy eigenfunctions of the 1D harmonic oscillator potential are written as $\psi_n(x)$.

Determine the energy of the **second excited state** and write down the possible full energy eigenfunctions (spatial and spin) for the two-particle system.

Figure 8.1: The question presents students with the state of a particle with both orbital and spin angular momentum expanded in the $|l^2, m_l; s^2, m_s\rangle$ eigenbasis. Students must determine if the value of J^2 for the particle is well-defined, the possible values of J_z that can be measured, and the possible value of J_x that can be measured. The question was initially administered on the first exam in Winter 2019 and readministered on the first exam in Winter 2024.

With this in mind, one of the skills we gave students practice in is the use and reading of Clebsch-Gordan tables for moving between these bases. The skill we are interested in this analysis is called “Change Bases - Separate to Total”. In this section, we will look at STEM Fluency data in this subcategory and see how performance in this skill changed as students practiced it.

8.1.2 Changes in STEM Fluency Performance on Skills Related to Addition of Angular Momenta

We present results from two quarters in which the STEM Fluency practice was given, although only one of the classes had the exam problem shown in Figure 8.1. Figure 8.2 contains graphs detailing the median accuracy and time per question for the “Change Bases - Separate to Total” subcategory across multiple administrations of the practice in various homework assignments.

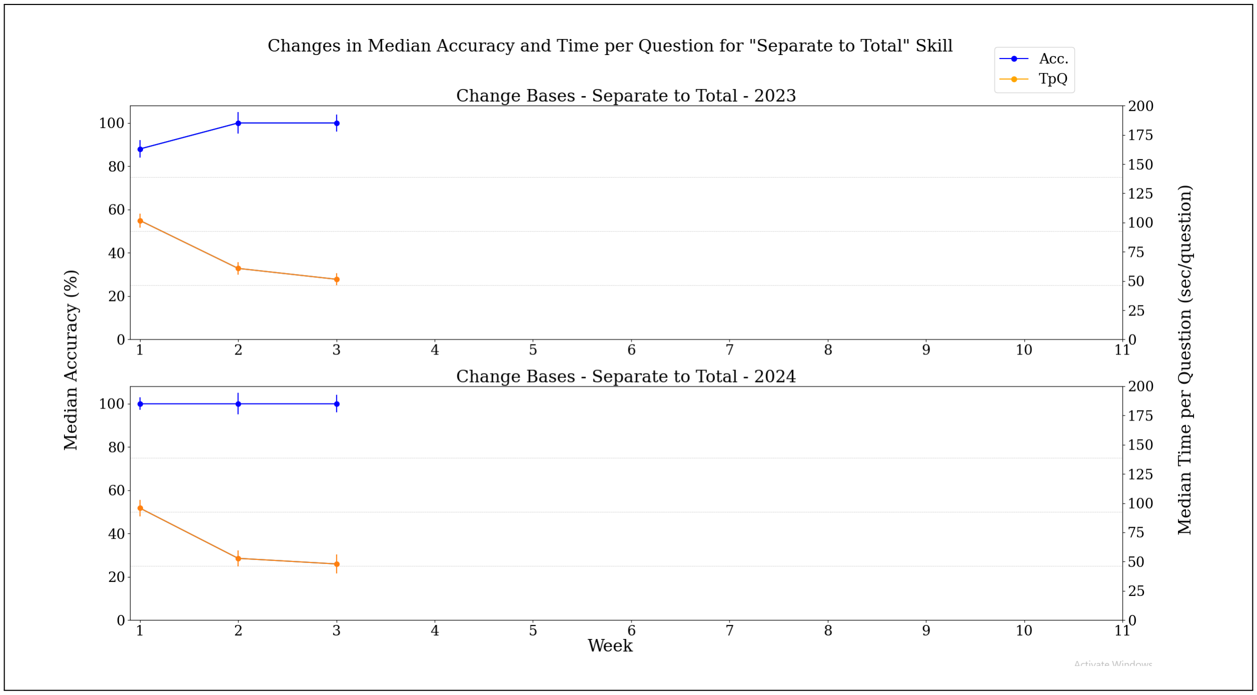


Figure 8.2: These graphs contain information on student performance on the skill “Change Bases - Separate to Total” in both the Winter 2023 and Winter 2024 administration. The blue line represents the median accuracy of the class, and the orange line represents the median time per question of the class. While students practiced this skill more than these administrations, we include only those weeks before the assessment using the exam question in Figure 8.3 given in week 5.

In Winter 2023, we observe statistically significant improvements in the median accuracy for the class, increasing from 88% to 100% between weeks 1 and 2. The accuracy stays at 100% in week 3. In Winter 2024, the median accuracy started and stayed at 100% for all weeks, meaning more than half of the students performed perfectly in this subcategory in that quarter for these administrations. In addition, we observe statistically significant improvements in the time per question in both Winter 2023 and Winter 2024. In both cases, the median time per question for the class significantly decreased from week 1 to week 2 and likewise from week 2 to week 3.

Overall, student performance on "Change Bases - Separate to Total" improved with additional homework practice. Students performed more accurately executing this skill and simultaneously got faster in doing so. Next, we analyze student responses to the exam question depicted in Figure 8.1 in order to assess if these performance improvements in STEM Fluency affected how they solved a problem requiring the execution of the skill.

8.1.3 Changes in Exam Data on a Question Involving Addition of Angular Momenta

The question in Figure 8.1 above was first administered to students in Winter 2019 to students who had not completed STEM Fluency practice. We readministered the question in Winter 2024 to students who had completed essential skills practice in order to assess the effect of the practice on student understanding of skills related to angular momentum. In both cases, this question was asked in the first midterm of the quarter in week 5. Table 8.1.3 contains data on the performance of students in both administrations on question a) in this problem.

Table 8.1: This table depicts difference in performance on an exam question that required students to execute the skill "Change Bases - Separate to Total". The question was first administered in Winter 2019 to students that did not complete the practice, and then readministered to students who completed the practice in Winter 2024.

Exam Performance on Question Requiring the Separate to Total Skill

Continued on next page

Table 8.1: This table depicts difference in performance on an exam question that required students to execute the skill “Change Bases - Separate to Total”. The question was first administered in Winter 2019 to students that did not complete the practice, and then readministered to students who completed the practice in Winter 2024. (Continued)

Skill Being Tested	Percent of Students with no SF Practice who Answered the Question Correctly	Percent of Students with SF Practice who Answered the Question Correctly	p-value
Change Bases - Separate to Total	35%	69%	<.001

Student performance was much better in Winter 2024 after students had completed the practice. In Winter 2019, only 35% of students were able to answer this question correctly and provide a correct explanation. In Winter 2024, after the practice, 69% of students were able to answer this question correctly with correct explanation. This increase in performance was statistically significant.

With these improvements in exam question performance in mind, we examined the student explanations to this question. To do this, we analyzed student responses and classified them according to whether or not students used a Clebsch-Gordan table in solving the question being analyzed. Table 8.1.3 contains information on the number of students who utilized Clebsch-Gordan tables when determining information about the allowed j values given the initial state in the separate basis.

Table 8.2: The table contains the percentage of students who explicitly and correctly used Clebsch-Gordan tables to translate from one basis to another in the exam problem depicted above. By observing the differences in the percentage of students that used Clebsch-Gordan tables between the populations of students that did and did not complete the practice, we can assess the effect that the practice had on how students approached the problem depicted above.

Exam Responses Using or Referencing Clebsch-Gordan Tables	
Percent of Students with no SF Practice who Referenced Clebsch-Gordan Tables	Percent of Students with SF Practice who Referenced Clebsch-Gordan Tables
15%	62%

In Winter 2019, only 15% of students explicitly referenced a Clebsch-Gordan table in determining which j values could be measured. This is a necessary step, since our state is in the separate basis, whereas information on what j values could be measured is in the total basis. A Clebsch-Gordan table allows one to translate between these bases in order to see what the state is in the total basis. In contrast, in Winter 2024, 62% of students directly used or referenced the use of a Clebsch-Gordan table. Thus, it appears that the practice had an effect on the way students approached this problem. In particular, students who completed the essential skills practice were more likely to use Clebsch-Gordan tables even when not prompted to do so.

8.1.4 *Discussion and Summary of Takeaways for Practice on Addition of Angular Momenta*

In both Winter 2023 and Winter 2024, we saw students become more accurate and quicker in translating from the separate basis to the total basis in the context of addition of angular momentum. Further, this improvement manifested as improvements on exam question performance as well: students were more able to identify the possible j values one could measure given a state in the separate basis. This improvement in exam performance was accompanied by changes in how students approached solving the problem. In particular, students were more likely to invoke the use of a Clebsch-Gordan table and utilize it to translate the given state to the total basis.

Putting these observations together, similar to the results in Chapter 7, this example provides another case in which the essential skills practice seems to have improved student performance on a common question by affecting how students approach the problem. As students completed the essential skills practice, students became more accurate and quicker in their execution of the skill involving translating from the separate basis to the total basis. In addition, the students that completed the practice were more likely to comment on and utilize Clebsch-Gordan tables in their solutions, and this was correlated with improvements on performance on the question at hand.

8.2 Assessing Student Performance on a Question Involving Degenerate Perturbation Theory

As mentioned in Chapter 2, there has been work studying student understanding of degenerate perturbation theory [45, 53]. For example, students often fail to identify a "good" basis and do not seem to understand the role such a basis plays in degenerate perturbation theory. Further, students often fail to identify the conditions under which degenerate perturbation theory is necessary. Finally, multiple works studying student ability to apply degenerate perturbation theory in various specific applications, *e.g.*, fine structure corrections and Zeeman effect correction, have found similar issues [66, 49, 65].

Many of the skills we identified were based on the difficulties described in those papers. We drew on these difficulties to develop a set of STEM Fluency questions to help improve student skill in multiple areas. A full list of the skills (and their respective categories) are the following.

- "DPT - Basic Examples": This category includes skills that have students identify the eigenvalues and eigenvectors of common, simple 2×2 and 3×3 matrices one often encounters in degenerate perturbation theory problems. There are four subcategories:

  - "Diagonalize - 2×2": Students must find the eigenvalues of common 2×2 matrices often encountered in degenerate perturbation theory problems.
  - "Diagonalize - 3×3": Students must find the eigenvalues of common 3×3 matrices often encountered in degenerate perturbation theory problems.
  - "Find Good Basis - 2×2": Students must find the eigenvectors of common 2×2 matrices often encountered in degenerate perturbation theory problems.
  - "Find Good Basis - 3×3": Students must find the eigenvectors of common 3×3 matrices often encountered in degenerate perturbation theory problems.

- "DPT - Find H' in Deg Subspace: This category includes skills involving students producing expressions for the matrix they must diagonalize in a degenerate perturbation theory problem. There is one subcategory for this category:
  - "Given Matrix": Students must identify the submatrix of H' that one must diagonalize in a degenerate perturbation theory problem.
- "DPT - Need to Use DPT?": This category contains skills asking students to identify, in various scenarios, whether or not one needs to use degenerate perturbation theory instead of normal time independent perturbation theory. There are two subcategories:
  - "Given H^0 and H': Given matrices for a Hamiltonian and a perturbation, students must identify whether or not degenerate perturbation theory is needed.
  - "Given System": Given a physical system and a perturbation in words, usually in the form of a description (*i.e.*, a 2D simple harmonic oscillator, or a 3D infinite square well, *etc.*), students must identify whether or not degenerate perturbation theory is needed.

Each of these skills both match the difficulties observed in the literature as well as our observations on exam questions involving degenerate perturbation theory.

Here, we will discuss our implementation of STEM Fluency practice for skills involving degenerate perturbation theory and present the exam question used to assess the practice. We also describe the skills tested for by this questions. In **??**, we present our assessment of the effect of the practice on student performance on an exam question on degenerate perturbation theory. In 8.2.2, we outline the trends in the STEM Fluency data for the relevant skills and our observations on student performance on the exam question.

8.2.1 Implementation Details for Degenerate Perturbation Theory Practice and Assessment

The skills involved with degenerate perturbation theory were practiced in PHYS 325 during both the Winter 2023 and Winter 2024 quarters. See Table 8.2.1 for the details of these administrations. Since the exam question only involves some of the skills, we begin by discussing the exam question used to assess the effectiveness of the practice in improving student understanding.

Table 8.3: A list of each administration of STEM Fluency administered to our upper-division quantum mechanics students relevant for this section. For each administration, we list the quarter of the administration, the course the administration occurred in, the number of students, N, in each course, and the total number of STEM Fluency homework assignments used.

List of Administrations of STEM Fluency Practice			
Quarter of Administration	Course of Administration	N for each Course	Number of STEM Fluency Assignments
Winter 2023	PHYS 325	43	10
Winter 2024	PHYS 325	45	9

Figure 8.3 contains an exam question that has students work through a standard degenerate perturbation theory problem. The problem was first administered in Winter 2022 to 43 students who did not complete STEM Fluency (as seen in Table 8.2.1). We readministered

it in Winter 2024 to 41 students who had completed STEM Fluency practice. The question gives students a two-dimensional quantum harmonic oscillator with a small perturbation and guides students through calculating the "good" basis in which to do perturbation theory on the (degenerate) first-excited states.

The Hamiltonian of a two-dimensional (2D) simple harmonic oscillator is given below:

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \frac{1}{2}m\omega^2\left(x^2 + y^2\right) \tag{8.1}$$

The corresponding energy eigenstates are $\psi_m(x)\psi_n(y) \equiv |m, n\rangle$.

(1) Using the form about for the eigenstates, list the ground state(s) and first excited state(s). What is the degeneracy of each?

(2) Consider adding a perturbation $H' = \gamma(x^4 + y^4)$, where γ is a small parameter. Write the new Hamiltonian for our system.

(3) Write expressions for the matrix elements of H' in the $\{|1, 0\rangle, |0, 1\rangle\}$ basis. Indicate explicitly how many matrix elements there should be, and organize these matrix elements into a matrix. Do not perform any calculations for this question.

(4) Are the off-diagonal matrix elements zero or non-zero? Explain. (Hint: use symmetry and parity!)

(5) Is the original basis of $\{|0, 1\rangle, |1, 0\rangle\}$ a "good basis" to use with perturbation theory for this problem? Explain why you made this choice. Why does your answer make sense?

Figure 8.3: This question guides students through a degenerate perturbation theory problem for the first excited state of a 2D quantum harmonic oscillator. The objective is to determine the "good" basis in which to do perturbation theory for this energy level. This question was administered for the first time in Winter 2022 on Exam 2 and then readministered in Winter 2024 on Exam 2.

For the purposes of this analysis, we focus on questions (3)-(5). To answer these questions, students need to reconstruct the matrix for H' in the degenerate subspace (in this case, the two-dimensional eigenspace corresponding to the first excited state energy). Students then need to evaluate the matrix elements and diagonalize the matrix to obtain the "good" basis. In this instance, the initial first excited states $|1, 0\rangle$ and $|0, 1\rangle$ (corresponding to excitations in the x and y directions, respectively) already form a "good" basis in which one can do perturbation theory.

In executing such a solution, one must first write down the matrix within H', the perturbation, to be diagonalized. This corresponds to the skill "Finding H' in the Degenerate Subspace - Given Matrix", which was part of the STEM Fluency practice. Then, after solving for the matrix elements, students must diagonalize the resulting matrix, which corresponds to the skill "Find Good Basis - 2 × 2". Finally, students must recognize that the resulting eigenvectors form a "good" basis in which to do perturbation theory, which is related to the skill "Given H^0 and H'". In the following sections, we discuss these three subcategories from our Winter 2023 and Winter 2024 administrations of STEM Fluency to see how they impacted students.

8.2.2 *Changes in STEM Fluency Performance on Skills Related to Degenerate Perturbation Theory*

Figures 8.4 and 8.5 contain two graphs illustrating the changes in the class-wide median accuracy and time per question for each skill relevant to the exam question, along with the standard error of the median, calculated using a bootstrapping procedure. The first graph shows the progression for the Winter 2023 administration, and the second contains the progression for the Winter 2024 administration.

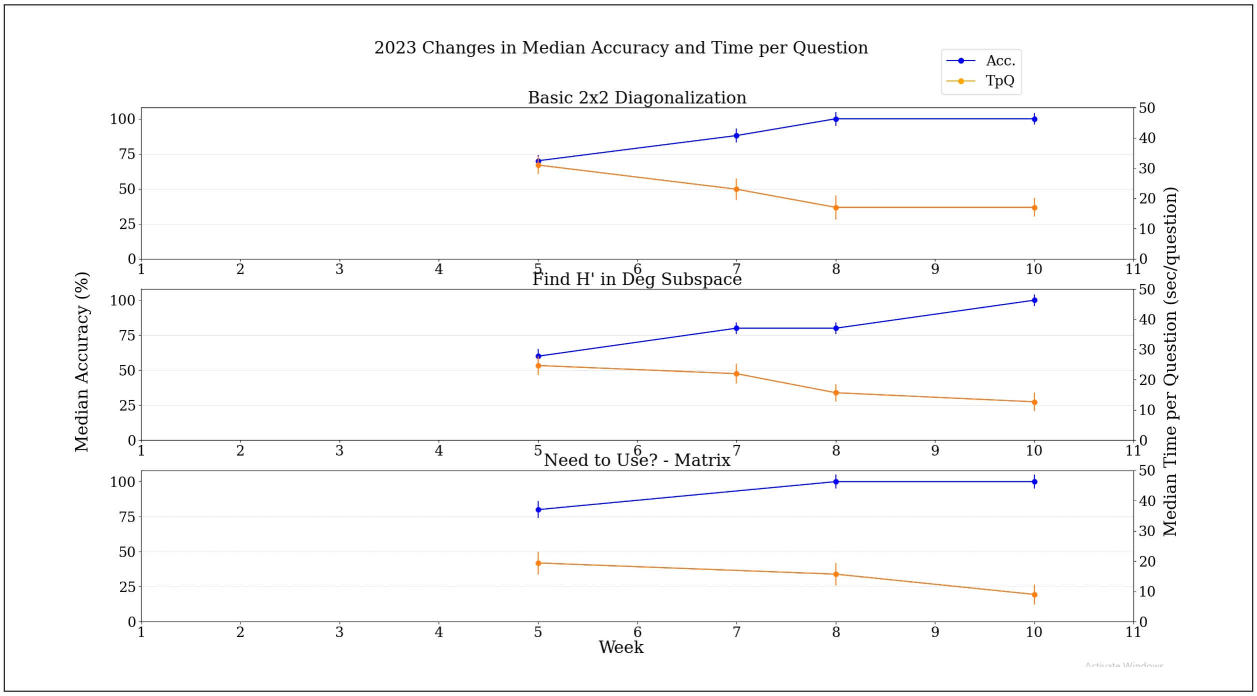


Figure 8.4: These graphs depict the median accuracy and time per question for three subcategories required to solve the exam problem in 8.3 in the Winter 2023 quarter. The blue lines represent the median accuracy for the class, and the orange line represents the median time per question for the class. Generally, we see that the median accuracy for the class improves with more practice, and the median time per question for the class decreases with more practice.

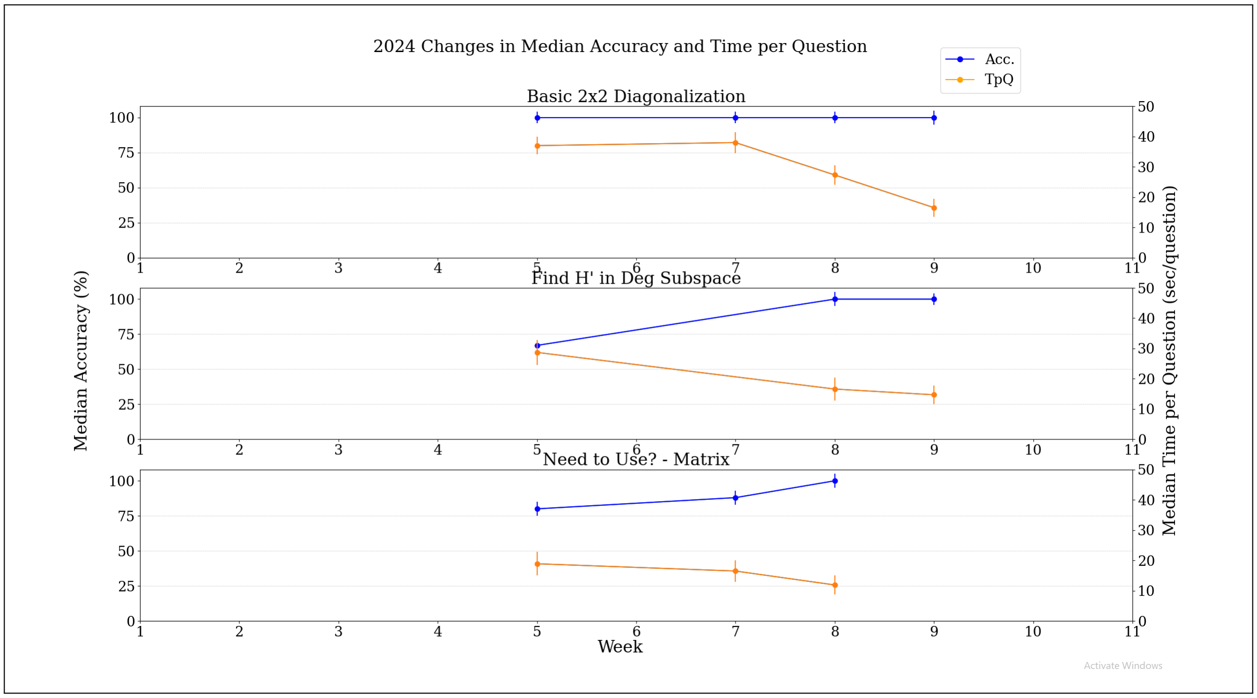


Figure 8.5: These graphs depict the median accuracy and time per question for three subcategories required to solve the exam problem in 8.3 in the Winter 2024 quarter. The blue lines represent the median accuracy for the class, and the orange line represents the median time per question for the class. Generally, we see that the median accuracy for the class improves with more practice, and the median time per question for the class decreases with more practice.

In each quarter, for each subcategory, we observe improvements in accuracy and speed. For example, in 2023, students practiced diagonalizing 2x2 matrices in weeks 5, 7, 8, and 10 of the course. The class-wide median accuracy started at 70%, increasing to 88% in week 7, and reaching 100% in week 8. Further, the median time per question for the class started at 31 seconds per question in week 5, decreasing to 23 seconds per question in week 7, and finally reaching 17 seconds per question in week 8. Overall, we see a general trend of students becoming more accurate and faster in applying these three skills. The only case where we do not observe this trend is in Winter 2024 on the “FindGoodBasis - 2x2” subcategory, where the median accuracy for the class *started* at 100%.

Notice that for both administrations of the practice in these skills, we see roughly identical progressions in the median accuracy and time per question of the class. As such, it seems that the results of the practice are relatively robust.

In the next section, we will present evidence that, even with these improvements in performance within STEM Fluency, student performance on related exam problems did not appear to reflect this.

8.2.3 Changes in Exam Performance Data on a Question Involving Degenerate Perturbation Theory

Figure 8.3 above details the exam problem originally administered in Winter 2022 to students who did not complete STEM Fluency and then readministered in Winter 2024 to students who did complete the practice. In both cases, this question was included on the second midterm of the quarter in week 9 of the course. We wished to see if the presence of the practice had any effect on student performance on the problem.

Table 8.2.3 contains information on student performance on the relevant questions in the problem, namely questions (3)-(5). In each case, it appears that the performance on each question stayed roughly the same. Further, the ways students responded to these questions was similar in Winter 2022 and Winter 2024.

Table 8.4: Exam performance data for both pre-STEM Fluency courses and the post-STEM Fluency course, as well as the primary skill tested for each question in the exam problem outlined in Figure 8.3. This exam question was administered in Winter 2024 on the second midterm.

Performance on an Exam Question Requiring the use of DPT Skills					
Question	Skill being Tested	w/o STEM Fluency	w/ STEM Fluency	Change in Performance	p-value
(3)	Finding H' in the Degenerate Subspace - Given Matrix	74	78	4	.84
(4)	Find Good Basis - 2 × 2	37	45	8	.6
(5)	Given H^0 and H'	44	39	-5	.73

8.2.4 Discussion and Summary of Takeaways for Practice on Degenerate Perturbation Theory

All together, we observed that student accuracy and time per question improved as students practiced a handful of skills related to degenerate perturbation theory (Section 8.2.2). However, this did not seem to yield improved performance on the related exam questions (Section 8.2.3). Since the core of the Essential Skills Framework is to help improve students' ability to solve problems *through* improved accuracy and fluency, this seems to suggest that these do not always happen together. When it comes to degenerate perturbation theory, the practice did not seem to impact student performance on the exam problem.

There are various explanations as to why this could have occurred. First, students may not have received enough practice in these skills as to translate into solving multistep problems. In the class where the practice was administered, students had three homework assignments involving the related skills before the exam problem was given. Pergaps more practice was required.

Alternatively, these skills may not be as amenable to essential skills practice as others discussed earlier. Practicing these skills in isolation (within STEM Fluency) led to improvements in performance, but there might be a hurdle for students to recognize when and how to apply these skills in the context of a multistep problem.

An interesting implication is that there may be more to learn about student difficulties in reasoning about degenerate perturbation theory. If providing students with practice and seeing improvements in the use of the skills does not translate into improvements in solving multistep practice, then there may be difficulties we have not yet observed or articulated properly. Alternatively, the difficulties themselves may not be addressable with essential skills practice. In either case, the result of the essential skills practice plus the exam performance to assess the effect of the practice points towards the potential for new, notable implications about the structure of student difficulties in topics that have already been studied.

8.3 Assessing Student Performance on Practice Questions Involving Eigenbases

As another example of how essential skills practice can reveal new trajectories in physics education research, consider the following example involving eigenbases. The notion of eigentheory is both foundational to quantum mechanics and in addition a difficult topic for students to reason with. Singh *et al.* found that students will often expand states in eigenbases incorrectly. Common errors are not including coefficients or using the eigenvalues as the coefficients in the expansion [109]. Many students have difficulties conceptualizing the role the eigenstates play in measurements and the time dependence of expectation values [137, 68]. The errors they make seem to correlate with student difficulty in deciding which states are allowable states for particles [116]. Finally, many students have trouble with the formal aspects of expanding states in eigenbases, for example what this means, and why we

Let $|\psi\rangle$ be the state of a particle. Let H be a Hamiltonian with energy eigenstates $|\psi_n\rangle$ with corresponding energy eigenvalues E_n. Finally, recall that the position-space wave functions for our state and the energy eigenstates are $\psi(x) = \langle x|\psi\rangle$ and $\psi_n(x) = \langle x|\psi_n\rangle$, respectively.

Which of the following corresponds to, generally, $|\psi\rangle$ and/or $\psi(x)$ expanded in the energy eigenbasis? Select all that apply.

- ☐ $\psi(x) = c_1\psi_1(x) + c_2\psi_2(x) + ...$, where each c_n is complex.
- ☐ $|\psi\rangle = c_1|\psi_1\rangle + c_2|\psi_2\rangle + ...$, where each c_n is complex.
- ☐ $\psi(x) = E_1\psi_1(x) + E_2\psi_2(x) + ...$
- ☐ $|\psi\rangle = E_1|\psi_1\rangle + E_2|\psi_2\rangle + ...$
- ☐ $|\psi\rangle = c(|\psi_1\rangle + |\psi_2\rangle + ...)$, where c is complex.

Figure 8.6: An example question from the subcategory "Expanding a State in an Eigenbasis". Students are tasked with identifying a generic expression for the expansion of a given state in a particular eigenbasis.

can do so [112].

To aid students with the *mechanics* of using eigenbases, we designed two sets of questions for STEM Fluency that focus on two skills. One involves expanding a state given to students, abstractly, in a given eigenbasis. This amounts to simply identifying an expression corresponding to said expansion. The other skill involves expanding a state in a *generic* basis. We chose this skill since, according to Wawro *et al.*, students' ideas about the role of eigentheory are often nuanced, finding that students often had more to say about the role of eigenbases in various contexts [133]. This is to be expected, since eigenbases play a special role in quantum mechanics, so students might think about them differently compared to arbitrary bases, hence why we provided separate practice in eigenbases and arbitrary bases. Figure 8.6 contains a sample question from the "Eigenbases" subcategory

In this section, we will discuss student performance in both these essential skills practice. In particular, we will outline the trends found in median accuracy and time per question across several administrations of the practice and comment on what we believe the trends imply about our knowledge of student difficulties involving eigenbases.

8.3.1 Implementation Details for, and Changes in Performance on, Skills Involving Expanding States in a Basis

Questions involving expanding states in a basis were given in three administrations of STEM Fluency in PHYS 324. Table 8.3.1 below contains information on which weeks students practiced these skills in each administration. In what follows, we combine information on administrations for arbitrary bases and eigenbases because these skills were always offered simultaneously.

Table 8.5: Administration Information for each offering of practice in the category "Expanding a State in a Basis". Each quarter the category was practiced twice with both of the "Arbitrary Bases" and "Eigenbases" subcategories being offered. While the weeks of practice are slightly different, they were practiced at roughly the same point in the course where Dirac notation was first being introduced.

Administration Information for Expanding States in a Basis Category

Quarter	First Administration Week	Last Administration Week
Fall 2021	5	6
Fall 2022	4	5
Fall 2023	4	6

In each administration, students practiced both the skills "Expanding states in Arbitrary Bases" and "Expanding States in Eigenbases" in two homeworks. With this in mind, we can compare student performance on both skills on the first and second offerings of the practice in order to assess how it changed.

Figure 8.7 contains graphs depicting changes in the class-wide median accuracy and time

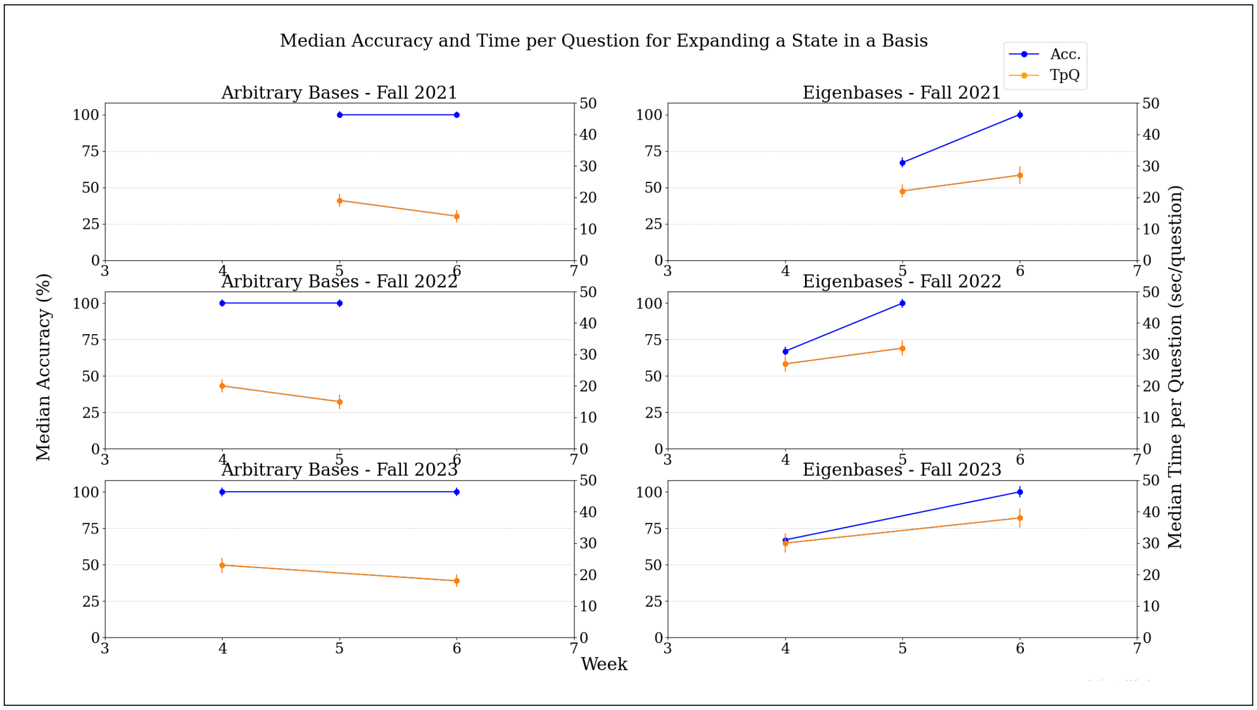


Figure 8.7: Class-wide median accuracy and time per question data for all three administrations of the “Arbitrary Bases” and “Eigenbases” subcategories. Note that each administration offered practice in each skill at different times, though roughly the same time.

per question for each administration of both the “Arbitrary Basis” and “Eigenbasis” subcategories. The first column of graphs contains data on the “Arbitrary Basis” subcategory, and the second columns contains data on the “Eigenbasis” subcategory. Each row is each administration in chronological order.

We can notice a few trends from these graphs. First, students performed fairly accurately on “Arbitrary Basis”, starting at 100% median accuracy and maintaining this accuracy with continued practice. Further, though not depicted, we observed the number of students below 100% median accuracy decrease with more practice (usually such students were in the bottom quartile of final course performance). This was seen in all three administrations of the practice. Second, students got significantly faster, with improved median time per question in each quarter. As such, students seemed to start and stay accurate at expanding states in arbitrary bases, and got quicker in doing so.

For "Eigenbases", though, we see a different story. First, students initially struggled to expand states in an eigenbasis, starting off with median class-wide accuracies of 67% in all three quarters. Students improved, though, reaching 100% median accuracy with continued practice. As for their median time per question, though, students got significantly *slower*. Each quarter shows median time per question increasing with continued practice, indicating that students are spending more time on average per question for questions involving expanding a state in specifically an eigenbasis.

Further, this pattern seems to manifest among students regardless of their final course performance. Figure 8.8 contains box and whisker plots conveying time per question data for the Fall 2023 quarter (similar trends were found in Fall 2021 and Fall 2022). The left two plots depict distributions of the time per question for the top quartile of students relative to final course grade, the middle two plots depict distributions of the time per question for the middle two quartiles of the course, and the left two plots depict distributions for the bottom quartile of the course.

For each student group, we see an increase in the median time per question (represented by an upward movement of the black bar in the middle of each plot) as well as an overall upward shift of the distribution in general. It's not just particular students who are slowing down. In Fall 2023, we see an upward shift in time per question for all groups of students in the class.

8.3.2 Discussion and Summary of Takeaways for Practice on Expanding States in a Basis

Different patterns in changes in accuracy and fluency were observed in STEM Fluency depending on whether or not students were expanding a state in an eigenbasis or not. With a few exceptions, students at all levels in all three administrations of the practice demonstrated the following patterns. When asked to expand a state in an arbitrary basis, each student

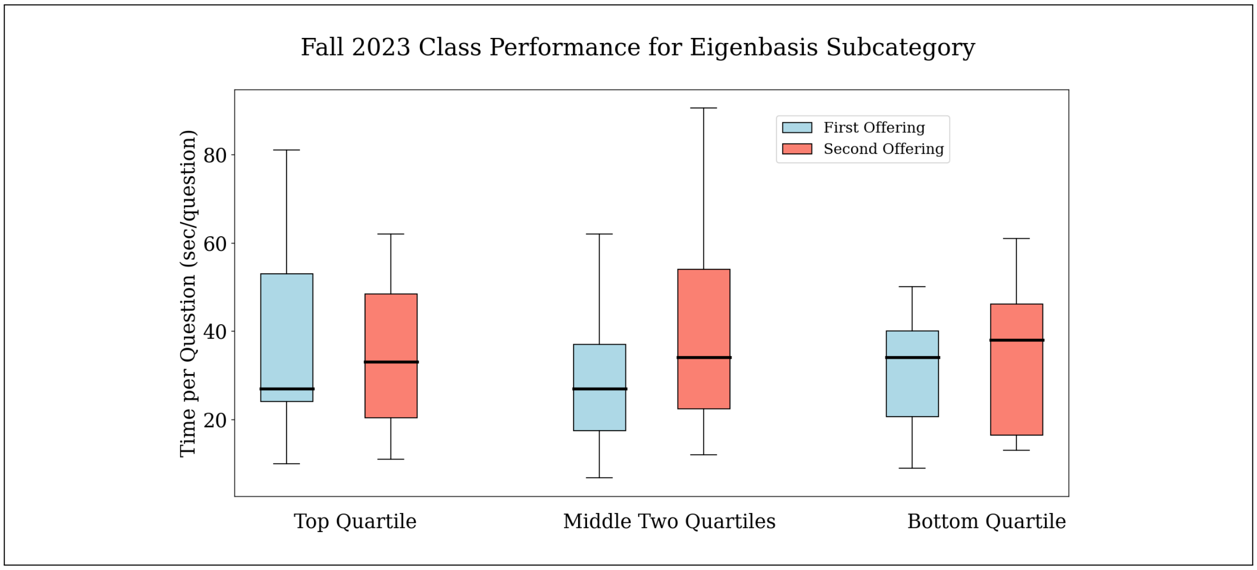


Figure 8.8: Box and whisker plots for the time per question for the “Eigenbases” subcategory in Fall 2023, split into three groups. The left group consists of the distributions of the initial and final administrations of the top quartile of students. The middle distributions consist of the data for the middle two quartiles of students. Finally, the left distributions consist of the data for the bottom quartile of students relative to final course performance.

group started off with 100% median accuracy and became quicker with more practice. When asked to expand a state *specifically* in an eigenbasis, though, each student group went from between 50% and 67% median accuracy to 100% median accuracy on their second practice (suggesting they struggled more with eigenbases at first compared to arbitrary bases). In addition, each student group got slower with more practice.

This seems to suggest that students think about eigenbases in specific, nuanced ways compared to arbitrary bases. That students slowed down with more practice means that students likely had to consider content and resources specific to eigenbases, causing them to slow down and be more reflective in their responses. Note that this behavior does not appear in the literature on student difficulties in quantum mechanics.

The results of the practice in these skills have seemingly revealed to us a new potential direction in research that could be pursued. For example, it could be that dual process theory provides some insight on these results [126]. In particular, students might be more reflective on their use of eigenbases, which could account for the slower time per question we observed for this skill as students got more practice. Such nunaces were revealed to us through the results of the essential skills practice, and thus the practice seems to again serve as a potential research tool by giving us further insights into student understanding and ideas about topics in physics.

8.4 Assessing Student Behaviors when Utilizing Completeness Relations

Compared to other mathematical tools, completeness relations see much less focus in quantum mechanics courses. That being said, there are a few studies that demonstrate that students have difficulties implementing them correctly as well as deciding when to use them. For example, Singh demonstrates that in both continuous and discrete cases, students in both undergraduate and graduate settings struggle to state completeness relations, and when they do, they often fail to utilize them correctly [109]. Further, Singh finds that students do not recognize how to use completeness relations to translate from Dirac notation to wave func-

Let x be the position operator, and let $|A\rangle$ be a ket. The completeness relation for position is $\int |x\rangle\langle x|\mathrm{d}x = 1$.

What is the result of inserting the completeness relation for position into $|A\rangle$?

○ $\int |x\rangle\langle x|A\rangle\mathrm{d}x$

○ $\int |A\rangle|x\rangle\langle x|\mathrm{d}x$

○ $\int |x\rangle|A\rangle\langle x|\mathrm{d}x$

○ $\int |x\rangle\langle x|\mathrm{d}x|A\rangle$

Figure 8.9: An example question from the subcategory "Completeness Relations - Continuous Basis". Students are given an initial expression (a bra, ket, inner product, *etc.*) and are tasked with identifying an expression that corresponds to the result of having inserted a completeness relation into the expression.

tion notation and vice versa [118]. Since students often work in one notation translate to another when solving problems, the conscious use of completeness relations for this purpose may be a helpful skill to develop.

With these observations in mind, we identified skills that involve the employment and recognition of completeness relations in both discrete and continuous contexts. The hope is that as students become more fluent in the use of completeness relations, they will utilize them more when solving problems and in particular when translating between notations. We identified two skills in category of "Completeness Relations": "Discrete Bases" and "Continuous Bases". "Discrete Bases" involves inserting a completeness relation for an operator with discrete eigenbasis into an expression, whereas "Continuous Basis" is likewise for operators with continuous eigenbases. Figure 8.9 contains an example question outlining the tasks students must perform when answering a question in the "Continuous Basis" subcategory.

8.4.1 Implementation Details for Completeness Relations

Practice involving completeness relations was administered in PHYS 324 in three different quarters. Details involving its implementation in these quarters is given in Table 8.4.1.

Table 8.6: Administration Information for each offering of practice in the category "Completeness Relations". Each quarter the category was practiced one or more times with both of the "Discrete Spectrum" and "Continuous Spectrum" subcategories being offered. While the weeks of practice are slightly different, they were practiced at roughly the same point in the course. In addition, the number of practice sessions for the category was changed from quarter to quarter based off of the results of previous quarters (namely, the better students in the course did, the less practice we offered them next time).

Administration Information for Completeness Relations Category		
Quarter	First Administration Week	Last Administration Week
Fall 2021	7	9
Fall 2022	7	10
Fall 2023	7	7

In Fall 2021, three administrations of practice for the completeness relations category were offered in weeks 7, 8, and 9. Fall 2022 offered two administrations in weeks 7 and 10. Finally, in Fall 2023, only one practice session was administered in week 7. Fewer practice sessions were offered from quarter to quarter based on student performance in previous quarters. For example, if students did well skills involving completeness relations in a previous quarter, the next administration would administer less practice in that category to make room for other skills students needed practice in.

Finally, although we did not have an exam question that explicitly assessed student ability to use completeness relations, an exam question from Fall 2021 (for students who completed the practice) showed that many students attempted to use completeness relations spontaneously. This problem asked students to produce mathematical expressions for the probability of two different measurements. What we had noticed on exam data in Fall 2021 is that on the energy probability question, students who completed the practice were more likely to explicitly mention the use of completeness relations when translating their energy probability in Dirac notation to that in wave function notation. As such, we will compare student responses to this question between those that did not complete the practice and those that did in order to see if the practice affected the ways students were using completeness relations in their solutions.

8.4.2 Changes in STEM Fluency Performance on Skills Related to Completeness Relations

Figure 8.10 depicts changes in class-wide median accuracy and time per question for the "Completeness Relations" category for all quarters it was administered (Fall 2021, Fall 2022, and Fall 2023). Notice that the practice occurred roughly at the same time in each quarter, though with different amounts of practice in each administration.

In general, we notice that students started off with 100% median accuracy in the course and maintained this median accuracy in each case. Further, we can see significant decreases in the median time per question, indicating that students got quicker after practicing this skill more. Overall, students started off fairly accurate and got quicker with more practice.

Further than this, even students who performed in the bottom quartile of final course performance generally reached 100% median accuracy as well. Figure 8.11 shows box and whisker plots for the bottom quartile of performers in PHYS 324 for the Fall 2021 and Fall 2022 administrations of the "Completeness Relations" practice (the two administrations where the practice in the category was repeated).

Initially, students in both Fall 2021 and Fall 2022 in the bottom quartile had a median accuracy less than 100%. This is in contrast to the top quartile of students and middle two quartiles in each administration, who had 100% median accuracies and distributions condensed around 100% accuracy (not shown). As such, the bottom quartile of students are really the only students who had room for improvement. After the practice, though, median accuracy for the bottom quartiles reached 100% median accuracy, and the distributions in

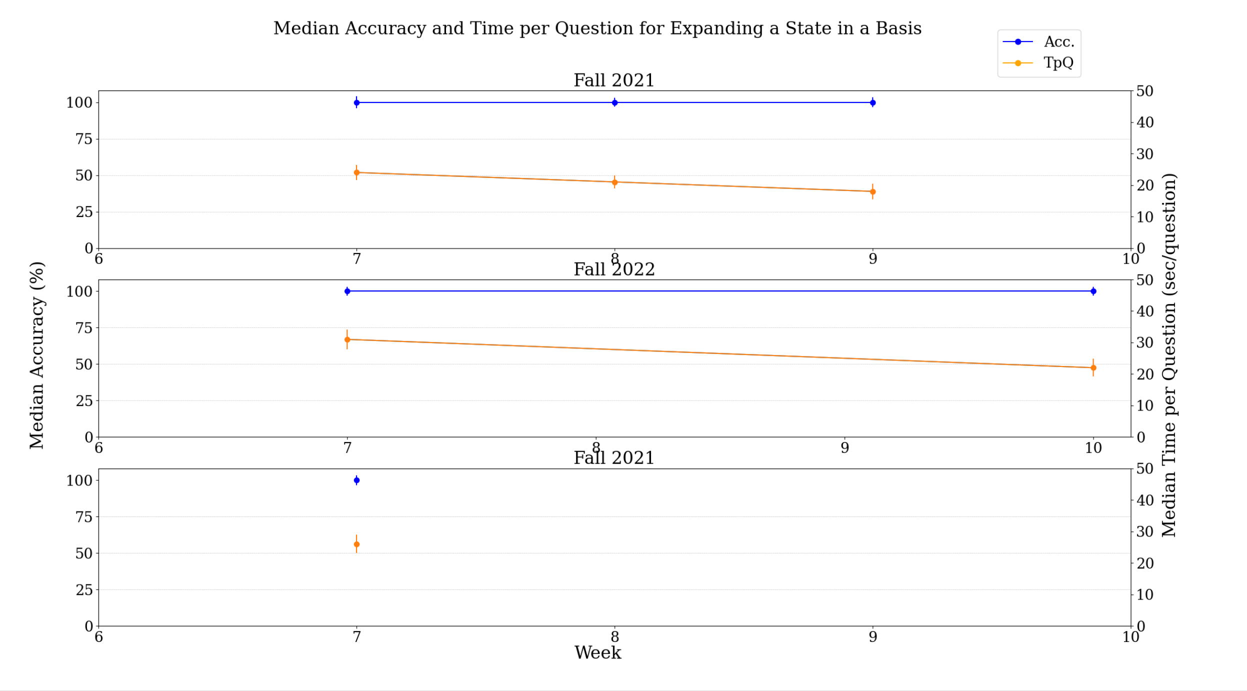


Figure 8.10: Class-wide median accuracy and time per question for all three administrations of the "Completeness Relations" category with continued practice. Note that the practice occurred in different amounts at different times.

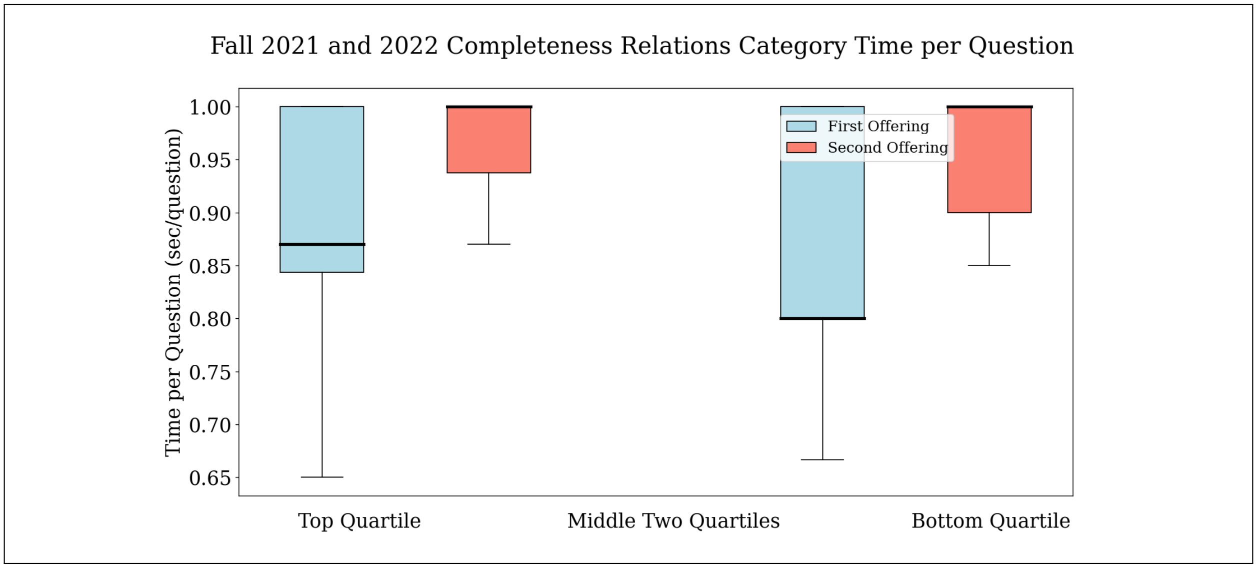


Figure 8.11: Box and whisker plots for the accuracy of the bottom quartile of students in the Fall 2021 and Fall 2022 administration of the "Completeness Relations" category for the initial and final practice in each administration. The solid black line represents the median of the group, the extent of the colored box the inner-quartile range, and the extent of the whiskers the full distribution of data.

their accuracies moved upwards, indicating most students improved their accuracy with more performance.

In total, students seemed to perform very well in this category on the practice. After more practice, students almost uniformly in the course reached 100% accuracy, indicating that they by the end of the practice, students were rarely answering completeness relations questions incorrectly. Overall, the practice seemed to make students very accurate as well as quick in their execution of skills related to completeness relations. As we will see, though, this does not necessarily translate to improved performance on exam questions.

8.4.3 Changes in Exam Response Patterns that Utilized Completeness Relations

To assess the effect of the practice on student ability to use completeness relations in exam questions, we will analyze some exam data where students spontaneously utilized completeness relations when solving exam problems. We will investigate exam problem A.1, administered first in Fall 2018 on the second exam of the course in Week 9, and readministered in Fall 2021 on the second exam of the course in Week 9 as well. We will focus on question A, which asks students to produce an expression for the probability of measuring a particular energy. In this question in Fall 2021, it was common for students to first answer in Dirac notation, and then convert their answer to wave function notation using a completeness relation.

Table 8.4.3 contains information on the use of completeness relations on this question. It details, for both Fall 2018 and Fall 2021, the percentage of students that got the question correct, the percentage and number of students that attempted the use a completeness relation, and the percentage of and number of those students who got the question correct.

Table 8.7: Percentages of students in Fall 2018 and Fall 2021 who explicitly utilized completeness relations on question A in Exam A.1. We include the percentage of students who got the question correct, the percentage and number of students who used completeness relations, and the percentage of number of students who *correctly* used completeness relations.

Quarter	Percentage of Students who Answered Correctly	Percentage (Number) of Students who Used Completeness Relations	Percentage (Number) of Students who *Correctly* Used Completeness Relations
Fall 2018	34%	1% (1)	1% (1)
Fall 2021	66%	9% (8)	0% (0)

In Fall 2018, 34% of students answered the question correctly, in contrast to 66% in Fall 2021 (as discussed in Chapter 6). In addition to this, although completeness relations were not explicitly queued in this question, many students who completed the practice in Fall 2021 decided to use completeness relations in their solution (up from 1% to 9%). Although the numbers are small, this difference is significant ($p = .007$ under a chi-square test of significance). As such, the practice seemed to have an effect on student ability to utilize completeness relations when solving this question even though completeness relations were not explicitly probed for in the question.

That being said, every student that decided to use completeness relations in Fall 2021 did so incorrectly. In comparison, the single student who used a completeness relation in Fall 2018 used it correctly. As such, even though more students utilized completeness relations after practicing them, this did not translate to correct use in the context of a multi-step problem.

8.4.4 Discussion and Summary of Takeaways for Practice on Completeness Relations

In all three quarters where completeness relations were practiced, we generally observed improvements in class-wide median accuracy and median time per question. Further, students at the bottom of the class relative to course performance, although initially struggling, managed to attain 100% median accuracy with continued practice when it was offered. As such, the practice seemed to have a positive impact on student accuracy and speed for the use of completeness relations.

Further, students that completed the practice were more likely to explicitly use completeness relations when converting a mathematical expression from Dirac notation to wave function notation on an exam question, even when not explicitly primed to do so. That being said, most students that did so did not utilize completeness relations correctly. As such, the practice seemed to have a tangible effect on how students responded to problems, but they still had trouble applying these skills correctly in the context of a multi-step problem.

Once again, it could mean that students need more practice, but students were already performing quite accurately in this particular skill at all levels of the class. As such, it is also possible that there's something more to student difficulties with completeness relations beyond their identification and use of them in scenarios documented in the literature. More work likely needs to be done to get a more holistic view of student understanding of completeness relations. That being said, it is a positive outcome that students recognize a use for completeness relations as a result of the practice, namely that of algebraically converting from Dirac notation to wave function notation. So while the practice suggests that more work likely needs to be done, the practice had some effect on student use of completeness relations.

8.5 Summary of Additional Analyses

In this chapter, we discussed additional assessments of essential skills practice on two more sets of skills. First, an assessment of practice on a skill related to the use of Clebsch-Gordan tables demonstrated, as in Chapter 7, that the practice benefits student learning. Practicing the skill requiring students to use Clebsch-Gordan tables to translate between bases in problems with multiple angular momenta resulted in more students consciously utilizing Clebsch-Gordan tables on an exam question that required their use, which resulted in better performance on said question.

In addition, we saw instances where the practice didn't produce the improvements observed with angular momenta and translating concepts to mathematics but still conveyed important information on the nature of student understanding of various topics practice. For example, we saw that despite improvements in performance in accuracy and speed in practice questions in skills utilized in degenerate perturbation theory, student performance on an exam question utilizing these skills did not see any differences. Further, we saw that as students practiced expanding states in a basis, students became more accurate and *slower* when specifically eigenbases were being practiced. Finally, we saw that despite improvements in accuracy and speed on the use of completeness relations (when possible) and more students consciously using them on a particular exam problem, students still tended to use them incorrectly.

The results in this chapter demonstrate a twofold utility of essential skills practice. First, corroborated by the results in Chapter 7, the practice seems to be a beneficial tool in helping improve student reasoning with the mathematics of quantum mechanics. Improvements in accuracy and speed often happened alongside changes in student response patterns as well as improved performance on exam questions requiring the use of these skills. Second, the practice seems to be a helpful research tool in that it can produce new potential directions of inquiry into student difficulties. That students can have improved performance in STEM Fluency but that may not yield improvements in performance on problems. This indicates that there may be more to learn about student difficulties in skills that fall under this umbrella. These new potential directions in research are provided by the results of the STEM Fluency practice.

All together, STEM Fluency has the ability to serve two purposes: 1) as an instructional tool to help improve student understanding, and 2) a research tool to help identify new potential directions into research on student difficulties. We believe this utility makes essential skills practice a valuable tool for instructors and researchers alike.

Chapter 9

9. CONCLUSION

Research on the learning and teaching of quantum mechanics has revealed that many students fail to develop a functional understanding of quantum mechanics after typical instruction [120, 25, 17]. Some of the difficulties that students encounter appear to be conceptual in nature and there have been curricula developed to address them [19, 115, 106, 94]. However, some of the errors that students make appear to be related to a lack of familiarity and fluency with the mathematical formalism inherent in the study of quantum mechanics and may require a different approach to instruction.

In this dissertation, we have discussed how we used the Essential Skills Framework to construct, implement, and assess a set of practice question sequences to improve student accuracy and fluency in applying basic quantum mechanical skills. The construction and implementation of this practice involved an in-depth review of the research literature on student understanding. About 120 skills were identified that students need to be able to apply in solving many of the problems posed in typical quantum mechanics courses. More than 1500 questions and homework assignments were then designed to promote student fluency and accuracy in applying these skills. These were implemented on homework assignments using STEM Fluency, developed by Andrew Heckler.

We found that the practice had beneficial effects on student learning in various contexts. For example, the practice in translating from verbal descriptions of quantities to mathematical expressions of these quantities improved student accuracy and speed in these skills with continued practice. Ultimately, they had better exam performance on questions that required the execution of these skills. In addition, students received practice in the use of Clebsch-Gordan tables when translating between the different bases commonly used in systems with multiple angular momenta. Their performance on the practice improved as they completed additional homework, and their performance on an exam question requiring the use of this skill was better than students who did not have the practice. More students consciously and correctly used Clebsch-Gordan tables on the exam. Overall, it appeared that the practice helped students become more fluent in various skills when solving problems, which results in improved performance on exam questions that utilize these skills. For many of the skills students at all levels (top, middle, and bottom relative to their final course grade) improved as well, thus narrowing the gap between the groups.

It's important to note that not all skills saw similar improvement. Within those skills that did see class-wide improvements, not all student groups benefited equally. For example, a few instances saw students at the top of the course benefiting from the practice, while those who performed at the bottom of the course did not (for example, skills involving the time independent and time dependent Schrodinger equation). This suggests that some students may need more practice in these skills or that modifications to the practice need to be made in order to make these more effective for all students. In general, though, we observed improvements in accuracy and speed for students across most of the skills administered to students.

The results from the assessments of the Essential Skills homework practice suggest its utility as an instructional tool. However, we have also found that the practice has utility as a research tool. In some cases our analyses of the results suggested new perspectives on student difficulties. One example was in the case of degenerate perturbation theory. We observed that student performance on the practice improved with continued practice, but we did not see any accompanying improvements on an exam question that required the use of the skills they practiced. While it could be that they simply needed more practice, another possibility is that there is more to student difficulties in degenerate perturbation theory than we were aware of. As another example, students received practice in expanding states in arbitrary bases and eigenbases. For arbitrary bases, students started off very accurate and got quicker the more they practiced. For eigenbases, though, students improved their accuracy but got slower the more they practiced. Ultimately, this suggests that students may be thinking about eigenbases in ways that are not documented in the research literature. Finally, we saw students improve their performance on practice involving the use of completeness relations. In the context of an exam problem, though, while students were more likely to explicitly use completeness relations, they often did so incorrectly. This suggests that while the practice may have benefited the students in recognizing when to use completeness relations, practice may not guarantee they use them correctly in a multi-step problem.

To reemphasize, the results discussed in this dissertation are not an exhaustive account of the findings. Many skills were practiced by students across various administrations of the practice, and many trends in the changes in accuracy and speed of the students were noted. Overall, though, the general picture seems to be that as students practiced the skills that we had identified using the research literature and previous exam data they became more accurate and quicker in their execution. This often translated to improved performance on exam questions utilizing these skills. In the cases that these trends did not occur, this potentially tells us something about student difficulties in these skills not previously seen in the literature. This is significant because our work shows these improvements and insights can be found at the upper-division level and not just the introductory level as previous studies have shown [82].

We do not claim that the essential skills practice should replace any other component of a course. The “Tutorials in Physics: Quantum Mechanics” from UW, for example, aid students in understanding material conceptually, which is quite different from what the Essential Skills practice provides. The “Tutorials” also have been shown to be effective means at improving student understanding of quantum mechanics [106, 19, 20, 131, 90, 94, 9]. We merely show that activities like the practice also benefit student learning. In some sense, the practice feels like a natural complement to many of these tutorial activities, which have historically focused on conceptual ideas as opposed to mathematical ideas, the focus of the essential skills practice.

With these successes in mind, there is more work that can be done based on the results of this project. The Essential Skills Framework's fourth step is to iterate: to use results from previous administrations along with new published research to adjust and modify the skills, questions, and organization of the practice. The goal of this, of course, is to make the practice increasingly more effective at helping promote student learning. In doing so, we wish to further refine the practice so as to find a minimal set of skills and optimal practice regimens. While these conclusions may not carry to other institutions that may utilize our practice, it may be that finding optimal routines for our students may inform future implementers of optimal routines in their institutions.

An additional way to assess the practice is by asking for student thoughts and perspectives on the practice. After all, it is desirable to not only provide effective practice, but to design the practice so students appreciate its benefits. Self-efficacy for essential skills practice is a desirable outcome for this reason. In a small sample of students early into the project, many students commented that they believed the practice was beneficial and many even enjoyed the practice. Some students, though, found the practice repetitive, mundane, and lacking nuance. In some sense, though, this outcome is to be expected if those same students improved from the practice. After all, the goal of the practice is to help students execute these skills like second-nature, so lacking nuance is, in some sense, to be expected.

We did find some of the feedback students gave on the structure of the practice itself useful. As an example, modifications to the organization of the question hierarchy were performed based on comments from students. In early versions of the practice, the category-subcategory structure for translating between concepts and mathematics was reversed: categories were the quantities in question (*e.g.*, expectation values, probabilities, probability densities, *etc.*), and subcategories were "concept to mathematics", "mathematics to concepts", and other special topics associated to each quantity. After Fall 2021, though, students commented that when answering practice questions in these categories, they knew which answer choices to select on various questions, since the name of the object was given to them on the list of categories being practiced. This led us to reverse the ordering of the categories and subcategories to their modern forms, with categories being the mode of translation and subcategories the quantities themselves.

Something that could be done in the future is interviewing students as they do the practice to make sure the practice they are receiving is the practice we intended to give them. As an example, it could be that when students practiced completeness relations, as mentioned above, students were actually practicing something different not matching what our intended practice was. By asking students their thoughts on what it is they believe they're practicing, how they're interpreting the questions given to them, and if they match what we intended our practice to elicit, then we are provided with a potentially extra avenue down which we can refine and improve our practice.

This dissertation discusses the construction and implementation of practice questions sequences designed to help students become more accurate and fluent in their use of mathematical and representational skills in quantum mechanics. Overall, the practice was an effective towards this end. We found improvements in the accuracy and speed of students in skills such as identifying mathematical expressions for quantities, addition of angular momentum, degenerate perturbation theory, *etc.*, indicating that the practice helped students become stronger in these skills. Accompanied with this was improvements on performance on exam questions requiring the use of these skills as well as changes in the ways students responded to the questions more in line with the outcomes of the practice. Although in many cases the practice was beneficial to student learning, there were some instances where the improvements within the practice did not transfer to better performance on exam questions. In these instances, though, the results indicate that there are other research questions one could pursue, such as in the case of expanding states in eigenbases. Overall, the practice served as an effective instructional and research tool, aiding in improving student learning of quantum mechanics and revealing new potential avenues for research into student difficulties in quantum mechanics.

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Appendix A

EXAM QUESTIONS

Here, we state the exam questions relevant to this study. For each question, we have included the year(s), course, exam number, and week the particular question was administered. Some exam questions were administered post-STEM Fluency multiple times: these dates will be marked.

Consider a particle in the quantum mechanical infinite square well of width a. At $t = 0$, the normalized wave function for the particle is given by

$$\Psi(x, t=0) = \begin{cases} \frac{\sqrt{15}}{a^{5/2}}\left(x^2 - ax\right) & 0 < x < a \\ 0 & \text{otherwise} \end{cases} \tag{A.1}$$

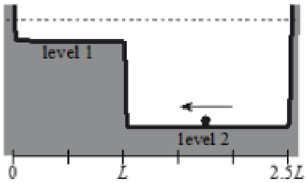
You may receive full credit on this problem without evaluating any integrals; however, writing down an integral without explaining where it comes from will not result in credit.

A. Suppose you were to measure the energy of this particle at $t = 0$. Determine the probability that the energy is equal to E_1, the energy of the ground state. Explain your reasoning.

B. Suppose you were to measure the position of this particle at $t = 0$ (no other measurements have been made). Determine the probability that the particle is measured to be between $x = 0$ and $x = a/3$. Explain your reasoning.

Figure A.1: This question gives students the state for a particle in an infinite square well as a nontrivial superposition of energy eigenstates. Students are then tasked with finding the probability of two measurements: one energy measurement, and one position measurement. The question was first administered in PHYS 324 in Exam 2 in Fall 2018, and readministered in PHYS 324 in Exam 2 in Fall 2021, both in week 9 of their respective quarters.

A (classical) ball moves back and forth in a track with steep sides. The track consists of two levels of unequal length joined by a steep ramp. Assume that the time spent in the steep portions is negligible, that there is no friction or energy loss in the system, and that the ball rolls smoothly, without bouncing, forever. Level 1 has length L, and level 2 has a length $1.5L$.



The ball has enough energy to reach both levels, and the speed of the ball on level 2 is 3 times the speed of the ball on level 1.

(1) Determine the probability density for both levels.

(2) What is the probability of finding the ball between $x = 3L/4$ and $x = 1.5L$?

(3) Determine the average position of the ball.

Figure A.2: This question presents students with a scenario in which a ball rolls (classically) along two levels back and forth. Students are tasked with first finding the probability density for both levels, and then using this to find the probability of finding the ball in a particular region of space and the average position of the ball. This question was first administered in Fall 2018 in week 5 on Exam 1 in PHYS 324 and then readministered in Fall 2023 in week 5 on Exam 1 in PHYS 324.

Consider a particle in the one-dimensional harmonic oscillator potential, $V(x) = \frac{1}{2}m\omega^2 x^2$. The initial wave function for this system is known to be $\Psi(x,0) = N\frac{\sin(2\pi x/b)}{x}$, where N is a normalization constant and b is a constant.

In each part below, write an expression for the given quantity. Explain your reasoning in each case for full credit. If you do not have enough information to answer, state so explicitly. Note: you do not need to evaluate any expression.

(1) The probability of measuring the position to be $-b \leq x \leq b$.

(2) The probability of measuring the ground state energy, E_0.

Figure A.3: This question gives students the state for a particle in a quantum harmonic oscillator as a nontrivial superposition of energy eigenstates. Students are then tasked with finding the probability of two measurements: one energy measurement, and one position measurement. The question was first administered in PHYS 324 in week 9 on Exam 2 in Fall 2015, and readministered in PHYS 324 in week 9 on Exam 2 in Fall 2023.

The wave function for an election in a hydrogen atom is given by:

$$\Psi(\vec{r}, t = 0) = \frac{1}{\sqrt{2}} \left(R_{21}(r) Y_1^{-1}(\theta, \varphi) - R_{21}(r) Y_1^{1}(\theta, \varphi) \right).$$

(1) Does the probability density in position space of this electron depend on time? Explain.

(2) Determine the expectation value of the L_z operator. Show your work.

(3) Is the given state an eigenstate of the Hamiltonian operator? Is the given state an eigenstate of the L^2 operator? Explain why or why not.

Figure A.4: This question gives students the state of an electron in a hydrogen atom as a superposition of (degenerate) energy eigenstates. Students are tasked with assessing whether or not the probability density for the particle depends on time, calculating the expectation value of the z-component of orbital angular momentum, and determining if the state is an eigenstate of either the energy or the magnitude-squared of the total angular momentum. The question was first administered in PHYS 324 in week 11 on the Final Exam in Fall 2018 and readministered in PHYS 324 in week 11 on the final exam in Fall 2023.

(1) An electron is in the state $|l, m_l\rangle|s, m_s\rangle = |2, -2\rangle|\frac{1}{2}, \frac{1}{2}\rangle$. Here $\hat{J} = \hat{L} + \hat{S}$.

  (a) Is the square of the total angular momentum ($\hat{J}^2$) of the electron well-defined? If yes, write down the eigenvalue. If no, write down the possible eigenvalues. In either case, explain how you arrive at your answer.

  (b) What are the possible outcomes if you measure the z-component of the total angular momentum, $\hat{J}_z$, of the electron? Explain.

  (c) What are the possible outcomes if you instead measure the x-component of the total angular momentum, $\hat{J}_x$, of the electron? Explain.

(2) Consider two non-interacting, identical spin-$\frac{1}{2}$ particles in the one-dimensional harmonic oscillator potential. The energy of each particle is independent of its spin state. The energy eigenfunctions of the 1D harmonic oscillator potential are written as $\psi_n(x)$.

Determine the energy of the **second excited state** and write down the possible full energy eigenfunctions (spatial and spin) for the two-particle system.

Figure A.5: The question presents students with the state of a particle with both orbital and spin angular momentum expanded in the $|l, m_l; s, m_s\rangle$ eigenbasis. Students must determine if the value of J^2 for the particle is well-defined, the possible values of J_z that can be measured, and the possible value of J_x that can be measured. The question was initially administered in week 5 in PHYS 325 on the first exam in Winter 2019 and readministered in week 5 in PHYS 325 on the first exam in Winter 2024.

The Hamiltonian of a two-dimensional (2D) simple harmonic oscillator is given below:

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \frac{1}{2}m\omega^2\left(x^2 + y^2\right) \tag{A.2}$$

The corresponding energy eigenstates are $\psi_m(x)\psi_n(y) \equiv |m, n\rangle$.

(1) Using the form about for the eigenstates, list the ground state(s) and first excited state(s). What is the degeneracy of each?

(2) Consider adding a perturbation $H' = \gamma(x^4 + y^4)$, where γ is a small parameter. Write the new Hamiltonian for our system.

(3) Write expressions for the matrix elements of H' in the $\{|1, 0\rangle, |0, 1\rangle\}$ basis. Indicate explicitly how many matrix elements there should be, and organize these matrix elements into a matrix. Do not perform any calculations for this question.

(4) Are the off-diagonal matrix elements zero or non-zero? Explain. (Hint: use symmetry and parity!)

(5) Is the original basis of $\{|0, 1\rangle, |1, 0\rangle\}$ a "good basis" to use with perturbation theory for this problem? Explain why you made this choice. Why does your answer make sense?

Figure A.6: This question guides students through a degenerate perturbation theory problem for the first excited state of a 2D quantum harmonic oscillator. The objective is to determine the "good" basis in which to do perturbation theory for this energy level. This question was administered for the first time in PHYS 325 in week 9 in Winter 2022 on Exam 2 and then readministered in PHYS 325 in week 9 in Winter 2024 on Exam 2.

Appendix B

STEM FLUENCY ORGANIZATIONAL HIERARCHY

This appendix outlines the hierarchy of topics, categories, and subcategories within STEM Fluency, and provides descriptions for the questions encountered in each subcategory. Examples of the questions from each subcategory are given in Appendix C.

Table B.1: This table contains the list of categories in the Topic “Translating Representations”, their respective subcategories, and descriptions of each subcategory.

Topic 1: Translating Representations		
Category Name	Subcategory Name and Number of Questions	Description of Subcategory
Concepts to Mathematics		
	Expectation Values	Given a state and an observable, identifying expressions for the expectation value of said observable for the given state in both Dirac notation and wave function notation.
	Probabilities - Dirac - 324	Given a state, an observable, and its eigenstates and eigenvalues, identifying an expression for the probability of a particular measurement in our state in Dirac notation for operators present in a first course in an upper-division sequence.
	Probabilities - Wave Functions	Given a state, an observable, and its eigenstates and eigenvalues, identifying an expression for the probability of a particular measurement in our state in wave function notation.
	Probabilities - Mixed - 324	Given a state, an observable, and its eigenstates and eigenvalues, identifying expressions for the probability of a particular measurement in our state in Dirac notation and/or wave function notation for operators present in a first course in an upper-division sequence.
	Probabilities - Mixed - 325	Given a state, an observable, and its eigenstates and eigenvalues, identifying expressions for the probability of a particular measurement in our state in Dirac notation and/or wave function notation for operators present in a second course in an upper-division sequence.

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Table B.1: This table contains the list of categories in the Topic "Translating Representations", their respective subcategories, and descriptions of each subcategory. (Continued)

	Probability Density	Given an observable with continuous spectrum and its eigenstates in Dirac notation, identifying expressions for the probability density in that operator's basis for the given state in both Dirac and wave function notation.
	Eigenvalue Equations	Given an operator, its eigenstates, and their corresponding eigenvalues, identifying an expression for the eigenvalue equation for a particular eigenvalue.
	Wave Functions/ States	Given a particle and an operator (with either discrete or continuous spectrum), identifying appropriate expressions for the state of the particle in Dirac notation or both Dirac *and* wave function notation.
	Stationary States	Given a Hamiltonian and its energy eigenstates, identifying expressions for states that are also stationary states in both wave function and Dirac notation.
	Inner Products	Given two states in either Dirac or wave function notation, identifying expressions for the inner product between those two states in either Dirac or wave function notation.
Mathematics to Concepts		
	Expectation Values	Given an expression for the expectation value of a particular observable for a state (not being told it is an expectation value), having to identify that this expression is, in fact, an expectation value.
	Probabilities - Dirac - 324	Given an expression for the probability of a particular measurement in Dirac notation (using operators introduced in the first course of an upper-division quantum mechanics sequence), and not being told it is a probability, having to identify that this expression is, in fact, a probability.

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Table B.1: This table contains the list of categories in the Topic "Translating Representations", their respective subcategories, and descriptions of each subcategory. (Continued)

	Probabilities - Wave Functions	Given an expression for the probability of a particular measurement in wave function notation, and not being told it is a probability, having to identify that this expression is, in fact, a probability.
	Probabilities - Mixed - 324	Given an expression for the probability of a particular measurement in either Dirac or wave function notation (using operators introduced in the first course of an upper-division quantum mechanics sequence), and not being told it is a probability, having to identify that this expression is, in fact, a probability.
	Probabilities - Mixed - 325	Given an expression for the probability of a particular measurement in either Dirac or wave function notation (using operators introduced in a second course of an upper-division quantum mechanics sequence), and not being told it is a probability, having to identify that this expression is, in fact, a probability.
	Probability Density	Given an expression for the probability density of a state in a particular continuous operator's eigenbasis in either Dirac or wave function notation, and not being told it is a probability density, having to identify that this expression is, in fact, a probability density.
	Eigenvalue Equations	Given an operator, its eigenstates, and an expression corresponding to an eigenvalue equation for one of its eigenvalues (and not being told its an eigenvalue equation), having to identify that this expression is, in fact, an eigenvalue equation. This is done in both Dirac and wave function notation.
	Wave Function/ State	Recognizing that a state in Dirac notation, or a wave function in a basis corresponding to a continuous operator, could be viewed as the state of a particle.

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Table B.1: This table contains the list of categories in the Topic “Translating Representations”, their respective subcategories, and descriptions of each subcategory. (Continued)

	Stationary State	Given a Hamiltonian, its eigenstates, and a state expanded in the energy eigenbasis of this Hamiltonian, having to identify whether or not this state is, in fact, able to be described as a stationary state. This is done in both Dirac and wave function notation.
	Inner Products	Given two states, an expression corresponding to the inner product of these states in Dirac or wave function notation, and not being told that this expression is an inner product, having to identify that this expression is, in fact, an inner product.
Special Topics		
	Interpret Expectation Values	Given a verbal description of an expectation value (namely, that of an ensemble average of measurements of a particular operator), recognizing that this results in the expectation value of said operator.
	Inner Products - What Object?	For an inner product, interpreting the resulting mathematical object as a complex number.
	Probability Density vs Probability	Given a state expressed in the basis of a continuous operator, choosing an expression for the probability density or probability as the desired quantity.
	Wave Function - Interpret	Given a wave function, interpreting the quantity as the coefficients of a state expanded in a continuous operator’s eigenbasis.
	Use of Eigenvalue Equations	Given an expression containing terms where an operator is acting on one of its eigenstates, producing expressions that utilize an eigenvalue equation to simplify the expression.
Translate Between Dirac and Wave Function Notation		

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Table B.1: This table contains the list of categories in the Topic "Translating Representations", their respective subcategories, and descriptions of each subcategory. (Continued)

	Probabilities - WF to D - Discrete	Given a state in wave function notation and an expression for the probability of a particular measurement (whose corresponding observable has a discrete spectrum) in wave function notation, producing an expression for the same probability in Dirac notation.
	Probabilities - D to WF - Discrete	Given a state in Dirac notation and an expression for the probability of a particular measurement (whose corresponding observable has a discrete spectrum) in Dirac notation, producing an expression for the same probability in wave function notation.
	Probabilities - D to WF - Continuous	Given a state in Dirac notation and an expression for the probability of a particular measurement (whose corresponding observable has a continuous spectrum) in Dirac notation, producing an expression for the same probability in wave function notation.

Table B.2: This table contains the list of categories in the Topic "Mathematical Procedures and Linear Algebra", their respective subcategories, and descriptions of each subcategory.

Topic 2: Mathematical Procedures and Linear Algebra		
Category Name	Subcategory Name and Number of Questions	Description of Subcategory
Allowed Operations on Bras and Kets		
	Stay in Hilbert Space	Given various bras or kets in a Hilbert space (or its dual), identifying which expressions using those bras/kets are also elements of that Hilbert space.
	What Can This Do?	Given a bra, a ket, an outer product of a bra and ket, or an inner product of a bra and ket, identifying which operations you can perform with these objects on other elements of the Hilbert space (or its dual).
Completeness Relations		
	Discrete Spectrum	Given a ket, a bra, or an inner product of a bra and ket, and given an operator with discrete spectrum (and its eigenstates), identifying an expression for the result of inserting the completeness relation for this operator into the expression correctly.
	Continuous Spectrum	Given a ket, a bra, or an inner product of a bra and ket, and given an operator with continuous spectrum (and its eigenstates), identifying an expression for the result of inserting the completeness relation for this operator into the expression correctly.
Complex Conjugation		
	Ket ↔ Bra (Single)	Given a bra or ket (with potentially real or complex coefficients), identifying an expression for the corresponding dual ket or bra.

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Table B.2: This table contains the list of categories in the Topic "Mathematical Procedures and Linear Algebra", their respective subcategories, and descriptions of each subcategory. (Continued)

	Inner Products	Given two states in Dirac notation and given their inner product, identifying an expression for the complex conjugate of this inner product.
	Complex Numbers and Variables	Given a complex number or a variable that is complex, identifying its complex conjugate.
	Ket ↔ Bra (Linear Combination) (Variables)	Given a linear combination of bras or kets with complex *number* coefficients, identifying the expression corresponding to its dual ket or bra.
	Ket ↔ Bra (Linear Combination) (Numbers)	Given a linear combination of bras or kets with complex *variable* coefficients, identifying the expression corresponding to its dual ket or bra.
	Wave Functions	Given a wave function (written in Dirac notation) in a continuous operator's eigenbasis, identifying an expression for the conjugate of this wave function also in Dirac notation.
Expanding a State in a Basis		
	Arbitrary Basis	Given a state in either Dirac or wave function notation, and given a basis (either finite or infinite dimensional), identifying an expression for the state expanded in that basis.
	Eigenbasis	Given a state in either Dirac notation or wave function notation and an observable and its eigenstates, identifying an expression for the state expanded in that eigenbasis. In addition, given some probabilities of measuring particular eigenvalues, reconstructing a state in the corresponding eigenbasis relative to those probabilities.
	Eigenbasis - Angular Momentum	Given a state and either one or two spin operators on this state, identifying expressions for the state expanded, in the single spin case, in the eigenbasis for that spin operator, or, in the two spin operator case, in the coupled or uncoupled basis.

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Table B.2: This table contains the list of categories in the Topic “Mathematical Procedures and Linear Algebra”, their respective subcategories, and descriptions of each subcategory. (Continued)

Orthogonality		
	Of Kets	Given two kets, give a mathematical expression for the verbal description of the kets being orthogonal and *vice versa.*
	Of Eigenbases	Given an eigenbasis for an operator, identify the expressions that are true between the various basis elements.
	Orthonormality of Bases	Given a basis that you are told is orthonormal, identify the expressions that are true between the various basis elements.

Table B.3: This table contains the list of categories in the Topic "Pre-Dirac Notation", their respective subcategories, and descriptions of each subcategory.

Topic 3: Pre-Dirac Notation		
Category Name	Subcategory Name and Number of Questions	Description of Subcategory
Continuous Probability		
	Probability Density to Probability	Given a probability density, produce an expression for the probability of a given event via integration.
	Probability Density and Probability Amplitude - Units	Given a probability density or a probability amplitude, give its units.
Time Dependence in Quantum Mechanics		
	Adding Time Dependence	Given a wave function at $t = 0$, expanded in the energy eigenbasis, produce an expression for this wave function for all time t.
	Is it Time Dependent?	Given a quantity like a wave function, probability density, probability for a particular event, or an expectation value, determine whether or not the quantity depends on time.
The Schrodinger Equation		

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Table B.3: This table contains the list of categories in the Topic “Pre-Dirac Notation”, their respective subcategories, and descriptions of each subcategory. (Continued)

	Stationary States	Given a state that you are told is in a stationary state for a given Hamiltonian, identify the various properties of this state that are true due to being a stationary state. Alternatively, given a state that is not a stationary state, again identify the properties of this state that are true in spite of not being in a stationary state
	Time Independent Schrodinger Equation	Identifying expressions that are equivalent to the time-independent Schrodinger equation.
	Time Dependent Schrodinger Equation	Identifying expressions that are equivalent to the time-dependent Schrodinger equation.
Graphs of States for Common Systems		
	Good Wave Function for Hamiltonian - ISW	After being told our particle is in an infinite square well, identifying expressions that could potentially be the state of the particle.
	Graphs of Wave Functions - ISW	After being told our particle is in an infinite square well, identifying graphs of functions that could potentially be the state of the particle.
	Good Wave Function for Hamiltonian - QHO	After being told our particle is in a quantum harmonic oscillator, identifying expressions that could potentially be the state of the particle.
	Graphs of Wave Functions - QHO	After being told our particle is in a quantum harmonic oscillator, identifying graphs of functions that could potentially be the state of the particle.

Table B.4: This table contains the list of categories in the Topic "Mathematization", their respective subcategories, and descriptions of each subcategory.

Topic 4: Mathematization		
Category Name	Subcategory Name and Number of Questions	Description of Subcategory
Choosing a Bra for a Probability Calculation		
	Continuous - Dirac	Given, in words, a probability that is being calculated, selecting the bra used for such a calculation. The operator in this case has ***continuous*** spectrum.
	Discrete - Dirac	Given, in words, a probability that is being calculated, selecting the bra used for such a calculation. The operator in this case has a ***discrete*** spectrum.
	Continuous - Wave Function	Given, in words, a probability that is being calculated, selecting the wave function used for such a calculation. The operator in this case has a ***continuous*** spectrum.
Choosing Which Notation		
	Matrix Notation	Given a generic problem context, have students decide to use matrix notation and assess the benefits of such a decision.
	Dirac Notation	Given a generic problem context, have students decide to use Dirac notation and assess the benefits of such a decision.
	Wave Function Notation	Given a generic problem context, have students decide to use Wave Function notation and assess the benefits of such a decision.
Error Checking Methods		

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Table B.4: This table contains the list of categories in the Topic “Mathematization”, their respective subcategories, and descriptions of each subcategory. (Continued)

	Probability - 0 to 1	Given a generic problem context and a hypothetical result, have students recognize that such an answer cannot be a probability since it is not between 0 and 1.
	Probability - Units	Given a generic problem context and a hypothetical result, have students recognize that such an answer cannot be a probability since it is not unitless.
	Expectation Value - Units	Given a generic problem context and a hypothetical result, have students recognize that such an answer cannot be an expectation value since it does not have the right units.
	Eigenvalues - Imaginary vs Real	Given a generic problem context and a hypothetical result, have students recognize that such an answer cannot be an eigenvalue since eigenvalues of Hermitian operators are real.
	Probability - Imaginary vs Real	Given a generic problem context and a hypothetical result, have students recognize that such an answer cannot be a probability since probabilities are always real.
Integration Shortcuts		
	Even vs Odd	Given an integral, without performing a calculation, assessing whether or not the integral is 0 or non-zero based on symmetry considerations of (components of) the integrand.
	Orthogonality	Given an integral, without performing a calculation, assessing whether or not the integral is 0 or non-zero based on knowledge of the form and properties of eigenfunctions of Hermitian operators.
Error Correcting		
	No Error	Given a generic problem context and a hypothetical solutions, having students determine there’s nothing to be fixed with the given result.

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Table B.4: This table contains the list of categories in the Topic "Mathematization", their respective subcategories, and descriptions of each subcategory. (Continued)

	Ortho vs Non-Ortho	Given a generic problem context and a step using orthonormality in a hypothetical solution, have students determine it was an improper use of orthonormality and give the correct result of the step.
	Wrong Norm-Square	Given a generic problem context and a hypothetical solution, having students determine that the norm-square in the expression is in the wrong location and determine where it should be.
	Wrong Integrand	Given a generic problem context and a hypothetical solution, having students determine that the integrand for the integral solution is incorrect and determine what the correct expression is.
	Time Dependence	Given a generic problem context and a step adding time dependence in a hypothetical solution, have students determine that this addition was done incorrectly and correcting the result of this step.

Table B.5: This table contains the list of categories in the Topic "Addition of Angular Momentum", their respective subcategories, and descriptions of each subcategory.

Topic 5: Addition of Angular Momentum		
Category Name	Subcategory Name and Number of Questions	Description of Subcategory
Change of Basis		
	Separate to Total	Given a state written in the uncoupled basis (for two angular momentum operators), identify the expression for the state written in the coupled basis using a Clebsch-Gordan table.
	Total to Separate	Given a state written in the coupled basis (for two angular momentum operators), identify the expression for the state written in the uncoupled basis using a Clebsch-Gordan table.
Dimension of Hilbert Space		
	Single	Given a single angular momentum operator of spin quantum number s, determine the dimension of the Hilbert space that operator acts on.
	Multiple	Given two angular momentum operators of (potentially different) spin quantum numbers s_1 and s_2, determine the dimension of the Hilbert space their product acts on.
Probabilities Between Bases		
	Total to Separate Basis - Given	Given a coupled eigenstate for two angular momentum operators, and given its expansion in the uncoupled basis, determine the probability of measuring a particular uncoupled eigenstate.

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Table B.5: This table contains the list of categories in the Topic “Addition of Angular Momentum”, their respective subcategories, and descriptions of each subcategory. (Continued)

	Total to Separate Basis - Build	Given a coupled eigenstate for two angular momentum operators, and *not* given its expansion in the uncoupled basis (meaning students must use a Clebsch-Gordan table to construct it themselves), determine the probability of measuring a particular uncoupled eigenstate.
	Separate to Total Basis - Given	Given an uncoupled eigenstate for two angular momentum operators, and given its expansion in the coupled basis, determine the probability of measuring a particular coupled eigenstate.
	Separate to Total Basis - Build	Given an uncoupled eigenstate for two angular momentum operators, and *not* given its expansion in the coupled basis (meaning students must use a Clebsch-Gordan table to construct it themselves), determine the probability of measuring a particular coupled eigenstate.

Table B.6: This table contains the list of categories in the Topic "Multiple Particles and Degeneracy", their respective subcategories, and descriptions of each subcategory.

Topic 6: Multiple Particles and Degeneracy		
Category Name	Subcategory Name and Number of Questions	Description of Subcategory
Degeneracy		
	What is the Degeneracy?	Given a Hamiltonian, determine the degeneracy of a particular energy level.
	List States	Given a Hamiltonian, determine the states that have a particular energy.
Identical Particles - Simple		
	Energy - Boson	Given a Hamiltonian and its energy eigenstates, writing the correct symmetrized state for our two particles given our knowledge that one has a particular energy and another has a different energy.
	Spin - Boson	Given two particles with spin as well as the spin eigenstates, writing the correct symmetrized state for our two particles given our knowledge that one has a particular z-component of spin and another has a different z-component of spin.
	Energy - Fermion	Given a Hamiltonian and its energy eigenstates, writing the correct anti-symmetrized state for our two particles given our knowledge that one has a particular energy and another has a different energy.
	Spin - Fermion	Given two particles with spin as well as the spin eigenstates, writing the correct anti-symmetrized state for our two particles given our knowledge that one has a particular z-component of spin and another has a different z-component of spin.
	Boson or Fermion	Given two particles and the state of the particles (either energy or spin), determining whether the particles are bosons, fermions, or neither.

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Table B.6: This table contains the list of categories in the Topic "Multiple Particles and Degeneracy", their respective subcategories, and descriptions of each subcategory. (Continued)

Identical Particles - Compound		
	Boson Compound	Given two particles (with spin), a Hamiltonian, and its energy eigenstates, identifying all states that are appropriately symmetrized given our knowledge of the energies and z-spins of the particles.
	Fermion Compound	Given two particles (with spin), a Hamiltonian, and its energy eigenstates, identifying all states that are appropriately anti-symmetrized given our knowledge of the energies and z-spins of the particles.
	Boson or Fermion	Given two particles and the state of the particles (a *compound* state with both energy and spin states), determining whether the particles are bosons, fermions, or neither.
	Given One, What is Other?	Given two particles, one of which is either a boson or a fermion, and given one component of the full compound state (with energy and spin states), determining which expression is the state of the other component given symmetry considerations.

Table B.7: This table contains the list of categories in the Topic “Perturbation Theory”, their respective subcategories, and descriptions of each subcategory.

Topic 7: Perturbation Theory		
Category Name	Subcategory Name and Number of Questions	Description of Subcategory
Non-Deg TIPT		
	First Order Energy	Given a Hamiltonian, a small perturbation to this Hamiltonian, the eigenstates of the base Hamiltonian, and the eigenstates of the corrected Hamiltonian, identifying the expression for the first order correction to the n^{th} energy eigenvalue.
	First Order State	Given a Hamiltonian, a small perturbation to this Hamiltonian, the eigenstates of the base Hamiltonian, and the eigenstates of the corrected Hamiltonian, identifying the expression for the first order correction to the n^{th} energy eigenstate.
	Second Order Energy	Given a Hamiltonian, a small perturbation to this Hamiltonian, the eigenstates of the base Hamiltonian, and the eigenstates of the corrected Hamiltonian, identifying the expression for the second order correction to the n^{th} energy eigenvalue.
DPT - Find H' in Deg Subspace		
	Given Matrix	Given a Hamiltonian and a small perturbation to it in matrix form, identifying the submatrix of the perturbation making up H' in the degenerate subspace.
DPT - Basic Examples		
	Diagonalize - 2x2	Given common 2×2 matrices that could represent a small perturbation to a system, solving for the eigenvalues and identifying them with the first order corrections to the first energy eigenvalues associated with that energy level.

Continued on next page

Table B.7: This table contains the list of categories in the Topic “Perturbation Theory”, their respective subcategories, and descriptions of each subcategory. (Continued)

	Diagonalize - 3x3	Given common 3×3 matrices that could represent a small perturbation to a system, solving for the eigenvalues and identifying them with the first order corrections to the first energy eigenvalues associated with that energy level.
	Find Good Basis - 2x2	Given common 2×2 matrices that could represent a small perturbation to a system, solving for the eigenstates of this perturbation in either matrix notation or Dirac notation.
	Find Good Basis - 3x3	Given common 3×3 matrices that could represent a small perturbation to a system, solving for the eigenstates of this perturbation in either matrix notation or Dirac notation.
DPT - Need to Use DPT?		
	Given H^0 and H'	Given matrices for a Hamiltonian and a small correction to it, assessing whether or not degenerate perturbation theory needs to be done and why.
	Given System	Given a base Hamiltonian (such as a 2D or 3D infinite square well) and a perturbation, assessing whether or not degenerate perturbation theory needs to be done and why.

Appendix C

STEM FLUENCY EXAMPLE QUESTIONS FOR EACH SUBCATEGORY

This appendix will follow the hierarchy in Appendix B and, for each subcategory, give an example question or two written in the STEM Fluency system. The choice of question is arbitrary and simply meant to provide a template for what each question is typically formatted like. For cases where multiple questions are presented for a particular subcategory, the reason is because there is enough variation in the answer choices so as to justify having multiple questions to provide a holistic description.

Some categories/subcategories have not been administered and thus will have no data on student performance mentioned in the dissertation itself. This is due to various circumstances, but largely because they were either 1) not fleshed out enough to justify their use, or 2) they could not fit into the schedule of skills used for the quarters.

Questions with boxes (*i.e.*, ☐) next to their answer choices indicate that the question is multiple choice, multiple select. Questions with circles (namely, ○) indicate that the question is multiple choice, single select.

Topic 1: Translating Representations

Category 1: Concepts to Mathematics

Subcategory 1: Expectation Values

Consider a particle in a state $|\psi\rangle$ with Hamiltonian H whose energy eigenvalues are E_n with corresponding energy eigenstates $|\psi_n\rangle$. Let the position space wave function for the state of our particle and for the energy eigenstates be $\psi(x) = \langle x|\psi\rangle$ and $\psi_n(x) = \langle x|\psi_n\rangle$, respectively.

Which expressions correspond to the expectation value of energy for our state? Select all that apply.

- ☐ $\int_{-\infty}^{\infty} \psi^*(x) H \psi(x) \mathrm{d}x$
- ☐ $\langle\psi|H|\psi\rangle$
- ☐ $|\langle\psi_n|\psi\rangle|^2$
- ☐ $\int_{-\infty}^{\infty} \psi_n^*(x)\psi(x)\mathrm{d}x$
- ☐ $\left|\int_{-\infty}^{\infty} \psi_n^*(x)\psi(x)\mathrm{d}x\right|^2$

Figure C.1: This question asks student to identify a mathematical expression for the desired expectation value in both wave function and Dirac notation. Some questions only have answers for one or the other notation. Alternative answer choices in this subcategory include probabilities, probability amplitudes, alternative integrands, such as ones containing eigenvalues.

Subcategory 2: Probabilities - Dirac - 324

Let H be a Hamiltonian with energy eigenstates $|n\rangle$ for $n = 1, 2, ...$, with associated energy eigenvalues E_n. In addition, consider a particle in the state $|\phi\rangle$.

What is the probability of measuring the energy of our particle to be E_3?

○ $|\langle 3|\phi\rangle|^2$

○ $|\langle 3|H|\phi\rangle|^2$

○ $\langle 3|\phi\rangle$

○ $\langle 3|H|\phi\rangle$

Figure C.2: This question asks students to identify a mathematical expression for the desired probability of an energy measurement in just Dirac notation. The answers for this question consist of expectation values (a common student response for probability questions), and misplacing/ neglecting the norm-square. Other answer choices contain other mathematical objects, such as eigenvalue equations, eigenvalues, *etc.* Questions in this category deal with observables seen in a first course, such as energy, position, and momentum.

Let x be the position operator with eigenstates $|x\rangle$. In addition, consider a particle in the state $|\phi\rangle$.

What is the probability of measuring our particle to be between the positions $x = -a$ and $x = a$ for some constant a?

- ○ $\int_{-a}^{a} |\langle x|\phi\rangle|^2 \mathrm{d}x$
- ○ $\langle a|\phi\rangle$
- ○ $\langle a|\phi\rangle - \langle -a|\phi\rangle$
- ○ $\langle x|\phi\rangle$ where $-a \leq x \leq a$
- ○ $\int_{-a}^{a} \langle x|\phi\rangle \mathrm{d}x$

Figure C.3: This question asks students to identify a mathematical expression for the desired probability of a position measurement in just Dirac notation. The answers for this question consist of expectation values (a common student response for probability questions), and misplacing/ neglecting the norm-square. Other answer choices contain other mathematical objects, such as eigenvalue equations, eigenvalues, *etc.* Questions in this category deal with observables seen in a first course, such as energy, position, and momentum.

Subcategory 3: Probabilities - WF

Let H be a Hamiltonian with position-space energy eigenstates $\psi_n(x)$ for $n = 0, 1, 2, ...,$ with corresponding energy eigenvalues E_n. Consider, at $t = 0$, a particle in the state $\Psi(x, 0)$.

What is the probability of measuring the particle to have energy E_4?

- ○ $|\int_{-\infty}^{\infty} \psi_4^*(x)\Psi(x,0)\mathrm{d}x|^2$
- ○ $\int_{-\infty}^{\infty} |\Psi(x,0)|^2\mathrm{d}x$
- ○ $\int_{-\infty}^{\infty} |\psi_4^*\Psi(x,0)|^2\mathrm{d}x$
- ○ $\int_{-\infty}^{\infty} \psi_4^*(x)\psi_4(x)\mathrm{d}x$
- ○ $|\int_{-\infty}^{\infty} \psi_4^*(x)\mathrm{d}x|^2$

Figure C.4: This question asks students to identify a mathematical expression for the desired probability of an energy measurement in just wave function notation. The answers for this question consist of expectation values (a common student response for probability questions), misplacing/ neglecting the norm-square, and a corresponding position probability expression.

Let H be a Hamiltonian with position-space energy eigenstates $\psi_n(x)$ for $n = 0, 1, 2, ...,$ with corresponding energy eigenvalues E_n. Consider, at $t = 0$, a particle in the state $\Psi(x, 0)$.

What is the probability of measuring the particle to have energy E_1?

- ○ $|\int_{-\infty}^{\infty} \psi_1^*(x)\Psi(x,0)\mathrm{d}x|^2$
- ○ $\int_{-\infty}^{\infty} \psi_1^*(x)\Psi(x,0)\mathrm{d}x$
- ○ $\int_{-\infty}^{\infty} |\psi_1^*(x)\Psi(x,0)|^2\mathrm{d}x$
- ○ $\int_{-\infty}^{\infty} \psi_1^*(x)H\Psi(x,0)\mathrm{d}x$
- ○ $|\int_{-\infty}^{\infty} \psi_1^*(x)H\Psi(x,0)\mathrm{d}x|^2$

Figure C.5: This question asks students to identify a mathematical expression for the desired probability of an energy measurement in just wave function notation. The answers for this question consist of expectation values (a common student response for probability questions), and misplacing/ neglecting the norm-square. Other answer choices contain other mathematical objects, such as eigenvalue equations, eigenvalues, *etc.* Questions in this category deal with observables seen in a first course, such as energy, position, and momentum.

Let x be the position operator, and let $\Psi(x,0)$ be the position-space wavefunction that describes the state of our particle at $t=0$.

What is the probability of measuring the position of the particle to be between $x=a$ and $x=b$, where a,b are constants?

- ○ $\int_a^b |\Psi(x,0)|^2 \mathrm{d}x$
- ○ $\int_a^b \Psi(x,0) \mathrm{d}x$
- ○ $\Psi(b,0)-\Psi(a,0)$
- ○ $\int_a^b x\Psi(x,0) \mathrm{d}x$
- ○ $\int_a^b x|\Psi(x,0)|^2 \mathrm{d}x$

Figure C.6: This question asks students to identify a mathematical expression for the desired probability of a position measurement in just wave function notation. The answers for this question consist of expectation values (a common student response for probability questions), evaluating the wave function at the bounds of the interval in question, and including just the wave function in the integrand. Other answer choices contain other mathematical objects, such as eigenvalue equations, eigenvalues, *etc.* Questions in this category deal with observables seen in a first course, such as energy, position, and momentum.

Subcategory 4: Probabilities - Mixed - 324

Consider a particle in a state $|\psi\rangle$ with Hamiltonian H, whose energies eigenvalues are E_n with corresponding energy eigenstates $|\phi_n\rangle$. Let $\psi(x) = \langle x|\psi\rangle$ be the position-space wave function for $|\psi\rangle$.

What expressions correspond to the probability of measuring the energy of our particle to be E_4? Select all that apply.

- ☐ $|\langle\phi_4|\psi\rangle|^2$
- ☐ $|\int_{-\infty}^{\infty} \phi_4^*(x)\psi(x)\mathrm{d}x|^2$
- ☐ $\int_{-\infty}^{\infty} |\phi_4^*(x)\psi(x)|^2\mathrm{d}x$
- ☐ $\langle\phi_4|H|\psi\rangle$
- ☐ $|\langle\phi_4|H|\psi\rangle|^2$

Figure C.7: This question asks students to identify a mathematical expression for the desired probability of an energy measurement in Dirac notation, wave function notation, or both. The answers for this question consist of expectation values (a common student response for probability questions), and misplacing/ neglecting the norm-square. Other answer choices contain other mathematical objects, such as eigenvalue equations, eigenvalues, *etc.* Questions in this category deal with observables seen in a first course, such as energy, position, and momentum.

Consider a particle in a state $|\psi\rangle$ in an infinite square well from 0 to a. Let $\psi(x) = \langle x|\psi\rangle$ be the position-space wave function for $|\psi\rangle$.

What expressions correspond to the probability of measuring the position of the particle to be on the right half of the well? Select all that apply.

- ☐ $\int_{a/2}^{a} |\psi(x)|^2 \mathrm{d}x$
- ☐ $\int_{a/2}^{a} |\langle x|\psi\rangle|^2 \mathrm{d}x$
- ☐ $\int_{a/2}^{a} \psi(x) \mathrm{d}x$
- ☐ $\int_{a/2}^{a} \langle x|\psi\rangle \mathrm{d}x$
- ☐ $\int_{a/2}^{a} \langle\psi|\psi\rangle \mathrm{d}x$

Figure C.8: This question asks students to identify a mathematical expression for the desired probability of a position measurement in Dirac notation, wave function notation, or both. The answers for this question consist of expectation values (a common student response for probability questions), and misplacing/ neglecting the norm-square. Other answer choices contain other mathematical objects, such as eigenvalue equations, eigenvalues, *etc.* Questions in this category deal with observables seen in a first course, such as energy, position, and momentum.

Subcategory 5: Probabilities - Mixed - 325

Let p be the momentum operator with eigenstates $|p\rangle$. In addition, consider a particle in the state $|\psi\rangle$ with momentum-space wave function $\psi(p,0)$ and position-space wavefunction $\phi(x,0)$ at $t=0$.

What is the probability of measuring our particle to be moving to the left (*i.e.*, has momentum between $-\infty$ and 0)? Select all that apply.

- ☐ $\int_{-\infty}^{0} |\langle p|\psi\rangle|^2 \mathrm{d}p$
- ☐ $\int_{-\infty}^{0} |\psi(p,0)|^2 \mathrm{d}p$
- ☐ $\langle p|\psi\rangle$
- ☐ $\langle 0|\psi\rangle - \langle -\infty|\psi\rangle$
- ☐ $\int_{-\infty}^{\infty} e^{-ipx/\hbar}\psi(p,0)\mathrm{d}p$
- ☐ $\int_{-\infty}^{0} \left|\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ipx}\phi(x,0)\mathrm{d}x\right|^2 \mathrm{d}p$

Figure C.9: This question asks students to identify a mathematical expression for the desired probability of a momentum measurement in Dirac notation, wave function notation, or both. The answers for this question consist of expectation values (a common student response for probability questions), and misplacing/ neglecting the norm-square. Other answer choices contain other mathematical objects, such as eigenvalue equations, eigenvalues, *etc.*

Consider two spin-$\frac{1}{2}$ particles, and let $\hat{\boldsymbol{S}} = \hat{\boldsymbol{S}}_1 + \hat{\boldsymbol{S}}_2$ be the total spin operator. Let $|s, m\rangle$ be the eigenstates for the coupled basis, where s is the quantum number for $\hat{\boldsymbol{S}}^2$ and m is the quantum number for $\hat{\boldsymbol{S}}_z$. Let the state of our particle be $|\chi\rangle$.

What is the probability of measuring out particles to be in the state $|1, -1\rangle$?

○ $|\langle 1, -1|\chi\rangle|^2$

○ $\langle 1, -1|\chi\rangle$

○ $\langle 1, -1|1, -1\rangle$

○ $|\langle 1, -1|1, -1\rangle|^2$

Figure C.10: This question asks students to identify a mathematical expression for the desired probability of a spin measurement in Dirac notation. The answers for this question consist of missing the norm-square, using the wrong state as the ket, or both.

Subcategory 6: Probability Density

Consider a particle in the state $|\psi\rangle$, and let x be the position operator whose eigenstates are $|x\rangle$. Let the position-space wave function for our particle be $\psi(x)$. Which of the following expressions correspond to the position-space probability density of our particle? Select all that apply.

☐ $|\psi(x)|^2$

☐ $|\langle x|\psi\rangle|^2$

☐ $\langle\psi|\psi\rangle$

☐ $\psi(x)$

☐ $\langle x|\psi\rangle$

Figure C.11: This question asks students to identify a mathematical expression for the desired probability density of a state in position space in both Dirac and wave function notation. The incorrect answers to this question include just the wave function in both notations, and the norm of the state in Dirac notation.

Consider a particle in the state $|\psi\rangle$, and let p be the position operator whose eigenstates are $|p\rangle$. Let the momentum-space wave function for our particle be $\psi(p)$. Which of the following expressions correspond to the position-space probability density of our particle? Select all that apply.

- ☐ $|\psi(p)|^2$
- ☐ $|\langle p|\psi\rangle|^2$
- ☐ $\langle\psi|\psi\rangle$
- ☐ $\langle p|p|\psi\rangle$
- ☐ $\psi(p)^2$

Figure C.12: This question asks students to identify a mathematical expression for the desired probability density of a state in momentum space in both Dirac and wave function notation. Incorrect answers include forgetting the norm-square, just squaring, and the norm of the state.

Subcategory 7: Eigenvalue Equations

Consider a Hamiltonian H whose energy eigenvalues are E_n with corresponding energy eigenstates $|\psi_n\rangle$. Let the position-space wavefunctions for the energy eigenstates be $\psi_n(x) = \langle x|\psi_n\rangle$.

Which expressions correspond to the eigenvalue equation for E_1? Select all that apply.

- ☐ $H\psi_1(x) = E_1\psi_1(x)$
- ☐ $H\psi_1(x) = E_1$
- ☐ $\int_{-\infty}^{\infty} \psi_1^*(x) H \psi_1(x) \mathrm{d}x$
- ☐ $\int_{-\infty}^{\infty} H\psi_1(x) \mathrm{d}x = E_1$

Figure C.13: This question asks students to identify a mathematical expression for the eigenvalue equation for a desired energy in both wave function and Dirac notation. Common incorrect answers involve forgetting a piece of the equation, or some other quantity.

Let S_z represent the spin-$\frac{1}{2}$ operator in the z-direction. Let $|\uparrow\rangle_z$ be the spin-up in the z-direction eigenstate of S_z with eigenvalue $\frac{\hbar}{2}$.

What is the eigenvalue equation for $|\uparrow\rangle_z$?

- ○ $S_z|\uparrow\rangle_z = \frac{\hbar}{2}|\uparrow\rangle_z$
- ○ $S_z|\uparrow\rangle_z = \frac{\hbar}{2}$
- ○ $\langle S_z\rangle = \frac{\hbar}{2}$
- ○ $S_z|\uparrow\rangle_z = |\uparrow\rangle_z$

Figure C.14: This question asks students to identify a mathematical expression for the eigenvalue equation for a desired z-spin in Dirac notation. Common incorrect answers involve forgetting a piece of the equation, or some other quantity such as expectation values or probabilities.

Subcategory 8: Wave Functions/ States

Let $|\psi\rangle$ be the state of a particle, and let x be the position operator with eigenstates $|x\rangle$. Which of the following best describes the position-space wave function for the state $|\psi\rangle$? Select all that apply.

- ☐ $\langle x|\psi\rangle$
- ☐ $|\psi(x)\rangle$
- ☐ $|\psi\rangle$
- ☐ $\langle\psi|x\rangle$
- ☐ $x|\psi\rangle$

Figure C.15: This question asks students to identify a mathematical expression for the state of a particle in position-space wave function notation and Dirac notation. Common incorrect answers involve inserting the wave function in a ket, simply doing just Dirac notation, or treating the position x as an operator or a bra.

Subcategory 9: Inner Products

Consider a particle in a state $|\psi\rangle$, and let $|\alpha\rangle$ be another ket in our Hilbert space. Let the position-space wave function for the state of our particle and for the other state be $\psi(x) = \langle x|\psi\rangle$ and $\alpha(x) = \langle x|\alpha\rangle$, respectively.

Which expressions correspond to the inner product of our state with $|\alpha$? Select all that apply.

- ☐ $\langle\alpha|\psi\rangle$
- ☐ $\int_{-\infty}^{\infty} \alpha^*(x)\psi(x)\mathrm{d}x$
- ☐ $\langle\alpha|x|\psi\rangle$
- ☐ $|\langle\alpha|\psi\rangle|^2$
- ☐ $|\int_{-\infty}^{\infty} \alpha^*(x)\psi(x)\mathrm{d}x|^2$

Figure C.16: This question asks students to identify a mathematical expression for the inner product of two states in both Dirac and wave function notation. Common incorrect answers involve using norm-squares, taking an expectation value, or multiplying kets instead of a bra and a ket.

Consider a particle in a state $|\psi\rangle$ with Hamiltonian H whose energy eigenvalues are E_n with corresponding energy eigenstates $|\psi_n\rangle$. Let the position-space wave functions for the state of our particle and for the energy eigenstates be $\psi(x) = \langle x|\psi\rangle$ and $\psi_n(x) = \langle x|\psi_n\rangle$, respectively.

Which expressions correspond to the inner product of our state with the n^{th} energy eigenstate? Select all that apply.

- ☐ $\langle\psi_n|\psi\rangle$
- ☐ $\int_{-\infty}^{\infty} \psi_n^*(x)\psi(x)\mathrm{d}x$
- ☐ $H|\psi_n\rangle = E_n|\psi_n\rangle$
- ☐ $\psi_n(x)\psi(x)$
- ☐ $\psi_n^*(x)\psi(x)$

Figure C.17: This question asks students to identify a mathematical expression for the inner product of a state with a particular energy eigenstate in both Dirac and wave function notation. Common incorrect answers involve simply multiplying states, using an eigenvalue equation, or some other mathematical expression.

Category 2: Mathematics to Concepts

Subcategory 1: Expectation Values

Consider a particle in a state $|\psi\rangle$. Let the momentum-space wave function for the state of our particle be $\psi(p) = \langle p|\psi\rangle$.
What description best characterizes the expression $\int_{-\infty}^{\infty} \psi^*(p) p \psi(p) \mathrm{d}p$? Select all that apply.

- ☐ The expectation value of the momentum of our particle.
- ☐ The probability that the momentum of our particle is p.
- ☐ The probability density for the momentum of our state.
- ☐ The average of the results of many measurements of the momentum for an ensemble of identically prepared particles in the state $|\psi\rangle$.
- ☐ The inner product of the momentum with our state $|\psi\rangle$.

Figure C.18: The question gives students the mathematical expression corresponding to the expectation value of some observable, and students must identify it as such. Incorrect answers consist of other common mathematical objects.

Subcategory 2: Probabilities - Dirac - 324

Let x be the position operator with eigenstates $|x\rangle$, and let $|\psi\rangle$ be the state of a particle. What do you call/ how do you interpret the quantity given by $\int_0^\infty |\langle x|\psi\rangle|^2 \mathrm{d}x$? Select all that apply.

- ☐ The probability of measuring the position of $|\psi\rangle$ to be between 0 and ∞?
- ☐ The expectation value of the position for a particle in the state $|\psi\rangle$
- ☐ The result of measuring the position of $|\psi\rangle$
- ☐ The wave function of $|\psi\rangle$ in the position basis
- ☐ The probability density for $|\psi\rangle$ in the position basis

Figure C.19: The question gives students the mathematical expression corresponding to the probability of a particular position measurement in Dirac notation, and students must identify it as such. Incorrect answers consist of other common mathematical objects.

Subcategory 3: Probabilities - WF

Let H be a Hamiltonian with energy eigenstates $|n\rangle$ for $n = 0, 1, 2, ...$ (with $\psi_n(x) = \langle x|\psi_n\rangle$ in position-space), with corresponding energy eigenvalues E_n. Consider a particle in the state $|\psi\rangle$ with position-space wave function $\psi(x, 0)$ at $t = 0$.
What do you call/ how do you interpret the quantity given by $\left|\int_{-\infty}^{\infty} \psi_3^*(x)\psi(x,0)\mathrm{d}x\right|^2$?
Please select all that apply.

- ☐ The probability of measuring the energy of $|\psi\rangle$ to be E_3.
- ☐ The expectation value of the energy of the state $|\psi\rangle$.
- ☐ The result of measuring the energy of the state $|\psi\rangle$.
- ☐ The wave function of $|\psi\rangle$ in the energy basis.
- ☐ The probability density for $|\psi\rangle$ in the energy basis.

Figure C.20: The question gives students the mathematical expression corresponding to the probability of a particular energy measurement in wave function notation, and students must identify it as such. Incorrect answers consist of other common mathematical objects.

Subcategory 4: Probabilities - Mixed - 324

Consider a particle in a state $|\psi\rangle$ with Hamiltonian H, whose energy eigenvalues are E_n with corresponding energy eigenstates $|\psi_n\rangle$. Let the position-space wave functions for the state and the energy eigenstates be $\psi(x) = \langle x|\psi\rangle$ and $\psi_n(x) = \langle x|\psi_n\rangle$.
Which of the following does the expression $\left|\int_{-\infty}^{\infty} \psi_2^*(x)\psi(x)\mathrm{d}x\right|^2$ correspond to? Select all that apply.

- ☐ The probability of measuring the energy of our particle to be E_2.
- ☐ The expectation value of the energy of the particle.
- ☐ The state of our particle written in the energy eigenbasis.
- ☐ The probability density for the energy of our particle.
- ☐ The average of the results of many energy measurements on an ensemble of particles prepared in the state $|\psi\rangle$.

Figure C.21: The question gives students the mathematical expression corresponding to the probability of a particular measurement in either Dirac notation or wave function notation, and students must identify it as such. Incorrect answers consist of other common mathematical objects. Question in this subcategory consist of observables encountered in a first quarter quantum mechanics course.

Subcategory 5: Probabilities - Mixed - 325

Consider a spin-$\frac{1}{2}$ particle with orbital angular momentum operator $\hat{\boldsymbol{L}}$ and associated magnitude squared quantum number $l = 1$. Define the eigenstates for the uncoupled basis for total angular momentum $\hat{\boldsymbol{J}} = \hat{\boldsymbol{L}} + \hat{\boldsymbol{S}}$ as $|l, m_l; s, m_s\rangle$, and define the eigenstates of the coupled basis be $|j, m\rangle$. Let the state of our particle be $|\chi\rangle$.
What do you call/ how do you interpret the quantity given by $|\langle 1, -1; \frac{1}{2}, -\frac{1}{2}|\chi\rangle|^2$?
Select all that apply.

☐ The probability of measuring $m_l = -1$ and $m_s = -\frac{1}{2}$ for our state $|\chi\rangle$.
☐ The expectation value of the z-components for our state $|\chi\rangle$.
☐ The result of measuring the z-component of angular momentum and spin for $|\chi\rangle$.
☐ The wave function of $|\chi\rangle$ in the uncoupled basis.
☐ The probability of measuring $m_l = 1$ and $m_s = \frac{1}{2}$ for our state $|\chi\rangle$.

Figure C.22: The question gives students the mathematical expression corresponding to the probability of a particular measurement in either Dirac notation or wave function notation, and students must identify it as such. Incorrect answers consist of other common mathematical objects. Question in this subcategory consist of observables encountered in a second quarter quantum mechanics course.

Subcategory 6: Probability Density

Let x be the position operator with eigenstates $|x\rangle$, and let $|\psi\rangle$ be a state for a particle. What is the best description for the quantity $|\langle x|\psi\rangle|^2$?

- ○ The position-space probability density for a particle described by the state $|\psi\rangle$.
- ○ The probability of measuring the position of a particle described by the state $|\psi\rangle$ to be at x.
- ○ The result of measuring the position of a particle described by the state $|\psi\rangle$.
- ○ The current position of a particle described by the state $|\psi\rangle$.

Figure C.23: The question gives students the mathematical expression corresponding to the probability density of a state in Dirac notation or wave function notation, and students must identify it as such. Incorrect answers consist of other common mathematical objects.

Subcategory 7: Eigenvalue Equations

Let H be a Hamiltonian with energy eigenstates $|n\rangle$ and energy eigenvalues E_n. Which of the following best describes the equation $H|j\rangle = E_j|j\rangle$?

- ○ The eigenvalue equation for $|j\rangle$.
- ○ A statement that the expectation value of H is E_j.
- ○ A statement that the probability of measuring $|j\rangle$ is E_j.
- ○ A statement that measuring the energy of a particle in the state $|j\rangle$ returns the state $E_j|j\rangle$.

Figure C.24: The question gives students the mathematical expression corresponding to the eigenvalue equation for a particular energy eigenvalue, and students must identify it as such. Incorrect answers consist of other common mathematical objects.

Subcategory 8: Wave Functions/ States

Let $|\psi\rangle$ be the state of a particle, and let x be the position operator with eigenstates $|x\rangle$. How can we interpret the expression $\langle x|\psi\rangle$? Select all that apply.

☐ The coefficient of $|x\rangle$ in the position basis expansion of $|\psi\rangle$.

☐ The position-space wave function corresponding to $|\psi\rangle$, $\psi(x)$.

☐ The action of the position operator on $|\psi\rangle$.

☐ The eigenvalue of $|\psi\rangle$ under the position operator.

☐ The position x times the position-space wave function $\psi(x)$.

Figure C.25: The question gives students the mathematical expression corresponding to the state of a particular in either wave function notation or Dirac notation, and students must identify it as such. Incorrect answers consist of other common mathematical objects.

Subcategory 9: Inner Products

Consider a particle in a state $|\psi\rangle$ with Hamiltonian H whose energy eigenvalues are E_n with corresponding energy eigenstates $|\psi_n\rangle$. Let the position-space wave function for the state of our particle and for the energy eigenstates be $\psi(x) = \langle x|\psi\rangle$ and $\psi_n(x) = \langle x|\psi_n\rangle$, respectively.

What description best characterizes the expression $\int_{-\infty}^{\infty} \psi_n^*(x)\psi(x)\mathrm{d}x$? Select all that apply.

- ☐ The inner product of our state with the n^{th} energy eigenstate.
- ☐ The expectation value of the energy for a particle in the state $|\psi\rangle$.
- ☐ The probability of measuring the energy of our particle to be E_n.
- ☐ The probability density of $|\psi\rangle$ in energy space.
- ☐ The probability *amplitude* of measuring the energy of our particle to be E_n.

Figure C.26: The question gives students the mathematical expression corresponding to the inner product of two states in either Dirac notation or wave function notation, and students must identify it as such. Incorrect answers consist of other common mathematical objects.

Category 3: Special Topics

Subcategory 1: Interpret Expectation Values

Let p be the momentum operator with eigenstates $|p\rangle$, and let $|\psi\rangle$ be a ket. Let's say we prepare an ensemble of identically prepared particles in the state $|\psi\rangle$, and I measure the momentum of each and average the results. What quantity do I end up with?

○ The expectation value of p for the state $|\psi\rangle$.
○ The probability of measuring the momentum p for our state $|\psi\rangle$.
○ The inner product of $|p\rangle$ with $|\psi\rangle$.
○ The action of the momentum operator on $|\psi\rangle$.
○ The result of measuring the momentum of $|\psi\rangle$.

Figure C.27: This question gives students a state and momentum eigenstates in Dirac notation and describes the ensemble average of an expectation value of momentum. Students must recognize that the quantity being described is, in fact, an expectation value. Common incorrect answers involve simply other mathematical objects, such as probabilities, acting on the state with the momentum operator, and so on.

Subcategory 2: Inner Products - What Object?

Let $|\psi\rangle$ and $|\phi\rangle$ be two kets. Which of the following best describes that kind of mathematical object $\langle\phi|\psi\rangle$ is?

- ○ A complex number.
- ○ A ket.
- ○ A bra.
- ○ An observable.

Figure C.28: This question gives students two states and asks what kind of mathematical object the inner product is. Incorrect answers involve kets, bras, and other non-numerical mathematical objects.

Subcategory 3: Probability Density VS Probability

Let $\psi(x)$ be a wave function for a particle. What best describes the difference between the two quantities $|\psi(x)|^2$ and $|\psi(x)|^2 \mathrm{d}x$?

- ○ $|\psi(x)|^2$ is the probability density for the position for our particle, and $|\psi(x)|^2 \mathrm{d}x$ is the probability that our particle will be located in a small interval around the position x.
- ○ $|\psi(x)|^2$ is the probability that the position of our particle is x, and $|\psi(x)|^2 \mathrm{d}x$ is the probability that our particle will be located in a small interval around x.
- ○ $|\psi(x)|^2$ is the probability density for the position of our particle, and $|\psi(x)|^2 \mathrm{d}x$ describes the inner product of $\psi(x)$ with itself.
- ○ $|\psi(x)|^2$ is the probability that the position of our particle is x, and $|\psi(x)|^2 \mathrm{d}x$ describes the inner product of $\psi(x)$ with itself.
- ○ There is no difference: the quantities describe the same physical object.

Figure C.29: This question gives students a state and mathematical expressions for its associated probability density and the probability of a particular measurement. Students must interpret the difference between the probability and probability density in this context.

Subcategory 4: Wave Function - Interpret

Let $\psi(x)$ denote the position-space wave function of a state $|\psi\rangle$. Which of the following expressions are true? Select all that apply.

- ☐ $\psi(x) = \langle x|\psi\rangle$
- ☐ $\psi(x) = |\psi\rangle$
- ☐ $\psi(x) = |\psi(x)\rangle$
- ☐ $\langle x|\psi\rangle = x\psi(x)$

Figure C.30: This questions asks students to choose correct mathematical expressions related to the state and wave function of a particle. Incorrect answers involve inserting the wave function in a ket, saying the ket is equal to the wave function, and saying the wave function in Dirac notation is simply the position operator times the wave function.

Subcategory 5: Use of Eigenvalue Equations

Consider a particle in the state $|\psi\rangle$ and let $\hat{\boldsymbol{S}}_z$ be the operator for spin-$\frac{1}{2}$ in the z-direction, with eigenstates $|\uparrow\rangle_z$ and $|\downarrow\rangle_z$ and corresponding eigenvalues $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$.

Let $|\psi\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle_z + \frac{i}{\sqrt{2}}|\downarrow\rangle_z$ be the eigenbasis expansion for spin-$\frac{1}{2}$ in the z-direction for our particle's state.

Which of the following is the expression $\hat{\boldsymbol{S}}_z \left(\frac{1}{\sqrt{2}}|\uparrow\rangle_z + \frac{i}{\sqrt{2}}|\downarrow\rangle_z\right)$ equivalent to?

○ $\frac{\hbar}{2}\frac{1}{\sqrt{2}}|\uparrow\rangle_z - \frac{\hbar}{2}\frac{i}{\sqrt{2}}|\downarrow\rangle_z$

○ $\frac{\hbar}{2}\frac{1}{\sqrt{2}}|\uparrow\rangle_z + \frac{\hbar}{2}\frac{i}{\sqrt{2}}|\downarrow\rangle_z$

○ Either $\frac{\hbar}{2}$ or $-\frac{\hbar}{2}$ depending on what the spin of the particle is.

Either $|\uparrow\rangle_z$ or $|\downarrow\rangle_z$ depending on what the result of a measurement corresponding to the action of $\hat{\boldsymbol{S}}_z$ is.

Figure C.31: This question asks students to utilize the eigenvalue equations for a spin operator in order to simplify a mathematical expression. Incorrect answers, involve using only one eigenvalue or interpreting the action as a measurement.

Category 4: Translate Between Dirac and Wave Function Notation

Subcategory 1: Probabilities - WF to D - Discrete

Let H be the Hamiltonian for the infinite square well from 0 to a, with energy eigenkets $|n\rangle$ for $n = 1, 2, 3, ...$ Let $\psi(x) = \langle x|\psi\rangle$ be the position-space wave function for a particle, and define $\langle x|n\rangle = \psi_n(x)$. Recall that the probability of measuring the third excited state energy E_4 is $|\int_0^a \psi_4^*(x)\psi(x)\mathrm{d}x|^2$. What is the corresponding expression for the probability of measuring the particle's energy to be E_4 in Dirac notation?

- ○ $|\langle 4|\psi\rangle|^2$
- ○ $\langle 4|\psi\rangle$
- ○ $\langle \psi|4\rangle$
- ○ $\langle 4|H|\psi\rangle$
- ○ $|\langle 4|H|\psi\rangle|^2$

Figure C.32: This question presents students with a wave function for a particle in position space, as well as key components involved in the translation of the probability of measuring the energy in position-space wave function notation. Students are then tasked with identifying the corresponding expression for the probability of the same measurement in Dirac notation. Common incorrect answer choices involve inserting the operator between the bra and ket, and misplacing norm-squares.

Subcategory 2: Probabilities - D to WF - Discrete

Let H be a Hamiltonian with energy eigenkets $|\psi_n\rangle$ for $n = 0, 1, 2, ...$ Assume that a particle is in the state $|\psi\rangle$. The probability of measuring the fifth excited state energy E_5 in Dirac notation is given by $|\langle\psi_5|\psi\rangle|^2$. Defining $\langle x|\psi_n\rangle = \psi_n(x)$ and $\langle x|\psi\rangle = \psi(x)$, what is the corresponding expression for this probability in position-space wave function notation?

- ○ $|\int_{-\infty}^{\infty} \psi_5^*(x)\psi(x)\mathrm{d}x|^2$
- ○ $|\int_{-\infty}^{\infty} \psi_5(x)\psi(x)\mathrm{d}x|^2$
- ○ $\int_{-\infty}^{\infty} \psi_5^*(x)\psi(x)\mathrm{d}x$
- ○ $\int_{-\infty}^{\infty} \psi_5(x)\psi(x)\mathrm{d}x$

Figure C.33: This question presents students with a state in Dirac notation, as well as key components involved in the translation of the probability of measuring the energy in Dirac notation. Students are then tasked with identifying the corresponding expression for the probability of the same measurement in position-space wave function notation. Common incorrect answer choices involve misplacing norm-squares, and not complex-conjugating particular terms.

Subcategory 3: Probabilities - D to WF - Continuous

Assume that we have a particle described by the momentum-space wave function $\phi(p) = \langle p|\psi\rangle$. Recall that the probability of the particle moving to the right in Dirac notation is given by the expression $\int_0^\infty |\langle p|\psi\rangle|^2 \mathrm{d}p$. If we define $\psi(x) = \langle x|\psi\rangle$ and $\langle x|p\rangle = e^{ipx/\hbar}$, what is the corresponding expression for this probability in position-space wavefunction notation?

- ○ $\int_0^\infty |\int_{-\infty}^\infty e^{-ipx/\hbar}\psi(x)\mathrm{d}x|^2\mathrm{d}p$
- ○ $\int_0^\infty e^{-ipx/\hbar}\psi(x)\mathrm{d}x$
- ○ $\int_0^\infty |\int_{-\infty}^\infty e^{ipx/\hbar}\psi(x)\mathrm{d}x|^2\mathrm{d}p$
- ○ $\int_0^\infty \int_{-\infty}^\infty |e^{-ipx/\hbar}\psi(x)|^2\mathrm{d}x\mathrm{d}p$

Figure C.34: This question presents students with a wave function for a particle in momentum space, as well as key components involved in the translation of the probability of measuring the momentum in Dirac notation. Students are then tasked with identifying the corresponding expression for the probability of the same measurement in position-space wave function notation. Common incorrect answer choices involve simply including pieces of the expression (the complex exponential, the position space wave-function), and misplacing the norm-squares.

Topic 2: Mathematical Procedures and Linear Algebra

Category 1: Allowed Operations on Bras and Kets

Subcategory 1: Stay in Hilbert Space

Let $|\psi\rangle$ and $|\phi\rangle$ be two kets in a Hilbert space $\mathcal{H}$. Which of the following quantities are also inside $\mathcal{H}$? Select all that apply.

- ☐ $|\psi\rangle + |\phi\rangle$
- ☐ $|\psi\rangle|\phi\rangle$
- ☐ $c|\psi\rangle$ for any complex number c
- ☐ $|\psi\rangle\langle\phi|$
- ☐ $\frac{|\phi\rangle}{|\psi\rangle}$

Figure C.35: This question provides students with a Hilbert space (or its dual) and some elements of this Hilbert space. Students are asked to identify which of the given quantities are also elements of this Hilbert space. Incorrect answers involve common operations observed in student responses, such as multiplying kets or dividing kets. Others involve adding kets and bras, or using bras as elements of the Hilbert space instead of its dual.

Subcategory 2: What Can This Do?

Let $|\psi\rangle$ be a ket inside a Hilbert space $\mathcal{H}$. What mathematical operations can this object perform? Select all that apply.

- ☐ You can add $|\psi\rangle$ to another ket in $\mathcal{H}$ to produce a new ket in $\mathcal{H}$.
- ☐ You can multiply $|\psi\rangle$ on the right of a bra in the dual space of $\mathcal{H}$ to produce a complex number.
- ☐ You can multiply $|\psi\rangle$ on the right or left of another ket in $\mathcal{H}$ to produce a new ket in $\mathcal{H}$.
- ☐ You can divide $|\psi\rangle$ by another ket in $\mathcal{H}$ to produce a complex numner.
- ☐ You can multiply $|\psi\rangle$ on the right or left of another ket in $\mathcal{H}$ to produce a complex number.

Figure C.36: This question gives students a Hilbert space (or its dual) and an element of this space. Students are asked to correctly identify the actions one can perform with this ket/bra (or combination of the two) as well as where the result of the operation is an element of. Incorrect answers involve the use of division, the use of multiplication, or the use of addition to produce objects that are described to be elements of space they are not.

Let $|\psi\rangle$ be a ket in a Hilbert space $\mathcal{H}$ and $\langle\phi|$ be a bra in the dual space of $\mathcal{H}$. Consider the mathematical object $|\psi\rangle\langle\phi|$. What mathematical operations can this perform? Select all that apply.

- ☐ You can act $|\psi\rangle\langle\phi|$ on the left of a ket in $\mathcal{H}$ to produce another ket in $\mathcal{H}$.
- ☐ You can act $|\psi\rangle\langle\phi|$ on the right of a bra in the dual space of $\mathcal{H}$ to produce another bra.
- ☐ You can act $|\psi\rangle\langle\phi|$ on the left of a ket in $\mathcal{H}$ to produce a complex number.
- ☐ You can act $|\psi\rangle\langle\phi|$ on the right of a bra in the dual space of $\mathcal{H}$ to produce a complex number.

Figure C.37: This question gives students a Hilbert space (or its dual) and an element of this space. Students are asked to correctly identify the actions one can perform with this ket/bra (or combination of the two) as well as where the result of the operation is an element of. This example has incorrect answers involving inappropriate uses scalar multiplication or scalars as the result of operations.

Category 2: Completeness Relations

Subcategory 1: Discrete Spectrum

Let A be a Hermitian operator with discrete spectrum with eigenkets $|a_n\rangle$ and eigenvalues a_n for $n = 1, 2, 3, ...$ What is the completeness relation for A?

○ $\sum_{n=1}^{\infty} |a_n\rangle\langle a_n|$

○ $\sum_{n=1}^{\infty} a_n|a_n\rangle$

○ $\sum_{n=1}^{\infty} a_n|a_n\rangle\langle a_n|$

○ $\sum_{n=1}^{\infty} |a_n\rangle$

Figure C.38: This question gives students a Hermitian operator with a discrete spectrum as well as all of its eigen-information. Students are tasked to identify an expression corresponding to the identity operator/ completeness relation associated with this operator. Incorrect answers involve the use of eigenvalues or writing the operator as a simple superposition of kets.

Let $|\alpha\rangle$ and $|\beta\rangle$ be kets, and let H be a Hamiltonian with eigenkets $|n\rangle$ for $n = 1, 2, 3, ...$ The completeness relation for H is $\sum_{n=1}^{\infty} |n\rangle\langle n| = 1$.

What is the result of inserting this completeness relation into the expression $\langle\alpha|\beta\rangle$?

- ○ $\sum_{n=1}^{\infty}\langle\alpha|n\rangle\langle n|\beta\rangle$
- ○ $\sum_{n=1}^{\infty}\langle\alpha|\beta\rangle\langle n|n\rangle$
- ○ $\langle\alpha|n\rangle + \langle n|\beta\rangle$
- ○ $\langle\alpha|n\rangle\langle n|\beta\rangle$

Figure C.39: This question gives students a Hermitian operator with a discrete spectrum, all of its eigen-information, and two kets inside of a Hilbert space. Students are asked to identify the expression corresponding to inserting the completeness relation for our operator into the inner product of these two kets. Other questions ask to insert this relation into simply a ket or simply a bra. Incorrect answers involve placing the given ket/ bra in incorrect locations, simply having one outer product in the completeness relation, or placing a sum between the kets and bras making up the completeness relation.

Subcategory 2: Continuous Spectrum

Let x be the position operator, and let $|A\rangle$ be a ket. The completeness relation for position is $\int |x\rangle\langle x|\mathrm{d}x = 1$.

What is the result of inserting the completeness relation for position into $|A\rangle$?

- ○ $\int |x\rangle\langle x|A\rangle\mathrm{d}x$
- ○ $\int |A\rangle|x\rangle\langle x|\mathrm{d}x$
- ○ $\int |x\rangle|A\rangle\langle x|\mathrm{d}x$
- ○ $\int |x\rangle\langle x|\mathrm{d}x|A\rangle$

Figure C.40: This question gives students a Hermitian operator with a continuous spectrum, all of its eigen-information, and a ket inside of a Hilbert space. Students are asked to identify the expression corresponding to inserting the completeness relation for our operator into this ket. Other questions ask to insert this relation into simply a bra or into an inner product of a bra and a ket. Incorrect answers involve not placing the original ket in the correct location (inside or outside the integral).

Category 3: Complex Conjugation

Subcategory 1: Ket ↔ Bra (Single)

Let $|\beta\rangle$ be a ket in a Hilbert space $\mathcal{H}$, and let c, d be real. What is the corresponding bra to the ket $(c+id)|\beta\rangle$?

- ☐ $(c+id)\langle\beta|$
- ☐ $(c-id)\langle\beta|$
- ☐ $-(c+id)\langle\beta|$
- ☐ $(-c+id)\langle\beta|$

Figure C.41: This question gives students a Hilbert space (or its dual), an element of this space, and a combination of real/ complex numbers acting as a coefficient for the vector. Students must identify the expression corresponding to the given vector's dual (either a bra or a ket), correctly conjugating the coefficient attached to it. Incorrect answers involve incorrect conjugations of the coefficient, with incorrect negative signs. The coefficients can either be variables or numbers.

Subcategory 2: Inner Products

Let $|\psi\rangle$ and $|\phi\rangle$ be two kets in a Hilbert space $\mathcal{H}$, and consider their inner product $\langle\psi|\phi\rangle$. Which expression is equivalent to $\langle\psi|\phi\rangle^*$?

- ○ $\langle\phi|\psi\rangle$
- ○ $\langle\psi|\phi\rangle$
- ○ $-\langle\psi|\phi\rangle$
- ○ $-\langle\phi|\psi\rangle$
- ○ $|\psi\rangle\langle\phi|$

Figure C.42: This question gives students two kets in a Hilbert space and asks them to identify an expression corresponding to the complex conjugate of their inner product. Incorrect answers involve no change, changing the inner product to an outer product, or the inclusion of a negative sign.

Subcategory 3: Complex Numbers and Variables

What is the complex conjugate of $ie^{i\pi/6}$?

- ☐ $-ie^{-i\pi/6}$
- ☐ $ie^{-i\pi/6}$
- ☐ $-ie^{i\pi/6}$
- ☐ $ie^{i\pi/6}$

Figure C.43: This question gives students a complex number (either as a number or a collection of variables), as asks students to identify the expression corresponding to its complex conjugate. Incorrect answers include various observed ways students mis-conjugate, including using negative signs inappropriately, and not recognizing all of the components that do receive conjugation individually.

Subcategory 4: Ket ↔ Bra (Linear Combination) (Variables)

Let $|\omega\rangle$, $|\mu\rangle$, and $|\nu\rangle$ be kets in a Hilbert space $\mathcal{H}$, and let r be a position real, θ, A, B be real, and z be complex. Define $|\lambda\rangle = \left(\frac{A-iB}{\sqrt{3}}\right)|\omega\rangle + z|\mu\rangle - re^{i\theta/3}|\nu\rangle$.

What is the bra corresponding to the ket $|\lambda\rangle$?

○ $\langle\lambda| = \left(\frac{A+iB}{\sqrt{3}}\right)\langle\omega| + z^*\langle\mu| - re^{-i\theta/3}\langle\nu|$

○ $\langle\lambda| = \left(\frac{A-iB}{\sqrt{3}}\right)\langle\omega| + z\langle\mu| - re^{i\theta/3}\langle\nu|$

○ $\langle\lambda| = -\left(\frac{A-iB}{\sqrt{3}}\right)\langle\omega| - z\langle\mu| + re^{i\theta/3}\langle\nu|$

○ $\langle\lambda| = -\left(\frac{A+iB}{\sqrt{3}}\right)\langle\omega| - z^*\langle\mu| + re^{-i\theta/3}\langle\nu|$

Figure C.44: This question gives students a Hilbert space (or its dual), a collection of vectors in this space, and a series of complex or real variables utilized as coefficients in a linear combination of these vectors. Students are tasked with identifying the expression for the dual (a linear combination of bras or kets), appropriately conjugating the coefficients. Incorrect answers involve incorrect conjugation of the coefficients, using negative signs inappropriately or forgetting to conjugate particular components.

Subcategory 5: Ket ↔ Bra (Linear Combination) (Numbers)

Let $|a\rangle$ and $|b\rangle$ be two kets in a Hilbert space $\mathcal{H}$. Define $|c\rangle = e^{i/2}(|a\rangle + 2i|b\rangle)$. What is the bra corresponding to the ket $|c\rangle$?

○ $\langle c| = e^{-i/2}(\langle a| - 2i\langle b|)$

○ $\langle c| = e^{i/2}(\langle a| + 2i\langle b|)$

○ $\langle c| = -e^{-i/2}(\langle a| + 2i\langle b|)$

○ $\langle c| = -e^{-i/2}(\langle a| - 2i\langle b|)$

Figure C.45: This question gives students a Hilbert space (or its dual), a collection of vectors in this space, and a series of complex or real numbers utilized as coefficients in a linear combination of these vectors. Students are tasked with identifying the expression for the dual (a linear combination of bras or kets), appropriately conjugating the coefficients. Incorrect answers involve incorrect conjugation of the coefficients, using negative signs inappropriately or forgetting to conjugate particular components.

Subcategory 6: Wave Functions

Let $\psi(x)$ be the position-space wave function for a state $|\psi\rangle$. Which of the following depicts the complex conjugate of $\psi(x)$? Select all that apply.

- ☐ $\psi^*(x)$
- ☐ $\langle\psi|x\rangle$
- ☐ $\langle x|\psi\rangle$
- ☐ $(|\psi\rangle)^*$
- ☐ $\langle\psi\rangle$

Figure C.46: This question gives students a wave function for a particle (either in wave function or Dirac notation) and asks students to identify the expressions that correspond to its complex conjugate (again, in both Dirac or wave function notation). Incorrect answers involve incorrect employments of complex conjugation such as the use of negative signs, as well as associating the state with the wave function.

Category 4: Expanding a State in a Basis

Subcategory 1: Arbitrary Basis

Let $|\psi\rangle$ be a ket inside an infinite dimensional Hilbert space $\mathcal{H}$, and let $|1\rangle$, $|2\rangle$, $|3\rangle$, ... be a basis for $\mathcal{H}$. Which of the following best describes $|\psi\rangle$ expanded in this basis? Select all that apply.

- ☐ $|\psi\rangle = \sum_{n=1}^{\infty} c_n|n\rangle$, where c_n is complex for all n.
- ☐ $|\psi\rangle = c_1|1\rangle + c_2|2\rangle + c_3|\rangle + ...$, where c_n is complex for all n.
- ☐ $|\psi\rangle = \sum_{n=1}^{\infty} |n\rangle$
- ☐ $|\psi\rangle = |1\rangle + |2\rangle + |3\rangle + ...$

Figure C.47: This question gives students a state in a Hilbert space (either infinite or finite dimensional), and an arbitrary basis. Often this basis is an eigenbasis, but not always. Some questions are in Dirac notation, others are in wave function notation. Students must identify expressions corresponding to the expansion of this state in this basis. Incorrect answers involve the use of real coefficients instead of complex coefficients, no coefficients, *etc.*

Subcategory 2: Eigenbasis

Let $|\psi\rangle$ be the state of a particle. Let H be a Hamiltonian with energy eigenstates $|\psi_n\rangle$ with corresponding energy eigenvalues E_n. Finally, recall that the position-space wave functions for our state and the energy eigenstates are $\psi(x) = \langle x|\psi\rangle$ and $\psi_n(x) = \langle x|\psi_n\rangle$, respectively.

Which of the following corresponds to, generally, $|\psi\rangle$ and/or $\psi(x)$ expanded in the energy eigenbasis? Select all that apply.

- ☐ $\psi(x) = c_1\psi_1(x) + c_2\psi_2(x) + ...$, where each c_n is complex.
- ☐ $|\psi\rangle = c_1|\psi_1\rangle + c_2|\psi_2\rangle + ...$, where each c_n is complex.
- ☐ $\psi(x) = E_1\psi_1(x) + E_2\psi_2(x) + ...$
- ☐ $|\psi\rangle = E_1|\psi_1\rangle + E_2|\psi_2\rangle + ...$
- ☐ $|\psi\rangle = c(|\psi_1\rangle + |\psi_2\rangle + ...)$, where c is complex.

Figure C.48: This question gives students a state in a Hilbert space (either infinite or finite dimensional), and an arbitrary basis. Often this basis is an eigenbasis, but not always. Some questions are in Dirac notation, others are in wave function notation. Students must identify expressions corresponding to the expansion of this state in this basis. Incorrect answers involve the use of real coefficients instead of complex coefficients, no coefficients, the use of the eigenvalues as the coefficients, *etc.*

Subcategory 3: Eigenbasis - Angular Momentum

Consider a spin-$\frac{3}{2}$ particle in the state $|\chi\rangle$. Define the eigenstates of the magnitude squared of the spin-$\frac{3}{2}$ operator and its z-component as $|s, m\rangle$, where $s = \frac{3}{2}$ and $m = \pm\frac{3}{2}, \pm\frac{1}{2}$.

What is the result of expanding $|\chi\rangle$ in the total spin-$\frac{3}{2}$ basis?

- ○ $|\chi\rangle = c_{\frac{3}{2}}|\frac{3}{2}, \frac{3}{2}\rangle + c_{\frac{1}{2}}|\frac{3}{2}, \frac{1}{2}\rangle + c_{-\frac{1}{2}}|\frac{3}{2}, -\frac{1}{2}\rangle + c_{-\frac{3}{2}}|\frac{3}{2}, -\frac{3}{2}\rangle$, where $c_{\frac{3}{2}}, c_{\frac{1}{2}}, c_{-\frac{1}{2}}$, and $c_{-\frac{3}{2}}$ are undetermined complex coefficients.
- ○ $|\chi\rangle = |\frac{3}{2}, \frac{3}{2}\rangle + |\frac{3}{2}, \frac{1}{2}\rangle + |\frac{3}{2}, -\frac{1}{2}\rangle + |\frac{3}{2}, -\frac{3}{2}\rangle$
- ○ $|\chi\rangle = \frac{1}{2}|\frac{3}{2}, \frac{3}{2}\rangle + \frac{1}{2}|\frac{3}{2}, \frac{1}{2}\rangle + \frac{1}{2}|\frac{3}{2}, -\frac{1}{2}\rangle + \frac{1}{2}|\frac{3}{2}, -\frac{3}{2}\rangle$
- ○ $|\chi\rangle = \frac{3\hbar}{2}|\frac{3}{2}, \frac{3}{2}\rangle + \frac{\hbar}{2}|\frac{3}{2}, \frac{1}{2}\rangle - \frac{\hbar}{2}|\frac{3}{2}, -\frac{1}{2}\rangle - \frac{3\hbar}{2}|\frac{3}{2}, -\frac{3}{2}\rangle$

Figure C.49: This questions gives students a particle with spin as well as the eigen-information for the spin operator associated to this spin. Students must identify the expression corresponding to this state expanded in the spin eigenbasis. Incorrect answers utilize no coefficients, the use of eigenvalues or the z-spin quantum number as a coefficient, or equal probabilities for each state.

Category 5: Orthogonality

Subcategory 1: Of Kets

Let $|\alpha\rangle$ and $|\beta\rangle$ be kets in a Hilbert space $\mathcal{H}$. What expression does the statement "$|\alpha\rangle$ and $|\beta\rangle$ are orthogonal" correspond to?

○ $\langle\alpha|\beta\rangle = 0$.
○ $\langle\alpha|\beta\rangle = 1$.
○ $\langle\alpha|\beta\rangle = \alpha^* \times \beta$.
○ $\langle\alpha|\beta\rangle = |\alpha|^2|\beta|^2$.

Figure C.50: This question gives students two kets in Hilbert space and tells students these kets are orthogonal. Students must identify which mathematical expressions are true given this information. Incorrect answers involve saying their inner product is 1 instead of 0, and confusing how kets and bras are multiplied.

Let $\psi(x)$ and $\phi(x)$ be wave functions in a Hilbert space $\mathcal{H}$. What expression does the statement "$\psi(x)$ and $\phi(x)$ are orthogonal" correspond to?

○ $\int_{-\infty}^{\infty} \psi^*(x)\phi(x)\mathrm{d}x = 0$.
○ $\int_{-\infty}^{\infty} \psi^*(x)\phi(x)\mathrm{d}x = 1$.
○ $\psi^*(x)\phi(x) = 0$.
○ $\psi^*(x)\phi(x) = 1$.

Figure C.51: This question gives students two wave functions in a Hilbert space and tells students these states are orthogonal. Students must identify which mathematical expressions are true given this information. Incorrect answers involve orthogonality implying the inner product is 1 instead of 0, and merely multiplying instead of integrating.

Subcategory 2: Of Eigenbases

Let H be a Hamiltonian with energy eigenkets $|n\rangle$ with $n = 0, 1, 2, 3, ...$ on a Hilbert space $\mathcal{H}$. Which of the following is/are true?

- ☐ $\langle 0|1\rangle = 0$.
- ☐ $\langle 2|2\rangle = 0$.
- ☐ $\langle 2|3\rangle = 0$.
- ☐ $\langle 4|4\rangle = 0$.
- ☐ $\langle 2|3\rangle = |4\rangle$.

Figure C.52: This question gives students an operator (in this case a Hamiltonian) as well as its eigenkets or eigenstates. Students must use this information (namely, the orthonormality of eigenbases of Hermitian operators) to identify which mathematical expressions are true. Incorrect answers involve confusing whether the inner product of the same (different) state(s) are 0 (or 1), or treating the inner product as a ladder operation (observed in a few student responses).

Let H be a Hamiltonian with energy eigenstates $\psi_n(x)$ with $n = 0, 1, 2, 3, ...$ on a Hilbert space $\mathcal{H}$. Which of the following is/are true?

- ☐ $\int_{-\infty}^{\infty} \psi_1^*(x)\psi_2(x)\mathrm{d}x = 0.$
- ☐ $\int_{-\infty}^{\infty} \psi_3^*(x)\psi_3(x)\mathrm{d}x = 1$
- ☐ $\psi_1^*(x)\psi_1(x) = 1.$
- ☐ $\psi_4^*(x)\psi_2(x) = 0.$
- ☐ $\int_{-\infty}^{\infty} \psi_6^*(x)\psi_6(x)\mathrm{d}x = 0$

Figure C.53: This question gives students an operator (in this case a Hamiltonian) as well as its eigenkets or eigenstates. Students must use this information (namely, the orthonormality of eigenbases of Hermitian operators) to identify which mathematical expressions are true. Incorrect answers involve confusing whether the inner product of the same (different) state(s) are 0 (or 1), or simply multiplying wave functions instead of integrating.

Subcategory 3: Orthonormality of Bases

Let $|\alpha\rangle, |\beta\rangle$, and $|\gamma\rangle$ form an orthonormal basis for the Hilbert basis $\mathcal{H}$. Which of the following is/are true?

☐ $\langle\alpha|\alpha\rangle = 1$.

☐ $\langle\alpha|\gamma\rangle = 0$.

☐ $\langle\alpha|\beta\rangle = 1$.

☐ $\langle\beta|\gamma\rangle = 1$.

☐ $\langle\gamma|\gamma\rangle = 0$.

Figure C.54: This question gives students a basis that they are explicitly told is orthonormal. Students must use this information to identify which mathematical expressions are true. Incorrect answers involve confusing whether the inner product of the same (different) state(s) are 0 (or 1), or treating the inner product as a ladder operation (observed in a few student responses).

Let $\psi_i(x)$ be an orthonormal basis for the Hilbert space $\mathcal{H}$, where $i = 1, 2, 3, ...$ Which of the following is/are true?

- ☐ $\int_{-\infty}^{\infty} \psi_1^*(x)\psi_2(x)\mathrm{d}x = 0.$
- ☐ $\int_{-\infty}^{\infty} \psi_3^*(x)\psi_3(x)\mathrm{d}x = 1.$
- ☐ $\psi_2^*(x)\psi_2(x) = 0.$
- ☐ $\psi_1^*(x)\psi_3(x) = 0.$
- ☐ $\int_{-\infty}^{\infty} \psi_3^*(x)\psi_4(x)\mathrm{d}x = 1.$

Figure C.55: This question gives students a basis that they are explicitly told is orthonormal. Students must use this information to identify which mathematical expressions are true. Incorrect answers involve confusing whether the inner product of the same (different) state(s) are 0 (or 1), or simply multiplying wave functions instead of integrating.

Topic 3: Pre-Dirac Notation

Category 1: Continuous Probability

Subcategory 1: Probability Density to Probability

Let $\Psi(x,t)$ be the wave function for a particle in a quantum mechanical system so that its probability density is $|\Psi(x,t)|^2$. What is the probability of measuring the particle to be between a and b?

- ○ $\int_a^b |\Psi(x,t)|^2 \mathrm{d}x$.
- ○ $\int_a^b \Psi(x,t) \mathrm{d}x$.
- ○ $|\Psi(b,t)|^2 - |\Psi(a,t)|^2$.
- ○ $|\Psi(b,t) - \Psi(a,t)|^2$.

Figure C.56: This question gives students a wave function for a particle and its associated probability density. Students must identify the mathematical expression corresponding to the probability of a particular event. Incorrect answers involve taking the difference between the probability density evaluate at the end points, the norm-square of the difference between the state evaluated at the end points, and integrating the state instead of the probability density.

Subcategory 2: Probability Density and Probability Amplitude - Units

Let $\Psi(x,t)$ be the wavefunction for a particle in a quantum mechanical system, so that its position-space probability density is $|\Psi(x,t)|^2$. Here, x refers to a one-dimensional position variable. What are the units of the wavefunction $\Psi(x,t)$?

○ $\frac{1}{[\text{meters}]^{1/2}}$.

○ Unitless.

○ $\frac{1}{[\text{meters}]}$.

○ [meters].

○ $[\text{meters}]^{1/2}$.

Figure C.57: This question gives students a wave function or probability density in either one spatial dimension or three spatial dimensions and acts students what the units of this quantity is. Incorrect answers involve having no units, not squaring (when applicable), or taking the reciprocal.

Category 2: Time Dependence in Quantum Mechanics

Subcategory 1: Adding Time Dependence

Let H be an arbitrary Hamiltonian with energy eigenstates $\psi_n(x)$ with energies E_n. Let $\Psi(x,0) = \frac{1}{2}\psi_1(x) - \sqrt{\frac{3}{4}}\psi_5(x)$. What is $\Psi(x,t)$?

- ○ $\Psi(x,t) = \frac{1}{2}\psi_1(x)e^{-\frac{i}{\hbar}E_1 t} - \sqrt{\frac{3}{4}}\psi_5(x)e^{-\frac{i}{\hbar}E_5 t}$
- ○ $\Psi(x,t) = \frac{1}{2}\psi_1(x)e^{-\frac{1}{\hbar}E_1 t} - \sqrt{\frac{3}{4}}\psi_5(x)e^{-\frac{1}{\hbar}E_5 t}$
- ○ $\Psi(x,t) = H\Psi(x,0)$
- ○ It is impossible to say: the state will be an arbitrary superposition of eigenfunctions of H.
- ○ $\Psi(x,t) = e^{-\frac{i}{\hbar}Et}(\frac{1}{2}\psi_1(x) - \sqrt{\frac{3}{4}}\psi_5(x))$ where E is the actual energy of our system.

Figure C.58: This question gives students a Hamiltonian, all of its eigen-information, and a state expanded in this energy eigenbasis at $t = 0$. Students are asked to identify the expression corresponding to the state at an arbitrary time t. Incorrect answers involve applying measurement when applicable, saying time dependence is impossible to determine, having an overall phase, and using decaying exponentials instead of complex exponentials.

Subcategory 2: Is it Time Dependent?

Let H be a Hamiltonian for a system with energy eigenstates $\psi_n(x)$, and let $\Psi(x) = \frac{1}{2}\psi_2(x) + \frac{1}{\sqrt{2}}\psi_3(x) - \frac{1}{2}\psi_5(x)$ be the state of a particle in position-space wave function notation.

Does the probability of measuring the particle to be between a and b, namely $\int_a^b |\Psi(x)|^2 dx$, change over time? Explain why or why not.

- ○ Yes: the integrand is time dependent, so the integral itself will be time dependent.
- ○ No: the integrand is not time dependent, so the integral itself will be not time dependent.
- ○ Yes: probabilities of position measurements always depend on time.
- ○ No: probabilities of position measurements never depend on time.

Figure C.59: This question gives students a Hamiltonian, its eigen-information, and the state of a particle expanded in this energy eigenbasis. Students must identify whether or not a particular quantity (a probability, a probability density, a state, or an expectation value) depends on time for this state. Incorrect answers depend on the quantity being asked about, but typically involve memorized responses. For example, an incorrect answer may be that the probability of a position probability may depend on time because the state depends on time.

Category 3: The Schrodinger Equation

Subcategory 1: Stationary States

Let H_{QHO} be the Hamiltonian for a quantum harmonic oscillator with energy eigenfunctions $\psi_n(x)$ and energies E_n. Assume that we have a particle in the state $\Psi(x,0) = \frac{1}{\sqrt{2}}\left(\psi_0(x) - \psi_3(x)\right)$. Is our particle in a stationary state? If so, why, and if not, why not?

- ○ Yes: the probability density of our particle is not time dependent.
- ○ Yes: it is a superposition of energy eigenfunctions.
- ○ No: the full wavefunction $\Psi(x,t)$ is time dependent.
- ○ No: the probability density for our particle depends on time.

Figure C.60: This question gives students a particle in a Hamiltonian, its eigen-information, and the state of a particle expanded in this energy eigenbasis. Students are asked to identify which statements are true given that the state is (or is not) a stationary state. Incorrect answers depend on whether the state is or is not described with a stationary state. For example, if the particle is in a stationary state, an incorrect answer involved the expectation value of position depending of time.

Subcategory 2: Time Independent Schrodinger Equation

Let H_{ISW} be the Hamiltonian for an infinite square well, and let $\psi_1(x)$, $\psi_2(x)$, and $\psi_4(x)$ be energy eigenfunctions of H_{ISW} with energies E_1, E_2, and E_4, respectively. Define $\Psi(x,0) = \frac{1}{2}\left(\psi_1(x) - \sqrt{2}\psi_2(x) + \psi_4(x)\right)$. What is $H_{\text{ISW}}\Psi(x,0)$ equal to?

○ $(E_1 - \sqrt{2}E_2 + E_4)\Psi(x,0)$.

○ $E_1 - \sqrt{2}E_2 + E_4$.

○ Either E_1, E_2, or E_4, with probabilities $1/4, 1/2$, and $1/4$, respectively.

○ $\frac{1}{2}(E_1\psi_1(x) - \sqrt{2}E_2\psi_2(x) + E_4\psi_4(x))$.

Figure C.61: This question gives students a Hamiltonian, its eigen-information, and the state of a particle expanded in this energy eigenbasis. Students are asked to identify the expression for the action of the Hamiltonian on this state, implicitly using the time-independent Schordinger equation. Incorrect answers involve summing eiegnvalues, simply giving the eigenvalues, and choosing one depending on what a measurement yields.

Subcategory 3: Time Dependent Schrodinger Equation

Let H be the Hamiltonian for a quantum harmonic oscillator with energy eigenstates $\psi_n(x)$ and energies E_n. Which of the following corresponds to the time dependent Schrodinger equation?

- ☐ $H\Psi(x,t) = i\hbar\frac{\partial\Psi(x,t)}{\partial t}$
- ☐ $H\psi_n(x) = E_n\psi_n(x)$ for all n.
- ☐ $H\Psi(x,t) = E_n\Psi(x,t)$ for $\Psi(x,t) = \psi_n(x)e^{-\frac{i}{\hbar}E_n t}$.
- ☐ $H\psi_n(x) = i\hbar\frac{\partial\psi_n(x)}{\partial t}$ for all n.

Figure C.62: This question gives students a Hamiltonian and its eigen-information. Students are asked to identify which expressions involving this information are true, utilizing knowledge of the time-dependent Schrodinger equation. Incorrect answers involve using the time-independent Schrodinger equation, or treating the eigenstates as states that can depend on time.

Let H be a Hamiltonian with energy eigenfunctions $\psi_n(x)$ and energies E_n. Let $\Psi(x,t)$ be an arbitrary wavefunction for a particle. Which of the following are always true?

☐ $H\Psi(x,t) = i\hbar\frac{\partial\Psi(x,t)}{\partial t}$

☐ $H\Psi(x,t) = E_n\Psi(x,t)$ where E_n is the energy of our particle.

☐ $H\Psi(x,t) = E_n\psi_n(x)$ where E_n and $\psi_n(x)$ are the energy and state of our particle after measuring its energy.

☐ $H\psi_n(x) = E_n\psi_n(x)$ for all n.

☐ $H\psi_n(x) = i\hbar\frac{\partial\psi_n(x)}{\partial t}$ for all n.

Figure C.63: This question gives students a Hamiltonian and its eigen-information. Students are asked to identify which expressions involving this information are true, utilizing knowledge of the time-dependent Schrodinger equation. Incorrect answers for this version involve treating action with the Hamiltonian as measurement or as finding the “real” energy of the particle.

Category 4: Graphs of States for Common Systems

Subcategory 1: Good Wave Function for Hamiltonian - ISW

Which of the following could be wave functions at $t = 0$ for a particle in an infinite square well from 0 to a? Assume that A in each answer choice normalizes the proposed wavefunctions, if possible.

☐ $\Psi(x,0) = \begin{cases} A\left(\sqrt{\frac{2}{5}}\sin\left(\frac{\pi x}{a}\right) + \sqrt{\frac{3}{5}}\sin\left(\frac{2\pi x}{a}\right)\right) & 0 \leq x \leq a \\ 0 & \text{otherwise} \end{cases}$

☐ $\Psi(x,0) = Ae^{-((x-a/2)/a)^2}$

☐ $\Psi(x,0) = A\sin^3\left(\frac{\pi x}{a}\right)$

☐ $\Psi(x,0) = A\left(\sqrt{\frac{2}{5}}\sin\left(\frac{\pi x}{a}\right) + \sqrt{\frac{3}{5}}\sin\left(\frac{2\pi x}{a}\right)\right)$

☐ $\Psi(x,0) = \begin{cases} A\sin^3\left(\frac{\pi x}{a}\right) & 0 \leq x \leq a \\ 0 & \text{otherwise} \end{cases}$

Figure C.64: This question gives students a particle in an infinite square well and asks students to identify which states could be possible states of the given particle? Incorrect answers for this version of the question involve not using the right boundary conditions, and not taking the state to be 0 outside of the well.

Let H_{ISW} be the Hamiltonian for the infinite square well from 0 to a, and define

$$\Psi(x,0) = \begin{cases} A\sin^3\left(\frac{\pi x}{a}\right) & 0 \leq x \leq a \\ 0 & \text{otherwise} \end{cases}$$

where A is a constant that normalizes our function. Could this be a wavefunction for an electron in this infinite square well?

- ☐ Yes: It satisfies the boundary conditions for the infinite square well.
- ☐ No: it is not a superposition of stationary states.
- ☐ Yes: it is a stationary state.
- ☐ No: it is not a stationary state.

Figure C.65: This question gives students a particle in an infinite square well and asks students to identify whether or not the given state could be a possible state of the particle. Incorrect answers for this version of the question involve not recognizing it's a superposition of eigenstates, and determining it is or is not because it is or is not a stationary state.

Subcategory 2: Graphs of Wave Functions - ISW

Let H_{ISW} be the Hamiltonian for the infinite square well between 0 and 1, and let $\Psi(x, 0)$ be the function given by the graph

Could $\Psi(x, 0)$ describe the state of a particle inside this well? If so, why, and if not, why not?

☐ Yes: it is smooth in the regions where the potential is not infinite.
☐ Yes: it satisfies the boundary conditions required by the infinite square well.
☐ No: it is not a stationary state.
☐ Yes: it is a stationary state.
☐ No: it does not satisfy the boundary conditions required by the infinite square well.

Figure C.66: This question gives students a particle in an infinite square well and a graph of a function. Students are asked to determine whether or not the given function could be the state of the particle. Incorrect answers involve not accounting for boundary conditions, smoothness, and whether or not the state is a stationary state.

Subcategory 3: Good Wave Function for Hamiltonian - QHO

Let H_{QHO} be the Hamiltonian for a quantum harmonic oscillator, and let $\Psi(x,0) = \sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)$, where a is a constant. Can this function be a wavefunction for an electron in our potential?

- ☐ Yes: it is normalized/ normalizable.
- ☐ No: it does not satisfy the boundary conditions required by the potential.
- ☐ No: it is not normalized/ normalizable.
- ☐ Yes: it is a stationary state.

Figure C.67: This question gives students a particle in a quantum harmonic oscillator and asks students to identify whether or not the given state could be a possible state of the particle. Incorrect answers for this version of the question involve not recognizing it's a superposition of eigenstates, and determining it is or is not because it is or is not a stationary state.

Subcategory 4: Graphs of Wave Functions - QHO

Let H_{QHO} be the Hamiltonian for a quantum harmonic oscillator, and let $\Psi(x, 0)$ be the function given by the graph

Could $\Psi(x, 0)$ describe the state of a particle inside this oscillator? If so, why, and if not, why not?

- ☐ Yes: it is smooth in the regions where the potential is not infinite.
- ☐ Yes: it satisfies the boundary conditions required by the quantum harmonic oscillator.
- ☐ No: it is not a stationary state.
- ☐ No: it does not satisfy the boundary conditions required by the quantum harmonic oscillator.

Figure C.68: This question gives students a particle in a quantum harmonic oscillator and a graph of a function. Students are asked to determine whether or not the given function could be the state of the particle. Incorrect answers involve not accounting for boundary conditions, smoothness, and whether or not the state is a stationary state.

Topic 4: Mathematization:

Category 1: Choosing a Bra in a Probability Calculation

Subcategory 1: Discrete - Dirac

Let Q be an arbitrary Hermitian operator with eigenstates $|q_i\rangle$ for $i = 0, 1, 2, 3, ...$ and eigenvalues q_i, and assume that you have a particle in the state $|\psi\rangle$. Recall that in calculating the probability of measuring a particular value of q_i, you take the inner product of a particular bra with $|\psi\rangle$. If you wanted to find the probability of measuring q_0, what bra do you take the inner product with?

- ○ $\langle q_0|$
- ○ $\langle Q|$
- ○ $\langle q_i|$
- ○ $\langle q_1|$

Figure C.69: This question gives students a state and a Hermitian operator with a discrete spectrum along with its eigen-information, and it reminds students heuristically on how to calculate the probability that a particular eigenvalue is measured. Students must identify which bra, in Dirac notation, is used in this calculation. Incorrect answers contain various observed responses students give when solving for probabilities, such as labeling the bra with the operator in question, keeping the index of the bra arbitrary, *etc.*

Subcategory 2: Continuous - Dirac

Assume you have a particle in the state $|\psi\rangle$, and let p be the momentum operator with eigenkets $|p\rangle$. Recall that calculating the probability of measuring the momentum to be in a small interval around a particular momentum p involves calculating an inner product of some bra with $|\psi\rangle$. If you wanted to calculate the momentum to be within a small interval around the momentum p_i, what bra do you take the inner product with?

○ $\langle p_i|$

○ $\langle p|$

○ $\langle p_0|$

○ $\langle p|p$

Figure C.70: This question gives students a state and a Hermitian operator with a continuous spectrum along with its eigen-information, and it reminds students heuristically on how to calculate the probability that a particular range of eigenvalues is measured. Students must identify which bra, in Dirac notation, is used in this calculation. Incorrect answers contain various observed responses students give when solving for probabilities, such as labeling the bra with the operator in question, keeping the index of the bra arbitrary, *etc.*

Subcategory 3: Continuous - Wave Function

Assume you have a particle in the state $\psi(x)$, and let H be Hamiltonian with eigenstates $\psi_n(x)$. Recall that calculating the probability of measuring the energy to be E_n involves calculating an inner product of some state with $\psi(x)$. If you wanted to calculate the energy to be E_3, what state do you take the inner product with?

- ○ $\psi_3(x)$
- ○ $\psi(x)$
- ○ $|\psi(x)|^2$
- ○ $|\psi_3(x)|^2$

Figure C.71: This question gives students a state and a Hermitian operator with a continuous spectrum along with its eigen-information, and it reminds students heuristically on how to calculate the probability that a particular range of eigenvalues is measured. Students must identify which state, in wave function notation, is used in this calculation. Incorrect answers contain various observed responses students give when solving for probabilities, such using the original state as the secondary state, and extra norm-squares.

Category 2: Error Checking Methods

Subcategory 1: Probability - 0 to 1

Let the state of a particle be $\psi(x)$ in a Hamiltonian H with energy eigenstates $\psi_n(x)$ and corresponding energy eigenvalues E_n. When calculating the probability of measuring the energy to be E_3, you conclude with the answer:

$\frac{3}{2}E_3$.

Is there anything wrong with this answer? Why or why not?

- ○ The answer must be incorrect: probabilities are between 0 and 1, and E_3 is (likely) not in that range of values.
- ○ The answer could be correct: the answer is proportional to E_3 and thus obeys an eigenvalue equation.
- ○ The answer must be incorrect: there's no state multiplying the E_3, so it does not obey an eigenvalue equation.
- ○ The answer could be correct: the units of the probability are energy units.

Figure C.72: This question gives students an operator and its eigen-information. Students are told that, when solving for the probability of a particular measurement, they find a particular answer that is given to them (in this case, one involving an eigenvalue). Students are asked whether or not this answer makes sense, given that they were solving for a probability. Incorrect answers involve comments made by students not recognizing the need for a probability to be between 0 and 1.

Subcategory 2: Probability - Units

Let the state of a particle be $\psi(x)$ in a Hamiltonian H with energy eigenstates $\psi_n(x)$ and corresponding energy eigenvalues E_n. When calculating the probability of measuring the energy to be E_3, you conclude with the answer:
$\frac{3}{2}E_3$.
Is there anything wrong with this answer, and why?

- ○ The answer must be incorrect: probabilities are unitless, but this has units of energy.
- ○ The answer could be correct: the answer is proportional to E_3 and thus obeys an eigenvalue equation.
- ○ The answer must be incorrect: there's no state multiplying the E_3, so it does not obey an eigenvalue equation.
- ○ The answer could be correct: the units of the probability are energy units.

Figure C.73: This question gives students an operator and its eigen-information. Students are told that, when solving for the probability of a particular measurement, they find a particular answer that is given to them (in this case, one involving an eigenvalue). Students are asked whether or not this answer makes sense, given that they were solving for a probability. Incorrect answers involve comments made by students not recognizing the need for a probability to be unitless.

Subcategory 3: Expectation Values - Units

Let the state of a particle be $\psi(x)$ in a Hamiltonian H with energy eigenstates $\psi_n(x)$ and corresponding energy eigenvalues E_n. When calculating the expectation value of the energy for our state, you conclude with the answer:

$\frac{3}{2}$.

Is there anything wrong with this answer, and why?

- ○ The answer must be incorrect: the expectation value of energy must have units of energy, and this appears unitless.
- ○ The answer could be correct: it is unitless.
- ○ The answer must be incorrect: it must be between 0 and 1.
- ○ The answer could be correct: it is a unitless number, which is what results from an integral.

Figure C.74: This question gives students an operator, its eigen-information, and a state. Students are told that, when solving for the expectation value of this operator for the given state, they find a particular answer that is given to them (in this case, a unitless number). Students are asked whether or not this answer makes sense, given that they were solving for an expectation value. Incorrect answers involve comments made by students not recognizing the need for an expectation value to have the same units as the eigenvalues (in this case, units of energy).

Subcategory 4: Eigenvalues - Real vs Imaginary

Let the state of a particle be $\psi(x)$ in a Hamiltonian H. When diagonalizing this Hamiltonian to find its energy eigenstates and eigenvalues, you find the following expression for the eigenvalues:

$\frac{i\pi n\hbar}{ma^2}$,

where a is a distance scale in the problem. Is there anything wrong with this answer, and why?

- ○ The answer must be incorrect: eigenvalues for observables must always be real.
- ○ The answer could be correct: it has units of energy.
- ○ The answer must be incorrect: it must be between 0 and 1.
- ○ The answer could be correct: it is quantized in n.

Figure C.75: This question gives students an operator and a state. Students are told that, when solving for the eigenvalues of this operator, they find a particular answer that is given to them (in this case, a number with an imaginary value). Students are asked whether or not this answer makes sense, given that they were solving for an eigenvalue. Incorrect answers involve comments made by students not recognizing the need for an eigenvalue of a Hermitian operator to be real.

Subcategory 5: Probability - Real vs Imaginary

Let the state of a particle be $\psi(x)$ in a Hamiltonian H with energy eigenstates $\psi_n(x)$ and corresponding energy eigenvalues E_n. When solving for the probability of measuring the energy to be E_1, you find the following expression:

$\frac{i}{\sqrt{2}}$.

Is there anything wrong with this answer, and why?

- ○ The answer must be incorrect: probabilities are real and this is imaginary.
- ○ The answer could be correct: it is unitless.
- ○ The answer must be incorrect: it's unitless, and needs to have units of energy.
- ○ The answer could be correct: it is between 0 and 1, as required of probabilites.

Figure C.76: This question gives students an operator, its eigen-information, and a state. Students are told that, when solving for the probability for a measurement, they find a particular answer that is given to them (in this case, a number with an imaginary value). Students are asked whether or not this answer makes sense, given that they were solving for a probability. Incorrect answers involve comments made by students not recognizing the need for a probability to be real.

Category 3: Error Correcting

Subcategory 1: No Error

Let $\psi(x)$ be the state of a particle, and let H be a Hamiltonian with eigenfunctions $\psi_n(x)$ and corresponding eigenvalues E_n. When trying to calculate the inner product of our state with $\psi_2(x)$, you produce the following expression:

$\int_{-\infty}^{\infty} \psi_2^*(x)\psi(x)dx$.

What is wrong, if anything, with this expression?

- ○ There is nothing wrong with this expression.
- ○ There should be a norm-square outside the integral.
- ○ There should be a norm-square inside the integral.
- ○ You should be taking the product of the functions, so that the inner product is $\psi_2^*(x)\psi(x)$.

Figure C.77: This question gives students a state, a Hamiltonian and its eigen-information, and tells students that in calculating an inner product of two states, they end up with a particular expression. Students must determine if this expression is correct or not using heuristics known about various mathematical objects like inner products, expectation values, probabilities, *etc.* This question involves nothing incorrect about the given expression. Incorrect answers involve common patterns in incorrect responses from students. For example, students often don't integrate when taking the inner product of two states in wave function notation.

Subcategory 2: Ortho vs Non-Ortho

Let $\psi(x)$ be the state of a particle. Let H be the Hamiltonian for the infinite square well from 0 to a, with eigenfunctions $\psi_n(x)$ and corresponding eigenvalues E_n. When trying to calculate the probability of the position of the particle to be on the left half of the well, you produce the following expression:

$\int_{a/2}^{a} \psi_2^*(x)\psi_4(x)dx$.

You then evaluate this to be 0 due to orthonormality. What is wrong, if anything, with this evaluation?

- ○ The integral in orthonormality must be over the entire range of the well.
- ○ The integrand is even, which means the integral must be non-zero.
- ○ The integrand is missing a norm-square.
- ○ There is nothing wrong: $\psi_2(x)$ and $\psi_3(x)$ are orthonormal.

Figure C.78: This question gives students a state, a Hamiltonian and its eigen-information, and tells students that in calculating a probability of measuring a particular state, they end up with a particular expression. Students must determine if this expression is correct or not using heuristics known about orthonormality. The expressions in this subcategory are incorrect. Incorrect answers involve saying nothing is wrong with the expression (which is incorrect here due to the bounds of integration), not integrating, misusing symmetry.

Subcategory 3: Wrong Norm-Square

Let $\psi(x)$ be the state of a particle, and let H be a Hamiltonian with eigenfunctions $\psi_n(x)$ and corresponding eigenvalues E_n. When trying to calculate the probability of measuring the energy to be $\psi_2(x)$, you produce the following expression:

$\int_{-\infty}^{\infty} |\psi_2^*(x)\psi(x)|^2 dx$.

What is wrong, if anything, with this expression?

○ The norm-square is in the wrong place: it should be outside of the integral.

○ The $\psi(x)$ should be $\psi_2(x)$, since we want the norm-square of $\psi_2(x)$ as the integrand.

○ The $\psi_2^*(x)$ should be $\psi^*(x)$, since for probability calculations, the integrand is always $|\psi(x)|^2$.

○ There is nothing wrong with the expression.

Figure C.79: This question gives students a state, a Hamiltonian and its eigen-information, and tells students that in calculating a probability of measuring a particular state, they end up with a particular expression. Students must determine if this expression is correct or not using heuristics known about where norm-squares are (if at all) for various quantities. The expressions in this subcategory are incorrect. Incorrect answers involve common responses students give for the respective objects. For example, when responding with probabilities, students often use the norm-square of the eigenstate in the integrand.

Category 4: Choosing a Notation

Subcategory 1: Matrix Notation

You are given a state for a spin-$\frac{1}{2}$ particle written in the spin-z basis and are asked to calculate the expectation value of spin-x for this particle. You are given the state in Dirac notation, and the matrix representations of the S_x, S_y, and S_z operators in the z-basis.

Which notation would be a good option to use for this problem and why?

- ○ Matrix notation, because you aren't given how S_x acts on the spin-z eigenstates in Dirac notation, and are given the spin-matrices in the z-basis.
- ○ Wave function notation, because you're trying to find the expectation value of the spin in a particular direction, x.
- ○ Dirac notation, because the state is given in Dirac notation.

Figure C.80: This question gives students a scenario (possibly a state, an operator and its eigen-information, *etc.*) and asks students, using the given information, which notation might best suit the problem. Incorrect answers are tailored to the scenario in question. For example, this question gives students the operators for S_i as matrices in the z-basis, and so while the state is given in Dirac notation, it's likely easier to turn your Dirac state into matrix notation as opposed to the matrices into relationships in Dirac notation.

Subcategory 2: Dirac Notation

You are given a state for a spin-$\frac{1}{2}$ particle written in the spin-z basis and are asked to calculate the expectation value of spin-x for this particle. You are given the state in Matrix notation, and you are told how the S_x, S_y, and S_z operators act on the spin-z eigenstates.

Which notation would be a good option to use for this problem and why?

○ Dirac notation, because you do not immediately have the matrices for the spin-operators.

○ Wave function notation, because you are finding the expectation value of the spin in a particular spatial direction, x.

○ Matrix notation, since you are given the state in matrix notation.

Figure C.81: This question gives students a scenario (possibly a state, an operator and its eigen-information, *etc.*) and asks students, using the given information, which notation might best suit the problem. Incorrect answers are tailored to the scenario in question. For example, this question gives students the information that the state is written in the z-basis in matrix notation, and are told how the matrices of the spin operators act on the z-eigenstates.

Subcategory 3: Wave Function Notation

While working on homework, you are assigned a problem that involves calculating the expecation value of position x for a state given to you.

Which notation would be a good option to use for this problem and why?

- ○ Wave function notation, because the position operator has continuous spectrum and thus would be easier to calculate in this representation.
- ○ Matrix notation, because the process for calculating expectation values in matrix notation is simply a row vector, a matrix, and a column vector.
- ○ Dirac notation, because Dirac notation is the most compact and will require the least amount of space to perform the calculation.

Figure C.82: This question gives students a scenario (possibly a state, an operator and its eigen-information, *etc.*) and asks students, using the given information, which notation might best suit the problem. Incorrect answers are tailored to the scenario in question. For example, this question asks the students to calculate the value of position.

Category 5: Integration Shortcuts

Subcategory 1: Even vs Odd

Let $\psi(x)$ be the state of a particle in a quantum harmonic oscillator. Recall that the energy eigenstates of this Hamiltonian are $\psi_n(x) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)$, where $H_n(x)$ are Hermite polynomials (which alternate being even and odd about $x = 0$). When calculating the expectation value of the position of the particle, you encounter the following integral:

$\int_{-\infty}^{\infty} xe^{-\frac{m\omega x^2}{\hbar}} H_0\left(\sqrt{\frac{m\omega}{\hbar}}x\right) H_1\left(\sqrt{\frac{m\omega}{\hbar}}x\right) dx$

Without performing any calculations, what can you say about this integral?

- ○ It is non-zero, because the integrand is even: x is odd, the exponential is even, $H_0(x)$ is even, and $H_1(x)$ is odd.
- ○ It is 0, because x is odd and thus the integrand is odd.
- ○ It is 0, because both x and $H_1(x)$ are odd.
- ○ We cannot say anything about this integral: we need to calculate it.

Figure C.83: This question gives students a particle in a particular, known Hamiltonian, as well as its eigen-information. Students are then given an integral involving the eigenstates of the integral and asked if they can deduce the value of the integral based on that information. Incorrect answers involve incorrect symmetry considerations (wrong point of symmetry, incorrect symmetries of given functions), and that no information can be deduced.

Subcategory 2: Orthonormality

Let $\psi(x)$ be the state of a particle in an infinite square well from 0 to a. Recall that the energy eigenstates of this Hamiltonian are $\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$. When calculating the probability of measuring the energy to be E_3, you encounter the following integral:
$\frac{2}{a}\int_0^a \sin\left(\frac{3\pi x}{a}\right)\sin\left(\frac{3\pi x}{a}\right)dx$
Without performing any calculations, what can you say about this integral?

- ○ It is 1, because it is the integral of an eigenstate with itself.
- ○ It is 0, because it is the integral of two eigenstates, which are always 0 due to orthonormality.
- ○ It is 0, because both sin functions are even.
- ○ We cannot say anything about this integral, we have to calculate it.

Figure C.84: This question gives students a particle in a particular, known Hamiltonian, as well as its eigen-information. Students are then given an integral involving the eigenstates of the integral and asked if they can deduce the value of the integral based on that information. Incorrect answers involve incorrect orthonormality considerations (not integrating over the full interval, not recognizing that eigenstates are orthogonal, *etc.*

Topic 5: Addition of Angular Momentum

Category 1: Change of Basis

Subcategory 1: Separate to Total

Let $\hat{\boldsymbol{S}}_1$ and $\hat{\boldsymbol{S}}_2$ be two angular momentum operators, with quantum numbers s_1, m_1, and s_2, m_2, respectively, and define the total angular momentum to be $\hat{\boldsymbol{S}} = \hat{\boldsymbol{S}}_1 + \hat{\boldsymbol{S}}_2$, with quantum numbers s, m. Recall that any state in the Hilbert space on which $\hat{\boldsymbol{S}}$ acts can be written in either the uncoupled basis $|s_1, m_1; s_2, m_2\rangle$ or the coupled basis $|s, m\rangle$, and we can write one basis in terms of the other Clebsch-Gordan coefficients.

Using a Clebsch-Gordan table, what is $|1, 0; \frac{1}{2}, -\frac{1}{2}\rangle$ written in the coupled basis?

- ○ $|1, 0; \frac{1}{2}, -\frac{1}{2}\rangle = \sqrt{\frac{2}{3}}|\frac{3}{2}, -\frac{1}{2}\rangle + \sqrt{\frac{1}{3}}|\frac{1}{2}, -\frac{1}{2}\rangle$
- ○ $|1, 0; \frac{1}{2}, -\frac{1}{2}\rangle = \sqrt{\frac{2}{3}}|\frac{3}{2}, -\frac{1}{2}\rangle - \sqrt{\frac{1}{3}}|\frac{1}{2}, -\frac{1}{2}\rangle$
- ○ $|1, 0; \frac{1}{2}, -\frac{1}{2}\rangle = \sqrt{\frac{1}{2}}|\frac{3}{2}, -\frac{1}{2}\rangle + \sqrt{\frac{1}{2}}|\frac{1}{2}, -\frac{1}{2}\rangle$
- ○ $|1, 0; \frac{1}{2}, -\frac{1}{2}\rangle = \sqrt{\frac{1}{3}}|\frac{3}{2}, -\frac{1}{2}\rangle - \sqrt{\frac{2}{3}}|\frac{1}{2}, -\frac{1}{2}\rangle$

Figure C.85: This question gives student two particles with spin as well as all of the eigen-information for sum of the spins in both the coupled and uncoupled bases. Students must, given a state in the uncoupled basis, use a Clebsch-Gordan table to write this state in the coupled basis. Incorrect answers involve using equal probabilities as the coefficients, and missing square roots and negative signs.

Subcategory 2: Total to Separate

Let $\hat{\boldsymbol{S}}_1$ and $\hat{\boldsymbol{S}}_2$ be two angular momentum operators, with quantum numbers s_1, m_1, and s_2, m_2, respectively, and define the total angular momentum to be $\hat{\boldsymbol{S}} = \hat{\boldsymbol{S}}_1 + \hat{\boldsymbol{S}}_2$, with quantum numbers s, m. Recall that any state in the Hilbert space on which $\hat{\boldsymbol{S}}$ acts can be written in either the uncoupled basis $|s_1, m_1; s_2, m_2\rangle$ or the coupled basis $|s, m\rangle$, and we can write one basis in terms of the other Clebsch-Gordan coefficients.

Assume that $s_1 = 1$ and $s_2 = 1$. Using a Clebsch-Gordan table, what is $|1, 0\rangle$ written in the coupled basis?

○ $|1,0\rangle = \sqrt{\frac{1}{2}}|1,1;1,-1\rangle + \sqrt{\frac{1}{2}}|1,-1;1,1\rangle$
○ $|1,0\rangle = \sqrt{\frac{1}{2}}|1,1;1,-1\rangle - \sqrt{\frac{1}{2}}|1,-1;1,1\rangle$
○ $|1,0\rangle = \sqrt{\frac{1}{3}}|1,1;1,-1\rangle + \sqrt{\frac{1}{3}}|1,0;1,0\rangle + \sqrt{\frac{1}{3}}|1,-1;1,1\rangle$
○ $|1,0\rangle = \sqrt{\frac{1}{3}}|1,1;1,-1\rangle - \sqrt{\frac{1}{3}}|1,0;1,0\rangle + \sqrt{\frac{1}{3}}|1,-1;1,1\rangle$

Figure C.86: This question gives student two particles with spin as well as all of the eigen-information for sum of the spins in both the coupled and uncoupled bases. Students must, given a state in the coupled basis, use a Clebsch-Gordan table to write this state in the uncoupled basis. Incorrect answers involve using equal probabilities as the coefficients, and missing square roots and negative signs.

Category 2: Dimension of Hilbert Space

Subcategory 1: Single

Consider a spin-1 particle with spin angular momentum operator $\hat{\boldsymbol{S}}$. What is the dimension of the Hilbert space that $\hat{\boldsymbol{S}}$ acts on?

- ○ 1
- ○ 2
- ○ 3
- ○ 4

Figure C.87: This question gives students a particle with a particular spin and asks them to identify the dimension of the Hilbert space the spin operator corresponding to that spin acts on. Incorrect answers involve simply the spin itself, not counting negative z-spins, and neglecting the z-spin being 0.

Subcategory 2: Multiple

Consider two particles, the first a spin-1 particle with spin angular momentum operator $\hat{\boldsymbol{S}}_1$, the second a spin-$\frac{1}{2}$ particle with spin angular momentum operator $\hat{\boldsymbol{S}}_2$. Consider the total spin $\hat{\boldsymbol{S}} = \hat{\boldsymbol{S}}_1 + \hat{\boldsymbol{S}}_2$. What is the dimension of the Hilbert space on which $\hat{\boldsymbol{S}}$ acts?

○ 2
○ 3
○ 5
○ 6

Figure C.88: This question gives students two particles with particular (but potentially different spins) and asks them to consider the total spin associated with the two particles. Alternative questions give students one particle with two different spin operators, such as spin angular momentum and orbital angular momentum. The question asks students to identify the dimension of the Hilbert space associated with the total spin operator. Incorrect answers include simply adding the spins, neglecting negative z-spins, and taking the difference between the spin values.

Category 3: Probabilities Between Bases

Subcategory 1: Total to Separate Basis - Given

Consider two particles. One is a spin-1 particle with angular momentum operator $\hat{\boldsymbol{S}}_1$, and the other is a spin-$\frac{1}{2}$ particle with angular momentum operator $\hat{\boldsymbol{S}}_2$. Consider the total angular momentum $\hat{\boldsymbol{S}} = \hat{\boldsymbol{S}}_1 + \hat{\boldsymbol{S}}_2$. The basis states for the uncoupled basis are $|s_1, m_1; s_2, m_2\rangle$, and the basis states for the coupled basis are $|s, m\rangle$.
If the state of the particles is $|\frac{3}{2}, -\frac{1}{2}\rangle = \sqrt{\frac{2}{3}}|1, 0; \frac{1}{2}, -\frac{1}{2}\rangle + \sqrt{\frac{1}{3}}|1, -1; \frac{1}{2}, \frac{1}{2}\rangle$, what is the probability of measuring both $m_1 = 0$ and $m_2 = -\frac{1}{2}$?

- ○ $\frac{2}{3}$
- ○ 0
- ○ 1
- ○ $\frac{1}{3}$
- ○ $\frac{1}{2}$

Figure C.89: This question gives students two particles of particular spins and gives them the eigen-information for the total spin for both the coupled and uncoupled bases for it. Students are given an eigenstate in the coupled basis, already expanded in the uncoupled basis, and are asked to find the probability of as particular outcome best found using the uncoupled basis. Incorrect answers involve 0 or 1 (common student heuristics), an equal probability, or reading the Clebsch-Gordan table wrong (if they needed to read it).

Subcategory 2: Total to Separate Basis - Build

Consider two particles. One is a spin-1 particle with angular momentum operator $\hat{\boldsymbol{S}}_1$, and the other is also a spin-1 particle with angular momentum operator $\hat{\boldsymbol{S}}_2$. Consider the total angular momentum $\hat{\boldsymbol{S}} = \hat{\boldsymbol{S}}_1 + \hat{\boldsymbol{S}}_2$. The basis states for the uncoupled basis are $|s_1, m_1; s_2, m_2\rangle$, and the basis states for the coupled basis are $|s, m\rangle$.

If the state of the particles is $|2, 0\rangle$, what is the probability of measuring both $m_1 = 1$ and $m_2 = -1$?

- ○ $\frac{1}{6}$
- ○ 0
- ○ $\frac{2}{3}$
- ○ $\frac{1}{3}$
- ○ $\frac{1}{2}$

Figure C.90: This question gives students two particles of particular spins and gives them the eigen-information for the total spin for both the coupled and uncoupled bases for it. Students are given an eigenstate in the coupled basis, *not* expanded in the uncoupled basis, and are asked to find the probability of as particular outcome best found using the uncoupled basis. Since the basis expansion for the state is not given, students must use a Clebsch-Gordan table. Incorrect answers involve 0 or 1 (common student heuristics), an equal probability, or reading the Clebsch-Gordan table wrong.

Subcategory 3: Separate to Total Basis - Given

Consider two particles. One is a spin-1 particle with angular momentum operator $\hat{\boldsymbol{S}}_1$, and the other is a spin-$\frac{1}{2}$ particle with angular momentum operator $\hat{\boldsymbol{S}}_2$. Consider the total angular momentum $\hat{\boldsymbol{S}} = \hat{\boldsymbol{S}}_1 + \hat{\boldsymbol{S}}_2$. The basis states for the uncoupled basis are $|s_1, m_1; s_2, m_2\rangle$, and the basis states for the coupled basis are $|s, m\rangle$.
If the state of the particles is $|1, -1; \frac{1}{2}, \frac{1}{2}\rangle = \sqrt{\frac{1}{3}}|\frac{3}{2}, -\frac{1}{2}\rangle - \sqrt{\frac{2}{3}}|\frac{1}{2}, -\frac{1}{2}\rangle$, what is the probability of measuring both $s = \frac{1}{2}$ and $m = -\frac{1}{2}$?

- ○ $\frac{2}{3}$
- ○ 0
- ○ 1
- ○ $\frac{1}{3}$
- ○ $\frac{1}{2}$

Figure C.91: This question gives students two particles of particular spins and gives them the eigen-information for the total spin for both the coupled and uncoupled bases for it. Students are given an eigenstate in the uncoupled basis, already expanded in the coupled basis, and are asked to find the probability of as particular outcome best found using the coupled basis. Incorrect answers involve 0 or 1 (common student heuristics), an equal probability, or reading the Clebsch-Gordan table wrong (if they needed to read it).

Subcategory 4: Separate to Total Basis - Build

Consider two particles. One is a spin-1 particle with angular momentum operator $\hat{\boldsymbol{S}}_1$, and the other is a spin-$\frac{1}{2}$ particle with angular momentum operator $\hat{\boldsymbol{S}}_2$. Consider the total angular momentum $\hat{\boldsymbol{S}} = \hat{\boldsymbol{S}}_1 + \hat{\boldsymbol{S}}_2$. The basis states for the uncoupled basis are $|s_1, m_1; s_2, m_2\rangle$, and the basis states for the coupled basis are $|s, m\rangle$.
If the state of the particles is $|1, 1; \frac{1}{2}, -\frac{1}{2}\rangle$, what is the probability of measuring both $s = \frac{1}{2}$ and $m = \frac{1}{2}$?

○ $\frac{2}{3}$
○ 0
○ 1
○ $\frac{1}{2}$
○ $\frac{1}{3}$

Figure C.92: This question gives students two particles of particular spins and gives them the eigen-information for the total spin for both the coupled and uncoupled bases for it. Students are given an eigenstate in the uncoupled basis, *not* expanded in the coupled basis, and are asked to find the probability of as particular outcome best found using the coupled basis. Since the basis expansion for the state is not given, students must use a Clebsch-Gordan table. Incorrect answers involve 0 or 1 (common student heuristics), an equal probability, or reading the Clebsch-Gordan table wrong.

Topic 6: Multiple Particles and Degeneracy

Category 1: Degeneracy

Subcategory 1: What is the Degeneracy?

Let H be the Hamiltonian for a 3-dimensional infinite square well in the x, y, and z directions of equal length a in all three directions. Recall that the energy eigenvalues for this system (due to the equal length) can be written as $E = E_{n,x} + E_{m,y} + E_{p,z}$, where $n, m, p = 1, 2, ...$, and all three of $E_{n,x}$, $E_{m,y}$, $E_{p,z}$ are energy eigenvalues for a 1-dimensional infinite square well with quantum numbers n, m, and p, respectively. What is the degeneracy of the first excited state of H?

- ○ 1
- ○ 3
- ○ 5
- ○ 2

Figure C.93: This question gives students a known Hamiltonian and all of its eigen-information. Students are asked to identify the degeneracy of a particular energy level. Incorrect answers consist of “no degeneracy”, and various ways of miscounting/ accumulating states contributing to the degeneracy.

Subcategory 2: List States

Let H be the Hamiltonian for a 3-dimensional infinite square well in the x, y, and z directions of equal length a in all three directions. Recall that the energy eigenvalues for this system (due to the equal length) can be written as $E = E_{n,x} + E_{m,y} + E_{p,z}$, where $n, m, p = 1, 2, ...$, and all three of $E_{n,x}$, $E_{m,y}$, $E_{p,z}$ are energy eigenvalues for a 1-dimensional infinite square well with quantum numbers n, m, and p, respectively. Let the energy eigenstates of this Hamiltonian in Dirac notation be represented as $|n, m, p\rangle$. What are the (degenerate?) states for this first excited state of H? Please select all that apply.

- ☐ $|2, 1, 1\rangle$
- ☐ $|1, 2, 1\rangle$
- ☐ $|1, 1, 2\rangle$
- ☐ $|2, 2, 2\rangle$
- ☐ $|2, 3, 1\rangle$

Figure C.94: This question gives students a known Hamiltonian and all of its eigen-information. Students are asked to identify each of the eigenstates corresponding to a particular energy level. There are multiple possibilities considering degeneracy. Incorrect answers include ideas such as bumping all indices up one, or forgetting the dependence of the energy level on n, n^2, *etc.*

Category 2: Identical Particles - Single

Subcategory 1: Energy - Boson

Consider two **bosons** in a quantum harmonic oscillator with Hamiltonian H. The energy eigenstates of the two particle system are $\psi_n(x_1)\psi_m(x_2)$, with $n, m = 0, 1, 2, ...$ If one particle is in the ground state (n or $m = 0$) and the other is in the third excited state (n or $m = 3$), what is the normalized state for these two particles?

○ $\frac{1}{\sqrt{2}}\left(\psi_0(x_1)\psi_3(x_2) + \psi_3(x_1)\psi_0(x_2)\right)$
○ $\frac{1}{\sqrt{2}}\left(\psi_0(x_1)\psi_3(x_2) - \psi_3(x_1)\psi_0(x_2)\right)$
○ $\psi_3(x_1)\psi_0(x_2)$
○ $\psi_0(x_1)\psi_3(x_2)$

Figure C.95: This question gives students two particles (bosons) in a Hamiltonian and all of the eigen-information for it. After being given information on the energy content of the particles, students must identify an appropriately symmetrized state of the particle. Incorrect answers involve anti-symmetrization or no symmetrization.

Subcategory 2: Spin - Boson

Consider two spin-1 **bosons**. The two-particle, uncoupled spin-1 eigenstates in the z-direction are denoted $|s_1, s_2\rangle$, where $s_i = -1$, 0, or 1.

If one particle has spin 1 (s_1 or $s_2 = 1$ in the z-direction, and the other has spin 0 (s_1 or $s_2 = 0$) in the z-direction, what is the normalized state for these two particles?

- ○ $\frac{1}{\sqrt{2}}\left(|1,0\rangle + |0,1\rangle\right)$
- ○ $\frac{1}{\sqrt{2}}\left(|1,0\rangle - |0,1\rangle\right)$
- ○ $|1,0\rangle$
- ○ $|0,1\rangle$

Figure C.96: This question gives students two particles (bosons) with spin and all of the spin eigen-information. After being given information on the z-spins of the particles, students must identify an appropriately symmetrized state of the particle. Incorrect answers involve anti-symmetrization or no symmetrization.

Subcategory 3: Energy - Fermion

Consider two **fermions** in an infinite square well with Hamiltonian H. The energy eigenstates of the two particle system are $\psi_n(x_1)\psi_m(x_2)$, with $n, m = 1, 2, ...$

If one particle is in the first excited state (n or $m = 2$) and the other is in the second excited state (n or $m = 3$), what is the normalized state for these two particles, if it exists?

○ $\frac{1}{\sqrt{2}}\left(\psi_2(x_1)\psi_3(x_2) + \psi_3(x_1)\psi_2(x_2)\right)$
○ $\frac{1}{\sqrt{2}}\left(\psi_2(x_1)\psi_3(x_2) - \psi_3(x_1)\psi_2(x_2)\right)$
○ $\psi_3(x_1)\psi_2(x_2)$
○ $\psi_2(x_1)\psi_3(x_2)$
○ Not possible.

Figure C.97: This question gives students two particles (fermions) in a Hamiltonian and all of the eigen-information for it. After being given information on the energy content of the particles, students must identify an appropriately anti-symmetrized state of the particle. Incorrect answers involve symmetrization or no symmetrization.

Subcategory 4: Spin - Fermion

Consider two spin-$\frac{1}{2}$ **fermions**. The two-particle, uncoupled spin-$\frac{1}{2}$ eigenstates in the z-direction are denoted $|s_1, s_2\rangle$, where $s_i = \uparrow$ or $\downarrow$ (spin up or spin down).

If one particle is spin up (s_1 or $s_2 = \uparrow$) in the z-direction, and the other particle is spin down (s_1 or $s_2 = \downarrow$) in the z-direction, what is the normalized state for these two particles, if it exists?

○ $|\downarrow, \uparrow\rangle$
○ $|\uparrow, \downarrow\rangle$
○ $\frac{1}{\sqrt{2}}(|\uparrow, \downarrow\rangle + |\downarrow, \uparrow\rangle)$
○ $\frac{1}{\sqrt{2}}(|\uparrow, \downarrow\rangle - |\downarrow, \uparrow\rangle)$
○ Not possible.

Figure C.98: This question gives students two particles (fermions) with spin and all of the spin eigen-information. After being given information on the z-spins of the particles, students must identify an appropriately anti-symmetrized state of the particle. Incorrect answers involve symmetrization or no symmetrization.

Subcategory 5: Boson or Fermion

Consider two spin-$\frac{1}{2}$ particles. The eigenstates for the z-components of spin for both particles can be written as $|s_1, s_2\rangle$, where $s_i = \pm\frac{1}{2}$.

Let $|\chi\rangle = \frac{1}{\sqrt{2}}\left(|\frac{1}{2}, -\frac{1}{2}\rangle + |-\frac{1}{2}, \frac{1}{2}\rangle\right)$ be the state of the two particles. Considering the exchange symmetry properties of this state, these particles are:

○ Bosons

○ Fermions

○ Neither bosons nor fermions

Figure C.99: This question gives students two particles in a particular Hamiltonian (or of a particular spin) and gives them the state of the particle. Students must use symmetry considerations to determine whether or not these particles are bosons, fermions, or neither.

Consider two particles in a system whose two particle Hamiltonian is H, with energy eigenstates $\psi_n(x_1)\psi_m(x_2)$, with $n, m = 0, 1, 2, ...$

Let $\Psi(x_1, x_2) = \psi_0(x_1)\psi_2(x_2)$ be the state of the two particles. Considering the exchange symmetry properties of this state, these particles are:

○ Bosons

○ Fermions

○ Neither bosons nor fermions

Figure C.100: This question gives students two particles in a particular Hamiltonian (or of a particular spin) and gives them the state of the particle. Students must use symmetry considerations to determine whether or not these particles are bosons, fermions, or neither. This question is an example of the case where the particles are neither, since their is not (anti-)symmetry under exchange.

Category 3: Identical Particles - Compound

Subcategory 1: Boson Compound

Consider two spin-1 **bosons** in a system with Hamiltonian H. Let the energy eigenstates of our two particles be $\psi_n(x_1)\psi_m(x_2)$, where $n = 1, 2, 3, ...$ is the quantum number of the first particle, and $m = 1, 2, 3, ...$ is the quantum number of the second. Let the spin-1 eigenstates for spin in the z-direction for these two particles be labeled $|s_1, s_2\rangle$, where $s_i = \pm 1, 0$. Recall that when considering the composite energy and spin system, we can combine our eigenstates with a product: $\psi_n(x_1)\psi_m(x_2) \otimes |s_1, s_2\rangle$. One particle has z-component of spin 1 (s_1 or $s_2 = 1$), and the other has z-component of spin -1 (s_1 or $s_2 = -1$). In addition, one particle is in the ground state (n or $m = 1$), and the other is in the first excited state (n or $m = 2$). What are allowed states for our particles? Please choose all that apply.

- ☐ $\left(\frac{\psi_1(x_1)\psi_2(x_2)+\psi_2(x_1)\psi_1(x_2)}{\sqrt{2}}\right) \otimes \left(\frac{1,-1\rangle+|-1,1\rangle}{\sqrt{2}}\right)$
- ☐ $\psi_1(x_1)\psi_2(x_2) \otimes |1, -1\rangle$
- ☐ $\left(\frac{\psi_1(x_1)\psi_2(x_2)-\psi_2(x_1)\psi_1(x_2)}{\sqrt{2}}\right) \otimes \left(\frac{|1,-1\rangle-|-1,1\rangle}{\sqrt{2}}\right)$
- ☐ $\left(\frac{\psi_1(x_1)\psi_2(x_2)-\psi_2(x_1)\psi_1(x_2)}{\sqrt{2}}\right) \otimes \left(\frac{|1,-1\rangle+|-1,1\rangle}{\sqrt{2}}\right)$
- ☐ $\left(\frac{\psi_1(x_1)\psi_2(x_2)+\psi_2(x_1)\psi_1(x_2)}{\sqrt{2}}\right) \otimes \left(\frac{|1,-1\rangle-|-1,1\rangle}{\sqrt{2}}\right)$

Figure C.101: This question gives students two bosons with spin in a particular Hamiltonian. After telling students the spin and energy eigenvalues for the particles, they must identify valid state(s) for the particles under appropriate full state symmetry considerations.

Subcategory 2: Fermion Compound

Consider two spin-$\frac{1}{2}$ **fermions** in a system with Hamiltonian H. Let the energy eigenstates of our two particles be $\psi_n(x_1)\psi_m(x_2)$, where $n = 1, 2, 3, ...$ is the quantum number of the first particle, and $m = 1, 2, 3, ...$ is the quantum number of the second. Let the spin-$\frac{1}{2}$ eigenstates for these two particles be labeled $|s_1, s_2\rangle$, where $s_i = \pm\frac{1}{2}$. Recall that when considering the composite energy and spin system, we can combine our eigenstates with a product: $\psi_n(x_1)\psi_m(x_2) \otimes |s_1, s_2\rangle$.

Both particles are spin-up (s_1 and $s_2 = \frac{1}{2}$). In addition, one particle is in the ground state (n or $m = 1$), and the other is in the second excited state (n or $m = 3$). What are allowed states for our particles? Please choose all that apply.

- ☐ $(\,) \otimes |\frac{1}{2}, \frac{1}{2}\rangle$
- ☐ $\psi_1(x_1)\psi_3(x_2) \otimes |\frac{1}{2}, \frac{1}{2}\rangle$
- ☐ $\psi_1(x_1)\psi_3(x_2) - |\frac{1}{2}, \frac{1}{2}\rangle$
- ☐ $\left(\frac{\psi_1(x_1)\psi_3(x_2) - \psi_3(x_1)\psi_1(x_2)}{\sqrt{2}}\right) \otimes |\frac{1}{2}, \frac{1}{2}\rangle$
- ☐ This state is not possible for fermions.

Figure C.102: This question gives students two fermions with spin in a particular Hamiltonian. After telling students the spin and energy eigenvalues for the particles, they must identify valid state(s) for the particles under appropriate full state symmetry considerations.

Subcategory 3: Boson or Fermion

Consider two spin-1 particles in a system with Hamiltonian H. The energy eigenstates of H are $|n, m\rangle$, where $n, m = 0, 1, 2, ...$ are the energy quantum numbers of particle 1 and particle 2, respectively. The spin-1 in the z-direction eigenstates are $|s_1, s_2\rangle$, where $s_i = \pm 1, 0$. Recall that the combined eigenstates for energy and spin-1 can be written as $|n, m\rangle \otimes |s_1, s_2\rangle$.

Let the state of our two particles be $\left(\frac{|2,0\rangle - |0,2\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|0,-1\rangle - |-1,0\rangle}{\sqrt{2}}\right)$. Considering the exchange symmetry properties of this state, these particles are:

○ Bosons

○ Fermions

○ Neither bosons nor fermions

Figure C.103: This question gives students two particles of a particular spin in a Hamiltonian, and gives students the state of the particle (both the spin and energy components of the state). Students must identify whether or not the particles are bosons, fermions, or neither based on the given state.

Subcategory 4: Given One, What is Other?

Consider two spin-$\frac{1}{2}$ **fermions** in a system with Hamiltonian H. Let the energy eigenstates of the system be $\psi_n(x_1)\psi_m(x_2)$, where $n, m = 0, 1, 2, ...$ are the energy quantum numbers for the first and second particles, respectively. We will denote the spin-$\frac{1}{2}$ in the z-direction eigenstates as $|s_1, s_2\rangle$, where $s_i = \pm\frac{1}{2}$.

If it is known that the energy portion of the state of the particle is described by $\frac{\psi_2(x_1)\psi_0(x_2)-\psi_0(x_1)\psi_2(x_2)}{\sqrt{2}}$, what are possibilities for the spin part of the states? Please choose all that apply.

- ☐ $|\frac{1}{2}\rangle$
- ☐ $|-\frac{1}{2}, -\frac{1}{2}\rangle$
- ☐ $\frac{|\frac{1}{2},-\frac{1}{2}\rangle+|-\frac{1}{2},\frac{1}{2}\rangle}{\sqrt{2}}$
- ☐ $\frac{|\frac{1}{2},-\frac{1}{2}\rangle-|-\frac{1}{2},\frac{1}{2}\rangle}{\sqrt{2}}$
- ☐ $|\frac{1}{2}, -\frac{1}{2}\rangle$

Figure C.104: This question gives students two fermions of a particular spin in a Hamiltonian, and gives them the energy component of the full wave function (either symmetrized or anti-symmetrized). Students must identify the spin-component of the full wave function under symmetry considerations, given that the particles are fermions. Various states, including both the state under the opposite symmetry condition, make up the incorrect answer choices.

Topic 7: Perturbation Theory

Category 1: Non-Deg TIPT

Subcategory 1: First Order Energy

Let H be a Hamiltonian with energy eigenstates $|n\rangle$ ($\psi_n(x)$ in position space), and corresponding eigenvalues E_n. Assume that H has no degeneracies. Write $H = H^0 + H'$, where H^0 is a Hamiltonian that we know how to solve: it has energy eigenstates $|n^{(0)}\rangle$ ($\psi_n^{(0)}(x)$ in position space) and energy eigenvalues $E_n^{(0)}$. Taking H' to be a small correction to H^0, we can use perturbation theory to solve for E_n in terms of $E_n^{(0)}$ and various corrections: $E_n = E_n^{(0)} + E_n^{(1)} + E_n^{(2)} + ...$

What is the corresponding expression for $E_2^{(1)}$, the first order correction to the 2^{nd} energy eigenvalue? Select all that apply.

- ☐ $E_2^{(1)} = \int \psi_2^{*(0)}(x) H' \psi_2^{(0)} \mathrm{d}x$
- ☐ $E_2^{(1)} = \langle 2^{(0)}|H'|2^{(0)}\rangle$
- ☐ $E_2^{(1)} = \int \langle 2^{(0)}|H'|2^{(0)}\rangle \mathrm{d}x$
- ☐ $E_2^{(1)} = \langle 2|H^0|2\rangle$
- ☐ $E_2^{(1)} = \int \psi_2^{*(0)}(x) H \psi_2(x) \mathrm{d}x$

Figure C.105: This question presents students typical preliminary information in a (non-degenerate) perturbation theory problem and asks them to identify the expression corresponding to the first order correction to a particular energy eigenvalue in both Dirac and wave function notation. Incorrect answers involve mixing the two notations, using the perturbed eigenstates instead of the unperturbed eigenstates, and using H or H^0 instead of H'.

Subcategory 2: First Order State

Let H be a Hamiltonian with energy eigenstates $|n\rangle$ ($\psi_n(x)$ in position space), and corresponding eigenvalues E_n. Assume that H has no degeneracies. Write $H = H^0 + H'$, where H^0 is a Hamiltonian that we know how to solve: it has energy eigenstates $|n^{(0)}\rangle$ ($\psi_n^{(0)}(x)$ in position space) and energy eigenvalues $E_n^{(0)}$. Taking H' to be a small correction to H^0, we can use perturbation theory to solve for $|n\rangle$ in terms of $|n^{(0)}\rangle$ and various corrections: $|n\rangle = |n^{(0)}\rangle + |n^{(1)}\rangle + |n^{(2)}\rangle + ...$

What is the corresponding expression for $|5^{(1)}\rangle$, the first order correction to the 5th energy eigenstate?

○ $|5^{(1)}\rangle = \sum_{m\neq 5} \frac{\langle m^{(0)}|H'|5^{(0)}\rangle}{E_5^{(0)} - E_m^{(0)}} |m^{(0)}\rangle$

○ $|5^{(1)}\rangle = \sum_{m\neq 5} \frac{\langle m^{(0)}|H|5^{(0)}\rangle}{E_5^{(0)} - E_m^{(0)}} |m^{(0)}\rangle$

○ $|5^{(1)}\rangle = \sum_{m\neq 5} \frac{\langle m^{(0)}|H'|5^{(0)}\rangle}{E_m^{(0)} - E_5^{(0)}} |m^{(0)}\rangle$

○ $|5^{(1)}\rangle = \sum_{m\neq 5} \frac{\langle m^{(0)}|H|5^{(0)}\rangle}{E_m^{(0)} - E_5^{(0)}} |m^{(0)}\rangle$

○ $|5^{(1)}\rangle = \sum_{m\neq 5} \frac{\langle m^{(0)}|H'|5^{(0)}\rangle}{E_m^{(0)} - E_5^{(0)}} |5^{(0)}\rangle$

Figure C.106: This question presents students typical preliminary information in a (non-degenerate) perturbation theory problem and asks them to identify the expression corresponding to the first order correction to a particular energy eigenstate in Dirac notation. Incorrect answers typically involve the use of H or H^0 instead of H' in the numerator, using the perturbed eigenstates instead of the unperturbed eigenstates in the numerator, reversing the bra and ket in the numerator, and reversing the order of the m and corrected index in the denominator.

Subcategory 3: Second Order Energy

Let H be a Hamiltonian with energy eigenstates $|n\rangle$ ($\psi_n(x)$ in position space), and corresponding eigenvalues E_n. Assume that H has no degeneracies. Write $H = H^0 + H'$, where H^0 is a Hamiltonian that we know how to solve: it has energy eigenstates $|n^{(0)}\rangle$ ($\psi_n^{(0)}(x)$ in position space) and energy eigenvalues $E_n^{(0)}$. Taking H' to be a small correction to H^0, we can use perturbation theory to solve for E_n in terms of $E_n^{(0)}$ and various corrections: $E_n = E_n^{(0)} + E_n^{(1)} + E_n^{(2)} + ...$

What is the corresponding expression for $E_1^{(2)}$, the second order correction to the 1$^{\text{st}}$ energy eigenvalue?

○ $E_1^{(2)} = \sum_{m\neq 1} \frac{\left|\int \psi_m^{*(0)}(x) H' \psi_1^{(0)}(x) \mathrm{d}x\right|^2}{E_1^{(0)} - E_m^{(0)}}$

○ $E_1^{(2)} = \sum_{m\neq 1} \frac{\left|\int \psi_m^{*}(x) H' \psi_1(x) \mathrm{d}x\right|^2}{E_1^{(0)} - E_m^{(0)}}$

○ $E_1^{(2)} = \sum_{m\neq 1} \frac{\left|\int \psi_m^{*(0)}(x) H' \psi_1^{(0)}(x) \mathrm{d}x\right|^2}{E_m^{(0)} - E_1^{(0)}}$

○ $E_1^{(2)} = \sum_{m\neq 1} \frac{\left|\int \psi_m^{*}(x) H' \psi_1(x) \mathrm{d}x\right|^2}{E_m^{(0)} - E_1^{(0)}}$

Figure C.107: This question presents students typical preliminary information in a (non-degenerate) perturbation theory problem and asks them to identify the expression corresponding to the second order correction to a particular energy eigenvalue in wave function notation. Incorrect answers involve reversing the order of the sum index and corrected index in the denominator, and whether or not the inner product in the numerator uses the unperturbed or perturbed eigenstates. Other incorrect answer choices involve using H or H^0 instead of H'.

Category 2: DPT - Find H' in Deg Subspace

Subcategory 1: Given Matrix

In a perturbation theory problem, you are working with a perturbed Hamiltonian $H = H^0 + H' = \begin{pmatrix} b+\epsilon_1 & 0 & 0 \\ 0 & a & \epsilon_1 \\ 0 & \epsilon_1 & a \end{pmatrix}$ ($a \neq b$ real constants) with $H' = \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & 0 & \epsilon_1 \\ 0 & \epsilon_1 & 0 \end{pmatrix}$ since it is known that $\epsilon_1 > 0$ is small. You notice that H^0 has a degenerate energy level, so you know that you need to perform degenerate perturbation theory to find the first order corrections to the associated energies and states.

What matrix do you need to diagonalize in order to do this?

○ $\begin{pmatrix} \epsilon_1 & 0 \\ 0 & 0 \end{pmatrix}$

○ $\begin{pmatrix} b+\epsilon_1 & 0 & 0 \\ 0 & a & \epsilon_1 \\ 0 & \epsilon_1 & a \end{pmatrix}$

○ $\begin{pmatrix} 0 & \epsilon_1 \\ \epsilon_1 & 0 \end{pmatrix}$

○ $\begin{pmatrix} \epsilon_1 & \epsilon_1 \\ \epsilon_1 & 0 \end{pmatrix}$

○ $\begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & 0 & \epsilon_1 \\ 0 & \epsilon_1 & 0 \end{pmatrix}$

Figure C.108: This question gives students a matrix for $H = H^0 + H'$ and a candidate choice for H', and asks students to identify H' in the degenerate subspace. Incorrect answers involve merely the wrong matrices, the full perturbation, and the full Hamiltonian.

Category 3: DPT - Basic Examples

Subcategory 1: Diagonalize - 2x2

Let $H = H^0 + H'$ be a Hamiltonian, where H^0 is an exactly solvable Hamiltonian and H' a small perturbation. Assume that H^0 has a degeneracy amongst the energy eigenstates $\psi^{(0)}_{n_1}(x)$ and $\psi^{(0)}_{n_2}(x)$, and that H' in the degenerate subspace for these states takes the form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (where $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ corresponds to $\psi^{(0)}_{n_1}(x)$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to $\psi^{(0)}_{n_2}(x)$). After diagonalizing H' in this degenerate subspace, what are the first order corrections to the energies of the states $\psi^{(0)}_{n_1}(x)$ and $\psi^{(0)}_{n_2}(x)$, namely $E^{(1)}_{n_1}$ and $E^{(1)}_{n_2}$?

- ○ $E^{(1)}_{n_1} = 1$ and $E^{(1)}_{n_2} = 1$.
- ○ $E^{(1)}_{n_1} = 1$ and $E^{(1)}_{n_2} = -1$.
- ○ $E^{(1)}_{n_1} = 1$ and $E^{(1)}_{n_2} = 0$.
- ○ $E^{(1)}_{n_1} = 2$ and $E^{(1)}_{n_2} = 0$.

Figure C.109: This question gives students a 2×2 matrix for H' and asks students to determine the first order corrections to the energy eigenvalues after using degenerate perturbation theory. The incorrect answer choices range from “no correction”, the new diagonal elements, and confusing which elements get summed (namely, not diagonalizing, but rather applying believed relationships).

Subcategory 2: Diagonalize - 3x3

Let $H = H^0 + H'$ be a Hamiltonian, where H^0 is an exactly solvable Hamiltonian and H' a small perturbation. Assume that H^0 has a degeneracy amongst the energy eigenstates $\psi_{n_1}^{(0)}(x)$, $\psi_{n_2}^{(0)}(x)$, and $\psi_{n_3}^{(0)}(x)$, and that H' in the degenerate subspace for these states takes the form $\begin{pmatrix} a & b & 0 \\ b & a & 0 \\ 0 & 0 & c \end{pmatrix}$ (where $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ corresponds to $\psi_{n_1}^{(0)}(x)$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ to $\psi_{n_2}^{(0)}(x)$, and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ to $\psi_{n_3}^{(0)}(x)$), where a, b, and c are constants.

After diagonalizing H' in this degenerate subspace, what are the first order corrections to the energies of the states $\psi_{n_1}^{(0)}(x)$, $\psi_{n_2}^{(0)}(x)$, and $\psi_{n_3}^{(0)}(x)$, namely $E_{n_1}^{(1)}$, $E_{n_2}^{(1)}$, and $E_{n_3}^{(1)}$?

○ $E_{n_1}^{(1)} = a$, $E_{n_2}^{(1)} = b$, and $E_{n_3}^{(1)} = c$.

○ $E_{n_1}^{(1)} = c$, $E_{n_2}^{(1)} = a + b$, and $E_{n_3}^{(1)} = a - b$.

○ $E_{n_1}^{(1)} = a$, $E_{n_2}^{(1)} = b + c$, and $E_{n_3}^{(1)} = b - c$.

○ $E_{n_1}^{(1)} = a + c$, $E_{n_2}^{(1)} = b$, and $E_{n_3}^{(1)} = a - c$.

Figure C.110: This question gives students a 3×3 matrix for H' and asks students to determine the first order corrections to the energy eigenvalues after using degenerate perturbation theory. The incorrect answer choices range from "no correction" to permutations on which energies get corrected.

Subcategory 3: Find Good Basis - 2x2

Let $H = H^0 + H'$ be a Hamiltonian, where H^0 is an exactly solvable Hamiltonian and H' a small perturbation. Assume that H^0 has a degeneracy amongst the energy eigenstates $\psi_{n_1}^{(0)}(x)$ and $\psi_{n_2}^{(0)}(x)$, and that H' in the degenerate subspace for these states takes the form $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ (where $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ corresponds to $\psi_{n_1}^{(0)}(x)$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to $\psi_{n_2}^{(0)}(x)$).
In matrix notation, what is a "good" basis to use for this perturbation on these states in the degenerate subspace?

○ $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

○ $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

○ $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

○ $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Figure C.111: This question gives students a 2×2 matrix for H' and asks students to determine the "good" basis in which one is to do perturbation theory in, in matrix notation. The answer choices correspond simply to "no change", and only correcting one of the basis states.

Let $H = H^0 + H'$ be a Hamiltonian, where H^0 is an exactly solvable Hamiltonian and H' a small perturbation. Assume that H^0 has a degeneracy amongst the energy eigenstates $\psi_{n_1}^{(0)}(x)$ and $\psi_{n_2}^{(0)}(x)$, and that H' in the degenerate subspace for these states takes the form $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ (where $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ corresponds to $|n_1^{(0)}\rangle$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to $|n_2^{(0)}\rangle$), where a and b are constants.

In Dirac notation, what is a "good" basis to use for this perturbation on these states in the degenerate subspace?

○ $|n_1^{(0)}\rangle$ and $|n_2^{(0)}\rangle$

○ $\frac{1}{\sqrt{2}}(|n_1^{(0)}\rangle + |n_2^{(0)}\rangle)$ and $\frac{1}{\sqrt{2}}(|n_1^{(0)}\rangle - |n_2^{(0)}\rangle)$

○ $|n_1^{(0)}\rangle$ and $\frac{1}{\sqrt{2}}(|n_1^{(0)}\rangle + |n_2^{(0)}\rangle)$

○ $|n_1^{(0)}\rangle$ and $\frac{1}{\sqrt{2}}(|n_1^{(0)}\rangle - |n_2^{(0)}\rangle)$

Figure C.112: This question gives students a 2 × 2 matrix for H' and asks students to determine the "good" basis in which one is to do perturbation theory in, in Dirac notation. The answer choices correspond simply to "no change", and only correcting one of the basis states.

Subcategory 4: Find Good Basis - 3x3

Let $H = H^0 + H'$ be a Hamiltonian, where H^0 is an exactly solvable Hamiltonian and H' a small perturbation. Assume that H^0 has a degeneracy amongst the energy eigenstates $\psi_{n_1}^{(0)}(x)$, $\psi_{n_2}^{(0)}(x)$, and $\psi_{n_3}^{(0)}(x)$, and that H' in the degenerate subspace for these states takes the form $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (where $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ corresponds to $\psi_{n_1}^{(0)}(x)$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ to $\psi_{n_2}^{(0)}(x)$, and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ to $\psi_{n_3}^{(0)}(x)$).

In wave function notation, what is a "good" basis to use for this perturbation on these states in the degenerate subspace?

○ $\psi_{n_1}^{(0)}(x)$, $\psi_{n_2}^{(0)}(x)$, and $\psi_{n_3}^{(0)}(x)$.

○ $\frac{1}{\sqrt{2}}(\psi_{n_1}^{(0)}(x) + \psi_{n_2}^{(0)}(x))$, $\frac{1}{\sqrt{2}}(\psi_{n_1}^{(0)}(x) - \psi_{n_2}^{(0)}(x))$, and $\psi_{n_3}^{(0)}(x)$.

○ $\frac{1}{\sqrt{2}}(\psi_{n_1}^{(0)}(x) + \psi_{n_3}^{(0)}(x))$, $\psi_{n_2}^{(0)}(x)$, and $\frac{1}{\sqrt{2}}(\psi_{n_1}^{(0)}(x) - \psi_{n_3}^{(0)}(x))$.

○ $\psi_{n_1}^{(0)}(x)$, $\frac{1}{\sqrt{2}}(\psi_{n_2}^{(0)}(x) + \psi_{n_3}^{(0)}(x))$, and $\frac{1}{\sqrt{2}}(\psi_{n_2}^{(0)}(x) - \psi_{n_3}^{(0)}(x))$.

Figure C.113: This question gives students a 3 × 3 matrix for H' and asks students to determine the "good" basis in which one is to do perturbation theory in, in wave function notation. The answer choices correspond simply to "no change" and permutations on which basis elements are the ones being corrected.

Category 4: DPT - Need to Use DPT?

Subcategory 1: Given H^0 and H'

When doing a perturbation theory problem, you are given a Hamiltonian $H = H^0 + H' = \begin{pmatrix} a+\epsilon_1 & 0 & 0 \\ 0 & b+\epsilon_2 & \epsilon_3 \\ 0 & \epsilon_3 & b \end{pmatrix}$ ($a \neq b$ real constants), and you have chosen to treat $H' = \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & \epsilon_3 \\ 0 & \epsilon_3 & 0 \end{pmatrix}$ as your perturbation since you know that $\epsilon_1, \epsilon_2, \epsilon_3 > 0$ are small.

Is **degenerate** perturbation theory necessary? Why or why not?

- ○ Yes: H^0 has degeneracies.
- ○ No: H^0 has no degeneracies.
- ○ Yes: H' has degeneracies.
- ○ No: H' has no degeneracies.

Figure C.114: This question gives students a particular Hamiltonian and asks students to determine if degenerate perturbation theory needs to be used. The answer choices run through the permutations of whether H^0 or H' has a degeneracy or not.

Subcategory 2: Given System

Consider a 3-dimensional infinite square well whose lengths in each dimension are all a (namely, equal), with Hamiltonian H^0. Let H' be a small perturbation, and say you want to perform perturbation theory.

When calculating the first order correction to the second excited state energy, do we need to perform degenerate perturbation theory?

○ Yes: the unperturbed energy has degeneracy larger than 1.
○ No: the unperturbed energy has degeneracy 1.
○ Yes: the perturbed energy has degeneracy larger than 1.
○ No: the perturbed energy has degeneracy 1.

Figure C.115: This question gives students a particular Hamiltonian known to them and a particular energy level, and asks them to determine if degenerate perturbation theory needs to be used. The answer choices run through the permutations of whether H^0 or H' has a degeneracy at that energy level or not.

Appendix D

STEM FLUENCY SCHEDULES FOR EACH ADMINISTRATION

This appendix contains tables outlining each assignment administered during each offering of STEM Fluency in our upper-division quantum mechanics courses. Each tables corresponds to a single administration and contains all of the categories selected for the assignment, their corresponding mastery levels, and the subcategories for each category chosen to be a part of the assignment.

Before outlining the assignments for each administration, note that many of the categories and subcategories early on are different from those located in Appendix B. This is due to the modifications to the structure and dropping various skills and questions we believed were not suitable to essential skills practice. The tables containing the assignments for Fall 2023 and Winter 2024 match the final structure we determined and outlined in Appendix B.

Table D.1: List of assignments for the Fall 2021 administration of STEM Fluency practice in Fall 2021. For each assignment, the categories, its mastery level, and their respective subcategories are given.

Fall 2021: PHYS 324			
Week	Category Name	Subcategory Name	Category Mastery
Week 1			
	Complex Conjugation		4
		Complex Numbers	
	Continuous Probability		4
		ProbDens - Units	
		ProbDens to Prob	
	Stationary States		3
		Idea to Math	
		Implication to Idea	
		Implication to Math	
		Math to Idea	
	The Schrodinger Equation		3
		Stationary States	
Week 2			
	Representations of Wave Functions		5
		Graphs of Wave Functions - ISW	
		Good Wave Function for Hamiltonian - ISW	
	The Schrodinger Equation		5
		Time Dependent Schrodinger Equation	

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Table D.1: List of assignments for the Fall 2021 administration of STEM Fluency practice in Fall 2021. For each assignment, the categories, its mastery level, and their respective subcategories are given. (Continued)

		Time Independent Schrodinger Equation	
Week 3			
	Representations of Wave Functions		5
		Graphs of Wave Functions - ISW	
		Good Wave Function for Hamiltonian - ISW	
		Graphs of Wave Functions - QHO	
		Good Wave Function for Hamiltonian - QHO	
	Time Dependence of Wave Functions		5
		Add Time Dependence to Wave Functions	
Week 4			
	Allowed Operations on Bras and Kets		4
		Stay in Hilbert Space	
		What Can This Do?	
	Complex Conjugation		5
		Ket ¡-¿ Bra (Single)	
		Ket ¡-¿ Bra (Linear Combinations) (Numbers)	
		Ket ¡-¿ Bra (Linear Combinations) (Variables)	
	Expectation Values		3

Continued on next page

Table D.1: List of assignments for the Fall 2021 administration of STEM Fluency practice in Fall 2021. For each assignment, the categories, its mastery level, and their respective subcategories are given. (Continued)

		Math to Idea	
		Math to Interpretation	
		Idea to Math	
		Idea to Idea	
	Inner Products		3
		Math to Idea	
		Idea to Math	
		What Object is it?	
Week 5			
	Complex Conjugation		4
		Inner Products	
	Expanding a State in a Basis		3
		Eigenbasis	
		Arbitrary Basis	
	Orthogonality		3
		Of Eigenbases	
		Of Kets	
		Orthonormality of Bases	
		Of Bases	
	Statement of Eigenvalue Equation		4
		NameThis	
		GivenEigenWhatIs?	
Week 6			
	Expanding a State in a Basis		2

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Table D.1: List of assignments for the Fall 2021 administration of STEM Fluency practice in Fall 2021. For each assignment, the categories, its mastery level, and their respective subcategories are given. (Continued)

		Eigenbasis	
		Arbitrary Basis	
	Orthogonality		2
		Of Eigenbases	
		Of Kets	
		Orthonormality of Bases	
		Of Bases	
	Probabilities		5
		Math to Idea	
		Idea to Math	
	Probability Density		4
		Idea to Math	
		Math to Idea	
		Comparisons B/W PD and P	
Week 7			
	Completeness Relations		5
		Discrete Spectrum	
		Continuous Spectrum	
	Complex Conjugation		4
		Wave Functions	
	Probabilities		3
		Math to Idea	
		Idea to Math	
	Probability Density		3

Continued on next page

Table D.1: List of assignments for the Fall 2021 administration of STEM Fluency practice in Fall 2021. For each assignment, the categories, its mastery level, and their respective subcategories are given. (Continued)

		Idea to Math	
		Math to Idea	
		Comparisons B/W PD and P	
	Wave Functions		4
		Math to Idea	
		Idea to Math	
		WF ¡-¿ Dirac (Math)	
Week 8			
	Completeness Relations		3
		Discrete Spectrum	
		Continuous Spectrum	
	Choosing a Bra for a Probability Calculation		6
		Discrete	
		Continuous	
	Turning Dirac Expressions into Wave Function Expressions and Vice Versa		6
		Probabilities - D to WF - Discrete	
		Probabilities - D to WF - Continuous	
		Probabilities - WF to D - Continuous	
Week 9			

Continued on next page

Table D.1: List of assignments for the Fall 2021 administration of STEM Fluency practice in Fall 2021. For each assignment, the categories, its mastery level, and their respective subcategories are given. (Continued)

	Completeness Relations		3
		Discrete Spectrum	
		Continuous Spectrum	
	Choosing a Bra for a Probability Calculation		5
		Discrete	
		Continuous	
	Turning Dirac Expressions into Wave Function Expressions and Vice Versa		5
		Probabilities - D to WF - Discrete	
		Probabilities - D to WF - Continuous	
		Probabilities - WF to D - Continuous	

Table D.2: List of assignments for the Fall 2021 administration of STEM Fluency practice in Fall 2022. For each assignment, the categories, its mastery level, and their respective subcategories are given. Note that there are slight differences in this assignment, such as the inclusion of a “context” category.

Fall 2022: PHYS 324			
Week	Category Name	Subcategory Name	Category Mastery
Week 1			
	Context Questions		3
		Continuous Probability	
		TSE: TISE	
		TSE: TDSE	
		Stationary States	
	Continuous Probability		3
		ProbDens - Unis	
		ProbDens to Prob	
	Stationary States		3
		Idea to Math	
		Implication to Idea	
		Implication to Math	
		Math to Idea	
	The Schrodinger Equation		3
		Stationary States	
		Time Independent Schrodinger Equation	
		Time Dependent Schrodinger Equation	
Week 2			
	Context Questions		3
		Continuous Probability	
		TSE: TISE	

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Table D.2: List of assignments for the Fall 2021 administration of STEM Fluency practice in Fall 2022. For each assignment, the categories, its mastery level, and their respective subcategories are given. Note that there are slight differences in this assignment, such as the inclusion of a “context” category. (Continued)

		TSE: TDSE	
		Stationary States	
	Continuous Probability		3
		ProbDens - Unis	
		ProbDens to Prob	
	Stationary States		3
		Idea to Math	
		Implication to Idea	
		Implication to Math	
		Math to Idea	
	The Schrodinger Equation		3
		Stationary States	
		Time Independent Schrodinger Equation	
		Time Dependent Schrodinger Equation	
Week 3			
	Allowed Operations on Bras and Kets		3
		Stay in Hilbert Space	
		What Can This Do?	
	Complex Conjugation		3
		Ket ¡-¿ Bra (Linear Combination) (Variables)	
	Context Questions		3

Continued on next page

Table D.2: List of assignments for the Fall 2021 administration of STEM Fluency practice in Fall 2022. For each assignment, the categories, its mastery level, and their respective subcategories are given. Note that there are slight differences in this assignment, such as the inclusion of a “context” category. (Continued)

		Inner Products	
		Ket ¡-¿ Bra	
		Allowed Operations on Bras and Kets	
	Inner Products		3
		Math to Idea	
		Idea to Math	
		What Object is it?	
Week 4			
	Context Questions		3
		Expectation Values	
		Expanding a State in a Basis	
		Statement of Eigenvalue Equations	
	Expanding a State in a Basis		3
		Arbitrary Basis	
		Eigenbasis	
	Expectation Values		3
		Math to Idea	
		Math to Interpretation	
		Idea to Math	
		Idea to Idea	
	Statement of Eigenvalue Equation		3
		GivenEigenWhatIs?	

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Table D.2: List of assignments for the Fall 2021 administration of STEM Fluency practice in Fall 2022. For each assignment, the categories, its mastery level, and their respective subcategories are given. Note that there are slight differences in this assignment, such as the inclusion of a "context" category. (Continued)

		NameThis	
Week 5			
	Context Questions		3
		Statement of Eigenvalue Equation	
		Expanding a State in a Basis	
		Orthogonality	
	Expanding a State in a Basis		3
		Eigenbasis	
		Arbitrary Basis	
	Orthogonality		3
		Of Kets	
		Of Bases	
		Of Eigenbases	
		Orthonormality of Bases	
	Statement of Eigenvalue Equation		3
		GivenEigenWhatIs?	
		NameThis	
Week 6			
	Context Questions		3
		Orthogonality	
		Probabilities	
		Probability Density	
	Orthogonality		3

Continued on next page

Table D.2: List of assignments for the Fall 2021 administration of STEM Fluency practice in Fall 2022. For each assignment, the categories, its mastery level, and their respective subcategories are given. Note that there are slight differences in this assignment, such as the inclusion of a “context” category. (Continued)

		Of Kets	
		Of Bases	
		Of Eigenbases	
		Orthonormality of Bases	
	Probabilities		3
		Math to Idea	
		Idea to Math	
	Probability Density		3
		Math to Idea	
		Idea to Math	
		Comparisons B/W PD and P	
Week 7			
	Completeness Relations		3
		Discrete Spectrum	
		Continuous Spectrum	
	Context Questions		3
		Wave Functions	
		Probabilities	
		Completeness Relations	
	Probabilities		3
		Math to Idea	
		Idea to Math	
	Wave Functions		3
		Math to Idea	
		Idea to Math	

Continued on next page

Table D.2: List of assignments for the Fall 2021 administration of STEM Fluency practice in Fall 2022. For each assignment, the categories, its mastery level, and their respective subcategories are given. Note that there are slight differences in this assignment, such as the inclusion of a “context” category. (Continued)

		Wf ¡-¿ Dirac - Math	
Week 8			
	Context Questions		3
		Wave Functions	
	Choosing a Bra for a Probability Calculation		3
		Discrete	
		Continuous	
	Turning Dirac Expressions into Wave Function Expressions and Vice Versa		3
		Probabilities - D to WF - Continuous	
		Probabilities - D to WF - Discrete	
		Probabilities - WF to D - Discrete	
	Wave Functions		3
		Math to Idea	
		Idea to Math	
		WF ¡-¿ Dirac - Math	
Week 9			
	Choosing a Bra for a Probability Calculation		3
		Discrete	
		Continuous	

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Table D.2: List of assignments for the Fall 2021 administration of STEM Fluency practice in Fall 2022. For each assignment, the categories, its mastery level, and their respective subcategories are given. Note that there are slight differences in this assignment, such as the inclusion of a “context” category. (Continued)

	Probabilities		3
		Math to Idea	
		Idea to Math	
	Turning Dirac Expressions into Wave Function Expressions and Vice Versa		3
		Probabilities - D to WF - Continuous	
		Probabilities - D to WF - Discrete	
		Probabilities - WF to D - Discrete	
	Wave Functions		3
		Math to Idea	
		Idea to Math	
		WF ¡-¿ Dirac - Math	
Week 10			
	Completeness Relations		3
		Discrete Spectrum	
		Continuous Spectrum	
	Choosing a Bra for a Probability Calculation		4
		Discrete	
		Continuous	
	Probabilities		4

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Table D.2: List of assignments for the Fall 2021 administration of STEM Fluency practice in Fall 2022. For each assignment, the categories, its mastery level, and their respective subcategories are given. Note that there are slight differences in this assignment, such as the inclusion of a "context" category. (Continued)

		Math to Idea	
		Idea to Math	
	Turning Dirac Expressions into Wave Function Expressions and Vice Versa		4
		Probabilities - D to WF - Continuous	
		Probabilities - D to WF - Discrete	
		Probabilities - WF to D - Discrete	

Table D.3: List of assignments for the Fall 2021 administration of STEM Fluency practice in Winter 2023. For each assignment, the categories, its mastery level, and their respective subcategories are given.

Winter 2023: PHYS 325			
Week	Category Name	Subcategory Name	Category Mastery
Week 1			
	Addition of Angular Momentum - Changing Bases		3
		Separate to Total	
		Total to Separate	
	Addition of Angular Momentum - Dimension of Hilbert Space		3
		Single	
		Multiple	
	Expanding a State in a Basis		3
		Eigenbasis - Angular Momentum	
	Probabilities		4
		Dirac - Math to Idea - 325	
		Dirac - Idea to Math - 325	
		WF - Math to Idea - 325	
		WF - Idea to Math - 325	
Week 2			
	Addition of Angular Momentum - Changing Bases		3
		Separate to Total	

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Table D.3: List of assignments for the Fall 2021 administration of STEM Fluency practice in Winter 2023. For each assignment, the categories, its mastery level, and their respective subcategories are given. (Continued)

		Total to Separate	
	Addition of Angular Momentum - Probabilities		3
		Separate to Total Basis - Given	
		Separate to Total Basis - Build	
		Total to Separate Basis - Given	
		Total to Separate Basis - Build	
	Identical Particles - Simple		4
		Energy - Fermion	
		Energy - Boson	
		Spin - Fermion	
		Spin - Boson	
		BosonOrFermion?	
	Probabilities		4
		Dirac - Math to Idea - 325	
		Dirac - Idea to Math - 325	
		WF - Math to Idea - 325	
		WF - Idea to Math - 325	
Week 3			
	Addition of Angular Momentum - Changing Bases		3

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Table D.3: List of assignments for the Fall 2021 administration of STEM Fluency practice in Winter 2023. For each assignment, the categories, its mastery level, and their respective subcategories are given. (Continued)

		Separate to Total	
		Total to Separate	
	Addition of Angular Momentum - Probabilities		4
		Separate to Total Basis - Given	
		Separate to Total Basis - Build	
		Total to Separate Basis - Given	
		Total to Separate Basis - Build	
	Identical Particles - Simple		3
		Energy - Fermion	
		Energy - Boson	
		Spin - Fermion	
		Spin - Boson	
		BosonOrFermion?	
	Identical Particles - Compound		3
		FermionCompound	
		BosonCompound	
		BosonOrFermion?	
Week 4			
	Degeneracy		3
		ListStates	
		WhatIsDegen?	

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Table D.3: List of assignments for the Fall 2021 administration of STEM Fluency practice in Winter 2023. For each assignment, the categories, its mastery level, and their respective subcategories are given. (Continued)

	Identical Particles - Compound		4
		FermionCompound	
		BosonCompound	
		BosonOrFermion?	
		GivenOneWhatIsOther?	
	Non-Deg-TIPT		4
		First Order Energy	
		First Order State	
		Second Order Energy	
	Probabilities		3
		Mixed - Math to Idea	
		Mixed - Idea to Math	
Week 5			
	DPT - Basic Examples		3
		Diagonalize - 2x2	
		Diagonalize - 3x3	
		FindGoodBasis - 2x2	
		FindGoodBasis -3x3	
	DPT - Find H' in Deg Subspace		3
		Given Matrix	
	DPT - Need to Use DPT?		3
		Given H0 and H' (Matrices)	
		Given System	
	Probabilities		4
		Mixed - Math to Idea	

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Table D.3: List of assignments for the Fall 2021 administration of STEM Fluency practice in Winter 2023. For each assignment, the categories, its mastery level, and their respective subcategories are given. (Continued)

		Mixed - Idea to Math	
Week 6			
	Addition of Angular Momentum - Changing Bases		4
		Separate to Total	
		Total to Separate	
	Addition of Angular Momentum - Probabilities		3
		Separate to Total Basis - Given	
		Separate to Total Basis - Build	
		Total to Separate Basis - Given	
		Total to Separate Basis - Build	
	Degeneracy		3
		ListStates	
		WhatIsDegen?	
	Identical Particles - Compound		3
		FermionCompound	
		BosonCompound	
		BosonOrFermion?	
		GivenOneWhatIsOther?	
Week 7			

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Table D.3: List of assignments for the Fall 2021 administration of STEM Fluency practice in Winter 2023. For each assignment, the categories, its mastery level, and their respective subcategories are given. (Continued)

	DPT - Basic Examples		3
		Diagonalize - 2x2	
		Diagonalize - 3x3	
		FindGoodBasis - 2x2	
		FindGoodBasis -3x3	
	DPT - Find H' in Deg Subspace		3
		Given Matrix	
	Non-Deg-TIPT		3
		First Order Energy	
		First Order State	
		Second Order Energy	
	Probabilities		3
		Mixed - Math to Idea	
		Mixed - Idea to Math	
Week 8			
	DPT - Basic Examples		3
		Diagonalize - 2x2	
		Diagonalize - 3x3	
		FindGoodBasis - 2x2	
		FindGoodBasis -3x3	
	DPT - Find H' in Deg Subspace		3
		Given Matrix	
	DPT - Need to Use DPT?		3
		Given H0 and H' (Matrices)	

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Table D.3: List of assignments for the Fall 2021 administration of STEM Fluency practice in Winter 2023. For each assignment, the categories, its mastery level, and their respective subcategories are given. (Continued)

		Given System	
	Degeneracy		3
		ListStates	
		WhatIsDegen?	
Week 9			
	Addition of Angular Momentum - Changing Bases		3
		Separate to Total	
		Total to Separate	
	Addition of Angular Momentum - Probabilities		3
		Separate to Total Basis - Given	
		Separate to Total Basis - Build	
		Total to Separate Basis - Given	
		Total to Separate Basis - Build	
	Probabilities		3
		Mixed - Math to Idea	
		Mixed - Idea to Math	
	Degeneracy		3
		ListStates	
		WhatIsDegen?	
Week 10			

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Table D.3: List of assignments for the Fall 2021 administration of STEM Fluency practice in Winter 2023. For each assignment, the categories, its mastery level, and their respective subcategories are given. (Continued)

	DPT - Basic Examples		3
		Diagonalize - 2x2	
		Diagonalize - 3x3	
		FindGoodBasis - 2x2	
		FindGoodBasis -3x3	
	DPT - Find H' in Deg Subspace		3
		Given Matrix	
	DPT - Need to Use DPT?		3
		Given H0 and H' (Matrices)	
		Given System	
	Identical Particles - Compound		3
		FermionCompound	
		BosonCompound	
		BosonOrFermion?	
		GivenOneWhatIsOther?	

Table D.4: List of assignments for the Fall 2021 administration of STEM Fluency practice in Fall 2023. For each assignment, the categories, its mastery level, and their respective subcategories are given.Notice that there are slight changes to this structure compared to before, such as the rearranging of the Topic "Translating Representations" to have the categories be "Translating Concepts to Mathematics" and *vice versa* instead of being the various mathematical quantities, such as "Expectation Values".

Fall 2023: PHYS 324			
Week	Category Name	Subcategory Name	Category Mastery
Week 1			
	Allowed Operations on Bras and Kets		3
		Stay in Hilbert Space	
		What Can This Do?	
	Continuous Probability		3
		ProbDens - Units	
		ProbDens to Prob	
	Stationary States		3
		Idea to Math	
		Math to Idea	
		Implication to Idea	
		Implication to Math	
	The Schrodinger Equation		3
		Stationary States	
		Time Dependent Schrodinger Equation	
		Time Independent Schrodinger Equation	
Week 2			

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Table D.4: List of assignments for the Fall 2021 administration of STEM Fluency practice in Fall 2023. For each assignment, the categories, its mastery level, and their respective subcategories are given.Notice that there are slight changes to this structure compared to before, such as the rearranging of the Topic "Translating Representations" to have the categories be "Translating Concepts to Mathematics" and *vice versa* instead of being the various mathematical quantities, such as "Expectation Values". (Continued)

	Allowed Operations on Bras and Kets		3
		Stay in Hilbert Space	
		What Can This Do?	
	Continuous Probability		3
		ProbDens - Units	
		ProbDens to Prob	
	Stationary States		3
		Idea to Math	
		Math to Idea	
		Implication to Idea	
		Implication to Math	
	The Schrodinger Equation		3
		Stationary States	
		Time Dependent Schrodinger Equation	
		Time Independent Schrodinger Equation	
Week 3			
	Complex Conjugation		3
		Inner Products	
		Ket ¡-¿ Bra (Linear Combination) (Variables)	

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Table D.4: List of assignments for the Fall 2021 administration of STEM Fluency practice in Fall 2023. For each assignment, the categories, its mastery level, and their respective subcategories are given.Notice that there are slight changes to this structure compared to before, such as the rearranging of the Topic "Translating Representations" to have the categories be "Translating Concepts to Mathematics" and *vice versa* instead of being the various mathematical quantities, such as "Expectation Values". (Continued)

	Idea to Math		3
		Inner Products	
		Statement of Eigenvalue Equation	
	Math to Idea		3
		Inner Products	
		Statement of Eigenvalue Equation	
	Special Topics		3
		Inner Products - What Objects?	
Week 4			
	Expanding a State in a Basis		3
		Arbitrary Basis	
		Eigenbasis	
	Math to Idea		3
		Expectation Values	
		Probabilities - WF	
	Idea to Math		3
		Expectation Values	
		Probabilities - WF	
	Special Topic		3
		Expectation Values - Interpretation	
Week 5			

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Table D.4: List of assignments for the Fall 2021 administration of STEM Fluency practice in Fall 2023. For each assignment, the categories, its mastery level, and their respective subcategories are given.Notice that there are slight changes to this structure compared to before, such as the rearranging of the Topic “Translating Representations” to have the categories be “Translating Concepts to Mathematics” and *vice versa* instead of being the various mathematical quantities, such as “Expectation Values”. (Continued)

	Math to Idea		3
		Inner Products	
		Statement of Eigenvalue Equation	
	Idea to Math		3
		Inner Products	
		Statement of Eigenvalue Equation	
	Orthogonality		3
		Of Kets	
		Of Bases	
		Of Eigenbases	
		Orthonormality of Bases	
	Special Topic		3
		Inner Products - What Object?	
Week 6			
	Expanding a State in a Basis		3
		Arbitrary Basis	
		Eigenbasis	
	Math to Idea		3
		Expectation Values	
		Probabilities - Dirac - 324	
	Idea to Math		3
		Expectation Values	

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Table D.4: List of assignments for the Fall 2021 administration of STEM Fluency practice in Fall 2023. For each assignment, the categories, its mastery level, and their respective subcategories are given.Notice that there are slight changes to this structure compared to before, such as the rearranging of the Topic "Translating Representations" to have the categories be "Translating Concepts to Mathematics" and *vice versa* instead of being the various mathematical quantities, such as "Expectation Values". (Continued)

		Probabilities - Dirac - 324	
	Orthogonality		3
		Of Kets	
		Of Bases	
		Of Eigenbases	
		Orthonormality of Bases	
Week 7			
	Completeness Relations		3
		Continuous Spectrum	
		Discrete Spectrum	
	Math to Idea		3
		Probabilities - Mixed - 324	
		Probability Density	
	Idea to Math		3
		Probabilities - Mixed - 324	
		Probability Density	
	Special Topic		3
		Probability Density - Comparisons	
Week 8			
	Idea to Math		3
		Expectation Values	
		Wave Functions	
	Math to Idea		3

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Table D.4: List of assignments for the Fall 2021 administration of STEM Fluency practice in Fall 2023. For each assignment, the categories, its mastery level, and their respective subcategories are given.Notice that there are slight changes to this structure compared to before, such as the rearranging of the Topic “Translating Representations” to have the categories be “Translating Concepts to Mathematics” and *vice versa* instead of being the various mathematical quantities, such as “Expectation Values”. (Continued)

		Expectation Values	
		Wave Functions	
	Special Topic		3
		Wave Functions - WF ¡-¿ Dirac - Math	
		Expectation Values - Interpretation	
	Turning Dirac Expressions into Wave Function Expressions and Vice Versa		3
		Probabilities - D to WF - Continuous	
		Probabilities - D to WF - Discrete	
		Probabilities - WF to D - Discrete	
Week 9			
	Idea to Math		3
		Wave Functions	
		Probabilities - Mixed - 324	
	Math to Idea		3
		Wave Functions	
		Probabilities - Mixed - 324	

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Table D.4: List of assignments for the Fall 2021 administration of STEM Fluency practice in Fall 2023. For each assignment, the categories, its mastery level, and their respective subcategories are given.Notice that there are slight changes to this structure compared to before, such as the rearranging of the Topic "Translating Representations" to have the categories be "Translating Concepts to Mathematics" and *vice versa* instead of being the various mathematical quantities, such as "Expectation Values". (Continued)

hline	Choosing a Bra for a Probability Calculation	Discrete	3
		Continuous	
	Turning Dirac Expressions into Wave Function Expressions and Vice Versa		3
		Probabilities - D to WF - Continuous	
		Probabilities - D to WF - Discrete	
		Probabilities - WF to D - Discrete	
Week 10			
	Idea to Math		3
		Wave Functions	
		Probabilities - Mixed - 324	
	Math to Idea		3
		Wave Functions	
		Probabilities - Mixed - 324	
hline	Choosing a Bra for a Probability Calculation	Discrete	3

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Table D.4: List of assignments for the Fall 2021 administration of STEM Fluency practice in Fall 2023. For each assignment, the categories, its mastery level, and their respective subcategories are given.Notice that there are slight changes to this structure compared to before, such as the rearranging of the Topic "Translating Representations" to have the categories be "Translating Concepts to Mathematics" and *vice versa* instead of being the various mathematical quantities, such as "Expectation Values". (Continued)

		Continuous	
	Turning Dirac Expressions into Wave Function Expressions and Vice Versa		3
		Probabilities - D to WF - Continuous	
		Probabilities - D to WF - Discrete	
		Probabilities - WF to D - Discrete	

Table D.5: List of assignments for the Fall 2021 administration of STEM Fluency practice in Winter 2024. For each assignment, the categories, its mastery level, and their respective subcategories are given.

Winter 2024: PHYS 325			
Week	Category Name	Subcategory Name	Category Mastery
Week 1			
	Addition of Angular Momentum - Changing Bases		3
		Separate to Total	
		Total to Separate	
	Addition of Angular Momentum - Dimension of Hilbert Space		3
		Single	
		Multiple	
	Idea to Math		3
		Mixed - Probabilities - 325	
	Math to Idea		3
		Mixed - Probabilities - 325	
Week 2			
	Addition of Angular Momentum - Changing Bases		3
		Separate to Total	
		Total to Separate	
	Addition of Angular Momentum - Probabilities		3

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Table D.5: List of assignments for the Fall 2021 administration of STEM Fluency practice in Winter 2024. For each assignment, the categories, its mastery level, and their respective subcategories are given. (Continued)

		Separate to Total - Given	
		Separate to Total - Build	
		Total to Separate - Given	
		Total to Separate - Build	
	Idea to Math		3
		Mixed - Probabilities - 325	
	Identical Particles - Simple		3
		Energy - Fermion	
		Energy - Boson	
		Spin - Fermion	
		Spin - Boson	
		BosonOrFermion?	
Week 3			
	Addition of Angular Momentum - Changing Bases		3
		Separate to Total	
		Total to Separate	
	Addition of Angular Momentum - Probabilities		3
		Separate to Total - Given	
		Separate to Total - Build	
		Total to Separate - Given	
		Total to Separate - Build	
	Identical Particles - Compound		3

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Table D.5: List of assignments for the Fall 2021 administration of STEM Fluency practice in Winter 2024. For each assignment, the categories, its mastery level, and their respective subcategories are given. (Continued)

		BosonCompound	
		FermionCompound	
		BosonOrFermion?	
	Identical Particles - Simple		3
		Energy - Fermion	
		Energy - Boson	
		Spin - Fermion	
		Spin - Boson	
		BosonOrFermion?	
Week 4			
	Degeneracy		3
		WhatIsDegen?	
		ListStates	
	Idea to Math		3
		Mixed - Probabilities - 325	
	Math to Idea		3
		Mixed - Probabilities - 325	
	Identical Particles - Compound		3
		BosonCompound	
		FermionCompound	
		BosonOrFermion?	
		GivenOneWhatIsOther?	
	Non-Deg-TIPT		3
		First Order Energy	
		First Order State	
		Second Order Energy	

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Table D.5: List of assignments for the Fall 2021 administration of STEM Fluency practice in Winter 2024. For each assignment, the categories, its mastery level, and their respective subcategories are given. (Continued)

Week 5			
	Idea to Math		3
		Mixed - Probabilities - 325	
	Math to Idea		3
		Mixed - Probabilities - 325	
	DPT - Basic Examples		3
		Diagonalize - 2x2	
		Diagonalize - 3x3	
		FindGoodBasis - 2x2	
		FindGoodBasis -3x3	
	DPT - Find H' in Deg Subspace		3
		Given Matrix	
	DPT - Need to Use DPT?		3
		Given H0 and H' (Matrices)	
		Given System	
Week 6			
	Addition of Angular Momentum - Changing Bases		3
		Separate to Total	
		Total to Separate	
	Addition of Angular Momentum - Probabilities		3
		Separate to Total - Given	

Continued on next page

Table D.5: List of assignments for the Fall 2021 administration of STEM Fluency practice in Winter 2024. For each assignment, the categories, its mastery level, and their respective subcategories are given. (Continued)

		Separate to Total - Build	
		Total to Separate - Given	
		Total to Separate - Build	
	Identical Particles - Compound		3
		BosonCompound	
		FermionCompound	
		BosonOrFermion?	
		GivenOneWhatIsOther?	
	Degeneracy		3
		WhatIsDegen?	
		ListStates	
Week 7			
	DPT - Basic Examples		3
		Diagonalize - 2x2	
		Diagonalize - 3x3	
		FindGoodBasis - 2x2	
		FindGoodBasis -3x3	
	DPT - Need to Use DPT?		3
		Given H0 and H' (Matrices)	
		Given System	
	Idea to Math		3
		Mixed - Probabilities - 325	
	Math to Idea		3
		Mixed - Probabilities - 325	
	Non-Deg-TIPT		3
		First Order Energy	

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Table D.5: List of assignments for the Fall 2021 administration of STEM Fluency practice in Winter 2024. For each assignment, the categories, its mastery level, and their respective subcategories are given. (Continued)

		First Order State	
		Second Order Energy	
Week 8			
	DPT - Basic Examples		3
		Diagonalize - 2x2	
		Diagonalize - 3x3	
		FindGoodBasis - 2x2	
		FindGoodBasis -3x3	
	DPT - Find H' in Deg Subspace		3
		Given Matrix	
	DPT - Need to Use DPT?		3
		Given H0 and H' (Matrices)	
		Given System	
	Degeneracy		3
		WhatIsDegen?	
		ListStates	
Week 9			
	Addition of Angular Momentum - Probabilities		3
		Separate to Total - Given	
		Separate to Total - Build	
		Total to Separate - Given	
		Total to Separate - Build	
	DPT - Basic Examples		3

Continued on next page

Table D.5: List of assignments for the Fall 2021 administration of STEM Fluency practice in Winter 2024. For each assignment, the categories, its mastery level, and their respective subcategories are given. (Continued)

		Diagonalize - 2x2	
		Diagonalize - 3x3	
		FindGoodBasis - 2x2	
		FindGoodBasis -3x3	
	DPT - Find H' in Deg Subspace		3
		Given Matrix	
	Idea to Math		3
		Mixed - Probabilities - 325	
	Math to Idea		3
		Mixed - Probabilities - 325	