# Partial differential equations and volume-minimization problems for Lagrangian submanifolds

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## Abstract

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We study several problems relating to calibrated geometry and volume-minimizing Lagrangian submanifolds. We prove a rigidity theorem for calibrated submanifolds with flat normal bundles, showing that a submanifold which is calibrated by a parallel form and has commuting shape operators is totally geodesic. In joint work with Yu Yuan, we derive constant rank theorems for the special Lagrangian equation, an application of which is a rigidity result for special Lagrangian cones, and the quadratic Hessian equation. In joint work with Arunima Bhattacharya, we consider regularity of Hamiltonian stationary Lagrangian graphs, proving that a Hamiltonian stationary graph which is Hölder continuous with exponent larger than 1/3 and has supercritical Lagrangian phase is smooth. We then construct singular solutions showing that 1/3 is the optimal Hölder exponent. Finally, in joint work with Micah Warren, we construct a properly immersed translating solution to curve diffusion flow, which is a fourth order analogue of curve shortening flow, and the one-dimensional case of the gradient flow for volume within a Hamiltonian isotopy class.

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# DEDICATION

to Sarah and Charlie

# Chapter 1

# INTRODUCTION

This thesis compiles the author's works on calibrated geometry and partial differential equations related to volume-minimization problems for Lagrangian submanifolds, including joint works with Yu Yuan, Arunima Bhattacharya, and Micah Warren.

In chapter 2 we provide some introductory material. We briefly discuss extrinsic geometry of submanifolds in order to establish conventions for the second fundamental form. We then provide a quick overview of calibrations, followed by a discussion of Lagrangian submanifolds in which we introduce all the PDEs which we study in later chapters. Finally, we describe the techniques of Legendre-Lewy transformation and Lewy-Yuan rotation, which are used to prove the main results in chapters 4 and 5.

Chapter 3 presents the results of [54]. The main result here is a rigidity theorem for submanifolds in Euclidean space which are calibrated by constant-coefficient differential forms. We prove that if such a submanifold has a flat normal bundle, then it must be a plane. A similar rigidity result holds in curved ambient manifolds, provided the calibration is a parallel form and any two shape operators of the submanifold commute (as endomorphisms of the tangent space). Under these hypotheses, we conclude that the submanifold must be totally geodesic. This work is motivated by work of Li [41], who showed that special Lagrangian cones in $\mathbb{C}^n$, with $n \leq 7$, with flat normal bundles are planes. The main technical tool here is a partial differential equation for the tangent plane of a minimal submanifold when viewed as a section of the restriction of the bundle of $k$-planes in the tangent bundle of the ambient space, which originates in the work of Fischer-Colbrie [26]. This PDE simplifies to a useful form under the flat normal bundle assumption, and coupling it with the calibration condition produces an identity which forces the second fundamental form of the submanifold to vanish.

In chapter 4, we present joint work with Yu Yuan on constant rank theorems for special Lagrangian equations [56]. Here, we adapt a general constant rank theorem of Caffarelli-Guan-Ma [12] to saddle solutions of the special Lagrangian equation and quadratic Hessian equation. The Caffarelli-Guan-Ma theorem asserts that if $u$ is a smooth convex solution of an elliptic PDE, and the PDE satisfies a condition known as inverse-convexity, then the Hessian $D^2u$ has constant rank. This is a strong minimum principle for the eigenvalues of the Hessian. We show that if $u$ is a smooth solution of the special Lagrangian equation

$$\sum_{i=1}^{n} \arctan \lambda_i(D^2u) = \Theta$$

with $\lambda_{\min}(D^2u) \geq \frac{\Theta-\pi}{n}$, then a strong minimum principle holds for $\lambda_{\min}$. Consequences of this result include a new Liouville theorem for entire solutions and new rigidity results for special Lagrangian cones. The current frontier of the regularity theory for the special Lagrangian equation is the problem of whether $C^{1,1}$ solutions are necessarily smooth; this problem is equivalent to classification of graphical special Lagrangian cones. For the quadratic Hessian equation

$$\sigma_2(D^2u) = \sum_{1\leq i<j\leq n} \lambda_i(D^2u)\lambda_j(D^2u) = 1,$$

we prove that a strong minimum principle holds for $\lambda_{\min}$ provided that $\lambda_{\min} \geq -\sqrt{\frac{2}{n(n-1)}}$. This result can be thought of as a local counterpart to the corresponding Liouville theorem proven by Chang-Yuan [14]. The proofs of these results use Lewy-Yuan rotation and Legendre-Lewy transform to transform the solutions and PDEs so that they verify the hypotheses of the Caffarelli-Guan-Ma constant rank theorem. In the last section of chapter 4, we provide a proof of the Caffarelli-Guan-Ma constant rank theorem based on a viscosity approach.

In chapter 5, we present joint work with Arunima Bhattacharya on regularity for Hamiltonian stationary submanifolds [6]. We show that a Hölder continuous Hamiltonian stationary graph whose Lagrangian phase is above the critical value $(n-2)\frac{\pi}{2}$ is smooth when its Hölder exponent is above $\frac{1}{3}$. This extends work of Chen-Warren [18], who proved that Lipschitz continuous Hamiltonian stationary graphs with supercritical phase are smooth. Furthermore, we show that our assumption on the Hölder regularity is optimal by constructing Hamiltonian stationary submanifolds which are graphs of $C^{\frac{1}{3}}$ functions with arbitrarily large phase. The technique to prove the regularity result is to use a Lewy-Yuan rotation to transform a $C^{1,\alpha}$ potential function to a $C^{1,1}$ potential, and then apply Chen-Warren's regularity theorem, showing the original graph is a smooth submanifold. Rotating back to the original configuration, we juxtapose the assumed Hölder regularity with the Taylor expansion over a supposed vertical tangent plane to rule out singularities. Our construction of singular solutions follows a strategy used by several authors to construct singular solutions in other settings such as: [52, 78, 7, 47, 48]. We first use the Cauchy-Kovalevskaya theorem to construct an analytic solution with certain properties, and then apply a Lewy-Yuan rotation to produce a singularity.

Finally, in chapter 6, we present joint work with Micah Warren in which we construct a properly immersed translating solution to curve diffusion flow, which is a fourth order version of curve shortening flow [55]. Our motivation to study curve diffusion flow stems from the gradient flow for volume of Lagrangian submanifolds within a given Hamiltonian isotopy class, which reduces to curve diffusion flow in dimension 1 [20]. Our solution is a natural analogue of the famous grim reaper curve (the translating solution of curve shortening flow) for this fourth order flow, being asymptotic to a line parallel to the direction of translation. We apply a shooting method to a nonlinear ODE to produce our desired solution. The arguments here are based on delicate, yet elementary, ODE estimates.

# Chapter 2

# PRELIMINARIES

## *2.1 Extrinsic geometry of submanifolds*

Let $(M, g)$ be a Riemannian manifold and $\Sigma \subset M$ a submanifold. The restriction to $\Sigma$ of the tangent bundle $TM$ decomposes as the tangent and normal bundles of $\Sigma$:

$$TM|_\Sigma = T\Sigma \oplus N\Sigma.$$

The Levi-Civita connection $\bar{\nabla}$ on $M$ induces connections on $T\Sigma$ and $N\Sigma$ defined by

$$\nabla_X Y = (\bar{\nabla}_X Y)^T, \quad \nabla_X V = (\bar{\nabla}_X V)^N,$$

respectively, for $X \in T\Sigma$, $Y$ a local section of $T\Sigma$, and $V$ a local section of $N\Sigma$. Here, $(\cdot)^T$ and $(\cdot)^N$ are the orthogonal projections onto $T\Sigma$ and $N\Sigma$. The connections $\bar{\nabla}$ and $\nabla$ are related by the second fundamental form, the section of $T^*\Sigma \otimes T^*\Sigma \otimes N\Sigma$ defined by

$$B(X, Y) = (\bar{\nabla}_X Y)^N.$$

For a tangent field $Y$ and normal field $V$, differentiating the equation $\langle Y, V \rangle = 0$ gives

$$0 = X\langle Y, V \rangle = \langle \bar{\nabla}_X Y, V \rangle + \langle Y, \bar{\nabla}_X V \rangle = \langle B(X, Y), V \rangle + \langle Y, \bar{\nabla}_X V \rangle.$$

Thus, for a normal vector $V$, we define the shape operator $A_V$ as a section of $T^*\Sigma \otimes T\Sigma$ implicitly by

$$\langle A_V(X), Y \rangle = -\langle B(X, Y), V \rangle, \tag{2.1}$$

or equivalently, $A_V(X) = (\bar{\nabla}_X V)^T$.

Suppose $\{E_i\}$ is an orthonormal frame for $T\Sigma$ and $\{V_\alpha\}$ an orthonormal frame for $N\Sigma$. We denote the components of the second fundamental form by $h_{ij}^\alpha = \langle B(E_i, E_j), V_\alpha \rangle$.

The mean curvature of $\Sigma$ is defined as the trace of $B$,

$$H = \sum_i B(E_i, E_i) = \sum_{i,\alpha} h_{ii}^\alpha V_\alpha.$$

We shall always use $D$ to denote the standard Euclidean gradient and $\nabla$ to denote the gradient operator induced by the Riemannian metric, which will be clear from the context.

## 2.2 *Calibrations*

A differential $k$-form $\omega$ on a Riemannian manifold $(M^n, g)$ is called a calibration if

1. $d\omega = 0$ and

2. $\omega(\xi_1, \ldots, \xi_k) \leq |\xi_1 \wedge \cdots \wedge \xi_k|$ for all vectors $\xi_1, \ldots, \xi_k \in T_pM$.

We say the oriented $k$-plane spanned by the vectors $\xi_1, \ldots, \xi_k$ is calibrated by $\omega$ if $\omega(\xi_1, \ldots, \xi_k) = |\xi_1 \wedge \cdots \wedge \xi_k|$. A submanifold whose tangent planes are all calibrated by $\omega$ is called a calibrated submanifold.

Calibrations are of fundamental importance in the theory of minimal submanifolds as a device for showing that a minimal submanifold is in fact volume-minimizing.

**Theorem 2.2.1** (Federer [25])**.** *Suppose $M$ is a Riemannian manifold and $\omega$ is a calibration on $M$. If $\Sigma^k \subset M$ is calibrated by $\omega$, then $\Sigma$ minimizes volume in its homology class.*

A consequence is that calibrated submanifolds automatically have zero mean curvature, hence are minimal submanifolds.

A classical argument demonstrates that when $u : \Omega \to \mathbb{R}$ satisfies the minimal surface equation and $\Omega$ is convex, then the graph of $u$ is a volume-minimizer relative to its boundary. The key is that the minimal surface equation is equivalent to the volume form of the graph of $u$ being closed, so the form obtained on the cylinder $\Omega \times \mathbb{R}$ by translating the volume form of the graph vertically is a calibration.

There are more interesting calibrations which admit large families of calibrated submanifolds, in contrast to the example above which is adapted to a particular submanifold. Classically, Wirtinger's inequality [85] showed that the symplectic form on a Kähler manifold and certain constant multiples of its exterior powers calibrate all complex submanifolds. In the seminal work [30] (see also [29]), Harvey and Lawson introduced several new families of constant-coefficient calibrations on Euclidean spaces which admit large families of calibrated submanifolds. These are the special Lagrangian calibrations, the associative and coassociative calibrations, and the Cayley calibration. These calibrations provided the first nontrivial examples of volume-minimizing submanifolds of high codimension other than complex submanifolds. Like the calibrations derived from the symplectic form on a Kähler manifold, the calibrations introduced by Harvey and Lawson also exist naturally as parallel forms on manifolds of special holonomy [35], for instance, the special Lagrangian calibrations are natural objects on Calabi-Yau manifolds and are relevant in string theory.

Here, we shall be concerned primarily with the special Lagrangian calibrations, the $n$-forms $\mathrm{Re}(e^{-i\Theta}dz_1 \wedge \cdots \wedge dz_n)$ on $\mathbb{C}^n$.

### *2.3 Lagrangian submanifolds*

In a symplectic manifold $(M^{2n}, \sigma)$, a submanifold $\Sigma^n$ is Lagrangian if $\sigma|_\Sigma = 0$. We are primarily interested in Lagrangian submanifolds in the Euclidean space $\mathbb{C}^n = \mathbb{R}^n \times \mathbb{R}^n$ with the standard symplectic form $\sigma = \sum_{i=1}^n dx_i \wedge dy_i$. An equivalent characterization of Lagrangian submanifolds is that a submanifold $\Sigma \subset \mathbb{C}^n$ is Lagrangian if $J(T\Sigma) = N\Sigma$, where $J$ is the standard complex structure map. When $\Sigma$ is given as the graph of a function $F : \mathbb{R}^n \to \mathbb{R}^n$, the Lagrangian property is equivalent to the existence of a potential for the vector-valued function $F$, a scalar-valued function $u$ such that

$F = Du$. Often, the gradient graph characterization of Lagrangian submanifolds will be the most useful for us. Using the gradient graph parametrization, we can relate geometric information about a submanifold to the potential function $u$. On the submnaifold $\Sigma = \{(x, Du(x))\} \subset \mathbb{C}^n$, for instance, the induced Riemannian metric from the standard Euclidean metric is $g = I + (D^2u)^2$.

A real $n$-plane $\zeta$ in $\mathbb{C}^n$ is Lagrangian if there is a unitary transformation taking the standard basis (over $\mathbb{C}$) vectors to an orthonormal basis for $\zeta$. This gives a natural map from the set of Lagrangian planes to the group $\mathrm{U}(1)$ by choosing such a unitary transformation and applying the canonical homomorphism $\mathrm{U}(n) \to \mathrm{U}(1)$. This map is the (complex) determinant of the unitary transformation. This is just evaluation of the $n$-form $dz_1 \wedge \cdots \wedge dz_n$ on $\zeta$. We can thus define the phase of a Lagrangian plane $\zeta$ by

$$\theta = \arg(dz_1 \wedge \cdots \wedge dz_n(\zeta)) \pmod{2\pi}.$$

For a Lagrangian submanifold, we define the phase as a function on the submanifold by taking the phase of each tangent plane. The Lagrangian phase turns out to be of geometric significance, due to the following result of Harvey and Lawson.

**Theorem 2.3.1** (Harvey-Lawson [30])**.** *If $\Sigma$ is a Lagrangian submanifold of $\mathbb{C}^n$, then*

$$H = J\nabla\theta.$$

This shows that minimal Lagrangian are precisely the Lagrangian submanifolds with constant phase. In fact, minimal Lagrangian submanifolds are precisely the submanifolds calibrated by the special Lagrangian calibrations.

**Theorem 2.3.2** (Harvey-Lawson [30])**.** *A submanifold $\Sigma$ is calibrated by the form* $\mathrm{Re}(e^{-i\Theta}dz_1 \wedge \cdots \wedge dz_n)$ *if and only if $\Sigma$ is Lagrangian with constant phase $\Theta$.*

A Lagrangian submanifold with constant phase is called a special Lagrangian submanifold.

If $\Sigma = \{(x, Du(x))\}$, then after applying an orthogonal transformation on both factors of $\mathbb{R}^n \times \mathbb{R}^n$ to diagonalize $D^2u(x)$, an orthogonal basis for the tangent space at $x$ is given by the vectors

$$E_i = \partial_{x_i} + \lambda_i \partial_{y_i}.$$

The determinant of the complex linear map sending the standard basis of $\mathbb{C}^n$ to $E_i$ is $(1 + \sqrt{-1}\lambda_1)\dots(1 + \sqrt{-1}\lambda_n)$, where $\lambda_i$ are the eigenvalues of $D^2u$. Taking the argument, the Lagrangian phase is

$$\theta = \sum_{i=1}^{n} \arctan \lambda_i.$$

Therefore $u \in C^2$ is the potential for a minimal (and hence volume-minimizing) Lagrangian submanifold if and only if $u$ satisfies the special Lagrangian equation

$$\sum_{i=1}^{n} \arctan \lambda_i = \Theta = \text{constant}. \tag{2.2}$$

The special Lagrangian equation is an elliptic Hessian equation. An equivalent algebraic formulation is

$$\cos\Theta(\sigma_1(D^2u) - \sigma_3(D^2u) + \dots) - \sin\Theta(1 - \sigma_2(D^2u) - \sigma_4(D^2u) + \dots) = 0, \tag{2.3}$$

where $\sigma_j(D^2u)$ is the elementary symmetric polynomial of degree $j$ in the eigenvalues of $D^2u$ (by definition, $\sigma_j(D^2u) = 0$ if $j > n$). Thus, other interesting Hessian equations arise as special cases of the special Lagrangian equation. When $n = 2$ and $\Theta = 0$, (2.3) becomes $\sigma_1(D^2u) = \triangle u = 0$. When $n = 2, 3$ and $\Theta = \frac{\pi}{2}$, (2.3) becomes the quadratic Hessian equation:

$$\sigma_2(D^2u) = 1,$$

and when $n = 3$, $\Theta = k\pi$, (2.3) becomes

$$\triangle u = \det D^2u.$$

The parabolic equation associated to the special Lagrangian equation is also of geometric significance. The gradient flow for the volume functional, mean curvature flow, was shown by Smoczyk [71] to preserve Lagrangian submanifolds. Therefore the evolution can be studied at the level of the Lagrangian potential via the Lagrangian mean curvature flow equation

$$u_t = \sum_{i=1}^{n} \arctan \lambda_i. \tag{2.4}$$

Special Lagrangian submanifolds are volume-minimizing, hence they are critical points of the volume functional under any choice of variation vector field along the submanifold. A different volume-minimization problem is to look for Lagrangian submanifolds which are critical points of the volume functional under variations of the potential function. As variation vector fields, these are Hamiltonian variations, vector fields along $\Sigma$ of the form $J\nabla\varphi$. From the first variation formula and Theorem 2.3.1, the Euler-Lagrange equation is then

$$\int \langle J\nabla\theta, J\nabla\varphi\rangle \, dV_\Sigma = 0, \tag{2.5}$$

so smooth critical points satisfy the fourth order Hamiltonian stationary equation

$$\triangle_g \theta = 0, \tag{2.6}$$

which was introduced by Oh [58]. If $\Sigma$ minimizes volume in its Hamiltonian isotopy class, the set of submanifolds to which $\Sigma$ can be deformed by flowing along Hamiltonian vector fields, then $\Sigma$ satisfies the Hamiltonian stationary equation.

The gradient flow for volume within a Hamiltonian isotopy class is a fourth order parabolic equation

$$\frac{dX}{dt} = -J\nabla\triangle_g\theta, \tag{2.7}$$

where $X : M \times [0,T)$ is a family of immersions of $M$ into $\mathbb{C}^n$, which has been studied by Schäfer [64] and Chen-Warren [20]. In the case of plane curves, this equation reduces to the curve diffusion flow

$$\frac{dX}{dt} = -\kappa_{ss} N \tag{2.8}$$

where $N$ is the unit normal and $\kappa$ is the curvature. These flows do not preserve the graphical property of the evolving submanifold in general [22], which limits the usefulness of working at the level of the potential.

## 2.4 Legendre-Lewy transformation

The Legendre transform is a natural operation in the study of Lagrangian submanifolds, as it produces the potential function for a gradient graph $\{(x, Du(x))\}$ when viewed as a graph over the $y$-plane. The Legendre transform $u^*$ is given by

$$u^*(p) = \sup_x p \cdot x - u(x).$$

Indeed, if $u$ is a $C^2$ strictly convex function, and $p$ is in the image of $Du$, then the supremum is attained at $Du^{-1}(p)$, so

$$u^*(p) = p \cdot Du^{-1}(p) - u(Du^{-1}(p)).$$

Differentiating with respect to $p$,

$$\begin{aligned} D_p u^*(p) &= Du^{-1}(p) + p \cdot (D^2u)^{-1}(Du^{-1}(p)) - Du(Du^{-1}(p)) \cdot (D^2u)^{-1}(Du^{-1}(p)) \\ &= Du^{-1}(p). \end{aligned}$$

Geometrically, this inversion of the gradient of $u$ corresponds to a reflection of the gradient graph across the plane $x = y$ in $\mathbb{C}^n$, or equivalently, rotation by an angle of $-\frac{\pi}{2}$ in each $x_i y_i$-coordinate plane, followed by a reflection across the $x$-plane.

We shall be interested in other (unitary) rotations of Lagrangian graphs, so in this section we discuss the properties of a class of rotations and relate them to the Legendre transform.

While the Legendre transform is defined for convex functions, a modification introduced by Lewy [39] allows this tool to be usefully adapted for semiconvex functions.

We say a function $u$ is $K$-semiconvex if $u(x)+\frac{1}{2}K|x|^2$ is a convex function. We define the Legendre-Lewy transform of a $K$-semiconvex function $u$ by $(u+\frac{1}{2}K|\cdot|^2)^*$. See [53] for a beautiful application of this technique to the classical Bernstein theorem.

Up to scalings and addition of linear functions, Legendre-Lewy transformation is simply a rotation of the gradient graph of $u$ by angle $\operatorname{arccot} K$. This rotation technique, known as Lewy-Yuan rotation, was introduced by Yuan to demonstrate rigidity of convex entire solutions to the special Lagrangian equation [87] and later, rigidity of entire solutions with phase above the critical value $(n-2)\frac{\pi}{2}$ [88]. This is a central technique in geometric PDEs for Lagrangian submanifolds; here we recall some of its fundamental properties which can be found, for instance, in [18].

**Proposition 2.4.1** ([88, 18])**.** *Assume $u\in C^1$ is $K$-semiconvex and $\varphi\in(-\operatorname{arccot} K,0)$. Then, the Lagrangian submanifold parametrized by*

$$x\mapsto((\cos\varphi)x-(\sin\varphi)Du(x),(\cos\varphi)Du(x)+(\sin\varphi)x)$$

*is the gradient graph of a potential function $\bar{u}$ which satisfies:*

1. *$\bar{u}\in C^{1,1}$, in particular, $D^2\bar{u}\le\tan(\frac{\pi}{2}+\varphi)I$ a.e.*

2. *if $D^2u(x)$ exists, then $\sum_{i=1}^n\arctan\lambda_i(D^2\bar{u}(\bar{x}))=\sum_{i=1}^n\arctan\lambda_i(D^2u(x))+n\varphi$, where $\bar{x}=)\cos\varphi)x-(\sin\varphi)Du(x)$.*

3. *$\bar{u}(\bar{x})=(u-\frac{1}{2}(\cot\varphi)|\cdot|^2)^*(-(\csc\varphi)\bar{x})-\frac{1}{2}(\cot\varphi)|\bar{x}|^2$.*

*Proof.* We first verify that the rotated Lagrangian submanifold is a graph, i.e. that after the rotation, the projection onto the $x$-plane is nondegenerate and injective. Let $w(x)=u(x)+\frac{1}{2}K|x|^2$, and note $w$ is convex by assumption. That means

$$(x_1-x_2)\cdot(Dw(x_1)-Dw(x_2))\ge 0$$

for any $x_1, x_2$. Then,

$$\begin{aligned}
\frac{|\bar{x}_1-\bar{x}_2|^2}{\cos^2\varphi} &= |x_1 - x_2 - \tan\varphi(Du(x_1) - Du(x_2))|^2 \\
&= |x_1 - x_2 + K\tan\varphi(x_1-x_2) - \tan\varphi(Du(x_1) + Kx_1 - Du(x_2) - Kx_2)|^2 \\
&= |x_1 - x_2 + K\tan\varphi(x_1-x_2) - \tan\varphi(Dw(x_1) - Dw(x_2))|^2 \\
&= (1+K\tan\varphi)^2|x_1-x_2|^2 + \tan^2\varphi|Dw(x_1)-Dw(x_2)|^2 \\
&\quad - 2\tan\varphi(1+K\tan\varphi)(x_1-x_2)\cdot(Dw(x_1)-Dw(x_2)).
\end{aligned} \tag{2.9}$$

Since $\varphi \in (-\operatorname{arccot} K, 0)$, we have $K\tan\varphi > -1$, so the last term is nonnegative and the coefficient of the first term is positive, and we conclude $|\bar{x}_1-\bar{x}_2|^2 \geq C|x_1-x_2|^2$, so the rotated submanifold is a graph.

Next we verify that the rotated submanifold is a Lipschitz graph, i.e. that $\bar{y} = (\cos\varphi)(Du(x) + \sin\varphi)x$ is a Lipschitz function of $\bar{x}$. We estimate the difference quotient $\frac{|\bar{y}_1-\bar{y}_2|}{|\bar{x}_1-\bar{x}_2|}$. We consider two cases; the first is when $|Du(x_1) - Du(x_2)| \geq 2|\cot\varphi||x_1-x_2|$. Rearranging this hypothesis, we then have $-(\sin\varphi)|x_1-x_2| \leq \frac{1}{2}\frac{\sin^2\varphi}{\cos\varphi}|Du(x_1)-Du(x_2)|$. Then,

$$\begin{aligned}
|x_1 - x_2| &= |(\cos\varphi)(x_1-x_2) - (\sin\varphi)(Du(x_1)-Du(x_2))| \\
&\geq -(\sin\varphi)|Du(x_1)-Du(x_2)| - (\cos\varphi)|x_1-x_2| \\
&\geq -\frac{1}{2}(\sin\varphi)|Du(x_1)-Du(x_2)|,
\end{aligned}$$

and

$$\begin{aligned}
|\bar{y}_1-\bar{y}_2| &= |(\cos\varphi)(Du(x_1)-Du(x_2)) + (\sin\varphi)(x_1-x_2)| \\
&\leq (\cos\varphi)|Du(x_1)-Du(x_2)| - (\sin\varphi)(|x_1-x_2| \\
&\leq (\cos\varphi)|Du(x_1)-Du(x_2)| + \frac{1}{2}(\sin\varphi\tan\varphi)|Du(x_1)-Du(x_2)| \\
&= \cos\varphi(1+\frac{1}{2}\tan^2\varphi)|Du(x_1)-Du(x_2)|.
\end{aligned}$$

Therefore

$$\frac{|\bar{y}_1-\bar{y}_2|}{|\bar{x}_1-\bar{x}_2|} \leq -2\tan\varphi(1+\frac{1}{2}\tan^2\varphi).$$

In the other case, when $|Du(x_1) - Du(x_2)| \leq 2|\cot\varphi||x_1 - x_2|$, we have

$$\begin{aligned}|\bar{y}_1 - \bar{y}_2| &\leq (\cos\varphi)|Du(x_1) - Du(x_2)| - (\sin\varphi)|x_1 - x_2| \\ &= (-2\cos\varphi\cot\varphi - \sin\varphi)|x_1 - x_2|.\end{aligned}$$

Using (2.9),

$$|\bar{y}_1 - \bar{y}_2| \leq \frac{1}{\sqrt{C}}(-2\cos\varphi\cot\varphi - \sin\varphi)|\bar{x}_1 - \bar{x}_2|.$$

In either case, we have a uniform estimate for the Lipschitz constant. This shows that $\bar{u} \in C^{1,1}$.

Now, at a point $x$ where $u$ is twice-differentiable, we have

$$D^2_{\bar{x}}\bar{u} = \frac{\partial\bar{y}}{\partial\bar{x}} = \left(\frac{\partial\bar{y}}{\partial x}\right)\left(\frac{\partial\bar{x}}{\partial x}\right)^{-1}.$$

We have

$$\frac{\partial\bar{y}}{\partial x} = (\cos\varphi)D^2u + (\sin\varphi)I$$

and

$$\frac{\partial\bar{x}}{\partial x} = \cos\varphi I - \sin\varphi D^2u.$$

Noting that the matrices $\frac{\partial\bar{y}}{\partial x}$ and $\frac{\partial\bar{x}}{\partial x}$ can be simultaneously diagonalized, we see that $D^2_{\bar{x}}\bar{u}$ has eigenvalues

$$\bar{\lambda}_i = \frac{\lambda_i\cos\varphi + \sin\varphi}{\cos\varphi - \lambda_i\sin\varphi} = \frac{\lambda_i + \tan\varphi}{1 - \lambda_i\tan\varphi} = \tan(\arctan\lambda_i + \varphi) \leq \tan(\frac{\pi}{2} + \varphi). \tag{2.10}$$

Because $\bar{u}$ is $C^{1,1}$, Alexandrov's theorem implies that $D^2_{\bar{x}}\bar{u}$ exists almost everywhere. This means the gradient graph of $\bar{u}$ has a tangent plane almost everywhere. Undoing the rotation, these tangent planes come from rotating (possibly vertical) tangent planes of the gradient graph of $u$, which shows the upper bound on $D^2\bar{u}$ holds almost everywhere. From (2.10), it follows that that $\sum_i \arctan\bar{\lambda}_i = \sum_i \arctan\lambda i + n\varphi$ whenever $D^2u$ exists at $x$.

Finally, we prove the third statement, connecting the rotation to the Legendre transform. Note that by the $K$-semiconvexity of $u$, the function $w(x) = u(x) -$

$\frac{1}{2}\cot\varphi|x|^2$ is uniformly convex. We have $Dw(x) = Du(x) - \cot\varphi x$, so $\bar{x} = -\sin\varphi Dw(x)$. Inverting via Legendre transform,

$$Dw^*\left(\frac{\bar{x}}{-\sin\varphi}\right) = x.$$

Because $\bar{y} = (\cos\varphi)Du(x) + (\sin\varphi)x$, we then have

$$\bar{y} = (\cos\varphi)(Dw(x) + (\cot\varphi)x) + (\sin\varphi)x = (\cos\varphi)Dw(x) + (\csc\varphi)x.$$

Rewriting in terms of $\bar{x}$,

$$\bar{y} = -(\cot\varphi)\bar{x} + (\csc\varphi)Dw^*(-(\csc\varphi)\bar{x}).$$

Integrating this expression produces the potential $\bar{u}$ up to an additive constant:

$$\begin{aligned}\bar{u}(\bar{x}) &= \int -(\cot\varphi)\bar{x} + (\csc\varphi)Dw^*(-(\csc\varphi)\bar{x})\, d\bar{x} \\ &= -\frac{1}{2}(\cot\varphi)|\bar{x}|^2 - w^*(-(\csc\varphi)\bar{x}). \quad \square\end{aligned}$$

When $u$ is semiconvex, Alexandrov's theorem implies that $D^2u$ exists almost everywhere (with respect to Lebesgue measure $dx$). While the identity $\sum_{i=1}^n \arctan\lambda_i(D^2\bar{u}(\bar{x})) = \sum_{i=1}^n \arctan\lambda_i(D^2u(x)) + n\varphi$ holds whenever $D^2u(x)$ exists, it may not hold a.e. with respect to the measure $d\bar{x}$, or, more importantly, with respect to the $n$-dimensional Hausdorff measure on the gradient graph of $u$.

As an example, take $u$ to be the antiderivative of the Cantor function $c(x)$ on $[0,1]$. Then $D^2u = 0$ almost everywhere with respect to $dx$. Rotating by $\varphi = -\frac{\pi}{4}$, we obtain a new potential function $\bar{u}$ defined on the interval $[0,\sqrt{2}]$. Under the change $(x,y) \mapsto \frac{1}{\sqrt{2}}(x+y, y-x)$, each horizontal line segment on the graph of $c(x)$ is taken to a line segment of slope $-1$. The projection onto the horizontal axis then decreases the length of these segments by a factor of $\frac{1}{\sqrt{2}}$. The horizontal line segments on the graph of the Cantor function have total length 1, so the rotated graph has slope $-1$ on a union of intervals of total length $\frac{1}{\sqrt{2}}$, which is half of the domain. Indeed, $D\bar{u}$

cannot have derivative $-1$ a.e., as since $D\bar{u}$ is Lipschitz, it is absolutely continuous, so $D\bar{u}(\sqrt{2}) - D\bar{u}(0) = \int_0^2 D^2\bar{u}(\bar{x})\, d\bar{x}$, yet $D\bar{u}(\sqrt{2}) = D\bar{u}(0) = 0$.

By Rademacher's theorem, the graph of $D\bar{u}$ has a tangent line a.e. Fix such a tangent line, and invert the rotation. If the tangent line has slope different from $-1$, it must be rotated to a vertical line when inverting the rotation, since $c(x)$ must not be differentiable at the resulting point of tangency. Therefore, on the complement of the set where $D^2\bar{u} = -1$, we have $D^2\bar{u} = 1$ a.e.

This phenomenon of a non-minimal vertical piece of a Lagrangian submanifold possessing positive measure which projects onto a nontrivial subset after a rotation is the mechanism behind the non-volume-minimizing singular examples of viscosity solutions to the special Lagrangian equation discovered by Mooney-Savin [47] and Mooney-Shankar [48].

The next proposition allows us to avoid such pathologies provided $u$ has additional regularity.

**Proposition 2.4.2.** *If $u \in C^1 \cap W^{2,n}$ is $K$-semiconvex, then for any $\varphi \in (-\operatorname{arccot} K, 0)$, the Lewy-Yuan rotation $\bar{u}$ satisfies*

$$\sum_{i=1}^{n} \arctan \lambda_i(D^2\bar{u}(\bar{x})) = \sum_{i=1}^{n} \arctan \lambda_i(D^2 u(x)) + n\varphi$$

*almost everywhere with respect to the measures $d\bar{x}$ and $d\mathcal{H}^n \llcorner \{(x, Du(x))\}$.*

*Proof.* First note that in rotated coordinates $(\bar{x}, \bar{y})$, because we have that $D\bar{u}$ is Lipschitz continuous, the push-forward of $n$-dimensional Hausdorff measure on the gradient graph under projection onto the $\bar{x}$ plane is absolutely continuous with respect to $d\bar{x}$. Therefore it suffices to show that the measure of the set of points $x$ where $D^2u$ does not exist is preserved under the transformation $x \mapsto \bar{x}$. This transformation is the gradient map of the uniformly convex function $w(x) = \frac{1}{2}\cos\varphi|x|^2 - \sin\varphi u(x)$, and has a Lipschitz inverse by (2.9). Reshetnyak's theorem [63] establishes that $W^{1,n}$ homeomorphisms send measure-zero sets to measure-zero sets, proving the proposition. $\square$

**Remark.** Because of the convexity of $w$ above, the regularity hypotheses may be weakened slightly due to a result of Šverák [74, Theorem 6]: it is sufficient that $w \in W^{2,p}$ where $p \geq n-1$ and $\sigma_{n-1}(D^2 w) \in L^q$ where $q \geq \frac{p}{p-1}$.

# Chapter 3

# CALIBRATED SUBMANIFOLDS WITH FLAT NORMAL BUNDLES

The content of this chapter first appeared in [54].

## *3.1 Minimal submanifolds with flat normal bundles*

In the theory of minimal submanifolds, there is a stark contrast between hypersurfaces (submanifolds of codimension 1) and submanifolds of high codimension. In codimension 1, objects such as variation fields and the mean curvature can be identified with scalars, while the vectorial nature of these objects cannot be avoided in high codimension. Several landmark results in the field, such as Simons' proof of the Bernstein conjecture in dimensions up to 7 [70] and the curvature estimates of Schoen-Simon-Yau [65] depend crucially on the fact that hypersurfaces have rank-1 normal bundles.

The situation for high-codimension minimal submanifolds is more complicated. Lawson and Osserman [38] gave an example of a nonflat graphical minimal cone in dimension 4 and codimension 3. Their example is the cone over the graph of a scaled Hopf map $S^3 \to S^2$. Barbosa [3], and later Fischer-Colbrie [26], proved rigidity for graphical minimal cones of dimension 3 in arbitrary codimension. While graphical minimal hypersurfaces over convex domains are volume-minimizing, Osserman [60] gave an example of an entire codimension-2 minimal graph which is not stable.

In recent decades, there has been success extending results for hypersurfaces to high codimension under the simplifying assumption that the minimal submanifold has a flat normal bundle. Motivation for this extra hypothesis can be found by considering

Simons' equation for the second fundamental form of a minimal submanifold [70]. In codimension 1, the equation simplifies to the incredibly useful form

$$\triangle A = -|A|^2 A,$$

while in high codimension, the equation is much more complicated and involves commutators of shape operators. In Euclidean space, submanifolds with flat normal bundles have shape operators which commute, so the commutator terms in Simons' equation vanish. This allows one to recover useful tools such as an improved Kato inequality. Using this improved Kato inequality, Xin [86] proved that a complete $n$-dimensional, $n \leq 5$, minimal submanifold with flat normal bundle in $\mathbb{R}^{n+m}$ which is locally a graph over a fixed $n$-plane and satisfies growth hypotheses for the volume and volume element is a plane. Later, the dimension restriction $n \leq 5$ was removed by Smoczyk, Wang, and Xin [72]. Curvature estimates of Schoen-Simon-Yau were also extended to the setting of normal-flat high-codimension minimal submanifolds by Smoczyk, Wang, and Xin [72] and by M.-T. Wang [80], who also proved that minimal graphs with flat normal bundles are stable. Fu [27, 28] and Seo [68] proved Bernstein-type theorems for minimal submanifolds with flat normal bundles under curvature decay and superstability assumptions.

There is much interest in determining the extent to which Liouville-Bernstein rigidity theorems extend to calibrated submanifolds. The Lawson-Osserman Hopf cone is in fact coassociative [30], giving an immediate counterexample to a general Bernstein theorem for calibrated submanifolds. Under various additional hypotheses, rigidity theorems for special Lagrangian graphs hold ([87, 77, 88, 82, 56]), but explicit entire solutions constructed by Warren [81] and Li [40] show that the general Bernstein theorem cannot hold for special Lagrangian graphs. It is not known whether nonflat graphical special Lagrangian cones exist. Bernstein-type results for other classes of calibrated submanifolds were obtained recently by Lien and Tsai [44].

Li [41] studied special Lagrangian cones with flat normal bundles and proved that graphical special Lagrangian cones in $\mathbb{R}^n \times \mathbb{R}^n$ with flat normal bundles are planes for $n \leq 7$. Inspired by this "global" work, we investigate submanifolds with commuting shape operators which are calibrated by parallel calibrations. Our main "local" result is:

**Theorem 3.1.1.** *Assume $(M^{n+m}, g)$ is a Riemannian manifold, $\omega$ is a parallel calibration on $M$, and $\Sigma^n \subset M$ is calibrated by $\omega$. If for any two vectors $V, W \in N_p\Sigma$, the shape operators $A_V$ and $A_W$ commute, then $\Sigma$ is totally geodesic.*

In the case that $M$ is flat, the assumption that the shape operators of $\Sigma$ commute is equivalent to the normal curvature of $\Sigma$ vanishing. In particular we have the following for submanifolds of Euclidean space.

**Corollary 3.1.2.** *If $\Sigma^n \subset \mathbb{R}^{n+m}$ is calibrated by a constant-coefficient calibration and the normal bundle $N\Sigma$ is flat, then $\Sigma$ is a plane.*

We point out that our result does not rely on completeness. Additionally, the assumption that the calibration $\omega$ is parallel is necessary, as demonstrated by codimension-1 minimal graphs in $\mathbb{R}^n$. It is well-known that the volume form of a codimension-1 minimal graph extends to a calibration, and the normal bundle of a hypersurface is trivially flat, but there are nonplanar minimal graphs.

The proof is based on a well-known formula for the Laplacian of the volume element. In the cases of complex submanifolds and special Lagrangian submanifolds, we also provide simple algebraic proofs without taking the Laplacian.

Before proceeding with the proof of Theorem 3.1.1, we derive the basic facts we will need about the normal curvature and volume form of a minimal submanifold.

Let $\Sigma^n \subset (M^{n+m}, g)$ be a submanifold. The normal connection on $\Sigma$ can be used to define a curvature operator on sections of $N\Sigma$:

$$R^N_{X,Y}V = \nabla_X \nabla_Y V - \nabla_Y \nabla_X V - \nabla_{[X,Y]}V.$$

**Definition 3.1.1.** *A submanifold* $\Sigma$ *has a flat normal bundle if for any* $X, Y \in T_p\Sigma$, $V \in N_p\Sigma$,

$$R^N_{X,Y} V = 0.$$

Let $\bar{R}$ denote the Riemann curvature of $M$. The normal curvature of $\Sigma$ is related to the curvature of $M$ by the Ricci equation for $\Sigma$ [73]:

**Lemma 3.1.3.** *If* $X, Y \in T_p\Sigma$ *and* $V, W \in N_p\Sigma$*, then*

$$\langle R^N_{XY} V, W\rangle = \langle \bar{R}_{XY} V, W\rangle + \langle X, [A_W, A_V](Y)\rangle.$$

*Proof.* Extend $V$ to a local section of $N\Sigma$, and extend $X$ and $Y$ to local sections of $T\Sigma$ with $[X, Y] = 0$. Using the definitions of the normal connection and shape operator,

$$\begin{aligned}
\nabla_X \nabla_Y V - \nabla_Y \nabla_X V &= \nabla_X(\bar{\nabla}_Y V - (\bar{\nabla}_Y V)^T) - \nabla_Y(\bar{\nabla}_X V - (\bar{\nabla}_X V)^T) \\
&= \nabla_X(\bar{\nabla}_Y V - A_V(Y)) - \nabla_Y(\bar{\nabla}_X V - A_V(X)) \\
&= [\bar{\nabla}_X(\bar{\nabla}_Y V - A_V(Y))]^N - [\bar{\nabla}_Y(\bar{\nabla}_X V - A_V(X))]^N \\
&= (\bar{R}_{XY} V)^N - B(X, A_V(Y)) + B(Y, A_V(X)).
\end{aligned}$$

Testing against $W$,

$$\begin{aligned}
\langle R^N_{XY} V, W\rangle &= \langle \bar{R}_{XY} V, W\rangle - \langle B(X, A_V(Y)), W\rangle + \langle B(Y, A_V(X)), W\rangle \\
&= \langle \bar{R}_{XY} V, W\rangle + \langle X, A_W(A_V(Y))\rangle - \langle A_W(Y), A_V(X)\rangle
\end{aligned}$$

using (2.1). Finally, use the fact that $A_V$ is self-adjoint to obtain

$$\langle R^N_{XY} V, W\rangle = \langle \bar{R}_{XY} V, W\rangle + \langle X, [A_W, A_V](Y)\rangle. \quad \square$$

From Lemma 3.1.3, we see that if $M$ is flat ($\bar{R} = 0$), then $\Sigma$ has flat normal bundle if and only if $[A_W, A_V] = 0$ for any $V, W \in N_p\Sigma$.

The useful fact for our purposes is that commuting endomorphisms can be simultaneously diagonalized, so submanifolds whose shape operators all commute admit local frames in which all of the shape operators are diagonal.

We conclude this section by computing the Laplacian of the tangent plane of $\Sigma$ as a section of the bundle $\bigwedge^n(TM|_\Sigma)$. See [86, 80] for a similar calculation or [26] for a dual version. Here we assume that $E_1, \ldots, E_n$ are an orthonormal frame for the tangent bundle of $\Sigma$, and $V_1, \ldots, V_m$ are an orthonormal frame for the normal bundle of $\Sigma$.

**Lemma 3.1.4.** *If $\Sigma \subset M$ is minimal, then*

$$\triangle(E_1 \wedge \cdots \wedge E_n) = -|B|^2 E_1 \wedge \cdots \wedge E_n + \sum_i \sum_{j \neq k, \alpha, \beta} h^\alpha_{ij} h^\beta_{ik} E_1 \wedge \cdots \wedge V_\alpha \wedge \cdots \wedge V_\beta \wedge \cdots \wedge E_n,$$

*where $V_\alpha$ and $V_\beta$ appear in the $j^{th}$ and $k^{th}$ positions in the wedge product, respectively.*

*Proof.* We calculate at a fixed point $p$ with the orthonormal frames $\{E_i\}$ and $\{V_\alpha\}$ chosen so that $\nabla E_i = 0$ and $\nabla V_\alpha = 0$ at $p$. First note that in an orthonormal frame,

$$2\langle E_j, \nabla_{E_i} E_j \rangle = E_i \langle E_j, E_j \rangle = 0. \tag{3.1}$$

Differentiating the tangent plane once,

$$\begin{aligned} \bar{\nabla}_{E_i}(E_1 \wedge \cdots \wedge E_n) = \sum_j \big[ & E_1 \wedge \cdots \wedge (\bar{\nabla}_{E_i} E_j)^T \wedge \cdots \wedge E_n \\ & + E_1 \wedge \cdots \wedge (\bar{\nabla}_{E_i} E_j)^N \wedge \cdots \wedge E_n \big]. \end{aligned}$$

For the first term, only the $E_j$ component of $(\bar{\nabla}_{E_i} E_j)^T$ produces a nonzero wedge product, but by (3.1), the $E_j$ component vanishes. For the terms involving the normal part, we substitute $(\bar{\nabla}_{E_i} E_j)^N = h^\alpha_{ij} V_\alpha$ (here we use the Einstein summation convention for Greek indices). Now we take another covariant derivative:

$$\begin{aligned} \bar{\nabla}_{E_i}(E_1 \wedge \cdots \wedge (h^\alpha_{ij} V_\alpha) \wedge \cdots \wedge E_n) = {} & E_1 \wedge \cdots \wedge \bar{\nabla}_{E_i}(h^\alpha_{ij} V_\alpha) \wedge \cdots \wedge E_n \\ & + \sum_{k \neq j} E_1 \wedge \cdots \wedge h^\alpha_{ij} V_\alpha \wedge \cdots \wedge h^\beta_{ik} V_\beta \wedge \cdots \wedge E_n. \end{aligned} \tag{3.2}$$

We have

$$\begin{aligned}
\bar{\nabla}_{E_i}(h^\alpha_{ij}V_\alpha) &= E_i\langle\bar{\nabla}_{E_i}E_j, V_\alpha\rangle V_\alpha + h^\alpha_{ij}\bar{\nabla}_{E_i}V_\alpha \\
&= E_i\langle\bar{\nabla}_{E_j}E_i, V_\alpha\rangle V_\alpha - \sum_k h^\alpha_{ij}h^\alpha_{ik}E_k \\
&= -E_i\langle E_i, \bar{\nabla}_{E_j}V_\alpha\rangle V_\alpha - \sum_k h^\alpha_{ij}h^\alpha_{ik}E_k \\
&= -\langle\bar{\nabla}_{E_i}E_i, \bar{\nabla}_{E_j}V_\alpha\rangle - \langle E_i, \bar{\nabla}_{E_i}\bar{\nabla}_{E_j}V_\alpha\rangle V_\alpha - \sum_k h^\alpha_{ij}h^\alpha_{ik}E_k \\
&= -\langle E_i, \bar{\nabla}_{E_j}\bar{\nabla}_{E_i}V_\alpha\rangle V_\alpha - \sum_k h^\alpha_{ij}h^\alpha_{ik}E_k \\
&= -E_j\langle E_i, \bar{\nabla}_{E_i}V_\alpha\rangle V_\alpha + \langle\bar{\nabla}_{E_j}E_i, \bar{\nabla}_{E_i}V_\alpha\rangle V_\alpha - \sum_k h^\alpha_{ij}h^\alpha_{ik}E_k \\
&= -E_j(h^\alpha_{ii}) - \sum_k h^\alpha_{ij}h^\alpha_{ik}E_k.
\end{aligned}$$

Substituting this expression into the wedge product in (3.2), only the $E_j$ term of the sum survives, so

$$\begin{aligned}
\bar{\nabla}_{E_i}(E_1 \wedge \cdots \wedge (h^\alpha_{ij}V_\alpha) \wedge \cdots \wedge E_n) = &-\sum_\alpha (h^\alpha_{ii} + (h^\alpha_{ij})^2)E_1 \wedge \cdots \wedge E_n \\
&+ \sum_{k \neq j} E_1 \wedge \cdots \wedge h^\alpha_{ij}V_\alpha \wedge \cdots \wedge h^\beta_{ik}V_\beta \wedge \cdots \wedge E_n.
\end{aligned} \tag{3.3}$$

Upon summing (3.3) over $i$, we have

$$\sum_i h^\alpha_{ii} = \langle H, V_\alpha\rangle = 0$$

since $\Sigma$ is minimal. Now since $|B|^2 = \sum_{i,j,\alpha}(h^\alpha_{ij})^2$, summing over $j$ proves the lemma.

□

### 3.2 Proof of Theorem 3.1.1

*Proof.* Since $\omega$ calibrates $\Sigma$,

$$\omega(E_1 \wedge \cdots \wedge E_n) = 1.$$

Combining this with the assumption that $\omega$ is parallel,

$$\begin{aligned} 0 &= E_i[\omega(E_1 \wedge \cdots \wedge E_n)] \\ &= \bar{\nabla}_{E_i}(\omega)(E_1 \wedge \cdots \wedge E_n) + \omega(\bar{\nabla}_{E_i}(E_1 \wedge \cdots \wedge E_n)) \\ &= \omega(\bar{\nabla}_{E_i}(E_1 \wedge \cdots \wedge E_n)). \end{aligned}$$

Differentiating again and summing,

$$\triangle[\omega(E_1 \wedge \cdots \wedge E_n)] = \omega(\triangle(E_1 \wedge \cdots \wedge E_n)).$$

By Lemma 3.1.4,

$$\triangle(E_1 \wedge \cdots \wedge E_n) = \sum_i \sum_{j \neq k, \alpha, \beta} h_{ij}^{\alpha} h_{ik}^{\beta} (E_1 \wedge \cdots \wedge V_\alpha \wedge \cdots \wedge V_\beta \wedge \cdots \wedge E_n) - |B|^2 (E_1 \wedge \cdots \wedge E_n).$$

Since $\Sigma$ has commuting shape operators, the frame $\{E_i\}$ can be chosen so that $[h_{ij}^{\alpha}]$ is diagonal for all $\alpha$. But since $j \neq k$ in the summation, one of $h_{ij}^{\alpha}$ and $h_{ik}^{\beta}$ must be zero in each term. Therefore

$$0 = \triangle[\omega(E_1 \wedge \cdots \wedge E_n)] = -|B|^2,$$

so $\Sigma$ is totally geodesic. □

For the cases of complex submanifolds or minimal Lagrangian submanifolds of a Kähler manifold, this rigidity result can be observed directly from algebraic consequences of the interaction between the normal bundle and the ambient complex structure without taking the Laplacian of the tangent plane.

Let $J$ denote the ambient complex structure. If $M$ is Kähler and $\Sigma$ is a complex submanifold, then $J$ preserves the tangent and normal bundles of $\Sigma$. If the shape operators of $\Sigma$ commute, then for any choice of normal vector $V$, the operators $A_V$ and $A_{JV}$ can be simultaneously diagonalized, but

$$\langle \bar{\nabla}_{E_i} V, E_i \rangle = \langle \bar{\nabla}_{E_i} JV, JE_i \rangle = \langle \bar{\nabla}_{JE_i} JV, E_i \rangle,$$

using the fact that $J$ is parallel, so $A_{JV}$ is diagonal only if $A_V = 0$.

If $\Sigma$ is instead Lagrangian, then $J$ is an isomorphism between $T\Sigma$ and $N\Sigma$, so we may translate any calculation involving normal vectors to an intrinsic calculation. In particular, $J(B(X,Y))$ takes values in $T\Sigma$. We can then define a section $\tilde{B}$ of $\bigotimes^3 T^*\Sigma$ by

$$\tilde{B}(X,Y,Z) = \langle B(X,Y), JZ\rangle.$$

Since

$$\langle B(X,Y), JZ\rangle = \langle \bar{\nabla}_X Y, JZ\rangle = -\langle Y, J\bar{\nabla}_X Z\rangle = \langle JY, B(X,Z)\rangle,$$

the tensor $\tilde{B}$ is symmetric in all three arguments. This means that $h^\alpha_{ij} = h^j_{i\alpha}$. But if $[h^\alpha_{ij}]$ is diagonal for each $\alpha$, then the only nonzero components are $h^i_{ii}$, which can be seen by choosing $i \neq \alpha$ in the previous identity. By minimality,

$$0 = \sum_i h^\alpha_{ii} = h^\alpha_{\alpha\alpha},$$

so the entire second fundamental form vanishes.

McLean [45, Prop. 4.2] showed that the normal bundle of a coassociative submanifold $\Sigma$ of a $G_2$ manifold is isomorphic to the bundle of antiselfdual 2-forms, and hence is also an intrinsic object. It is therefore natural to ask whether there is a simple algebraic proof that coassociative submanifolds with commuting shape operators are totally geodesic.

# Chapter 4

# CONSTANT RANK THEOREMS

The content of this chapter first appeared in the author's joint work with Yu Yuan [56].

## *4.1 Statement of main results*

In this chapter, we establish constant rank results for the special Lagrangian equation

$$\sum_{i=1}^{n} \arctan \lambda_i = \Theta, \tag{4.1}$$

and the quadratic Hessian equation on the positive branch

$$\sigma_2\left(D^2 u\right) = \sum_{1\leq 1<j\leq n} \lambda_i \lambda_j = 1 \text{ with } \lambda_1 + \cdots + \lambda_n > 0, \tag{4.2}$$

where $\lambda_1 \leq \cdots \leq \lambda_n$ are the eigenvalues of the Hessian $D^2u$ and $\Theta$ is constant.

A constant rank theorem asserts that the minimum or maximum eigenvalue of the Hessian of a solution to a certain elliptic equation obeys the strong minimum or maximum principle. In other words, the Hessian shifted by an appropriate scalar matrix, $D^2u - aI$ or $aI - D^2u$, has constant rank. That certain condition for fully nonlinear elliptic equations is the inverse-convexity of Alvarez-Lasry-Lions in [1], employed to obtain the constant rank result for those various equations by Caffarelli-Guan-Ma in [12].

By connecting the convexity of the level sets of equation (4.1) at critical or supercritical phase $|\Theta| \geq (n-2)\pi/2$ [88], to the inverse-convexity of (4.1) at subcritical phase $|\Theta| < (n-2)\pi/2$, we obtain the following constant rank theorem (strong minimum principle for the smallest Hessian eigenvalue) for the special Lagrangian equation (4.1), which is new for saddle solutions.

**Theorem 4.1.1.** *If $u$ is a smooth solution of* (4.1) *in a (connected) domain with* $\arctan \lambda_{\min} \geq (\Theta - \pi)/n$ *and $\lambda_{\min}$ attains a local minimum at an interior point, then $\lambda_{\min}$ is constant. Furthermore, if in addition the local minimum* $\arctan \lambda_{\min} > (\Theta - \pi)/n$, *then there is a fixed direction $e$ such that the Hessian $D^2u$ splits as $\lambda_{\min} \equiv u_{ee} \equiv const$. If $\lambda_{\max}$ attains a local interior maximum, the corresponding conclusions for $\lambda_{\max}$ also hold, provided* $\arctan \lambda_{\max} \leq (\Theta + \pi)/n$ *and* $\arctan \lambda_{\max} < (\Theta + \pi)/n$ *respectively.*

One application of Theorem 4.1.1 is the following rigidity result for homogeneous degree 2 solutions.

**Corollary 4.1.2.** *If $u$ is a smooth degree 2 homogeneous solution of* (4.1) *on* $\mathbb{R}^n \backslash \{0\}$, *and* $\arctan \lambda_{\min} > (\Theta - \pi)/n$ *or* $\arctan \lambda_{\max} < (\Theta + \pi)/n$, *then $u$ is a quadratic polynomial.*

As a more general byproduct of the proof of Theorem 4.1.1, we find a Liouville type result for entire solutions of (4.1), which is new at subcritical and critical phases $|\Theta| \leq (n-2)\pi/2$ in dimension $n \geq 5$.

**Theorem 4.1.3.** *If $u$ is an entire smooth solution of* (4.1) *on* $\mathbb{R}^n$ *with* $\arctan \lambda_{\min} \geq \gamma > (\Theta - \pi)/n$ *or* $\arctan \lambda_{\max} \leq \gamma < (\Theta + \pi)/n$, *then $u$ is a quadratic polynomial.*

Relying on the same inverse-convexity used to derive the Liouville type result for almost convex entire solutions to (4.2) in [14], we have the following constant rank result for the quadratic Hessian equation (4.2).

**Theorem 4.1.4.** *If $u$ is a smooth solution of* (4.2) *in a (connected) domain with* $\lambda_{\min} \geq -\sqrt{2/[n(n-1)]}$ *and $\lambda_{\min}$ attains a local minimum at interior point, then $\lambda_{\min}$ is constant. Furthermore, if in addition* $\lambda_{\min} > -\sqrt{2/[n(n-1)]}$, *then there is a fixed direction $e$ such that the Hessian $D^2u$ splits as $\lambda_{\min} \equiv u_{ee} \equiv const$.*

A sufficient condition for fully nonlinear elliptic equations to have a lower constant rank theorem is convexity. But (4.1) with convex solutions and the equivalent equation of (4.2), $\Lambda\left(D^2u\right)=\sqrt{\sigma_2\left(D^2u\right)}=1$, are concave, not convex. A relaxed convexity, as alluded to earlier, the inverse-convexity of a fully nonlinear elliptic equation $G\left(D^2u\right)=c$, that is, convexity of $G\left(A^{-1}\right)$ in terms of $A$, permits a strong minimum principle for the minimum eigenvalue of the Hessian of solutions. In terms of the Legendre transform $u^*=w\left(y\right)$ of strongly convex solution $u\left(x\right)$ to $G\left(D^2u\right)=c$, as

$$(x, Du\left(x\right))=\left(Dw\left(y\right), y\right) \text{ and then } D^2u=\left(D^2w\right)^{-1},$$

the equation $G\left(D^2u\right)=c$ becomes a concave fully nonlinear elliptic equation

$$\Lambda\left(D^2w\right)=-G\left(\left(D^2w\right)^{-1}\right)=-c.$$

Now given $\lambda_{\min}\left(D^2u\right)$ attains its local minimum, we have $\lambda_{\max}\left(D^2w\right)=\lambda_{\min}^{-1}\left(D^2u\right)$ reaching its local maximum, and say, $\lambda_{\max}\left(D^2w\right)=w_{11}$ at the local maximum point. By the strong maximum principle, subsolution $w_{11}\equiv l\equiv\lambda_{\max}\left(D^2w\right)$. The gradient graphs split off a line

$$\left(x_1, x', u_1(x), D'u(x)\right)=\left(ly_1, D'w(y'), y_1, y'\right).$$

Accordingly, $u_1\left(x\right)=l^{-1}x_1$, hence $u_{11}\equiv\lambda_{\min}\left(D^2u\right)$ is constant.

The subtlety lies in the case $\lambda_{\min}\left(D^2u\right)=0$, where the Legendre transformation is not possible. The strong minimum principle for $\lambda_{\min}$ was achieved in [12] by realizing its superharmonicity in a hidden smooth form of $\triangle_G\lambda_{\min}\leq B\left(x\right)\cdot D\lambda_{\min}$. Moreover, the Hessian $D^2u$ may no longer split, that is, the the minimum-eigendirection may not be fixed, as counterexamples by Korevaar-Lewis in [37, p.31] and also by Székelyhidi-Weinkove in [75, p.6530] indicate. A quantitative version of the constant rank theorem was discovered in [76].

The lower constant rank results for convex solutions in Theorem 4.1.1 and Theorem 4.1.4 are contained in [12, p.1772], as the two corresponding equations are inverse-convex then. The inverse convexity was also employed in deriving the upper, or equivalently lower, constant rank result for solutions of (4.1) with the eigenvalues of the Hessian between -1 and 1 by Bhattacharya-Shankar in [7, p.18]. The connection to the inverse-convexity of (4.1) with saddle solutions in Theorem 4.1.1 and [7] is via the Lewy-Yuan rotation technique described in section 2.4.

Note that the lower bound in Theorem 4.1.1 with $\Theta = \pi/2$ and $n = 3$ is exactly the one $\lambda_{\min} > \tan(-\pi/6) = -1/\sqrt{3} = -m$ in Theorem 4.1.4 with $n = 3$. This is because here the algebraic form of (4.1) is (4.2). In this instance, the new rotated equation $\sum_{i=1}^{n} \arctan \bar{\lambda}_i = -\pi/2$ is equivalent to $\sigma_2(D^2\bar{u}) = 1$ on the negative branch $\bar{\lambda}_1 + \bar{\lambda}_2 + \bar{\lambda}_3 < 0$. In terms of $\mu_i = (\lambda_i + m)^{-1}$, the equation $\sigma_2(\bar{\lambda}) = 1$ becomes $\Lambda(\mu) = \sigma_2(\mu)/\sigma_1(\mu) = 1/(2m)$, which is concave. In other words, $D^2u + mI$ satisfies an inverse-convex equation, with abused notation, $-\Lambda\left((D^2u + mI)^{-1}\right) = -1/(2m)$. This particular three dimensional case inspired the Legendre-Lewy transformation, $(u + m|x|^2/2)^*$ with $m = \sqrt{2/[n(n-1)]}$, used to derive the Liouville type result in [14], and the constant rank result Theorem 4.1.4 for almost convex solutions of (4.2) with $D^2u > -mI$ in general dimensions. The rigidity result for general semiconvex entire solutions to (4.2) needed more effort in [69].

The above rotation used in proving Theorem 4.1.1 turns an entire solution to (4.1) with the specific lower or upper Hessian bound into a new entire one with bounded Hessian on a concave/convex level set. The Evans-Krylov theorem then leads to the rigidity result Theorem 4.1.3. Coupled with the quadratic rigidity of 2-homogeneous analytic solutions to a uniformly elliptic Hessian equation in $\mathbb{R}^4 \setminus \{0\}$ in [51], the argument in [87, p.124–125] also shows the Liouville type result for entire solutions to (4.1) with arbitrary lower or upper Hessian bound now in four dimensions. A Bernstein type result, namely a quadratic rigidity for entire solutions to (4.1) with supercritical phase $|\Theta| > (n-2)\pi/2$ was derived in [88]. Precious entire solutions $W(x) = (x_1^2 + x_2^2 - 1)e^{x_3} + e^{-x_3}/4$ and $L(x) = x_1^2 x_2 - \frac{2}{3}x_2^3 - x_2 x_3$ to (4.1) with critical $\Theta = \pi/2$ or $\sigma_2(D^2W) = 1$ and (4.1) with subcritical $\Theta = 0$ or $\triangle L = \det D^2 L$ were constructed by Warren in [81] and by C.-Y. Li in [40], respectively.

While there are singular $C^{1,\alpha}$ or Lipschitz viscosity solutions to (4.1) with subcritical phase [52, 78, 47], regularity for $C^{1,1}$ solutions in five dimensions and higher is an open question, which is equivalent to the quadratic rigidity of 2-homogeneous solutions to (4.1) with subcritical phase. Nontrivial 2-homogeneous solutions were found to a saddle uniformly elliptic Hessian equation in five dimensions [50]. Should a nontrivial 2-homogeneous solution exist to (4.1) with subcritical phase, say for example $\Theta = 0$, the lower bound for the Hessian must be less than or equal to $-\tan(\pi/5)$. Otherwise, a splitting of the Hessian by Corollary 4.1.2 implies the triviality of the supposed example. It would be marvelous that the lower bound of the Hessian is exactly the $-\tan(\pi/5)$ in Theorem 4.1.1.

In the final section of this chapter, we present another proof of a general constant rank theorem for inverse-convex elliptic Hessian equations, using viscosity approach, as opposed to the smooth [12, 10] and approximation [75, 76] proofs. The argument is to demonstrate the next smallest nonidentically vanishing eigenvalue is a supersolution of a linear equation in the viscosity sense. In particular, the demonstration of the constancy of the smallest eigenvalue is simple. After our this work was finished, we found a similar viscosity approach for the constant rank theorem had appeared in the work of Bryan-Ivaki-Scheuer [11].

## 4.2 Proof of constant rank and rigidity theorems for special Lagrangian equations

*Proof of Theorem 4.1.1.* We show the assertions for $\lambda_{\min}$. These assertions applied to the solution $v = -u$ of the equation $\sum_{i=1}^{n} \arctan \lambda_i (D^2 v) = -\Theta$ yield the corresponding conclusions for $\lambda_{\max}(D^2 u)$. We consider three cases: $\Theta \leq (2-n)\pi/2$, $\Theta \in ((2-n)\pi/2, \pi]$, and $\Theta \in (\pi, n\pi/2)$.

Case $\Theta \leq (2-n)\pi/2$ : By [88, Lemma 2.1], the level set of equation $F(D^2 u) = \sum_{i=1}^{n} \arctan \lambda_i = \Theta$ is concave as a graph along the gradient $\nabla_\lambda \sum_{i=1}^{n} \arctan \lambda_i$, then $\triangle_F u_{ee} = F_{ij} \partial_{ij} u_{ee} \leq 0$. Therefore, a strong minimum principle holds for $u_{ee}$. If $\lambda_1$

reaches its local minimum ($> -\infty$), then after change of $x$-coordinates if necessary, $u_{11}$ reaches a local minimum of the same value of $\lambda_1$, at the same point, so $u_{11} \equiv \lambda_1$ is constant.

Case $\Theta \in (\pi, n\pi/2)$ : The hypothesis $\arctan \lambda_1 \geq \frac{\Theta-\pi}{n}$ implies $\lambda_n \geq \cdots \geq \lambda_1 > 0$, so $u$ is strongly convex. Let $u^*$ be the Legendre transform of $u$. The eigenvalues $\mu_i$ of $D^2 u^* = (D^2 u)^{-1}$ satisfy $\mu_i = \lambda_i^{-1} > 0$ and (1.1) becomes

$$\sum_{i=1}^{n} \arctan \mu_i = n\frac{\pi}{2} - \Theta.$$

Since $\mu_i > 0$, this elliptic equation for $u^*$ is concave, so we have a strong maximum principle for pure second derivatives of $u^*$. In turn, this also yields a strong minimum principle for pure second derivatives of $u$. The conclusion of Theorem 4.1.1 follows.

Case $\Theta \in ((2-n)\pi/2, \pi]$ : Let $\alpha = (\Theta - \pi)/n \in (-\pi/2, 0]$ and $\beta = \frac{\pi}{2} + \alpha \in (0, \pi/2]$. First we assume that the attained local minimum is strictly larger than the assumed lower bound, that is

$$\frac{\pi}{2} > \theta_n \geq \cdots \geq \theta_1 > \alpha.$$

where $\theta_i = \arctan \lambda_i$. We apply a Lewy-Yuan rotation as in Proposition 2.4.1 with angle $-\beta$ such that in the new coordinates $(\bar{x}, \bar{y})$,

$$\bar{\theta}_i = \theta_i - \beta = \theta_i - \frac{\pi}{2} - \alpha \in (-\pi/2, \pi/2)$$

and

$$\bar{F}\left(D^2 \bar{u}\right) = \sum_{i=1}^{n} \arctan \bar{\lambda}_i = (2-n)\frac{\pi}{2}, \tag{4.3}$$

where $\bar{\theta}_i = \arctan \bar{\lambda}_i$ and $\bar{\lambda}_i' s$ are the eigenvalues of the Hessian $D^2 \bar{u}$ of the new potential $\bar{u}$.

Again by [88, Lemma 2.1], the level set of the equation (4.3) is concave as a graph along the gradient $\nabla_{\bar{\lambda}} \sum_{i=1}^{n} \arctan \bar{\lambda}_i$, then $\triangle_{\bar{F}} \bar{u}_{\bar{e}\bar{e}} = \bar{F}_{ij} \partial_{ij} \bar{u}_{\bar{e}\bar{e}} \leq 0$. Thus a strong minimum principle holds for $\bar{u}_{\bar{e}\bar{e}}$.

Now $\lambda_{\min} = \lambda_1$ reaches its local minimum, which is strictly larger than $\tan(\alpha - \beta) = -\infty$. Correspondingly, $\bar{\theta}_{\min} = \bar{\theta}_1 = \theta_1 - \beta$ reaches its local minimum. By the strong minimum principle, $\bar{u}_{\bar{1}\bar{1}} \equiv \tan\bar{\delta} \equiv \bar{\lambda}_1$, under change of $\bar{x}$-coordinates if necessary. Then the new Lagrangian graph admits a parametrization of the form

$$\begin{aligned}&\left(cx_1 + su_1(x), cx' + sD'u(x), -sx_1 + cu_1(x), -sx' + cD'u(x)\right)\\ &\quad = \left(\bar{x}_1, \bar{x}', \tan\bar{\delta}\ \bar{x}_1, D'\bar{u}(\bar{x}')\right)\end{aligned}$$

where $c = \cos\beta,\ s = \sin\beta$. Accordingly, $u_1(x) = \tan\left(\bar{\delta} + \beta\right)x_1$, hence $u_{11} \equiv \lambda_1$ is constant.

Next we consider the borderline situation, the attained local minimum $\arctan\lambda_1 = \alpha = (\Theta - \pi)/n$. The previous new representation by rotation is no longer valid on all of the domain, because $\bar{\theta}_1 = -\pi/2$, or equivalently $\bar{\lambda}_1 = -\infty$. Whenever this rotation is possible, that is, when $\arctan\lambda_1 > \alpha$, we have

$$\begin{aligned}\bar{\lambda}_i = \tan(\theta_i - \beta) = -a - \frac{1 + a^2}{\lambda_i - a} \quad &\text{for } \tan\beta = -\tan^{-1}\alpha = -a^{-1}\\ \text{or} \qquad D^2\bar{u} = -aI - &\left(1 + a^2\right)\left(D^2u - aI\right)^{-1}.\end{aligned}$$

By the above mentioned concavity of the level set of $\bar{F}\left(D^2\bar{u}\right) = (2 - n)\pi/2$, which is preserved under translation and scaling, the level set of

$$\tilde{F}(A) \overset{\text{def}}{=} \bar{F}\left(-aI - \left(1 + a^2\right)A\right) = (2 - n)\frac{\pi}{2}$$

is also concave as a graph along $\nabla_A\tilde{F}$. Hence the condition for constant rank in [12, (2.2)] with $A = D^2u - aI$ and $X = D^2u_e$ holds. Effectively, the condition (4.9) for constant rank with $F(A)$ and $u(x)$ replaced by $\tilde{F}\left(A^{-1}\right)$ and $u(x) - a|x|^2/2$ respectively holds. Therefore, by [12, Theorem 1.1] or Theorem 4.1, $D^2u - aI$ has constant rank, and in turn, $\lambda_{\min} = \lambda_1 \equiv \tan\alpha = a$. $\square$

The Hessian splitting conclusion of Theorem 4.1.1 is the main tool used to prove Corollary 4.1.2. The idea is to split off a quadratic function in one variable from $u$

and then observe that the remaining part of the solution also satisfies the hypotheses of Theorem 4.1.1.

*Proof of Corollary 4.1.2.* We prove the quadratic rigidity under the assumption $\arctan\lambda_{\min} > (\Theta-\pi)/n$. This result applied to the solution $v=-u$ of the equation $\sum_{i=1}^n \arctan\lambda_i(D^2v) = -\Theta$ yields the same rigidity when $\arctan\lambda_{\max}(D^2u) < (\Theta+\pi)/n$.

Since $u$ is homogeneous of degree 2, $D^2u$ is homogeneous of degree 0, so $\arctan\lambda_{\min}$ achieves its minimum value, $\arctan\lambda_{\min} \geq \arctan m_1 > (\Theta-\pi)/n$. By Theorem 4.1.1, $\lambda_{\min}$ is constant and the Hessian $D^2u$ splits such that, after a change of coordinates if necessary, $u_{11}\equiv m_1$. Then,

$$u = \frac{1}{2}m_1x_1^2 + \tilde{u}(x_2,\dots,x_n).$$

The function $\tilde{u}$ satisfies the equation

$$\sum_{i=1}^{n-1}\arctan\lambda_i(D^2\tilde{u}) = \Theta - \arctan m_1, \tag{4.4}$$

and $\arctan\lambda_i(D^2\tilde{u}) \geq \arctan m_1 > (\Theta-\pi)/n$ for all $1\leq i\leq n-1$.

By Theorem 4.1.1 in dimension $n-1$, if $\arctan\lambda_{\min}(D^2\tilde{u}) > (\tilde{\Theta}-\pi)/(n-1)$, with $\tilde{\Theta} = \Theta - \arctan m_1$, then $\lambda_{\min}(D^2\tilde{u})$ is constant and the Hessian $D^2\tilde{u}$ splits. Indeed, by the assumption $\arctan m_1 > (\Theta-\pi)/n$, or equivalently

$$\frac{\Theta-\pi}{n} > \frac{\Theta - \arctan m_1 - \pi}{n-1},$$

we have

$$\lambda_{\min}(D^2\tilde{u}) \geq \arctan m_1 > \frac{\Theta-\pi}{n} > \frac{\tilde{\Theta}-\pi}{n-1}.$$

Inductively, we conclude that $u$ is a quadratic polynomial. $\square$

**Remark.** The assumption that $u$ is homogeneous degree 2 is used to ensure that every $\lambda_i$ attains an interior local minimum. In fact, if $u$ is a smooth solution of (4.1) with $\arctan\lambda_1 \geq (\Theta-\pi)/n$, and every $\lambda_i$ for $1\leq i\leq k$, attains an interior local minimum, then $\lambda_i$ is constant for $1 \leq i \leq k$. If all the local minima $\arctan \lambda_i > (\Theta - \pi)/n$, then the Hessian splits as $\lambda_i \equiv u_{ii} \equiv const.$ for $1 \leq i \leq k$.

We now proceed with the proof of Theorem 4.1.3, which is an application of the Evans-Krylov theorem to the equation obtained by the rotation in the proof of Theorem 4.1.1.

*Proof of Theorem 4.1.3.* We show the quadratic rigidity under the condition $\arctan \lambda_{\min} \geq \gamma > (\Theta - \pi)/n$. The same rigidity for the solution $v = -u$ to the equation $\sum \arctan \lambda_i \left(D^2 v\right) = -\Theta$ under the condition $\arctan \lambda_{\min} \left(D^2 v\right) \geq -\gamma > (-\Theta - \pi)/n$ or equivalently $\arctan \lambda_{\max} \left(D^2 u\right) \leq \gamma < (\Theta + \pi)/n$ yields the corresponding conclusion under the $\lambda_{\max}$ condition.

The result is known if $\Theta \geq \pi$ as then $\lambda_{\min} \geq 0$ [87, Theorem 1.1], or if $\Theta \geq \pi - n\pi/6 - \varepsilon'(n)$ as then $\lambda_{\min} \geq -1/\sqrt{3} - \varepsilon(n)$ [84, p.924] [42, p.22], or if $\Theta < (2-n)\pi/2$ [88, Theorem 1.1]. We consider more than the remaining range, $\Theta \in [(2-n)\pi/2, \pi)$, where our unified argument is in the spirit of [88].

First, each Lagrangian angle $\theta_i = \arctan \lambda_i$ satisfies

$$\frac{\pi}{2} > \theta_i \geq \gamma > \frac{\Theta - \pi}{n} = \alpha \in \left[-\frac{\pi}{2}, 0\right),$$

where we may and do assume $\gamma \leq 0$. In order to include the negative critical phase, set $\delta = (\gamma - \alpha)/2 > 0$. We apply Proposition 2.4.1 with angle $-\tilde{\beta} = -(\pi/2 + \alpha + \delta) \in (-\pi/2, -\delta)$ to obtain a new potential $\bar{u}$ such that

$$\bar{\theta}_i = \theta_i - \frac{\pi}{2} - \alpha - \delta \in \left(-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta\right)$$

and

$$\bar{F}\left(D^2 \bar{u}\right) = \sum_{i=1}^{n} \arctan \bar{\lambda}_i = (2-n)\frac{\pi}{2} - n\delta. \tag{4.5}$$

The new potential $\bar{u}$ is an entire solution of (4.5), because of the uniform convexity of $u + \frac{\cot \tilde{\beta}}{2}|x|^2$ (the Legendre transform takes uniformly convex entire functions to entire functions).

Then we have an entire solution $\bar{u}$ with bounded Hessian on the concave level set [88, Lemma 2.1] of the now uniformly elliptic equation (4.5). Scaling Evans-Krylov's Hölder seminorm estimate for the Hessian $D^2\bar{u}$ and then taking limit yields the constancy of $D^2\bar{u}$. Thus $(\bar{x}, D\bar{u}(\bar{x}))$, or equivalently $(x, Du(x))$, is a plane, and in turn, the original potential $u$ is a quadratic polynomial. $\square$

### 4.3 Proof of constant rank theorem for quadratic Hessian equation

*Proof of Theorem 4.1.4.* First we assume the attained local minimum is strictly larger than the assumed lower bound, $\lambda_{\min} > -\sqrt{2/[n(n-1)]} \overset{\text{def}}{=} m$. Let $w(y)$ be the Legendre-Lewy transform, $(u(x)+m|x|^2/2)^*$. In terms of gradient graphs in $\mathbb{R}^n \times \mathbb{R}^n$,

$$(x, Du(x) + mx) = (Dw(y), y). \tag{4.6}$$

A tangent plane of $(x, Du(x) + mx)$ is formed by tangent vectors $(e, (D^2u(x) + mI)e)$ with $e \in \mathbb{R}^n$; correspondingly, tangent vectors $\left((D^2u(x) + mI)^{-1} e, e\right)$ form the tangent plane of $(Dw(y), y)$ at $y = Du(x) + mx$. Thus

$$D^2w(y) = \left(D^2u(x) + mI\right)^{-1},$$

and in terms of eigenvalues,

$$\mu_i = (\lambda_i + m)^{-1} > 0.$$

It follows that $\mu$ satisfies

$$1 = \sum_{1 \le i < j \le n} (\mu_i^{-1} - m)(\mu_j^{-1} - m) = \frac{\sigma_{n-2}(\mu)}{\sigma_n(\mu)} - (n-1)m\frac{\sigma_{n-1}(\mu)}{\sigma_n(\mu)} + 1$$

or

$$\Lambda\left(D^2w\right) = \Lambda(\mu) \overset{\text{def}}{=} \frac{\sigma_{n-1}(\mu)}{\sigma_{n-2}(\mu)} = \frac{1}{(n-1)m}, \tag{4.7}$$

where $m = \sqrt{2/[n(n-1)]}$ is used. The function $\Lambda(\mu)$ is concave and $\nabla_\mu \Lambda(\mu) > 0$ componentwise [43, Theorem 15.18], therefore the level set $\Lambda(\mu) = 1/[(n-1)m]$ is convex as a graph along $\nabla_\mu \Lambda$. Hence

$$\triangle_\Lambda w_{ee} = \Lambda_{ij}\partial_{ij} w_{ee} \geq 0.$$

Consequently, a strong maximum principle holds for $w_{ee}$.

Now $\mu_{\max} = (\lambda_{\min} + m)^{-1}$ reaches its local maximum. By the strong maximum principle, $w_{11} \equiv l \equiv \mu_{\max}$, under change of $y$-coordinates if necessary. Then the gradient graph (4.6) reads

$$(x_1, x', u_1(x) + mx_1, D'u(x) + mx') = (ly_1, D'w(y'), y_1, y') .$$

Accordingly, $u_1(x) = (l^{-1} - m)\, x_1$, hence $u_{11} \equiv \lambda_{\min}$ is constant.

Next we consider the borderline situation, the attained local minimum of $\lambda_{\min} = m$. The previous Legendre-Lewy transformation is no longer valid. Whenever the transformation is possible, that is, when $\lambda_{\min}(D^2u) > m$, we have (4.7). By the above proved convexity of the level set of $\Lambda(D^2w) = 1/\left[(n-1)\, m\right]$ as a graph along $\nabla_{D^2w}\Lambda$, the level set of

$$\tilde{F}(D^2u) = -\Lambda\left(\left(D^2u + mI\right)^{-1}\right) = -1/\left[(n-1)\, m\right]$$

is concave as a graph along $\nabla_{D^2u}\tilde{F}\left(D^2u\right)$. As in the proof of Theorem 4.1.1, the condition for constant rank in [12, (2.2)] with $A = D^2u + mI$ and $X = D^2u_e$ holds. Effectively, the condition (4.9) for constant rank with $F\left(A\right)$ and $u\left(x\right)$ replaced by $\tilde{F}\left(A^{-1}\right)$ and $u\left(x\right) + m\left|x\right|^2/2$ respectively holds. Therefore, by [12, Theorem 1.1] or Theorem 4.1, $D^2u + mI$ has constant rank, and in turn, $\lambda_{\min} = \lambda_1$ is constant. $\square$

### *4.4 Viscosity proof of constant rank theorem*

In this section we present a viscosity approach to prove the following constant rank theorem for nonlinear elliptic Hessian equations. As mentioned in the introduction, a similar viscosity proof appeared in [11]. For completeness, we include our proof here. For convenience, we state and prove the theorem under the assumption that $F(D^2u) = f(\lambda_1, \ldots, \lambda_n)$ where $f$ is a symmetric function of its arguments, but the proof can be adapted to accommodate a general Hessian equation.

**Theorem 4.4.1.** *Assume $u \in C^3$ is a convex solution of the elliptic equation $F(D^2u) = 0$ where $F(M^{-1})$ is a convex function on $\mathrm{Sym}^+(n)$. Then $D^2u$ has constant rank.*

One of the obstacles in proving a constant rank theorem is that the eigenvalues of the Hessian matrix need not be differentiable. Previous works deal with this issue by applying strong minimum principle arguments to smooth quantities [12, 10], or by approximations coupled with weak Harnack inequalities [75, 76]. The current approach is to compute directly with the eigenvalues of the Hessian in order to apply the strong minimum principle in the viscosity sense. In particular, this approach gives a simple argument for establishing the constancy of the smallest eigenvalue.

We begin with a lemma establishing a formula for 1-sided derivatives of eigenvalues at matrices in $\mathrm{Sym}(n)$ with repeated eigenvalues.

**Lemma 4.4.2.** *Assume $M = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ with $\lambda_1 \leq \cdots \leq \lambda_n$. Furthermore, assume $\lambda_{j-1} < \lambda_j = \cdots = \lambda_k < \lambda_{k+1}$. Then, if $j \leq i \leq k$ and $A \in \mathrm{Sym}(n)$, then*

$$\left.\frac{d^+}{dt}\right|_{t=0} \lambda_i(M + tA) = \lim_{t \to 0^+} \frac{\lambda_i(M + tA) - \lambda_i}{t}$$

*exists and*

$$\left.\frac{d^+}{dt}\right|_{t=0} \lambda_i(M + tA) = \lambda_{i-j+1}(A|_E),$$

*where $E = (e_j, \ldots, e_k)$ is the $\lambda_i$-eigenspace of $M$ and $A|_E$ is the $(k-j+1)\times(k-j+1)$ matrix with entries $(A_{rs})_{r,s=j}^k$.*

*Proof.* We may choose an orthogonal basis for $E$ to diagonalize $A|_E$. Once $A|_E$ is diagonal, it is clear that for small nonzero $t$, the eigenvalues of the perturbed matrix $(M + tA)|_E$ satisfy

$$\left.\frac{d^+}{dt}\right|_{t=0} \lambda_{i-j+1}((M + tA)|_E) = \lambda_{i-j+1}(A|_E).$$

The entries of $tA$ outside of the block $tA|_E$ make contributions on the order of $t^2$ to $\lambda_i(M + tA)$, so this proves the lemma. $\square$

A consequence of Lemma 4.4.2 is that if $M$ has repeated eigenvalues $\lambda_j = \cdots = \lambda_k$, then $\lambda_i$ $(j \leq i \leq k)$ is differentiable in the direction $A$ at $M$ if and only if $\lambda_{i-j+1}(A|_E) = \lambda_{k-i+1}(A|_E)$. To see this, note that if $\frac{d}{dt}|_{t=0}\lambda_i(M+tA)$ exists, then

$$\begin{aligned}
\left.\frac{d^+}{dt}\right|_{t=0} \lambda_i(M+tA) &= \left.\frac{d^-}{dt}\right|_{t=0} \lambda_i(M+tA) \\
&= \lim_{t\to 0^-} \frac{\lambda_i(M+tA)-\lambda_i}{t} \\
&= -\lim_{t\to 0^+} \frac{\lambda_i(M-tA)-\lambda_i}{t} \\
&= -\left.\frac{d^+}{dt}\right|_{t=0} \lambda_i(M-tA) \\
&= -\lambda_{i-j+1}(-A|_E).
\end{aligned}$$

Note that $-\lambda_{i-j+1}(-A|_E) = \lambda_{k-i+1}(A|_E)$.

In particular, $\lambda_j$ is differentiable in the direction $A$ if and only if $A|_E$ is a scalar matrix.

*Proof of Theorem 4.4.1.* Recall that $\lambda_j$ is a Lipschitz function, $C^1$ on the set $\{p : \lambda_{j-1} < \lambda_j < \lambda_{j+1}\}$. We assume that $0 = \lambda_1(0) = \cdots = \lambda_{n-r}(0)$, so that the rank of $D^2u$ reaches its minimum at 0. All computations will be performed at a fixed point $p$, assuming that $D^2u(p)$ is diagonal with $\lambda_j(p) = u_{jj}(p)$. Note that the assumption that $F$ is a symmetric function of the eigenvalues implies that $F_{ij}(D^2u(p)) = f_{\lambda_i}\delta_{ij}$. For a positive integer $t \leq n$, let

$$I_t = \{(i,j,k,l) \,:\, 1 \leq i,j,k,l \leq n, \text{ at least one of } i,j,k,l \leq t\}.$$

Step 1: Superharmonicity of $\lambda_1$.

Assume first that $\lambda_1(p) < \lambda_2(p)$. At $p$, we have

$$\begin{aligned}
\partial_i \lambda_1 &= \partial_i u_{11}, \\
\partial_{ii}\lambda_1 &= \partial_{ii}u_{11} + \sum_{j>1} \frac{2}{\lambda_1-\lambda_j}(\partial_i u_{1j})^2.
\end{aligned} \tag{4.8}$$

Differentiating the equation $F(D^2u)=0$, we obtain

$$0=\partial_{11}F(D^2u)=\sum_{i,j,k,l}F_{ij,kl}u_{ij1}u_{kl1}+\sum_{i,j}F_{ij}u_{ij11},$$

so

$$\begin{aligned}\triangle_F u_{11}&=-\sum_{i,j,k,l}F_{ij,kl}u_{ij1}u_{kl1}\\&=-\sum_{i,j,k,l>1}F_{ij,kl}u_{ij1}u_{kl1}-\sum_{(i,j,k,l)\in I_1}F_{ij,kl}u_{ij1}u_{kl1}.\end{aligned}$$

where $\triangle_F=\sum_{i,j}F_{ij}\partial_{ij}$.

By the inverse-convexity we have

$$-\sum_{i,j,k,l>1}F_{ij,kl}u_{ij1}u_{kl1}\leq 2\sum_{i,j>1}F_{ii}\lambda_j^{-1}u_{ij1}^2. \tag{4.9}$$

The sum of terms with at least one index equal to 1 is of the form $B\cdot Du_{11}$, and at $p$, $Du_{11}=D\lambda_1$ by (4.8). Therefore, at $p$

$$\triangle_F u_{11}\leq 2\sum_{i,j>1}F_{ii}\lambda_j^{-1}u_{ij1}^2+B\cdot D\lambda_1.$$

Then,

$$\begin{aligned}\triangle_F\lambda_1&=\triangle_F u_{11}+2\sum_i\sum_{j>1}F_{ii}\frac{1}{\lambda_1-\lambda_j}u_{ij1}^2\\&\leq B\cdot D\lambda_1+2\sum_{i,j>1}F_{ii}\left(\frac{1}{\lambda_j}+\frac{1}{\lambda_1-\lambda_j}\right)u_{ij1}^2+2\sum_{j>1}F_{11}\frac{1}{\lambda_1-\lambda_j}u_{1j1}^2\\&\leq B\cdot D\lambda_1\end{aligned} \tag{4.10}$$

because $\frac{1}{\lambda_j}-\frac{1}{\lambda_j-\lambda_1}\leq 0$.

Now assume that $\lambda_1(p)=\cdots=\lambda_s(p)<\lambda_{s+1}(p)$. Suppose $\phi$ is a smooth function supporting $\lambda_1$ from below at $p$. Then

$$\phi\leq\lambda_1\leq\frac{1}{s}\sum_{m\leq s}\lambda_m$$

with equality at $p$. Observe that $\sum_{m\leq s}\lambda_m$ is differentiable in a neighborhood of $p$. It follows from the assumption that $\phi$ supports $\lambda_1$ from below that $\lambda_1$ is also differentiable at $p$. Therefore Lemma 4.4.2 implies that $u_{ije}(p) = 0$ if $i \neq j$ for $i, j \leq s$.

We now compute $\triangle_F \sum_{m \leq s} \lambda_m$. We have

$$\begin{aligned}
\triangle_F \sum_{m \leq s} \lambda_m &= \triangle_F \sum_{m \leq s} u_{mm} + 2 \sum_i \sum_{j>s} \sum_{m \leq s} F_{ii} \frac{u_{ijm}^2}{\lambda_m - \lambda_j} \\
&= - \sum_{i,j,k,l} \sum_{m \leq s} F_{ij,kl} u_{ijm} u_{klm} + 2 \sum_i \sum_{j>s} \sum_{m \leq s} F_{ii} \frac{u_{ijm}^2}{\lambda_m - \lambda_j} \\
&= - \sum_{i,j,k,l>s} \sum_{m \leq s} F_{ij,kl} u_{ijm} u_{klm} + 2 \sum_i \sum_{j>s} \sum_{m \leq s} F_{ii} \frac{u_{ijm}^2}{\lambda_m - \lambda_j} \\
&\quad - \sum_{(i,j,k,l) \in I_s} \sum_{m \leq s} F_{ij,kl} u_{ijm} u_{klm} \\
&\leq 2 \sum_{i,j>s} \sum_{m \leq s} F_{ii} \frac{u_{ijm}^2}{\lambda_j} + 2 \sum_i \sum_{j>s} \sum_{m \leq s} F_{ii} \frac{u_{ijm}^2}{\lambda_m - \lambda_j} \\
&\quad + \sum_{m \leq s} B_m \cdot D\lambda_m.
\end{aligned}$$

Lemma 4.4.2 also implies that $D\lambda_i = D\lambda_j$ for $i, j \leq s$, so

$$\triangle_F \sum_{m \leq s} \lambda_m \leq B \cdot D\lambda_1. \tag{4.11}$$

Combining (4.10) with (4.11) shows that $\lambda_1$ is a viscosity supersolution of the equation $\triangle_F w - B \cdot Dw = 0$, so by the strong minimum principle and the assumption that $\lambda_1(0) = 0$ it follows that $\lambda_1 \equiv 0$.

Step 2: Superharmonicity of $\lambda_{a+1}$ given $\lambda_1 = \cdots = \lambda_a = 0$.

Assume that for some $a < r$, $\lambda_a \equiv 0$. Fix $p \in \Omega$ and assume that $\lambda_{a+1}(p) = \cdots = \lambda_{a+s}(p) < \lambda_{a+s+1}(p)$. Let $\mu = \sum_{a<m\leq a+s} \lambda_m$.

If $\lambda_{a+1}(p) > 0$, then we compute

$$
\begin{aligned}
\triangle_F \mu &= \triangle_F \mu + 2\triangle_F \sum_{m\le a} \lambda_m \\
&= \triangle_F \sum_{a<m\le a+s} u_{mm} + 2\sum_i \sum_{j>a+s} \sum_{a<m\le a+s} F_{ii} \frac{u_{ijm}^2}{\lambda_m - \lambda_j} \\
&\quad + 2\sum_i \sum_{j\le a} \sum_{a<m\le a+s} F_{ii} \frac{u_{ijm}^2}{\lambda_m} + 2\triangle_F \sum_{m\le a} u_{mm} \\
&\quad + 4\sum_i \sum_{j>a} \sum_{m\le a} F_{ii} \frac{u_{ijm}^2}{-\lambda_j}
\end{aligned}
$$

$$
\begin{aligned}
&= -\sum_{i,j,k,l} \sum_{a<m\le a+s} F_{ij,kl} u_{ijm} u_{klm} - 2\sum_{i,j,k,l} \sum_{m\le a} F_{ij,kl} u_{ijm} u_{klm} \\
&\quad + 2\sum_i \sum_{j>a+s} \sum_{a<m\le a+s} F_{ii} \frac{u_{ijm}^2}{\lambda_m - \lambda_j} + 2\sum_i \sum_{j\le a} \sum_{a<m\le a+s} F_{ii} \frac{u_{ijm}^2}{\lambda_m} \\
&\quad + 4\sum_i \sum_{j>a} \sum_{m\le a} F_{ii} \frac{u_{ijm}^2}{-\lambda_j}
\end{aligned}
$$

By the inverse-convexity assumption,

$$
\begin{aligned}
-\sum_{i,j,k,l} \sum_{a<m\le a+s} F_{ij,kl} u_{ijm} u_{klm} \le {} & 2\sum_{i,j>a+s} \sum_{a<m\le a+s} F_{ii} \frac{1}{\lambda_j} u_{ijm}^2 \\
& - \sum_{(i,j,k,l)\in I_{a+s}} \sum_{a<m\le a+s} F_{ij,kl} u_{ijm} u_{klm}
\end{aligned}
$$

and

$$
\begin{aligned}
-2\sum_{i,j,k,l} \sum_{m\le a} F_{ij,kl} u_{ijm} u_{klm} \le {} & 4\sum_{i,j>a+s} \sum_{m\le a} F_{ii} \frac{1}{\lambda_j} u_{ijm}^2 \\
& - 2\sum_{(i,j,k,l)\in I_{a+s}} \sum_{m\le a} F_{ij,kl} u_{ijm} u_{klm}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\triangle_F \mu \le\, & 2 \sum_{i,j>a+s} \sum_{a<m\le a+s} F_{ii} \frac{1}{\lambda_j} u_{ijm}^2 - \sum_{(i,j,k,l)\in I_{a+s}} \sum_{a<m\le a+s} F_{ij,kl} u_{ijm} u_{klm} \\
& + 4 \sum_{i,j>a+s} \sum_{m\le a} F_{ii} \frac{1}{\lambda_j} u_{ijm}^2 - 2 \sum_{(i,j,k,l)\in I_{a+s}} \sum_{m\le a} F_{ij,kl} u_{ijm} u_{klm} \\
& + 2 \sum_i \sum_{j>a+s} \sum_{a<m\le a+s} F_{ii} \frac{u_{ijm}^2}{\lambda_m - \lambda_j} + 2 \sum_i \sum_{j\le a} \sum_{a<m\le a+s} F_{ii} \frac{u_{ijm}^2}{\lambda_m} \\
& + 4 \sum_i \sum_{j>a} \sum_{m\le a} F_{ii} \frac{u_{ijm}^2}{-\lambda_j} \\
=\, & 2 \sum_{i,j>a+s} \sum_{a<m\le a+s} F_{ii} \frac{1}{\lambda_j} u_{ijm}^2 - \sum_{(i,j,k,l)\in I_{a+s}} \sum_{a<m\le a+s} F_{ij,kl} u_{ijm} u_{klm} \\
& + 4 \sum_{i,j>a+s} \sum_{m\le a} F_{ii} \frac{1}{\lambda_j} u_{ijm}^2 - 2 \sum_{(i,j,k,l)\in I_{a+s}} \sum_{m\le a} F_{ij,kl} u_{ijm} u_{klm} \\
& + 2 \sum_i \sum_{j>a+s} \sum_{a<m\le a+s} F_{ii} \frac{u_{ijm}^2}{\lambda_m - \lambda_j} + 2 \sum_i \sum_{j\le a} \sum_{a<m\le a+s} F_{ii} \frac{u_{ijm}^2}{\lambda_m} \\
& + 4 \sum_i \sum_{a<j\le a+s} \sum_{m\le a} F_{ii} \frac{u_{ijm}^2}{-\lambda_j} + 4 \sum_i \sum_{j>a+s} \sum_{m\le a} F_{ii} \frac{u_{ijm}^2}{-\lambda_j} \\
\le & - \sum_{(i,j,k,l)\in I_{a+s}} \sum_{a<m\le a+s} F_{ij,kl} u_{ijm} u_{klm} - 2 \sum_{(i,j,k,l)\in I_{a+s}} \sum_{m\le a} F_{ij,kl} u_{ijm} u_{klm} \\
& + 2 \sum_i \sum_{j>a} \sum_{m\le a} F_{ii} \frac{u_{ijm}^2}{-\lambda_j},
\end{aligned}
\tag{4.12}
$$

because

$$
\sum_{i,j>a+s} \sum_{a<m\le a+s} F_{ii} \frac{1}{\lambda_j} u_{ijm}^2 + \sum_i \sum_{j>a+s} \sum_{a<m\le a+s} F_{ii} \frac{u_{ijm}^2}{\lambda_m - \lambda_j} \le 0
$$

and

$$
\sum_{i,j>a+s} \sum_{m\le a} F_{ii} \frac{1}{\lambda_j} u_{ijm}^2 + \sum_i \sum_{j>a+s} \sum_{m\le a} F_{ii} \frac{u_{ijm}^2}{-\lambda_j} \le 0.
$$

We now estimate the terms $F_{ij,kl} u_{ijm} u_{klm}$. We use the same technique as [75, p.6527]. First,

$$
- \sum_{(i,j,k,l)\in I_{a+s}} \sum_{a<m\le a+s} F_{ij,kl} u_{ijm} u_{klm} - 2 \sum_{(i,j,k,l)\in I_{a+s}} \sum_{m\le a} F_{ij,kl} u_{ijm} u_{klm}
$$

$$\leq C(\|F\|_{C^2}, \|u\|_{C^3}, n, a, s) \sum_i \sum_{j,k \leq a+s} |u_{ijk}|$$
$$\leq C \left( \sum_i \sum_{j<k\leq a+s} |u_{ijk}| + \sum_i \sum_{j \leq a+s} |u_{ijj}| \right).$$

For $j \leq a$, $\partial_i \lambda_j = 0$. If $\phi$ is a smooth function supporting $\lambda_{a+1}$ from below at $p$, then $\lambda_{a+1}$ is differentiable at $p$ because $\mu/s$ supports $\lambda_{a+1}$ from above. Therefore Lemma 4.4.2 implies that $D\lambda_{a+1}(p) = \cdots = D\lambda_{a+s}(p)$. We then have $u_{ijj}(p) = \partial_i \lambda_{a+1}(p)$ if $a < j \leq a+s$.

For the remaining terms, we use the Cauchy-Schwarz inequality. It follows from Lemma 4.4.2 that the matrix $(\partial_i u_{jk})_{j,k}$ is 0 in the upper left $a \times a$ block and the $s \times s$ block $(\partial_i u_{jk})_{j,k=a+1}^{a+s}$ is diagonal. Therefore the only nonzero terms of the sum $\sum_i \sum_{j<k\leq a+s} |u_{ijk}|$ are those with $j \leq a < k$. By assumption, $\mu > 0$ at $p$, so

$$|u_{ijk}| \leq \frac{u_{ijk}^2}{M\mu} + M\mu$$

where $M$ is a large constant to be chosen. By ellipticity,

$$\frac{u_{ijk}^2}{M\mu} \leq C \sum_i F_{ii} \frac{u_{ijk}^2}{M\mu} \frac{\mu}{\lambda_k - \lambda_j}.$$

It follows that

$$\sum_i \sum_{j\leq a} \sum_{a<k\leq a+s} |u_{ijk}| \leq \frac{C}{M} \sum_i \sum_{j\leq a} \sum_{a<k\leq a+s} F_{ii} \frac{u_{ijk}^2}{\lambda_k - \lambda_j}. \tag{4.13}$$

Combining (4.13) with the result of (4.4), we have

$$\begin{aligned}\triangle_F \mu &\leq 2 \sum_i \sum_{j>a} \sum_{m\leq a} F_{ii} \frac{u_{ijm}^2}{\lambda_m - \lambda_j} + \frac{C}{M} \sum_i \sum_{j\leq a} \sum_{a<k\leq a+s} F_{ii} \frac{u_{ijk}^2}{\lambda_k - \lambda_j} \\ &\quad + CM\mu + \sum_i \sum_{j\leq a+s} |u_{ijj}| \\ &= 2 \sum_i \sum_{j>a} \sum_{m\leq a} F_{ii} \frac{u_{ijm}^2}{\lambda_m - \lambda_j} + \frac{C}{M} \sum_i \sum_{j\leq a} \sum_{a<k\leq a+s} F_{ii} \frac{u_{ijk}^2}{\lambda_k - \lambda_j} \\ &\quad + CM\mu + B|D\mu|.\end{aligned} \tag{4.14}$$

Provided $M$ is large enough, the sum of the first two terms in (4.14) is nonpositive, so

$$\triangle_F \mu \leq CM\mu + B|D\mu|.$$

It then follows that

$$\triangle_F \lambda_{a+1} \leq C\lambda_{a+1} + B|D\lambda_{a+1}|$$

in the viscosity sense on the set $\{\lambda_{a+1} > 0\}$. That the differential inequality holds also on the set $\{\lambda_{a+1} = 0\}$ follows from (4.11).

Now the theorem follows by the strong minimum principle and induction on $a$. $\square$

# Chapter 5

# OPTIMAL REGULARITY FOR HÖLDER CONTINUOUS HAMILTONIAN STATIONARY LAGRANGIAN GRAPHS

The content of this chapter first appeared in the author's joint work with Arunima Bhattacharya [6].

## 5.1 Statement of main results

In this chapter, we address the question of regularity of Hölder continuous Hamiltonian stationary Lagrangian graphs. Recall that a smooth function $u$ has a Hamiltonian stationary gradient graph if equation (2.6) is satisfied, that is, if

$$\triangle_g \theta = 0. \tag{5.1}$$

Before we state our main results, we first clarify the notion of a weak solution to the Hamiltonian stationary equation. For a $C^1$ function $u$, we denote the gradient graph $\{(x, Du(x))\} \subset \mathbb{R}^n \times \mathbb{R}^n$ by $L_u$.

**Definition 5.1.1.** *We say that $u \in C^1(\Omega)$ is a weak solution of* (5.1) *if*

1. $u \in W^{2,1}(\Omega)$,
2. $\sqrt{g} \in L^1(\Omega)$,
3. $\theta \in W^{1,2}(\Omega)$, *and*
4. *for all compactly supported smooth functions $\eta$ on the cylinder $\Omega \times \mathbb{R}^n$, $u$ satisfies*

$$\int \langle \nabla\theta, \nabla\eta \rangle \, d(\mathcal{H}^n \llcorner L_u) = 0.$$

Note that, under the above conditions, $\nabla\eta$ is well defined a.e. by projecting the Euclidean gradient onto the tangent plane of $L_u$.

Our first result is the following.

**Theorem 5.1.1.** *Let $u \in C^{1,\beta}$ be a weak solution of (5.1) on the unit ball $B_1 \subset \mathbb{R}^n$ with $\beta \in (\frac{1}{3}, 1)$ and $|\theta| \geq (n-2)\frac{\pi}{2} + \delta$. Then $u$ is smooth in $B_{1/2}$ and satisfies the estimate*

$$\|u\|_{C^{k,\alpha}(B_{1/2})} \leq C(\alpha, k, n, \delta, \|u\|_{C^{1,\beta}(B_1)})$$

*for $k \geq 2$.*

We prove the sharpness of the above Hölder exponent $\beta$ by constructing the following counterexamples, which hold even under the strongest convexity condition on the Lagrangian phase, namely, $|\theta| > (n-1)\frac{\pi}{2}$.

**Theorem 5.1.2.** *On $B_1 \subset \mathbb{R}^n$ $(n \geq 3)$, for each positive integer $k$ there exists a weak solution $u^{(n,k)}$ to (5.1) with $|\theta| > (n-1)\frac{\pi}{2}$ such that $u^{(n,k)} \in C^{1,\frac{1}{2k+1}} \setminus C^{1,\frac{1}{2k+1}+\varepsilon}$ for all $\varepsilon > 0$.*

The above result shows that a convex solution $u \in C^{1,\frac{1}{3}}$ to the Hamiltonian stationary equation need not be any more regular. This is a notable departure from the second order theory of special Lagrangian submanifolds, for which Chen-Shankar-Yuan [17] established that convex viscosity solutions of the special Lagrangian equation must always be smooth.

The singular solutions constructed in this work have gradient graphs that are smooth submanifolds of $\mathbb{C}^n$. It is an interesting question whether there exists a Hamiltonian stationary Lagrangian graph with a hypercritical phase or with a convex potential that has a genuine geometric singularity, i.e., a gradient graph with unbounded curvature.

It is worth noting that in higher codimensions Lawson-Osserman [38] constructed examples of Lipschitz minimal submanifolds, including graphical ones, that fail to be

$C^1$. Combined with the absence of a rigidity result for special Lagrangian cones in dimensions $n > 4$, this indicates that imposing restrictions on the phase, or on the minimal or maximal eigenvalues of the Hessian of $u$, may be essential in establishing higher regularity for Lipschitz continuous Hamiltonian stationary Lagrangian graphs.

We would also like to point out that without the condition $|\theta| \geq (n-2)\frac{\pi}{2} + \delta$, the result of Theorem 5.1.1 remains valid provided that one assumes $\arctan(\lambda_{\min}) > (\theta - \pi)/n + \delta$. This follows from [56] and [4].

To highlight the distinction between our results and those established in special Lagrangian geometry, we first briefly describe how the range of the Lagrangian phase, $\theta$, affects the concavity properties of the arctangent operator, which plays a key role in regularity. The phase $(n-2)\frac{\pi}{2}$ is referred to as critical: indeed, the level set $\{\lambda = (\lambda_1, .., \lambda_n) \in \mathbb{R}^n \mid \Theta = \sum_{i=1}^n \arctan \lambda_i\}$ is concave precisely when $|\Theta| \geq (n-2)\frac{\pi}{2}$ [88, Lemma 2.2]. When $|\Theta| \geq (n-1)\frac{\pi}{2}$, the phase is termed hypercritical. In this range, the concavity of the level set is immediate, as the condition enforces $\lambda_i \geq 0$ for all $i$, making the arctangent operator concave. Finally, when $|\Theta| \geq (n-2)\frac{\pi}{2} + \delta$, the phase is called supercritical. In this range, one can show that the Hessian of the potential $u$ admits a uniform negative lower bound dependent on $\delta$; as a consequence, the potential is semi-convex.

For the special Lagrangian equation (4.1) with phase $|\Theta| \geq (n-2)\frac{\pi}{2}$, Warren-Yuan [83, 84] and Wang-Yuan [79] established Hessian estimates showing that even $C^0$ viscosity solutions are in fact analytic. When passing to the Hamiltonian stationary setting, our results reveal the following: Theorem 5.1.1 demonstrates that one can still expect full regularity for $C^{1,\frac{1}{3}+}$ solutions to (5.1) provided the phase satisfies $|\theta| \geq (n-2)\frac{\pi}{2} + \delta$. However, Theorem 5.1.2 shows that singular $C^{1,\frac{1}{3}}$ solutions to (5.1) exist even under the strongest convexity assumption on the phase $|\theta| > (n-1)\frac{\pi}{2}$. This stands in sharp contrast with the special Lagrangian equation, where singular solutions arise only in the subcritical range $|\theta| < (n-2)\frac{\pi}{2}$, that is, when the arctangent operator loses all convexity properties. In this case, Nadirashvili-Vlăduţ

[52] constructed $C^{1,\frac{1}{3}}$ solutions in dimension three, while Wang-Yuan [78] proved the existence of solutions $u \in C^{1,\frac{1}{2m+1}} \setminus C^{1,\frac{1}{2m+1}+\varepsilon}$ for every integer $m \geq 1$ in dimension three.

Hamiltonian stationary submanifolds can be viewed as a natural extension of minimal Lagrangian submanifolds in Kähler-Einstein manifolds; in particular, they include special Lagrangians arising in Calabi-Yau manifolds. The existence and stability problem has been studied by many authors via different approaches (cf. [57], [13], [66], [32], [2], [33], [34], and references therein). Every special Lagrangian submanifold is Hamiltonian stationary. However, the converse does not hold: for example, a circle in $\mathbb{C}$ and the Clifford torus in $\mathbb{C}^2$ are both Hamiltonian stationary but not special Lagrangian. Moreover, there exist non-flat cones that are Hamiltonian stationary without being special Lagrangian (see the work of Schoen-Wolfson [67]). For a recent survey of progress on regularity theory for special Lagrangian and Hamiltonian stationary equations, we refer the reader to [15].

Another source of difficulty in the study of Hamiltonian stationary submanifolds arises from the fact that the maximum principle, a key tool used in the study of minimal surface equations, does not apply to the fourth order equations governing these submanifolds. Moreover, even in the simplest setting of $\mathbb{C}^2$ equipped with the standard symplectic form, Hamiltonian stationary submanifolds lack a monotonicity property (see the work of Minicozzi [46] and Schoen-Wolfson [66]), creating significant challenges in understanding singularities. Recent progress in this direction can be found in the works of Pigati-Rivière [61] and Orriols [59].

In two dimensions, Schoen-Wolfson [67] established regularity results in the setting of general Kähler manifolds, where singularities are known to occur; such singularities are non-graphical (see [66, Section 7]). In the Euclidean setting, they showed that any $C^{2,\alpha}$ solution of (5.1) is smooth [66, Proposition 4.6]. In [18], Chen-Warren proved that weak $C^{1,1}$ solutions to (5.1) are smooth as long as one of these conditions is met: The Lagrangian phase, $\theta$, satisfies $|\theta| \geq (n-2)\frac{\pi}{2} + \delta$; the potential $u$ is strongly convex; or the Hessian of $u$ is bounded by a constant less than 1. Smoothness of $C^{1,1}$ solutions to (5.1) without restrictions on the phase or the Hessian was shown in two dimensions by Bhattacharya-Warren [9] and in dimensions up to four by Bhattacharya [4]. The smoothness of $C^1$ regular Hamiltonian stationary submanifolds of a symplectic manifold was established by Bhattacharya-Chen-Warren [5] through an analysis of the double divergence form of the fourth order nonlinear scalar equation arising from studying the stationary points of the volume of $L_u = (x, Du(x))$ in an open ball $B \subset \mathbb{R}^{2n}$ equipped with a Riemannian metric among nearby competing gradient graphs $L_t = (x, Du(x) + tD\eta(x))$ for compactly supported smooth functions $\eta$. This fourth order double divergence form of the Hamiltonian stationary equation was also studied by Bhattacharya-Skorobogatova [8] to prove a partial regularity result for Lipschitz continuous Hamiltonian stationary Lagrangian graphs in all dimensions. For results on the compactness of Hamiltonian stationary submanifolds in $\mathbb{C}^n$ and more general symplectic manifolds, based on curvature and smoothness estimates derived from the regularity theory of the governing fourth order equations, we refer the reader to the works of Chen-Warren [19] and Chen-Ma [16].

The regularity of Hölder continuous Hamiltonian stationary Lagrangian graphs has long been an open problem. By developing new techniques, inspired in part by the methods of Wang-Yuan [78], we resolve this question. Let us now provide a brief overview of the main ideas used to prove our results. Note that throughout this paper, we assume $\theta > 0$, since the case $\theta < 0$ can be handled analogously by symmetry.

## 5.2 Regularity for Hölder continuous HSL graphs

In this section, we prove Theorem 5.1.1. First, we clarify some notations.

*Notation*

Throughout this paper, we use $B_r$ to denote a ball of radius $r$ centered at the origin unless specified otherwise. We use $D$ to denote the derivative with respect to the ambient Euclidean metric, and $\nabla_g$ to denote the covariant derivative on the Lagrangian submanifold, with induced metric $g$.

We start by deriving properties of graphical smooth submanifolds of $\mathbb{R}^n \times \mathbb{R}^m$.

**Lemma 5.2.1.** *Suppose that $F : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ is $C^\beta$ where $\beta > \frac{1}{l+1}$ and $l \geq 1$ is an integer. If the graph of $F$ is a smooth submanifold of $\mathbb{R}^n \times \mathbb{R}^m$, then $F \in C^{\frac{1}{l}}_{\text{loc}}(\Omega)$.*

*Proof.* Let us fix $K \Subset \Omega$. For the sake of contradiction, suppose that $F \notin C^{\frac{1}{l}}(K)$. Then there exists a sequence $\{x^{(i)}\}_{i=1}^\infty \subset K$ such that $x^{(i)} \to x \in K$ and

$$\limsup \frac{|F(x^{(i)}) - F(x)|}{|x^{(i)} - x|^{\frac{1}{l}}} = \infty.$$

Without loss of generality, we assume that $x = 0$ and $F(0) = 0$. Since the graph of $F$ is a smooth submanifold, the above necessarily implies that the graph of $F$ has a vertical tangent plane at 0. Let $P$ denote this tangent plane and $\pi_x$ be the projection onto the $x$-plane. After rotations of the $x$ and $y$ coordinates, suppose that $\{\partial_{y_1}, \ldots, \partial_{y_k}\}$ forms a basis for $\ker \pi_x|_P$ and $\{\partial_{x_{k+1}}, \ldots, \partial_{x_n}\}$ forms a basis for $\pi_x(P)$. By the Implicit Function Theorem, the graph of $F$ can be represented as a graph of a smooth function $G(y_1, \ldots, y_k, x_{k+1}, \ldots, x_n)$. We write

$$x = (x', x'') \text{ and } y = (y', y'')$$

where $x' = (x_1, \ldots, x_k)$, $x'' = (x_{k+1}, \ldots, x_n)$, $y' = (y_1, \ldots, y_k)$, and $y'' = (y_{k+1}, \ldots, y_m)$ (note, it is possible that $k = m$ or $k = n$). Thus the graph of $F$ is locally the set

$$\{(x', x'', y', y'') \mid (x', y'') = G(y', x'')\}.$$

Now since the vectors $\partial_{y_1}, \ldots, \partial_{y_k}$ are tangent to the graph of $F$, we get $D_{y'}G(0) = 0$.

We fix a unit vector $e$ in the $(y', x'')$-plane and consider the curve $(te, G(te))$ for sufficiently small values of $t$. Next decompose $e$ into $x$ and $y$ components, $e = e_x + e_y$. Then we get

$$|e_y||t| \leq |y(t)|.$$

Assume $|e_y| \geq \frac{1}{2}$. This yields $|y(t)| \sim |t|$. By the assumed Hölder continuity of $F$, we have

$$|y(t)| \lesssim |x(t)|^{\beta}.$$

By Taylor's Theorem, we get $|x(t)| \lesssim |t|^{j}$ for some integer $j$, which may depend on $e$. Let's suppose that, for some choice of $e$, $j > l$. Then we get

$$|t| \lesssim |y(t)| \lesssim |x(t)|^{\beta} \lesssim |t|^{(l+1)\beta},$$

but this yields a contradiction since $(l+1)\beta > 1$.

Therefore we must have

$$|y(t)|^{l} \lesssim |t|^{l} \lesssim |x(t)|.$$

On the other hand, if $|e_x| > \frac{1}{2}$, then $|x(t)| \sim |t|$ and $|y(t)| \lesssim t$, which gives

$$|y(t)|^{l} \lesssim |t|^{l} \lesssim |x(t)|.$$

This contradicts the assertion that

$$\frac{|y|}{|x|^{\frac{1}{l}}} \to \infty$$

as $x \to 0$. Thus $F \in C^{\frac{1}{l}}(K)$. □

Next, we establish smoothness of a Hölder continuous Hamiltonian stationary Lagrangian graph when the Hölder exponent exceeds $\frac{1}{2}$ and the phase satisfies $\theta \geq (n-2)\frac{\pi}{2} + \delta$. The significance of this supercriticality of phase is that it permits a downward rotation of the gradient graph, under which the rotated submanifold can be shown to be smooth. By subsequently performing an upward rotation of the new submanifold, one recovers the smoothness of the original submanifold.

**Proposition 5.2.2.** *Let $u \in C^{1,\beta}$ be a weak solution of* (5.1) *on the unit ball $B_1 \subset \mathbb{R}^n$ with $\beta \in (\frac{1}{2}, 1)$ and $\theta \geq (n-2)\frac{\pi}{2} + \delta$. Then $u$ is smooth inside $B_1$.*

*Proof.* Since the phase satisfies $\theta \geq (n-2)\frac{\pi}{2} + \delta$, we can apply Proposition 2.4.1 to perform a downward rotation of the gradient graph $L_u = \{(x, Du(x)) \mid x \in B_1\} \subset \mathbb{R}^n \times \mathbb{R}^n$ by an angle $\varphi \in (-\frac{\delta}{n}, 0)$. The new rotated coordinates are given by

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

This yields a new potential $\bar{u}(\bar{x})$, which is a $C^{1,1}$ solution of

$$\triangle_g \bar{\theta} = 0 \text{ on } B_r$$

where $B_r \subset \bar{x}(B_1)$ and

$$\bar{\theta} = \theta - n\varphi > \theta - \delta.$$

The rotated phase remains in the supercritical range, and hence by [18, Theorem 1.2], the new potential $\bar{u}$ is smooth. We point out that although our definition of a weak solution is slightly different than that in [18], the proof of their result does not use the condition $u \in W^{2,n}$. Consequently, the gradient graph $L_{\bar{u}} = \{(\bar{x}, D\bar{u}(\bar{x})) \,|\, \bar{x} \in B_r\}$ is a smooth submanifold of $\mathbb{C}^n$. Rotating it back up by the same angle $\varphi$, and using the fact that rotations preserve smooth manifolds, we conclude that the resulting submanifold, $L_u = \{(x, Du(x)) \,|\, x \in B_{r'}\}$ where $B_{r'} \subset \bar{x}^{-1}(B_r)$, is also smooth.

Now applying Lemma 5.2.1 we get $u \in C^{1,1}_{\text{loc}}$. Then $u$ is smooth by [18, Theorem 1.2]. □

Next, we prove the following Lemma to push the Hölder exponent to $\frac{1}{2}$.

**Lemma 5.2.3.** *If $F : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth map with partial derivatives vanishing up to order $2k-1$ ($k \geq 1$) and there exists a constant $c > 0$ such that $|F(x)| \geq c|x|^{2k}$ for $x$ in $B_r$, then $F$ is not injective on any neighborhood of 0.*

*Proof.* First we observe that since $|F(x)| \geq c|x|^{2k}$, the map $F$ has an isolated zero at 0. If $F$ were injective in a neighborhood of 0, its local degree must be $\pm 1$ [31]. Let $Q$ denote the homogeneous degree $2k$ part of the Taylor expansion of $F$. Since $Q$ is of even degree, $Q(x) = Q(-x)$.

For $s < r$, consider the map $\Phi_s : S^{n-1} \to S^{n-1}$ given by

$$\Phi_s(x) = \frac{F(sx)}{|F(sx)|}.$$

Letting $s \to 0$, we see that $\Phi_s \to \frac{Q(x)}{|Q(x)|}$ uniformly, so $\Phi_s$ is homotopic to $\frac{Q(x)}{|Q(x)|}$. By Sard's Theorem, there exists $y \in S^{n-1}$ such that $\left(\frac{Q(\cdot)}{|Q(\cdot)|}\right)^{-1}(\{y\})$ is a discrete set. Since $Q$ is even, the preimage of $y$ has even cardinality and therefore $\frac{Q(\cdot)}{|Q(\cdot)|}$ has even degree. Since degree is homotopy invariant, $\Phi_s$ has even degree, and therefore $F$ has even local degree at 0, contradicting injectivity. $\square$

**Proposition 5.2.4.** *Let $u \in C^{1,\frac{1}{2}}$ be a weak solution of* (5.1) *on the unit ball $B_1 \subset \mathbb{R}^n$ with $\theta \geq (n-2)\frac{\pi}{2} + \delta$. Then $u$ is smooth inside $B_1$.*

*Proof.* As in the proof of Proposition 5.2.2, we perform a downward rotation of the gradient graph $L_u = \{(x, Du(x)) \mid x \in B_1\} \subset \mathbb{R}^n \times \mathbb{R}^n$. By [18], we get that the gradient graph $L_u$ is a smooth submanifold of $\mathbb{C}^n$.

Let's assume $u$ is singular at 0 and that $Du(0) = 0$. Let $\pi_x$ be the orthogonal projection onto the $x$-plane in $\mathbb{C}^n$. After rotating coordinates on the $y$-plane, we assume that $\{\partial_{y_1}, \dots, \partial_{y_k}\}$ $(k \geq 1)$ forms a basis for $\ker \pi_x|_P$. After another coordinate change on the $x$-plane, we assume that $\{\partial_{x_{k+1}}, \dots, \partial_{x_n}\}$ forms a basis for $\pi_x(P)$. Then $P$ projects nondegenerately onto the plane spanned by the vectors $\partial_{y_1}, \dots, \partial_{y_k}, \partial_{x_{k+1}}, \dots, \partial_{x_n}$. So $L_u$ is a graph over this plane:

$$L_u = \{(y_1, \dots, y_k, x_{k+1}, \dots, x_n, F(y_1, \dots, y_k, x_{k+1}, \dots, x_n))\}$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$. Since the vectors $\partial_{y_1}, \dots, \partial_{y_k}$ are tangent to $L_u$ at 0, we get $\partial_{y_i} F(0) = 0$ for $i = 1, \dots, k$. Therefore, $F$ has a Taylor expansion of the form

$$F(y', x'') = F_1(x'') + F_2(y', x'') + \dots \tag{5.2}$$

where $F_i$ is the degree $i$ homogeneous term of the Taylor series of $F$.

Next we consider the map $F' : \mathbb{R}^k \to \mathbb{R}^k$ given by projecting

$$F(y_1, \ldots, y_k, 0, \ldots, 0)$$

onto the $x_1, \ldots, x_k$ directions. Since $Du \in C^{\frac{1}{2}}$, points on $L_u$ satisfy $|y|^2 \leq C|x|$, which implies

$$|F'(y_1, \ldots, y_k)| \geq C|(y_1, \ldots, y_k)|^2.$$

Combining this with (5.2) shows that $F'$ is quadratic to leading order and verifies the hypothesis of Lemma 5.2.3. Therefore, $F'$ is not injective. So there exist distinct points $(y_1^{(1)}, \ldots, y_k^{(1)})$ and $(y_1^{(2)}, \ldots, y_k^{(2)})$ such that $F'(y_1^{(1)}, \ldots, y_k^{(1)}) = F'(y_1^{(2)}, \ldots, y_k^{(2)})$. This means that two different points of $L_u$ have the $x$-coordinate given by

$$(F'(y_1^{(1)}, \ldots, y_k^{(1)}), 0, \ldots, 0),$$

contradicting the assumption that $L_u$ is a graph. Therefore, $L_u$ does not have a vertical tangent plane in the interior of $B_1$. Thus $u \in C^{1,1}_{\text{loc}}$ and is therefore smooth in the interior by [18, Theorem 1.2]. $\square$

**Proposition 5.2.5.** *Let $u \in C^{1,\beta}$ be a weak solution of* (5.1) *on the unit ball $B_1 \subset \mathbb{R}^n$ with $\beta \in (\frac{1}{3}, \frac{1}{2})$ and $\theta \geq (n-2)\frac{\pi}{2} + \delta$. Then $u$ is smooth inside $B_1$.*

*Proof.* The proof follows the same strategy as in Proposition 5.2.2. Since the gradient graph $L_u$ is a smooth submanifold, Lemma 5.2.1 implies that $u \in C^{1,\frac{1}{2}}_{\text{loc}}$. Applying Proposition 5.2.4 then yields smoothness of $u$ in the interior. $\square$

Now we have all the ingredients to prove the main result of this section.

*Proof of Theorem 5.1.1.* Smoothness of $u$ is obtained by combining Propositions 5.2.2, 5.2.4, and 5.2.5, after which the desired estimate follows from [18, Theorem 1.2]. $\square$

In fact, note that combining Lemma 5.2.1 with Lemma 5.2.3 and using the argument in the proof of Proposition 5.2.4, we get the following.

**Lemma 5.2.6.** *Suppose that $F : \mathbb{R}^n \to \mathbb{R}^n$ is $C^\beta$ where $\beta > \frac{1}{2l+1}$ and $l \geq 1$ is an integer. If the graph of $F$ is a smooth submanifold of $\mathbb{R}^n \times \mathbb{R}^n$, then $F \in C^{\frac{1}{2l-1}}_{\text{loc}}$.*

This shows that singular Hölder continuous solutions of (5.1) with phase $\theta \geq (n-2)\frac{\pi}{2} + \delta$ are precisely $C^{1,\frac{1}{2l+1}}$ for some integer $l \geq 1$. In the next section, we show that each possible degree of Hölder regularity is achieved by a weak solution with hypercritical phase.

We conclude this section with an interesting observation on the Hölder continuity of the Lagrangian phase.

**Proposition 5.2.7.** *Let $u \in C^{1,\beta}$ be a semi-convex weak solution of (5.1) on the unit ball $B_1 \subset \mathbb{R}^n$. Then there exists a $\gamma$ such that the Lagrangian phase, $\theta$, is $C^\gamma$ inside $B_1$.*

*Proof.* Since $u$ is semi-convex, we can rotate the gradient graph of $u$ down by an angle $\varphi$, which is dependent on the negative lower bound of $D^2u$, to obtain a new potential $\bar{u} \in C^{1,1}$. The new rotated coordinates are given by

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The new phase $\bar{\theta}(\bar{x}) = \theta(x) - n\varphi$ is a weak solution of the equation $\triangle_{\bar{g}}\bar{\theta} = 0$, which is uniformly elliptic since $\bar{u} \in C^{1,1}$ as a function of $\bar{x}$. By the De Giorgi-Nash Theorem, $\bar{\theta}$ is $C^\alpha$ in $\bar{x}$ for some $\alpha$.

Since $Du \in C^\beta$ by assumption, we get

$$\begin{aligned} |\bar{x}_1 - \bar{x}_2| &= |\cos\varphi(x_1 - x_2) + \sin\varphi(Du(x_1) - Du(x_2))| \\ &\leq |x_1 - x_2| + |Du(x_1) - Du(x_2)| \\ &\leq |x_1 - x_2| + C_1|x_1 - x_2|^\beta \\ &\leq C_2|x_1 - x_2|^\beta. \end{aligned}$$

Then

$$\frac{|\theta(x_1) - \theta(x_2)|}{|x_1 - x_2|^{\alpha\beta}} \leq C_2^{\alpha} \frac{|\bar{\theta}(\bar{x}_1) - \bar{\theta}(\bar{x}_2)|}{|\bar{x}_1 - \bar{x}_2|^{\alpha}},$$

so $\theta$ is $C^{\alpha\beta}$ in $x$. □

Note that the above proposition is different from [18, Corollary 4.6], which proves that for semi-convex $C^1$ weak solutions of (5.1), the phase $\theta$ is Hölder continuous with respect to the intrinsic metric on $L_u$.

### *5.3 Singular solutions*

In this section, we prove Theorem 5.1.2. We begin with a lemma establishing conditions under which a Lagrangian graph can be rotated upward while preserving the property of being a graph.

**Lemma 5.3.1.** *Let $w \in C^2(B_1)$ with $\lambda_{\max}(D^2 w) < 1$ everywhere except possibly on a discrete set, where $\lambda_{\max}$ denotes the largest eigenvalue of $D^2 w$. Then the submanifold of $\mathbb{R}^n \times \mathbb{R}^n$ obtained by rotating the gradient graph of $w$,*

$$L_w = \{(x, Dw(x)) \mid x \in B_1\},$$

*upward by an angle of $\frac{\pi}{4}$ is again a graph.*

*Proof.* We perform an upward rotation on $L_w$ by an angle of $\frac{\pi}{4}$. Then the rotated submanifold has the following parametrization

$$\Phi(x) = \frac{1}{\sqrt{2}}(x - Dw(x), x + Dw(x)).$$

We assume there are points $x, y \in B_1$ such that

$$\Phi(x) - \Phi(y) = \sqrt{2}(0, V)$$

is vertical, i.e., the image of $\Phi$ is not a graph. Then

$$x - y = Dw(x) - Dw(y) = V.$$

But observe that

$$\begin{aligned} |V|^2 &= \langle V, Dw(x) - Dw(y) \rangle \\ &= \int_0^1 \langle V, D^2 w(y + tV)V \rangle \, dt \\ &< |V|^2, \end{aligned}$$

since $\lambda_{\max}(D^2 w(y + tV)) < 1$ for almost every $t \in [0, 1]$. This is a contradiction. Therefore, the rotated submanifold must be a graph. $\square$

Before constructing singular solutions to prove Theorem 5.1.2, we proceed with a useful lemma which will allow us to take advantage of rotational symmetry when analyzing the Cauchy problem for (5.1).

Note that for the rest of this section, we will denote a point $x \in \mathbb{R}^n$, by $x = (x', x_n)$ where $x' \in \mathbb{R}^{n-1}$.

**Lemma 5.3.2.** *Let $u$ be a smooth function defined on a neighborhood of the origin that is axisymmetric, i.e.,*

$$u(x', x_n) = v(|x'|, x_n).$$

*Then the Hessian $D^2 u$ has the eigenvalue $\frac{v_r}{r}$ with multiplicity $n - 2$, and its two remaining eigenvalues are precisely the eigenvalues of the $2 \times 2$ matrix $D^2 v$.*

*Proof.* Let $r = |x'|$. An elementary computation shows that

$$D^2 u = \begin{pmatrix} \frac{v_r}{r} I_{n-1} + (v_{rr} - \frac{v_r}{r}) \frac{x' \otimes x'}{r^2} & v_{rn} \frac{x'}{r} \\ v_{rn} \frac{(x')^T}{r} & v_{nn} \end{pmatrix}.$$

For a vector $\xi \in \mathbb{R}^{n-1}$ with $\xi \perp x'$, we have

$$D^2 u(\xi, 0) = \frac{v_r}{r} (\xi, 0).$$

Also, observe that

$$D^2 u(\partial_r) = v_{rr} \partial_r + v_{rn} \partial_n$$

$$D^2 u(\partial_n) = v_{rn} \partial_r + v_{nn} \partial_n.$$

This proves the claim. $\square$

Now we are ready to prove Theorem 5.1.2.

*Proof of Theorem 5.1.2.* Fix a positive integer $k$. We construct a solution $w$ to (5.1) by solving the Cauchy problem

$$\begin{aligned}\triangle_g \theta &= 0,\\ w(x',0) &= \frac{1}{2}|x'|^2 - |x'|^{2k+2},\\ w_n(x',0) &= 0,\\ w_{nn}(x',0) &= -2|x'|^2,\\ w_{nnn}(x',0) &= 0.\end{aligned}$$

The Cauchy-Kovalevskaya Theorem guarantees the existence of a unique analytic solution $w$ to the Cauchy problem. Observe that $w$ must be axisymmetric, since the Cauchy data is invariant under rotations in the $x'$-plane.

Furthermore, the Cauchy data ensures that $w$ has a Taylor expansion of the form

$$w(x) = \frac{1}{2}r^2 - r^{2k+2} - x_n^2 r^2 + x_n^4 W(r, x_n)$$

where $W$ is an analytic function (and an even function in $r$).

We have

$$\begin{aligned}\frac{w_r(x)}{r} &= 1 - (2k+2)r^{2k} - 2x_n^2 + x_n^4\frac{W_r(r,x_n)}{r}, \qquad (5.3)\\ w_{rr}(x) &= 1 - (2k+2)(2k+1)r^{2k} - 2x_n^2 + x_n^4 W_{rr}(r,x_n),\\ w_{rn}(x) &= -4x_n r + O(x_n^3).\end{aligned}$$

So near the origin, we get

$$\frac{w_r(x)}{r} \leq 1 - (2k+2)r^{2k} - x_n^2.$$

Therefore $D^2w$ has eigenvalues $\lambda_1, \ldots, \lambda_{n-2}$ which reach a local maximum value of 1 at the origin by Lemma 5.3.2. Moreover, observe that near the origin, we have

$$\begin{aligned}\lambda_{n-1}(D^2w) &= w_{rr} + O(w_{rn}^2) \\ &= 1 - (2k+2)(2k+1)r^{2k} - 2x_n^2 + Cx_n^2r^2 + O(x_n^4) \\ &\leq 1 - (2k+2)(2k+1)r^{2k} - x_n^2.\end{aligned} \tag{5.4}$$

So $\lambda_{n-1}$ also reaches a local maximum value of 1 at the origin. The final eigenvalue $\lambda_n$ is 0 at the origin.

Since $n-1$ eigenvalues of $D^2w$ have local maximum values of 1 at the origin, we can rotate the gradient graph of $w$ so that the tangent plane at the origin becomes vertical.

Using Lemma 5.3.1, we rotate the gradient graph $L_w$ upward by an angle of $\frac{\pi}{4}$ to obtain a new Lagrangian graph which is the gradient graph of a new potential $u^{(n,k)}$. After a rescaling, the new gradient graph, $L_{u^{(n,k)}}$, is the image of the following parametrization

$$\begin{aligned}\Phi(x) &= (x - Dw(x), x + Dw(x)) \\ &= ([(2k+2)|x'|^{2k} + 2x_n^2]x', [1 + 2|x'|^2]x_n, \\ &\quad [2 - (2k+2)|x'|^{2k} - 2x_n^2]x', [1 - 2|x'|^2]x_n) + O(x_n^3).\end{aligned} \tag{5.5}$$

Set $\bar{x} = x - Dw(x)$ and $\bar{y} = x + Dw(x)$.

To show that the image of $\Phi$ is the graph of a $C^{\frac{1}{2k+1}}$ function, we must show that $\bar{y}$ is $C^{\frac{1}{2k+1}}$ as a function of $\bar{x}$. If not, then there would be a point $\bar{x}_0$ such that

$$\limsup_{\bar{x} \to \bar{x}_0} \frac{|\bar{y}(\bar{x}) - \bar{y}(\bar{x}_0)|}{|\bar{x} - \bar{x}_0|^{\frac{1}{2k+1}}} = \infty.$$

Necessarily, $\bar{y}$ is not differentiable at $\bar{x}_0$. By our construction, the only point where $\bar{y}$ is not differentiable is the origin. Therefore, it suffices to establish Hölder continuity of $\bar{y}$ at the origin. Since $\bar{y} = -\bar{x} + 2x(\bar{x})$, it is enough to show that $x$ is a $C^{\frac{1}{2k+1}}$ function of $\bar{x}$ at the origin. From (5.5), we get

$$|\bar{x}|^2 \geq (2k+2)|x'|^{4k+2} + 4|x'|^4x_n^2 \geq C|x|^{4k+2}$$

near the origin. Therefore $u^{(n,k)} \in C^{\frac{1}{2k+1}}$, as desired. Considering the image of the $x_1$ axis under $\Phi$ shows that $Du$ has no higher Hölder continuity.

Since $Du^{(n,k)}\bar{x} = \bar{y}(\bar{x})$, to show that $u^{(n,k)} \in W^{2,1}$, it is enough to show that $x$ is a $W^{1,1}$ function of $\bar{x}$. We have

$$D_{\bar{x}}x = (I - D^2w)^{-1}.$$

Thus, it suffices to show that

$$\int_{B_\rho} |(I - D^2w)|^{-1}\, d\bar{x} < \infty.$$

Using the standard change of variables formula, we get

$$\int_{B_\rho} |(I - D^2w)^{-1}|\, d\bar{x} = \int_{x^{-1}(B_\rho)} |(I - D^2w)|^{-1} \det(I - D^2w)\, dx.$$

By equations (5.3)-(5.4), there are positive constants $C_1, C_2, C_3, C_4$ such that

$$|(I - D^2w)^{-1}| \sim (1 - \lambda_{n-1}(D^2w))^{-1} \leq \frac{1}{C_1|x'|^{2k} + C_2x_n^2}$$

and

$$1 - \lambda_i(D^2w) \leq C_3|x'|^{2k} + C_4x_n^2$$

for $1 \leq i \leq n-2$. Note that $\frac{C_3|x'|^{2k}+C_4x_n^2}{C_1|x'|^{2k}+C_2x_n^2}$ is bounded, and hence integrable, so $u^{(n,k)} \in W^{2,1}$.

Similarly, to show that $\sqrt{g}$ is integrable, note that $\sqrt{g} \sim \det(I - D^2w)^{-1}$. After changing variables from $\bar{x}$ back to $x$, the integrand becomes bounded.

Since $w$ is a smooth solution of (5.1), $\theta$ is a smooth function on $L_w$. Therefore, after performing the rotation, the phase $\bar{\theta}$ is still a smooth function on the submanifold $L_{u^{(n,k)}}$. Moreover, $\bar{\theta}(\bar{x}) = \theta(x) + n\frac{\pi}{4}$. Therefore

$$D_{\bar{x}}\bar{\theta} = (I - D^2w)^{-1}D_x\theta.$$

Hence to show that $\bar{\theta} \in W^{1,2}(B_1)$, it is enough to show that

$$\int_{B_1} |(I - D^2w)^{-1}|^2\, d\bar{x} < \infty.$$

Changing variables back to $x$, the integrand becomes bounded again.

Finally, we see that $u^{(n,k)}$ satisfies (5.1) in the integral sense because $L_{u^{(n,k)}}$ is a smooth submanifold and the phase is a smooth harmonic function on $L_{u^{(n,k)}}$. $\square$

We conclude this section by characterizing the Sobolev regularity of the weak solutions constructed above.

**Lemma 5.3.3.** *For $q, k > 0$, the function $\frac{1}{(|x'|^{2k}+|x_n|^2)^q} \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ if and only if*

$$q < \frac{1}{2} + \frac{n-1}{2k}.$$

*Proof.* Let $r = |x'|$ and $s = |x_n|$. Using polar coordinates in the $x'$-plane, we need to determine when

$$\int_0^1 \int_0^1 \frac{r^{n-2}}{(r^{2k}+s^2)^q}\, dr\, ds < \infty.$$

Split the integral into the regions where $s \leq r^k$ and $s \geq r^k$.

In the region $s \leq r^k$, we have

$$\begin{aligned}\int_0^1 \int_0^{r^k} \frac{r^{n-2}}{(r^{2k}+s^2)^q}\, ds\, dr &\leq \int_0^1 \int_0^{r^k} r^{n-2-2kq}\, ds\, dr \\ &= \int_0^1 r^{n-2-2kq+k}\, dr.\end{aligned}$$

The final integral converges if and only if $n-2-2kq+k > -1$, or equivalently, $q < \frac{1}{2} + \frac{n-1}{2k}$.

In the region $s \geq r^k$, we have

$$\begin{aligned}\int_0^1 \int_0^{s^{\frac{1}{k}}} \frac{r^{n-2}}{(r^{2k}+s^2)^q}\, dr\, ds &\leq \int_0^1 \int_0^{s^{\frac{1}{k}}} r^{n-2} s^{-2q}\, dr\, ds \\ &= \frac{1}{n-1}\int_0^1 s^{\frac{n-1}{k}-2q}\, dr.\end{aligned}$$

Again, the last integral is finite if and only if $\frac{n-1}{k} - 2q > -1$, or equivalently $q < \frac{1}{2} + \frac{n-1}{2k}$. This shows that the condition on $q$ is sufficient to ensure $\frac{1}{(|x'|^{2k}+|x_n|^2)^q} \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

To show the condition is necessary, again consider the region where $s \geq r^k$. Here we have

$$\int_0^1 \int_0^{r^k} \frac{r^{n-2}}{(r^{2k}+s^2)^q}\, ds\, dr \geq \int_0^1 \int_0^{s^{\frac{1}{k}}} r^{n-2-2kq}\, dr\, ds.$$

The inner integral converges if and only if $q < \frac{n-1}{2k}$, and in this case we have

$$\int_0^1 \int_0^{s^{\frac{1}{k}}} r^{n-2-2kq}\, dr\, ds = \frac{1}{n-1-2kq} \int_0^1 s^{\frac{n-1-2kq}{k}}\, ds$$

which converges if and only if $q < \frac{1}{2} + \frac{n-1}{2k}$. $\square$

**Proposition 5.3.4.** *The solution $u^{(n,k)}$ constructed above is in $W^{2,p}$ for $p < n - \frac{1}{2} + \frac{n-1}{2k}$. In particular, $u^{(n,k)} \in W^{2,n}$ for $k < n-1$.*

*Proof.* By the proof of Theorem 5.1.2, $u^{(n,k)} \in W^{2,p}$ whenever

$$\int_{B_\rho} |(I - D^2 w)^{-1}|^p \, d\bar{x} < \infty.$$

After changing variables and applying equations (5.3)-(5.4), convergence of the integral is equivalent to convergence of

$$\int_{B_\rho} \frac{1}{(|x'|^{2k} + x_n^2)^{p-n+1}}\, dx.$$

Applying Lemma 5.3.3 shows that this integral converges if and only if $p - n + 1 < \frac{1}{2} + \frac{n-1}{2k}$. $\square$

# Chapter 6

# A TRANSLATING SOLUTION TO CURVE DIFFUSION FLOW

The content of this chapter first appeared in the author's joint work with Micah Warren [55].

## *6.1 Curve diffusion flow*

For a one dimensional manifold $M$, a map

$$X(\omega, t) : M \times I \to \mathbb{R}^2$$

which satisfies

$$\left(\frac{dX}{dt}\right)^{\perp} = -\kappa_{ss} N \tag{6.1}$$

is a solution to the curve diffusion flow. Here $\kappa$ is the curvature, which can be defined as the derivative of the angle $\theta$ that the unit tangent vector makes with the $x$-axis with respect to the arc length parameter $s$. The unit normal vector $N$ is the complex rotation of the oriented unit tangent vector of the curve.

This flow has a long history of interest in a wide variety of fields in both pure and applied mathematics, going back to Mullins [49] in 1957; see [21] for a more complete list of references. In higher dimensions, the curve diffusion flow generalizes to the surface diffusion flow for hypersurfaces [24], and to the gradient flow for volume of Lagrangian submanifolds within a given Hamiltonian isotopy class [20].

For compact $M$ it is easy to check using the first variation formula that curve diffusion flow is the gradient flow for arc length on the space of curves with a fixed enclosed area, where the tangent space consists of variations of the form

$$\frac{dX}{dt} = J\nabla f$$

for $f$ a function on $M$, where $J$ is the standard complex rotation and the tangent space is equipped with the $L^2$ metric

$$\|f\|^2 = \int_M f^2\, ds.$$

Here, we present a construction of a nontrivial properly immersed translating solution to curve diffusion flow.

In [62] it was shown that singularities occur for the flow. In particular, an immersed figure-8 curve must develop singularities under the flow. Other examples of curves which develop singularities are provided in [23]. It is a natural question to consider self-similar solutions, as these are models for formation of singularities.

In [21] it was shown that:

- (Corollary 4) The only open, properly immersed stationary solutions to the flow are lines;

- There are many stationary solutions that are not properly immersed (Euler spirals);

- (Proposition 11) If an open translator has curvature vanishing at $+\infty$ and $-\infty$ and the angle the translating vector makes with the tangent is the same at both $+\infty$ and $-\infty$, then the translator is a stationary line;

- (Proposition 12) If $X : \mathbb{R} \to \mathbb{R}^2$ is an open properly immersed translator satisfying
$$\lim_{\rho\to\infty} \frac{1}{\rho^2} \int_{X^{-1}(B_\rho)} \kappa^2\, ds + \frac{1}{\rho} \int_{X^{-1}(B_{2\rho})} |\langle \vec{E}, T\rangle|\, ds = 0,$$
where $\vec{E}$ is the translating velocity and $T$ is the oriented unit tangent vector, then $X(\mathbb{R})$ is a straight line.

The authors asked whether the growth condition or properness are necessary in Proposition 12. Our main result partially answers this question by showing that properness by itself does not imply the translator is a line.

Our main result is:

**Theorem 6.1.1.** *There exists a properly immersed translating solution to the curve diffusion flow with nonconstant curvature.*

Our strategy is to use a shooting method together with symmetry to show that a bounded yet nontrivial solution to a certain third order nonlinear ODE (6.2) exists. The solution of the ODE will describe the angle of the tangent vector of the profile curve for the translating solution of the flow.

Similar methods were used to study a prescribed-contact-angle free boundary problem for translating solutions of curve diffusion flow in [36].

### *6.1.1 Translating solutions*

Suppose that the unprojected flow satisfies

$$\frac{dX}{dt} = \vec{E}$$

for some fixed unit vector $\vec{E}$. From (6.1) we have

$$\vec{E} \cdot N = -\kappa_{ss}.$$

Now, without loss of generality, let

$$\vec{E} = (0,1) = J(1,0)$$

so that

$$\begin{aligned}\vec{E} \cdot N &= J(1,0) \cdot JT \\ &= (1,0) \cdot T \\ &= \cos\theta,\end{aligned}$$

and the equation becomes

$$\cos\theta = -\kappa_{ss}$$

or

$$\theta_{sss} = -\cos\theta. \tag{6.2}$$

An arc-length parameterized path

$$\gamma : \mathbb{R} \to \mathbb{R}^2$$

whose angle $\theta$ satisfies (6.2) is the profile of a translating solution to the curve diffusion flow. If, in addition

$$\theta \to \frac{\pi}{2} \text{ as } s \to \infty,$$
$$\theta \to -\frac{\pi}{2} \text{ as } s \to -\infty$$

then the solution will be properly immersed and have nonconstant curvature.

Before constructing a solution with the desired properties, we give a simple criterion which provides a large family of solutions to (6.2) which correspond to translators which are not properly immersed, as their curvatures grow as asymptotically linear functions of arc length (which is also the case for the stationary Euler spiral solutions to the curve diffusion flow).

**Proposition 6.1.2.** *Suppose that $\theta$ satisfies* (6.2) *with $\theta_s(0) \geq a > 0$ and $\theta_{ss}(0) > \frac{2}{a}$. Then the corresponding curve $\gamma$ is not properly immersed.*

*Proof.* We claim that $\theta_{ss}$ is uniformly bounded away from 0 as $s \to \infty$. First observe that $\theta_{ss}$ can never reach 0. For contradiction, assume that $s_0 > 0$ is the first time such that $\theta_{ss}(s_0) = 0$. Then for any $s \in [0, s_0]$,

$$\begin{aligned} |\theta_{ss}(s) - \theta_{ss}(0)| &= \left| \int_0^s \cos(\theta(\sigma))\, d\sigma \right| \\ &= \left| \int_0^s \frac{1}{\theta_s(\sigma)} \frac{d}{d\sigma} \sin(\theta(\sigma))\, d\sigma \right|. \end{aligned} \tag{6.3}$$

Integrating by parts,

$$\begin{aligned}\left|\int_0^s \frac{1}{\theta_s(\sigma)}\frac{d}{d\sigma}\sin(\theta(\sigma))\,d\sigma\right| &\leq \left|\frac{1}{\theta_s(s)}\sin(\theta(s)) - \frac{1}{\theta_s(0)}\sin(\theta(0))\right| \\ &\quad + \left|\int_0^s \frac{d}{d\sigma}\frac{1}{\theta_s(\sigma)}\sin(\theta(\sigma))\,d\sigma\right| \\ &\leq \frac{1}{\theta_s(0)} + \frac{1}{\theta_s(s)} + \left|\int_0^s \frac{d}{d\sigma}\frac{1}{\theta_s(\sigma)}\,d\sigma\right| \\ &= \frac{1}{\theta_s(0)} + \frac{1}{\theta_s(s)} + \left|\frac{1}{\theta_s(s)} - \frac{1}{\theta_s(0)}\right| \\ &= \frac{2}{a},\end{aligned}$$

where we used the fact that $\theta_s$ is monotone increasing. This is a contradiction since $\theta_{ss}(0) > \frac{2}{a}$, so we must have $\theta_{ss} > 0$ for all $s > 0$. Considering (6.3) again now shows that

$$\theta_{ss}(s) \geq \theta_{ss}(0) - \frac{2}{a} > 0$$

for all $s > 0$, so $\theta_{ss}$ is uniformly bounded below by a positive constant $\varepsilon$. Then

$$\kappa(s) - \kappa(0) = \int_0^s \theta_{ss}\,ds \geq \varepsilon s,$$

so the curvature of $\gamma$ grows at least linearly as $s \to \infty$. (Note also that

$$\kappa(s) - \kappa(0) \leq \left(\theta_{ss}(0) + \frac{2}{a}\right) s,$$

so the curvature also has a linear upper bound.) Since the curvature of $\gamma$ is increasing, the piece $\{\gamma(s) : s > 0\}$ is contained in the osculating circle at 0. Since $\gamma$ has infinite arc length inside a compact set, it is not properly immersed. $\square$

### *6.1.2 Strategy of construction*

Our strategy for constructing a properly immersed translating solution is:

1. Modify the ODE (6.2) to obtain a monotone ODE (6.4). Show that any solution to the modified equation which stays bounded for $s > 0$ must converge to $\pi/2$ as $s \to \infty$.

2. Show that any solution to the modified equation with suitable initial conditions (6.5) which stays bounded for $s > 0$ is a solution to the original equation (6.2).

3. Demonstrate the existence of a bounded solution $\theta(s)$ to the modified equation.

4. Reflect the solution in Step 3 across $s = 0$ to get an entire solution which also converges to $-\pi/2$ as $s \to -\infty$.

5. With $\theta$ determined as a function of arc length, construct the path $\gamma$ forwards and backwards. The tangent directions approach $\pm\pi/2$ so the solution must be properly immersed.

## 6.2 *Proof of Theorem 6.1.1*

### *6.2.1 Step 1*

We start by modifying the ODE (6.2) slightly so that the right-hand side is monotone: Consider the ODE

$$u''' = f(u) \tag{6.4}$$

with

$$f(u) = \begin{cases} -\cos u & \text{if } 0 \leq u \leq \pi \\ -1 & \text{if } u \leq 0 \\ +1 & \text{if } u \geq \pi. \end{cases}$$

It is more straightforward to apply shooting methods to monotone ODE. If we have a solution whose values lie within the range $[0, \pi]$ this will also be a solution of the original ODE (6.2). Our goal is then to shoot to find a solution to the modified ODE, and then argue that this solution satisfies the original ODE.

The following claim sets up a shooting method.

**Lemma 6.2.1.** *Let $u_a$ be the solution to the ODE* (6.4)

$$u''' = f(u)$$

*with initial conditions*

$$\begin{aligned} u(0) &= 0, \\ u'(0) &= a, \\ u''(0) &= 0. \end{aligned} \tag{6.5}$$

*These have the following order preservation property: If $a < b$ then*

$$u_a(s) < u_b(s) \textit{ for all } s > 0.$$

*Proof.* Let

$$v(s) = u_b(s) - u_a(s).$$

Then

$$\begin{aligned} v(0) &= 0, \\ v'(0) &= b - a, \\ v''(0) &= 0, \\ v'''(0) &= 0. \end{aligned}$$

So $v > 0$, and $v' > 0$ for small values of $s$. Let $s_0$ be the first value such that $v'(s_0) = 0$. Then

$$0 > \int_0^{s_0} v''(\tau)\, d\tau = \int_0^{s_0} \int_0^{\tau} \left(f(u_b) - f(u_a)\right)\, d\sigma\, d\tau$$

but this is a contradiction, because on the whole interval we had $f(u_b) - f(u_a) \geq 0$. There is no such $s_0$; we may conclude $v' > 0$ and hence $u_b(s) > u_a(s)$ for all positive $s$. □

The following is a straightforward application of the fundamental theorem of calculus. Because we refer to this argument frequently in the sequel we give it a name.

**Lemma 6.2.2.** *(Trifecta). Suppose that $v$ is a solution to a monotone third order ODE of the form $v''' = h(v)$ for some monotone nondecreasing function $h(v)$. If at any point $s$, $h(v(s)), v'(s)$ and $v''(s)$ are all nonnegative and at least one of these is positive, the solution $v$ will increase forever without reaching a future critical point, and $s$ cannot be a local maximum itself.*

We now begin analyzing the behavior of bounded solutions to (6.4).

For cleaner presentation, let $v = u - \pi/2$, and consider instead the following ODE:

$$v''' = g(v) \tag{6.6}$$

with

$$g(v) = \begin{cases} \sin v & \text{if } -\pi/2 \le v \le \pi/2 \\ -1 & \text{if } v \le -\pi/2 \\ +1 & \text{if } v \ge \pi/2. \end{cases}$$

**Lemma 6.2.3.** *Suppose that $v$ is a solution to* (6.6). *Suppose that at $s_{\max}$ the solution $v$ has a positive local maximum and at $s_{\min}$, the next critical point, $v$ has a negative local minimum. Then $v(s_{\max}) > |v(s_{\min})|$ .*

*Proof.* We know $v'(s_{\max}) = 0$ and $v''(s_{\max}) < 0$; if $v''(s_{\max}) = 0$ the trifecta (Lemma 6.2.2) would preclude further critical points. Similarly, we have $v''(s_{\min}) > 0$. There must be a point $s_0 \in (s_{\max}, s_{\min})$ where $v''(s_0) = 0$. To avoid the trifecta we must have $v(s_0) \ge 0$. Also, $s_0$ is unique, as if there were another point $s_1 > s_0$ with $v''(s_1) = 0$, then $v'''$ would vanish at a point in $(s_0, s_1)$, but this happens only when $v$ vanishes, therefore $v(s_1) < 0$, creating the trifecta. Thus

$$\begin{aligned} 0 \le v''(s_{\min}) - v''(s_0) &= \int_{s_0}^{s_{\min}} v'''(s)\, ds \\ &= \int_{s_0}^{s_{\min}} g(v(s))\, ds = \int_{v(s_0)}^{v(s_{\min})} g(v) \frac{ds}{dv}\, dv \\ &= \int_{v(s_{\min})}^{v(s_0)} g(v) \left(-\frac{ds}{dv}\right)\, dv \end{aligned}$$

using a change of variables $s \mapsto v(s)$.

Now let

$$\rho(v) = \left(-\frac{ds}{dv}\right) = -\frac{1}{v'(s)}.$$

Compute

$$\begin{aligned}
\frac{d}{dv}\rho(v) &= \frac{d}{ds}\left(\frac{-1}{v'(s)}\right)\frac{ds}{dv} \\
&= \frac{v''}{(v')^3} \leq 0 \text{ on } [s_0, s_{\min})
\end{aligned}$$

and strictly negative on the interior. Thus, as a function of $v$, $\rho$ is decreasing and we have for any $0 < w < \min\{v(s_0), |v(s_{min})|\}$

$$\rho(w) < \rho(-w). \tag{6.7}$$

Now for purposes of contradiction assume that

$$v(s_0) \leq |v(s_{\min})|.$$

Then

$$\begin{aligned}
0 &\leq \int_{v(s_{\min})}^{v(s_0)} g(v)\rho(v)\, dv \\
&= \int_{v(s_{\min})}^{-v(s_0)} g(v)\rho(v)\, dv + \int_{-v(s_0)}^{v(s_0)} g(v)\rho(v)\, dv
\end{aligned}$$

with the first term nonpositive, so we have

$$\begin{aligned}
0 &\leq \int_{-v(s_0)}^{v(s_0)} g(v)\rho(v)\, dv \\
&= \int_0^{v(s_0)} g(v)\rho(v)\, dv + \int_{-v(s_0)}^{0} g(v)\rho\,(v)\ dv \\
&= \int_0^{v(s_0)} g(v)\rho(v)\, dv + \int_{v(s_0)}^{0} g\,(-v)\,\rho\,(-v)\ d(-v) \\
&= \int_0^{v(s_0)} g(v)\left[\rho(v) - \rho\,(-v)\right]\ dv < 0
\end{aligned}$$

using the fact that $g$ is odd, and with the last inequality following from (6.7). This contradicts $v(s_0) \leq |v(s_{\min})|$. Thus $v(s_0) > |v(s_{\min})|$ which implies that $v(s_{\max}) > v(s_0) > |v(s_{\min})|$ and proves the Lemma. □

Noting that an order-reversed proof holds for a maximum subsequent to a minimum, it follows that if a solution bounces between positive local maxima and negative local minima, the distance between these must strictly be decreasing. We would like to show it decreases all the way to zero.

Next we quantify the difference between $v(s_{\max})$ and $v(s_0)$ in the above proof to get a more precise decay estimate.

**Lemma 6.2.4.** *Suppose that $v$ is a solution to* (6.6). *Suppose that $v(s_{\max}) \leq \pi$ is a positive local maximum and at $s_{\min}$, the next critical point, $v$ has a negative local minimum. Then*

$$|v(s_{\min})| < \delta(v(s_{\max}))v(s_{\max})$$

*where*

$$\delta(z) = 1 - \frac{z^2}{192} < 1.$$

*If $v(s_{\max}) > \pi$, and $v$ has a negative local minimum at $s_{\min}$, then*

$$|v(s_{\min})| < v(s_{\max}) - \frac{\pi^3}{192}.$$

*Proof.* Consider solutions which begin from a positive local maximum at $s = 0$; $v$ is the solution of the initial value problem with

$$v(0) = v(s_{\max}) = k > 0,$$
$$v'(0) = 0,$$
$$v''(0) = -b < 0.$$

By reasoning in the previous proof, there is some $s_0 > 0$ such that $v''(s_0) = 0$ and $v(s_0) > 0$.

Suppose that $v(s_{\max}) \leq \pi$. First we claim that the hypothesis that a negative minimum occurs subsequently implies that $b > \frac{k}{4}$: Assume not. It follows that

$$v(s) \geq k - \frac{k}{4}\frac{s^2}{2} \geq k\left(1 - \frac{s^2}{8}\right)$$

while $v$ remains positive. So $v(s) \geq k/2$ while $s \in [0,2]$ and we estimate

$$\begin{aligned} v'(2) &= \int_0^2 v''(s)\,ds \\ &= \int_0^2 \left(-b + \int_0^s v'''(t)\,dt\right)\,ds \\ &\geq -2\frac{k}{4} + 2\sin\left(\frac{k}{2}\right) > 0 \end{aligned}$$

as $2\sin x > x$ for $x \in (0, \pi/2)$. Thus $v$ must have achieved a local minimum while its value was positive, and Lemma 6.2.2 then contradicts the hypothesis. Thus $b > \frac{k}{4}$.

Next we claim that $s_0 \geq b$ :

$$\begin{aligned} b &= v''(s_0) - v''(0) \\ &= \int_0^{s_0} v'''(s)\,ds \\ &\leq \int_0^{s_0} ds = s_0. \end{aligned}$$

Now since $v''' \leq 1$, we have

$$v(s) \leq k - \frac{b}{2}s^2 + \frac{1}{6}s^3.$$

As $b \in (0, s_0]$ and $v$ is decreasing, we have

$$\begin{aligned} v(s_0) \leq v(b) &\leq k - \frac{b^3}{2} + \frac{1}{6}b^3 \\ &= k - \frac{b^3}{3} \\ &< k - \frac{1}{3}\left(\frac{k}{4}\right)^3 \\ &= k\left(1 - \frac{1}{192}k^2\right). \end{aligned}$$

Therefore

$$v(s_0) \le \left(1 - \frac{1}{192}k^2\right) v(s_{\max}).$$

As shown in the proof of the previous Lemma, $|v(s_{\min})| < v\,(s_0)$, which now implies the result.

Now assume $v(s_{\max}) > \pi$. The same argument used above implies that in order for $v$ to achieve a negative local minimum, $b > \frac{\pi}{4}$. Continuing with the same reasoning, we find

$$\begin{aligned} v(s_0) \le v(b) &\le k - \frac{b^3}{3} \\ &< k - \frac{1}{3}\left(\frac{\pi}{4}\right)^3 \\ &= k - \frac{\pi^3}{192}. \end{aligned}$$

Therefore $|v(s_{\min})| < v(s_{\max}) - \frac{\pi^3}{192}$. □

**Proposition 6.2.5.** *Suppose a solution $v$ to* (6.6) *is bounded for $s > 0$. Then $v(s) \to 0$ as $s \to +\infty$.*

*Proof.* Let $v$ be a bounded solution. Either there are infinitely many critical points, or finitely many. If there are finitely many (or none), the solution must be monotone on an unbounded interval, so it converges to a limit. This limit can't be anything other than 0, otherwise the third derivative would be uniformly positive or uniformly negative, contradicting boundedness.

So assume that $v$ has infinitely many critical points. Local maxima must be positive and local minima must be negative, by the trifecta argument (Lemma 6.2.2). By Lemma 6.2.3, the maxima must have strictly decreasing values and the minima are strictly increasing, so the sequences of values of local maxima and local minima each have a limit. We claim the only value they can converge to is 0. Suppose that an infinite sequence of local maxima has $k_0 > 0$ as a limit point. Then somewhere in the sequence we have $v(s_{\max}) < \min(\frac{1}{\delta(k_0)}k_0, k_0 + \frac{\pi^3}{192})$. Lemma 6.2.4 then forces subsequent maxima below $k_0$ to obtain a contradiction. □

### *6.2.2 Step 2*

Now we need to show that a solution to (6.4) which is bounded must actually be a solution to (6.2). This is a modification of the proof of Lemma 6.2.3.

**Proposition 6.2.6.** *A solution to* (6.4) *with initial conditions* (6.5) *that reaches* 0 *or* $\pi$ *at some* $s > 0$ *escapes to* $-\infty$ *or* $\infty$*, respectively, never returning to* $[0, \pi]$*. In particular, any solution to* (6.4) *with these initial conditions which remains bounded is also a solution to* (6.2)*.*

*Proof.* We break this into three claims and then draw the conclusion.

**Claim 6.2.7.** *A solution to* (6.4) *with initial conditions* (6.5) *that reaches a critical point with critical value in* $(0, \pi)$ *and then exceeds* $\pi$ *and reaches a local maximum violates Lemma 6.2.3.*

**Claim 6.2.8.** *A solution to* (6.4) *with initial conditions* (6.5) *that reaches* $\pi$ *without first finding a critical point will not achieve a subsequent local maximum.*

**Claim 6.2.9.** *A solution to* (6.4) *with initial conditions* (6.5) *that reaches a critical point less than* $\pi$*, crosses* 0 *and subsequently achieves a local minimum violates Lemma 6.2.3*

Proof of Claim 6.2.7. Let $s_1$ be the first time $u$ crosses $\pi$ and then let $s_0 < s_1$ be the last critical point before this. If $u(s_0) > \pi/2$ this is a trifecta; no further critical points occur. Now if $u(s_0) \leq \pi/2$ we can translate Lemma 6.2.3; this corresponds to solution $v$ which achieves a minimum with a value in $(-\pi/2, 0]$. By the assumption, the subsequent maximum would have a value strictly larger than $\pi/2$; this is impossible by Lemma 6.2.3.

Proof of Claim 6.2.8. As we are assuming that $u''(0) = 0$, we can repeat the proof of Lemma 6.2.3. Let $s_1$ be where $u$ crosses $\pi$. Note that if $u''(s_1) \geq 0$, then $u$ will enjoy the trifecta and will increase indefinitely afterwards. So assuming that $u$ is to achieve a critical value larger than $\pi$ leads us to $u''(s_1) < 0$. After translating $v = u - \pi/2$ we get

$$\begin{aligned} 0 &> u''(s_1) - u''(0) \\ &= v''(s_1) - v''(0) = \int_0^{s_1} v'''(s)\, ds \\ &= \int_0^{s_1} g(v(s))\, ds = \int_{-\pi/2}^{\pi/2} g(v) \frac{ds}{dv}\, dv. \end{aligned} \tag{6.8}$$

Then we repeat the argument in the proof of Lemma 6.2.3; let

$$\rho(v) = \frac{ds}{dv} = \frac{1}{v'(s)}$$

so that

$$\frac{d}{dv}\rho(v) = \frac{-v''}{(v')^3}.$$

As before, $v''(s) < 0$ for all $s \in (0, s_1]$. It follows that

$$\frac{d}{dv}\rho(v) = \frac{-v''}{(v')^3} \geq 0 \text{ for all } s \in [0, s_1].$$

Integrating the increasing odd function $g$ against the increasing density function $\rho$ we get

$$\int_{-\pi/2}^{\pi/2} g\,(v)\, \rho(v) dv > 0$$

contradicting (6.8). Thus $u$ cannot have a further critical value larger than $\pi$, it must increase forever.

Proof of Claim 6.2.9. Translating $v = u - \pi/2$, look at the last critical point $v$ achieves before crossing $-\pi/2$. If the critical value is negative, this represents a negative trifecta. If the critical value is positive but less than $\pi/2$, the subsequent minimum which we are assuming is less than $-\pi/2$ violates Lemma 6.2.3 .

The conclusion of the Lemma quickly follows: Any solution reaching $\pi$ is covered by Claim 6.2.7 or Claim 6.2.8. Any solution reaching 0 cannot have reached $\pi$ first by Claim 6.2.7 and Claim 6.2.8, thus Claim 6.2.9 applies. $\square$

*6.2.3 Step 3.*

We now demonstrate that a bounded solution to (6.4) with the initial conditions (6.5) exists.

**Lemma 6.2.10.** *There exists $a > 0$ such that the solution to*

$$u''' = f(u)$$

*with*

$$\begin{aligned} u(0) &= 0, \\ u'(0) &= a, \\ u''(0) &= 0 \end{aligned}$$

*is bounded for all $s > 0$.*

*Proof.* Fix any $m \in \mathbb{N}$. It's easy to check that there are values of $a$ such that $u_a(m) > 10$ and $u_a(m) < -10$. By the order preservation property Lemma 6.2.1 and continuous dependence on initial conditions, there is a set $I_m = [a_m, b_m]$ such that

$$\begin{aligned} u_{a_m}(m) &= 0, \\ u_{b_m}(m) &= \pi. \end{aligned}$$

By the order preservation property,

$$I_m = \left\{ a \in \mathbb{R}^+ \mid 0 \leq u_a(m) \leq \pi \right\}.$$

We claim that

$$I_{m+1} \subset I_m.$$

This follows immediately from Proposition 6.2.6: If $u_a(m+1) \in [0, \pi]$ then it cannot have left the region prior to that, so $u_a(m) \in (0, \pi)$.

Now we conclude that

$$I_M = \bigcap_{m \le M} I_m$$

and we can take

$$A = \bigcap_{m \in \mathbb{N}} I_m$$

which is the intersection of compact sets. Now the intersection of compact sets either is nonempty or must be empty at a finite stage. By the shooting, each $[a_m, b_m]$ is nonempty. It follows that there must be at least one point $a_\infty \in A$. The solution shooting from $a_\infty$ stays bounded for all time. □

### *6.2.4 Step 4.*

We may now take advantage of the odd initial conditions and the symmetry of the equation to reflect the solution, extending it to a bounded entire solution.

**Corollary 6.2.11.** *There exists a such that the solution to the ODE*

$$\theta''' = -\cos\theta$$

*with initial conditions*

$$\theta(0) = 0,$$
$$\theta'(0) = a,$$
$$\theta''(0) = 0$$

*is bounded for all time and converges to* $\pi/2$ *as* $s \to \infty$ *and* $-\pi/2$ *as* $s \to -\infty$.

*Proof.* By Lemma 6.2.10, there exists $a$ such that the initial conditions produce a solution $\theta(s)$ to (6.4) which is bounded for $s > 0$. By Proposition 6.2.6, $\theta(s)$ is a solution to (6.2), and it converges to $\frac{\pi}{2}$ as $s \to \infty$ by Proposition 6.2.5. The initial conditions for $\theta$ given in (6.5) are odd. The third order equation is even, so we may reflect the solution $\theta(s) = -\theta(-s)$ for $s < 0$ to obtain a solution for all $s$. Reflecting the solution gives the desired properties in the negative direction. □

### 6.2.5 Step 5.

Finally, we can integrate the tangent directions to construct the profile curve. Given a tangent direction $\theta(s)$ we are left to solve the standard ODE

$$\begin{aligned}\frac{d}{ds}\gamma(s) &= T(s) = (\cos\theta(s), \sin\theta(s))\,,\\ \gamma(0) &= (0,0)\end{aligned}$$

which follows from basic ODE theory. Given that $\theta$ solves (6.2), the result is a curve whose flow under (6.1) results in translation.

The solution $\gamma$ is clearly an immersion, the tangent vector is nonvanishing by construction. To see that it must be a proper immersion, note that because $\theta(s) \to \pi/2$ as $s \to \infty$ we can find a value $\sigma_0$ such that $\theta(s) \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$ for $s \geq \sigma_0$. Letting $\gamma(s) = (x(s), y(s))$ we have

$$y\,(s) - y(\sigma_0) = \int_{\sigma_0}^{s} \sin\theta(s)\,ds \geq \frac{\sqrt{2}}{2}(s - \sigma_0) \text{ for } s > \sigma_0.$$

Repeating this argument for negative values of $s$, we see the immersion must be proper. This proves Theorem 6.1.1.

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