INTERSECTION LOCAL TIME
FOR POINTS OF INFINITE MULTIPLICITY

Richard F. Bass
Krzysztof Burdzy
Davar Khoshnevisan
University of Washington

Abstract. For each $a \in (0, 1/2)$, there exists a random measure β_a which is supported on the set of points where two-dimensional Brownian motion spends a units of local time. The measure β_a is carried by a set which has Hausdorff dimension equal to $2 - a$. A Palm measure interpretation of β_a is given.

1. Introduction and main results. Let $U(x, \varepsilon) \subset \mathbb{R}^2$ denote the circle with center x and radius ε. Let X be 2-dimensional Brownian motion starting from 0 and killed upon hitting $U(0, 1)$. Let N_ε^x be the number of excursions of X from x which hit $U(x, \varepsilon)$. For "most" points x we have $N_\varepsilon^x = 0$. If

(1.1) $$\lim_{\varepsilon \to 0} N_\varepsilon^x / |\log \varepsilon| = a$$

then we might say that "Brownian motion X spends a units of local time at x." Note that the normalization in (1.1) is different from that used in the definition of the local time for 1-dimensional Brownian motion. One of the things we will show is that points x with the property (1.1) do exist for some a.

Let $\mathrm{Dim}(A)$ denote the Hausdorff dimension of the set A. The carrying dimension of a measure μ is α if α is the infimum of γ's for which one can find a set $A = A_\gamma$ such that $\mu(A^c) = 0$ and the Hausdorff dimension of A is equal to γ.

1980 *Mathematics Subject Classification* (1985 *Revision*). 60G17, 60G57, 60J55, 60J65.
Key words and phrases. Brownian motion, local time, intersection local time, excursions, exit system.
Research partially supported by NSF grant DMS 91-00244 .
Running head: Points of infinite multiplicity

For $a \in (0, 2)$, let A_a be the set of x for which (1.1) holds. Our main results are contained in the following

Theorem 1.1. *(i) Let* $a \in (0, 1/2)$. *With probability* 1 *there exists a random measure* β_a, *which is carried by* A_a *and whose carrying dimension is equal to* $2 - a$, *a.s.*

(ii) For every $a > 0$, $\mathrm{Dim}(A_a) \leq 2 - a/e$, *a.s.*

See Theorems 5.1, 6.1 and 6.2 and Corollary 5.1 for more precise statements. We believe that Theorem 1.1 (i) remains true for all $a \in (0, 2)$ and that the sharp bound in (ii) is $2 - a$ although we are not able to prove this.

Theorem 5.2 contains a "Palm measure" decomposition of β_a which may be presented at the heuristic level as follows. Let Q_a^x be the distribution of X contitioned by the event that x is in the support of β_a. The trajectory of a process under Q_a^x is continuous and consists of three independent parts. The first part is an h-process in the unit disc starting from 0 and converging to x at its lifetime. The last part is a Brownian motion starting from x and killed upon hitting of the unit circle. The middle part consists of an infinite number of excursions from x. They are generated by a Poisson Point Process whose mean measure is the product of the Lebesgue measure on $[0, a]$ and an excursion law which may be described as the distribution of an h-process in the unit disc starting from x and converging to x.

We show (Corollary 5.1) that β_a is supported on the set of points visited infinitely often by the Brownian motion, and so we call β_a an "intersection local time for points of infinite multiplicity." It should be pointed out however that we use the term "intersection local time" in a way quite different from the usage for points of multiplicity k. Here β_a is a random measure on the state space while intersection local time for points of multiplicity k refers to a random measure on the Cartesian product of k copies of $[0, \infty)$.

Let us briefly address the question of "uniqueness." There may be, and in fact there are, many measures supported on the set of points satisfying (1.1). However there is only one measure satisfying Theorem 5.2 (see also Remark 5.2 (i)). The local times L_t^x of one-dimensional Brownian motion, viewed as a function of x, are the (unique) density of the occupation measure. Theorem 5.2 presents a result which is similar in spirit. It gives an integral formula which any "intersection local time" should satisfy.

For a rectangle D, the random variable $\beta_a(D)$ is constructed as the L^2-limit of a sequence $\beta_a^\varepsilon(D)$ as $\varepsilon \to 0$. For every ε, β_a^ε is a measure supported by points in the lattice with mesh ε. The measure β_a^ε has an atom of size ε^{2-a} at a point x in such a lattice if and only if the event

$$B_\varepsilon^x \stackrel{\text{df}}{=} \{L^{x,\varepsilon}(\tau) \geq a\varepsilon \log^2 \varepsilon - \varepsilon |\log \varepsilon| \log |\log \varepsilon|\}$$

occurs where $L^{x,\varepsilon}$ denotes the local time of X on $U(x,\varepsilon)$. The proof requires very accurate estimates for the probability of $B_{\varepsilon_1}^{x_1} \cap B_{\varepsilon_2}^{x_2}$ for various x_1, x_2, ε_1 and ε_2.

In order to simplify the estimates, we will ignore in our construction of β_a all points that satisfy (1.1) but lie outside $\{z : |z| \leq 1/16\}$. Hence β_a is really "intersection local time truncated to $\{z : |z| \leq 1/16\}$." Extending the definition to the whole path is not difficult (see Remark 5.1 (i)).

The measure β_a represents the amount of local time spent at points satisfying (1.1) up until the hitting time of $U(0,1)$. We also indicate how to define $\beta_a(dx,t)$, the amount of local time up until time t; see Remark 5.1 (ii).

The existence of points of infinite multiplicity is due to Dvoretzky, et al. (1958); see also Le Gall (1992). There is a large literature concerning intersection local times for points of finite multiplicity. See Dynkin (1988), Le Gall (1992), and Bass and Khoshnevisan (1992) and the references therein.

This paper owes a lot in terms of substance and presentation to lecture notes of Le Gall (1992).

We would like to thank the referee for a detailed report which helped us to clarify some important aspects of the paper.

2. Preliminary estimates. We will identify $\mathbb{R}^2$ and $\mathbb{C}$ and use both vector notation and complex analytic notation. All constants will be assumed strictly positive and finite unless specified otherwise. Their value may change from one proof to another.

Let $C_*[0,\infty)$ denote the space of functions which are continuous on some interval

$[0,b)$, $b \le \infty$, and then jump to an isolated coffin state Δ. Since the paths in $C_*[0,\infty)$ are not continuous, we will equip this space with the Skorohod topology and a compatible metric.

See Doob (1984) for the definitions and properties of Brownian motion, h-processes, harmonic functions, etc. Recall that harmonic functions, the Green function and 2-dimensional Brownian motion are invariant under conformal mappings (see Durrett (1984)).

The distribution of 2-dimensional Brownian motion starting from x will be denoted P^x and the corresponding expectation will be denoted E^x. The notation for the h-processes will be P_h^x and E_h^x. Most of the time X will denote a process with distribution P^x. The hitting time of a set A will be denoted T_A, i.e.,

$$T_A = T(A) = \inf\{t > 0 : X_t \in A\}.$$

We present a short review of some of the properties of h-processes. The proofs may be found in Doob (1984) and Meyer, Smyth and Walsh (1972).

Let $D \subset \mathbb{C}$ be a Greenian domain and h be a positive superharmonic function in D. Let $p_t^D(x,y)$ be the transition density for Brownian motion killed at $T(D^c)$ and

$$p_t^h(x,y) = p_t^D(x,y)h(y)/h(x).$$

Any process with the p_t^h-transition densities will be called an h-process (conditioned Brownian motion).

Let σ be the lifetime of X. Suppose that M is a closed subset of D and let $L = \sup\{t < \sigma : X(t) \in M\}$ be the last exit time from M. Let

$Y^1(t) = X(t), \quad t \in [0, T(M)),$

$Y^2(t) = X(T(M)+t), \quad t \in [0, \sigma - T(M)),$

$Y^3(t) = X(t), \quad t \in [0, L),$

$Y^4(t) = X(L+t), \quad t \in [0, \sigma - L),$

$Y^5(t) = X(\sigma - t), \quad t \in (0, \sigma).$

Under P_h^x, each process Y^k is an h_k-process in a domain D_k. We have

$D_1 = D_4 = D \backslash M$,

$D_2 = D_3 = D_5 = D$.

$h_1 = h_2 = h$.

h_3 is a potential supported by ∂M.

h_4 has the boundary values 0 on ∂M and the same boundary values as h on $\partial D \backslash M$.

h_5 is the Green function $G_D(x, \cdot\,)$ if $x \in D$ or a harmonic function with a pole at x if $x \in \partial D$.

The initial distributions of Y^1 and Y^3 are concentrated on $\{x, \Delta\}$. For the remaining initial distributions see Doob (1984).

Let $U(x, \varepsilon) \stackrel{\text{df}}{=} \{y \in \mathbb{R}^2 : |x - y| = \varepsilon\}$ and $\tau \stackrel{\text{df}}{=} T(U(0, 1))$. Let $U^+(x, \varepsilon)$ and $U^-(x, \varepsilon)$ denote the complement and interior, respectively, of the closed disc of radius ε about x. We will start with some estimates of the hitting probabilities. Recall that if $\varepsilon < |z - x| < r$ then

$$P^z(T(U(x, \varepsilon)) < T(U(x, r))) = \frac{\log(|z - x|/r)}{\log(\varepsilon/r)}. \tag{2.1}$$

Lemma 2.1. *(i) For every $x \in U^-(0, 1) \setminus \{0\}$ we have*

$$\lim_{\varepsilon \to 0} P^0(T(U(x, \varepsilon)) < \tau) \frac{\log \varepsilon}{\log |x|} = 1.$$

The convergence is uniform over every compact set $K \subset U^-(0, 1) \setminus \{0\}$.

(ii) For all $\varepsilon \in (0, 1/4)$ and x such that $|x| \in (\varepsilon, 1/4)$ we have

$$\log |x|/2 \log \varepsilon \leq P^0(T(U(x, \varepsilon)) < \tau) \leq 2 \log |x|/ \log \varepsilon.$$

(iii) For all x, y, $|x| < 1/16$, $|y| < 1/16$, and $\varepsilon < |x - y|/2$ we have

$$\log |x - y|/2 \log \varepsilon \leq P^z(T(U(y, \varepsilon)) < \tau) \leq 4 \log |x - y|/ \log \varepsilon$$

for each $z \in U(x, \varepsilon)$.

Proof. (i) By the symmetry of Brownian motion we may assume that x is real. The function $f_x(z) = (z - x)/(zx - 1)$ maps the unit disc onto itself, is one-to-one, $f_x(x) = 0$ and $f_x(0) = x$. By the conformal invariance of Brownian motion,

$$P^0(T(U(x, \varepsilon)) < \tau) = P^x(T(f_x(U(x, \varepsilon))) < \tau).$$

We have $|f_x'(z)| = |(x^2 - 1)/(zx - 1)^2|$ and in particular $|f_x'(x)| = |1 - x^2|^{-1}$. It is easy to see that $(f_x(x + w) - f_x(x))/w$ converges to $f_x'(x)$ as $w \to 0$ uniformly in $x \in K$. Hence, for each $\delta > 0$ there is $\varepsilon_0(\delta, K) > 0$ such that if $\varepsilon < \varepsilon_0$ and $x \in K$ then

$$U\left(0, \varepsilon(1 - \delta)/|1 - x^2|\right) \subset f_x(U(x, \varepsilon)) \subset U\left(0, \varepsilon(1 + \delta)/|1 - x^2|\right).$$

This and (2.1) imply that

$$\begin{aligned} P^0(T(U(x, \varepsilon)) < \tau) \frac{\log \varepsilon}{\log |x|} &= P^x(T(f_x(U(x, \varepsilon))) < \tau) \frac{\log \varepsilon}{\log |x|} \\ &\leq \frac{\log |x|}{\log(\varepsilon(1 + \delta)/|1 - x^2|)} \frac{\log \varepsilon}{\log |x|} \end{aligned}$$

for $\varepsilon < \varepsilon_0$. Thus

$$\limsup_{\varepsilon \to 0} \sup_{x \in K} P^0(T(U(x, \varepsilon)) < \tau) \frac{\log \varepsilon}{\log |x|} \leq 1.$$

The lower bound for the lim inf is also 1 for similar reasons.

(ii) For $x < 1/2$ we have by (2.1)

$$\begin{aligned} P^0(T(U(x, \varepsilon)) < \tau) &\leq P^0(T(U(x, \varepsilon)) < T(U(x, 2))) \\ &= \log |x/2| / \log(\varepsilon/2) \leq 2 \log |x| / \log \varepsilon. \end{aligned}$$

Similarly,

$$\begin{aligned} P^0(T(U(x, \varepsilon)) < \tau) &\geq P^0(T(U(x, \varepsilon)) < T(U(x, 1/2))) \\ &= \log |2x| / \log(2\varepsilon) \geq \log |x| / 2 \log \varepsilon. \end{aligned}$$

(iii) We apply (2.1) again to obtain

$$\begin{aligned}P^z(T(U(y,\varepsilon)) < \tau) &\le P^z(T(U(y,\varepsilon)) < T(U(y,2))) \\ &= \log(|z-y|/2)/\log(\varepsilon/2) \le \log(|x-y|/4)/\log(\varepsilon/2) \\ &\le 4\log|x-y|/\log\varepsilon\end{aligned}$$

and

$$\begin{aligned}P^z(T(U(y,\varepsilon)) < \tau) &\ge P^z(T(U(y,\varepsilon)) < T(U(y,3/4))) \\ &= \log(4|z-y|/3)/\log(4\varepsilon/3) \ge \log(8|x-y|/3)/\log(4\varepsilon/3) \\ &\ge \log|x-y|/2\log\varepsilon. \quad \square\end{aligned}$$

We offer now a brief review of excursion theory (see Blumenthal (1992), Burdzy (1987) or Maisonneuve (1975) for more information).

An excursion law H^x is a σ-finite measure on $C_*[0,\infty)$ which is strong Markov with the transition probabilities of a killed Brownian motion or an h-process. Unless specified otherwise, we will assume that the transition probabilities are those of a killed Brownian motion. Moreover, the H^x-measure of the set of trajectories not starting from x is equal to 0.

Now we will discuss a special case of an exit system. Let $L_t^{x,\varepsilon}$ denote the local time of X on $U(x,\varepsilon)$ (see Itô and McKean (1974) or Karatzas and Shreve (1988) for the definition). We will normalize $L_t^{x,\varepsilon}$ later in such a way that it satisfies the "exit system formula" (2.3) below. Let excursion laws $H^y = H^y_{x,\varepsilon}$ from $U(x,\varepsilon)$ be defined by

$$H^y(A) = \lim_{\substack{z\to y\\ z\in U^+(x,\varepsilon)}} \frac{P_*^z(A)}{\operatorname{dist}(z,U(x,\varepsilon))} + \lim_{\substack{z\to y\\ z\in U^-(x,\varepsilon)}} \frac{P_*^z(A)}{\operatorname{dist}(z,U(x,\varepsilon))} \tag{2.2}$$

where P_*^z denotes the distribution of Brownian motion killed upon hitting $U(x,\varepsilon)$. Let $\mu(t) = \inf\{s>0 : L_s^{x,\varepsilon} > t\}$, $\eta_t = \inf\{s>t : X(s)\in U(x,\varepsilon)\} - t$ and

$$e_t(s) = \begin{cases} X(t+s) & \text{if } s<\eta_t \text{ and } X_t \in U(x,\varepsilon), \\ \Delta & \text{otherwise.}\end{cases}$$

Then

$$(2.3) \qquad E^z \sum_{0<t<\infty} Z_t f(e_t) = E^z \int_0^\infty Z_t H^{X(t)}(f) dL_t^{x,\varepsilon} = E^z \int_0^\infty Z_{\mu(t)} H^{X(\mu(t))}(f) dt$$

for all $z \in \mathbb{C}$, all positive predictable processes Z and positive measurable functions f defined on $C_*[0,\infty)$ which vanish on paths equal identically to Δ. We will call (2.3) the "exit system formula" and (L, H) an exit system from $U(x,\varepsilon)$. In particular, the local time $L_t^{x,\varepsilon}$ will be always normalized (unless specified otherwise) so that $(L^{x,\varepsilon}, H)$ forms an exit system from $U(x,\varepsilon)$ and the excursions laws H satisfy (2.2).

Formula (2.2) can be used to define an excursion law for an h-process. The exit system formula also holds for h-processes. It is not hard to check that the normalization of the local time imposed by the exit system formula is identical for any h-process (including Brownian motion).

The following lemma lists a few consequences of the exit system formula which will be used repeatedly throughout the paper. Similar lemma holds also for a Poisson point process of excursions from a single point, which is discussed in Section 5.

Lemma 2.2. *(i) Suppose that for some $A \subset C_*[0,\infty)$ and every $y \in U(x,\varepsilon)$ we have $H^y(A) \in (c_1, c_2)$. Then the number of excursions $e_t \in A$ such that $L_t^{x,\varepsilon} < c_3$ is minorized by a Poisson distribution with the mean $c_1 c_3$ and majorized by a Poisson distribution with the mean $c_2 c_3$.*

(ii) The expected amount of time spent by excursions e_t such that $L_t^{x,\varepsilon} < c$ in a set D is majorized by

$$c \sup_{y \in U(x,\varepsilon)} \int_D G(y,z) dz$$

where $G(y, \cdot)$ is the density of the expected occupation time for H^y. In the case when $D = \mathbb{C}$, we have an estimate for the expected sum of lifetimes of excursions.

Proof. The lemma follows easily from (2.3). □

Lemma 2.3. *Suppose $(L^{x,\varepsilon}, H)$ is an exit system from $U(x,\varepsilon)$ normalized as in (2.2).*

(i) For each compact set $K \subset U^-(0,1)$ and each $\delta > 0$

$$\liminf_{\varepsilon\to 0} \inf_{x\in K} \inf_{y\in U(x,\varepsilon)} H^y(\tau < \infty)(\varepsilon|\log((1-\delta)\varepsilon/|1-x^2|)|) \geq 1 \tag{2.4}$$

and

$$\limsup_{\varepsilon\to 0} \sup_{x\in K} \sup_{y\in U(x,\varepsilon)} H^y(\tau < \infty)(\varepsilon|\log((1+\delta)\varepsilon/|1-x^2|)|) \leq 1. \tag{2.5}$$

(ii) For all $x \in U^-(0,1/2)$, $\varepsilon < 1/2$ and $y \in U(x,\varepsilon)$ we have

$$\frac{1}{\varepsilon|\log(\varepsilon/2)|} \leq H^y(\tau < \infty) \leq \frac{1}{\varepsilon|\log(2\varepsilon)|}.$$

(iii) For all $y \in U(x,\varepsilon)$ and $r > 1$ we have

$$H^y(T(U(x,r\varepsilon)) < \infty) = \frac{1}{\varepsilon \log r}.$$

Proof. (i) Recall the function $f_x(z) = (z-x)/(zx-1)$ from the proof of Lemma 2.1. It maps circles onto circles so $f_x(U(x,\varepsilon))$ is a circle with radius r_0 which contains 0 in its interior, although its center z_0 is not necessarily 0. The function f_x is analytic in $U^-(0,1)$ so for each compact set K there is $\varepsilon_0 > 0$ and $M < \infty$ such that for $\varepsilon < \varepsilon_0$ and $z \in U^-(x,2\varepsilon)$ we have $|f'_x(z) - f'_x(x)| < \varepsilon M$. Hence, for small ε,

$$\varepsilon(|f'_x(x)| - \varepsilon M) \leq r_0 \leq \varepsilon(|f'_x(x)| + \varepsilon M) \tag{2.6}$$

and in particular the radius r_0 is less than $2\varepsilon|f'_x(x)|$. It follows that

$$U^-(0,1) \subset U^-(z_0, 1 + 2\varepsilon|f'_x(x)|).$$

This and (2.1) imply for $|z - x| > \varepsilon$, $z \in U^-(0,1)$,

$$\begin{aligned} P^z(\tau < T(U(x,\varepsilon))) &= P^{f_x(z)}(\tau < T(f_x(U(x,\varepsilon)))) \\ &\geq \left| \frac{\log(|f_x(z) - z_0|/r_0)}{\log(r_0/(1+2\varepsilon|f'_x(x)|))} \right|. \end{aligned}$$

For fixed x, ε and $z_1 \in U(x,\varepsilon)$ we have

$$\lim_{\substack{z \to z_1 \\ z \in U^+(x,\varepsilon)}} \frac{|\log(|f_x(z) - z_0|/r_0)|}{\operatorname{dist}(f_x(z), f_x(U(x,\varepsilon)))} = 1/r_0$$

and, therefore,

$$\begin{aligned} \lim_{\substack{z \to z_1 \\ z \in U^+(x,\varepsilon)}} \frac{P^z(\tau < T(U(x,\varepsilon)))}{\operatorname{dist}(z, U(x,\varepsilon))} &= \lim_{\substack{z \to z_1 \\ z \in U^+(x,\varepsilon)}} \frac{P^{f_x(z)}(\tau < T(f_x(U(x,\varepsilon))))}{\operatorname{dist}(z, U(x,\varepsilon))} \\ &= \lim_{\substack{z \to z_1 \\ z \in U^+(x,\varepsilon)}} \frac{|f'_x(z_1)| P^{f_x(z)}(\tau < T(f_x(U(x,\varepsilon))))}{\operatorname{dist}(f_x(z), f_x(U(x,\varepsilon)))} \\ &\geq \liminf_{\substack{z \to z_1 \\ z \in U^+(x,\varepsilon)}} \frac{|f'_x(z_1)|}{\operatorname{dist}(f_x(z), f_x(U(x,\varepsilon)))} \left| \frac{\log(|f_x(z) - z_0|/r_0)}{\log(r_0/(1+2\varepsilon|f'_x(x)|))} \right| \\ &\geq \frac{|f'_x(z_1)|}{r_0 |\log(r_0/(1+2\varepsilon|f'_x(x)|))|}. \end{aligned}$$

This and (2.6) imply that for small ε and $z_1 \in U(x,\varepsilon)$

$$\begin{aligned} H^{z_1}(\tau < \infty)(\varepsilon|\log((1-\delta)\varepsilon/|1-x^2|)|) &\geq \frac{|f'_x(z_1)|\varepsilon|\log((1-\delta)\varepsilon/|1-x^2|)|}{r_0|\log(r_0/(1+2\varepsilon|f'_x(x)|))|} \\ &\geq \frac{(|f'_x(x)| - \varepsilon M)\varepsilon|\log((1-\delta)\varepsilon/|1-x^2|)|}{\varepsilon(|f'_x(x)| + \varepsilon M)|\log(\varepsilon(|f'_x(x)| - \varepsilon M)/(1+2\varepsilon|f'_x(x)|))|} \\ &= \frac{(|f'_x(x)| - \varepsilon M)}{(|f'_x(x)| + \varepsilon M)} \frac{|\log((1-\delta)\varepsilon|f'_x(x)|)|}{|\log(\varepsilon(|f'_x(x)| - \varepsilon M)/(1+2\varepsilon|f'_x(x)|))|} \\ &\geq (1 - 3\varepsilon M/|f'_x(x)|) \left(1 + \frac{|\log(1-\delta/2)|}{|\log(\varepsilon(|f'_x(x)| - \varepsilon M)/(1+2\varepsilon|f'_x(x)|))|} \right). \end{aligned}$$

The last expression is greater than 1 for all $\varepsilon < \varepsilon_1$ and all $x \in K$. This clearly implies (2.4). The proof of (2.5) is analogous.

(ii) By (2.1) we have

$$P^z(\tau < T(U(x,\varepsilon))) \le P^z(T(U(x,1/2)) < T(U(x,\varepsilon))) = (\log(2\varepsilon) - \log(2|x-z|))/\log(2\varepsilon).$$

It follows that

$$H^y(\tau < \infty) = \lim_{\substack{z\to y\\ z\in U^+(x,\varepsilon)}} (|z-x|-\varepsilon)^{-1} P^z(\tau < T(U(x,\varepsilon))) \le 1/(\varepsilon|\log(2\varepsilon)|).$$

The other inequality may be proved in an analogous way using the fact that $U(0,1) \subset U^-(x,2)$.

(iii) Recall that

$$P^z(T(U(x,r\varepsilon)) < T(U(x,\varepsilon))) = \frac{|\log|z-x| - \log\varepsilon|}{|\log(r\varepsilon) - \log\varepsilon|}$$

if $\varepsilon < |z-x| < r\varepsilon$. Hence

$$\begin{aligned} H^y(T(U(x,r\varepsilon)) < \infty) &= \lim_{\substack{z\to y\\ z\in U^+(x,\varepsilon)}} (|z-x|-\varepsilon)^{-1} P^z(T(U(x,r\varepsilon)) < T(U(x,\varepsilon))) \\ &= 1/(\varepsilon \log r). \quad \square \end{aligned}$$

Lemma 2.4. *(i) Suppose that (L^j, H_j) is an exit system from $U(x^j,\varepsilon_j)$, $j=1,2$ normalized as in (2.2). For each $\eta > 0$ there is $c > 0$ such that*

$$\frac{H_1^{y^1}(X(T(U(0,r))) \in dv)}{H_2^{y^2}(X(T(U(0,r))) \in dv)} \in \left((1-\eta)\frac{\varepsilon_2 \log\varepsilon_2}{\varepsilon_1\log\varepsilon_1}, (1+\eta)\frac{\varepsilon_2\log\varepsilon_2}{\varepsilon_1\log\varepsilon_1}\right)$$

for all $y^j \in U(x^j,\varepsilon_j)$ and $v \in U(0,r)$ provided r is chosen so that $\varepsilon_j/r < c$ and $|x^j|/r < c$ for $j=1,2$.

(ii) Let h_1 be the positive harmonic function in the annulus $A \stackrel{\text{df}}{=} U^-(x,2) \cap U^+(x,\varepsilon)$ which has boundary values equal to 1 on $U(x,\varepsilon)$ and 0 elsewhere. For $y \in U(x,\varepsilon)$ let H_1^y

be the excursion law in A with the transition probabilities of an h_1-process. For each $\eta > 0$ there is $c < \infty$ such that

$$\frac{H_1^{y^1}(X(T(U(x, r_1\varepsilon))) \in dv)}{H_1^{y^2}(X(T(U(x, r_1\varepsilon))) \in dv)} \in (1 - \eta, 1 + \eta)$$

for all $y^j \in U(x, \varepsilon)$, $j = 1, 2$, and $v \in U(x, r_1\varepsilon)$ provided $r_1 \geq c$ and $r_1\varepsilon < 2$.

Proof. (i) By scaling, we may assume that $r = 1$. Suppose that $\delta > 0$. Using the Harnack principle find $c_1 > 0$ so small that

$$\frac{P^{z^1}(X(\tau) \in dv)}{P^{z^2}(X(\tau) \in dv)} \in (1 - \delta, 1 + \delta) \tag{2.7}$$

for all $v \in U(0, r)$ and $z^j \in U^-(0, c_1)$, $j = 1, 2$.

Brownian motion conditioned to hit $v \in U(0, 1)$ is an h_v-process, where h_v is a positive harmonic function in $U^-(0, 1)$ which vanishes everywhere on the boundary except for a pole at v. By the Harnack principle,

$$\frac{\sup_{z \in U^-(0, c_1)} h_v(z)}{\inf_{z \in U^-(0, c_1)} h_v(z)} \leq c_2 < \infty.$$

Find $c_3 < c_1/8$ so small that

$$P^z(T(U(x, \varepsilon)) < \tau) < \delta/c_2$$

for all $z \in A_1 \stackrel{\text{df}}{=} U^-(0, c_1) \cap U^+(0, c_1/4)$, $|x| < c_3$ and $\varepsilon < c_3$. Then

$$P^z_{h_v}(T(U(x, \varepsilon)) < \tau) \leq \sup_{y \in U(x, \varepsilon)} h_v(y) P^z(T(U(x, \varepsilon)) < \tau)/h_v(z) < c_2\delta/c_2 = \delta$$

for all $v \in U(0, r)$, $z \in A_1$, $|x| < c_3$ and $\varepsilon < c_3$. This and (2.7) imply that

$$\frac{P^{z^1}(\tau < T(U(x^1, \varepsilon_1)), X(\tau) \in dv)}{P^{z^2}(\tau < T(U(x^2, \varepsilon_2)), X(\tau) \in dv)} \in ((1 - \delta)^2, (1 + \delta)/(1 - \delta)) \tag{2.8}$$

for all $v \in U(0,r)$, $z^j \in A_1$, $|x^j| < c_3$ and $\varepsilon_j < c_3$, $j = 1,2$.

The circles $U(x^j, c_1/2)$ lie in A_1 provided $|x^j| < c_3 < c_1/8$ for $j = 1,2$. According to Lemma 2.3 (iii), we have

$$\frac{H_1^{y^1}(T(U(x^1,c_1/2)) < \infty)}{H_2^{y^2}(T(U(x^2,c_1/2)) < \infty)} = \frac{\varepsilon_2 \log(c_1/2\varepsilon_2)}{\varepsilon_1 \log(c_1/2\varepsilon_1)}$$

for $y^j \in U(x^j, \varepsilon_j)$, $\varepsilon_j < c_3$, $j = 1,2$. This, (2.8) and the strong Markov property applied at the hitting time of $U(x^j, c_1/2)$ imply that

$$\frac{H_1^{y^1}(X(T(U(0,1))) \in dv)}{H_2^{y^2}(X(T(U(0,1))) \in dv)} \in \left((1-\delta)^2 \frac{\varepsilon_2 \log(c_1/2\varepsilon_2)}{\varepsilon_1 \log(c_1/2\varepsilon_1)}, \frac{(1+\delta)}{(1-\delta)} \frac{\varepsilon_2 \log(c_1/2\varepsilon_2)}{\varepsilon_1 \log(c_1/2\varepsilon_1)}\right).$$

Given $\eta > 0$, we can take δ and c_1 small enough so that

$$\left((1-\delta)^2 \frac{\varepsilon_2 \log(c_1/2\varepsilon_2)}{\varepsilon_1 \log(c_1/2\varepsilon_1)}, \frac{(1+\delta)}{(1-\delta)} \frac{\varepsilon_2 \log(c_1/2\varepsilon_2)}{\varepsilon_1 \log(c_1/2\varepsilon_1)}\right) \subset \left((1-\eta)\frac{\varepsilon_2 \log \varepsilon_2}{\varepsilon_1 \log \varepsilon_1}, (1+\eta)\frac{\varepsilon_2 \log \varepsilon_2}{\varepsilon_1 \log \varepsilon_1}\right),$$

and hence the proof of (i) is complete.

(ii) Let H_2^y be the excursion law from $U(x,\varepsilon)$ with the transition probabilities of Brownian motion killed upon exiting A. It follows easily from part (i) that there is $c < \infty$ such that

$$\frac{H_2^{y^1}(X(T(U(x,r_1\varepsilon))) \in dv)}{H_2^{y^2}(X(T(U(x,r_1\varepsilon))) \in dv)} \in (1-\delta, 1+\delta) \tag{2.9}$$

for all $y^j \in U(x,\varepsilon)$, $j = 1,2$, and $v \in U(x, r_1\varepsilon)$ provided $r_1 \geq c$ and $r_1\varepsilon < 2$. Let P_*^z denote the distribution of Brownian motion killed at the hitting time of A^c. We have

$$\begin{aligned} H_1^y(X(T(U(x,r_1\varepsilon))) \in dv) &= \lim_{\substack{z \to y \\ z \in U^+(x,\varepsilon)}} \frac{P_{h_1}^z(X(T(U(0,r_1\varepsilon))) \in dv)}{\operatorname{dist}(z, U(x,\varepsilon))} \\ &= \lim_{\substack{z \to y \\ z \in U^+(x,\varepsilon)}} \frac{h_1(v) P_*^z(X(T(U(0,r_1\varepsilon))) \in dv)/h_1(z)}{\operatorname{dist}(z, U(x,\varepsilon))} \\ &= h_1(v) H_2^y(X(T(U(x,r_1\varepsilon))) \in dv)/h_1(y). \end{aligned}$$

This, (2.9) and the fact that h_1 is constant on circles with center x imply part (ii) of the lemma. $\square$

Lemma 2.5. *Let (L, H) denote the exit system from $U(x, \varepsilon)$ normalized as in (2.2). There exist $c_1 > 0$ and $c_2 < \infty$ such that if $|x| \le 1/4$, $|y| \le 1/4$, $|x - y| > 2\varepsilon$ and $\varepsilon/|x - y| < c_1$ then*

$$H^z(T(U(y, \varepsilon)) < \tau \mid \tau < \infty) < c_2 \log|x - y| / \log \varepsilon$$

for every $z \in U(x, \varepsilon)$. Moreover, for each $\delta > 0$ there is $c_3 > 0$ such that if $\varepsilon/|x - y| < c_3$ and $|x - y| < c_3$ then

$$H^z(T(U(y, \varepsilon)) < \tau \mid \tau < \infty) < (1 + \delta) \log|x - y| / \log \varepsilon$$

for every $z \in U(x, \varepsilon)$.

Proof. We have by (2.1)

$$P^x(T(U(y, \varepsilon)) < \tau) < P^x(T(U(y, \varepsilon)) < T(U(y, 2))) = \frac{\log(|x - y|/2)}{\log(\varepsilon/2)}.$$

It follows that the probability that the first excursion from $U(x, \varepsilon)$ which reaches $U(0, 1)$ also hits $U(y, \varepsilon)$ is less than $\log(|x - y|/2)/\log(\varepsilon/2)$. It follows from the exit system formula that for at least one $z \in U(x, \varepsilon)$

$$H^z(T(U(y, \varepsilon)) < \tau \mid \tau < \infty) < \log(|x - y|/2)/\log(\varepsilon/2). \tag{2.10}$$

Let $U_1 \stackrel{\mathrm{df}}{=} U(x, |x - y|/2)$ and let T_1 be the hitting time of U_1. By the strong Markov property applied at T_1,

$$H^z(T(U(y, \varepsilon)) < \tau \mid \tau < \infty) = \frac{\int_{U_1} H^z(X(T_1) \in dv) P^v(T(U(y, \varepsilon)) < \tau < T(U(x, \varepsilon)))}{\int_{U_1} H^z(X(T_1) \in dv) P^v(\tau < T(U(x, \varepsilon)))}.$$

This and (2.10) combined with Lemma 2.4 (i) show that for each $\delta > 0$ there is $c_1 > 0$ such that if $\varepsilon/|x - y| < c_1$ then

$$H^z(T(U(y, \varepsilon)) < \tau \mid \tau < \infty) < (1 + \delta)\log(|x - y|/2)/\log(\varepsilon/2) \tag{2.11}$$

for all $z \in U(x, \varepsilon)$. The quantity on the right hand side of (2.11) is bounded by $c_2 \log|x - y|/\log \varepsilon$ for all $\varepsilon < 1/2$ and $|x - y| < 1/2$. It is bounded by $(1 + 2\delta)\log|x - y|/\log \varepsilon$ provided ε and $|x - y|$ are sufficiently small. □

Lemma 2.6. *For $x \in U^-(0,1)$ let h_x denote the positive harmonic function in $U^-(0,1) \setminus \{x\}$ which vanishes on $U(0,1)$ and has a pole at x. Let (L,H) be an exit system from $U(z,\varepsilon)$ (normalized as in (2.2)) for the h_x-process and let σ denote the lifetime of an excursion. For each $\delta > 0$ there are $r > 0$ and $\varepsilon_0 > 0$ such that if $x \in U^-(0,1/2)$, $\varepsilon < \varepsilon_0$, $y \in U(z,\varepsilon)$ and $|z-x| < r\varepsilon$ then*

$$H^y(X(\sigma-) = x) \in (1/((1+\delta)(\varepsilon|\log\varepsilon|)), 1/((1-\delta)(\varepsilon|\log\varepsilon|))).$$

Proof. Suppose that $\gamma > 0$ is small and $|v-z| = \varepsilon - \gamma$. Then

$$P^v(T(U(z,\varepsilon/2)) < T(U(z,\varepsilon))) = \frac{|\log(\varepsilon-\gamma) - \log\varepsilon|}{|\log(\varepsilon/2) - \log\varepsilon|} = \frac{|\log(\varepsilon-\gamma) - \log\varepsilon|}{\log 2}.$$

Note that $|v-x| \geq (1/2 - r)\varepsilon$ for $v \in U(z,\varepsilon/2)$. Since $U^-(z,\varepsilon) \subset U^-(x,(1+r)\varepsilon)$ we have for small $\rho > 0$ and $v \in U(z,\varepsilon/2)$

$$\begin{aligned} P^v(T(U(x,\rho)) < T(U(z,\varepsilon))) &\leq P^v(T(U(x,\rho)) < T(U(x,(1+r)\varepsilon))) \\ &= \frac{|\log(|v-x|) - \log((1+r)\varepsilon)|}{|\log\rho - \log((1+r)\varepsilon)|} \\ &\leq \frac{|\log((1/2-r)\varepsilon) - \log((1+r)\varepsilon)|}{|\log\rho - \log((1+r)\varepsilon)|} \\ &= \frac{|\log(1/2-r) - \log(1+r)|}{|\log\rho - \log((1+r)\varepsilon)|}. \end{aligned}$$

The strong Markov property applied at $T(U(z,\varepsilon/2))$ implies for $v \in U(z,\varepsilon-\gamma)$, small ρ and $r < r_0$

$$\begin{aligned} P^v(T(U(x,\rho)) < T(U(z,\varepsilon))) &\leq \frac{|\log(\varepsilon-\gamma) - \log\varepsilon|}{\log 2} \frac{|\log(1/2-r) - \log(1+r)|}{|\log\rho - \log((1+r)\varepsilon)|} \\ &\leq (1+\delta)\frac{|\log(\varepsilon-\gamma) - \log\varepsilon|}{|\log\rho|}. \end{aligned} \tag{2.12}$$

We have $\lim_{y\to x} h_x(y)/|\log|y-x|| = c_1$ for some $c_1 > 0$. It is easy to check (see Doob (1984) or Durrett (1984)) that for $x \in U^-(0,1/2)$ and $y \in U^-(0,1)$

$$c_1|\log(2|y-x|)| \leq h_x(y) \leq c_1|\log(|y-x|/2)|.$$

This and (2.12) imply for $v \in U(z, \varepsilon - \gamma)$

$$\begin{aligned} P^v_{h_x}(T(U(x,\rho)) < T(U(z,\varepsilon))) &\le \frac{\sup_{y\in U(x,\rho)} h_x(y)}{h_x(v)} (1+\delta) \frac{|\log(\varepsilon-\gamma) - \log\varepsilon|}{|\log\rho|} \\ &\le \frac{c_1 |\log(\rho/2)|}{c_1 |\log(2(\varepsilon-\gamma))|} (1+\delta) \frac{|\log(\varepsilon-\gamma) - \log\varepsilon|}{|\log\rho|}. \end{aligned}$$

Recall that σ denotes the lifetime of the process. Let $\rho \to 0$ in the last formula to obtain

$$P^v_{h_x}(\sigma < T(U(z,\varepsilon))) \le \frac{(1+\delta)|\log(\varepsilon-\gamma) - \log\varepsilon|}{|\log(2(\varepsilon-\gamma))|}.$$

Hence for $y \in U(z, \varepsilon)$ and $\varepsilon < \varepsilon_0$

$$\begin{aligned} H^y(X(\sigma-) = x) &\le \limsup_{\gamma\to 0} \gamma^{-1} \sup_{v\in U(z,\varepsilon-\gamma)} P^v_{h_x}(\sigma < T(U(z,\varepsilon))) \\ &\le \lim_{\gamma\to 0} \frac{\gamma^{-1}(1+\delta)|\log(\varepsilon-\gamma) - \log\varepsilon|}{|\log(2(\varepsilon-\gamma))|} \\ &= \frac{(1+\delta)}{\varepsilon|\log(2\varepsilon)|} \le \frac{(1+2\delta)}{\varepsilon|\log\varepsilon|}. \end{aligned}$$

This proves the upper bound in the lemma. The lower bound may be derived in a similar way. □

The following definitions are needed for Lemma 2.7 and Proposition 5.1 below. Let $(L^{x,\varepsilon}, H_1)$ be an exit system from $U(x, \varepsilon)$ for (unconditioned) Brownian motion normalized as in (2.2) and

$$\overline{H}^y_1 \stackrel{\text{df}}{=} (a\varepsilon \log^2\varepsilon - \varepsilon|\log\varepsilon| \log|\log\varepsilon|) H^y_1.$$

Let $\widetilde{H}^y_1$ be the excursion law $\overline{H}^y_1$ truncated to excursions which do not hit $U(0,1)$.

Lemma 2.7. *Let $\widetilde{G}_\varepsilon(y, \cdot\,)$ be the density of the expected occupation measure for the excursion law $\widetilde{H}^y_1$. Then*

$$\int_{U^-(x,r)} \widetilde{G}_\varepsilon(y,z) dz$$

goes to 0 as $r \to 0$ uniformly in $\varepsilon < r$ and $y \in U(x, \varepsilon)$.

Proof. The lemma involves estimates for Martin kernels, i.e., Poisson kernels. Instead of using known formulae we will provide our own estimates to ensure that we have the correct normalization.

Let $G_D(\,\cdot\,,\,\cdot\,)$ be the Green function for a domain D. The density of the expected occupation time for Brownian motion starting at z and killed at the hitting time of D^c is equal to $c_1 G_D(z,\,\cdot\,)$. If z is real then

$$G_{U^-(0,1)}(z, y) = \left| \log \left| \frac{z - y}{zy - 1} \right| \right|.$$

Hence, by scaling, if $\operatorname{Im} z = \operatorname{Im} x$ then

$$G_{U^-(x,\varepsilon)}(z, y) = \left| \log \left| \varepsilon \frac{z - y}{(z - x)(y - x) - \varepsilon^2} \right| \right|.$$

Thus for $y \in U^-(x, \varepsilon)$,

$$\begin{aligned}
\frac{\widetilde{G}_\varepsilon(x + \varepsilon, y)/c_1}{a\varepsilon \log^2 \varepsilon - \varepsilon |\log \varepsilon| \log |\log \varepsilon|} &= \lim_{\delta \to 0+} \delta^{-1} G_{U^-(x,\varepsilon)}(x + \varepsilon - \delta, y) \\
&= \lim_{\delta \to 0+} \delta^{-1} \left| \log \left| \varepsilon \frac{x + \varepsilon - \delta - y}{(x + \varepsilon - \delta - x)(y - x) - \varepsilon^2} \right| \right| \\
&= \lim_{\delta \to 0+} \delta^{-1} \left| \log \left| 1 + \delta \frac{y - x + \varepsilon}{\varepsilon(y - x) - \varepsilon^2 - \delta(y - x)} \right| \right| \\
&= \varepsilon^{-1} \left| \operatorname{Re} \frac{y - x + \varepsilon}{y - x - \varepsilon} \right|.
\end{aligned}$$

Simple calculations then show that

$$\widetilde{G}_\varepsilon(x + \varepsilon, y) \le 2ac_1 \varepsilon \log^2 \varepsilon / |y - x - \varepsilon|.$$

It follows that

$$\int_{U^-(x,\varepsilon)} \widetilde{G}_\varepsilon(x + \varepsilon, y) dy \le 8\pi a c_1 \varepsilon^2 \log^2 \varepsilon. \tag{2.13}$$

The formula for the Green function in $U^+(0,1)$ has the same form as the one for $U^-(0,1)$, i.e., for real z

$$G_{U^+(0,1)}(z,y) = \left|\log\left|\frac{z-y}{zy-1}\right|\right|$$

and, therefore, for z such that $\operatorname{Im} z = \operatorname{Im} x$,

$$G_{U^+(x,\varepsilon)}(z,y) = \left|\log\left|\varepsilon\frac{z-y}{(z-x)(y-x)-\varepsilon^2}\right|\right|.$$

Now we are dealing with an excursion law which is a truncated version of the one with Brownian transition probabilities, so the same calculation as before gives us only an upper bound, i.e.,

$$\frac{\widetilde{G}_\varepsilon(x+\varepsilon,y)/c_1}{a\varepsilon\log^2\varepsilon - \varepsilon|\log\varepsilon|\log|\log\varepsilon|} \le \varepsilon^{-1}\left|\operatorname{Re}\frac{y-x+\varepsilon}{y-x-\varepsilon}\right|.$$

This yields the following bounds

$$\widetilde{G}_\varepsilon(x+\varepsilon,y) \le ac_1(1+2\varepsilon/|y-x-\varepsilon|)\log^2\varepsilon$$

and

$$\int_{U^-(x,\sqrt{\varepsilon})\cap U^+(x,\varepsilon)} \widetilde{G}_\varepsilon(x+\varepsilon,y)dy \le ac_1(\pi\varepsilon + 4\pi\varepsilon^{3/2})\log^2\varepsilon. \tag{2.14}$$

Suppose that $U(x,\varepsilon) \subset U^-(0,1)$ and let h_1 be the positive harmonic function in $A_\varepsilon \stackrel{\text{df}}{=} U^-(0,1)\cap U^+(x,\varepsilon)$ which vanishes on $U(0,1)$ and is equal to 1 on $U(x,\varepsilon)$. Since

$$G_{U^-(x,r)}(0,y) = |\log(|y-x|/r)|,$$

the Harnack principle implies that

$$G_{U^-(x,r)}(z,y) \le c_2|\log(|y-x|/r)|$$

if $|y-x| \ge \sqrt{\varepsilon}$ and $|z-x| \le \sqrt{\varepsilon}/2$. The Green function $G^{h_1}_{U^-(x,r)}$ for the h_1-process (which is defined by $G^{h_1}_{U^-(x,r)}(z,y) = h_1(y)G_{U^-(x,r)}(z,y)/h_1(z)$) must therefore satisfy

$$G^{h_1}_{U^-(x,r)}(z,y) \le h_1(y)c_2|\log(|y-x|/r)|/h_1(z).$$

It is easy to check that

$$c_3 \log |y-x| / \log \varepsilon \le h_1(y) \le c_4 \log |y-x| / \log \varepsilon$$

if $|y| < (1-|x|)/2$ and so

$$G^{h_1}_{U^-(x,r)}(z,y) \le (2c_4 \log |y-x| / \log \varepsilon) c_2 |\log(|y-x|/r)| / c_3 \log 2 \tag{2.15}$$

for $z \in U(x, \sqrt{\varepsilon}/2)$. For $y \in U(x,\varepsilon)$, the $\overline{H}^y_1$-probability of hitting $U(x, \sqrt{\varepsilon}/2)$ is equal to

$$(a\varepsilon \log^2 \varepsilon - \varepsilon |\log \varepsilon| \log |\log \varepsilon|)/(\varepsilon |\log(2\sqrt{\varepsilon})|)$$

by Lemma 2.3 (iii) and the corresponding probability for $\widetilde{H}^y_1$ cannot be larger than that. By the strong Markov property applied at $T(U(x, \sqrt{\varepsilon}/2))$ and (2.15),

$$\widetilde{G}_\varepsilon(x+\varepsilon, y) \le c_1 \frac{a\varepsilon \log^2 \varepsilon - \varepsilon |\log \varepsilon| \log |\log \varepsilon|}{\varepsilon |\log(2\sqrt{\varepsilon})|} \frac{c_5 |\log |y-x| \log(|y-x|/r)|}{|\log \varepsilon|}.$$

This leads to

$$\int_{U^-(x,r)\cap U^+(x,\sqrt{\varepsilon})} \widetilde{G}_\varepsilon(x+\varepsilon, y) dy \le c_6 r.$$

This, (2.13) and (2.14) prove the lemma for $z = x+\varepsilon$. The estimates hold for all $z \in U(x,\varepsilon)$ by symmetry. □

3. Moment estimates. Recall that $(L^{x,\varepsilon}, H)$ denotes an exit system of X from $U(x,\varepsilon)$ normalized as in (2.2). For any $a > 0$ let $Y^{x,\varepsilon}_a$ be the indicator function of the event

$$\{L^{x,\varepsilon}(\tau) \ge a\varepsilon \log^2 \varepsilon - \varepsilon |\log \varepsilon| \log |\log \varepsilon|\}.$$

Let $D_* \stackrel{\text{df}}{=} U^-(0, 1/16)$. We will denote $\{z \in \varepsilon\mathbb{Z}^2 : 2\varepsilon \le |z| \le 1/16\}$ by $\mathbb{Z}^2_\varepsilon$ or $\mathbb{Z}^2(\varepsilon)$ and for a set D we will let

$$\beta^\varepsilon_a(D) \stackrel{\text{df}}{=} \varepsilon^{2-a} \sum_{x \in \mathbb{Z}^2_\varepsilon \cap D} Y^{x,\varepsilon}_a.$$

We will abbreviate $\beta^\varepsilon_a(\mathbb{C})$ to β^ε_a.

Theorem 3.1. *For each $a > 0$ and every non-empty open rectangle D*

$$\lim_{\varepsilon\to 0} E^0\beta_a^\varepsilon(D) = \int_{D\cap D_*} |1-x^2|^a|\log|x||dx. \tag{3.1}$$

Proof. Consider some $x \in \mathbb{Z}_\varepsilon^2 \cap D$. Suppose that for $y \in U(x,\varepsilon)$ we have

$$P^0(T(U(x,\varepsilon)) < \tau) \le c_1 \log|x|/\log\varepsilon \tag{3.2}$$

and

$$H^y(\tau < \infty) \ge \frac{1}{\varepsilon|\log(\varepsilon/c_2)|} \tag{3.3}$$

where c_1 and c_2 may depend on x. Choose any small $\delta > 0$. Then by the strong Markov property and Lemma 2.2 (i) we have for $\varepsilon < \varepsilon_0 = \varepsilon_0(c_1, c_2)$

$$\begin{aligned}
P^0(Y_a^{x,\varepsilon} = 1) &= P^0(L^{x,\varepsilon}(\tau) \ge a\varepsilon\log^2\varepsilon - \varepsilon|\log\varepsilon|\log|\log\varepsilon|) \\
&= P^0(T(U(x,\varepsilon)) < \tau)\times \\
&\quad \times E^0(P^{X(T(U(x,\varepsilon)))}(L^{x,\varepsilon}(\tau) \ge a\varepsilon\log^2\varepsilon - \varepsilon|\log\varepsilon|\log|\log\varepsilon|) \mid T(U(x,\varepsilon)) < \tau) \\
&\le c_1(\log|x|/\log\varepsilon)\exp\left(-(a\varepsilon\log^2\varepsilon - \varepsilon|\log\varepsilon|\log|\log\varepsilon|)\frac{1}{\varepsilon|\log(\varepsilon/c_2)|}\right) \\
&= c_1(\log|x|/\log\varepsilon)\times \\
&\quad \times \exp\left(-a|\log\varepsilon| + a\log c_2 + \frac{a\log^2 c_2}{\log\varepsilon - \log c_2} + \log|\log\varepsilon| + \frac{\log c_2\log|\log\varepsilon|}{\log\varepsilon - \log c_2}\right) \\
&\le (1+\delta)c_1(\log|x|/\log\varepsilon)c_2^a\varepsilon^a|\log\varepsilon| = (1+\delta)c_1c_2^a\varepsilon^a|\log|x||.
\end{aligned} \tag{3.4}$$

Let $D_1 = U^-(0,\delta)$. According to Lemmas 2.1 (i)-(ii) and 2.3 (i)-(ii), formulae (3.2) and (3.3) hold with $c_1 = c_2 = 2$ for $x \in D_1$ and $c_1 = 1+\delta$, $c_2 = |1-x^2|/(1-\delta)$ for $x \in D\setminus D_1$, provided $\varepsilon < \varepsilon_1$. It follows that for small ε

$$
\begin{aligned}
&E^0 \beta_a^\varepsilon(D) = E^0 \varepsilon^{2-a} \sum_{x \in \mathbb{Z}_\varepsilon^2 \cap D} Y_a^{x,\varepsilon} \\
&\le \varepsilon^{2-a} \sum_{x \in \mathbb{Z}_\varepsilon^2 \cap D_1} (1+\delta) 2 \cdot 2^a \varepsilon^a |\log|x|| + \varepsilon^{2-a} \sum_{x \in \mathbb{Z}_\varepsilon^2 \cap D \setminus D_1} (1+\delta)^2 \left(\frac{|1-x^2|}{1-\delta} \right)^a \varepsilon^a |\log|x|| \\
&= \varepsilon^2 \sum_{x \in \mathbb{Z}_\varepsilon^2 \cap D_1} (1+\delta) 2^{a+1} |\log|x|| + \varepsilon^2 \sum_{x \in \mathbb{Z}_\varepsilon^2 \cap D \setminus D_1} (1+\delta)^2 \left(\frac{|1-x^2|}{1-\delta} \right)^a |\log|x||.
\end{aligned}
$$

We see that

$$
\limsup_{\varepsilon \to 0} E^0 \beta_a^\varepsilon(D) \le (1+\delta) 2^{a+1} \int_{D_1} |\log|x|| dx + (1+\delta)^2 (1-\delta)^{-a} \int_{(D \cap D_*) \setminus D_1} |1-x^2|^a |\log|x|| dx,
$$

and since δ is arbitrarily small and $\int_{U(0,1)} |\log|x|| dx < \infty$,

$$
\limsup_{\varepsilon \to 0} E^0 \beta_a^\varepsilon(D) \le \int_{D \cap D_*} |1-x^2|^a |\log|x|| dx.
$$

The lower bound may be obtained in a similar way. □

Theorem 3.2. *For each $a \in (0,1)$ there is $c_1 = c_1(a) < \infty$ such that*

$$
\limsup_{\varepsilon \to 0} E^0 (\beta_a^\varepsilon)^2 < c_1.
$$

Proof. First we will estimate $E^0(Y_a^{x,\varepsilon} Y_a^{y,\varepsilon})$ for $x, y \in \mathbb{Z}_\varepsilon^2$, $x \ne y$. Let (L, H) be an exit system from $U(x,\varepsilon) \cup U(y,\varepsilon)$ normalized so that $L = L^{x,\varepsilon} + L^{y,\varepsilon}$. If $(L^{x,\varepsilon}, H_1)$ is an exit system from $U(x,\varepsilon)$, then for each $z \in U(x,\varepsilon)$ the excursion law H^z may be obtained from H_1^z by killing the H_1^z-excursions at the hitting time of $U(y,\varepsilon)$. The same remark applies to the excursion laws from $U(y,\varepsilon)$.

Let θ denote the usual shift operator for a Markov process and

$$
\begin{aligned}
A_0 &= \{T(U(x,\varepsilon)\cup U(y,\varepsilon)) < \tau\},\\
T_x &= \inf\{t>0 : L_t^{x,\varepsilon} = a\varepsilon\log^2\varepsilon - \varepsilon|\log\varepsilon|\log|\log\varepsilon|\},\\
T_y &= \inf\{t>0 : L_t^{y,\varepsilon} = a\varepsilon\log^2\varepsilon - \varepsilon|\log\varepsilon|\log|\log\varepsilon|\},\\
T &= \min(T_x, T_y),\\
A_1^x &= \{T = T_x < \tau\},\\
A_1^y &= \{T = T_y < \tau\},\\
A_2^x &= \{T(U(y,\varepsilon))\circ\theta_T < \tau\circ\theta_T\},\\
A_2^y &= \{T(U(x,\varepsilon))\circ\theta_T < \tau\circ\theta_T\},\\
A_3^x &= \{L_\tau^{y,\varepsilon} - L_T^{y,\varepsilon} \ge a\varepsilon\log^2\varepsilon - \varepsilon|\log\varepsilon|\log|\log\varepsilon| - L_T^{y,\varepsilon}\},\\
A_3^y &= \{L_\tau^{x,\varepsilon} - L_T^{x,\varepsilon} \ge a\varepsilon\log^2\varepsilon - \varepsilon|\log\varepsilon|\log|\log\varepsilon| - L_T^{x,\varepsilon}\}.
\end{aligned}
$$

In order to have $Y_a^{x,\varepsilon}Y_a^{y,\varepsilon} = 1$ one of the following two events must happen: either $A_0 \cap A_1^x \cap A_2^x \cap A_3^x$ or $A_0 \cap A_1^y \cap A_2^y \cap A_3^y$.

Lemma 2.1 (ii) implies

$$P^0(A_0) \le 2(\log|x| + \log|y|)/\log\varepsilon. \tag{3.5}$$

Recall from Lemma 2.5 that for $z \in U(x,\varepsilon)$

$$H_1^z(T(U(y,\varepsilon)) < \tau \mid \tau < \infty) \le \phi_\varepsilon(|x-y|)/\log\varepsilon \tag{3.6}$$

where

$$\phi_\varepsilon(\alpha) = \begin{cases} c_2\log\alpha & \text{if } \varepsilon/\alpha < c_3 \text{ and } \alpha \ge c_3,\\ (1+\delta)\log\alpha & \text{if } \varepsilon/\alpha < c_3 \text{ and } \alpha < c_3,\\ \log\varepsilon & \text{otherwise.}\end{cases}$$

Note that $\phi_\varepsilon(\alpha) < 0$ if $\alpha < 1/2$ and $\varepsilon < 1$. Lemma 2.3 (ii) contains the following estimate

$$H_1^z(\tau < \infty) \ge \frac{1}{\varepsilon|\log(\varepsilon/2)|}.$$

This and (3.6) yield for $z \in U(x,\varepsilon)$

$$H^z(\tau < \infty) = H_1^z(\tau < T(U(y,\varepsilon))) \geq \frac{1-\phi_\varepsilon(|x-y|)/\log\varepsilon}{\varepsilon|\log(\varepsilon/2)|}. \tag{3.7}$$

The analogous estimate holds for excursion laws from $U(y,\varepsilon)$. It follows from (3.7) and Lemma 2.2 (i) that given $A_0 \cap \{T = T_x, L_T^{y,\varepsilon} = b\}$, the P^0-probability of $\{T < \tau\}$ is less than or equal to

$$\exp\left(-\frac{1-\phi_\varepsilon(|x-y|)/\log\varepsilon}{\varepsilon|\log(\varepsilon/2)|}(b + a\varepsilon\log^2\varepsilon - \varepsilon|\log\varepsilon|\log|\log\varepsilon|)\right). \tag{3.8}$$

By Lemma 2.1 (iii) and the strong Markov property applied at T we have

$$P^0(A_2^x \mid A_0 \cap A_1^x \cap \{L_T^{y,\varepsilon} = b\}) \leq 4\log|x-y|/\log\varepsilon. \tag{3.9}$$

Lemma 2.2 (i) implies that

$$\begin{aligned} &P^0(A_3^x \mid A_0 \cap A_1^x \cap A_2^x \cap \{L_T^{y,\varepsilon} = b\}) \\ &\quad\leq \exp\left(-\frac{1}{\varepsilon|\log(\varepsilon/2)|}(-b + a\varepsilon\log^2\varepsilon - \varepsilon|\log\varepsilon|\log|\log\varepsilon|)\right) \\ &\quad\leq \exp\left(-\frac{1-\phi_\varepsilon(|x-y|)/\log\varepsilon}{\varepsilon|\log(\varepsilon/2)|}(-b + a\varepsilon\log^2\varepsilon - \varepsilon|\log\varepsilon|\log|\log\varepsilon|)\right). \end{aligned} \tag{3.10}$$

An upper bound on $P^0(A_0 \cap A_1^x \cap A_2^x \cap A_3^x)$ may be obtained by multiplying (3.5) and (3.8)-(3.10) since this product does not depend on b. Hence

$$\begin{aligned} &P^0(A_0 \cap A_1^x \cap A_2^x \cap A_3^x) \\ &\quad\leq 2\frac{\log|x| + \log|y|}{\log\varepsilon} 4\frac{\log|x-y|}{\log\varepsilon} \times \\ &\qquad\times \exp\left(-\frac{1-\phi_\varepsilon(|x-y|)/\log\varepsilon}{\varepsilon|\log(\varepsilon/2)|} 2(a\varepsilon\log^2\varepsilon - \varepsilon|\log\varepsilon|\log|\log\varepsilon|)\right) \\ &\quad= \frac{8(\log|x| + \log|y|)\log|x-y|}{\log^2\varepsilon} \times \\ &\qquad\times \exp\left(-2a|\log\varepsilon| - \frac{2a\log 2|\log\varepsilon|}{|\log\varepsilon - \log 2|} + 2\log|\log\varepsilon| + \frac{2\log 2\log|\log\varepsilon|}{|\log\varepsilon - \log 2|}\right. \\ &\qquad\qquad \left. + 2\phi_\varepsilon(|x-y|)\left[-a - \frac{a\log 2}{|\log\varepsilon - \log 2|} - \frac{\log|\log\varepsilon|}{\log\varepsilon} - \frac{\log 2\log|\log\varepsilon|}{\log\varepsilon|\log\varepsilon - \log 2|}\right]\right). \end{aligned} \tag{3.11}$$

Note that for small $\varepsilon > 0$

$$-a - \frac{a \log 2}{|\log \varepsilon - \log 2|} - \frac{\log|\log \varepsilon|}{\log \varepsilon} - \frac{\log 2 \log|\log \varepsilon|}{\log \varepsilon |\log \varepsilon - \log 2|} > -a$$

and so

$$\begin{aligned} P^0(A_0 \cap A_1^x \cap A_2^x \cap A_3^x) &\le \frac{8(\log|x| + \log|y|)\log|x-y|}{\log^2 \varepsilon} c_4 \varepsilon^{2a} \log^2 \varepsilon \exp(-2a\phi_\varepsilon(|x-y|)) \\ &\le c_5(\log|x| + \log|y|)\log|x-y|\varepsilon^{2a} \exp(-2a\phi_\varepsilon(|x-y|)) \\ &\le c_5(\log|x| + \log|y|)\log|x-y|\varepsilon^{2a}\psi_\varepsilon(|x-y|) \end{aligned}$$

where $\psi_\varepsilon(|x-y|) \stackrel{\text{df}}{=} \exp(-2a\phi_\varepsilon(|x-y|))$. A similar estimate holds for $P^0(A_0 \cap A_1^y \cap A_2^y \cap A_3^y)$ so

$$\begin{aligned} E^0(Y_a^{x,\varepsilon} Y_a^{y,\varepsilon}) &\le P^0((A_0 \cap A_1^x \cap A_2^x \cap A_3^x) \cup (A_0 \cap A_1^y \cap A_2^y \cap A_3^y)) \\ &\le 2c_5(\log|x| + \log|y|)\log|x-y|\varepsilon^{2a}\psi_\varepsilon(|x-y|). \end{aligned} \tag{3.12}$$

Now we will estimate $E^0(\beta_a^\varepsilon)^2$. Note that we always have $E^0(Y_a^{x,\varepsilon} Y_a^{y,\varepsilon}) \le 1$.

$$\begin{aligned} E^0(\beta_a^\varepsilon)^2 &= E^0\left(\varepsilon^{2-a} \sum_{x \in \mathbb{Z}_\varepsilon^2} Y_a^{x,\varepsilon}\right)^2 \\ &= \varepsilon^{4-2a} \sum_{x \in \mathbb{Z}_\varepsilon^2} E^0(Y_a^{x,\varepsilon})^2 + \varepsilon^{4-2a} \sum_{\substack{x,y \in \mathbb{Z}_\varepsilon^2 \\ x \ne y}} E^0(Y_a^{x,\varepsilon} Y_a^{y,\varepsilon}) \\ &= \varepsilon^{4-2a} \sum_{x \in \mathbb{Z}_\varepsilon^2} E^0 Y_a^{x,\varepsilon} + \varepsilon^{4-2a} \sum_{\substack{x,y \in \mathbb{Z}_\varepsilon^2 \\ x \ne y \\ |x-y| \le \varepsilon/c_3}} E^0(Y_a^{x,\varepsilon} Y_a^{y,\varepsilon}) + \varepsilon^{4-2a} \sum_{\substack{x,y \in \mathbb{Z}_\varepsilon^2 \\ x \ne y \\ |x-y| > \varepsilon/c_3}} E^0(Y_a^{x,\varepsilon} Y_a^{y,\varepsilon}) \\ &\le \varepsilon^{2-a} E^0 \beta_a^\varepsilon + \varepsilon^{4-2a} 16 c_3^{-2} \varepsilon^{-2} \\ &\qquad + \varepsilon^{4-2a} \sum_{\substack{x,y \in \mathbb{Z}_\varepsilon^2 \\ x \ne y \\ |x-y| > \varepsilon/c_3}} 2c_5(\log|x| + \log|y|)\log|x-y|\varepsilon^{2a}\psi_\varepsilon(|x-y|) \\ &\le \varepsilon^{2-a} E^0 \beta_a^\varepsilon + 16 c_3^{-2} \varepsilon^{2-2a} + \varepsilon^4 \sum_{\substack{x,y \in \mathbb{Z}_\varepsilon^2 \\ x \ne y \\ |x-y| > \varepsilon/c_3}} 2c_5(\log|x| + \log|y|)\log|x-y|\psi_\varepsilon(|x-y|). \end{aligned} \tag{3.13}$$

Suppose that $a < 1$. Find $\delta > 0$ so small that $a_1 \stackrel{\text{df}}{=} a(1+\delta) < 1$. Choose $c_3 > 0$ so that (3.6) is satisfied with this choice of δ and c_3 in the definition of ϕ_ε; such a choice is possible according to Lemma 2.5. Then

$$\psi_\varepsilon(|x-y|) \le \begin{cases} |x-y|^{c_6} & \text{if } \varepsilon/|x-y| < c_3, \\ |x-y|^{-2a_1} & \text{if } \varepsilon/|x-y| < c_3 \text{ and } |x-y| < c_3. \end{cases} \tag{3.14}$$

The last term on the right hand side of (3.13) is no larger than the Riemann sum approximation to the integral

$$\int_{D_*}\int_{D_*} 2c_5(\log|x| + \log|y|)\log|x-y|\psi_0(|x-y|)\,dx\,dy \tag{3.15}$$

where

$$\psi_0(|x-y|) \stackrel{\text{df}}{=} \begin{cases} |x-y|^{c_6} & \text{if } |x-y| \ge c_3, \\ |x-y|^{-2a_1} & \text{if } |x-y| < c_3. \end{cases} \tag{3.16}$$

The integral in (3.15) is finite for $a_1 < 1$ so the last term on the right hand side of (3.13) stays bounded as $\varepsilon \to 0$. Since $E^0\beta_a^\varepsilon$ stays bounded by Theorem 3.1, the other two terms are also bounded and the proof is complete. $\square$

Remark 3.1. The argument of the last proof shows that for any open set D

$$\limsup_{\varepsilon\to 0} E^0(\beta_a^\varepsilon(D))^2 \le \int_{D\cap D_*}\int_{D\cap D_*} c_1(\log|x| + \log|y|)\log|x-y|\psi_0(|x-y|)\,dx\,dy$$

where ψ_0 is defined in (3.16) and $a_1 > a$. For each $a_2 > a_1$ there exists $c_2 < \infty$ such that this double integral is not greater than $c_2\rho^{4-2a_2}$ for any square D with side length $\rho < 2$.

Theorem 3.3. *For each $a \in (0, 2/3)$ there is $c_1 = c_1(a) < \infty$ such that*

$$\limsup_{\varepsilon\to 0} E^0(\beta_a^\varepsilon)^3 < c_1.$$

Proof. The proof is very similar to that of Theorem 3.2 so we will only outline the main steps.

The first main step of the proof is an estimate of $E^0(Y_a^{x,\varepsilon} Y_a^{y,\varepsilon} Y_a^{z,\varepsilon})$ for $x, y, z \in \mathbb{Z}_\varepsilon^2$, $x \neq y \neq z \neq x$. Let (L, H) be an exit system from $U(x, \varepsilon) \cup U(y, \varepsilon) \cup U(z, \varepsilon)$ normalized so that $L = L^{x,\varepsilon} + L^{y,\varepsilon} + L^{z,\varepsilon}$. In order to have $Y_a^{x,\varepsilon} Y_a^{y,\varepsilon} Y_a^{z,\varepsilon} = 1$, the local time on each circle $U(x, \varepsilon), U(y, \varepsilon)$ and $U(z, \varepsilon)$ must reach

$$a\varepsilon \log^2 \varepsilon - \varepsilon |\log \varepsilon| \log |\log \varepsilon|. \tag{3.17}$$

We will consider two stopping times T_1 and T_2. The first one, T_1, is the time when the local time reaches the above level on one of the circles and T_2 is the time when the local time reaches the same level on one of the two remaining circles. The process must hit one of the circles in the first place and then hit one of the remaining circles after each of the stopping times T_1 and T_2. The probabilities of these three hits may be bounded as in (3.5) and (3.9) by

$$2(\log |x| + \log |y| + \log |z|)/ \log \varepsilon, \tag{3.18}$$

$$4(\log |x - y| + \log |x - z| + \log |z - y|)/ \log \varepsilon, \tag{3.19}$$

and again by

$$4(\log |x - y| + \log |x - z| + \log |z - y|)/ \log \varepsilon. \tag{3.20}$$

Let $(L^{x,\varepsilon}, H_1)$ denote the exit system from $U(x, \varepsilon)$ and suppose that $v \in U(x, \varepsilon)$. Then we have the following formula analogous to (3.7):

$$H^v(\tau < \infty) = H_1^v(\tau < T(U(y, \varepsilon))) \geq \frac{1 - (\phi_\varepsilon(|x - y|) + \phi_\varepsilon(|x - z|))/ \log \varepsilon}{\varepsilon |\log(\varepsilon/2)|}.$$

The additional term $\phi_\varepsilon(|x - z|)$ comes from the fact that now we have two circles rather than just one besides $U(x, \varepsilon)$. Thus we obtain for all $v \in U(x, \varepsilon) \cup U(y, \varepsilon) \cup U(z, \varepsilon)$

$$H^v(\tau < \infty) \geq \frac{1 - \eta_\varepsilon(x, y, z)/ \log \varepsilon}{\varepsilon |\log(\varepsilon/2)|}$$

where

$$\eta_\varepsilon(x,y,z) \stackrel{\text{df}}{=} \max(\phi_\varepsilon(|x-y|)+\phi_\varepsilon(|x-z|), \phi_\varepsilon(|x-y|)+\phi_\varepsilon(|y-z|), \phi_\varepsilon(|x-z|)+\phi_\varepsilon(|y-z|)).$$

This allows us to estimate the probability that the local time on each circle reaches the level (3.17) assuming that the process hits a circle and then jumps from one circle to another after T_1 and T_2. We can do this the same way as was done in (3.8) and (3.10). The upper bound for this probability is

$$\exp\left(-\frac{1-\eta_\varepsilon(x,y,z)/\log\varepsilon}{\varepsilon|\log(\varepsilon/2)|}3(a\varepsilon\log^2\varepsilon - \varepsilon|\log\varepsilon|\log|\log\varepsilon|)\right).$$

An upper bound for $E^0(Y_a^{x,\varepsilon}Y_a^{y,\varepsilon}Y_a^{z,\varepsilon})$ is obtained by multiplying this expression by (3.18)-(3.20). Calculations similar to those which lead from (3.11) to (3.12) give in the present case

$$E^0(Y_a^{x,\varepsilon}Y_a^{y,\varepsilon}Y_a^{z,\varepsilon}) \le c_2\alpha(x,y,z)\varepsilon^{3a}\exp(-3a\eta_\varepsilon(x,y,z))$$

where

$$\alpha(x,y,z) \stackrel{\text{df}}{=} (\log|x|+\log|y|+\log|z|)(\log|x-y|+\log|x-z|+\log|z-y|)^2.$$

We have

$$
\begin{aligned}
E^0(\beta_a^\varepsilon)^3 &= E^0\Bigg(\varepsilon^{2-a}\sum_{x\in\mathbb{Z}_\varepsilon^2} Y_a^{x,\varepsilon}\Bigg)^3 \\
&= \varepsilon^{6-3a}\sum_{x\in\mathbb{Z}_\varepsilon^2} E^0(Y_a^{x,\varepsilon})^3 + 3\varepsilon^{6-3a}\sum_{\substack{x,y\in\mathbb{Z}_\varepsilon^2\\ x\neq y}} E^0((Y_a^{x,\varepsilon})^2 Y_a^{y,\varepsilon}) + 6\varepsilon^{6-3a}\sum_{\substack{x,y,z\in\mathbb{Z}_\varepsilon^2\\ x\neq y\neq z\\ x\neq z}} E^0(Y_a^{x,\varepsilon}Y_a^{y,\varepsilon}Y_a^{z,\varepsilon}) \\
&= \varepsilon^{6-3a}\sum_{x\in\mathbb{Z}_\varepsilon^2} E^0 Y_a^{x,\varepsilon} + 3\varepsilon^{6-3a}\sum_{\substack{x,y\in\mathbb{Z}_\varepsilon^2\\ x\neq y}} E^0(Y_a^{x,\varepsilon}Y_a^{y,\varepsilon}) \\
&\quad + 6\varepsilon^{6-3a}\Bigg(\sum_{\substack{x,y,z\in\mathbb{Z}_\varepsilon^2\\ x\neq y\neq z\\ x\neq z\\ |x-y|\leq\varepsilon/c_3}} + \sum_{\substack{x,y,z\in\mathbb{Z}_\varepsilon^2\\ x\neq y\neq z\\ x\neq z\\ |x-z|\leq\varepsilon/c_3}} + \sum_{\substack{x,y,z\in\mathbb{Z}_\varepsilon^2\\ x\neq y\neq z\\ x\neq z\\ |z-y|\leq\varepsilon/c_3}}\Bigg) E^0(Y_a^{x,\varepsilon}Y_a^{y,\varepsilon}Y_a^{z,\varepsilon}) \\
&\quad + 6\varepsilon^{6-3a}\sum_{\substack{x,y,z\in\mathbb{Z}_\varepsilon^2\\ |x-y|>\varepsilon/c_3\\ |x-z|>\varepsilon/c_3\\ |z-y|>\varepsilon/c_3}} E^0(Y_a^{x,\varepsilon}Y_a^{y,\varepsilon}Y_a^{z,\varepsilon}) \\
&\leq \varepsilon^{4-2a}E^0\beta_a^\varepsilon + 3\varepsilon^{2-a}E^0(\beta_a^\varepsilon)^2 + 18\varepsilon^{6-3a}16c_4^{-2}\varepsilon^{-4} \\
&\quad + 6\varepsilon^{6-3a}\sum_{\substack{x,y,z\in\mathbb{Z}_\varepsilon^2\\ |x-y|>\varepsilon/c_3\\ |x-z|>\varepsilon/c_3\\ |z-y|>\varepsilon/c_3}} \alpha(x,y,z)\exp(-3a\eta_\varepsilon(x,y,z)).
\end{aligned} \tag{3.21}
$$

The first three terms of the above sum go to 0 as $\varepsilon\to 0$ by Theorems 3.1 and 3.2.

Suppose that $a<2/3$. Find $\delta>0$ so that $a_1 \stackrel{\text{df}}{=} a(1+\delta)<2/3$. By Lemma 2.5 we may find a constant c_3 such that

$$
\psi_\varepsilon(|x-y|) \leq \begin{cases} |x-y|^{c_5} & \text{if } \varepsilon/|x-y|<c_3, \\ |x-y|^{-3a_1} & \text{if } \varepsilon/|x-y|<c_3 \text{ and } |x-y|<c_3, \end{cases}
$$

where $\psi_\varepsilon(|x-y|)=\exp(-3a\phi_\varepsilon(|x-y|))$. Let

$$
\widetilde{\psi}_0(|x-y|) \stackrel{\text{df}}{=} \begin{cases} |x-y|^{c_6} & \text{if } |x-y|\geq c_3, \\ |x-y|^{-3a_1} & \text{if } |x-y|<c_3. \end{cases}
$$

The last term in (3.21) is bounded from above by the Riemann sum approximation to

$$\int_{D_*}\int_{D_*}\int_{D_*}\alpha(x,y,z)\Psi(x,y,z)\,dx\,dy\,dz \tag{3.22}$$

where

$$\begin{aligned}\Psi(x,y,z) &\stackrel{\text{df}}{=} \max(\widetilde{\psi}_0(|x-y|)\widetilde{\psi}_0(|x-z|),\widetilde{\psi}_0(|x-y|)\widetilde{\psi}_0(|y-z|),\widetilde{\psi}_0(|z-y|)\widetilde{\psi}_0(|x-z|))\\ &\le \widetilde{\psi}_0(|x-y|)\widetilde{\psi}_0(|x-z|)+\widetilde{\psi}_0(|x-y|)\widetilde{\psi}_0(|y-z|)+\widetilde{\psi}_0(|z-y|)\widetilde{\psi}_0(|x-z|).\end{aligned}$$

Since $a_1 < 2/3$, the integral in (3.22) is finite and the last term in (3.21) stays bounded as $\varepsilon \to 0$. □

Remark 3.2. One can find estimates for higher moments of β^ε_a in the same way as in the proofs of Theorems 3.2 and 3.3.

Theorem 3.4. *Let $D_x \stackrel{\text{df}}{=} (-\infty, Re\, x] \times (-\infty, Im\, x]$ and $\beta^\varepsilon_a(x) \stackrel{\text{df}}{=} \beta^\varepsilon_a(D_x)$. For each $a \in (0,1/2)$ there are $c_1 < \infty$ and $\gamma > 0$ such that*

$$\limsup_{\varepsilon\to 0} E^0(\beta^\varepsilon_a(x) - \beta^\varepsilon_a(y))^2 < c_1|x-y|^{2+\gamma}.$$

Proof. Let $D_{x,y}$ be the intersection of D_* and the symmetric set difference of D_x and D_y. It follows from Remark 3.1 that

$$\limsup_{\varepsilon\to 0} E^0(\beta^\varepsilon_a(x) - \beta^\varepsilon_a(y))^2 \le c_2 \int\limits_{D_{x,y}}\int\limits_{D_{x,y}} (\log|x| + \log|y|)\log|x-y|\psi_0(|x-y|)\,dx\,dy.$$

It is elementary to check that for every $a_1 \in (a, 1/2)$ there is $c_3 < \infty$ such that

$$\int_{\widetilde{D}}\int_{\widetilde{D}}(\log|x| + \log|y|)\log|x-y|\psi_0(|x-y|)\,dx\,dy \le c_3\delta^{3-2a_1}$$

for any set $\widetilde{D}$ which is the union of a pair of perpendicular strips of length 1 and width δ. The theorem now easily follows from this estimate and the fact that $D_{x,y}$ is a subset of such a set $\widetilde{D}$ provided $|x-y| \le \delta$. □

4. Convergence. We start with a few lemmas needed for the proof of our main convergence result. Let $A_{x,b}(k)$ denote the event that there are at most k crossings from $U(x,b/4)$ to $U(x,b/2)$ before τ. In other words, $A^c_{x,b}(k)$ holds if and only if there exist

$$s_1 < t_1 < \cdots < s_{k+1} < t_{k+1} < \tau$$

such that $X(s_j) \in U(x,b/4)$ and $X(t_j) \in U(x,b/2)$ for all $j \le k+1$.

Lemma 4.1. *For each $a > 0$, $\delta > 0$ and $b > 0$ there exist $k < \infty$ and $\varepsilon_0 > 0$ such that if $x, y \in U^-(0,1/16)$, $|x-y| > b$ and $\varepsilon < \varepsilon_0$ then*

$$P^0(\{Y_a^{y,\varepsilon} = 1\} \cap A_{x,b}(k)) \ge (1-\delta) P^0(Y_a^{y,\varepsilon} = 1).$$

Proof. Let $T_1 = T(U(y,\varepsilon))$ and let T_2 be the smallest t such that

$$L^{y,\varepsilon}(t) \ge a\varepsilon \log^2 \varepsilon - \varepsilon |\log \varepsilon| \log|\log \varepsilon|.$$

We will consider the process X under P^0 conditioned by $\{Y_a^{y,\varepsilon} = 1\}$. Its path is divided into three parts X^f, X^m and X^l by T_1 and T_2, i.e.,

$$X^f(t) = \begin{cases} X(t) & \text{if } t < T_1, \\ \Delta & \text{if } t \ge T_1, \end{cases}$$
$$X^m(t) = \begin{cases} X(t+T_1) & \text{if } 0 \le t < T_2 - T_1, \\ \Delta & \text{if } t \ge T_2 - T_1, \end{cases}$$
$$X^l(t) = \begin{cases} X(t+T_1+T_2) & \text{if } 0 \le t < \tau - T_2 - T_1, \\ \Delta & \text{if } t \ge \tau - T_2 - T_1. \end{cases} \tag{4.1}$$

First we will analyze X^m and we start with a bound on the number N^m of excursions of X^m from $U(y,\varepsilon)$ which hit $U(y,b/4)$. Suppose $(L^{y,\varepsilon}, H)$ is an exit system from $U(y,\varepsilon)$ for standard Brownian motion. We know from Lemma 2.3 (iii) that

$$H^z(T(U(y,b/4)) < \infty) = \frac{1}{\varepsilon \log(b/4\varepsilon)}$$

for $z \in U(y,\varepsilon)$ and $\varepsilon < b/4$. By Lemma 2.1 (iii) and the strong Markov property applied at $T(U(y,b/4))$,

$$H^z(T(U(y,b/4)) < \tau = \infty) \le \frac{1}{\varepsilon \log(b/4\varepsilon)} \frac{4\log(b/4)}{\log \varepsilon}.$$

This and Lemma 2.2 (i) imply that the distribution of N^m is stochastically bounded by a Poisson random variable with mean

$$\frac{4\log(b/4)}{\varepsilon \log(b/4\varepsilon)\log\varepsilon}(a\varepsilon \log^2 \varepsilon - \varepsilon |\log \varepsilon| \log|\log\varepsilon|).$$

The last quantity is bounded by $c_1 = c_1(b) < \infty$ for all $\varepsilon < \varepsilon_0$.

Next we will find a bound for the number of crossings between $U(x,b/4)$ and $U(x,b/2)$ during one excursion of X^m which hits $U(y,b/4)$. Let h be the positive harmonic function in $D \overset{\text{df}}{=} U^-(0,1) \cap U^+(y,\varepsilon)$ which is equal to 1 on $U(y,\varepsilon)$ and 0 elsewhere on the boundary. Suppose that $N^m \ge n$. The distribution of the n-th excursion of X^m which hits $U(y,b/4)$ after $T(U(y,b/4))$ is that of an h-process. We will show that the number of crossings between $U(x,b/4)$ and $U(x,b/2)$ is majorized by a geometric distribution for any h-process, where h refers to the just defined harmonic function. In order to do this it will suffice to show that there is a lower bound $c_2 > 0$ for the probability of not hitting of $U(x,b/4)$ for an h-process starting from any point of $U(x,b/2)$.

It is easy to see that

$$P^z(T(U(y,b/4)) < T(U(0,1) \cup U(x,b/4)) > c_3 > 0$$

for all $z \in U(x,b/2)$. This and the Harnack principle yield

$$\begin{aligned} &P^z_h(T(U(y,b/4)) < T(U(0,1) \cup U(x,b/4)) \\ &\ge P^z(T(U(y,b/4)) < T(U(0,1) \cup U(x,b/4))\frac{\min\{h(v) : v \in U(y,b/4)\}}{\max\{h(v) : v \in U(x,b/2)\}} \ge c_3c_4 > 0. \end{aligned} \tag{4.2}$$

The function $h_1(z) \stackrel{\text{df}}{=} |\log(|z-y|/2)/\log(\varepsilon/2)|$ is harmonic in D and its boundary values are greater than or equal to those of h so

$$h(z) \le |\log(|z-y|/2)/\log(\varepsilon/2)|$$

for all $z \in D$. We have

$$P^z(T(U(y,\varepsilon)) < T(U(y,b/2))) = \frac{|\log(b/4) - \log(b/2)|}{|\log\varepsilon - \log(b/2)|}$$

and, therefore,

$$\begin{aligned} P^z_h(T(U(y,\varepsilon)) < T(U(y,b/2))) &= \frac{1}{h(z)} P^z(T(U(y,\varepsilon)) < T(U(y,b/2))) \\ &\ge \frac{|\log(\varepsilon/2)|}{|\log(|z-y|/2)|} \frac{|\log(b/4) - \log(b/2)|}{|\log\varepsilon - \log(b/2)|}. \end{aligned}$$

The last expression stays bounded from below by $c_5 > 0$ as $\varepsilon \to 0$. This, the strong Markov property and (4.2) show that there is a lower bound $c_2 > 0$ for the probability of not hitting of $U(x,b/4)$ for an h-process starting from any point of $U(x,b/2)$.

We see that the number of crossings between $U(x,b/4)$ and $U(x,b/2)$ during one excursion of X^m which hits $U(y,b/4)$ is majorized by a geometric distribution whose parameter stays bounded as $\varepsilon \to 0$. The excursions are independent given their initial values and their number is bounded by a Poisson distribution, so the number of crossings of X^m between $U(x,b/4)$ and $U(x,b/2)$ has a finite expectation which is bounded by a constant independent of ε. A similar analysis may be applied to X^f which is an h_2-process in D and to X^l which is a Brownian motion killed upon exiting $U(0,1)$. We conclude that the expectation of the total number of crossings between $U(x,b/4)$ and $U(x,b/2)$ by X conditioned by $\{Y_a^{y,\varepsilon} = 1\}$ is bounded by a constant independent of ε and the exact position of x and y. This easily implies the lemma. $\square$

Lemma 4.2. *For each $k < \infty$, $a > 0$, $\delta > 0$ and $b > 0$ there exist $b_0 > 0$ and $\varepsilon_0 > 0$ such that if $x, y_1, y_2 \in U^-(0,1/2)$, $|y_1 - y_2| < b_0$, $|x - y_1| > b$, $|x - y_2| > b$ and $\varepsilon_1, \varepsilon_2 < \varepsilon_0$ then*

$$\frac{E^0(Y_a^{x,\varepsilon_1} \mid \{Y_a^{y_1,\varepsilon_1} = 1\} \cap A_{x,b}(k))}{E^0(Y_a^{x,\varepsilon_1} \mid \{Y_a^{y_2,\varepsilon_2} = 1\} \cap A_{x,b}(k))} \in (1-\delta, 1+\delta). \tag{4.3}$$

Proof. We will consider X conditioned by $\{Y_a^{y_j,\varepsilon_j} = 1\}$. Decompose this process into three parts $X^{j,f}, X^{j,m}$ and $X^{j,l}$ in the same way as in (4.1). Let $N^{j,m}$ be the number of excursions of $X^{j,m}$ which hit $U(y_1, b/4)$ (we will assume that ε_2 and b_0 are so small that $U(y_2, \varepsilon_2) \subset U(y_1, b/4)$). Suppose that $N^{j,m} \geq n$ and let $T_0^{j,n}$ and $T_\infty^{j,n}$ be the start and the end of the n-th excursion. Then let

$$V_n^j(t) = \begin{cases} X(t + T_0^{j,n}) & \text{if } N^{j,m} \geq n \text{ and } t \in [0, T_\infty^{j,n} - T_0^{j,n}),\ n \geq 1, \\ \Delta & \text{otherwise,} \end{cases}$$

$$T_{2k+1}^{j,n} = \inf\{t > T_{2k}^{j,n} : X(t) \in U(x, b/4)\}, \quad k \geq 0,$$

$$T_{2k}^{j,n} = \inf\{t > T_{2k-1}^{j,n} : X(t) \in U(x, b/2)\}, \quad k \geq 1,$$

$$Z_k^{j,n}(t) = \begin{cases} X(t + T_{2k-1}^{j,n}) & \text{if } T_{2k-1}^{j,n} < \infty \text{ and } t \in [0, T_{2k}^{j,n} - T_{2k-1}^{j,n}),\ k \geq 1, \\ \Delta & \text{otherwise.} \end{cases}$$

Let $N_n^{j,m}$ be the number of crossings between $U(x, b/4)$ and $U(x, b/2)$ by the n-th excursion of $X^{j,m}$ which hits $U(y_1, b/4)$. In other words, $N_n^{j,m}$ is the largest k such that $T_{2k-1}^{j,n} < \infty$. Let $N^{j,f}$ and $N^{j,l}$ be the numbers of such crossings by $X^{j,f}$ and $X^{j,l}$. The event $A_{x,b}(k)$ is the finite union of disjoint events of the form

$$\{N^{j,f} = n_f, N^{j,l} = n_l, N^{j,m} = n_m, N_1^{j,m} = k_1, \dots, N_{n_m}^{j,m} = k_{n_m}\}.$$

It will suffice to prove (4.3) for any such event in place of $A_{x,b}(k)$. We will now argue that the Radon-Nikodym derivative of the distribution of $(Z_1^{1,n}, \dots, Z_{k_n}^{1,n})$ given $\{N^{1,m} = n_m \geq n, N_n^{1,m} = k_n\}$ and the distribution of $(Z_1^{2,n}, \dots, Z_{k_n}^{2,n})$ given $\{N^{2,m} = n_m \geq n, N_n^{2,m} = k_n\}$ is arbitrarily close to 1 when ε_j and b_0 are sufficiently small. Conditioning of h-processes produces other h-processes (see Section 2 for a review of these properties) so $(Z_1^{j,n}, \dots, Z_{k_n}^{j,n})$ conditioned by $\{N^{j,m} = n_m \geq n, N_n^{j,m} = k_n\}$ is a vector of h-processes (here "h" is used in the generic sense). Such processes may be represented as mixtures of h-processes, each of which starts from a single point and converges a.s. to a single point at its lifetime. Thus it will be sufficient to prove that the Radon-Nikodym derivative of the distributions of the vectors of the initial and terminal values of $(Z_1^{1,n}, \dots, Z_{k_n}^{1,n})$ given $\{N^{1,m} = n_m \geq n, N_n^{1,m} = k_n\}$ and $(Z_1^{2,n}, \dots, Z_{k_n}^{2,n})$ given $\{N^{2,m} = n_m \geq n, N_n^{2,m} = k_n\}$ is arbitrarily close to 1 when ε_j and b_0 are sufficiently small.

Let h_j be a positive harmonic function in $U^-(0,1) \cap U^+(y_j, \varepsilon_j)$ which has zero boundary values on $U(0,1)$, is constant on $U(y_j, \varepsilon_j)$ and normalized such that $h_j(x) = 1$. The transition probabilities of V_n^j after hitting of $U(y_1, b/4)$ are those of an h_j-process. It is easy to see that $h_1(z)/h_2(z) \in (1-\delta, 1+\delta)$ for all $z \in U^-(x, b/2)$ provided b_0 and ε_0 are sufficiently small. The hitting distributions of $U(y_1, b/4)$ by V_n^1 and V_n^2 have a Radon-Nikodym derivative arbitrarily close to 1 provided ε_j and b_0 are small, according to Lemma 2.4 (i). The strong Markov property and the fact that h_1/h_2 is close to 1 on $U^-(x, b/2)$ may be used to show that the initial and terminal distributions of $(Z_1^{1,n}, \dots, Z_{k_n}^{1,n})$ given $\{N^{1,m} = n_m \geq n, N_n^{1,m} = k_n\}$ and $(Z_1^{2,n}, \dots, Z_{k_n}^{2,n})$ given $\{N^{2,m} = n_m \geq n, N_n^{2,m} = k_n\}$ have the Radon-Nikodym derivative arbitrarily close to 1 provided ε_j and b_0 are small.

Excursions of X^m from $U(y_j, \varepsilon_j)$ which hit $U(y_1, b/4)$ are independent given their initial values. Hence we can extend our claim about the Radon-Nikodym derivatives as follows. The Radon-Nikodym derivative of the distributions of the families
$(Z_1^{1,n}, \dots, Z_{k_n}^{1,n})_{1 \leq n \leq n_m}$ given

$$\{N^{1,m} = n_m, N_1^{1,m} = k_1, \dots, N_{n_m}^{1,m} = k_{n_m}\}$$

and $(Z_1^{2,n}, \dots, Z_{k_n}^{2,n})_{1 \leq n \leq n_m}$ given

$$\{N^{2,m} = n_m, N_1^{2,m} = k_1, \dots, N_{n_m}^{2,m} = k_{n_m}\}$$

lies in an arbitrarily small neighborhood of 1 if we assume that b_0 and ε_0 are small. This property can be further extended to take into account the crossings between $U(x, b/4)$ and $U(x, b/2)$ by $X^{j,f}$ and $X^{j,l}$.

Note that the event $\{Y_a^{x,\varepsilon_1} = 1\}$ is completely determined by the parts of the path of X between the successive hits of $U(x, b/4)$ and $U(x, b/2)$. Hence the ratio of the probabilities of $\{Y_a^{x,\varepsilon_1} = 1\}$ given $\{Y_a^{y_1,\varepsilon_1} = 1\} \cap A_{x,b}(k)$ and $\{Y_a^{y_2,\varepsilon_2} = 1\} \cap A_{x,b}(k)$ can be made arbitrarily close to 1 by choosing sufficiently small b_0 and ε_0. $\square$

Lemma 4.3. *For each $a > 0$, $b > 0$ and $\delta > 0$ there exist $b_0 > 0$ and $\varepsilon_0 > 0$ such that if*

$y_1, y_2 \in U^-(0,1/2) \setminus U^-(0,b)$, $|y_1 - y_2| < b_0$ *and* $\varepsilon_1, \varepsilon_2 < \varepsilon_0$ *then*

$$\frac{E^0(\varepsilon_1^{2-a} Y_a^{y_1,\varepsilon_1})}{E^0(\varepsilon_2^{2-a} Y_a^{y_2,\varepsilon_2})} \in ((1-\delta)(\varepsilon_1/\varepsilon_2)^2, (1+\delta)(\varepsilon_1/\varepsilon_2)^2).$$

Proof. Recall from the proof of Theorem 3.1 (see (3.4)) that for each $\delta > 0$ there is $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ and $y \in U^-(0,1/2) \setminus U^-(0,b)$

$$(1-\delta)\varepsilon^a |1-y^2|^a |\log|y|| \le P^0(Y_a^{y,\varepsilon} = 1) \le (1+\delta)\varepsilon^a |1-y^2|^a |\log|y||. \tag{4.4}$$

Let $b_0 > 0$ be so small that

$$(1-\delta) \le \frac{|1-y_1^2|^a |\log|y_1||}{|1-y_2^2|^a |\log|y_2||} \le (1+\delta)$$

for all $y_1, y_2 \in U^-(0,1/2) \setminus U^-(0,b)$, $|y_1 - y_2| < b_0$. We combine this and (4.4) to obtain

$$(\varepsilon_1/\varepsilon_2)^a (1-\delta)^2/(1+\delta) \le \frac{P^0(Y_a^{y_1,\varepsilon_1} = 1)}{P^0(Y_a^{y_2,\varepsilon_2} = 1)} \le (\varepsilon_1/\varepsilon_2)^a (1+\delta)^2/(1-\delta)$$

for all $y_1, y_2 \in U^-(0,1/2) \setminus U^-(0,b)$, $|y_1 - y_2| < b_0$ and $\varepsilon < \varepsilon_0$. Since $\delta > 0$ is arbitrarily small, the lemma easily follows. □

Theorem 4.1. *For each* $a \in (0,2/3)$ *and every rectangle* D, $\lim_{\varepsilon \to 0} \beta_a^\varepsilon(D)$ *exists in* $L^2(P^0)$.

Proof. Fix some $a \in (0,2/3)$. We will prove the theorem only for for $D = [-1,1]^2$. The proof for other rectangles D is analogous. It will suffice to show that

$$\lim_{\varepsilon_1,\varepsilon_2 \to 0} E^0(\beta_a^{\varepsilon_1} - \beta_a^{\varepsilon_2})^2 = 0.$$

Fix some small $b > 0$ and let $\mathcal{M}$ be the collection of all squares

$$M = [jb,(j+1)b) \times [kb,(k+1)b), \quad j,k \in \mathbb{Z},$$

which intersect D. We will denote by $\mathcal{M}_3$ the family of all squares M_1 such that there is $M \in \mathcal{M}$ with the same center as M_1 and the side of M is three times smaller than that of M_1. We will write $M_1 \sim M_2$ for $M_1, M_2 \in \mathcal{M}$ if $\text{dist}(M_1, M_2) < b$ and write $M_1 \nsim M_2$ otherwise.

We have

$$E^0(\beta_a^{\varepsilon_1} - \beta_a^{\varepsilon_2})^2 = E^0\Big(\sum_{x\in\mathbb{Z}^2(\varepsilon_1)} \varepsilon_1^{2-a} Y_a^{x,\varepsilon_1} - \sum_{y\in\mathbb{Z}^2(\varepsilon_2)} \varepsilon_2^{2-a} Y_a^{y,\varepsilon_2}\Big)^2$$

$$= E^0\Big(\sum_{x\in\mathbb{Z}^2(\varepsilon_1)} (\varepsilon_1^{2-a} Y_a^{x,\varepsilon_1})^2 + \sum_{x\in\mathbb{Z}^2(\varepsilon_2)} (\varepsilon_2^{2-a} Y_a^{x,\varepsilon_2})^2$$
$$+ 2 \sum_{\substack{x\in\mathbb{Z}^2(\varepsilon_1)\\ y\in\mathbb{Z}^2(\varepsilon_1)\\ x\neq y}} \varepsilon_1^{2-a} Y_a^{x,\varepsilon_1} \varepsilon_1^{2-a} Y_a^{y,\varepsilon_1} + 2 \sum_{\substack{x\in\mathbb{Z}^2(\varepsilon_2)\\ y\in\mathbb{Z}^2(\varepsilon_2)\\ x\neq y}} \varepsilon_2^{2-a} Y_a^{x,\varepsilon_2} \varepsilon_2^{2-a} Y_a^{y,\varepsilon_2}$$
$$- 2 \sum_{x\in\mathbb{Z}^2(\varepsilon_1)} \sum_{y\in\mathbb{Z}^2(\varepsilon_2)} \varepsilon_1^{2-a} Y_a^{x,\varepsilon_1} \varepsilon_2^{2-a} Y_a^{y,\varepsilon_2}\Big)$$

$$
\begin{aligned}
=&E^0\Bigg(\sum_{x\in\mathbb{Z}^2(\varepsilon_1)}\varepsilon_1^{4-2a}Y_a^{x,\varepsilon_1}+\sum_{x\in\mathbb{Z}^2(\varepsilon_2)}\varepsilon_2^{4-2a}Y_a^{x,\varepsilon_2}\Bigg)\\
&+E^0\Bigg[\sum_{\substack{M_1,M_2\in\mathcal{M}\\ M_1\sim M_2}}2\Bigg(\sum_{\substack{x\in\mathbb{Z}^2(\varepsilon_1)\cap M_1\\ y\in\mathbb{Z}^2(\varepsilon_1)\cap M_2\\ x\neq y}}\varepsilon_1^{2-a}Y_a^{x,\varepsilon_1}\varepsilon_1^{2-a}Y_a^{y,\varepsilon_1}+\sum_{\substack{x\in\mathbb{Z}^2(\varepsilon_2)\cap M_1\\ y\in\mathbb{Z}^2(\varepsilon_2)\cap M_2\\ x\neq y}}\varepsilon_2^{2-a}Y_a^{x,\varepsilon_2}\varepsilon_2^{2-a}Y_a^{y,\varepsilon_2}\\
&\qquad-\sum_{x\in\mathbb{Z}^2(\varepsilon_1)\cap M_1}\sum_{\substack{y\in\mathbb{Z}^2(\varepsilon_2)\cap M_2\\ x\neq y}}\varepsilon_1^{2-a}Y_a^{x,\varepsilon_1}\varepsilon_2^{2-a}Y_a^{y,\varepsilon_2}\\
&\qquad-\sum_{x\in\mathbb{Z}^2(\varepsilon_2)\cap M_1}\sum_{\substack{y\in\mathbb{Z}^2(\varepsilon_1)\cap M_2\\ x\neq y}}\varepsilon_2^{2-a}Y_a^{x,\varepsilon_2}\varepsilon_1^{2-a}Y_a^{y,\varepsilon_1}\Bigg)\Bigg]\\
&+E^0\Bigg[\sum_{\substack{M_1,M_2\in\mathcal{M}\\ M_1\nsim M_2}}2\Bigg(\sum_{\substack{x\in\mathbb{Z}^2(\varepsilon_1)\cap M_1\\ y\in\mathbb{Z}^2(\varepsilon_1)\cap M_2}}\varepsilon_1^{2-a}Y_a^{x,\varepsilon_1}\varepsilon_1^{2-a}Y_a^{y,\varepsilon_1}+\sum_{\substack{x\in\mathbb{Z}^2(\varepsilon_2)\cap M_1\\ y\in\mathbb{Z}^2(\varepsilon_2)\cap M_2}}\varepsilon_2^{2-a}Y_a^{x,\varepsilon_2}\varepsilon_2^{2-a}Y_a^{y,\varepsilon_2}\\
&\qquad-\sum_{x\in\mathbb{Z}^2(\varepsilon_1)\cap M_1}\sum_{y\in\mathbb{Z}^2(\varepsilon_2)\cap M_2}\varepsilon_1^{2-a}Y_a^{x,\varepsilon_1}\varepsilon_2^{2-a}Y_a^{y,\varepsilon_2}\\
&\qquad-\sum_{x\in\mathbb{Z}^2(\varepsilon_2)\cap M_1}\sum_{y\in\mathbb{Z}^2(\varepsilon_1)\cap M_2}\varepsilon_2^{2-a}Y_a^{x,\varepsilon_2}\varepsilon_1^{2-a}Y_a^{y,\varepsilon_1}\Bigg)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
\leq& E^0(\varepsilon_1^{2-a}\beta_a^{\varepsilon_1}+\varepsilon_2^{2-a}\beta_a^{\varepsilon_2})\\
+&E^0\Bigg[\sum_{M\in\mathcal{M}_3}2\Bigg(\sum_{\substack{x\in\mathbb{Z}^2(\varepsilon_1)\cap M\\ y\in\mathbb{Z}^2(\varepsilon_1)\cap M\\ x\neq y}}\varepsilon_1^{2-a}Y_a^{x,\varepsilon_1}\varepsilon_1^{2-a}Y_a^{y,\varepsilon_1}+\sum_{\substack{x\in\mathbb{Z}^2(\varepsilon_2)\cap M\\ y\in\mathbb{Z}^2(\varepsilon_2)\cap M\\ x\neq y}}\varepsilon_2^{2-a}Y_a^{x,\varepsilon_2}\varepsilon_2^{2-a}Y_a^{y,\varepsilon_2}\\
&\quad+\sum_{x\in\mathbb{Z}^2(\varepsilon_1)\cap M}\sum_{\substack{y\in\mathbb{Z}^2(\varepsilon_2)\cap M\\ x\neq y}}\varepsilon_1^{2-a}Y_a^{x,\varepsilon_1}\varepsilon_2^{2-a}Y_a^{y,\varepsilon_2}\\
&\quad+\sum_{x\in\mathbb{Z}^2(\varepsilon_2)\cap M}\sum_{\substack{y\in\mathbb{Z}^2(\varepsilon_1)\cap M\\ x\neq y}}\varepsilon_2^{2-a}Y_a^{x,\varepsilon_2}\varepsilon_1^{2-a}Y_a^{y,\varepsilon_1}\Bigg)\Bigg]\\
+&E^0\Bigg[\sum_{\substack{M_1,M_2\in\mathcal{M}\\ M_1\nsim M_2}}2\Bigg(\sum_{x\in\mathbb{Z}^2(\varepsilon_1)\cap M_1}\varepsilon_1^{2-a}Y_a^{x,\varepsilon_1}\Bigg(\sum_{y\in\mathbb{Z}^2(\varepsilon_1)\cap M_2}\varepsilon_1^{2-a}Y_a^{y,\varepsilon_1}-\sum_{y\in\mathbb{Z}^2(\varepsilon_2)\cap M_2}\varepsilon_2^{2-a}Y_a^{y,\varepsilon_2}\Bigg)\\
&\quad+\sum_{x\in\mathbb{Z}^2(\varepsilon_2)\cap M_1}\varepsilon_2^{2-a}Y_a^{x,\varepsilon_2}\Bigg(\sum_{y\in\mathbb{Z}^2(\varepsilon_2)\cap M_2}\varepsilon_2^{2-a}Y_a^{y,\varepsilon_2}-\sum_{y\in\mathbb{Z}^2(\varepsilon_1)\cap M_2}\varepsilon_1^{2-a}Y_a^{y,\varepsilon_1}\Bigg)\Bigg)\Bigg].
\end{aligned}
$$

Let us call the three expectations in the last expression I_1, I_2 and I_3. Choose an arbitrarily small $\eta > 0$. Recall from Theorem 3.1 that $E^0\beta_a^\varepsilon$ remains bounded as $\varepsilon \to 0$. Hence I_1 is less than η provided ε_1 and ε_2 are small.

Suppose that $a_1 \in (a, 1)$. Then use Remark 3.1 to obtain the following estimate for small ε_1 and ε_2.

$$\begin{aligned}
I_2 &\le E^0\Big[\sum_{M\in\mathcal{M}_3}\Big(\sum_{x\in\mathbb{Z}^2(\varepsilon_1)\cap M}\varepsilon_1^{2-a}Y_a^{x,\varepsilon_1} + \sum_{y\in\mathbb{Z}^2(\varepsilon_2)\cap M}\varepsilon_2^{2-a}Y_a^{y,\varepsilon_2}\Big)^2\Big]\\
&\le E^0\Big[\sum_{M\in\mathcal{M}_3} 2\Big(\Big(\sum_{x\in\mathbb{Z}^2(\varepsilon_1)\cap M}\varepsilon_1^{2-a}Y_a^{x,\varepsilon_1}\Big)^2 + \Big(\sum_{y\in\mathbb{Z}^2(\varepsilon_2)\cap M}\varepsilon_2^{2-a}Y_a^{y,\varepsilon_2}\Big)^2\Big)\Big]\\
&= E^0\Big[\sum_{M\in\mathcal{M}_3} 2((\beta_a^{\varepsilon_1}(M))^2 + (\beta_a^{\varepsilon_2}(M))^2)\Big]\\
&\le \sum_{M\in\mathcal{M}_3} 2(c_1 b^{4-2a_1} + c_1 b^{4-2a_1})\\
&\le c_2 b^{-2} b^{4-2a_1} = c_2 b^{2-2a_1}.
\end{aligned}$$

Now choose $b > 0$ so that $I_2 < \eta$.

The third term I_3 may be further split into two pieces as follows. Let $\widetilde{\mathcal{M}}$ be the family of 4 squares in $\mathcal{M}$ which have 0 on their borders, let $\mathcal{M}' = \mathcal{M}\setminus\widetilde{\mathcal{M}}$ and let $\widetilde{M}$ be the union of the squares in $\widetilde{\mathcal{M}}$. Then

$$I_3 \le E^0\Big[\sum_{\substack{M_1,M_2\in\mathcal{M}'\\ M_1\not\sim M_2}} \cdots\Big] + 2E^0\Big[\sum_{\substack{M_1\in\widetilde{\mathcal{M}}\\ M_2\in\mathcal{M}\\ M_1\not\sim M_2}} \cdots\Big] \stackrel{\text{df}}{=} I_4 + I_5.$$

We bound I_5 using Remark 3.1 and Theorem 3.2.

$$\begin{aligned}
I_5 &\le 2E^0\Big[\sum_{\substack{M_1\in\widetilde{\mathcal{M}}\\ M_2\in\mathcal{M}\\ M_1\nsim M_2}} 2\Big(\sum_{x\in\mathbb{Z}^2(\varepsilon_1)\cap M_1}\varepsilon_1^{2-a}Y_a^{x,\varepsilon_1}\sum_{y\in\mathbb{Z}^2(\varepsilon_1)\cap M_2}\varepsilon_1^{2-a}Y_a^{y,\varepsilon_1}\\
&\qquad+\sum_{x\in\mathbb{Z}^2(\varepsilon_2)\cap M_1}\varepsilon_2^{2-a}Y_a^{x,\varepsilon_2}\sum_{y\in\mathbb{Z}^2(\varepsilon_2)\cap M_2}\varepsilon_2^{2-a}Y_a^{y,\varepsilon_2}\Big)\Big]\\
&\le 4E^0(\beta_a^{\varepsilon_1}(\widetilde{M})\beta_a^{\varepsilon_1}+\beta_a^{\varepsilon_2}(\widetilde{M})\beta_a^{\varepsilon_2})\\
&\le 4(E^0(\beta_a^{\varepsilon_1}(\widetilde{M}))^2)^{1/2}(E^0(\beta_a^{\varepsilon_1})^2)^{1/2}+4(E^0(\beta_a^{\varepsilon_2}(\widetilde{M}))^2)^{1/2}(E^0(\beta_a^{\varepsilon_2})^2)^{1/2}\\
&\le 4c_1b^{(4-2a_1)/2}c_2+4c_1b^{(4-2a_1)/2}c_2.
\end{aligned}$$

We will choose $b>0$ so that $I_5<\eta$.

It remains to show that I_4 is smaller than η when $\varepsilon_1,\varepsilon_2\to 0$. Fix some $\delta>0$ and choose $k_0<\infty$ so that Lemma 4.1 holds for every $k>k_0$ and some $\varepsilon_0=\varepsilon_0(k)>0$.

Let $B(k)$ denote the event that there exist

$$s_1<t_1<\cdots<s_k<t_k<\tau$$

such that $|X(s_i)-X(t_i)|>b/8$ for all $i\le k$. For a fixed $b>0$, the P^0-probability of $B(k)$ goes to 0 as $k\to\infty$ by continuity of Brownian paths. It follows from Theorem 3.3 that there is an upper bound for the 3/2-th moments of the random variables $(\beta_a^{\varepsilon_1}+\beta_a^{\varepsilon_2})^2$ which is uniform in ε_j, so these random variables are uniformly integrable. Hence for large k we have

$$2E^0\left[(\beta_a^{\varepsilon_1}+\beta_a^{\varepsilon_2})^2\mathbf{1}_{B(k)}\right]\le\eta/2 \tag{4.5}$$

for all $\varepsilon_1,\varepsilon_2<1/4$. Let us fix a k which satisfies this property and is greater than k_0.

Let $\mathcal{K}$ be the family of all squares

$$M=[jb_1,(j+1)b_1)\times[nb_1,(n+1)b_1)$$

for integer j and n. We will choose $b_1 > 0$ so that b is a large integer multiple of b_1. We have

$$I_4 = E^0\Bigg[\sum_{\substack{M_1,M_2\in\mathcal{M}'\\ M_1\nsim M_2}}\ \sum_{\substack{M_3,M_4\in\mathcal{K}\\ M_3\subset M_1\\ M_4\subset M_2}} 2\Bigg(\sum_{x\in\mathbb{Z}^2(\varepsilon_1)\cap M_3}\varepsilon_1^{2-a}Y_a^{x,\varepsilon_1}\Bigg(\sum_{y\in\mathbb{Z}^2(\varepsilon_1)\cap M_4}\varepsilon_1^{2-a}Y_a^{y,\varepsilon_1}-\sum_{y\in\mathbb{Z}^2(\varepsilon_2)\cap M_4}\varepsilon_2^{2-a}Y_a^{y,\varepsilon_2}\Bigg)$$

$$(4.6)\qquad +\sum_{x\in\mathbb{Z}^2(\varepsilon_2)\cap M_3}\varepsilon_2^{2-a}Y_a^{x,\varepsilon_2}\Bigg(\sum_{y\in\mathbb{Z}^2(\varepsilon_2)\cap M_4}\varepsilon_2^{2-a}Y_a^{y,\varepsilon_2}-\sum_{y\in\mathbb{Z}^2(\varepsilon_1)\cap M_4}\varepsilon_1^{2-a}Y_a^{y,\varepsilon_1}\Bigg)\Bigg)\Bigg].$$

Consider x, y_1 and y_2 such that $x \in M_3 \cap \mathbb{Z}^2(\varepsilon_1)$, $y_1 \in M_4 \cap \mathbb{Z}^2(\varepsilon_1)$, $y_2 \in M_4 \cap \mathbb{Z}^2(\varepsilon_2)$, $M_3, M_4 \in \mathcal{K}$, $M_3 \subset M_1$, $M_4 \subset M_2$, $M_1, M_2 \in \mathcal{M}'$ and $M_1 \nsim M_2$. Note that $|x - y_j| \geq b$. Recall events $A_{x,b}(k)$ defined at the beginning of this section. It follows from Lemmas 4.1-4.3 that there are $b_1 > 0$ and $\varepsilon_0 > 0$ such that for $\varepsilon_1, \varepsilon_2 < \varepsilon_0$

$$\frac{E^0(Y_a^{x,\varepsilon_1} \mid \{Y_a^{y_1,\varepsilon_1} = 1\} \cap A_{x,b}(k))}{E^0(Y_a^{x,\varepsilon_1} \mid \{Y_a^{y_2,\varepsilon_2} = 1\} \cap A_{x,b}(k))} \in (1-\delta, 1+\delta),$$

$$\frac{E^0(\varepsilon_1^{2-a}Y_a^{y_1,\varepsilon_1})}{E^0(\varepsilon_2^{2-a}Y_a^{y_2,\varepsilon_2})} \in ((1-\delta)(\varepsilon_1/\varepsilon_2)^2, (1+\delta)(\varepsilon_1/\varepsilon_2)^2)$$

and

$$P^0(\{Y_a^{y_1,\varepsilon_1} = 1\} \cap A_{x,b}(k)) \geq (1-\delta)P^0(Y_a^{y_1,\varepsilon_1} = 1).$$

We use these three properties to obtain

$$
\begin{aligned}
&E^0(\varepsilon_1^{2-a} Y_a^{x,\varepsilon_1} \varepsilon_1^{2-a} Y_a^{y_1,\varepsilon_1}) \\
&\quad = E^0(\varepsilon_1^{2-a} Y_a^{x,\varepsilon_1} \varepsilon_1^{2-a} Y_a^{y_1,\varepsilon_1} \mathbf{1}_A) + E^0(\varepsilon_1^{2-a} Y_a^{x,\varepsilon_1} \varepsilon_1^{2-a} Y_a^{y_1,\varepsilon_1} \mathbf{1}_{A^c}) \\
&\quad = E^0(\varepsilon_1^{2-a} Y_a^{y_1,\varepsilon_1} \mathbf{1}_A) E^0(\varepsilon_1^{2-a} Y_a^{x,\varepsilon_1} \mid Y_a^{y_1,\varepsilon_1} \mathbf{1}_A = 1) \\
&\qquad + E^0(\varepsilon_1^{2-a} Y_a^{x,\varepsilon_1} \varepsilon_1^{2-a} Y_a^{y_1,\varepsilon_1} \mathbf{1}_{A^c}) \\
&\quad \le E^0(\varepsilon_1^{2-a} Y_a^{y_1,\varepsilon_1}) E^0(\varepsilon_1^{2-a} Y_a^{x,\varepsilon_1} \mid Y_a^{y_1,\varepsilon_1} \mathbf{1}_A = 1) \\
&\qquad + E^0(\varepsilon_1^{2-a} Y_a^{x,\varepsilon_1} \varepsilon_1^{2-a} Y_a^{y_1,\varepsilon_1} \mathbf{1}_{A^c}) \\
&\quad \le (1+\delta)^2 (\varepsilon_1/\varepsilon_2)^2 E^0(\varepsilon_2^{2-a} Y_a^{y_2,\varepsilon_2}) E^0(\varepsilon_1^{2-a} Y_a^{x,\varepsilon_1} \mid Y_a^{y_2,\varepsilon_2} \mathbf{1}_A = 1) \\
&\qquad + E^0(\varepsilon_1^{2-a} Y_a^{x,\varepsilon_1} \varepsilon_1^{2-a} Y_a^{y_1,\varepsilon_1} \mathbf{1}_{A^c}) \\
&\quad \le (1+\delta)^3 (\varepsilon_1/\varepsilon_2)^2 E^0(\varepsilon_2^{2-a} Y_a^{y_2,\varepsilon_2} \mathbf{1}_A) E^0(\varepsilon_1^{2-a} Y_a^{x,\varepsilon_1} \mid Y_a^{y_2,\varepsilon_2} \mathbf{1}_A = 1) \\
&\qquad + E^0(\varepsilon_1^{2-a} Y_a^{x,\varepsilon_1} \varepsilon_1^{2-a} Y_a^{y_1,\varepsilon_1} \mathbf{1}_{A^c}) \\
&\quad = (1+\delta)^3 (\varepsilon_1/\varepsilon_2)^2 E^0(\varepsilon_1^{2-a} Y_a^{x,\varepsilon_1} \varepsilon_2^{2-a} Y_a^{y_2,\varepsilon_2} \mathbf{1}_A) \\
&\qquad + E^0(\varepsilon_1^{2-a} Y_a^{x,\varepsilon_1} \varepsilon_1^{2-a} Y_a^{y_1,\varepsilon_1} \mathbf{1}_{A^c}) \\
&\quad \le (1+\delta)^3 (\varepsilon_1/\varepsilon_2)^2 E^0(\varepsilon_1^{2-a} Y_a^{x,\varepsilon_1} \varepsilon_2^{2-a} Y_a^{y_2,\varepsilon_2}) \\
&\qquad + E^0(\varepsilon_1^{2-a} Y_a^{x,\varepsilon_1} \varepsilon_1^{2-a} Y_a^{y_1,\varepsilon_1} \mathbf{1}_{A^c})
\end{aligned}
$$

where $A = A_{x,b}(k)$. In order to simplify the notation let $1 + \delta_1 \stackrel{\text{df}}{=} (1+\delta)^3$. It is easy to see that $(\varepsilon_1/\varepsilon_2)^2$ times the ratio of the number of $x \in \mathbb{Z}^2(\varepsilon_1) \cap M$ to the number of $x \in \mathbb{Z}^2(\varepsilon_2) \cap M$ goes to 1 as $\varepsilon_1, \varepsilon_2 \to 0$ uniformly in $M \in \mathcal{K}$. Note that we always have

$A^c_{x,b}(k) \subset B(k)$. We obtain from the last inequality and (4.6)

$$\begin{aligned} I_4 \le E^0\Bigg[& \sum_{\substack{M_1,M_2\in\mathcal{M}'\\ M_1\nsim M_2}} \sum_{\substack{M_3,M_4\in\mathcal{K}\\ M_3\subset M_1\\ M_4\subset M_2}} \\ & 2\Bigg(2\delta_1 \sum_{x\in\mathbb{Z}^2(\varepsilon_1)\cap M_3} \varepsilon_1^{2-a} Y_a^{x,\varepsilon_1} \sum_{y\in\mathbb{Z}^2(\varepsilon_1)\cap M_4} \varepsilon_1^{2-a} Y_a^{y,\varepsilon_1} \\ & + \sum_{x\in\mathbb{Z}^2(\varepsilon_1)\cap M_3} \varepsilon_1^{2-a} Y_a^{x,\varepsilon_1} \sum_{y\in\mathbb{Z}^2(\varepsilon_1)\cap M_4} \varepsilon_1^{2-a} Y_a^{y,\varepsilon_1} \mathbf{1}_{A^c_{x,b}(k)} \\ & + 2\delta_1 \sum_{x\in\mathbb{Z}^2(\varepsilon_2)\cap M_3} \varepsilon_2^{2-a} Y_a^{x,\varepsilon_2} \sum_{y\in\mathbb{Z}^2(\varepsilon_2)\cap M_4} \varepsilon_2^{2-a} Y_a^{y,\varepsilon_2} \\ & + \sum_{x\in\mathbb{Z}^2(\varepsilon_2)\cap M_3} \varepsilon_2^{2-a} Y_a^{x,\varepsilon_2} \sum_{y\in\mathbb{Z}^2(\varepsilon_2)\cap M_4} \varepsilon_2^{2-a} Y_a^{y,\varepsilon_2} \mathbf{1}_{A^c_{x,b}(k)} \Bigg)\Bigg] \\ \le E^0\Bigg[& \sum_{\substack{M_1,M_2\in\mathcal{M}'\\ M_1\nsim M_2}} \sum_{\substack{M_3,M_4\in\mathcal{K}\\ M_3\subset M_1\\ M_4\subset M_2}} \\ & 2\Bigg(2\delta_1 \sum_{x\in\mathbb{Z}^2(\varepsilon_1)\cap M_3} \varepsilon_1^{2-a} Y_a^{x,\varepsilon_1} \sum_{y\in\mathbb{Z}^2(\varepsilon_1)\cap M_4} \varepsilon_1^{2-a} Y_a^{y,\varepsilon_1} \\ & + \mathbf{1}_{B(k)} \sum_{x\in\mathbb{Z}^2(\varepsilon_1)\cap M_3} \varepsilon_1^{2-a} Y_a^{x,\varepsilon_1} \sum_{y\in\mathbb{Z}^2(\varepsilon_1)\cap M_4} \varepsilon_1^{2-a} Y_a^{y,\varepsilon_1} \\ & + 2\delta_1 \sum_{x\in\mathbb{Z}^2(\varepsilon_2)\cap M_3} \varepsilon_2^{2-a} Y_a^{x,\varepsilon_2} \sum_{y\in\mathbb{Z}^2(\varepsilon_2)\cap M_4} \varepsilon_2^{2-a} Y_a^{y,\varepsilon_2} \\ & + \mathbf{1}_{B(k)} \sum_{x\in\mathbb{Z}^2(\varepsilon_2)\cap M_3} \varepsilon_2^{2-a} Y_a^{x,\varepsilon_2} \sum_{y\in\mathbb{Z}^2(\varepsilon_2)\cap M_4} \varepsilon_2^{2-a} Y_a^{y,\varepsilon_2} \Bigg)\Bigg]. \end{aligned}$$

Elementary calculations with these sums similar to those given at the beginning of the proof show that

$$I_4 \le 4\delta_1 E^0(\beta_a^{\varepsilon_1} + \beta_a^{\varepsilon_2})^2 + 2E^0\left[(\beta_a^{\varepsilon_1} + \beta_a^{\varepsilon_2})^2 \mathbf{1}_{B(k)}\right]. \tag{4.7}$$

We may assume that the first term is bounded by $\eta/2$ since δ_1 is arbitrarily small and $E^0(\beta_a^\varepsilon)^2$ is bounded as $\varepsilon \to 0$ by Theorem 3.2. The second term is bounded by $\eta/2$ in view of (4.5). $\square$

5. Intersection local time.

Theorem 5.1. *For each $a \in (0, 1/2)$ there exists P^0-a.s. a random measure β_a on $\mathbb{C}$ such that for every rectangle $D \subset \mathbb{C}$*

$$\beta_a(D) = \lim_{\varepsilon \to 0} \beta_a^\varepsilon(D) \tag{5.1}$$

in L^2-norm.

Proof. The existence of the limit in (5.1) has been proved in Theorem 4.1. The proof of Theorem 4.1 can be obviously generalized to any finite union of rectangles D. The resulting set function β_a is evidently non-negative and finitely additive on the field of finite unions of rectangles with rational corner coordinates. In order to prove that it can be extended to a measure on the Borel σ-field it would suffice to show that the function

$$(x_1, x_2, y_1, y_2) \to \beta_a([x_1, y_1] \times [x_2, y_2]) \tag{5.2}$$

is continuous. It follows from Theorem 3.4 that for some $\gamma > 0$,

$$E^0(\beta_a(D_x) - \beta_a(D_y))^2 < c_1 |x - y|^{2+\gamma}$$

where $D_x = (-\infty, x_1] \times (-\infty, x_2]$ for $x = (x_1, x_2)$. The Kolmogorov lemma implies that the function

$$x = (x_1, x_2) \to \beta_a(D_x) = \beta_a((-\infty, x_1] \times (-\infty, x_2])$$

has a continuous version. Thus (5.2) also has a continuous version and the proof is complete. $\square$

Remark 5.1. (i) We will argue that β_a is a measurable function of the path and, moreover, it is defined locally. First, the local time on a circle $U(y, \varepsilon)$ may be determined pathwise and locally and this observation extends to β_a^ε. For a fixed rectangle D, we have obtained $\beta_a(D)$ as an L^2 limit of β_a^ε's, but we can instead take an a.s. limit by passing to a subsequence if necessary. The measure β_a is uniquely determined by the values of $\beta_a(D)$ for the countable family of rectangles D with rational corner coordinates.

Hence, β_a is well defined for the paths of any process whose distribution is (locally) mutually absolutely continuous with that of a 2-dimensional Brownian motion. In particular, it is possible to extend β_a to the part of the path which lies in $U^-(0,1) \setminus D_*$. Unless stated otherwise, we will use the original definition of β_a.

(ii) Let $\beta_a(D,t)$ be the β_a-measure of D defined relative to $X([0,t])$. The local time spent on $U(x,\varepsilon)$ by X up to time t is a non-decreasing function of t, so our construction of β_a shows that for a fixed set D, the function $t \to \beta_a(D,t)$ is non-decreasing. Since $\beta_a(\mathbb{R}^2,t) = \beta_a(D,t) + \beta_a(D^c,t)$ we have

$$\sup_{D \subset \mathbb{R}^2} (\beta_a(D,t+s) - \beta_a(D,t)) \le \beta_a(\mathbb{R}^2,t+s) - \beta_a(\mathbb{R}^2,t).$$

The function $\beta_a(\mathbb{R}^2,\,\cdot\,)$ is continuous and even Hölder continuous, a.s. Let us outline a proof of this property. The piece $X([t,t+\Delta t])$ of the Brownian path has diameter of order $(\Delta t)^{1/2}$. Hence, the measures $\beta_a(\,\cdot\,,t+\Delta t)$ and $\beta_a(\,\cdot\,,t)$ differ only on a set of diameter $(\Delta t)^{1/2}$. By Remark 3.1, the second moment of the difference $\beta_a(\mathbb{R}^2,t+\Delta t) - \beta_a(\mathbb{R}^2,t)$ is of order $(\Delta t)^{(1/2)(4-2a)} = (\Delta t)^{2-a}$. Kolmogorov's theorem may now be used to show that $\beta_a(\mathbb{R}^2,\,\cdot\,)$ is a continuous and, moreover, Hölder function, since $2-a>1$ for $a \in (0,1/2)$. $\square$

For each $x \in U^-(0,1)$ we will define a process Z^x_a which may be thought of as "Brownian motion conditioned to spend a units of local time at x."

Fix some $x \in U^-(0,1)$ and let h be a positive harmonic function in $D \stackrel{\text{df}}{=} U^-(0,1) \setminus \{x\}$ which has a pole at x and zero boundary values on $U(0,1)$. We will normalize h so that $\lim_{z \to x} h(z)/|\log|z-x|| = 1$. Recall the space $C_*[0,\infty)$ from Section 2. Let H^x be an excursion law (i.e., a σ-finite measure on $C_*[0,\infty)$) with the transition probabilities of an h-process and such that the H^x-measure of paths that do not start at x is zero. The existence of H^x may be proved in the same way as in the Brownian case (see Burdzy (1987)). We will choose a normalization of H^x based on Lemma 5.1 (i) below. Let λ denote Lebesgue measure on $[0,\infty)$ and let W be a Poisson point process on $\mathcal{W} \stackrel{\text{df}}{=} [0,\infty) \times C_*[0,\infty)$ with mean measure $\lambda \times H$. The process W is a random collection of pairs (t, e_t) where $t \in [0,\infty)$ and $e_t \in C_*[0,\infty)$.

Lemma 5.1.

(i) $\lim_{\varepsilon\to 0} H^x(T(U(x,\varepsilon)) < \infty)/|\log \varepsilon| = c \in (0,\infty)$.

(ii) Let $\sigma(e_t)$ be the lifetime of the excursion e_t. For every $t_0 < \infty$

$$\sum_{\substack{(t,e_t)\in W\\ t\le t_0}} \sigma(e_t) < \infty \quad \text{a.s.}$$

Proof. (i) Recall that

$$\lim_{z\to x} h(z)/|\log|z-x|| = 1. \tag{5.3}$$

Fix small $\delta > 0$ and find $r = r(\delta) > 0$ such that

$$h(z)/|\log|z-x|| \in (1-\delta, 1+\delta)$$

for all z such that $|z-x| \le r$. For all $\varepsilon \le r$, $y \in U(x,\varepsilon)$ and small δ,

$$\begin{aligned} P_h^y(T(U(x,r)) < \infty) &= \int_{U(x,r)} \frac{h(v)}{h(y)} P^y(X(T(U(x,r))) \in dv) \\ &\in \left(\frac{(1-\delta)\log r}{(1+\delta)\log\varepsilon}, \frac{(1+\delta)\log r}{(1-\delta)\log\varepsilon}\right) \\ &\subset ((1-3\delta)\log r/\log\varepsilon, (1+3\delta)\log r/\log\varepsilon). \end{aligned}$$

By the strong Markov property applied at the hitting time of $U(x,\varepsilon)$

$$H^x(T(U(x,r)) < \infty) = \int_{U(x,\varepsilon)} P_h^y(T(U(x,r)) < \infty) H^x(X(T(U(x,\varepsilon))) \in dy).$$

Let $c_1 = c_1(r) = H^x(T(U(x,r)) < \infty)$. Then

$$H^x(T(U(x,\varepsilon)) < \infty) \in (c_1 \log\varepsilon/(1+3\delta)\log r, c_1\log\varepsilon/(1-3\delta)\log r)$$

and, therefore,

$$c_2/(1+3\delta) \le \liminf_{\varepsilon\to 0} \frac{H^x(T(U(x,\varepsilon))<\infty)}{|\log\varepsilon|} \le \limsup_{\varepsilon\to 0} \frac{H^x(T(U(x,\varepsilon))<\infty)}{|\log\varepsilon|} \le c_2/(1-3\delta).$$

Since δ is arbitrarily small, the result follows.

(ii) First we will estimate the density of the expected occupation measure $G_H(y)$ for the excursion law H^x. The Green function $G(y,z)$ for the standard Brownian motion killed upon leaving $U(0,1)$ is bounded by $c_1|\log(|y-z|/2)|$. Hence the Green function $G_h(y,z)$ for the h-process is bounded by $c_1 h(z)|\log(|y-z|/2)|/h(y)$. By (5.3), for y and z close to x,

$$G_h(y,z) \le c_2 \frac{|\log|x-z|\log(|y-z|/2)|}{|\log|x-y||}.$$

By the strong Markov property applied at $T(U(x,\varepsilon))$ and part (i) of the lemma

$$\begin{aligned}
G_H(z) &= \int_{U(x,\varepsilon)} G_h(y,z) H^x(X(T(U(x,\varepsilon))) \in dy) \\
&\le H^x(T(U(x,\varepsilon))<\infty) \sup_{y\in U(x,\varepsilon)} c_2 \frac{|\log|x-z|\log(|y-z|/2)|}{|\log|x-y||} \\
&\le \sup_{y\in U(x,\varepsilon)} c_3 |\log|x-z|\log(|y-z|/2)| \\
&\le c_3|\log|x-z|\log(|x-z|/4)|
\end{aligned}$$

for $|x-z|>2\varepsilon$. Since ε does not appear on the right hand side of the last formula, the inequality is valid for all z close to x, say $|z-x|<\varepsilon_0$. Hence $\int_{U(0,1)} G_H(z)dz = c_4 < \infty$. The expected value of $\sum_{\substack{(t,e_t)\in W\\ t\le t_0}} \sigma(e_t)$ is equal to $t_0 \int_{U(0,1)} G_H(z)dz$ and, therefore, it is finite. □

Trajectories of Z_a^x will be assembled from three parts. The first part is an h-process $\{Z_1(t), 0\le t<t_1\}$ in D which starts from 0 and approaches x at t_1.

Let H^x be normalized so that $c=a$ in Lemma 5.1 (i). For $u>0$ let

$$T(u) = \sup\{t : \sum_{s<t} \sigma(e_s) \le u\}.$$

By Lemma 5.1 (ii), $T(u)$ is well-defined for all $u < \infty$ a.s. It is easy to see that a.s. $T(u) < \infty$ for each u and, moreover, a.s. for almost all u there is a point $(s, e_s) \in W$ such that $s = T(u)$. For such u let

$$Z_2(u) = e_{T(u)}\Big(u - \sum_{s<T(u)} \sigma(e_s)\Big).$$

For the remaining u let $Z_2(u) = x$. The process Z_2 does not have jumps because excursions e_t are continuous. We will argue that it does not have discontinuities of the second kind. The H^x-measure of the paths which have diameter greater than any fixed $\varepsilon > 0$ is finite by Lemma 5.1 (i). Since the total mass of H^x is infinite, every two excursions e_s, e_t such that $(s, e_s), (t, e_t) \in W$, $s < t$ and with diameter greater than ε must be separated by $(u, e_u) \in W$, $s < u < t$ where the diameter of e_u is less than ε. It follows that excursions of Z_2 from x with diameter greater than ε do not cluster and, therefore, Z_2 is continuous. Let $t_2 \stackrel{\text{df}}{=} \sum_{s<1} \sigma(e_s)$.

Let $\{Z_3(t), 0 \le t \le t_3\}$ be a Brownian motion starting from x and killed at the hitting time of $U(0,1)$. We may and do assume that Z_1, Z_2 and Z_3 are independent. Let

$$Z_a^x(t) = \begin{cases} Z_1(t) & \text{for } 0 \le t \le t_1, \\ Z_2(t - t_1) & \text{for } t_1 \le t \le t_1 + t_2, \\ Z_3(t - (t_1 + t_2)) & \text{for } t_1 + t_2 \le t \le t_1 + t_2 + t_3. \end{cases}$$

The distribution of the process $\{Z_a^x(t), 0 \le t \le t_1 + t_2 + t_3\}$ will be denoted Q_a^x.

Proposition 5.1. *For each fixed $x \in U^-(0,1)$, the distributions of X under P^0 conditioned by $\{Y_a^{x,\varepsilon} = 1\}$ converge to Q_a^x as $\varepsilon \to 0$.*

Proof. Let $(L^{x,\varepsilon}, H_1)$ be an exit system from $U(x,\varepsilon)$ under P^0 normalized as in (2.2). We will use another exit system $(\overline{L}^{x,\varepsilon}, \overline{H}_1)$ where

$$\overline{L}^{x,\varepsilon}(t) = L^{x,\varepsilon}(t)/(a\varepsilon \log^2 \varepsilon - \varepsilon |\log \varepsilon| \log |\log \varepsilon|)$$

and

$$\overline{H}_1^y = (a\varepsilon \log^2 \varepsilon - \varepsilon |\log \varepsilon| \log |\log \varepsilon|) H_1^y.$$

Note that $\overline{L}^{x,\varepsilon}$ exceeds 1 if and only if $\{Y_a^{x,\varepsilon} = 1\}$ occurs.

Let h_1 be the positive harmonic function in $A_\varepsilon \stackrel{\text{df}}{=} U^-(0,1) \cap U^+(x,\varepsilon)$ which vanishes on $U(0,1)$ and is equal to 1 on $U(x,\varepsilon)$. Let $\widetilde{H}_1^y$ be the excursion law $\overline{H}_1^y$ truncated to excursions which do not hit $U(0,1)$. The transition probabilities for $\widetilde{H}_1^y$-excursions which lie in A_ε are those of an h_1-process.

Step 1. In this step, we will analyze the point process of excursions of the process X under P^0 conditioned by $\{Y_a^{x,\varepsilon} = 1\}$. We start with a preliminary result on processes conditioned to spend a given amount of local time on a circle.

Fix some x. Recall that $(L^{x,\varepsilon}, H_1)$ is an exit system from $U(x,\varepsilon)$ under P^0 normalized as in (2.2). We will show that for each $\delta > 0$ there exists $\varepsilon_0 > 0$ such that if $\varepsilon \in (0, \varepsilon_0)$, $y_1, y_2 \in U(x,\varepsilon)$ and $b > 0$ then

$$\frac{P^{y_1}(L^{x,\varepsilon}(\tau) \geq b)}{P^{y_2}(L^{x,\varepsilon}(\tau) \geq b)} \in (1-\delta, 1+\delta). \tag{5.4}$$

It follows from Lemma 2.4 (i) that the hitting distribution of $U(x,\sqrt{\varepsilon})$ for the first excursion from $U(x,\varepsilon)$ which hits $U(x,\sqrt{\varepsilon})$ and starts from any $z \in U(x,\varepsilon)$ has a density whose ratio with the uniform density (properly normalized) lies within $(1-\delta, 1+\delta)$ provided ε is small. The amount of local time spent on $U(x,\varepsilon)$ before hitting $U(x,\sqrt{\varepsilon})$ does not depend on where on the circle $U(x,\varepsilon)$ the process starts, by rotation invariance. Let S be the hitting time of $U(x,\sqrt{\varepsilon})$. Then for all $y_1, y_2 \in U(x,\varepsilon)$, $s > 0$ and $z \in U(x,\sqrt{\varepsilon})$,

$$\frac{P^{y_1}(L^{x,\varepsilon}(S) \in ds, X(S) \in dz)}{P^{y_2}(L^{x,\varepsilon}(S) \in ds, X(S) \in dz)} \in ((1-\delta)^2, (1+\delta)^2).$$

An application of the strong Markov property at S shows that

$$\frac{P^{y_1}(L^{x,\varepsilon}(\tau) \geq b, L^{x,\varepsilon}(S) \leq b)}{P^{y_2}(L^{x,\varepsilon}(\tau) \geq b, L^{x,\varepsilon}(S) \leq b)} \in ((1-\delta)^2, (1+\delta)^2). \tag{5.5}$$

Since $\tau > S$,

$$\frac{P^{y_1}(L^{x,\varepsilon}(\tau) \geq b, L^{x,\varepsilon}(S) > b)}{P^{y_2}(L^{x,\varepsilon}(\tau) \geq b, L^{x,\varepsilon}(S) > b)} = \frac{P^{y_1}(L^{x,\varepsilon}(S) > b)}{P^{y_2}(L^{x,\varepsilon}(S) > b)} = 1.$$

This and (5.5) imply (5.4) except that we may have to change the value of δ.

Let $V_t = (X_t, L_t^{x,\varepsilon})$. We will sometimes refer to X as the first component of V. Note that the process V_t is Markov and let $\Pi^{y,b}$ denote the distribution of V starting from (y, b). Fix some $b_0 > 0$ and let $\Pi_g^{y,b}$ be the distribution of V under $\Pi^{y,b}$ conditioned to exit $U^-(0,1) \times (0, b_0)$ through $U^-(0,1) \times \{b_0\}$ (here g is an appropriate Π-harmonic function). The distribution of X_t under P^y conditioned by $\{L^{x,\varepsilon}(\tau) > b_0\}$ is the same as the distribution of the first component of V under $\Pi_g^{y,0}$.

Suppose that $V_0 = (z, b_1)$ for some $(z, b_1) \in U^-(0,1) \times (0, b_0)$. Let S be the first hitting time of $B_\varepsilon \stackrel{\text{df}}{=} U(0,1) \cup U(x,\varepsilon)$ by X. The local time $L^{x,\varepsilon}$ does not increase between 0 and S. Hence, the distribution of V between 0 and S under Π^{z,b_1} given $\{X(S) = y\}$ is independent of the σ-field generated by the second component of V. The density of the distribution of $X(S)$ under Π_g^{z,b_1} at a point $y \in B_\varepsilon$ is proportional to the corresponding density under Π^{z,b_1} multiplied by $P^y(L^{x,\varepsilon}(\tau) > b_0 - b_1)$. Note that since $b_0 - b_1 > 0$, the last probability is equal to 0 for $y \in U(0,1)$. The distribution of $\{X(t), 0 \le t < S\}$ under Π_g^{z,b_1} is therefore that of an h_2-process in A_ε starting from z. The conditioning harmonic function h_2 is equal to 0 on $U(0,1)$ and its values on $U(x,\varepsilon)$ lie in $(1-\delta, 1+\delta)$ where $\delta > 0$ may be assumed to be arbitrarily small provided ε is chosen sufficiently small, by (5.4).

The process V under $\Pi_g^{y,0}$ is Markov and the excursion theory applies to it just like to the standard Brownian motion. We will consider excursions of V under $\Pi_g^{y,0}$ from $U(x,\varepsilon) \times [0,\infty)$. Note that the second component of V does not change during such excursions. The first component of an excursion of V corresponds to an excursion of X from $U(x,\varepsilon)$. We have shown that the transition probabilities for X under Π_g^{z,b_1} away from B_ε are those of an h_2-process. This implies that the point process of excursions of X under $\Pi_g^{y,0}$ has the intensity bounded below by $(1-\delta)\widetilde{H}_1^y$ and above by $(1+\delta)\widetilde{H}_1^y$. It remains to check whether this normalization of excursion laws matches that of the local time $\overline{L}^{x,\varepsilon}$ on $U(x,\varepsilon)$. Since $\{L^{x,\varepsilon}(\tau) > b_0\}$ is a condition of strictly positive probability, the a.s. path properties of the conditioned process are the same as those for the unconditioned process. Hence, with and without conditioning, the number of excursions from $U(x,\varepsilon)$ of diameter ρ divided by ρ converges a.s. to a constant multiple of the local time as $\rho \to 0$. It follows that in order to use $\overline{L}^{x,\varepsilon}$ as the local time in the exit system for V, we may have to slightly renormalize the excursion laws of V but the intensity of the point process of excursions of

X under $\Pi_g^{y,0}$ will remain bounded below by $(1-\delta_1)\widetilde{H}_1^y$ and above by $(1+\delta_1)\widetilde{H}_1^y$ for some $\delta_1 = \delta_1(\varepsilon)$ which goes to 0 as $\varepsilon \to 0$.

Step 2. This is the main part of the proof. Suppose that X has the distribution P^0 conditioned by $\{Y_a^{x,\varepsilon} = 1\}$. Recall the decomposition (4.1) of X into X^f, X^m and X^l.

Recall the exit system $(\overline{L}^{x,\varepsilon}, \overline{H}_1)$ defined at the beginning of the proof. Note that $\overline{L}^{x,\varepsilon}$ is equal to 1 at the lifetime of X^m.

Suppose that $X(t_1) \in U(x,\varepsilon)$, $X(t_2) \in U(x,\varepsilon)$ and $X(t) \notin U(x,\varepsilon)$ for $t \in (t_1,t_2)$. Let $u = \overline{L}^{x,\varepsilon}(t_1)$. Then let

$$e_u^\varepsilon(s) = \begin{cases} X(s+t_1) & \text{if } s \in [0, t_2-t_1), \\ \Delta & \text{if } s \geq t_2 - t_1. \end{cases}$$

The collection of pairs $W_\varepsilon \stackrel{\text{df}}{=} \{(u, e_u^\varepsilon), u \leq 1\} \subset \mathcal{W}$ is the point process of excursions of X^m from $U(x,\varepsilon)$. Recall that the analogous point process of excursions for Z_2 is called W.

Let $\mathcal{W}_r$ be the subset of $\mathcal{W}$ consisting of all pairs (t,e) such that the path e hits $U(x,r)$. Let $\mathcal{D} \stackrel{\text{df}}{=} D(C_*[0,\infty))$ be the space of functions defined on $[0,1]$ which are right continuous, have left limits and take values in $C_*[0,\infty)$. We equip it with the Skorohod topology and a compatible metric ρ. This is possible since $C_*[0,\infty)$ is a metric space. If $V \subset \mathcal{W}$ is finite, non-empty and consists of pairs (t, e_t) such that there are no two distinct pairs with the same first coordinate, we will identify V with a function $f \stackrel{\text{df}}{=} \pi(V) \in \mathcal{D}$ such that $f(s) = e_t$ where $t = \max\{u \leq s : \exists (u, e_u) \in V\}$. For all other $V \subset \mathcal{W}$ we will let $\pi(V)$ be the function which is identically equal to the zero function in $C_*[0,\infty)$. Let $\mathcal{L}^r$ and $\mathcal{L}_\varepsilon^r$ be the distributions of $\pi(W \cap \mathcal{W}_r)$ and $\pi(W_\varepsilon \cap \mathcal{W}_r)$, resp. We will prove that the $\mathcal{L}_\varepsilon^r$ converge weakly to $\mathcal{L}^r$ as $\varepsilon \to 0$. It will suffice to show that the intensity of the point process of excursions of X^m that hit $U(x,r)$ converges to that of Z_2 and that the distribution of the k-th excursion of X^m that hits $U(x,r)$ converges to that of the k-th excursion of Z_2 that hits $U(x,r)$.

We have already defined h_1 as the positive harmonic function in $A_\varepsilon = U^-(0,1) \cap U^+(x,\varepsilon)$ which vanishes on $U(0,1)$ and is equal to 1 on $U(x,\varepsilon)$. Recall that $\widetilde{H}_1^y$ is the excursion law $\overline{H}_1^y$ truncated to excursions which do not hit $U(0,1)$. The transition probabilities for $\widetilde{H}_1^y$-excursions which lie in A_ε are those of an h_1-process.

Fix some small $\delta > 0$. For small ε and $y \in U(x,\varepsilon)$ we have

$$\frac{P^y(X(T(U(x,r))) \in dz)}{|dz|/(2\pi r)} \in (1-\delta, 1+\delta)$$

and so

$$\frac{h(y)P_h^y(X(T(U(x,r))) \in dz)}{h(z)|dz|/(2\pi r)} \in (1-\delta, 1+\delta).$$

Recall that $\lim_{z\to x} h(z)/|\log|z-x|| = 1$. This, Lemma 5.1 (i) and the strong Markov property applied at the hitting time of $U(x,\varepsilon)$ imply that

$$\frac{H^x(X(T(U(x,r))) \in dz)}{ah(z)|dz|/(2\pi r)} \in (1-\delta, 1+\delta).$$

Since δ is arbitrarily small,

$$H^x(X(T(U(x,r))) \in dz) = ah(z)|dz|/(2\pi r).$$

It follows from Lemma 2.3 (iii), scaling and Lemma 2.4 (i) that for $y \in U(x,\varepsilon)$ the ratio of $\overline{H}_1^y$-probability of hitting $dz \subset U(x,r)$ to

$$\frac{1}{\varepsilon|\log(\varepsilon/r)|}(a\varepsilon\log^2\varepsilon - \varepsilon|\log\varepsilon|\log|\log\varepsilon|)\frac{|dz|}{2\pi r} \tag{5.6}$$

lies in $(1-\delta, 1+\delta)$ provided ε is small. In order to obtain the analogous result for $\widetilde{H}_1^y$, we have to multiply the expression in (5.6) by $P^z(T(U(x,\varepsilon)) < \tau)$ which is equal to $h_1(z)$. Hence

$$\frac{\widetilde{H}_1^y(X(T(U(x,r))) \in dz)\varepsilon|\log(\varepsilon/r)|2\pi r}{h_1(z)(a\varepsilon\log^2\varepsilon - \varepsilon|\log\varepsilon|\log|\log\varepsilon|)|dz|} \in (1-\delta, 1+\delta)$$

for small ε. It is routine to check that $|\log\varepsilon|h_1(z)$ converges to $h(z)$ when $\varepsilon \to 0$. Thus

$$\widetilde{H}_1^y(X(T(U(x,r))) \in dz) \to ah(z)|dz|/(2\pi r) = H^x(X(T(U(x,r))) \in dz) \tag{5.7}$$

when $\varepsilon \to 0$ uniformly in $y \in U(x, \varepsilon)$. In particular

$$\widetilde{H}_1^y(T(U(x,r)) < \infty) \to H^x(T(U(x,r)) < \infty)$$

uniformly in $y \in U(x, \varepsilon)$ as $\varepsilon \to 0$. We conclude from this and Step 1 that the intensity of the process of excursions of X^m that hit $U(x,r)$ converges to that of Z_2 as $\varepsilon \to 0$.

The distribution of the part of the k-th $\widetilde{H}_1^y$-excursion which hits $U(x,r)$ after the hitting time of $U(x,r)$ is that of an h_1-process starting from the hitting place of $U(x,r)$ and as $\varepsilon \to 0$, it converges to the distribution of an h-process which is the distribution of the analogous part of the k-th H^y-excursion assuming it hits $U(x,r)$ at the same place. The time-reversed part of the $\widetilde{H}_1^y$-excursion between its start and the hitting time of $U(x,r)$ is that of an h_2-process starting from the hitting place of $U(x,r)$. Here h_2 is a positive harmonic function in $U^-(x,r) \cap U^+(x,\varepsilon)$ which vanishes on $U(x,r)$. When $\varepsilon \to 0$, this distribution converges to the distribution of an h-process which is the same as for the first part of the H^x-excursion. Finally, we note that by the strong Markov property in the case of $\widetilde{H}_1^y$- and H^y-excursions, the two parts of the excursion are independent given the hitting point of $U(x,r)$. This, Step 1 and the convergence of the hitting distributions in (5.7) imply that the $\mathcal{L}_\varepsilon^r$ converge weakly to $\mathcal{L}^r$ as $\varepsilon \to 0$.

Let us equip $\mathcal{W}$ with the metric

$$\nu(V_1, V_2) = \sum_{n=1}^{\infty} 2^{-n} \min(1, \rho(\pi(V_1 \cap \mathcal{W}_{1/n}), \pi(V_2 \cap \mathcal{W}_{1/n}))).$$

We see that W_ε converge in distribution to W on $(\mathcal{W}, \nu)$ as $\varepsilon \to 0$. By the Skorohod representation we may assume that X^m and Z are defined on a probability space such that for each $r > 0$ the processes $\pi(W_\varepsilon \cap \mathcal{W}_r)$ converge to $\pi(W \cap \mathcal{W}_r)$ a.s.

Let $\widetilde{G}_\varepsilon(y, \cdot\,)$ be the density of the expected occupation measure of $\widetilde{H}_1^y$. By Step 1 and Lemma 2.2 (ii), the expected amount of time spent by X^m in $U^-(x,r)$ is bounded by

$$(1+\delta_1) \sup_{y \in U(x,\varepsilon)} \int_{U(x,r)} \widetilde{G}_\varepsilon(y,z) dz$$

and this goes to 0 as $r \to 0$ uniformly in $\varepsilon < r$ by Lemma 2.7. Now standard arguments may be applied to show that this and the a.s. convergence of $\pi(W_\varepsilon \cap \mathcal{W}_r)$ to $\pi(W \cap \mathcal{W}_r)$ for each r imply the weak convergence of X^m to Z_2.

Let T_1 be the first time when $\overline{L}^{x,\varepsilon}$ reaches the level 1. The processes X^m and X^l are independent given $\{X(T_1) = y\}$ and the distribution of X^l given this condition is that of a Brownian motion starting from y and killed upon hitting of $U(0,1)$. Since $y \in U(x,\varepsilon)$, it is clear that X^l converges in distribution to Brownian motion starting from x as $\varepsilon \to 0$ and the processes X^m and X^l are asymptotically independent. The first part of the trajectory (i.e., X^f) may be treated in a similar way. $\square$

Proposition 5.2. *For every $x \in U^-(0,1)$, the measures Q_a^y converge weakly to Q_a^x as $y \to x$.*

Proof. The proof is similar to that of Proposition 5.1 and so is omitted. $\square$

Theorem 5.2. *For every $a \in (0,1/2)$ and each non-negative measurable function f on $\mathbb{C} \times C_*[0,\infty)$ we have*

$$E^0 \int_{\mathbb{C}} f(y,X)\beta_a(dy) = \int_{D_*} Q_a^y(f(y,X))|1-y^2|^a|\log|y||dy. \tag{5.8}$$

Proof. Suppose

$$f(y,X) = \mathbf{1}_D(y)\mathbf{1}_A(X) \tag{5.9}$$

for some rectangle D and $A \subset C_*[0,\infty)$. Then

$$\begin{aligned}\left|E^0 \int_{\mathbb{C}} f(y,X)\beta_a(dy) - E^0 \int_{\mathbb{C}} f(y,X)\beta_a^\varepsilon(dy)\right| &\le E^0(\mathbf{1}_A(X)|\beta_a(D) - \beta_a^\varepsilon(D)|) \\ &\le (E^0\mathbf{1}_A(X))^{1/2}(E^0(\beta_a(D) - \beta_a^\varepsilon(D))^2)^{1/2} \to 0\end{aligned}$$

as $\varepsilon \to 0$. Hence

$$E^0 \int_{\mathbb{C}} f(y,X)\beta_a(dy) = \lim_{\varepsilon\to 0} E^0 \int_{\mathbb{C}} f(y,X)\beta_a^\varepsilon(dy)$$

for functions of the form (5.9). This can be easily extended to the class of all bounded non-negative continuous functions f. Note that it will suffice to prove (5.8) for this class of functions. Now we interchange the order of integration and summation in the last expression to obtain

$$E^0 \int_{\mathbb{C}} f(y, X)\beta_a(dy) = \lim_{\varepsilon \to 0} \sum_{y \in \mathbb{Z}_\varepsilon^2} \varepsilon^{2-a} E^0 Y_a^{y,\varepsilon} E^0(f(y, X) \mid Y_a^{y,\varepsilon} = 1). \tag{5.10}$$

By Propositions 5.1 and 5.2, $E^0(f(z, X) \mid Y_a^{z,\varepsilon} = 1)$ converge to $Q_a^y(f(y, X))$ as $\varepsilon \to 0$ and $z \to y$. It follows from this and Theorem 3.1 that the limit in (5.10) is equal to the right hand side of (5.8). □

Remark 5.2. (i) There is only one random measure β_a which satisfies (5.8) and is a measurable functional of the Brownian path. Indeed, if there were another measure $\widetilde{\beta}_a$ with the same property we would have

$$E^0 \int_{\mathbb{C}} f(y, X)(\beta_a(dy) - \widetilde{\beta}_a(dy)) = 0$$

for all non-negative functions f and $\widetilde{\beta}_a$ would be identically equal to β_a.

(ii) It follows from Theorem 5.2 that if $Q_a^x(A_x) = 1$ for all x, then with probability 1, the event A_x holds for β_a-almost all x.

Corollary 5.1. *Let N_ε^x be the number of excursions of X from x which hit $U(x, \varepsilon)$. The measure β_a is P^0-a.s. supported on the set of points x such that*

$$\lim_{\varepsilon \to 0} N_\varepsilon^x / |\log \varepsilon| = a.$$

In particular, the measure β_a is supported on points x which are visited infinitely often by the Brownian motion X.

Proof. Fix some $x \in U(0, 1)$ and let Z have the distribution Q_a^x. Let $\widetilde{N}_\varepsilon^x$ be the number of excursions of Z from x which hit $U(x, \varepsilon)$ and let $M_k^x = \widetilde{N}_{e^{-k}}^x - \widetilde{N}_{e^{-k+1}}^x$. By Lemma 5.1 (i),

$$\lim_{k \to \infty} H^x(T(U(0, e^{-k})) < T(U(0, e^{-k+1})) = \infty) = a.$$

It follows from Lemma 2.2 (i) that M_k^x is minorized and majorized by Poisson random variables with expectations $a - \delta_k$ and $a + \delta_k$, resp., where $\delta_k \to 0$ as $k \to \infty$. The random variables M_k^x represent the numbers of excursions in disjoint subsets of $C_*[0, \infty)$ for different k. Since Q_a^x-excursions form a Poisson point process (by definition), the M_k^x's are independent for different k. The strong law of large numbers shows that Q_a^x-a.s.

$$\lim_{j \to \infty} \sum_{k=1}^{j} M_k^x / j = a$$

and this implies that

$$\lim_{j \to \infty} \widetilde{N}_{e^{-j}}^x / j = a.$$

It follows that

$$\lim_{\varepsilon \to 0} \widetilde{N}_\varepsilon^x / |\log \varepsilon| = a.$$

Now the corollary follows easily from Remark 5.2 (ii). □

Corollary 5.2. *With P^0-probability 1, $\beta_a(D) > 0$ for every open set $D \subset D_*$ such that $T(D) < \infty$. Hence $\beta_a(D_*) > 0$ P^0-a.s.*

Proof. Let $r \in (0, \infty)$ and let $Y_{a,r}^{x,\varepsilon}$ be the indicator function of the event

$$\{L^{x,\varepsilon}(\tau) \geq a\varepsilon \log^2(r\varepsilon) - \varepsilon |\log(r\varepsilon)| \log |\log(r\varepsilon)|\}.$$

We would like to argue that if we replaced $Y_a^{x,\varepsilon}$ by $Y_{a,r}^{x,\varepsilon}$ in the definition of β_a^ε and in the definition of β_a, then we would obtain a measure $\widetilde{\beta}_a = \widetilde{\beta}_{a,r}$ such that

$$\widetilde{\beta}_a = r^{2a} \beta_a. \tag{5.11}$$

To prove the existence of $\widetilde{\beta}_a$ we repeat the arguments of Sections 3-5 with $Y_{a,r}^{x,\varepsilon}$ in place of $Y_a^{x,\varepsilon}$. Since this is a tedious and routine task, we will omit the details. However we will verify (5.11) by proving an inequality analogous to (3.4). Under the assumptions (3.2) and (3.3) we have for small ε

$$
\begin{aligned}
P^0(Y^{x,\varepsilon}_{a,r}=1) &= P^0(L^{x,\varepsilon}(\tau)\geq a\varepsilon\log^2(r\varepsilon)-\varepsilon|\log(r\varepsilon)|\log|\log(r\varepsilon)|) \\
&= P^0(T(U(x,\varepsilon))<\tau)E^0P^{X(T(U(x,\varepsilon)))}(L^{x,\varepsilon}(\tau)\geq a\varepsilon\log^2(r\varepsilon)-\varepsilon|\log(r\varepsilon)|\log|\log(r\varepsilon)|) \\
&\leq c_1(\log|x|/\log\varepsilon)\exp\left(-(a\varepsilon\log^2(r\varepsilon)-\varepsilon|\log(r\varepsilon)|\log|\log(r\varepsilon)|)\frac{1}{\varepsilon|\log(\varepsilon/c_2)|}\right) \\
&= c_1\frac{\log|x|}{\log\varepsilon}\exp\left(-a|\log\varepsilon|+a\log c_2+\frac{a\log^2 c_2}{\log\varepsilon-\log c_2}+2a\log r+\frac{2a\log r\log c_2}{\log\varepsilon-\log c_2}\right. \\
&\qquad\left.-\frac{a\log^2 r}{|\log\varepsilon-\log c_2|}+\log|\log(r\varepsilon)|+\frac{\log c_2\log|\log(r\varepsilon)|}{\log\varepsilon-\log c_2}-\frac{\log r\log|\log(r\varepsilon)|}{|\log\varepsilon-\log c_2|}\right) \\
&\leq (1+\delta)c_1(\log|x|/\log\varepsilon)r^{2a}c_2^a\varepsilon^a|\log(r\varepsilon)|\leq(1+2\delta)c_1r^{2a}c_2^a\varepsilon^a|\log|x||.
\end{aligned}
$$

The only essential difference between this formula and (3.4) is the presence of the factor r^{2a} on its right hand side. Thus (3.1) holds also for $Y^{x,\varepsilon}_{a,r}$ except that the right hand side is multiplied by r^{2a}. By repeating the arguments of Sections 3-5 we may construct a measure $\widetilde{\beta}_a$ corresponding to $Y^{x,\varepsilon}_{a,r}$ which satisfies (5.8) except for a factor of r^{2a}. By the uniqueness of β_a (see Remark 5.2 (i)) we see that (5.11) must hold.

Let $X^r(t)\stackrel{\text{df}}{=}X(tr^2)/r$. If X is a Brownian motion starting from 0 and killed upon hitting of $U(0,1)$ then X^r is a Brownian motion starting from 0 and killed at the hitting time of $U(0,1/r)$. Let $L^{x,\varepsilon}_r$ be the local time of X^r on $U(x,\varepsilon)$. Note that

$$rL^{x,\varepsilon}_r(T(U(0,1/r)))=L^{rx,r\varepsilon}(\tau).$$

Hence

$$
\begin{aligned}
&\{L^{x,\varepsilon}_r(T(U(0,1/r)))/r\geq a\varepsilon\log^2(r\varepsilon)-\varepsilon|\log(r\varepsilon)|\log|\log(r\varepsilon)|\} \\
&=\{rL^{x,\varepsilon}_r(T(U(0,1/r)))\geq ar\varepsilon\log^2(r\varepsilon)-r\varepsilon|\log(r\varepsilon)|\log|\log(r\varepsilon)|\}
\end{aligned}
$$

occurs if and only if

$$\{L^{rx,r\varepsilon}(\tau)\geq ar\varepsilon\log^2(r\varepsilon)-r\varepsilon|\log(r\varepsilon)|\log|\log(r\varepsilon)|\}$$

holds. This identity of events leads via the constructions of β_a and $\widetilde{\beta}_{a,r}$ to $\widetilde{\beta}_{a,r}^{X^r}(D/r) = r^2 \beta_a^X(D)$. Since $\widetilde{\beta}_a$ is just a constant multiple of β_a, we see that

$$P(\beta_a^{X^r}(D/r) > 0) = P(\beta_a^X(D) > 0). \tag{5.12}$$

Let $\beta_a(s,t,D)$ be the β_a-measure of D defined relative to $X([s,t])$ (see Remark 5.1 (i)). Let $T_r = T(U(0,r))$. Then (5.12) implies that

$$P^0(\beta_a(0,T_r,U^-(0,r)) > 0) \stackrel{\text{df}}{=} p > 0. \tag{5.13}$$

Let X^1 and X^2 be independent Brownian motions starting from 0. If the event

$$A_r \stackrel{\text{df}}{=} \{\beta_a^{X^1}(0,T_r,U^-(0,r)) > 0\}$$

holds then there is a (random) point x^r such that

$$\beta_a^{X^1}(0,T_r,U^-(x^r,\delta)) > 0$$

for every $\delta > 0$. Here is one way of choosing x^r in a measurable way. Let

$$B_r = \{x : \beta_a^{X^1}(0,T_r,U^-(x,\delta)) > 0 \ \ \forall \delta > 0\}.$$

Note that B_r is non-empty if A_r holds, and B_r is necessarily closed. Let

$$\begin{aligned} x_1^r &= \inf\{x_1 : \exists x_2 \text{ such that } (x_1,x_2) \in B_r\}, \\ x_2^r &= \inf\{x_2 : (x_1^r, x_2) \in B_r\}, \\ x^r &= (x_1^r, x_2^r). \end{aligned}$$

One can easily modify this definition of x^r to make sure that $x^r \neq X^1(T_r)$.

By (5.13), the event

$$\bigcap_{r_1>0} \bigcup_{r<r_1} \{\beta_a^{X^1}(0,T_r,U^-(0,r)) > 0\}$$

has probability $p > 0$. Since it belongs to the σ-field $\mathcal{F}_{0+}$, its probability is 1, by Blumenthal's 0-1 law. It follows that a.s. there are sequences $\{y^k\}_{k\geq 1}$ and $\{r_k\}_{k\geq 1}$ such that $y^k \to 0$ as $k \to \infty$ and for each k and $\delta > 0$,

$$\beta_a^{X^1}(0, T_{r_k}, U^-(y^k, \delta)) > 0.$$

Let $\beta_a^{X^1,X^2}(s,t,u,v,D)$ be the β_a-measure of D defined relative to $X^1([s,t]) \cup X^2([u,v])$. Since X^1 and X^2 are independent and X^2 would not hit a fixed point, for every fixed k, $y^k \notin X^2([0,1])$ and, therefore, $U^-(y^k, \delta_k) \not\subseteq X^2([0,1])$ for some $\delta_k > 0$. For a similar reason, using the strong Markov property at T_{r_k}, we obtain $U^-(y^k, \delta_k) \not\subseteq X^1([T_{r_k}, 1])$ for some $\delta_k > 0$. Since β_a is defined locally and

$$\beta_a^{X^1}(0, T_{r_k}, U^-(y^k, \delta_k)) > 0$$

we obtain

$$\beta_a^{X^1,X^2}(0,1,0,1,U^-(y^k, \delta_k)) > 0$$

a.s. for every k. We see that with probability 1, $\beta_a^{X^1,X^2}(0,1,0,1,U^-(0,r)) > 0$ for all $r > 0$.

For a fixed $t > 0$, the P^0-distribution of $\{(X(s+t)-X(t), X(t-s)-X(t)), s \in [0,t)\}$ is locally mutually absolutely continuous with that of (X^1, X^2) so a.s. $\beta_a(U(X(t), r)) > 0$ for all $r > 0$. Thus a.s. for all rational $t, r > 0$ such that $X(t) \in D_*$ we have $\beta_a(U(X(t), r)) > 0$. This clearly implies our claim. $\square$

6. Hausdorff dimension. In this section we will interpret β_a in the spirit of Remark 5.1 (i), i.e., as a measure defined locally and relative to the whole path of a process. We start with two technical lemmas.

Lemma 6.1. *For every $a \in (0, 1/2)$ and $\delta > 0$ there is an integer $M < \infty$ with the following property. For β_a-almost every point x there exists a (random) m such that for* *all $k > m$, $\varepsilon = 2^{-k}$, for some integers $i_1, i_2 \in [0, M]$ and some $z \in \mathbb{Z}^2_\varepsilon + (i_1\varepsilon/M, i_2\varepsilon/M)$ (here i_1, i_2 and z may depend on k) we have $x \in U^-(z, \varepsilon)$ and $Y^{z,\varepsilon}_{(1-\delta)a} = 1$.*

Proof. Fix some $x \in U^-(0, 1/2)$. Suppose Z has distribution Q^x_a. Choose $r \in (0, 1)$ and $\varepsilon_0 > 0$ so that Lemma 2.6 holds with $\delta/4$ in place of δ. Assume that $\varepsilon < \varepsilon_0$. Then choose $M < \infty$ so that for each x and ε one can find $i_1, i_2 \in [0, M]$ and $z \in \mathbb{Z}^2_\varepsilon + (i_1\varepsilon/M, i_2\varepsilon/M)$ such that $|z - x| < r\varepsilon$. For each k, the numbers i_1, i_2 and $z = z_k$ will be chosen so that the last condition is satisfied.

Note that $U^-(z, \varepsilon) \subset U^-(x, 2\varepsilon)$. Lemma 5.1 (i) and Lemma 2.2 (i) imply that the number of excursions of Z from x which hit $U(z, \varepsilon)$ and return to x is minorized by a Poisson random variable with expectation greater than $c_1 a|\log \varepsilon|$, where c_1 may be chosen arbitrarily close to 1 provided ε is small. We will assume that $c_1(1 - \delta/4) > (1 - \delta/2)$.

Let $h_x(\,\cdot\,)$ be the positive harmonic function in $U^-(0, 1)\setminus\{x\}$ which vanishes on $U(0, 1)$ and has a pole at x. The part of an excursion of Z from the hitting time of $U(z, \varepsilon)$ until its return to x has the distribution of an h_x-process, i.e., conditioned Brownian motion in $U^-(0, 1) \setminus \{x\}$ starting from the hitting point of the circle $U(z, \varepsilon)$ and converging to x. Lemma 2.6 and Lemma 2.2 (i) imply that the amount of local time spent on $U(z, \varepsilon)$ by this excursion is stochastically minorized by an exponential random variable with mean $(1 - \delta/4)\varepsilon|\log \varepsilon|$.

Suppose that N and V_k, $k \geq 1$, are independent random variables, N has a Poisson distribution with mean $\rho \stackrel{\text{df}}{=} c_1 a|\log \varepsilon|$, and the V_k's are i.i.d. with exponential distribution with mean $\lambda^{-1} \stackrel{\text{df}}{=} (1 - \delta/4)\varepsilon|\log \varepsilon|$. Let $R = \sum_{k=1}^N V_k$. For $b > -\lambda$ one has $Ee^{-bV_k} = 1/(1 + b/\lambda)$. By the independence of the V_k's,

$$E(\exp(-bR) \mid N) = (E\exp(-bV_k))^N$$

and so

$$
\begin{aligned}
E\exp(-bR) &= \sum_{k=0}^{\infty} e^{-\rho}\frac{\rho^k}{k!}(E\exp(-bV_k))^k \\
&\le \sum_{k=0}^{\infty} e^{-\rho}\frac{\rho^k}{k!}\left(\frac{1}{1+b/\lambda}\right)^k \\
&= \exp\left(-\rho + \frac{\rho}{1+b/\lambda}\right) \\
&= \exp\left(-\frac{\rho b/\lambda}{1+b/\lambda}\right). \qquad (6.1)
\end{aligned}
$$

Substituting the values of ρ and λ gives

$$
\begin{aligned}
E\exp(-bR) &= \exp\left(-\frac{abc_1(1-\delta/4)\varepsilon\log^2\varepsilon}{1+b(1-\delta/4)\varepsilon|\log\varepsilon|}\right) \\
&\le \exp\left(-\frac{ab(1-\delta/2)\varepsilon\log^2\varepsilon}{1+b(1-\delta/4)\varepsilon|\log\varepsilon|}\right).
\end{aligned}
$$

The Chebyshev inequality yields for $b > 0$

$$
\begin{aligned}
P(R \le (1-\delta)a\varepsilon\log^2\varepsilon) &\le P(\exp(-bR) \ge \exp(-b(1-\delta)a\varepsilon\log^2\varepsilon) \\
&\le E\exp(-bR)\exp(b(1-\delta)a\varepsilon\log^2\varepsilon) \\
&\le \exp\left(-\frac{ab(1-\delta/2)\varepsilon\log^2\varepsilon}{1+b(1-\delta/4)\varepsilon|\log\varepsilon|}\right)\exp(b(1-\delta)a\varepsilon\log^2\varepsilon). \qquad (6.2)
\end{aligned}
$$

Let

$$
b = \frac{\delta}{4(1-\delta)(1-\delta/4)\varepsilon|\log\varepsilon|}
$$

and $c_2 = c_2(a,\delta) = a\delta^2/(3\delta^2 - 16\delta + 16)$. Then $c_2 > 0$ for $\delta \in (0,1)$ and the right hand side of (6.2) is equal to ε^{c_2}.

We have proved that for $\varepsilon = 2^{-k}$, the probability that the process Z spends less than $(1-\delta)a\varepsilon\log^2\varepsilon$ units of local time on $U(z_k, 2^{-k})$ is less than 2^{-kc_2}. By the Borel-Cantelli lemma, for all large k the process Z spends more than $(1-\delta)a\varepsilon\log^2\varepsilon$ units of local time on $U(z_k, 2^{-k})$ Q_a^x-a.s. It remains to use Remark 5.2 (ii) to complete the proof of the lemma. $\square$

Lemma 6.2. *For every $a \in (0, 1/2)$, $\delta > 0$, and for β_a-almost every point x there exists a (random) m such that for all $k > m$, $\varepsilon = 2^{-k}$, and every disc $U^-(z,\varepsilon)$ such that $z \in \mathbb{Z}^2_\varepsilon$ and $x \in U^-(z,\varepsilon)$ we have $\beta_a(U^-(z,\varepsilon)) \le \varepsilon^{2-a-\delta}$.*

Remark 6.1. For each x with $3\varepsilon < |x| < 1/16 - \varepsilon$ there is at least one disc $U^-(z,\varepsilon)$ such that $z \in \mathbb{Z}^2_\varepsilon$ and $x \in U^-(z,\varepsilon)$.

Proof. We fix some $x \in U^-(0,1/2)$ and let h be a positive harmonic function in $U^-(x,2) \setminus \{x\}$ which has a pole at x and zero boundary values on $U(x,2)$. Suppose Z has distribution analogous to Q^x_a except that the first part of the process is an h-path from 0 to x, excursions of Z from x have the transition densities of the h-process in $U^-(x,2) \setminus \{x\}$, the excursion law is normalized so that

$$\lim_{\varepsilon\to 0} H^x(T(U(x,\varepsilon)) < \infty)/|\log\varepsilon| = a, \tag{6.3}$$

and the last part of Z is a Brownian motion starting at x and killed upon hitting $U(x,2)$.

Let β^Z_a be the intersection local time for Z. The construction of β_a was performed only for Brownian motion, but the existence of β^Z_a is assured in light of Remark 5.1 (i). Although Z and X do not have mutually absolutely continuous distributions, this is true locally and the existence of β^Z_a may be proved using a localizing argument.

First we obtain an upper estimate for the total local time (say, L^Z_ε) spent by Z on $U(x,\varepsilon)$ by repeating the argument presented in the proof of Lemma 6.1. Fix some $\gamma > 0$. Formula (6.3) and Lemma 2.2 (i) imply that the number of excursions of Z from x which hit $U(z,\varepsilon)$ and return to x is majorized by a Poisson random variable with the expectation smaller than $(1+\gamma)a|\log\varepsilon|$ provided ε is small.

It is easy to check that Lemma 2.6 applies also in the present context. The part of an excursion of Z after hitting of $U(x,\varepsilon)$ until its return to x has the distribution of an h-process. Hence Lemma 2.6 and Lemma 2.2 (i) imply that the amount of local time spent on $U(x,\varepsilon)$ by this excursion is stochastically majorized by an exponential random variable with mean $(1+\gamma)\varepsilon|\log\varepsilon|$.

Suppose that N and V_k, $k \geq 1$, are independent random variables, N has a Poisson distribution with the mean $\rho \stackrel{\text{df}}{=} (1+\gamma)a|\log\varepsilon|$, V_k's are i.i.d. with the exponential distribution with the mean $\lambda^{-1} \stackrel{\text{df}}{=} (1+\gamma)\varepsilon|\log\varepsilon|$. Let $R = \sum_{k=1}^N V_k$. By (6.1), for $b > -\lambda$,

$$E\exp(-bR) = \exp\left(-\frac{\rho b/\lambda}{1+b/\lambda}\right)$$

and hence

$$E\exp(-bR) = \exp\left(-\frac{ab(1+\gamma)^2\varepsilon\log^2\varepsilon}{1+b(1+\gamma)\varepsilon|\log\varepsilon|}\right).$$

The Chebyshev inequality yields for $b \in (-\lambda, 0)$

$$\begin{aligned} P(R \geq (1+3\gamma)a\varepsilon\log^2\varepsilon) &\leq P(\exp(-bR) \geq \exp(-b(1+3\gamma)a\varepsilon\log^2\varepsilon) \\ &\leq E\exp(-bR)\exp(b(1+3\gamma)a\varepsilon\log^2\varepsilon) \\ &\leq \exp\left(-\frac{ab(1+\gamma)^2\varepsilon\log^2\varepsilon}{1+b(1+\gamma)\varepsilon|\log\varepsilon|}\right)\exp(b(1+3\gamma)a\varepsilon\log^2\varepsilon). \end{aligned} \tag{6.4}$$

Now we let

$$b \stackrel{\text{df}}{=} -\frac{\gamma}{2(1+\gamma)(1+3\gamma)a\varepsilon|\log\varepsilon|} > -\lambda$$

and

$$c_1 \stackrel{\text{df}}{=} -\frac{\gamma^2(2\gamma-1)}{10\gamma^2+14\gamma+4} > 0.$$

The last two inequalities hold for small $\gamma > 0$. With this choice of b and c_1, the right hand side of (6.4) is equal to ε^{c_1}. Hence

$$P(L_\varepsilon^Z \geq (1+3\gamma)a\varepsilon\log^2\varepsilon) \leq \varepsilon^{c_1}. \tag{6.5}$$

Let L_ε^X be the local time spent by the process X with distribution P^0 on $U(x,\varepsilon)$ before hitting $U(x,2)$. An estimate analogous to (3.4) gives for small ε

$$P^0(L_\varepsilon^X \geq (1+4\gamma)a\varepsilon\log^2\varepsilon) \geq c_2\varepsilon^{(1+5\gamma)a}$$

where $c_2 < \infty$ may depend on x. It follows from (5.8) that for any fixed x and $r > 1$ and all ε,

$$E^0 \beta_a(U^-(x, r\varepsilon)) \le c_3 \varepsilon^2$$

and this combined with the previous inequality yields

$$\begin{aligned} E^0(\beta_a(U^-(x, r\varepsilon)) \mid L_\varepsilon^X \ge (1+4\gamma) a\varepsilon \log^2 \varepsilon) &= \frac{E^0(\beta_a(U^-(x, r\varepsilon)) \mathbf{1}_{\{L_\varepsilon^X \ge (1+4\gamma)a\varepsilon \log^2 \varepsilon\}})}{P^0(L_\varepsilon^X \ge (1+4\gamma) a\varepsilon \log^2 \varepsilon)} \\ &\le c_3 \varepsilon^2 / (c_2 \varepsilon^{(1+5\gamma)a}) = c_4 \varepsilon^{2-(1+5\gamma)a}. \end{aligned} \tag{6.6}$$

Let h_1 be the positive harmonic function in the annulus $A \stackrel{\text{df}}{=} U^-(x, 2) \cap U^+(x, \varepsilon)$ which has boundary values equal to 1 on $U(x, \varepsilon)$ and 0 elsewhere. Note that h_1 is a constant multiple of h in A. The process X conditioned by $\{L_\varepsilon^X = c\}$ may be decomposed into three parts. The first one is a Brownian motion starting from 0, conditioned to hit $U(x, \varepsilon)$ before hitting $U(x, 2)$ and killed upon hitting $U(x, \varepsilon)$. The second part consists of excursions from $U(x, \varepsilon)$ which return to this circle and do not hit $U(x, 2)$. The excursions which stay in A are governed by excursion laws whose transition probabilities are those of an h_1-process. The last part is a Brownian motion starting from a random point of $U(x, \varepsilon)$ and conditioned to hit $U(x, 2)$ before returning to $U(x, \varepsilon)$. This decomposition makes it clear that

$$E^0(\beta_a(U^-(x, r\varepsilon)) \mid L_\varepsilon^X = c)$$

is an increasing function of c. This and (6.6) imply that

$$E^0(\beta_a(U^-(x, r\varepsilon)) \mid L_\varepsilon^X = c) \le c_4 \varepsilon^{2-(1+5\gamma)a} \tag{6.7}$$

for all $c \le (1+4\gamma) a\varepsilon \log^2 \varepsilon$.

Now consider the process Z conditioned by $\{L_\varepsilon^Z = c\}$. It can be decomposed into three parts and the description of the decomposition of X given in the previous paragraph applies to Z. The only difference is that the excursion laws governing the excursions within $U^-(x, \varepsilon)$ are different in both cases. Consequently, the "boundary" processes on $U(x, \varepsilon)$ are different for X and Z. However, the distribution of the number of excursions from $U(x,\varepsilon)$ in A which hit a fixed circle $U(x,r_1\varepsilon)$, $r_1 \in (1, 2/\varepsilon)$, is the same for both conditioned processes X and Z, namely it has a Poisson distribution with mean $c\alpha$, where α depends on ε and r_1. We are using here the fact that both X and Z are killed on exiting $U^-(x,2)$, i.e., x plays the role of the origin and the processes are invariant under rotation.

By Lemma 2.4 (ii), for an arbitrary $\eta > 0$, the Radon-Nikodym derivative of the hitting distributions of $U(x,r_1\varepsilon)$ for excursion laws in A starting at any two points of $U(x,\varepsilon)$ and with the transition probabilities of an h_1-process lies within $(1-\eta, 1+\eta)$ provided r_1 is sufficiently large. Now suppose that Z is conditioned by $\{L^Z_\varepsilon = c\}$ and X is conditioned by $\{L^X_\varepsilon = c(1+4\gamma)/(1+3\gamma)\}$. If we assume that r_1 is sufficiently large and, therefore, η may be assumed to be very small, then the hitting points of $U(x,r_1\varepsilon)$ of excursions of X from $U(x,\varepsilon)$ form a (not necessarily Poisson) point process on $U(x,r_1\varepsilon)$ which has a greater intensity than that for Z. Since the part of each excursion after the hitting time of $U(x,r_1\varepsilon)$ is an h_1-process independent of the past, the intersection local time β^Z_a of $U^-(x,r\varepsilon)\cap U^+(x,r_1\varepsilon)$ is stochastically dominated by β_a for the conditioned processes. In view of (6.7),

$$E(\beta^Z_a(U^-(x,r\varepsilon)\cap U^+(x,r_1\varepsilon)) \mid L^Z_\varepsilon = c) \le c_4\varepsilon^{2-(1+5\gamma)a}$$

for all $c \le (1+3\gamma)a\varepsilon\log^2\varepsilon$. Hence

$$E(\beta^Z_a(U^-(x,r\varepsilon)\cap U^+(x,r_1\varepsilon)) \mid L^Z_\varepsilon \le (1+3\gamma)a\varepsilon\log^2\varepsilon) \le c_4\varepsilon^{2-(1+5\gamma)a}.$$

Let $r = 2r_1$ and $B(\varepsilon) = U^-(x,r\varepsilon)\cap U^+(x,r\varepsilon/2)$. Then

$$P(\beta^Z_a(B(\varepsilon)) \ge \varepsilon^{2-(1+6\gamma)a} \mid L^Z_\varepsilon \le (1+3\gamma)a\varepsilon\log^2\varepsilon) \le c_4\varepsilon^{2-(1+5\gamma)a}/\varepsilon^{2-(1+6\gamma)a} = c_4\varepsilon^{\gamma a}$$

and, by (6.5),

$$\begin{aligned}
&P(\beta^Z_a(B(\varepsilon)) \ge \varepsilon^{2-(1+6\gamma)a}) \\
&\quad \le P(\beta^Z_a(B(\varepsilon)) \ge \varepsilon^{2-(1+6\gamma)a} \mid L^Z_\varepsilon \le (1+3\gamma)a\varepsilon\log^2\varepsilon) + P(L^Z_\varepsilon \ge (1+3\gamma)a\varepsilon\log^2\varepsilon) \\
&\quad \le c_4\varepsilon^{\gamma a} + \varepsilon^{c_1} \le c_5\varepsilon^{c_6}
\end{aligned}$$

where $c_6 > 0$. Now let $\varepsilon = 2^{-k}$. We obtain

$$P(\beta_a^Z(B(2^{-k})) \geq 2^{-k(2-(1+6\gamma)a)}) \leq c_5 2^{-kc_6}.$$

The Borel-Cantelli Lemma implies that a.s. for large k we have

$$\beta_a^Z(B(2^{-k})) \leq 2^{-k(2-(1+6\gamma)a)}.$$

Since

$$\sum_{k=n}^{\infty} 2^{-k(2-(1+6\gamma)a)} \leq c_7 2^{-n(2-(1+6\gamma)a)}$$

and

$$\beta_a^Z(U^-(x, 2^{-n})) \leq \sum_{k=n}^{\infty} \beta_a^Z(B(2^{-k})),$$

we have

$$\beta_a^Z(U^-(x, 2^{-n})) \leq c_7 2^{-n(2-(1+6\gamma)a)}$$

a.s. for large n. Since γ is arbitrarily small, this implies that

$$\beta_a^Z(U^-(x, \varepsilon)) \leq c_8 \varepsilon^{2-a-\delta/2} \tag{6.8}$$

a.s. for $\varepsilon < \varepsilon_0$, where ε_0 may be random.

Now we will modify the process Z. First we remove all excursions from x which return to x and hit $U(0,1)$. We also kill the last part of the process at the hitting time of $U(0,1)$. The intersection local time for the resulting process (say, $\widetilde{Z}$) is obviously less than β_a^Z and, therefore, satisfies (6.8). Finally, replace the first part of $\widetilde{Z}$ with the Brownian motion in $U^-(0,1)$ starting from 0 and conditioned to go to x. This new path has a distribution absolutely continuous with respect to the original one, so (6.8) holds also for this process. Let the new process be called V and the corresponding intersection local time β_a^V. The distribution of V is Q_a^x. We conclude that Q_a^x-a.s. for small ε

$$\beta_a^V(U^-(x, \varepsilon)) \leq c_8 \varepsilon^{2-a-\delta/2}.$$

If $x \in U^-(z,\varepsilon)$ then $U^-(z,\varepsilon) \subset U(x,2\varepsilon)$. Since $\delta > 0$ is arbitrary, we obtain

$$\beta_a^V(U^-(z,\varepsilon)) \le \varepsilon^{2-a-\delta}$$

a.s. for all small ε and all z such that $x \in U^-(z,\varepsilon)$. The lemma now follows from Remark 5.2 (ii). $\square$

Recall that the Hausdorff dimension of a set A is equal to α if α is the infimum of all $\gamma > 0$ with the following property. For each $\delta > 0$ there exists a covering of A with discs $U^-(x^k, r_k)$, $k \ge 1$, such that $\sum_{k\ge 1} r_k^\gamma < \delta$.

The carrying dimension of a measure μ is α if α is the infimum of γ's for which one can find a set $A = A_\gamma$ such that $\mu(A^c) = 0$ and the Hausdorff dimension of A is equal to γ.

Theorem 6.1. *The carrying dimension of β_a is equal to $2-a$.*

Proof. Our argument is more or less standard (see, e.g., Rogers and Taylor (1961)).

(i) First we will prove that the carrying dimension of β_a is not greater than $2-a$.

Fix some $a \in (0,1/2)$ and an arbitrary $\delta > 0$. Suppose that $M < \infty$ satisfies Lemma 6.1. For integers $i_1, i_2 \in [0,M]$ let $\Lambda_\varepsilon^{i_1,i_2}$ be the collection of all discs $U^-(z,\varepsilon)$ such that $z \in \mathbb{Z}_\varepsilon^2 + (i_1\varepsilon/M, i_2\varepsilon/M)$ and $Y_{a-\delta}^{z,\varepsilon} = 1$. Let $|\Lambda_\varepsilon^{i_1,i_2}|$ denote the cardinality of $\Lambda_\varepsilon^{i_1,i_2}$. Note that

$$|\Lambda_\varepsilon^{0,0}| = \varepsilon^{a-\delta-2}\beta_{a-\delta}^\varepsilon.$$

By Theorem 4.1, $\beta_{a-\delta}^\varepsilon$ converges in L^2 so $|\Lambda_\varepsilon^{0,0}|\varepsilon^{2-a+2\delta}$ converges to 0 in L^2 as $\varepsilon \to 0$. It is also true that for any $i_1, i_2 \in [0,M]$, $|\Lambda_\varepsilon^{i_1,i_2}|\varepsilon^{2-a+2\delta}$ converges to 0 in L^2 as $\varepsilon \to 0$. This may be proved by repeating the arguments that lead to Theorem 4.1 for the families of discs shifted by $(i_1\varepsilon/M, i_2\varepsilon/M)$.

Let $\Gamma_k^{i_1,i_2}$ be equal to $\Lambda_\varepsilon^{i_1,i_2}$ with $\varepsilon = 2^{-k}$. Note that $\Gamma_k^{i_1,i_2} 2^{-k(2-a+2\delta)}$ converges a.s. to 0 when $k \to \infty$ through a subsequence. Hence one can choose an increasing sequence of integers $\{k_n\}_{n\geq 1}$ such that for every pair i_1, i_2,

$$\sum_{n=1}^{\infty} |\Gamma_{k_n}^{i_1,i_2}| 2^{-k_n(2-a+2\delta)} < \infty \quad \text{a.s.}$$

It follows that there exists a random j such that

$$\sum_{i_1,i_2=0}^{M} \sum_{n=j}^{\infty} |\Gamma_{k_n}^{i_1,i_2}| 2^{-k_n(2-a+2\delta)} < \delta. \tag{6.9}$$

Lemma 6.1 shows that for β_a-almost every point x, there is $m = m(x)$ such that for every $k > m$ the point x is covered by a disc from the family $\bigcup_{i_1,i_2=0}^{M} \Gamma_k^{i_1,i_2}$. This implies that β_a-almost all points are covered by discs from $\bigcup_{i_1,i_2=0}^{M} \bigcup_{n\geq j} \Gamma_{k_n}^{i_1,i_2}$. The sum of their radii raised to the power $2 - a + 2\delta$ is less than δ by (6.9). Since δ is arbitrarily small, the carrying dimension of β_a is not greater than $2 - a$.

(ii) Now we will prove the lower bound for the Hausdorff dimension using Frostman's method. We will assume that the carrying dimension α of β_a is less than $2 - a$ and we will show that this assumption leads to a contradiction.

Let $\mathcal{M}_k$ be the family of all discs $U^-(x, \varepsilon)$, $x \in \mathbb{Z}^2_\varepsilon$, where $\varepsilon = 2^{-k}$. Note that each disc with radius $r \in (2^{-k}(1 - \sqrt{2}/2)/2, 2^{-k}(1 - \sqrt{2}/2))$ is contained in a disc from the family $\mathcal{M}_k$. This easily implies that we can assume that all the discs in the definition of the Hausdorff dimension belong to $\bigcup_{k\geq 1} \mathcal{M}_k$. Since $\delta \stackrel{\text{df}}{=} 2 - a - \alpha > 0$, for each $m \geq 1$ there is a subfamily $\{M_j^m\}_{j\geq 1}$ of $\bigcup_{k\geq m} \mathcal{M}_k$ such that the β_a-measure of the complement of $\bigcup_{j\geq 1} M_j^m$ is equal to 0 and

$$\sum_{j=1}^{\infty} \text{diam}(M_j^m)^{\alpha+\delta/2} < 2^{-m}.$$

Let $\{N_j\}_{j\geq 1}$ be a sequence containing all the elements of all sequences $\{M_j^m\}_{j\geq 1}$, $m \geq 1$. Then

$$\sum_{j=1}^{\infty} \text{diam}(N_j)^{\alpha+\delta/2} < 1.$$

By Corollary 5.2, $\beta_a(U^-(0,1)) > 0$ a.s. so there is a random integer n such that

$$\sum_{j=n}^{\infty} \operatorname{diam}(N_j)^{\alpha+\delta/2} < \beta_a(U^-(0,1)). \tag{6.10}$$

Now eliminate from the sequence $\{N_j\}_{j\geq n}$ all the elements such that $\beta_a(N_j) > \operatorname{diam}(N_j)^{2-a-\delta/2}$ to obtain a sequence $\{R_j\}_{j\geq 1}$. Lemma 6.2 implies that β_a-almost every point is covered by at least one element of $\{R_j\}_{j\geq 1}$. It follows that

$$\sum_{j=1}^{\infty} \operatorname{diam}(R_j)^{\alpha+\delta/2} \geq \sum_{j=1}^{\infty} \beta_a(R_j) \geq \beta_a(U^-(0,1))$$

which contradicts (6.10). $\square$

Recall that N_ε^x denotes the number of excursions of X from x which hit $U(x,\varepsilon)$.

Theorem 6.2. *For every $a > 0$ the Hausdorff dimension of the set of all points x such that*

$$\liminf_{\varepsilon\to 0} N_\varepsilon^x / |\log \varepsilon| \geq a \tag{6.11}$$

is a.s. less than or equal to $2 - a/e$ with the convention that a negative dimension signifies the empty set.

Proof. Standard arguments show that it is enough to prove that the Hausdorff dimension of the set of all points $x \in U^-(0,1/32)$ satisfying (6.11) is a.s. less than or equal to $2-a/e$. Fix some $a > 0$ and (small) $\delta > 0$. If x satisfies (6.11) then

$$N_\varepsilon^x > (1-\delta)a|\log \varepsilon| \tag{6.12}$$

for all ε less than some (random) ε_0. For each $\varepsilon > 0$ find $z = z_x^\varepsilon \in \mathbb{Z}_\varepsilon^2$ such that $x \in U^-(z,\varepsilon)$. Suppose $r > \varepsilon$ and let

$$\begin{aligned}
T_1 &= T(U(z,\varepsilon)) \\
T_{2n} &= \inf\{t > T_{2n-1} : X(t) \in U(z,r)\}, \quad n \geq 1, \\
T_{2n+1} &= \inf\{t > T_{2n} : X(t) \in U(z,\varepsilon)\}, \quad n \geq 1, \\
M &= M(z,\varepsilon,r) = \max\{n : T_{2n+1} < \infty\}.
\end{aligned}$$

If x satisfies (6.12) then there exists $r_0 > 0$ such that

$$M(z, \varepsilon, r) > (1 - 2\delta)a|\log r| \tag{6.13}$$

for all $r < r_0$, $\varepsilon < r$ and $z = z_x^\varepsilon$. Let $A(\delta, k)$ be the set of all x such that (6.13) holds for all $r < 1/k$, $\varepsilon < r$ and $z = z_x^\varepsilon$ and let $F(z, \varepsilon, r)$ denote the event in (6.13).

The probability that Brownian motion starting from a point on $U(z, r)$ will hit $U(x, \varepsilon)$ before hitting $U(0, 1)$ is bounded by $\log(r/2)/\log(\varepsilon/2)$ (here we need the assumption that $x \in U^-(0, 1/32)$). A repeated application of the strong Markov property at the stopping times T_{2n} gives the following bound

$$P(F(z, \varepsilon, r)) \le \left(\frac{\log(r/2)}{\log(\varepsilon/2)}\right)^{(1-2\delta)a|\log r|}.$$

Now we let $r = \varepsilon^b$ for some $b \in (0, 1)$. We obtain for small ε

$$P(F(z, \varepsilon, \varepsilon^b)) \le \left(\frac{\log(\varepsilon^b/2)}{\log(\varepsilon/2)}\right)^{(1-2\delta)ab|\log \varepsilon|} \le \varepsilon^{(1-3\delta)ab|\log b|}.$$

The best bound is obtained when we let $b = 1/e$. Then

$$P(F(z, \varepsilon, \varepsilon^{1/e})) \le \varepsilon^{(1-3\delta)a/e}.$$

Let K_ε be the number of $z \in \mathbb{Z}_\varepsilon^2$ such that $F(z, \varepsilon, \varepsilon^{1/e})$ holds. Then

$$EK_\varepsilon \le c\varepsilon^{(1-3\delta)a/e-2}.$$

The Chebyshev inequality yields

$$P(K_\varepsilon \ge c\varepsilon^{(1-4\delta)a/e-2}) \le \frac{c\varepsilon^{(1-3\delta)a/e-2}}{c\varepsilon^{(1-4\delta)a/e-2}} = \varepsilon^{\delta a/e},$$

and in particular

$$P(K_{2^{-j}} \ge c(2^{-j})^{(1-4\delta)a/e-2}) \le (2^{-j})^{\delta a/e}.$$

Since $\sum_j (2^{-j})^{\delta a/e} < \infty$, the Borel-Cantelli Lemma implies that a.s.

$$K_{2^{-j}} \leq c(2^{-j})^{(1-4\delta)a/e-2}$$

for all large j. It follows that a.s. for all large j the set $A(\delta, k)$ may be covered by at most $c(2^{-j})^{(1-4\delta)a/e-2}$ discs of radius 2^{-j}. Hence, the Hausdorff dimension of $A(\delta, k)$ is less than or equal to $2-(1-5\delta)a/e$. The same bound is true for the Hausdorff dimension of $\bigcup_k A(\delta, k)$ and, therefore, for the set of points satisfying (6.12). Since δ is arbitrarily small, the result follows.

As for the case when $2 - a/e < 0$, note that for small δ the quantity $c(2^{-j})^{(1-4\delta)a/e-2}$ is less than 1 and, therefore, $K_{2^{-j}} = 0$ for all large j. □

References

1. Bass, R.F. and Khoshnevisan, D., *Intersection local times and Tanaka formulas* Ann. l'Institut H. Poincaré (1993), (to appear).
2. Blumenthal, R., *Excursions of Markov Processes*, Birkhäuser, Boston, 1992.
3. Burdzy, K., *Multidimensional Brownian Excursions and Potential Theory*, Longman, Harlow, Essex, 1987.
4. Doob, J.L., *Classical Potential Theory and Its Probabilistic Counterpart*, Springer, New York, 1984.
5. Durrett, R., *Brownian Motion and Martingales in Analysis*, Wadsworth, Belmont, CA, 1984.
6. Dvoretzky, A., Erdös, P. and Kakutani, S., *Points of multiplicity c of plane Brownian paths*, Bull. Res. Council Israel Sect. F **7** (1958), 175–180.
7. Dynkin, E.B., *Self–intersection gauge for random walks and for Brownian motion*, Ann. Probab. **16** (1988), 1–57.
8. Itô, K. and McKean, H.P., *Diffusion Processes and Their Sample Paths*, Springer, New York, 1974.
9. Karatzas, I. and Shreve, S.E., *Brownian Motion and Stochastic Calculus*, Springer, New York, 1988.
10. Le Gall, J.-F., *Some properties of planar Brownian motion*, Ecole d'Eté de Probabilités de Saint-Flour XX — 1990, Lecture Notes in Mathematics 1527 (P.L. Hennequin, ed.), Springer, New York, 1992.
11. Maisonneuve, B., *Exit systems*, Ann. Probab. **3** (1975), 399–411.
12. Meyer, P.A., Smythe, R.T. and Walsh, J.B., *Birth and death of Markov processes*, Proc. Sixth Berkeley Symp., vol. III Univ. of California Press, Berkeley (1972), 295–305.
13. Rogers, C.A. and Taylor, S.J., *Functions continuous and singular with respect to a Hausdorff measure*, Mathematika **8** (1961), 1–31.

Department of Mathematics, GN-50, Seattle, WA 98195