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Empirical Analysis on U.S. Real Output

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Abstract

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The relative importance of permanent (trend) versus cyclical shocks to GDP has been a central issue in macroeconomics since the work of Nelson and Plosser (1982). Morley et al. (2003) find large trend shocks. In contrast, Perron and Wada (2009) argue for a onetime change in the mean growth rate at 1973:1 to be the only trend shock to the post-war U.S. real output.

Chapter 1 presents a joint work with Richard Startz. We re-estimate the Perron and Wada (2009) model conditional on a trend break having occurred at any one quarter. We then average the conditional estimates of the trend variance over the probability that the break occurred in a specified quarter. We do this both by an approximate Bayesian model average in which the conditional estimates are done by maximum likelihood and the date probabilities are found using the Schwarz (1978) approximation to the Bayesian marginal likelihood, and an exact Bayesian analysis which incorporates break date uncertainty into a trend-cycle decomposition of U.S. real GDP. The weight of the evidence supports the Perron and Wada (2009)'s finding of a fairly small trend variance, but the data does not provide very strong evidence against the alternative.

As confirmed in Chapter 1, little evidence has been found for the stochastic trends when researchers allow for adequate number of structural breaks in the growth rates. Therefore deterministic (linear) trends with structural breaks are often proposed to describe the trend component for U.S. real output. In Chapter 2, we examine the effect of unknown structural

breaks, including those in the mean growth rate and the covariance matrix, on the evidence of the stochastic trend for the U.S. postwar quarterly real GDP. We use Bayesian approach to compare the stochastic trend models with the deterministic (linear) trend models, allowing for up to four unknown structural breaks in the mean growth rate and/or up to one break in the shocks' covariance matrix. We find evidence for two structural breaks in mean: one around early 1970s, and the other after 2000. Data also identify early 1980s as the date for a volatility reduction. Conditional on the selected break dates, data favors the stochastic trend models over deterministic trend models. Exclusions of the stochastic trends and the effect of ongoing real shocks reported in the literature could be misleading if one ignores the structural breaks in the error variances and covariances.

In Chapter 3, we present evidence for the changing correlation between U.S. trend and cycle GDP in the post-WWII period. Researchers usually assume constant trend-cycle correlation when using unobserved component models to decompose U.S. real output. We introduce the time varying correlation into a UC model with a random walk mean growth rate and stochastic volatilities. We find that the estimated correlation is negative but could be close to zero before 1980s. And it has become more negative since the 1980s till the end of the sample (2012:4). By allowing the correlation to change over time, we are able to reconcile some of the debating results from earlier work. Through counterfactual studies, we show that the change in correlation contributes equally with the reduction in the cycle volatility to the great moderation. As a by-product, we find evidence for a stochastic trend and ongoing permanent shocks. We also find some signs of the grow rate slowdown around 1970 and further reduction around 2005.

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Chapter 1

IS IT ONE BREAK OR ONGOING PERMANENT SHOCKS THAT EXPLAINS U.S. REAL GDP? A BAYESIAN ANALYSIS USING AN UNOBSERVED COMPONENT MODEL

The relative importance of a nonstationary component in real output has been a key issue in our understanding of the business cycle at least since Nelson and Plosser (1982). Using an unobserved component model, Morley et al. (2003) (hereafter MNZ) find that the trend component of U.S. real GDP has a unit root and accounts for most of the fluctuations in output. However, Perron and Wada (2009) (hereafter PW) find that once a structural break in the mean growth rate of real GDP—exogenously set as occurring in 1973:1—is incorporated in MNZ’s specification, the nonstationary component essentially disappears. One can think of the dispute as being between a one-time, very large permanent change in the growth rate versus ongoing permanent shocks to the level of real output.

In this paper we take the MNZ/PW model and allow a break in the GDP process to occur at any date, rather than in a pre-specified quarter¹. It may be useful to think of the analysis that follows as taking place in several steps. First, we estimate the probability that a trend break occurred on a specified date for each date in the sample. We then estimate the probability distribution for the model parameters, notably for the trend variance, conditional on a break having occurred on a given date. Finally, we integrate the conditional probability distributions with respect to the probability of the break date to obtain an unconditional distribution.

For break dates close to PW’s 1973:1 choice, the conditional distributions show a very small trend variance. For break dates far from 1973:1, the conditional distributions show a large trend variance—one consistent with the MNZ findings. We do find that the bulk of

¹PW also estimate a Clark (1987) type model to support their arguments derived from the fixed break date model. This alternative model uses a random walk process to approximate an endogenized uncertain break date. However, it explicitly assumes zero correlation among all shocks and may be subject to the over-identification critique pointed out by MNZ and Oh and Zivot (2006).

the probability distribution for the date of the break is close to the date chosen by PW. As a result, when we integrate the conditional distributions across break dates the mode of the unconditional distribution is close to the PW finding of a very small trend variance. However, the probability weight on dates far from PW's choice is not negligible. This means that the unconditional distribution is bimodal with the lower mode being fairly close to the MNZ finding. Or to say it in a different way, the evidence weighs in the direction of a small trend variance, but the evidence is not strong enough to be conclusive.

Estimation requires two separable steps: estimation of the conditional distributions and estimation of the probability that the break occurred on a particular date. In section 1.2 of the paper we estimate the conditional distributions by maximum likelihood and the break-date probabilities using the Schwarz (1978) approximation to the Bayesian marginal likelihood. The latter can be thought of as using the SIC (also referred to as BIC) for model weighting instead of as a criterion for model choice. We call this "approximate Bayesian model averaging". In this section, the conditional distributions are derived without requiring specification of a prior.

In section 1.3, we execute a complete Bayesian model averaging, which does require priors for the conditional distributions. (In both sections the prior for the break date is that all dates are ex ante equally likely.) The cost of the complete Bayesian approach is that the results are dependent on priors for the conditional distributions, which is not true for the approximate Bayesian approach. (The complete Bayesian approach is also more computationally expensive.) The advantage of the complete approach is that it eliminates the approximation error in using the Schwarz criterion. As it turns out, the results are fairly similar for the two approaches and the results are not sensitive to the choice of prior.

Our primary interest is in the unconditional distribution for trend variance. However, one might wish to compare directly the PW model and the MNZ model with our unknown break date model. Our model and the PW model both assume that a break occurred at some point. Perhaps the data prefer the no-break MNZ version. Or, conditional on there being a break, we can ask whether the data clearly identify 1973:1 as the correct choice of break. PW conducted extensive robustness checks and concluded that the evidence strongly favors a break and that the break occurred in 1973:1 or at least in a nearby quarter. However, model

comparisons can be problematic when one parameter is on the edge of the parameters space (the trend variance equaling zero). Non-standard tests are required to compare different break dates².

We avoid the complication of classical tests by implementing Bayesian model comparisons. It is straight forward to compare models in a Bayesian framework by simply comparing the marginal likelihoods of different models. With our approximate Bayesian model averaging, we find modest evidence that a break occurred, although the PW's choice of the break date is positively supported if there is assumed to be a break. In our exact Bayesian estimation, we find that the evidence supports a break and further that it supports PW's choice of when the break occurred. However, the evidence is not decisive. In other words, part of the reason for the dispute in the literature over the size of the variance of the GDP trend component is that the data does not speak clearly enough to settle the issue.

The remainder of our paper is organized as follows. Section 1.1 specifies our model with an uncertain break date in the mean growth rate. Section 1.2 describes the approximate Bayesian model averaging approach and shows results from it. Section 1.3 presents our exact Bayesian approach and the corresponding results. Section 1.4 concludes our paper.

1.1 The benchmark model: an unobserved component model with an uncertain break date

Both MNZ and PW adopt an unobserved component (UC) model with a random walk trend component, an AR(2) cycle component and correlation between the trend and cycle shocks.

The unobserved component, trend-cycle decomposition model is:

$$y_t = \tau_t + c_t \quad (1.1)$$

$$\tau_t = \mu_t + \tau_{t-1} + \eta_t \quad (1.2)$$

$$\mu_t = \mu + \mathbf{1}(t > Tb)d \quad (1.3)$$

$$c_t = \phi_1 c_{t-1} + \phi_2 c_{t-2} + \epsilon_t \quad (1.4)$$

In the above model, y_t is the logarithm of real GDP, which is the sum of the trend component τ_t and the cyclical component c_t . τ_t follows a random walk with drift μ_t . μ_t is

²See, for instance, Morley and Eo (2013).

the mean growth rate of the real output, which may or may not be constant. As PW argue for the possibility of a structural break in the mean growth rate, we also allow for such a structural change here. As shown by (1.3), there is a permanent change in the mean growth rate μ_t one period after the break date denoted by Tb , with the size of change denoted by d . $\mathbf{1}(t > Tb)$ is an indicator function that is zero until the break date, and takes the value 1 afterwards. In other words, the mean growth rate equals μ in the earlier sample periods, and $\mu + d$ after the break date. The cyclical component c_t is assumed to be a stationary AR(2) process. η_t and ϵ_t are the shocks to the trend and cycle respectively. We allow for contemporaneous correlation (ρ) between trend and cycle shocks in the model as follows³:

$$\begin{bmatrix} \eta_t \\ \epsilon_t \end{bmatrix} \sim i.i.d.N \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_\eta^2 & \rho\sigma_\eta\sigma_\epsilon \\ \rho\sigma_\eta\sigma_\epsilon & \sigma_\epsilon^2 \end{bmatrix} \right) \quad (1.5)$$

The model given in (1.1)-(1.5) nests both the model of MNZ, with $d = 0$ and $\sigma_\eta \gg 0$, and the PW specification, with $d \neq 0$ and $\sigma_\eta \approx 0$. PW find that once we allow for a break in the mean growth rate of the trend component at 1973:1, the stochastic variation for trend GDP becomes insignificant except for periods around the break date. In contrast to the results in MNZ, the standard deviation of the trend shock is estimated to be close to zero. Specifically, the standard deviation of the trend shock is estimated to be 1.2368 in MNZ, while the estimate of the same parameter in PW is 0.104 after allowing for the one-time trend break. PW argue that their estimation captures a growth rate slowdown around 1973:1 and a deterministic broken-trend model is a better description for the quarterly U.S. real GDP.

Opinions from the literature vary considerably on the existence and timing of this structural break in the mean growth rate. For instance, Ben-David and Papell (1998) conduct a series of classical hypothesis tests and reject the significance of trend breaks in U.S. real GDP between 1950 and 1990. Chen and Zivot (2010) conduct Bayesian estimation with 130 years of annual data and find that the only possible structural break after 1947 takes place between 1947 and 1952. Perron and Wada (2009) obtain evidence for the structural trend break occurring around 1973 using the unobserved component model presented above.

³In this UC representation with an AR(2) cycle, the correlation of η_t and ϵ_t is identified if neither σ_η nor σ_ϵ equals zero.

In the succeeding sections, we will let the break date be uncertain and estimated by the data. For the purpose of comparison, we use the same set of data as MNZ and PW to generate results reported in this paper. The dataset includes quarterly real GDP data from 1947:1 to 1998:2⁴. We take the model with an uncertain break date as our benchmark. The fixed break date case (PW) and the no break date case (MNZ) will be considered and estimated separately.

1.2 An approximate Bayesian model averaging

We propose an approximate Bayesian model averaging (BMA) approach to construct the approximate Bayesian posterior distribution for parameters of interest. Schwarz (1978) shows that the approximation of the Bayesian log marginal likelihood consists of the maximized log-likelihood and a penalty term for model complexity, with the approximation error bounded. Our approach takes advantage of this approximation to construct the approximate posterior inferences in our benchmark model.

1.2.1 Methodology

An approximate Bayesian model averaging

Hereafter, we let $f(\cdot)$ denote the probability for discrete variables, and the probability density for continuous variables without distinction. The method we propose takes the following steps:

1. Let T denote the total number of observations we have. Conduct exact maximum likelihood estimation to estimate the model we set up in section 1.1 with every possible break date. We then obtain a series of maximum likelihood estimators and the maximized log likelihoods $\{\hat{\ell}_t\}$ for the MLEs given $Tb = t \in [1, T-1]$.
2. Approximate the Bayesian marginal likelihood given a break date at time t ($S_t =$

⁴We also conduct our analysis using an updated dataset running through 2008:2. No substantial differences are found. Details are reported in Appendix A.1

$f(Y|Tb = t)$ according to Schwarz (1978)'s approximation:

$$\log S_t \approx \hat{\ell}_t - \frac{k}{2} \log(T) \quad (1.6)$$

where $Y = \{y_1, y_2, \dots, y_T\}$, and k represents the total number of the free parameters in the model⁵.

3. According to Bayes rule,

$$f(Tb = t|Y) \propto f(Tb = t)S_t$$

If we assume a flat prior $f(Tb = t) = 1/(T - 1)$ for all break dates (equivalently all possible models in the model space), we can simplify the above posterior to the following:

$$f(Tb = t|Y) \propto S_t \quad (1.7)$$

Thus, we can approximate the posterior of different break dates using

$$f(Tb = t|Y) = \frac{f(Tb = t)S_t}{\sum_{t=1}^{T-1} f(Tb = t)S_t} = \frac{S_t}{\sum_{t=1}^{T-1} S_t} \quad (1.8)$$

where S_t is approximated by (1.6) given the MLE in each model.

4. We are particularly interested in the trend shock volatility σ_η , which determines the existence and importance of the random walk component for trend GDP. We illustrate below how to construct its approximate posterior distribution. Similar approaches can be applied to other parameters of interest.

According to the Bayes theorem and the law of total probability $f(\sigma_\eta|Y) = \sum_{t=1}^{T-1} f(Tb = t|Y)f(\sigma_\eta|Tb = t, Y)$. The posterior density of σ_η is the mixture density of its posterior under each model. While $f(Tb = t|Y)$ can be approximated using (1.8), $f(\sigma_\eta|Tb = t, Y)$ can be approximated by the square root of a non-central χ^2 distribution as explained below.

⁵ $k = 7$ for the model with a given break date t . $k = 6$ for the no break model, as d drops out.

In order to ensure non-negative variances and $-1 \leq \rho \leq 1$, we need to put constraints on the parameter space. Instead of a direct restriction on the MLE procedure, we adopt the following reparameterization:

$$\begin{bmatrix} \sigma_\eta^2 & \rho\sigma_\eta\sigma_\epsilon \\ \rho\sigma_\eta\sigma_\epsilon & \sigma_\epsilon^2 \end{bmatrix} = \begin{bmatrix} Q_1 & 0 \\ Q_2 & Q_3 \end{bmatrix} \begin{bmatrix} Q_1 & Q_2 \\ 0 & Q_3 \end{bmatrix}$$

Unconstrained MLE can be conducted with respect to $[Q_1, Q_2, Q_3]$ and transformed to obtain valid estimates for $[\sigma_\eta, \sigma_\epsilon, \rho]$ that guarantee a positive semidefinite covariance matrix. Within each possible model given $Tb = t \in [1, T-1]$, we obtain Q_1 's maximum likelihood estimator \hat{Q}_1^t and its asymptotic variance $\hat{\Sigma}_{Q_1}^t$. According to Heyde and Johnstone (1979) and Chen (1985), the posterior distribution of Q_1 can be asymptotically approximated by a normal distribution $N(\hat{Q}_1^t, \hat{\Sigma}_{Q_1}^t)$. Therefore, the conditional posterior of $\sigma_\eta = \sqrt{Q_1^2}$ asymptotically follows $\sqrt{\chi_1^2(\frac{\hat{Q}_1^{t2}}{\hat{\Sigma}_{Q_1}^t})}$.

We thus propose an approximation to the posterior sampling of σ_η by first sampling Tb from $f(Tb = t|Y)$, and then (conditional on the sampled Tb) sampling Q_1 from $N(\hat{Q}_1^{Tb}, \hat{\Sigma}_{Q_1}^{Tb})$. Posterior samples of σ_η can be obtained by $\sigma_\eta = |Q_1|$.

5. Given the equal prior probability for each break date, we compute the marginal likelihood for the benchmark model with an uncertain structural break according to the following:

$$\begin{aligned} f(Y|M) &= \sum_{t=1}^{T-1} f(Y|Tb = t)f(Tb) \\ &= \frac{1}{T-1} \sum_{t=1}^{T-1} f(Y|Tb = t) \end{aligned} \tag{1.9}$$

Model comparison

We compare two models based on their posterior odds. To be specific, posterior odds of Model_{*i*} versus Model_{*j*} is defined as

$$\frac{f(M_i|Y)}{f(M_j|Y)} = \frac{f(M_i)}{f(M_j)} \times \frac{f(Y|M_i)}{f(Y|M_j)} \quad (1.10)$$

If we assume equal prior probability for two models, the posterior odds can be simplified as the ratio of marginal likelihoods, also known as the Bayes factor $B_{ij} = f(Y|M_i)/f(Y|M_j)$. We approximate $f(Y|M_i)$ and $f(Y|M_j)$ using (1.6) and (1.9). For convenience, we will construct $2\log(B_{ij})$, which is a monotonic transformation of the Bayes factor and commonly referred as the difference between two BIC statistics. If $2\log(B_{ij})$ is positive (negative), we prefer M_i (M_j).

Kass and Raftery (1995) and Raftery (1995) suggest using the following criteria for significance of model comparison.

$2\log(B_{ij})$	Evidence for M_i , against M_j
0 to 2	Not worth more than a bare mention
2 to 6	Positive
6 to 10	Strong
> 10	Very strong

Table 1.1: Criteria for model comparisons based on twice the log Bayes factor

As a metric for what follows, a BIC difference of 2 corresponds to model odds of 2.7 to 1 and a BIC difference of 6 corresponds to odds of 20 to 1. While there is not an exact frequentist comparison, note the usual 5 percent significance level implies that if the null is true the odds are 19 to 1 against rejecting while a 1 percent significance level implies 99 to 1 odds.

1.2.2 Results from the approximate BMA

Break date posterior

The upper panel of Figure 1.1 presents the posterior distribution for all possible break dates in the mean growth rate from 1947:1 to 1998:1, which is in fact a monotonic transformation

of the log likelihood given each break date. The most likely break date we find is 1973:1, which has an approximate posterior probability of 4.51% among all considered. This finding coincides with PW's choice of break date. The periods that are relatively more likely to be the break date mostly locate between 1965 and 1980, which is in line with PW's findings that the mean growth rate starts to fall in the late 1960s and becomes stable after the late 1970s with the main changes occurring in 1973-1974.

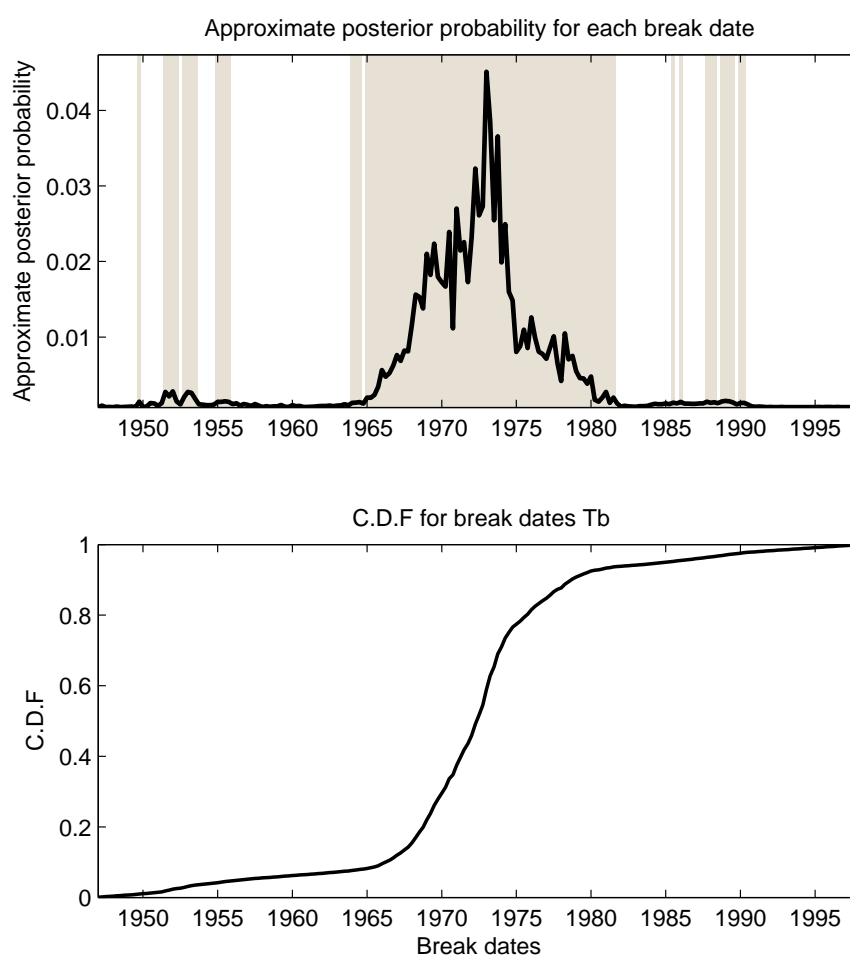


Figure 1.1: Approximate posterior distribution of break dates within [1947:1, 1998:1].

Note: Shaded areas in the upper panel show the 90% highest posterior density region.

Approximate posterior for the trend shock volatility

As we are interested in understanding how an uncertain break date affects our inferences on the trend and cycle of U.S. GDP, we place our focus on the standard deviation σ_η of the trend shock η_t . If the estimated σ_η is significantly far from zero, we find evidence supporting the stochastic trend in GDP. If σ_η is estimated to be small and close to zero, GDP may be better described as a broken linear trend process as argued by PW.

We generate one million posterior samples of the trend shock volatility σ_η according to the procedure described in section 1.2.1 and construct the approximate posterior density as presented in Figure 1.2. Comparing to PW's results, it is unsurprising that the uncertainty of σ_η increases when we incorporate the break date uncertainty. However, as the uncertainty of σ_η increases, a second mode occurs in its posterior sample density. As shown in Figure 1.2, the bimodal posterior distribution has the first mode at about 0.08 (close to PW's estimates) and the second mode at about 1.2 (close to MNZ's estimates)⁶.

As shown in Figure 1.3, the ML estimate of the trend standard deviation is indeed highly sensitive to the specified break date⁷. The point estimate of σ_η stays quite stable at about 1.2 except for the cases with break dates from mid 1960s to early 1980s. Notice that data assigns about 15%-20% posterior probability mass to the break dates at the two ends of our sample periods, leading to the occurrence of the second mode in the posterior density of σ_η . Although the posterior of σ_η suggests a small stochastic trend component, the existence of the second mode makes such a conclusion less than certain.

Model comparison

While we are primarily interested in the results of model averaging, we can also conduct model comparisons. Three models are considered in this section:

- I. Benchmark model: uncertain break date, i.e. $Tb \in [1947:1, 1998:1]$.

⁶The overall posterior median is at 0.2829 and the posterior mean is at 0.4211. The 90% highest posterior density intervals are $(0.03, 0.68) \cup (1.08, 1.33)$.

⁷The apparent knife-edge appearance of the point estimates in Figure 3 reflects the fact that the likelihood function is multi-modal, and which mode is the global maximum switches abruptly with the break date

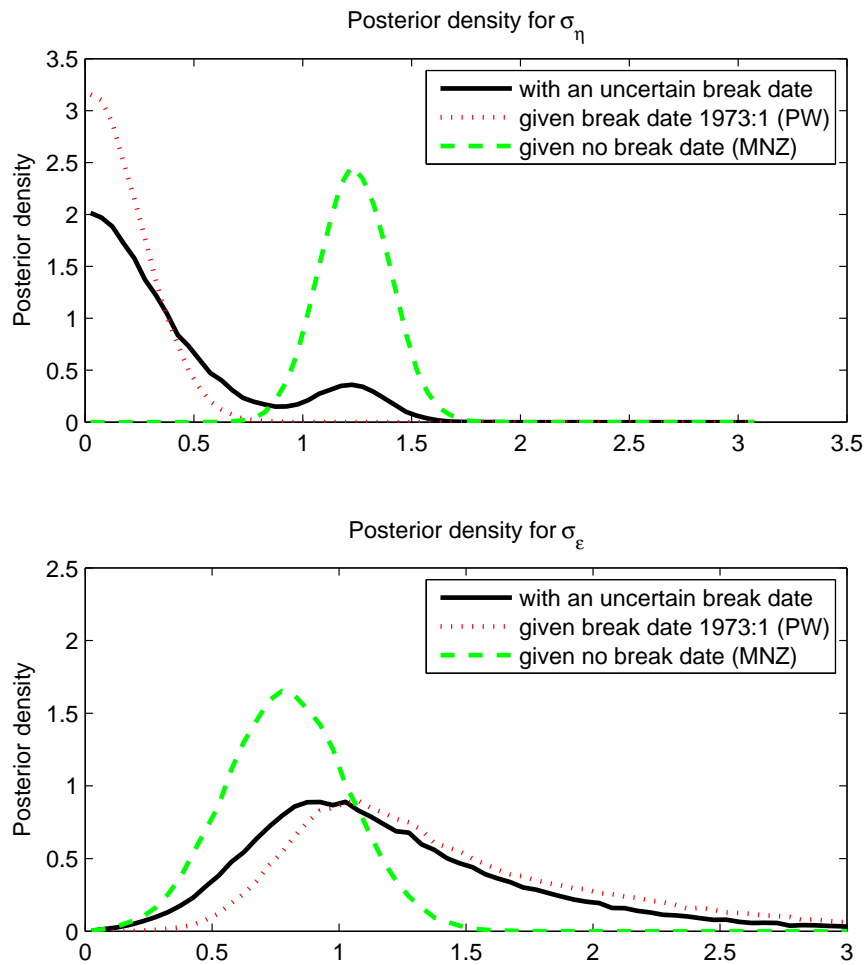


Figure 1.2: Approximate posterior density for σ_η (upper panel) and σ_ϵ (lower panel).

II. PW model: fixed break date at 1973:1.

III. MNZ model: no structural break.

Approximation to the log marginal likelihood of these models are computed according to the methodology described in Section 1.2.1. Results are shown in Table 1.2.

The model that best fits the data is the PW model. Our benchmark model turns out to have the lowest approximate log marginal likelihood. The MNZ model falls between PW's

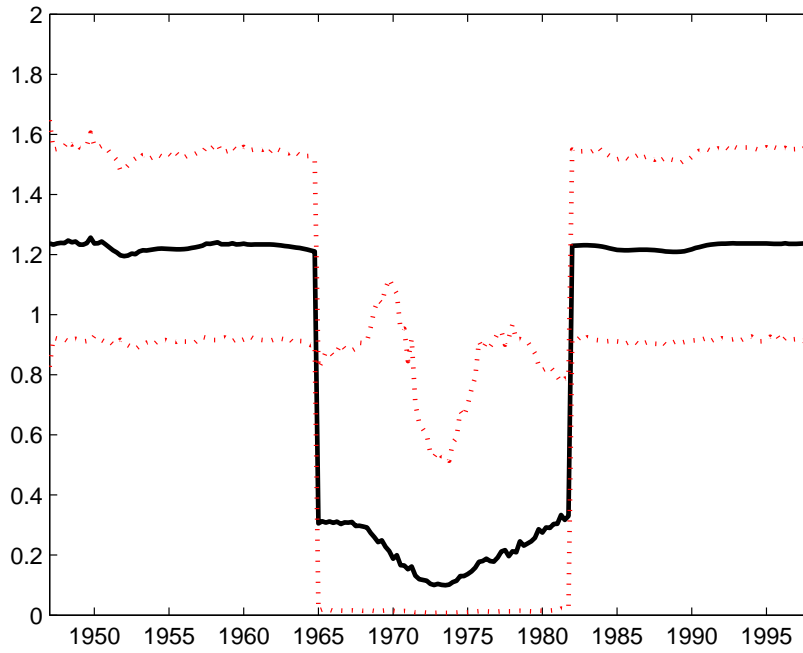


Figure 1.3: Point estimate of σ_η for $Tb \in [1947 : 1, 1998 : 1]$.

Note: 1. Dotted lines show the 95% confidence intervals. 2. For comparisons, the MLE estimate of σ_η is 0.1042 assuming the PW break date and 1.2368 assuming no break date.

Model	Approx. log marginal likelihood	$2\log(B_{ij})$	
		Row over column PW	MNZ
uncertain break date (Ours)	-301.3601	-4.4492	-1.4810
$Tb = 1973 : 1$ (PW)	-299.1355		2.9682
no break (MNZ)	-300.6196		

Table 1.2: Approximate log marginal likelihoods for different models

model and ours.

Comparing PW’s model with a fixed break date at 1973:1 to MNZ’s no break date model, “positive” evidence is found to support the PW model with twice the log Bayes factor equal to 2.9682. One may conclude that a deterministic trend model with a trend break at 1973:1 is the better description for U.S. GDP, although the evidence is not decisive.

However, once we take into account the break date uncertainty, evidence for the occurrence of a trend break is reversed. The data favors MNZ’s no break model over ours with twice the log Bayes factor at 1.4810 though the difference in the log marginal likelihood is “minor” according to Table 1.1.

Based on the approximate marginal likelihood, we find positive evidence for PW’s choice of break date conditional on there being a break. But regarding to the existence of the trend break, evidence is conflicting and dependent on whether the break date is assumed to be uncertain. Furthermore, Weakliem (1999) points out and shows examples of the limitation of BIC based studies. The Schwarz (1978)’s approximation is asymptotically equivalent to the log marginal likelihood obtained under specific diffuse normal priors⁸ which may not be consistent with the actual priors researchers have. To overcome the potential limitation and reduce the approximation error by BIC, we provide an exact Bayesian analysis in the next section.

1.3 Exact Bayesian estimation

In this section, we estimate the benchmark model defined by (1.1)-(1.5) using an exact Bayesian approach. We allow the break date $Tb \in [1, T - 1]$ (i.e. [1947:1,1998:1]) to be estimated together with other parameters. Our Bayesian estimation is conducted using the MCMC Gibbs sampling approach.

Comparing to our approximate BMA approach proposed in the previous section, the exact Bayesian approach requires slightly more computation. It, however, provides several

⁸According to Raftery (1995) and Chow (1981), ignoring the approximation error automatically implies a diffuse parameter prior around the maximum likelihood point estimates. The implied prior is $N(\hat{\theta}, (-\frac{\hat{H}}{T})^{-1})$, where $\hat{\theta}$ is the parameters’ maximum likelihood estimates, \hat{H} is the Hessian matrix evaluated at $\hat{\theta}$, and T is the sample size. In our case, this prior turns out to be highly diffuse over the reasonable parameter space.

advantages. First, priors are directly specified and so are not subject to critiques of the Schwarz (1978) approximation. Secondly, Bayesian approaches provide direct finite sample inference. Finally, the trend-cycle decomposition can be obtained as one of the direct outputs of the Bayesian Gibbs sampling.

1.3.1 Bayesian estimation

The model (1.1)-(1.5) can be rewritten into the state space form:

$$y_t = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} x_t \quad (1.11)$$

$$x_t = \begin{bmatrix} \mu \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} d \\ 0 \\ 0 \end{bmatrix} 1(t > Tb) + \begin{bmatrix} 1 & 0 & 0 \\ 0 & \phi_1 & \phi_2 \\ 0 & 1 & 0 \end{bmatrix} x_{t-1} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \eta_t \\ \epsilon_t \end{bmatrix} \quad (1.12)$$

where $x_t = [\tau_t, c_t, c_{t-1}]'$.

In order to ensure that the estimated covariance matrix is positive semidefinite, we decompose the covariance matrix in the following way:

$$\begin{bmatrix} \sigma_\eta^2 & \rho\sigma_\eta\sigma_\epsilon \\ \rho\sigma_\eta\sigma_\epsilon & \sigma_\epsilon^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \quad (1.13)$$

and directly estimate $\{\sigma_1, \sigma_2, b\}$ instead of the covariance matrix parameters $\{\sigma_\eta, \sigma_\epsilon, \rho\}$. The posterior samples for the covariance parameters are obtained through transformation.

We specify independent proper priors for all parameters estimated. Inverse gamma priors $IG(100, 0.5)$ ⁹ are assumed for σ_1^2 and σ_2^2 . These priors are diffuse and do not have finite moments. Therefore, a heavy weight will be put on sample information. We assume somewhat informative normal priors for $\phi_1 \sim N(1, 1)$, $\phi_2 \sim N(0, 1)$ and $b \sim N(0, 1)$. As noted in de Pooter et al. (2008), μ and d are nearly unidentified when the samples of ϕ_1 and ϕ_2 get very close to the non-stationary region. In this case, arbitrary real values for μ

⁹We follow Koop (2003) for the definition of inverse gamma (IG) distribution. If $x > 0$ follows inverse gamma distribution $IG(s^{-2}, \nu)$, the probability density function of x is defined as:

$$f(x; s^{-2}, \nu) = \left(\frac{2s^{-2}}{\nu}\right)^{-\frac{\nu}{2}} \frac{1}{\Gamma(\frac{\nu}{2})} x^{-\frac{\nu}{2}-1} \exp\left(-\frac{\nu}{2s^{-2}x}\right)$$

where $\Gamma(\cdot)$ is the gamma function.

and d can be drawn and cause the Gibbs sampler to have difficulty in moving away from the nonstationary region. To avoid such situations, we impose truncations for $\mu \in [0, 2]$ and $d \in [-0.5, 0.5]$. We assume uniform priors for μ and d over the truncated areas and develop truncated normal posteriors accordingly. While the above priors are broadly consistent with estimates from the literature, we emphasize that our estimation results are robust to more diffuse priors¹⁰. Lastly, we assume a flat proper prior for Tb such that all dates from 1947:1 to 1998:1 have equal probability to be the break date in the mean growth rate. Therefore, the joint prior density is the product of all the above marginal prior densities. We present prior moments and quantiles in Table 1.3.

We use the Gibbs sampling approach to draw posterior samples for parameters. We run the Gibbs sampler for 300,000 times and save every 10th draw to reduce the autocorrelation within samples. We thus obtain 30,000 draws from the Gibbs procedures and discard the first 10,000 to avoid the effect of the initial values. To guarantee the convergence of the Gibbs sampler, we divide our samples, excluding the burn-in draws, into three sets—a first set of 6000 draws, a middle set of 7000 draws and a last set of 7000 draws. We find that the posterior distribution and estimates based on the three subsets don't vary much, suggesting the convergence of our MCMC samples.

Define $\theta = [\mu, \phi_1, \phi_2, \sigma_\eta, \sigma_\epsilon, \rho, d]$. Let $(\cdot)^{(k)}$ denote the the k^{th} posterior draw of the latent variable x_t or the parameters. Y denotes all the observed quarterly log real GDP $\{y_1, y_2, \dots, y_T\}$. The k^{th} step in our Gibbs sampler involves the following blocks:

1. Draw $\{x_t^{(k)} : t = 1, \dots, T\} \sim f(x_1, \dots, x_T | Y, \theta^{(k-1)}, Tb^{(k-1)})$ using the simulation smoother developed by Durbin and Koopman (2002).
2. Draw $[\phi_1^{(k)}, \phi_2^{(k)}] \sim f(\phi_1, \phi_2 | Y, x_t^{(k)}, \sigma_\epsilon^{(k-1)})$ given that the second row in (1.12) has the following regression form:

$$c_t = \begin{bmatrix} c_{t-1} & c_{t-2} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} + \epsilon_t \quad (1.14)$$

¹⁰As robustness check, we set the variance for the normal priors to be 10 and priors for σ_1^2 and σ_2^2 to be $IG(100, 0.1)$, and get essentially the same results.

The posterior samples for $[\phi_1^{(k)}, \phi_2^{(k)}]$ must guarantee the stationarity of the process $\{c_t : t = 1, \dots, T\}$. Therefore, we discard nonstationary draws and regenerate new ones until they meet the stationary requirement.

3. Draw $[\mu^{(k)}, d^{(k)}] \sim f(\mu, d|Y, x_t^{(k)}, \sigma_\eta^{(k-1)})$ given the regression in the first row of (1.12):

$$\tau_t - \tau_{t-1} = \begin{bmatrix} 1 & 1(t > Tb) \end{bmatrix} \begin{bmatrix} \mu \\ d \end{bmatrix} + \eta_t \quad (1.15)$$

4. Draw $[\sigma_1^{(k)}, \sigma_2^{(k)}] \sim f(\sigma_1, \sigma_2|Y, x_t^{(k)}, \mu^{(k)}, d^{(k)}, \phi_1^{(k)}, \phi_2^{(k)}, b^{(k-1)})$. Residual terms $[\hat{\eta}_t, \hat{\epsilon}_t]'$ can be obtained from the simulation smoother in the first step. Define $\eta_t^* = \eta_t \sim N(0, \sigma_1^2)$ and $\epsilon_t^* = -b\eta_t + \epsilon_t \sim N(0, \sigma_2^2)$, and we have the following:

$$B^{-1} \begin{bmatrix} \hat{\eta}_t \\ \hat{\epsilon}_t \end{bmatrix} = \begin{bmatrix} \hat{\eta}_t \\ -b\hat{\eta}_t + \hat{\epsilon}_t \end{bmatrix} = \begin{bmatrix} \hat{\eta}_t^* \\ \hat{\epsilon}_t^* \end{bmatrix} \sim N \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} \right) \quad (1.16)$$

where

$$B = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$$

We can then draw $[\sigma_1^2, \sigma_2^2]$ separately from two inverse Gamma distributions.

5. Draw $b^{(k)} \sim f(b|Y, x_t^{(k)}, \mu^{(k)}, d^{(k)}, \phi_1^{(k)}, \phi_2^{(k)}, \sigma_2^{(k)})$. Given the second row in (1.16), we have a standard regression to sample b :

$$\hat{\epsilon}_t = \hat{\eta}_t b + \epsilon_t^* \quad (1.17)$$

where $\epsilon_t^* \sim N(0, \sigma_2^2)$. We can construct the posterior samples for $[\sigma_\eta, \sigma_\epsilon, \rho]$ according to (1.13).

6. Draw $Tb^{(k)} \sim f(Tb|Y, \theta^{(k)})$. According to Wang and Zivot (2000), given the flat proper prior assumed for Tb ,

$$\begin{aligned} f(Tb|Y, \theta) &= \frac{f(Y|Tb, \theta)f(Tb|\theta)}{f(Y|\theta)} \\ &\propto f(Y|Tb, \theta)f(Tb) \\ &\propto f(Y|Tb, \theta) \end{aligned} \quad (1.18)$$

More details on the Gibbs sampling are summarized in Appendix A.2.

Given that the PW and the MNZ models are both nested models for our benchmark, we use same priors for the unrestricted parameters in the nested models as we do for the benchmark model. Bayesian model comparisons can be conducted using the Bayes factor and criteria defined in Section 1.2.1. The Bayesian marginal likelihood of each model can be numerically computed using Chib (1995)'s method.

1.3.2 Bayesian estimation results

We present the Bayesian estimation results in Table 1.3. Posterior statistics after discarding the initial burn-in samples are shown. 90% highest posterior density intervals are also provided in Table 1.3.

Our Bayesian inferences are in line with our approximate results reported earlier. The most likely break dates center around 1965-1980 period as shown in Figure 1.4. The posterior mode is 1973:1 which has a posterior probability of 2.8%.

The estimated standard deviation for trend shock is slightly smaller than that for the cycle shocks, but contains more uncertainty. The size of the structural trend break d has a posterior mean of -0.2755, significantly non-zero.

Figure 1.5 reports the exact Bayesian posterior density for σ_η . The posterior density under the PW and the MNZ specifications are also reported for comparison. Our Bayesian results show an even more significant bimodal distribution and stronger evidence for the nonstationary component. The first mode is at about 0.45, which is a little further away from zero when compared to PW's estimate. The second mode is at about 1. There are about 65% of posterior samples for σ_η centering around the first mode and 35% around the second one. We believe that the finite sample results reveal the limited power of the current data to clearly identify whether or not the stochastic component is important for the U.S. real output.

Model comparisons are constructed using the estimated log marginal likelihoods. We again use twice the log Bayes factor and the criteria in Table 1.1 to compare different models. The same three models are considered and all estimated by full Bayesian approaches.

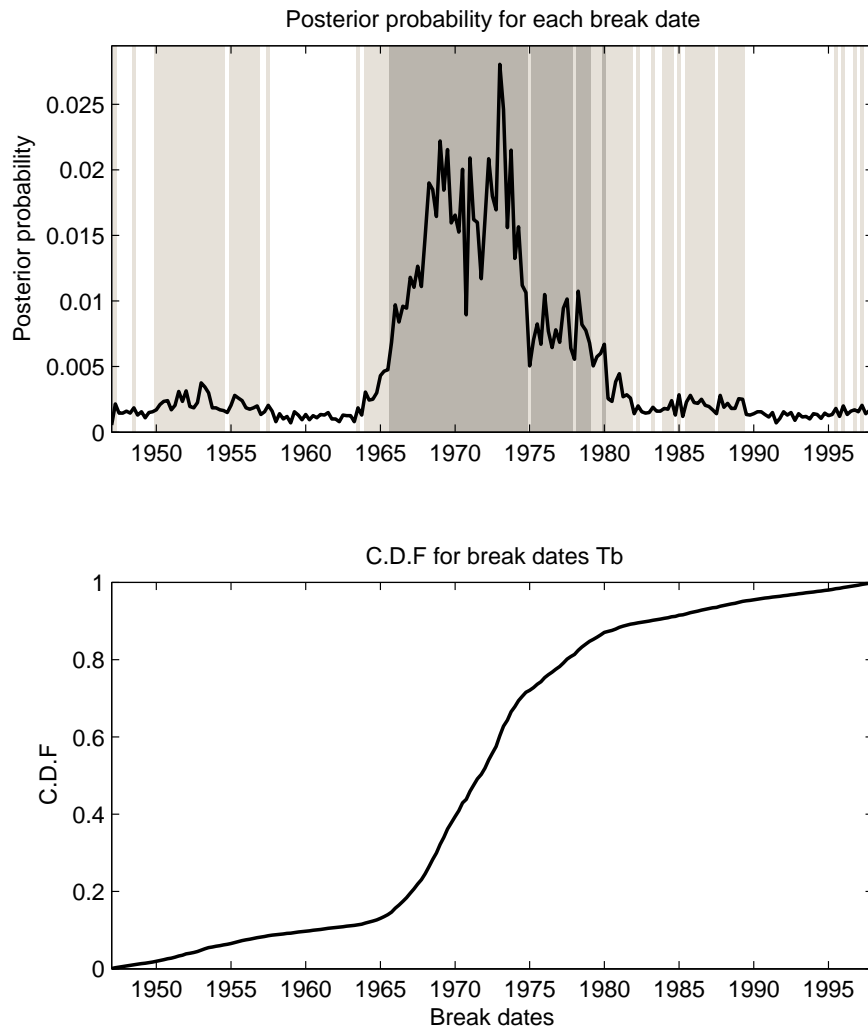


Figure 1.4: Posterior distribution of break dates.

Note: In the upper panel, all (including both dark and light) shaded areas represent the 90% HPD intervals, while the dark ones represent the 70% HPD intervals.

Different from our approximate results, we find consistent and stronger evidence for the existence of a structural break (as reported in Table 1.4). PW's model and ours are "positively" supported by data against MNZ's no break date model. However, the marginal likelihood of our model is only slightly lower than PW's model. Twice the log Bayes factor

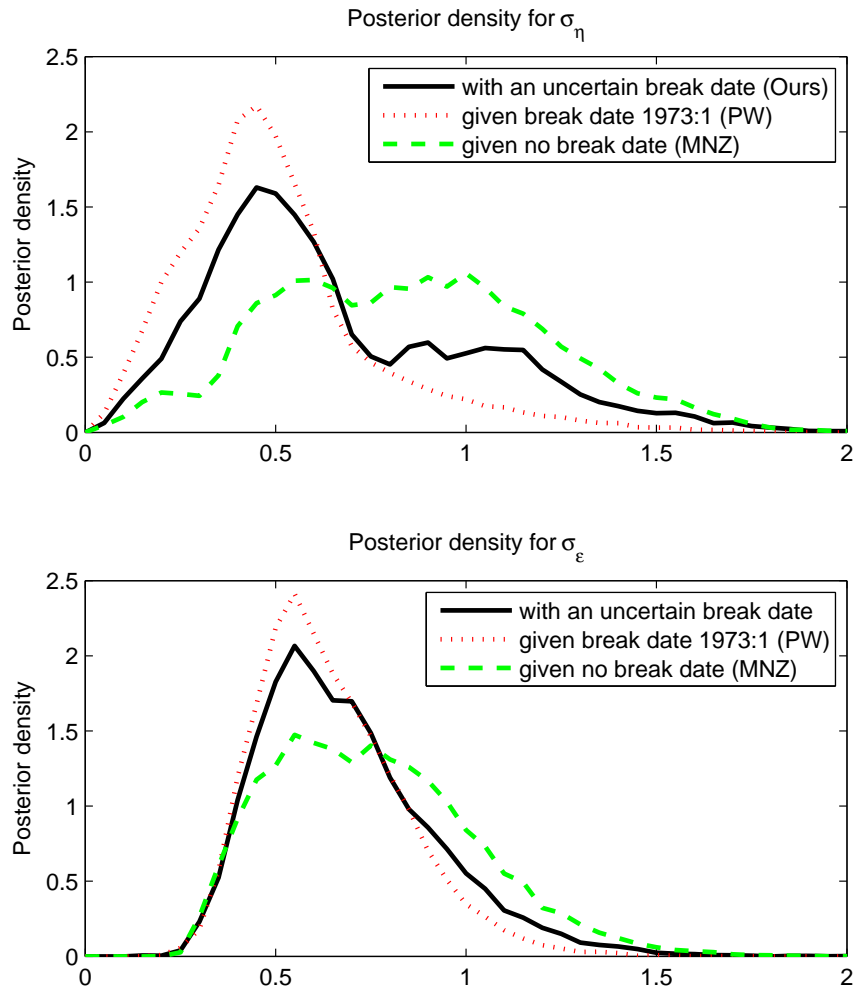


Figure 1.5: Posterior density of σ_η (upper panel) and σ_ϵ (lower panel).

is 1.7314 for the PW model vs. the benchmark model, favoring the PW's choice of break date but not decisively.

To more clearly observe how the uncertain σ_η affect the trend-cycle decomposition, we report our estimated trend and cycle in Figure 1.6. Based on the samples obtained from the Gibbs samplers, we can estimate the trend component of U.S. real GDP by the posterior sample mean with no reliance on the point estimate of parameters. Our trend estimate

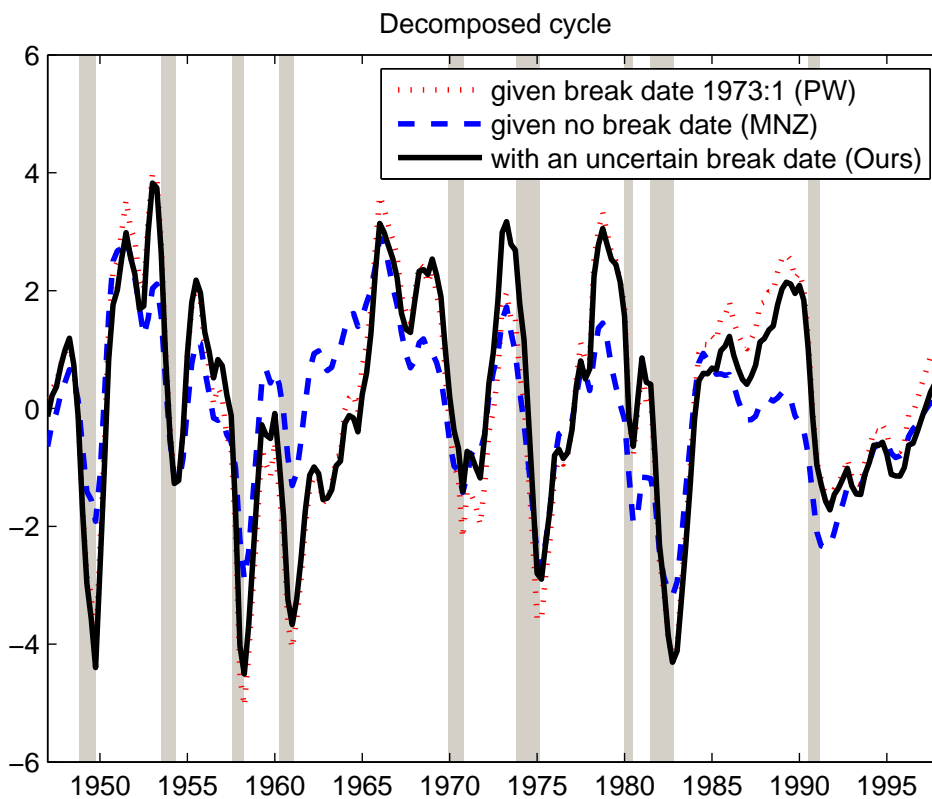


Figure 1.6: Posterior estimate of cycles for benchmark model.

Note: Shaded areas represent the NBER recession periods.

suggests a stochastic trend but much smoother than what MNZ suggest. The estimated cycles mostly lie between PW and MNZ's cycles.

1.4 Conclusion

We conduct both an approximate BMA and an exact Bayesian estimation to endogenize the break date uncertainty for the trend-cycle decomposition of U.S. real GDP. We find positive evidence for a structural break in the mean growth rate of the U.S. real output, most likely taking place around 1970s, based on our Bayesian estimation.

While our estimation results are mostly in line with those reported in PW, the data

does not definitively settle the question. When we allow the break date to vary rather than being fixed at 1973:1, we are more uncertain about the relative importance of the permanent trend shock η_t than either PW or MNZ are. We show, by both methods, that the posterior distribution of the trend shock volatility σ_η exhibits a bimodal distribution where there is a significant probability, although not larger than the alternative, for it to be larger than 1. The estimated trend component from our Bayesian approach shows some volatility.

In summary, the evidence favors the position of PW that once a single structural break is accounted for the variance of the trend component of GDP is relatively small. However, the evidence is much less than decisive when uncertainty about the break date is accounted for properly.

Benchmark model: with an uncertain break date					
Log marginal likelihood:-297.2713					
	Prior		Posterior		
	median	90% quantile	mean	median	90% HPD*
μ	1	(0.1, 1.9)	0.9442	0.9494	(0.77, 1.09)
ϕ_1	1	(-0.64, 2.64)	1.4395	1.4712	(1.24,1.69)
ϕ_2	0	(-1.64, 1.64)	-0.5610	-0.5750	(-0.77,-0.35)
σ_η	0.24	(0.05, 17.31)	0.6714	0.5711	(0.14,1.24)
σ_ϵ	0.61	(0.08, 65.65)	0.6885	0.6515	(0.36,1.05)
ρ	0	(-1, 1)	0.2377	0.4139	(-0.90,0.15) \cup (0.45,1)
d	0	(-0.48, 0.48)	-0.2513	-0.2755	(-0.49,-0.08)
	median	90% quantile	median	mode	90% quantile**
Tb	1972:3	(1949:3, 1995:4)	1971:4	1973:1	(1953:2, 1989:2)

Posterior***						
	PW model: fixed break date			MNZ model: no break date		
	Log marginal likelihood:-296.4056			Log marginal likelihood:-299.2063		
	mean	median	90% HPD	mean	median	90% HPD
μ	0.9621	0.9654	(0.89,1.02)	0.8238	0.8283	(0.71,0.92)
ϕ_1	1.4559	1.4688	(1.30,1.57)	1.4188	1.4702	(1.14,1.71)
ϕ_2	-0.5567	-0.5640	(-0.67,-0.39)	-0.4467	-0.5806	(-0.78,-0.34)
σ_η	0.5019	0.4602	(0.04,0.6)	0.8525	0.8458	(0.30,1.42)
σ_ϵ	0.6439	0.6147	(0.44,0.98)	0.7566	0.7306	(0.34,1.14)
ρ	0.5297	0.8129	(-0.16,0.98)	-0.1001	-0.3782	(-0.95,0.50)
d	-0.3062	-0.3146	(-0.39,-0.18)		—	—
Tb		1973:1 (fixed)			—	—

* HPD refers to highest posterior density interval.

** We report the 70% and 90% HPD for Tb in Figure 4.

*** Priors for the unrestricted parameters in PW and MNZ models are the same as those in the benchmark.

Table 1.3: Bayesian Inferences.

Model	Log marginal likelihood	$2\log(B_{ij})$	
		Row over column	
		PW	MNZ
uncertain break date(Ours)	-297.2713	-1.7314	3.8702
Tb=1973:1(PW)	-296.4056		5.6016
no break(MNZ)	-299.2064		

Table 1.4: Log marginal likelihoods and model comparisons.

Chapter 2

STOCHASTIC TRENDS AND STRUCTURAL BREAKS

Evidence on the stochastic trend component, or the implied unit roots, in the macroeconomic time series is usually found to be sensitive with respect to the assumption of structural breaks. For U.S. real output data, it is usually reported that one can reject the unit root hypothesis with adequate numbers of trend breaks incorporated¹. In other words, except for a number of large but infrequent permanent shocks, shocks affecting the real output are temporary whose effect will eventually vanish.

Less attention is given to comparing how the breaks in variances of shocks affect the evidence for stochastic trends. Using real GDP data, Murray and Nelson (2000) find that the Augmented Dickey-Fuller test rejects unit roots too often if researchers fail to allow for heteroskedasticity. Morley et al. (2012) reject the stationary hypothesis when allowing for pre-specified breaks in both mean and variances using a likelihood ratio test. However, Meligkotsidou et al. (2011) find that ignoring structural breaks occurred at unknown dates in the error variances may be responsible for not rejecting the unit root hypothesis with international data of interest rates, exchange rates and CPI.

This paper aims to examine the effect of unknown structural breaks, including those in mean growth rates and the covariance matrices of shocks, on the evidence of stochastic trend for the U.S. postwar quarterly real GDP. We use Bayesian approach to compare the stochastic trend models with the deterministic (linear) trend models within an unobserved component framework. We formally incorporate up to four unknown structural breaks in the mean growth rate and/or up to one break in the shocks' covariance matrix. The

¹Perron (1989) assumes a level shift at 1929 and a change in mean growth rate at 1973. He reverses most unit root rejections in Nelson and Plosser (1982), although his conclusions are partially reversed later by Zivot and Andrews (1992) who considered the breaks as unknown. Ben-David et al. (2003) extend Zivot and Andrews (1992)'s approach to incorporate a second unknown break of each kind and reject unit roots on more series than Zivot and Andrews (1992). More comprehensive reviews include, for instance, Maddala and Kim (1999) and Haldrup et al. (2012).

most appropriate number of breaks are selected by Bayesian marginal likelihood. And the corresponding break dates are estimated by data.

Specifically, we consider four settings for structural breaks: no break, breaks only in mean growth rates, breaks only in covariance matrices and the general models containing both breaks. In each setting, we estimate the stochastic trend models and their counterparts with deterministic trends by restricting the trend variances to be zero. The marginal likelihoods are computed for each model and used as the criteria for the model selections. A Bayesian “unit root testing” can therefore be conducted as comparing the best-fit stochastic trend model with the best-fit deterministic trend model using their marginal likelihoods.

As a preview of our findings, we identify two structural breaks in mean and one structural break in the covariance matrix for both stochastic and deterministic trend models. In addition, using the selected break dates, the estimated trend variances for the stochastic trend models are smaller after allowing for mean breaks. However, the best-fit one is still slightly favored against the best-fit deterministic alternative. Our results are close to the Morley et al. (2012) findings but the evidence against deterministic trends is weaker.

Furthermore, we conduct model comparisons within the same structural break setting. We find that the deterministic trend is favored only when there are breaks in mean but no break in variances and covariances. It confirms the usual finding that allowing for adequate amount of mean breaks provides evidence against stochastic trends. However, such rejections of unit roots may be misleading if one ignores the break in the covariance matrix.

The remainder of the paper is organized as follows. Section 2.1 sets up the model we use. Section 2.2 describes the Bayesian methodology. We report the empirical results in Section 2.3 and conclude in Section 2.4.

2.1 An unobserved component model

2.1.1 model

The unobserved component (hereafter UC) models are used to decompose the real output into the trend and cycle components to study the relative importance of the permanent

shocks. The idea is that permanent shocks can affect the long-run trend of GDP, while the effect of transitory shocks will eventually vanish thus only affect the “cycle” of the real output. In this section, we introduce a basic UC model which nests both stochastic trend and deterministic trend models.

Following Morley et al. (2003) and Perron and Wada (2009), the unobserved component, trend-cycle decomposition model is:

$$y_t = \tau_t + c_t \quad (2.1)$$

$$\tau_t = \mu_t + \tau_{t-1} + \eta_t \quad (2.2)$$

$$c_t = \phi_1 c_{t-1} + \phi_2 c_{t-2} + \epsilon_t \quad (2.3)$$

In the above model, y_t is the logarithm of real GDP, which is the sum of the trend component τ_t and the cyclical component c_t . τ_t follows a random walk with drift μ_t . μ_t is the mean growth rate of the real output, which may or may not be constant. The cyclical component c_t is assumed to be a stationary AR(2) process. η_t and ϵ_t are the shocks to the trend and cycle respectively. We allow for contemporaneous correlation (ρ_t) between trend and cycle shocks in the model as the following²:

$$\begin{bmatrix} \eta_t \\ \epsilon_t \end{bmatrix} \sim i.i.d.N \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \Sigma_t \right) \quad (2.4)$$

where

$$\Sigma_t = \begin{bmatrix} \sigma_{\eta t}^2 & \rho_t \sigma_{\eta t} \sigma_{\epsilon t} \\ \rho_t \sigma_{\eta t} \sigma_{\epsilon t} & \sigma_{\epsilon t}^2 \end{bmatrix} \quad (2.5)$$

It has been found in the literature that structural breaks can influence the evidence for the unit root thus evidence for the stochastic trends. Assume there are M structural breaks in the mean growth rate μ_t occurring at Tm_1, Tm_2, \dots, Tm_M .

$$\mu_t = \mu + \sum_{m=1}^{m=M} d_m \mathbf{1}(t > Tm_m) \quad (2.6)$$

²In this UC representation with an AR(2) cycle, the correlation of σ_η and σ_ϵ is identified if neither σ_η nor σ_ϵ equals zero.

$\{d_m : m = 1, 2, \dots, M\}$ being significantly non-zero suggests the significance of the structural change in the mean growth rate.

Moreover, Kim and Nelson (1999a) and McConnell and Perez-Quiros (2000) find empirical evidence for the so-called “Great Moderation”—a significant reduction in the real output volatility in the US economy after the early 1980s. Assume there are N structural breaks in the covariance matrix Σ_t occurring at Tv_1, Tv_2, \dots, Tv_N .

$$\sigma_{\eta t} = \sum_{j=1}^{N+1} \sigma_{\eta, j} \mathbf{1}(Tv_{j-1} \leq t \leq Tv_j) \quad (2.7)$$

$$\sigma_{\epsilon t} = \sum_{j=1}^{N+1} \sigma_{\epsilon, j} \mathbf{1}(Tv_{j-1} \leq t \leq Tv_j) \quad (2.8)$$

$$\rho_t = \sum_{j=1}^{N+1} \rho_j \mathbf{1}(Tv_{j-1} \leq t \leq Tv_j) \quad (2.9)$$

where $Tv_0 = 0$ and $Tv_{N+1} = T$.

The parameters of the covariance matrix in equation (2.5) are now replaced with the ones defined by (2.7)-(2.9).

If $\sigma_{\eta t}$ is non-zero, the trend component τ_t is a random walk with drift. We denote the model as the Stochastic-Trend (hereafter ST) representation. We call the above model with an explicit zero restriction on $\sigma_{\eta t}$ (as well as ρ_t accordingly) the Deterministic-Trend (hereafter DT) representation. Note that allowing for a random walk component does not impose a stochastic trend. If $\sigma_{\eta t}$ is estimated to be near zero, τ_t becomes near deterministic overtime. In this case, the assumption of the stochastic trend may become unnecessary and thus penalized by the Bayesian marginal likelihood.

2.1.2 Settings of structural breaks and selection of break dates

We consider the following four settings of structural breaks and accordingly 8 types of models:

As notations for model types, ST and DT represent Stochastic-Trend and Deterministic-Trend representation as mentioned earlier. “0” denotes the models without any structural breaks. “M” suggests that there are structural breaks in the mean growth rate μ_t in the

	ST: unrestricted $\sigma_{\eta t}$	DT: $\sigma_{\eta t} = \rho_t = 0$
S1: No structural breaks	ST0	DT0
S2: only breaks in μ_t	STM	DTM
S3: only breaks in Σ_t	STV	DTV
S4: breaks in both μ_t and Σ_t	STMV	DTMV

form of (2.6). “V” suggests the assumption of structural breaks in the covariance matrix Σ_t set up by (2.7)-(2.9).

We allow for up to 4 structural breaks in the mean growth rate and up to 1 structural break in the covariance matrix³. Posterior sampling and computation of the marginal likelihoods are constructed according to the approaches described in Section 2.2.

Within each type of models, the number of structural breaks is selected according to the highest marginal likelihood except that we exclude models with boundary break dates estimation. To be specific, the break dates conditional on a specific number of breaks are estimated by the posterior modes. The particular model is excluded if there is one break date estimated to be within 3 quarters from the beginning or the end of the sample period, or within 3 quarters from the adjacent break date.

2.1.3 Testing for the existence of stochastic trends

The tests of stochastic trends can be viewed as model comparisons in the Bayesian framework.

Given the selected break date under the ST and DT specification, the posterior odds of ST against DT representation can be defined as the ratio of their posterior model probability. If the ratio is larger than one, the ST representation is more favored by the data. On the contrary, we have evidence for the DT representation if the ratio is smaller than one. We evaluate the significance of the evidence according to the criteria described in Section 2.2.3.

³In unreported exercise, we also allow for two structural breaks in the covariance matrices. However, the posterior modes of the second break dates turn out to be either close to the end of the sample period or very close to the posterior modes of the first break dates. Estimations point to the single structural break case even though we allow for more.

For the purpose of comparisons with some of the earlier results, we can also restrict our tests within a certain setting simply by comparing the best-fit ST and DT models within the same setting.

2.2 Approaches for Bayesian estimation and model comparisons

The model (2.1)-(2.5) can be rewritten into the state space form:

$$y_t = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} x_t \quad (2.10)$$

$$x_t = \begin{bmatrix} \mu_t \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & \phi_1 & \phi_2 \\ 0 & 1 & 0 \end{bmatrix} x_{t-1} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \eta_t \\ \epsilon_t \end{bmatrix} \quad (2.11)$$

where $x_t = [\tau_t, c_t, c_{t-1}]'$. μ_t and the covariance matrix Σ_t are set up as in Section 2.1.1, which may have M and N structural breaks.

In order to ensure that the estimated covariance matrix is positive semidefinite, we decompose the covariance matrix in the following way:

$$\begin{bmatrix} \sigma_{\eta,j}^2 & \rho_j \sigma_{\eta,j} \sigma_{\epsilon,j} \\ \rho_j \sigma_{\eta,j} \sigma_{\epsilon,j} & \sigma_{\epsilon,j}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ b_j & 1 \end{bmatrix} \begin{bmatrix} \sigma_{1,j}^2 & 0 \\ 0 & \sigma_{2,j}^2 \end{bmatrix} \begin{bmatrix} 1 & b_j \\ 0 & 1 \end{bmatrix} \quad (2.12)$$

with $j = 1, 2, \dots, N + 1$. We directly estimate $\{\sigma_{1,j}, \sigma_{2,j}, b_j\}$ instead of the covariance matrix parameters $\{\sigma_{\eta,j}, \sigma_{\epsilon,j}, \rho_j\}$. The posterior samples for the covariance parameters are obtained through transformation.

2.2.1 Prior specifications

We assume independent priors for all parameters.

Conditional on M breaks in mean (N breaks Σ_t), we assume a discrete uniform prior for the break dates over all ordered subsequences of length M (N). This prior assumes that all combinations of break dates are equally likely.

Inverse gamma priors $IG(100, 0.5)^4$ are assumed for σ_{1n}^2 and σ_{2n}^2 with $n = 1, 2, \dots, N + 1$. These priors are diffuse and do not have finite moments. Therefore, a heavy weight will be put on sample information.

⁴We follow Koop (2003) for the definition of inverse gamma (IG) distribution. If $x > 0$ follows inverse

We assume somewhat informative normal priors for $\phi_1 \sim N(1, 1)$, $\phi_2 \sim N(0, 1)$ and $b_n \sim N(0, 1)$ for $n = 1, 2, \dots, N + 1$. And we assume truncated uniform priors for $\mu \sim U[0, 2]$ and $d_m \sim U[-1, 1]$ for $m = 1, 2, \dots, M$.

Therefore, the joint prior density is the product of all the above marginal prior densities.

2.2.2 Gibbs approach

We use the Gibbs sampling approach to draw posterior samples for parameters. We run the Gibbs sampler for 200,000 times and save every 10th draw to reduce the autocorrelation within samples. We thus obtain 20,000 draws from the Gibbs procedures and discard the first 10,000 to avoid the effect of the initial values. To guarantee the convergence of the Gibbs sampler, we divide our samples, excluding the burn-in draws, into three sets—a first set of 3000 draws, a middle set of 4000 draws and a last set of 3000 draws. We find that the posterior distribution and estimates based on the three subsets don't vary much, suggesting the convergence of our MCMC samples.

Define $\theta = [\mu, \phi_1, \phi_2, \sigma_{\eta_n}, \sigma_{\epsilon_n}, \rho_n, d_m]$. \tilde{x} denotes the series of variable x for all time periods. The Gibbs sampling for the most general model contains the following blocks.

1. $\{\tau_t, c_t : t = 1, \dots, T\} | \theta, \tilde{y}$.
2. $[\phi_1^{(k)}, \phi_2^{(k)}] | \tilde{c}$.
3. $[\mu, d_1, d_2, \dots, d_M] | \tilde{\tau}$.
4. $\Sigma_1, \Sigma_2, \dots, \Sigma_{N+1} | \tilde{y}$ and other parameters.
5. $Tm_1, \dots, Tm_M, Tv_1, \dots, Tv_N | \tilde{y}$.

For the nested models, simply skip the irrelevant blocks.

More details on the Gibbs sampling are summarized in Appendix B.2.

gamma distribution $IG(s^{-2}, \nu)$, the probability density function of x is defined as:

$$f(x; s^{-2}, \nu) = \left(\frac{2s^{-2}}{\nu}\right)^{-\frac{\nu}{2}} \frac{1}{\Gamma(\frac{\nu}{2})} x^{-\frac{\nu}{2}-1} \exp\left(-\frac{\nu}{2s^{-2}x}\right)$$

where $\Gamma(\cdot)$ is the gamma function.

2.2.3 Marginal likelihood and Bayes factor

We compare two models based on their posterior odds. To be specific, posterior odds of Model_{*i*} versus Model_{*j*} is defined as

$$\frac{f(M_i|Y)}{f(M_j|Y)} = \frac{f(M_i)}{f(M_j)} \times \frac{f(Y|M_i)}{f(Y|M_j)} \quad (2.13)$$

where *f* represents probability mass or density and *Y* stands for data information. If we assume equal prior probability for two models, the posterior odds can be simplified as the ratio of marginal likelihoods, also known as the Bayes factor $B_{ij} = f(Y|M_i)/f(Y|M_j)$.

For convenience, we will construct $2\log(B_{ij})$. If $2\log(B_{ij})$ is positive (negative), we prefer M_i (M_j).

Kass and Raftery (1995) and Raftery (1995) suggest using the following criteria for significance of model comparison.

$2\log(B_{ij})$	Evidence for M_i , against M_j
0 to 2	Not worth more than a bare mention
2 to 6	Positive
6 to 10	Strong
> 10	Very strong

Table 2.1: Criteria for model comparisons based on twice the log Bayes factor.

2.3 Results

2.3.1 Break dates selections

The break dates selected for all types of models are presented in Table 2.2. For both ST and DT representations, assuming two breaks in mean and one break in the covariance matrix yield the highest marginal likelihood. And the number of structural breaks chosen

are consistent across models although the estimated break dates are not. Except for the deterministic trend model with a break only in the covariance matrix (the DTV model), all models identify 1983:2 as the break date for the covariance matrix, around when the Great Moderation is considered to occur⁵. As for the breaks in mean, the first structural break is estimated to occur from 1968:3 to 1973:4, consistent with the timing for the growth rate slowdown found by, for instance, Perron and Wada (2009). The second mean break is estimated to be at 2006:1 with the stochastic trend models and at 2002 (quarter 2 or 4) with the deterministic trend models.

More detailed results conditional on numbers of breaks and settings are reported in Appendix B.1.

2.3.2 Testing for the stochastic trends

From the previous section, we select the STMV and DTMV models with two breaks in mean and one break in the covariance matrix as the best-fit stochastic trend and deterministic trend representations. According to the marginal likelihood, the stochastic trend model is better supported by the data as it has a higher log marginal likelihood being compared with the deterministic trend model. However, the twice log Bayes factor for STMV over DTMV is 0.68, suggesting that the evidence is not particularly strong.

The posterior estimates for the best-fit STMV and DTMV models are shown in Table 2.3. We find evidence for two growth rate slowdowns and one volatility reduction. The evidence is similar whether we allow the trend component to be stochastic or not. Specifically, the posterior samples for the size of change in the mean growth rates (d_1 and d_2) mostly lie below zero. As for the change in the variance and covariance matrix, the volatility reduction is quite significant if we assume broken deterministic trend. However, it is less significant when we allow for a stochastic trend. Although the estimated variances in the chosen stochastic trend model decrease after the structural break, the 90% highest posterior density regions for variances in the pre-break and post-break period overlap.

⁵For model DTV, the break date estimated is 1947:1, the first quarter in our sample period. Together with the low marginal likelihood, it in fact suggests that no structural break in variance should be considered if we don't allow for breaks in mean. We report it simply as it is the only model within the type.

While the stochastic trend is slightly preferred, the trend shock volatility $\sigma_{\eta t}$ is estimated to be 0.54 before 1983:2 and 0.32 afterwards. The estimates are much smaller than what are usually reported by stochastic trend models.

To compare with the literature, we also conduct model comparisons within each setting. Table 2.4 shows the results comparing the best-fit ST and DT models within the same setting of breaks. If twice the log Bayes factor is positive (negative), data favors stochastic trend (deterministic trend) models. Deterministic trend is preferred only when the mean growth rate but not the covariance matrix exhibits structural breaks. Stochastic trends are preferred in other settings.

It's been widely reported in the literature that the unit-root hypothesis can be rejected if all the large and infrequent permanent shocks are accounted for by mean breaks. However, our results show that the seemingly strong rejections could result from not incorporating the break in the covariance matrix. And such rejections may not be robust after we incorporate the changes in the covariance matrix.

2.4 Conclusion

In this paper, we examine the effect of unknown structural breaks, including those in mean growth rates and the covariance matrix, on the evidence of the stochastic trend for the U.S. postwar quarterly real GDP. We use Bayesian approach to compare the stochastic trend (ST) models with the deterministic (linear) trend (DT) models, allowing for up to four unknown structural breaks in the mean growth rate and/or up to one break in the shocks' covariance matrix. We find evidence for two structural breaks in mean, one around early 1970s, and the other after 2000. Data also clearly identifies early 1980s as the date for volatility reductions.

Conditional on the chosen break dates, we find slight evidence for the stochastic trend models although the estimated trend variances are smaller comparing to earlier estimates reported in the literature. The bottom line is, the stochastic trend and the ongoing real shocks, even if not dominating, are at least not ignorable.

Models with stochastic trends					
	Breaks in mean μ_t		Breaks in covariance		LML
			matrix Σ_t		
	# of breaks	Break dates	# of breaks	Break dates	
ST0	—	—	—	—	-360.02
STM	2	1968:3, 2006:1	—	—	-356.35
STV	—	—	1	1983:2	-334.06
STMV*	2	1972:2, 2006:1	1	1983:2	-330.46

Models with deterministic trends					
	Breaks in mean μ_t		Breaks in covariance		LML
			matrix Σ_t		
	# of breaks	Break dates	# of breaks	Break dates	
DT0	—	—	—	—	-362.94
DTM	2	1973:2, 2002:4	—	—	-354.50
DTV	—	—	1	1947:1	-399.57
DTMV*	2	1973:4, 2002:2	1	1983:2	-330.80

* Best-fit ST and DT models.

LML—Log marginal likelihood.

Notations for models:

0—There is no break. M—There are breaks in the mean growth rate.

V—There are breaks in the covariance matrix.

Table 2.2: Selected break dates for each type of models.

	STMV	DTMV
M	2	2
Tm_1	1972:2 (1952:4, 2003:3)	1973:4 (1954:2, 1999:4)
Tm_2	2006:1 (2000:4, 2008:3)	2002:2 (1988:4, 2007:4)
N	1	1
Tv_1	1983:2 (1982:1, 1984:1)	1983:2 (1982:1, 1984:3)
μ	0.92 (0.80, 1.02)	0.92 (0.83, 1.02)
d_1	-0.18 (-0.45, 0.03)	-0.17 (-0.33, -0.01)
d_2	-0.31 (-0.55, -0.09)	-0.34 (-0.56, -0.07)
ϕ_1	1.48 (1.34, 1.61)	1.37 (1.27, 1.46)
ϕ_2	-0.53 (-0.66, -0.40)	-0.41 (-0.50, -0.31)
$\sigma_{\eta 1}$	0.54 (0.17, 0.98)	—
$\sigma_{\epsilon 1}$	0.92 (0.58, 1.29)	1.11 (1.01, 1.23)
ρ_1	0.13 (-0.56, 0.92)	—
$\sigma_{\eta 2}$	0.32 (0.11, 0.75)	—
$\sigma_{\epsilon 2}$	0.51 (0.30, 0.80)	0.54 (0.48, 0.61)
ρ_2	-0.33 (-0.96, 0.22)	—
LML	-330.46	-330.80

Note:

Numbers in the parentheses are 90% highest posterior density regions.

LML-Log marginal likelihood.

STMV-Stochastic trend model with breaks in both mean and the covariance matrix.

DTMV-Deterministic trend model with breaks in both mean and the covariance matrix.

Table 2.3: Bayesian estimation for the best-fit STMV and DTMV models.

	Model	2 log Bayes Factor
S1	ST0 vs. DT0	5.84
S2	STM vs. DTM	-3.70
S3	STV vs. DTV	131.02
S4	STMV vs. DTMV	0.68

Notations for settings and models:

ST–Stochastic Trend. DT–Deterministic Trend.

S1: No structural break. S2: There are breaks in μ_t .

S3: There are breaks in Σ_t . S4: There are breaks in both μ_t and Σ_t .

Table 2.4: Model comparisons within each setting.

Chapter 3

**HAS THE TREND-CYCLE CORRELATION OF GDP CHANGED? A
TIME-VARYING PARAMETER ESTIMATION FOR THE
POST-WWII U.S. REAL GDP**

Nelson and Plosser (1982) first find evidence for the existence of a stochastic trend in US real output. A stochastic trend suggests that there are ongoing permanent shocks affecting the business cycles, leaving a possibility of interactions between the permanent shocks and the transitory shocks of the real output. Morley et al. (2003) show that the trend-cycle decompositions (thus the estimated output gaps) based on unobserved component (UC) models are sensitive to the specification of the trend-cycle correlation.

Attempts have been made to estimate the correlation between trend and cycle GDP. However, reported estimates for the trend-cycle correlation of US real output vary from around zero to around -0.9. Most of the existing studies assume constant trend-cycle correlation. And the point estimates for the correlation tend to be more negative in studies using more recent data¹. For instance, Clark (1989) reports a slightly negative point estimate (-0.12) for the correlation with wide confidence interval expanding from -0.4 to +0.3. But Morley et al. (2003), Oh and Zivot (2006), and Sinclair (2009) find evidence for the correlation being around -0.9.

This paper revisits the issue by introducing a time varying correlation into a flexible UC model for the post-WWII quarterly US real GDP. We find that some of the contradicting results in the existing literature can be reconciled by allowing the trend-cycle correlation to change overtime.

We introduce the time-varying correlation into the otherwise standard UC model, which was first introduced Harvey and Todd (1983) and Clark (1987) to account for the changes in the mean growth rate. Our model also assumes stochastic volatility to take account of

¹Advocates for the deterministic GDP trend, such as Perron and Wada (2009), automatically imply zero trend-cycle correlations.

the “Great Moderation”—empirical findings about volatility reductions in a lot of macroeconomic series after early 1980s. Notice that there is no theoretical exclusion for the time-varying correlation between trend and cycle GDP. In general, different underlying shocks or interacting mechanisms imply different levels of correlations². And we do not have any reason to believe *a priori* that the factors affecting the business cycle does not change over time, nor should a certain type of mechanism dominate in all periods.

Based on our flexible framework, we find that the correlation mostly lie below zero for the postwar period. Our estimates show significant changes in the trend-cycle correlation. Specifically, the estimated correlation is negative but could be close to zero before 1980s. And the correlation has become significantly more negative since the 1980s till the end of the sample (2012:4). Our estimation seems to be a synthesis of some earlier contradicting results. The estimated correlation before 1980s is close to what is reported in Clark (1989), and the level after 1980s is consistent with the large and negative estimates by more recent papers (such as Morley et al. (2003) and Sinclair (2009)) which incorporate recent data. The major changes we find in the correlation occurs around 1981-1984.

Kim and Nelson (1999a) and McConnell and Perez-Quiros (2000) first find evidence for the reduction in the U.S. real output volatility in the early 1980s. Our UC model allow us to further identify whether both the trend and cycle components experience such volatility changes. Our results suggest significant volatility reduction in cyclical GDP around the early 1980s while the estimated trend variance only drops slightly. A counterfactual study suggests that the changes in the correlation and cycle volatility accounts (equally) for almost all volatility reductions in the Real GDP. Further more, an interesting implication is that the volatility stays at a relatively low level till the end of the sample (2012:4). Although the Great Recession from 2007 to 2009 created some turmoils, the volatility returns to a low level after the recession.

Our estimation also shed some light on some of the debates regarding the trend of U.S.

²For example, Clark (1989) suggests that a surge in business investment could generate positive correlations—a cyclical upturn together with improving longer-run output and capacity. On the contrary, an increase in disability benefits might increase consumption temporarily but degrade long-run output, causing negative correlations. Negative correlation can also occur due to the “time to built” effect or the slow adjustment process after a permanent productivity shock (as suggested in Morley (2007)). Weber (2011) has a review on the theoretical interpretations of the correlation between the trend and cycle GDP

real output. We find some signs of the grow rate slowdown around 1970 and a further reduction around 2005. However, distinct from the argument made by Perron and Wada (2009), the permanent changes in the mean growth rate alone are not enough to explained all of the permanent changes in the real output. We find evidence for ongoing permanent shocks and a stochastic trend.

The remaining of the paper is organized as follows. We present the model set-up in Section 3.1. Section 3.2 shows the estimation methodologies. We discuss the results and implications in Section 3.3. We conclude in Section 3.4.

3.1 A UC model with time varying parameters

A lot of researchers use unobserved component (UC) models to decompose the real output into the trend and cycle components and study the relative importance of the permanent shocks. The logic is that permanent shocks can affect the long-run trend of GDP, while the effect of transitory shocks will eventually vanish thus only affect the “cycle” of the real output.

Following Harvey and Todd (1983) and Clark (1987), the basic unobserved component we use for trend-cycle decomposition is:

$$y_t = \tau_t + c_t \quad (3.1)$$

$$\tau_t = \mu_{t-1} + \tau_{t-1} + \eta_t \quad (3.2)$$

$$\mu_t = \mu_{t-1} + \zeta_t \quad (3.3)$$

$$c_t = \phi_1 c_{t-1} + \phi_2 c_{t-2} + \epsilon_t \quad (3.4)$$

In the above model, y_t is the logarithm of real GDP, which is the sum of the trend component τ_t and the cyclical component c_t . τ_t follows a random walk with drift. μ_t is the mean growth rate of the real output, which also follows a random walk when ζ_t is non-zero. The cyclical component c_t is assumed to be a stationary AR(2) process. η_t and ϵ_t are the shocks to the trend and cycle respectively.

Given (3.2)-(3.3), both the level and the slope (i.e. the mean growth rate) of the trend GDP are allowed to change over time by a random walk mechanism. Alternative specifications for a changing slope in the literature are to extend the constant mean growth rate to

allow for structural breaks or Markov switching mechanisms. The benefit of our set-up is that we do not need to specify break dates, number of structural changes required in the structural break models, nor the number of states needed in the Markov switching models³. It is also a convenient approximation for the cases where multiple changes and levels may occur⁴.

The covariance matrix of $[\eta_t, \zeta_t, \epsilon_t]'$ has three variances and three correlations to be identified. According to Oh and Zivot (2006), the correlation between η_t and ζ_t needs to be specified for identification, as well as one of the other two correlations. We choose to leave the correlation between η_t and ϵ_t being estimated while restricting the remaining two correlations to be zero. Interactions between η_t and ϵ_t are usually of interests in the literature. It corresponds to the use of (3.3) to capture the relatively rare permanent changes in the mean growth rate while there may be ongoing trend shocks (such as productivity shocks) affecting the level of the growth rate and interacting with the cycle disturbance. In practice, it may be difficult to identify the correlation between ζ_t and ϵ_t if it is not pre-specified. The standard deviation for ζ_t is usually estimated to be very small for post war U.S. quarterly output (see, for instance, Clark (1987), Perron and Wada (2009) and Kim and Nelson (1999b)). Weak identification is quite likely due to one of the variance being close to zero. Therefore, we set up the covariance matrix in the following way:

$$\begin{bmatrix} \eta_t \\ \epsilon_t \\ \zeta_t \end{bmatrix} \sim i.i.d.N \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_{\eta t}^2 & \sigma_{\eta \epsilon t}^2 & 0 \\ \sigma_{\eta \epsilon t}^2 & \sigma_{\epsilon t}^2 & 0 \\ 0 & 0 & \sigma_{\zeta t}^2 \end{bmatrix} \right) \quad (3.5)$$

Note that the covariance matrix is time-varying. Therefore, the model allows us to incorporate the empirical findings of “Great Moderation” and examine whether the correlation

³However, a random walk mechanism implies that the change in the parameter is smooth while a structural break or Markov switching usually implies an abrupt change.

⁴A non-zero ζ_t implies that GDP is I(2) which is usually not supported by popular diagnostics such as ADF test. As a matter of fact, we find that the estimation of σ_{ζ} is very small such that it can be hard to distinguish it from an I(1) process from the statistical point of view. But a small yet nonzero σ_{ζ} allows μ_t to evolve relatively smoothly over time. This result is consistent with what is usually reported for post war quarterly output, for instance, Clark (1987), Kim and Nelson (1999b), Oh and Zivot (2006) and Perron and Wada (2009).

has changed or not. Following the literature of multivariate stochastic volatility such as Cogley and Sargent (2005) and Primiceri (2005), assume that

$$\Sigma_t = \begin{bmatrix} \sigma_{\eta t}^2 & \sigma_{\eta\epsilon t}^2 & 0 \\ \sigma_{\eta\epsilon t}^2 & \sigma_{\epsilon t}^2 & 0 \\ 0 & 0 & \sigma_{\zeta t}^2 \end{bmatrix} \quad (3.6)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ b_t & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^{h_{1t}} & 0 & 0 \\ 0 & e^{h_{2t}} & 0 \\ 0 & 0 & e^{h_{3t}} \end{bmatrix} \begin{bmatrix} 1 & b_t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (3.7)$$

$$= B_t H_t B_t' \quad (3.8)$$

where

$$h_{it} = h_{it-1} + \nu_{it}, \quad \nu_{it} \sim N(0, \sigma_{\nu_i}^2), i = 1, 2, 3 \quad (3.9)$$

$$b_t = b_{t-1} + w_t, \quad w_t \sim N(0, \sigma_w^2) \quad (3.10)$$

The decomposition of the variance covariance matrix Σ_t is a convenient device. The “structural” shocks $[\eta_t, \epsilon_t, \zeta_t]'$ are considered as linear combinations of three independent shocks whose variances are governed by the diagonal elements in matrix H_t . The interactions among the three independent shocks are controlled by matrix B_t , which also affects the covariances among $[\eta_t, \epsilon_t, \zeta_t]'$. Specifications in (3.9) and (3.10) allow all elements in the covariance matrix to change freely over time.

A recent paper, Weber (2011), also examines the change in the trend-cycle correlation of U.S. real output and attempts to study the causal structural for the monthly IP data. He finds no significant changes in the correlation, which is distinct from the results reported in this paper. While the quarterly real GDP data we use may exhibit different statistical properties, it may worth noting that he only allows for one pre-specified structural break at February 1984. And allowing for more than one structural break in his model would yield unclear form of overidentification restrictions. Such overidentification framework may not be appropriate for the flexible structures we assume in this paper, especially when we

find evidence for more than just a simple structural break for the correlation changes with quarterly GDP data.

While the empirical finding of a changing trend-cycle correlation is useful to find out the major mechanisms driving the business cycles, it is impossible to discuss the causal economic structures without placing further structural assumptions on our UC model as what Weber (2011) does. Theoretical or structural implications of our findings should be further explored but beyond the scope of this paper.

3.2 Methodology

A convenient way to estimate the models with time-varying parameters and stochastic volatilities is to use Bayesian estimation. A Bayesian Gibbs sampling approach will be used for estimation.

The UC model consisting of (3.1)-(3.4) can be rewritten into the state space form:

$$y_t = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} x_t \quad (3.11)$$

$$x_t = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \phi_1 & \phi_2 \\ 0 & 0 & 1 & 0 \end{bmatrix} x_{t-1} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \eta_t \\ \epsilon_t \\ \zeta_t \end{bmatrix} \quad (3.12)$$

where $x_t = [\tau_t, \mu_t, c_t, c_{t-1}]'$ and the the disturbances $[\eta_t, \epsilon_t, \zeta_t]$ are assumed to follow a multivariate normal distribution with their covariance matrix specified by (3.5)-(3.10).

Priors for parameters are assume to be independent with each other. We assume normal priors $N(1.3, 1)$ and $N(-0.7, 1)$ for the AR coefficients ϕ_1 and ϕ_2 . Priors for variances σ_{vi}^2 and σ_w^2 are assumed to follow independent inverse gamma distribution $IG(10^2, 0.1)$. For random walk state variables τ_t , μ_t , h_{it} and b_t , the simulation smoother approach we use requires specifying the prior distributions for the initial states τ_1 , μ_1 , h_{i1} and b_1 . Initial states are assumed to follow independent normal distributions. The mean values are given according to the estimation of the subsample from 1947:1 to 1959:3 (50 sample points) assuming no parameters instability. We impose large variance (10) such that the priors

are only weakly informative. Robustness checks are conducted with different priors for the initial states and sensitivity is only found for the estimates in the early sample periods.

We use the Gibbs sampler approach to draw posterior samples for parameters. The Gibbs sampler has a sample size of 20,000 while the first half of the posterior samples are dropped for convergence.

Define $\theta = [\phi_1, \phi_2, \sigma_{\eta t}, \sigma_{\epsilon t}, \rho_t, \sigma_{\zeta t} : t = 1, 2, \dots, T]$. Let $(\cdot)^{(k)}$ denote the k^{th} posterior draw of the latent variable x_t or the parameters. Y denotes all the observed quarterly log real GDP $\{y_1, y_2, \dots, y_T\}$. The k^{th} step in our Gibbs sampler involves the following blocks:

1. Draw $\{x_t^{(k)} : t = 1, \dots, T\} \sim f(x_1, \dots, x_T | Y, \theta^{(k-1)})$ using the simulation smoother developed by Durbin and Koopman (2002).
2. Draw $[\phi_1^{(k)}, \phi_2^{(k)}] \sim f(\phi_1, \phi_2 | Y, x_t^{(k)}, \sigma_\epsilon^{(k-1)})$ given that the third row in (3.12) has the following regression form:

$$c_t = \begin{bmatrix} c_{t-1} & c_{t-2} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} + \epsilon_t \quad (3.13)$$

The posterior samples for $[\phi_1^{(k)}, \phi_2^{(k)}]$ must guarantee the stationarity of the process $\{c_t : t = 1, \dots, T\}$. Therefore, we discard nonstationary draws and regenerate new ones until they meet the stationary requirement.

3. Draw $[\sigma_{\nu i}^{(k)}, i = 1, 2, 3]$ and $\{h_{it} : i = 1, 2, 3, t = 1, 2, \dots, T\}$. Residual terms $[\hat{\eta}_t, \hat{\epsilon}_t, \hat{\zeta}_t]'$ can be obtained from the simulation smoother in the first step. Define $\eta_t^* = \eta_t \sim N(0, \sigma_{1t}^2)$, $\epsilon_t^* = -b\eta_t + \epsilon_t \sim N(0, \sigma_{2t}^2)$ and $\zeta_t^* = \zeta_t \sim N(0, \sigma_{3t}^2)$, and we have the following:

$$B_t^{-1} \begin{bmatrix} \hat{\eta}_t \\ \hat{\epsilon}_t \\ \hat{\zeta}_t \end{bmatrix} = \begin{bmatrix} \hat{\eta}_t^* \\ \hat{\epsilon}_t^* \\ \hat{\zeta}_t^* \end{bmatrix} \sim N \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_{1t}^2 & 0 & 0 \\ 0 & \sigma_{2t}^2 & 0 \\ 0 & 0 & \sigma_{3t}^2 \end{bmatrix} \right) \quad (3.14)$$

where

$$B_t = \begin{bmatrix} 1 & 0 & 0 \\ b_t & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

While $\{h_{it} = \log(\sigma_{it}^2), i = 1, 2, 3\}$ follow random walks, we can use the method proposed by Kim et al. (1998) to sample $\{h_{it} : i = 1, 2, 3, t = 1, 2, \dots, T\}$. And given inverse Gamma priors, $[\sigma_{\nu_i}^{(k)}, i = 1, 2, 3]$ have inverse Gamma posteriors and can be easily sampled.

4. Draw $b_t^{(k)}$ and σ_w . Given the second row in (3.14) and the random walk assumption on the evolving of b_t , we have a state space regression to sample b_t as the state variable and σ_w as the hyperparameter using similar approach as in Step 1:

$$\hat{\epsilon}_t = \hat{\eta}_t b_t + \epsilon_t^* \quad \epsilon_t^* \sim N(0, \sigma_{2t}^2) \quad (3.15)$$

$$b_t = b_{t-1} + w_t, \quad w_t \sim N(0, \sigma_w^2) \quad (3.16)$$

We can then construct the posterior samples for $[\sigma_{\eta_t}, \sigma_{\epsilon_t}, \sigma_{\zeta_t}, \rho_t]$ according to (3.8).

More details on the Gibbs sampling are summarized in Appendix C.1.

3.3 Results

We use the US quarterly real GDP data from 1947:1 to 2012:4. The parameters are estimated by the posterior median and the 16th-84th posterior percentiles are reported. Results of the time invariant parameters and their posterior percentiles are shown in Table 3.1. The rest are shown in Figures (3.1)-(3.5).

3.3.1 Has the correlation changed?

We find time varying correlations between the trend and cycle GDP (Figure 3.1). The estimated correlation stays negative throughout the whole sample periods. Only small proportion of the posterior samples in the early sample periods are positive, which are,

Parameters	Prior	Posterior median	16-84th percentiles
ϕ_1	$N(1.3, 1)$	1.264	[1.089,1.391]
ϕ_2	$N(-0.7, 1)$	-0.438	[-0.547,-0.311]
$\sigma_{\nu 1}$	$\sqrt{IG(10^2, 0.1)}$	0.030	[0.009,0.147]
$\sigma_{\nu 2}$	$\sqrt{IG(10^2, 0.1)}$	0.242	[0.148,0.368]
$\sigma_{\nu 3}$	$\sqrt{IG(10^2, 0.1)}$	0.014	[0.007,0.038]
σ_w	$\sqrt{IG(10^2, 0.1)}$	0.011	[0.006,0.022]

Table 3.1: Posterior estimates for time invariant parameters and the 16th-84th percentiles.

however, excluded by the 16-84th percentiles reported⁵. In most of the time before 1980s, the estimated correlation is slightly negative with posterior median around -0.5. Starting from the late 1980s, the estimated correlation stays stable around -0.85.

Although the posterior median of the correlation show variations overtime with the major changes in the early 1980s, the 16th-84th percentiles for periods before and after the early 1980s overlap as shown in Figure 3.1. However, it is more informative to examine the joint distribution of the correlations from two periods rather than the marginal distribution of one correlation. To this end, we construct the difference between the samples for correlation at 1980:1 and the correlation at every quarter after it. Specifically, for the k^{th} iteration in the Gibbs sampler, we compute

$$\Delta\rho_t^{(k)} = \rho_{1980:1}^{(k)} - \rho_t^{(k)}, t \in [1980:1, 2012:4]. \quad (3.17)$$

The median and the 16th-84th percentiles of $\Delta\rho_t$ are presented in Figure 3.2. The posterior samples mostly lie above zero, showing significant evidence for the change in correlation occurring at the early 1980s.

⁵For the samples in the first few sample points, confidence intervals may include zero if we change the initial values.

3.3.2 Implications on the Great Moderations

Empirical evidence has been found in the literature for a real output volatility reduction after 1984. Within our framework, we can investigate the volatility of which component has contributed to the “Great Moderation”. We find only mild reduction in the trend volatility $\sigma_{\eta t}$ (Figure 3.3) but significant drop in the cyclical volatility $\sigma_{\epsilon t}$ (Figure 3.4) after 1984. The major changes occur around early 1980s, consistent with the estimated date by, for instance, Kim and Nelson (1999a) and McConnell and Perez-Quiros (2000).

Many empirical studies on the “Great Moderation” are based on reduced form ARIMA models for real GDP. It is interesting to compare our results with those in the literature by converting our UC model into the implied ARIMA(2,2,3) representation:

$$(1 - \phi_1 L - \phi_2 L^2)(1 - L)^2 y_t = (1 + \theta_1 L + \theta_2 L^2 + \theta_3 L^3) u_t \quad (3.18)$$

where $u_t \stackrel{i.i.d.}{\sim} N(0, \sigma_{ut}^2)$. The parameters of the above ARIMA(2,2,3) model are functions of the parameters of our UC model. Oh and Zivot (2006) show the equations for parameter conversion. Not surprisingly, the variation in the volatility parameter σ_{ut} (as shown in Figure 3.6) is in line with those findings of great moderation after 1984 (e.g. Kim and Nelson (1999a)).

Different from the pure reduced form studies, the UC-model-implied ARIMA representation provide a chance to conduct a counterfactual study to understand the impact of changes in the volatility of the trend and cycle shocks and changes in the correlation between them. Specifically, we want to investigate whether we could have observed the “Great Moderation” if only one of the above three components changes as estimated after 1980:1. And We compute the implied σ_u in the ARIMA representation holding the other two of ρ_t , $\sigma_{\eta t}$ and $\sigma_{\epsilon t}$ at the levels of 1980:1. Counterfactual results are shown in Figure 3.7. The changes in ρ_t and $\sigma_{\epsilon t}$ alone can respectively reduce σ_{ut} by about half of the actual volatility reduction. But changing $\sigma_{\eta t}$ alone barely changes σ_{ut} after 1984.

Another interesting finding is that σ_u remains at low level after 1984 except for some turmoils created by the Great Recession from 2007-2009.

3.3.3 Other implications

Is there growth rate slowdown?

Perron (1989) and Perron and Wada (2009) identify a productivity slowdown for US in the 1970s. We find some signs of the growth rate slowing down, although not smoothly. Our estimated μ_t (Figure 3.5) is similar to the one reported in Kim and Nelson (1999b). There are two high growth rate periods: early 1950s and early 60s, while the mean growth rate started to fall after 1965. The average level of μ_t seems to be lower after 1970s and further decreases after about 2005.

As the estimated μ_t suggests a reduction in the mean growth rate during the Great Recession, the implied output gap was not as large as what people would have expected. Figure 3.8 presents the estimated output gap by our model. Magnitudes of the cycles remain relatively small until the end of the sample.

Is the trend GDP stochastic?

Samples for $\sigma_{\eta t}$ (Figure 3.3) stays significantly above zero over time. Distinct from what is claimed in Perron and Wada (2009) who assumes the correlation is constant overtime, the change in mean growth rate does not seem to be the only source for the permanent changes in the US real GDP.

On the other hand, the standard deviation for ζ_t is estimated to be around 0.1 and stable over time. It's slightly larger than what is usually reported in the literature but still quite small. Therefore, it is not likely that the changes in $m\mu_t$ can account for most of the permanent changes in the trend component.

3.4 Concluding remarks

In this paper, we present evidence for the changing correlation between the trend and cycle GDP in postwar U.S. by introducing the time varying correlation into a UC model with random walk mean growth rate and stochastic volatility. We find that the estimated correlation is negative but could be close to zero before 1980s. And it has become more negative since the 1980s till the end of the sample (2012:4). By allowing the correlation to

change over time, we are able to synthesize some of the debating results from earlier work. Through counterfactual study, we show that the change in correlation contributes equally with the reduction in the cycle volatility to the great moderation.

As a by product, we find evidence for a stochastic trend and ongoing permanent shocks. We also find some signs of the grow rate slowdown around 1970 and further reduction around 2005.

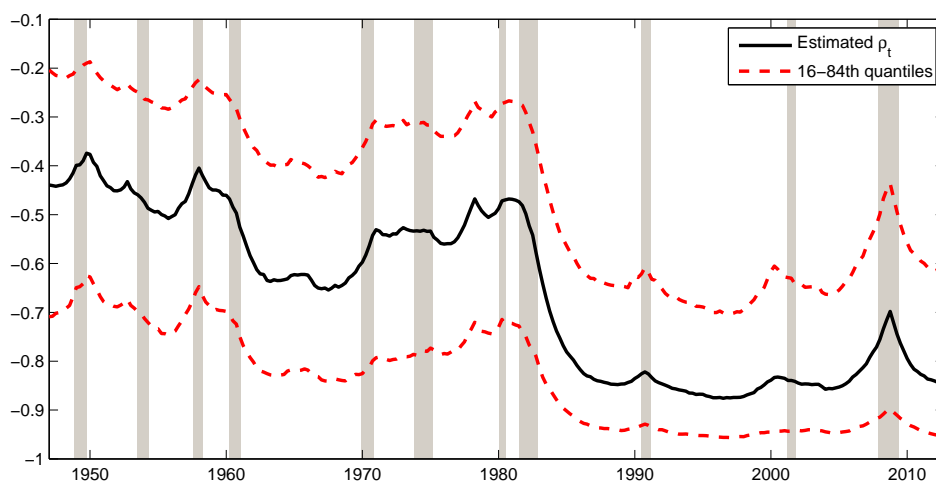


Figure 3.1: Estimated correlation between trend and cycle: ρ_t

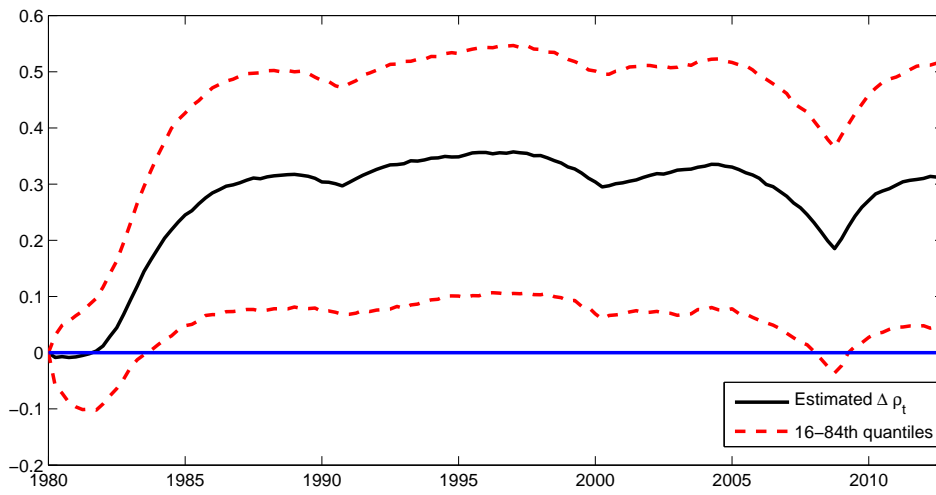


Figure 3.2: Estimation of difference between correlation at 1980:1 and correlation after that.

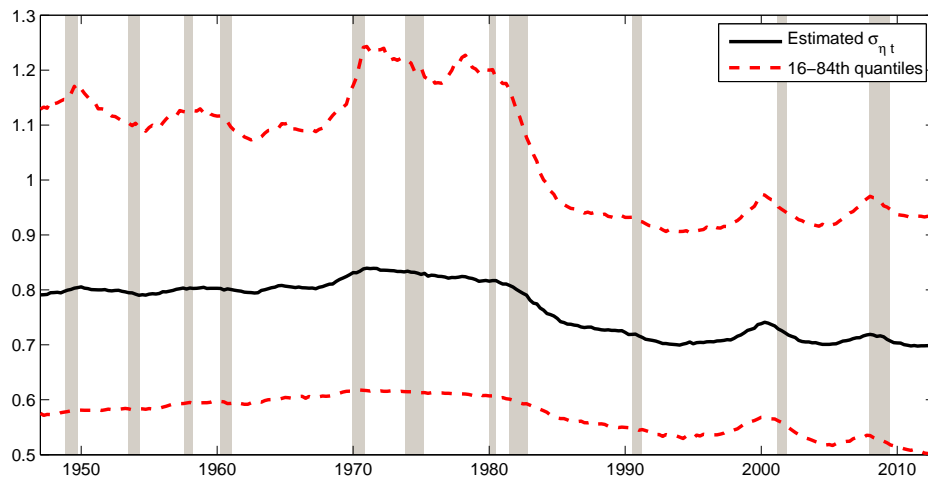


Figure 3.3: Estimated standard deviation for trend shock: $\sigma_{\eta t}$

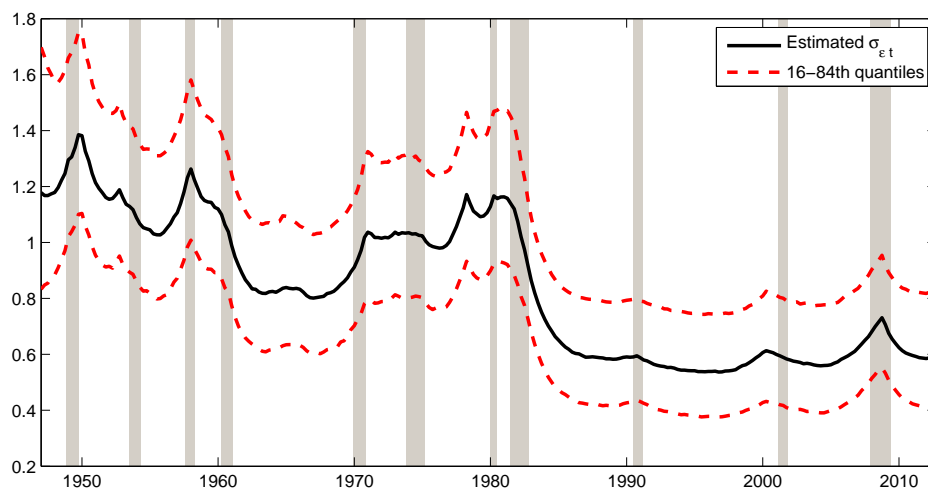


Figure 3.4: Estimated standard deviation for cycle shock: σ_{ϵ_t}

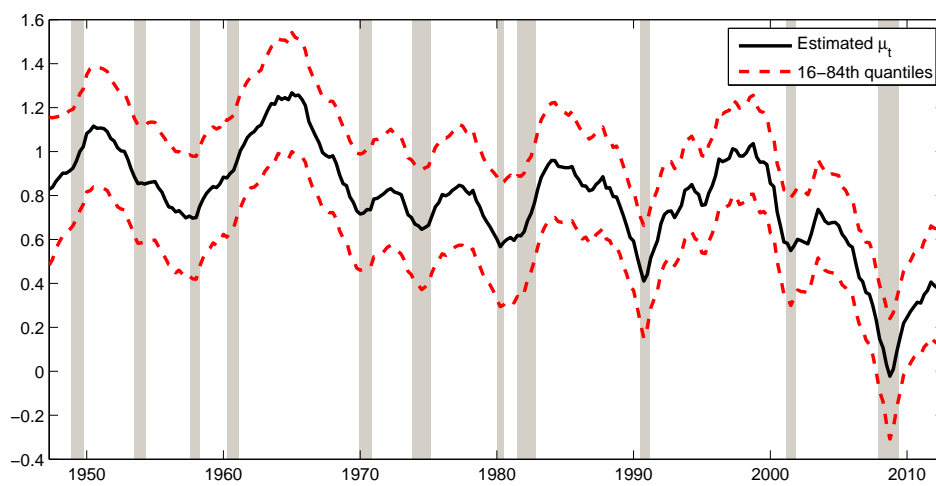


Figure 3.5: Estimated mean growth rate: μ_t

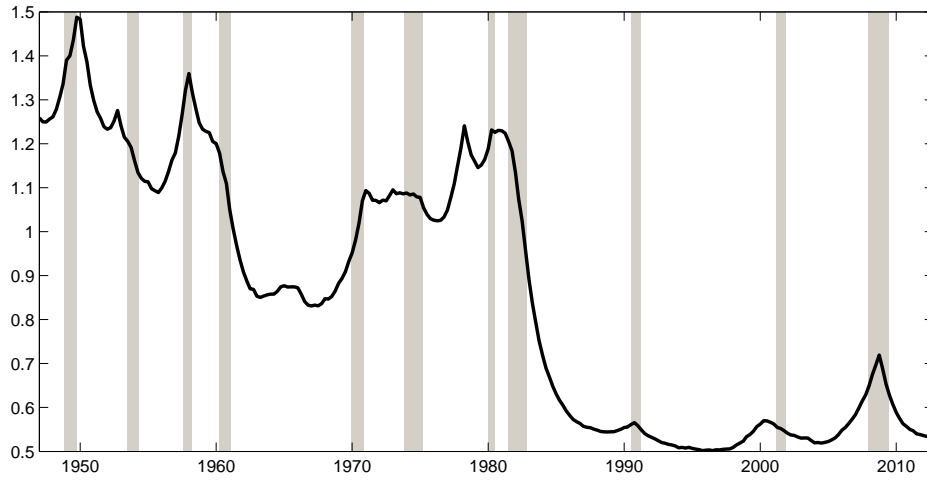


Figure 3.6: Volatility for disturbance in the equivalent ARIMA(2,2,3): σ_{ut}

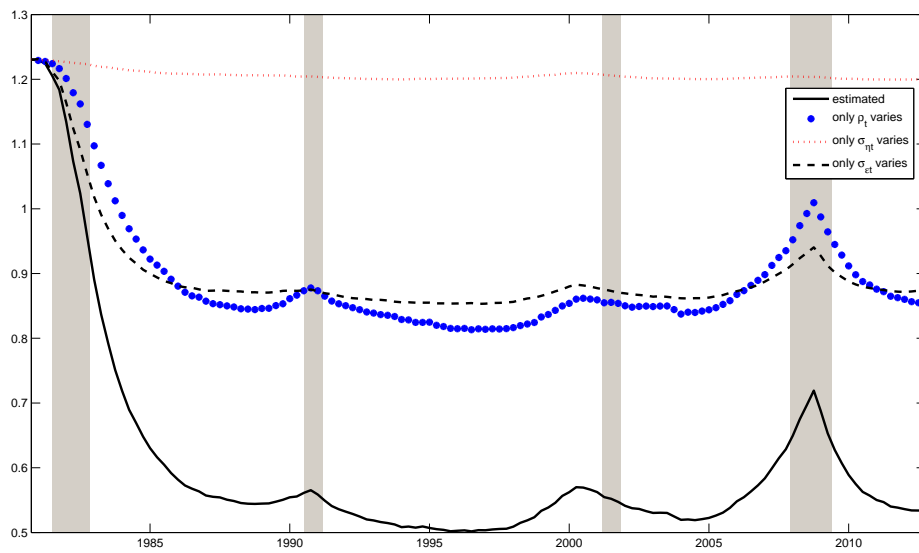


Figure 3.7: Counterfactual and actual σ_{ut}

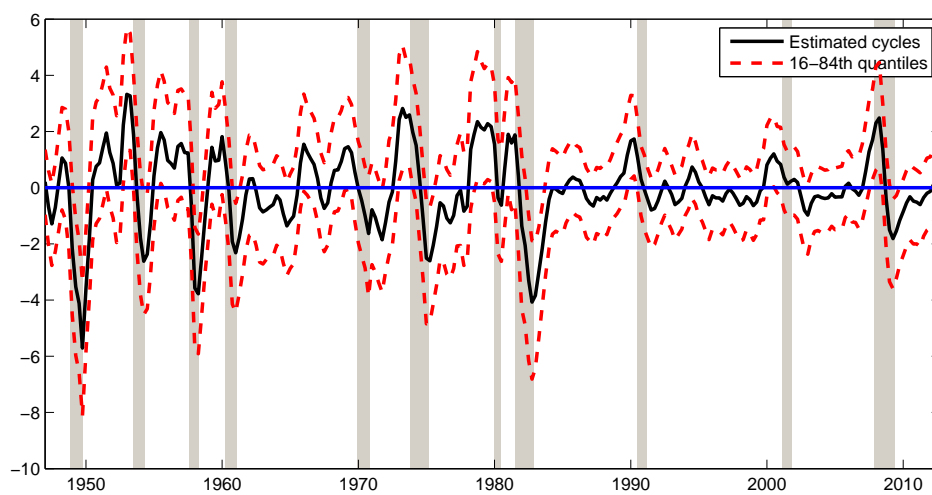


Figure 3.8: Estimated cyclical GDP

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Appendix A

APPENDIX TO CHAPTER 1

A.1 Results with updated data

We repeat our analysis on the updated data to 2008:2. Results are mostly in line with those reported earlier except for quantitative changes.

The most likely break date is 1969:3, with the posterior probability of 1.99%. The break dates within the 70% HPD region mostly locate from 1965 to 1980, as shown in Figure A.1. The Bayesian estimated results are presented in Table A.1.

The upper panel in Figure A.2 presents the Bayesian posterior density for σ_η . The bimodal distribution is similar to that reported in Section 1.3.2, with two modes at 0.45 and 1.05. There are around 70% posterior samples centering around the first mode and the remaining 30% around the second mode. The estimated GDP cycles are reported in Figure A.3.

Model comparison results, reported in Table A.2, do not vary much qualitatively compared to our earlier results.

A.2 The Gibbs sampling used in Section 1.3

Define $\theta = [\mu, \phi_1, \phi_2, \sigma_\eta, \sigma_\epsilon, \rho, d]$. Let $(\cdot)^{(k)}$ denote the the k^{th} posterior draw of the latent variable x_t or parameter. Y denotes all the observed quarterly log real GDP $\{y_1, y_2, \dots, y_T\}$. The k^{th} step in our Gibbs sampler used in Section 1.3 involves the following blocks:

- Draw $\{x_t^{(k)} : t = 1, \dots, T\} \sim f(x_1, \dots, x_T | Y, \theta^{(k-1)}, Tb^{(k-1)})$ obtained from the simulation smoother developed by Durbin and Koopman (2002). We then obtain $\tau_t^{(k)}$ and $c_t^{(k)}$ as the first two elements in $x_t^{(k)}$.
- Draw $[\phi_1^{(k)}, \phi_2^{(k)}] \sim f(\phi_1, \phi_2 | Y, x_t^{(k)}, \sigma_\epsilon^{(k-1)})$.

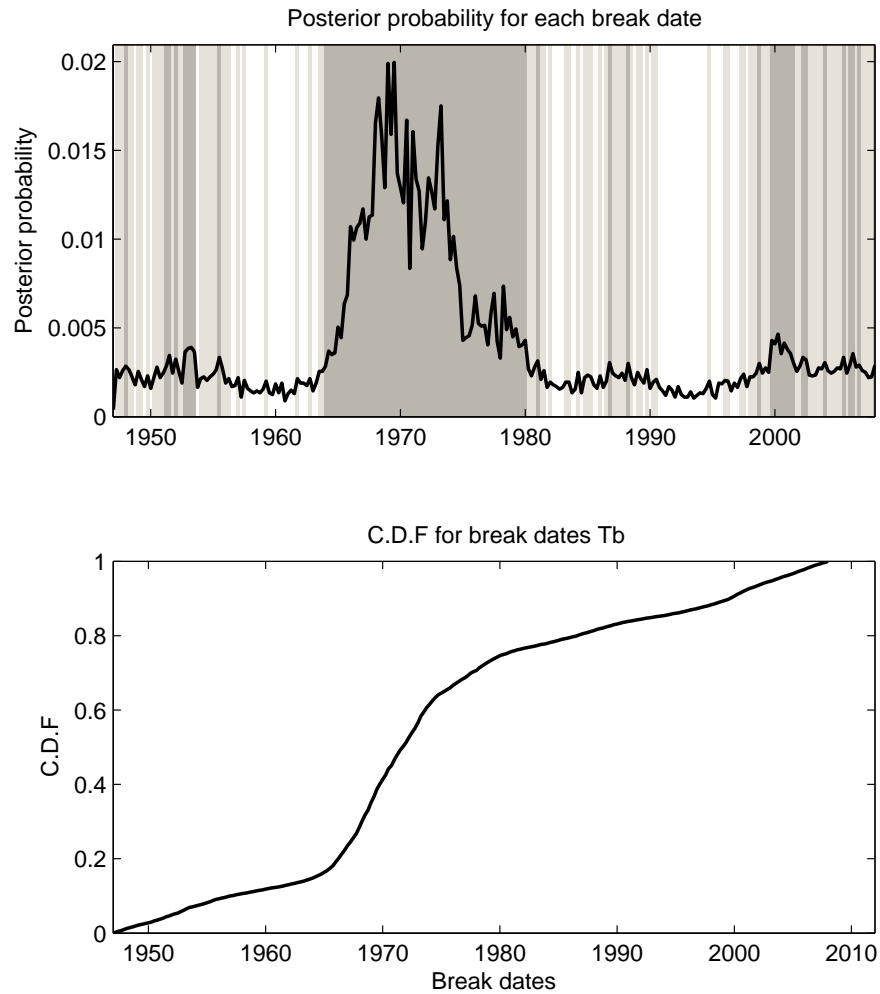


Figure A.1: Posterior distribution of break dates with data up to 2008:2.

Note: In the upper panel, all (including both dark and light) shaded areas represent the 90% HPD intervals, while the dark ones represent the 70% HPD intervals.

Stack (1.14) by time, we have

$$Y_c = C\Phi + \epsilon$$

where $Y_c = \{c_3^{(k)}, \dots, c_T^{(k)}\}'$, $\epsilon = \{\epsilon_3, \dots, \epsilon_T\}' \stackrel{i.i.d.}{\sim} N(0, 1/h_\epsilon)$ with $h_\epsilon = (\sigma_\epsilon^{(k-1)})^{-2}$,

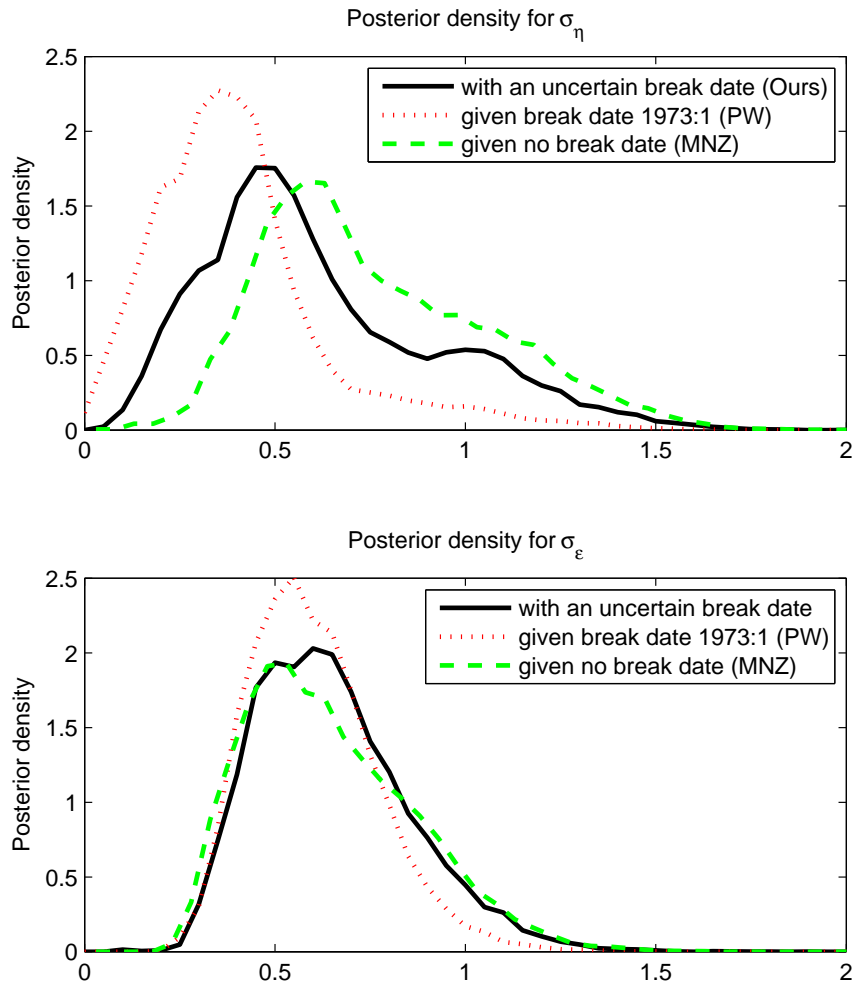


Figure A.2: Posterior density of σ_η (upper panel) and σ_ϵ (lower panel) with data up to 2008:2.

$\Phi = \{\phi_1, \phi_2\}'$ and

$$C = \begin{bmatrix} c_2 & c_1 \\ \vdots & \\ c_{t-1} & c_{t-2} \\ \vdots & \\ c_{T-1} & c_{T-2} \end{bmatrix}$$

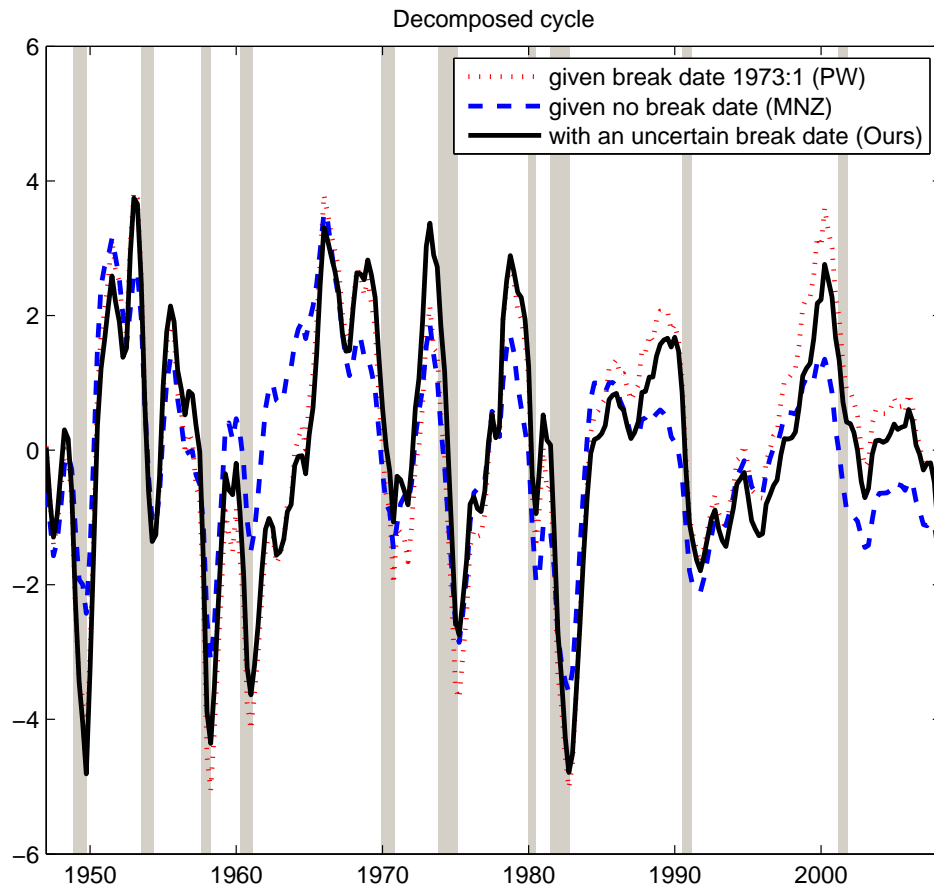


Figure A.3: Posterior estimate of trend and cycle for benchmark model.

Note: Shaded areas represent the NBER recession periods.

Given multivariate normal prior $N(\Phi_0, V_{\Phi_0})$ for Φ , the posterior distribution follows a multivariate normal distribution $N(\tilde{\Phi}, \tilde{V}_{\Phi})$, where

$$\tilde{V}_{\Phi} = (V_{\Phi_0}^{-1} + h_{\epsilon} C' C)^{-1} \quad (\text{A.1})$$

$$\tilde{\Phi} = \tilde{V}_{\Phi} (V_{\Phi_0}^{-1} \Phi_0 + h_{\epsilon} C' Y_c) \quad (\text{A.2})$$

The posterior samples for $[\phi_1^{(k)}, \phi_2^{(k)}]$ must guarantee the stationarity of the process. Therefore, we discard nonstationary draws and regenerate new ones until they meet

the stationary requirement.

- Draw $[\mu^{(k)}, d^{(k)}] \sim f(\mu, d|Y, x_t^{(k)}, \sigma_\eta^{(k-1)})$.

Stack (1.15) by time, we have

$$Y_\tau = D \begin{bmatrix} \mu \\ d \end{bmatrix} + \eta$$

where $Y_\tau = \{\tau_2^{(k)} - \tau_1^{(k)}, \dots, \tau_T^{(k)} - \tau_{T-1}^{(k)}\}'$, $\eta = \{\eta_2, \dots, \eta_T\}' \stackrel{i.i.d.}{\sim} N(0, 1/h_\eta)$ with $h_\eta = (\sigma_\eta^{(k-1)})^{-2}$ and

$$D = \begin{bmatrix} 1 & 0 \\ \vdots & \\ 1 & 1(t > Tb) \\ \vdots & \\ 1 & 1 \end{bmatrix}$$

Given uniform priors $\mu \sim [0, 2]$ and $d \sim [-0.5, 0.5]$, the posterior distribution follows a truncated multivariate normal distribution. It's equivalent to sample the posterior from $N(\tilde{M}, \tilde{V}_M)$ and discard the samples out of the truncated region, where

$$\tilde{V}_M = (h_\eta D' D)^{-1} \quad (\text{A.3})$$

$$\tilde{M} = h_\eta \tilde{V}_M D' Y_\tau \quad (\text{A.4})$$

- Draw $[\sigma_1^{(k)}, \sigma_2^{(k)}] \sim f(\sigma_1, \sigma_2|Y, x_t^{(k)}, \mu^{(k)}, d^{(k)}, \phi_1^{(k)}, \phi_2^{(k)}, b^{(k-1)})$.

According to (1.16),

$$\begin{bmatrix} \eta_t^* \\ \epsilon_t^* \end{bmatrix} \sim N \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} \right) \quad (\text{A.5})$$

We assume independent priors for $h_i^{-1} = \sigma_i^2 \sim IG(s_{i0}^{-2}, \nu_{i0})$ with $i = 1, 2$. It's equivalent to assume a Gamma prior $G(s_{i0}^{-2}, \nu_{i0})$ for h_i . The posterior of h_i , in this case, is

$G(\tilde{s}_i^{-2}, \tilde{\nu}_i)$, where for $i = 1, 2$

$$\tilde{\nu}_i = T + \nu_{i0} \quad (\text{A.6})$$

$$\tilde{s}_1^2 = \frac{\eta^{*'} \eta^* + \nu_{10} s_{10}^2}{\tilde{\nu}_1} \quad (\text{A.7})$$

$$\tilde{s}_2^2 = \frac{\epsilon^{*'} \epsilon^* + \nu_{20} s_{20}^2}{\tilde{\nu}_2} \quad (\text{A.8})$$

$$\eta^* = [\eta_1^{*(k)}, \dots, \eta_T^{*(k)}]' \quad (\text{A.9})$$

$$\epsilon^* = [\epsilon_1^{*(k)}, \dots, \epsilon_T^{*(k)}]' \quad (\text{A.10})$$

- Draw $b^{(k)} \sim f(b|Y, x_t^{(k)}, \mu^{(k)}, d^{(k)}, \phi_1^{(k)}, \phi_2^{(k)}, \sigma_2^{(k)})$. Stack (1.17) by time, we have

$$\hat{\epsilon} = Eb + e^*$$

where $\hat{\epsilon} = \{\hat{\epsilon}_1^{(k)}, \dots, \hat{\epsilon}_T^{(k)}\}'$, $E = \{\hat{\eta}_1^{(k)}, \dots, \hat{\eta}_T^{(k)}\}'$ and $e^* = \{\epsilon_1^*, \dots, \epsilon_T^*\}' \stackrel{i.i.d.}{\sim} N(0, 1/h_{s2})$ with $h_{s2} = (\sigma_2^{(k)})^{-2}$. $\hat{\eta}_t^{(k)}$ and $\hat{\epsilon}_t^{(k)}$ are residuals in the first two rows in (1.12).

Given normal prior $N(b_0, V_{b0})$ for b , the posterior distribution follows a normal distribution $N(\tilde{b}, \tilde{V}_b)$, where

$$\tilde{V}_b = (V_{b0}^{-1} + h_{s2} E' E)^{-1} \quad (\text{A.11})$$

$$\tilde{b} = \tilde{V}_b (V_{b0}^{-1} b_0 + h_{s2} E' \hat{\epsilon}) \quad (\text{A.12})$$

- Draw $Tb^{(k)} \sim f(Tb|Y, \theta^{(k)})$. According to (1.18), we can draw Tb from a multinomial distribution where $f(Tb|Y, \theta) = \frac{f(Y|Tb, \theta)}{\sum_{t=1}^{T-1} f(Y|Tb=t, \theta)}$

Note that the fixed break date and the no break date model can be also estimated by the above Gibbs sampler with corresponding blocks skipped for the restricted parameters.

Benchmark model: with an uncertain break date						
Log marginal likelihood:-336.9226						
	Prior		Posterior			
	median	90% quantile	mean	median	90% HPD*	
μ	1	(0.10, 1.90)	0.9278	0.9315	(0.77, 1.08)	
ϕ_1	1	(-0.64, 2.64)	1.4300	1.4581	(1.24,1.67)	
ϕ_2	0	(-1.64, 1.64)	-0.5505	-0.5613	(-0.75,-0.37)	
σ_η	0.24	(0.05, 17.31)	0.6185	0.5439	(0.16,1.02)	
σ_ϵ	0.61	(0.08, 65.65)	0.6505	0.6261	(0.34,0.98)	
ρ	0	(-1.00, 1.00)	0.2030	0.3292	(-0.90,-0.40) \cup (-0.15,1.00)	
d	0	(-0.48, 0.48)	-0.1839	-0.1929	(-0.43,0)	
	median	90% quantile	median	mode	90% quantile**	
Tb	1977:2	(1949:4, 2004:4)	1971:4	1969:3	(1952:2, 2003:3)	

Posterior***						
	PW model: fixed break date			MNZ model: no break date		
	Log marginal likelihood:-336.6105			Log marginal likelihood:-338.2757		
	mean	median	90% HPD	mean	median	90% HPD
μ	0.9563	0.9572	(0.88,1.04)	0.8325	0.8351	(0.75,0.91)
ϕ_1	1.4518	1.4609	(1.28,1.65)	1.4466	1.4866	(1.23,1.72)
ϕ_2	-0.5443	-0.5474	(-0.72,-0.36)	-0.5674	-0.5809	(-0.77,-0.40)
σ_η	0.4538	0.4193	(0.08,0.79)	0.7149	0.6483	(0.27,1.19)
σ_ϵ	0.5991	0.5803	(0.35,0.86)	0.6456	0.6110	(0.31,0.98)
ρ	0.5838	0.8614	(-0.4,1.00)	0.0951	0.0302	(-0.90,0.20) \cup (0.60,1.00)
d	-0.2200	-0.2212	(-0.33,-0.11)		—	
Tb		1973:1 (fixed)			—	

* HPD refers to highest posterior density interval.

** We report the 70% and 90% HPD for Tb in Figure 4.

*** Priors for the unrestricted parameters in PW and MNZ models are the same as those in the benchmark.

Table A.1: Bayesian Inferences with data up to 2008:2.

Model	Log marginal likelihood	BIC difference	
		Row over column	
		PW	MNZ
uncertain break date(Ours)	-337.2264	-1.6492	2.1190
Tb=1973:1(PW)	-336.6105		3.7682
no break(MNZ)	-334.1752		

Table A.2: Log marginal likelihoods and model comparisons with data up to 2008:2.

Appendix B
APPENDIX TO CHAPTER 2

B.1 Estimation of break dates for S2 and S4.

Model		M=1	M=2	M=3	M=4
STM	LML	-358.77	-356.35	-356.32	-357.41
	Tm	2002:2	1968:3, 2006:1	1973:2, 2006:1,2012:3	1969:1,2006:1, 2008:2,2012:3
DTM	LML	-355.01	-354.50	-354.87	-354.93
	Tm	1973:2	1973:2, 2002:4	1969:1, 1978:4, 2005:3	1947:4, 1969:1, 2002:2, 2005:3

Note: LML-log marginal likelihood.

Sample Period: 1947:1-2012:4

Table B.1: Estimated break dates for S2.

Model		M=1	M=2	M=3	M=4
STMV	LML	-331.33	-330.46	-331.47	-334.99
	T _m	2002:2	1972:2, 2006:1	1969:3, 2006:1, 2006:1	1969:1, 2002:2, 2002:2, 2006:1
	T _v	1983:2	1983:2	1983:2	1983:2
DTMV	LML	-331.44	-330.80	-332.40	-334.04
	T _m	2002:2	1973:4, 2006:1	1969:1, 2002:2, 2006:1	1969:1,2002:2, 2002:2, 2006:1
	T _v	1983:2	1983:2	1983:2	1983:2

Note: LML-log marginal likelihood.

Sample Period: 1947:1-2012:4

Table B.2: Estimated break dates for S4 with N=1.

B.2 Details on the Gibbs sampler

Let θ denote all the parameters except for the break dates. Let $(\cdot)^{(k)}$ denote the the k^{th} posterior draw of the latent variable x_t or parameters. Y denotes all the observed quarterly log real GDP $\{y_1, y_2, \dots, y_T\}$. The k^{th} step in our Gibbs sampler used in Section 2.2 involves the following blocks:

- Draw $\{x_t^{(k)} : t = 1, \dots, T\} \sim f(x_1, \dots, x_T | Y, \theta^{(k-1)}, Tb^{(k-1)})$ obtained from the simulation smoother developed by Durbin and Koopman (2002). We then obtain $\tau_t^{(k)}$ and $c_t^{(k)}$ as the first two elements in $x_t^{(k)}$.
- Draw $[\phi_1^{(k)}, \phi_2^{(k)}] \sim f(\phi_1, \phi_2 | Y, x_t^{(k)}, \sigma_\epsilon^{(k-1)})$.

Stack the second row of (2.11) by time, we have

$$Y_c = C\Phi + \epsilon$$

where $Y_c = \{c_3^{(k)}, \dots, c_T^{(k)}\}'$, $\epsilon = \{\epsilon_3, \dots, \epsilon_T\}' \stackrel{i.i.d.}{\sim} N(0, 1/h_{\epsilon t})$ with $h_{\epsilon t} = (\sigma_{\epsilon t}^{(k-1)})^{-2}$, $\Phi = \{\phi_1, \phi_2\}'$ and

$$C = \begin{bmatrix} c_2 & c_1 \\ \vdots & \\ c_{t-1} & c_{t-2} \\ \vdots & \\ c_{T-1} & c_{T-2} \end{bmatrix}$$

Given multivariate normal prior $N(\Phi_0, V_{\Phi_0})$ for Φ , the posterior distribution follows a multivariate normal distribution $N(\tilde{\Phi}, \tilde{V}_\Phi)$, where

$$\tilde{V}_\Phi = (V_{\Phi_0}^{-1} + h_\epsilon C' C)^{-1} \tag{B.1}$$

$$\tilde{\Phi} = \tilde{V}_\Phi (V_{\Phi_0}^{-1} \Phi_0 + h_\epsilon C' Y_c) \tag{B.2}$$

The posterior samples for $[\phi_1^{(k)}, \phi_2^{(k)}]$ must guarantee the stationarity of the process. Therefore, we discard nonstationary draws and regenerate new ones until they meet the stationary requirement.

- Draw $[\mu^{(k)}, d_1^{(k)}, d_2^{(k)}, \dots, d_M^{(k)}] \sim f(\mu, d|Y, x_t^{(k)}, \sigma_{\eta t}^{(k-1)})$.

Stack the first row of (2.11) by time, we have

$$Y_\tau = D \begin{bmatrix} \mu \\ d_1 \\ \vdots \\ d_M \end{bmatrix} + \eta$$

where $Y_\tau = \{\tau_2^{(k)} - \tau_1^{(k)}, \dots, \tau_T^{(k)} - \tau_{T-1}^{(k)}\}'$, $\eta_t = \{\eta_2, \dots, \eta_T\}' \stackrel{i.i.d.}{\sim} N(0, 1/h_{\eta t})$ with $h_{\eta t} = (\sigma_{\eta t}^{(k-1)})^{-2}$.

Given uniform priors $\mu \sim [0, 2]$ and $d \sim [-1, 1]$, the posterior distribution follows a truncated multivariate normal distribution. It's equivalent to sample the posterior from $N(\tilde{M}, \tilde{V}_M)$ and discard the samples out of the truncated region, where

$$\tilde{V}_M = (h_\eta D' D)^{-1} \quad (\text{B.3})$$

$$\tilde{M} = h_\eta \tilde{V}_M D' Y_\tau \quad (\text{B.4})$$

- Draw $[\sigma_{1,j}^{(k)}, \sigma_{2,j}^{(k)}, b_j] \sim f(\sigma_1, \sigma_2|Y, x_t^{(k)}, \mu^{(k)}, d^{(k)}, \phi_1^{(k)}, \phi_2^{(k)}, b^{(k-1)})$ for $j = 1, 2, \dots, N + 1$. The following steps are based on sample periods from Tv_{j-1} to Tv_j of length of T_j .

Residual terms $[\hat{\eta}_t, \hat{\epsilon}_t]'$ can be obtained from the simulation smoother in the first step. Define $\eta_t^* = \eta_t \sim N(0, \sigma_{1,j}^2)$ and $\epsilon_t^* = -b_j \eta_t + \epsilon_t \sim N(0, \sigma_{2,j}^2)$, and we have the following:

$$B_j^{-1} \begin{bmatrix} \hat{\eta}_t \\ \hat{\epsilon}_t \end{bmatrix} = \begin{bmatrix} \hat{\eta}_t \\ -b_j \hat{\eta}_t + \hat{\epsilon}_t \end{bmatrix} = \begin{bmatrix} \hat{\eta}_t^* \\ \hat{\epsilon}_t^* \end{bmatrix} \sim N \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_{1,j}^2 & 0 \\ 0 & \sigma_{2,j}^2 \end{bmatrix} \right) \quad (\text{B.5})$$

where

$$B_j = \begin{bmatrix} 1 & 0 \\ b_j & 1 \end{bmatrix}$$

We assume independent priors for $h_{i,j}^{-1} = \sigma_{i,j}^2 \sim IG(s_{i0}^{-2}, \nu_{i0})$ with $i = 1, 2$. It's equivalent to assume a Gamma prior $G(s_{i0}^{-2}, \nu_{i0})$ for $h_{i,j}$. The posterior of $h_{i,j}$, in this

case, is $G(\tilde{s}_{i,j}^{-2}, \tilde{\nu}_{i,j})$, where for $i = 1, 2$

$$\tilde{\nu}_i = T_j + \nu_{i0} \quad (\text{B.6})$$

$$\tilde{s}_1^2 = \frac{\eta^{*'} \eta^* + \nu_{10} s_{10}^2}{\tilde{\nu}_1} \quad (\text{B.7})$$

$$\tilde{s}_2^2 = \frac{\epsilon^{*'} \epsilon^* + \nu_{20} s_{20}^2}{\tilde{\nu}_2} \quad (\text{B.8})$$

$$\eta^* = [\eta_{Tv_{j-1}+1}^{*(k)}, \dots, \eta_{Tv_j}^{*(k)}]' \quad (\text{B.9})$$

$$\epsilon^* = [\epsilon_{Tv_{j-1}+1}^{*(k)}, \dots, \epsilon_{Tv_j}^{*(k)}]' \quad (\text{B.10})$$

For b_j , the posterior sample can be obtained by the second row in (B.5) as a standard regression problem.

We can then construct the posterior samples for $[\sigma_{\eta,j}, \sigma_{\epsilon,j}, \rho_j]$ according to (2.12).

- Draw $Tm_j^{(k)} \in (Tm_{j-1}, Tm_{j+1})$ for $j = 1, 2, \dots, M$. We can draw $Tm_j^{(k)}$ from a multinomial distribution where

$$f(Tm_j | Y, \theta, Tm_{j-1}, Tm_{j+1}) = \frac{f(Y | Tm_j, Tm_{j-1}, Tm_{j+1}, \theta)}{\sum_{t=Tm_{j-1}+1}^{Tm_{j+1}-1} f(Y | Tm_j = t, Tm_{j-1}, Tm_{j+1}, \theta)}$$

and Y are observations between Tm_{j-1} and Tm_{j+1} .

$Tv_j^{(k)} \in (Tv_{j-1}, Tv_{j+1})$ for $j = 1, 2, \dots, N + 1$ can be sampled in a similar way.

Appendix C

APPENDIX TO CHAPTER 3

C.1 The Gibbs sampling used in Section 3.2

1. Draw $\{x_t^{(k)} : t = 1, \dots, T\} \sim f(x_1, \dots, x_T | Y, \theta^{(k-1)})$ using the simulation smoother developed by Durbin and Koopman (2002). We then obtain $\tau_t^{(k)}$, $\mu_t^{(k)}$ and $c_t^{(k)}$ as the first three elements in $x_t^{(k)}$.
2. Draw $[\phi_1^{(k)}, \phi_2^{(k)}] \sim f(\phi_1, \phi_2 | Y, x_t^{(k)}, \sigma_\epsilon^{(k-1)})$. Stack (3.13) by time, we have

$$Y_c = C\Phi + u$$

where $Y_c = \{c_t^{(k)}/\sigma_{\epsilon t}^{(k-1)}, t = 3, 4, \dots, T\}'$, $u = \{u_3, \dots, u_T\}' \stackrel{i.i.d.}{\sim} N(0, 1)$, $\Phi = \{\phi_1, \phi_2\}'$ and

$$C = \begin{bmatrix} c_2/\sigma_{\epsilon 3}^{(k-1)} & c_1/\sigma_{\epsilon 3}^{(k-1)} \\ \vdots & \vdots \\ c_{t-1}/\sigma_{\epsilon t}^{(k-1)} & c_{t-2}/\sigma_{\epsilon t}^{(k-1)} \\ \vdots & \vdots \\ c_{T-1}/\sigma_{\epsilon T}^{(k-1)} & c_{T-2}/\sigma_{\epsilon T}^{(k-1)} \end{bmatrix}$$

Given multivariate normal prior $N(\Phi_0, V_{\Phi_0})$ for Φ , the posterior distribution follows a multivariate normal distribution $N(\tilde{\Phi}, \tilde{V}_{\Phi})$, where

$$\tilde{V}_{\Phi} = (V_{\Phi_0}^{-1} + C'C)^{-1} \tag{C.1}$$

$$\tilde{\Phi} = \tilde{V}_{\Phi}(V_{\Phi_0}^{-1}\Phi_0 + C'Y_c) \tag{C.2}$$

3. Draw $[\sigma_{\nu i}^{(k)}, i = 1, 2, 3]$ and $\{h_{it} : i = 1, 2, 3, t = 1, 2, \dots, T\}$. Residual terms $[\hat{\eta}_t, \hat{\epsilon}_t, \hat{\zeta}_t]'$ can be obtained from the simulation smoother in the first step. According to (3.14), $\{h_{it} = \log(\sigma_{it}^2), i = 1, 2, 3\}$ follow random walks. We can use the method proposed by Kim et al. (1998) to sample $\{h_{it} : i = 1, 2, 3, t = 1, 2, \dots, T\}$.

We assume independent priors for $\sigma_{\nu_i}^2 \sim IG(s_{i0}^{-2}, \nu_{i0})$ with $i = 1, 2, 3$. It's equivalent to assume a Gamma prior $G(s_{i0}^{-2}, \nu_{i0})$ for $\sigma_{\nu_i}^{-2}$. The posterior of $\sigma_{\nu_i}^{-2}$, in this case, is $G(\tilde{s}_i^{-2}, \tilde{\nu}_i)$, where for $i = 1, 2$

$$\tilde{\nu}_i = T + \nu_{i0} \quad (\text{C.3})$$

$$\tilde{s}_i^2 = \frac{\hat{\nu}'\hat{\nu} + \nu_{i0}s_{i0}^2}{\tilde{\nu}_i} \quad (\text{C.4})$$

$$\hat{\nu} = [h_{i2}^{(k)} - h_{i1}^{(k)}, \dots, h_{iT}^{(k)} - h_{i(T-1)}^{(k)}]'. \quad (\text{C.5})$$

4. Draw $b_t^{(k)}$ and σ_w . Given the second row in (3.14) and the random walk assumption on the evolving of b_t , we have a state space regression to sample b_t as the state variable using similar approach as in Step 1. And we can sample σ_w given draws of $b_t^{(k)}$ using inverse Gamma posterior similar to how we sample σ_{ν_i} , $i = 1, 2, 3$:

$$\hat{\epsilon}_t = \hat{\eta}_t b_t + \epsilon_t^* \quad \epsilon_t^* \sim N(0, \sigma_{2t}^2) \quad (\text{C.6})$$

$$b_t = b_{t-1} + w_t, \quad w_t \sim N(0, \sigma_w^2) \quad (\text{C.7})$$

We can then construct the posterior samples for $[\sigma_{\eta_t}, \sigma_{\epsilon_t}, \sigma_{\zeta_t}, \rho_t]$ according to (3.8).