Local and Global Convergence for Convex-Composite Optimization

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Abstract

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Convex-composite optimization seeks to minimize $f(x) := h(c(x))$ over $x \in \mathbb{R}^n$, where h is closed, proper, and convex, and c is smooth. Such problems include nonlinear programming, mini-max optimization, estimation of nonlinear dynamics with non-Gaussian noise as well as many modern approaches to large-scale data analysis and machine learning. Almost all methods for solving this problem involve direction finding subproblems based on linearizing the smooth function c at some current iterate. When h is the identity function on the real line, these direction finding subproblems correspond to steepest descent, prox-gradient descent, Newton's method, or quasi-Newton methods. When h is infinite-valued piecewise linear convex, the subproblems are quadratic programs, one class of which corresponds to sequential quadratic programming of nonlinear programming.

This thesis is divided into two parts. The first part is devoted to globalization strategies including line search and trust region methods. The second part is devoted to local analysis in the case where h is piecewise linear-quadratic convex, where the subproblems correspond to a Newton-like algorithm for an associated generalized equation describing the optimality conditions.

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DEDICATION

To my mother Nasreen Esmail, my family, and my friends

Chapter 1

INTRODUCTION

The convex-composite optimization problem [3] is

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad h(c(x)), \tag{1.1}$$

where $h : \mathbb{R}^m \to \overline{\mathbb{R}}$ is closed, proper and convex, and $c : \mathbb{R}^n \to \mathbb{R}^m$ is sufficiently[1] smooth. Almost all methods for solving 1.1 use a direction-finding subproblem based at an iterate $x^k \in \mathbb{R}^n$ similar to

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad h(c(x^k) + \nabla c(x^k)[x - x^k]) + \frac{1}{2}[x - x^k]^\top H_k[x - x^k], \tag{1.2}$$

where H_k is or approximates the Hessian of a Lagrangian for (1.1) introduced by Burke [4].

Convex-composite optimization has a long history with investigations in the 1970s [33,34], 1980s [3, 4, 25, 39, 41, 48, 50], and 1990s [7, 8, 14, 42], where much of the emphasis was on a calculus for compositions and its relationship to nonlinear programming (NLP) and exact penalization [21]. Recently, there has been a resurgence of interest in local [18, 19] and global [1, 11, 13, 16, 18, 20, 28] algorithms for this class of problems especially with respect to establishing the iteration complexity of first-order methods for (1.1) in part due to numerous modern applications in machine learning and nonlinear dynamics.

The rest of this introduction provides concrete examples that justify its ubiquity along with describing the two projects that comprise this thesis.

Example 1 (Nonlinear Least Squares)**.** When $h = \frac{1}{2}\|\cdot\|^2$, the problem (1.1) reads

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad \frac{1}{2}\left\|c(x)\right\|_2^2. \tag{1.3}$$

---

[1] at least $\mathcal{C}^1$ for global methods, and at least $\mathcal{C}^2$ for local methods

An iterative scheme for the nonlinear least squares problem dates back to Gauss and Newton [47, Chapter 10.3] and proceeds in the manner of the iterative subproblems (1.2) by setting $H_k = 0$ and solving a linear least squares problem at every iteration:

$$\underset{d \in \mathbb{R}^n}{\text{minimize}} \quad \frac{1}{2}\left\| c(x^k) + \nabla c(x^k) d \right\|_2^2. \tag{1.4}$$

Modifications to this include the Levenberg-Marquardt method [47, Chapter 10.3] that proceeds by choosing $\alpha_k > 0$ and setting $H_k = \alpha_k I$. The subproblems are now

$$\underset{d \in \mathbb{R}^n}{\text{minimize}} \quad \frac{1}{2}\left\| c(x^k) + \nabla c(x^k) d \right\|_2^2 + \frac{\alpha_k}{2}\|d\|_2^2. \tag{1.5}$$

The first half of this thesis argues that, near a solution $\overline{x}$, one should instead take $H_k \approx \sum_{i=1}^m c_i(x^k)\nabla^2 c_i(x^k)$.

Example 2 (Additive Composite)**.** When $h(y, x) = y + g(x)$ and $c(x) = (c(x), x)$ the problem (1.1) reads

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad c(x) + g(x). \tag{1.6}$$

When c has β-Lipschitz gradient, the proximal gradient method arises from the quadratic majorization at some iterate x^k:

$$c(x^k) + \nabla c(x^k)^\top [x - x^k] + \frac{\beta}{2}\|x - x_k\|_2^2 \geq c(x), \text{ for all } x \in \mathbb{R}^n.$$

Consequently, the update based on solving

$$\min_x c(x^k) + \nabla c(x^k)^\top [x - x^k] + \frac{\beta}{2}\|x - x_k\|_2^2 + g(x)$$

at every iteration gives $x^{k+1} := \text{prox}_{\beta^{-1}g}\left(x^k - \frac{1}{\beta}\nabla c(x^k)\right)$ and can be viewed as an instance of (1.2) with $H_k = \beta I$.

Example 3 (Newton's Method)**.** Continuing the previous example, the assumption of β-smoothness is global in that it is a uniform upper-bound on the eigenvalues of $\nabla^2 c(x)$. The local theory presented in Chapter 4 instead suggests solving the quadratic expansion based at x^k. Namely,

$$\min_x c(x^k) + \nabla c(x^k)^\top [x - x^k] + \frac{1}{2}[x - x^k]^\top \nabla^2 c(x^k)[x - x^k] + g(x).$$

When $g = 0$ and $\nabla^2 c(x^k)$ is positive definite, the update corresponds to the classical Newton method.

Example 4 (Nonlinear Programming)**.** As a final example, given smooth functions $g_0, g_1, \ldots, g_m$ on $\mathbb{R}^n$, consider the inequality-contrained nonlinear programming problem

$$\begin{array}{ll} \underset{x \in \mathbb{R}^n}{\text{minimize}} & g_0(x) \\ \text{subject to} & g_i(x) \leq 0, \;\; i = 1, \ldots, m \end{array}$$

By defining

$$h(c) = \begin{cases} c_0 & \text{if } c_i \leq 0, \;\; i = 1, 2, \ldots, m \\ \infty & \text{else,} \end{cases}$$

$$c(x) = (g_0(x), g_1(x), \ldots, g_m(x)),$$

we obtain another instance of infinite-valued convex-composite optimization. The Hessian of the classical Lagrangian $L : \mathbb{R}^n \times \mathbb{R}^m_+ \to \mathbb{R}$ defined by

$$L(x, y) = g_0(x) + \sum_{i=1}^{m} y_i g_i(x)$$

is used as the curvature term appearing in sequential quadratic programming [47, Chapter 18.1], and the subproblems of 1.2 using the full Hessian again capture this method:

$$\begin{array}{ll} \underset{d \in \mathbb{R}^n}{\text{minimize}} & g_0(x^k) + \nabla g_0(x^k)^\top d + \frac{1}{2} d^\top \nabla^2_{xx} L(x^k, y^k) d \\ \text{subject to} & g_i(x^k) + \nabla g_i(x^k)^\top d \leq 0, \;\; i = 1, \ldots, m. \end{array}$$

Numerous other applied problems fall within this class including mini-max optimization, estimation of nonlinear dynamics with non-Gaussian noise as well as many modern approaches to large-scale data analysis and machine learning [1, 2, 12].

Our study of globalization strategies in Chapter 3 assumes that the objective f has the form $f(x) = h(c(x)) + g(x)$, with $h : \mathbb{R}^m \to \mathbb{R}$ and $g : \mathbb{R}^n \to \overline{\mathbb{R}}$ convex. In this case, we allow arbitrary infinite-valued convex functions, rather than only piecewise linear-quadratic, but restrict to compositions only through the finite-valued h. The problem is

$$(\mathcal{P}) \qquad \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad f(x) := h(c(x)) + g(x).$$

Drusvyatskiy and Lewis [18] have established local and global convergence of proximal-based methods for solving $\mathcal{P}$, and Drusvyatskiy and Paquette [16] have established iteration complexity results for proximal methods to locate first-order stationary points for $\mathcal{P}$. However, the contribution of the second half of this thesis focuses on descent methods based on search directions generated from (1.2) taking $H_k = 0$. We relate the objective of the subproblem to a surrogate for the directional derivative of f as done in [3] (see (2.3). With this we offer a generalization of the following weak Wolfe conditions.

For a $\mathcal{C}^1$-smooth function $f : \mathbb{R}^n \to \mathbb{R}$, a direction d such that $f'(x; d) < 0$, and constants $0 < \sigma_1 < \sigma_2 < 1$, the step size $t > 0$ satisfies the weak Wolfe conditions if

$$f(x + td) \leq f(x) + t\sigma_1 f'(x; d)$$
$$f'(x + td; d) \geq \sigma_2 f'(x; d)$$

The first condition is a sufficient decrease condition on function value. It guarantees the function value at the new iterate $x + td$ is almost lower than the first-order expansion of f at x evaluated at the new point $x + td$. For all $t > 0$ sufficiently small, the sufficient decrease condition holds. However, if $\sigma_1 \geq 1$ and f is convex, the condition is unenforceable. The second condition is a curvature condition that guarantees nontermination at points with "strongly negative slopes" [47, Chapter 3.1], consequently encouraging larger steps. We also include a trust region algorithm and numerical experiments.

Chapter 4 is devoted to generalizing the local theory described in the examples above by considering the case where the function h is piecewise linear-quadratic convex. The work is due in part to recent breakthroughs in broader field of variational analysis contained in the work of Cibulka et. al. [9] and Lewis [26].

1.1 Notation

These sections summarize the relevant notation and tools of convex and variational analysis used in the thesis. Unless otherwise stated, we follow the notation in [15, 26, 42, 47].

1.1.1 Preliminaries

We work in $(\mathbb{R}^n, \langle\cdot,\, \cdot\rangle)$ with the standard inner product $\langle x,\, y\rangle = x^\top y = \sum_{i=1}^n x_i y_i$ and $\|x\|^2 = x^\top x$. Throughout, we switch between the notations $\langle x,\, y\rangle$ and $x^\top y$ for clarity considerations. Let $\mathbb{B} := \left\{x \in \mathbb{R}^n \;\middle|\; \|x\| \leq 1\right\}$ be the closed unit ball. For $A \in \mathbb{R}^{m\times n}$, its *range*, *null space*, and *transpose* are $\operatorname{Ran}(A)\,, \operatorname{Null}(A)\,, A^\top$ respectively, and for a finite collection of mappings $\{A_k\}_{k\in J}$ with index set J, let $\operatorname{diag} A_k$ denote the block diagonal matrix with kth block A_k. Let $e_j \in \mathbb{R}^\ell$ denote the standard unit coordinate vector. For any two points $x, x' \in \mathbb{R}^n$, denote the line segment connecting x and x' by $[x, x'] := \left\{(1-\lambda)x + \lambda x' \;|\; 0 \leq \lambda \leq 1\right\}$.

1.1.2 Convex Analysis

A set $C \subset \mathbb{R}^m$ is *locally closed* at a point $\bar{c}$, not necessarily in C, if there exists a closed neighborhood V of $\bar{c}$ such that $C \cap V$ is closed. Any closed set is locally closed at all of its points, and the closure and interior of C is denoted by $\operatorname{cl} C$ and $\operatorname{int} C$, respectively.

For a closed convex set $C \subset \mathbb{R}^m$, let $\operatorname{aff} C$ denote the *affine hull* of C and $\operatorname{par}(C)$ the *subspace parallel* to C. Then, for any $c \in C$, $\operatorname{par}(C) := \operatorname{aff} C - c = \mathbb{R}(C - C)$, where we employ *Minkowski set algebra* for addition of sets: for sets $C_1, C_2 \subset \mathbb{R}^m$ and $t \in \mathbb{R}$, define $C + C' := \left\{c + c' \;|\; c \in C,\; c' \in C'\right\}$ and $\Lambda C := \left\{\lambda c \;|\; \lambda \in \Lambda,\; c \in C\right\}$. When $C = \{c\}$, we omit the set braces and write $c + C'$. The *relative interior* of C is given by

$\operatorname{ri}(C) = \left\{x \in \operatorname{aff} C \;\middle|\; \exists\,(\epsilon > 0)\ (x + \epsilon\mathbb{B}) \cap \operatorname{aff} C \subset C\right\}$. For a sequence of sets $\{C_n\}_{n\in\mathbb{N}}$, with $C_n \subset \mathbb{R}^m$, the *outer and inner limits* are defined, respectively, as

$$\limsup_{n\to\infty} C_n := \left\{x \;\middle|\; \exists\,(\text{infinite } K \subset \mathbb{N},\ x^k \underset{K}{\to} x)\ \forall\,(k \in K)\ x^k \in C_k\right\}$$
$$\liminf_{n\to\infty} C_n := \left\{x \;\middle|\; \exists\,(n_0 \in \mathbb{N},\ x^n \to x)\ \forall\,(n \geq n_0)\ x^n \in C_n\right\}.$$

The sets C_n *converge* to a set C if the two limits agree and equal C:

$$\limsup_{n\to\infty} C_n = \liminf_{n\to\infty} C_n = C.$$

1.1.3 Variational Analysis

The functions in this paper take values in the extended reals $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$. For $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, the *domain* of f is $\operatorname{dom}(f) := \left\{x \in \mathbb{R}^n \;\middle|\; f(x) < \infty\right\}$, and the *epigraph* of f is $\operatorname{epi} f := \left\{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} \;\middle|\; f(x) \leq \alpha\right\}$.
We say f is *closed* if $\operatorname{epi} f$ is a closed subset of $\mathbb{R}^{n+1}$, f is *proper* if $\operatorname{dom}(f) \neq \emptyset$ and $f(x) > -\infty$ for all $x \in \mathbb{R}^n$, and f is *convex* if $\operatorname{epi} f$ is a convex subset of $\mathbb{R}^{n+1}$. For a set $X \subset \operatorname{dom}(f)$ and $\overline{x} \in X$, the function f is *strictly continuous at $\overline{x}$ relative to X* if

$$\limsup_{\substack{x,x' \underset{X}{\to} \overline{x} \\ x \neq x'}} \frac{\left\|f(x) - f(x')\right\|}{\|x - x'\|} < \infty,$$

where $x, x \underset{X}{\to} \overline{x} \iff x, x' \in X$ and $x, x' \to \overline{x}$ represents *converegence within* X. This finiteness property is equivalent to f being locally Lipschitz at $\overline{x}$ relative to X (see [42, Section 9.A]). By [40, Theorem 10.4], proper and convex functions $g : \mathbb{R}^n \to \overline{\mathbb{R}}$ are strictly continuous relative to $\operatorname{ri}\big(\operatorname{dom}(g)\big)$. To each nonempty closed convex set C, we associate the closed, proper, and convex *indicator function* defined by

$$\delta\left(x \mid C\right) := \begin{cases} 0 & x \in C, \\ +\infty & x \notin C. \end{cases}$$

Suppose $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is finite at $\overline{x}$ and $w, v \in \mathbb{R}^n$. The *subderivative* $\mathrm{d}f(\overline{x}) : \mathbb{R}^n \to \overline{\mathbb{R}}$ and *one-sided directional derivative* $f'(\overline{x}; \cdot)$ at $\overline{x}$ for w are

$$\mathrm{d}f(\overline{x})(w) := \liminf_{\substack{t \searrow 0 \\ w' \to w}} \frac{f(\overline{x} + tw) - f(\overline{x})}{t}, \qquad f'(\overline{x}; w) := \lim_{t \searrow 0} \frac{f(\overline{x} + tw) - f(\overline{x})}{t}.$$

At points $w \in \mathbb{R}^n$ such that $f'(\overline{x}; w)$ exists and is finite, the *one-sided second directional derivative* is

$$f''(\overline{x}; w) := \lim_{t \searrow 0} \frac{f(\overline{x} + tw) - f(\overline{x}) - tf'(\overline{x}; w)}{\frac{1}{2}t^2}.$$

For any $w, v \in \mathbb{R}^n$, the *second subderivative at $\overline{x}$ for v and $w \in \mathbb{R}^n$* is

$$\mathrm{d}^2 f(\overline{x}|v)(w) := \liminf_{\substack{t \searrow 0 \\ w' \to w}} \Delta_t^2 f(\overline{x}|v)(w), \text{ where } \Delta_t^2 f(\overline{x}|v)(w) := \frac{f(\overline{x} + tw') - f(\overline{x}) - t \langle v, w' \rangle}{\frac{1}{2}t^2}.$$

The structure of our problem class allows the classical one-sided first and second directional derivatives $f'(\overline{x}; \cdot)$ and $f''(\overline{x}; \cdot)$ to entirely capture the variational properties of their more general counterparts.

Suppose $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is finite at $\overline{x}$. Define the *(Fréchet) regular subdifferential*

$$\widehat{\partial} f(\overline{x}) := \left\{ v \in \mathbb{R}^n \;\middle|\; f(x) \geq f(\overline{x}) + \langle v, x - \overline{x} \rangle + o(\|x - \overline{x}\|) \right\},$$

and the *(limiting or Mordukhovich) subdifferential* by

$$\partial f(\overline{x}) := \left\{ v \in \mathbb{R}^n \;\middle|\; \exists\, (x^n \underset{f}{\to} \overline{x})\ \exists\, (v^n \to v)\ \forall\, (n \in \mathbb{N})\ v^n \in \widehat{\partial} f(x^n) \right\}, \tag{1.7}$$

where $x^n \underset{f}{\to} \overline{x}$ denotes *f-attentive convergence*, i.e., that $x^n \to \overline{x}$, with $f(x^n) \to f(\overline{x})$. In the case of a closed, proper, convex function f, the set $\partial f(\overline{x})$ is the usual subdifferential of convex analysis. The tools of first and second subderivative functions and subdifferential sets allow us to concisely write first-order necessary conditions and second-order necessary and sufficient conditions for local minima.

Theorem 1 (First-order necessity, second-order necessity and sufficiency)**.** [42, Theorems 10.1, 13.24] For a proper function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, consider the problem $\min_x f(x)$.

(a) If f has a local minimum at $\overline{x}$, then $0 \in \partial f(\overline{x})$ and for all $w \in \mathbb{R}^n$, $\mathrm{d}f(\overline{x})(w) \geq 0$ and $\mathrm{d}^2 f(\overline{x}|0)(w) \geq 0$.

(b) If $0 \in \partial f(\overline{x})$ and $\mathrm{d}^2 f(\overline{x}|0)(w) > 0$ for $w \neq 0$, then $\overline{x}$ is a local minimizer of f.

(c) The statement $0 \in \partial f(\overline{x})$ and $\mathrm{d}^2 f(\overline{x}|0)(w) > 0$ for $w \neq 0$ is equivalent to $\overline{x}$ being a *strong local minimizer* of f, i.e., there exists a neighborhood U of $\overline{x}$ and a constant $\gamma > 0$ such that

$$f(x) \geq f(\overline{x}) + \gamma \|x - \overline{x}\|^2 \text{ for all } x \in U \cap \operatorname{dom}(f). \tag{1.8}$$

A set-valued mapping $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is a mapping from $\mathbb{R}^n$ into the power set of $\mathbb{R}^m$, so for each $x \in \mathbb{R}^n$, $S(x) \subset \mathbb{R}^m$. The *graph* and *domain* of S are defined to be

$$\operatorname{gph} S := \left\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \;\middle|\; y \in S(x)\right\} \text{ and } \operatorname{dom}(S) := \left\{x \in \mathbb{R}^n \;\middle|\; S(x) \neq \emptyset\right\},$$

and S is *graph-convex* whenever $\operatorname{gph} S$ is a convex subset of $\mathbb{R}^n \times \mathbb{R}^m$. For a point $(\overline{x}, \overline{y}) \in \operatorname{gph} S$, and neighborhoods U of $\overline{x}$ and V of $\overline{y}$, a *graphical localization* of S at $\overline{x}$ for $\overline{y}$ is a set-valued mapping $\widetilde{S}$ defined by $\operatorname{gph} \widetilde{S} = \operatorname{gph} S \cap (U \times V)$. A *single-valued localization* of S *at* $\overline{x}$ for $\overline{y}$ is a graphical localization that is also function. If the domain of $\widetilde{S}$ is a neighborhood of $\overline{x}$, $\widetilde{S}$ is called a single-valued localization of S *around* $\overline{x}$ for $\overline{y}$. The mapping S is *outer semicontinuous* at $\overline{x}$ relative to $X \subset \mathbb{R}^n$ if

$$\limsup_{x \xrightarrow[X]{} \overline{x}} S(x) := \left\{ u \;\middle|\; \exists\, (x^n \xrightarrow[X]{} \overline{x})\; \exists\, (u^n \to u)\; \forall\, (n \in \mathbb{N})\; u^n \in S(x^n) \right\} \subset S(\overline{x}),$$

and is *inner semicontinous* relative to $X \subset \mathbb{R}^n$ if

$$S(\overline{x}) \subset \liminf_{x \xrightarrow[X]{} \overline{x}} S(x) := \left\{ u \;\middle|\; \forall\, (x^n \xrightarrow[X]{} \overline{x})\; \exists\, (N \in \mathbb{N},\; u^n \to u)\; \forall\, (n \geq N)\; u^n \in S(x^n) \right\},$$

where $x^n \xrightarrow[X]{} \overline{x} \iff x^n \to \overline{x}$ with $x^n \in X$. Then, (1.7) is $\partial f(\overline{x}) := \limsup_{x \xrightarrow[f]{} \overline{x}} \widehat{\partial} f(x)$. The last notion employed from variational analysis is that of normal and tangent vectors. Let

$C \subset \mathbb{R}^n$, and let $\bar{c} \in C$. Define the *normal cone* to C at $\bar{c}$ as

(1.9)
$$N\left(\bar{c}\,|\,C\right) := \limsup_{c \xrightarrow[C]{} \bar{c}} \widehat{N}(c\,|\,C), \text{ where } \widehat{N}(c\,|\,C) := \left\{ v \;\middle|\; \forall\,(c' \in C)\;\; \left\langle v,\, c'-c \right\rangle \leq o(\left\|c'-c\right\|) \right\},$$

and the *tangent cone* to C at $\bar{c}$ as $T\left(\bar{c}\,|\,C\right) := \limsup_{t \searrow 0} t^{-1}(C - \bar{c})$. A set C is *Clarke regular* at $\bar{c} \in C$ if C is locally closed at $\bar{c}$ and $N\left(\bar{c}\,|\,C\right) = \widehat{N}(\bar{c}\,|\,C)$. A nonempty, closed, convex set C is Clarke regular at all $\bar{c} \in C$, with $N\left(\bar{c}\,|\,C\right) = \left\{ v \mid \langle v,\, c - \bar{c}\rangle \leq 0 \text{ for all } c \in C \right\}$, and $T\left(\bar{c}\,|\,C\right) = \left\{ v \;\middle|\; \langle v,\, w\rangle \leq 0 \text{ for all } w \in N\left(\bar{c}\,|\,C\right) \right\} = \operatorname{cl}\left\{\mathbb{R}_{++}(C - \bar{c})\right\}$ [42, Theorem 6.9]. We refer the reader to [42, Chapter 6] for a thorough exposition.

Suppose $g : \mathbb{R}^n \to \mathbb{R}^m$ is $\mathcal{C}^1$-smooth, $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is a set-valued mapping with closed graph and $\{\mathbf{B}_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^{m\times n}$. Consider the *generalized equation* $0 \in g(z) + G(z)$. The *Newton method* for $g + G$ is the iteration

$$\text{(1.10)} \qquad \text{find } z^{k+1} \text{ such that } 0 \in g(z^k) + \nabla g(z^k)(z^{k+1} - z^k) + G(z^{k+1}), \text{ for } k \in \mathbb{N},$$

and the *quasi-Newton method* for $g + G$ is the iteration

$$\text{(1.11)} \qquad \text{find } z^{k+1} \text{ such that } 0 \in g(z^k) + \mathbf{B}_k(z^{k+1} - z^k) + G(z^{k+1}), \text{ for } k \in \mathbb{N}.$$

Finally, a sequence of functions $f^k : \mathbb{R}^m \to \overline{\mathbb{R}}$ *epigraphically converges* to $f : \mathbb{R}^m \to \overline{\mathbb{R}}$, written $f^k \xrightarrow{e} f$, if and only if $\operatorname{epi} f^k \to \operatorname{epi} f$ (see [42, Section 7.B]).

Chapter 2

CONVEX-COMPOSITE THEORY

In this chapter we recall the basic ingredients of convex-composite optimization and the associated variational structures.

Definition 2.0.1 (Convex-composite functions)**.** Let $h : \mathbb{R}^m \to \overline{\mathbb{R}}$ be a closed, proper, convex function and $c : \mathbb{R}^n \to \mathbb{R}^m$ a smooth function. Define $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ by $f(x) := h(c(x))$. We say the function f is convex-composite.

2.1 Finite-Valued Convex-Composite

In Chapter 3, we assume $f(x) = h(c(x)) + g(x)$, where h is finite-valued and convex and g is closed, proper, and convex. In this case, the calculus simplifies dramatically. As in [4], we have $\operatorname{dom}(f) = \operatorname{dom}(g)$ and

$$f(x+d) = h(c(x) + \nabla c(x)d) + g(x+d) + o(\|d\|). \tag{2.1}$$

Consequently, at any $x \in \operatorname{dom}(g)$ and $d \in \mathbb{R}^n$, f is directionally differentiable, with

$$\mathrm{d}f(x)(d) = f'(x; d) = h'(c(x); \nabla c(x)d) + g'(x; d).$$

This motivates defining the *subdifferential* of f at any $x \in \operatorname{dom}(g)$ by setting

$$\partial f(x) := \nabla c(x)^\top \partial h(c(x)) + \partial g(x). \tag{2.2}$$

Within the context of variational analysis [42], we have that f is *subdifferentially regular* on its domain and the subdifferential of f as defined above agrees with the *regular and limiting subdifferentials* of variational analysis. In particular, $f'(x; d) = \sup_{v \in \partial f(x)} \langle v,\, d\rangle$.

Following [3], we define an approximation to the directional derivative that is key to our algorithmic development.

Definition 2.1.1. Let f be as in $\mathcal{P}$ and $x \in \operatorname{dom}(g)$. Define $\Delta f(x; \cdot) : \mathbb{R}^n \to \overline{\mathbb{R}}$ by

$$\Delta f(x; d) = h(c(x) + \nabla c(x)d) + g(x+d) - h(c(x)) - g(x). \tag{2.3}$$

The next lemma records the interplay between $\Delta f(x; d)$ and its infinitesimal counterpart $f'(x; d)$ and is a consequence of (2.1) and the definitions.

Lemma 2.1.1. Let f be given as in $\mathcal{P}$ and let $x \in \operatorname{dom}(g)$. Then

(a) the function $d \mapsto \Delta f(x; d)$ is convex;

(b) for any $d \in \mathbb{R}^n$, the difference quotients $\frac{\Delta f(x;td)}{t}$ are nondecreasing in $t > 0$, with

$$f'(x; d) = \inf_{t>0} \frac{\Delta f(x; td)}{t},$$

and in particular;

(c) for any $d \in \mathbb{R}^n$, $f'(x; d) \leq \Delta f(x; d)$;

(d) for any $d \in \mathbb{R}^n$, $t \in [0, 1]$, $\Delta f(x; td) \leq t\Delta f(x; d)$.

We now state equivalent first-order necessary conditions for a local minimizer $\overline{x}$ of $\mathcal{P}$, emphasizing that $f'(x; d)$ and $\Delta f(x; d)$ are interchangeable with respect to these conditions. The proof of this result parallels that given in [4] using (2.1) and (2.2).

Theorem 2 (First-order necessary conditions for $\mathcal{P}$)**.** [4, Theorem 2.6] Let h, c, and g be as given in $\mathcal{P}$. If $\overline{x} \in \operatorname{dom}(g)$ is a local minimizer of $\mathcal{P}$, then $f'(\overline{x}; d) \geq 0$, for all $d \in \mathbb{R}^n$. Moreover, the following conditions are equivalent for any $x \in \operatorname{dom}(g)$,

(a) $0 \in \partial f(x)$;

(b) for all $d \in \mathbb{R}^n$, $0 \leq f'(x; d)$;

(c) for all $d \in \mathbb{R}^n$, $0 \leq \Delta f(x; d)$;

(d) for all $\eta > 0$, $d = 0$ solves $\min\{\Delta f(x; d) \mid \|d\| \leq \eta\}$.

The next lemma shows that if the sequence $\{(x^k, d^k)\} \subset \mathbb{R}^n \times \mathbb{R}^n$ is such that d^k is an approximate solution to $\mathcal{P}_k$ for all k with $\Delta f(x^k; d^k) \to 0$, then cluster points of $\{x^k\}$ are first-order stationary for $\mathcal{P}$.

Lemma 2.1.2. Let h, c, and g be as in $\mathcal{P}$ and $\alpha \in \mathbb{R}$. Set $\mathcal{L} := \operatorname{lev}_f(\alpha)$. Let $\{(x^k, \eta_k)\} \subset \mathcal{L} \times \mathbb{R}_+$, with $(x^k, \eta_k) \to (\overline{x}, \overline{\eta}) \in \mathbb{R}^n \times \mathbb{R}_+$ and $0 < \overline{\eta} < \infty$. Define

$$\begin{aligned} \Delta_k f(d) &:= \Delta f(x^k; d) + \delta_{\eta_k \mathbb{B}}(d), \text{ and} \\ \overline{\Delta}_k f &:= \min_d \Delta_k f(d) \end{aligned} \tag{2.4}$$

If, for each $k \geq 1$, $d^k \in \eta_k \mathbb{B}$ satisfies

$$\Delta f(x^k; d^k) \leq \beta \overline{\Delta}_k f \leq 0, \tag{2.5}$$

with $\Delta f(x^k; d^k) \to 0$, then $0 \in \partial f(\overline{x})$.

Proof. Since f is closed, $f(\overline{x}) \leq \alpha$, which implies $\overline{x} \in \operatorname{dom}(g)$. Define the functions

$$\begin{aligned} h_k(d) &:= h(c(x^k) + \nabla c(x^k) d) - h(c(x^k)), \\ h_\infty(d) &:= h(c(\overline{x}) + \nabla c(\overline{x}) d) - h(c(\overline{x})), \\ g_k(d) &:= g(x^k + d) - g(x^k), \text{ and} \\ g_\infty(d) &:= g(\overline{x} + d) - g(\overline{x}). \end{aligned}$$

Since $0 < \eta_k \to \overline{\eta}$, with $\overline{\eta} > 0$, and since $\delta\left(d \mid \eta_k \mathbb{B}\right) = \delta\left(\frac{1}{\eta_k} d \mid \mathbb{B}\right)$, [42, Proposition 7.2] implies

$$\delta\left(\cdot \mid \eta_k \mathbb{B}\right) \xrightarrow{e} \delta\left(\cdot \mid \overline{\eta} \mathbb{B}\right).$$

By [42, Exercise 7.8(d)], $g_k \xrightarrow{e} g_\infty$, so [42, Exercise 7.47] implies $g_k + \delta\left(\cdot \mid \eta_k \mathbb{B}\right) \xrightarrow[k \to \infty]{e} g_\infty + \delta\left(\cdot \mid \overline{\eta} \mathbb{B}\right)$, and applying [42, Exercise 7.47] again yields

$$h_k + g_k + \delta\left(\cdot \mid \eta_k \mathbb{B}\right) \xrightarrow{e} h_\infty + g_\infty + \delta\left(\cdot \mid \overline{\eta} \mathbb{B}\right).$$

Equivalently,

$$\Delta f(x^k; \cdot) + \delta\left(\cdot \mid \eta_k \mathbb{B}\right) \xrightarrow{e} \Delta f(\overline{x}; \cdot) + \delta\left(\cdot \mid \overline{\eta} \mathbb{B}\right).$$

By [42, Proposition 7.30] and (2.5),

$$0 = \limsup_k \overline{\Delta}_k f \leq \min_{\|d\| \leq \overline{\eta}} \Delta f(\overline{x}; d) \leq 0,$$

so Theorem 2 implies $0 \in \partial f(\overline{x})$. □

The approximate solution condition (2.5) is described in [3]. It can be satisfied by employing the trick described in [5, Remark 6, page 343]. Specifically, any solution technique solving the convex subproblems $\mathcal{P}_k$ that also generates lower bounds $\ell_{k,j} \in \mathbb{R}$ such that $\ell_{k,j} \nearrow \overline{\Delta}_k f$ and $\Delta f(x^k; d^{k,j}) \searrow \overline{\Delta}_k f$ as $j \to \infty$. If $\overline{\Delta}_k f < 0$, then the condition

$$\Delta f(x^k; d^{k,j}) \leq \beta \ell_{k,j}$$

is finitely satisfied, and

$$\Delta f(x^k; d^{k,j}) \leq \beta \overline{\Delta}_k f.$$

We conclude this section with a mean-value theorem for $\mathcal{P}$.

Theorem 3 (Mean-Value for convex-composite)**.** [42, Theorem 10.48] Let f be as in $\mathcal{P}$, g strictly continuous relative to its domain, and $x_0, x_1 \in \operatorname{dom}(g)$. Then there exists $t \in (0,1)$, $x_t := (1-t)x_0 + tx_1$ and $v \in \partial f(x_t)$ such that

$$f(x_1) - f(x_0) = \langle v,\, x_1 - x_0 \rangle \,.$$

Proof. Let $F(t) := (1-t)x_0 + tx_1$ and let $\varphi(t) = f(F(t)) - (1-t)f(x_0) - tf(x_1)$. Then

$$\varphi(t) = h(c(F(t))) + g(F(t)) - (1-t)f(x_0) - tf(x_1)$$

is an instance of $\mathcal{P}$, since $g \circ F$ is convex. Consequently, the chain rules for φ and $-\varphi$ on $[0,1]$ are

$$\begin{aligned} \partial\varphi(t) &= F'(t)^\top \nabla c(F(t))^\top \partial h(c(F(t))) + F'(t)^\top \partial g(F(t)) + f(x_0) - f(x_1) \\ &= \Big\{ \langle v,\, x_1 - x_0 \rangle \;\Big|\; v \in \partial f(F(t)) \Big\} + f(x_0) - f(x_1), \text{ and} \\ \partial(-\varphi)(t) &= F'(t)^\top \nabla c(F(t))^\top \partial(-h)(c(F(t))) + F'(t)^\top \partial(-g)(F(t)) + f(x_1) - f(x_0) \\ &= \Big\{ \langle -v,\, x_1 - x_0 \rangle \;\Big|\; v \in \partial f(F(t)) \Big\} + f(x_1) - f(x_0). \end{aligned}$$

As g is continuous on its domain, φ is continuous on $[0,1]$ with $\varphi(0) = \varphi(1) = 0$. Therefore, φ attains either its minimum or maximum value at some $\bar{t} \in (0,1)$, and $0 \in \partial\varphi(\bar{t})$ or $0 \in \partial(-\varphi)(\bar{t})$ respectively. ☐

2.2 Infinite-Valued Convex-Composite

In Chapter 4, we study fully infinite-valued convex-composite optimization. This section studies the calculus associated with convex-composite optimization in the infinite-valued case.

Definition 2.2.1 (Convex-composite Lagrangian)**.** [4] For any $y \in \mathbb{R}^m$, define the function $(yc) : \mathbb{R}^n \to \mathbb{R}$ by $(yc)(x) := \big\langle y,\, c(x)\big\rangle$. The Lagrangian for the convex-composite f is defined by $L(x, y) := (yc)(x) - h^\star(y)$, where $h^\star : \mathbb{R}^m \to \overline{\mathbb{R}}$ denotes the *Fenchel conjugate* of the convex function h defined by $h^\star(y) := \sup_{z \in \mathbb{R}^m} \langle z,\, y\rangle - h(z)$. The Hessian of L in its first variables is denoted

$$\nabla^2_{xx} L(x, y) = \nabla^2(yc)(x) = \sum_{i=1}^{m} y_i \nabla^2 c_i(x). \tag{2.6}$$

Definition 2.2.2 (Convex-composite multiplier sets)**.** Suppose f is convex-composite. Define the set of multipliers at $\overline{x} \in \operatorname{dom}(f)$ for $v \in \mathbb{R}^n$ as in [42, Theorem 13.14] by

$$Y(\overline{x}, v) := \left\{ y \;\middle|\; \begin{pmatrix} v \\ 0 \end{pmatrix} \in \begin{pmatrix} \partial_x L(\overline{x}, y) \\ \partial_y(-L)(\overline{x}, y) \end{pmatrix} \right\} = \left\{ y \in \partial h(c(\overline{x})) \;\middle|\; \nabla c(\overline{x})^\top y = v \right\}, \tag{2.7}$$

and define the set of multipliers at $\overline{x}$ for 0 by

$$M(\overline{x}) := Y(\overline{x}, 0) = \operatorname{Null}\left(\nabla c(\overline{x})^\top\right) \cap \partial h(c(\overline{x})). \tag{2.8}$$

A calculus for convex-composite functions at a point $\overline{x} \in \operatorname{dom}(f)$ requires various types of "constraint qualifications." Stronger versions of the *basic constraint qualification* (BCQ) will be employed to ensure uniqueness of the multiplier and underlying strict complementarity properties in later sections.

Definition 2.2.3 (Convex-composite constraint qualifications)**.** Suppose f is convex-composite and $\overline{x} \in \operatorname{dom}(f)$. We say f satisfies the

- *basic constraint qualification* at $\overline{x}$ if

$$\text{(BCQ)} \qquad \operatorname{Null}\left(\nabla c(\overline{x})^\top\right) \cap N\left(c(\overline{x}) \,|\, \operatorname{dom}(h)\right) = \{0\},$$

- *transversality condition* at $\overline{x}$ if

$$\text{(TC)} \qquad \operatorname{Null}\left(\nabla c(\overline{x})^\top\right) \cap \operatorname{par}\left(\partial h(c(\overline{x}))\right) = \{0\},$$

- *strict criticality condition* at $\overline{x} \in \operatorname{dom}(f)$ for $\overline{y}$ if

$$\text{(SC)} \qquad \operatorname{Null}\left(\nabla c(\overline{x})^\top\right) \cap \operatorname{ri}\left(\partial h(c(\overline{x}))\right) = \{\overline{y}\}.$$

Remark 1. Following [42, Definition 10.23], one says that a convex-composite function f is strongly amenable at $\overline{x} \in \operatorname{dom}(f)$ if f satisfies (BCQ) at $\overline{x}$. One says that f is fully amenable at $\overline{x} \in \operatorname{dom}(f)$ if f satisfies (BCQ) at $\overline{x}$ and the function h is PLQ convex. Here, we make use of the underlying assumption that c is $\mathcal{C}^2$-smooth.

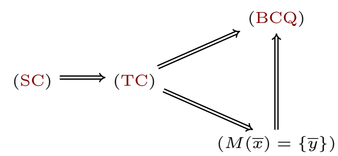
Notice the basic constraint qualification is a *local property* in the following sense. If f satisfies (BCQ) at $\overline{x}$, then there exists a neighborhood U of $\overline{x}$ such that f satisfies (BCQ) at all $x \in [U \cap c^{-1}(\operatorname{dom}(h))]$. Moreover, the basic constraint qualification ensures that the chain rule applies in the subdifferential calculus for convex-composite functions and establishes a foundation for the application of tools from variational analysis.

Theorem 4 (Convex-composite first order necessary conditions)**.** Suppose f is convex-composite and $\overline{x} \in \operatorname{dom}(f)$ is such that f satisfies (BCQ) at $\overline{x}$. Then, $\partial f(\overline{x}) = \nabla c(\overline{x})^\top \partial h(c(\overline{x}))$, and for any $d \in \mathbb{R}^n$, $\mathrm{d}f(\overline{x})(d) = h'(c(\overline{x}); \nabla c(\overline{x})d)$. Suppose, in addition, that $\overline{x}$ is a local solution to **P**. Then, $M(\overline{x}) := \operatorname{Null}\left(\nabla c(\overline{x})^\top\right) \cap \partial h(c(\overline{x})) \neq \emptyset$, or equivalently, $0 \in \partial f(\overline{x})$, and for any $d \in \mathbb{R}^n$, $h'(c(\overline{x}); \nabla c(\overline{x})d) \geq 0$.

Proof. This follows from Theorem 2 and [42, Proposition 8.21, Exercise 10.26(b)]. □

We now establish a relationship between the various notions of a constraint qualification given in Definition 2.2.3.

Lemma 2.2.1. Suppose f is convex-composite, $\overline{x} \in \operatorname{dom}(f)$, and $\overline{y} \in \mathbb{R}^m$. Then, the following implications hold:



Proof. [(TC)$\Longrightarrow$(BCQ)] By [42, Proposition 8.12], at any point $\bar{c} \in \operatorname{dom}(\partial h)$, $N\left(\bar{c} \,|\, \operatorname{dom}(h)\right) \subset \operatorname{par}\left(\partial h(\bar{c})\right)$. The implication follows.

[$(M(\overline{x}) = \{\overline{y}\}) \Longrightarrow$(BCQ)]

Let $M(\overline{x}) = \{\overline{y}\}$ and suppose there exists

$$0 \neq v \in \operatorname{Null}\left(\nabla c(\overline{x})^\top\right) \cap N\left(c(\overline{x}) \,|\, \operatorname{dom}(h)\right) \subset \operatorname{Null}\left(\nabla c(\overline{x})^\top\right) \cap \operatorname{par}\left(\partial h(c(\overline{x}))\right).$$

Then, by the subgradient inequality, $v + \overline{y} \in \operatorname{Null}\left(\nabla c(\overline{x})^\top\right) \cap \partial h(c(\overline{x})) = M(\overline{x})$, which is a contradiction.

The rest of the proof appears in Lemma 4.6.1 in the appendix as general facts about closed convex sets C and linear maps A. $\square$

Gauss-Newton methods for iteratively solving **P** are based on finding a search direction that approximates a solution to subproblems of the form

$$(\widehat{\mathbf{P}}) \qquad \underset{d \in \mathbb{R}^n}{\text{minimize}} \quad h(c(\widehat{x}) + \nabla c(\widehat{x})d) + \frac{1}{2} d^\top \widehat{H} d.$$

Local rates of convergence for algorithms of this type, where the function h is assumed to be finite-valued and piecewise linear convex were developed by Womersley [46] based on tools developed for classical nonlinear programming. More recently, Cibulka et. al. [9] successfully applied a modern approach through generalized equations to obtain similar and stronger results again in the piecewise linear convex case. Inspired by these results and the existence of a sophisticated first- and second-order subdifferential calculus for piecewise linear-quadratic convex functions [42], we develop a convergence theory in the piecewise linear-quadratic case from the generalized equations perspective. The basic notational objects for our development are given in the next definition.

Definition 2.2.4 (Convex-composite generalized equations)**.** Let f be convex-composite, and define the set-valued mapping $g + G : \mathbb{R}^{n+m} \rightrightarrows \mathbb{R}^{n+m}$ by

$$(2.9) \qquad g(x,y) = \begin{pmatrix} \nabla c(x)^\top y \\ -c(x) \end{pmatrix}, \quad G(x,y) = \begin{pmatrix} \{0\}^n \\ \partial h^\star(y) \end{pmatrix}.$$

For a fixed $(\overline{x}, \overline{y}) \in \mathbb{R}^n \times \mathbb{R}^m$, define the linearization mapping

$$(2.10) \qquad \mathcal{G} : (x,y) \mapsto g(\overline{x},\overline{y}) + \nabla g(\overline{x},\overline{y}) \begin{pmatrix} x - \overline{x} \\ y - \overline{y} \end{pmatrix} + G(x,y),$$

where $\nabla g(\overline{x},\overline{y}) = \begin{pmatrix} \nabla^2(\overline{y}c)(\overline{x}) & \nabla c(\overline{x})^\top \\ -\nabla c(\overline{x}) & 0 \end{pmatrix}$.

Observe that for any $\overline{x} \in \operatorname{dom}(f)$ where f satisfies (BCQ), $\overline{x}$ satisfies the first-order necessary conditions of Theorem 2 for the problem **P** if and only if there exists $\overline{y}$ such that $(\overline{x},\overline{y})$ solves the generalized equation $g + G \ni 0$. More precisely, we have

$$(2.11) \qquad 0 \in g(\overline{x},\overline{y}) + G(\overline{x},\overline{y}) \Leftrightarrow \nabla c(\overline{x})^\top \overline{y} = 0 \text{ and } \overline{y} \in \partial h(c(\overline{x})) \Leftrightarrow M(\overline{x}) \neq \emptyset.$$

The relationship between the linearization of the generalized equation described in (2.10) and the subproblems $\widehat{\mathbf{P}}$ is described in the following lemma. The proof follows from Theorem 4.

Lemma 2.2.2. Let f be convex-composite and $(\widehat{x},\widehat{y}) \in \mathbb{R}^n \times \mathbb{R}^m$ be such that f satisfies (BCQ) at $\widehat{x}$, and define $\widehat{H} := \nabla^2(\widehat{y}c)(\widehat{x})$. Then, $(\tilde{d},\tilde{y}) \in \mathbb{R}^n \times \mathbb{R}^m$ satisfy the optimality conditions for

$$(\widehat{\mathbf{P}}) \qquad \underset{d \in \mathbb{R}^n}{\text{minimize}} \quad h(c(\widehat{x}) + \nabla c(\widehat{x})d) + \frac{1}{2} d^\top \widehat{H} d$$

if and only if $(\widehat{x}+\tilde{d},\tilde{y})$ solves the Newton equations for $g+G$: $0 \in g(\widehat{x},\widehat{y}) + \nabla g(\widehat{x},\widehat{y}) \begin{pmatrix} x - \widehat{x} \\ y - \widehat{y} \end{pmatrix} + G(x,y)$.

Chapter 3

STEP SIZE METHODS

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Abstract This chapter considers descent methods for solving non-finite valued nonsmooth convex-composite optimization problems that employ Gauss-Newton subproblems to determine the iteration update. Specifically, we establish the global convergence properties for descent methods that use a backtracking line-search, a weak Wolfe line-search, or a trust-region update. All of these approaches are designed to exploit the structure associated with convex-composite problems.

3.1 Introduction

This chapter considers three descent methods for solving the convex-composite optimization problem

$$(\mathcal{P}) \qquad \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad f(x) := h(c(x)) + g(x),$$

where $h : \mathbb{R}^m \to \mathbb{R}$ is convex, $g : \mathbb{R}^n \to \overline{\mathbb{R}}$ is proper, strictly continuous relative to its domain and convex, and $c : \mathbb{R}^n \to \mathbb{R}^m$ is $\mathcal{C}^1$-smooth. Our focus is on methods that employ search directions or steps $d^k \in \mathbb{R}^n$ that approximate solutions to Gauss-Newton subproblems

$$(\mathcal{P}_k) \qquad \begin{aligned} &\underset{d \in \mathbb{R}^n}{\text{minimize}} \quad \Delta f(x^k; d) := h(c(x^k) + \nabla c(x^k)d) + g(x^k + d) - f(x^k) \\ &\text{subject to} \quad \|d\| \leq \eta_k, \end{aligned}$$

where $\{x^k\} \subset \operatorname{dom}(g)$ are the iterates generated by the algorithm, $\{\eta_k\} \subset (0, \infty]$, and $\Delta f(x; d)$ is an approximation to the directional derivative $f'(x; d)$ as in [3] (also see Lemma 2.1.1). By *descent*, we mean that d^k satisfies $\Delta f(x^k; d^k) < 0$ at each iteration k. Two of the approaches are line search methods based on backtracking and a weak Wolfe conditions. The third approach is a trust-region method.

Previously, the backtracking line search was studied in finite-valued case and in the absence of the function g [3]. In recent work, Lewis and Wright [28] utilized a similar backtracking line search in the context of infinite-valued prox-regular composite optimization. Lewis and Overton [27] developed a weak Wolfe algorithm using directional derivatives for finite-valued nonsmooth functions f that are absolutely continuous along the line segment of interest, with finite termination in particular when the function f is semi-algebraic. The method of Lewis and Overton can be applied in the finite-valued convex-composite case where $g = 0$. Here, we develop a weak Wolfe algorithm for infinite-valued problems that uses the approximation $\Delta f(x; d)$ to the directional derivative which exploits the structure associated with convex-composite problems.

The function g in $\mathcal{P}$ is typically nonsmooth and is used to induce structure in the solution $\overline{x}$. For example, it can be used to introduce sparsity or group sparsity in the solution $\overline{x}$ as well as bound constraints $\overline{x}$. Drusvyatskiy and Lewis [18] have established local and global convergence of proximal-based methods for solving $\mathcal{P}$, and Drusvyatskiy and Paquette [16] have established iteration complexity results for proximal methods to locate first-order stationary points for $\mathcal{P}$.

While the assumptions we use are similar to those in [16, 18], our algorithmic approach differs significantly. In particular, we use either a backtracking or an adaptive weak Wolfe line search, or a trust-region strategy, to induce objective function descent at each iteration. In addition, we do not exclusively use proximal methods to generate search directions or employ the backtracking line search to estimate Lipschitz constants as in [16, 28]. Moreover, all of the methods discussed here make explicit use of the structure in $\mathcal{P}$, thereby differing from the method developed in [27].

3.2 Backtracking for Convex-Composite Minimization

The simplest and most well established line search is the Armijo-Goldstein backtracking procedure [47]. It has been adapted for the convex-composite setting in [3, 35, 49] where it takes the form

$$f(x + td) \leq f(x) + \sigma_1 t \Delta f(x; d)$$

and enforces *sufficient decrease* of f along the ray $\left\{x + td \mid t > 0\right\}$, with Δf acting as a surrogate for the directional derivative. Existence of step sizes $t > 0$ satisfying the sufficient decrease follows immediately from Lemma 2.1.1. The method of proof to follow adapts the step-size arguments given in Royer and Wright [43] to the convex-composite setting. Similar ideas on convex majorants for the composite $\mathcal{P}$ are employed in [16, 28].

Algorithm 1 Global Backtracking

procedure BACKTRACKINGGLOBAL(x^0, σ_1, θ)
    $k \leftarrow 0$
    repeat
        Find $d^k \in \mathbb{R}^n$ such that $\Delta f(x^k; d^k) < 0$
        if no such d^k **then**
            $0 \in \partial f(x^k)$ **return**
        end if
        $t \leftarrow 1$
        while $f(x^k + td^k) > f(x^k) + \sigma_1 t \Delta f(x^k; d^k)$ **do**
            $t \leftarrow \theta t$
        end while
        $t_k \leftarrow t$
        $x^k \leftarrow x^k + t_k d^k$
        $k \leftarrow k + 1$
    until
end procedure

Theorem 5. Let f be as in $\mathcal{P}$, $x^0 \in \operatorname{dom}(g)$, $0 < \sigma_1 < 1$, and $0 < \theta < 1$. Set $\mathcal{L} := \operatorname{lev}_f(f(x^0))$. Suppose there exists $M > 0$ and $\widetilde{M} > 0$ such that $\|d^k\| \leq M$, $\sup_{x \in \mathcal{L}} \|\nabla c(x)\| \leq \widetilde{M}$, and that

(i) ∇c is $L_{\nabla c}$-Lipschitz on $\mathcal{L} + M\mathbb{B}_n$;

(ii) h is L_h-Lipschitz on $c(\mathcal{L} + M\mathbb{B}) + \widetilde{M}M\mathbb{B}_m$.

Let $\{x^k\}$ be a sequence initialized at x^0 and generated by Algorithm 1: Then one of the following must occur:

(a) the algorithm terminates finitely at a first-order stationary point for f;

(b) $f(x^k) \searrow -\infty$;

(c) $\displaystyle\sum_{k=0}^{\infty} \frac{\Delta f(x^k; d^k)^2}{\left\|d^k\right\|_2^2} < \infty$, in particular, $\Delta f(x^k; d^k) \to 0$.

Proof. We assume (a) - (b) do not occur and show (c) occurs. Since (a) does not occur, the sequence $\{x^k\}$ is infinite, and $\Delta f(x^k; d^k) < 0$ for all $k \geq 0$. The sufficient decrease (WWI) obtained by the backtracking subroutine gives a strict descent method, so the function values $\{f(x^k)\}$ are strictly decreasing, with $\{x^k\} \subset \mathcal{L}$ for all $k \geq 0$. In particular, $f(x^k) \searrow \overline{f} > -\infty$.

We first show that for each $k \geq 0$, the step size $0 < t_k \leq 1$ satisfies

$$t_k \geq \min\left\{1, \frac{\mu(1-\sigma_2)|\Delta f(x^k; d^k)|}{L_{\nabla c} L_h \left\|d^k\right\|^2}\right\}, \tag{3.1}$$

by considering two cases.

If the unit step $t_k = 1$ is accepted, the bound is immediate. Following [43], suppose now that the unit step length is not accepted. Then $\widehat{t} := \theta^j \in (0, 1]$ does not satisfy the decrease condition for some $j \geq 0$. Using the Lipschitz condition on h, the quadratic bound lemma, and Lemma 2.1.1, we obtain

$$\begin{aligned}\sigma_1 \widehat{t} \Delta f(x^k; d^k) < f(x^k + \widehat{t} d^k) - f(x) &\leq \Delta f(x^k; \widehat{t} d^k) + \frac{L_{\nabla c} L_h}{2} \left\|\widehat{t} d^k\right\|_2^2 \\ &\leq \widehat{t} \Delta f(x^k; d^k) + (\widehat{t})^2 \frac{L_{\nabla c} L_h}{2} \left\|d^k\right\|^2\end{aligned}$$

After dividing both sides by $\widehat{t} > 0$ and rearranging,

$$\widehat{t} \geq \frac{2(1-\sigma_1)|\Delta f(x^k; d^k)|}{L_{\nabla c} L_h \left\|d^k\right\|_2^2}. \tag{3.2}$$

Consequently, when the backtracking algorithm terminates at $t_k > 0$,

$$t_k \geq \frac{2\theta(1-\sigma_1)|\Delta f(x^k; d^k)|}{L_{\nabla c} L_h \left\|d^k\right\|_2^2}. \tag{3.3}$$

Therefore, t_k satisfying (WWI) implies

$$\sigma_1 \min\left\{1, \theta \frac{2(1-\sigma_1)|\Delta f(x; d)|}{L_{\nabla c} L_h \|d\|_2^2}\right\} |\Delta f(x; d)| \leq \sigma_1 t_k |\Delta f(x^k; d^k)| \leq f(x^k) - f(x^{k+1}).$$

Using the boundedness of the search directions and arguing as in the proof of Theorem 6, the bound (3.3) holds for all $k \geq k_0$. Summing the previous display,

$$0 < \sum_{k \geq k_0} \theta \frac{2\sigma_1(1-\sigma_1)\Delta f(x^k; d^k)^2}{L_{\nabla c} L_h \left\| d^k \right\|_2^2} < f(x^0) - \lim_{k \to \infty} f(x^k).$$

Since (b) does not occur, $\lim_{k\to\infty} f(x^k) > -\infty$, so (c) must occur. □

Remark 2. When h is the identity on $\mathbb{R}$ and $g = 0$, we recover the convergence analysis of backtracking for smooth minimization.

The following corollary is an immediate consequence of Lemma 2.1.2.

Corollary 3.2.1. Let the hypotheses of Theorem 5 hold. If $0 < \beta < 1$ and the directions $\{d^k\}$ are chosen to satisfy

$$\Delta f(x^k; d^k) \leq \beta \overline{\Delta}_k f < 0,$$

then the occurrence of (c) in Theorem 5 implies that cluster points of $\{x^k\}$ are first-order stationary for $\mathcal{P}$.

3.3 Weak Wolfe for Convex-Composite Minimization

Definition 3.3.1. Weak Wolfe in the convex composite case is defined at $x \in \operatorname{dom}(g)$ with $\Delta f(x; d) < 0$ by choosing $0 < \sigma_1 < \sigma_2 < 1$ and $\mu > 0$ and requiring

$$\text{(WWI)} \qquad f(x+td) \leq f(x) + \sigma_1 t \Delta f(x; d), \text{ and}$$

$$\text{(WWII)} \qquad \sigma_2 \Delta f(x; d) \leq \frac{\Delta f(x+td; \mu d)}{\mu}.$$

Remark 3. The second condition (WWII) is a *curvature condition* that parallels the classical weak Wolfe [44, 45] curvature condition for smooth, unconstrained minimization:

$$\sigma_2 f'(x; d) \leq f'(x+td; d),$$

which prevents the line search early termination at "strongly negative" slopes [47, Section 3.1].

Remark 4. The strong Wolfe conditions require $|f'(x+td;d)| \le -\sigma_2 f'(x;d)$, whenever f is smooth. However, in nonsmooth minimization, kinks and upward cusps at local minimizers make this condition unworkable.

The following lemma shows that the set of points satisfying (WWI) and (WWII) has nonempty interior.

Lemma 3.3.1. Let f be as in $\mathcal{P}$, $x \in \operatorname{dom}(g)$, and d chosen so that $\Delta f(x;d) < 0$. Suppose f is bounded below on the ray $\{x + td : t > 0\}$, and $\mu \in \mathbb{R}$. Then, the set

$$C(\mu) := \left\{ t > 0 \;\middle|\; \begin{array}{c} f(x+td) \le f(x) + \sigma_1 t \Delta f(x;d), \\ \sigma_2 \Delta f(x;d) \le \dfrac{\Delta f(x+td;\mu d)}{\mu} \end{array} \right\}$$

has nonempty interior for any $\mu > 0$.

Proof. Define

$$K(y,z,t) := h(y) + g(z) - [f(x) + \sigma_1 t \Delta f(x;d)],$$

$$G(t) := \begin{pmatrix} c(x+td) \\ x+td \\ t \end{pmatrix}, \text{ with } G'(t) = \begin{pmatrix} \nabla c(x+td)d \\ d \\ 1 \end{pmatrix},$$

and set $\phi(t) := K(G(t)) = f(x+td) - [f(x) + \sigma_1 t \Delta f(x;d)]$. Then, $\phi(t)$ is convex-composite,

$$\begin{aligned} \Delta\phi(t;\mu) &= K(G(t) + G'(t)\mu) - K(G(t)) \\ &= h(c(x+td) + \nabla c(x+td)\mu d) + g(x + (t+\mu)d) - [f(x) + \sigma_1(t+\mu)\Delta f(x;d)] \\ &- (h(c(x+td)) + g(x+td) - [f(x) + \sigma_1 t \Delta f(x;d)]) \\ &= \Delta f(x+td;\mu d) - \mu\sigma_1 \Delta f(x;d), \end{aligned}$$

and, by Lemma 2.1.1,

$$\begin{aligned} \phi'(t;\mu) &= f'(x+td;\mu d) - \mu\sigma_1 \Delta f(x;d) \\ &\le \mu \Delta\phi(t;1). \end{aligned}$$

Consequently, $\phi'(0;1) \le (1-\sigma_1)\Delta f(x;d) < 0$, so there exists $\bar{t} > 0$ such that for all $t \in (0,\bar{t})$, $\phi(t) < 0$. This is equivalent to (WWI) being satisfied on $(0,\bar{t})$.

Since ϕ is bounded below on the ray, $\phi(t) \nearrow \infty$. Let $\widehat{t} := \sup\{t > \bar{t} : \phi(s) < 0 \text{ for all } s \in (0,t)\}$. Then, since g is closed and h is finite-valued, $\phi(\widehat{t}) = \liminf_{t \nearrow \widehat{t}} \phi(t)$, which implies

$$\begin{aligned} h(c(x+\widehat{t}d)) + g(x+\widehat{t}d) &= -[f(x) + \sigma_1 \widehat{t}\Delta f(x;d)] + \liminf_{t \nearrow \widehat{t}} \phi(t) \\ &\le -[f(x) + \sigma_1 \widehat{t}\Delta f(x;d)] < \infty, \end{aligned}$$

so $x + \widehat{t}d \in \mathrm{dom}\,(g)$. Since g is continuous relative to its domain, ϕ is continuous relative to its domain, so $\phi(\widehat{t}) \le 0$. We now consider two cases on the value of $\phi(\widehat{t})$.

Suppose $\phi(\widehat{t}) < 0$. We aim to show that f satisfies (WWI) and (WWII) on the interval $((\widehat{t}-\mu)_+, \widehat{t}]$. To prove this, we show that if $\phi(\widehat{t}) < 0$, then $t > \widehat{t}$ implies $x+td \notin \mathrm{dom}\,(g)$ and, as a consequence, $\widehat{t} \ge 1$. Suppose to the contrary that there exists $t > \widehat{t}$ with $x+td \in \mathrm{dom}\,(g)$. Then, the definition of $\widehat{t}$, convexity of $\mathrm{dom}\,(\phi)$, and the intermediate value theorem imply there exists $\tilde{t}$ such that $\phi(\tilde{t}) = 0$ and $t \ge \tilde{t} > \widehat{t}$. But relative continuity of ϕ at $\tilde{t}$ with respect to $\mathrm{dom}\,(\phi)$ means there exist points in $(\widehat{t},\tilde{t})$ which contradict the definition of $\widehat{t}$. This proves the claim. Consequently, if $\phi(\widehat{t}) < 0$, then f satisfies both (WWI) and (WWII) on the interval $((\widehat{t}-\mu)_+, \widehat{t}]$, as the right-hand side of (WWII) is $+\infty$.

Otherwise, $\phi(\widehat{t}) = 0$. Let $\tilde{t} \in \arg\min_{t \in [0,\widehat{t}]} \phi(t)$. Then, $\tilde{t} \in (0,\widehat{t})$, with $0 \le \phi'(\tilde{t};\mu) \le \Delta\phi(\tilde{t},\mu)$ for all μ, equivalently

$$\frac{\Delta f(x+\tilde{t}d;\mu d)}{\mu} \ge \sigma_1 \Delta f(x;d) > \sigma_2 \Delta f(x;d) \quad \forall\, \mu > 0,$$

so (WWI) and (WWII) hold with strict inequality at $\tilde{t}$. We now consider two cases based on whether $x + (\tilde{t}+\mu)d \in \mathrm{dom}\,(g)$.

First, for all sufficiently small $\mu > 0$, $x + (\tilde{t}+\mu)d \in \mathrm{dom}\,(g)$. Because the inequalities in (WWI) and (WWII) are strict at $\tilde{t}$, relative continuity of f and of $t \mapsto \Delta f(x+td;d)$ at $t = \tilde{t}$ imply there exists an open interval $\mathcal{I}$ with $\tilde{t} \in \mathcal{I}$ and $x + \mathcal{I}d \subset \mathrm{dom}\,(g)$ where both (WWI) and (WWII) hold.

For those $\mu > 0$ for which $x + (\tilde{t} + \mu)d \notin \operatorname{dom}(g)$, (WWI) and (WWII) hold for all $t \in ((\tilde{t} - \mu)_+, \widehat{t})$ as argued in the previous case where $\phi(\widehat{t}) < 0$. □

Next, we prove finite termination of a bisection algorithm to point $\bar{t} \geq 0$ satisfying the weak Wolfe conditions. The algorithm is analogous to the weak Wolfe bisection method for finite-valued nonsmooth minimization in [27].

Algorithm 2 Weak Wolfe Bisection Method

Require: $x \in \operatorname{dom}(g), d \in \mathbb{R}^n$ with $\Delta f(x; d) < 0$, and $0 < \sigma_1 < \sigma_2 < 1$, $\mu > 0$.

procedure WWBISECT$(x, d, \sigma_1, \sigma_2)$
  $\alpha \leftarrow 0$;
  $t \leftarrow 1$;
  $\beta \leftarrow \infty$;
  while (WWI) and (WWII) fail **do**
    if $f(x + td) > f(x) + \sigma_1 t \Delta f(x; d)$ **then** ▷ If not sufficient decrease
      $\beta \leftarrow t$
    else if $\sigma_2 \Delta f(x; d) > \frac{\Delta f(x+td; \mu d)}{\mu}$ **then** ▷ Else if not curvature
      $\alpha \leftarrow t$
    else
      return t
    end if
    if $\beta = \infty$ **then** ▷ Doubling Phase
      $t \leftarrow 2t$
    else ▷ Bisection Phase
      $t \leftarrow \frac{1}{2}(\alpha + \beta)$
    end if
  end while
end procedure

Lemma 3.3.2. Let f be given as in $\mathcal{P}$ with g strictly continuous relative to its domain, and suppose $x \in \operatorname{dom}(g)$ and d is chosen such that $\Delta f(x; d) < 0$. Then, one of the following must occur in Algorithm 2:

(a) the doubling phase does not terminate finitely, with the parameter β never set to a finite value, the parameter α becoming positive on the first iteration and doubling every iteration thereafter, with $f(x + t_k d) \searrow -\infty$;

(b) both the doubling phase and the bisection phase terminate finitely to a $\bar{t} \geq 0$ for which the weak Wolfe conditions are satisfied.

Proof. Suppose the procedure does not terminate finitely. If the parameter β is never set to a finite value, then the doubling phase does not terminate. Then the parameter α becomes positive on the first iteration and doubles on each subsequent iteration k, with t_k satisfying

$$f(x + t_k d) \leq f(x) + \sigma_1 t_k \Delta f(x; d), \quad \forall\, k \geq 1.$$

Therefore, since $\Delta f(x; d) < 0$, the function values $f(x + t_k d) \searrow -\infty$, so the first option occurs.

Otherwise, the procedure does not terminate finitely, and β is eventually finite. Therefore, the doubling phase terminates finitely, but the bisection phase does not terminate finitely. This implies there exists $\bar{t} \geq 0$ such that

$$\alpha_k \nearrow \bar{t}, \quad t_k \to \bar{t}, \quad \beta_k \searrow \bar{t}. \tag{3.4}$$

We now consider two cases. First, suppose that the parameter α is never set to a positive number. Then, $\alpha_k = 0$ for all $k \geq 1$, and $t_k, \beta_k \to 0$, so the first `if` statement is entered in each iteration. This implies

$$\sigma_1 \Delta f(x; d) < \frac{f(x + t_k d) - f(x)}{t_k}, \quad \forall\, k \geq 1.$$

Since $[x, x + d] \subset \operatorname{dom}(g)$, Lemma 2.1.1 yields the chain of inequalities

$$\sigma_1 \Delta f(x; d) \leq f'(x; d) \leq \Delta f(x; d) < 0,$$

which contradicts $\sigma_1 \in (0,1)$.

Otherwise, the parameter α is eventually positive. Then, the bisection phase does not terminate, and the algorithm generates infinite sequences $\{\alpha_k\}, \{t_k\}$, and $\{\beta_k\}$ satisfying (3.4) such that, for all k large, $0 < \alpha_k < t_k < \beta_k < \infty$, and

$$f(x+\alpha_k d) \leq f(x) + \sigma_1 \alpha_k \Delta f(x;d), \tag{3.5}$$

$$f(x+\beta_k d) > f(x) + \sigma_1 \beta_k \Delta f(x;d), \tag{3.6}$$

$$\sigma_2 \Delta f(x;d) > \frac{\Delta f(x+\alpha_k d; \mu d)}{\mu}, \tag{3.7}$$

$$[x, x+\max\{\alpha_k+\mu, \beta_k\} d] \subset \operatorname{dom}(g). \tag{3.8}$$

Letting $k \to \infty$ in (3.7) and using lower semicontinuity of g gives

$$\sigma_2 \Delta f(x;d) \geq \frac{\Delta f(x+\bar{t}d; \mu d)}{\mu}. \tag{3.9}$$

By Theorem 3, for sufficiently large k there exists $\tau_k \in (0,1)$ so that the vectors

$$x^k := (1-\tau_k)(x+\alpha_k d) + \tau_k(x+\beta_k d) = x + [(1-\tau_k)\alpha_k + \tau_k \beta_k] d,$$
$$v^k \in \partial f(x^k)$$

yield an extended form of the mean-value theorem

$$f(x+\beta_k d) - f(x+\alpha_k d) = \left\langle v^k,\, (\beta_k - \alpha_k) d \right\rangle. \tag{3.10}$$

Let $\gamma_k := (1-\tau_k)\alpha_k + \tau_k \beta_k \in (\alpha_k, \beta_k)$, so that $x^k = x + \gamma_k d$. Then, $\gamma_k \to \bar{t}$ as $k \to \infty$. Combining (3.5) and (3.6) and using (3.10) gives

$$\sigma_1(\beta_k - \alpha_k)\Delta f(x;d) < f(x+\beta_k d) - f(x+\alpha_k d) = \left\langle v^k,\, (\beta_k - \alpha_k) d \right\rangle.$$

Dividing by $\beta_k - \alpha_k > 0$ gives

$$\begin{aligned} \sigma_1 \Delta f(x;d) &\leq f'(x+\gamma_k d; d) \\ &\leq \frac{\Delta f(x+\gamma_k d; \mu d)}{\mu}. \end{aligned}$$

As $k \to \infty$, using (3.9), we obtain the string of inequalities

$$\frac{\Delta f(x + \bar{t}d; \mu d)}{\mu} \leq \sigma_2 \Delta f(x; d) < \sigma_1 \Delta f(x; d) \leq \frac{\Delta f(x + \bar{t}d; \mu d)}{\mu},$$

which is a contradiction. Therefore, either the doubling phase never terminates or the procedure terminates finitely at some $\bar{t}$ at which f satisfies both weak Wolfe conditions. □

A global convergence result for the weak Wolfe line search that parallels [3, Theorem 2.4] now follows under standard Lipschitz assumptions, which hold, in particular, if the initial set $\mathrm{lev}_f(f(x^0))$ is compact.

Algorithm 3 Global Weak Wolfe

procedure WeakWolfeGlobal($x^0, \sigma_1, \sigma_2, \mu$)
    $k \leftarrow 0$
    repeat
        Find $d^k \in \mathbb{R}^n$ such that $\Delta f(x^k; d^k) < 0$
        if no such d^k **then**
            $0 \in \partial f(x^k)$ **return**
        end if
        Let t_k be a step size satisfying (WWI) and (WWII)
        if no such t_k **then**
            f unbounded below. **return**
        end if
        $x^k \leftarrow x^k + t_k d^k$
        $k \leftarrow k + 1$
    until
end procedure

Theorem 6. Let f be as in $\mathcal{P}$ with g strictly continuous relative to its domain, $x^0 \in \mathrm{dom}\,(g)$, $0 < \sigma_1 < \sigma_2 < 1$, and $0 < \mu < 1$. Set $\mathcal{L} := \mathrm{lev}_f(f(x^0))$. Suppose there exists $M, \widetilde{M} > 0$ such that $\|d^k\| \leq M$ for all $k \geq 0$, $\sup_{x \in \mathcal{L}} \|\nabla c(x)\| \leq \widetilde{M}$, and

(i) c is L_c-Lipschitz on $\mathcal{L}$;

(ii) ∇c is $L_{\nabla c}$-Lipschitz on $\mathcal{L}$;

(iii) g is L_g-Lipschitz on $(\mathcal{L} + M\mu\mathbb{B}) \cap \operatorname{dom}(g)$;

(iv) h is L_h-Lipschitz on $c(\mathcal{L}) + M\widetilde{M}\mu\mathbb{B}$.

Let $\{x^k\}$ be a sequence initialized at x^0 and generated by Algorithm 3: Then at least one of the following must occur:

(a) the algorithm terminates finitely at a first-order stationary point for f;

(b) for some k the step size selection procedure generates a sequence of trial step sizes $t_{k_n} \overset{n\uparrow\infty}{\longrightarrow} \infty$ such that $f(x^k + t_{k_n} d^k) \to -\infty$;

(c) $f(x^k) \searrow -\infty$;

(d) $\displaystyle\sum_{k=0}^{\infty} \frac{\Delta f(x^k; d^k)^2}{\|d^k\| + \|d^k\|^2} < \infty$, in particular, $\Delta f(x^k; d^k) \to 0$.

Proof. We assume (a) - (c) do not occur and show (d) occurs. Since (a) does not occur, the sequence $\{x^k\}$ is infinite, and $\Delta f(x^k; d^k) < 0$ for all $k \geq 0$. Since (b) does not occur, Lemma 3.3.2 implies that the weak Wolfe bisection method terminates finitely at every iteration $k \geq 0$. The sufficient decrease condition (WWI) gives a strict descent method, so the function values $\{f(x^k)\}$ are strictly decreasing, with $\{x^k\} \subset \mathcal{L}$ for all $k \geq 0$. By the nonoccurrence of (c), $f(x^k) \searrow \overline{f} > -\infty$.

We first show that for each $k \geq 0$, the step size t_k satisfies

$$t_k \geq \min\left\{1 - \mu, \frac{\mu(1-\sigma_2)|\Delta f(x^k; d^k)|}{K\left(\|d^k\| + \|d^k\|^2\right)}\right\}, \tag{3.11}$$

by considering two cases.

First, suppose $\Delta f(x^{k+1}; \mu d^k) = \infty$. Then $x^{k+1} + \mu d^k = x^k + (t_k + \mu)d^k \notin \operatorname{dom}(g)$. Since $x^k + d^k \in \operatorname{dom}(g)$, $t_k + \mu > 1$, and by assumption $0 < \mu < 1$. Therefore, $t_k \geq 1 - \mu$.

Otherwise, $\Delta f(x^{k+1}; \mu d^k) < \infty$. Then

$$\begin{aligned}\Delta f(x^{k+1}; \mu d^k) - \Delta f(x^k; \mu d^k) &= h(c(x^{k+1}) + \nabla c(x^{k+1})\mu d^k) - h(c(x^{k+1})) + g(x^{k+1} + \mu d^k) - g(x^{k+1}) \\ &\quad - [h(c(x^k) + \nabla c(x^k)\mu d^k) - h(c(x^k)) + g(x^k + \mu d^k) - g(x^k)] \\ &= h(c(x^k)) - h(c(x^{k+1})) \\ &\quad + h(c(x^{k+1}) + \nabla c(x^{k+1})\mu d^k) - h(c(x^k) + \nabla c(x^k)\mu d^k) \\ &\quad + g(x^k) - g(x^{k+1}) + g(x^{k+1} + \mu d^k) - g(x^k + \mu d^k) \\ &\leq 2L_h L_c t_k \left\| d^k \right\| + L_h L_{\nabla c} \mu t_k \left\| d^k \right\|^2 + 2L_g t_k \left\| d^k \right\| \\ &\leq K t_k \left(\left\| d^k \right\| + \left\| d^k \right\|^2 \right),\end{aligned}$$

for some $K \geq 0$. Adding and subtracting in (WWII) gives

$$\begin{aligned}\sigma_2 \Delta f(x^k; d^k) &\leq \frac{\Delta f(x^k + t_k d^k; \mu d^k)}{\mu} \\ &= \frac{\Delta f(x^k; \mu d^k)}{\mu} + \left[\frac{\Delta f(x^k + t_k d^k; \mu d^k)}{\mu} - \frac{\Delta f(x^k; \mu d^k)}{\mu} \right] \\ &\leq \Delta f(x^k; d^k) + \frac{K}{\mu} t_k \left(\left\| d^k \right\| + \left\| d^k \right\|^2 \right) \text{ (since } 0 < \mu < 1),\end{aligned}$$

which rearranges to

$$0 < \frac{\mu(1 - \sigma_2)|\Delta f(x^k; d^k)|}{K \left(\|d^k\| + \|d^k\|^2 \right)} \leq t_k, \tag{3.12}$$

so (3.11) holds. Next, (WWI) and (3.11) imply

$$\sigma_1 \min \left\{ 1 - \mu, \frac{\mu(1 - \sigma_2)|\Delta f(x^k; d^k)|}{K \left(\|d^k\| + \|d^k\|^2 \right)} \right\} |\Delta f(x^k; d^k)| \leq \sigma_1 t_k |\Delta f(x^k; d^k)| \leq f(x^k) - f(x^{k+1}). \tag{3.13}$$

We aim to show that the bound (3.12) holds for all large k by showing $\Delta f(x^k; d^k) \to 0$ and using boundedness of the search directions $\{d^k\}$. Suppose there exists a subsequence $J_1 \subset \mathbb{N}$

for which $\Delta f(x^k; d^k) \underset{J_1}{\not\to} 0$. Let $\gamma > 0$ be such that $\sup_{k\in J_1} \Delta f(x^k; d^k) \le -\gamma < 0$. Then, since $\{d^k\} \subset M\mathbb{B}$,

$$\frac{\mu(1-\sigma_2)|\Delta f(x^k; d^k)|}{K\left(\|d^k\| + \|d^k\|^2\right)} \underset{J_1}{\not\to} 0. \tag{3.14}$$

If there exists a further subsequence $J_2 \subset J_1$ with

$$\frac{\mu(1-\sigma_2)|\Delta f(x^k; d^k)|}{K\left(\|d^k\| + \|d^k\|^2\right)} \ge 1-\mu, \quad \forall\, k \in J_2,$$

then by expanding the recurrence given by (WWI), and writing $J_2 = \{k_1, k_2, \dots\}$, we have

$$\begin{aligned} f(x^{k_n}) &\le f(x^{k_{n-1}}) - \sigma_1(1-\mu)\gamma \\ &\le f(x^{k_1}) - C(k_n)\sigma_1(1-\mu)\gamma \end{aligned} \tag{3.15}$$

with $C(k_n) \to \infty$ as $n \to \infty$. This contradicts the nonoccurence of (c). By (3.14), there exists a subsequence $J_2 \subset J_1$ and $\delta > 0$ so that

$$0 < \delta \le \frac{\mu(1-\sigma_2)|\Delta f(x^k; d^k)|}{K\left(\|d^k\| + \|d^k\|^2\right)} < 1-\mu$$

for all large $k \in J_2$. Repeating the argument at (3.15) with δ in place of $1-\mu$, we conclude

$$\frac{\mu(1-\sigma_2)|\Delta f(x^k; d^k)|}{K\left(\|d^k\| + \|d^k\|^2\right)} \underset{J_1}{\to} 0,$$

and consequently $\Delta f(x^k; d^k) \underset{J_1}{\to} 0$, which is a contradiction. Therefore, (3.12) holds for all $k \ge k_0$. Summing over $k \in \mathbb{N}$ in (3.13) gives

$$0 < \sum_{k\ge k_0} \frac{\sigma_1\mu(1-\sigma_2)\Delta f(x^k; d^k)^2}{K\left(\|d^k\| + \|d^k\|^2\right)} < f(x^0) - \lim_{k\to\infty} f(x^k).$$

Since (c) does not occur, $\lim_{k\to\infty} f(x^k) > -\infty$, so (d) must occur. □

Remark 5. When h is the identity on $\mathbb{R}$ and $g = 0$, we recover the convergence analysis of weak Wolfe for smooth minimization given in [47, Theorem 3.2].

Remark 6. The hypotheses of Theorem 6 simplify if h is globally Lipschitz. In that case, the boundedness condition on $\left\{\left\|\nabla c(x)\right\| \mid x \in \mathcal{L}\right\}$ is not necessary. Alternatively, if $\left\|\nabla c(x)\right\|$ is bounded on the closed convex hull of $\mathcal{L}$, then the Lipschitz condition of c on $\mathcal{L}$ is immediate.

The following corollary is an immediate consequence of Lemma 2.1.2.

Corollary 3.3.1. Let the hypotheses of Theorem 6 hold. If $0 < \beta < 1$ and the directions $\left\{d^k\right\}$ are chosen to satisfy

$$\Delta f(x^k; d^k) \leq \beta \overline{\Delta}_k f < 0,$$

then the occurrence of (d) in Theorem 6 implies that cluster points of $\left\{x^k\right\}$ are first-order stationary for $\mathcal{P}$.

3.4 Trust Region Subproblems

In this section, we let $\|\cdot\|$ denote an arbitrary norm on $\mathbb{R}^n$.

Definition 3.4.1. For $\delta > 0$ and $x \in \operatorname{dom}(g)$, define the set of Cauchy steps $D_\delta^C(x)$ by

$$D_\delta^C(x) := \operatorname*{arg\,min}_{\|d\| \leq \delta} \Delta f(x; d),$$

and set

$$\Delta_\delta^C f(x) := \inf_{\|d\| \leq \delta} \Delta f(x; d).$$

Observe that $\Delta_{\eta_k}^C f(x^k) = \overline{\Delta}_k f$ where $\overline{\Delta}_k f$ is defined in Lemma 2.1.2. Our pattern of proof in this section follows a path similar to the standard approaches to such results given in [10].

Throughout this section we make the following basic Lipschitz continuity assumptions.

Assumption:

A1: c, ∇c, and g are Lipschitz continuous on $\operatorname{dom}(g)$ with Lipschitz constants L_c, $L_{c'}$, and L_g, respectively.

A2: h is Lipschitz continuous on $g(\operatorname{dom}(g))$ with Lipschitz constant L_h.

These assumptions imply the Lipschitz continuity of $\Delta_\delta^C f$ on $\mathrm{dom}\,(g)$.

Lemma 3.4.1. Let the assumptions A1 and A2 hold. Then, for $\delta > 0$, the mapping $x \mapsto \Delta_\delta^C f(x)$ is Lipschitz continuous on $\mathrm{dom}\,(g)$ with constant $L_\Delta := L_h(2L_c + \delta L_{c'}) + 2L_g$.

Proof. Observe that for any $x^1, x^2 \in \mathrm{dom}\,(g)$ and $d \in \delta\mathbb{B}$ we have

$$\begin{aligned}\Delta f(x^1; d) - \Delta f(x^2; d) =& [h(c(x_1) + \nabla c(x_1)d) - h(c(x_2) + \nabla c(x_2)d)] + [h(c(x_2)) - h(c(x_1))]\\ & [g(x^1 + d) - g(x^2 + d)] + [g(x^2) - g(x^1)].\end{aligned}$$

Hence,

$$\Delta f(x^1; d) - \Delta f(x^2; d) \leq L_\Delta \left\| x^1 - x^2 \right\|,$$

and, by symmetry,

$$\Delta f(x^2; d) - \Delta f(x^1; d) \leq L_\Delta \left\| x^1 - x^2 \right\|.$$

Taking $d \in D_\delta^C(x^2)$ in the first of these inequalities and $d \in D_\delta^C(x^1)$ in the second gives the result. □

Definition 3.4.2. (Sufficient Decrease Condition for f) We say that a direction choice method satisfies the sufficient decrease condition for f if for all $\epsilon > 0$ and $\delta_k > 0$, if $x^k \in \mathbb{R}^n$ satisfies $|\Delta_1^C f(x^k)| > \epsilon$, then there exists constants $\kappa_1, \kappa_2 > 0$ depending only on ϵ such that the direction choice $d^k \in \mathbb{R}^n$ satisfies

$$\Delta f(x^k; d^k) + \frac{1}{2} d^{k\top} H_k d^k < -\kappa_1 \min(\kappa_2, \delta_k). \tag{3.16}$$

Remark 7. The sufficient decrease condition can be defined using $|\Delta_{\widehat{\delta}}^C f(x^k)|$ for any $\widehat{\delta} > 0$, but this choice of $\widehat{\delta}$ must then remain constant throughout the iteration process; $\widehat{\delta} = 1$ is chosen for simplicity.

Lemma 3.4.2. Let $x \in \mathrm{dom}\,(g)$, $H \in \mathbb{R}^{n\times n}$, $\delta > 0$ and let $\sigma > 0$ be such that $\|d\|_2 \leq \sigma\|d\|$ for all $d \in \mathbb{R}^n$. If $\widehat{d} \in D_1^C(x)$ with $\Delta_1^C f(x) < 0$, then there exists $\widehat{t} \in (0, \min(1, \delta)]$ such that

$$\Delta f(x; \widehat{t}\widehat{d}) + \frac{\widehat{t}^2}{2} \widehat{d}^\top H \widehat{d} \leq \frac{1}{2} \Delta_1^C f(x) \min\left(\frac{|\Delta_1^C f(x)|}{\sigma^2 \|H\|_2}, 1, \delta \right).$$

Proof. For any $t \in (0, \min(1,\delta)]$, Lemma 2.1.1 and Hölder's inequality implies

$$\Delta f(x; t\widehat{d}) + \frac{t^2}{2}\widehat{d}^\top H \widehat{d} \le t\Delta f(x;\widehat{d}) + \frac{t^2}{2}\sigma^2 \|H\|_2 .$$

Set $\alpha = \Delta f(x;\widehat{d}) < 0$, $\beta = \sigma^2\|H\|_2 > 0$, and

$$\widehat{t} = \operatorname*{arg\,min}_{t\in[0,\min(1,\delta)]} \alpha t + \beta t^2/2 \qquad = \min(1,\delta,-\alpha/\beta).$$

There are two cases to consider. If $-\alpha/\beta \le \min(1,\delta)$, then $\widehat{t} = -\alpha/\beta$ and

$$\begin{aligned}
\Delta f(x;\widehat{t}\widehat{d}) + \frac{\widehat{t}^2}{2}\widehat{d}^\top H\widehat{d} &\le \widehat{t}\Delta f(x;\widehat{d}) + \frac{t^2}{2}\widehat{d}^\top H \widehat{d} \\
&= -\frac{\Delta f(x;\widehat{d})^2}{\sigma^2\|H\|_2} + \frac{\Delta f(x;\widehat{d})^2}{2\sigma^4\|H\|_2^2}\widehat{d}^\top H \widehat{d} \\
&= -\frac{\Delta f(x;\widehat{d})^2}{2\sigma^2\|H\|_2} \\
&= \frac{1}{2}\Delta_1^C f(x)\left(\frac{|\Delta_1^C f(x)|}{\sigma^2\|H\|_2}\right).
\end{aligned}$$

Otherwise, $\min(1,\delta) \le -\alpha/\beta$. Setting $\widehat{t} = \min(1,\delta)$ gives

$$\begin{aligned}
\Delta f(x;\widehat{t}\widehat{d}) + \frac{\widehat{t}^2}{2}\widehat{d}^\top H\widehat{d} &\le \widehat{t}\Delta f(x;\widehat{d}) + \frac{\widehat{t}^2}{2}\sigma^2\|H\|_2 \\
&= \widehat{t}\Delta f(x;\widehat{d}) - \frac{\widehat{t}}{2}\Delta f(x;\widehat{d}) \\
&= \frac{1}{2}\Delta_1^C f(x)\min(1,\delta).
\end{aligned}$$

The result follows. □

Lemma 3.4.3. Let the assumptions A1 and A2 hold. Suppose $\sigma > 0$ satisfies $\|d\|_2 \le \sigma\|d\|$ for all $d \in \mathbb{R}^n$, $H \in \mathbb{R}^{n\times n}$, $0 < \overline{\beta}_1 \le \overline{\beta}_2 < 1$, and $\alpha, \kappa_1, \kappa_2 > 0$. Choose $\overline{\delta} > 0$ so that

$$\kappa_1(1-\overline{\beta}_2)\min(\kappa_2,\delta) \ge \frac{1}{2}\delta^2 L_h L_{\nabla c} + \frac{1}{2}\sigma^2\delta^2\|H\|_2$$

for all $\delta \in [0,\overline{\delta}]$. Then, for every $\delta \in [0,\overline{\delta}]$, $x \in \operatorname{dom}(g)$, and $d \in \delta\mathbb{B}$ for which

$$\Delta f(x;d) + \frac{1}{2}d^\top H d \le -\kappa_1\min(\kappa_2,\delta),$$

one has

$$f(x+d) - f(x) \leq \overline{\beta}_2[\Delta f(x;d) + \frac{1}{2}d^\top Hd] \leq \overline{\beta}_1[\Delta f(x;d) + \frac{1}{2}d^\top Hd].$$

Proof. The Lipschitz assumptions on h and ∇c imply

$$f(x+d) - f(x) \leq \Delta f(x;d) + \frac{L_h L_{\nabla c}}{2}\|d\|_2^2.$$

Since $\frac{1}{2}d^\top Hd + \frac{1}{2}\sigma^2\delta^2\|H\|_2 \geq 0$, it follows that

$$\begin{aligned}
f(x+d) - f(x) &\leq \Delta f(x;d) + \frac{1}{2}\delta^2 L_h L_{\nabla c} + \frac{1}{2}d^\top Hd + \frac{1}{2}\sigma^2\delta^2\|H\|_2 \\
&\leq \Delta f(x;d) + \frac{1}{2}d^\top Hd + \kappa_1(1-\overline{\beta}_2)\min(\kappa_2, \delta) \\
&\leq \Delta f(x;d) + \frac{1}{2}d^\top Hd - (1-\overline{\beta}_2)[\Delta f(x;d) + \frac{1}{2}d^\top Hd] \\
&= \overline{\beta}_2[\Delta f(x;d) + \frac{1}{2}d^\top Hd] \\
&< \overline{\beta}_1[\Delta f(x;d) + \frac{1}{2}d^\top Hd].
\end{aligned}$$

□

Algorithm 4 Trust Region

procedure TRS($H_0 \in \mathbb{R}^{n\times n},\ \delta_0 > 0,\ x^0 \in \mathbb{R}^n, 0 < \gamma_1 \leq \gamma_2 < 1 \leq \gamma_3, 0 < \beta_2 < \beta_3 < 1$ and $0 \leq \beta_1 \leq \beta_2$.)

    $k \leftarrow 1$

    repeat

        Find $d^k \in D_k := \left\{d \;\middle|\; \|d\| \leq \delta_k, \Delta f(x^k; d) + \frac{1}{2}d^\top H_k d < 0\right\}$ ▷ Step 1

        if no such d^k **then return**

        end if

        $r_k \leftarrow \frac{f(x^k+d^k)-f(x^k)}{\Delta f(x^k;d^k)+\frac{1}{2}d^{k\top}H_k d^k}$

        if $r_k > \beta_3$ **then** ▷ Step 2

            Choose $\delta_{k+1} \in [\delta_k, \gamma_3\delta_k]$

        else if $\beta_2 \leq r_k \leq \beta_3$ **then**

            Set $\delta_{k+1} = \delta_k$

        else

            Choose $\delta_{k+1} \in [\gamma_1\delta_k, \gamma_2\delta_k]$

        end if

        if $r_k < \beta_1$ **then** ▷ Step 3

            $x^{k+1} \leftarrow x^k$

            $H_{k+1} \leftarrow H_k$

        else

            $x^{k+1} \leftarrow x^k + d^k$

            Choose H_{k+1}

        end if

        $k \leftarrow k + 1$

    until

end procedure

Theorem 7. Let the hypotheses of Lemma 3.4.3 be satisfied. If the sequence $\{H_k\}$ is bounded and the choice of search directions d^k satisfy the sufficient decrease condition in Definition 3.4.2, then at least one of the following must occur:

(i) $D_k = \emptyset$ for some k,

(ii) $f(x^k) \searrow -\infty$,

(iii) $|\Delta_1^C f(x^k)| \to 0$.

Proof. We assume that none of (i) - (iii) occur and derive a contradiction. First observe that Lemma 3.4.2 tells us that if $D_k \neq \emptyset$, the there is an element of D_k that satisfies the sufficient decrease condition, and so the algorithm is well-defined and the hypotheses of the theorem can be satisfied. Consequently, since (1) does not occur, the sequence $\left\{x^k\right\}$ is infinite. Since (iii) does not occur, there exists $\zeta > 0$ such that the set

$$J := \left\{ k \in \mathbb{N} \,\middle|\, 2\zeta < |\Delta_1^C f(x^k)| \right\}$$

is an infinite subsequence of $\mathbb{N}$. By the sufficient decrease condition, there exists $\kappa_1, \kappa_2 > 0$ such that

$$\Delta f(x^k; d^k) + \frac{1}{2} d^{k\top} H_k d^k \leq -\kappa_1 \min(\kappa_2, \delta_k) \quad \text{whenever} \quad |\Delta_1^C f(x^k)| > \zeta. \tag{3.17}$$

In particular, (3.17) holds for all $k \in J$. Lemma 3.4.3 and Lipschitz continuity of ∇c imply the existence of a $\overline{\delta}$ such that

$$r_k \geq \beta_2 \text{ and } x^{k+1} = x^k + d^k \quad \text{whenever} \quad |\Delta_1^C f(x^k)| > \zeta \quad \text{and} \quad \delta_k \leq \overline{\delta}. \tag{3.18}$$

By Lemma 3.4.1,

$$|\Delta_1^C f(z^1) - \Delta_1^C f(z^2)| \leq \zeta \quad \text{whenever} \quad \left\|z^1 - z^2\right\| \leq \overline{\epsilon} \quad \text{with} \quad z^1, z^2 \in \operatorname{dom}(g), \tag{3.19}$$

where $\overline{\epsilon} := \zeta / L_\Delta$.

For each $k \in J$, let $\nu(k)$ be the first integer $p \geq k$ such that either

$$\left\|x^{p+1} - x^k\right\| \leq \overline{\epsilon}, \tag{3.20}$$

or

$$\delta_p \le \overline{\delta} \tag{3.21}$$

is violated. We need to show that $\nu(k)$ is well-defined for every $k \in J$. Assume that $\nu(k)$ is not well-defined for some $k_0 \in J$. That is,

$$\left\| x^{p+1} - x^{k_0} \right\| \le \overline{\epsilon} \quad \text{and} \quad \delta_p \le \overline{\delta} \quad \forall\, p \ge k_0. \tag{3.22}$$

For every $k \in J$ with $k \ge k_0$, observe that (3.19) and the first condition in (3.22) imply that $|\Delta_1^C f(x^p)| \ge \zeta$ for all $p > k_0$. Combining this with (3.17), (3.18), the second condition in (3.22) and Step 2 of the algorithm gives

$$\begin{aligned} f(x^{p+1}) - f(x^p) &\le \beta_2[\Delta f(x^p; d^p) + \frac{1}{2}(d^p)^T H_p) d^p] \\ &\le -\beta_2 \kappa_1 \min(\kappa_2, \delta_p) \\ &= -\beta_2 \kappa_1 \min(\kappa_2, \delta_{k_0}) \end{aligned} \tag{3.23}$$

for all $p \ge k_0$. But then $f(x^k) \downarrow -\infty$, which contradicts our working hypotheses. Therefore, $\nu(k)$ is well-defined for every $k \in J$.

Let $k \in J$ and suppose $\nu(k)$ is such that (3.20) is violated but (3.21) is not violated at $x^{\nu(k)+1}$. Then either $\nu(k) = k$ or, as in (3.23),

$$\begin{aligned} f(x^{p+1}) - f(x^p) &\le \beta_2[\Delta f(x^p; d^p) + \frac{1}{2}(d^p)^T H_p) d^p] \\ &\le -\beta_2 \kappa_1 \min(\kappa_2, \delta_p), \end{aligned} \tag{3.24}$$

for $p = k, \ldots, \nu(k) - 1$. If $\nu(k) = k$, then, by (3.18), $r_k \ge \beta_2$ and so

$$f(x^{\nu(k)+1}) - f(x^k) \le -\beta_2 \kappa_1 \min(\kappa_2, \delta_k) \le -\beta_2 \kappa_1 \min(\kappa_2, \overline{\epsilon});$$

otherwise, summing (3.24) over p gives

$$\begin{aligned} f(x^{\nu(k)+1}) - f(x^k) &\le (f(x^{\nu(k)+1}) - f(x^{\nu(k)})) - \sum_{p=k}^{\nu(k)-1} \beta_1 \kappa_1 \min(\kappa_2, \delta_p) \\ &\le -\beta_1 \kappa_1 \min(\kappa_2, \sum_{p=k}^{\nu(k)-1} \delta_p) \\ &\le -\beta_1 \kappa_1 \min(\kappa_2, \overline{\epsilon}), \end{aligned}$$

since $\sum_{p=k}^{\nu(k)-1} \delta_p \geq \left\| x^{\nu(k)+1} - x^k \right\| \geq \bar{\epsilon}$ (here, the second inequality uses the elementary fact that for any three non-negative numbers τ_1, τ_2, τ_3, we have $\min(\tau_1,\, \tau_2 + \tau_3) \leq \min(\tau_1,\, \tau_2) + \min(\tau_1,\, \tau_3)$). Hence,

$$f(x^{\nu(k)+1}) - f(x^k) \leq -\beta_2\kappa_1 \min(\kappa_2, \bar{\epsilon}) \tag{3.25}$$

when (3.20) is violated but (3.21) is not violated.

Next suppose that (3.21) is violated at $\nu(k)$, and let $s(k)$ be the smallest positive integer for which $x^{\nu(k)+s(k)} \neq x^{\nu(k)}$. The integer $s(k)$ is well defined since, by (3.18), $x^{\nu(k)}$ is eventually updated. Also note that $x^{(\nu(k)+i)}$ satisfies (3.18) for $i = 0, 1, \ldots, s(k)-1$. Therefore, by (3.17) and (3.18),

(3.26)

$$\begin{aligned} f(x^{\nu(k)+s(k)}) - f(x^{\nu(k)}) &\leq \beta_2[\Delta f(x^{\nu(k)}; d^{\nu(k)+s(k)-1}) + \frac{1}{2} d^{(\nu(k)+s(k)-1)\top} H_{\nu(k)} d^{(\nu(k)+s(k)-1)}] \\ &\leq -\beta_2\kappa_1 \min(\kappa_2,\, \delta_{\nu(k)+s(k)-1}) \\ &\leq -\beta_2\kappa_1 \min(\kappa_2,\, \gamma_1\bar{\delta}), \end{aligned}$$

since $x^{(\nu(k)+i)} = x^{\nu(k)}$ so that $\delta_{(\nu(k)+i)+1} = \gamma_3\delta_{(\nu(k)+i)}$, $i = 0, 1, \ldots, s(k)-1$, and, in particular, $\delta_{\nu(k)+s(k)-1} = \gamma_3^{-1}\delta_{\nu(k)+s(k)}$. Set $s(k) = 1$ if (3.20) is violated but (3.21) is not violated at $x^{\nu(k)+1}$. Putting (3.25) together with (3.26), gives

$$f(x^{\nu(k)+s(k)}) - f(x^k) \leq -\beta_2\kappa_1 \min(\kappa_2,\, \gamma_1\bar{\delta}, \bar{\epsilon}) \quad \forall\, k \in J.$$

But then, $f(x^k) \downarrow -\infty$ which contradicts our working hypotheses. This establishes the theorem. □

3.5 *Numerical Experiments*

We conclude with a proof of concept for the step size methods developed in the previous section. Nesterov [22] introduced variants of the Chebyshev-Rosenbrock functions (NCR functions), a smooth version

$$\tilde{f}(x) := \frac{1}{4}(x_1 - 1)^2 + \sum_{i=1}^{n-1}(x_{i+1} - 2x_i^2 + 1)^2,$$

along with a nonsmooth version

$$f(x) := \frac{1}{4}(x_1 - 1)^2 + \sum_{i=1}^{n-1} |x_{i+1} - 2x_i^2 + 1|.$$

The unique minimizer of both functions is $\overline{x} := (1, \ldots, 1)^\top$, and the active manifold in the sense of [26] for the functions is $\mathcal{M} := \left\{ x \,\middle|\, x_{i+1} = 2x_i^2 - 1,\ i = 1, \ldots, n-1 \right\}$ which contains $\overline{x}$ and the point $\widehat{x} = (-1, 1, \ldots, 1)^\top$. Gürbüzbalaban and Overton [22, Section 3] studied the BFGS algorithm applied to both functions. The authors are able to minimize the smooth function starting from $\widehat{x}$ and running BFGS initialized with the identity matrix. The authors succeed at reducing the smooth objective below 10^{-15} for $n = 8$ after 6700 iterations and for $n = 10$ after 50,000 iterations.

For the nonsmooth function, the BFGS algorithm cannot be initialized at $\widehat{x}$; instead, the authors initialize randomly about the origin and run BFGS initialized with the identity matrix. This initialization scheme works only for $n = 2$.

Our experiments are focused on the nonsmooth variation recognized as an instance convex-composite optimization. We consider two random initialization schemes. We either initialize each component uniformly on $[-0.2, 0.2]$ or initialize each component uniformly on $[0.5, 1.5]$. The former initialization scheme is farther from the global minimizer and thus presents a greater challenge for all algorithms to follow.

Defining

$$\begin{aligned} h(c_1, c_2, \ldots, c_n) &= \frac{1}{4}c_1^2 + |c_2| + \cdots + |c_n|, \\ c_1(x) &= x_1 - 1, \\ c_i(x) &= x_i - 2x_{i-1}^2 + 1, \quad i = 2, \ldots, n, \\ c(x) &= (c_1(x), c_2(x), \ldots, c_n(x)), \end{aligned}$$

gives $f(x) = h(c(x))$. After block decomposing $c = (c_1, c_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, the objective is

$$h\left(\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} \nabla c_1^\top \\ \nabla c_2 \end{pmatrix} d \right) = \frac{1}{4}(c_1 + \nabla c_1^\top d)^2 + \|c_2 + \nabla c_2 d\|_1 ,$$

and the direction finding subproblem $\mathcal{P}_k$

$$\underset{d}{\text{minimize}} \quad \frac{1}{4}(c_1(x) + \nabla c_1(x)^\top d)^2 + \left\| c_2(x) + \nabla c_2(x) d \right\|_1 \tag{3.27}$$

is equivalent to the quadratic program

$$\begin{array}{ll} \underset{d,\, s \,\in\, \mathbb{R}^n \times \mathbb{R}^{n-1}}{\text{minimize}} & \frac{1}{4}(\nabla c_1(x)^T d)^2 + \frac{1}{2} c_1(x) \nabla c_1(x)^T d + \mathbf{1}^\top s \\ \text{subject to} & \begin{pmatrix} \nabla c_2(x) & -I_{n-1} \\ -\nabla c_2(x) & -I_{n-1} \end{pmatrix} \begin{pmatrix} d \\ s \end{pmatrix} \leq \begin{pmatrix} -c_2(x) \\ c_2(x) \end{pmatrix} \end{array}$$

All direction finding subproblems are solved using CPLEX Optimization Studio V12.8.0 for MATLAB R2018a using a 2.7 GHz Intel Core i5 machine with 8 GB of memory. Each figure below is generated using 100 randomly sampled starting points.

3.5.1 Line Search Methods

We implement the backtracking (GNBT) and weak Wolfe (GNWW) algorithms using search directions generated from Cauchy steps of (3.27) by constraining $\|d\|_\infty \leq 10$. The line search parameters are $\sigma_1 = 10^{-4}$ and $\theta = 0.5$ for Algorithm 1 and $\sigma_1 = 10^{-4}$, $\sigma_2 = 0.8$ and $\mu = 0.9$ for Algorithm 3.

Hard Initializations

The line search methods are able to reduce f below 10^{-10} at hard starting points routinely when $n = 7$, but only 80% of the time when $n = 8$ and not routinely whenever $n \geq 9$.

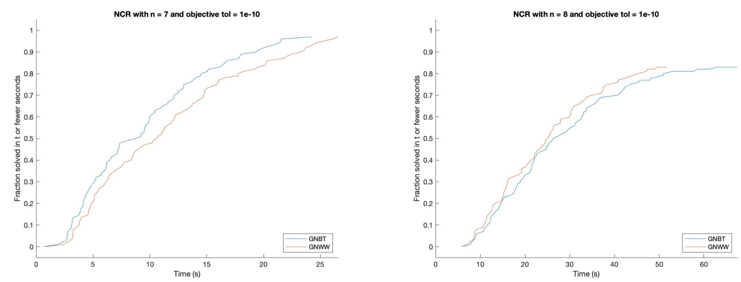

Figure 3.1: Line Search Efficiency Curves for $n = 7, 8$ at Hard Initializations

Easy Initializations

Like the hard starting points, we are unable to reduce f below 10^{-10} on these points for $n \geq 9$ even after 5,000 iterations.

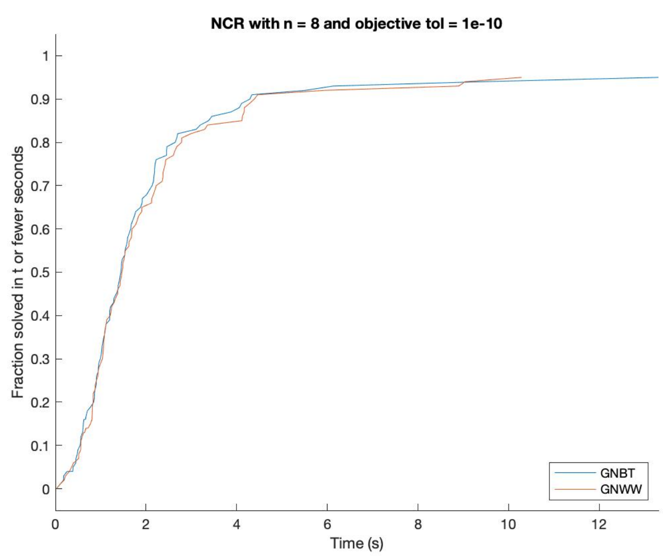

Figure 3.2: Line Search Efficiency Curves for $n = 8$ at Easy Initializations

3.5.2 *Trust Region Method*

For the trust region method of Algorithm 4, we set $\gamma_1 = \gamma_2 = 0.5$, $\gamma_3 = 1.1$ and $\beta_1 = 10^{-6}$, $\beta_2 = 10^{-5}$ and $\beta_3 = 0.8$ with initial trust region radius $\delta_0 = 1$. We let $H_k = 0$ in every iteration, so the search directions correspond to Cauchy steps.

Hard Initializations

For hard initial points, we are able to routinely reduce f below 10^{-10} for $n \leq 4$. However, the method fails to reduce f below 10^{-8} at these starting points even when $n = 5$ after 5,000 iterations.

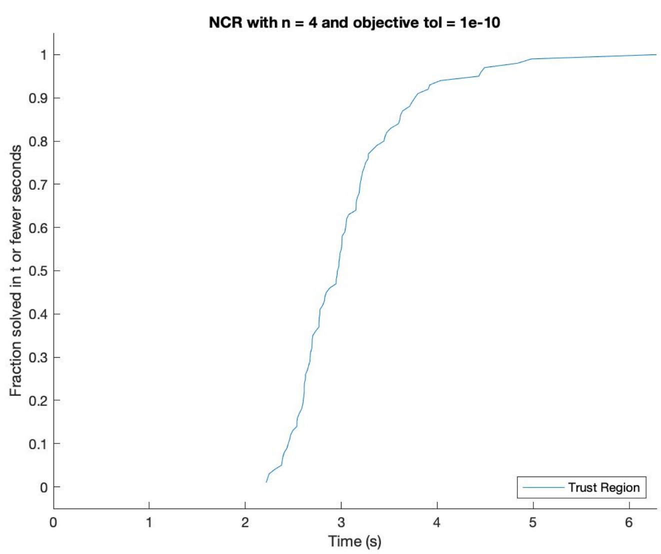

Figure 3.3: TRS Efficiency Curve for $n = 4$ at Hard Initializations

Easy Initializations

The trust region method is able to reduce f below 10^{-10} past dimension $n = 5000$ at the easy starting points.

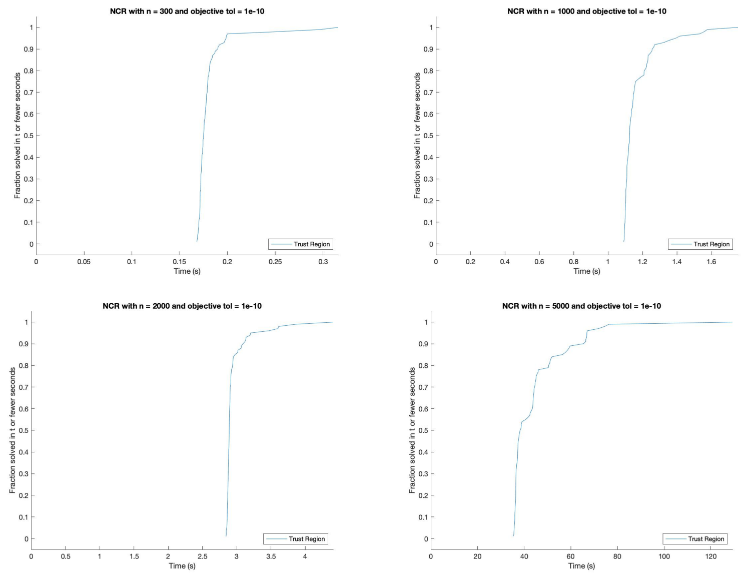

Figure 3.4: TRS Efficiency Curves for Various Dimensons n at Easy Initializations

Chapter 4

NEWTON AND QUASI-NEWTON METHODS

Authors

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Abstract This chapter concerns the local convergence theory of Newton and quasi-Newton methods for *convex-composite* optimization: minimize $f(x) := h(c(x))$, where h is an infinite-valued proper convex function and c is $\mathcal{C}^2$-smooth. We focus on the case where h is infinite-valued piecewise linear-quadratic and convex. Our approach embeds the optimality conditions for convex-composite optimization problems into a generalized equation. We establish conditions for strong metric subregularity and strong metric regularity of the corresponding set-valued mappings. This allows us to extend classical convergence of Newton and quasi-Newton methods to the broader class of non-finite valued piecewise linear-*quadratic* convex-composite optimization problems. In particular we establish local quadratic convergence of the Newton method under conditions that parallel those in nonlinear programming when h is non-finite valued piecewise linear.

4.1 Introduction

This chapter concerns local convergence theory of Newton and quasi-Newton methods for the solution of the *convex-composite* problem:

$$(\mathbf{P}) \qquad \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad f(x) := h(c(x)),$$

where $h : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ is piecewise linear-quadratic (PLQ) and convex, and $c : \mathbb{R}^n \to \mathbb{R}^m$ is $\mathcal{C}^2$-smooth.

These, and almost all other methods for solving **P**, use a direction-finding subproblem similar to

$$(\mathbf{P}_k) \qquad \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad h(c(x^k) + \nabla c(x^k)[x - x^k]) + \frac{1}{2}[x - x^k]^\top H_k [x - x^k],$$

where H_k is the Hessian of a Lagrangian for **P** [4]. When the Hessian H_k is used in the subproblems, the method corresponds to a Newton method (1.10), and when H_k is approximated by a matrix B_k, it corresponds to a quasi-Newton method (1.11). In either case, the subproblems $\mathbf{P}_k$ may or may not be convex depending on whether H_k, $B_k \succeq 0$. In the context of the broader class of prox-regular h, Lewis and Wright [28] take $B_k = \mu_k I$ at each iteration, thereby guaranteeing existence and uniqueness of the "proximal step" and a global descent algorithm. Instead, our focus is on developing methods possessing fast local rates of convergence by taking advantage of second-order information together with the convex geometry of $\operatorname{dom}(h)$ developed by Rockafellar [41].

When h is assumed to be a finite-valued piecewise linear convex function, Womersley [46] established second-order rates of convergence for these algorithms under conditions comparable to those used in NLP, i.e., linear independence of the active constraint gradients, strict complementarity, and strong second-order sufficiency. Notwithstanding this correspondence to NLP, the method of proof differs significantly from the standard methodology to establishing such results in the NLP case developed by Robinson [36, 37]. Notably, in the case of NLP, the function h is piecewise linear but not finite-valued. In subsequent work, Robinson [38] introduced the revolutionary idea of *generalized equations*, whose variational properties can be used to establish local rates of convergence for Newton's method for NLP. By employing the techniques of generalized equations, Cibulka et. al. [9] recently connected classical second-order necessary and sufficient conditions for a local minimizer of **P** with strong metric subregularity (see Definition 4.3.1) of the underlying KKT mapping when h is piecewise linear convex but not necessarily finite-valued. However, their analysis relies heavily on the fact that h is piecewise linear. And so, the old question of what conditions imply local quadratic convergence when h is not piecewise linear remains open. However, their technique created the possibility of an extension to the case where h is a member of the PLQ class. This extension is our goal. It is hoped that the methods and techniques developed in this paper provide insight into how to extend these results beyond the PLQ class.

As noted above, we couch the analysis in the context *Newton's method* for generalized equations. The first-order necessary conditions of a local minimum of **P** are encoded through a generalized equation of the form $g(x, y) + G(x, y) \ni 0$, where $g : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ is a $\mathcal{C}^1$-smooth function, $G : \mathbb{R}^{n+m} \rightrightarrows \mathbb{R}^{n+m}$ is a set-valued mapping, (x, y) represents a primal-dual pair, and the function $\nabla g(x, y)$ is a KKT matrix for **P** (see Definition 2.2.4). Newton's method (1.10) for solving this generalized equation corresponds to solving the optimality conditions for $\mathbf{P}_k$. The Newton iterate at (x^k, y^k) is obtained by solving the following linearized generalized equation:

$$\text{(4.1)} \quad \text{Find } (x^{k+1}, y^{k+1}) \text{ such that } g(x^k, y^k) + \nabla g(x^k, y^k) \begin{pmatrix} x^{k+1} - x^k \\ y^{k+1} - y^k \end{pmatrix} + G(x^{k+1}, y^{k+1}) \ni 0.$$

The details of this derivation appear in Section 2.2.

The goal of this chapter is to establish local convergence rates for algorithms based on iteratively solving $\mathbf{P}_k$ in the case where h is a PLQ convex function. We do this by augmenting the strategy of Cibulka et. al. [9] with additional innovations by Lewis [26] and Rockafellar [41]. In particular, we are able to establish conditions under which these algorithms are locally quadratically convergent. The first phase of our analysis involves extensive application of the first- and second-order PLQ calculus [41,42] to establish conditions under which the underlying generalized equation is strongly metrically subregular. This allows us to establish sufficient conditions for the superlinear convergence of quasi-Newton methods for algorithms whose direction finding subproblems are based on $\mathbf{P}_k$. The second phase of our analysis employs the technique of partly smooth functions in the sense of [23,26] to establish conditions under which a local approximation to the underlying generalized equation is strongly metrically regular (see Definition 4.5.1). This allows us to give conditions for the local quadratic convergence of the Newton method based on $\mathbf{P}_k$.

We also note that recent work by Drusvyatskiy and Lewis [18] considers similar types of results for convex-composite optimization problems of the form $\varphi(x) = h(c(x)) + g(x)$, where h is finite-valued and $L-$Lipschitz, ∇c is β-Lipschitz, and g is closed, proper, convex, but infinite-valued. One of their goals is to understand the convergence of prox-linear type methods through either the subregularity [18, Theorems 5.10 and 5.11] or strong regularity [18, Theorem 6.2] of $\partial\varphi$ at *stable* strong minima or sharp minima of φ [18, Theorems 7.1 and 7.2].

When h is only assumed to be finite-valued convex and g is zero, the first result on the local quadratic convergence for convex-composite problems was that of Burke and Ferris [7]. In that work, the authors established a constraint qualification for the inclusion $c(\overline{x}) \in \arg\min h$ that ensures the local quadratic convergence of constrained Gauss-Newton methods. In [7], the authors assumed $\arg\min h$ was a set of *weak sharp minima* [6]. However, it was observed by Li and Wang [30] that the sharpness hypothesis was not required. Rather, a local quadratic growth condition [30, Theorem 2] was sufficient for the proof techniques in [7] to succeed. The authors continued research [29] in relaxations of the constraint qualification on $c(\overline{x}) \in \arg\min h$ and studied proximal methods [24] for their convergence.

Our focus on the PLQ class is motivated by the great variety of modern problems in data analysis, estimation of dynamical systems, inverse problems, and machine learning that are posed within this class. The key to the success of the convex-composite structure is that it separates the data associated to the problem, the function c, from the model within which we wish to explore the data, the function h. Consequently, the broader the class of functions h available, the greater the variety of ways within which we can explore underlying extremal properties of the input function c, e.g., sparsity, robustness, network structure, dynamics, influence of hyperparameters, etc. Importantly, we have learned that features of the data can be more readily extracted by imposing nonsmoothness in the function h.

The roadmap of this chapter is as follows. Section 4.2 discusses the convex geometry and differential theory of piecewise linear-quadratic functions collected in [42]. The second-order theory of [42] allows us to rewrite the general second-order necessary and sufficient conditions for a local minimum of **P**. We extract a crucial result from [42] that highlights natural candidates for manifolds of partial smoothness [26] inherent to the function h. Section 4.3 extends the result [9, Theorem 7.1] relating the strong metric subregularity of (2.9) to the second-order sufficient conditions of local minima and ends with a convergence study of quasi-Newton methods for **P**. Section 4.4 establishes conditions for the partly smooth structure of PLQ convex functions and sets the stage for Section 4.5, where we analyze the local quadratic convergence of Newton's method as in [15].

4.2 Geometry of PLQ Functions and Their Domains

In this section, unless otherwise stated, we let $f := h \circ c$ where h is piecewise linear-quadratic convex and c is $\mathcal{C}^2$-smooth.

Definition 4.2.1 (piecewise linear-quadratic)**.** A proper function $h : \mathbb{R}^m \to \overline{\mathbb{R}}$ is called piecewise linear-quadratic (PLQ) if $\operatorname{dom}(h) \neq \emptyset$ and $\operatorname{dom}(h)$ can be represented as the union of $\mathcal{K} \geq 1$ polyhedral sets of the form

$$C_k = \left\{ c \;\middle|\; \langle a_{kj},\, c\rangle \leq \alpha_{kj},\ \text{for all } j \in \{1, \ldots, s_k\} \right\} \tag{4.2}$$

relative to each of which $h(c)$ is given by an expression of the form $\frac{1}{2}\langle c,\, Q_k c\rangle + \langle b_k,\, c\rangle + \beta_k$ for some scalar $\beta_k \in \mathbb{R}$, vector $b_k \in \mathbb{R}^n$, and symmetric matrix Q_k.

Remark 8. The sets C_k do not necessarily form a partition of the set C.

The following lemma is straightforward.

Lemma 4.2.1. Suppose h is piecewise linear-quadratic convex. Then, for any $k \in \mathcal{K}$, the matrices Q_k satisfy $\langle c, Q_k c\rangle \geq 0$ for all $c \in \operatorname{par}(C_k)$.

For the sake of reference we recall the normal and tangent cone structure for polyhedral sets.

Definition 4.2.2 (Active indices)**.** For a piecewise linear-quadratic function h and a point $\bar{c} \in \operatorname{dom}(h)$, define the set $\mathcal{K}(\bar{c}) := \left\{k \in \mathcal{K} \mid \bar{c} \in C_k\right\}$, and write $\bar{k} := |\mathcal{K}(\bar{c})|$, so that $\mathcal{K}(\bar{c}) = \left\{k_1, k_2, \ldots, k_{\bar{k}}\right\}$.

Theorem 8 (Normal and Tangent Cones to Polyhedra)**.** [42, Theorem 6.46] Suppose $c \in C_k$ with C_k polyhedral as in (4.2). Let $I_k(c) = \left\{j \in \{1, \ldots, s_k\} \;\middle|\; \langle a_{kj}, c\rangle = \alpha_{kj}\right\}$, and let $\ell_k = |I_k(c)|$. Then,

(4.3)

$$N\left(c \,|\, C_k\right) = \left\{\sum_{j \in I_k(c)} \lambda_j a_{kj} \mid \lambda_j \geq 0,\ j \in I_k(c)\right\} \text{ and } T\left(c \,|\, C_k\right) = \left\{v \;\middle|\; \langle a_{kj}, v\rangle \leq 0,\ j \in I_k(c)\right\}.$$

Our first- and second-order analysis in the PLQ case heavily depends on the following results from [42].

Proposition 4.2.1. [42, Propositions 10.21, 13.9] If $h : \mathbb{R}^m \to \overline{\mathbb{R}}$ is piecewise linear-quadratic, then $\operatorname{dom}(h)$ is closed, h is continuous relative to $\operatorname{dom}(h)$. Consequently, h is closed. At any point $\bar{c} \in \operatorname{dom}(h)$, $h'(\bar{c}; \cdot) = \mathrm{d}h(\bar{c})$, and $h'(\bar{c}; \cdot)$ is piecewise linear with $\operatorname{dom}\left(h'(\bar{c}; \cdot)\right) = \bigcup_{k \in K(\bar{c})} T\left(\bar{c} \,|\, C_k\right) = T\left(\bar{c} \,|\, \operatorname{dom}(h)\right)$. In particular, for $k \in \mathcal{K}(\bar{c})$ and $w \in T\left(\bar{c} \,|\, C_k\right)$,

$$h'(\bar{c}; w) = \langle Q_k \bar{c} + b_k,\, w\rangle . \tag{4.4}$$

If, in addition, h is convex, then $\operatorname{dom}(h)$ is polyhedral,

$$\emptyset \neq \partial h(\bar{c}) = \bigcap_{k \in \mathcal{K}(\bar{c})} \left\{y \;\middle|\; y - Q_k \bar{c} - b_k \in N\left(\bar{c} \,|\, C_k\right)\right\}, \tag{4.5}$$

$h''(\overline{c};\cdot)$ is piecewise linear-quadratic, but not necessarily convex, and for any $w \in \mathbb{R}^m$,

$$(4.6) \qquad 0 \le h''(\overline{c}; w) = \begin{cases} \langle w,\, Q_k w\rangle & \text{when } w \in T\left(\overline{c}\,|\,C_k\right), \\ \infty & \text{when } w \notin T\left(\overline{c}\,|\,\mathrm{dom}\,(h)\right). \end{cases}$$

For every $y \in \partial h(\overline{c})$, $\mathrm{d}^2 h(\overline{c}|y)$ is piecewise linear-quadratic and convex. Let $K(\overline{c}, y) := \left\{w \;\middle|\; h''(\overline{c}; w) = \langle y,\, w\rangle \right\}$. Then, $K(\overline{c}, y)$ is a polyhedral cone, and

$$(4.7) \qquad \mathrm{d}^2 h(\overline{c}|y)(w) = \lim_{\tau \searrow 0} \Delta_\tau^2 h(\overline{c}|y)(w) = \begin{cases} h''(\overline{c}; w) & w \in K(\overline{c}, y), \\ +\infty & \text{otherwise.} \end{cases}$$

Moreover, there exists a neighborhood V of $\overline{c}$ such that

$$(4.8) \qquad h(c) = h(\overline{c}) + h'(\overline{c}; c - \overline{c}) + \frac{1}{2} h''(\overline{c}; c - \overline{c}) \text{ for } c \in V \cap \mathrm{dom}\,(h).$$

Theorem 9. [42, Theorem 13.14] Let $f = h \circ c$ for a $\mathcal{C}^2$ mapping $c : \mathbb{R}^n \to \mathbb{R}^m$ and a piecewise linear-quadratic convex $h : \mathbb{R}^m \to \overline{\mathbb{R}}$. Let $\overline{x} \in \mathrm{dom}\,(f)$ and suppose f satisfies (BCQ) at $\overline{x}$. Then, for any $v \in \partial f(\overline{x})$, the set $Y(\overline{x}, v)$ given by (2.7) is compact as well as convex and nonempty, and for any $w \in \mathbb{R}^n$

$$(4.9) \qquad \mathrm{d}^2 f(\overline{x}|v)(w) = \mathrm{d}^2 \overline{f}(\overline{x}|v)(w) + \max\left\{\left\langle w,\, \nabla^2 (yc)(\overline{x}) w\right\rangle \;\middle|\; y \in Y(\overline{x}, v)\right\},$$

with $\overline{f}(x) := h(c(\overline{x}) + \nabla c(\overline{x})[x - \overline{x}])$ piecewise linear-quadratic convex.

The standard development of first- and second-order optimality conditions requires the notion of directions of non-ascent.

Definition 4.2.3. Let the *directions of non-ascent* for any proper $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ at $x \in \mathrm{dom}\,(f)$ be denoted by $D(x) := \left\{d \in \mathbb{R}^n \;\middle|\; \mathrm{d}f(x)(d) \le 0\right\}$. By Theorem 4, if f is convex-composite and f satisfies (BCQ) at x, then

$$(4.10) \qquad D(x) = \left\{d \in \mathbb{R}^n \;\middle|\; h'(c(x); \nabla c(x) d) \le 0\right\}$$

In the PLQ convex case, (BCQ) ensures that we have the following convenient representation of the set $D(\overline{x})$.

Lemma 4.2.2. Let f be as in **P**, and let $\overline{x} \in \mathbb{R}^n$ be such that f satisfies (BCQ) at $\overline{x}$. Set $\overline{c} := c(\overline{x})$. Then, $D(\overline{x})$ is convex and the union of finitely many polyhedral closed convex sets with following the representation:

$$
\begin{aligned}
D(\overline{x}) &= \bigcup_{k\in\mathcal{K}(\overline{c})} \left\{ d \,\middle|\, \nabla c(\overline{x})d \in T\left(\overline{c}\,|\,C_k\right), \left\langle Q_k\overline{c} + b_k,\, \nabla c(\overline{x})d\right\rangle \leq 0 \right\} \\
&= \bigcup_{k\in\mathcal{K}(\overline{c})} \left\{ d \,\middle|\, \begin{array}{l} \left\langle Q_k\overline{c} + b_k,\, \nabla c(\overline{x})d\right\rangle \leq 0 \\ \left\langle a_{kj},\, \nabla c(\overline{x})d\right\rangle \leq 0, j \in I_k(\overline{c}) \end{array} \right\}
\end{aligned}
\tag{4.11}
$$

Proof. ($\subset$) Suppose $d \in D(\overline{x})$. By (4.10), $\nabla c(\overline{x})d \in \operatorname{dom}\left(h'(\overline{c};\cdot)\right)$. In particular, by Proposition 4.2.1, $\nabla c(\overline{x})d \in T\left(\overline{c}\,|\,C_k\right)$ for some $k \in \mathcal{K}(\overline{c})$. By (4.4), we also have $\left\langle Q_k\overline{c} + b_k,\, \nabla c(\overline{x})d\right\rangle = h'(c(\overline{x}); \nabla c(\overline{x})d) \leq 0$.
($\supset$) If $d \in \bigcup_{k\in\mathcal{K}(\overline{c})} \left\{ d \,\middle|\, \nabla c(\overline{x})d \in T\left(c\,|\,C_k\right), \left\langle Q_k\overline{c} + b_k,\, \nabla c(\overline{x})d\right\rangle \leq 0 \right\}$, then for some $k \in \mathcal{K}(\overline{c})$, $\nabla c(\overline{x})d \in T\left(\overline{c}\,|\,C_k\right)$. Then, again by Proposition 4.2.1, $h'(c(\overline{x}); \nabla c(\overline{x})d) = \left\langle Q_k\overline{c} + b_k,\, \nabla c(\overline{x})d\right\rangle \leq 0$, so $d \in D(\overline{x})$. □

We now have the tools necessary to rewrite Theorem 1 in the context of piecewise linear-quadratic convex functions h.

Theorem 10 (PLQ second-order necessary and sufficient conditions)**.** [42, Theorems 13.24(b), 13.14], [41, Theorem 3.4]. Let $h : \mathbb{R}^m \to \overline{\mathbb{R}}$ be piecewise linear-quadratic and convex with $\overline{x} \in \operatorname{dom}(f)$ such that f satisfies (BCQ) at $\overline{x}$.

(a) If f has a local minimum at $\overline{x}$, then $0 \in \nabla c(\overline{x})^\top \partial h(c(\overline{x}))$ and

$$h''(c(\overline{x}); \nabla c(\overline{x})d) + \max\left\{ \left\langle d,\, \nabla^2(yc)(\overline{x})d\right\rangle \,\middle|\, y \in M(\overline{x}) \right\} \geq 0$$

for all $d \in D(\overline{x})$.

(b) If $0 \in \nabla c(\overline{x})^\top \partial h(c(\overline{x}))$ and

$$h''(c(\overline{x}); \nabla c(\overline{x})d) + \max\left\{\langle d,\, \nabla^2(yc)(\overline{x})d\rangle \;\middle|\; y \in M(\overline{x})\right\} > 0$$

for all $d \in D(\overline{x}) \setminus \{0\}$, then $\overline{x}$ is a strong local minimizer (see (1.8)) of f.

4.3 Strong Metric Subregularity of the KKT Mapping

In this section we establish conditions under which the set-valued mapping Definition 2.2.4 satisfies strong metric subregularity.

Definition 4.3.1 (Strong metric subregularity)**.** A set-valued mapping $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is *strongly metrically subregular* at $\overline{x}$ for $\overline{y}$ if $(\overline{x}, \overline{y}) \in \operatorname{gph} S$ and there exists $\kappa \geq 0$ and a neighborhood U of $\overline{x}$ such that $\|x - \overline{x}\| \leq \kappa \operatorname{dist}\left(\overline{y} \;\middle|\; S(x)\right)$ for all $x \in U$.

Our discussion of strong metric subregularity only requires f to satisfy (BCQ) at $\overline{x} \in \operatorname{dom}(f)$.

Lemma 4.3.1. Consider the KKT mapping $g + G$ and the mapping $\mathcal{G}$ given in Definition 2.2.4. Then, strong metric subregularity of $g + G$ at $(\overline{x}, \overline{y})$ for 0 is equivalent to the property that $(\overline{x}, \overline{y})$ is an isolated point of $\mathcal{G}^{-1}(0)$.

Proof. By [15, Corollary 3I.10], strong metric subregularity of $g+G$ at $(\overline{x}, \overline{y})$ for 0 is equivalent to strong metric subregularity of the linearization $\mathcal{G}$ (2.10) at $(\overline{x}, \overline{y})$.

By [42, Theorem 11.14, Proposition 12.30] the mapping $G(x, y)$ is polyhedral; that is, $\operatorname{gph} G$ is the union of finitely many polyhedral sets. Then [15, Corollary 3I.11] establishes the equivalence of strong metric subregularity of $\mathcal{G}$ at $(\overline{x}, \overline{y})$ for 0 and $(\overline{x}, \overline{y})$ being an isolated point of $\mathcal{G}^{-1}(0)$. □

The main result of this section now follows.

Theorem 11. Suppose $h : \mathbb{R}^m \to \overline{\mathbb{R}}$ is piecewise linear-quadratic and convex with $\overline{x} \in \operatorname{dom}(f)$ such that f satisfies (BCQ) at $\overline{x}$. Then, the following are equivalent:

1. The set $M(\overline{x}) := \mathrm{Null}\left(\nabla c(\overline{x})^\top\right) \cap \partial h(c(\overline{x}))$ in (2.8) is a singleton and the second-order sufficient conditions of Theorem 10 are satisfied at $\overline{x}$;

2. The mapping $g + G$ is strongly metrically subregular at $(\overline{x}, \overline{y})$ for 0 and $\overline{x}$ is a strong local minimizer of f.

Proof. For a point $x \in \mathrm{dom}\,(f)$, define $\Delta f(x; d) := h(c(x) + \nabla c(x)d) - h(c(x))$.

($\Rightarrow$) By Lemma 4.3.1 we argue strong metric subregularity of $g + G$ at $(\overline{x}, \overline{y})$ for 0 by showing that there is a neighborhood of $(\overline{x}, \overline{y})$ on which $(\overline{x}, \overline{y})$ is the unique solution to the generalized equation $\mathcal{G} \ni 0$ (2.10). After the change of variables $d := x - \overline{x}$, we show that there is a neighborhood U of $(0, \overline{y})$ such that $(d, y) = (0, \overline{y})$ is the unique solution to the generalized equation

$$Hd + \nabla c(\overline{x})^\top y = 0 \tag{4.12}$$

$$c(\overline{x}) + \nabla c(\overline{x})d \in \partial h^\star(y) \quad (\Leftrightarrow y \in \partial h(c(\overline{x}) + \nabla c(\overline{x})d)), \tag{4.13}$$

where $H := \nabla^2_{xx} L(\overline{x}, \overline{y})$. Suppose there is no such neighborhood. Then, there exists a sequence of vectors $\{(d^i, y^i)\}_{i\in\mathbb{N}}$ converging to $(0, \overline{y})$ with $(d^i, y^i) \neq (0, \overline{y})$ that solve the generalized equation (4.12), (4.13). First assume $d^i \neq 0$ for all $i \in \mathbb{N}$. Define for each $i \in \mathbb{N}$, $t_i := \left\|d^i\right\|$, $v^i := d^i/\left\|d^i\right\|$, and assume without loss of generality that $v^i \to \overline{v}$ and that

$$\left\{c(\overline{x}) + \nabla c(\overline{x})d^i\right\}_{i\in\mathbb{N}} \subset C_{k_0} \text{ for some } k_0 \in K(c(\overline{x}) + \nabla c(\overline{x})d^i) \subset K(\overline{c}), \tag{4.14}$$

since $d^i \to 0$. Taking the inner product on both sides of (4.12) with d^i, we obtain

$$0 = \left\langle d^i,\, Hd^i\right\rangle + \left\langle d^i,\, \nabla c(\overline{x})^\top y^i\right\rangle \text{ for all } i \in \mathbb{N}. \tag{4.15}$$

The subgradient inequality for h at $c(\overline{x}) + \nabla c(\overline{x})d^i$ with subgradient y_i gives

$$\Delta f(\overline{x}; d^i) \le \left\langle d^i,\, \nabla c(\overline{x})^\top y_i\right\rangle = -\left\langle d^i,\, Hd^i\right\rangle. \tag{4.16}$$

Dividing through by $t_i > 0$ and letting $i \to \infty$, $\mathrm{d}f(\overline{x})(\overline{v}) \le \liminf_i \frac{\Delta f(\overline{x}; t_i v^i)}{t_i}$. Hence by (BCQ), Theorem 9 and (4.16), $h'(c(\overline{x}); \nabla c(\overline{x})\overline{v}) = \mathrm{d}f(\overline{x})(\overline{v}) \le \lim_i -\left\langle v^i,\, Hd^i\right\rangle = 0$, and so

$\overline{v} \in D(\overline{x}) \setminus \{0\}$. By second-order sufficiency, $h''(c(\overline{x}); \nabla c(\overline{x})\overline{v}) + \overline{v}^\top H \overline{v} > 0$. We now show $\nabla c(\overline{x})\overline{v} \in T\left(\overline{c} \,|\, C_{k_0}\right)$. By (4.14) and the computation $\frac{c(\overline{x})+\nabla c(\overline{x})d^i - c(\overline{x})}{t_i} = \nabla c(\overline{x})v^i \to \nabla c(\overline{x})\overline{v} \in T\left(\overline{c} \,|\, C_{k_0}\right)$. Then by (4.6), $h''(c(\overline{x}); \nabla c(\overline{x})\overline{v}) = \overline{v}^\top \nabla c(\overline{x})^\top Q_{k_0} \nabla c(\overline{x})\overline{v}$, so that

$$\overline{v}^\top H \overline{v} + \overline{v}^\top \nabla c(\overline{x})^\top Q_{k_0} \nabla c(\overline{x})\overline{v} > 0. \tag{4.17}$$

On the other hand, by (4.5),

$$y^i \in \partial h(c(\overline{x}) + \nabla c(\overline{x})d^i) = \bigcap_{k \in \mathcal{K}(c(\overline{x}) + \nabla c(\overline{x})d^i)} \left\{ y \,\middle|\, y - Q_k(c(\overline{x}) + \nabla c(\overline{x})d^i) - b_k \in N\left(c(\overline{x}) + \nabla c(\overline{x})d^i \,|\, C_k\right) \right\},$$

and so $y^i - Q_{k_0}(c(\overline{x}) + \nabla c(\overline{x})d^i) - b_{k_0} \in N\left(c(\overline{x}) + \nabla c(\overline{x})d^i \,|\, C_{k_0}\right)$ for all $i \in \mathbb{N}$. Since $c(\overline{x}) \in C_{k_0}$, we have

$$\begin{aligned}
0 &\geq \left\langle y^i - [Q_{k_0}(c(\overline{x}) + \nabla c(\overline{x})d^i) + b_{k_0}],\, c(\overline{x}) - [c(\overline{x}) + \nabla c(\overline{x})d^i] \right\rangle \\
&= \left\langle y^i - Q_{k_0}(c(\overline{x}) + \nabla c(\overline{x})d^i) - b_{k_0},\, -\nabla c(\overline{x})d^i \right\rangle \\
&= -\left\langle d^i,\, \nabla c(\overline{x})^\top y^i \right\rangle + \left\langle Q_{k_0}(c(\overline{x}) + \nabla c(\overline{x})d^i) + b_{k_0},\, \nabla c(\overline{x})d^i \right\rangle.
\end{aligned}$$

Together with (4.15),

$$\begin{aligned}
0 &\geq \left\langle d^i,\, Hd^i \right\rangle + \left\langle Q_{k_0}(c(\overline{x}) + \nabla c(\overline{x})d^i) + b_{k_0},\, \nabla c(\overline{x})d^i \right\rangle \\
&= \left\langle d^i,\, Hd^i \right\rangle + \left\langle \nabla c(\overline{x})d^i,\, Q_{k_0} \nabla c(\overline{x})d^i \right\rangle + \left\langle Q_{k_0}c(\overline{x}) + b_{k_0},\, \nabla c(\overline{x})d^i \right\rangle \\
&= \left\langle d^i,\, Hd^i \right\rangle + \left\langle \nabla c(\overline{x})d^i,\, Q_{k_0} \nabla c(\overline{x})d^i \right\rangle + h'(c(\overline{x}); \nabla c(\overline{x})d^i) \quad \text{(by (4.4))} \\
&\geq \left\langle d^i,\, Hd^i \right\rangle + \left\langle \nabla c(\overline{x})d^i,\, Q_{k_0} \nabla c(\overline{x})d^i \right\rangle,
\end{aligned}$$

where the final inequality follows from Theorem 2, Theorem 10, and the observation that $\nabla c(\overline{x})d^i \in C_{k_0} - c(\overline{x}) \subset T\left(c(\overline{x}) \,|\, C_{k_0}\right)$. Next, divide the inequality $0 \geq \left\langle d^i,\, Hd^i \right\rangle + \left\langle \nabla c(\overline{x})d^i,\, Q_{k_0} \nabla c(\overline{x})d^i \right\rangle$ by t_i^2 and let $i \to \infty$ to yield the contradiction $0 \geq \overline{v}^\top H \overline{v} + \overline{v}^\top \nabla c(\overline{x})^\top Q_k \nabla c(\overline{x})\overline{v} > 0$. Consequently, $d^i = 0$ for all i sufficiently large, so without loss of generality, we now suppose $d^i = 0$ for all $i \in \mathbb{N}$. Hence by hypothesis, and $y^i \neq \overline{y}$ for all $i \in \mathbb{N}$. But then we contradict uniqueness of $M(\overline{x})$.

(⇐) By Lemma 4.3.1, $(\overline{x}, \overline{y})$ is an isolated point of $\mathcal{G}^{-1}(0)$. That is, there is a neighborhood U of $(\overline{x}, \overline{y})$ on which $(\overline{x}, \overline{y})$ is the unique solution to the generalized equation

$$\begin{aligned} &H(x-\overline{x}) + \nabla c(\overline{x})^\top y = 0 \\ &c(\overline{x}) + \nabla c(\overline{x})(x-\overline{x}) \in \partial h^\star(y). \end{aligned}$$

For $x = \overline{x}$, this implies there is a neighborhood $U_{\overline{y}}$ about $\overline{y}$ such that

$$U_{\overline{y}} \cap M(\overline{x}) = \{\overline{y}\}. \tag{4.18}$$

Suppose there is $y \in \big(M(\overline{x})\big) \setminus U_{\overline{y}}$. Then $y_t = (1-t)\overline{y} + ty \in M(\overline{x})$ for $t \in [0,1]$. But for t small, $y_t \in U_{\overline{y}} \cap M(\overline{x})$, which contradicts (4.18), so $M(\overline{x})$ is the singleton $\{\overline{y}\}$. Therefore, it only remains to show that the second-order sufficient conditions of Theorem 10 are satisfied at $\overline{x}$.

Since $\overline{x}$ is local minimizer of f at which f satisfies (BCQ), Theorem 4 gives $0 \in \nabla c(\overline{x})^\top \partial h(c(\overline{x}))$ and $h'(c(\overline{x}); \nabla c(\overline{x})d) \geq 0$ for all $d \in \mathbb{R}^n$. Let $\overline{d} \in \mathbb{R}^n \setminus \{0\}$ with $h'(c(\overline{x}); \nabla c(\overline{x})d) = 0$, or equivalently, $\overline{d} \in D(\overline{x})$. Without loss of generality, suppose $\left\|\overline{d}\right\| = 1$. In particular, by (4.11), there exists $k_0 \in K(\overline{c})$ such that

$$\nabla c(\overline{x})\overline{d} \in T\left(\overline{c} \,|\, C_{k_0}\right) \text{ and } \left\langle Q_{k_0}\overline{c} + b_{k_0},\, \nabla c(\overline{x})\overline{d}\right\rangle = h'(c(\overline{x}); \nabla c(\overline{x})\overline{d}) = 0 \tag{4.19}$$

Since h is PLQ convex, the second-order necessary conditions of Theorem 10 imply $h''(c(\overline{x}); \nabla c(\overline{x})\overline{d}) + \overline{d}^\top H\overline{d} \geq 0$.

We show this inequality is strict to complete the proof. Suppose to the contrary that

$$h''(c(\overline{x}); \nabla c(\overline{x})\overline{d}) + \overline{d}^\top H\overline{d} = 0. \tag{4.20}$$

Then, $\overline{d} \neq 0$ solves the program

$$\begin{aligned} &\underset{d}{\text{minimize}} \quad h'(c(\overline{x}); \nabla c(\overline{x})d) + \frac{1}{2}h''(c(\overline{x}); \nabla c(\overline{x})d) + \frac{1}{2}d^\top Hd \\ &\text{subject to} \quad d \in D(\overline{x}). \end{aligned}$$

By (4.8) and continuity of $d \mapsto c(\overline{x}) + \nabla c(\overline{x})d$, there exists $\epsilon > 0$ so that

$$\Delta f(\overline{x}; d) = h'(c(\overline{x}); \nabla c(\overline{x})d) + \frac{1}{2}h''(c(\overline{x}); \nabla c(\overline{x})d) \text{ for } d \in \epsilon\mathbb{B} \cap \left\{d \mid c(\overline{x}) + \nabla c(\overline{x})d \in \operatorname{dom}(h)\right\}.$$

By (4.19) and polyhedrality, $c(\overline{x}) + t\nabla c(\overline{x})\overline{d} \in \operatorname{dom}(h)$ for sufficiently small $t > 0$. It follows, after shrinking $\epsilon > 0$ if necessary, that

$$\Delta f(\overline{x}; t\overline{d}) + \frac{t^2}{2}\overline{d}^\top H\overline{d} = 0 \text{ for all } 0 \leq t < \epsilon. \tag{4.21}$$

Since $0 \in \partial f(\overline{x})$ and f satisfies (BCQ) at $\overline{x}$, (4.9) with $v = 0$, $y = \overline{y}$, and $w \in \mathbb{R}^n$ gives $d^2 f(\overline{x}|0)(w) = d^2\overline{f}(\overline{x}|0)(w) + w^\top Hw$, where $\overline{f}$ is also piecewise linear-quadratic by the discussion following (4.9). Since $\overline{x}$ is a strong local minimizer,

$$d^2 f(\overline{x}|0)(w) = \liminf_{\substack{\tau\searrow 0\\ w'\to w}} \frac{f(\overline{x} + tw') - f(\overline{x})}{\frac{1}{2}\tau^2} \geq \liminf_{\substack{\tau\searrow 0\\ w'\to w}} \gamma\|w'\|^2 = \gamma\|w\|^2 \text{ (see Theorem 1).}$$

Then, we have $d^2 f(\overline{x}|0)(w) = d^2\overline{f}(\overline{x}|0)(w) + w^\top Hw \geq \gamma\|w\|^2$. By (4.7) the lim inf defining $d^2\overline{f}(\overline{x}|0)(w)$ is also expressed as a limit only in τ (because $\overline{f}$ is piecewise linear-quadratic), so

$$d^2\overline{f}(\overline{x}|0)(w) = \lim_{\tau\searrow 0} \frac{\overline{f}(\overline{x} + \tau w) - \overline{f}(\overline{x})}{\frac{1}{2}\tau^2} = \lim_{\tau\searrow 0} \frac{\Delta f(\overline{x}; \tau w)}{\frac{1}{2}\tau^2}.$$

Putting the last two observations together, $d^2 f(\overline{x}|0)(w) = \lim_{\tau\searrow 0} \frac{\Delta f(\overline{x};\tau w)}{\frac{1}{2}\tau^2} + w^\top Hw \geq \gamma\|w\|^2$. But, for $0 < \tau < \epsilon$ and $w = \overline{d}$, (4.21) gives the contradiction $0 = \lim_{\tau\searrow 0}\left\{\frac{\Delta f(\overline{x};\tau\overline{d}) + \frac{\tau^2}{2}\overline{d}^\top H\overline{d}}{\frac{1}{2}\tau^2}\right\} = d^2 f(\overline{x}|0)(\overline{d}) \geq \gamma\left\|\overline{d}\right\|^2 = \gamma > 0.$ □

4.3.1 Application: superlinear convergence of quasi-Newton methods

Let f and $g + G$ be given by Definition 2.2.4 and consider the corresponding quasi-Newton method (1.11) initialized at (x^0, y^0). In this section, we assume the $\mathbf{B}_k$ defined in (1.11) take the form

$$\mathbf{B}_k = \begin{pmatrix} B_k & \nabla c(x^k)^\top \\ -\nabla c(x^k) & 0 \end{pmatrix}. \tag{4.22}$$

This choice allows us to relate the optimality conditions for the subproblems $\mathbf{Q}_k$ defined in Lemma 4.3.2 for solving **P** to the quasi-Newton method of (1.11). As in Section 2.2, the following is immediate:

Lemma 4.3.2. Let f be convex-composite, and let $(x^k, y^k) \in \mathbb{R}^n \times \mathbb{R}^m$ be such that f satisfies (BCQ) at x^k, let $B_k \in \mathbb{R}^{n\times n}$. Then, $(d^k, y^{k+1}) \in \mathbb{R}^n \times \mathbb{R}^m$ satisfy the optimality conditions for

$$(\mathbf{Q}_k) \qquad \underset{d \in \mathbb{R}^n}{\text{minimize}} \quad h(c(x^k) + \nabla c(x^k)d) + \frac{1}{2}d^\top B_k d$$

if and only if (x^{k+1}, y^{k+1}) satisfy the quasi-Newton update for $g+G$ given by Definition 2.2.4, with the choice (4.22). Namely, $0 \in g(x^k, y^k) + \mathbf{B}_k \begin{pmatrix} x^{k+1} - x^k \\ y^{k+1} - y^k \end{pmatrix} + G(x^{k+1}, y^{k+1})$, where $x^{k+1} := x^k + d^k$.

As a consequence of strong metric subregularity of the linearization $\mathcal{G}$ given by (2.10), we have the following convergence result:

Theorem 12. [15, Dennis-Moré Theorem for Generalized Equations] Let $(\overline{x}, \overline{y})$ be a solution of $g+G \ni 0$ given by Definition 2.2.4 and let U be a neighborhood of $(\overline{x}, \overline{y})$. For some starting point $(x^0, y^0) \in U$ consider a sequence $\{(x^k, y^k)\}_{k\in\mathbb{N}}$ generated by (1.11) which remains in U for all $k \in \mathbb{N}$ and satisfies $(x^k, y^k) \neq (\overline{x}, \overline{y})$ for all $k \in \mathbb{N}$. Define $\mathbf{E}_k := \mathbf{B}_k - \nabla g(\overline{x}, \overline{y})$ and $s^k := (x^{k+1} - x^k, y^{k+1} - y^k)$. If the linearization mapping $\mathcal{G}$ given by (2.10) is strongly metrically subregular at $(\overline{x}, \overline{y})$ for 0 and the sequence $\{(x^k, y^k)\}_{k\in\mathbb{N}}$ satisfies $(x^k, y^k) \to (\overline{x}, \overline{y})$ and $\mathbf{E}_k s^k = o(||s^k||)$ then $(x^k, y^k) \to (\overline{x}, \overline{y})$ superlinearly.

Remark 9. Suppose the function g is $\mathcal{C}^1$-smooth and $(x^k, y^k) \to (\overline{x}, \overline{y})$. Then, $\mathbf{E}_k s^k = o(||s^k||) \iff [\mathbf{B}_k - \nabla g(x^k, y^k)]s^k = o(||s^k||)$.

The following corollary is of algorithmic significance.

Corollary 4.3.1. Let f be as in **P**. Suppose $M(\overline{x}) = \{\overline{y}\}$ and the second-order sufficient conditions of Theorem 10 are satisfied at $\overline{x}$. Then, $(\overline{x}, \overline{y})$ solves $0 \in g(\overline{x}, \overline{y}) + G(\overline{x}, \overline{y})$.

Moreover, there exists a neighborhood U of $(\overline{x}, \overline{y})$ such that if $(x^0, y^0) \in U$, the sequence $\{(x^k, y^k)\}_{k\in\mathbb{N}}$ generated from the optimality conditions for $\mathbf{Q}_k$ remains in U with $(x^k, y^k) \neq (\overline{x}, \overline{y})$ for all $k \in \mathbb{N}$, and

$$(x^k, y^k) \to (\overline{x}, \overline{y}) \text{ and } (B_k - \nabla^2(y^k c)(x^k))[x^{k+1} - x^k] = o(||s^k||),$$

then $(x^k, y^k) \to (\overline{x}, \overline{y})$ superlinearly.

Remark 10. Consequently, the sufficient conditions for superlinear convergence of quasi-Newton methods require us to choose B_k as an approximation to the Hessian of the Lagrangian $\nabla^2_{xx} L(x^k, y^k) = \nabla^2(y^k c)(x^k)$ in the update direction $x^{k+1} - x^k$ at every iteration.

4.4 Partial Smoothness

The notion of partial smoothness, introduced by Lewis [26], generalizes classical notions of nondegeneracy, strict complementarity, and active constraint identification by illuminating the appropriate underlying manifold geometry of optimization problems. This allows for a more thorough understanding of the convergence behavior of algorithms applied to nonsmooth optimization problems, where solutions lie on well-defined submanifolds of the parameter space on which the function behaves smoothly and off of which it behaves nonsmoothly. Partial smoothness in the context of **P** allows us in Section 4.5 to establish metric regularity properties of the solution mapping.

Definition 4.4.1. Define a set $\mathcal{M} \subset \mathbb{R}^m$ to be a *manifold* of codimension ℓ around $\overline{c} \in \mathbb{R}^m$ if $\overline{c} \in \mathcal{M}$, and there exists an open set $V \subset \mathbb{R}^m$ containing $\overline{c}$ and a $\mathcal{C}^2$-smooth function $F : V \to \mathbb{R}^\ell$ with surjective derivative throughout V such that $\mathcal{M} \cap V = \{c \in V : F(c) = 0\}$. In which case (see [26]), the *tangent space* to $\mathcal{M}$ at $\overline{c}$ is $T\left(\overline{c} \,|\, \mathcal{M}\right) = \mathrm{Null}\left(\nabla F(\overline{c})\right)$, the *normal space* to $\mathcal{M}$ at $\overline{c}$ is $N\left(\overline{c} \,|\, \mathcal{M}\right) = \mathrm{Ran}\left(\nabla F(\overline{c})^\top\right)$, both independent of the choice of F. Moreover, the set $\mathcal{M}$ is Clarke regular at $\overline{c}$, and $N\left(\overline{c} \,|\, \mathcal{M}\right)$ equals the normal cone defined in (1.9).

Definition 4.4.2 (Partial smoothness for closed, convex functions)**.** Suppose $h : \mathbb{R}^m \to \overline{\mathbb{R}}$ is a closed, proper, convex function and that $\overline{c} \in \mathcal{M} \subset \mathbb{R}^m$. The function h is *partly smooth* at $\overline{c}$ relative to $\mathcal{M}$ if $\mathcal{M}$ is a manifold around $\overline{c}$ and the following four properties hold:

(a) (restricted smoothness) the restriction $h|_{\mathcal{M}}$ is smooth around $\overline{c}$, in that there exists a neighborhood V of $\overline{c}$ and a $\mathcal{C}^2$-smooth function g defined on V such that $h = g$ on $V \cap \mathcal{M}$;

(b) (existence of subgradients) at every point $c \in \mathcal{M}$ close to $\overline{c}$, $\partial h(c) \neq \emptyset$;

(c) (normals and subgradients parallel) $\mathrm{par}\left(\partial h(\overline{c})\right) = N\left(\overline{c}\,|\,\mathcal{M}\right)$;

(d) (subgradient inner semicontinuity) the subdifferential map ∂h is inner semicontinuous at $\overline{c}$ relative to $\mathcal{M}$.

We say that h is *partly smooth relative to* $\mathcal{M}$ if $\mathcal{M}$ is a manifold and h is partly smooth at each point in $\mathcal{M}$ relative to $\mathcal{M}$.

Remark 11. By [26, Proposition 2.4], requiring (a) - (d) in the definition is equivalent to requiring (a), (b), (d), and *normal sharpness*:

$$h'(\overline{c}; -w) > -h'(\overline{c}; w), \quad \forall\, w \in N\left(\overline{c}\,|\,\mathcal{M}\right) \setminus \{0\}\,, \tag{4.23}$$

and is also equivalent to requiring (a), (b), (d), and *lineality and tangent equality*:

$$\left\{ w \in \mathbb{R}^m \,\middle|\, -h'(\overline{c}; w) = h'(\overline{c}; -w) \right\} =: \mathrm{lin}\; h'(\overline{c}; \cdot) = T\left(\overline{c}\,|\,\mathcal{M}\right). \tag{4.24}$$

In the context of the PLQ functions given in Definition 4.2.1, a natural choice for the active manifold at a point $\overline{c} \in \mathrm{dom}\,(h)$ for **P** is the set given by

$$\mathcal{M}_{\overline{c}} := \mathrm{ri}\left(\bigcap_{k \in \mathcal{K}(\overline{c})} C_k \right), \tag{4.25}$$

where $\mathcal{K}(\overline{c})$ are the active indices at $\overline{c}$ (see Definition 4.2.2). The analysis of the manifold $\mathcal{M}_{\overline{c}}$ requires a more thorough understanding of the structure of $\mathrm{dom}\,(h)$, which we obtain from the following key result due to Rockafellar and Wets.

Lemma 4.4.1. [42, Lemma 2.50] Suppose C is a convex set which is the union of a finite collection of polyhedral sets C_k. If the polyhedral sets $\{C_k\}_{k=1}^{\mathcal{K}}$ are represented in terms of a single family of non-constant affine functions $l_i(x) = \langle a_i,\, x\rangle - \alpha_i$ indexed by $i = 1, \ldots, s$, then for each k there is a subset I_k of $\{1, \ldots, s\}$ such that $C_k = \left\{x \;\middle|\; l_i(x) \leq 0 \text{ for all } i \in I_k\right\}$. Let I denote the set of indices $i \in \{1, \ldots, s\}$ such that $l_i \leq 0$ for all $x \in C$. Then, $C = \left\{x \;\middle|\; l_i(x) \leq 0 \text{ for all } i \in I\right\}$. If $\operatorname{int} C \neq \emptyset$, then C can be written as the union of a finite collection of polyhedral sets $\left\{D_j\right\}_{j\in J}$ such that

(a) each set D_j is included in one of the sets C_k,

(b) $\operatorname{int} D_j \neq \emptyset$, so $D_j = \operatorname{cl}\operatorname{int} D_j$,

(c) $\operatorname{int} D_{j_1} \cap \operatorname{int} D_{j_2} = \emptyset$ when $j_1 \neq j_2$.

This result implies that the domain of h has a finite stratification [17, Definition 3.1] for which h is a stratifiable function [17, Definition 3.2]. This stratification is central to our discussion of partial smoothness and is referred to as the Rockafellar-Wets PLQ Representation.

Theorem 13 (Rockafellar-Wets PLQ Representation)**.** Suppose h is piecewise linear-quadratic convex and $\operatorname{int}\operatorname{dom}(h) \neq \emptyset$. Then, without loss of generality, we may assume the polyhedral sets $\{C_k\}_{k=1}^{\mathcal{K}}$ defining h are given in terms of a common set of $s > 0$ hyperplanes $\mathcal{H} := \left\{(a_j, \alpha_j)\right\}_{j=1}^{s} \subset (\mathbb{R}^m \setminus \{0\}) \times \mathbb{R}$, so that for all $k \in \{1, \ldots, \mathcal{K}\}$,

$$C_k = \left\{c \;\middle|\; \left\langle \omega_{kj} a_j,\, c\right\rangle \leq \omega_{kj}\alpha_j, \text{ for all } j \in \{1, \ldots, s\}\right\},$$

with $\omega_{kj} \in \{\pm 1\}$,

$$I_k(c) = \left\{j \;\middle|\; \left\langle \omega_{kj} a_j,\, c\right\rangle = \omega_{kj}\alpha_j\right\} = \left\{j \;\middle|\; \left\langle a_j,\, c\right\rangle = \alpha_j\right\} \subset \{1, \ldots, s\}, \tag{4.26}$$

and

(a) $\emptyset \neq \operatorname{int} C_k = \left\{c \;\middle|\; \left\langle \omega_{kj} a_j,\, c\right\rangle < \omega_{kj}\alpha_j, \text{ for all } j \in \{1, \ldots, s_k\}\right\}$, for all $k \in \{1, \ldots, \mathcal{K}\}$,

(b) $\operatorname{int} C_{k_1} \cap \operatorname{int} C_{k_2} = \emptyset$ when $k_1 \neq k_2$.

Condition (b) implies that if $c \in C_{k_1} \cap C_{k_2}$, then $c \in \operatorname{bdry} C_{k_1} \cap \operatorname{bdry} C_{k_2}$ when $k_1 \neq k_2$.

Proof. The proof of the previous lemma shows that for every polyhedron D_j and every $i \in \{1, \ldots, s\}$, either $l_i(x) \leq 0$ for all $x \in D_j$ or $l_i(x) \geq 0$ for all $x \in D_j$. Therefore each affine function is used in the definition of D_j, and D_j is contained entirely within one of the sets C_k, relative to which h takes the form $\frac{1}{2}\langle c, Q_k c\rangle + \langle b_k, c\rangle + \beta_k$. □

The basic assumptions employed for the remainder of this section are listed below.

Assumption 1.

(a) The function h is PLQ convex with $\operatorname{dom}(h)$ given by the Rockafellar-Wets PLQ representation described in Theorem 13,

(b) $\bar{c} \in \operatorname{dom}(h)$ satisfies $\bar{k} := |\mathcal{K}(\bar{c})| \geq 2$,

Remark 12. Whenever $\mathcal{K}(\bar{c}) = \{k_0\}$, h is continuously differentiable on $\operatorname{int} C_{k_0}$. Therefore, we assume that $\bar{k} \geq 2$ and delay the discussion of $\bar{k} = 1$ to Section 4.5.2

The following lemma further supports the choice for the manifold $\mathcal{M}_{\bar{c}}$.

Lemma 4.4.2. Let $\mathcal{M}_{\bar{c}}$ be as in (4.25) and let Assumption 1 hold. Then, for any $c \in \mathcal{M}_{\bar{c}}$, $\mathcal{K}(c) = \mathcal{K}(\bar{c})$, and so $\mathcal{M}_c = \mathcal{M}_{\bar{c}}$. Moreover, for any $k \in \mathcal{K}(\bar{c})$, the active index sets $I_k(c)$ satisfy $I_k(c) = I_k(\bar{c})$

Proof. Suppose $\mathcal{K}(c) \neq \mathcal{K}(\bar{c})$. Since the definition of $\mathcal{M}_{\bar{c}}$ implies $\mathcal{K}(\bar{c}) \subset \mathcal{K}(c)$, there exists $j \in \mathcal{K}(c') \setminus \mathcal{K}(\bar{c})$. By (b) in Theorem 13, we necessarily have $c \in \operatorname{bdry} C_j$.
We first argue the existence of $\epsilon > 0$ such that that $(\bar{c} + \epsilon\mathbb{B}) \cap C_k = \emptyset$ for all $k \notin \mathcal{K}(\bar{c})$. If no such ϵ exists, since there are only finitely many $k \in K \setminus \mathcal{K}(\bar{c})$, there would exist an index $k_0 \notin \mathcal{K}(\bar{c})$ and an infinite sequence $c^n \to \bar{c}$ with $\{c^n\} \subset C_{k_0}$. By closedness of the set C_{k_0}, $\bar{c} \in C_{k_0}$, which is a contradiction.

Since $c, \overline{c} \in \mathcal{M}_{\overline{c}}$, by [40, Theorem 6.4] there exists a $\mu > 1$ such that $\tilde{c} := (1-\mu)\overline{c} + \mu c \in \bigcap_{k \in \mathcal{K}(\overline{c})} C_k$.

Since $c \in \operatorname{bdry} C_j$, there exists a $z \in \operatorname{int} C_j$ sufficiently close to c so that the ray $\mathcal{R} := \left\{ \tilde{c} + \lambda(z - \tilde{c}) \mid 0 \le \lambda \right\}$ meets $\overline{c} + \epsilon\mathbb{B}$. We consider two cases. To set the stage, for any two points $x, y \in \mathbb{R}^m$, denote the line segment connecting them by $[x, y] = \left\{ (1-\lambda)x + \lambda y \mid 0 \le \lambda \le 1 \right\}$.

Case 1. There is a point $x \in \mathcal{R} \cap (\overline{c} + \epsilon\mathbb{B}) \cap C$. Then $z \in [\tilde{c}, x] \subset C_k$ for some $k \in \mathcal{K}(\overline{c})$. But then $z \in (\operatorname{int} C_j) \cap C_k$, a contradiction.

Case 2. We have $\mathcal{R} \cap (\overline{c} + \epsilon\mathbb{B}) \cap C = \emptyset$. Then there is a point $x \in (\overline{c} + \epsilon\mathbb{B}) \setminus C$ such that $z \in [\tilde{c}, x]$. Since $x \notin C$, there is a first point, which we denote by $\widehat{z}$, in C_j on this line segment as one moves from x to $\tilde{c}$. Then the line segment $[\widehat{z}, \overline{c}] \subset C$. The point $\widehat{z}$ is not on the line segment $[\tilde{c}, \overline{c}]$ since then both c' and z would be on the line segment $[\tilde{c}, \overline{c}]$ and so $\operatorname{int} C_j \cap \operatorname{bdry} C_k \neq \emptyset$ for some $k \in \mathcal{K}(\overline{c})$, a contradiction. Consequently, the points $\tilde{c}, \overline{c}$ and $\widehat{z}$ are not all collinear and hence form a triangle inside of C. Let $\tilde{z}$ be on the boundary of $\overline{c} + \epsilon\mathbb{B}$ and on the line segment $[\widehat{z}, \overline{c}]$. Then the line segment $[\tilde{z}, \tilde{c}]$ passes through $\operatorname{int} C_j$. This is again a contradiction.

Therefore, no such c exists, and $\mathcal{K}(c) = \mathcal{K}(\overline{c})$ for all $c \in \mathcal{M}_{\overline{c}}$.

For the second claim, suppose there exists $k \in \mathcal{K}(\overline{c}),\ c \in \mathcal{M}_{\overline{c}}$ and $j \in \{1, \dots, s\}$ with

$$\left\langle c,\, \omega_{kj} a_j \right\rangle < \omega_{kj}\alpha_j \text{ and } \left\langle \overline{c},\, \omega_{kj} a_j \right\rangle = \omega_{kj}\alpha_j. \tag{4.27}$$

Again by [40, Theorem 6.4], we may choose $\mu > 1$ so that $\mu\overline{c} + (1-\mu)c \in \mathcal{M}_{\overline{c}}$. In particular, $\mu\overline{c} + (1-\mu)c \in C_k$. But writing $\mu = 1 + \epsilon$ with $\epsilon > 0$ gives the contradiction

$$\begin{aligned} \omega_{kj}\alpha_j &\ge \left\langle \mu\overline{c} + (1-\mu)c,\, \omega_{kj} a_j \right\rangle \\ &= (1+\epsilon)\left\langle \overline{c},\, \omega_{kj} a_j \right\rangle - \epsilon \left\langle c,\, \omega_{kj} a_j \right\rangle > \omega_{kj}\alpha_j \text{ by (4.27)}. \end{aligned}$$

Therefore $I_k(\overline{c}) \subset I_k(c)$. Reversing the roles of c and $\overline{c}$ in (4.27) gives the other inclusion. □

The previous lemma tells us distinct points $c, c' \in \mathcal{M}_{\overline{c}}$ have the same active indices $\mathcal{K}(c)$ and $\mathcal{K}(c')$. Moreover, for any active polyhedron C_k, the active hyperplanes for that polyhedron, $I_k(c)$ and $I_k(c')$, at c and c' are the same. This observation offers a global description of $\mathcal{M}_{\overline{c}}$ in terms of the active hyperplanes at $\overline{c}$ alone.

Lemma 4.4.3. Let $\mathcal{M}_{\overline{c}}$ be as in (4.25), and let Assumption 1 hold. Then,

$$\mathcal{M}_{\overline{c}} = \left\{ c \;\middle|\; \begin{array}{c} \left\langle c,\, a_j \right\rangle = \alpha_j \text{ for all } k \in \mathcal{K}(\overline{c}), j \in I_k(\overline{c}) \\ \left\langle c,\, \omega_{kj} a_j \right\rangle < \omega_{kj}\alpha_j \text{ for all } k \in \mathcal{K}(\overline{c}), j \notin I_k(\overline{c}) \end{array} \right\}.$$

In particular, $I_{k_1}(c) = I_{k_2}(c)$ for all $c \in \mathcal{M}_{\overline{c}}$ and $k_1, k_2 \in \mathcal{K}(\overline{c})$. Moreover, for any $k \in \mathcal{K}(\overline{c})$ and $c \in \mathcal{M}_{\overline{c}}$, $T\left(c \,|\, \mathcal{M}_{\overline{c}}\right) = \operatorname{Null}\left(A_k(\overline{c})^\top\right)$, and $N\left(c \,|\, \mathcal{M}_{\overline{c}}\right) = \operatorname{Ran}\left(A_k(\overline{c})\right)$, where $A_k(\overline{c})$ is the matrix whose columns are the gradients of the active constraints at $\overline{c} \in C_{\overline{k}}$ in some ordering.

Remark 13. By Lemma 4.4.2 and Lemma 4.4.3, for all $c \in \mathcal{M}_{\overline{c}}$, $k \in \mathcal{K}(\overline{c})$, and $j \in \mathcal{K}(c)$, $\operatorname{Ran}\left(A_k(\overline{c})\right) = \operatorname{Ran}\left(A_j(c)\right)$. This observation becomes important in a structural definition to follow.

Proof. Define

$$\mathcal{C}_1 := \bigcap_{k \in \mathcal{K}(\overline{c})} C_k, \quad \mathcal{C}_2 := \left\{ c \;\middle|\; \begin{array}{c} \left\langle c,\, \omega_{kj} a_j \right\rangle = \omega_{kj}\alpha_j \text{ for all } k \in \mathcal{K}(\overline{c}), j \in I_k(\overline{c}) \\ \left\langle c,\, \omega_{kj} a_j \right\rangle \le \omega_{kj}\alpha_j \text{ for all } k \in \mathcal{K}(\overline{c}), j \notin I_k(\overline{c}) \end{array} \right\}.$$

We aim to show $\operatorname{ri}(\mathcal{C}_1) \supset \operatorname{ri}(\mathcal{C}_2)$. For $k \in \mathcal{K}(\overline{c})$ and $j \in I_k(\overline{c})$ define $\mathcal{C}_{k,j} := \left\{ c \;\middle|\; \left\langle c,\, \omega_{kj} a_j \right\rangle = \omega_{kj}\alpha_j \right\}$, and for $k \in \mathcal{K}(\overline{c})$ and $j \notin I_k(\overline{c})$, let $\mathcal{D}_{k,j} := \left\{ c \;\middle|\; \left\langle c,\, \omega_{kj} a_j \right\rangle \le \omega_{kj}\alpha_j \right\}$. Then by defintion of $I_k(\overline{c})$,

$$\overline{c} \in \bigcap_{\substack{k \in \mathcal{K}(\overline{c}) \\ j \in I_k(\overline{c})}} \operatorname{ri}\left(\mathcal{C}_{kj}\right) \cap \bigcap_{\substack{k \in \mathcal{K}(\overline{c}) \\ j \notin I_k(\overline{c})}} \operatorname{ri}\left(\mathcal{D}_{kj}\right),$$

so [40, Theorem 6.5] gives

$$\operatorname{ri}(\mathcal{C}_2) = \left\{ c \;\middle|\; \begin{array}{c} \left\langle c,\, \omega_{kj} a_j \right\rangle = \omega_{kj}\alpha_j \text{ for all } k \in \mathcal{K}(\overline{c}), j \in I_k(\overline{c}) \\ \left\langle c,\, \omega_{kj} a_j \right\rangle < \omega_{kj}\alpha_j \text{ for all } k \in \mathcal{K}(\overline{c}), j \notin I_k(\overline{c}) \end{array} \right\}.$$

Moreover, $\mathcal{C}_1 \supset \mathcal{C}_2$ with $\mathcal{C}_2$ not entirely contained within the relative boundary of $\mathcal{C}_1$ because $\overline{c} \in \mathcal{C}_2 \cap \mathcal{M}_{\overline{c}}$. By [40, Corollary 6.5.2], $\mathcal{M}_{\overline{c}} := \mathrm{ri}\,(\mathcal{C}_1) \supset \mathrm{ri}\,(\mathcal{C}_2)$. Lemma 4.4.2 shows $\mathcal{M}_{\overline{c}} := \mathrm{ri}\,(\mathcal{C}_1) \subset \mathrm{ri}\,(\mathcal{C}_2)$ because $I_k(c) = I_k(\overline{c})$ throughout $\mathcal{M}_{\overline{c}}$.
For the second claim, the structure of $\mathcal{M}_{\overline{c}}$ implies that if $\left\langle c, \omega_{k_1 j} a_j \right\rangle = \omega_{k_1 j}\alpha_j$ for some $k_1 \in \mathcal{K}(\overline{c})$, then $\left\langle c, \omega_{k_2 j} a_j \right\rangle = \omega_{k_2 j}\alpha_j$ for any other $k_2 \in \mathcal{K}(\overline{c})$ as $\omega_{kj} \in \{\pm 1\}$. Hence $I_{k_2}(c) \supset I_{k_1}(c)$, and this argument is symmetric in k_1 and k_2.

The tangent and normal cone formulas hold throughout $\mathcal{M}_{\overline{c}}$ by Theorem 8. □

Based on Lemma 4.4.3 and Remark 13, we now establish the notational tools required for our analysis.

Definition 4.4.3. Let $\mathcal{M}_{\overline{c}}$ be as in (4.25), and let Assumption 1 hold. Define $A_{\overline{k}}(c)$ to be the matrix whose columns are the gradients of the active constraints at $c \in C_{\overline{k}}$ in some ordering. By Theorem 13 and Lemma 4.4.3, without loss of generality, we can define $A := A_{\overline{k}}(c)$ independent of the choice of $c \in \mathcal{M}_{\overline{c}}$, and for any $j \in \left\{1, \ldots, \overline{k}\right\}$, there exists a diagonal matrix P_j with entries ± 1 on the diagonal such that

$$AP_j = A_{k_j}(c) \text{ independent of } c \in \mathcal{M}_{\overline{c}}. \tag{4.28}$$

We let ℓ be the common number of columns $\ell := |I_k(\overline{c})| = |I_{k'}(\overline{c})|$ for all $k, k' \in \mathcal{K}(\overline{c})$, so that $A \in \mathbb{R}^{m\times\ell}, P_j \in \mathbb{R}^{\ell\times\ell}, P_{\overline{k}} = I_\ell$, and define the following block matrices $\widehat{\mathcal{Q}} := \mathrm{diag}(Q_k), \widehat{\mathcal{A}} := \mathrm{diag} AP_j$

(4.29)

$$\mathcal{A} := \begin{pmatrix} (1-\overline{k})AP_1 & AP_2 & \cdots & A \\ AP_1 & (1-\overline{k})AP_2 & \cdots & A \\ \vdots & \ddots & \ddots & \vdots \\ AP_1 & AP_2 & \cdots & (1-\overline{k})A \end{pmatrix}, \quad \mathcal{Q} := \begin{bmatrix} Q_{k_1} \\ Q_{k_2} \\ \vdots \\ Q_{k_{\overline{k}}} \end{bmatrix}, \quad \mathcal{B} := \begin{bmatrix} b_{k_1} \\ b_{k_2} \\ \vdots \\ b_{k_{\overline{k}}} \end{bmatrix}, \quad J := \begin{bmatrix} I_m \\ I_m \\ \vdots \\ I_m \end{bmatrix}$$

and averaged quantities

$$\overline{Q} = (1/\overline{k})J^\top \widehat{\mathcal{Q}} J, \quad \overline{A} = (1/\overline{k})J^\top \widehat{\mathcal{A}}, \quad \overline{b} = (1/\overline{k})J^\top \mathcal{B}, \quad \lambda_0(\overline{c}) = \overline{Q}\overline{c} + \overline{b}.$$

In a fashion similar to the *structure functional* approach of [31,32,46], we give a formula for the subdifferential in terms of the active manifold structure previously laid out.

Lemma 4.4.4. Let $\mathcal{M}_{\bar{c}}$ be as in (4.25), let Assumption 1 hold, and recall the notation of Definition 4.4.3. For any $c \in \mathcal{M}_{\bar{c}}$, $\partial h(c)$ can be given by two equivalent formulations:

$$\partial h(c) = \left\{ y \;\middle|\; \begin{array}{l} \exists\, \mu = (\mu_1^\top, \ldots, \mu_{\bar{k}}^\top)^\top \geq 0 \\ \text{such that } Jy = \mathcal{Q}c + \mathcal{B} + \widehat{\mathcal{A}}\mu \end{array} \right\} = \lambda_0(c) + \overline{A}\mathcal{U}(c), \tag{4.30}$$

where

$$\mathcal{U}(c) := \left\{ \mu \geq 0 \;\middle|\; \mathcal{A}\mu = \bar{k}\left[\mathcal{Q}c + \mathcal{B} - J(\overline{Q}c + \bar{b})\right] \right\}. \tag{4.31}$$

Proof. By (4.5) and Lemma 4.4.2, $y \in \partial h(c)$ if and only if $y \in Q_{k_j}c + b_{k_j} + N\left(c \,|\, C_{k_j}\right)$ for all $j \in \left\{1, \ldots, \bar{k}\right\}$. In terms of the active indices at c for the polyhedron C_{k_j}, (4.3) and (4.28) imply

$$y = Q_{k_j}c + b_{k_j} + AP_j\mu_j, \text{ where } j \in \left\{1, \ldots, \bar{k}\right\}, \mu_j \geq 0.$$

Hence $y \in \partial h(c)$ if and only if there exists $\mu = (\mu_1^\top, \ldots, \mu_{\bar{k}}^\top)$ such that (y, μ) satisfies the system

$$Jy = \mathcal{Q}c + \mathcal{B} + \widehat{\mathcal{A}}\mu, \quad \mu = (\mu_1^\top, \ldots, \mu_{\bar{k}}^\top)^\top \geq 0.$$

Since $J^\top J = \bar{k}I_m$, multiplying both sides of the first equation in (4.30) by $(1/\bar{k})J^\top$ gives $y = \overline{Q}c + \bar{b} + \overline{A}\mu$, where μ satisfies

$$\overline{Q}c + \bar{b} + \overline{A}\mu = AP_j\mu_j + Q_{k_j}c + b_{k_j}, \text{for all } j \in \left\{1, \ldots, \bar{k}\right\}, \mu \geq 0.$$

The set of μ that satisfy the display defines membership in $\mathcal{U}(c)$, so $\partial h(c) = \lambda_0(c) + \overline{A}\mathcal{U}(c)$. □

The notion of nondegeneracy that we use imposes linear independence of the columns of A.

Definition 4.4.4 (Nondegeneracy)**.** Let $\mathcal{M}_{\bar{c}}$ be as in (4.25), let Assumption 1 hold, and recall the notation of Definition 4.4.3. We say that $\mathcal{M}_{\bar{c}}$ satisfies the *nondegeneracy condition* if $\mathrm{Null}\,(A) = \{0\}$.

Nondegenerency yields a uniqueness property of the multipliers $\mu \in \mathcal{U}(c)$.

Lemma 4.4.5. Let $\mathcal{M}_{\overline{c}}$ be as in (4.25), let Assumption 1 hold, and recall the notation of Definition 4.4.3. Suppose $\mathcal{M}_{\overline{c}}$ satisfies the nondegeneracy condition of Definition 4.4.4, $c \in \mathcal{M}_{\overline{c}}$, and $y \in \partial h(c)$. Then, there is a unique $\mu \in \mathcal{U}(c)$, given by $\mu(c,y)_j = P_j(A^\top A)^{-1}A^\top(y - (Q_{k_j}c + b_{k_j})), j \in \left\{1, \ldots, \overline{k}\right\}$ so that $y = \lambda_0(c) + \overline{A}\mu(c,y)$.

Proof. For any $j \in \left\{1, \ldots, \overline{k}\right\}$, Lemma 4.4.4 implies there exists $\mu_j \geq 0$ such that $y = Q_{k_j}c + b_{k_j} + AP_j\mu_j$. Nondegeneracy implies μ_j is given uniquely by the equation $\mu(c,y)_j = P_j(A^\top A)^{-1}A^\top(y - (Q_{k_j}c + b_{k_j}))$. □

A corresponding notion of strict complementarity is provided by the next lemma.

Lemma 4.4.6. Let $\mathcal{M}_{\overline{c}}$ be as in (4.25), let Assumption 1 hold, and recall the notation of Definition 4.4.3. Suppose $c \in \mathcal{M}_{\overline{c}}$ and $\operatorname{ri}\left(\partial h(c)\right) \neq \emptyset$. Then $y \in \operatorname{ri}\left(\partial h(c)\right)$ if and only if $\mu(c,y)_i > 0$ for all $i \in \left\{1, \ldots, \overline{k}\right\}$.

Proof. By [40, Theorem 6.4], $y \in \operatorname{ri}\left(\partial h(c)\right)$ if and only if for all $y' \in \partial h(c)$, there exists $t > 1$ so that $ty + (1-t)y' \in \partial h(c)$. Choose a $y' \in \partial h(c)$ with $y' \neq y$.
($\Rightarrow$) If there exists $i_0 \in \left\{1, \ldots, \overline{k}\right\}$ and $j \in \{1, \ldots, \ell\}$, with $(\mu(c,y)_{i_0})_j = 0$, then, by (4.30),

$$\partial h(c) \ni ty + (1-t)y' = Q_{i_0}c + b_{i_0} + AP_{i_0}[t\mu(c,y)_{i_0} + (1-t)\mu(c,y')_{i_0}].$$

By Lemma 4.4.5, $\mu(c, ty + (1-t)y')_{i_0} = t\mu(c,y)_{i_0} + (1-t)\mu(c,y')_{i_0}$. By assumption, the right-hand side has its jth component is negative for all $t > 1$, a contradiction.
($\Leftarrow$) We must show there exists $\epsilon > 0$ such that if $t := 1+\epsilon$ then $t\mu(c,y)_{i_0} + (1-t)\mu(c,y')_{i_0} > 0$. After rearranging, this is equivalent to finding $\epsilon > 0$ so that $\mu(c,y)_{i_0} + \epsilon[\mu(c,y)_{i_0} - \mu(c,y')_{i_0}] > 0$. If $\mu(c,y)_{i_0} - \mu(c,y')_{i_0} \geq 0$, the claim is immediate. Otherwise, we choose ϵ via

$$0 < \epsilon < \min\left\{\frac{(\mu(c,y)_{i_0})_j}{(\mu(c,y')_{i_0})_j - (\mu(c,y)_{i_0})_j} \;\middle|\; (\mu(c,y)_{i_0})_j - (\mu(c,y')_{i_0})_j < 0, j \in \{1, \ldots, \ell\}\right\}.$$

Then $y \in \operatorname{ri}\left(\partial h(c)\right)$. □

However, a weaker notion of strict complementarity in conjunction with nondegeneracy suffices to show that $\operatorname{ri}\big(\partial h(c)\big) \neq \emptyset$ throughout $\mathcal{M}_{\bar{c}}$.

Definition 4.4.5 (k-strict complementarity)**.** Let $\mathcal{M}_{\bar{c}}$ be as in (4.25), let Assumption 1 hold, and recall the notation of Definition 4.4.3. We say *k-strict complementarity* holds at (c, y) for $\mu = (\mu_1^\top, \ldots, \mu_{\bar{k}}^\top)^\top$ if

(a) $c \in \mathcal{M}_{\bar{c}},\ y \in \partial h(c)$,

(b) There exists $k \in \mathcal{K}(\bar{c})$ with $\mu_k > 0$,

(c) Whenever there exists $j \in \mathcal{K}(c) \setminus \{k\}$ and $i \in \{1, \ldots, \ell\}$ with $(\mu_j)_i = 0$, then the scalars $(P_{j'})_{ii} = 1$ for all $j' \in \mathcal{K}(c)$,

(d) (y, μ) satisfies (4.30).

Remark 14. When k-strict complementarity holds at a pair (c, y) and an index j satisfies (c), the active polyhedra $\{C_k\}_{k \in \mathcal{K}(\bar{c})}$ are all within the same closed half-space of the corresponding hyperplane. Also observe that $y \in \operatorname{ri}\big(\partial h(c)\big)$ implies k-strict complementarity at (c, y).

A requirement of partial smoothness is that the normal space to $\mathcal{M}_{\bar{c}}$ and $\operatorname{par}\big(\partial h(c)\big)$ are equal. The nondegeneracy condition allows us to describe $\operatorname{par}\big(\partial h(c)\big)$ using the vectors in $\mathcal{U}(c)$ rather than the subgradients in $\partial h(c)$.

Lemma 4.4.7. Let $\mathcal{M}_{\bar{c}}$ be as in (4.25), let Assumption 1 hold, and recall the notation of Definition 4.4.3. Suppose $\mathcal{M}_{\bar{c}}$ satisfies the nondegeneracy condition. Then, for any $c \in \mathcal{M}_{\bar{c}}$,

$$\operatorname{par}\big(\partial h(c)\big) = \operatorname{Ran}(A) \iff \operatorname{par}\big(\mathcal{U}(c)\big) = \operatorname{Null}(\mathcal{A}). \tag{4.32}$$

Proof. By Lemma 4.4.3, $N\left(c \,|\, \mathcal{M}_{\bar{c}}\right) = \operatorname{Ran}(A)$, and by Lemma 4.4.4, $\partial h(c) = \lambda_0(c) + \overline{A}\mathcal{U}(c)$. The system of linear equations (4.31) in $\mathcal{U}(c)$ has coefficient matrix $\mathcal{A}$ defined in (4.29) which is block-circulant and can be block row-reduced to

$$\begin{pmatrix} AP_1 & 0 & 0 & \cdots & -A \\ 0 & AP_2 & 0 & \cdots & -A \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & AP_{\overline{k}-1} & -A \\ 0 & \cdots & \cdots & 0 & 0 \end{pmatrix}. \tag{4.33}$$

We now compute Null$(\mathcal{A})$. Suppose $\mu = (\mu_1^\top, \ldots, \mu_{\overline{k}}^\top)^\top \in$ Null$(\mathcal{A})$. Then (4.33) and nondegeneracy imply that $\mu \in$ Null$(\mathcal{A})$ if and only if $\mu_j = P_j \mu_{\overline{k}}$ for all $j \in \left\{1, \ldots, \overline{k}-1\right\}$, i.e.,

(4.34)

$$\text{Null}\,(\mathcal{A}) = \left\{ \begin{pmatrix} P_1 \mu_{\overline{k}} \\ \vdots \\ P_{\overline{k}-1}\mu_{\overline{k}} \\ \mu_{\overline{k}} \end{pmatrix} \;\middle|\; \mu_{\overline{k}} \in \mathbb{R}^\ell \right\}, \text{with basis} \left\{ \begin{pmatrix} P_1 e_p \\ \vdots \\ P_{\overline{k}-1} e_p \\ e_p \end{pmatrix} \;\middle|\; p \in \{1, \ldots, \ell\} \right\} =: \{\zeta_1, \ldots, \zeta_\ell\}.$$

By (4.31),

$$\text{par}\,\big(\mathcal{U}(c)\big) := \mathbb{R}(\mathcal{U}(c) - \mathcal{U}(c)) \subset \text{Null}\,(\mathcal{A}), \tag{4.35}$$

and since $\overline{A} = \frac{1}{k}\begin{bmatrix} AP_1 & \cdots & AP_{\overline{k}-1} & A \end{bmatrix}$, (4.30) implies

$$\text{par}\,\big(\partial h(c)\big) = \text{par}\left(\overline{A}\mathcal{U}(c)\right) = \overline{A}\text{par}\,\big(\mathcal{U}(c)\big) \subset \overline{A}\text{Null}\,(\mathcal{A}) = \left\{ A\mu_k \;\middle|\; \mu_k \in \mathbb{R}^\ell \right\} = \text{Ran}\,(A),$$

so ($\Leftarrow$) in (4.32) is clear as "$\subset$" becomes an equation. For ($\Rightarrow$), suppose strict containment: par$\big(\mathcal{U}(c)\big) \subsetneq$ Null$(\mathcal{A})$. Then there exists $p \in \{1, \ldots, \ell\}$ such that $\zeta_p \notin$ par$\big(\mathcal{U}(c)\big)$. This implies that the pth column of A is not in par$\big(\partial h(c)\big)$ which we have assumed equal to Ran(A). This contradiction establishes (4.32). □

We now show that nondegeneracy and k-strict complementarity together imply that the normal space and subdifferential are parallel.

Lemma 4.4.8. Let $\mathcal{M}_{\overline{c}}$ be as in (4.25), let Assumption 1 hold, and recall the notation of Definition 4.4.3. Suppose $\mathcal{M}_{\overline{c}}$ satisfies the nondegeneracy condition, and the k-strict complementarity of Definition 4.4.5 holds at (c, y) for μ. Then,

$$\operatorname{par}\left(\partial h(c)\right) = N\left(c \,|\, \mathcal{M}_{\overline{c}}\right), \tag{4.36}$$

where it is shown in Lemma 4.4.3 that $N\left(c \,|\, \mathcal{M}_{\overline{c}}\right) = \operatorname{Ran}(A)$. Moreover, (4.36) holds throughout $\mathcal{M}_{\overline{c}}$, and ∂h is inner semicontinuous relative to $\mathcal{M}_{\overline{c}}$.

Proof. We first show that a sufficient condition to guarantee the right-hand side of (4.32) is (c, v) satisfying the k-strict complementarity condition of Definition 4.4.5 for $\mu \in \mathcal{U}(c)$. To see this note that, by relabeling the active polyhedral sets if necessary, we can assume without loss of generality that the index k in k-strict complementarity is $\overline{k}$. Let $p \in \{1, \ldots, \ell\}$, $t \in \mathbb{R}$, and consider the step given by $\mu + t\zeta_p$, where ζ_p is the pth basis element of $\operatorname{Null}(\mathcal{A})$ given in (4.34), i.e.,

$$\mu + t\zeta_p := \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_{\overline{k}-1} \\ \mu_{\overline{k}} \end{pmatrix} + t \begin{pmatrix} P_1 e_p \\ \vdots \\ P_{\overline{k}-1} e_p \\ e_p \end{pmatrix}, \tag{4.37}$$

We consider two cases. If, for all $j \in \left\{1, \ldots, \overline{k}\right\}$, $(\mu_j)_p > 0$, then for sufficiently small t, $\mu + t\zeta_p \geq 0$, and $\mathcal{A}(\mu + t\zeta_p) = \mathcal{A}\mu$. That is, both $\mu \in \mathcal{U}(c)$ and $\mu + t\zeta_p \in \mathcal{U}(c)$, which implies $\zeta_p \in \operatorname{par}\left(\mathcal{U}(c)\right)$. Otherwise, there exists $j \in \left\{1, \ldots, \overline{k}\right\}$ with $(\mu_j)_p = 0$. By part (c) of k-strict complementarity, the scalars $P_{j'} e_p = 1$ for all $j' \in \left\{1, \ldots, \overline{k}\right\}$, so repeating the previous argument with $t > 0$ gives $\zeta_p \in \operatorname{par}\left(\mathcal{U}(c)\right)$. Since $p \in \{1, \ldots, \ell\}$ was arbitrary, k-strict complementarity is a sufficient condition guaranteeing $\operatorname{par}\left(\mathcal{U}(c)\right) = \operatorname{Null}(\mathcal{A})$.
This argument shows, under nondegeneracy, that

$$k\text{-strict complementarity at } (c, y) \text{ for } \mu \Longrightarrow \operatorname{ri}\left(\partial h(c)\right) \neq \emptyset, \tag{4.38}$$

because, given any $\mu \in \mathcal{U}(c)$, the fact that $\operatorname{par}\left(\mathcal{U}(c)\right) = \operatorname{Null}(\mathcal{A})$ together with (4.30) implies there exists a strictly positive $\tilde{\mu} \in \mathcal{U}(c)$ and a $\tilde{y} \in \partial h(c)$ given by $\tilde{y} = \lambda_0(c) + \overline{A}\tilde{\mu}$, with $\mu(c,\tilde{y}) = \tilde{\mu}$. By Lemma 4.4.6, $\tilde{y} \in \operatorname{ri}\big(\partial h(c)\big)$.

We now argue that if, for some $c \in \mathcal{M}_{\overline{c}},\ y \in \partial h(c),\ k$-strict complementarity holds at (c,y) for μ, then $\operatorname{ri}\big(\partial h(c)\big) \neq \emptyset$ throughout $\mathcal{M}_{\overline{c}}$. This will imply (4.36) holds throughout $\mathcal{M}_{\overline{c}}$ as well. By (4.38), suppose $y \in \operatorname{ri}\big(\partial h(c)\big)$ so that $\mu(c,y) > 0$ by Lemma 4.4.6.

Choose any other $c' \in \mathcal{M}_{\overline{c}}$. Since $\mathcal{M}_{\overline{c}}$ is relatively open, there exists $c'' \in \mathcal{M}_{\overline{c}}$ and $\lambda \in (0,1)$ so that $c' = \lambda c + (1-\lambda)c''$. Let $y'' \in \partial h(c'')$. By Lemma 4.4.5, there exists a unique vector $\mu(c'',y'')$ associated with (c'',y''). Since $c,\ c'' \in \mathcal{M}_{\overline{c}}$ and $\mu(c,y) > 0$, $\lambda\mu(c',y') + (1-\lambda)\mu(c,y) > 0$. It follows from (4.30) that for all $j \in \left\{1,\ldots,\overline{k}\right\}$ and $\lambda \in (0,1)$,

$$\lambda y + (1-\lambda)y'' = Q_{k_j}c' + b_{k_j} + AP_j(\lambda\mu(c,y) + (1-\lambda)\mu(c'',y'')). \tag{4.39}$$

Define $y' := \lambda y + (1-\lambda)y''$. Then (4.39) implies that the equations (4.30) defining membership $y' \in \partial h(c')$ are satisfied, with $\mu(c',y') = \lambda\mu(c,y) + (1-\lambda)\mu(c'',y'') > 0$, so $y' \in \operatorname{ri}\big(\partial h(c')\big)$ by Lemma 4.4.6. Since $c' \in \mathcal{M}_{\overline{c}}$ was arbitrary, $\operatorname{ri}\big(\partial h(c)\big) \neq \emptyset$ for all $\mathcal{M}_{\overline{c}}$.

We lastly establish $\partial h(c)$ is inner semicontinuous relative to $\mathcal{M}_{\overline{c}}$. The previous paragraph and (4.39) showed $\partial h|_{\mathcal{M}_{\overline{c}}}$ is graph-convex. By defining $S(c) = \partial h(c)$ for $c \in \mathcal{M}_{\overline{c}}$ and $S(c) = \emptyset$ otherwise and noting the convex sets $\{c\}$ and $\mathcal{M}_{\overline{c}}$ cannot be separated, [42, Theorem 5.9(b)] gives inner semicontinuity of ∂h at all $c \in \mathcal{M}_{\overline{c}}$ relative to $\mathcal{M}_{\overline{c}}$. □

The main result of this section shows that partial smoothness follows from nondegeneracy and k-strict complementarity.

Theorem 14. Let $\mathcal{M}_{\overline{c}}$ be as in (4.25), let Assumption 1 hold, and recall the notation of Definition 4.4.3. Suppose $\mathcal{M}_{\overline{c}}$ satisfies the nondegeneracy condition, and $c \in \mathcal{M}_{\overline{c}}$ and $y \in \partial h(c)$ are such that (c,y) satisfies the k-strict complementarity condition of Definition 4.4.5. Then h is partly smooth relative to $\mathcal{M}_{\overline{c}}$.

Proof. By definition of $\mathcal{M}_{\overline{c}}$, for any $k \in \mathcal{K}(\overline{c})$ and any $c \in \mathcal{M}_{\overline{c}},\ h(c) = \frac{1}{2}\langle c,\, Q_k c\rangle + \langle b_k,\, c\rangle + \beta_k$, so $h|_{\mathcal{M}_{\overline{c}}}$ is smooth. By Proposition 4.2.1, $\operatorname{dom}(\partial h) = \operatorname{dom}(h) \supset \mathcal{M}_{\overline{c}}$, so existence of subgradients holds throughout $\mathcal{M}_{\overline{c}}$ as well. The normal cone and subdifferential being parallel along with subdifferential inner semicontinuity relative to $\mathcal{M}_{\overline{c}}$ are the content of Lemma 4.4.8. $\square$

Remark 15. Observe that if the hypotheses of Theorem 14 are satisfied, the assumption that f satisfies (TC) at $\overline{x}$ is equivalent to requiring

$$\operatorname{Null}\left(\nabla c(\overline{x})^{\top}\right) \cap \operatorname{Ran}(A) = \{0\}. \tag{4.40}$$

This condition and the nondegeneracy condition imply the $n \times \ell$ matrix $\nabla c(\overline{x})^{\top} A$ has full rank equal to $\ell \leq n$, i.e., $\operatorname{Null}\left(\nabla c(\overline{x})^{\top} A\right) = \{0\}$.

We now show the assumptions of Theorem 14 allow us to write the cone of non-ascent directions as a subspace at strictly critical points.

Lemma 4.4.9 (Non-ascent directions)**.** Let $\mathcal{M}_{\overline{c}}$ be as in (4.25), let Assumption 1 hold, and recall the notation of Definition 4.4.3. Suppose f satisfies (BCQ) at $\overline{x}$, $\overline{y} \in M(\overline{x})$, and $\overline{c} := c(\overline{x})$. Then, $D(\overline{x}) \supset \operatorname{Null}\left(A^{\top} \nabla c(\overline{x})\right)$. If, in addition, f satisfies (SC) at $\overline{x}$ for $\overline{y}$ and $\mathcal{M}_{\overline{c}}$ satisfies the nondegeneracy condition, then $D(\overline{x}) \subset \operatorname{Null}\left(A^{\top} \nabla c(\overline{x})\right)$.

Proof. Since f satisfies (BCQ) at $\overline{x}$, Theorem 4 gives $D(\overline{x}) = \left\{ d \in \mathbb{R}^n \;\middle|\; h'(c(\overline{x}); \nabla c(\overline{x}) d) \leq 0 \right\}$.
($\supset$) Since $\overline{y} \in M(\overline{x})$, by (4.30), there exists $\overline{\mu} \in \mathcal{U}(\overline{c})$ so that $J\overline{y} = \mathcal{Q}\overline{c} + \mathcal{B} + \widehat{\mathcal{A}}\overline{\mu}$. Then, for any $j \in \left\{1, \ldots, \overline{k}\right\}$,

$$\begin{aligned}
D(\overline{x}) &= \bigcup_{j=1}^{\overline{k}} \left\{ d \;\middle|\; \begin{array}{l} \left\langle Q_{k_j} \overline{c} + b_{k_j},\, \nabla c(\overline{x}) d \right\rangle \leq 0 \\ P_j A^{\top} \nabla c(\overline{x}) d \leq 0 \end{array} \right\} && \text{by (4.11), Definition 4.4.3} \\
&= \bigcup_{j=1}^{\overline{k}} \left\{ d \;\middle|\; \begin{array}{l} \left\langle \overline{y} - A P_j \overline{\mu}_j,\, \nabla c(\overline{x}) d \right\rangle \leq 0 \\ P_j A^{\top} \nabla c(\overline{x}) d \leq 0 \end{array} \right\} && \text{since } \overline{y} \in M(\overline{x}) \\
&= \bigcup_{j=1}^{\overline{k}} \left\{ d \;\middle|\; \begin{array}{l} \left\langle \overline{\mu}_j,\, P_j A^{\top} \nabla c(\overline{x}) d \right\rangle \geq 0 \\ P_j A^{\top} \nabla c(\overline{x}) d \leq 0 \end{array} \right\}.
\end{aligned}$$

The inclusion follows.

($\subset$) Let $0 \neq d \in D(\overline{x})$, and suppose to the contrary that $d = d_1 + d_2$, where $d_1 \in \mathrm{Null}\left(A^\top \nabla c(\overline{x})\right)$ and $d_2 = \nabla c(\overline{x})^\top A w$, $w \neq 0$. By Lemma 4.4.8, $\mathrm{Ran}\,(A) \subset \mathrm{par}\left(\partial h(\overline{c})\right)$. Since $\overline{y} \in \mathrm{ri}\left(\partial h(\overline{c})\right)$, there exists $\epsilon > 0$ so that $\overline{y} + \epsilon A w \in \partial h(\overline{c})$. Then,

$$\begin{aligned}
0 &\geq h'(c(\overline{x}); \nabla c(\overline{x}) d) \\
&= \sup_{y \in \partial h(\overline{c})} \left\langle \nabla c(\overline{x})^\top y,\, d \right\rangle \\
&\geq \left\langle \overline{y} + \epsilon A w,\, \nabla c(\overline{x})(d_1 + \nabla c(\overline{x})^\top A w) \right\rangle \\
&\geq \left\langle \nabla c(\overline{x})^\top \overline{y},\, d \right\rangle + \epsilon \left\| \nabla c(\overline{x})^\top A w \right\|^2 \\
&= \epsilon \left\| \nabla c(\overline{x})^\top A w \right\|^2,
\end{aligned}$$

so $w = 0$ (see Remark 15). □

By a continuity argument in (x, y), we have the following result which is important for our discussion of the metric regularity of Newton's iteration in the next section. It states that, in the presence of partial smoothness, (TC) and the curvature condition are local properties.

Lemma 4.4.10. Suppose (4.40) holds and that for all $j \in \mathcal{K}(\overline{c})$ and

$$d^\top \nabla c(\overline{x})^\top Q_j \nabla c(\overline{x}) d + d^\top \nabla^2 (\overline{y} c)(\overline{x}) d > 0, \quad \forall\, d \in \mathrm{Null}\left(A^\top \nabla c(\overline{x})\right) \setminus \{0\}\,.$$

Then, there exists a neighborhood $\mathcal{N}$ of $(\overline{x}, \overline{y})$ such that if $(x, y) \in \mathcal{N}$ then for all $j \in \mathcal{K}(\overline{c})$,

$$d^\top \nabla c(x)^\top Q_j \nabla c(x) d + d^\top \nabla^2 (y c)(x) d > 0, \quad \forall\, d \in \mathrm{Null}\left(A^\top \nabla c(x)\right) \setminus \{0\}\,. \tag{4.41}$$

and $\mathrm{Null}\left(\nabla c(x)^\top\right) \cap \mathrm{Ran}\,(A) = \{0\}\,.$

The following examples are inspired by the discussion in [26].

Example 5. In $\mathbb{R}^2$, let $h_a(c) = \|c\|_1^2$, so h is piecewise linear-quadratic convex. If $\mathcal{M} := \{0\}$, then h_a is not partly smooth relative to $\mathcal{M}$ because $\partial h_a(0) = \{0\}$ while $N\left(0 \,|\, \mathcal{M}\right) = \mathbb{R}^n$.

On the other hand, if $h_b(c) = \|c\|_1$ with the same domain representation, then $\partial h(0) = \mathbb{B}_\infty$, in which case h_b is partly smooth relative to $\mathcal{M}$.
Suppose we represent the domain of h_a and h_b as the four quadrants in the plane, relative to each of which h_a, h_b are linear-quadratic. This representation meets the criteria of the Rockafellar-Wets PLQ representation of Theorem 13. For both h_a and h_b, the nondegeneracy condition for $\mathcal{M}$ holds since A can be taken to be I_2.

Example 6. In $\mathbb{R}^2$, the domain of h_a and h_b in the previous example can be presented in the following way. Take each of the four quadrants in the plane and split them along their respective diagonal. Define h_a as usual on each of the pieces. Then this presentation describes $\operatorname{dom}(h_a)$ using 4 hyperplanes and also meets the Rockafellar-Wets PLQ representation theorem. However, the nondegeneracy condition fails for $\mathcal{M}$ in this representation.
On the manifold $\mathcal{M}$ given by an "artificial" diagonal, the matrix A is comprised of a single column, with $N\left(c \,|\, \mathcal{M}\right) = \operatorname{Ran}(A)$ for any $c \in \mathcal{M}$. However, h_a is smooth on $\mathcal{M}$ with $\operatorname{par}\left(\partial h(c)\right) = \{0\}$.

We end this section with a relationship between partial smoothness and the convergence analysis of quasi-Newton methods studied in 4.3.1. The following result is a finite identification property for any algorithm solving **P** in the presence of an active manifold at a solution.

Theorem 15. [28, Theorem 4.10] Suppose the closed, proper, convex function $h : \mathbb{R}^m \to \overline{\mathbb{R}}$ is partly smooth at the point $\bar{c} \in \mathbb{R}^m$ relative to a manifold $\mathcal{M} \subset \mathbb{R}^m$. Consider a subgradient $\bar{y} \in \operatorname{ri}\left(\partial h(\bar{c})\right)$. Suppose the sequence $\{\widehat{c}_k\} \subset \mathbb{R}^m$ satisfies $\widehat{c}_k \to \bar{c}$ and $h(\widehat{c}_k) \to h(\bar{c})$. Then, $\widehat{c}_k \in \mathcal{M}_{\bar{c}}$ for all large k if and only if $\operatorname{dist}\left(\bar{y} \,\middle|\, \partial h(\widehat{c}_k)\right) \to 0$.

Combining Corollary 4.3.1 and Theorem 15, we have the following relationship between the sufficient conditions for superlinear convergence of the quasi-Newton method $\mathbf{Q}_k$ and the finite identification of an active manifold at a solution.

Corollary 4.4.1. Let $\mathcal{M}_{\overline{c}}$ be as in (4.25), let Assumption 1 hold, and recall the notation of Definition 4.4.3 Let $\overline{x} \in \operatorname{dom}(f)$ and $\overline{c} := c(\overline{x})$.
Suppose

(a) $\mathcal{M}_{\overline{c}}$ satisfies the nondegeneracy condition,

(b) the k-strict complementarity condition of Definition 4.4.5 holds at $(c, y) \in \mathbb{R}^m \times \mathbb{R}^m$,

(c) $M(\overline{x}) = \{\overline{y}\}$, and

(d) the second-order sufficient conditions of Theorem 10 are satisfied at $\overline{x}$.

Consider the neighborhood U of $(\overline{x}, \overline{y})$ of Corollary 4.3.1, and a starting point $(x^0, y^0) \in U$. Suppose the sequence $\left\{(x^k, y^k)\right\}_{k\in\mathbb{N}}$ is generated from the optimality conditions for $\mathbf{Q}_k$, remains in U for all $k \in \mathbb{N}$, and satisfies $(x^k, y^k) \neq (\overline{x}, \overline{y})$ for all $k \in \mathbb{N}$. Then, the sufficient conditions for superlinear convergence of Corollary 4.3.1 imply $c(x^k)+\nabla c(x^k)[x^{k+1}-x^k] \in \mathcal{M}_{\overline{c}}$ for all large k.

Proof. Since $x^k \to \overline{x}$, $d^k \to 0$. By continuity, $\widehat{c}_k := c(x^k)+\nabla c(x^k)[x^{k+1}-x^k] \to \overline{c}$. The quasi-Newton method (1.11) with $\mathbf{B}_k$ given by (4.22) implies $y^{k+1} \in \partial h(\widehat{c}_k)$, so $\{\widehat{c}_k\} \subset \operatorname{dom}(h)$. By Proposition 4.2.1, $h(\widehat{c}_k) \to h(\overline{c})$. Since $y^k \to \overline{y}$, $\operatorname{dist}\Big(\overline{y} \;\Big|\; \partial h(\widehat{c}_k)\Big) \leq \left\|\overline{y} - y^{k+1}\right\| \to 0$. Then, by partial smoothness and Theorem 15, $\widehat{c}_k \in \mathcal{M}_{\overline{c}}$ for all large k. ☐

4.5 Strong Metric Regularity and Local Quadratic Convergence of Newton's Method

The point of this section is to marry the partial smoothness hypothesis to the hypotheses used to establish strong metric subregularity in Section 4.4 to establish strong metric regularity of a solution mapping that is an appropriately defined local version of $g + G$ in (2.9). In addition, we establish the local quadratic convergence of the Newton method for $g + G$.

Definition 4.5.1 (Metric regularity)**.** A set-valued mapping $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is *metrically regular* at $\overline{x}$ for $\overline{y}$ when $\overline{y} \in S(\overline{x})$, the graph of S is locally closed at $(\overline{x}, \overline{y})$, and there exists $\kappa \geq 0$

and neighborhoods U of $\overline{x}$ and V of $\overline{y}$ such that $\operatorname{dist}\left(x \;\middle|\; S^{-1}(y)\right) \leq \kappa \operatorname{dist}\left(y \;\middle|\; S(x)\right)$ for all $(x, y) \in U \times V$. The infimum of κ over all $(\kappa,\, U,\, V)$ satisfying the display is called the metric regularity modulus of S at $\overline{x}$ for $\overline{y}$, and is denoted $\operatorname{reg}(S; \overline{x}|\overline{y})$.

Definition 4.5.2 (Strong metric regularity)**.** A set-valued mapping $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is *strongly metrically regular* at $\overline{x}$ for $\overline{y}$ when it is metrically regular at $\overline{x}$ for $\overline{y}$ and S^{-1} has a single-valued localization at $\overline{y}$ for $\overline{x}$. Equivalently, when S^{-1} has a Lipschitz continuous single-valued localization around $\overline{y}$ for $\overline{x}$.

4.5.1 Partly Smooth Problems

In this section, we make the following assumptions:

Assumption 2. Let f be as in **P**, $(\overline{x}, \overline{y}) \in \operatorname{dom}(f) \times \mathbb{R}^m$, $\overline{c} := c(\overline{x})$, $\overline{k} = |\mathcal{K}(\overline{c})|$, where $\mathcal{K}(\overline{c})$ are the active indices given in Definition 4.2.2. Let $\mathcal{M}_{\overline{c}}$ be the active manifold defined in (4.25) and let $\overline{\mu}_j \in \mathbb{R}^\ell$ for $j \in \left\{1, \ldots, \overline{k}\right\}$, where $\ell = |I_k(\overline{c})|$ for any $k \in \mathcal{K}(\overline{c})$ with $I_k(\overline{c})$ defined in (4.26). Recall that ℓ is well-defined by Lemma 4.4.2. With these specifications, we assume that

(a) $\operatorname{dom}(h)$ is given by the Rockafellar-Wets PLQ representation of Theorem 13,

(b) c is $\mathcal{C}^3$-smooth,

(c) $\mathcal{M}_{\overline{c}}$ satisfies the nondegeneracy condition (in particular, $\overline{k} \geq 2$),

(d) f satisfies (SC) at $\overline{x}$ for $\overline{y}$; i.e., $\operatorname{Null}\left(\nabla c(\overline{x})^\top\right) \cap \operatorname{ri}\left(\partial h(\overline{c})\right) = \{\overline{y}\}$, so that in particular, as in (4.30), $J\overline{y} = \mathcal{Q}\overline{c} + \mathcal{B} + \widehat{\mathcal{A}}\overline{\mu}$, where $\overline{\mu} = (\overline{\mu}_1^\top, \ldots, \overline{\mu}_{\overline{k}}^\top)^\top > 0$ by Lemma 4.4.6,

(e) $\overline{x}$ satisfies the second-order sufficient conditions of Theorem 10, i.e.,

$$h''(c(\overline{x}); \nabla c(\overline{x})d) + \left\langle d,\, \nabla^2(\overline{y}c)(\overline{x})d\right\rangle > 0 \quad \forall\, d \in \operatorname{Null}\left(A^\top \nabla c(\overline{x})\right) \setminus \{0\},$$

where, by Lemma 2.2.1, $M(\overline{x}) = \{\overline{y}\}$, and by Lemma 4.4.9, $D(\overline{x}) = \operatorname{Null}\left(A^\top \nabla c(\overline{x})\right)$.

The conditions (c) - (e) in Assumption 2 can be interpreted in terms of similar assumptions employed in classical NLP. Condition (c) corresponds to the linear independence of the active constraint gradients, (d) corresponds to strict complementary slackness, and (e) corresponds to the strong second-order sufficiency condition. The convergence results developed in this section subsume those known for NLP, since they follow from the case in which h is non finite-valued piecewise linear convex.
We begin with a key technical lemma important for establishing metric regularity.

Lemma 4.5.1. In the notation of Definition 4.4.3, for any $i, j \in \left\{1, \ldots, \overline{k}\right\}$, $(Q_{k_i} - Q_{k_j})\mathrm{Null}\left(A^\top\right) \subset \mathrm{Ran}\,(A)$.

Proof. Let $w \in \mathrm{Null}\left(A^\top\right)$. By polyhedrality, there exists $|t| > 0$ such that $c_t := \overline{c} + tw \in \mathcal{M}_{\overline{c}}$. By Proposition 4.2.1, $\mathrm{dom}\,(\partial h) = \mathrm{dom}\,(h)$, so there exists $v \in \partial h(c_t)$ and $\overline{v} \in \partial h(\overline{c})$. By (4.30), $(v, \mu(c_t, v))$ and $(\overline{v}, \overline{\mu})$ satisfy $Jv = \mathcal{Q}c_t + \mathcal{B} + \widehat{\mathcal{A}}\mu(c_t, v)$ and $J\overline{v} = \mathcal{Q}\overline{c} + \mathcal{B} + \widehat{\mathcal{A}}\overline{\mu}$. Then for any $i, j \in \mathcal{K}(\overline{c})$,

$$0 = (Q_{k_i} - Q_{k_j})c_t + A(P_i\mu(c_t, v)_i - P_j\mu(c_t, v)_j) + b_{k_i} - b_{k_j},$$
$$0 = (Q_{k_i} - Q_{k_j})\overline{c} + A(P_i\overline{\mu}_i - P_j\overline{\mu}_j) + b_{k_i} - b_{k_j}.$$

Subtracting the second equation from the first and rearranging gives

$$(Q_{k_i} - Q_{k_j})w = t^{-1}A\left\{P_j(\mu(c_t, v)_j - \overline{\mu}_j) - P_i(\mu(c_t, v)_i - \overline{\mu}_i)\right\}. \tag{4.42}$$

□

We now define a family of local approximations to $g+G$ for which strong metric regularity is established.

Definition 4.5.3. For a point $\overline{c} \in \mathcal{M}_{\overline{c}}$ and each $j \in \left\{1, \ldots, \overline{k}\right\}$, define $g_j : \mathbb{R}^{n+m+\ell} \to$

$\mathbb{R}^{n+m+\ell+\ell}$.

$$g_j(x,y,\mu_j) := \begin{pmatrix} \nabla c(x)^\top y \\ y - Q_{k_j} c(x) - b_{k_j} - AP_j\mu_j \\ A^\top[c(x) - \overline{c}] \\ -\mu_j \end{pmatrix}, \quad G_0 := \begin{pmatrix} \{0\}^n \\ \{0\}^m \\ \{0\}^\ell \\ \mathbb{R}^\ell_+. \end{pmatrix}$$

and set $\overline{\boldsymbol{x}}_j := (\overline{x}, \overline{y}, \overline{\mu}_j) \in \mathbb{R}^{n+m+\ell}$, where $\overline{x}, \overline{y}, \overline{\mu}_j$ are as in Assumption 2. Then

$$\nabla g_j(x,y,\mu_j) = \begin{pmatrix} \nabla^2(yc)(x) & \nabla c(x)^\top & 0 \\ -Q_{k_j}\nabla c(x) & I & -AP_j \\ A^\top \nabla c(x) & 0 & 0 \\ 0 & 0 & -I_\ell \end{pmatrix}, \quad g_j(\overline{\boldsymbol{x}}_j) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\overline{\mu}_j \end{pmatrix} \in -G_0 \text{ (see Assumption 2 (d)).}$$

In parallel to the study in Section 4.3, we introduce the linearization of these mappings.

Definition 4.5.4 ($\mathcal{M}_{\overline{c}}$-restricted KKT Mappings)**.** Let $\overline{c}$ and $\overline{k}$ be given by Assumption 2, and g_j and G_0 be as in Definition 4.5.3. For all $j \in \left\{1, \ldots, \overline{k}\right\}$, define the linearization of $g_j + G_0$ at $\boldsymbol{u} = (\widehat{x}, \widehat{y}, \widehat{\mu}_j)$

$$\mathcal{G}^j_{\boldsymbol{u}}(\boldsymbol{x}) := g_j(\boldsymbol{u}) + \nabla g_j(\boldsymbol{u})(\boldsymbol{x} - \boldsymbol{u}) + G_0, \text{ or equivalently,} \tag{4.43}$$

$$\mathcal{G}^j_{(\widehat{x},\widehat{y},\widehat{\mu}_j)}(x,y,\mu_j) := g_j(\widehat{x},\widehat{y},\widehat{\mu}_j) + \nabla g_j(\widehat{x},\widehat{y},\widehat{\mu}_j) \begin{pmatrix} x - \widehat{x} \\ y - \widehat{y} \\ \mu_j - \widehat{\mu}_j \end{pmatrix} + G_0.$$

For any $\boldsymbol{u} = (\widehat{x}, \widehat{y}, \widehat{\mu}_j)$, define the function

$$F_{\boldsymbol{u}}(\boldsymbol{x},\boldsymbol{z}) := g_j(\boldsymbol{u}) + \nabla g_j(\boldsymbol{u})(\boldsymbol{x} - \boldsymbol{u}) - \boldsymbol{z} = \begin{pmatrix} \nabla c(\widehat{x})^\top y + \nabla^2(\widehat{y}c)(\widehat{x})[x - \widehat{x}] - z_1 \\ y - Q_{k_j}[c(\widehat{x}) + \nabla c(\widehat{x})[x - \widehat{x}]] - b_{k_j} - AP_j\mu_j - z_2 \\ A^\top[c(\widehat{x}) + \nabla c(\widehat{x})[x - \widehat{x}] - \overline{c}] - z_3 \\ -\mu_j - z_4 \end{pmatrix}. \tag{4.44}$$

Then,

$$\operatorname{gph} \mathcal{G}^j_{\boldsymbol{u}} = \left\{ (\boldsymbol{x}, \boldsymbol{z}) \;\middle|\; F_{\boldsymbol{u}}(\boldsymbol{x}, \boldsymbol{z}) \in -G_0 \right\}, \tag{4.45}$$

with dom $\left(\mathcal{G}^j_{\boldsymbol{u}}\right) = \mathbb{R}^{n+m+\ell}$. Explicitly,

$$\operatorname{gph} \mathcal{G}^j_{(\widehat{x},\widehat{y},\widehat{\mu}_j)} = \left\{ (x, y, \mu_j, z_1, z_2, z_3, z_4) \;\middle|\; \begin{array}{l} z_1 = \nabla c(\widehat{x})^\top y + \nabla^2(\widehat{y}c)(\widehat{x})[x - \widehat{x}] \\ z_2 = y - Q_{k_j}[c(\widehat{x}) + \nabla c(\widehat{x})[x - \widehat{x}]] - b_{k_j} - AP_j\mu_j \\ z_3 = A^\top[c(\widehat{x}) + \nabla c(\widehat{x})[x - \widehat{x}] - \overline{c}] \\ z_4 \in -\mu_j + \mathbb{R}^\ell_+ \end{array} \right\}. \tag{4.46}$$

The next lemma shows that the error in the Newton iterates can be measured in terms of (x, y) alone, independent of the vectors μ_j.

Lemma 4.5.2. Let $\overline{x}, \overline{y}, \overline{\mu}, \overline{c}, \overline{k}$, and $\mathcal{Q}$ be as in Assumption 2, and g_j and G_0 be as in Definition 4.5.3. For any $j \in \left\{1, \ldots, \overline{k}\right\}$, define $\eta_j : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n+m+\ell}$ by

$$\eta_j(x, y) := \begin{pmatrix} \nabla c(x)^\top y \\ Q_{k_j}(\overline{c} - c(x)) \\ A^\top(c(x) - \overline{c}) \end{pmatrix}. \tag{4.47}$$

Observe that for any $(x, y, \mu_j) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^\ell$,

$$g_j(x, y, \mu_j) = \begin{pmatrix} \eta_j(x, y) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ y - \overline{y} + AP_j(\overline{\mu}_j - \mu_j) \\ 0 \\ -\mu_j \end{pmatrix} \text{ and } \nabla g_j(x, y, \mu_j) = \begin{pmatrix} \nabla\eta_j(x, y) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & I & -AP_j \\ 0 & 0 & 0 \\ 0 & 0 & -I \end{pmatrix}$$

Set $\overline{\boldsymbol{x}}_j := (\overline{x}, \overline{y}, \overline{\mu}_j)$. Then, for any $\boldsymbol{u} := (\widehat{x}, \widehat{y}, \widehat{\mu}_j) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^\ell$,

$$\left\| F_{\boldsymbol{u}}(\overline{\boldsymbol{x}}_j, g_j(\overline{\boldsymbol{x}}_j)) \right\| = \left\| \eta_j(\widehat{x}, \widehat{y}) + \nabla\eta_j(\widehat{x}, \widehat{y}) \begin{pmatrix} \overline{x} - \widehat{x} \\ \overline{y} - \widehat{y} \end{pmatrix} - \eta_j(\overline{x}, \overline{y}) \right\|, \tag{4.48}$$

since $\eta_j(\overline{x}, \overline{y}) = 0$.

The following lemma uses the strict criticality assumption to show the normal cone to the graph of these linearization are captured by the range of $\nabla F_{\overline{\boldsymbol{x}}_j}$.

Lemma 4.5.3. Let $\overline{x}, \overline{y}, \overline{\mu}, \overline{c}, \overline{k}$, and $\mathcal{Q}$ be as in Assumption 2 and set $\overline{\boldsymbol{x}}_j := (\overline{x}, \overline{y}, \overline{\mu}_j)$. Then, for all $j \in \left\{1, \ldots, \overline{k}\right\}$, the mapping $\mathcal{G}^j_{\overline{\boldsymbol{x}}_j}$ in (4.45) has $N\left((\overline{\boldsymbol{x}}_j, \mathbf{0}) \mid \operatorname{gph} \mathcal{G}^j_{\overline{\boldsymbol{x}}_j}\right) = \operatorname{Ran}(W)$, where

$$W := \begin{pmatrix} \nabla^2(\overline{y}c)(\overline{x}) & -\nabla c(\overline{x})^\top Q_{k_j} & \nabla c(\overline{x})^\top A \\ \nabla c(\overline{x}) & I_m & 0 \\ 0 & -P_j A^\top & 0 \\ -I_n & 0 & 0 \\ 0 & -I_m & 0 \\ 0 & 0 & -I_\ell \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.49}$$

Proof. The set $\operatorname{gph} \mathcal{G}^j_{\overline{\boldsymbol{x}}_j} = \left\{(\boldsymbol{x}, \boldsymbol{z}) \;\middle|\; F_{\overline{\boldsymbol{x}}_j}(\boldsymbol{x}, \boldsymbol{z}) \in -G\right\}$ defined in (4.45) is closed with $(\overline{\boldsymbol{x}}_j, \mathbf{0}) \in \operatorname{gph} \mathcal{G}^j_{\overline{\boldsymbol{x}}_j}$. In addition, $\overline{\mu}_j > 0$, $N\left(F_{\overline{\boldsymbol{x}}_j}(\overline{\boldsymbol{x}}_j, \mathbf{0}) \mid -G_0\right) = \mathbb{R}^{n+m+\ell} \times \{0\}^\ell$, and

$$\nabla F_{\overline{\boldsymbol{x}}_j}(\overline{\boldsymbol{x}}_j, \mathbf{0})^\top = \begin{pmatrix} \nabla^2(\overline{y}c)(\overline{x}) & -\nabla c(\overline{x})^\top Q_{k_j} & \nabla c(\overline{x})^\top A & 0 \\ \nabla c(\overline{x}) & I_m & 0 & 0 \\ 0 & -P_j A^\top & 0 & I_\ell \\ -I_n & 0 & 0 & 0 \\ 0 & -I_m & 0 & 0 \\ 0 & 0 & -I_\ell & 0 \\ 0 & 0 & 0 & I_\ell \end{pmatrix} = \Big(W \mid R\Big),$$

where the matrix R is being defined by this expression. Combining the facts in the previous two sentences, the constraint qualification (4.70) in Theorem 20 (see appendix), for $N\left((\overline{\boldsymbol{x}}_j, \mathbf{0}) \mid \operatorname{gph} \mathcal{G}^j_{\overline{\boldsymbol{x}}_j}\right)$ is the requirement that $\operatorname{Null}(W) = \{0\}$. If we verify $\operatorname{Null}(W) = \{0\}$, then $N\left((\overline{\boldsymbol{x}}_j, \mathbf{0}) \mid \operatorname{gph} \mathcal{G}^j_{\overline{\boldsymbol{x}}_j}\right) = \operatorname{Ran}(W)$ by Theorem 20. But the presence of the identity matrices in W immediately give $\operatorname{Null}(W) = \{0\}$. $\square$

The metric regularity of the mappings $g_j + G_0$ follow from the second-order sufficient conditions of Theorem 10.

Lemma 4.5.4. Let $\overline{x}, \overline{y}, \overline{\mu}, \overline{c}, \overline{k}$, and $\mathcal{Q}$ be as in Assumption 2, W as in (4.49) and set $\overline{\boldsymbol{x}}_j := (\overline{x}, \overline{y}, \overline{\mu}_j)$. For all $j \in \left\{1, \ldots, \overline{k}\right\}$,

$$(\mathbf{0}, -\boldsymbol{z}) \in N\left((\overline{\boldsymbol{x}}_j, \mathbf{0}) \mid \operatorname{gph} \mathcal{G}^j_{\overline{\boldsymbol{x}}_j}\right) \iff \boldsymbol{z} = 0,$$

where $\mathcal{G}^j_{\overline{\boldsymbol{x}}_j}$ is given by (4.45). Then, $\mathcal{G}^j_{\overline{\boldsymbol{x}}_j}$ is metrically regular at $\overline{\boldsymbol{x}}_j$ for $\mathbf{0}$ and

$$\begin{pmatrix} \nabla^2(\overline{y}c)(\overline{x}) & \nabla c(\overline{x})^\top & 0 \\ -Q_{k_j}\nabla c(\overline{x}) & I_m & -AP_j \\ A^\top \nabla c(\overline{x}) & 0 & 0 \end{pmatrix}$$

is nonsingular.

Proof. By Lemma 4.5.3, $N\left((\overline{\boldsymbol{x}}_j, \mathbf{0}) \mid \operatorname{gph} \mathcal{G}^j_{\overline{\boldsymbol{x}}_j}\right) = \operatorname{Ran}(W)$, and so the statement

$$(\mathbf{0}, -\boldsymbol{z}) \in N\left((\overline{\boldsymbol{x}}_j, \mathbf{0}) \mid \operatorname{gph} \mathcal{G}^j_{\overline{\boldsymbol{x}}_j}\right) \iff \boldsymbol{z} = 0$$

is equivalent to

$$\left[\begin{pmatrix} 0 \\ 0 \\ 0 \\ -z_1 \\ -z_2 \\ -z_3 \\ -z_4 \end{pmatrix} = \begin{pmatrix} \nabla^2(\overline{y}c)(\overline{x}) & -\nabla c(\overline{x})^\top Q_{k_j} & \nabla c(\overline{x})^\top A \\ \nabla c(\overline{x}) & I & 0 \\ 0 & -P_j A^\top & 0 \\ -I_n & 0 & 0 \\ 0 & -I_m & 0 \\ 0 & 0 & -I_\ell \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} d \\ v \\ w \end{pmatrix} \text{ for some } \begin{pmatrix} d \\ v \\ w \end{pmatrix}\right] \iff \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = 0. \tag{4.50}$$

Since ($\Leftarrow$) is trivial, we only establish ($\Rightarrow$). Define $H := \nabla^2(\overline{y}c)(\overline{x})$. Then the left-hand side of (4.50) becomes

$$0 = Hd - \nabla c(\overline{x})^\top Q_{k_j} v + \nabla c(\overline{x})^\top A w, \tag{4.51}$$

$$0 = \nabla c(\overline{x}) d + v, \tag{4.52}$$

$$0 = -P_j A^\top v, \tag{4.53}$$

$$z_1 = d,\ \ z_2 = v,\ \ z_3 = w,\ \ z_4 = 0.$$

Since $z_4 = 0$, we need only show $z_1 = z_2 = z_3 = 0$, which we establish by showing $d = v = w = 0$. First suppose $d \neq 0$. From (4.53) and Definition 4.4.3, $v \in \text{Null}\left(A^\top\right)$. Then (4.52) and gives $\nabla c(\overline{x}) d = -v \in \text{Null}\left(A^\top\right)$. By Lemma 4.4.9, $d \in D(\overline{x}) \setminus \{0\}$. Taking the inner product on both sides of (4.51) with d and using (4.52) gives $d^\top H d = d^\top \nabla c(\overline{x})^\top Q_{k_j} v = -d^\top \nabla c(\overline{x})^\top Q_{k_j} \nabla c(\overline{x}) d$, so

$$d^\top \nabla c(\overline{x})^\top Q_{k_j} \nabla c(\overline{x}) d + d^\top H d = 0.$$

But the second-order sufficient conditions of Theorem 10 imply that for any $j \in \left\{1, \ldots, \overline{k}\right\}$,

$$d^\top \nabla c(\overline{x})^\top Q_{k_j} \nabla c(\overline{x}) d + d^\top H d > 0.$$

This contradiction implies $d = 0$. But then $v = 0$ by (4.52). Finally, (4.51) states that w must satisfy $Aw \in \text{Null}\left(\nabla c(\overline{x})^\top\right) \cap \text{Ran}\,(A) = \{0\}$. By the nondegeneracy condition of Definition 4.4.4, $w = 0$. Equation (4.45) gives local closedness of $\mathcal{G}^j_{\overline{\boldsymbol{x}}_j}$ at $(\overline{\boldsymbol{x}}_j, \mathbf{0})$, so the coderivative criterion for metric regularity [15, Theorem 4C.2] implies $\mathcal{G}^j_{\overline{\boldsymbol{x}}_j}$ is metrically regular at $\overline{\boldsymbol{x}}_j$ for $\mathbf{0}$, as required. □

The metric regularity of the mappings $\mathcal{G}^j_{\overline{\boldsymbol{x}}_j}$ imply a parameterized uniform version of metric regularity, where we allow $\overline{\boldsymbol{x}}_j$ to move.

Lemma 4.5.5. Let $\overline{x}, \overline{y}, \overline{\mu}, \overline{c}, \overline{k}$, and $\mathcal{Q}$ be as in Assumption 2, set $\overline{\boldsymbol{x}}_j := (\overline{x}, \overline{y}, \overline{\mu}_j)$, and let $\mathcal{G}^j_{\overline{\boldsymbol{x}}_j}$ be given by (4.45). For all $j \in \left\{1, \ldots, \overline{k}\right\}$, there exists a neighborhood $U_j \subset \mathbb{R}^{n+m+\ell}$ of $\overline{\boldsymbol{x}}_j$ and a neighborhood $V_j \subset \mathbb{R}^{n+m+\ell+\ell}$ of $\mathbf{0}$ such that the mapping

$$(\boldsymbol{u}, \boldsymbol{z}) \mapsto \mathcal{G}^{-j}_{\boldsymbol{u}}(\boldsymbol{z}) := \left(\mathcal{G}^j_{\boldsymbol{u}}\right)^{-1}(\boldsymbol{z}) \text{ for } (\boldsymbol{u}, \boldsymbol{z}) \in U_j \times V_j$$

is single-valued with $\mathcal{G}_{\boldsymbol{u}}^{-j}(\mathbf{0}) \in U_j$.

Proof. Fix $j \in \left\{1, \dots, \overline{k}\right\}$. By Lemma 4.5.4 and [15, Theorem 6D.1], for every $\lambda >$ $\operatorname{reg}(\mathcal{G}^j_{\overline{\boldsymbol{x}}_j}; \overline{\boldsymbol{x}}_j|\mathbf{0})$ there exists $a > 0$ and $b > 0$ such that

$$\operatorname{dist}\left(\boldsymbol{x} \,\middle|\, \mathcal{G}_{\boldsymbol{u}}^{-j}(\boldsymbol{z})\right) \leq \lambda \operatorname{dist}\left(\boldsymbol{z} \,\middle|\, \mathcal{G}_{\boldsymbol{u}}^{j}(\boldsymbol{x})\right), \quad \text{for every } \boldsymbol{u}, \boldsymbol{x} \in \overline{\boldsymbol{x}}_j + a\mathbb{B}, \boldsymbol{z} \in b\mathbb{B}. \tag{4.54}$$

By reducing a, if necessary, we may assume the conclusion of Lemma 4.4.10 holds on $\overline{\boldsymbol{x}}_j + a\mathbb{B}$. We follow the argument given in [15, Theorem 6D.2] by recalling (4.47) and choosing

$$L > \operatorname{lip}(\nabla \eta_j; (\overline{x}, \overline{y})) := \limsup_{\substack{(x,y),(x',y') \to (\overline{x},\overline{y}) \\ (x,y) \neq (x',y')}} \frac{\left\| \nabla \eta_j(x,y) - \nabla \eta_j(x',y') \right\|}{\left\| (x,y) - (x',y') \right\|}, \text{ and } \gamma > \frac{1}{2}\lambda L.$$

Define $\overline{a} := \min\left\{\frac{1}{\gamma}, a\right\} > 0$, $U_j := \overline{\boldsymbol{x}}_j + \overline{a}\mathbb{B}$, and $V_j := b\mathbb{B}$. We first establish nonemptiness of $\mathcal{G}_{\boldsymbol{u}}^{-j}(\boldsymbol{z})$. Fix $\boldsymbol{x} = \overline{\boldsymbol{x}}_j$, and choose any $(\boldsymbol{u}, \boldsymbol{z}) \in U_j \times V_j$, and consider two cases in (4.54). If $\operatorname{dist}\left(\boldsymbol{z} \,\middle|\, \mathcal{G}_{\boldsymbol{u}}^{j}(\overline{\boldsymbol{x}}_j)\right) = 0$, then by closedness of the set $\mathcal{G}_{\boldsymbol{u}}^{j}(\overline{\boldsymbol{x}}_j)$, it follows that $\overline{\boldsymbol{x}}_j \in \mathcal{G}_{\boldsymbol{u}}^{-j}(\boldsymbol{z})$. On the other hand, if $0 < \operatorname{dist}\left(\boldsymbol{z} \,\middle|\, \mathcal{G}_{\boldsymbol{u}}^{j}(\overline{\boldsymbol{x}}_j)\right) < \infty$, where finiteness is guaranteed because $\operatorname{dom}\left(\mathcal{G}_{\boldsymbol{u}}^{j}\right) = \mathbb{R}^{m+n+\ell}$. Then the implication

$$\operatorname{dist}\left(\overline{\boldsymbol{x}}_j \,\middle|\, \mathcal{G}_{\boldsymbol{u}}^{-j}(z)\right) \leq \lambda \operatorname{dist}\left(\boldsymbol{z} \,\middle|\, \mathcal{G}_{\boldsymbol{u}}^{j}(\overline{\boldsymbol{x}}_j)\right) \Longrightarrow \operatorname{dist}\left(\overline{\boldsymbol{x}}_j \,\middle|\, \mathcal{G}_{\boldsymbol{u}}^{-j}(z)\right) < \infty$$

holds, so in both cases $\mathcal{G}_{\boldsymbol{u}}^{-j}(z) \neq \emptyset$.

We now show single-valuedness. For the same j, $\boldsymbol{u}$, and $\boldsymbol{z}$, write $\boldsymbol{u} = (\widehat{x}, \widehat{y}, \widehat{\mu}_j)$, and suppose there are two points $\boldsymbol{x}_1 = (x_1, y_1, \mu_{j_1})$, $\boldsymbol{x}_2 = (x_2, y_2, \mu_{j_2})$ satisfying $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathcal{G}_{\boldsymbol{u}}^{-j}(\boldsymbol{z})$. Then subtracting the equations in (4.46) gives

$$0 = \nabla^2(\widehat{y}c)(\widehat{x})[x_2 - x_1] + \nabla c(\widehat{x})^\top (y_2 - y_1) \tag{4.55}$$

$$y_2 - y_1 = Q_{k_j} \nabla c(\widehat{x})[x_2 - x_1] + AP_j(\mu_{j_2} - \mu_{j_1}) \tag{4.56}$$

$$0 = A^\top \nabla c(\widehat{x})[x_2 - x_1]. \tag{4.57}$$

Then $\nabla c(\widehat{x})[x_2 - x_1] \in \operatorname{Null}\left(A^\top\right)$. Suppose $x_2 \neq x_1$. Taking the inner product on both sides of (4.55) and using the choice of $\overline{a}$ in accordance with Lemma 4.4.10,

$$\begin{aligned} 0 &= [x_2 - x_1]^\top \nabla^2(\widehat{y}c)(\widehat{x})[x_2 - x_1] + [x_2 - x_1]^\top \nabla c(\widehat{x})^\top (y_2 - y_1) && \text{by (4.55)} \\ &= [x_2 - x_1]^\top \nabla^2(\widehat{y}c)(\widehat{x})[x_2 - x_1] + [x_2 - x_1]^\top \nabla c(\widehat{x})^\top [Q_{k_j} \nabla c(\widehat{x})[x_2 - x_1] + AP_j(\mu_{j_2} - \mu_{j_1})] && \text{by (4.56)} \\ &= [x_2 - x_1]^\top \nabla^2(\widehat{y}c)(\widehat{x})[x_2 - x_1] + [x_2 - x_1]^\top \nabla c(\widehat{x})^\top Q_{k_j} \nabla c(\widehat{x})[x_2 - x_1] && \text{by (4.57)} \\ &> 0, \end{aligned}$$

so $x_2 = x_1$. But then (4.55), (4.56), and Lemma 4.4.10 imply

$$y_2 - y_1 \in \operatorname{Null}\left(\nabla c(\widehat{x})^\top\right) \cap \operatorname{Ran}(A) = \{0\},$$

so $y_2 = y_1$. The nondegeneracy condition of Definition 4.4.4 and (4.56) together imply

$$0 = AP_j(\mu_{j_2} - \mu_{j_1}) \Longrightarrow \mu_{j_2} = \mu_{j_1},$$

so single-valuedness is established. We conclude the proof by following the proof given in [15, Theorem 6D.2] and write $(x, y, \mu_j) = \boldsymbol{x} = \mathcal{G}_{\boldsymbol{u}}^{-j}(0)$. Then the quadratic bound lemma and the choice of γ gives

$$\begin{aligned} \left\| \begin{pmatrix} x - \overline{x} \\ y - \overline{y} \end{pmatrix} \right\| &\le \|\boldsymbol{x} - \overline{\boldsymbol{x}}_j\| \\ &= \operatorname{dist}\left(\overline{\boldsymbol{x}}_j \,\middle|\, \mathcal{G}_{\boldsymbol{u}}^{-j}(0)\right) \\ &\le \lambda \operatorname{dist}\left(\boldsymbol{0} \,\middle|\, \mathcal{G}_{\boldsymbol{u}}^{j}(\overline{\boldsymbol{x}}_j)\right) \\ &\le \frac{2\gamma}{L} \operatorname{dist}\left(\boldsymbol{0} \,\middle|\, \mathcal{G}_{\boldsymbol{u}}^{j}(\overline{\boldsymbol{x}}_j)\right) \\ &\le \frac{2\gamma}{L} \|g_j(\boldsymbol{u}) + \nabla g_j(\boldsymbol{u})(\overline{\boldsymbol{x}}_j - \boldsymbol{u}) - g_j(\overline{\boldsymbol{x}}_j)\| && \text{by (4.43) and } -g_j(\overline{\boldsymbol{x}}_j) \in G_0 \\ &= \frac{2\gamma}{L} \|F_{\boldsymbol{u}}(\overline{\boldsymbol{x}}_j, g_j(\overline{\boldsymbol{x}}_j))\| && \text{by (4.44)} \\ &= \frac{2\gamma}{L} \left\| \eta_j(\widehat{x}, \widehat{y}) + \nabla \eta_j(\widehat{x}, \widehat{y}) \begin{pmatrix} \overline{x} - \widehat{x} \\ \overline{y} - \widehat{y} \end{pmatrix} - \eta_j(\overline{x}, \overline{y}) \right\| && \text{by (4.48)} \end{aligned}$$

$$\leq \gamma \left\| \begin{pmatrix} \widehat{x} - \overline{x} \\ \widehat{y} - \overline{y} \end{pmatrix} \right\|^2$$
$$\leq \gamma \left\| \boldsymbol{u} - \overline{\boldsymbol{x}}_j \right\|^2$$
$$< \overline{a},$$

so $\boldsymbol{x} = \mathcal{G}^j_{\boldsymbol{u}}(\mathbf{0}) \in U_j$. □

Our work so far implies that Newton's method applied to the individual mappings $\mathcal{G}^j_{\overline{\boldsymbol{x}}_j}$ exhibit local quadratic convergence.

Theorem 16. Let $\overline{x}, \overline{y}, \overline{\mu}, \overline{c}, \overline{k}$, and $\mathcal{Q}$ be as in Assumption 2, set $\overline{\boldsymbol{x}}_j := (\overline{x}, \overline{y}, \overline{\mu}_j)$, and let $\mathcal{G}^j_{\overline{\boldsymbol{x}}_j}$ be given by (4.45). Then, the mappings $\left\{\mathcal{G}^j_{\overline{\boldsymbol{x}}_j}\right\}_{j=1}^{\overline{k}}$ are strongly metrically regular (see Definition 4.5.2) at $\overline{\boldsymbol{x}}_j$ for $\mathbf{0}$. Moreover, for all $j \in \left\{1, \ldots, \overline{k}\right\}$, there exists a neighborhood U_j of $\overline{\boldsymbol{x}}_j$ such that, for every $\boldsymbol{x}^0 \in U_j$, there is a unique sequence $\boldsymbol{x}^k_j = (x^k, y^k, \mu^k_j) \subset U_j$ generated by Newton's method for $g_j + G_0$ (1.10). Both this sequence, and the sequence (x^k, y^k), converge at a quadratic rate to $\overline{\boldsymbol{x}}_j$ and $(\overline{x}, \overline{y})$ respectively.

Proof. The metric regularity at $\overline{\boldsymbol{x}}_j$ for $\mathbf{0}$ was established in Lemma 4.5.4. Lemma 4.5.5 with $u = \overline{\boldsymbol{x}}_j$ shows $\mathcal{G}^{-j}_{\overline{\boldsymbol{x}}_j}$ has a single-valued localization around $\mathbf{0}$ for $\overline{\boldsymbol{x}}_j$, so the strong metric regularity of $\mathcal{G}^j_{\overline{\boldsymbol{x}}_j}$ at $\overline{\boldsymbol{x}}_j$ for $\mathbf{0}$ follows.

For the second claim, we again follow the proof in [15, Theorem 6D.2] by taking U_j as in Lemma 4.5.5, and choosing any $\boldsymbol{x}^0 \in U_j$. Following the proof of the final claim of Lemma 4.5.5, we find, for every $k \geq 1$, the existence and uniqueness of $\boldsymbol{x}^k$ given $\boldsymbol{x}^{k-1}$ satisfying

$$\mathbf{0} \in \mathcal{G}^j_{\boldsymbol{x}^{k-1}}(\boldsymbol{x}^k), \quad \left\| \begin{pmatrix} x^k - \overline{x} \\ y^k - \overline{y} \end{pmatrix} \right\| \leq \left\| \boldsymbol{x}^k - \overline{\boldsymbol{x}}_j \right\| \leq \gamma \left\| \begin{pmatrix} x^{k-1} - \overline{x} \\ y^{k-1} - \overline{y} \end{pmatrix} \right\|^2 \leq \gamma \left\| \boldsymbol{x}^{k-1} - \overline{\boldsymbol{x}}_j \right\|^2, \text{ and } \boldsymbol{x}^k \in U_j.$$

Moreover, since $\theta := \gamma \left\| \boldsymbol{x}^0 - \overline{\boldsymbol{x}}_j \right\| < \gamma \overline{a} < 1$, $\left\| \boldsymbol{x}^k - \overline{\boldsymbol{x}}_j \right\| \leq \theta^{2^k - 1} \left\| \boldsymbol{x}^0 - \overline{\boldsymbol{x}}_j \right\|^2$ for all $k \geq 1$, which completes the proof of quadratic convergence of both sequences. □

We now move from an isolated analysis of the mappings $\mathcal{G}_{\boldsymbol{u}}^j$ to how they behave as a whole. The goal is to guarantee the y obtained by solving $\mathbf{0} \in \mathcal{G}_{\boldsymbol{u}}^j(\boldsymbol{x})$ at some $\boldsymbol{u} = (\widehat{x}, \widehat{y}, \widehat{\mu}_j)$ for $\boldsymbol{x} = (x, y, \mu_j)$ has $y \in \partial h(c(\widehat{x}) + \nabla c(\widehat{x})[x - \widehat{x}])$.

Theorem 17. Let $\overline{x}, \overline{y}, \overline{\mu}, \overline{c}, \overline{k}$, and $\mathcal{Q}$ be as in Assumption 2, set $\overline{\boldsymbol{x}}_j := (\overline{x}, \overline{y}, \overline{\mu}_j)$, and let $\mathcal{G}_{\overline{\boldsymbol{x}}_j}^j$ be given by (4.45). Suppose $i \neq j$ and $i, j \in \left\{1, \ldots, \overline{k}\right\}$. There exists a neighborhood $\mathcal{N}$ of $(\overline{x}, \overline{y}, \overline{\mu}_1, \ldots, \overline{\mu}_{\overline{k}}) =: (\overline{x}, \overline{y}, \overline{\mu}) \in \mathbb{R}^{n+m+\overline{k}\ell}$ such that, if $(\widehat{x}, \widehat{y}, \widehat{\mu}_1, \ldots, \widehat{\mu}_{\overline{k}}) \in \mathcal{N}$ and $\boldsymbol{u}_j := (\widehat{x}, \widehat{y}, \widehat{\mu}_j)$, $\boldsymbol{u}_i := (\widehat{x}, \widehat{y}, \widehat{\mu}_i)$, with $\widehat{\mu}_i > 0$ and $\widehat{\mu}_j > 0$, then

(4.58)
$$\boldsymbol{x}_j := \mathcal{G}_{\boldsymbol{u}_j}^{-j}(\mathbf{0}) = \begin{pmatrix} x_j \\ y_j \\ \mu_j \end{pmatrix}, \quad \boldsymbol{x}_i := \mathcal{G}_{\boldsymbol{u}_i}^{-i}(\mathbf{0}) = \begin{pmatrix} x_i \\ y_i \\ \mu_i \end{pmatrix} \text{ satisfy } \begin{pmatrix} x_j \\ y_j \end{pmatrix} = \begin{pmatrix} x_i \\ y_i \end{pmatrix} \text{ for all } i, j \in \left\{1, \ldots, \overline{k}\right\}.$$

That is, there exists $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ such that $(x, y) = (x_i, y_i)$ for all $i \in \left\{1, \ldots, \overline{k}\right\}$. Moreover,

(i) $c(\widehat{x}) + \nabla c(x)[x - \widehat{x}] \in \mathcal{M}_{\overline{c}}$,

(ii) $\mu(c(\widehat{x}) + \nabla c(\widehat{x})[x - \widehat{x}], y)_j = \mu_j > 0$ for all $j \in \left\{1, \ldots, \overline{k}\right\}$,

(iii) $y \in \operatorname{ri}\left(\partial h(c(\widehat{x}) + \nabla c(\widehat{x})[x - \widehat{x}])\right)$,

where the mapping $\mu(c, y)$ is defined in Lemma 4.4.5.

Proof. For $j \in \left\{1, \ldots, \overline{k}\right\}$, define $\pi_j : \mathbb{R}^{n+m+\overline{k}\ell} \to \mathbb{R}^{n+m+\ell}$ by $\pi_j(x, y, \mu_1, \ldots, \mu_j, \ldots, \mu_{\overline{k}}) := (x, y, \mu_j)$. We first show there exists a neighborhood $\mathcal{N}$ of $(\overline{x}, \overline{y}, \overline{\mu}_1, \ldots, \overline{\mu}_{\overline{k}})$ such that, for all $j \in \left\{1, \ldots, \overline{k}\right\}$ and all $(\widehat{x}_j, \widehat{y}_j, \widehat{\mu}_j) = \boldsymbol{u}_j \in \mathcal{N}_j := \pi_j(\mathcal{N})$,

(a) the mappings $\left\{\mathcal{G}_{\boldsymbol{u}_j}^{-j}(\mathbf{0})\right\}_{j=1}^{\overline{k}}$ are single-valued with $\mathcal{G}_{\boldsymbol{u}_j}^{-j}(\mathbf{0}) \in \mathcal{N}_j$,

(b) μ_j associated to $\mathcal{G}_{\boldsymbol{u}_j}^{-j}(\mathbf{0})$ has $\mu_j > 0$,

(c) the condition (4.41) is satisfied at all $(x, y, \mu_j) \in \mathcal{N}_j$, and

(d) $c(\widehat{x}_j) + \nabla c(\widehat{x}_j)[x_j - \widehat{x}_j] \in \mathcal{M}_{\overline{c}}$, where $(x_j, y_j, \mu_j) = \mathcal{G}^{-j}_{(\widehat{x},\widehat{y},\widehat{\mu}_j)}(\mathbf{0})$.

Parts (a), (b), and (c) are a consequence of Lemma 4.5.5. We now justify (d). For any $j \in \left\{1, \ldots, \overline{k}\right\}$, the definition of $(x_j, y_j, \mu_j) = \mathcal{G}^{-j}_{(\widehat{x},\widehat{y},\widehat{\mu}_j)}(\mathbf{0})$ implies, in particular, $A^\top[c(\widehat{x}_j) + \nabla c(\widehat{x}_j)[x_j - \widehat{x}_j] - c(\overline{x})] = 0$. By the polyhedral structure of $\mathcal{M}_{\overline{c}}$, for any $w \in \operatorname{Null}\left(A^\top\right) = T\left(\overline{c}\,|\,\mathcal{M}_{\overline{c}}\right)$, there exists $\tau > 0$ such that $\overline{c} + tw \in \mathcal{M}_{\overline{c}}$ for all $|t| < \tau$. Lemma 4.5.5 argued that, for all sufficiently small $\epsilon > 0$,

$$\mathcal{G}^{-j}_{\boldsymbol{u}}(\mathbf{0}) \in (\overline{\boldsymbol{x}}_j + \epsilon\mathbb{B}) \text{ for all } \boldsymbol{u} \in \overline{\boldsymbol{x}}_j + \epsilon\mathbb{B} \text{ (see (a))}. \tag{4.59}$$

The continuity of c and (4.59) imply that for $\boldsymbol{u}_j$ sufficiently close to $\overline{\boldsymbol{x}}_j$, $c(\widehat{x}_j)+\nabla c(\widehat{x}_j)[x_j-\widehat{x}_j]$ can be made as close to $c(\overline{x})$ as desired. Then there exists a neighborhood of $(\overline{x}, \overline{y}, \overline{\mu}_j)$ such (d) holds. The neighborhood $\mathcal{N}$ also exists because there are only finitely many indices j in consideration.

Now let $\boldsymbol{u}_j := (\widehat{x}, \widehat{y}, \widehat{\mu}_j) \in \mathcal{N}_j$, $\boldsymbol{u}_i := (\widehat{x}, \widehat{y}, \widehat{\mu}_i) \in \mathcal{N}_i$, with $\widehat{\mu}_i > 0$ and $\widehat{\mu}_j > 0$, and denote

$$\mathcal{G}^{-j}_{\boldsymbol{u}_j}(\mathbf{0}) = \begin{pmatrix} x_j \\ y_j \\ \mu_j \end{pmatrix}, \quad \mathcal{G}^{-i}_{\boldsymbol{u}_i}(\mathbf{0}) = \begin{pmatrix} x_i \\ y_i \\ \mu_i \end{pmatrix}.$$

By (4.46),

$$0 = \nabla^2(\widehat{y}c)(\widehat{x})[x_j - x_i] + \nabla c(\widehat{x})^\top(y_j - y_i) \tag{4.60}$$

$$y_i = Q_{k_i}(c(\widehat{x}) + \nabla c(\widehat{x})[x_i - \widehat{x}]) + AP_i\mu_i + b_{k_i} \tag{4.61}$$

$$y_j = Q_{k_j}(c(\widehat{x}) + \nabla c(\widehat{x})[x_j - \widehat{x}]) + AP_j\mu_j + b_{k_j} \tag{4.62}$$

$$0 = A^\top\nabla c(\widehat{x})[x_j - x_i] \tag{4.63}$$

Define $\widehat{c}_i := c(\widehat{x}) + \nabla c(\widehat{x})[x_i - \widehat{x}] \in \mathcal{M}_{\overline{c}}$ by (d). By Assumption 3, $\overline{y} = Q_{k_i}\overline{c} + b_{k_i} + AP_i\overline{\mu}_i = Q_{k_j}\overline{c} + b_{k_j} + AP_j\overline{\mu}_j$, and in particular,

$$Q_{k_i}\overline{c} + b_{k_i} - b_{k_j} = Q_{k_j}\overline{c} + AP_j\overline{\mu}_j - AP_i\overline{\mu}_i. \tag{4.64}$$

Then (4.42) with $w := \widehat{c}_i - \overline{c} \in \text{Null}\left(A^\top\right)$, $t = 1$, and any $y \in \partial h(\widehat{c}_i)$ gives

$$\begin{aligned}
y_i &= Q_{k_i} w + Q_{k_i}\overline{c} + b_{k_i} + AP_i\mu_i \\
&= \left(Q_{k_j} w + A\left\{P_j(\mu(\widehat{c}_i, y)_j - \overline{\mu}_j) - P_i(\mu(\widehat{c}_i, y)_i - \overline{\mu}_i)\right\}\right) + Q_{k_i}\overline{c} + b_{k_i} + AP_i\mu_i + b_{k_j} - b_{k_j} \\
&= Q_{k_j} w + b_{k_j} + [Q_{k_i}\overline{c} + b_{k_i} - b_{k_j}] + AP_i\mu_i + A\left\{P_j(\mu(\widehat{c}_i, y)_j - \overline{\mu}_j) - P_i(\mu(\widehat{c}_i, y)_i - \overline{\mu}_i)\right\} \\
&= Q_{k_j}[\widehat{c}_i - \overline{c}] + b_{k_j} + [Q_{k_j}\overline{c} + AP_j\overline{\mu}_j - AP_i\overline{\mu}_i] + AP_i\mu_i + A\left\{P_j(\mu(\widehat{c}_i, y)_j - \overline{\mu}_j) - P_i(\mu(\widehat{c}_i, y)_i - \overline{\mu}_i)\right\} \\
&= Q_{k_j}\widehat{c}_i + b_{k_j} + AP_i[\mu_i - \mu(\widehat{c}_i, y)_i] + AP_j\mu(\widehat{c}_i, y)_j \\
&\in y_j + Q_{k_j}\nabla c(\widehat{x})[x_i - x_j] + \text{Ran}\,(A)
\end{aligned}$$

where the fourth equivalence follows from (4.64). This implies

$$y_j - y_i - Q_{k_j}\nabla c(\widehat{x})[x_j - x_i] \in \text{Ran}\,(A)\,. \tag{4.65}$$

Taking the inner product on both sides of (4.60) with $x_j - x_i$ gives

$$\begin{aligned}
0 &= [x_j - x_i]^\top \nabla^2(\widehat{y}c)(\widehat{x})[x_j - x_i] + [x_j - x_i]^\top \nabla c(\widehat{x})^\top (y_j - y_i) \\
&= [x_j - x_i]^\top \nabla^2(\widehat{y}c)(\widehat{x})[x_j - x_i] + [x_j - x_i]^\top \nabla c(\widehat{x})^\top Q_{k_j}\nabla c(\widehat{x})[x_j - x_i] \text{ by (4.65), (4.63).}
\end{aligned}$$

By Lemma 4.4.10 and (4.63), $x_i = x_j$. Then (4.65), (4.60), and (c) imply $y_i - y_j \in \text{Ran}\,(A) \cap \text{Null}\left(\nabla c(\widehat{x})^\top\right) = \{0\}$, which proves (4.58).

Since i and j were arbitrary, letting x and y denote the common values of the first two components of $\mathcal{G}^{-j}_{\boldsymbol{u}_j}(\mathbf{0})$ for each $j \in \left\{1, \ldots, \overline{k}\right\}$. Then $Jy = \mathcal{Q}(c(\widehat{x}) + \nabla c(\widehat{x})[x - \widehat{x}]) + \mathcal{B} + \widehat{\mathcal{A}}\mu$, with $c(\widehat{x}) + \nabla c(\widehat{x})[x - \widehat{x}] \in \mathcal{M}_{\overline{c}}$, and $\mu_1, \ldots, \mu_{\overline{k}} > 0$. By (4.30) and Lemma 4.4.6, $\mu(c(\widehat{x}) + \nabla c(\widehat{x})[x - \widehat{x}], y)_j = \mu_j > 0$, with $y \in \text{ri}\left(\partial h(c(\widehat{x}) + \nabla c(\widehat{x})[x - \widehat{x}])\right)$. □

Our final theorem integrates the ideas from Section 4.4 and our work in this section to establish the local quadratic convergence of Newton's method for **P**.

Theorem 18. Let $\overline{x}, \overline{y}, \overline{\mu}, \overline{c}, \overline{k}$, and $\mathcal{Q}$ be as in Assumption 2, set $\overline{\boldsymbol{x}}_j := (\overline{x}, \overline{y}, \overline{\mu}_j)$, and let $\mathcal{G}^j_{\overline{\boldsymbol{x}}_j}$ be given by (4.45). There exists a neighborhood $\mathcal{N}$ of $(\overline{x}, \overline{y}, \overline{\mu})$ on which the conclusions of Lemma 4.4.10 are satisfied such that if $(x^0, y^0, \mu^0) \in \mathcal{N}$, then there exists a unique sequence $\left\{(x^k, y^k, \mu^k)\right\}_{k\in\mathbb{N}}$ satisfying the optimality conditions of $\mathbf{P}_k$ for all $k \in \mathbb{N}$, with

(a) $c(x^{k-1}) + \nabla c(x^{k-1})[x^k - x^{k-1}] \in \mathcal{M}_{\overline{c}}$,

(b) $\mu(c(x^{k-1}) + \nabla c(x^{k-1})[x^k - x^{k-1}], y^k)_j > 0$ for all $j \in \left\{1, \ldots, \overline{k}\right\}$,

(c) $y^k \in \mathrm{ri}\left(\partial h(c(x^{k-1}) + \nabla c(x^{k-1})[x^k - x^{k-1}])\right)$,

(d) $H_{k-1}[x^k - x^{k-1}] + \nabla c(x^{k-1})^\top y^k = 0$,

(e) $x^k - x^{k-1}$ is a strong local minimizer of the model function $\phi_{(x^{k-1}, y^{k-1})}$, given by Definition 4.6.1.

Moreover, the sequence (x^k, y^k) converges to $(\overline{x}, \overline{y})$ at a quadratic rate.

Proof. All claims except (e) follow from Theorem 16 and Theorem 17. By Lemma 4.6.3, Lemma 4.6.4, and (d), claim (e) is equivalent to showing

(4.66)
$$h''(c(x^{k-1}) + \nabla c(x^{k-1})[x^k - x^{k-1}]; \nabla c(x^{k-1})\delta) + \delta^\top H_{k-1}\delta > 0 \;\forall\, \delta \in \mathrm{Null}\left(A^\top \nabla c(x^{k-1})\right) \setminus \{0\}\,.$$

Using (4.6) and partial smoothness,

$$h''(c(x^{k-1}) + \nabla c(x^{k-1})[x^k - x^{k-1}]; \nabla c(x^{k-1})\delta) = \delta^\top \nabla c(x^{k-1})^\top Q_j \nabla c(x^{k-1})\delta, \quad \forall\, j \in \mathcal{K}(\overline{c}),$$

so (4.41) gives (4.66). □

Remark 16. The fact that $\left\{x^k - x^{k-1}\right\}$ is a strong local minimizer of $\phi_{(x^{k-1}, y^{k-1})}$ does not mean that there are not other critical points for the model function outside the neighborhood of interest. It may be that at any iteration the problem $\widehat{\mathbf{P}}$ does not have a finite optimal value, in particular, should there exist directions of negative curvature orthogonal to the manifold.

4.5.2 Smooth Problems

In this section, we make the following assumptions:

Assumption 3. Let f be as in **P** and $(\bar{x}, \bar{y}) \in \mathrm{dom}\,(f) \times \mathbb{R}^m$, $\bar{c} := c(\bar{x})$, $\bar{k} = |\mathcal{K}(\bar{c})|$, where $\mathcal{K}(\bar{c})$ are the active indices given in Definition 4.2.2. Let $\mathcal{M}_{\bar{c}}$ be the active manifold defined in (4.25). We assume that

(a) $\mathrm{dom}\,(h)$ is given by the Rockafellar-Wets PLQ representation of Theorem 13,

(b) c is $\mathcal{C}^3$-smooth,

(c) $\mathcal{K}(\bar{c}) = \{k_0\}$,

(d) $\bar{x}$ satisfies the second-order sufficient conditions of Theorem 10,

Remark 17. Since $\bar{k} = 1$, we omit reference to the index k_0 for the rest of this section.

Remark 18. By (a) and (c), $c(\bar{x}) \in \mathrm{int}\,\mathrm{dom}\,(h)$ and $\partial h(\bar{c}) = \{\bar{y}\}$. Then, (d) becomes

$\bar{y} = Q\bar{c} + b,\ \nabla c(\bar{x})^\top \bar{y} = 0,\ d^\top \nabla c(\bar{x})^\top Q \nabla c(\bar{x}) d + d^\top \nabla^2 (\bar{y}c)(\bar{x}) d > 0\ \forall\, d \in \mathbb{R}^n \setminus \{0\}$, where $D(\bar{x}) = \mathbb{R}^n$.

As in Lemma 4.4.10, we have the following stability result.

Lemma 4.5.6. Suppose $d^\top \nabla c(\bar{x})^\top Q \nabla c(\bar{x}) d + d^\top \nabla^2 (\bar{y}c)(\bar{x}) d > 0$ for all $d \in \mathbb{R}^n \setminus \{0\}$. Then, there exists a neighborhood $\mathcal{N}$ of $(\bar{x}, \bar{y})$ such that if $(x, y) \in \mathcal{N}$ then,

$$d^\top \nabla c(x)^\top Q \nabla c(x) d + d^\top \nabla^2 (yc)(x) d > 0, \quad \forall\, d \in \mathbb{R}^n \setminus \{0\}\,, \tag{4.67}$$

and $c(x) \in \mathrm{int}\,\mathrm{dom}\,(h)$.

Our local analogue of the KKT mapping (2.9) is the following.

Definition 4.5.5. Define $g : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ by

$$g(x, y) := \begin{pmatrix} \nabla c(x)^\top y \\ y - Qc(x) - b \end{pmatrix}, \quad G := \{0\}^{n+m}\,,$$

and set $\bar{\boldsymbol{x}} := (\bar{x}, \bar{y})$. Then,

$$\nabla g(x, y) = \begin{pmatrix} \nabla^2 (yc)(x) & \nabla c(x)^\top \\ -Q\nabla c(x) & I_m \end{pmatrix}, \quad g(\bar{x}, \bar{y}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Assumption 3 (d) implies $\nabla g(\overline{x}, \overline{y})$ is nonsingular. Consequently, and the Newton method (1.10) corresponds to the classical Newton's method for solving the *equation* $g(x, y) = 0$. Namely,

$$\text{Find } (x^{k+1}, y^{k+1}) \text{ such that } g(x^k, y^k) + \nabla g(x^k, y^k) \begin{pmatrix} x^{k+1} - x^k \\ y^{k+1} - y^k \end{pmatrix} = 0. \tag{4.68}$$

The local quadratic convergence of the iteration (4.68) near $(\overline{x}, \overline{y})$ with $\nabla g(\overline{x}, \overline{y})$ is nonsingular is well-known, with (4.68) corresponding to the optimality conditions for $\mathbf{P}_k$. We conclude with the following theorem, which parallels Theorem 18.

Theorem 19. Let $\overline{x}, \overline{y}, \overline{c} := c(\overline{x})$, and $\mathcal{M}_{\overline{c}}$ be as in Assumption 3. Then, there exists a neighborhood $\mathcal{N}$ of $(\overline{x}, \overline{y})$ on which the conclusions of Lemma 4.5.6 are satisfied such that if $(x^0, y^0) \in \mathcal{N}$, then there exists a unique sequence $\{(x^k, y^k)\}_{k \in \mathbb{N}}$ satisfying the optimality conditions of $\mathbf{P}_k$ for all $k \in \mathbb{N}$, with

(a) $c(x^{k-1}) + \nabla c(x^{k-1})[x^k - x^{k-1}] \in \mathcal{M}_{\overline{c}}$,

(b) $\partial h(c(x^{k-1}) + \nabla c(x^{k-1})[x^k - x^{k-1}]) = \{y^k\}$,

(c) $H_{k-1}[x^k - x^{k-1}] + \nabla c(x^{k-1})^\top y^k = 0$,

(d) $x^k - x^{k-1}$ is a strong local minimizer of the model function $\phi_{(x^{k-1}, y^{k-1})}$, given by Definition 4.6.1.

Moreover, the sequence (x^k, y^k) converges to $(\overline{x}, \overline{y})$ at a quadratic rate.

4.6 Appendix

Lemma 4.6.1. Suppose $C \subset \mathbb{R}^m$ is a nonempty, closed, convex set and $A \in \mathbb{R}^{n \times m}$. Consider the following equations:

(a) $$\mathrm{Null}\,(A) \cap \mathrm{ri}\,(C) = \{\overline{y}\}\,,$$

(b) $$\mathrm{Null}\,(A) \cap \mathrm{par}\,(C) = \{0\},$$

(c) $$\mathrm{Null}\,(A) \cap C = \{\overline{y}\}\,.$$

Then $(a) \implies (b) \implies (c)$.

Proof. $[(a) \Rightarrow (b)]$ Since $\overline{y} \in C$, there exists an integer $n \geq 1$ and points $\{y_1, \ldots, y_n\} \subset \mathrm{aff}\, C$ that span $\{y_i - \overline{y}\}_{i=1}^n = \mathrm{par}\,(C)$. By convexity and the assumption $\overline{y} \in \mathrm{ri}\,(C)$, we can further assume $\{y_1, \ldots, y_n\} \subset \mathrm{ri}\,(C)$. By [40, Theorem 6.4], there exists $\{z_1, \ldots, z_n\} \subset \mathrm{ri}\,(C)$ and $t_i > 0$ such that, for all $i \in \{1, \ldots, n\}$, $y_i - \overline{y} = -t_i(z_i - \overline{y})$. Then, after relabeling, we may suppose $\{y_1, \ldots, y_n\} \subset \mathrm{ri}\,(C)$ satisfies

$$\mathrm{par}\,(C) = \left\{ \sum_{i=1}^{n} \mu_i(y_i - \overline{y}) \;\middle|\; \mu_i \geq 0, i \in \{1, \ldots, n\} \right\} \tag{4.69}$$

Now suppose (b) does not hold. Then, there exists $0 \neq z \in \mathrm{Null}\,(A) \cap \mathrm{par}\,(C)$. By (4.69), $z = \sum_i \mu_i(y_i - \overline{y})$ with $\mu_i \geq 0$ and $\sum_i \mu_i \neq 0$. Define $t := \frac{1}{\sum_i \mu_i}$, and for $i \in \{1, \ldots, n\}$, define $\lambda_i := t\mu_i$. Then $\lambda_i \geq 0$ for all $i \in \{1, 2 \ldots, n\}$, with $\sum_{i=1}^n \lambda_i = 1$. Then by [40, Theorem 6.1] $\overline{y} + tz = \overline{y} + \sum_i \lambda_i(y_i - \overline{y}) = \sum_i \lambda_i y_i \in \mathrm{ri}\,(C)$. But then $\overline{y}$ and $\overline{y} + tz$ are two points in $\mathrm{Null}\,(A) \cap \mathrm{ri}\,(C)$, so (b) must hold.

$[(b) \Rightarrow (c)]$ Suppose (b) and that there exists $y_1, y_2 \in \mathrm{Null}\,(A) \cap C$. Then $y_1 - y_2 \in \mathrm{Null}\,(A) \cap \mathrm{par}\,(C) = \{0\}$, so $y_1 = y_2$. □

Theorem 20 (Normals Cones to Sets with Constraint Structure)**.** [42, Theorem 6.14] Let $C = \left\{ x \in X \;\middle|\; F(x) \in Z \right\}$ for closed convex sets $X \subset \mathbb{R}^n$ and $Z \subset \mathbb{R}^m$ and a $\mathcal{C}^1$-mapping $F : \mathbb{R}^n \to \mathbb{R}^m$. Suppose $\overline{x} \in C$ satisfies the constraint qualification

$$[y \in N\left(F(\overline{x}) \,|\, Z\right),\; -\nabla F(\overline{x})^\top y \in N\left(\overline{x} \,|\, X\right)] \iff y = (0, \ldots, 0). \tag{4.70}$$

Then $N\left(\overline{x}\,|\,C\right) = \left\{\nabla F(\overline{x})^\top y + v \;\middle|\; y \in N\left(F(\overline{x})\,|\,Z\right), v \in N\left(\overline{x}\,|\,X\right)\right\}$.

Definition 4.6.1 (The model function at $\widehat{x}$)**.** Let f be as in **P** and $\widehat{x} \in \mathrm{dom}\,(f)$. Suppose f satisfies (BCQ) at $\widehat{x}$. Define $\boldsymbol{u} := (\widehat{x}, \widehat{y})$, $\widehat{H} := \nabla^2(\widehat{y}c)(\widehat{x})$,

$$\psi(v,w) := h(v) + w, \text{ and } \Phi_{\boldsymbol{u}}(d) := \begin{pmatrix} c(\widehat{x}) + \nabla c(\widehat{x})d \\ \frac{1}{2}d^\top \widehat{H} d \end{pmatrix}.$$

Then, for any $(v,w) \in \mathrm{dom}\,(h) \times \mathbb{R}$ and $(d,s) \in \mathbb{R}^n \times \mathbb{R}$,

$$\nabla\Phi_{\boldsymbol{u}}(d) = \begin{pmatrix} \nabla c(\widehat{x}) \\ d^\top \widehat{H} \end{pmatrix}, \quad \psi'((v,w);(d,s)) = h'(v;d) + s, \quad \psi''((v,w);(d,s)) = h''(v;d).$$

Set $\phi_{\boldsymbol{u}}(d) := \psi(\Phi_{\boldsymbol{u}}(d)) = h(c(\widehat{x}) + \nabla c(\widehat{x})d) + \frac{1}{2}d^\top \widehat{H} d$. By Theorem 9, $\phi_{\boldsymbol{u}}$ is piecewise linear-quadratic, though not necessarily convex because $\widehat{H}$ may not be positive semi-definite. However, $\phi_{\boldsymbol{u}}$ is convex-composite with ψ piecewise linear-quadratic convex.

The following lemma shows that if f satisfies (BCQ) at $\widehat{x}$, then the model function at $\widehat{x}$ satisfies its (BCQ) throughout its domain.

Lemma 4.6.2. Let f be as in **P**, and suppose f satisfies (BCQ) at $\widehat{x}$. Then, $\phi_{\boldsymbol{u}}$ given in Definition 4.6.1 satisfies (BCQ) at all points $\overline{d} \in \mathrm{dom}\,(\phi_{\boldsymbol{u}}) = \left\{d \;\middle|\; c(\widehat{x}) + \nabla c(\widehat{x})d \in \mathrm{dom}\,(h)\right\}$.

Proof. Let $\overline{d} \in \left\{d \;\middle|\; c(\widehat{x}) + \nabla c(\widehat{x})d \in \mathrm{dom}\,(h)\right\}$. By definition,

$$\mathrm{Null}\left(\nabla\Phi_{\boldsymbol{u}}(\overline{d})^\top\right) = \mathrm{Null}\left(\begin{pmatrix}\nabla c(\widehat{x})^\top & \widehat{H}d\end{pmatrix}\right) \text{ and } N\left(\Phi_{\boldsymbol{u}}(\overline{d})\,|\,\mathrm{dom}\,(\psi)\right) = N\left(c(\widehat{x}) + \nabla c(\widehat{x})\overline{d}\,|\,\mathrm{dom}\,(h)\right) \times \{0\}$$

Suppose $v = (v_1, v_2) \in \mathrm{Null}\left(\nabla\Phi_{\boldsymbol{u}}(\overline{d})^\top\right) \cap N\left(\Phi_{\boldsymbol{u}}(\overline{d})\,|\,\mathrm{dom}\,(\psi)\right)$. Then $v_2 = 0$, and

$$v_1 \in \mathrm{Null}\left(\nabla c(\widehat{x})^\top\right) \cap N\left(c(\widehat{x}) + \nabla c(\widehat{x})\overline{d}\,|\,\mathrm{dom}\,(h)\right) \subset \mathrm{Null}\left(\nabla c(\widehat{x})^\top\right) \cap N\left(c(\widehat{x})\,|\,\mathrm{dom}\,(h)\right) = \{0\},$$

where the inclusion follows since $\left\langle v_1,\, \nabla c(\widehat{x})\overline{d}\right\rangle = 0$. ∎

Lemma 4.6.3. Let $\phi_{\boldsymbol{u}}$ be as in Definition 4.6.1, and suppose f satisfies (BCQ) at $\widehat{x}$. Consider the problem

$$(\mathcal{P}_{\phi_{\boldsymbol{u}}}) \qquad \underset{d}{\text{minimize}} \quad \phi_{\boldsymbol{u}}(d)$$

Then, the cone of non-ascent directions $D_{\phi_{\boldsymbol{u}}}(\overline{d})$ at any $\overline{d} \in \operatorname{dom}(\phi_{\boldsymbol{u}})$ is given by

$$D_{\phi_{\boldsymbol{u}}}(\overline{d}) = \left\{ \delta \;\middle|\; h'(c(\widehat{x}) + \nabla c(\widehat{x})\overline{d}; \nabla c(\widehat{x})\delta) + \overline{d}^{\top} \widehat{H}\delta \le 0 \right\}. \tag{4.71}$$

Moreover, the second-order necessary and sufficient conditions of Theorem 10 applied to $\phi_{\boldsymbol{u}}$ are

1. If $\phi_{\boldsymbol{u}}$ has a local minimum at $\overline{d}$, then $0 \in \widehat{H}\overline{d} + \nabla c(\widehat{x})^{\top}\partial h(c(\widehat{x}) + \nabla c(\widehat{x})\overline{d})$ and

   $$h''(c(\widehat{x}) + \nabla c(\widehat{x})\overline{d}; \nabla c(\widehat{x})\delta) + \delta^{\top}\widehat{H}\delta \ge 0,$$

   for all $\delta \in D_{\phi_{\boldsymbol{u}}}(\overline{d})$.

2. If $0 \in H\overline{d} + \nabla c(\widehat{x})^{\top}\partial h(c(\widehat{x}) + \nabla c(\widehat{x})\overline{d})$ and

   $$h''(c(\widehat{x}) + \nabla c(\widehat{x})\overline{d}; \nabla c(\widehat{x})\delta) + \delta^{\top}\widehat{H}\delta > 0,$$

   for all $\delta \in D_{\phi_{\boldsymbol{u}}}(\overline{d}) \setminus \{0\}$, then $\overline{d}$ is a strong local minimizer of $\phi_{\boldsymbol{u}}$.

Proof. Since (BCQ) is satisfied at all points $d \in \operatorname{dom}(\phi_{\boldsymbol{u}})$, the chain rule of Theorem 4 gives

$$\partial\phi_{\boldsymbol{u}}(d) = \widehat{H}d + \nabla c(\widehat{x})^{\top}\partial h(c(\widehat{x}) + \nabla c(\widehat{x})d),$$

$$\mathrm{d}\phi_{\boldsymbol{u}}(d)(\delta) = h'(c(\widehat{x}) + \nabla c(\widehat{x})d; \nabla c(\widehat{x})\delta) + d^{\top}\widehat{H}\delta,$$

which is (4.71). The set of Lagrange multipliers for $\phi_{\boldsymbol{u}}$ becomes

$$\begin{aligned} M_{\phi_{\boldsymbol{u}}}(d) &:= \operatorname{Null}\left(\nabla\Phi_{\boldsymbol{u}}(d)^{\top}\right) \cap \partial\psi(\Phi_{\boldsymbol{u}}(d)) \\ &= \operatorname{Null}\left(\begin{pmatrix}\nabla c(\widehat{x})^{\top} & \widehat{H}d\end{pmatrix}\right) \cap (\partial h(c(\widehat{x}) + \nabla c(\widehat{x})d) \times \{1\}), \end{aligned} \tag{4.72}$$

so that $\begin{pmatrix} y_1 & y_2 \end{pmatrix} \in M_{\phi_{\boldsymbol{u}}}(d) \iff \left\{ \widehat{H}d + \nabla c(\widehat{x})^\top y_1,\ y_1 \in \partial h(c(\widehat{x}) + \nabla c(\widehat{x})d),\ y_2 = 1. \right.$ The Lagrangian [4] is $L(d, y) := \langle y,\, \Phi_{\boldsymbol{u}}(d)\rangle - \psi^\star(y), \quad y = (y_1, y_2) \in \mathbb{R}^m \times \mathbb{R}$, with $\nabla^2(y\Phi_{\boldsymbol{u}})(d) = y_2\widehat{H}$. Then, from Theorem 10, for any $\delta \in \mathbb{R}^n$,

$$\psi''(\Phi_{\boldsymbol{u}}(d); \nabla\Phi_{\boldsymbol{u}}(d)\delta) + \max\left\{\left\langle \delta,\, \nabla^2(y\Phi_{\boldsymbol{u}})(d)\delta\right\rangle \,\middle|\, y \in M_{\phi_{\boldsymbol{u}}}(d)\right\} = h''(c(\widehat{x}) + \nabla c(\widehat{x})\overline{d}; \nabla c(\widehat{x})\delta) + \delta^\top \widehat{H}\delta.$$

□

The following lemma relates an active manifold at a solution to **P** to the directions of non-ascent for the model function Definition 4.6.1. It is an immediate consequence of Theorem 14, Lemma 4.4.9, and (4.71), and the proof is identical to Lemma 4.4.9.

Lemma 4.6.4 (Model non-ascent directions)**.** Let f be as in **P**, $\overline{x} \in \operatorname{dom}(f)$, $\overline{c} := c(\overline{x})$, $\overline{k} = |\mathcal{K}(\overline{c})|$, where $\mathcal{K}(\overline{c})$ are the active indices given in Definition 4.2.2. Let $(\widehat{x}, \widehat{y})$ and $\phi_{\boldsymbol{u}}$ be as in Definition 4.6.1, and let the active manifold $\mathcal{M}_{\overline{c}}$ be as in (4.25), with $\operatorname{dom}(h)$ given by the Rockafellar-Wets PLQ representation theorem. Suppose $0 = \widehat{H}\overline{d} + \nabla c(\widehat{x})^\top \overline{y}$, $c(\widehat{x}) + \nabla c(\widehat{x})\overline{d} \in \mathcal{M}_{\overline{c}}$, and $\overline{y} \in \operatorname{ri}\left(\partial h(c(\widehat{x}) + \nabla c(\widehat{x})\overline{d})\right)$. Then, $\phi_{\boldsymbol{u}}$ satisfies (SC) at $\overline{d}$ for $(\overline{y}, 1)$, and

if $\overline{k} \geq 2$, then, in the notation of Definition 4.4.3, $D_{\phi_{\boldsymbol{u}}}(\overline{d}) = \operatorname{Null}\left(A^\top \nabla c(\widehat{x})\right)$.

if $\overline{k} = 1$, then, $D_{\phi_{\boldsymbol{u}}}(\overline{d}) = \mathbb{R}^n$.

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