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# The Vanishing of the Brauer Group of a del Pezzo Surface of Degree 4

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**Abstract**

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of Degree 4

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A del Pezzo surface  $X$  of degree 4 over a field  $k$  can be thought of as the smooth complete intersection of 2 quadrics in  $\mathbb{P}^4$ . Arithmetic geometers are interested in computing the quotient of its Brauer group  $\text{Br } X / \text{Br}_0 X$ , where  $\text{Br}_0 X = \text{im}(\text{Br } k \rightarrow \text{Br } X)$ . Several algorithms have been implemented to compute this quotient; see [2], [30]. In [30], the algorithm relies on a specific arithmetic input related to the solvability of certain quadrics associated to the pencil determined by  $X$ . We explicitly construct a del Pezzo surface  $X$  of degree 4 over a field  $k$  such that  $H^1(k, \text{Pic } \overline{X}) \simeq \mathbb{Z}/2\mathbb{Z}$  while  $\text{Br } X / \text{Br } k$  is trivial. This proves that the algorithm to compute the Brauer group in [30] cannot be generalized in some cases.

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# DEDICATION

to my parents

## Chapter 1

## INTRODUCTION

The Brauer group of a field  $k$ , denoted by  $\text{Br } k$ , was defined by Richard Brauer in 1929. It classifies central simple algebras over a field  $k$  with respect to the Morita equivalence. Two central simple algebras  $\mathcal{A}$  and  $\mathcal{B}$  are said to be Morita equivalent over  $k$  if there exist two positive integers  $n$  and  $m$  such that  $\mathcal{A} \otimes M_n(k) \simeq \mathcal{B} \otimes M_m(k)$  as  $k$ -algebras. Moreover,  $\text{Br } k$  is isomorphic to the Galois cohomology group  $H^2(k, k_s^\times)$ ; we say this is the cohomological definition of the Brauer group. The cohomological definition of the Brauer group of a field can be generalized to define the Brauer group of a scheme  $X$  denoted by  $\text{Br } X$  as the étale cohomology  $H_{\text{ét}}^2(X, \mathbb{G}_m)$ , where  $\mathbb{G}_m$  is the sheaf of units on  $X$ . When  $X$  is projective, geometrically integral and smooth,  $\text{Br } X$  is realized concretely as the equivalence classes of Azumaya algebras over  $X$  (see Chapter 2). This has opened the door to many arithmetic and geometric applications of the Brauer group.

One application introduced by Manin in 1971 [19] is the Brauer-Manin obstruction to the Hasse principle over a global field  $k$ . Let  $\Omega_k$  denote the set of places on  $k$ . The existence of a  $k$ -point on  $X$  implies the existence of a  $k_v$ -point on  $X$  for all  $v \in \Omega_k$ . The class of varieties that satisfy the converse is said to satisfy the Hasse principle. The Hasse principle does not hold in general. Each element  $\mathcal{A} \in \text{Br } X$  determines a map

$$\begin{aligned} X(\mathbb{A}_k) &\rightarrow \mathbb{Q}/\mathbb{Z} \\ (x_v) &\mapsto (\mathcal{A}, (x_v)), \end{aligned}$$

where  $X(\mathbb{A}_k)$  is the set of adelic points. In [19], Manin defines the Brauer-Manin set  $X(\mathbb{A}_k)^{\text{Br}} = \bigcap_{\mathcal{A} \in \text{Br } X} X(\mathbb{A}_k)^{\mathcal{A}}$  where

$$X(\mathbb{A}_k)^{\mathcal{A}} = \{(x_v) \in X(\mathbb{A}_k) : (\mathcal{A}, (x_v)) = 0\}.$$

One can show that the Brauer-Manin set lies within the chain of inclusions

$$X(k) \subset X(\mathbb{A}_k)^{\text{Br}} \subset X(\mathbb{A}_k),$$

where  $X(k)$  is the set of rational points. So if  $X(\mathbb{A}_k)^{\text{Br}}$  is empty then  $X(k)$  is empty. If furthermore  $X(\mathbb{A}_k) \neq \emptyset$ , we say that  $X$  admits a Brauer-Manin obstruction to the Hasse principle. It is conjectured by Colliot-Thélène and Sansuc [3] that the Brauer-Manin obstruction is the only obstruction to the Hasse principle if  $X$  is geometrically rational.

Various theorems and algorithms have been implemented to compute the quotient of the Brauer group  $\text{Br } X / \text{im}(\text{Br } k \rightarrow \text{Br } X)$  for certain schemes. For example, such algorithms have been implemented for most cubic surfaces; See [4], [8] and [9]. Algorithms have also been implemented for some del Pezzo surfaces of degree 2 as in [6], and for most del Pezzo surfaces of degree 4 as in [2], and [30].

In this thesis we highlight the algorithm to compute  $\text{Br } X / \text{im}(\text{Br } k \rightarrow \text{Br } X)$  for a del Pezzo surface of degree 4 as in [30]. Furthermore, we prove that it cannot be generalized in some cases.

For the reader's convenience, we rewrite below the main theorem [30, Theorem 3.4] that leads to the algorithm. By embedding  $X$  anticanonically into  $\mathbb{P}^4$ , we view it as the intersection of two quadrics as in [31, Proposition 3.26]. The two quadrics  $Q$  and  $Q'$  define a pencil  $\{\lambda Q + \mu Q' : [\lambda, \mu] \in \mathbb{P}^1\}$ . By [31, Proposition 3.26], the pencil has five degenerate geometric fibers which are rank 4 quadrics. Let  $\mathcal{S}$  be the degree 5 subscheme of  $\mathbb{P}^1$  representing the degeneracy locus of the pencil. For every closed point  $T \in \mathcal{S}$ , denote by  $k(T)$  the residue field of  $T$  and by  $Q_T$  the corresponding quadric in the pencil. Let  $\epsilon_T$  be the discriminant of a smooth rank 4 quadric obtained by restricting  $Q_T$  to a hyperplane  $H_T$  in  $\mathbb{P}^4$  not containing the vertex of  $Q_T$ . By [31, Section 3.4.1], the square class of  $\epsilon_T$  does not depend on the choice of  $H_T$ . So we consider  $\epsilon_T$  as an element in  $k(T)/k(T)^{\times 2}$ .

We say that a subscheme  $\mathcal{T} \subset \mathcal{S}$  satisfies (\*) if it satisfies the conditions

$$(*) \quad \deg(\mathcal{T}) = 2, \quad \prod_{T \in \mathcal{T}} N_{k(T)/k}(\epsilon_T) \in k^{\times 2}, \quad \text{and } \epsilon_T \notin k(T)^{\times 2} \text{ for every } T \in \mathcal{T}.$$

**Theorem 1.0.1.** [30, Theorem 3.4] *If  $\text{Br } X \neq \text{Br}_0 X$  then there exists a subscheme  $\mathcal{T} \subset \mathcal{S}$  that satisfies (\*) and there exists a  $T \in \mathcal{S} - \mathcal{T}$  such that  $\epsilon_T \notin k(T)^{\times 2}$ . If  $\#\text{Br } X / \text{Br}_0 X = 4$  then there exists three degree 1 points  $\{T_0, T_1, T_2\} \subset \mathcal{S}$  such that each pair satisfies (\*).*

*If each quadric  $Q_T$  in the pencil associated to  $X$  corresponding to the points  $T \in \mathcal{S}$  of degree at most 2 has a smooth  $k(T)$ -point then the converse of the above statements hold.*

*When  $\#\text{Br } X / \text{Br}_0 X = 4$ , we have  $\text{Br } X / \text{Br}_0 X \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ .*  $\square$

The proofs of the above algorithms use the Hochschild-Serre spectral sequence

$$0 \rightarrow \text{Pic } X \rightarrow (\text{Pic } \overline{X})^{G_k} \rightarrow \text{Br } k \rightarrow \text{Br}_1 X \rightarrow \text{H}^1(k, \text{Pic } \overline{X}) \rightarrow \text{H}^3(k, \mathbb{G}_m),$$

where  $G_k$  is the absolute Galois group of  $k$  and  $\text{Br}_1 X = \ker(\text{Br } X \rightarrow \text{Br } \overline{X})$ , to establish the isomorphism

$$\frac{\text{Br}_1 X}{\text{Br}_0 X} \simeq \text{H}^1(k, \text{Pic } \overline{X})$$

under some arithmetic assumptions like the assumption of solvability of the quadrics in Theorem 1.0.1. Depending on the field  $k$  and on  $X$ , the map  $\text{H}^1(k, \text{Pic } \overline{X}) \rightarrow \text{H}^3(k, \mathbb{G}_m)$  might be non trivial as we prove for del Pezzo surfaces of degree 4 in this thesis. In other words, this proves that the arithmetic assumption in Theorem 1.0.1 is necessary; changing the arithmetic input of the algorithm leads to a different outcome. In particular, we prove the following theorem.

**Theorem 1.0.2.** *Let  $k = \mathbb{Q}^{\text{cycl}}(a, b, c)$  where  $a, b$  and  $c$  are independent transcendental elements. Let  $X$  be the del Pezzo surface of degree 4 in  $\mathbb{P}_k^4$  defined by the intersection of the following two quadrics*

$$Q : ax_0^2 + bx_1^2 + x_2^2 + cx_4^2 = 0$$

$$Q' : bcx_0^2 + x_1^2 + x_2^2 + ax_3^2 = 0.$$

*Then  $\text{H}^1(k, \text{Pic } \overline{X}) \simeq \mathbb{Z}/2\mathbb{Z}$  while  $\frac{\text{Br } X}{\text{Br } k}$  is trivial.*

Uematsu has done similar work for an affine diagonal quadric [27] and for a diagonal cubic surface [26]. In contrast to his proofs, our proof does not rely on computing the boundary

map  $d_2^{1,1} : H^1(k, \text{Pic } \overline{X}) \rightarrow H^3(k, \mathbb{G}_m)$  of the Hochschild-Serre spectral sequence. Instead it relies on the algorithm in [30, Section 4.1] and the work done by Harpaz [14]. In [14], Harpaz proves that for a specific example of an affine diagonal quadric  $U$  over a field  $F$ ,  $\text{Br } U / \text{Br } F$  vanishes while  $H^1(F, \text{Pic } \overline{U}) \simeq \mathbb{Z}/2\mathbb{Z}$ . In his argument, he base extends  $X$  to a field  $L$ ; over this field he shows that  $\text{Br } X_L / \text{Br } L \simeq \mathbb{Z}/2\mathbb{Z}$ . Finally, he proves that the generating class  $\mathcal{A}$  of  $\text{Br } X_L / \text{Br } L$  is not in the image of  $\text{Br } X / \text{Br } k$ . We carry a similar proof to Harpaz's argument; we base extend to the field  $L = k(\sqrt{a})$ . For the explicit computation of the class of the algebra  $\mathcal{A}$  that generates  $\text{Br } X_L / \text{Br } L$ , we use the algorithm in [30, Section 4.1].

**Outline:** Chapter 2 provides some background about Brauer groups and del Pezzo surfaces of degree 4. In Chapter 3, we explain and summarize the algorithm in [30] to compute  $\text{Br } X / \text{Br}_0 X$ . We prove the main theorem in Chapter 4.

**Notation:** Throughout this thesis, we use  $k$  to denote a field and  $X$  to denote a scheme. We denote by  $\overline{k}$  a fixed algebraic closure of  $k$  and by  $k_s$  the separable closure of  $k$  in  $\overline{k}$ . We denote by  $X_s$  the base change of  $X$  to  $k_s$  and by  $\overline{X}$  the base change of  $X$  to  $\overline{k}$ . Let  $G_k$  denote the absolute Galois group of  $k$ . Denote by  $H^i(k, A)$  the  $i$ -th group cohomology of the Galois group  $G_k$  and the  $G_k$ -module  $A$ .

## Chapter 2

# BACKGROUND

This chapter is divided into two main parts; the first part introduces the Brauer group of a field and the Brauer group of a scheme and the second part reviews some results about the geometry of a del Pezzo surface of degree 4.

### 2.1 Brauer groups

Brauer groups, both of a field and of a scheme, are an important tool for both geometric and arithmetic purposes. One main use of the Brauer group in arithmetic geometry is to explain the non existence of rational points on schemes [19]. In this section we define the Brauer group and review some of its properties.

#### 2.1.1 Brauer group of a field

We first define the Brauer group of a field concretely in terms of central simple algebras. Then we define the Brauer group in terms of Galois cohomology. For more details see [10].

**Definition 2.1.1.** Let  $k$  be a field. A  $k$ -algebra is central if its center is exactly the field  $k$ , and simple if it does not have any proper two sided ideals. A central simple algebra over  $k$  is a finite dimensional  $k$ -algebra that is central and simple.

**Example 2.1.2.** The ring of matrices  $M_n(k)$  where  $n$  is any positive integer is a central simple algebra.

**Example 2.1.3.** The Hamiltonion algebra  $\mathbb{H}$  is a central simple algebra over  $\mathbb{R}$  generated by the 4 elements  $\{1, i, j, ij\}$  that satisfy the multiplicative relations  $i^2 = -1$ ,  $j^2 = -1$ , and  $ji = -ij$ . Generalizations of  $\mathbb{H}$  are the quaternion algebras over any field  $k$  generated by

$\{1, i, j, ij\}$  that satisfy the multiplicative relations  $i^2 = a$ ,  $j^2 = b$ , and  $ji = -ij$  for some  $a, b \in k^\times$ . We denote such quaternion algebras by  $(a, b)$ .

**Example 2.1.4.** Let  $L$  be a degree  $n$  cyclic extension of  $k$ . Fix  $a \in k^\times$  and  $\sigma$  a generator of  $\text{Gal}(L/k)$ . Let  $L[y]_\sigma$  be the twisted polynomial ring which is defined as the set of elements in the polynomial ring  $L[y]$  with a non commutative multiplication extending the multiplication on  $L$  and such that  $yl = (\sigma l)y$  for every  $l \in L$ . We define the cyclic algebra denoted by  $(\sigma, b)$  as the following quotient

$$(\sigma, a) := \frac{L[y]_\sigma}{y^n - a}.$$

If  $k$  contains a primitive  $n$ -th root of unity then, by Kummer theory, any cyclic Galois extension of  $k$  is of the form  $k(\sqrt[n]{a})/k$  for some  $a \in k^\times$ . So in this case the cyclic algebras are  $(\sigma, b)$  where  $\sigma$  is a generator of  $\text{Gal}(k(\sqrt[n]{a})/k)$  and  $a, b \in k^\times$ . If  $\sigma \in \text{Gal}(k(\sqrt[n]{a})/k)$  is such that  $\sigma: \sqrt[n]{a} \mapsto \omega \sqrt[n]{a}$  then  $(\sigma, b) = (a, b)_n$  where  $(a, b)_n$  is the  $k$ -algebra given by the presentation

$$(a, b)_n = \{x, y : x^n = a, y^n = b, xy = \omega yx\}.$$

The algebras  $(a, b)_n$  are called symbols. In the special case when  $n = 2$  they coincide with the quaternion algebras defined in Example 2.1.3. When  $k$  contains a primitive  $n$ -th root of unity the symbols are of special importance as they are the building blocks of the Brauer group as we will see later in Theorem 2.1.16.

**Theorem 2.1.5.** [10, Theorem 2.2.1] *Let  $k$  be a field and  $\mathcal{A}$  be a finite dimensional  $k$ -algebra. Then  $\mathcal{A}$  is a central simple algebra if and only if there exists an integer  $n > 0$  and a finite field extension  $K/k$  so that  $\mathcal{A} \otimes K$  is isomorphic to the matrix ring  $M_n(K)$ .  $\square$*

**Definition 2.1.6.** Let  $\mathcal{A}$  be a central simple algebra over  $k$ . A field extension  $K/k$  such that  $\mathcal{A} \otimes K \simeq M_n(K)$  for some positive integer  $n$  is called a splitting field of  $\mathcal{A}$ .

**Example 2.1.7.** The quaternion algebra  $(a, b)$  defined in Example 2.1.3 admits  $k(\sqrt{a})$  as a splitting field.

By Theorem 2.1.5, it follows that the dimension of a central simple algebra over  $k$  is a square. So we may define the degree as follows.

**Definition 2.1.8.** Let  $\mathcal{A}$  be a central simple algebra over a field  $k$ . The degree of  $\mathcal{A}$  is  $\sqrt{\dim_k(\mathcal{A})}$ .

**Example 2.1.9.** The cyclic algebra  $(\sigma, a)$  where  $\sigma$  is a generator of  $\text{Gal}(L/k)$ , as in Example 2.1.4, has degree  $[L : k]$ .

Given two central simple  $k$ -algebras  $\mathcal{A}$  and  $\mathcal{B}$ , we consider their tensor product  $\mathcal{A} \otimes \mathcal{B}$ . By applying Theorem 2.1.5 in both directions we deduce that  $\mathcal{A} \otimes \mathcal{B}$  is a central simple algebra as well. Let  $\mathcal{A}^{\text{opp}}$  be the opposite algebra of  $\mathcal{A}$  which is defined to be the same  $k$ -vector space as  $\mathcal{A}$  with multiplication defined in the reverse order to that of  $\mathcal{A}$ . Consider the  $k$ -linear map  $\phi: \mathcal{A} \otimes \mathcal{A}^{\text{opp}} \rightarrow \text{End}(\mathcal{A})$  given by  $\sum a_i \otimes b_i \mapsto (x \mapsto \sum a_i x b_i)$ . The map  $\phi$  is nonzero and  $\mathcal{A} \otimes \mathcal{A}^{\text{opp}}$  is simple. So  $\phi$  is injective. By dimension reasons,  $\phi$  is an isomorphism.

**Definition 2.1.10.** Let  $k$  be a field. We say that the central simple algebras  $\mathcal{A}$  and  $\mathcal{B}$  over  $k$  are Brauer or Morita equivalent, denoted  $\mathcal{A} \sim \mathcal{B}$ , if there exists two positive integers  $m$  and  $n$  such that  $\mathcal{A} \otimes M_n(k) \simeq \mathcal{B} \otimes M_m(k)$ . By the above discussion we may define the Brauer group of  $k$ , denoted by  $\text{Br } k$ , as

$$\text{Br } k := \frac{\{\text{central simple algebras}/k\}}{\sim}$$

with the tensor product of algebras as its operation. The neutral element of  $\text{Br } k$  is the Brauer class of the matrix algebra. Moreover by properties of the tensor product of central simple algebras, the tensor product is commutative on the Brauer classes. So  $\text{Br } k$  is an abelian group.

**Definition 2.1.11.** Let  $K/k$  be a finite Galois extension. We define  $\text{Br}(K/k)$  to be the subgroup of  $\text{Br } k$  that consists of all central simple algebras split by  $K$ . By Theorem 2.1.5,  $\text{Br } k$  is the union of all  $\text{Br}(K/k)$  over all finite Galois extensions of  $K/k$ .

Another way of characterizing Brauer classes is using division algebras.

**Theorem 2.1.12.** *(Wedderburn)[10, Theorem 2.1.3] Let  $k$  be a field and  $\mathcal{A}$  a finite dimensional simple  $k$ -algebra. There exists a positive integer  $n$  and a  $k$ -division algebra  $D$  such that  $\mathcal{A} \simeq M_n(D)$ . Furthermore  $D$  is uniquely determined up to isomorphism.  $\square$*

**Example 2.1.13.** Let  $k$  be an algebraically closed field. By Theorem 2.1.12 and the fact that there is no finite dimensional division algebra containing  $k$  other than  $k$ , any finite dimensional simple  $k$ -algebra  $\mathcal{A}$  is isomorphic to  $M_n(k)$  for some positive integer  $n$ . Hence  $\text{Br } k = 0$ .

**Example 2.1.14.** Let  $k$  be a finite field. Wedderburn's Little Theorem states that every finite domain is a field. By Wedderburn's Little Theorem and by Theorem 2.1.12,  $\text{Br } k$  is trivial.

**Example 2.1.15.** The only division algebras over  $\mathbb{R}$  are  $\mathbb{R}$ ,  $\mathbb{H}$ , and  $\mathbb{C}$ . Since  $\mathbb{C}$  is not central, and by Theorem 2.1.12,  $\text{Br } \mathbb{R} \simeq \mathbb{Z}/2\mathbb{Z}$  and is generated by  $\mathbb{H}$ .

A central result in the theory of Brauer groups of fields is the Merkurjev-Suslin Theorem.

**Theorem 2.1.16.** *(Merkurjev-Suslin)[10, Theorem 2.5.7]*

*Assume that  $k$  contains a primitive  $m$ -th root of unity. Then a central simple  $k$ -algebra whose class has order dividing  $m$  in  $\text{Br } k$  is Brauer equivalent to the tensor product of cyclic algebras of degree  $m$  of the form  $(a_i, b_i)_m$  for some  $a_i, b_i \in k^\times$ .*

Definition 2.1.10 of the Brauer group is in terms of central simple algebras. Furthermore, there is a cohomological definition of the Brauer group of a field  $k$  given by the Galois cohomology  $H^2(k, k_s^\times)$ . The first step in linking both definitions is the following theorem that follows from Galois descent.

**Theorem 2.1.17.** *Let  $k$  be a field and  $K$  be an extension of  $k$ . Let  $\text{CSA}_K(n)$  be the isomorphism classes of central simple  $k$ -algebras of degree  $n$  split by  $K$ . Let  $\text{PGL}_n(K)$  be the*

invertible  $n \times n$  matrices of determinant 1. There is a base point preserving bijection

$$\text{CSA}_K(n) \longleftrightarrow \text{H}^1(\text{Gal}(K/k), \text{PGL}_n(K))$$

that maps an algebra  $\mathcal{A}$  to the cocycle that associates to  $\sigma \in \text{Gal}(K/k)$  the map  $g^{-1} \circ \sigma(g) \in \text{Aut}(M_n(K)) \simeq \text{PGL}_n(K)$  where  $g$  is an isomorphism  $\mathcal{A} \otimes K \simeq M_n(K)$ . The base points of  $\text{CSA}_K(n)$  and  $\text{H}^1(\text{Gal}(K/k), \text{PGL}_n(K))$  are the matrix algebra  $M_n(k)$  and the trivial cocycle respectively.

*Proof.* Here we sketch the proof; for more details see [10, Section 2.3]. Given a vector space  $V$  with an associated bilinear form  $\Phi$ , we say that  $(W, \Psi)$  is a twist of  $(V, \Phi)$  over  $K$  if  $W_K \simeq V_K$  and this isomorphism respects the associated bilinear forms. Let  $TF(V, \Phi)$  denote the pointed set of all twists of  $V$  with base point  $(V, \Phi)$ . By Theorem 2.1.5,  $\text{CSA}_K(n)$  are the twists of  $M_n(k)$  over  $K$  preserving the tensor product. By Galois descent [10, Theorem 2.3.3], we have a base point preserving bijection  $\text{CSA}_K(n) = TF(M_n(k), \otimes) \longleftrightarrow \text{H}^1(\text{Gal}(K/k), \text{Aut}_K(M_n(K)))$  that maps an algebra  $\mathcal{A}$  to the cocycle that associates to  $\sigma \in \text{Gal}(K/k)$  the map  $g^{-1} \circ \sigma(g) \in \text{Aut}(M_n(K))$  where  $g$  is an isomorphism  $\mathcal{A} \otimes K \simeq M_n(K)$ . Choosing a different isomorphism  $\mathcal{A} \otimes K \simeq M_n(K)$  results in an equivalent cocycle. Moreover,  $\text{Aut}(M_n(K)) = \text{PGL}_n(K)$  because the map  $\text{GL}_n(K) \rightarrow \text{Aut}(M_n(K))$  sending  $C$  to  $M \mapsto CMC^{-1}$  is surjective with kernel the subgroup of scalar matrices. Therefore there is a point preserving bijection  $\text{CSA}_K(n) \longleftrightarrow \text{H}^1(\text{Gal}(K/k), \text{PGL}_n(K))$ .  $\square$

Note that the tensor product on central simple algebras induces the product

$$\text{CSA}_K(n) \times \text{CSA}_K(m) \rightarrow \text{CSA}_K(mn)$$

for  $n$  and  $m$  two positive integers. By Theorem 2.1.17, this product induces a product on the Galois cohomology

$$\text{H}^1(\text{Gal}(K/k), \text{PGL}_n(K)) \times \text{H}^1(\text{Gal}(K/k), \text{PGL}_m(K)) \rightarrow \text{H}^1(\text{Gal}(K/k), \text{PGL}_{mn}(K)).$$

Define the maps

$$\lambda_{mn} : \text{CSA}_n(K) \rightarrow \text{CSA}_{nm}(K)$$

that send  $\mathcal{A} \mapsto \mathcal{A} \otimes M_m(k)$  for  $n$  and  $m$  positive integers. Using Wedderburn's Theorem 2.1.12 one can show that  $\lambda_{mn}$  are injective for all positive integers  $m, n$ ; see [10, Lemma 2.4.5]. By injectivity of the  $\lambda_{nm}$  for all positive integers  $m, n$ , the equivalence induced by the direct limit on  $\varinjlim \text{CSA}_K(n)$  via the maps  $\lambda_{nm}$  is the same as the Brauer equivalence of central simple algebras defined in Definition 2.1.10. Hence  $\text{Br}(K/k) = \varinjlim \text{CSA}_K(n)$ . By Theorem 2.1.17 the map corresponding to  $\lambda_{mn}$  on the cohomology

$$\lambda_{mn} : \text{H}^1(\text{Gal}(K/k), \text{PGL}_n(K)) \rightarrow \text{H}^1(\text{Gal}(K/k), \text{PGL}_{nm}(K))$$

is induced by the inclusion  $\text{PGL}_n \rightarrow \text{PGL}_{mn}$  which sends each matrix  $M$  to the block matrix with  $M$  as its diagonal blocks  $m$  times. Let  $\text{H}^1(\text{Gal}(K/k), \text{PGL}_\infty(K)) = \varinjlim \text{H}^1(\text{Gal}(K/k), \text{PGL}_n(K))$ . By Theorem 2.1.17 and by the equality  $\text{Br}(K/k) = \varinjlim \text{CSA}_K(n)$ , we deduce that

$$\text{Br}(K/k) \simeq \text{H}^1(\text{Gal}(K/k), \text{PGL}_\infty(K)).$$

Let  $\text{H}^1(k, \text{PGL}_\infty)$  be the inverse limit  $\varprojlim \text{H}^1(\text{Gal}(K/k), \text{PGL}_\infty(K))$  over all finite Galois extensions  $K/k$ . By the previous statement, by  $\text{Br}(K/k) \simeq \text{H}^1(\text{Gal}(K/k), \text{PGL}_\infty(K))$  and by the fact that  $\text{Br } k$  is the union of  $\text{Br}(K/k)$  for all finite Galois extensions  $K/k$ , we get that  $\text{Br}(k) \simeq \text{H}^1(k, \text{PGL}_\infty)$ .

So we have

$$\text{Br}(K/k) \simeq \text{H}^1(\text{Gal}(K/k), \text{PGL}_\infty(K)), \text{ and } \text{Br}(k) \simeq \text{H}^1(k, \text{PGL}_\infty).$$

**Theorem 2.1.18.** [10, Theorem 4.4.7] *Let  $k$  be a field,  $K/k$  a finite Galois extension of  $k$  and  $k_s$  a separable closure of  $k$ . There exists natural isomorphisms of abelian groups*

$$\text{Br}(K/k) \simeq \text{H}^2(\text{Gal}(K/k), K^\times), \text{ and } \text{Br } k \simeq \text{H}^2(k, k_s^\times).$$

*Proof.* Here we sketch the proof. For more details see [10, Lemma 4.4.3], [10, Lemma 4.4.4], and [10, Theorem 4.4.5]. By the isomorphism  $\text{Br}(K/k) \simeq \text{H}^1(\text{Gal}(K/k), \text{PGL}_\infty(K))$ , it is enough to prove that  $\text{H}^1(\text{Gal}(K/k), \text{PGL}_\infty(K)) \simeq \text{H}^2(\text{Gal}(K/k), K^\times)$  as groups with the group operation on  $\text{H}^1(\text{Gal}(K/k), \text{PGL}_\infty(K))$  that induced by the tensor product of central

simple algebras for the first isomorphism. The second isomorphism follows by taking the union over all finite Galois extensions.

The long exact sequence of cohomology associated to the sequence

$$1 \rightarrow K^\times \rightarrow \mathrm{GL}_m(K) \rightarrow \mathrm{PGL}_m(K) \rightarrow 1$$

yields a map  $\delta_m : \mathrm{H}^1(\mathrm{Gal}(K/k), \mathrm{PGL}_m(K)) \rightarrow \mathrm{H}^2(\mathrm{Gal}(K/k), K^\times)$  for every  $m$ . The boundary map  $\delta_m$  takes every 1-cocycle  $\sigma \rightarrow c_\sigma$  to a 2-cocycle  $a_{\sigma,\tau} = b_\sigma \sigma(b_\tau) b_{\sigma\tau}^{-1}$  where  $b_\sigma$  is an invertible matrix that is a lift of  $c_\sigma \in \mathrm{PGL}_m(K)$  to  $\mathrm{GL}_m(K)$ . So  $a_{\sigma,\tau} = \mu_{\sigma,\tau} I_m$  where  $\mu_{\sigma,\tau} \in K^\times$  and  $I_m$  is the identity matrix. Changing the image of a 1-cocycle  $\sigma \rightarrow c_\sigma$  in the following diagram

$$\begin{array}{ccc} \mathrm{H}^1(\mathrm{Gal}(K/k), \mathrm{PGL}_m(K)) & \xrightarrow{\delta_m} & \mathrm{H}^2(\mathrm{Gal}(K/k), K^\times) \\ \downarrow \lambda_{mn} & & \downarrow id \\ \mathrm{H}^1(\mathrm{Gal}(K/k), \mathrm{PGL}_{mn}(K)) & \xrightarrow{\delta_{mn}} & \mathrm{H}^2(\mathrm{Gal}(K/k), K^\times), \end{array}$$

we get that the image by  $\delta_m$  is the 2-cocycle  $a_{\sigma,\tau} = \mu_{\sigma,\tau} I_m$  and the image by  $\delta_{mn} \circ \lambda_{mn}$  is  $\mu_{\sigma,\tau} I_{mn}$ . So the above diagram commutes. Taking the direct limit with respect to  $\lambda_{mn}$  we get a map

$$\delta_\infty : \mathrm{H}^1(\mathrm{Gal}(K/k), \mathrm{PGL}_\infty(K)) \rightarrow \mathrm{H}^2(\mathrm{Gal}(K/k), K^\times).$$

The image under  $\delta_\infty$  of the product of two cocycles whose images are represented by the 2-cocycles say  $a_{\sigma,\tau} = \mu_{\sigma,\tau} I_m$  and  $a'_{\sigma,\tau} = \nu_{\sigma,\tau} I_n$  is the product  $\mu_{\sigma,\tau} \nu_{\sigma,\tau}$ . Therefore the map  $\delta_\infty$  is a group homomorphism. Furthermore, by Hilbert Theorem 90,  $\mathrm{H}^1(\mathrm{Gal}(K/k), \mathrm{GL}_m(K))$  is trivial. So each map  $\delta_m$  is injective. Hence  $\delta_\infty$  is injective. By the proof of [10, Theorem 4.4.5] the map  $\delta_n$  is surjective when  $n$  is the degree of the extension  $K/k$ . So  $\delta_\infty$  is an isomorphism.  $\square$

**Corollary 2.1.19.** *Let  $k$  be a field. The Brauer group  $\mathrm{Br} k$  is a torsion group.*

*Proof.* Let  $K/k$  be a finite extension of degree  $n$ . The map

$$\mathrm{Cor} \circ \mathrm{Res} : \mathrm{H}^1(\mathrm{Gal}(K/k), \mathrm{PGL}_\infty) \rightarrow \mathrm{H}^1(\mathrm{Gal}(K/k), \mathrm{PGL}_\infty)$$

is multiplication by  $n$  and the zero map simultaneously. Hence  $\text{Br}(K/k)$  is a torsion group with every element of order dividing  $n$ . Then  $\text{Br } k$  which is the union of all groups  $\text{Br}(K/k)$  is also a torsion group.  $\square$

**Corollary 2.1.20.** *For a cyclic Galois extension  $K/k$ , we have*

$$\text{Br}(K/k) \simeq \frac{k^\times}{N_{K/k}(K^\times)}.$$

*Proof.* Let  $\sigma$  be the generator of  $\text{Gal}(K/k) := G$ . We have a free resolution

$$\dots \xrightarrow{N} \mathbb{Z}[G] \xrightarrow{\sigma-1} \mathbb{Z}[G] \xrightarrow{N} \mathbb{Z}[G] \xrightarrow{\sigma-1} \mathbb{Z}[G] \rightarrow \mathbb{Z} \rightarrow 0.$$

Using this resolution we get that  $H^2(\text{Gal}(K/k), k^\times) = \frac{k^\times}{N_{K/k}(K^\times)}$ . By Theorem 2.1.18 we deduce that  $\text{Br}(K/k) \simeq \frac{k^\times}{N_{K/k}(K^\times)}$ .  $\square$

*Remark 2.1.21.* The explicit isomorphism in 2.1.20 from right to left maps  $b \in k^\times$  to the class of the cyclic algebra  $(\sigma, b)$  (see [10, Corollary 4.7.8] for the proof).

### 2.1.2 Brauer group of a scheme

In this section we generalize the definition of the Brauer group of a field to the Brauer group of a scheme  $X$  and we state some important tools used to compute it. For the remainder of this section we consider a projective, geometrically integral, smooth scheme  $X$  over a field  $k$ ; we say  $X$  is nice.

**Definition 2.1.22.** The Brauer group of  $X$  denoted by  $\text{Br } X$  is the second étale cohomology  $H_{\text{ét}}^2(X, \mathbb{G}_m)$  where  $\mathbb{G}_m$  is the sheaf of units on  $X$ .

**Definition 2.1.23.** An Azumaya algebra  $\mathcal{A}$  on  $X$  is a coherent  $\mathcal{O}_X$ -sheaf of algebras on  $X$  such that  $\mathcal{A}_x \neq 0$  for all  $x \in X$  and  $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{O}_{U_i} \simeq M_{r_i}(\mathcal{O}_{U_i})$  for some étale cover  $\{U_i\}_i$  of  $X$  and  $\{r_i\}_i \subset \mathbb{Z}$ . We say that two Azumaya algebras  $\mathcal{A}$  and  $\mathcal{A}'$  on  $X$  are Morita equivalent if there exists locally free coherent  $\mathcal{O}_X$ -modules  $\mathcal{E}$  and  $\mathcal{E}'$  of positive rank on each  $x \in X$  such that  $\mathcal{A} \otimes_{\mathcal{O}_X} \text{End}_{\mathcal{O}_X}(\mathcal{E}) \simeq \mathcal{A}' \otimes_{\mathcal{O}_X} \text{End}_{\mathcal{O}_X}(\mathcal{E}')$ .

We define the Azumaya Brauer group to be the group of equivalence classes of Azumaya algebra. Note that this reduces to our previous definition of  $\text{Br } k$  when  $X = \text{Spec } k$  for a field  $k$ . For a nice scheme  $X$ , by a result by Gabber, [7], we get that the Azumaya Brauer group is isomorphic to the Brauer group. This gives a more concrete description of the Brauer group of a nice scheme. There are two main tools to compute  $\text{Br } X$  which are the residue sequence and the Hochschild-Serre spectral sequence; we recall both tools in the the first part of the remainder of this section.

We start by constructing the residue maps associated to a scheme; we first define the residue map associated to a prime divisor. Let  $D$  be a prime Weil divisor on  $X$  and  $x$  the corresponding codimension one point. Denote by  $v_x$  the valuation induced by  $x$ ,  $R$  the associated discrete valuation ring inside its field of fractions, and  $k(x)$  the associated residue field. We replace  $R$  by its completion and we denote by  $K$  the field of fractions of the completion of  $R$ . Note that the valuation  $v_x$  uniquely extends to a valuation  $v$  on the maximal unramified extension  $K_{un}$  of  $K$ . This valuation induces a homomorphism  $f_1: \text{H}^2(\text{Gal}(K_{un}/K), K_{un}^\times) \rightarrow \text{H}^2(\text{Gal}(K_{un}/K), \mathbb{Z})$ . By [24, Theorem 10], each algebra in  $\text{Br } K$  is split by  $K_{un}$ . So  $\text{Br}(K_{un}/K) = \text{Br } K$ . Hence  $f_1$  can be thought as a homomorphism from  $\text{Br } K$ . Moreover, the long exact sequence on cohomology associated to the short exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z}$$

induces an isomorphism  $f_2: \text{H}^2(\text{Gal}(K_{un}/K), \mathbb{Z}) \xrightarrow{\sim} \text{H}^1(\text{Gal}(K_{un}/K), \mathbb{Q}/\mathbb{Z})$ . Composing the homomorphisms  $f_1$  and  $f_2$ , and using the fact that  $\text{Gal}(K_{un}/K) \simeq G_{k(x)}$  we get a map

$$f_2 \circ f_1: \text{Br } K \rightarrow \text{H}^1(k(x), \mathbb{Q}/\mathbb{Z})$$

called the residue map associated to  $x$  and denoted by  $\partial_x$ .

**Proposition 2.1.24.** *[12, Proposition 2.1] Let  $D$  be a prime Weil divisor on  $X$  and  $x$  the associated codimension 1 point. Let  $R$  be the completion of the discrete valuation ring associated to  $v_x$ ,  $K$  its field of fractions, and  $k(x)$  the associated residue field. We have an*

exact sequence

$$0 \rightarrow \text{Br } R \rightarrow \text{Br } K \xrightarrow{\partial_x} \text{H}^1(k(x), \mathbb{Q}/\mathbb{Z}),$$

where  $\partial_x$  is the residue map defined before, after excluding the  $p$ -primary parts of all groups if  $k(x)$  is imperfect of characteristic  $p$ .

Now we consider all such residue maps. Denote by  $X^{(1)}$  the set of codimension one points on  $X$ .

**Theorem 2.1.25.** (Residue sequence)[13, 11, 12]

For each  $x \in X^{(1)}$  the group homomorphisms  $\partial_x: \text{Br } k(X) \rightarrow \text{H}^1(k(x), \mathbb{Q}/\mathbb{Z})$  defined above satisfy the exact sequence

$$0 \rightarrow \text{Br } X \rightarrow \text{Br } k(X) \xrightarrow{\oplus \partial_x} \bigoplus_{x \in X^{(1)}} \text{H}^1(k(x), \mathbb{Q}/\mathbb{Z})$$

after excluding the  $p$ -primary parts of all groups if  $\dim X \leq 1$  and  $k(x)$  is imperfect of characteristic  $p$  for some  $x$ , or if  $\dim X \geq 2$  and  $k(x)$  has characteristic  $p$  for some  $x$ .  $\square$

The elements of  $\text{Br } X$ , i.e., the algebras in  $\text{Br } k(X)$  that are in the kernel of every residue map  $\partial_x$  where  $x \in X^{(1)}$  are called unramified. The image of a cyclic algebra  $(\sigma, b)$  of degree  $n$  under  $\partial_x$  in  $\text{H}^1(k(x), \mathbb{Q}/\mathbb{Z}) = \text{Hom}(\text{Gal}(k(x)_s/k(x)), \mathbb{Q}/\mathbb{Z})$  has kernel  $\text{Gal}(k(x)_s/L)$  where  $L$  is a degree  $n$  cyclic Galois extension of  $k(x)$ . So  $\partial_x((\sigma, b))$  is determined by a choice of a generator of  $\text{Gal}(L/k(x))$  that is mapped to  $\frac{1}{n} + \mathbb{Z}$ . In the following theorem we give an explicit description of  $\partial_x((a, b)_m)$  where  $(a, b)_m$  is a symbol algebra of degree  $m$  assuming  $k$  is separably closed. The proof uses Milnor-Theory and can be found in [10, 7.5.1].

**Theorem 2.1.26.** [10, 7.5.1] Let  $k$  be separably closed field. Consider  $(a, b)_m$  a symbol algebra of degree  $m$  in  $\text{Br } k(X)$  where  $a, b \in k(X)^\times$ . Let  $x \in X^{(1)}$ . The image of  $(a, b)_m$  by  $\partial_x$  has kernel  $\text{Gal}(k(x)_s/L)$  where  $L = k(x)(\sqrt[m]{a^{v_x(b)}b^{-v_x(a)}})$ . So  $\partial_x((a, b)_m)$  is determined by a choice of a generator of  $\text{Gal}(L/k(x))$  that is mapped to  $\frac{1}{m} + \mathbb{Z}$ .  $\square$

**Example 2.1.27.** Consider the projective line  $\mathbb{P}_{\mathbb{Q}}^1$ . By Theorem 2.1.25 we have an exact sequence

$$0 \rightarrow \text{Br } \mathbb{P}_{\mathbb{Q}}^1 \rightarrow \text{Br } \mathbb{Q}(t) \xrightarrow{\oplus \partial_x} \bigoplus_{x \in \mathbb{P}^1(1)} \text{H}^1(\mathbb{Q}(x), \mathbb{Q}/\mathbb{Z}).$$

Consider the quaternion algebra  $(2, t) \in \mathbb{Q}(t)$  and the codimension one point  $x$  associated to  $t = 0$ . The field extension  $L = \mathbb{Q}(\sqrt{2^{v_x(t)}t^{-v_x(2)}}) = \mathbb{Q}(\sqrt{2})$  is a non trivial extension of  $\mathbb{Q}$ . Hence by Theorem 2.1.26,  $\partial_x((2, t))$  is non trivial. Thus  $(2, t)$  is ramified at  $x$ .

Another tool for computing the Brauer group relies on the Hochschild-Serre spectral sequence. For background about spectral sequences see [21, Appendix B]. A spectral sequence  $E_2^{p,q} \implies L^{p+q}$  yields an exact sequence

$$0 \rightarrow E^{1,0} \rightarrow L^1 \rightarrow E^{0,1} \rightarrow E^{2,0} \rightarrow \ker(L^2 \rightarrow E^{0,2}) \rightarrow E^{1,1} \rightarrow E^{3,0}.$$

**Theorem 2.1.28.** (*Hochschild-Serre*) *Let  $X$  be a nice scheme. Denote by  $G$  the absolute Galois group  $\text{Gal}(k_s/k)$ . There exists an exact sequence*

$$0 \rightarrow \text{Pic } X \rightarrow (\text{Pic } X_s)^G \rightarrow \text{Br } k \rightarrow \text{Br}_1 X \rightarrow \text{H}^1(k, \text{Pic } X_s) \rightarrow \text{H}^3(k, \mathbb{G}_m)$$

where  $X_s$  is the base change of  $X$  to the separable closure of  $k$ , and  $\text{Br}_1 X$  is the kernel of  $\text{Br } X \rightarrow \text{Br } X_s$ .

*Proof.* Here we give a sketch of the proof. Let  $G$  be the absolute Galois group of  $k$ . The Hochschild-Serre spectral sequence is

$$\text{H}^p(G, \text{H}_{\text{ét}}^q(X_s, \mathbb{G}_m)) \implies \text{H}_{\text{ét}}^{p+q}(X, \mathbb{G}_m);$$

see [23, Theorem 6.4.5] for the proof. By the discussion above the theorem, we get an exact sequence

$$\begin{aligned} 0 \rightarrow \text{H}^1(G, \text{H}_{\text{ét}}^0(X_s, \mathbb{G}_m)) \rightarrow \text{H}_{\text{ét}}^1(X, \mathbb{G}_m) \rightarrow \text{H}^0(G, \text{H}_{\text{ét}}^1(X_s, \mathbb{G}_m)) \rightarrow \text{H}^2(G, \text{H}_{\text{ét}}^0(X_s, \mathbb{G}_m)) \\ \rightarrow \ker(\text{H}_{\text{ét}}^2(X, \mathbb{G}_m) \rightarrow \text{H}^0(G, \text{H}_{\text{ét}}^2(X_s, \mathbb{G}_m))) \rightarrow \text{H}^1(G, \text{H}_{\text{ét}}^1(X, \mathbb{G}_m)) \rightarrow \text{H}^3(G, \text{H}_{\text{ét}}^0(X_s, \mathbb{G}_m)). \end{aligned}$$

We then apply the chomological definitions of the Brauer group of a field and a scheme, Hilbert theorem 90, and the facts that  $\text{H}_{\text{ét}}^0(X, \mathbb{G}_m) = k^\times$  by [23, Proposition 2.2.22] and  $\text{H}_{\text{ét}}^1(X, \mathbb{G}_m) = \text{Pic } X$  by [23, Proposition 6.6.1] to get the required exact sequence.  $\square$

*Remark 2.1.29.* In the case when the base field  $k$  is a local or a global field we have  $H^3(k, \mathbb{G}_m) = 0$ ; see [22, Corollary 7.2.2, Proposition 8.3.11(iv)] for the proofs. So over a local or global field  $k$  we have an isomorphism

$$\mathrm{Br} X / \mathrm{Br}_0 X \simeq H^1(k, \mathrm{Pic} X_s)$$

where  $\mathrm{Br}_0 X = \mathrm{im}(\mathrm{Br} k \rightarrow \mathrm{Br} X)$ .

## 2.2 Geometry of a del Pezzo surface of degree 4

The goal of this section is to review some geometric facts about a del Pezzo surface of any degree. For more details see [29]. We then prove some special facts about a del Pezzo surface of degree 4. Furthermore, we compute the Picard group of a del Pezzo surface of degree 4 and we study the Galois action on the Picard group.

Recall that, by [15, Theorem 1.1], on a surface there exists a pairing of divisors

$$(\cdot, \cdot): \mathrm{Pic} X \times \mathrm{Pic} X \rightarrow \mathbb{Z}$$

that is bilinear. A property that it satisfies is that it extends the intersection multiplicity of two nonsingular curves intersecting transversally; the intersection multiplicity of two such curves is the number of points in their intersection.

**Definition 2.2.1.** Let  $X$  be a nice surface over a field  $k$ . We say that  $X$  is a del Pezzo surface over  $k$  if  $X$  admits an ample anticanonical divisor  $-K_X$ . We define the degree of a del Pezzo surface as  $d := (K_X, K_X)$ .

We say that  $r$  points on  $\mathbb{P}^2$  are in general position if no 3 lie on a line, no 6 lie on a conic, and no 8 lie on a singular cubic with a point on the singularity.

**Theorem 2.2.2.** [29, Theorem 1.6] *Let  $X$  be a del Pezzo surface of degree  $d$  over a field  $k$ . Then  $X_s$  is isomorphic to the blowup of  $\mathbb{P}_{k^s}^2$  at  $9 - d$  points in general position where  $0 \leq d \leq 8$ , or to  $\mathbb{P}_{k^s}^1 \times \mathbb{P}_{k^s}^1$ . In the latter case  $d = 8$ .*

**Proposition 2.2.3.** [5, Proposition 1.7] *Any birational morphism  $f: X \rightarrow Y$  where  $X$  and  $Y$  are smooth projective surfaces over a separably closed field  $k$  factors as*

$$X = X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_r = Y,$$

where each map  $X_i \rightarrow X_{i+1}$  is a blow-up of a  $k$ -closed point of  $X_{i+1}$ .  $\square$

We say that a curve  $C$  on a surface is exceptional if its self intersection,  $(C, C)$ , is  $-1$ . We say that a  $k$ -surface is minimal if there it does not admit non intersecting exceptional curves that we can blow-down simultaneously to get a birational map to another surface.

**Proposition 2.2.4.** [5, Theorem 1.3] *The minimal smooth projective rational surfaces over a separably closed field are  $\mathbb{P}^2$  and the Hirzebruch surfaces  $\mathbb{F}_n := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$  where either  $n = 0$  or  $n \geq 2$ .*  $\square$

**Lemma 2.2.5.** *Let  $X$  be del Pezzo surface of degree  $d$ . Then every geometrically irreducible curve with negative self intersection is exceptional.*

*Proof.* The adjunction formula applied on  $C$  is

$$2p_a(C) - 2 = (C, C) - (C, -K_X).$$

The arithmetic genus of  $C$  satisfies  $p_a(C) \geq 0$  since  $C$  is irreducible. On the other hand,  $(C, C) < 0$  by assumption, and since  $-K_X$  is ample  $(C, -K_X) > 0$ . The only way this can happen is when  $p_a(C) = 0$  and  $(C, C) = (C, K_X) = -1$ . Hence  $C$  is exceptional.  $\square$

*proof of Theorem 2.2.2.* This proof can be found in [29, Theorem 1.6]; we restate it for the reader's convenience. We consider  $Y$  a minimal model of  $X_s$  and the corresponding birational morphism  $f: X_s \rightarrow Y$ . By Proposition 2.2.3,  $f$  can be factored as

$$X_s = X_{(0)} \rightarrow X_{(1)} \rightarrow \cdots \rightarrow X_{(r)} = Y.$$

By Proposition 2.2.4 we only need to consider two cases:  $Y = \mathbb{P}_{k_s}^2$ , and  $Y = (\mathbb{F}_n)_{k_s}$  for  $n = 0$  or  $n \geq 2$ .

$\mathbf{Y} = \mathbb{P}_{k_s}^2$ : No point that is blown up in one step lies on the exceptional curve of a previous exceptional curve, or else  $X_{\bar{k}}$  will have an irreducible curve of degree less than  $-1$  which contradicts Lemma 2.2.5. So  $X_s$  is the blow-up of  $\mathbb{P}_{k_s}^2$  at  $r$  distinct points. The degree is  $d = K_X^2 = (\mathcal{O}(-3), \mathcal{O}(-3)) - r = 9 - r$ . Note that  $d = (K_X, K_X) \geq 1$  because  $-K_X$  is ample; So  $0 \leq r \leq 8$ . It remains to show that the points are in general position. If three points lie on a line  $L$  then  $(f^{-1}L, f^{-1}L) = (L, L) - 3 < -1$  where  $f^{-1}L$  is the strict transform of  $L$  which contradicts Lemma 2.2.5. Similarly when 6 blown up points lie on a conic or 8 on a singular cubic with one point at the singularity, we get a curve of self intersection  $< -1$  a contradiction.

$\mathbf{Y} = \mathbb{F}_{\mathbf{n}} := \mathbb{P}(\mathcal{O}_{\mathbb{P}_{k_s}^1} \oplus \mathcal{O}_{\mathbb{P}_{k_s}^1}(\mathbf{n}))$  for  $\mathbf{n} = \mathbf{0}$  or  $\mathbf{n} \geq \mathbf{2}$ : If  $n = 0$ , then  $Y = \mathbb{P}_{k_s}^1 \times \mathbb{P}_{k_s}^1$ . In the case when  $X_s = Y$ ,  $X_s$  is a del Pezzo surface of degree  $(K_X, K_X) = 2(\mathcal{O}(-2), \mathcal{O}(-2)) = 8$ . In the case when  $X_s \neq Y$ , any two non intersecting exceptional curves in  $X_{r-1}$  can be contracted resulting in a birational map  $X_{(r-1)}: \rightarrow \mathbb{P}_{k_s}^2$ . Hence we get a new birational map  $X: \rightarrow \mathbb{P}_{k_s}^2$ . So we return to the above case. If  $n \geq 2$  then there is a curve on  $Y$  that satisfies  $(C, C) < -1$ . Its strict transform  $f^{-1}(C)$  via the base extension also has self intersection  $< -1$ ; this contradicts Lemma 2.2.5.

□

**Corollary 2.2.6.** *Let  $X$  be a del Pezzo surface of degree  $d$  over a field  $k$ . Assume that  $X_s$  is not isomorphic to  $\mathbb{P}_{k_s}^1 \times \mathbb{P}_{k_s}^1$ . Then the Picard group of  $X_s$ ,  $\text{Pic } X_s$ , is isomorphic to  $\mathbb{Z}^{10-d}$ .*

*Proof.* The del Pezzo surface base extended to  $k_s$ ,  $X_s$ , is not isomorphic to  $\mathbb{P}_{k_s}^1 \times \mathbb{P}_{k_s}^1$ . So by Theorem 2.2.2,  $X_s$  is isomorphic to the blow-up of  $\mathbb{P}_{k_s}^2$  at  $r = 9 - d$  points  $\{P_1, \dots, P_r\}$ . So by [15, Proposition V.3.2],  $\text{Pic } X_s \simeq \text{Pic } \mathbb{P}_{k_s}^2 \oplus \mathbb{Z}^{9-d} \simeq \mathbb{Z}^{10-d}$ . □

Let  $\{e_1, \dots, e_r\}$  be the classes of the exceptional divisors associated to the blown up points  $\{P_1, \dots, P_r\}$  respectively. Let  $l$  be the class of the pullback of a line in  $\mathbb{P}_{k_s}^2$  that does not pass through any of the points  $\{P_1, \dots, P_r\}$ . Then  $\text{Pic } X_s$  admits  $\{e_1, \dots, e_r, l\}$  as a basis.

Moreover, the canonical class  $K_X$  can be written as  $K_X = -3l + \Sigma e_i$  in terms of this basis because  $K_{\mathbb{P}^2} = -3l$ .

We turn our attention to del Pezzo surfaces of degree 4 which we prove can be embedded in  $\mathbb{P}^4$  and they are concretely given by a smooth complete intersection of two quadrics in  $\mathbb{P}^4$ .

**Theorem 2.2.7.** *A scheme  $X$  is a del Pezzo surface of degree 4 if and only if it is isomorphic to a smooth complete intersection of two quadrics in  $\mathbb{P}^4$ .*

**Lemma 2.2.8.** *Let  $X$  over  $k_s$  be a del Pezzo surface of degree 4. The linear system  $|-K_X|$  separates points and tangent vectors.*

*Proof.* In this proof we work over  $k_s$ . By Theorem 2.2.2,  $X$  is the blow-up of  $\mathbb{P}^2$  at 5 points  $P_1, P_2, \dots, P_5$ . Let  $R_1$  and  $R_2$  be two distinct points in  $X$ . First we consider the case when  $R_1$  and  $R_2$  are in  $\mathbb{P}^2$ . In  $\mathbb{P}^2$  there exists a conic passing through 4 given points and not through another point. Consider the conic passing through  $R_2$  and three of the blown-up points but not  $R_1$ . The union of the conic and the line passing through the remaining 2 blown-up points is a cubic that passes through  $R_2$  but not  $R_1$ . The strict transform of this cubic is a divisor in the linear system  $|-K_X|$  that separates  $R_1$  and  $R_2$ . We also need to consider the case when the points  $R_1$  and  $R_2$  are on the same exceptional curve that lies above some blown-up point say  $P_1$ . In this case we consider the line passing through  $P_1$  of a certain slope so that its strict transform passes through  $R_1$  on the exceptional curve. The strict transform of the union of this line and the conic passing through the remaining blown-up points  $P_2, \dots, P_5$  is a divisor in  $|-K_X|$  that separates  $R_1$  and  $R_2$ . Hence  $|-K_X|$  separates points. What remains to prove is that  $|-K_X|$  separates tangent vectors. Consider a point  $R$  and a tangent direction  $t_R$  at  $R$ . If the tangent direction is in  $\mathbb{P}^2$ , then we consider a line passing through  $R$  and some blown-up point say  $P_1$  such that  $t_R$  is not along the line. The strict transform of the union of this line and the conic passing through the remaining blown-up points  $P_2, \dots, P_5$  separates  $R$  and  $t_R$ . We now consider the case when  $t_R$  is along an exceptional curve that lies above a blown-up point  $P_1$ . Then we consider the union of a line through  $P_1$  of a certain slope such that its strict transform intersects the exceptional

curve at  $R$  and a conic that passes through the remaining blown-up points  $P_2, \dots, P_5$  but not  $P_1$ . The strict transform of the constructed cubic is a divisor that separates  $R$  and  $t_R$ . Therefore the linear system  $| -K_X |$  separates points and tangent vectors.  $\square$

*Proof of Theorem 2.2.7.* The proof is standard; however we use some ideas from the proof of [25, Theorem 1.9].

Let  $\omega_X$  be the canonical sheaf associated to the canonical divisor  $K_X$ . First we compute  $h^0(X, -\omega_X)$ . By applying the Riemann-Roch theorem to the sheaf  $m\omega_X$ , we have

$$h^0(X, m\omega_X) - h^1(X, m\omega_X) + h^0(X, (1-m)\omega_X) = \frac{1}{2}(m\omega_X, (m-1)\omega_X) + \chi(X, \mathcal{O}_X). \quad (2.1)$$

By Serre duality and the fact that  $-\omega_X$  is ample, we have  $h^2(X, \mathcal{O}_X) = h^0(X, \omega_X) = 0$ . Furthermore, by the Kodaira vanishing theorem it follows that  $h^1(X, \omega_X) = 0$ . So the Euler characteristic is

$$\chi(X, \mathcal{O}_X) = h^0(X, \mathcal{O}_X) = 1.$$

We consider  $m < 0$ . The second term  $h^1(X, m\omega_X)$  of (2.1) vanishes by the Kodaira vanishing theorem. The third term  $h^0(X, (1-m)\omega_X)$  of (2.1) vanishes because  $-\omega_X$  is ample and so its inverse does not have global sections. Hence for  $m < 0$ , we have

$$h^0(X, m\omega_X) = \frac{1}{2}m(m-1)(4) + 1. \quad (2.2)$$

For  $m = -1$ ,  $h^0(X, -\omega_X) = 5$ . Let  $i: X \rightarrow \mathbb{P}^4$  be the morphism of  $X$  to  $\mathbb{P}^4$  given by the linear system  $| -K_X |$ . By Lemma 2.2.8, the linear system  $| -K_X |$  separates points and tangent vectors over  $k_s$ . So  $i_s: X_s \rightarrow \mathbb{P}_{k_s}^4$  is an embedding. Hence  $i: X \rightarrow \mathbb{P}^4$  is an embedding as well. The embedding before yields the exact sequence of sheaves

$$0 \rightarrow \mathcal{I}_X \rightarrow \mathcal{O}_{\mathbb{P}^4} \rightarrow i_*\mathcal{O}_X \rightarrow 0.$$

Twisting by  $\mathcal{O}_{\mathbb{P}^4}(2)$  we get the exact sequence

$$0 \rightarrow \mathcal{I}_X(2) \rightarrow \mathcal{O}_{\mathbb{P}^4}(2) \rightarrow i_*\mathcal{O}_X(-2\omega_X) \rightarrow 0.$$

This yields the long exact sequence of cohomology

$$0 \rightarrow H^0(\mathbb{P}^4, \mathcal{I}_X(2)) \rightarrow H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) \rightarrow H^0(X, -2\omega_X) \rightarrow \dots$$

We have  $h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) = \binom{4+2}{2} = 15$ . To compute  $h^0(X, -2\omega_X)$ , we apply the Riemann-Roch theorem again. Applying (2.2) with  $m = -2$ , we deduce that

$$h^0(X, -2\omega_X) = \frac{1}{2}(2)(3)(4) + 1 = 13.$$

Hence by the exact sequence of cohomology above,  $h^0(X, \mathcal{I}_X(2))$  is at least of dimension 2. So  $X$  is contained in the intersection of two linearly independent quadrics  $Q$  and  $Q'$ . Assume for the sake of contradiction that one of the quadrics say  $Q'$  is reducible. By Klein's theorem applied to the quadric  $Q$  and the codimension 1 irreducible subscheme  $X$ , there exists a hypersurface  $V \subset \mathbb{P}^4$  such that  $X = Q \cap V$  is a complete intersection. Since  $Q'$  is assumed to be irreducible then  $V$  is a hyperplane. However the degree of  $X$  is 4. So we get a contradiction by applying Bezout's theorem to  $Q \cap V$ . Hence  $Q$  and  $Q'$  are irreducible. So  $Q \cap Q'$  is pure of dimension 2. Further,  $X \subset Q \cap Q'$  is of dimension 2. So  $X = Q \cap Q'$  in  $\mathbb{P}^4$ .

Conversely, let  $Q \cap Q'$  be a smooth complete intersection of two quadrics. Let  $i: Q \cap Q' \rightarrow Q$  and  $j: Q \rightarrow \mathbb{P}^4$ . By the adjunction formula we have

$$\omega_{Q \cap Q'} = (j \circ i)^*(\omega_{\mathbb{P}^4} \otimes \mathcal{O}_{\mathbb{P}^4}(Q) \otimes \mathcal{O}_{\mathbb{P}^4}(Q')) = (j \circ i)^*\mathcal{O}_{\mathbb{P}^4}(-5 + 2 + 2) = \mathcal{O}_{Q \cap Q'}(-1).$$

Hence  $-\omega_{Q \cap Q'}$  is ample. So  $Q \cap Q'$  is a del Pezzo surface. To determine the degree of  $Q \cap Q'$  as defined for a del Pezzo surface, we use Riemann-Roch. By the same above reasoning of (2.2), we have

$$h^0(X, -\omega_{Q \cap Q'}) = \frac{1}{2}(-1)(-1-1)((\omega_{Q \cap Q'}, \omega_{Q \cap Q'})) + 1.$$

Further,  $h^0(X, -\omega_{Q \cap Q'}) = 5$ . So  $(\omega_{Q \cap Q'}, \omega_{Q \cap Q'}) = 4$ . □

By using Theorem 2.2.7, we embed a del Pezzo surface  $X$  of degree 4 over  $k$  in  $\mathbb{P}^4$  and we define it as the smooth complete intersection of two quadrics  $Q$  and  $Q'$  in  $\mathbb{P}^4$ . We define

a pencil of quadrics

$$\{\lambda Q + \mu Q' : [\lambda : \mu] \in \mathbb{P}^1\}.$$

For every  $t \in \mathbb{P}^1$ , we denote by  $Q_t$  the corresponding quadric in the pencil. Let  $q$  and  $q'$  be the quadratic forms associated to the quadrics  $Q$  and  $Q'$  respectively. Let  $A$  and  $A'$  be the matrices associated to the quadratic forms  $q$  and  $q'$  respectively. We define the characteristic polynomial associated to the pencil of quadrics to be

$$f(\lambda, \mu) = \det(\lambda A + \mu A').$$

The polynomial  $f(\lambda, \mu)$  is a degree 5 polynomial in  $k[\lambda, \mu]$ .

The following lemma is used to prove Proposition 2.2.10 and is of importance on its own.

**Lemma 2.2.9.** [31, Lemma 3.27] *The point  $t \in \mathbb{P}_k^1$  is a simple root of the polynomial  $f(\lambda, \mu)$  if and only if the quadric  $Q_t$  has a unique singular point and this singular point does not belong to  $X$ .  $\square$*

**Proposition 2.2.10.** [31, Proposition 3.26] *Let  $X$  be the intersection of two quadrics  $Q$  and  $Q'$  in  $\mathbb{P}^4$  over a field  $k$  of characteristic not 2. We use the same notation as above. The following are equivalent.*

1. *The variety  $X$  is smooth of codimension 2 in  $\mathbb{P}^4$ .*
2. *The polynomial  $f(\lambda, \mu) \in k[\lambda, \mu]$  is non zero and is separable.*
3. *The quadratic forms  $q$  and  $q'$  are not proportional, and for every  $t \in \mathbb{P}^2$  the singular locus of  $Q_t$  is disjoint from  $X$ .*

*Further, any of the equivalent conditions above will imply that the quadratic forms  $\lambda q + \mu q'$  where  $(\lambda, \mu) \in \bar{k}^{\times 2}$  are of rank  $\geq 4$  with at least one quadratic form of rank 5.  $\square$*

By Proposition 2.2.10 we deduce that the degeneracy locus of the pencil of quadrics associated to  $X$  is a reduced degree 5 subscheme in  $\mathbb{P}^2$  which we denote by  $\mathcal{S}$ . Let  $\mathcal{S}(\bar{k}) =$

$\{t_0, \dots, t_4\}$  where  $t_0, \dots, t_4 \in \mathbb{P}^1(\bar{k})$ . Denote by  $k(t_i)$  the smallest field contained in  $\bar{k}$  and containing  $t_i$ , and by  $Q_{t_i}$  the corresponding quadric. Let  $\epsilon_{t_i}$  be the discriminant of the smooth rank 4 quadric obtained by restricting  $Q_{t_i}$  to a hyperplane  $H_i$  in  $\mathbb{P}^4$  not containing the vertex of  $Q_{t_i}$ . By [31, Section 3.4.1], the square class of  $\epsilon_{t_i}$  does not depend on the choice of  $H_i$ . So we consider  $\epsilon_{t_i}$  as an element in  $k(t_i)/k(t_i)^{\times 2}$ .

Now we turn our attention to the Picard group of  $\bar{X}$  and the Galois action on its classes. We will describe  $\text{Pic } \bar{X}$  in terms of some conic classes that will be useful for the algorithm in Chapter 3.

**Theorem 2.2.11.** [2, Theorem 2] *Let  $X$  be a del Pezzo surface of degree 4 over a field  $k$ . There are 10 families of conics on  $X$ . Moreover, their classes  $C_0, \dots, C_4, C'_0, \dots, C'_4$  in  $\text{Pic } \bar{X}$  can be written in terms of the basis of  $\text{Pic } \bar{X}$  as  $C_i = l - e_i$  and  $C'_i = H - C_i$  where  $H$  is the hyperplane class of  $X$ . Over  $\bar{k}$  the conics in each class form a pencil.  $\square$*

As in [30, Section 2.3], we define the conic classes  $C_0, \dots, C_4, C'_0, \dots, C'_4$  in  $\text{Pic } \bar{X}$  as follows. For  $i = 0, \dots, 4$ , let  $k_i$  be a finite extension of  $k$  such that the rank 4 quadric  $Q_{t_i}$  has a smooth  $k_i$ -point and such that  $[k_i(\sqrt{\epsilon_{t_i}}) : k_i] = [k(t_i)(\sqrt{\epsilon_{t_i}}) : k(t_i)]$ . Let  $P_i$  be any smooth  $k_i$ -point on  $Q_{t_i}$  and  $H_{P_i}$  be the hyperplane tangent to  $Q_{t_i}$  at  $P_i$ . By [30, Lemma 2.1], we have  $Q_{t_i} \cap H_{P_i} = L_{P_i} \cup L'_{P_i}$  for some planes  $L_{P_i}$  and  $L'_{P_i}$  defined over  $k_i(\sqrt{\epsilon_{t_i}})$ . We define  $C_{P_i} = X \cap L_{P_i}$  and  $C'_{P_i} = X \cap L'_{P_i}$ . We have,

$$C_{P_i} = W \cap Q_{t_i} \cap L_{P_i} = W \cap L_{P_i} \text{ and } C'_{P_i} = W \cap Q_{t_i} \cap L'_{P_i} = W \cap L'_{P_i}$$

for some smooth quadric  $W$  in the pencil associated to  $X$ . Hence  $C_{P_i}$  and  $C'_{P_i}$  are conics on  $X$ . A different choice of  $k_i$ -smooth point  $\tilde{P}_i$  on  $Q_{t_i}$  leads to two different planes  $L_{\tilde{P}_i}$  and  $L'_{\tilde{P}_i}$ , and hence two different conics. Because  $Q_{t_i}$  is a singular rank 4 quadric over  $\bar{k}$ , it is a cone over a smooth quadric  $Q'$  in  $\mathbb{P}^3$ . The two families of lines on  $Q'$  induce two families of planes on  $Q_{t_i}$ . So  $L_{P_i}, L'_{P_i}, L_{\tilde{P}_i}$  and  $L'_{\tilde{P}_i}$  belong to two pencils. So without loss of generality we may assume that  $C_{P_i} \sim C_{\tilde{P}_i}$  and  $C'_{P_i} \sim C'_{\tilde{P}_i}$ . We denote the classes of these conics by  $C_i$  and  $C'_i$  which are independent of the chosen point on  $Q_{t_i}$ . By Theorem 2.2.11, the classes of these

conics are the 10 possible classes of conics on  $X$ , and  $C'_i = H - C_i$  for every  $i \in \{0, \dots, 4\}$  where  $H$  is the hyperplane class of  $X$ .

**Proposition 2.2.12.** [30, Proposition 2.2] *After possibly interchanging the  $C_i$  and the  $C'_i$  for some indices  $i$ , we may assume that the Picard group  $\text{Pic } \bar{X} \cong \mathbb{Z}^6$  is freely generated by the following classes*

$$\frac{1}{2}(H + C_0 + C_1 + C_2 + C_3 + C_4), C_0, C_1, C_2, C_3, C_4$$

where  $H$  is the hyperplane class of  $X$ . □

Consider  $\sigma \in G_k$ , and let  $\sigma' : \text{Spec } \bar{k} \rightarrow \text{Spec } \bar{k}$  be the corresponding morphism on schemes. After base changing to  $\bar{k}$  we get

$$\text{id}_X \times \sigma' : \bar{X} = X \times_k \text{Spec } \bar{k} \rightarrow \bar{X}.$$

The morphism above induces an automorphism on  $\text{Pic } \bar{X}$ . This defines the action of  $G_k \rightarrow \text{Aut}(\text{Pic } \bar{X})$ . This action fixes the canonical class and the intersection multiplicity [20, Theorem 23.8].

Let  $\Gamma$  be the graph of ten vertices indexed by  $C_i$  and  $C'_i$  whose edges join  $C_i$  and  $C'_i$ . The group  $\text{Aut}(\Gamma)$  is the semi-direct product  $(\mathbb{Z}/2\mathbb{Z})^5 \rtimes S_5$  where the  $\mathbb{Z}/2\mathbb{Z}$  entries represent the exchanges for each  $\{C_i, C'_i\}$  and the  $S_5$  entry represents a permutation of the sets  $\{C_i, C'_i\}$  for  $i \in \{0, \dots, 4\}$ . Let  $\mathcal{O}(K_X^\perp)$  be the subgroup of  $\text{Aut}(\Gamma)$  that fixes the orthogonal complement of  $K_X$  in  $\text{Pic } \bar{X}$ . By the discussion in [18, p:8-10], there is a natural embedding of index 2  $\mathcal{O}(K_X^\perp) \hookrightarrow \text{Aut}(\Gamma)$ . Moreover by [18, p:8-10] the image of this embedding constitutes of all automorphisms that are the product of an even number of exchanges and an element of  $S_5$ .

Fix  $T \in \mathcal{S}$ . Let  $\Gamma_T$  be subgraph with  $2 \deg(T)$  vertices indexed by  $C_i$  and  $C'_i$  for  $t_i \in T(\bar{K})$ .

**Proposition 2.2.13.** [30, Proposition 2.3] *The action of  $G_K$  on  $\text{Pic } \bar{X}$  induces an action on  $\Gamma$  that factors through*

$$\prod_{T \in \mathcal{S}} \text{Aut}(\Gamma_T) \cap \mathcal{O}(K_{\bar{Y}}^\perp) \subset \text{Aut}(\Gamma).$$

Moreover,  $G_K$  acts transitively on  $\{C_i, C'_i : t_i \in T(\bar{K})\}$  if and only if  $\epsilon_T \notin k(T)^{\times 2}$ .

*Proof.* By the discussion above, the action of  $G_k$  factors through  $\prod_{T \in \mathcal{S}} \text{Aut}(\Gamma_T) \cap \mathcal{O}(K_X^\perp)$ . Moreover, by definition the pair  $\{C_i, C'_i\}$  is defined over  $k(t_i)$  and each individual conic over  $k(t_i, \sqrt{\epsilon_T})$ . So  $\epsilon_T \notin k(T)^{\times 2}$  is exactly the condition for a transitive action of  $G_k$  on  $\{C_i, C'_i\}$ .  $\square$

## Chapter 3

**A KNOWN ALGORITHM TO COMPUTE THE BRAUER  
GROUP OF A DEL PEZZO SURFACE OF DEGREE 4**

In this section we review an algorithm to compute  $\text{Br } X/\text{Br}_0 X$  where  $X$  is a del Pezzo surface of degree 4 over a field  $k$  of characteristic not 2 under some arithmetic assumptions related to the solvability of the quadrics in the pencil determined by  $X$  that we state below. This algorithm can be found in [30, Section 4.1].

We start by fixing some notation. By embedding  $X$  anticanonically into  $\mathbb{P}^4$ , we view it as the smooth complete intersection of two quadrics as in Theorem 2.2.7. The two quadrics  $Q$  and  $Q'$  define a pencil  $\{\lambda Q + \mu Q' : [\lambda, \mu] \in \mathbb{P}^1\}$ . We use the same notation as in Section 2.2 for the characteristic polynomial  $f(\lambda, \mu)$  and the degeneracy locus  $\mathcal{S} \subset \mathbb{P}^1$  associated to the pencil. For every closed point  $T \in \mathcal{S}$ , denote by  $k(T)$  the residue field of  $T$  and by  $Q_T$  the corresponding quadric in the pencil. Let  $\epsilon_T$  be the discriminant of the smooth rank 4 quadric that is obtained by restricting  $Q_T$  to a hyperplane  $H_T$  in  $\mathbb{P}^4$  not containing the vertex of  $Q_T$ . By [31, Section 3.4.1], the square class of  $\epsilon_T$  does not depend on the choice of  $H_T$ . Therefore we consider  $\epsilon_T$  as an element in  $k(T)/k(T)^{\times 2}$ .

Over  $\bar{k}$ ,  $\mathcal{S}(\bar{k}) = \{t_0, \dots, t_4\}$  where  $t_0, \dots, t_4 \in \mathbb{P}^1(\bar{k})$ . For every  $t_i \in \mathcal{S}(\bar{k})$ , denote by  $k(t_i)$  the smallest field contained in  $\bar{k}$  and containing  $k$  and  $t_i$ , and by  $Q_{t_i}$  the corresponding quadric in the pencil over  $\bar{k}$ . Let  $\epsilon_{t_i}$  be the discriminant of the degree 4 quadric obtained by restricting  $Q_{t_i}$  to a hyperplane  $H_i$  in  $\mathbb{P}^4$  not containing the vertex of  $Q_{t_i}$ . By [31, Section 3.4.1], the square class of  $\epsilon_{t_i}$  does not depend on the choice of  $H_i$ . Therefore we consider  $\epsilon_{t_i}$  as an element in  $k(t_i)/k(t_i)^{\times 2}$ .

We say that a subscheme  $\mathcal{T} \subset \mathcal{S}$  satisfies (\*) if it satisfies the conditions

$$(*) \quad \deg(\mathcal{T}) = 2, \quad \prod_{T \in \mathcal{T}} N_{k(T)/k}(\epsilon_T) \in k^{\times 2}, \quad \text{and } \epsilon_T \notin k(T)^{\times 2} \text{ for every } T \in \mathcal{T}.$$

### 3.1 Lifting cocycles in $H^1(k, \text{Pic } \overline{X})$ to Brauer classes in $\text{Br } X/\text{Br}_0 X$

**Lemma 3.1.1.** [30, Lemma 3.1] *If a subscheme  $\mathcal{T} \subset \mathcal{S}$  satisfies (\*), then*

$$\epsilon_T \in \text{im}(k^\times/k^{\times 2} \rightarrow k(T)^\times/k(T)^{\times 2}) \quad \text{for every } T \in \mathcal{T}.$$

*Proof.* We repeat the proof of [30, Lemma 3.1]. If  $T$  has degree 1 then  $k(T) = k$  and there is nothing to prove. Assume  $T$  has degree 2. Consider the short exact sequence of  $G_k$  modules

$$0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \text{Ind}_{\text{id}}^{\text{Gal}(k(T)/k)}(\mathbb{Z}/2\mathbb{Z}) \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0.$$

A part of the long exact sequence on cohomology is

$$H^1(k, \mathbb{Z}/2\mathbb{Z}) \rightarrow H^1(k, \text{Ind}_{\text{id}}^{\text{Gal}(k(T)/k)}(\mathbb{Z}/2\mathbb{Z})) = H^1(k(T), \mathbb{Z}/2\mathbb{Z}) \rightarrow H^1(k, \mathbb{Z}/2\mathbb{Z}).$$

Since  $\mu_2 \subset k$ ,  $H^1(k, \mathbb{Z}/2\mathbb{Z}) \simeq H^1(k, \mu_2)$ . The cohomology group  $H^1(k, \mu_2)$  is isomorphic to  $k^\times/k^{\times 2}$  by Kummer Theory. Moreover,  $N_{k(T)/k}(\epsilon_T) \equiv 0 \pmod{k(T)^{\times 2}}$  because  $\mathcal{T}$  satisfies (\*). Hence  $\epsilon_T \in \ker(H^1(k(T), \mathbb{Z}/2\mathbb{Z}) \rightarrow H^1(k, \mathbb{Z}/2\mathbb{Z})) = \text{im}(H^1(k, \mathbb{Z}/2\mathbb{Z}) \rightarrow H^1(k(T), \mathbb{Z}/2\mathbb{Z}))$ . Hence the result follows.  $\square$

For  $\mathcal{T} \subset \mathcal{S}$  satisfying (\*) and  $T \in \mathcal{T}$ , we define  $k_{\mathcal{T}} := k(\sqrt{\epsilon_T})$  which is independent of  $T$  because  $\deg(\mathcal{T}) = 2$  and  $\prod_{T \in \mathcal{T}} N_{k(T)/k}(\epsilon_T) \in k^{\times 2}$ . By Lemma 3.1.1 and since  $\epsilon_T \notin k(T)^{\times 2}$ ,  $k_{\mathcal{T}}$  is a quadratic extension of  $k$ .

**Proposition 3.1.2.** *Let  $\mathcal{T} \subset \mathcal{S}$  satisfy (\*). The cocycle in  $H^1(k, \text{Pic } \overline{X})$  given by*

$$\sigma \mapsto \begin{cases} -H + \sum_{t_i \in \mathcal{T}(\overline{k})} C_i & \text{if } \sigma \notin G_{k_{\mathcal{T}}} \\ 0 & \text{otherwise} \end{cases}$$

*is non trivial if and only if there exists  $T \in \mathcal{S} - \mathcal{T}$  such that  $\epsilon_T \notin k(T)^{\times 2}$ .*

*Proof.* This proof can be found in [30, Section 3.2, Proposition 3.3] assuming that  $Q_T$  has a smooth  $k(T)$ -point for all  $T \in \mathcal{T}$ . However the proof works without the assumption that  $Q_T$  has a smooth  $k(T)$ -point for all  $T \in \mathcal{T}$ . We repeat the proof for the reader's convenience.

The long exact sequence on cohomology associated to the short exact sequence

$$0 \rightarrow \text{Pic } \bar{X} \xrightarrow{\times 2} \text{Pic } \bar{X} \rightarrow (\text{Pic } \bar{X}/2\text{Pic } \bar{X}) \rightarrow 0$$

induces an isomorphism

$$\frac{(\text{Pic } \bar{X}/2\text{Pic } \bar{X})^{G_k}}{((\text{Pic } \bar{X})^{G_k}/2(\text{Pic } \bar{X})^{G_k})} \xrightarrow{\sim} \text{H}^1(k, \text{Pic } \bar{X})[2]; \quad D \mapsto (\sigma \mapsto \frac{1}{2}(d - \sigma d)) \quad (3.1)$$

where  $d$  is a lift of  $D$  to  $\text{Pic } \bar{X}$ .

We prove that the divisor  $\sum_{t_i \in \mathcal{S}(\bar{k})} C_i \in (\text{Pic } \bar{X}/2\text{Pic } \bar{X})^{G_k}$  and that its image of under the isomorphism (3.1), is the cocycle  $\alpha: G_k \rightarrow \text{Pic } \bar{X}$  defined by

$$\sigma \mapsto \begin{cases} -H + \sum_{t_i \in \mathcal{S}(\bar{k})} C_i & \text{if } \sigma \notin G_{k, \mathcal{S}} \\ 0 & \text{otherwise.} \end{cases} \quad (3.2)$$

By definition, for every  $t_i \in \mathcal{S}(\bar{k})$  the pair of conics  $\{C_i, C'_i\}$  is defined over  $k(t_i)$  and each individual conic is defined over  $k(t_i, \sqrt{\epsilon_{t_i}})$ . Hence  $\sum_{t_i \in \mathcal{S}(\bar{k})} C_i$  is fixed by  $\sigma$  for every  $\sigma \in G_{k, \mathcal{S}}$ . Furthermore, if  $\sigma \notin G_{k, \mathcal{S}}$  then

$$\sum_{t_i \in \mathcal{S}(\bar{k})} C_i - \sigma(\sum_{t_i \in \mathcal{S}(\bar{k})} C_i) = \sum_{t_i \in \mathcal{S}(\bar{k})} C_i - \sum_{t_i \in \mathcal{S}(\bar{k})} C'_i = -2H + 2\sum_{t_i \in \mathcal{S}(\bar{k})} C_i.$$

By the previous two statements,  $\sum_{t_i \in \mathcal{S}(\bar{k})} C_i \in (\text{Pic } \bar{X}/2\text{Pic } \bar{X})^{G_k}$ . Moreover, by the explicit description of isomorphism (4.2),  $\alpha \in \text{H}^1(k, \text{Pic } \bar{X})$  as defined above is the image of  $\sum_{t_i \in \mathcal{S}(\bar{k})} C_i$  by the isomorphism (3.1).

Therefore  $\alpha$  is trivial if and only if  $\sum_{t_i \in \mathcal{S}(\bar{k})} C_i \in 2\text{Pic } \bar{X} + (\text{Pic } \bar{X})^{G_k}$ . We will prove that  $\sum_{t_i \in \mathcal{S}(\bar{k})} C_i \notin 2\text{Pic } \bar{X} + (\text{Pic } \bar{X})^{G_k}$  if and only if there exists  $T \in \mathcal{S} - \mathcal{S}$  such that  $\epsilon_T \notin k(T)^{\times 2}$ . We determine an equivalent criterion to  $\sum_{t_i \in \mathcal{S}(\bar{k})} C_i \notin 2\text{Pic } \bar{X} + (\text{Pic } \bar{X})^{G_k}$  by using the generators of  $\text{Pic } \bar{X}$  to . By Equation (3.2),  $\sum_{t_i \in \mathcal{S}(\bar{k})} C_i \notin (\text{Pic } \bar{X})^{G_k}$ . Moreover, by the same argument as in the proof of Lemma 4.2.1 any combination of the generators of  $\text{Pic } \bar{X}$  involving an odd coefficient of  $\frac{H + \sum_i C_i}{2}$  is not fixed by  $G_k$ . Therefore  $\sum_{t_i \in \mathcal{S}(\bar{k})} C_i \in 2\text{Pic } \bar{X} + (\text{Pic } \bar{X})^{G_k}$  if and only if there exists a choice of signs such that

$$\sum_{t_i \in (\mathcal{S} - \mathcal{S})(\bar{k})} \pm C_i \in (\text{Pic } \bar{X})^{G_k}.$$

If there exists  $T \in \mathcal{S} - \mathcal{T}$  such that  $\epsilon_T \notin k(T)^{\times 2}$  then by Proposition 2.2.13  $G_k$  acts transitively on each pair  $\{C_i, C'_i : t_i \in T(\bar{k})\}$ . Then for some  $t_j \in T(\bar{k})$  and by using the fact that  $C'_j = H - C_j$ , there exists  $\sigma \in G_k$  such that

$$\sigma(\pm C_j + \Sigma_{\{t_i \neq t_j, t_i \in (\mathcal{S} - \mathcal{T})(\bar{k})\}} \pm C_i) = \mp C_j + D$$

where  $D$  is a linear combination of the  $C_i$ 's excluding  $C_j$ . Then for any choice of signs

$$\Sigma_{t_i \in (\mathcal{S} - \mathcal{T})(\bar{k})} \pm C_i \notin (\text{Pic } \bar{X})^{G_k}.$$

This proves that  $\alpha$  is a non trivial cocycle in  $H^1(k, \text{Pic } \bar{X})$ .

Conversely, if for every  $T \in \mathcal{S} - \mathcal{T}$  we have  $\epsilon_T \in k(T)^{\times 2}$  then for any choice of signs

$$\Sigma_{t_i \in (\mathcal{S} - \mathcal{T})(\bar{k})} \pm C_i \in (\text{Pic } \bar{X})^{G_k}.$$

So  $\alpha$  is trivial in  $H^1(k, \text{Pic } \bar{X})$ . □

Let  $Y$  be a nice scheme over a field  $k$ . Let  $L$  be a cyclic extension of  $k$ ; denote by  $\sigma$  the generator of  $\text{Gal}(L/k)$ . Let  $f$  be any element in  $k(Y)^\times$ . We denote by  $\text{Br}_{cyc}(Y, L)$  the set of classes of algebras  $[(L/k, f)]$  in the image of  $\text{Br } Y / \text{Br}_0 Y \rightarrow \text{Br } k(Y) / \text{Br}_0 Y$ . We view  $1 - \sigma$  as endomorphisms of  $\text{Div } Y_L$ . We let  $N_{L/k}: \text{Div } Y_L \rightarrow \text{Div } Y_k$  and  $\bar{N}_{L/k}: \text{Pic } Y_L \rightarrow \text{Pic } Y_k$  be the usual norm maps.

**Theorem 3.1.3.** *There exists an injection  $\text{Br}_{cyc}(Y, L) \hookrightarrow H^1(k, \text{Pic } \bar{Y})$  given by the composition of the maps*

$$\text{Br}_{cyc}(X, L) \xrightarrow{\sim} \frac{\ker(\bar{N}_{L/k})}{\text{im}(1 - \sigma)} \hookrightarrow H^1(k, \text{Pic } \bar{Y}). \quad (3.3)$$

*The image of the class  $[(L/k, f)]$  under the above composition is the cocycle*

$$\sigma \mapsto \begin{cases} D & \text{if } \sigma \notin G_L \\ 0 & \text{otherwise} \end{cases},$$

*where  $D$  is the divisor such that  $(f) = N_{L/k}(D)$ .* □

*Proof.* The first isomorphism follows from [28, Theorem 3.3]. By [28, Theorem 3.3], the isomorphism maps the class of the algebra  $[(L/k, f)]$  to the divisor  $D$  such that

$$N_{L/k}(D) = (f).$$

The extension  $L/k$  is cyclic; so by using the explicit resolution we compute

$$\frac{\ker(\overline{N}_{L/k})}{\text{im}(1 - \sigma)} \simeq H^1(\text{Gal}(L/k), \text{Pic } Y_L).$$

The image of  $D$  under this isomorphism is the cocycle  $\alpha$  that maps  $\sigma \mapsto D$ .

Furthermore, the first part of the inflation-restriction exact sequence yields the injection

$$H^1(\text{Gal}(L/k), \text{Pic } Y_L) \hookrightarrow H^1(k, \text{Pic } \overline{Y}).$$

The image of  $\alpha$  under the above inflation map is the required cocycle.  $\square$

Applying the map in Theorem 3.1.3 to the del Pezzo surface  $X$  and the cyclic extension  $k_{\mathcal{T}}/k$ , we get the map

$$\text{Br}_{\text{cyc}}(X, k_{\mathcal{T}}) \xrightarrow{\simeq} \frac{\ker(\overline{N}_{k_{\mathcal{T}}/k})}{\text{im}(1 - \sigma)} \hookrightarrow H^1(k, \text{Pic } \overline{X}). \quad (3.4)$$

For the remainder of this section, we assume that for every subscheme  $\mathcal{T} \subset \mathcal{S}$  satisfying  $(*)$ , the quadric  $Q_T$  has a smooth  $k(T)$ -point for every  $T \in \mathcal{T}$ . Let

$$\mathcal{A}_{\mathcal{T}} := (k_{\mathcal{T}}/k, l^{-2} \prod_{T \in \mathcal{T}} N_{k(T)/k}(l_T)),$$

where  $l_T$  is a  $k(T)$ -linear form such that the associated hyperplane is tangent to  $Q_T$  at a smooth point for every  $T \in \mathcal{T}$  and  $l$  is any linear form.

**Proposition 3.1.4.** *Let  $\mathcal{T} \subset \mathcal{S}$  satisfy  $(*)$ . The cyclic algebra  $\mathcal{A}_{\mathcal{T}}$  is in the image of  $\text{Br } X \rightarrow \text{Br } k(X)$ . Further, if there exists  $T \in \mathcal{S} - \mathcal{T}$  such that  $\epsilon_T \notin K(T)^{\times 2}$  then the algebra  $\mathcal{A}_{\mathcal{T}}$  is non trivial. In particular, it maps under the map 3.4 to the nontrivial cocycle*

$$\sigma \mapsto \begin{cases} -H + \sum_{t_i \in \mathcal{T}(\overline{k})} C_i & \text{if } \sigma \notin G_{k_{\mathcal{T}}} \\ 0 & \text{otherwise} \end{cases}$$

in  $H^1(k, \text{Pic } \overline{X})$ .

*Proof.* To prove that  $\mathcal{A}_{\mathcal{T}} \in \text{im}(\text{Br } X \rightarrow \text{Br } k(X))$ , we prove that  $\mathcal{A}_{\mathcal{T}}$  is unramified at every codimension 1 point  $x \in X^{(1)}$ , i.e.,  $\partial_x(\mathcal{A}) = 0$  by Theorem 2.1.25. The prime divisors which correspond to a valuation such that the function  $l^{-2}\prod_{T \in \mathcal{T}} N_{k(T)/k}(l_T)$  has an odd valuation are  $C_T$  and  $C'_T$  for every  $T \in \mathcal{T}$ . However, for every  $T \in \mathcal{T}$ ,  $k_{\mathcal{T}} \subset \kappa(C_T \cup C'_T)$  because  $C_T$  and  $C'_T$  are conjugate over  $k_{\mathcal{T}}$ . So by Theorem 2.1.26,  $\mathcal{A}$  is unramified at  $C_T$  and  $C'_T$  for every  $T \in \mathcal{T}$ . Hence  $\mathcal{A} \in \text{im}(\text{Br } X \rightarrow \text{Br } k(X))$  by Theorem 2.1.25.

By definition of the maps defining 3.4, the class of the algebra  $\mathcal{A}_{\mathcal{T}}$  gets mapped to a cocycle  $\alpha$  that maps  $G_{k_{\mathcal{T}}}$  to the identity and maps any element  $\sigma \notin G_{k_{\mathcal{T}}}$  to a divisor  $D$  such that  $N_{k_{\mathcal{T}}/k}(D) = \text{div}(l^{-2}\prod_{T \in \mathcal{T}} N_{k(T)/k}(l_T)) = -2H + \sum_{t_i \in \mathcal{T}(\bar{k})} C_i + \sum_{t_i \in \mathcal{T}(\bar{k})} C'_i$ . So  $D$  can be chosen as  $-H + \sum_{t_i \in \mathcal{T}(\bar{k})} C_i$ . The cocycle  $\alpha$  is non trivial by Proposition 4.2.2. So  $\mathcal{A}_{\mathcal{T}}$  is non trivial as well.  $\square$

### 3.2 The algorithm

Let  $X$  be a del Pezzo surface of degree 4 over a field  $k$  of  $\text{char}(k) \neq 2$ . We describe the algorithm to compute  $\text{Br } X / \text{Br}_0 X$  that is discussed in [30, Sections 3.3-4.1]. First we recall a result by Manin about all possibilities of  $\text{Br } X / \text{Br}_0 X$ .

**Theorem 3.2.1.** [20] *For a del Pezzo surface  $X$  over a field  $k$ ,  $\text{Br } X / \text{Br}_0 X$  is isomorphic to one of the following  $0$ ,  $\mathbb{Z}/2\mathbb{Z}$ , or  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ .*  $\square$

The algorithm we describe determines which of the possibilities in Theorem 3.2.1 is  $\text{Br } X / \text{Br}_0 X$  according to the vanishing locus  $\mathcal{S}$  of the pencil associated to  $X$ .

**Proposition 3.2.2.** *The cardinality of  $H^1(k, \text{Pic } \bar{X})$  divides 4. Moreover, the cardinality  $\#H^1(k, \text{Pic } \bar{X}) \geq 2$  if and only if there exists a subscheme  $\mathcal{T} \subset \mathcal{S}$  that satisfies (\*) and  $T \in \mathcal{S} - \mathcal{T}$  such that  $\epsilon_T \notin k(T)^{\times 2}$ . The cardinality  $\#H^1(k, \text{Pic } \bar{X}) = 4$  if and only if there exist at least three degree 1 points  $\{T_0, T_1, T_2\} \subset \mathcal{S}$  such that each pair of these points satisfies (\*).*

*Proof.* The proof depends on counting the cardinality of  $\frac{(\text{Pic } \bar{X}/2\text{Pic } \bar{X})^{G_K}}{((\text{Pic } \bar{X})^{G_K}/2(\text{Pic } \bar{X})^{G_K})} \simeq H^1(k, \text{Pic } \bar{X})$  and using Proposition 2.2.13. The non trivial cocycles are constructed similar to Proposition

3.1.4. The details of one case are done in Lemma 4.2.1. To verify the claims we use the Magma [1] script in the arXiv distribution of [30].  $\square$

**Theorem 3.2.3.** [30, Theorem 3.4] *If  $\text{Br } X \neq \text{Br}_0 X$  then there exists a subscheme  $\mathcal{T} \subset \mathcal{S}$  that satisfies (\*) and there exists a  $T \in \mathcal{S} - \mathcal{T}$  such that  $\epsilon_T \notin k(T)^{\times 2}$ . If  $\#\text{Br } X / \text{Br}_0 X = 4$  then there exists three degree 1 points  $\{T_0, T_1, T_2\} \subset \mathcal{S}$  such that each pair satisfies (\*).*

*If each quadric  $Q_T$  in the pencil associated to  $X$  corresponding to the points  $T \in \mathcal{S}$  of degree at most 2 has a smooth  $k(T)$ -point then the converse of the above statements hold. When  $\#\text{Br } X / \text{Br}_0 X = 4$ , we have  $\text{Br } X / \text{Br}_0 X \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ . Furthermore, the nontrivial elements in  $\text{Br } X / \text{Br}_0 X$  are of the form  $\mathcal{A}_{\mathcal{T}}$  where  $\mathcal{T}$  runs over degree 2 subschemes of  $\mathcal{S}$  satisfying (\*).*

*Proof.* We rewrite the proof in [30, Theorem 3.4] for the reader's convenience. It follows from the Hochschild-Serre spectral sequence that we have an injection

$$\frac{\text{Br } X}{\text{Br}_0 X} \hookrightarrow \text{H}^1(k, \text{Pic } \bar{X}).$$

So if  $\text{Br } X \neq \text{Br}_0 X$  then  $\text{H}^1(k, \text{Pic } \bar{X})$  is not trivial. Hence by Proposition 3.2.2, there exists a degree 2 subscheme  $\mathcal{T} \subset \mathcal{S}$  satisfying (\*) and there exists a  $T \in \mathcal{S} - \mathcal{T}$  such that  $\epsilon_T$  is not a square in  $k(T)^{\times}$ . If  $\#\text{Br } X / \text{Br}_0 X = 4$  then  $\#\text{H}^1(k, \text{Pic } \bar{X}) = 4$  because  $\text{Br } X / \text{Br}_0 X$  injects into  $\text{H}^1(k, \text{Pic } \bar{X})$  and  $\#\text{H}^1(k, \text{Pic } \bar{X})$  divides 4 by Proposition 3.2.2. Hence by Proposition 3.2.2, if  $\#\text{Br } X / \text{Br}_0 X = 4$  then there exists three degree 1 points  $\{T_0, T_1, T_2\} \subset \mathcal{S}$  such that each pair satisfies (\*).

For the remainder of the proof we assume that each quadric  $Q_T$  in the pencil associated to  $X$  corresponding to the points  $T \in \mathcal{S}$  of degree at most 2 have a smooth  $k(T)$ -point. If  $\mathcal{T}$  is a subscheme of  $\mathcal{S}$  that satisfies (\*) and  $T \in \mathcal{S} - \mathcal{T}$  is such that  $\epsilon_T \notin k(T)^{\times 2}$ , and there are no three degree 1 points  $\{T_0, T_1, T_2\} \subset \mathcal{S}$  such that each pair satisfies (\*) then by Proposition 3.2.2  $\#\text{H}^1(k, \text{Pic } \bar{X}) = 2$ . By Proposition 3.1.4,  $\text{Br } X / \text{Br}_0 X = \{\text{id}, \mathcal{A}_{\mathcal{T}}\}$ . If there exists three degree 1 points  $\{T_0, T_1, T_2\} \subset \mathcal{S}$  such that each pair satisfies (\*) then by Proposition 3.2.2,  $\#\text{H}^1(k, \text{Pic } \bar{X}) = 4$ . In this case by Proposition 3.1.4, there exists three nontrivial classes

of algebras  $\mathcal{A}_{\{T_0, T_1\}}$ ,  $\mathcal{A}_{\{T_1, T_2\}}$ , and  $\mathcal{A}_{\{T_0, T_2\}}$  in  $\text{Br } X / \text{Br}_0 X$ . Furthermore, by the construction of these algebra we notice that  $\mathcal{A}_{\{T_0, T_2\}} = \mathcal{A}_{\{T_0, T_1\}} + \mathcal{A}_{\{T_1, T_2\}}$ . So  $\text{Br } X / \text{Br}_0 X \simeq \mathbb{Z}/2\mathbb{Z}$ , and  $\text{Br } X / \text{Br}_0 X = \{\text{id}, \mathcal{A}_{\{T_0, T_1\}}, \mathcal{A}_{\{T_1, T_2\}}, \mathcal{A}_{\{T_0, T_2\}}\}$ .

Otherwise  $\text{Br } X = \text{Br}_0 X$ . □

Now putting all this together we deduce a practical algorithm to compute  $\text{Br } X / \text{Br}_0 X$ . This algorithm can be found in [30, Section 4.1]. It takes as input the two quadrics  $Q$  and  $Q'$  defining  $X$  over a field  $k$  of  $\text{char}(k) \neq 2$  and gives as output the elements of  $\text{Br } X / \text{Br}_0 X$ .

1. Compute the characteristic polynomial

$$f(\lambda, \mu) = \det(\lambda A + \mu A'),$$

where  $A$  and  $A'$  are the matrices corresponding to the quadratic forms associated to  $Q$  and  $Q'$ .

2. If  $f(\lambda, \mu) \in k[\lambda, \mu]$  is irreducible or has an irreducible quartic factor then

$$\text{Br } X = \text{Br}_0 X.$$

3. If there exists three degree 1 points  $\{T_0, T_1, T_2\}$  such that each pair satisfies (\*). Then  $\text{Br } X / \text{Br}_0 X \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  and

$$\frac{\text{Br } X}{\text{Br}_0 X} = \{\text{id}, \mathcal{A}_{\{T_0, T_1\}}, \mathcal{A}_{\{T_1, T_2\}}, \mathcal{A}_{\{T_0, T_2\}}\}.$$

4. At this step  $\# \text{Br } X / \text{Br}_0 X \leq 2$ . If there exists a subscheme  $\mathcal{T} \subset \mathcal{S}$  that satisfies (\*) and there exists a  $T \in \mathcal{S} - \mathcal{T}$  such that  $\epsilon_T \notin k(T)^{\times 2}$  then

$$\frac{\text{Br } X}{\text{Br}_0 X} = \{\text{id}, \mathcal{A}_{\mathcal{T}}\}.$$

5. At this step  $\text{Br } X = \text{Br}_0 X$ .

**Example 3.2.4.** Let  $k = \mathbb{C}(a, b)$  where  $a, b$  are independent transcendental variables over  $\mathbb{C}$ . Consider the del Pezzo surface  $X$  of degree 4 defined by the intersection of the two quadrics in  $\mathbb{P}_k^4$

$$Q : ax_0^2 + x_1^2 + x_2^2 + bx_4^2 = 0$$

$$Q' : x_0^2 + abx_1^2 + x_2^2 + x_3^2 = 0.$$

The characteristic polynomial is  $f(\lambda, \mu) = 2^5 b \lambda \mu (\lambda + \mu) (\lambda a + \mu) (\lambda + \mu a)$ . It is separable and split. A direct computation shows that the only pair satisfying (\*) is

$$\mathcal{T} = \{T_0 = [0 : 1], T_1 = [1 : 0]\}$$

with  $\epsilon_{T_1} = \epsilon_{T_2} = 2^4 ab$ . Furthermore, the point  $T_3 = [1 : -1] \in \mathcal{S} - \mathcal{T}$  has discriminant  $\epsilon_{T_3} = 2^4 b(a-1)(1-ab) \notin k(T_3)^{\times 2}$ . Therefore  $\text{Br } X / \text{Br}_0 X \simeq \mathbb{Z}/2\mathbb{Z}$ . Moreover,  $\text{Br } X / \text{Br}_0 X$  is generated by the class of the algebra

$$\mathcal{A}_{\mathcal{T}} = \left( ab, \frac{(2x_1 + 2ix_2)(2x_0 + 2ix_2)}{(x_0 + x_1)^2} \right),$$

where  $2x_1 + 2ix_2$  and  $2x_0 + 2ix_2$  are the linear forms with associated hyperplanes tangent to  $Q$  at  $P_1 = [0 : 1 : i : 0 : 0]$  and to  $Q'$  at  $P_2 = [1 : 0 : i : 0 : 0]$  respectively.

## Chapter 4

**AN EXAMPLE OF A DEL PEZZO SURFACE OF DEGREE 4  
WITH TRIVIAL BRAUER GROUP AND NON TRIVIAL  
 $H^1(K, \text{Pic } \overline{X})$**

In this section we prove the main theorem.

**Theorem 4.0.1.** *Let  $k = \mathbb{Q}^{\text{cycl}}(a, b, c)$  where  $a, b$  and  $c$  are independent transcendental elements. Let  $X$  be the del Pezzo surface of degree 4 in  $\mathbb{P}_k^4$  defined by the intersection of the following two quadrics*

$$\begin{aligned} Q &: ax_0^2 + bx_1^2 + x_2^2 + cx_4^2 = 0 \\ Q' &: bcx_0^2 + x_1^2 + x_2^2 + ax_3^2 = 0. \end{aligned}$$

Then  $H^1(k, \text{Pic } \overline{X}) \simeq \mathbb{Z}/2\mathbb{Z}$  while  $\frac{\text{Br } X}{\text{Br } k}$  is trivial.

The following corollary follows immediately by the algorithm in Chapter 3.

**Corollary 4.0.2.** *The assumption that  $Q_T$  has a smooth  $k(T)$ -point for every  $T \in \mathcal{T}$  in Theorem 3.2.3 cannot be omitted entirely to apply the algorithm for computing  $\text{Br } X/\text{Br } k$ .*

□

The proof of Theorem 4.0.1 relies on the functoriality with respect to base extension from  $k$  to  $L = k(\sqrt{a})$  of the Hochschild-Serre spectral sequence:

$$0 \rightarrow \text{Pic } X \rightarrow (\text{Pic } \overline{X})^{G_k} \rightarrow \text{Br } k \rightarrow \text{Br}_1 X \rightarrow H^1(k, \text{Pic } \overline{X}) \rightarrow H^3(k, \mathbb{G}_m),$$

where  $\text{Br}_1 X = \ker(\text{Br } X \rightarrow \text{Br } \overline{X})$ . Since  $X$  is rational then  $\text{Br}_1 X = \text{Br } X$ . Later in this chapter, we prove that the algorithm in [30, Section 4.1], that is explained in Chapter 3, can be applied over  $L$  and that the generating Brauer class of  $\text{Br } X_L/\text{Br } L$  is not in the image of  $\text{Br } X/\text{Br } k$ .

### 4.1 Degeneracy locus of $X$

Let  $A$  and  $A'$  be the matrices associated to the quadratic forms of the quadrics  $Q$  and  $Q'$  respectively. The characteristic polynomial of the pencil of quadrics  $\{\lambda Q + \mu Q' : [\lambda : \mu] \in \mathbb{P}^1\}$  is

$$f(\lambda, \mu) = \det(\lambda A + \mu A') = 32\mu\lambda(\lambda + \mu)(b\lambda + \mu)(a\lambda + bc\mu). \quad (4.1)$$

Since all the irreducible factors of  $f(\lambda, \mu)$  are distinct, it is separable in  $k[\lambda, \mu]$ . Therefore by Proposition 2.2.10,  $X$  is smooth. Moreover, the degeneracy locus  $\mathcal{S}$  of this del Pezzo surface consists of the five degree 1 points

$$T_0 = [1 : 0], T_1 = [0 : 1], T_2 = [1 : -1], T_3 = [1 : -b], \text{ and } T_4 = [bc : -a]$$

corresponding to the linear factors of  $f(\lambda, \mu)$ .

For each point in the degeneracy locus  $\mathcal{S}$ , we compute the corresponding quadric and discriminant as explained in Chapter 3. We show the case corresponding to  $T_0$  and summarize the rest in an array below. The singular locus of the quadric  $Q_{T_0} = V(ax_0^2 + bx_1^2 + x_2^2 + cx_4^2)$  is  $[0 : 0 : 0 : 1 : 0]$ . So we may choose  $H_{T_0}$  to be  $V(x_3)$ . By a direct computation,

$$\epsilon_{T_0} = 16abc \sim abc \pmod{k(T_0)^{\times 2}} = k^{\times 2}.$$

The quadrics and discriminants corresponding to all the points  $\{T_0, \dots, T_4\}$  are summarized below.

$$\begin{aligned} Q_{T_0} : ax_0^2 + bx_1^2 + x_2^2 + cx_4^2 &= 0, & \epsilon_{T_0} &\sim abc \\ Q_{T_1} : bcx_0^2 + x_1^2 + x_2^2 + ax_3^2 &= 0, & \epsilon_{T_1} &\sim abc \\ Q_{T_2} : (a - bc)x_0^2 + (b - 1)x_1^2 - ax_3^2 + cx_4^2 &= 0, & \epsilon_{T_2} &\sim ac(b - 1)(a - bc) \\ Q_{T_3} : (a - b^2c)x_0^2 + (1 - b)x_2^2 - abx_3^2 + cx_4^2 &= 0, & \epsilon_{T_3} &\sim abc(1 - b)(a - b^2c) \\ Q_{T_4} : (b^2c - a)x_1^2 + (bc - a)x_2^2 - a^2x_3^2 + bc^2x_4^2 &= 0, & \epsilon_{T_4} &\sim b(b^2c - a)(bc - a) \end{aligned}$$

### 4.2 Computing $\mathbf{H}^1(k, \text{Pic } \overline{X})$ and $\mathbf{H}^1(L, \text{Pic } \overline{X})$

The first part of the following Lemma is used to compute  $\mathbf{H}^1(k, \text{Pic } \overline{X})$  and  $\mathbf{H}^1(L, \text{Pic } \overline{X})$  and the second part will be used later in the proof of Theorem 4.0.1 to replace  $\text{Br}_0 X$  and

$\text{Br}_0 X_L$  by  $\text{Br } k$  and  $\text{Br } L$  respectively.

**Lemma 4.2.1.** *Let  $K = k$  or  $L$ , where  $k$  and  $L$  are the fields defined before. For  $X$  as before we have,*

1.  $\frac{(\text{Pic } \bar{X}/2 \text{ Pic } \bar{X})^{G_K}}{((\text{Pic } \bar{X})^{G_K}/2(\text{Pic } \bar{X})^{G_K})} = \langle C_0 + C_1 \rangle$
2.  $(\text{Pic } \bar{X})^{G_K} = \langle H \rangle$ .

*Proof.* We start by proving that  $(\text{Pic } \bar{X}/2 \text{ Pic } \bar{X})^{G_K} = \langle H, C_0 + C_1 \rangle$ . By Proposition 2.2.12, classes in  $\text{Pic } \bar{X}/2 \text{ Pic } \bar{X}$  can be represented by divisors of the form  $D = \beta(\frac{H+\sum_i C_i}{2}) + \sum_i \alpha_i C_i$  where  $\beta$  and  $\alpha_i$  are either 0 or 1 for all  $i \in \{0, \dots, 4\}$ . Let  $\sigma$  be any element in  $G_K$ . By Proposition 2.2.13 and since each  $T \in \mathcal{S}$  has degree 1,  $\sigma$  is the product of an even number of exchanges between  $C_i$  and  $C'_i = H - C_i$  for  $i \in \{0, \dots, 4\}$ . Let  $I \subset \{0, \dots, 4\}$  be the set of indices of the exchanges that are factors of  $\sigma$ . Since  $\sigma$  is the product of an even number of exchanges,  $I$  has even cardinality. The Galois element  $\sigma$  is determined by the set of indices  $I$ ; so we denote it by  $\sigma_I$ . We are interested in characterizing  $D \in (\text{Pic } \bar{X}/2 \text{ Pic } \bar{X})^{G_K}$  or equivalently  $D \in \text{Pic } \bar{X}$  such that  $D - \sigma_I D \in 2 \text{ Pic } \bar{X}$  for every  $\sigma_I \in G_K$ . First we compute  $\sigma_I D$  for any  $\sigma_I \in G_K$ . Let  $E := \frac{H+\sum_i C_i}{2}$  and  $\gamma := \sum_{i \in I} \alpha_i$ .

$$\begin{aligned}
\sigma_I D &= \sigma(\beta(E) + \sum_i \alpha_i C_i) \\
&= \beta \left( \frac{H + \sum_{i \in I} (H - C_i) + \sum_{i \notin I} C_i}{2} \right) + \sum_{i \notin I} \alpha_i C_i + \sum_{i \in I} \alpha_i (H - C_i) \\
&= \beta \left( (1 + |I|)E - \frac{2 + |I|}{2} \sum_{i \in I} C_i - \frac{|I|}{2} \sum_{i \notin I} C_i \right) + \sum_{i \notin I} \alpha_i C_i + 2\gamma E \\
&\quad - \sum_{i \in I} (\gamma + \alpha_i) C_i - \gamma \sum_{i \notin I} C_i \\
&= (\beta + \beta|I| + 2\gamma)E - \sum_{i \in I} \left( \frac{\beta|I| + 2\gamma}{2} + \beta + 3\alpha_i \right) C_i - \frac{\beta|I| + 2\gamma + 2\alpha_i}{2} \sum_{i \notin I} C_i.
\end{aligned}$$

We expand and arrange  $D - \sigma_I D$  as

$$D - \sigma_I D = -(\beta|I| + 2\gamma)E + \sum_{i \in I} \left( \frac{\beta|I| + 2\gamma}{2} + \beta + 2\alpha_i \right) C_i + \frac{\beta|I| + 2\gamma}{2} \sum_{i \notin I} C_i.$$

By transitivity of the action of  $G_K$ , Proposition 2.2.13, we may assume  $I$  to be non trivial.

By Considering the coefficients of  $D - \sigma_I D$  we get

$$\frac{\beta|I| + 2\gamma}{2} \equiv 0 \pmod{2}, \quad \frac{\beta|I| + 2\gamma}{2} + \beta + 2\alpha_j \equiv 0 \pmod{2}.$$

Hence  $\beta = 0$ . So  $\gamma$  is even.

Now we consider the possibilities of the even cardinality sets  $I \subset \{0, \dots, 4\}$ . Since  $C_0, C_1$  are defined over  $K(\sqrt{\epsilon_{T_0}}) = K(\sqrt{\epsilon_{T_1}})$ , then either both exchanges between  $\{C_0, C'_0\}$  and  $\{C_1, C'_1\}$  are factors of  $\sigma \in G_K$  or both are not. Hence  $\#(I \cap \{0, 1\}) \neq 1$ . By the computation of  $D - \sigma_I D$  before when  $I = \{0, 1\}$ , we deduce that  $\alpha_0 + \alpha_1 \equiv 0 \pmod{2}$ . Since  $I$  has even cardinality and  $\#(I \cap \{0, 1\}) \neq 1$  and by transitivity of the action of  $G_K$  on  $\{C_2, C'_2\}$ ,  $\{C_3, C'_3\}$ , and  $\{C_4, C'_4\}$ , the nontrivial possibilities of  $I - \{0, 1\} \cap I$  are  $\{2, 3\}$ ,  $\{2, 4\}$ , and  $\{3, 4\}$ . The class of the divisor  $D \pmod{2}$  is fixed by  $\sigma_I$  where  $I - \{0, 1\} \cap I$  is  $\{2, 3\}$ ,  $\{2, 4\}$ , or  $\{3, 4\}$  if and only if  $\alpha_2 + \alpha_3$ ,  $\alpha_2 + \alpha_4$ , or  $\alpha_3 + \alpha_4$  are even respectively. From the discussion before we deduce

$$\alpha_0 + \alpha_1 \equiv 0, \quad \alpha_2 + \alpha_3 \equiv 0, \quad \alpha_2 + \alpha_4 \equiv 0, \quad \text{and} \quad \alpha_3 + \alpha_4 \equiv 0 \pmod{2}.$$

Since  $\alpha_i \in \{0, 1\}$ , from the above congruences we deduce that  $D$  is  $0$ ,  $C_0 + C_1$ ,  $C_2 + C_3 + C_4$ , or  $\Sigma_i C_i$ . Hence  $(\text{Pic } \overline{X}/2 \text{ Pic } \overline{X})^{G_K} = \langle C_0 + C_1, C_2 + C_3 + C_4 \rangle$ . Further,

$$C_2 + C_3 + C_4 = C_0 + C_1 + 2\left(\frac{1}{2}(H + \Sigma C_i)\right) - 2C_0 - 2C_1 - H.$$

So we may rewrite the generators as  $(\text{Pic } \overline{X}/2 \text{ Pic } \overline{X})^{G_K} = \langle H, C_0 + C_1 \rangle$ .

We prove that  $\frac{(\text{Pic } \overline{X}/2 \text{ Pic } \overline{X})^{G_K}}{((\text{Pic } \overline{X})^{G_K}/2(\text{Pic } \overline{X})^{G_K})} = \langle C_0 + C_1 \rangle$  or equivalently that  $(\text{Pic } \overline{X})^{G_K} = \langle H \rangle$ . Let  $\Sigma_i a_i C_i$  be a nontrivial combination of the  $C_i$ 's. Let  $j \in \{0, \dots, 4\}$  be an arbitrary element such that  $a_j \neq 0$ . By Proposition 2.2.13 and the fact that  $\epsilon_j$  is not a square in  $K(T_j)^\times = K^\times$ , there exists a  $\tau \in G_K$  such that  $\tau C_j = C'_j = H - C_j$ . Therefore,  $\tau(\Sigma_i a_i C_i) = -\alpha_j C_j + D'$  where  $D'$  is a linear combination of  $C_i$   $i \neq j$ . Hence  $\Sigma_i a_i C_i$  is not fixed by  $G_K$ . Therefore  $(\text{Pic } \overline{X})^{G_K} = \langle H \rangle$  and this proves (1) and (2).  $\square$

**Proposition 4.2.2.** *For  $X$  and  $k$ , and  $L$  as before, we have  $H^1(k, \text{Pic } \overline{X}) \simeq H^1(L, \text{Pic } \overline{X}) \simeq \mathbb{Z}/2\mathbb{Z}$  and  $\text{Res}: H^1(k, \text{Pic } \overline{X}) \rightarrow H^1(L, \text{Pic } \overline{X})$  is an isomorphism.*

*Proof.* Consider the following short exact sequence

$$0 \rightarrow \text{Pic } \bar{X} \xrightarrow{\times 2} \text{Pic } \bar{X} \rightarrow (\text{Pic } \bar{X}/2\text{Pic } \bar{X}) \rightarrow 0.$$

The connecting morphism in the induced long exact sequence yields

$$\frac{(\text{Pic } \bar{X}/2\text{Pic } \bar{X})^{G_K}}{((\text{Pic } \bar{X})^{G_K}/2(\text{Pic } \bar{X})^{G_K})} \xrightarrow{\sim} \text{H}^1(K, \text{Pic } \bar{X})[2]; \quad [D] \mapsto (\sigma \mapsto \frac{1}{2}(d - \sigma d)) \quad (4.2)$$

where  $K = k$  or  $L$ , and  $d$  is a lift of  $D$  to  $\text{Pic } \bar{X}$ . By Lemma 4.2.1 (1) and by the isomorphism (4.2), we deduce that  $\text{H}^1(K, \text{Pic } \bar{X})[2] \simeq \mathbb{Z}/2\mathbb{Z}$ . Further,  $\text{H}^1(K, \text{Pic } \bar{X})$  is 2-torsion by Proposition 3.2.2; so  $\text{H}^1(k, \text{Pic } \bar{X}) \simeq \text{H}^1(L, \text{Pic } \bar{X}) \simeq \mathbb{Z}/2\mathbb{Z}$ . Moreover, the image of the cocycle corresponding to  $C_0 + C_1$ , that generates  $\frac{(\text{Pic } \bar{X}/2\text{Pic } \bar{X})^{G_K}}{((\text{Pic } \bar{X})^{G_K}/2(\text{Pic } \bar{X})^{G_K})}$  by Lemma 4.2.1, under the restriction map  $\text{Res}: \text{H}^1(k, \text{Pic } \bar{X}) \rightarrow \text{H}^1(L, \text{Pic } \bar{X})$  is the cocycle corresponding to  $C_0 + C_1$ . So  $\text{Res}: \text{H}^1(k, \text{Pic } \bar{X}) \rightarrow \text{H}^1(L, \text{Pic } \bar{X})$  is an isomorphism.  $\square$

### 4.3 The Brauer group of $X$ over $L = k(\sqrt{a})$

We use the algorithm in [30, Section 4.1], which we have explained in Chapter 3, to prove that  $\text{Br } X_L/\text{Br } L \simeq \mathbb{Z}/2\mathbb{Z}$  and to explicitly construct a non trivial algebra  $\mathcal{A}$  whose class generates  $\text{Br } X_L/\text{Br } L$ . This algebra will be used in the proof of Theorem 4.0.1.

**Proposition 4.3.1.** *The Brauer group  $\text{Br } X_L/\text{Br } L$  is isomorphic to  $\mathbb{Z}/2\mathbb{Z}$  and is generated by the class of the quaternion algebra:*

$$\mathcal{A} = \left( \epsilon_{T_0}, \frac{(2aix_0 + 2\sqrt{a}x_2)(2ix_1 + 2x_2)}{(x_0 + x_1)^2} \right).$$

Moreover,  $\mathcal{A} - \sigma\mathcal{A} \sim (c, b) \in \text{Br } L$ .

*Proof.* The same computation of the degeneracy locus, associated quadrics, and discriminants of  $X$  in Section 4.1 still work over  $L$ . Moreover  $\mathcal{T} = \{T_0, T_1\}$  is a degree two subscheme of  $\mathcal{S}$  such that  $\prod_{T \in \mathcal{T}} N_{L(T)/L} = \epsilon_{T_0}\epsilon_{T_1} = a^2b^2c^2 \in L(T)^{\times 2} = L^{\times 2}$ , and the element  $\epsilon_{T_0} = \epsilon_{T_1} = bc$  is non trivial in  $L^{\times}/L^{\times 2}$ . Hence  $\mathcal{T}$  satisfies (\*) as defined in Chapter 3. By direct computations we show  $\{T_0, T_1\}$  is the only subscheme of  $\mathcal{S}$  that satisfies (\*). Moreover, the quadrics  $Q_{T_0} : ax_0^2 + bx_1^2 + x_2^2 + cx_4^2 = 0$  and  $Q_{T_1} : bcx_0^2 + x_1^2 + x_2^2 + ax_3^2 = 0$  have

smooth  $L$ -points  $P_{T_0} = [i : 0 : \sqrt{a} : 0 : 0]$  and  $P_{T_1} = [0 : i : 1 : 0 : 0]$  respectively. Further,  $\epsilon_{T_2}$  is not a square in  $L^\times$ . So by Theorem 3.2.3  $\text{Br } X_L / \text{Br } L \simeq H^1(L, \text{Pic } \overline{X}) \simeq \mathbb{Z}/2\mathbb{Z}$  and is generated by the algebra

$$\mathcal{A} = \left( \epsilon_{T_0}, \frac{l_{T_0} l_{T_1}}{l^2} \right)$$

where  $l_T$  is the  $L(T)$ -linear form such that the associated hyperplane is tangent to  $Q_T$  at a smooth point  $P_T$  of  $Q_T$  for  $T \in \mathcal{T}$ , and  $l$  is any linear form.

Computing the linear forms we get,  $l_{T_0} : 2aix_0 + 2\sqrt{a}x_2$ , and  $l_{T_1} : 2ix_1 + 2x_2$ . Substituting these into  $\mathcal{A}$  yields the required algebra.

We have

$$\begin{aligned} \mathcal{A} - \sigma\mathcal{A} &= \left( bc, \frac{(2aix_0 + 2\sqrt{a}x_2)(2ix_1 + 2x_2)}{(x_0 + x_1)^2} \right) \left( bc, \frac{(2aix_0 - 2\sqrt{a}x_2)(2ix_1 + 2x_2)}{(x_0 + x_1)^2} \right) \\ &= \left( bc, \frac{-4(\sqrt{a})^2(ax_0^2 + x_2^2)(2ix_1 + 2x_2)^2}{(x_0 + x_1)^4} \right) \\ &\sim \left( bc, \frac{-(ax_0^2 + x_2^2)}{(x_0 + x_1)^2} \right) \end{aligned}$$

By the defining equation of  $Q'$ ,  $ax_0^2 + x_2^2 = -bx_1^2 - cx_4^2 = -b(x_1^2 - cb(i\frac{x_4}{b})^2)$ . So  $\mathcal{A} - \sigma\mathcal{A} \sim (bc, b) \sim (c, b)$ .  $\square$

#### 4.4 Characterizing $\text{im}(\text{Br } X / \text{Br } k \rightarrow \text{Br } X_L / \text{Br } L)$

Let  $\text{Gal}(L/k) = \langle \sigma \rangle$ .

**Lemma 4.4.1.** *Let  $\text{Gal}(L/k) = \langle \sigma \rangle$ . If  $\mathcal{A} - \sigma\mathcal{A} \neq x - \sigma x$  for every  $x \in \text{Br } L$  then  $[\mathcal{A}] \notin \text{im}(\text{Br } X / \text{Br } k \rightarrow \text{Br } X_L / \text{Br } L)$ .*

*Proof.* By [10, Proposition 3.3.17], the generalized inflation-restriction sequence for the field extension  $L(X)/k(X)$  is

$$0 \rightarrow H^2(\text{Gal}(L/k), L(X)^\times) \xrightarrow{\text{inf}} H^2(k(X), \overline{k(X)}^\times) \xrightarrow{\text{Res}} H^2(L(X), \overline{k(X)}^\times)^{\text{Gal}(L/k)}.$$

If the class of  $\mathcal{A}$  is in the image of  $\text{Br } X/\text{Br } k \rightarrow \text{Br } X_L/\text{Br } L$ , then there exists  $x \in \text{Br } L$  such that  $\mathcal{A} - x \in \text{im}(\text{Br } X \rightarrow \text{Br } X_L)$ . Hence  $\mathcal{A} - x \in \text{im}(\text{Br } k(X) \rightarrow \text{Br } L(X))$ . By the generalized inflation-restriction sequence above,  $\mathcal{A} - x$  is fixed by  $\text{Gal}(L/k) = \langle \sigma \rangle$ . Rearranging we get that  $\mathcal{A} - \sigma\mathcal{A} = x - \sigma x$ .  $\square$

#### 4.5 Proof of Theorem 4.0.1

First we may replace  $\text{Br}_0 X$  and  $\text{Br}_0 X_L$  by  $\text{Br } k$  and  $\text{Br } L$  respectively because the map  $(\text{Pic } \bar{X})^{G_K} = \langle H \rangle \rightarrow \text{Br } K$  is trivial by the exact sequence that follows from the Hochschild-Serre spectral sequence where  $K = k$  or  $L$ .

For the sake of contradiction we assume that  $\text{Br } X/\text{Br } k$  is non trivial. Since there is an injection  $\text{Br } X/\text{Br } k \hookrightarrow H^1(k, \text{Pic } \bar{X}) \simeq \mathbb{Z}/2\mathbb{Z}$  by the exact sequence that follows from the Hochschild-Serre spectral sequence and  $\text{Br } X/\text{Br } k$  is nontrivial, there is a unique nontrivial class in  $\text{Br } X/\text{Br } k$ ; denote this class by  $[\mathcal{B}]$ . By Proposition 4.2.2, Proposition 4.3.1 and the functoriality of the Hochschild-Serre spectral sequence we have

$$\begin{array}{ccc} \text{Br } X/\text{Br } k & \xrightarrow{\simeq} & H^1(k, \text{Pic } \bar{X}) \\ \downarrow & & \downarrow \simeq \\ \text{Br } X_L/\text{Br } L & \xrightarrow{\simeq} & H^1(L, \text{Pic } \bar{X}). \end{array}$$

So  $[\mathcal{B}] \in \text{Br } X/\text{Br } k$  gets mapped to  $[\mathcal{A}] \in \text{Br } X_L/\text{Br } L$  as defined in Proposition 4.3.1 by the field extension map. We will show that any algebra in the class of  $[\mathcal{A}] \in \text{Br } X_L/\text{Br } L$  is not in the image of the map  $\text{Br } X \rightarrow \text{Br } X_L$ , thus resulting in a contradiction. By Lemma 4.4.1 it is enough to prove that  $\mathcal{A} - x$  is not fixed by  $\sigma$  for all  $x \in \text{Br } L$ .

Suppose that there exists  $x \in \text{Br } L$  such that  $\mathcal{A} - x$  is fixed by  $\sigma$ , i.e.,  $\mathcal{A} - \sigma\mathcal{A} = x - \sigma x$ . By Proposition 4.3.1, both sides of the equation  $\mathcal{A} - \sigma\mathcal{A} = x - \sigma x$  are in  $\text{Br } L$ . Let  $Y = \text{Spec } \mathbb{Q}^{\text{cycl}}(b, c)[\sqrt{a}]$ . Let  $P : \sqrt{a} = 0$  be a divisor on  $Y$ . Because  $L$  is the function field of  $Y$  and  $P$  is a divisor on  $Y$ , we have the residue sequence, ([13], [11],[12]),

$$\text{Br } \mathcal{O}_{Y,\eta} \rightarrow \text{Br } L \xrightarrow{\partial_P} H^1(\mathbb{Q}^{\text{cycl}}(b, c)(P), \mathbb{Q}/\mathbb{Z}).$$

If  $x$  is cyclic of degree  $n$  then  $\partial_P(x)$  is determined by a degree  $n$  cyclic extension of  $\mathbb{Q}^{cycl}(b, c)$  and a choice of a generator of the Galois group of the extension. By Kummer Theory the cyclic extension determined by  $\partial_P(x)$  is of the form  $\mathbb{Q}^{cycl}(b, c)(\sqrt[n]{\alpha})/\mathbb{Q}^{cycl}(b, c)$  where  $\alpha \in \mathbb{Q}^{cycl}(b, c)^\times$ . Let  $f = f(b, c) \in L$  be a function of  $b$ , and  $c$ . By Theorem 2.1.26 the residue  $\partial_P$  of the cyclic algebra  $(\sqrt[n]{a}, f(b, c))_n \in \text{Br } L$  is determined by the cyclic extension

$$\mathbb{Q}^{cycl}(b, c)(\sqrt[n]{(\sqrt{a})^{-v_P(f)} f^{v_P(\sqrt{a})}})/\mathbb{Q}^{cycl}(b, c)$$

and a choice of generator of the Galois group of this extension. Therefore we may choose  $f$  such that  $\partial_P((\sqrt{a}, f(b, c))_n) = \partial_P(x)$ . Furthermore  $\sigma((\sqrt{a}, f(b, c))_n) = (-\sqrt{a}, f(b, c))_n = (\sqrt{a}, f(b, c))_n$  because  $-1$  is an  $n$ -th root of unity in  $L$ . Since  $(\sqrt{a}, f(a, b))_n$  is fixed by  $\sigma$ , subtracting  $(\sqrt{a}, f(b, c))$  from  $x$  will not change  $x - \sigma x$ . Moreover,  $(\sqrt{a}, f(b, c))$  has the same residue as  $x$  at  $P$ ; so we may assume without loss of generality that  $x$  is unramified at  $P$ . By Murkurjev-Suslin Theorem, 2.1.16, and the fact that  $\partial_P$  is a homomorphism we extend this argument to any central simple algebra  $x \in \text{Br } L$ . So we assume  $x$  is unramified at  $P$  in general. We claim that  $\mathcal{A} - \sigma\mathcal{A}$  is also unramified at  $P$  and we prove it later. Hence we may specialize the equation  $\mathcal{A} - \sigma\mathcal{A} = x - \sigma x$  at the divisor  $P$ ,

$$(\mathcal{A} - \sigma\mathcal{A})|_P = x|_P - (\sigma x)|_P.$$

The action of  $\sigma$  on  $x$  commutes with specialization. Furthermore,  $P$  is invariant under  $\sigma$ . Hence  $(\sigma x)|_P = \sigma x|_P = x|_P$ . So  $(\mathcal{A} - \sigma\mathcal{A})|_P$  is trivial in  $\text{Br } \kappa$  where  $\kappa$  is the residue field of  $Y$  at  $P$ . In our example  $\kappa = \mathbb{Q}^{cycl}(b, c)$ . Since  $\mathcal{A} - \sigma\mathcal{A} \sim (b, c)$ , by Proposition 4.3.1, is a constant Azumaya algebra in  $\text{Br } \mathcal{O}_{Y, \eta}$ ,  $(\mathcal{A} - \sigma\mathcal{A})|_P = \mathcal{A} - \sigma\mathcal{A} \sim (b, c)$ . Hence  $(b, c)$  is trivial in  $\text{Br } \mathbb{Q}^{cycl}(b, c)$ . This is a contradiction because the extension  $\mathbb{Q}^{cycl}(b)(\sqrt{c^{-v_D(b)} b^{v_D(c)}})/\mathbb{Q}^{cycl}(b)$  associated to  $\partial_{c=0}((c, b))$  by Theorem 2.1.26 is non trivial.  $\square$

Now we prove the claim that the algebra  $\mathcal{A} - \sigma\mathcal{A}$  is unramified at  $P$ :  $\sqrt{a} = 0$ . We have the residue sequence, ([13], [11],[12])

$$\text{Br } \mathcal{O}_{Y, \eta} \rightarrow \text{Br } L \xrightarrow{\partial_P} \text{H}^1(\mathbb{Q}^{cycl}(b, c)(P), \mathbb{Q}/\mathbb{Z}).$$

By Theorem 2.1.26 the residue  $\partial_P$  of the cyclic algebra  $\mathcal{A} - \sigma\mathcal{A} \sim (b, c) \in \text{Br } L$  is determined by the cyclic extension  $\mathbb{Q}^{cycl}(b, c)(\sqrt{b^{-v_P(c)}c^{v_P(b)}})/\mathbb{Q}^{cycl}(b, c)$  and a choice of a generator of the Galois group of this extension. The valuations  $v_P(c) = v_P(b) = 0$ . By the last two sentences, we deduce that  $\partial_P(\mathcal{A} - \sigma\mathcal{A}) = 0$ . Therefore  $\mathcal{A} - \sigma\mathcal{A}$  is unramified at  $P$ .  $\square$

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