Some Inverse Problems in Analysis and Geometry

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Abstract

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The aim of a typical inverse problem is to recover the interior properties of a medium by making measurements only on the boundary. These types of problems are motivated by geophysics, medical imaging and quantum mechanics among other fields. In this thesis, we consider two inverse problems arising in partial differential equations and geometry.

The first part is devoted to the Calderón's problem with partial data. We consider the problem of developing a method to reconstruct a potential q from the partial data Dirichlet-to-Neumann map for the Schrödinger equation $(-\Delta_g+q)u = 0$ on a fixed admissible manifold (M, g), where Δ_g is the Laplace-Beltrami operator. If the part of the boundary that is inaccessible for measurements satisfies a certain flatness condition in one direction, then we reconstruct the local attenuated geodesic ray transform of the one-dimensional Fourier transform of the potential q. This allows us to reconstruct q locally, if the local (unattenuated) geodesic ray transform is constructively invertible. We also reconstruct q globally, if M satisfies certain concavity condition and if the global geodesic ray transform can be inverted constructively.

In the second part, we study the Gaussian thermostat ray transforms on both closed Riemannian surfaces and compact Riemannian surfaces with boundary. We establish results on the injectivity of the thermostat ray transform, under certain conditions, and the surjectivity of its adjoint.

TABLE OF CONTENTS

GLOSSARY

M: a smooth manifold;

M^{int}: the interior of a smooth manifold M;

∂M: the boundary of a smooth manifold M ;

TM: the tangent bundle of a smooth manifold M;

$TM \setminus \{0\}$: the tangent bundle of a smooth manifold M excluding the zero section;

(M, g): a Riemannian manifold with metric g;

SM: the unit sphere bundle of a Riemannian manifold (M, g);

$\operatorname{supp} f$: the support of a function or distribution f;

$H^s(M)$: Sobolev space of order s over (M, g);

$C^k(M)$: the space of functions on M with continuous derivatives up to order k.

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DEDICATION

To my mother Zamzagul F. Surtaeva,

and

to the memory of my father Mukhtarkhan T. Assylbekov.

Chapter 1

INTRODUCTION

In inverse problems one aims to recover the interior properties of a medium by making measurements only on the boundary. Such problems are motivated by geophysics, medical imaging and quantum mechanics among other fields. In this thesis, we consider two inverse problems arising in partial differential equations and geometry.

In the first part, we focus on the Calderón's problem with partial and local data. We consider the problem of developing a method to reconstruct a potential q from the partial Dirichlet-to-Neumann map for the Schrödinger equation $(-\Delta_g+q)u = 0$ on a fixed admissible manifold (M, g). If the part of the boundary that is inaccessible for measurements satisfies a certain flatness condition in one direction, then we reconstruct the local attenuated geodesic ray transform of the one-dimensional Fourier transform of the potential q. This allows us to reconstruct q locally, if the local (unattenuated) geodesic ray transform is constructively invertible. The constructive inversion problem for local or global geodesic ray transforms is one of the major topics of interest in integral geometry. We also reconstruct q globally, if M satisfies certain concavity condition and if the global geodesic ray transform can be inverted constructively. We derive a certain boundary integral equation which involves the given partial data and describes the traces of complex geometrical optics solutions. For the construction of complex geometrical optics solutions we use a new family of Green's functions for the Laplace-Beltrami operator and the corresponding single layer potentials.

In the second part, as a joint work with Hanming Zhou, we study Gaussian thermostat ray transforms on both closed Riemannian surfaces and compact Riemannian surfaces with boundary. We establish certain results on the injectivity of the thermostat ray transform and the surjectivity of its adjoint. The injectivity result assumes certain condition involving conjugate points for a modified Jacobi equation. Surjectivity of the adjoints is expressed in terms of distributions invariant under the thermostat flow. The proofs are based on subelliptic type estimates and Pestov identity.

In this introduction we describe these problems in more details. We also give a short survey on earlier results and techniques. The results of this thesis are stated in Chapter 2 and Chapter 3.

1.1 *The Calderón problem with partial and local data*

In 1980, Alberto Calderón [8] proposed the problem whether one can determine the electrical conductivity of a medium from voltage and current measurements at the boundary. In the mathematical literature, this problem is known as Calderón's inverse conductivity problem. A more precise mathematical formulation of the problem is as follows:

Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded domain with C^∞ boundary $\partial\Omega$. The objective of Calderón problem is to recover the electrical conductivity γ of the conductivity equation

$$\operatorname{div}(\gamma \nabla u) = 0 \quad \text{in} \quad \Omega,$$

from the Dirichlet-to-Neumann map $\Lambda_\gamma(u|_{\partial\Omega}) = \gamma \partial_\nu u|_{\partial\Omega}$ where ν is the outer unit normal to $\partial\Omega$. This map, also known as voltage-to-current map, encodes all possible voltage and current measurements at the boundary.

Calderón's problem can be reduced to the problem of determining an electric potential q from the Dirichlet-to-Neumann map associated to the Schrödinger operator $-\Delta + q$, where $q = \gamma^{-1/2}\Delta\gamma^{1/2}$.

In the fundamental paper by Sylvester and Uhlmann [61] it was shown that C^2 isotropic conductivities can be uniquely determined from the knowledge of the Dirichlet-to-Neumann map. The main contribution of this paper is the construction of *complex geometrical optics* (CGO) solutions for the conductivity equation. This method led to many results in Calderón's problem. Moreover, the construction of CGO solutions were generalized to other types of partial differential equations to solve several inverse problems. The reader is referred to the recent survey article by Uhlmann [63] for an exposition on the progress made in Calderón's problem and applications of CGO solutions to other inverse problems.

The uniqueness result in [61] was extended to a reconstruction procedure by Nachman [39] and independently by Novikov [41].

The technique of CGO solutions was generalized to so-called admissible manifolds [17] using Carleman estimate approach following [7, 31]. A compact Riemannian manifold (M, g) with boundary of dimension $n \geq 3$, is said to be *admissible* if it is conformal to a submanifold with boundary of $(\mathbb{R} \times M_0, e \oplus g_0)$ where (M_0, g_0) is simple $(n-1)$-dimensional manifold and e is Euclidean metric on $\mathbb{R}$. The uniqueness result of Sylvester and Uhlmann [61] was then generalized to the setting of admissible manifolds. Calderón problem on admissible geometries is closely connected to the invertibility of the attenuated geodesic ray transforms on the transversal manifold M_0. The reconstruction procedure of Nachman [39] was extended to admissible geometries [30].

Making measurements only on part of the boundary is important in real life applications. Inverse problems with such restrictions are more difficult. When the Dirichlet-to-Neumann map is known only on part of the boundary, the first uniqueness result is due to Bukhgeim and Uhlmann [7]. They prove that a C^2 isotropic conductivity can be uniquely determined in Ω if the Dirichlet-to-Neumann map is restricted to, roughly speaking, slightly more than half of the boundary. This result has been improved significantly by Kenig, Sjöstrand and Uhlmann [31] where they show that the knowledge of the Dirichlet-to-Neumann map on a possibly very small open subset of the boundary determines the isotropic conductivity in Ω uniquely. The approaches of [7, 31] are based on Carleman estimates with boundary terms. Constructive proof of the result in [31] is given by Nachman and Street [40]. For recent results on Calderón's inverse problem with partial data, see [28].

There is also a result by Isakov [26] where he proves a uniqueness result when $\Gamma_- = \Gamma_+ = \Gamma$ and the inaccessible part of the boundary for measurements is either part of a hyperplane or part of a sphere. This work is based on a reflection argument.

In the current part of the thesis we consider partial data Calderón's problem on manifolds. The methods of [31, 26] were unified and extended to so-called admissible manifolds (which will be described below) by Kenig and Salo [29] obtaining improved results. The latter was possible due to improved Carleman estimates with boundary terms and the invertibility of local and global geodesic ray transforms. In Chapter 2 of this thesis, we develop a reconstruction procedure for the partial data Calderón problem on admissible geometries. This is a constructive version of a proof of the uniqueness result of Kenig and Salo [29], and extends the constructive proof of Nachman and Steet [40] in the Euclidean setting. For the proof, we derive a certain boundary integral equation which involves the given partial data and describes the traces of CGO solutions. Then we construct of CGO solutions, following [40], using a new family of Green's functions for the Laplace-Beltrami operator Δ_g and the corresponding single layer potentials.

1.2 Invariant distributions and tensor tomography for Gaussian thermostats

Let (M, g) be a compact oriented Riemannian manifold (with or without boundary) and E be a smooth vector field on M (called the *external field*). A parameterized curve $\gamma(t)$ on M satisfying the equation

$$D_t\dot{\gamma} = E(\gamma) - \frac{\langle E(\gamma), \dot{\gamma}\rangle}{|\dot{\gamma}|^2}\dot{\gamma}. \tag{1.1}$$

is called a *thermostat geodesic*. Here and in what follows D_t denotes the covariant derivative along γ. This differential equation defines a flow $\phi_t = (\gamma(t), \dot{\gamma}(t))$ on SM (the unit sphere bundle of M) which is called a *Gaussian thermostat* (or *isokinetic dynamics*, see [25]). The flow ϕ reduces to the geodesic flow when $E = 0$. As in the case of geodesic flows, Gaussian thermostats are reversible in the sense that the flip $(x, v) \mapsto (x, -v)$ changes ϕ_t with ϕ_{-t}. We denote the Gaussian thermostat by (M, g, E) and the generating vector of the thermostat flow by $\mathbf{G}_E$, which is a vector field on SM.

In this thesis we will consider the case when M is a surface (i.e. a 2-dimensional manifold). Then for $(x, v) \in SM$ we can write

$$E(x) = \langle E(x), v\rangle v + \langle E(x), iv\rangle iv,$$

where i indicates the rotation by $\pi/2$ according to the orientation of M. Thus on surfaces, the equation (1.1) can be rewritten as

$$D_t\dot{\gamma} = \lambda(\gamma, \dot{\gamma})i\dot{\gamma}, \tag{1.2}$$

where

$$\lambda(x, v) := \langle E(x), iv\rangle. \tag{1.3}$$

Notice that for Gaussian thermostats, λ corresponds to a 1-form on M. If λ is a smooth function on M, (1.2) defines the magnetic flow on surfaces associated with the magnetic field $\Omega = \lambda\, d\mathrm{Vol}_g$, where $d\mathrm{Vol}_g$ is the area form of M. One can consider a general function $\lambda \in C^\infty(SM)$, and call the induced flow a *generalized thermostat*.

In dynamical systems, Gaussian thermostats provide interesting models in non-equilibrium statistical mechanics [19, 20, 53]. Gaussian thermostats also arise in geometry as the flows of metric connections with non-zero torsion; see [65].

Tensor tomography problem

Given a Gaussian thermostat (M, g, E), we define the *thermostat ray transform* of a smooth function φ on SM to be

$$I\varphi(\gamma) := \int_0^T \varphi(\gamma(t), \dot{\gamma}(t))\, dt.$$

When M is closed, γ is a closed thermostat geodesic with period T. A basic question of integral geometry is whether the ray transform is injective. Of course, this question makes sense only in the case when the flow has sufficiently many closed orbits. Anosov flows constitute a wide class of flows with sufficiently many closed orbits. Recall that a Gaussian thermostat (M, g, E) is said to be *Anosov* if there is a continuous invariant splitting $T(SM) = \mathbb{R}\mathbf{G}_E \oplus E^u \oplus E^s$ in such a way that there are constants $C > 0$ and $0 < \rho < 1 < \eta$ such that for all $t > 0$ we have

$$\|d\phi_{-t}|_{E^u}\| \le C\,\eta^{-t} \quad \text{and} \quad \|d\phi_t|_{E^s}\| \le C\,\rho^t,$$

where the norms are taken with respect to the Sasaki Riemannian metric on SM.

There is a natural obstruction to the injectivity of the ray transform, i.e. the functions of the type $\varphi = \mathbf{G}_E u$ with $u \in C^\infty(SM)$. However, in applications one often needs to invert the ray transform of functions on SM arising from symmetric tensor fields. Therefore, we consider this particular case which is known as the tensor tomography problem.

Let $\varphi = \varphi_{i_1 \dots i_m}\, dx^{i_1} \otimes \cdots \otimes dx^{i_m}$ be a smooth symmetric m-tensor field on M. φ induces a smooth function $\hat{\varphi} \in C^\infty(SM)$ defined by

$$\hat{\varphi}(x, v) := \varphi_{i_1 \dots i_m}(x)\, v^{i_1} \cdots v^{i_m}, \quad (x, v) \in SM.$$

In what follows we will drop the hat, and we hope that it will be clear from the context when we mean the function on SM induced by the tensor. By $C^\infty(S_m(M))$ we denote the bundle of smooth symmetric m-tensor fields on M.

If $m = 0$, the transform I_0 (I acting on functions on M) is said to be *s-injective* if it is injective. If $m \geq 1$, we say that I_m (I acting on m-tensors) is *s-injective* if $I_m \varphi \equiv 0$ implies that $\varphi = \mathbf{G}_E h$ for some $h \in C^\infty(S_{m-1}(M))$. The tensor tomography problem asks under what conditions I_m is s-injective. The tensor tomography problem on Anosov surfaces (the case of geodesic flows) was studied in [15, 57, 45, 24, 9, 23], and [11, 2] for magnetic Anosov surfaces. In this part of the thesis, we will focus on the tensor tomography problem for Gaussian thermostats. In [13] Dairbekov and Paternain proved the s-injectivity of I_m for $m = 0, 1$, but considering more general Anosov thermostats. In [5] Assylbekov and Dairbekov extended this result to the case when the Riemannian metric is replaced by a Finsler metric. They have shown that for $m = 0$ injectivity result holds even when the flow is not Anosov, but has no conjugate points. When $m = 2$, Jane and Paternain [27] proved s-injectivity under the assumption that the external field is divergence free and the surface has negative Gaussian curvature.

Similarly there is a tensor tomography problem for Gaussian thermostats on compact Riemannian surfaces with boundary. In this case, the ray transform is along thermostat geodesics joining boundary points. For the boundary case, the tensor tomography problem for geodesic flows has been extensively studied, see e.g. [38, 3, 50, 56, 58, 59, 44, 37, 48] and the references therein. The case of magnetic flows was considered in [14, 1]. We will study the boundary case in the last section of Chapter 3.

Invariant distributions

One key ingredient in the proof of the s-injectivity of I_2 for the case of Anosov surfaces (the case of geodesic flows) by Paternain, Salo and Uhlmann [45] was the surjectivity of the adjoint of the geodesic ray transform acting on 1-forms. The problem of the surjectivity of the adjoint appears in range characterization problems [51, 46]. This is also an interesting problem in its own right. We also investigate the surjectivity of the adjoint of the thermostat ray transform. However, in the case of thermostat flows the surjectivity of I_1^* seems not enough for proving the s-injectivity of I_2. In general, thermostats do not preserve the Liouville measure on SM unless $E \equiv 0$ (see [12, 13]). This is a crucial difference with the case of geodesic flows and magnetic flows, and this makes the problem much harder.

Since the thermostat ray transforms I_0 and I_1 are s-injective for two-dimensional Anosov thermostats, one can consider the surjectivity of I_0^* and I_1^*. One of the aims of the current thesis is to show that I_0^* and I_1^* are indeed surjective. Using duality arguments, one can express surjectivity of the adjoints in terms of certain distributions that are invariant under the Gaussian thermostat flow; see Section 3.1.

Chapter 2

RECONSTRUCTION IN THE PARTIAL DATA CALDERÓN PROBLEM

2.1 Statement of results

Let (M, g) be a compact oriented Riemannian manifold with boundary. Following Bukhgeim and Uhlmann [7], we work with the following Hilbert space which is the largest domain of the Laplace-Beltrami operator Δ_g:

$$H_{\Delta_g}(M) = \{u \in L^2(M) : \Delta_g u \in L^2(M)\}.$$

The trace maps $\mathtt{tr}(u) = u|_{\partial M}$ and $\mathtt{tr}_\nu(u) = \frac{\partial u}{\partial \nu}\big|_{\partial M}$ defined on $C^\infty(M)$ have extensions to a bounded operators $H_{\Delta_g}(M) \to H^{-1/2}(\partial M)$ and $H_{\Delta_g}(M) \to H^{-3/2}(\partial M)$, respectively; see Proposition 2.2.1.

Now we introduce the following space on the boundary ∂M:

$$\mathcal{H}_g(\partial M) = \{\mathtt{tr}(u) : u \in H_{\Delta_g}(M)\} \subset H^{-1/2}(\partial M).$$

The topology on $\mathcal{H}_g(\partial M)$ is defined in Section 2.2, right before Proposition 2.2.4. Under this topology, the operator $\mathtt{tr} : H_{\Delta_g}(M) \to \mathcal{H}_g(\partial M)$ is bounded.

Suppose that $q \in L^\infty(M)$ such that 0 is not a Dirichlet eigenvalue of $-\Delta_g + q$ in M. Then for $f \in \mathcal{H}_g(\partial M)$, the Dirichlet problem has a unique solution $u \in H_{\Delta_g}(M)$

$$\begin{aligned} (-\Delta_g + q)u = 0 \quad &\text{in} \quad M, \\ \mathtt{tr}(u) = f \quad &\text{on} \quad \partial M. \end{aligned}$$

The Dirichlet-to-Neumann map is defined by

$$\Lambda_{g,q}(f) = \mathtt{tr}_\nu(u).$$

By the results of Section 2.2, $\Lambda_{g,q}$ is a bounded linear operator $\Lambda_{g,q} : \mathcal{H}_g(\partial M) \to H^{-3/2}(\partial M)$. Given two open subsets $\Gamma_-, \Gamma_+ \subset \partial M$. The partial data inverse problem is to determine q from the knowledge of $\Lambda_{g,q} f$ on Γ_- for all $f \in \mathcal{H}_g(\partial M)$ supported in Γ_+.

We need to introduce the notion of admissible manifolds.

Definition. A compact Riemannian manifold (M, g) with boundary, of dimension $n \geq 3$, is said to be *admissible* if $(M, g) \subset\subset \mathbb{R} \times (M_0, g_0)$, $g = c(e \oplus g_0)$ where $c > 0$ smooth function on M, e is the Euclidean metric and (M_0, g_0) is a simple $(n-1)$-dimensional manifold. We say that a manifold with boundary (M_0, g_0) is *simple*, if ∂M_0 is strictly convex, and for any point $x \in M_0$ the exponential map $\exp_x$ is a diffeomorphism from its maximal domain in $T_x M_0$ onto M_0.

Compact submanifolds of Euclidean space, the sphere minus a point and hyperbolic space are all examples of admissible manifolds.

If (M, g) is admissible, points of M can written as $x = (x_1, x')$, where x_1 is the Euclidean coordinate. We define

$$\partial M_\pm = \{x \in \partial M : \pm \partial_\nu \varphi(x) > 0\},$$
$$\partial M_{\tan} = \{x \in \partial M : \partial_\nu \varphi(x) = 0\},$$

where $\varphi(x) = x_1$. The function φ is a natural limiting Carleman weight in (M, g); see [17]. In the results below we assume that there is a part which inaccessible for measurements $\Gamma_{\mathrm{i}} \subset \partial M_{\tan}$, and the accessible part will be denoted by $\Gamma_{\mathrm{a}} = \partial M_{\tan} \setminus \Gamma_{\mathrm{i}}$.

The first main result of our thesis, says that one can reconstruct the local attenuated geodesic ray transform of the one-dimensional Fourier transform (with respect to x_1-variable) of the potential q from the partial knowledge of the Dirichlet-to-Neumann map with $\Gamma_+ \supset \partial M_+ \cup \Gamma_{\mathrm{a}}$ and $\Gamma_- \supset \partial M_- \cup \Gamma_{\mathrm{a}}$.

Theorem 2.1.1. *Let (M, g) be an admissible manifold, and suppose that $q \in C(M)$ such that 0 is not a Dirichlet eigenvalue of $-\Delta_g + q$. Let $\Gamma_{\mathrm{i}} \subset \partial M_{\tan}$ be closed such that for some open $E \subset \partial M_0$ one has*

$$\Gamma_{\mathrm{i}} \subset \mathbb{R} \times (\partial M_0 \setminus E).$$

Let $\Gamma_{\mathrm{a}} = \partial M_{\tan} \setminus \Gamma_{\mathrm{i}}$ *and let* $\Gamma_{\pm} \subset \partial M$ *be a neighborhood of* $\partial M_{\pm} \cup \Gamma_{\mathrm{a}}$*. Then for any given geodesic* $\gamma : [0, T] \to M_0$ *with endpoints on* E *and for any* $\lambda \in \mathbb{R}$*, the integral*

$$\int_0^T e^{-2\lambda t} \widehat{(cq)}(2\lambda, \gamma(t))\, dt$$

can be constructively recovered from the knowledge of $\Lambda_{g,q}(f)$ *on* Γ_- *for all* $f \in \mathcal{H}_g(\partial M)$ *supported in* Γ_+*. Here* q *is extended outside of* M *by zero, and* $\widehat{(cq)}$ *is the one-dimensional Fourier transform of* q *with respect to* x_1*-variable.*

This is a constructive version of the corresponding uniqueness result by Kenig and Salo [29, Theorem 2.1].

In the next result, we consider the local geodesic ray transform I_O in an open subset O of the transversal simple manifold (M_0, g_0) which is defined for $f \in C(M_0)$ as

$$I_O f(\gamma) := \int_\gamma f(\gamma(t))\, dt, \quad \gamma \text{ is a geodesic contained in } O \text{ with endpoints on } \partial M_0.$$

We say that I_O is *constructively invertible* in O, if any $f \in C(M_0)$ can be recovered in O from the knowledge of $I_O f$.

Using Theorem 2.1.1 one can constructively recover potentials in the set where the local geodesic ray transform is invertible.

Theorem 2.1.2. *Suppose that* (M, g)*,* $q \in C(M)$*,* $E \subset \partial M_0$ *and* $\Gamma_{\pm}$ *are as in Theorem 2.1.1. Assume that* $O \subset M_0$ *is open such that* $O \cap \partial M_0 \subset E$ *and the local ray transform is constructively invertible on* O*. Then* q *can be constructively determined in* $M \cap (\mathbb{R} \times O)$ *from the knowledge of* $\Lambda_{g,q}(f)$ *on* Γ_- *for all* $f \in \mathcal{H}_g(\partial M)$ *supported in* Γ_+*.*

This result gives a constructive proof of the corresponding uniqueness result by Kenig and Salo [29, Theorem 2.2]; the latter is the above mentioned generalization of the result of Isakov [26].

Constructive invertibility of the local ray transform, to the best of the author's knowledge, is known in the following cases: if M_0 has dimension $n \geq 3$ and if $p \in \partial M_0$ is such that ∂M_0 is strictly convex near p, then there is an open $O \subset M_0$ containing p on which I_O is constructively invertible; this result is due to Uhlmann and Vasy [64]. In two dimensions, no such result is known. Even injectivity of the local geodesic ray transform is an open question.

If $\partial M_{\tan}$ has zero measure in ∂M, we give the reconstruction procedure to determine potentials globally. The problem is reduced to the constructive invertibility of the global geodesic ray transform on the transversal simple manifold M_0.

Theorem 2.1.3. *Let (M, g) be an admissible manifold, and suppose that $q \in C(M)$ such that 0 is not a Dirichlet eigenvalue of $-\Delta_g + q$. Suppose that $\partial M_{\tan}$ is of zero measure in ∂M. If the global geodesic ray transform is constructively invertible in M_0, then q can be constructively determined in M from the knowledge of $\Lambda_{g,q}(f)$ on ∂M_- for all $f \in \mathcal{H}_g(\partial M)$ supported in ∂M_+.*

This is a generalization with refinements to admissible manifolds of the corresponding result by Nachman and Street [40] in Euclidean setting. More precisely, comparing to [40], we do not assume that the subsets of Dirichlet data inputs overlap with the subsets of Neumann data measurements. So our reconstruction procedure is new even in Eulidean space. The version of Theorem 2.1.3 was given by Kenig, Salo and Uhlmann [30] for full data case on admissible manifolds of dimension three.

Constructive invertibility of the global ray transform is known in the following cases:

- $(M_0, g_0) = (\overline{\Omega}, e)$ where $\Omega \subset \mathbb{R}^n$ is open and bounded with C^∞ boundary, and e is the Euclidean metric. In this case inversion formula is given in the book of Sharafutdinov [55, Section 2.12].

- (M_0, g_0) of dimension $n \geq 3$, have strictly convex boundary and is globally foliated by strictly convex hypersurfaces. For such case, there is a layer stripping type algorithm for reconstruction developed by Uhlmann and Vasy [64].

- (M_0, g_0) is a simple surface. In this case, there is a Fredholm type inversion formula which was derived by Pestov and Uhlmann [50]; see also the article of Krishnan [32].

The problem of constructive inversion of local or global geodesic ray transforms is of independent interest in integral geometry.

The structure of the chapter is as follows. In Section 2.2 we give some preliminaries about trace operators and Green's identity for the space $H_{\Delta_g}(M)$. We also consider the well-posedness of the Dirichlet problem for the Schrödinger equation $(-\Delta_g + q)u = 0$ with boundary condition in $\mathcal{H}_g(\partial M)$. Section 2.3, following the arguments of [40] and modifying them, is devoted to the construction of the new Green's operators for the Laplace-Beltrami operator, and in Section 2.4 the corresponding single layer potentials are constructed. The solvability of the required boundary integral equation is given in Section 2.5. Then we construct complex geometrical optics solutions in Section 2.6, and we use these solutions to give reconstruction procedures in Section 2.7.

2.2 Trace operators and the Dirichlet-to-Neumann map

Let (M, g) be a compact Riemannian manifold with boundary. We use the notation $d\,\mathrm{Vol}_g$ for the volume form of (M, g) and $d\sigma_{\partial M}$ for the induced volume form on the boundary ∂M. For any two functions u, v on M, define an inner product

$$(u|v)_{L^2(M)} := \int_M u(x)\overline{v(x)}\,d\,\mathrm{Vol}_g(x),$$

and the corresponding norm will be denoted by $\|\cdot\|_{L^2(M)}$. For any two functions f, h on $\Gamma \subset \partial M$, define an inner product

$$(f|h)_\Gamma := \int_\Gamma f(x)\overline{h(x)}\,d\sigma_{\partial M}(x),$$

and by $\|\cdot\|_\Gamma$ will be denoted the corresponding norm. We also write for short

$$\|\nabla u\|_{L^2(M)} = \left(\int_M |\nabla u(x)|_g^2\,d\,\mathrm{Vol}_g(x)\right)^{1/2}.$$

Following Bukhgeim and Uhlmann [7], we work with the following Hilbert space which is the largest domain of the Laplace-Beltrami operator Δ_g:

$$H_{\Delta_g}(M) = \{u \in L^2(M) : \Delta_g u \in L^2(M)\}.$$

The norm on $H_{\Delta_g}(M)$ is

$$\|u\|^2_{H_{\Delta_g}(M)} = \|u\|^2_{L^2(M)} + \|\Delta_g u\|^2_{L^2(M)}.$$

The proof of the following result is essentially the same as in [7] (see also, for example [34]). We include it here for the completeness and accuracy of the exposition.

Proposition 2.2.1. *The trace maps* $\mathtt{tr}(u) = u|_{\partial M}$ *and* $\mathtt{tr}_\nu(u) = \frac{\partial u}{\partial \nu}\big|_{\partial M}$ *defined on* $C^\infty(M)$ *have extensions to a bounded operators* $H_{\Delta_g}(M) \to H^{-1/2}(\partial M)$ *and* $H_{\Delta_g}(M) \to H^{-3/2}(\partial M)$*, respectively. Moreover, if* $u \in H_{\Delta_g}(M)$ *and* $\mathtt{tr}(u) \in H^{3/2}(\partial M)$*, then* $u \in H^2(M)$ *and* $\mathtt{tr}_\nu(u) \in H^{1/2}(\partial M)$*.*

Proof. First, we show that the trace map $\mathtt{tr}$ has an extension to a bounded operator $H_{\Delta_g}(M) \to H^{-1/2}(\partial M)$. Let $u \in C^\infty(M)$ and $w \in H^{1/2}(\partial M)$. By the surjectivity of the trace map on $H^2(M)$, there is $v \in H^2(M)$ such that

$$\mathtt{tr}(v) = 0, \quad \mathtt{tr}_\nu(v) = \frac{\partial v}{\partial \nu}\bigg|_{\partial M} = w, \quad \|v\|_{H^2(M)} \le C\|w\|_{H^{1/2}(\partial M)}.$$

Using Green's formula, we get

$$\begin{aligned}(\mathtt{tr}(u)|w)_{\partial M} &= \int_{\partial M} \mathtt{tr}(u)\overline{w}\, d(\partial M)_g \\ &= \int_{\partial M} \mathtt{tr}(u)\overline{\mathtt{tr}_\nu(v)}\, d(\partial M)_g = \int_M (u\overline{\Delta_g v} - \overline{v}\Delta_g u)\, d\,\mathrm{Vol}_g\,.\end{aligned}$$

Therefore,

$$|(\mathtt{tr}(u)|w)_{\partial M}| \le \|u\|_{H_{\Delta_g}(M)}\|v\|_{H^2(M)} \le C\|u\|_{H_{\Delta_g}(M)}\|w\|_{H^{1/2}(\partial M)}.$$

This proves that the map $\mathtt{tr} : C^\infty(M) \to H^{-1/2}(\partial M)$ is bounded and controlled by the $H_{\Delta_g}(M)$-norm. Since $C^\infty(M)$ is dense in $H_{\Delta_g}(M)$, we can extend $\mathtt{tr}$ to a bounded linear map $H_{\Delta_g}(M) \to H^{-1/2}(\partial M)$.

Next, we show that the trace map $\mathtt{tr}_\nu$ has an extension to a bounded operator $H_{\Delta_g}(M) \to H^{-3/2}(\partial M)$. Let $u \in C^\infty(M)$ and $w \in H^{3/2}(\partial M)$. By the surjectivity of the trace map on $H^2(M)$, there is $v \in H^2(M)$ such that

$$\mathtt{tr}(v) = w, \quad \mathtt{tr}_\nu(v) = \frac{\partial v}{\partial \nu}\bigg|_{\partial M} = 0, \quad \|v\|_{H^2(M)} \le C\|w\|_{H^{3/2}(\partial M)}.$$

Using Green's formula, we get

$$\begin{aligned}(\mathtt{tr}_\nu(u)|w)_{\partial M} &= \int_{\partial M} \mathtt{tr}_\nu(u)\overline{w}\, d(\partial M)_g \\ &= \int_{\partial M} \mathtt{tr}_\nu(u)\overline{\mathtt{tr}(v)}\, d(\partial M)_g = \int_M (\overline{v}\Delta_g u - u\overline{\Delta_g v})\, d\,\mathrm{Vol}_g\,.\end{aligned}$$

Therefore,

$$|(\mathtt{tr}_\nu(u)|w)_{\partial M}| \le \|u\|_{H_{\Delta_g}(M)}\|v\|_{H^2(M)} \le C\|u\|_{H_{\Delta_g}(M)}\|w\|_{H^{3/2}(\partial M)}.$$

This proves that the map $\mathtt{tr}_\nu : C^\infty(M) \to H^{-3/2}(\partial M)$ is bounded and controlled by the $H_{\Delta_g}(M)$-norm. Since $C^\infty(M)$ is dense in $H_{\Delta_g}(M)$, we can extend $\mathtt{tr}_\nu$ to a bounded linear map $H_{\Delta_g}(M) \to H^{-3/2}(\partial M)$.

Now, we give the proof of the last statement. First, we consider the case when $\mathtt{tr}(u) = 0$. Let $u \in C^\infty(M)$ with $\mathtt{tr}(u) = 0$. Using, Green's identity, we have

$$\begin{aligned}\|u\|^2_{H^1(M)} &= \|u\|^2_{L^2(M)} + \|\nabla u\|^2_{L^2(M)} \\ &= \|u\|^2_{L^2(M)} - (u, \Delta_g u)_{L^2(M)} \\ &\le \|u\|^2_{L^2(M)} + \frac{1}{2}\left(\|u\|^2_{L^2(M)} + \|\Delta_g u\|^2_{L^2(M)}\right) \\ &\le C\|u\|^2_{H_{\Delta_g}(M)},\end{aligned} \tag{2.1}$$

for some constant $C > 0$. By [62, Theorem 1.3] in Chapter 5, we have

$$\|u\|^2_{H^2(M)} \le C\|\Delta_g u\|^2_{L^2(M)} + C\|u\|^2_{H^1(M)},$$

for some another constant $C > 0$. Combining this with (2.1), we obtain

$$\|u\|^2_{H^2(M)} \le C\|u\|^2_{H_{\Delta_g}(M)}, \quad u \in C^\infty(M), \quad \mathtt{tr}(u) = 0.$$

By density arguments, we obtain

$$\|u\|^2_{H^2(M)} \le C\|u\|^2_{H_{\Delta_g}(M)}, \quad u \in H_{\Delta_g}(M), \quad \mathtt{tr}(u) = 0.$$

This proves the last statement for the case when $\mathtt{tr}(u) = 0$.

Suppose now that $u \in H_{\Delta_g}(M)$ with $\texttt{tr}(u) \in H^{3/2}(\partial M)$. By the surjectivity of the trace operator, there is $v \in H^2(M)$ such that $\texttt{tr}(v) = \texttt{tr}(u)$. Set $w := u - v$, then $w \in H_{\Delta_g}(M)$ with $\texttt{tr}(w) = 0$. By what we have proved above, $w \in H^2(M)$, and hence $u \in H^2(M)$. □

The proof of Proposition 2.2.1 gives the following.

Corollary 2.2.2. *For $u \in H_{\Delta_g}(M)$ and $v \in H^2(M)$ we have the generalized Green's identity*

$$
\begin{aligned}
&(u|(-\Delta_g)v)_{L^2(M)} - ((-\Delta_g)u|v)_{L^2(M)} \\
&\qquad = \langle \texttt{tr}_\nu(u), \texttt{tr}(v)\rangle_{H^{-3/2,3/2}(\partial M)} - \langle \texttt{tr}(u), \texttt{tr}_\nu(v)\rangle_{H^{-1/2,1/2}(\partial M)},
\end{aligned}
$$

where $\langle \cdot,\cdot\rangle_{H^{-s,s}(\partial M)}$ is the duality between $H^{-s}(\partial M)$ and $H^s(\partial M)$.

Now we introduce the following space on the boundary ∂M:

$$\mathcal{H}_g(\partial M) = \{\texttt{tr}(u) : u \in H_{\Delta_g}(M)\} \subset H^{-1/2}(\partial M).$$

Assume that $q \in L^\infty(M)$ and let us introduce the Bergman space $b_q(M)$ as follows

$$b_q(M) = \{u \in L^2(M) : (-\Delta_g + q)u = 0\} \subset H_{\Delta_g}(M).$$

The topology on this space is a subspace topology in $L^2(M)$. It is not difficult to check that $b_q(M)$ is a closed subspace of $L^2(M)$.

We need the following result to define a topology on $\mathcal{H}_g(\partial M)$:

Proposition 2.2.3. *If $q \in L^\infty(M)$ and 0 is not a Dirichlet eigenvalue of $-\Delta_g + q$ in M, then $\texttt{tr} : b_q(M) \to \mathcal{H}_g(\partial M)$ is one-to-one and onto.*

Proof. Let $u, v \in b_q(M)$ is such that $\texttt{tr}(u) = \texttt{tr}(v)$. Set $w = u - v$, then $w \in b_q(M)$ and $\texttt{tr}(w) = 0$. By the last statement of Proposition 2.2.1, $w \in H^2(M)$. By assumption, 0 is not a Dirichlet eigenvalue of $-\Delta_g + q$ in M. Therefore, $(-\Delta_g + q)w = 0$ with $w|_{\partial M} = 0$ imply that $w = 0$.

Let $h \in \mathcal{H}_g(\partial M)$. By definition of $\mathcal{H}_g(\partial M)$, there is $u \in H_{\Delta_g}(M)$ such that $\texttt{tr}(u) = h$. Take $v \in H_0^1(M)$ being the solution to the Dirichlet problem $(-\Delta_g + q)v = (-\Delta_g + q)u$,

$v|_{\partial M} = 0$. Set $w = u - v$. Then $w \in H_{\Delta_g}(M)$, $(-\Delta_g + q)w = 0$ and $\mathtt{tr}(w) = 0$. In other words, $w \in b_q(M)$ with $\mathtt{tr}(w) = h$. □

Let P_q denote the inverse of $\mathtt{tr} : b_q(M) \to \mathcal{H}_g(\partial M)$. We define the norm on $\mathcal{H}_g(\partial M)$ as

$$\|f\|_{\mathcal{H}_g(\partial M)} = \|P_0 f\|_{L^2(M)}.$$

In particular, by Proposition 2.2.3, this implies that $\mathtt{tr} : b_0 \to \mathcal{H}_g(\partial M)$ as well as $P_0 : \mathcal{H}_g(\partial M) \to b_0$ are bounded. Next, we give the following solvability result of the Dirichlet problem with boundary data in $\mathcal{H}_g(\partial M)$:

Proposition 2.2.4. *The operator* $\mathtt{tr} : H_{\Delta_g}(M) \to \mathcal{H}_g(\partial M)$ *is bounded. If* $q \in L^\infty(M)$ *and* 0 *is not a Dirichlet eigenvalue of* $-\Delta_g + q$ *in* M*, then* $\mathtt{tr} : b_q(M) \to \mathcal{H}_g(\partial M)$ *is a homeomorphism.*

Proof. Let $u \in H_{\Delta_g}(M)$. Consider $v \in H^1_0(M)$ being the solution to the Dirichlet problem $(-\Delta_g)v = (-\Delta_g)u$, $v|_{\partial M} = 0$. Set $w = u - v$. Note that $u \mapsto v$ is bounded $H_{\Delta_g}(M) \to L^2(M)$, and hence $u \mapsto w$ is bounded $H_{\Delta_g}(M) \to b_0(M)$ as well, since $(-\Delta_g)w = 0$. Since $\mathtt{tr} : b_0 \to \mathcal{H}_g(\partial M)$ is bounded and since $\mathtt{tr}(w) = \mathtt{tr}(u)$, we can conclude that the map $u \mapsto \mathtt{tr}(u)$ is bounded $H_{\Delta_g}(M) \to \mathcal{H}_g(\partial M)$.

Since the inclusion $b_q \hookrightarrow H_{\Delta_g}(M)$ is bounded, by the first part of the proposition, the map $\mathtt{tr} : b_q \to \mathcal{H}_g(M)$ is bounded. Bijectivity of the latter map, which follows from Proposition 2.2.3, together with Open Mapping Theorem, implies the last statement. □

We also extend the domain of the Dirichlet-to-Neumann map to $\mathcal{H}_g(\partial M)$:

Proposition 2.2.5. *Suppose that* $q \in L^\infty(M)$ *and* 0 *is not a Dirichlet eigenvalue of* $-\Delta_g + q$ *in* M*. Then* $(\Lambda_q - \Lambda_0)|_{\mathcal{H}(\partial M)}$ *is a bounded operator* $\mathcal{H}_g(\partial M) \to (\mathcal{H}_g(\partial M))^*$*. Moreover, the following integral identity holds*

$$\langle h, (\Lambda_{g,q} - \Lambda_{g,0})f\rangle_{H^{-1/2,1/2}(\partial M)} = (P_0(h)|qP_q(f))_{L^2(M)}, \tag{2.2}$$

for all $f, h \in \mathcal{H}_g(\partial M)$.

Proof. Suppose that $f, h \in \mathcal{H}_g(\partial M)$. Let $u \in H_{\Delta_g}(M)$ be the unique solution to the boundary value problem

$$(-\Delta_g + q)u = 0 \text{ in } \Omega, \quad \mathtt{tr}(u) = f,$$

and let u_0 be the unique solution to the boundary value problem

$$(-\Delta_g)u_0 = 0 \text{ in } \Omega, \quad \mathtt{tr}(u_0) = f.$$

Set $w := u - u_0$, then we have

$$(-\Delta_g)w = -qu \text{ in } \Omega, \quad \mathtt{tr}(w) = 0.$$

By the last statement of Proposition 2.2.1, we can conclude that $w \in H^2(M)$. Note that by Proposition 2.2.4, there is $v_h \in H_{\Delta_g}(M)$ such that $(-\Delta_g)v_h = 0$ and $\mathtt{tr}(v_h) = h$. Now, we can apply Corollary 2.2.2 and get

$$\begin{aligned}(v_h|qu)_{L^2(M)} &= -(v_h|(-\Delta_g)w)_{L^2(M)} \\ &= -((-\Delta_g)v_h|w)_{L^2(M)} + \langle h, \mathtt{tr}_\nu(w)\rangle_{H^{-1/2,1/2}(\partial M)}.\end{aligned}$$

Since $(-\Delta_g)v_h = 0$ and $\mathtt{tr}_\nu(w) = \mathtt{tr}_\nu(u - u_0) = (\Lambda_{g,q} - \Lambda_{g,0})f$, we obtain

$$\langle h, (\Lambda_{g,q} - \Lambda_{g,0})f\rangle_{H^{-1/2,1/2}(\partial M)} = (v_h|qu)_{L^2(M)}. \tag{2.3}$$

The right-hand side depends continuously on $f, h \in \mathcal{H}_g(\partial M)$. Hence, so does the left hand-side and this together with (2.3) implies that the result. □

2.3 The Green's operators

Let (M, g) be an admissible manifold and let $q \in L^\infty(M)$. Let us introduce certain notations which will be used thoughout the thesis. For $\tau \in \mathbb{R}$, we consider the following disjoint decomposition $\partial M = S_\tau^+ \cup S_\tau^-$, where

$$S_\tau^+ := \{x \in \partial M : \operatorname{sgn}(\tau)\partial_\nu\varphi(x) \geq |3\tau|^{-1}\}, \quad S_\tau^- := \partial M \setminus S_\tau^+.$$

For $\delta > 0$, we can write $S_\tau^- = S_{\tau,\delta}^- \cup S_{\tau,\delta}^0$, where

$$S_{\tau,\delta}^- := \{x \in \partial M : \operatorname{sgn}(\tau)\partial_\nu \varphi(x) \le -\delta\},$$
$$S_{\tau,\delta}^0 := \{x \in \partial M : -\delta < \operatorname{sgn}(\tau)\partial_\nu \varphi(x) < (3|\tau|)^{-1}\}.$$

Constructions of Green's operators and the corresponding single layer potentials, as well as construction of complex geometrical optics solutions are based on the following Carleman estimates with boundary terms for the conjugated operator

$$e^{\tau x_1}(-\Delta_g + q)e^{-\tau x_1}.$$

Proposition 2.3.1. *Let (M, g) be an admissible manifold and let $q \in L^\infty(M)$. There are constants $C_0, \tau_0 >$ such that for all $\tau \in \mathbb{R}$ with $|\tau| \ge \tau_0$ and $\delta > 0$, we have*

$$(\delta|\tau|)^{1/2}\|\partial_\nu u\|_{S_{\tau,\delta}^-} + \|\partial_\nu u\|_{S_{\tau,\delta}^0} + |\tau|\|u\|_{L^2(M)} + \|\nabla u\|_{L^2(M)}$$
$$\le C_0\|e^{\tau x_1}(-\Delta_g + q)e^{-\tau x_1}u\|_{L^2(M)} + C_0|\tau|^{1/2}\|\partial_\nu u\|_{S_\tau^+} \quad (2.4)$$

for all $u \in C^\infty(M)$ with $u|_{\partial M} = 0$.

Proof. This estimate was proven by Kenig and Salo; see [29, Proposition 4.2]. □

Define

$$\mathcal{D}_\tau^\pm = \{u \in C^\infty(M) : u|_{\partial M} = \mathtt{tr}_\nu(u)|_{S_\tau^\pm} = 0\}.$$

The aim of this section is to prove the following result.

Theorem 2.3.2. *Let (M, g) be an admissible manifold. There is a constant $\tau_0 > 0$ such that for all $\tau \in \mathbb{R}$ with $|\tau| \ge \tau_0$, there is a linear operator*

$$G_\tau : L^2(M) \to L^2(M)$$

such that

$$e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}G_\tau v = v, \quad v \in L^2(M)$$

and

$$G_\tau e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}u = u, \quad u \in \mathcal{D}^+_{-\tau}. \tag{2.5}$$

This operator satisfies

$$\|G_\tau f\|_{L^2(M)} \leq \frac{C_0}{|\tau|}\|f\|_{L^2(M)}, \quad f \in L^2(M),$$

where $C_0 > 0$ is independent of τ. Moreover, $G_\tau : L^2(M) \to e^{\tau x_1}H_{\Delta_g}(M)$ and for all $v \in L^2(M)$ support of $\mathtt{tr}(G_\tau v)$ is in S^+_τ.

Let π_τ be the orthogonal projection onto $\mathcal{L}_\tau$ the closure of $e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}\mathcal{D}^+_\tau$ in $L^2(M)$.

Lemma 2.3.3. *Let $\pi^\perp_\tau := \mathrm{Id} - \pi_\tau$. Then $\pi^\perp_\tau$ is the orthogonal projection onto*

$$\mathcal{A}_\tau = \{u \in L^2(M) : e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}u = 0 \text{ and } \mathtt{tr}(u) \text{ is supported in } S^+_\tau\}.$$

Proof. It is enough to show that u is orthogonal to $e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}\mathcal{D}^+_\tau$ if and only if u is in $\mathcal{A}_\tau$. Suppose that $u \in \mathcal{A}_\tau$. Then for $v \in \mathcal{D}^+_\tau$, we have

$$(u|e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}v)_{L^2(M)} = (e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}u|v)_{L^2(M)} - (\mathtt{tr}(u), \mathtt{tr}_\nu(v))_{H^{-1/2,1/2}(\partial M)}.$$

Since $\mathtt{tr}(u)$ supported in S^+_τ, $\mathtt{tr}_\nu(v) = 0$ in S^+_τ and $e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}u = 0$, we obtain

$$(u|e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}v)_{L^2(M)} = 0,$$

which means that u is orthogonal to $e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}\mathcal{D}^+_\tau$. Converse is as in [40, Lemma 3.3]. □

Proposition 2.3.4. *Let (M, g) be an admissible manifold. There is $\tau_0 > 0$ such that for all $\tau \in \mathbb{R}$ with $|\tau| \geq \tau_0$ and for a given $v \in L^2(M)$, there is a unique solution $u \in L^2(M)$ of the equation*

$$e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}u = v \quad in \quad M$$

such that $\mathtt{tr}(u)$ is supported in S^+_τ, $\pi_\tau u = u$ and $\|u\|_{L^2(M)} \leq C_0\frac{1}{|\tau|}\|v\|_{L^2(M)}$ with constant $C_0 > 0$ independent of τ.

Proof. First, we show the existence. Define a linear functional L on $e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}\mathcal{D}_\tau^+$ by

$$L(e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}w) = (v|w)_{L^2(M)}, \quad w \in \mathcal{D}_\tau^+.$$

Then we have

$$\begin{aligned}|L(e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}w)| &\le \|v\|_{L^2(M)}\|w\|_{L^2(M)} \\ &\le C_0\frac{1}{|\tau|}\|v\|_{L^2(M)}\|e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}w\|_{L^2(M)},\end{aligned}$$

where in the last step we have used the Carleman estimate (2.4). By the Hahn-Banach theorem, we may extend L to a linear continuous functional $\widetilde{L}$ on $\mathcal{L}_\tau$. On the orthogonal complement of $\mathcal{L}_\tau$ in $L^2(M)$ we define $\widetilde{L}$ to be zero. By the Riesz representation theorem, there exists $u \in L^2(M)$ such that

$$\widetilde{L}(f) = (u|f)_{L^2(M)}, \quad f \in L^2(M).$$

In particular,

$$\begin{aligned}(u|e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}w)_{L^2(M)} &= \widetilde{L}(e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}w) \\ &= (v|w)_{L^2(M)}, \qquad w \in \mathcal{D}_\tau^+.\end{aligned} \tag{2.6}$$

If we take $w \in C_0^\infty(M^{\text{int}})$ in the above equation, we obtain $e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}u = v$. Moreover,

$$\|u\|_{L^2(M)} \le C_0\frac{1}{|\tau|}\|v\|_{L^2(M)}.$$

Since $\widetilde{L} \equiv 0$ on the orthogonal complement of $\mathcal{L}_\tau$ in $L^2(M)$, we have that $u \in \mathcal{L}_\tau$ and hence $\pi_\tau u = u$.

To finish the proof, we need to show that $\mathtt{tr}(u)$ is supported in S_τ^+. For arbitrary $w \in \mathcal{D}_\tau^+$, using the generalized Green's identity from Corollary 2.2.2, we get

$$(u|e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}w)_{L^2(M)} = \langle \mathtt{tr}(u), \mathtt{tr}_\nu(w)\rangle_{H^{-1/2,1/2}(\partial M)} + (v|w)_{L^2(M)}.$$

According to (2.6), we have $\langle \mathtt{tr}_\nu(w), \mathtt{tr}(u)\rangle_{\partial M} = 0$. Since $w \in \mathcal{D}_\tau^+$ was arbitrary, we can conclude that $\mathtt{tr}(u)$ is supported in S_τ^+.

Now, we prove uniqueness. Suppose that $u' \in L^2(M)$ is another solution of the equation $e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}u' = v$ satisfying all the conditions of the proposition. Then $e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}(u-u') = 0$, $\mathtt{tr}(u-u')$ is supported in S_τ^+, and $\pi_\tau(u-u') = u-u'$. However, by Lemma 2.3.3, $\pi_\tau(u-u') = 0$. Thus, we obtain $u-u' = 0$ which finishes the proof. □

Let $H_\tau : L^2(M) \to L^2(M)$ be the solution operator obtained in the previous result. In other words, the operator H_τ is defined by $H_\tau v = u$, where u and v are as in Proposition 2.3.4. The following is an immediate corollary of the preceeding result.

Corollary 2.3.5. *Let (M, g) be an admissible manifold. There is $\tau_0 > 0$ such that for all $\tau \in \mathbb{R}$ with $|\tau| \geq \tau_0$, there is a linear operator*

$$H_\tau : L^2(M) \to L^2(M)$$

such that

$$e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}H_\tau v = v, \quad v \in L^2(M)$$

and $\pi_\tau H_\tau = H_\tau$. This operator satisfies

$$\|H_\tau f\|_{L^2(M)} \leq C_0 \frac{1}{|\tau|}\|f\|_{L^2(M)}$$

where $C_0 > 0$ is independent of τ. Moreover, $H_\tau : L^2(M) \to e^{\tau x_1}H_{\Delta_g}(M)$ and for all $v \in L^2(M)$ support of $\mathtt{tr}(H_\tau v)$ is in S_τ^+.

Thus, the operator H_τ satisfies Theorem 2.3.2 except (2.5). We shall accodingly modify H_τ to obtain (2.5). We need the technical result.

Lemma 2.3.6. *Let $T_\tau := H_\tau \pi_{-\tau}$. Then $T_\tau^* = T_{-\tau}$.*

Proof. Note that $T_\tau^* \pi_\tau^\perp = \pi_{-\tau}H_\tau^*(\mathrm{Id} - \pi_\tau) = \pi_{-\tau}H_\tau^* - \pi_{-\tau}H_\tau^*\pi_\tau = 0$, where in the first step we have used the fact that $\pi_{-\tau}^* = \pi_{-\tau}$ (since $\pi_{-\tau}$ is projection) and in the last step we have used that $H_\tau^*\pi_\tau = H_\tau^*$ (this follows from $\pi_\tau H_\tau = H_\tau$ which is true by Corollary 2.3.5). Also note that $T_{-\tau}\pi_\tau^\perp = H_{-\tau}\pi_\tau\pi_\tau^\perp = 0$. Thus, $T_\tau^*\pi_\tau^\perp = T_{-\tau}\pi_\tau^\perp = 0$, and hence, to prove the lemma it is sufficient to show that

$$T_\tau^* e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}v = H_{-\tau}e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}v$$

for all $\mathcal{D}_\tau^+$. Observe that $\pi_{-\tau}T_\tau^* = \pi_{-\tau}^2 H_\tau^* = \pi_{-\tau}H_\tau^* = T_{-\tau}^*$ (since $\pi_{-\tau}^2 = \pi_{-\tau}$). Therefore, it is enough to show that

$$\begin{aligned}(e^{\tau x_1}&(-\Delta_g)e^{-\tau x_1}w|T_\tau^* e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}v)_{L^2(M)} \\ &= (e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}w|H_{-\tau}e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}v)_{L^2(M)},\end{aligned}$$

for all $w \in \mathcal{D}_{-\tau}^+$ and for all $v \in \mathcal{D}_\tau^+$. We have

$$\begin{aligned}(e^{\tau x_1}&(-\Delta_g)e^{-\tau x_1}w|T_\tau^* e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}v)_{L^2(M)} \\ &= (e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}w|\pi_{-\tau}H_\tau^* e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}v)_{L^2(M)} \\ &= (H_\tau\pi_{-\tau}e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}w|e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}v)_{L^2(M)} \\ &= (H_\tau e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}w|e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}v)_{L^2(M)}.\end{aligned}$$

Since by Corollary 2.3.5 we know that $\mathtt{tr}(H_\tau e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}w)$ is supported in S_τ^+, we can use Green's identity and the fact that $v \in \mathcal{D}_\tau^+$ to get

$$\begin{aligned}(e^{\tau x_1}&(-\Delta_g)e^{-\tau x_1}w|T_\tau^* e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}v)_{L^2(M)} \\ &= (e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}H_\tau e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}w|v)_{L^2(M)}\end{aligned}$$

Since $e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}H_\tau = \mathrm{Id}$ by Corollary 2.3.5, we obtain

$$\begin{aligned}(e^{\tau x_1}&(-\Delta_g)e^{-\tau x_1}w|T_\tau^* e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}v)_{L^2(M)} \\ &= (e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}w|v)_{L^2(M)} \\ &= (w|e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}v)_{L^2(M)}.\end{aligned}$$

Here, in the last step we used the Green's identity and that $w|_{\partial M} = v|_{\partial M} = 0$.

Using that $\texttt{tr}(H_{-\tau}e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}v)$ is supported in $S^+_{-\tau}$, $w \in \mathcal{D}^+_{-\tau}$ and the Green's identity, we obtain

$$\begin{aligned}(e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}w|H_{-\tau}e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}v)_{L^2(M)}&\\ &= (w|e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}H_{-\tau}e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}v)_{L^2(M)}\\ &= (w|e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}v)_{L^2(M)}.\end{aligned}$$

In the last step we used the fact that $e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}H_{-\tau} = \mathrm{Id}$ (by Corollary 2.3.5). The proof of the lemma is thus complete. □

Proof of Theorem 2.3.2. Define $G_\tau = H_\tau + \pi^\perp_\tau H^*_{-\tau}$. By Corollary 2.3.5 and Lemma 2.3.3, it follows that

$$e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}G_\tau v = v, \quad v \in L^2(M),$$

$$\|G_\tau f\|_{L^2(M)} \le C_0\frac{1}{|\tau|}\|f\|_{L^2(M)}, \quad f \in L^2(M),$$

$G_\tau : L^2(M) \to e^{\tau x_1}H_{\Delta_g}(M)$ and that for all $v \in L^2(M)$ support of $\texttt{tr}(G_\tau v)$ is in S^+_τ.

It is left to prove (2.5). For this, we need first to show that $G^*_\tau = G_{-\tau}$. Using Lemma 2.3.6, we can show

$$G^*_\tau = H^*_\tau + H_{-\tau}\pi^\perp_\tau = (H_\tau\pi^\perp_{-\tau} + T_\tau)^* + H_{-\tau} - T_{-\tau} = H_{-\tau} + \pi^\perp_{-\tau}H^*_\tau = G_{-\tau}.$$

Using this, for $f \in L^2(M)$ and $u \in \mathcal{D}^+_{-\tau}$, we have

$$(f|G_\tau e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}u)_{L^2(M)} = (G_{-\tau}f|e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}u)_{L^2(M)}.$$

We have shown that $\texttt{tr}(G_{-\tau}f)$ is supported in $S^+_{-\tau}$. This fact together with $u \in \mathcal{D}^+_{-\tau}$ allows us to use the generalized Green's identity from Corollary 2.2.2 and get

$$(f|G_\tau e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}u)_{L^2(M)} = (e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}G_{-\tau}f|u)_{L^2(M)} = (f|u)_{L^2(M)}.$$

Here, in the last step we used the already proven fact that $e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}G_{-\tau} = \mathrm{Id}$. This finishes the proof. □

2.4 Single layer operators

The aim of this section is to construct the single layer operators S_τ corresponding to the Green's operators G_τ constructed in the previous section.

Let $\tau_0 > 0$ be as in Theorem 2.3.2. For $\tau \in \mathbb{R}$ with $|\tau| \geq \tau_0$, consider the operator

$$(\mathtt{tr} \circ G_\tau)^* : e^{-\tau x_1}(\mathcal{H}_g(\partial M))^* \to L^2(M).$$

In other words, $(\mathtt{tr} \circ G_\tau)^*$ defined for $h \in e^{-\tau x_1}(\mathcal{H}_g(\partial M))^*$ by

$$(f|(\mathtt{tr} \circ G_\tau)^* h)_{L^2(M)} = \langle (\mathtt{tr} \circ G_\tau) f, h \rangle_{H^{-1/2,1/2}(\partial M)}, \quad f \in L^2(M).$$

Proposition 2.4.1. *For all $h \in e^{-\tau x_1}(\mathcal{H}_g(\partial M))^*$ we have*

$$e^{-\tau x_1}(-\Delta_g) e^{\tau x_1} (\mathtt{tr} \circ G_\tau)^* h = 0$$

and the support of $\mathtt{tr}((\mathtt{tr} \circ G_\tau)^* h)$ is in $S^+_{-\tau}$. Moreover, suppose that B is a neighborhood of S^+_τ such that $\overline{B} \subset \partial M_{\mathrm{sgn}(\tau)}$, and that the support of h is in $\partial M \setminus B$. Then $(\mathtt{tr} \circ G_\tau)^* h = 0$.

Proof. Let $h \in e^{-\tau x_1}(\mathcal{H}_g(\partial M))^*$. Then for all $f \in \mathcal{D}^+_{-\tau}$, we obtain

$$\begin{aligned}(e^{\tau x_1}(-\Delta_g) e^{-\tau x_1} f | (\mathtt{tr} \circ G_\tau)^* h)_{L^2(M)} & \\ &= \langle (\mathtt{tr} \circ G_\tau) e^{\tau x_1}(-\Delta_g) e^{-\tau x_1} f, h \rangle_{H^{-1/2,1/2}(\partial M)} \\ &= \langle \mathtt{tr}(f), h \rangle_{H^{-1/2,1/2}(\partial M)} = 0.\end{aligned} \tag{2.7}$$

Here, we have used (2.5) and $\mathtt{tr}(f) = 0$. If we take $f \in C_0^\infty(M)$ in (2.7) and use the generalized Green's identity in Corollary 2.2.2, we can show that

$$(f|e^{-\tau x_1}(-\Delta_g) e^{\tau x_1} (\mathtt{tr} \circ G_\tau)^* h)_{L^2(M)} = (e^{\tau x_1}(-\Delta_g) e^{-\tau x_1} f | (\mathtt{tr} \circ G_\tau)^* h)_{L^2(M)} = 0.$$

Hence, we obtain $e^{-\tau x_1}(-\Delta_g) e^{\tau x_1} (\mathtt{tr} \circ G_\tau)^* h = 0$ for all $h \in e^{-\tau x_1}(\mathcal{H}_g(\partial M))^*$.

Let us now show that the support of $\mathtt{tr}((\mathtt{tr} \circ G_\tau)^* h)$ is in $S^+_{-\tau}$. For arbitrary $f \in \mathcal{D}^+_{-\tau}$, using the generalized Green's identity from Corollary 2.2.2, we get

$$
\begin{aligned}
&\langle \mathtt{tr}((\mathtt{tr}\circ G_\tau)^*h), \mathtt{tr}_\nu(f)\rangle_{H^{-1/2,1/2}(\partial M)} \\
&\qquad = \langle \mathtt{tr}((\mathtt{tr}\circ G_\tau)^*h), \mathtt{tr}_\nu(f)\rangle_{H^{-1/2,1/2}(\partial M)} \\
&\qquad\quad - \langle \mathtt{tr}_\nu((\mathtt{tr}\circ G_\tau)^*h), \mathtt{tr}(f)\rangle_{H^{-3/2,3/2}(\partial M)} \\
&\qquad = (e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}(\mathtt{tr}\circ G_\tau)^*h|f)_{L^2(M)} \\
&\qquad\quad - ((\mathtt{tr}\circ G_\tau)^*h|e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}f)_{L^2(M)} \\
&\qquad = -((\mathtt{tr}\circ G_\tau)^*h|e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}f)_{L^2(M)} = 0,
\end{aligned}
$$

where in the last step we have used (2.7).

Now, we prove the last statement of the proposition. If h is supported in $\partial M \setminus B$, then for all $f \in L^2(M)$ we have

$$(f|(\mathtt{tr}\circ G_\tau)^*h)_{L^2(M)} = \langle(\mathtt{tr}\circ G_\tau)f, h\rangle_{H^{-1/2,1/2}(S^+_\tau)} = 0.$$

This is because by the last statement of Theorem 2.3.2, $(\mathtt{tr}\circ G_\tau)f$ is supported in S^+_τ. The proof of the proposition is thus complete. □

For $\tau \in \mathbb{R}$ with $|\tau| \geq \tau_0$, define the operator S_τ for $h \in (\mathcal{H}_g(\partial M))^*$ by

$$S_\tau h = e^{-\tau x_1}(\mathtt{tr}\circ(\mathtt{tr}\circ G_\tau)^*)^*(e^{\tau x_1}h).$$

Proposition 2.4.2. *For $\tau \in \mathbb{R}$ with $|\tau| \geq \tau_0$, the operator S_τ is bounded $(\mathcal{H}_g(\partial M))^* \to \mathcal{H}_g(\partial M)$, and for $h \in (\mathcal{H}(\partial M))^*$, $S_\tau h$ depends only on $h|_{\partial M_{-\operatorname{sgn}(\tau)}}$ and supported in B.*

Proof. By Proposition 2.4.1, we have that the operator $(\mathtt{tr}\circ G_\tau)^* : e^{-\tau x_1}(\mathcal{H}_g(\partial M))^* \to e^{-\tau x_1}H_{\Delta_g}(M)$ is bounded, and hence the operator $\mathtt{tr}\circ(\mathtt{tr}\circ G_\tau)^* : e^{-\tau x_1}(\mathcal{H}_g(\partial M))^* \to e^{-\tau x_1}\mathcal{H}_g(\partial M)$ is bounded as well. This implies the boundedness of $(\mathtt{tr}\circ(\mathtt{tr}\circ G_\tau)^*)^* : e^{\tau x_1}(\mathcal{H}_g(\partial M))^* \to e^{\tau x_1}\mathcal{H}_g(\partial M)$. Therefore, S_τ is a bounded operator $(\mathcal{H}_g(\partial M))^* \to \mathcal{H}_g(\partial M)$.

We have by Proposition 2.4.1 that $\mathtt{tr}\circ(\mathtt{tr}\circ G_\tau)^*(e^{-\tau x_1}\tilde{h})$ is supported in $S^+_{-\tau}$ if $\tilde{h} \in (\mathcal{H}_g(\partial M))^*$. By duality, for $h \in (\mathcal{H}_g(\partial M))^*$, $S_\tau h = e^{-\tau x_1}(\mathtt{tr}\circ(\mathtt{tr}\circ G_\tau)^*)^*(e^{\tau x_1}h)$ depends only on $h|_{\partial M_{-\operatorname{sgn}(\tau)}}$.

By the last statement of Proposition 2.4.1, if $\tilde{h} \in (\mathcal{H}_g(\partial M))^*$ is supported in $\partial M \setminus B$ then $\mathtt{tr}((\mathtt{tr} \circ G_\tau)^*(e^{-\tau x_1}\tilde{h})) = 0$. By duality, for any $h \in (\mathcal{H}_g(\partial M))^*$, $e^{-\tau x_1}(\mathtt{tr} \circ (\mathtt{tr} \circ G_\tau)^*)^*(e^{\tau x_1}h)$ supported in B. ☐

2.5 Boundary integral equation

In the present section, we prove the solvability of the following boundary integral equation: for $\tau \in \mathbb{R}$ with $|\tau| \geq \tau_0$

$$(\mathrm{Id} + S_\tau(\Lambda_{g,q} - \Lambda_{g,0}))\, h = f, \quad f, h \in \mathcal{H}_g(\partial M). \tag{2.8}$$

To prove the solvability of (2.8), we need the following result on basic properties of the operator $S_\tau(\Lambda_{g,q} - \Lambda_{g,0})$.

Proposition 2.5.1. *Suppose that $q \in L^\infty(M)$ and 0 is not a Dirichlet eigenvalue of $-\Delta_g + q$ in M. There is $\tau_0 > 0$ such that for all $\tau \in \mathbb{R}$ with $|\tau| \geq \tau_0$, the operator $S_\tau(\Lambda_{g,q} - \Lambda_{g,0})$ is a bounded operator $\mathcal{H}_g(\partial M) \to \mathcal{H}_g(\partial M)$, and for $f \in \mathcal{H}_g(\partial M)$, $S_\tau(\Lambda_{g,q} - \Lambda_{g,0})f$ is supported in B and can be computed from the knowledge of $\Lambda_q f|_{\partial M_{-\operatorname{sgn}(\tau)}}$. Moreover, the following factorization identity holds*

$$S_\tau(\Lambda_{g,q} - \Lambda_{g,0}) = \mathtt{tr} \circ e^{-\tau x_1} G_\tau e^{\tau x_1} q P_q.$$

Proof. First part of the proposition is a consequence of Proposition 2.2.5 and Proposition 2.4.2. To prove the last statement, consider $h \in (\mathcal{H}_g(\partial M))^*$ and $f \in \mathcal{H}_g(\partial M)$. Then

$$\langle h, \mathtt{tr} \circ e^{-\tau x_1} G_\tau e^{\tau x_1} q P_q(f) \rangle_{H^{-1/2,1/2}(\partial M)} = (e^{\tau x_1}(\mathtt{tr} \circ G_\tau)^*(e^{-\tau x_1}h) | q P_q(f))_{L^2(M)}.$$

By Proposition 2.4.1, $e^{\tau x_1}(\mathtt{tr} \circ G_\tau)^*(e^{-\tau x_1}h)$ is in b_0. Using (2.2), we show that

$$\begin{aligned}
&(e^{\tau x_1}(\mathtt{tr} \circ G_\tau)^*(e^{-\tau x_1}h) | q P_q(f))_{L^2(M)} \\
&\qquad = \langle e^{\tau x_1} \mathtt{tr}\left((\mathtt{tr} \circ G_\tau)^*(e^{-\tau x_1}h)\right), (\Lambda_{g,q} - \Lambda_{g,0})f \rangle_{H^{-1/2,1/2}(\partial M)} \\
&\qquad = \langle h, e^{-\tau x_1}\left(\mathtt{tr} \circ (\mathtt{tr} \circ G_\tau)^*\right)^*(e^{\tau x_1}(\Lambda_{g,q} - \Lambda_{g,0})f) \rangle_{H^{-1/2,1/2}(\partial M)} \\
&\qquad = \langle h, S_\tau(\Lambda_{g,q} - \Lambda_{g,0})f \rangle_{H^{-1/2,1/2}(\partial M)}.
\end{aligned}$$

The proof is thus finished. ☐

The following result shows that the boundary integral equation is equivalent to the certain integral equation; compare with [30, Proposition 3.2].

Proposition 2.5.2. *Suppose that* $q \in L^\infty(M)$ *and* 0 *is not a Dirichlet eigenvalue of* $-\Delta_g + q$ *in* M. *There is* $\tau_0 > 0$ *such that for all* $\tau \in \mathbb{R}$ *with* $|\tau| \geq \tau_0$ *and for all* $f, h \in \mathcal{H}_g(\partial M)$, $(\mathrm{Id} + S_\tau(\Lambda_{g,q} - \Lambda_{g,0}))\, h = f$ *holds if and only if* $(\mathrm{Id} + e^{-\tau x_1} G_\tau e^{\tau x_1} q)\, P_q(h) = P_0(f)$.

Proof. Suppose that $f, h \in \mathcal{H}_g(\partial M)$ satisfies $(\mathrm{Id} + S_\tau(\Lambda_{g,q} - \Lambda_{g,0}))\, h = f$. Note that by Theorem 2.3.2, we can show that

$$\Delta_g \left(\mathrm{Id} + e^{-\tau x_1} G_\tau e^{\tau x_1} q\right) P_q(h) = q P_q(h) - q P_q(h) = 0.$$

Therefore, it is enough to prove that

$$\mathtt{tr}\left(\left(\mathrm{Id} + e^{-\tau x_1} G_\tau e^{\tau x_1} q\right) P_q(h)\right) = f,$$

or equivalently

$$h + \mathtt{tr}\left(e^{-\tau x_1} G_\tau e^{\tau x_1} q P_q(h)\right) = f.$$

Using the factorization identity in Proposition 2.5.1, we can see that the left hand-side is $(\mathrm{Id} + S_\tau(\Lambda_{g,q} - \Lambda_{g,0}))\, h$, which is equal to f by assumption.

The converse direction can be shown by applying $\mathtt{tr}$ to the both sides of the identity

$$\left(\mathrm{Id} + e^{-\tau x_1} G_\tau e^{\tau x_1} q\right) P_q(h) = P_0(f)$$

and using the factorization identity in Proposition 2.5.1. □

Corollary 2.5.3. *Suppose that* $q \in L^\infty(M)$ *and* 0 *is not a Dirichlet eigenvalue of* $-\Delta_g + q$ *in* M. *There is* $\tau_0 > 0$ *such that for all* $\tau \in \mathbb{R}$ *with* $|\tau| \geq \tau_0$, *the operator* $\mathrm{Id} + S_\tau(\Lambda_{g,q} - \Lambda_{g,0}) : \mathcal{H}_g(\partial M) \to \mathcal{H}_g(\partial M)$ *is an isomorphism if and only if so is the operator* $\mathrm{Id} + e^{-\tau x_1} G_\tau e^{\tau x_1} q : b_q(M) \to b_0(M)$.

The following proposition combined together with the above two results implies the solvability of the boundary integral equation (2.8).

Proposition 2.5.4. *Suppose that $q \in L^\infty(M)$ and 0 is not a Dirichlet eigenvalue of $-\Delta_g + q$ in M. There is $\tau_0 > 0$ such that for all $\tau \in \mathbb{R}$ with $|\tau| \geq \tau_0$, the operator $\operatorname{Id} + e^{-\tau x_1} G_\tau e^{\tau x_1} q : b_q(M) \to b_0(M)$ is an isomorphism.*

Proof. Since $\|G_\tau\|_{L^2(M)\to L^2(M)} \leq \mathcal{O}(|\tau|^{-1})$ by Theorem 2.3.2, the operator $\operatorname{Id} + G_\tau q : L^2(M) \to L^2(M)$ is an isomorphism for big enough $|\tau| \gg 1$. Then for such τ, the operator $\operatorname{Id} + e^{-\tau x_1} G_\tau e^{\tau x_1} q : L^2(M) \to L^2(M)$ is an isomorphism whose inverse is $e^{-\tau x_1}(\operatorname{Id} + G_\tau q)^{-1} e^{\tau x_1}$. Let $u \in b_0$ and $w = e^{-\tau x_1}(\operatorname{Id} + G_\tau q)^{-1} e^{\tau x_1} u$. We need to show that $w \in b_q$. Applying $\operatorname{Id} + e^{-\tau x_1} G_\tau e^{\tau x_1} q$ to w, we get that

$$w + e^{-\tau x_1} G_\tau e^{\tau x_1} q w = u.$$

Since $e^{\tau x_1}(-\Delta_g) e^{-\tau x_1} G_\tau = \operatorname{Id}$ (by Theorem 2.3.2), we get $(-\Delta_g) e^{-\tau x_1} G_\tau e^{\tau x_1} = \operatorname{Id}$ and hence $(-\Delta_g + q) w = 0$. ☐

2.6 Complex geometrical optics solutions

Let $q \in L^\infty(M)$ be such that 0 is not a Dirichlet eigenvalue of $-\Delta_g + q$ in M, and let $\tau \in \mathbb{R}$ with $|\tau| \geq \tau_0$. In this section we construct the complex geometrical optics solutions for the Schrödinger equation $(-\Delta_g + q)u = 0$ in M whose trace is supported in $\Gamma_{\operatorname{sgn}(\tau)}$.

2.6.1 Solution operator

To construct the complex geometrical optics solutions, we need to generalize Proposition 2.3.4 to the case when the solution is determined on S_τ^-.

Set $\mathcal{D} = \{\psi \in C^\infty(M) : \mathtt{tr}(\psi) = 0\}$ and define

$$M_\tau = \{(e^{-\tau x_1}(-\Delta_g) e^{\tau x_1} \psi, \mathtt{tr}_\nu(\psi)|_{S_\tau^+}) : \psi \in \mathcal{D}\} \subset L^2(M) \times L^2(S_\tau^+).$$

Lemma 2.6.1. *For $\tau \in \mathbb{R}$ with $|\tau| > 0$, let $(u, u_\tau^+) \in L^2(M) \times L^2(S_\tau^+)$. Then (u, u_τ^+) is in orthogonal to the closure of M_τ if and only if $e^{\tau x_1}(-\Delta_g) e^{-\tau x_1} u = 0$, $\mathtt{tr}(u)|_{S_\tau^-} = 0$ and $\mathtt{tr}(u)|_{S_\tau^+} = u_\tau^+$.*

Proof. Suppose that (u, u_τ^+) is orthogonal to M_τ. Then for $\psi \in \mathcal{D}$, we have

$$(u|e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}\psi)_{L^2(M)} + (u_\tau^+|\mathtt{tr}_\nu(\psi)|_{S_\tau^+})_{S_\tau^+} = 0.$$

Taking $\psi \in C_0^\infty(M^{\mathrm{int}})$, this gives $e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}u = 0$.

Now, consider arbitrary $\psi \in \mathcal{D}$. Using the generalized Green's identity from Corollary 2.2.2, we get

$$(u|e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}\psi)_{L^2(M)} = -\langle \mathtt{tr}(u), \mathtt{tr}_\nu(\psi)\rangle_{H^{-1/2,1/2}(\partial M)}.$$

Combining this together with the previous equality gives that $\mathtt{tr}(u)|_{S_\tau^-} = 0$ and $\mathtt{tr}(u)|_{S_\tau^+} = u_\tau^+$.

To prove the converse, suppose that (u, u_τ^+) is such that $e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}u = 0$, $\mathtt{tr}(u)|_{S_\tau^-} = 0$ and $\mathtt{tr}(u)|_{S_\tau^+} = u_\tau^+$. Then for $\psi \in \mathcal{D}$, we have

$$\begin{aligned}
&(u|e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}\psi)_{L^2(M)} \\
&\qquad = (e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}u|\psi)_{L^2(M)} - \langle \mathtt{tr}(u), \mathtt{tr}_\nu(\psi)\rangle_{H^{-1/2,1/2}(\partial M)} \\
&\qquad = -\langle \mathtt{tr}(u), \mathtt{tr}_\nu(\psi)\rangle_{H^{-1/2,1/2}(\partial M)} \\
&\qquad = -(u_\tau^+|\mathtt{tr}_\nu(\psi)|_{S_\tau^+})_{S_\tau^+},
\end{aligned}$$

which means that (u, u_τ^+) is orthogonal to M_τ. □

Let us denote by m_τ the operator of orthogonal projection onto the closure of M_τ in $L^2(M) \times L^2(S_\tau^+)$.

Proposition 2.6.2. *Let (M, g) be an admissible manifold. There are constants $C_0, \tau_0 > 0$ such that for all $\tau \in \mathbb{R}$ with $|\tau| \geq \tau_0$, $\delta > 0$ and for given $f \in L^2(M)$ and $f_\tau^- \in L^2(S_\tau^-)$, there exists a unique solution $u \in L^2(M)$ of the equation*

$$e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}u = f \quad \text{in} \quad M$$

such that $\mathtt{tr}(u)|_{S_\tau^-} = f_\tau^-$, $m_\tau(u, \mathtt{tr}(u)|_{S_\tau^+}) = (u, \mathtt{tr}(u)|_{S_\tau^+})$ and

$$\|u\|_{L^2(M)} \leq C_0 \frac{1}{|\tau|}\|f\|_{L^2(M)} + C_0 \frac{1}{(\delta|\tau|)^{1/2}}\|f_\tau^-\|_{L^2(S_{\tau,\delta}^-)} + C_0\|f_\tau^-\|_{L^2(S_{\tau,\delta}^0)}.$$

Proof. Define a linear functional $l : M_\tau \to \mathbb{C}$ by

$$l(e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}\psi, \mathtt{tr}_\nu(\psi)|_{S_\tau^+}) = (f|\psi)_{L^2(M)} - \left(f_\tau^-|\mathtt{tr}_\nu(\psi)\right)_{S_\tau^-}.$$

On the orthogonal complement of M_τ we define l to be zero. By the Carleman estimate (2.4), we have

$$\begin{aligned}
&|l(e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}\psi, \mathtt{tr}_\nu(\psi)|_{S_\tau^+})| \\
&\quad \le \|f\|_{L^2(M)}\|\psi\|_{L^2(M)} + \|f_\tau^-\|_{S_{\tau,\delta}^-}\|\mathtt{tr}_\nu(\psi)\|_{S_{\tau,\delta}^-} + \|f_\tau^-\|_{S_{\tau,\delta}^0}\|\mathtt{tr}_\nu(\psi)\|_{S_{\tau,\delta}^0} \\
&\quad \le C_0\left(\frac{1}{|\tau|}\|f\|_{L^2(M)} + \frac{1}{(\delta|\tau|)^{1/2}}\|f_\tau^-\|_{L^2(S_{\tau,\delta}^-)} + \|f_\tau^-\|_{L^2(S_{\tau,\delta}^0)}\right) \\
&\qquad \times \left(\|e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}\psi\|_{L^2(M)} + |\tau|^{1/2}\|\mathtt{tr}_\nu(\psi)\|_{S_\tau^+}\right).
\end{aligned}$$

By Riesz representation theorem, there is $(u, u_\tau^+) \in L^2(M) \times L^2(S_\tau^+)$ such that

$$l(w, w_\tau^+) = (u|w)_{L^2(M)} + (u_\tau^+|w_\tau^+)_{S_\tau^+},$$

for $(w, w_\tau^+) \in L^2(M) \times L^2(S_\tau^+)$. In particular, for $\psi \in \mathcal{D}$, we have

$$(u|e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}\psi)_{L^2(M)} + (u_\tau^+|\mathtt{tr}_\nu(\psi))_{S_\tau^+} = (f|\psi)_{L^2(M)} - \left(f_\tau^-|\mathtt{tr}_\nu(\psi)\right)_{S_\tau^-}.$$

Taking $\psi \in C_0^\infty(M^{\mathrm{int}})$, gives that $e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}u = f$. Moreover,

$$\|u\|_{L^2(M)} \le C_0\frac{1}{|\tau|}\|f\|_{L^2(M)} + C_0\frac{1}{(\delta|\tau|)^{1/2}}\|f_\tau^-\|_{L^2(S_{\tau,\delta}^-)} + C_0\|f_\tau^-\|_{L^2(S_{\tau,\delta}^0)}.$$

Since $l \equiv 0$ on the orthogonal complement of M_τ in $L^2(M)\times L^2(S_\tau^+)$, we have that $(u, \mathtt{tr}(u)|_{S_\tau^+})$ is in the closure of M_τ and hence $m_\tau(u, \mathtt{tr}(u)|_{S_\tau^+}) = (u, \mathtt{tr}(u)|_{S_\tau^+})$.

For arbitrary $\psi \in \mathcal{D}$, using the generalized Green's identity from Corollary 2.2.2, we get

$$(u|e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}\psi)_{L^2(M)} + \langle \mathtt{tr}(u), \mathtt{tr}_\nu(\psi)\rangle_{H^{-1/2,1/2}(\partial M)} = (f|\psi)_{L^2(M)}.$$

Comparing this with the previous equality, this gives that $\mathtt{tr}(u)|_{S_\tau^-} = f_\tau^-$ and $\mathtt{tr}(u)|_{S_\tau^+} = u_\tau^+$.

Now, we prove uniqueness. Suppose that $u' \in L^2(M)$ is another solution of the equation $e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}u' = f$ satisfying all the conditions of the proposition. Then $e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}(u-$

$u') = 0$, $\mathtt{tr}(u-u')|_{S_\tau^-} = 0$, $\mathtt{tr}(u-u')|_{S_\tau^+} = u_\tau^+ - u'^+_\tau$, and $(u-u', u_\tau^+ - u'^+_\tau)$ is in the closure of M_τ. However, by Lemma 2.6.1, $(u-u', u_\tau^+ - u'^+_\tau)$ is orthogonal to the closure of M_τ. Thus, we obtain $u - u' = 0$ which finishes the proof. □

Let $R_\tau : L^2(M) \times L^2(S_\tau^-) \to L^2(M)$ be the solution operator obtained in the previous result. In other words, the operator R_τ is defined by $R_\tau(f, f_\tau^-) = u$, where u, f, f_τ^- are as in Proposition 2.6.2.

2.6.2 Construction of complex geometrical optics solutions

Now, we are ready to construct complex geometrical optics solutions whose traces are supported in $\Gamma_{\mathrm{sgn}(\tau)}$. These are the solutions of the form

$$u = e^{-\tau x_1}(a + r_0), \tag{2.9}$$

where r_0 is a correction term and a is an amplitude.

Construction in the case $q = 0$

Recall that the transversal manifold (M_0, g_0) is assumed to be simple. Let $\gamma : [0, T] \to M_0$ be the given geodesic in (M_0, g_0) with endpoints on E. Choose another simple manifold $(\widetilde{M}_0, g_0)$ such that $(M_0, g_0) \subset\subset (\widetilde{M}_0^{\mathrm{int}}, g_0)$ and extend the geodesic γ in $\widetilde{M}_0$. Choose $\varepsilon > 0$ such that $\gamma(t) \in \widetilde{M}_0 \setminus M_0$ for all $t \in (-2\varepsilon, 0) \cup (T, 2\varepsilon)$ and set $p = \gamma(-\varepsilon)$ which is in $\widetilde{M}_0 \setminus M_0$. Simplicity of $(\widetilde{M}_0, g_0)$ implies that there are globally defined polar coordinates (r, θ) centered at p. In these polar coordinates γ corresponds to $r \mapsto (r, \theta_0)$ for some $\theta_0 \in S^{n-2}$. Following [17, Section 5.2], we choose the following specific a:

$$a(x_1, r, \theta) = e^{-i\tau r}|g|^{-1/4}c^{1/2}e^{i\lambda(x_1+ir)}b(\theta),$$

where $\lambda \in \mathbb{R}$ and $b \in C^\infty(S^{n-2})$ is fixed such that b is supported near θ_0 so that $a = 0$ near $\partial M_0 \setminus E$.

Assume now that u has the required form (2.9). Then the equation $(-\Delta_g)u_0 = 0$ is equivalent to

$$e^{\tau x_1}(-\Delta_g)e^{-\tau x_1} r_0 = f, \tag{2.10}$$

where $f := e^{\tau x_1}\Delta_g e^{-\tau x_1} a$. Set $\Phi = x_1 + ir$. Then a starightforward calculation shows that

$$\begin{aligned} f &= -e^{i\tau r} e^{\tau\Phi}(-\Delta_g)e^{-\tau\Phi}|g|^{-1/4}c^{1/2}e^{i\lambda(x_1+ir)}b(\theta) \\ &= -e^{i\tau r}[-\tau^2\langle d\Phi, d\Phi\rangle_g + \tau(2\langle d\Phi, d\cdot\rangle_g + \Delta_g\Phi) - \Delta_g](|g|^{-1/4}c^{1/2}e^{i\lambda(x_1+ir)}b(\theta)). \end{aligned}$$

Here the Riemannian inner product $\langle\cdot,\cdot\rangle_g$ was extended as a complex bilinear form acting on complex valued 1-forms. It was shown in [17, Section 5] that $\langle d\Phi, d\Phi\rangle_g = 0$ and $(2\langle d\Phi, d\cdot\rangle_g + \Delta_g\Phi)(|g|^{-1/4}e^{i\lambda(x_1+ir)}b(\theta)) = 0$. Hence, we get

$$f = -e^{i\tau r}(-\Delta_g)(|g|^{-1/4}c^{1/2}e^{i\lambda(x_1+ir)}b(\theta)). \tag{2.11}$$

This shows that $\|f\|_{L^2(M)} \lesssim 1$ as $\tau \to \infty$.

We want to ensure that $\mathtt{tr}(u_0)$ is supported in $\Gamma_{\mathrm{sgn}(\tau)}$ where $\Gamma_{\mathrm{sgn}(\tau)} \supset \partial M_{\mathrm{sgn}(\tau)} \cup \Gamma_{\mathrm{a}}$. To achieve this, following [29], we take a small parameter $\delta > 0$ to be chosen later, and define the following sets

$$V^{\tau,\delta} := \{x \in S_\tau^- : \mathrm{dist}_{\partial M}(x, \Gamma_{\mathrm{i}}) < \delta \text{ or } x \in \partial M_{-\mathrm{sgn}(\tau)}\}, \quad \Gamma_{\mathrm{a}}^{\tau,\delta} := S_\tau^- \setminus V^{\tau,\delta}.$$

Note that $\partial M_{\mathrm{sgn}(\tau)} \cup \partial M_{\tan} \subset (S_\tau^-)^{\mathrm{int}}$. For the boundary condition, we set

$$f_{\tau,\delta}^- := \begin{cases} -a & \text{on } V^{\tau,\delta}, \\ 0 & \text{on } \Gamma_{\mathrm{a}}^{\tau,\delta}. \end{cases}$$

Defining $f_{\tau,\delta}^-$ in such a way, we have $f_{\tau,\delta}^-|_{\Gamma_{\mathrm{a}}^{\tau,\delta}\cap\partial M_{\tan}} = 0$. Recall that $\Gamma_{\mathrm{i}} \subset \mathbb{R} \times (\partial M_0 \setminus E)$ and a was chosen in a way to satisfy $a = 0$ near $\partial M_0 \setminus E$. Therefore, $f_{\tau,\delta}^-|_{V^{\tau,\delta}\cap\partial M_{\tan}} = 0$, and hence we have

$$f_{\tau,\delta}^-|_{\partial M_{\tan}} = 0.$$

Since $\|f_{\tau,\delta}^-\|_{L^\infty(S_\tau^-)} \lesssim 1$, we obtain the following estimates

$$\|f_{\tau,\delta}^-\|_{L^2(S_{\tau,\delta}^-)} \lesssim \sigma_{\partial M}(S_{\tau,\delta}^-)$$

and

$$\|f^-_{\tau,\delta}\|_{L^2(S^0_{\tau,\delta})} \lesssim \sigma_{\partial M}\left(\{x \in \partial M : -\delta < \operatorname{sgn}(\tau)\partial_\nu\varphi(x) < 0\}\right) \\ + \sigma_{\partial M}\left(\{x \in \partial M : 0 < \operatorname{sgn}(\tau)\partial_\nu\varphi(x) < (3|\tau|)^{-1}\}\right).$$

If we set

$$r_0 = R_\tau(f, f^-_{\tau,\delta}),$$

then, by Proposition 2.6.2, r_0 solves (2.10) with $\mathtt{tr}(r_0)|_{S^-_\tau} = f^-_{\tau,\delta}$ and satisfies

$$\|r_0\|_{L^2(M)} \lesssim \frac{1}{|\tau|} + \frac{1}{(\delta|\tau|)^{1/2}}\sigma_{\partial M}(S^-_{\tau,\delta}) \\ + \sigma_{\partial M}\left(\{x \in \partial M : -\delta < \operatorname{sgn}(\tau)\partial_\nu\varphi(x) < 0\}\right) \\ + \sigma_{\partial M}\left(\{x \in \partial M : 0 < \operatorname{sgn}(\tau)\partial_\nu\varphi(x) < (3|\tau|)^{-1}\}\right).$$

Thus, there is constant $C_0 > 0$ such that

$$\|r_0\|_{L^2(M)} \leq C_0\left(\frac{1}{|\tau|} + \frac{1}{(\delta|\tau|)^{1/2}} + o_{\tau\to\infty}(1) + o_{\delta\to 0}(1)\right).$$

We choose δ such that $C_0 o_{\delta\to 0}(1) \leq \varepsilon/2$. Then we take $|\tau| \geq \tau_0$ large enough so that

$$C_0\left(\frac{1}{|\tau|} + \frac{1}{(\delta|\tau|)^{1/2}} + o_{\tau\to\infty}(1)\right) \leq \varepsilon/2.$$

Therefore, we get $\|r_0\|_{L^2(M)} \to 0$ as $\tau \to \infty$. This will give the complex geometrical optics solution $u_0 = e^{-\tau x_1}(a + r_0)$ to $(-\Delta_g)u_0 = 0$ whose trace is supported in $\Gamma_{\operatorname{sgn}(\tau)}$. Thus, we have proved the following proposition.

Proposition 2.6.3. *Let (M, g) be an admissible manifold. Suppose that $\gamma : [0, T] \to M_0$ is a given geodesic in (M_0, g_0) with endpoints on E, and let $\theta_0 \in S^{n-2}$ be as in the begining of Section 2.6.2. For $\tau \in \mathbb{R}$ with $|\tau| \geq \tau_0$ and $\delta > 0$, for any $\lambda \in \mathbb{R}$ and for any $b \in C^\infty(S^{n-2})$ supported sufficiently close to θ_0, there is a solution $u_0 \in H_{\Delta_g}(M)$ to the equation $(-\Delta_g)u_0 = 0$ of the form*

$$u_0 = e^{-\tau x_1}(a + r_0),$$

and satisfying

$$\operatorname{supp}(\mathtt{tr}(u_0)) \subset \Gamma_{\operatorname{sgn}(\tau)}$$

where

$$a = e^{-i\tau r}|g|^{-1/4}c^{1/2}e^{i\lambda(x_1+ir)}b(\theta),$$

and $\|r_0\|_{L^2(M)} \to 0$ *as* $\tau \to \infty$.

Remark 2.6.4. Modifying the above arguments in appropriate places, one can construct complex geometrical optics solutions whose traces are supported in $\partial M_{\operatorname{sgn}(\tau)}$ if $\partial M_{\tan}$ has zero measure in ∂M. Let us indicate these modifications. Up to (2.11) everything is same except that we do not put any restrictions on b, so that we do not require a to vanish on any part of the boundary. In order to ensure that $\operatorname{supp}(\mathtt{tr}(u)) \subset \partial M_{\operatorname{sgn}(\tau)}$, for fixed $\delta > 0$, we set

$$f^-_{\tau,\delta} := -a.$$

Since $\|f^-_{\tau,\delta}\|_{L^\infty(S^-_\tau)} \lesssim 1$ and $\sigma_{\partial M}(\partial M_{\tan}) = 0$, we obtain the following estimates

$$\|f^-_{\tau,\delta}\|_{L^2(S^-_{\tau,\delta})} \lesssim \sigma_{\partial M}(S^-_{\tau,\delta})$$

and

$$\begin{aligned}\|f^-_{\tau,\delta}\|_{L^2(S^0_{\tau,\delta})} \lesssim \sigma_{\partial M}\left(\{x \in \partial M : -\delta < \operatorname{sgn}(\tau)\partial_\nu\varphi(x) < 0\}\right) \\ + \sigma_{\partial M}\left(\{x \in \partial M : 0 < \operatorname{sgn}(\tau)\partial_\nu\varphi(x) < (3|\tau|)^{-1}\}\right).\end{aligned}$$

We use Proposition 2.6.2 to solve (2.10) for r_0 with $\mathtt{tr}(r_0)|_{S^-_\tau} = f^-_{\tau,\delta}$ and to show that r_0 satisfies the same estimate as before for some $C_0 > 0$ constant:

$$\|r_0\|_{L^2(M)} \leq C_0\left(\frac{1}{|\tau|} + \frac{1}{(\delta|\tau|)^{1/2}} + o_{\tau\to\infty}(1) + o_{\delta\to 0}(1)\right).$$

Thus, we have constructed the complex geometrical optics solution $u_0 \in H_{\Delta_g}(M)$ to $(-\Delta_g)u_0 = 0$ of the form

$$u_0 = e^{-\tau x_1}(a + r_0)$$

whose trace is supported in $\partial M_{\operatorname{sgn}(\tau)}$ and $\|r_0\|_{L^2(M)} \to 0$ as $\tau \to \infty$.

Construction for general q

Next, we construct complex geometrical optics solutions for the Schrödinger equation $(-\Delta_g + q)u = 0$ in M with $q \in L^\infty(M)$ such that $\operatorname{supp}(\mathtt{tr}(u)) \subset \Gamma_{\operatorname{sgn}(\tau)}$.

Proposition 2.6.5. *Let (M, g) be an admissible manifold and let $q \in L^\infty(M)$ be such that 0 is not a Dirichlet eigenvalue of $-\Delta_g + q$ in M. Suppose that $\gamma : [0, T] \to M_0$ is a given geodesic in (M_0, g_0) with endpoints on E. For any $\tau \in \mathbb{R}$ with $|\tau| \geq \tau_0$, there is a solution $u \in L^2(M)$ to the equation $(-\Delta_g + q)u = 0$ of the form*

$$u = u_0 + e^{-\tau x_1} r_1,$$

where u_0 is as in Proposition 2.6.3 and one has

$$(\operatorname{Id} + G_\tau \circ q) r_1 = -G_\tau q e^{\tau x_1} u_0,$$

and $\|r_1\|_{L^2(M)} \lesssim \frac{1}{|\tau|}$ as $\tau \to \infty$. Moreover, $\mathtt{tr}(u)$ is supported in $\Gamma_{\operatorname{sgn}(\tau)}$.

Proof. Consider the following integral equation

$$(\operatorname{Id} + G_\tau \circ q) r_1 = -G_\tau q e^{\tau x_1} u_0. \tag{2.12}$$

Since $q \in L^\infty(M)$ and $\|G_\tau\|_{L^2(M) \to L^2(M)} \lesssim \frac{1}{|\tau|}$ by Theorem 2.3.2, for sufficiently large this integral equation has a unique solution $r_1 = -(\operatorname{Id} + G_\tau \circ q)^{-1} G_\tau q e^{\tau x_1} u_0$ in terms of the convergent Neumann series. Then $\|G_\tau\|_{L^2(M) \to L^2(M)} \lesssim \frac{1}{|\tau|}$ implies that $\|r_1\|_{L^2(M)} \lesssim \frac{1}{|\tau|}$. Using the fact that $(-\Delta_g) e^{-\tau x_1} G_\tau = e^{-\tau x_1}$ (by Theorem 2.3.2) and that $(-\Delta_g) u_0 = 0$, and using (2.12), we can show

$$\begin{aligned}
(-\Delta_g + q)u &= (-\Delta_g + q)u_0 + (-\Delta_g + q) e^{-\tau x_1} r_1 \\
&= q u_0 + (-\Delta_g + q)(-e^{-\tau x_1} G_\tau q r_1 - e^{-\tau x_1} G_\tau q e^{\tau x_1} u_0) \\
&= q u_0 - e^{-\tau x_1} q r_1 - q u_0 - q e^{-\tau x_1} G_\tau q r_1 - q e^{-\tau x_1} G_\tau q e^{\tau x_1} u_0 \\
&= -e^{-\tau x_1} q (\operatorname{Id} + G_\tau \circ q) r_1 - q e^{-\tau x_1} G_\tau q e^{\tau x_1} u_0 \\
&= e^{-\tau x_1} q G_\tau q e^{\tau x_1} u_0 - q e^{-\tau x_1} G_\tau q e^{\tau x_1} u_0 \\
&= 0.
\end{aligned}$$

Let us now prove the last part of the proposition. By Proposition 2.6.3, we have that $\mathtt{tr}(u_0)$ is supported in $\Gamma_{\mathrm{sgn}(\tau)}$. Note that Theorem 2.3.2 implies that $\mathtt{tr}(G_\tau q r_1)$ is supported in S_τ^+. These, together with (2.12) imply that the trace of $u = u_0 + e^{-\tau x_1} r_1$ is supported in $\Gamma_{\mathrm{sgn}(\tau)}$. □

Remark 2.6.6. If $\partial M_{\tan}$ has zero measure in ∂M, one can replace u_0 in the above proposition with the one obtained in Remark 2.6.4. Then the proof of Proposition 2.6.5 shows that so-obtained complex geometrical optics solution $u \in H_{\Delta_g}(M)$ to $(-\Delta_g + q)u_0 = 0$ of the form

$$u = u_0 + e^{-\tau x_1} r_1$$

has $\mathrm{supp}(\mathtt{tr}(u)) \subset \partial M_{\mathrm{sgn}(\tau)}$ and $\|r_1\|_{L^2(M)} \lesssim \frac{1}{|\tau|}$ as $\tau \to \infty$.

2.7 Proofs of the main results

Proof of Theorem 2.1.1. Suppose that $q \in C(M)$ such that 0 is not a Dirichlet eigenvalue of $-\Delta_g + q$. Assume the knowledge of (M, g) and $\Lambda_{g,q} f$ on Γ_- for all $f \in \mathcal{H}(\partial M)$ supported in Γ_+. Then by Proposition 2.2.5, the following integral identity holds

$$\langle \mathtt{tr}(u_2), (\Lambda_{g,q} - \Lambda_{g,0})\mathtt{tr}(u_1)\rangle_{H^{-1/2,1/2}(\partial M)} = (u_2|qu_1)_{L^2(M)}, \tag{2.13}$$

where $u_1 \in H_{\Delta_g}(M)$ is a solution of $(-\Delta_g + q)u_1 = 0$ in M with $\mathtt{tr}(u_1)$ supported in Γ_+, and $u_2 \in H_{\Delta_g}(M)$ is a solution of $(-\Delta_g)u_2 = 0$ in M with $\mathtt{tr}(u_2)$ supported in Γ_-.

Let $\tau \geq \tau_0$ and let $\gamma : [0, T] \to M_0$ be the given geodesic in (M_0, g_0) with endpoints on E. By Proposition 2.6.5, there is $u_1 \in H_{\Delta_g}(M)$ solving $(-\Delta_g + q)u_1 = 0$ in M with $\mathtt{tr}(u_1)$ supported in Γ_+, and having the form

$$u_1 = e^{-\tau x_1}(e^{-i\tau r}|g|^{-1/4}c^{1/2}e^{i\lambda(x_1+ir)}b(\theta) + r' + r_0) = u_1' + e^{-\tau x_1} r_0$$

where $\|r_0\|_{L^2(M)} \lesssim \frac{1}{|\tau|}$ and $\|r'\|_{L^2(M)} \to 0$ as $\tau \to +\infty$ (here u_1' is a solution to $(-\Delta_g)u_1' = 0$ as in Proposition 2.6.3).

By Proposition 2.6.3, there is a $\overline{u}_2 \in H_{\Delta_g}(M)$ solving $(-\Delta_g)\overline{u}_2 = 0$ in M with $\mathtt{tr}(\overline{u}_2)$ supported in Γ_-, and having the form

$$\overline{u}_2 = e^{\tau x_1}(e^{i\tau r}|g|^{-1/4}c^{1/2}e^{i\lambda(x_1+ir)} + r'')$$

where $\|r''\|_{L^2(M)} \to 0$ as $\tau \to +\infty$. Then u_2 will be the complex geometrical optics solutions to $(-\Delta_g)u_2 = 0$ in M with $\mathtt{tr}(u_2)$ supported in Γ_-, and having the form

$$u_2 = e^{\tau x_1}(e^{-i\tau r}|g|^{-1/4}c^{1/2}e^{-i\lambda(x_1-ir)} + \overline{r''})$$

The important thing to note is that u_1' as well as u_2 depend only on (M,g), i.e. independent on q. Since $\mathtt{tr}(u_1)$ is supported in Γ_+ and $\mathtt{tr}(u_2)$ is supported in Γ_-, the left hand-side of (2.13) requires only the given partial data of $\Lambda_{g,q}$.

Now, we show that $\mathtt{tr}(u_1)$ can be reconstructed from the above mentioned partial knowledge of $\Lambda_{g,q}$. By Proposition 2.6.5 and Proposition 2.5.2, one can check that $\mathtt{tr}(u_1)$ satisfies the following boundary integral equation

$$(\mathrm{Id} + S_\tau(\Lambda_{g,q} - \Lambda_{g,0}))\mathtt{tr}(u_1) = \mathtt{tr}(u_1').$$

Then Corollary 2.5.3 and Proposition 2.5.4 imply solvability of the above boundary integral equation for sufficiently large τ. Substituting this solution $\mathtt{tr}(u_1)$ into the left hand-side of (2.13), we can determine

$$(u_2|qu_1)_{L^2(M)}$$

for all complex geometrical optics solutions u_1, u_2 of the above form.

Using the decay properties of $r', r'', \tilde{r}$ and taking limit as $\tau \to \infty$, we can reconstruct

$$\int_M cq|g|^{-1/2}e^{2i\lambda(x_1+ir)}b(\theta)\, d\,\mathrm{Vol}_g\,.$$

Now we extend q as zero to $\mathbb{R} \times M_0$. Since $d\,\mathrm{Vol}_g = |g|^{1/2}dx_1\,dr\,d\theta$, the above expression becomes

$$\int_{S^{n-2}}\int_0^\infty e^{-2\lambda r}\left(\int_{-\infty}^\infty e^{2i\lambda x_1}(cq)(x_1,r,\theta)\,dx_1\right)\,dr\,d\theta.$$

Varying $b \in C^\infty(S^{n-2})$ so that the support of b is sufficiently close to θ_0 and noting that the term in the brackets is the one-dimensional Fourier transform of q with respect to the x_1-variable, which we denote by $\widehat{q}$, we determine

$$\int_0^\infty e^{-2\lambda r}\,\widehat{(cq)}(2\lambda, r, \theta_0)\,dr.$$

Recalling that $r \mapsto (r, \theta_0)$ corresponds to the given geodesic $\gamma : [0, T] \to M_0$ with endpoints on E, we finish the proof. □

Proof of Theorem 2.1.2. Assume that $O \subset M_0$ is open such that $O \cap \partial M_0 \subset E$ and the local geodesic ray transform is invertible on O. According to Theorem 2.1.1, we can constructively determine

$$\int_0^T e^{-2\lambda t} \widehat{(cq)}(2\lambda, \gamma(t))\, dt \tag{2.14}$$

for all geodesics $\gamma : [0, T] \to O$ with $\gamma(0), \gamma(T) \in E$. This is the local attenuated geodesic ray transform of $\widehat{(cq)}(2\lambda, \cdot)$ in O, with attenuation -2λ. Setting $\lambda = 0$, we determine an unattenuated local geodesic ray transform of $\widehat{(cq)}(0, \cdot)$ in O. Then using the constructive invertibility assumption for the local geodesic ray transform, we recover $\widehat{(cq)}(0, \cdot)$ in O.

Now, we go back to (2.14) and differentiate it with respect to λ at $\lambda = 0$. Since we have reconstructed $\widehat{(cq)}(0, \cdot)$, we constructively determine the local geodesic ray transform of $\left(\frac{\partial}{\partial \lambda} \widehat{(cq)}\right)(0, \cdot)$ in O. Using the invertibility assumption for the local geodesic ray transform again, we obtain $\left(\frac{\partial}{\partial \lambda} \widehat{(cq)}\right)(0, \cdot)$ in O.

Using this argument iteratively by taking higher derivatives of (2.14) with respect to λ, we can reconstruct

$$\left(\frac{\partial^k}{\partial \lambda^k} \widehat{(cq)}\right)(0, \cdot) \ \text{ in } O \ \text{ for all integers } \ k \geq 0.$$

Since q is compactly supported in x_1-variable, its Fourier transform $\widehat{(cq)}(\lambda, \cdot)$ is analytic with respect to λ. Therefore, we have reconstructed the Taylor series expansion of $\widehat{(cq)}(\lambda, \cdot)$ in O. Then we determine q in $M \cap (\mathbb{R} \times O)$ by inverting the one-dimensional Fourier transform of cq with respect to the x_1-variable. □

Proof of Theorem 2.1.3. Let (M, g) be a known admissible manifold such that $\partial M_{\tan}$ is of measure zero in ∂M. Suppose that $q \in C(M)$ such that 0 is not a Dirichlet eigenvalue of $-\Delta_g + q$. Assume the knowledge of $\Lambda_{g,q} f$ on ∂M_- for all $f \in \mathcal{H}(\partial M)$ supported in ∂M_+.

Using Remark 2.6.4 and Remark 2.6.6, as in the proof of Theorem 2.1.1, we can construct $u_1 \in H_{\Delta_g}(M)$ and $u_2 \in H_{\Delta_g}(M)$ solving $(-\Delta_g + q)u_1 = 0$ in M with $\mathtt{tr}(u_1)$ supported in

∂M_+ and solving $(-\Delta_g)u_2 = 0$ in M with $\mathtt{tr}(u_2)$ supported in ∂M_-, respectively, and having the forms

$$u_1 = e^{-\tau x_1}(e^{-i\tau r}|g|^{-1/4}c^{1/2}e^{i\lambda(x_1+ir)}b(\theta) + r' + r_0) = u_1' + e^{-\tau x_1}r_0,$$
$$u_2 = e^{\tau x_1}(e^{-i\tau r}|g|^{-1/4}c^{1/2}e^{-i\lambda(x_1-ir)} + \overline{r''}),$$

where $\|r_0\|_{L^2(M)} \lesssim \frac{1}{|\tau|}$, $\|r'\|_{L^2(M)} \to 0$ and $\|r''\|_{L^2(M)} \to 0$ as $\tau \to +\infty$ (here u_1' is a solution to $(-\Delta_g)u_1' = 0$ as in Remark 2.6.4).

Continuing as in the proof of Theorem 2.1.1, but replacing $\Gamma_\pm$ by $\partial M_\pm$, for any $\lambda \in \mathbb{R}$, we can constructively determine

$$\int_0^T e^{-2\lambda t}\widehat{(cq)}(2\lambda, \gamma(t))\, dt$$

for all geodesics $\gamma : [0, T] \to M_0$ in (M_0, g_0). Using the constructive invertibility assumption for the global geodesic ray transform, we reconstruct q in M via similar steps as in the proof of Theorem 2.1.2. $\square$

Chapter 3

INVARIANT DISTRIBUTIONS AND TENSOR TOMOGRAPHY FOR GAUSSIAN THERMOSTATS

3.1 Statement of results

Injectivity results for I_m

For the case of Gaussian thermostats we obtain several injectivity results of the thermostat ray transform under various assumptions. In order to state these results we need to introduce some notations.

Let (M, g, E) be a two-dimensional Gaussian thermostat. Since M is assumed to be oriented there is a circle action on the fibres of SM with infinitesimal generator V called the vertical vector field. Let X denote the generator of the geodesic flow of g. We complete X, V to a global frame of $T(SM)$ by defining the vector field $X_\perp := [V, X]$, where $[\cdot, \cdot]$ is the Lie bracket for vector fields. In this global frame, the generating vector field $\mathbf{G}_E$ for a Gaussian thermostat (M, g, E) equals $X + \lambda V$.

Define the *thermostat curvature* to be the quantity $\mathbb{K} := K - \operatorname{div}_g E$, where K is the Gaussian curvature of the surface (M, g). The quantity $\mathbb{K}$ can also be written as $K + X_\perp\lambda + \lambda^2 + \mathbf{G}_E V\lambda$. Notice that $\mathbb{K}$ is a smooth function on M. Following [45], we introduce a definition involving a modified thermostat Jacobi equation.

Definition 3.1.1. Let (M, g, E) be a Gaussian thermostat on a closed oriented Riemannian surface. We say that (M, g, E) *has no* β*-conjugate points* if for any thermostat geodesic γ, all non-trivial solutions to the β-Jacobi equation along γ

$$\ddot{y} - V(\lambda)\dot{y} + (\beta\mathbb{K} - \mathbf{G}_E V(\lambda))y = 0 \tag{3.1}$$

vanish at most once. The *terminator value* of (M, g, E) is defined to be

$$\beta_{\text{ter}} = \sup\{\beta \in [0, \infty] : (M, g, E) \text{ has no } \beta\text{-conjugate points}\}.$$

It is clear that 1-conjugate points are the same as usual conjugate points for thermostat geodesics (see [4, 42] for more details on the thermostat Jacobi equation).

Theorem 3.1.2. *Let (M, g, E) be an Anosov Gaussian thermostat on a closed oriented Riemannian surface. Assume that $\beta_{\text{ter}} \geq (m+1)/2$ for some integer $m \geq 2$, then I_m is s-injective.*

Theorem 3.1.2 generalizes the corresponding injectivity result in [45] which is for the geodesic ray transform. In particular, [45] showed the s-injectivity of I_2 on Anosov surfaces, before which it was only known for Anosov surfaces without focal points [57]. Recently Guillarmou [23] settled the tensor tomography problem on Anosov surfaces for tensor fields of any order. It was proved that s-injectivity of I_2 also holds on 2D Anosov magnetic surfaces [2]. The problem of proving s-injectivity of I_2 for 2D Anosov Gaussian thermostats without the assumption on terminator values is still open. The difficulty comes from the fact that in general $V(\lambda)$ is nonzero for Gaussian thermostats, see Section 3.2 for details.

The condition on β_{ter} is closely related to the works [10, 49] where absence of β-conjugate points also appears in the case of geodesic flows on manifolds with boundary. When the thermostat curvature is non-positive, i.e. $\mathbb{K} \leq 0$, it is not difficult to see that $\beta_{\text{ter}} = \infty$. We get the following result as a corollary of Theorem 3.1.2, and it generalizs an earlier result [27] which is for $m = 2$.

Corollary 3.1.3. *Let (M, g, E) be an Anosov Gaussian thermostat on a closed oriented Riemannian surface of non-positive thermostat curvature. Then I_m is s-injective for any integer $m \geq 2$.*

According to the result of Wojtkowski [65, Theorem 5.2] a Gaussian thermostat on a closed surface with negative thermostat curvature is always Anosov.

Corollary 3.1.4. *Let (M, g, E) be a Gaussian thermostat on a closed oriented Riemannian surface of negative thermostat curvature. Then I_m is s-injective for any integer $m \geq 2$.*

At the end of this chapter, we apply the ideas from Anosov Gaussian thermostats to study the injectivity of the thermostat ray transform on compact surfaces (M, g) with smooth boundaries. We will focus on a class of Gaussian thermostats which are called *simple* Gaussian thermostats (see Section 3.8 of this chapter for precise definition). Roughly speaking, simple Gaussian thermostats are the analogues of Anosov Gaussian thermostats for manifolds with boundary.

Simplicity is related to the boundary rigidity problem [36] which is a motivation for the tensor tomography problem. It was shown by Pestov and Uhlmann [51] that simple surfaces are boundary rigid. Later this rigidity result was generalized to 2D simple magnetic systems [14] and 2D simple systems involving magnetic fields and potentials [6].

Theorem 3.1.5. *Let (M, g, E) be a simple Gaussian thermostat on a compact oriented Riemannian surface with boundary. Assume that $\beta_{\text{ter}} \geq (m+1)/2$ for some integer $m \geq 2$, then I_m is s-injective.*

In particular, $\beta_{\text{ter}} = \infty$ when the thermostat curvature is non-positive.

Corollary 3.1.6. *Let (M, g, E) be a simple Gaussian thermostat on a compact oriented Riemannian surface with boundary of non-positive thermostat curvature. Then I_m is s-injective for any integer $m \geq 2$.*

The tensor tomography problems for simple surfaces [44] and 2D simple magnetic systems [1] were proved without curvature assumptions, using a different method which was developed for the boundary case. It is an interesting problem to show s-injectivity of $I_m, m \geq 2$ for simple Gaussian thermostats on surfaces.

For manifolds with boundaries, there are also local tensor tomography problems, i.e. whether one can determine a symmetric tensor near a boundary point, up to the natural obstruction, from its integrals along curves near this point? For manifolds of dimension three and higher, there are recent works by Uhlmann and Vasy [64], Stefanov, Uhlmann and Vasy [60] for the geodesic case, and Zhou [64, Appendix] for general smooth curves, including the thermostats. However, the local problem for surfaces is still open.

Invariant distributions

To study the adjoints, let us briefly introduce distributions on SM. Let γ be a closed thermostat geodesic and δ_γ denote the measure on SM which corresponds to integrating over $(\gamma, \dot{\gamma})$ on SM. We can define the thermostat ray transform by the distributional pairing

$$I\varphi(\gamma) = \langle \delta_\gamma, \varphi \rangle.$$

Denote by $\mathcal{D}'(SM)$ the space of distributions on $C^\infty(SM)$. Both of these spaces are reflexive, so the dual of $\mathcal{D}'(SM)$ is $C^\infty(SM)$. Any differential operator P can act on a distribution $\mu \in \mathcal{D}'(SM)$ via duality, that is $\langle P\mu, \varphi \rangle := \langle \mu, P^*\varphi \rangle$ for any $\varphi \in C^\infty(SM)$. Since $\mathbf{G}_E = -(\mathbf{G}_E + V(\lambda))^*$ (see Section 3.2), we define the following subspace of $\mathcal{D}'(SM)$:

$$\mathcal{D}'_{\text{inv}}(SM) := \{\mu \in \mathcal{D}'(SM) : (\mathbf{G}_E + V(\lambda))\mu = 0\}.$$

Hence a distribution μ is in $\mathcal{D}'_{\text{inv}}(SM)$ if and only if $\langle \mu, \mathbf{G}_E\varphi \rangle = 0$ for all $\varphi \in C^\infty(SM)$. This agrees with the definition of the thermostat ray transform given by the distributional pairing.

Without loss of generality we can consider the thermostat ray transform I as the map

$$I : C^\infty(SM) \to L(\mathcal{D}'_{\text{inv}}(SM), \mathbb{R}), \quad I\varphi(\mu) = \langle \mu, \varphi \rangle \text{ for } \mu \in \mathcal{D}'_{\text{inv}}(SM).$$

By $L(F, \mathbb{R})$ we mean the space of continuous linear maps from a locally convex topological space F to $\mathbb{R}$. Equip this space with the weak* topology, then I becomes a continuous linear map from a Frechét space into $L(\mathcal{D}'_{\text{inv}}(SM), \mathbb{R})$ which is locally convex. Since $\mathcal{D}'_{\text{inv}}(SM)$ is a closed subspace of a reflexive space $\mathcal{D}'(SM)$, it is also reflexive. Therefore, the dual of $L(\mathcal{D}'_{\text{inv}}(SM), \mathbb{R})$ is the space of invariant distributions $\mathcal{D}'_{\text{inv}}(SM)$. This implies that the adjoint of the thermostat ray transform I is the map

$$I^* : \mathcal{D}'_{\text{inv}}(SM) \to \mathcal{D}'(SM), \quad \langle I^*\mu, \varphi \rangle = \langle \mu, I\varphi \rangle \text{ for } \varphi \in C^\infty(SM).$$

On an oriented surface any $u \in C^\infty(SM)$ admits a Fourier expansion $u = \sum_{m\in\mathbb{Z}} u_m$ (see Section 3.2) where

$$u_m(x,v) := \frac{1}{2\pi}\int_0^{2\pi} u(\rho_t(x,v))e^{-imt}\,dt,$$

and ρ_t is the flow generated by V. One can use duality to decompose a distribution into its Fourier components. That is, if $\mu \in \mathcal{D}'(SM)$ then $\langle \mu_k, \varphi\rangle = \langle \mu, \varphi_k\rangle$ for all $\varphi \in C^\infty(SM)$. Now we can give the statements of our results which express the surjectivities of I_0^* and I_1^* in terms of the existence of some invariant distributions.

Theorem 3.1.7. *Let (M, g, E) be an Anosov Gaussian thermostat on a closed oriented Riemannian surface. Given $f \in C^\infty(M)$, there exists $w \in H^{-1}(SM)$ with $(\mathbf{G}_E + V(\lambda))w = 0$ and $w_0 = f$.*

As was explained in [45], by the ergodicity of Anosov flows, the only L^2 solutions to $Xw = 0$ on geodesic flows are constants. Therefore, the optimal regularity that we can expect for solutions to $(\mathbf{G}_E + V(\lambda))w = 0$ is H^{-1}.

Theorem 3.1.8. *Let (M, g, E) be an Anosov Gaussian thermostat on a closed oriented Riemannian surface. For a given solenoidal 1-form α (i.e. divergence free), there exists $w \in H^{-1}(SM)$ with $(\mathbf{G}_E + V(\lambda))w = 0$ and $w_{-1} + w_1 = \alpha$.*

One can consider the surjectivity of I_m^* for $m \geq 2$, however the constraint on m-tensors may not have explicit geometric meanings as that in the geodesic case. One can also derive surjectivity results on surfaces with boundaries by similar techniques. For the boundary case one should expect to show the existence of smooth invariant functions. This is known for I_0^* and I_1^* on simple manifolds of any dimension, see [51] and [16]. For I_m^*, $m \geq 2$, there are results on simple surfaces [46].

Finally, it's also worth pointing out that recently Paternain, Salo and Uhlmann generalized the techniques for the study of I and I^* on Anosov surfaces to higher dimensional Anosov and simple manifolds [47].

3.2 Pestov identity

Note that we have a global frame $\{X, X_\perp, V\}$ for $T(SM)$, which satisfies the structure equations given by $X = [V, X_\perp]$, $X_\perp = [X, V]$ and $[X, X_\perp] = -KV$ where K is the Gaussian curvature of the surface. Using this frame we can define a Riemannian metric on SM by declaring $\{X, X_\perp, V\}$ to be an orthonormal basis and the volume form of this metric will be denoted by $d\Sigma^3$.

Recall the generating vector field of a Gaussian thermostat (M, g, E) is $\mathbf{G}_E = X + \lambda V$. The fact that $X, X_\perp, V$ are volume preserving implies the following lemma which was proved in [13, Lemma 3.2].

Lemma 3.2.1. *Let (M, g, E) be a Gaussian thermostat on a closed oriented Riemannian surface. Then the following hold:*

$$L_{\mathbf{G}_E}\, d\Sigma^3 = V(\lambda)\, d\Sigma^3, \quad L_{X_\perp}\, d\Sigma^3 = 0, \quad L_V\, d\Sigma^3 = 0,$$

where L_Z denotes the Lie derivative along the vector field Z.

For any two functions $u, v : SM \to \mathbb{C}$ define the L^2 inner product:

$$(u, v) := \int_{SM} u\bar{v}\, d\Sigma^3,$$

the corresponding norm will be denoted by $\|\cdot\|$.

The space $L^2(SM)$ decomposes orthogonally as a direct sum

$$L^2(SM) = \bigoplus_{k\in\mathbb{Z}} H_k$$

where H_k is the eigenspace of $-iV$ corresponding to the eigenvalue k. A function $u \in L^2(SM)$ has a Fourier series expansion

$$u = \sum_{k=-\infty}^{\infty} u_k$$

where $u_k \in H_k$, then $\|u\|^2 = \Sigma\|u_k\|^2$ with $\|u\|^2 = (u, u)^{1/2}$. We denote the subspace $\Omega_k := H_k \cap C^\infty(SM)$.

Consider the isothermal coordinates (x, y) on the surface (M, g) such that the metric can be written as $ds^2 = e^{2\rho}(dx^2 + dy^2)$ where $\rho \in C^\infty(M, \mathbb{R})$. This gives coordinates (x, y, φ) on SM where φ is the angle between a unit vector v and $\frac{\partial}{\partial x}$. In these coordinates, the elements in the Fourier expansion of $f = f(x, y, \varphi)$ are given by

$$f_k(x, y, \varphi) = \Big(\frac{1}{2\pi}\int_0^{2\pi} f(x, y, \varphi')e^{ik\varphi'}\, d\varphi'\Big)e^{ik\varphi}.$$

In particular, for a given symmetric tensor field f of order m, $f_k = 0$ for $|k| \geq m + 1$.

We define the H^1-norm of a function $u \in C^\infty(SM)$ as

$$\|u\|^2_{H^1(SM)} := \|\mathbf{G}_E u\|^2 + \|X_\perp u - V(\lambda)Vu\|^2 + \|Vu\|^2 + \|u\|^2.$$

Notice that $\|u\|^2_{H^1(SM)}$ is equivalent to the standard H^1-norm $\|u\|^2 + \|\nabla u\|^2$, where $\nabla u = (Xu, X_\perp u, Vu)$.

Lemma 3.2.2. *Let (M, g, E) be a Gaussian thermostat on a closed oriented Riemannian surface. For any two functions $u, v \in C^\infty(SM, \mathbb{C})$ the following hold*

$$(Vu, v) = -(u, Vv), \quad (X_\perp u, v) = -(u, X_\perp v)$$

and

$$(\mathbf{G}_E u, v) = -(u, \mathbf{G}_E v) - (V(\lambda)u, v).$$

Proof. We will use the following consequence of Stokes' theorem. Let N be a closed oriented manifold and Θ be a volume form. Let $\mathfrak{X}$ be a vector field on N and $f \in C^\infty(N)$. Then the following holds

$$\int_N \mathfrak{X}(f)\Theta = -\int_N fL_{\mathfrak{X}}\Theta. \tag{3.2}$$

Now, the statement of the lemma is the consquence of Lemma 3.2.1 and (3.2). □

In particular, Lemma 3.2.2 implies the following expressions for the adjoints

$$X_\perp^* = -X_\perp, \quad V^* = -V, \quad \mathbf{G}_E^* = -(\mathbf{G}_E + V(\lambda)).$$

The following integral identity will play a fundamental role in our arguments. Its proof can be found in [13, Theorem 3.3], which is valid for more general thermostats.

Theorem 3.2.3 (Pestov identity). *Let (M, g, E) be a Gaussian thermostat on a closed oriented Riemannian surface. If $u \in C^\infty(SM, \mathbb{C})$, then*

$$\|\mathbf{G}_E V u\|^2 - (\mathbb{K} V u, V u) = \|V \mathbf{G}_E u\|^2 - \|\mathbf{G}_E u\|^2.$$

Remark 3.2.4. The Pestov identity above also holds for Gaussian thermostats on a compact oriented surfaces with smooth boundaries provided that $u|_{\partial SM} = 0$.

3.3 α-controlled thermostats

For $\alpha \in [0, 1]$, we say that a Gaussian thermostat (M, g, E) on a closed surface is *α-controlled* if for any $u \in C^\infty(SM)$ (for compact surfaces with boundaries, we additionally assume $u|_{\partial(SM)} = 0$) the following holds

$$\|\mathbf{G}_E u\|^2 - (\mathbb{K} u, u) \geq \alpha \|\mathbf{G}_E u\|^2.$$

It is obvious that if $\mathbb{K} \leq 0$, then (M, g, E) is 1-controlled.

Theorem 3.3.1. *Let (M, g, E) be an Anosov Gaussian thermostat on a closed surface. Then there is an $\alpha > 0$ such that*

$$\|\mathbf{G}_E \varphi\|^2 - (\mathbb{K}\varphi, \varphi) \geq \alpha \left(\|\mathbf{G}_E \varphi\|^2 + \|\varphi\|^2 \right)$$

for all $\varphi \in C^\infty(SM)$.

Proof. Consider the following Riccati type equation

$$\mathbf{G}_E(r - V(\lambda)) + r(r - V(\lambda)) + \mathbb{K} = 0.$$

It was shown in [13] that for Anosov thermostats there are real-valued continuous solutions $r^\pm$ (on SM) to this equation, which are differentiable along the thermostat flow and satisfy $r^+ - r^- > 0$. We prove that the following integral identity holds

$$\|\mathbf{G}_E \varphi\|^2 - (\mathbb{K}\varphi, \varphi) = \|\mathbf{G}_E \varphi - r\varphi + V(\lambda)\varphi\|^2, \tag{3.3}$$

where $r = r^{\pm}$.

$$|\mathbf{G}_E\varphi - r\varphi + V(\lambda)\varphi|^2 = |\mathbf{G}_E(\varphi)|^2 + |r\varphi|^2 + |V(\lambda)\varphi|^2 - 2\mathfrak{Re}(r\mathbf{G}_E(\varphi)\,\overline{\varphi}) + 2\mathfrak{Re}(V(\lambda)\mathbf{G}_E(\varphi)\,\overline{\varphi}) - 2rV(\lambda)|\varphi|^2.$$

Since r satisfies the Ricatti equation,

$$|\mathbf{G}_E\varphi - r\varphi + V(\lambda)\varphi|^2 = |\mathbf{G}_E(\varphi)|^2 - \mathbb{K}|\varphi|^2 + |V(\lambda)\varphi|^2 - \mathbf{G}_E((r - V(\lambda))|\varphi|^2) - rV(\lambda)|\varphi|^2.$$

Integrate this over SM and use (3.2) together with Lemma 3.2.1 to derive (3.3).

Let $A := \mathbf{G}_E\varphi - r^+\varphi + V(\lambda)\varphi$ and $B := \mathbf{G}_E\varphi - r^-\varphi + V(\lambda)\varphi$, the equation (3.3) implies $\|A\| = \|B\|$. We obtain the following expressions for φ and $\mathbf{G}_E\varphi$

$$\varphi = (r^+ - r^-)^{-1}(A - B),$$
$$\mathbf{G}_E\varphi = (1 - c)A + cB,$$

where $c := \frac{r^+ - V\lambda}{r^+ - r^-}$. From these equations one concludes that there is an $\alpha > 0$ such that

$$2\alpha\|\varphi\|^2 \le \|A\|^2, \quad 2\alpha\|\mathbf{G}_E\varphi\|^2 \le \|A\|^2.$$

Combining above inequalities with (3.3), this completes the proof. □

Remark 3.3.2. The proof of Theorem 3.3.1 shows that the following more general statement holds: if there is a bounded measurable function $r : SM \to \mathbb{R}$ such that

$$\mathbf{G}_E(r - V(\lambda)) + r(r - V(\lambda)) + \beta\mathbb{K} \le 0,$$

then the Gaussian thermostat (M, g, E) is $(\beta - 1)/\beta$-controlled.

3.4 Surjectivity of I_0^*

This section is devoted to the surjectivity of the adjoint of the thermostat ray transform acting on functions, i.e. I_0^*. To prove the surjectivity of I_0^*, we need to study the properties of the operator $P := V\mathbf{G}_E$. Appling Lemma 3.2.2, it is easy to see that $P^* = (\mathbf{G}_E + V(\lambda))V$. If F is a subspace of $\mathcal{D}'(SM)$, we denote by $F_\diamond$ the subspace of those $v \in F$ such that $\langle v, 1\rangle = 0$.

Lemma 3.4.1. *Let (M, g, E) be an Anosov Gaussian thermostat on a closed oriented Riemannian surface. Then there is a positive constant C such that*

$$\|u\|_{H^1(SM)} \leq C\|Pu\|$$

for all $u \in C^\infty_\diamond(SM)$.

Proof. Apply Pestov identity and Theorem 3.3.1 for $u \in C^\infty(SM)$

$$\begin{aligned} \|V\mathbf{G}_E u\|^2 &= \|\mathbf{G}_E V u\|^2 - (\mathbb{K}Vu, Vu) + \|\mathbf{G}_E u\|^2 \\ &\geq \|\mathbf{G}_E u\|^2 + \alpha(\|\mathbf{G}_E V u\|^2 + \|Vu\|^2). \end{aligned} \tag{3.4}$$

Recall the commutation relation $[\mathbf{G}_E, V]u = X_\perp u - V(\lambda)Vu$, which implies that

$$\|X_\perp u - V(\lambda)Vu\|^2 \leq 2(\|\mathbf{G}_E V u\|^2 + \|V\mathbf{G}_E u\|^2).$$

Therefore,

$$\|\mathbf{G}_E V u\|^2 \geq \frac{1}{2}\|X_\perp u - V(\lambda)Vu\|^2 - \|V\mathbf{G}_E u\|^2. \tag{3.5}$$

Thus, there are constants $C', C'' > 0$ such that

$$C'\|\nabla u\|^2 \leq \|\mathbf{G}_E u\|^2 + \|X_\perp u - V(\lambda)Vu\|^2 + \|Vu\|^2 \leq C''\|Pu\|^2,$$

here $\nabla u = (Xu, X_\perp u, Vu)$. By the Poincaré inequality, there are constants $D, D' > 0$ satisfying

$$\|u\|^2 \leq D(\|\mathbf{G}_E u\|^2 + \|X_\perp u\|^2 + \|Vu\|^2) \leq D'\|\nabla u\|^2$$

for all $u \in C^\infty_\diamond(SM)$. Hence, there is $C > 0$ such that

$$\|u\|_{H^1(SM)} \leq C\|Pu\|$$

for all $u \in C^\infty_\diamond(SM)$. □

Lemma 3.4.1 implies a solvability result for the adjoint P^*.

Lemma 3.4.2. *Let (M, g, E) be an Anosov Gaussian thermostat on a closed oriented Riemannian surface. For any $f \in H^{-1}_\diamond(SM)$ there is $h \in L^2(SM)$ such that*

$$P^* h = f \quad in \quad SM.$$

Moreover, $\|h\| \leq C\|f\|_{H^{-1}(SM)}$ with $C > 0$ being independent of f.

Proof. Consider the subspace $PC^\infty_\diamond(SM)$ of $L^2(SM)$. By Lemma 3.4.1, any element w of $PC^\infty_\diamond(SM)$ has the form $w = Pu$ for some $u \in C^\infty_\diamond(SM)$. For a given $f \in H^{-1}_\diamond(SM)$, consider the linear functional

$$L : PC^\infty_\diamond(SM) \to \mathbb{C}, \quad L(Pu) = \langle u, f\rangle.$$

Lemma 3.4.1 implies that the functional L satisfies

$$|L(Pu)| \leq \|f\|_{H^{-1}(SM)} \|u\|_{H^1(SM)} \leq C\|f\|_{H^{-1}(SM)} \|Pu\|.$$

This says that L is continuous on $PC^\infty_\diamond(SM)$. Therefore, by Hahn-Banach Theorem, the operator L has a continuous extension

$$\mathcal{L} : L^2(SM) \to \mathbb{C}, \quad |\mathcal{L}(v)| \leq C\|f\|_{H^{-1}(SM)} \|v\|.$$

Now, we apply the Riesz Representation Theorem to find $h \in L^2(SM)$ satisfying

$$\mathcal{L}(v) = (v, h), \quad \|h\| \leq C\|f\|_{H^{-1}(SM)}.$$

If $u \in C^\infty_\diamond(SM)$, we have

$$\langle u, P^*h\rangle = \langle Pu, h\rangle = L(Pu) = \langle u, f\rangle.$$

It follows that $P^*h = f$, since f is orthogonal to constants. □

Now, we are ready to prove the surjectivity of I_0^*. Actually Theorem 3.1.7 is a particular case of the next result (let $a = 0$).

Theorem 3.4.3. *Let (M, g, E) be an Anosov Gaussian thermostat on a closed oriented Riemannian surface. Given $a \in H^{-1}_{\diamond}(SM)$ and $f \in L^2(M)$, there exists $w \in H^{-1}(SM)$ with $(\mathbf{G}_E + V(\lambda))w = a$ and $w_0 = f$.*

Proof. For a given $f \in C^\infty(M)$, by Lemma 3.4.2, there is $h \in L^2(SM)$ satisfying

$$P^* h = a - (\mathbf{G}_E + V(\lambda))f \quad \text{in} \quad SM.$$

Setting $w := Vh + f$, we get

$$\begin{aligned}(\mathbf{G}_E + V(\lambda))w &= (\mathbf{G}_E + V(\lambda))Vh + (\mathbf{G}_E + V(\lambda))f \\ &= P^* h + (\mathbf{G}_E + V(\lambda))f = a\end{aligned}$$

and it is easy to see that $w_0 = f$. □

3.5 Surjectivity of I_1^*

Let (M, g, E) be a Gaussian thermostat on a compact oriented surface. Consider the following first order differential operators introduced by Guillemin and Kazhdan [24]

$$\eta_+ = \frac{1}{2}(X + iX_\perp), \quad \eta_- = \frac{1}{2}(X - iX_\perp).$$

It was shown that $\eta_\pm : \Omega_k \to \Omega_{k\pm1}$ for $k \in \mathbb{Z}$, and that these operators are elliptic. We introduce the following differential operators $\mu_\pm : \Omega_k \to \Omega_{k\pm1}$ for $k \in \mathbb{Z}$, corresponding to the Gaussian thermostat (M, g, E), given by

$$\mu_+ = \eta_+ + \lambda_1 V, \quad \mu_- = \eta_- + \lambda_{-1} V, \tag{3.6}$$

where $\lambda = \lambda_1 + \lambda_{-1}$ (notice that λ corresponds to a 1-form). Thus $\mu_+ + \mu_- = \mathbf{G}_E = X + \lambda V$.

For fixed $m \geq 1$, we define the projection operator $T_m : C^\infty(SM) \to \bigoplus_{|k| \geq m+1} \Omega_k$ by

$$T_m u = \sum_{|k| \geq m+1} u_k.$$

We also consider the operator $Q_m : C^\infty(SM) \to \bigoplus_{|k| \geq m+1} \Omega_k$ defined by $Q_m u := T_m V \mathbf{G}_E u$.

The next proposition will be the key ingredient for the proofs of the main results.

Proposition 3.5.1. *Let (M, g, E) be an α-controlled Gaussian thermostat on a closed oriented Riemannian surface, and let $m \geq 1$ be an integer. Then for any given $u \in \bigoplus_{|k|\geq m} \Omega_k$ the following holds*

$$\|Q_m u\|^2 \geq (1-(m-1)^2+\alpha m^2)(\|\mu_- u_m\|^2 + \|\mu_+ u_{-m}\|^2) \\ + (1-m^2+\alpha(m+1)^2)(\|\mu_- u_{m+1}\|^2 + \|\mu_+ u_{-m-1}\|^2) + \|v\|^2 + \alpha\|w\|^2,$$

where $v = T_m \mathbf{G}_E u$ and $w = T_m \mathbf{G}_E V u$.

Proof. Let $u \in \bigoplus_{|k|\geq m} \Omega_k$. Since $\mathbf{G}_E = \mu_+ + \mu_-$,

$$\|\mathbf{G}_E u\|^2 = \|\mu_- u_{m+1}\|^2 + \|\mu_- u_m\|^2 + \|\mu_+ u_{-m-1}\|^2 + \|\mu_+ u_{-m}\|^2 + \|v\|^2.$$

Similarly

$$\|\mathbf{G}_E V u\|^2 = (m+1)^2\|\mu_- u_{m+1}\|^2 + m^2\|\mu_- u_m\|^2 + (m+1)^2\|\mu_+ u_{-m-1}\|^2 \\ + m^2\|\mu_+ u_{-m}\|^2 + \|w\|^2.$$

Since $V\mathbf{G}_E u = \sum_{|k|\leq m} ik(\mathbf{G}_E u)_k + Q_m u$, we have

$$\|V\mathbf{G}_E u\|^2 = m^2\|\mu_- u_{m+1}\|^2 + (m-1)^2\|\mu_- u_m\|^2 + m^2\|\mu_+ u_{-m-1}\|^2 \\ + (m-1)^2\|\mu_+ u_{-m}\|^2 + \|Q_m u\|^2.$$

By the Pestov identity and the hypotheses, we get

$$\|V\mathbf{G}_E u\|^2 \geq \alpha\|\mathbf{G}_E V u\|^2 + \|\mathbf{G}_E u\|^2.$$

Making the appropriate substitutions we obtain our result. □

Lemma 3.5.2. *Let (M, g, E) be an Anosov Gaussian thermostat. Suppose that there is a constant $C > 0$ such that*

$$\|\mathbf{G}_E u\| \leq C\|Q_m u\|$$

for all $u \in \bigoplus_{|k|\geq m} \Omega_k$. Then there exists another constant $D > 0$ such that

$$\|u\|_{H^1(SM)} \leq D\|Q_m u\|$$

for all $u \in \bigoplus_{|k|\geq m} \Omega_k$.

Proof. Let $u \in \bigoplus_{|k|\geq m} \Omega_k$. By the definitions of T_m and Q_m we have

$$\|Pu\|^2 = \sum_{|k|\leq m} k^2 \|(\mathbf{G}_E u)_k\|^2 + \|Q_m u\|^2 \leq C_1 \|\mathbf{G}_E u\|^2 + \|Q_m u\|^2$$

for some constant $C_1 > 0$. The hypothesis guarantees the existence of a constant $C_2 > 0$ such that

$$\|Pu\| \leq C_2 \|Q_m u\|$$

for any $u \in \bigoplus_{|k|\geq m} \Omega_k$. Now we apply Lemma 3.4.1 to finish the proof. □

Lemma 3.5.3. *Let (M, g, E) be an Anosov Gaussian thermostat which is α-controlled, for some $\alpha > (m-1)/(m+1)$ then there is a constant $C > 0$ such that*

$$\|u\|_{H^1(SM)} \leq C \|Q_m u\|$$

holds for any $u \in \bigoplus_{|k|\geq m} \Omega_k$.

Proof. By Proposition 3.5.1, for $\alpha > (m-1)/(m+1)$, there is a constant $C > 0$ satisfying

$$\|Q_m u\| \geq C \|\mathbf{G}_E u\|. \tag{3.7}$$

Now, one can conclude the proof by applying Lemma 3.5.2. □

Remark 3.5.4. As an immediate corollary of Lemma 3.5.3 and the smooth Livsic theorem (Lemma 3.5.5 below), one obtains that on an Anosov Gaussian thermostat which is α-controlled for $\alpha > (m-1)/(m+1)$, I_m is s-injective. In particular, an Anosov Gaussian thermostat with non-positive thermostat curvature is 1-controlled, this is enough for proving Corollary 3.1.3.

Let us state the smooth Livsic theorem from [35, Theorem 2.1] reformulated in a way convenient for our thesis.

Lemma 3.5.5. *Let (M, g, E) be an Anosov Gaussian thermostat on a closed oriented Riemannian surface, and let $f \in C^\infty(SM)$. Then $If \equiv 0$ if and only if $\mathbf{G}_E u = f$ for some $u \in C^\infty(SM)$.*

Proof. This result was proven in a greater generality assuming that the flow is transitive [35]. Transitivity of Anosov Gaussian thermostats on surfaces follows from the work of Ghys [21]. □

However, in Section 3.7 we will prove Theorem 3.1.2 which is a stronger version of the injectivity of I_m, namely $\alpha = (m-1)/(m+1)$.

Lemma 3.5.6. *Let (M, g, E) be an Anosov Gaussian thermostat which is α-controlled for some $\alpha > (m-1)/(m+1)$. For any $f \in H^{-1}(SM)$ with $f_k = 0$ for $|k| \le m-1$, there is $h \in L^2(SM)$ such that*

$$Q_m^* h = f \quad \text{in} \quad SM.$$

Moreover, $\|h\| \le C\|f\|_{H^{-1}(SM)}$ with $C > 0$ being independent of f.

Proof. Consider the subspace $Q_m \bigoplus_{|k|\ge m} \Omega_k$ of $L^2(SM)$. By Lemma 3.4.1, any element v of $Q_m \bigoplus_{|k|\ge m} \Omega_k$ has the form $v = Q_m u$ for some $u \in \bigoplus_{|k|\ge m} \Omega_k$. For a given $f \in H_\diamond^{-1}(SM)$, we consider the linear functional

$$L : Q_m \bigoplus_{|k|\ge m} \Omega_k \to \mathbb{C}, \quad L(Pu) = \langle u, f\rangle.$$

Lemma 3.5.3 implies that this functional satisfies

$$|L(Q_m u)| \le \|f\|_{H^{-1}(SM)} \|u\|_{H^1(SM)} \le C\|f\|_{H^{-1}(SM)} \|Q_m u\|.$$

This means that L is continuous on $\bigoplus_{|k|\ge m} \Omega_k$. Therefore, by Hahn-Banach theorem L has a continuous extension

$$\mathcal{L} : L^2(SM) \to \mathbb{C}, \quad |\mathcal{L}(v)| \le C\|f\|_{H^{-1}(SM)} \|v\|.$$

Now, we apply the Riesz representation theorem to find $h \in L^2(SM)$ satisfying

$$\mathcal{L}(v) = (v, h), \quad \|h\| \le C\|f\|_{H^{-1}(SM)}.$$

If $u \in C^\infty(SM)$, we have

$$\langle u, Q_m^* h\rangle = \langle Q_m u, h\rangle = \langle Q_m(u - \sum_{|k|\le m-1} u_k), h\rangle = L(Q_m(u - \sum_{|k|\le m-1} u_k))$$
$$= \langle u - \sum_{|k|\le m-1} u_k, f\rangle = \langle u, f\rangle.$$

The last equality holds because $f_k = 0$ for all k satisfying $|k| \le m-1$. □

Now, we give the proof of our main result on the surjectivity of I_1^*.

Proof of Theorem 3.1.8. Set $a := -(\mathbf{G}_E + V(\lambda))\alpha$. Since $\delta\alpha = 0$, by [45] this is equivalent to $\eta_+\alpha_{-1} + \eta_-\alpha_1 = 0$. On the other hand, $(\lambda_1 V + V(\lambda_1))\alpha_{-1} = (\lambda_{-1}V + V(\lambda_{-1}))\alpha_1 = 0$, which imples $a_0 = 0$. By Theorem 3.3.1, an Anosov thermostat is α-controlled for some $\alpha > 0$. Therefore, we can apply Lemma 3.5.6 with $m = 1$ to find $h \in L^2(SM)$ such that

$$Q_m^* h = (\mathbf{G}_E + V(\lambda))VTh = -(\mathbf{G}_E + V(\lambda))\alpha.$$

Set $w := VTh + \alpha$, then $(\mathbf{G}_E + V(\lambda))w = 0$ and $w_{-1} + w_1 = \alpha$. □

3.6 *Injectivity of operators* μ_+, μ_-

The following result on the injectivity of μ_+, μ_- is one of the crucial components in the proof of Theorem 3.1.2. It does generalize the corresponding result obtained in [24].

Proposition 3.6.1. *Let (M, g, E) be a Gaussian thermostat on a closed oriented Riemannian surface of genus ≥ 2. Consider the operators $\mu_\pm : \Omega_k \to \Omega_{k\pm1}$ defined as in (3.6), then $\mu_+ : \Omega_k \to \Omega_{k+1}$ is injective for $k \ge 1$ and $\mu_- : \Omega_k \to \Omega_{k-1}$ is injective for $k \le -1$.*

This is a consequence of the following lemmas. The first lemma says that the kernel of $\mu_\pm$ is invariant under the conformal change of the metric and the Gaussian thermostat: $(g, E) \mapsto (e^{2\sigma}g, e^{-2\sigma}E)$.

Lemma 3.6.2. *Let (M, g, E) be a Gaussian thermostat on an oriented surface, and let $u \in \Omega_m$ be such that $\mu_+ u = 0$. Then $\tilde{u} = e^{m\sigma}u$ satisfies $\tilde{\mu}_+\tilde{u} = 0$ for any smooth function*

$\sigma \in C^\infty(M, \mathbb{R})$. *Here* $\tilde{\mu}_+$ *denotes the operator defined as in* (3.6) *for the Gaussian thermostat* $(M, \tilde{g}, \tilde{E})$ *with* $\tilde{g} = e^{2\sigma} g$ *and* $\tilde{E} = e^{-2\sigma} E$.

Before giving the proof we introduce some conventions. If A is a notation for some object in the context of the thermostat (M, g, E), by $\tilde{A}$ we denote the same object but in the context of the thermostat $(M, \tilde{g}, \tilde{E})$. For example, since SM denotes the unit sphere bundle with respect to the metric g, then $\tilde{S}M$ denotes the unit sphere bundle with respect to the metric $\tilde{g}$. Another example, by α we denote the 1-form dual to the external vector field E with respect to the metric g. Then $\tilde{\alpha}$ denotes the 1-form dual to the external vector field $\tilde{E}$ with respect to the metric $\tilde{g}$.

Proof. Consider the isothermal coordinates (x, y) on (M, g) such that the metric can be written as $ds^2 = e^{2\rho}(dx^2 + dy^2)$ where $\rho \in C^\infty(M, \mathbb{R})$. This gives coordinates (x, y, φ) on SM where φ is the angle between a unit vector v and $\frac{\partial}{\partial x}$. In these coordinates, we have $V = \frac{\partial}{\partial \varphi}$ and

$$\begin{aligned} X &= e^{-\rho}\left(\cos\varphi\frac{\partial}{\partial x} + \sin\varphi\frac{\partial}{\partial y} + \left(-\frac{\partial \rho}{\partial x}\sin\varphi + \frac{\partial \rho}{\partial y}\cos\varphi\right)\frac{\partial}{\partial \varphi}\right), \\ X_\perp &= -e^{-\rho}\left(-\sin\varphi\frac{\partial}{\partial x} + \cos\varphi\frac{\partial}{\partial y} - \left(\frac{\partial \rho}{\partial x}\cos\varphi + \frac{\partial \rho}{\partial y}\sin\varphi\right)\frac{\partial}{\partial \varphi}\right). \end{aligned}$$

Consider $u \in \Omega_m$ and write $u(x, y, \varphi) = h(x, y)e^{im\varphi}$. Then a straightforward calculation, using these formulas, shows that

$$\eta_+(u) = e^{(m-1)\rho}\partial(he^{-m\rho})e^{i(m+1)\varphi}, \tag{3.8}$$

where $\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$.

In order to write μ_+ we set $\alpha_z := \frac{1}{2}(E^1 - iE^2)$ where E^1 and E^2 are coordinates of the vector field E, i.e. $E = (E^1, E^2)$. A straightforward calculation shows that

$$\alpha_+(x, y, \varphi) = \alpha_z(x, y)\, e^{\rho} e^{i\varphi}.$$

Combine this with (3.8) and (3.6), we obtain

$$\mu_+(u) = e^{(m-1)\rho}(\partial - me^{2\rho}\alpha_z)(he^{-m\rho})e^{i(m+1)\varphi}. \tag{3.9}$$

The same coordinates (x, y) will be isothermal on $(M, \tilde{g})$ and the metric $\tilde{g}$ can be written as $d\tilde{s}^2 = e^{2\rho+2\sigma}(dx^2 + dy^2)$. Then the coordinates on $\tilde{S}M$ will be (x, y, φ) where φ is as before. In these coordinates we have

$$\tilde{\mu}_+(\tilde{u}) = e^{(m-1)(\rho+\sigma)}(\partial - me^{2\rho+2\sigma}\tilde{\alpha}_z)(\tilde{h}e^{-m(\rho+\sigma)})e^{i(m+1)\varphi} \tag{3.10}$$

for any $\tilde{u} \in \tilde{\Omega}_m$ written as $\tilde{u}(x, y, \varphi) = \tilde{h}(x, y)e^{im\varphi}$.

Assume that $\mu_+ u = 0$, where $u \in \Omega_m$ is written as $u(x, y, \varphi) = h(x, y)e^{im\varphi}$. Then from (3.9) we conclude that $(\partial - me^{2\rho}\alpha_z)(he^{-m\rho}) = 0$.

Now consider $\tilde{u} = e^{m\sigma}u$. Then $\tilde{u} = \tilde{h}e^{im\varphi}$ with $\tilde{h} = e^{m\sigma}h$, and $\tilde{\alpha}_z = e^{-2\sigma}\alpha_z$. Therefore, by (3.10), we have

$$\begin{aligned}\tilde{\mu}_+(\tilde{u}) &= e^{(m-1)(\rho+\sigma)}(\partial - me^{2\rho+2\sigma}\tilde{\alpha}_z)(\tilde{h}e^{-m(\rho+\sigma)})e^{i(m+1)\varphi} \\ &= e^{(m-1)(\rho+\sigma)}(\partial - me^{2\rho}\alpha_z)(he^{-m\rho})e^{i(m+1)\varphi}.\end{aligned}$$

Thus, we conclude that $\tilde{\mu}_+\tilde{u} = 0$. □

Lemma 3.6.3. *Let (M, g, E) be a Gaussian thermostat on an oriented surface. If $(M, \tilde{g}, \tilde{E})$ is the conformal Gaussian thermostat, that is $\tilde{g} = e^{2\sigma}g$ and $\tilde{E} = e^{-2\sigma}E$, then $\operatorname{div}_{\tilde{g}} \tilde{E} = e^{-2\sigma} \operatorname{div}_g E$.*

Proof. The proof follows by straightforward computations in isothermal coordinates (x, y) on (M, g). The Christoffel symbols are

$$\Gamma^1_{11} = -\Gamma^1_{22} = \Gamma^2_{12} = \frac{\partial\rho}{\partial x}, \quad \Gamma^2_{22} = -\Gamma^2_{11} = \Gamma^1_{12} = \frac{\partial\rho}{\partial y}.$$

If $E = (E^1, E^2)$ in coordinates (x, y), then the expression for $\operatorname{div}_g E$ is

$$\operatorname{div}_g E = \nabla_1 E^1 + \nabla_2 E^2 = \frac{\partial E^1}{\partial x} + \frac{\partial E^2}{\partial y} + 2\left(\frac{\partial\rho}{\partial x}E^1 + \frac{\partial\rho}{\partial y}E^2\right).$$

Note that the metric $\tilde{g}$ can be written as $d\tilde{s}^2 = e^{2\rho+2\sigma}(dx^2 + dy^2)$. Therefore the Christoffel symbols for $\tilde{g}$ are

$$\tilde{\Gamma}^1_{11} = -\tilde{\Gamma}^1_{22} = \tilde{\Gamma}^2_{12} = \frac{\partial(\rho+\sigma)}{\partial x}, \quad \tilde{\Gamma}^2_{22} = -\tilde{\Gamma}^2_{11} = \tilde{\Gamma}^1_{12} = \frac{\partial(\rho+\sigma)}{\partial.}y,$$

Since $\tilde{E} = (e^{-2\sigma}E^1, e^{-2\sigma}E^2)$ in coordinates (x, y), the expression for $\operatorname{div}_{\tilde{g}} \tilde{E}$ is

$$\begin{aligned}\operatorname{div}_{\tilde{g}} \tilde{E} &= \tilde{\nabla}_1 E^1 + \tilde{\nabla}_2 E^2 = e^{-2\sigma}\left(\frac{\partial E^1}{\partial x} + \frac{\partial E^2}{\partial y} + 2\left(\frac{\partial \rho}{\partial x}E^1 + \frac{\partial \rho}{\partial y}E^2\right)\right) \\ &= e^{-2\sigma} \operatorname{div}_g E.\end{aligned}$$

□

Lemma 3.6.4. *Let (M, g, E) be a Gaussian thermostat on a closed oriented surface of genus ≥ 2, then there exists a function $\sigma \in C^\infty(M, \mathbb{R})$, such that the conformal Gaussian thermostat $(M, e^{2\sigma}g, e^{-2\sigma}E)$ has negative thermostat curvature.*

Proof. Let $K, \tilde{K}$ be the Gaussian curvatures of (M, g) and $(M, e^{2\sigma}g)$ respectively. It is well known that $\tilde{K} = e^{-2\sigma}(K - \Delta_g \sigma)$, here Δ_g is the Laplacian under the metric g. On the other hand, a straightforward calculation shows that the thermostat curvature of (M, g, E) has the form

$$\mathbb{K} = K - \operatorname{div}_g E.$$

Above discussion together with Lemma 3.6.3 implies that the thermostat curvature of $(M, e^{2\sigma}g, e^{-2\sigma}E)$ is

$$\tilde{\mathbb{K}} = \tilde{K} - \operatorname{div}_{\tilde{g}} \tilde{E} = e^{-2\sigma}(K - \Delta_g \sigma - \operatorname{div}_g E).$$

To prove the lemma, we need to find a real-valued smooth function σ and a constant $c < 0$ for the following equation

$$K - \Delta_g \sigma - \operatorname{div}_g E = c < 0. \tag{3.11}$$

Notice that on a closed connected Riemannian surface, the solvability condition for (3.11) is

$$0 = \int_M K - c - \operatorname{div}_g E \, d\mathrm{Vol}_g = \int_M K - c \, d\mathrm{Vol}_g.$$

By the Gauss-Bonnet theorem and the assumption that the genus ≥ 2 (i.e. the Euler characteristic $\chi(M) < 0$), we can choose

$$c = \frac{\int_M K \, d\mathrm{Vol}_g}{\mathrm{Vol}_g(M)} = \frac{2\pi\chi(M)}{\mathrm{Vol}_g(M)} < 0,$$

where $\text{Vol}_g(M)$ is the volume of M under the metric g.

Thus there exists $\sigma \in C^\infty(M, \mathbb{R})$ such that

$$\tilde{\mathbb{K}} = e^{-2\sigma} \frac{2\pi\chi(M)}{\text{Vol}_g(M)} < 0.$$

□

Lemma 3.6.2 and 3.6.4 imply that to prove Proposition 3.6.1, we only need to show that it's true for the case $\mathbb{K} < 0$.

Lemma 3.6.5. *Given a Gaussian thermostat (M, g, E) on a closed oriented surface with $\mathbb{K} = K - \text{div}_g E < 0$, where K is the Gaussian curvature of (M, g), then $\mu_+ : \Omega_k \to \Omega_{k+1}$ is injective for $k \geq 1$ and $\mu_- : \Omega_k \to \Omega_{k-1}$ is injective for $k \leq -1$.*

Proof. Let $u \in \Omega_k$, since $\mathbf{G}_E = \mu_+ + \mu_-$, the following expressions hold

$$\mathbf{G}_E u = \mu_+ u + \mu_- u, \quad \mathbf{G}_E V u = ik\mu_+ u + ik\mu_- u,$$
$$V\mathbf{G}_E u = i(k+1)\mu_+ u + i(k-1)\mu_- u.$$

Substituting these into the Pestov identity, we obtain an integral identity

$$2k\|\mu_- u\|^2 = 2k\|\mu_+ u\|^2 + k^2(\mathbb{K}u, u).$$

According to our hypothesis $\mathbb{K} < 0$, we come to the following inequality

$$2k\|\mu_- u\|^2 \leq 2k\|\mu_+ u\|^2. \tag{3.12}$$

Consider the case $k \geq 1$ and assume $\mu_+ u = 0$, we get

$$0 \leq \|\mu_- u\|^2 \leq 0,$$

Thus $u \equiv 0$ as desired. Using similar ideas for the case $k \leq -1$ one can prove that $\mu_- u = 0$ implies $u \equiv 0$. □

3.7 Injectivity of I_m

Before giving the proof of the s-injectivity of I_m, it is worth pointing out that if the terminator value of a Gaussian thermostat (M, g, E) is β_{ter}, then (M, g, E) is free of β_{ter}-conjugate points. Indeed assume that (M, g, E) has β_{ter}-conjugate points, i.e. there exists a thermostat geodesic γ and a non-trivial solution $y(t)$ to the β_{ter}-Jacobi equation along γ such that $y(0) = y(T) = 0$ for some $T > 0$. Notice that $\dot{y}(T) \neq 0$, thus there is a small neighborhood U of β_{ter}, such that for all $\beta \in U$ there are β-conjugate points. This contradicts the definition of the terminator values.

Since (M, g, E) has no β_{ter}-conjugate points, by Remark 3.3.2, it is $(\beta_{\text{ter}} - 1)/\beta_{\text{ter}}$-controlled. Notice that for Anosov Gaussian thermostats, there are no conjugate points in the usual sense, which means that $\beta_{\text{ter}} \geq 1$ (actually one can get $\beta_{\text{ter}} > 1$ for Anosov Gaussian thermostats).

The following injectivity result will imply Theorem 3.1.2.

Theorem 3.7.1. *Let (M, g, E) be a Gaussian thermostat on a closed surface of genus $\mathrm{g} \geq 2$ which is $(m-1)/(m+1)$-controlled for $m \geq 1$. Let φ be a symmetric m-tensor and suppose that there is a smooth solution h to the transport equation*

$$\mathbf{G}_E h = \varphi.$$

Then h is of degree $m - 1$.

Proof. Let $u = \sum_{|k| \geq m} h_k$, then $\mathbf{G}_E u$ has degree m and $Q_m u = 0$. By Proposition 3.5.1 and the assumption $\alpha = (m-1)/(m+1)$, we get that

$$\mu_- u_m = 0 \quad \text{and} \quad \mu_+ u_{-m} = 0.$$

Thus

$$\mathbf{G}_E u = \mu_- u_{m+1} + \mu_+ u_{-(m+1)}$$

and

$$\mathbf{G}_E V u = i(m+1)\mu_- u_{m+1} - i(m+1)\mu_+ u_{-(m+1)}.$$

Therefore,

$$
\begin{aligned}
X_\perp u - V(\lambda)Vu &= [\mathbf{G}_E, V]u \\
&= i(m+1)\mu_- u_{m+1} - i(m+1)\mu_+ u_{-(m+1)} - im\mu_- u_{m+1} + im\mu_+ u_{-(m+1)} \\
&= i\mu_- u_{m+1} - i\mu_+ u_{-(m+1)}.
\end{aligned}
$$

It is known that $X_\perp u - V(\lambda)Vu = i\mu_- u - i\mu_+ u$. Hence $\mu_- u = \mu_- u_{m+1}$ and $\mu_+ u = \mu_+ u_{-(m+1)}$, in particular, $\mu_+ u_k = 0$ and $\mu_- u_{-k} = 0$ for $k \geq m$. Then Proposition 3.6.1 implies that $u \equiv 0$, thus h is of degree $m-1$. □

Proof of Theorem 3.1.2. Let φ be a symmetric m-tensor, such that $I_m \varphi \equiv 0$. By Lemma 3.5.5, there is $h \in C^\infty(SM)$ such that $\mathbf{G}_E h = \varphi$.

On the other hand, a closed oriented surface whose unit sphere bundle carries an Anosov flow must have genus ≥ 2. Indeed, by a classic result of Plante and Thurston [52], if an S^1-bundle over a closed oriented surface carries an Anosov flow, the fundamental group of the bundle must grow exponentially. However the fundamental group of any S^1-bundle over a 2-sphere or torus only has polynomial growth.

Finally, by Remark 3.3.2 and the discussion about terminator values at the beginning of this section, (M, g, E) is $(m-1)/(m+1)$-controlled.

Now Theorem 3.1.2 is a direct consequence of Theorem 3.7.1. □

3.8 Results for surfaces with boundary

As mentioned in the introduction, some of the arguments above also work for compact surfaces with boundary. The main change when dealing with the boundary case is that the functions need to vanish on the boundary whenever appropriate.

In this section we assume that (M, g) is a compact oriented Riemannian surface with smooth boundary ∂M, we will prove Theorem 3.1.5 which is an injectivity result for Gaussian thermostats (M, g, E) on surfaces with boundary. Let Λ denote the second fundamental form of ∂M and $\nu(x)$ the inward unit normal to ∂M at x. We say that ∂M is *strictly thermostat* *convex* if

$$\Lambda(x,v) > \langle E(x) - \langle E(x), v\rangle v, \nu(x)\rangle \tag{3.13}$$

for all $(x,v) \in S(\partial M)$, here E is the external field.

For $x \in M$, we define the *thermostat exponential map* by

$$\exp_x^E(tv) = \pi \circ \phi_t(v), \quad t \geq 0, \; v \in S_x M$$

which is C^1-smooth on $T_x M$ and C^∞-smooth on $T_x M \setminus \{0\}$.

We say that (M, g, E) is *simple* if 1) ∂M is strictly thermostat convex and 2) the thermostat exponential map $\exp_x^E : (\exp_x^E)^{-1}(M) \to M$ is a diffeomorphism for every $x \in M$. These two conditions guarantee that every two points on M are connected by a unique thermostat geodesic and there is no conjugate points. In this case, M is diffeomorphic to the unit ball of $\mathbb{R}^n$, which is simply connected.

Results in Section 3.2 are still valid in the boundary case if the trace of u or v vanishes. The Pestov identity also holds:

Theorem 3.8.1. *Let (M, g, E) be a Gaussian thermostat on a compact oriented surface with boundary. If $u \in C^\infty(SM, \mathbb{C})$ and $u|_{\partial SM} = 0$, then*

$$\|\mathbf{G}_E V u\|^2 - (\mathbb{K} V u, V u) = \|V \mathbf{G}_E u\|^2 - \|\mathbf{G}_E u\|^2.$$

Notice that the estimate of Theorem 3.3.1 plays an important role in the arguments for the case of closed surfaces. To establish our result for the boundary case, we need a similar estimate. Given a Riemannian surface M with boundary, denoting ∂SM the boundary of SM, we define a subset of ∂SM,

$$\partial_+ SM := \{(x,v) \in \partial SM : \langle v, \nu(x)\rangle_g \geq 0\}.$$

Note that $\nu(x)$ is the inward unit normal to ∂M at x. We start with the following existence result of distinct solutions to the Riccati equation on 2D simple Gaussian thermostats.

Lemma 3.8.2. *Let (M, g, E) be a simple Gaussian thermostat on a compact oriented surface with boundary. Then there exist smooth nowhere equal solutions r^+ and r^- to the Riccati type equation*

$$\mathbf{G}_E r + r^2 - V(\lambda)r + \mathbb{K} - \mathbf{G}_E V(\lambda) = 0. \tag{3.14}$$

Proof. We embed M into larger compact surfaces $\tilde{M}$, $\widetilde{M}$ with boundary such that $M \subset \tilde{M}^{int} \subset \tilde{M} \subset \widetilde{M}^{int} \subset \widetilde{M}$, and extend g and E smoothly onto $\widetilde{M}$ such that $(\tilde{M}, g, E)$ and $(\widetilde{M}, g, E)$ are simple too.

We consider a maximum thermostat geodesic $\gamma_z : [0, l] \to \widetilde{M}$ with $z = (\gamma_z(0), \dot{\gamma}_z(0)) \in \partial_+ S\widetilde{M}$. Let y_z be the solution to the thermostat Jacobi equation

$$\ddot{y}_z - V(\lambda)\dot{y}_z + (\mathbb{K} - \mathbf{G}_E V(\lambda))y_z = 0$$

along γ_z satisfying $y_z(0) = 0$, $\dot{y}_z(0) = 1$. By the simplicity of $(\widetilde{M}, g, E)$, γ_z has no conjugate points, thus $r(z, t) = \frac{\dot{y}_z(t)}{y_z(t)}$ is a solution to the Riccati equation on $(0, l]$ with $\lim_{t\to 0} r(t) = +\infty$. Notice that $r(z, t)$ smoothly depends on $z \in \partial_+ S\widetilde{M}$. We do the same thing for all the thermostat geodesics on $\widetilde{M}$, which can be parametrized by $z \in \partial_+ S\widetilde{M}$, to get a well-defined smooth solution $r^+(x, \xi) = r(z(x, \xi), \tau^-(x, \xi))$ to the Riccati equation (3.14) on $S\widetilde{M}^{int}$, where $(x, \xi) = (\gamma_z(\tau^-(x, \xi)), \dot{\gamma}_z(\tau^-(x, \xi)))$, $\tau^-(x, \xi)$ is the length of the unique thermostat geodesic segment connecting $\pi(z)$ and x with $\xi \in S_x\widetilde{M}$ tangent to γ_z at x. It is not difficult to see that z and τ^- smoothly depend on $(x, \xi) \in S\widetilde{M}^{int}$. Moreover $\lim_{(x,\xi)\to\partial_+ S\widetilde{M}} r^+(x, \xi) = +\infty$.

Notice that by our definition of $\tilde{M}$ and $\widetilde{M}$, the restriction to $\tilde{M}$ of a thermostat geodesic γ of $(\widetilde{M}, g, E)$ (if nonempty), $\gamma|_{\tilde{M}}$, is a thermostat geodesic of $(\tilde{M}, g, E)$. By a similar approach as above with the initial condition $y_z(0) = 0$, $\dot{y}_z(0) = 1$ at $z \in \partial_+ S\tilde{M}$ for the thermostat Jacobi equation, one can get a smooth solution r^- to the Riccati equation (3.14) on $S\tilde{M}^{int}$ with $\lim_{(x,\xi)\to\partial_+ S\tilde{M}} r^-(x, \xi) = +\infty$.

Since $S\tilde{M} \subset S\widetilde{M}$ and $\partial_+ S\tilde{M}$ is compact, there exists $K > 0$ such that $\sup_{(x,\xi)\in\partial_+ S\tilde{M}} r^+(x, \xi) \leq K$. We can find a smaller compact surface U, whose boundary ∂U is uniformly, sufficiently close to $\partial\tilde{M}$, with $M \subset U \subset \tilde{M}^{int}$ and (U, g, E) is still simple. Then there exists $c > 0$ such that $\sup_{\partial_+ SU} r^+ < K + c$ and $\inf_{\partial_+ SU} r^- > K + c$, i.e. r^+ and r^- never coincide on $\partial_+ SU$.

Now we claim that $r^+ \neq r^-$ on SM (Actually $r^+ \neq r^-$ on $S\tilde{M}^{int}$). We prove by contradictions, assume that there exists $(x,\xi) \in SM$ such that $r^+(x,\xi) = r^-(x,\xi)$. Consider the restrictions of r^+ and r^- onto the thermostat geodesic $\gamma_{x,\xi} : [-l^-, l^+] \to U$, l^-, $l^+ > 0$, with $(\gamma_{x,\xi}(0), \dot{\gamma}_{x,\xi}(0)) = (x,\xi)$ and $\gamma_{x,\xi}(-l^-)$, $\gamma_{x,\xi}(l^+) \in \partial U$. Notice that the zeroth order term of the Riccati equation (3.14) is a polynomial with respect to r. Moreover, $[-l^-, l^+]$ is compact, thus the zeroth order term of (3.14) is Lipschitz continuous in r when it is restricted on $\gamma_{x,\xi}$. By the Picard-Lindelöf theorem of first order ODEs, one has the global existence and uniqueness of the solution to the Riccati equation on $\gamma_{x,\xi}$ with $r(0) = r^+(x,\xi) = r^-(x,\xi)$. This implies that $r^+ \equiv r^-$ along $\gamma_{x,\xi}$. In particular, there is $z \in \partial_+ SU$ such that $r^+(z) = r^-(z)$. However, since r^+ and r^- are never equal on $\partial_+ SU$, we reach a contradiction. Therefore, r^+ and r^- are two distinct solutions to the Riccati equation (3.14) on SM. □

The following is an analogue of Theorem 3.3.1 on compact surfaces with boundary.

Theorem 3.8.3. *Let (M,g,E) be a simple Gaussian thermostat on a compact oriented surface with boundary. Then there exists an $\alpha > 0$ such that*

$$\|\mathbf{G}_E\varphi\|^2 - (\mathbb{K}\varphi,\varphi) \geq \alpha\left(\|\mathbf{G}_E\varphi\|^2 + \|\varphi\|^2\right)$$

for all $\varphi \in C^\infty(SM,\mathbb{C})$ with $\varphi|_{\partial SM} = 0$.

Proof. Applying Lemma 3.8.2, the proof is almost identical to the proof of Theorem 3.3.1. □

Applying Theorem 3.8.1 and 3.8.3, the results of Sections 3.4 and 3.5 also hold for the boundary case. To prove Theorem 3.1.5, we need the following lemma on the injectivity of $\mu_\pm$ which is an analogue of Proposition 3.6.1.

Lemma 3.8.4. *Let (M,g,E) be a Gaussian thermostat on a compact oriented Riemannian surface with boundary. Consider the operators $\mu_\pm : \Omega_k \to \Omega_{k\pm1}$ defined as in (3.6). Let $k \geq 1$, if $\mu_+ u = 0$ where $u \in \Omega_k$, $u|_{\partial SM} = 0$, then $u = 0$; if $\mu_- u = 0$ where $u \in \Omega_{-k}$, $u|_{\partial SM} = 0$, then $u = 0$.*

Proof. Notice that M can be embedded into a closed surface of genus ≥ 2. By Lemma 3.6.4, we only need to show the injectivity of $\mu_\pm$ for Gaussian thermostats of negative thermostat curvature, which is straightforward by applying Theorem 3.8.1. $\square$

With the help of above lemma, we obtain the following injectivity result whose proof is similar to that for Theorem 3.7.1.

Proposition 3.8.5. *Let (M, g, E) be a Gaussian thermostat on a compact oriented surface with boundary which is $(m-1)/(m+1)$-controlled for $m \geq 1$. Let φ be a symmetric m-tensor and suppose that there is a smooth solution h, $h|_{\partial SM} = 0$, to the transport equation*

$$\mathbf{G}_E h = \varphi.$$

Then h is of degree $m-1$.

To prove Theorem 3.1.5, we need a version of Livsic Theorem for surfaces with boundary. Given a 2D simple Gaussian thermostat (M, g, E), let $\tau(x, v)$, $(x, v) \in SM$ be the time that the thermostat geodesic $\gamma_{x,v}$ starting at x in direction v exits M. The simplicity assumption implies that τ is finite for all $(x, v) \in SM$ and it is smooth on SM except $S(\partial M)$, the unit sphere bundle of the boundary ∂M.

Given f a smooth function on SM, it is easy to see that

$$u^f(x, v) = -\int_0^{\tau(x,v)} f(\gamma_{x,v}(t), \dot{\gamma}_{x,v}(t))\, dt \tag{3.15}$$

solves the transport equation

$$\mathbf{G}_E u = f$$

in SM. Moreover, if $If \equiv 0$, we obtain $u^f|_{\partial SM} = 0$. The ingredient is the following regularity statement.

Proposition 3.8.6. *Let (M, g, E) be a simple Gaussian thermostat on a compact oriented surface with boundary. Given $f \in C^\infty(SM)$ with $If \equiv 0$, let u^f be the function defined by (3.15), then $u^f \in C^\infty(SM)$ too.*

The proof of Proposition 3.8.6 for simple surfaces can be found in [43], a similar argument works for simple Gaussian thermostats, thus we leave it to the reader. Now Theorem 3.1.5 follows from Proposition 3.8.6 and 3.8.5.

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