Topics in Continuum Theory

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Abstract

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Mathematics

Continuum Theory is the study of compact, connected, metric spaces. These spaces arise naturally in the study of topological groups, compact manifolds, and in particular the topology and dynamics of one-dimensional and planar systems, and the area sits at the crossroads of topology and geometry. Major contributors to its development include, but are not limited to, Urysohn, Borsuk, Moore, Sierpiński, Menger, Mazurkiewicz, Ulam, Hahn, Whyburn, Kuratowski, Knaster, Moise, Čech and Bing. After the 1950's the area fell out of style, primarily due to the increased interest in the then-developing field of algebraic topology.

Current efforts in complex and topological dynamics, including many open problems, often fall within the purview of continuum theory. This paper covers a selection of the standard topics in continuum theory, as well as a number of topics not yet available in book form, e.g. Kelley continua and some classical results concerning the pseudo-arc. As well, many known results are given their first explicit proofs, and two new results are obtained, one concerning slc continua and the other a broad generalization of a result of Menger.

Basic Definitions

The following definitions are assumed throughout the text. Here X will refer to a topological space. We assume known the definition and basic properties of metric spaces, and will denote the distance on X by $d_X(\cdot,\cdot)$, or occasionally just by d_X. When the space X is understood, the subscript may be dropped.

(Interior, Boundary, Neighborhood, Regular) By the *interior* of a subset $A \subseteq X$ we mean the largest open subset of X contained in A. We will denote it by $\text{int}(A)$. The closure will be denoted $\overline{A}$, and the boundary $\overline{A} \cap (\overline{X \setminus A})$ by ∂A. For a point x (respectively, a subset A), by a *neighborhood* of x (of A) we mean a set whose interior contains x (contains A). If we mean an open neighborhood then we will always say so. A set is called a *regular open set* (respectively, *regular closed set*) if it is equal to the interior of its closure (resp. the closure of its interior).

(Complete) A metric space is *complete* if every Cauchy sequence in X has a convergent subsequence - that is to say, a sequence x_n of points such that for every $\epsilon > 0$ there is an N so that for all $m, n \geq N$ we have $d(x_n, x_m) < \epsilon$.

(Baire Category Theorem, Cech Complete) We assume the Baire Category Theorem, at least in so much as that it implies a complete metric space can't be written as a countable union of subsets with empty interior. A space is *Cech Complete* if every closed subset of it is complete. It is known that complete metric spaces are Cech Complete.

(Hausdorff, Normal, Hereditarily Normal) X is a *Hausdorff space* (resp. *normal* *space*), if points are closed and for every pair of distinct points $x, y \in X$ (resp. disjoint closed subsets A, B) there are disjoint open neighborhoods of x and y (resp. of A and B). A space is *hereditarily normal* if every subset of X is normal. We assume the basic properties of normal spaces are known.

(G_δ, F_σ, Perfectly Normal) A subset of X is a G_δ set if it's the intersection of countably many open sets. A subset of X is an F_σ set if it's the union of countably many closed sets. A space is *perfectly normal* if every open set in X is a G_δ set.

(Separable, First Countable, Second Countable) A subset of X is *dense* if its closure is all of X. A space is *separable* if X has a countable dense subset. A space is *first countable* if every point has a countable local basis. A space is *second countable* if its topology admits a countable basis.

(Compact, Sequentially Compact, Limit Point Compact) We assume known that the following are equivalent for a subset of a metric space to be compact: Every open cover has a finite subcover, or every sequence has a convergent subsequence, or every infinite subset has a limit point. The basic properties of compact sets, e.g. the intersection property, are assumed.

(Locally Compact, Precompact, σ-Compact) A space is *locally compact* if every point has a local neighborhood basis of compact sets. Equivalently, if every point has a local open neighborhood basis of *precompact* sets, i.e. sets whose closures are compact. Such sets are sometimes called *relatively compact*. A space is called *σ-compact* if it can be written as a countable union of compact subsets.

(Separated, Separation) If two subsets of X are such that $\overline{A} \cap B = A \cap \overline{B} = \varnothing$ and neither are empty, we say that they are *separated*. If $X = A \cup B$ where A and B are separated, then we say the pair A, B is a *separation* of X. Out of laziness, sometimes we may say that $X = A \cup B$ *is a separation of* X to mean that A, B are a separation of X.

(Connected, Component) A space X is *connected* if no pair of its subsets is a separation of X. Again, we assume the basic properties of connected sets, e.g. that the union of a collection of connected sets that all have a point in common is connected, and that if E is connected and $E \subseteq F \subseteq \overline{E}$ then F is connected. A maximal connected subset of X is called a *component*. A component of a space is always closed and contains any connected subset of X which it intersects non-trivially.

(Separates, Separated from) If C is a subset of X such that $X \setminus C$ is not connected, then we say that C *separates* X. If $X \setminus C = A \cup B$ is a separation and $E \subseteq A, F \subseteq B$ we sometimes say that C *separates* E *and* F. If $Y = X \setminus C$ as in the previous line, we might also say that E *is separated from* F *in* Y. It will always be obvious what is meant.

(Cutting, Cut Point) If C separates X we will also sometimes call C a *cutting* of X. If C is a single point, then we will call C a *cut point*.

($I, I^n, I^\omega, \mathbb{R}^n, \mathbb{S}^n, \mathbb{D}^n$, Manifold) We denote the closed arc $[0,1]$ by I, and its n-fold product by I^n. We denote the Hilbert Cube by I^ω. $\mathbb{R}^n$ is n-dimensional Euclidean space, and $\mathbb{D}^n$ is the n-dimensional open ball. A space is *locally Euclidean* if every point has a neighborhood homeomorphic to $\mathbb{R}^n$. A second-countable, Hausdorff, locally Euclidean space is an n-*manifold* if every point has an open neighborhood homeomorphic to $\mathbb{R}^n$.

(Path-Connected, Arc-Connected) A space is *path-connected* if for every pair of points $x, y \in X$ there exists a continuous function $f : I \to X$ with $f(0) = x$ and $f(1) = y$. We will call the image f a *path* from x to y, and will call x and y the end points of $f(I)$. A space is *arc-connected* if we assume that f is an *embedding*, i.e. a homeomorphism with its image, in which case we call $f(I)$ an *arc* from x to y. In either case, we may use the notation $[xy]$ to denote a path, or arc, from x to y.

(Locally Connected, Locally Path-Connected) A space is *locally connected* (resp. *lo-*

cally path-connected, *locally arc-connected*) if every point has a local basis of *open* connected (resp. open path-connected, open arc-connected) neighborhoods.

(Zero-Dimensional, Totally Disconnected, Scattered) A topological space X is *zero-dimensional* if it has a basis of clopen sets - sets that are both open and closed. A space is *totally disconnected* if its components are single points. X is *scattered* if every subset of X has an isolated point. For locally compact Hausdorff spaces, the first two concepts are equivalent.

(Cantor Set, Perfect Set) By the *Cantor Set* we mean the standard ternary Cantor Set, consisting of the remainder of the unit interval I after successively removing the open middle third of the remaining intervals. A set $A \subseteq X$ is *dense in itself* if every point of A is a limit point of A. The set A is *perfect* if it's closed and dense in itself.

We assume the following two equivalent statements of a theorem of Brouwer: Up to homeomorphism, the Cantor Set is the unique zero-dimensional, compact, Hausdorff space without isolated points - equivalently, is the unique perfect, compact, totally disconnected and metrizable space.

(Zorn's Lemma, Well-Ordering Theorem, Maximal Principle) There are several basic theorems of logic that we will use. *Zorn's Lemma* states that if $\mathcal{P}$ is a partially ordered set such that every totally ordered subset of $\mathcal{P}$ has an upper bound in $\mathcal{P}$, then $\mathcal{P}$ contains at least one maximal element. The *Well-Ordering Theorem* states that every ordered set can be well-ordered. The *Hausdorff Maximal Principle* states that every totally ordered subset of a partially ordered subset is contained in some maximal totally ordered subset.

(Brouwer Reduction Theorem) A set A is 'irreducible' with respect to a property P if no proper, non-empty subset of A satisfies P. A property is said to be 'inducible' if any nested intersection of compact sets A_n satisfying P implies that $\cap_n A_n$ also satisfies P. The *Brouwer Reduction Theorem* states that any non-empty compact set with an inducible property P contains a non-empty closed subset which is irreducible with respect to P.

($B_\epsilon(x), A_\epsilon$, Hausdorff Metric) We will denote the open ball of radius ϵ about a point $x \in X$ as $B_\epsilon(x)$. If $A \subseteq X$ then by A_ϵ we mean $\cup_{x \in A} B_\epsilon(x)$. We will often call this the ϵ-neighborhood of A. The *Hausdorff Metric* H_d is a metric on the non-empty closed subsets of a metric space (X, d) defined by $H_d(A, B) = \liminf\{\epsilon \mid A \subseteq B_\epsilon \text{ and } B \subseteq A_\epsilon\}$. We will often omit the subscript.

($\liminf(A_n)$, $\limsup(A_n)$, $\lim(A_n)$) If A_n is a sequence of subsets of X, then $\liminf A_n$ is the set of points x so that every $B_\epsilon(x)$ eventually intersects every A_n; similarly, $\limsup A_n$ are those points all of whose neighborhoods intersect infinitely many A_n. When the two agree, we denote both by $\lim A_n$. Note that both $\liminf A_n$ and $\limsup A_n$ are always closed, and that $\lim A_n = A$ if and only if $A_n \to A$ in terms of the Hausdorff Metric.

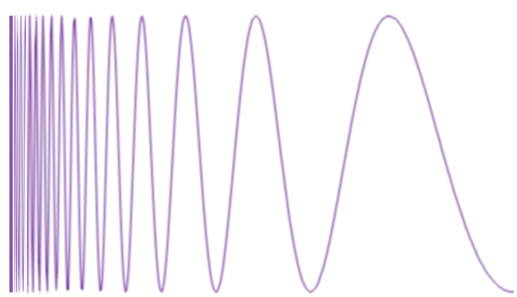
Contents

Chapter 1

Continuum Theory

1.1 Global Structure

By a *continuum* (pl. *continua*) we mean a compact, connected metric space. Examples include the unit interval, the circle, the Hilbert Cube, etc. The reader may be familiar with the continuum below, which we simply call the *sine continuum*. Many naturally occurring objects in topological and complex dynamics are most appropriately studied in the context of continuum theory.



The elementary properties are as follows. If X is a continuum, as a compact metric space X is second-countable and thus separable. Further, compact metric spaces are complete. Since X is a compact Hausdorff space, it is locally compact in the sense that every point has a local basis of precompact open neighborhoods.

If $A \subseteq X$ then by A_ϵ we mean $\cup_{x \in A} B_\epsilon(x)$. If A is a closed subset of X then for $\epsilon_n = {}^1\!/\!_n$, it is immediate that $A = \cap C_{\epsilon_n}$. Thus every closed subset is a G_δ set, i.e. X is perfectly normal. In particular, a continuum is normal and hereditarily normal - equivalently, any pair A, B of separated sets have disjoint neighborhoods.

By standard results, the countable product of metrizable spaces is metrizable, the product of connected spaces is connected, and the product of compact sets is compact, and thus a finite or countable product of continua is a continuum.

If $X_{n+1} \subseteq X_n$ are non-empty continua and $Y = \cap X_n$, then Y is a non-empty compact metric space. If $Y = C \cup D$ were a separation, then since C, D are closed in Y they are closed in X_1, and so for ϵ sufficiently small, we have $C_\epsilon \cap D_\epsilon = \varnothing$. Then $Y_\epsilon = C_\epsilon \cup D_\epsilon$ is a separation. Since $X_n \to Y$ in the Hausdorff sense, $X_n \subseteq Y_\epsilon$ for n sufficiently large. Suppose $X_N \subseteq Y_\epsilon$. Since it's connected and $C_\epsilon \cup D_\epsilon$ is a separation of Y_ϵ, then $X_N \subseteq C_\epsilon$, without loss of generality. But then $Y \subseteq X_N \subseteq C_\epsilon$ and so $D = \varnothing$, impossible. Thus the nested intersection of continua is a continuum.

In fact, with a slight modification we have shown more: If $Y = \limsup X_n$ (necessarily closed) and $\liminf X_n$ is not empty, then as before we may assume $\liminf X_N \subseteq C$ since *every* X_n has points converging to any point in $\liminf X_n$. Thus eventually all $X_k \subseteq C_\epsilon$, again implying $D = \varnothing$, i.e. $Y = \limsup X_n$ is connected. When the limit exists, it must then be a continuum.

Observation 1.1.1. *If X is a continuum, then*

(a) X is separable and second-countable,

(b) X is complete and perfectly normal,

(c) every $x \in X$ has a local basis of precompact open sets,

(d) a countable product of continua is a continuum,

(e) a nested intersection of continua is a continuum, and

(f) if $A_i \subseteq X$ are connected with $\liminf A_i \neq \varnothing$, then $\limsup A_i$ is a continuum.

Thus continua are relatively well-behaved, topologically. One property we did not mention: Local connectedness. Continua which are locally connected ('lc') are called *Peano*

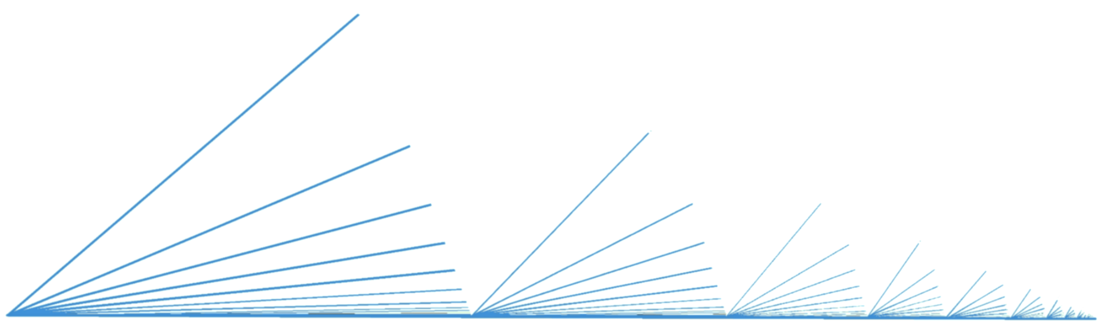
Continua and are the most well-understood; the sine continuum mentioned on the previous page is an example of a continuum which fails to be locally connected. In this vein, consider the following definition:

Definition 1.1.2. ***(Connected im Kleinen)*** *X is connected im kleinen ('cik') at $x \in X$ if for every neighborhood (equivalently, open neighborhood) U of x, there is a connected set $V \subseteq U$ with $x \in int(V) \subseteq U$.*

Equivalently, X is cik at x if for every neighborhood U of x, there is an open $V \subseteq U$ containing x so that any $y, z \in V$ are contained in a single connected subset of U.

The space X is said to be cik if it's cik at each of its points.

Being connected im kleinen at a point does not imply local connectedness there; the right-most point in the continuum below has closed, connected neighborhoods, but its proper open neighborhoods are all disconnected. Globally, however, the two properties are equivalent for continua. For other basic results see [24].



Proposition 1.1.3. *If X is connected im kleinen, then X is locally connected.*

Proof. Recall that a space is locally connected if and only if components of open sets are open. So let U be open with component C. If $x \in C$ then there is a connected $V_x \subseteq U$ with $x \in \text{int}(V_x) \subseteq U$. Since V_x is connected and $V_x \cap \{x\} \neq \varnothing$, we have $V_x \subseteq C$. Thus $x \in \text{int}(V_x) \subseteq C$, and so $C = \bigcup_{x \in C} \text{int}(V_x)$ is open. □

Corollary 1.1.4. *A continuum X is Peano if and only if X is cik.* □

Before exploring local pathologies, let's understand how continua are 'put together' out of larger pieces. This will give us an interface between topological and geometric arguments, which is the crux of continuum theory. By a *subcontinuum* of X we mean a subset which is a continuum.

Lemma 1.1.5. *Suppose X is a connected space and C is a proper closed subset with boundary ∂C. If $x \in C$ then there is no separation between x and ∂C in C.*

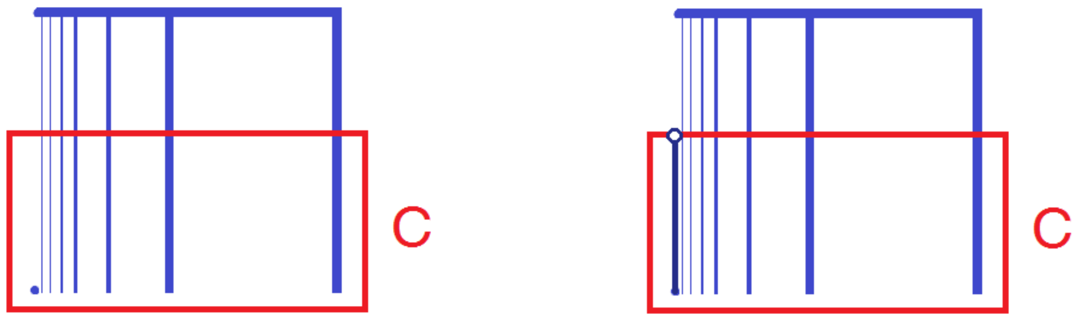
Proof. Suppose $C = A \cup B$ is a separation with $x \in A$ and $\partial C \subseteq B$. Then A and $C \setminus A$ are closed in C and thus in X, and $A \cup \big((C \setminus A) \cup \overline{X \setminus C}\big)$ is therefore a separation of X, a contradiction. ☐

Corollary 1.1.6. *If C is a non-empty, proper closed subset of a continuum X and D is a component of C, then $D \cap \partial C \neq \varnothing$.*

Proof. If $x \in D$ then by the above, x and ∂C are not separated, i.e. $\overline{D} \cap \partial C \neq \varnothing$ or $D \cap \overline{\partial C} \neq \varnothing$. But each is closed, so $D = \overline{D}$ and $\partial C = \overline{\partial C}$, and thus $D \cap \partial C \neq \varnothing$. ☐

Below is a σ-compact, connected metric space X for which the previous corollary fails. It is composed of the vertical segments $\{{}^1\!/_n \times [0,1] \mid n \in \mathbb{N}\}$, the horizontal segment $\{1\} \times [0,1]$ and the point $(0,0)$. Let C be X intersected with $\{y \leq {}^1\!/_2\}$. Then $(0,0)$ is a component of C but doesn't intersect its boundary, which is contained in $y = {}^1\!/_2$. Replacing the origin with the segment $\{0\} \times [0, {}^1\!/_2)$ makes X locally compact.

Another space, the Cantor Leaky Tent, and sometimes also called the *Knaster-Kuratowski Fan*, is a connected, separable metric space with a point p such that every component of $T \setminus \{p\}$ is a singleton [34] *(p. 145)*. The proof of its connectedness is technical so we omit an exposition.



Theorem 1.1.7. ***(Sierpinski)*** *[19] (p. 173) No continuum can be written as a countably infinite union of non-empty, pairwise-disjoint closed sets.*

Proof. We follow Kuratowski. Suppose $X = \cup X_n$ is a pairwise-disjoint union of (at least two) non-empty closed sets. By normality, there is a closed neighborhood F of X_2 disjoint from X_1. Let D be a component of F intersecting X_2. By the above corollary, $D \cap (X \setminus F) \neq \varnothing$ and since $D \cap X_1 = \varnothing$ we have that D intersects some X_k non-trivially, for $k \geq 3$.

Write $D_1 = D$ and note that $D_1 = \cup(D_1 \cap X_n)$ satisfies the same criteria as X, with at least two $(D_1 \cap X_k)$ being non-empty. Inductively define D_k, a decreasing sequence of compact subsets of X, so that $D_k \cap X_k = \varnothing$. Then $\cap D_k$ is a nested intersection of compact sets and therefore non-empty, yet $D_k \cap X_k = \varnothing$ for all k and therefore $(\cap D_k) \cap X = \varnothing$, a contradiction. □

Corollary 1.1.8. *If X is a compact metric space and $X = \cup X_n$ where each X_n is a non-empty continuum disjoint from the rest, then each X_n is a component of X.*

Proof. If it were false, then for some X_k, the component containing X_k would be strictly larger; call it D. But then D writes as $\cup(D \cap X_n)$ with at least two non-empty summands, wheras D is a continuum, impossible by the above theorem. □

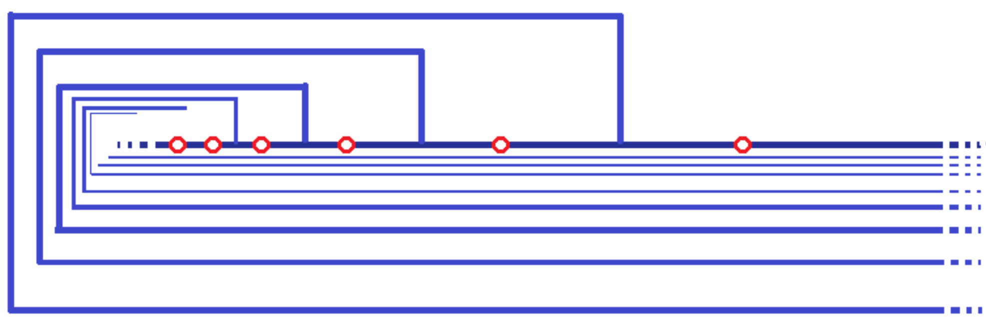
We say that a continuum with more than one point is *non-degenerate*.

Corollary 1.1.9. *A non-degenerate continuum X is uncountable.* □

Corollary 1.1.10. *A non-degenerate continuum X has the cardinality of $\mathbb{R}$.*

Proof. By the above corollary, X is at least uncountable. Since X is separable, let $\mathcal{E}$ be a countable dense subset. Then the set of sequences of elements of $\mathcal{E}$ has cardinality $\mathbb{N}^{\mathbb{N}} \simeq \mathbb{R}$ and each point in X corresponds to at least one such (convergent) sequence. Thus the cardinality of X is at most that of $\mathbb{R}$, completing the proof. $\square$

We note that the property of not being a disjoint union of at least two and at most countably many closed, non-empty subsets is called *σ-connected.* Thus continua are σ-connected. Compactness is essential; there are connected, σ-compact, locally compact (and thus topologically complete) metric spaces for which σ-connectedness fails [19] *(p. 175)*:



The space is composed of the positive reals minus the points $1/n$, a sequence of solid horizontal lines $L_n = \mathbb{R}^+ \times (-1/n)$ and paths as shown from L_n to the point between $1/n$ and $1/n+1$. It can be given a complete metric since it's locally compact. As an aside, by an elementary argument it can be shown that a connected, locally connected, complete metric space is also σ-connected [6] *(p. 87)*.

Recall that $a, b \in X$ are in the same *quasicomponent* of X if and only if there is no separation $X = A \cup B$ with $a \in A$ and $b \in B$. It is easy to show that this is an equivalence relation, and that if Q is the quasicomponent of $x \in X$, then Q is the intersection of all clopen sets containing x. It is harder, but still standard, to show that quasicomponents and components coincide for compact Hausdorff spaces [7] *(p. 357)*.

Lemma 1.1.11. *Suppose X is a compact Hausdorff space. If Q is a component of X with open neighborhood U, then there is a clopen $C \subseteq X$ such that $Q \subseteq C \subseteq U$.*

Proof. Since components are quasicomponents, pick a point $q \in Q$ and write $Q = \cap R_\alpha$, where R_α are the clopen neighborhoods of q. Then $\{R_\alpha^c\}$ is an open cover of $X \setminus U$ and thus contains a finite subcover $R_1^c, R_2^c, \ldots, R_n^c$. Write its union as R. Then R is disjoint from Q, covers $X \setminus U$ and is clopen, so R^c is the desired set C. □

Thus if Q and R are distinct quasicomponents of a compact Hausdorff space X, there is a clopen set C with $Q \subseteq C \subseteq R^c$ and vice-versa there is a clopen D with $R \subseteq D \subseteq Q^c$. Then $C \setminus D$ and $D \setminus C$ are disjoint, clopen subsets containing Q and R respectively and thus the quotient of X by (quasi)components is Hausdorff. Since X is compact, the identification map is closed and therefore a bona fide quotient.

Corollary 1.1.12. *If X is a compact metric space and $X = \cup X_n$ where each X_n is a continuum disjoint from the rest, then at least one X_n is open.*

Proof. By **1.1.8** each X_n is a component. If $f : X \to Y$ is the quotient by components, since Y is countable it contains an isolated point y corresponding to some X_y. But isolated points are open and f is continuous, so $f^{-1}(q) = X_q$ is open. □

Theorem 1.1.13. ***(Cut Wire Theorem)*** *[15] (p. 72) Suppose that X is a compact Hausdorff space and that A and B are non-empty, closed subsets of X. If no connected subset of X intersects both A and B non-trivially, then they are separated in X.*

Proof. If no connected set intersects both A and B, then no component does, so no quasicomponent does. Thus if Q is a quasicomponent intersecting A, we have $Q \subseteq X \setminus B$. Since A is compact, it is covered by finitely many such quasicomponents Q_k which don't intersect B, and by the lemma it is therefore also covered by finitely many clopen $L_k \subseteq X \setminus B$. But then $L = L_1 \cup \cdots \cup L_n$ is clopen with $A \subseteq L$ and $B \subseteq L^c$ non-empty, so (L, L^c) separates A and B. □

Though the usefulness of this theorem cannot be overstated, it should be viewed as an appetizer for the next two theorems, the Extension Theorem and the Boundary Bumping

Theorem. First, we mention the following two easy propositions, which are similar in flavor and apply in arbitrary connected spaces.

Proposition 1.1.14. *If X is a connected space and $C \subset X$ is connected such that $X \setminus C = A \cup B$ is a separation, then $A \cup C$ and $B \cup C$ are connected. If X and C are continua, so are $A \cup C$ and $B \cup C$.*

Proof. Suppose, by way of contradiction, that $E \cup F$ is a separation of $A \cup C$. Since C is connected, assume without loss of generality that $C \subseteq E$. We wish to show that $X = F \cup (E \cup B)$ is a separation. Since $F \cap C = \varnothing$ and $F \subseteq A \cup C$ we have $F \subseteq A$. Since A and B are separated in X, F is separated from B in X. Thus F is separated from $E \cap B$ and is also separated from $E \cap (A \cup C) = E \cap B^c$. Thus F is separated from E in X and so $X = F \cup (E \cup B)$ is a separation, impossible.

Similarly, $B \cup C$ is connected. If X and C are continua then $\overline{(A \cup C)} = \overline{A} \cup C$. Since A is separated from B, $\overline{A} \subseteq B^c = A \cup C$ and thus $A \cup C$ is closed. It's a continuum by the first part. □

Proposition 1.1.15. *Suppose X is a connected space and $C \subset X$ is connected. If D is a component of $X \setminus C$, then $X \setminus D$ is connected.*

Proof. Suppose that $X \setminus D = E \cup F$ is a separation. Then $D \cup E$ is connected by **1.1.14**. Since D is a component of $X \setminus C$ it is a maximal connected subset of $X \setminus C$, so $(D \cup E) \cap C \neq \varnothing$. Thus $C \cap E \neq \varnothing$; similarly, $C \cap F \neq \varnothing$. But $C \subseteq X \setminus D = E \cup F$ is connected whereas E and F are separated, impossible. □

Theorem 1.1.16. ***(Extension Theorem)*** *[15] (p. 74) If X is a continuum and C is a proper subcontinuum with neighborhood U, then there is a subcontinuum D with $C \subsetneq D \subseteq U$. Any non-degenerate continuum contains a non-degenerate, proper subcontinuum.*

Proof. By normality, there is a regular-open V with $C \subseteq V \subseteq \overline{V} \subseteq U$ and $V \neq X$. If D is the component of $\overline{V}$ containing C, by **1.1.6** $D \cap (X \setminus V) \neq \varnothing$ since V is regular-open. In particular, D is strictly larger than C. It's a continuum by construction. For the second part, just set $C = \{x\}$ for some $x \in X$. □

Corollary 1.1.17. *If X is a continuum with subcontinuum C and $\epsilon > 0$, there is a continuum D with $C \subsetneq D \subseteq X$ such that $\text{diam}(D) - \text{diam}(C) < \epsilon$. In particular, if X is non-degenerate and $x \in X$, it is contained in non-degenerate subcontinua of arbitrarily small diameter.*

Proof. Set $U = C_\delta$ for $\delta < \epsilon/2$ and apply the Extension Theorem. □

This shows that there are lots of subcontinua in a non-degenerate continuum. It might have appeared obvious, but we will encounter some exotic continua for which this is not immediately clear. The following theorem is also very useful:

Theorem 1.1.18. ***(Boundary Bumping Theorem)*** *[15] (p. 74) Suppose X is a continuum and A is a non-empty proper subset. If C is a component of A, then $\overline{C} \cap \partial A \neq \varnothing$.*

Proof. Suppose not. Then $\overline{C}$ is a subcontinuum and is contained in the proper, open set $U = X \setminus (\overline{X \setminus A})$. By the previous theorem, there is a continuum D with $C \subsetneq D \subseteq U$. But then $D \subseteq A$ is a connected set strictly larger than C, contradicting that C is a component. □

Most proofs thus far will work just as well for *Hausdorff Continua*, that is to say compact, connected, Hausdorff spaces (recall that compact Hausdorff spaces are normal) - they do, however, depend crucially on the equality between components and quasicomponents. The following corollaries are used almost as frequently as the theorem itself, so we record them here:

Corollary 1.1.19. *Suppose X is a continuum and that A is a proper subcontinuum. If D is a component of $X \setminus A$, then $A \cup D$ is a continuum.*

Proof. Since D is closed in $X \setminus A$ we have $\overline{D} \setminus D \subseteq A$. By the Boundary Bumping Theorem, $A \cap \overline{D} \neq \varnothing$ and thus $A \cup D$ is connected. Since A is closed and $\overline{D} \setminus D \subseteq A$ it is closed and thus a continuum. □

Corollary 1.1.20. *If C is a proper closed subset of a continuum X, the number of components of C is bounded from above by the number of components of ∂C.*

Proof. Since components of closed sets are closed in X, it's immediate by **1.1.18**. □

As we saw in the examples after **1.1.6**, it is very difficult for a space to satisfy the requirements of the Boundary Bumping Theorem. However, the following is true, indicating how strong local connectedness is [23] *(p. 273)*:

Proposition 1.1.21. *Suppose X is a connected and locally connected space, and A is a non-empty proper subset. If C is a component of A, then $\overline{C} \cap \partial A \neq \varnothing$.*

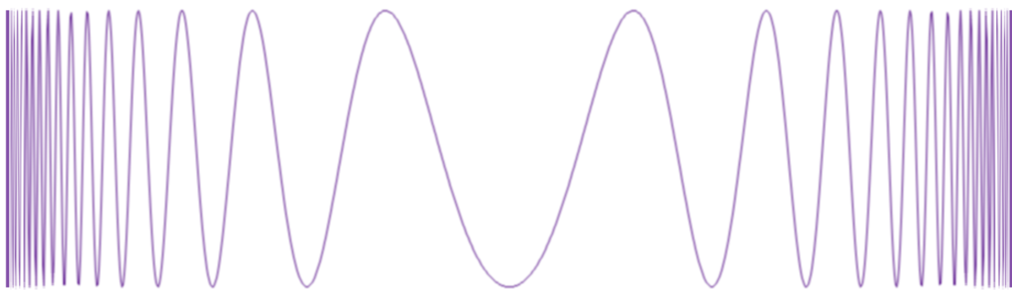
Proof. Suppose otherwise. Then $\overline{C} \cap \overline{X \setminus A} = \varnothing$ and thus $\overline{C} \subseteq A$. By maximality of components, $C = \overline{C}$ is closed. Again by maximality it must also be a component of $X \setminus (\overline{X \setminus A})$, an open set, so since X is locally connected we have that C is open. Thus C is clopen in X, a contradiction to X being connected. □

Thus we may assume the conclusion of the Boundary Bumping Theorem when working in open, connected subsets of Peano continua, for example. Up until now, the concepts used have been familiar. The following is special to continua, and is important for understanding the global properties of a space [19] *(p. 208)*.

Definition 1.1.22. ***(Composant)*** *If $x \in X$, then we define the composant*

$\kappa(x) = \{y \in X \mid$ *there exists a proper subcontinuum* $A \subsetneq X$ *with* $x, y \in A\}$.

The composant of a point is not always all of X. Take, for example, an end point of a closed interval. We say a composant is *trivial* if it's all of X. A continuum which contains a non-trivial composant is called *irreducible*. In the continuum pictured below, composed of two adjoined copies of the sine continuum (the *double sine continuum*), no point on one vertical segment is contained in the composant of a point on the other, as can easily be shown.



Proposition 1.1.23. *A composant in a continuum is connected and dense.*

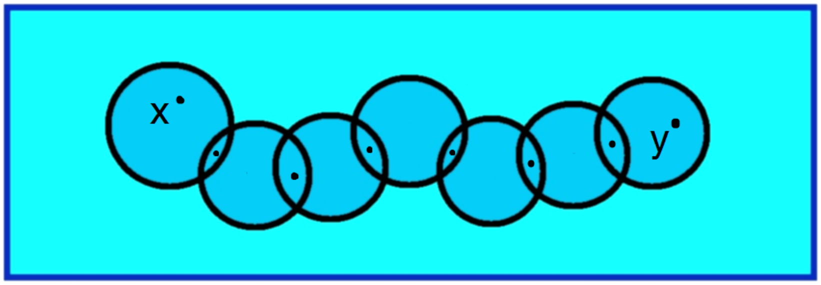
Proof. If $y \in \kappa(x)$ then there is a continuum A_y with $x, y \in A_y$. Thus $\kappa(x) = \cup A_y$ is connected. Since it's connected, $\overline{\kappa(x)}$ is a continuum, and assuming it is not all of X it is contained in $\kappa(x)$ by the definition of a composant. But by the Extension Theorem, there exists a proper subcontinuum D with $\kappa(x) = \overline{\kappa(x)} \subsetneq D \subsetneq X$, contradicting that $\kappa(x)$ is a composant. □

Lemma 1.1.24. *If X is a continuum, then each composant $\kappa(p)$ is the countable union of nested subcontinua $C_1 \subseteq C_2 \subseteq \cdots \subseteq X$ each containing p.*

Proof. We follow [15] *(p. 203)*. Let $U_1, U_2, \ldots$ be a countable open basis for $X \setminus p$, and let C_n be the component of $X \setminus U_n$ containing p; in particular each C_n is a proper subcontinuum of X. Thus $\cup C_n \subseteq \kappa(p)$. If $x \in \kappa(p)$ let A be a proper subcontinuum of X with $x, p \in A$. A is not dense and thus there is a U_j with $A \cap U_j = \varnothing$, i.e. $A \subseteq X \setminus U_j$. Thus $x \in A \subseteq C_k \subseteq \cup C_n$, so $\kappa(p) = \cup C_n$ as desired. To make them nested, take $C'_k = \cup_1^k C_j$. □

This speaks to large subcontinua. Can we 'travel' via small continua from one point to another? The answer is yes. Define an *ϵ-mesh* in X to be a finite set of n points $\{x_1, \ldots, x_n\}$

so that $d(x_i, x_{i+1}) < \epsilon$. We say this is an *$\epsilon$-mesh from x_1 to x_n*. If $x \in X$, we say that X is *well-meshed* at x if for every $\epsilon > 0$ and $y \in X$ there is an ϵ-mesh from x to y. We say X is well-meshed if it's well-meshed at each point.



Proposition 1.1.25. *If X is a continuum, it's well-meshed.*

Proof. Fix $x \in X$. Let E_ϵ be the set of points y for which there exists an ϵ-mesh from x to y. If $y \in E_\epsilon$ and $\{x_1, \ldots, x_n\}$ is an ϵ-mesh from x to y, then for any point z in an ϵ-neighborhood of y we have $\{x_1, \ldots, x_n, z\}$ is an ϵ-mesh from x to z, so E_ϵ is open. If $z \in \overline{E_\epsilon}$ then let $y_n \to z$. Then eventually some y_n is at a distance less than ϵ from z and so $\{x_1, \ldots, x_n, z\}$ is again an ϵ-mesh from x to z if $\{x_1, \ldots, x_n\}$ is an ϵ-mesh from x to y_n. Thus E_ϵ is clopen and therefore all of X. □

As such, we can apply the Extension Theorem in small neighborhoods of an ϵ-mesh to obtain a sequence of continua 'traveling' from one point to another. But be careful: The sine continuum shows that not every continuum can be written as a finite union of subcontinua with arbitrarily small diameter.

Definition 1.1.26. ***(Property S)*** *[37] (p. 20) A metric space has property S if for every $\epsilon > 0$ it has a finite cover by connected subsets each with diameter less than ϵ.*

Theorem 1.1.27. *A continuum X is Peano if and only if it has property S.*

Proof. Suppose that X is locally connected. Then for every $\epsilon > 0$ and every $x \in X$ there is a connected neighborhood U_x of x with $\text{diam}(U_x) < \epsilon$. Since X is compact it's covered by finitely many such sets, constituting the required covering.

Let $\epsilon > 0$. If X has property S then there is a finite cover $A_1, \ldots, A_n$ by connected sets with diameters less than $\epsilon/2$. Suppose $x \in X$ and let A be the union of those $\overline{A_k}$ satisfying $x \in \overline{A_k}$. As a union of connected sets with the point x in common, A is connected, and $\text{diam}(A) < \epsilon$ by the triangle inequality. Since the closure and union operations commute for finite collections of sets, $x \notin \overline{X \setminus A}$ and thus A is a neighborhood of x. Thus X is cik at x since ϵ was arbitrary, but x was arbitrary as well, so by **1.1.4** X is locally connected. □

Corollary 1.1.28. *A connected union of two Peano continua is a Peano continuum.*

Proof. Suppose that $X = A \cup B$ is connected and A and B are Peano continua. Both A and B have property S, so if $X = A \cup B$ then X also has property S. Clearly X is a continuum, so by the previous theorem X is a Peano continuum. □

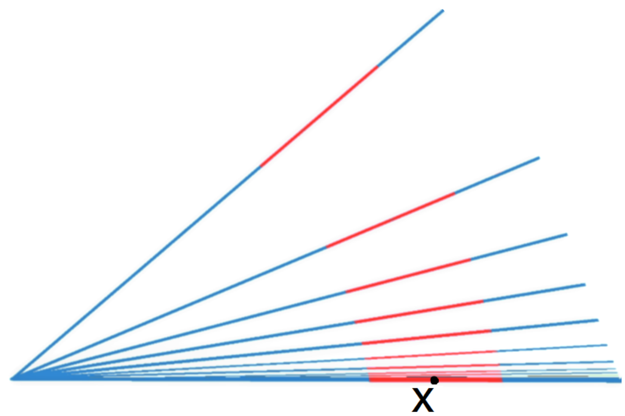
This property was first explored by Sierpiński [31] *(p.44)*. It follows that the sets A_k can be chosen to be continua, though it's not clear they can be chosen to be *Peano* continua. This is part of the basic theory of Peano continua, so we state the result without proof:

Theorem 1.1.29. *If X is a Peano continuum, then for any $\epsilon > 0$ it has a finite cover by locally connected subcontinua $A_1, \ldots, A_n$ satisfying* $\text{diam}(A) < \epsilon$. □

Returning to the notion of local connectedness, the final topic we will explore in this chapter is that of a *convergence continuum.* This will be pivotal in our understanding of how the 'locally connected pieces' are situated in a continuum.

Definition 1.1.30. ***(Convergence Continuum)*** *A non-degenerate subcontinuum $K \subseteq X$ of a metric space is called a convergence continuum if there is a sequence A_n of subcontinua with $A_n \cap K = \varnothing$ for all n and* $\lim A_n = K$.

Here, the limit is with respect to the Hausdorff metric. For example, the vertical line in the sine continuum is a convergence continuum, and is precisely the set of points where it fails to be cik. Some continua fail to have any convergence continua, e.g. an arc (due to the non-degeneracy requirement). Below is the *Harmonic Fan*; X is not cik at x but the sequence of red arcs A_n converge to a compact interval K on the bottom segment, which is then a convergence continuum containing x.



Proposition 1.1.31. *A convergence continuum K is nowhere-dense.*

Proof. If U is a neighborhood of $x \in K$ with $x \in \limsup A_n$, there is a sequence $a_n \in A_n$ with $a_n \to x$. Thus some A_n eventually intersects U, a contradiction. □

Proposition 1.1.32. *The A_n's as above may be taken to be pairwise-disjoint.*

Proof. A_1 intersects only finitely many of the others, because otherwise if $x_n \in A_1 \cap A_n$ then (x_n) would have a limit point in $A_1 \cap K = \varnothing$. Inductively define a subsequence A'_k so that A'_{k+1} intersects none of the previous ones. Then as a subsequence of a *convergent* sequence A_n, it too converges to K. □

Definition 1.1.33. *For a topological space X, define the following subsets:*

(a) $lc(X) = \{x \mid X$ is locally connected at $x\}$

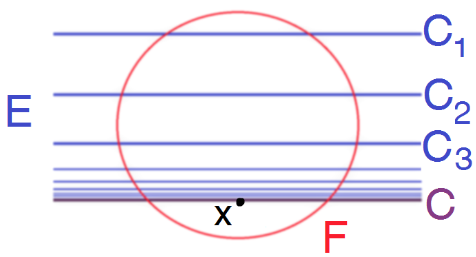
(b) $lc^*(X) = \{x \mid X$ is not locally connected at $x\}$

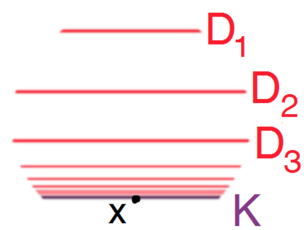
(c) $cik(X) = \{x \mid X$ is connected im kleinen at $x\}$

(d) $cik^*(X) = \{x \mid X$ is not connected im kleinen at $x\}$

Theorem 1.1.34. ***(Continuum of Convergence Theorem)*** *[19] (p. 245) If X is a continuum and $x \in cik^*(X)$, then there is a convergence continuum K in X with $x \in K \subseteq cik^*(X)$.*

Proof. By definition, if $x \in cik^*(X)$ it has a neighborhood E with component C such that $x \in C \setminus \text{int}(C)$. Pick a closed neighborhood F of x with $F \subseteq \text{int}(E)$, and let $x_n \to x$ be points of $F \setminus C$, possible since F is a neighborhood and C is not. Let C_n be the component of x_n in E, and let D_n be the component of x_n in F. Note that $C_n \subseteq E \setminus C$. Since D_n is closed in F it is closed in X. Their $\liminf$ contains x, so using **1.1.1(f)** it is elementary to show there is a convergent subsequence D'_n.



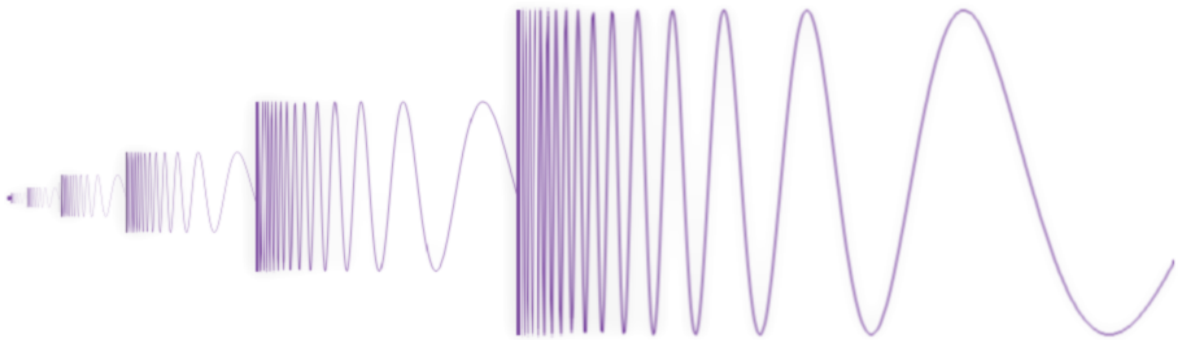


Let $K = \lim D'_n$. Then $x \in K \subseteq F \subseteq E$. Since K is connected and C is the component of x in E, we have that $K \subseteq C$. By the Boundary Bumping Theorem, D'_n intersects the boundary of F for all n, and since ∂F is closed, D'_n has a limit point in it. Thus $K \cap \partial F \neq \varnothing$, yet $x \in \text{int}(F)$. In particular, K is not degenerate and is thus a convergence continuum containing x.

If $y \in K$ then E is a neighborhood of y, with component C since both are contained in the connected set K. If X was cik at y then we would have $y \in \text{int}(C)$, but since $y \in \limsup C_n$, we have that $(\limsup C_n) \cap \text{int}(C) \neq \varnothing$. But $\text{int}(C)$ is open and thus some C_n eventually intersects it, a contradiction. Thus $K \subseteq cik^*(X)$. □

If $cik^*(X)$ is non-empty then it is dense in itself, since every point of it is contained in a non-degenerate subcontinuum of $cik^*(X)$. If C is the Cantor Set, take the cone over C, i.e. all straight line segments from points of C to the point $(1/2, 1/2)$ in $\mathbb{R}^2$. Then the complement of $cik^*(X)$ is a single point, and thus $cik^*(X)$ is open. Its components are in correspondence with C so are uncountable in number.

Unfortunately, $cik^*(X)$ need be neither open nor closed. It is closed in the case of the sine continuum, but is not closed in the infinite string of them as show below. Below, it's disconnected, but in the sine continuum it's connected. It may not separate X as in the case of the sine continuum, or may separate it into countably many pieces as below (or finitely many with an obvious modification).



There are continua for which $cik(X)$ has uncountably many components. Consider the standard middle-thirds Cantor Set $C \subseteq [0, 1]$ constructed by the removal of open intervals of lengths $1/3^n$ for $n \in \mathbb{N}$. If U is such an interval with length $1/3^k$, we attach to its end points a ladder of height $1/3^k$ with 'steps' converging to U, as shown below, creating the *Cantor Leaky Ladder*. If X is the unit interval unioned with these ladders, then $cik^*(X)$ is the union of those U's and thus $X \setminus cik^*(X) = cik(X)$ has components in correspondence with C, since the points of C can still be separated by half-planes as usual.

Corollary 1.1.35. *Suppose X is a continuum and $lc^*(X)$ is non-empty. Then*

(a) $lc^*(X)$ contains a non-degenerate subcontinuum of X, and is thus uncountable,

(b) $lc^*(X)$ is dense in itself.

Proof. (a) If $cik^*(X)$ is empty then by **1.1.4** $lc^*(X)$ would also be empty, so assume otherwise. Since lc implies cik at a point, we have $cik^*(X) \subseteq lc^*(X)$. Since the former contains a non-degenerate subcontinuum, so does $lc^*(X)$. It is thus uncountable.

(b) If x is an isolated point of $lc^*(X)$ then X is cik at x, lest it be isolated in $cik^*(X)$ (impossible). Let V_n be a basis of connected neighborhoods of x. Since x is isolated, it has an open local basis $\mathcal{U}$ such that X is locally connected at every other point in any $U \in \mathcal{U}$; pick an index n so that $V_n \subset \overline{V_n} \subset U$. For fixed V_n each point of $\partial(V_n)$ has an open connected neighborhood contained in $W_\alpha \subset U$.

Then the union Z_n of V_n and these neighborhoods is connected, and it's open since $V_n = \text{int}(V_n) \cup \partial V_n$, i.e. it can be written as the union of the open W_α and $\text{int}(V_n)$. Thus $Z_n \subset U$ is an open, connected neighborhood of x. But U was an arbitrary neighborhood of x, proving the proposition. □

Proposition 1.1.36. *$lc(X)$ and $cik(X)$ are G_δ sets for any metric space X.*

Proof. By definition, $lc(X) = \cap E_n$ and $cik(X) = \cap F_n$ defined as:

$$E_n = \{x \in X \mid \exists \text{ a connected open neighborhood } U_x \text{ of } x \text{ with diameter } < \frac{1}{n}\}$$

$$F_n = \{x \in X \mid \exists \text{ a connected neighborhood } V_x \text{ of } x \text{ with diameter } < \frac{1}{n}\}$$

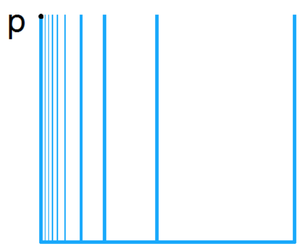
If $x \in E_n$, let U_x be as above with diameter less than $1/n$. Then, trivially, every point in U_x has a connected open neighborhood with diameter less than $1/n$, namely U_x, and thus $U_x \subseteq E_n$. Therefore E_n is open, so $lc(X)$ is a G_δ set. Similarly, if $x \in F_n$ then every point of $\text{int}(V_x) = W_x$ is contained in the connected neighborhood V_x and thus $F_n = \cup_{x \in F_n} W_x$ is open. □

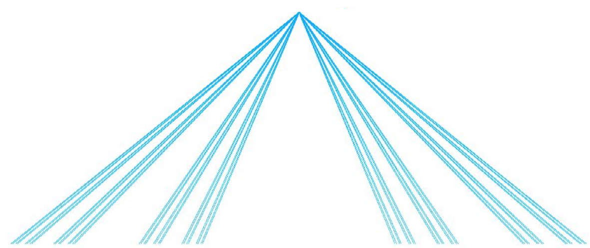
Thus we have developed a fairly good 'algebra' of subcontinua and their components in arbitrary continua. This is helpful in understanding the global aspects of these spaces. The local picture is the subject of the next section.

1.2 Local Structure

In Theorem **1.1.4** and Theorem **1.1.27** we saw two examples of what are often called 'local-to-global' theorems. That is to say, understanding how a continuum behaves locally can shed light on how it behaves globally. Differential topology is replete with such theorems. The first definition is due to Whyburn [38] *(p. 739)*.

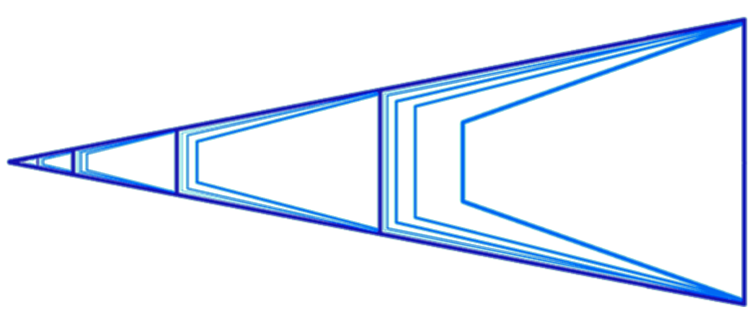
Definition 1.2.1. ***(Semi-Locally Connected)*** *If X is a continuum and $x \in X$ is a point, then X is semi-locally connected, written slc, at x if it has a local basis of neighborhoods V_n such that $X \setminus V_n$ has only finitely many components. We say X is slc if it's slc at each of its points.*





The continuum on the left, the *Harmonic Comb*, is not cik (or lc) at p, but is slc at p since it has neighborhoods whose removal cut X into only one component. The continuum on the right, the aforementioned *Cantor Fan*, is locally connected at its vertex v, but the removal of small neighborhoods about v disconnects X into uncountably many components.

Below is the *Trapezoid Basin Continuum* [10] *(p. 137)*. It is an infinite sequence of trapezoids each containing an infinite sequence of arcs, as shown, converging to a point. It is cik and slc at its vertex, but is not locally connected there.



Proposition 1.2.2. *In the definition of semi-local connectedness, it is equivalent to require that the neighborhoods V_n be regular open sets.*

Proof. One direction is immediate, so suppose X is slc at x with neighborhood basis V_n as proscribed. Then $W_n = \text{int}(\overline{V_n})$ is a regular-open neighborhood basis for x, so it's sufficient to show that $X \setminus W_n$ has finitely many components for every n.

Fix an n and suppose $C_1, \dots, C_k$ are the components of $X \setminus V_n$. Then the sets C_j are contained in at most k components of $X \setminus W_n$ by maximality; call these components $D_1, \dots, D_\ell$. If D were some other component of $X \setminus W_n$ then $D \subseteq \partial W_n$ and D is separated from $D_1 \cup \dots \cup D_\ell = X \setminus V_n$. But then there is a neighborhood of D contained in $\overline{V_n}$ and thus $D \subseteq \text{int}(\overline{V_n})$, i.e. $D \subseteq W_n$. Thus $D_1, \dots, D_\ell$ are the only components of $X \setminus W_n$, as desired. □

Now with a name like 'semi-locally connected', surely there is some connection between lc and slc. To this end, consider the following definition [9] *(p. 545)*:

Definition 1.2.3. ***(Aposyndesis)*** *If $x, y \in X$ are distinct points we say X is aposyndetic at x with respect to y if there is a continuum $K \subseteq X \setminus \{y\}$ whose interior contains x. If X is aposyndetic at x with respect to every $y \in X \setminus \{x\}$ we say X is aposyndetic at x. X is aposyndetic if it's aposyndetic at every point.*

Proposition 1.2.4. *If X is cik at x then X is aposyndetic at x.*

Proof. If $y \in X$ is distinct from x, then for some $\epsilon > 0$ we have that $y \notin B_\epsilon(x)$, a neighborhood of x. By definition of connected im kleinen and letting $r = \epsilon/2$, there is a connected neighborhood V of x contained in $B_r(x)$ whose closure is contained in $B_\epsilon(x)$ and thus $\overline{V} = K$ is the desired continuum. □

Corollary 1.2.5. *A Peano continuum is aposyndetic.* □

Aposyndesis has proven to be eminently useful in the study of planar continua and its development was a major research objective in the 1950's and 60's for topologists, though the definition is a bit clunky. The following is easier to remember:

Definition 1.2.6. ***(Freely Decomposable)*** *[9] (p. 545) A continuum X is* freely decomposable *if for every pair of distinct points $x, y \in X$ there are subcontinua A, B with $x \in A \setminus B$ and $y \in B \setminus A$ such that $X = A \cup B$.*

Theorem 1.2.7. *Suppose that X is a continuum. Then the following are equivalent: (a) X is freely decomposable.*

(b) X is aposyndetic.

(c) X is semi-locally connected.

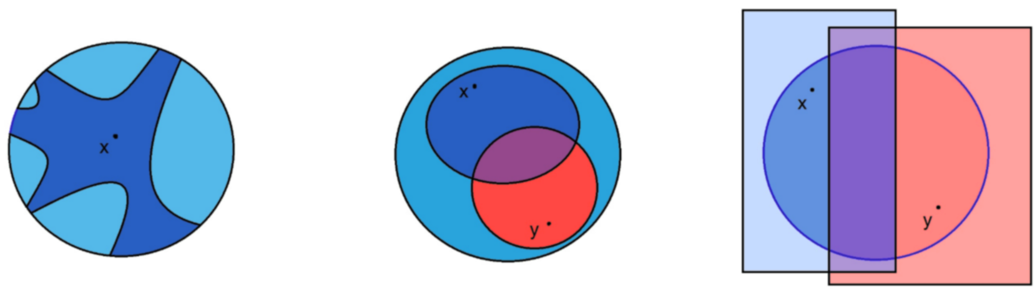
Proof. Assume X is freely decomposable and that $x, y \in X$ are distinct. We show that X is aposyndetic at x with respect to y. By assumption there are subcontinua $A \cup B = X$ satisfying $x \in A \setminus B$ and $y \in B \setminus A$. If $x \notin \mathrm{int}(A)$ then every neighborhood of x intersects $X \setminus A \subseteq B$ and thus $x \in \overline{B} = B$, a contradiction. Thus $A \subseteq X \setminus \{y\}$ is a neighborhood of x, as needed.

Now let X be aposyndetic. If U is a neighborhood of $x \in X$, then every $y \notin U$ is contained in a neighborhood continuum $K_y \subseteq X \setminus \{x\}$. Since $X \setminus U$ is compact it's contained in finitely many such continua $K_1, \ldots, K_n$. Then $V = X \setminus (\cup K_j)$ is a neighborhood of x and $X \setminus V = K_1 \cup \cdots \cup K_n$ has at most n components since each K_j is connected. Thus X is slc at x, but x was arbitrary, so X is slc.

Finally, assume X is slc and let $x, y \in X$ be distinct. Let U be an open neighborhood of x with $y \notin \overline{U}$ and such that $X \setminus U$ has only finitely many components $C_1, \ldots, C_k$ with $y \in C_1$. Since $y \notin \overline{U}$ it is contained in the interior of C_1, then because X is slc at y it has a neighborhood $V \subseteq C_1$ such that $X \setminus V$ has only finitely many components $D_1, \ldots, D_n$ with $x \in D_1$. Similarly, $x \in \mathrm{int}(D_1)$. Then $C_1, \ldots, C_k, D_1, \ldots, D_n$ is a finite cover of X by closed connected sets whose members separate x and y. Since X is connected, by finite ascent we may union all those but C_1 which D_1 is not separated from and vice-versa with C_1 back and forth to obtain two continua which cover X and neither of which contain both x and y. □

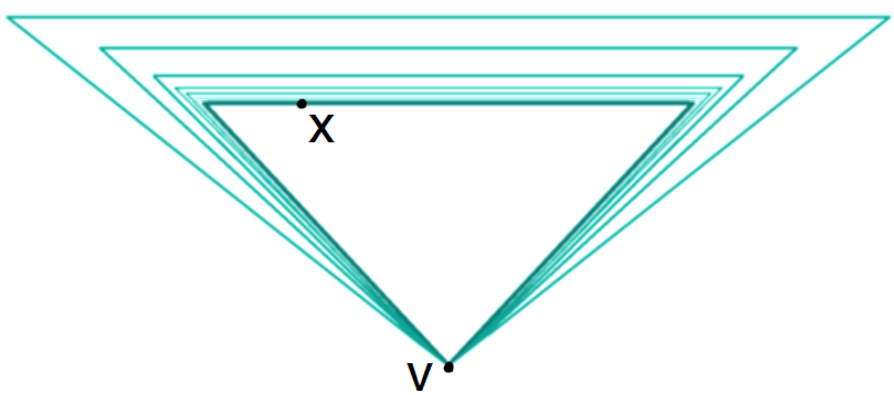
Corollary 1.2.8. *A Peano continuum is freely decomposable.* □

Corollary 1.2.9. *A Peano continuum is semi-locally connected.* □



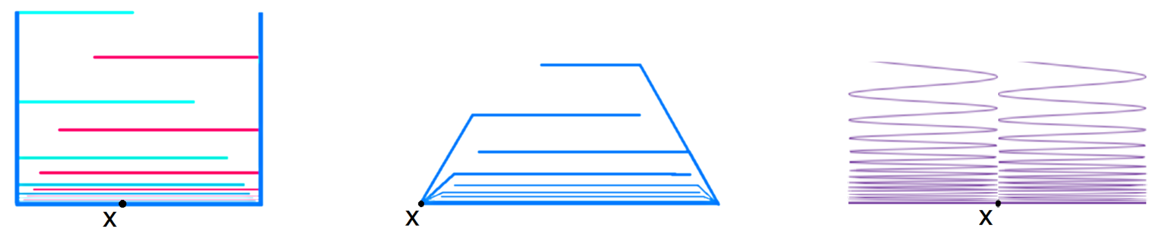
Semi-Local Connectedness Aposyndesis Free Decomposability

Aposyndesis mirrors the T_1 separation axiom, with neighborhood replaced by 'continuum neighborhood,' naturally placing aposyndesis in a spectrum of properties many of which have proven fruitful.

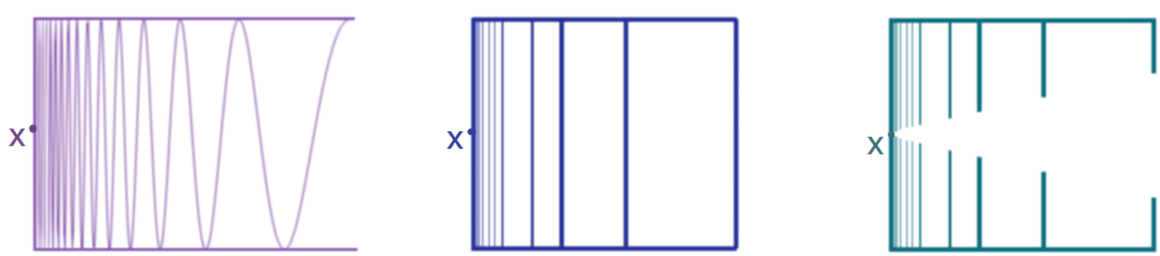


The continuum above, the *Triangle Basin*, is not slc at its vertex v. However, it is locally connected there and thus aposyndetic at v. It is composed of a simple closed curve in the plane shown in dark blue, with an infinite sequence of triangles converging to it at every point except a common vertex. It is slc at any other point x on the dark triangle, since complements of small open balls will be connected. But it is not aposyndetic at x with respect to v since every subcontinuum containing x with non-empty interior contains v.

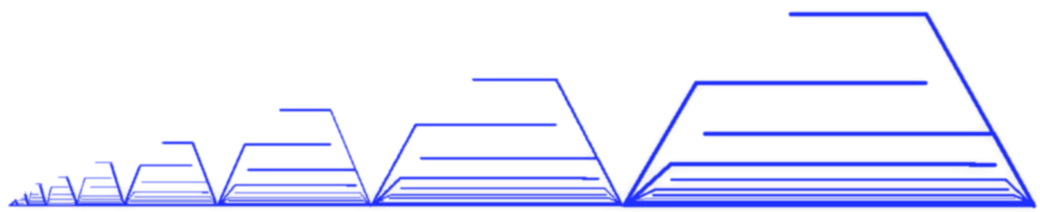
The continua pictured below are pathological examples. The *Jagged Box*, on the left, is composed of the bottom three edges of a square and a pair of intertwined, infinite sequences of arcs converging to the bottom edge. It is not aposyndetic at x with respect to any other point on the bottom edge. The middle, the *Double Fan Continuum*, is the quotient of the first by collapsing the vertical segments to points. On the right, no point on the bottom segment is aposyndetic with respect to any other, despite the presence of 'fat' subcontinua containing x. None are slc at x.



Below are three similar continua, the *Boxed Sin Continuum*, the *Infinite Ladder* and the *Leaky Ladder*. The first two are aposyndetic and slc at x but not connected im kleinen, whereas the last is slc but not aposyndetic. Of course, attaching a copy of the Cantor Fan to x makes any of them fail to be slc. Doing so can turn our examples of slc points into non-slc points without affecting the other properties.



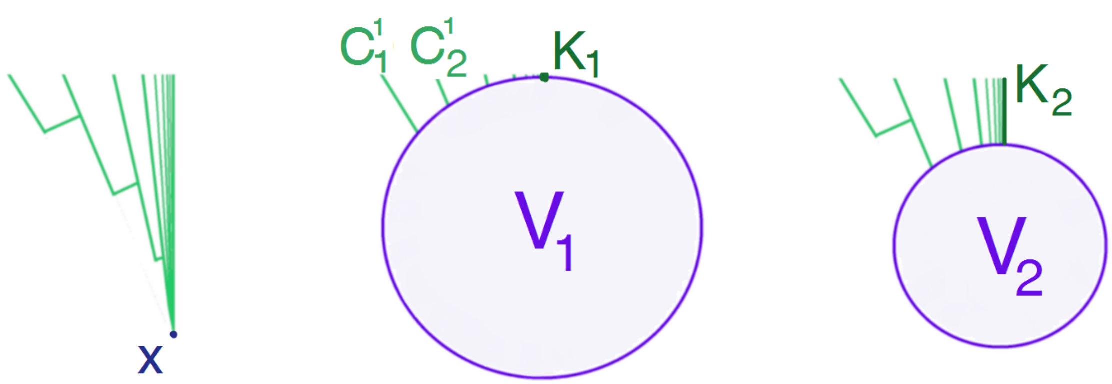
Thus, other than the fact that lc $\implies$ cik $\implies$ aposyndesis, we have exhibited points in continua which have every possible combination of our local properties. Finally, the continuum below shows that in the definition of semi-local connectedness it is *not* equivalent to assume the neighborhoods V_n are closed.



We have purposefully avoided efficiency with our examples, as it's useful to see as many as possible. Now some results to go along with them. First, notice in the case of the Triangle Basin that its vertex v is the only point at which it fails to be slc. Thus v is not contained in a subcontinuum of $slc^*(X)$, where we define $slc(X)$ to be the set of points where X is slc, and $slc^*(X)$ its complement, as in **1.1.33**. However, the inner-most triangle is *some* convergence continuum containing v.

Theorem 1.2.10. *If X is a continuum and if $x \in slc^*(X)$, then x is contained in some convergence continuum $K \subseteq X$ [37] (p. 19).*

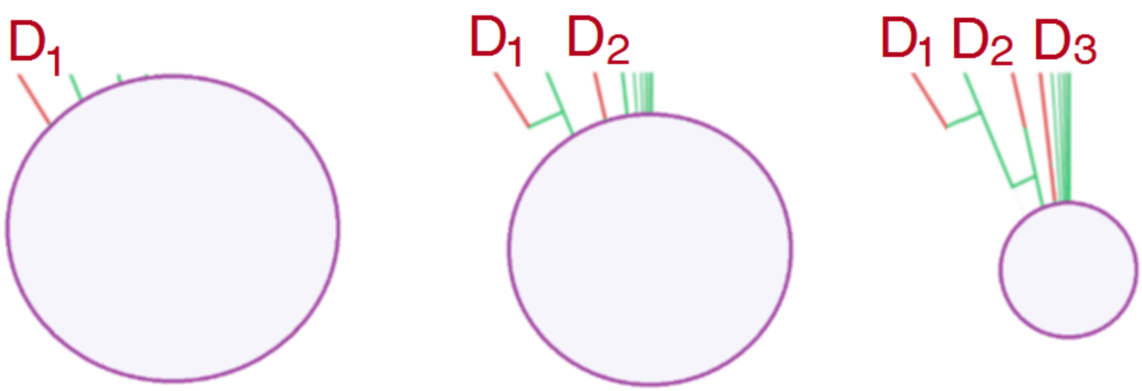
Proof. Suppose X is a continuum and $x \in X$. If $x \in slc^*(X)$ there is some neighborhood U of x such that every open neighborhood $x \in V \subseteq U$ has the property that $X \setminus V$ has infinitely many components. Let V_n be an open local basis for x of such neighborhoods $V_n \subseteq U$ satisfying $V_{j+1} \subseteq \overline{V_{j+1}} \subseteq V_j$ for all j.



If C_k^n are the components of $X \setminus V_n$, then each is closed in X since they're closed in the closed set $X \setminus V_n$. Thus C_k^n is a continuum for every k, n so for $n = 1$, since they're disjoint we can pick a $c_k \in C_k^1$ which will have a limit point in $\cup C_k^1$. Thus by a simple application of **1.1.1(f)** there is some continuum K_1 - which may be a single point - such that $C_k^1 \to K_1$ in the Hausdorff sense.

If the sets C_k^1 are contained in only finitely many components $D_1, \dots, D_m$ of $X \setminus V_2$ then $(D_1 \cup \cdots \cup D_m)^c$ is a neighborhood of x contained in $(\cup C_k^1)^c = V_1 \subseteq U$ whose complement has finitely many components, a contradiction. Thus there is an infinite collection of components C_j^2 each containing some C_k^1, and then as before some sequence from the collection C_j^2 converges to a continuum K_2 containing K_1.

By the Boundary Bumping Theorem, each C_j^2 intersects ∂V_2 and thus so does K_2. But $K_1 \subsetneq K_2$ since $V_1 \subseteq \overline{V_1} \subseteq V_2$ and thus K_2 is a convergence continuum. Repeating inductively, we can find convergence continua $K_j \subseteq K_{j+1}$ with $K_j \cap \partial V_j \neq \varnothing$. As a nested collection of connected sets, $J = \cup K_n$ is connected, and thus so is $K = \overline{J}$. But $\partial V_{j+1} \subseteq V_j$ and thus $\partial V_j \to x$, so $x \in K$. Similarly, since L is closed in complements of arbitrarily small neighborhoods of x, we have $K \setminus L = \{x\}$, and thus K is a non-degenerate continuum containing x.



Now we show that K is a convergence continuum. Since every C_k^j is contained in some C_ℓ^{j+1} and infinitely many C_ℓ^m contain some C_k^j we can pick some $D_2 = C_k^2$ containing a component other than C_1^1. We can inductively define a component $D_{j+1} = C_k^{j+1}$ disjoint from D_ℓ for $\ell \leq j$.

Then every D_j is a continuum containing some C_k^l for $\ell \leq j$. Since $\lim_k C_k^j \to K_j$ we thus have that $\cup K_j = K \subseteq \liminf D_k$. But each D_k is composed of some such components and $\lim_k C_k^j \subseteq K$ for all j, so $\limsup D_k \subseteq K$ and thus $\lim D_k = K$ is a continuum. □

The next two propositions are some of the most routinely used auxiliary results. The first gives a relation between aposyndesis and semi-local connectedness at the local level; the second shows that aposyndesis behaves well with respect to limits. Both were initially exploited by Jones.

Proposition 1.2.11. *If X is a continuum and $x \in X$, then X is slc at x if and only if X is aposyndetic at y with respect to x for every $y \in X \setminus \{x\}$.*

Proof. Suppose X is aposyndetic at every $y \neq x$ with respect to x; let K_y be a continuum neighborhood of y not containing x. Then the complement of every open neighborhood of x is compact and thus covered by finitely many K_y, i.e. has only finitely many components by maximality of components. Thus X is slc at x.

Now suppose that X is slc at x with open neighborhood basis $V_{n+1} \subseteq \overline{V_{n+1}} \subseteq V_n$ such that $X \setminus V_n$ has finitely many components for all n. If N is large enough so that $y \notin \overline{V_N}$ then the component of $X \setminus V_N$ containing y is a neighborhood of y and is a continuum, since V_N is open. Thus X is aposyndetic at y with respect to x. □

Proposition 1.2.12. *Suppose X is a continuum and $x_n, y_n \in X$ with $x_n \to x, y_n \to y$ and $x_n \neq y_n, x \neq y$. If X is not aposyndetic at x_n with respect to y_n for all n then X is not aposyndetic at x with respect to y.*

Proof. Suppose otherwise, i.e. let K be a neighborhood continuum of x not containing y. Then eventually it doesn't contain y_n, but does eventually contain x_n in its interior. Suppose N is large enough for both to be the case for x_N, y_N; but then K is a neighborhood continuum of x_N not containing y_N, a contradiction. □

Corollary 1.2.13. *If X is a continuum and $x \in X$, the set of points y such that X fails to be aposyndetic at x with respect to y is closed. The set of points y such that X fails to be* *aposyndetic at y with respect to x is closed.*

Proof. Set $x_n \equiv x$ and $y_n \equiv y$ respectively in the previous proposition. □

To this end, it is useful to have some notation. If $x \in X$ define

$$\mathrm{Ap}(x) = \{y \in X \mid X \text{ is aposyndetic at } x \text{ with respect to } y\}$$

$$\mathrm{Ap}_y(x) = \{y \in X \mid X \text{ is aposyndetic at } y \text{ with respect to } x\}$$

We can define $\mathrm{Ap}(X) = \{x \in X \mid X \text{ is aposyndetic at } x\}$ and $\mathrm{Ap}_y(X)$ as the set of points x so that X is aposyndetic at all other points with respect to x. We will denote their complements by $\mathrm{Ap}^*(x)$, $\mathrm{Ap}_y^*(x)$, $\mathrm{Ap}^*(X)$ and $\mathrm{Ap}_y^*(X)$ as before.

Corollary 1.2.14. *If X is a continuum, then $\mathrm{Ap}(x)$ and $\mathrm{Ap}_y(x)$ are open.* □

Compare the following with proposition **1.1.36**:

Proposition 1.2.15. *$\mathrm{Ap}(X)$, $\mathrm{Ap}_y(X)$ and $slc(X)$ are G_δ sets if X is a continuum.*

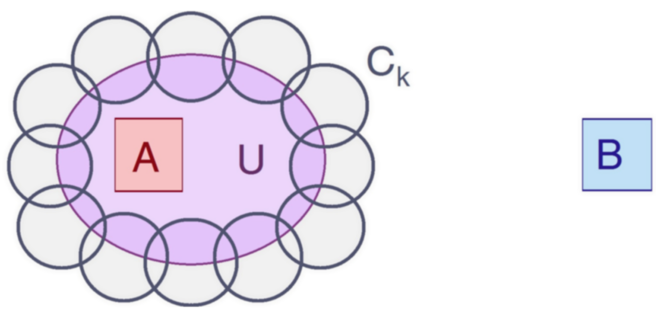
Proof. We show that $\mathrm{Ap}^*(X)$ is an F_σ set. Let A_n be the set of points x such that X is not aposyndetic at x with respect to some point y satisfying $d(x,y) \geq {}^1\!/\!_n$. Then $\mathrm{Ap}^*(X) = \cup A_n$, so it's sufficient to show that each A_n is closed. Fix an n and suppose $p_k \in A_n$ with $p_k \to p$. Then there are q_k so that X is not aposyndetic at p_k with respect to q_k for every k and $d(p_k, q_k) \geq {}^1\!/\!_n$ for all k. Then if q is some limit point of q_n we have $d(p,q) \geq {}^1\!/\!_n$ by continuity and X is not aposyndetic at p with respect to q by **1.2.12**. Thus $p \in A_n$ by definition of A_n.

Let B_n be the set of points x such that X is not aposyndetic at some y with respect to x satisfying $d(x,y) \geq {}^1\!/\!_n$. Then $\mathrm{Ap}_y^*(X) = \cup B_n$ so it's sufficient to show each B_n is closed. Let $x_k \in B_n$ and suppose $x_k \to x$. Let y_k satisfy $d(x_k, y_k) \geq {}^1\!/\!_n$ where X is not aposyndetic at y_k with respect to x_k and suppose y is a limit point of y_k. Then by **1.2.12** X is not aposyndetic at y with respect to x, so since $d(x,y) \geq {}^1\!/\!_n$ we have $x \in B_n$ and thus B_n is closed.

Now by **1.2.11** $x \in slc^*(X)$ if and only if there is a p such that X is not aposyndetic at p with respect to x. To this end, define C_n as the set of points x with some y satisfying $d(x,y) \geq {}^1\!/\!{}_n$ and X is not aposyndetic at y with respect to x. Then $slc^*(X) = \cup C_n$ so it's sufficient to show each C_n is closed. If $x_k \in C_k$ pick y_k so that X is not aposyndetic at y_k with respect to x_k for any k and $d(x_k, y_k) \geq {}^1\!/\!{}_n$. Then by **1.2.12** if $x_k \to x$ and y is a limit point of y_k then X is not aposyndetic at y with respect to x and $d(x,y) \geq {}^1\!/\!{}_n$. Thus $x \in C_n$, as desired. □

Proposition 1.2.16. *If X is a continuum and $x \in X$ then $Ap^*_y(x)$ is a continuum [20] (p. 145).*

Proof. By **1.2.14** $Ap^*_y(x)$ is closed, so it's sufficient to prove that it's connected. Suppose $Ap^*_y(x) = A \cup B$ is a separation and $x \in A$. Each is closed in X, so by normality we may pick an open neighborhood U of A such that $\overline{U} \cap B = \varnothing$. Then for every $u \in \partial U$, there is a continuum neighborhood C_u of u not containing x since $\partial U \cap Ap^*_y(x) = \varnothing$.



Since ∂U is compact it's covered by finitely many such sets, say $C_1, \ldots, C_n$. If $V = U \setminus (\cup C_k)$, then by the Boundary Bumping Theorem V^c has only finitely many components. If $y \in B$ and if D is the component of V^c containing y, then since $y \notin \overline{U}$ we have $y \in \text{int}(D)$. Since V is open V^c is closed and thus D is closed in X. Thus it's a continuum neighborhood of y and therefore X is aposyndetic at y with respect to x, a contradiction. □

The next result is surprisingly convenient:

Proposition 1.2.17. ***(Double Basis Theorem)*** *If X is a continuum and X is slc and cik (resp. lc) at x, then x has a local neighborhood basis of connected (resp. connected open) sets each of whose complement has only finitely many components.*

Proof. Let U be an arbitrary neighborhood of x and let $V \subseteq U$ be a neighborhood of x whose complement has finitely many components $C_1, \ldots, C_n$. Let $W \subseteq V$ be a connected neighborhood of x, and let $D_1, \ldots, D_n$ be the (not necessarily distinct) components of $X \setminus W$ containing $C_1, \ldots, C_n$ respectively. Let $Y = X \setminus (D_1 \cup \cdots \cup D_n)$.

Then Y is a neighborhood of x whose complement has at most n components and $Y = W \cup (\cup_\alpha A_\alpha)$ where each A_α is some component of W^c. But then by the Boundary Bumping Theorem, we have that Y is connected. Therefore Y is the desired neighborhood of x contained in U. If X is locally connected at x then we may take W to be open and thus each D_k is closed, i.e. Y is open, as desired. □

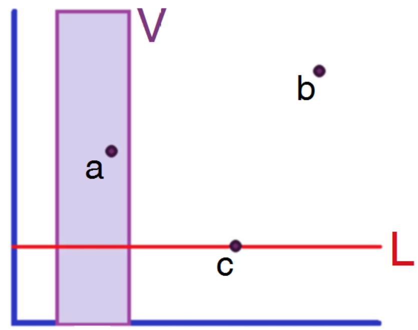
Proposition 1.2.18. *The union of two aposyndetic continua is aposyndetic.*

Proof. If $X = A \cup B$ where A and B are aposyndetic, assume $x \in X$ and let y be some other point. If $x \in A \setminus B$ and $y \in B \setminus A$ then A is a continuum neighborhood of x not containing y. If $y \in A$ then since A is aposyndetic there is a continuum neighborhood of x not containing y in A, and since $x \in A \setminus B$ it is a neighborhood in X as well. Thus in either case, X is aposyndetic at x with respect to y.

The same will hold if $x \in B \setminus A$, so assume $x \in A \cap B$. If $y \in A \setminus B$ (or $B \setminus A$) and C is a continuum neighborhood of x in A not containing y, then $B \cup C$ is a continuum neighborhood of x not containing y. Thus assume $y \in A \cap B$. Then there are continuum neighborhoods C, D of x in A, B respectively so that neither contain y, and thus $C \cup D$ is the desired continuum neighborhood of x in X. □

Theorem 1.2.19. *The product of two non-degenerate continua is aposyndetic [11] (p. 403).*

Proof. Let X and Y be non-degenerate and let $a = (a_1, a_2), b = (b_1, b_2) \in X \times Y$ be distinct. We show that $X \times Y$ is aposyndetic at a with respect to b. Without loss of generality assume that a and b differ in their first coordinate. If $d_X(a_1, b_1) > \epsilon$ let $V = \overline{B_\epsilon(x)}$. Then $b_1 \notin V$ so $b \notin V \times Y$, a closed neighborhood of a.



If $c = (c_1, c_2) \in X \times Y$ differs from both a and b at both coordinates then $L = X \times \{c_2\}$ also doesn't contain b and is connected since $L \simeq X$. Thus $(V \times Y) \cup L$ is a closed neighborhood of a, and it's connected since it can be written as the union $\cup_{v \in V}[(\{v\} \times Y) \cup L]$, a union of connected sets with a point c in common. Thus it's a continuum neighborhood of a not containing b, as desired. □

Corollary 1.2.20. *The product of two non-degenerate continua is slc. The product of two non-degenerate continua is freely decomposable.* □

Now we introduce our last local property of the section. It has proven to be a crucial tool, and is an alternative weakening of local connectedness. Once again, though the property is defined pointwise we can use it to obtain considerable global information about X.

Definition 1.2.21. ***(Property of Kelley)*** *Let X be a continuum and let $x \in X$. If for every subcontinuum $K \subseteq X$ containing x and for every sequence of points $x_n \to x$ there are subcontinua K_n with $x_n \in K_n$ and $K_n \to K$, then we say that X* has the property of Kelley *at x. If X has the property of Kelley at every $x \in X$ then we say that X* has the property of Kelley.

The property of Kelley was first introduced in [16] *(p. 22)* as *property* 3.2. If X has the property of Kelley we will just say that X is a Kelley continuum, or that X is Kelley. In some papers it is also denoted by *property* (κ). It was originally given with the following (seemingly stronger) definition:

Definition 1.2.22. *A continuum X is Kelley at $x \in X$ if for every $\epsilon > 0$ there is a $\delta > 0$ so that whenever $y \in B_\delta(x)$ and whenever K is a subcontinuum containing x, there is a subcontinuum Y containing y with $H(K, Y) < \epsilon$ [35] (p. 291).*

Proof. One direction is immediate, so assume that X is a continuum containing x and that X is Kelley in the first sense at x. By contradiction, suppose that there is an $\epsilon > 0$ such that $\delta_n \to 0$ and $d(x_n, x) < \delta_n$, yet there are subcontinua K_n containing x such that every continuum Y containing x_n satisfies $H(Y, K_n) \geq \epsilon$. Pick a convergent subsequence of K_n converging to K, which is thus a continuum containing x. But then $H(K_n, K) \to 0$ yet if Y_n is a continuum containing x_n we have $d(Y_n, K_n) < \epsilon$ and thus Y_n doesn't converge to K. But Y_n was an arbitrary sequence of continua containing x_n, respectively, a contradiction. □

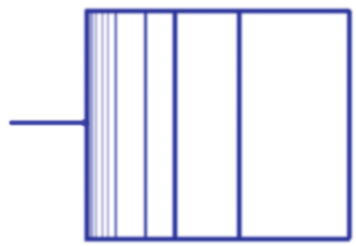
Since X is compact, we can use the second definition to obtain:

Corollary 1.2.23. *If X is Kelley and $\epsilon > 0$, there is a $\delta > 0$ such that for any $x \in X$, any $y \in B_\delta(x)$ and any subcontinuum K containing x, there is a subcontinuum Y containing y such that $H(K, Y) < \epsilon$.*

Proof. Let $\epsilon > 0$ and suppose that X is Kelley. Suppose by contradiction that there are $x_n \in X$ - which we may assume is convergent after taking a subsequence - and continua K_n containing x_n respectively such that there are points y with $d(x_n, y_n) \to 0$ such that any continuum Y_n containing y_n satisfies $H(K_n, Y_n) \geq \epsilon$.

Assuming $x_n \to x$ we have that $y_n \to x$, and we may pick a convergent subsequence of continua $K_n \to K$ converging to a subcontinuum K containing x. Then if Y_n contain y_n respectively we have $H(K_n, K) \to 0$ while $H(K_n, Y_n) \geq \epsilon$ and thus Y_n does not converge to K. But $y_n \to x$ and Y_n was an arbitrary sequence of subcontinua containing y_n, respectively, contradicting that X is Kelley at x. □

The first definition makes it clear that this property is a topological invariant, whereas the second property, though defined in terms of a metric, is easier to use. In particular, the Kelley property implies that the behavior of subcontinua is somewhat tame with respect to limits taken within the given continuum. The places where a continuum fails to be Kelley are places that are 'geometrically unnatural' relative to a continuum, which itself may 'typically' be pathological.



In the continuum above, X is not Kelley at the point x where the arc meets the ladder, yet X is globally aposyndetic. If we instead attach a harmonic or Cantor fan at x, then it will fail to be *slc* at x and the removal of X will leave countably (respectively uncountably) infinitely many components. If instead of attaching an arc to a single point we attach one to a countable set or a Cantor set along the left-most step, the set of points where X fails to be Kelley has countably (respectively uncountably) infinitely many components.

The harmonic fan itself is globally Kelley, yet fails to be aposyndetic or slc along its bottom edge. In the continuum above, the left midpoint is Kelley and aposyndetic, but not slc. Returning to the triangle basin, X is slc and Kelley at x but not aposyndetic. By attaching an arc or harmonic fan at x we obtain a continuum which fails to be aposyndetic and Kelley (and slc, respectively). Thus by themselves, these three properties have no relations, even if one of them is assumed globally.

Notice that in all of our examples, our continua were not cik at x. The infinite ladder itself is Kelley, slc and aposyndetic but not cik along its left rung. The vertex of the trapezoid basin is cik, slc and Kelley, but is not lc. Attaching the vertex of a harmonic fan to the vertex of the trapezoid basin produces a point which is cik and Kelley, but neither lc nor slc. However, the following is true:

Proposition 1.2.24. *If X is cik at x, then X is Kelley at x.*

Proof. Suppose that $x \in X$ and that K is a subcontinuum containing x. Suppose that $x_n \to x$. Since X is cik at x it has a continuum neighborhood Y_n with $\operatorname{diam}(Y_n) < {}^1\!/\!{}_n$. Since $x_n \to x$, after taking a subsequence assume that $x_n \in Y_n$. Then $K \cup Y_n$ is a continuum containing x_n with $H(K, K \cup Y_n) < {}^1\!/\!{}_n$, as required. □

Corollary 1.2.25. *Every Peano continuum is Kelley.* □

Definition 1.2.26. *If X is a continuum, let* $\operatorname{Kel}(X)$ *be the set of points at which X is Kelley, and let* $\operatorname{Kel}^*(X)$ *be its complement.*

The next result coincides with those for our other local properties.

Proposition 1.2.27. *If X is a continuum, then* $\operatorname{Kel}(X)$ *is a G_δ set.*

Proof. We show that its complement is an F_σ set. Note that $\operatorname{Kel}^*(X) = \cup_n \{x \in X \mid \exists x_n \to x$ and a continuum K containing x such that for every sequence of subcontinua Y_n containing x_n, we have $H(Y_n, K) \geq {}^1\!/\!{}_n\}$, so it suffices to show that each is closed. Pick an n and suppose that $x_k \to x$ are contained in continua K_k respectively so that for any sequence of points $y_k^n \to x_n$ and any subcontinua Y_k^n containing y_k^n we have that $H(Y_k^n, K_n) \geq {}^1\!/\!{}_n$. After taking a subsequence we may assume that K_k converge to some continuum K containing x. Then $y_k^k \to x$ and $K_k \to K$, but $H(K_k, Y_k^k) \geq {}^1\!/\!{}_k$ and thus $H(\limsup Y_k^k, K) \geq \epsilon$. But Y_k^k was an arbitrary sequence of continua containing y_k^k, and thus x also satisfies the required property for the sequence y_k^k and subcontinuum K. □

Definition 1.2.28. ***(Constituent)*** *If $A \subseteq B$ and A is connected, define the* constituent $\mathrm{Con}_B(A)$ *to be the component of B containing A.*

Definition 1.2.29. ***(Filament, Ample)*** *[27] (p. 1581) If X is a continuum and $A \subseteq X$ is a continuum, then A is a* filament continuum *if it has a neighborhood B such that $Con_B(A)$ has empty interior. A set $Y \subseteq X$ is called filament if every subcontinuum of Y is filament. A subcontinuum A is called* ample *if for every neighborhood U of A there is a continuum B with $A \subseteq int(B) \subseteq B \subseteq U$.*

Theorem 1.2.30. *If X is a Kelley continuum and $A \subseteq X$ is a subcontinuum, then A is ample if and only if A is not filament [27] (p. 1586).*

Proof. By definition, if A is ample then it is not filament, so to prove the other direction we need a lemma:

Lemma 1.2.31. *If X is a Kelley continuum and A is a non-filament subcontinuum, then for each open set U containing A the set $Con_U(A)$ is open in X.*

Proof. Suppose that $x \in \mathrm{Con}_U(A)$; then it it sufficient to show that for every sequence $x_n \to x$, eventually $x_n \in \mathrm{Con}_U(A)$. Pick an open neighborhood V of A such that $\overline{V} \subseteq U$ by normality, and since A is not filament $C = \mathrm{Con}_V(A)$ has non-empty interior. Since $\overline{C}$ is connected and contains A, it's contained in $\mathrm{Con}_U(A)$, and since X is Kelley there are continua $C_n \to \overline{C}$ such that $x_n \in C_n$.

Since U is a neighborhood of $\overline{C}$ the sets C_n are eventually contained in U, and since $\overline{C}$ has non-empty interior and $x_n \to x$ they eventually intersect $\overline{C}$. But if $C_n \subseteq U$ intersects $\overline{C}$ we have that $C_n \cup \overline{C}$ is connected, i.e. is contained in $\mathrm{Con}_U(A)$. Since $x_n \in C_n$ we have that $x_n \in U$ for n sufficiently large, as desired. □

Now let U and V be as before; then if $C = \mathrm{Con}_V(A)$ by the lemma $\mathrm{Con}_V(A)$ is an open neighborhood of A, and thus its closure is a continuum containing A in its interior. Since $\overline{V} \subseteq U$ and $\mathrm{Con}_V(A) \subseteq V$, we see that $\overline{C} \subseteq U$, as desired. □

Thus the subcontinua of Kelley continua fall into two categories: Ample and filament. It happens to be that, unlike aposyndesis, which might be satisfied at *no* points of a continuum, every continuum is Kelley on a dense G_δ set. We conclude this section with some of the most important auxillary results, also from [27].

Proposition 1.2.32. *Subcontinua of filament continua are filament.*

Proof. If $B \subseteq A$ are subcontinua of a continuum X and A is filament, let U be a neighborhood of B. If $\mathrm{Con}_U(B)$ has non-empty interior and if V is an open neighborhood of A, then the constituent of A in $U \cup V$ contains U by maximality of components and thus has non-empty interior, impossible. □

Corollary 1.2.33. *If $B \subseteq A$ are subcontinua of X and B is ample, then A is ample.*

Proof. Take the contrapositive and apply the previous theorem. □

Proposition 1.2.34. *If K is a filament subcontinuum of X there is a filament subcontinuum L properly containing K.*

Proof. Let U be an open neighborhood of K so that $\mathrm{Con}_U(K)$ has empty interior. Apply the Extension Theorem to K in U to produce a subcontinuum L properly containing K in U, which is contained in $\mathrm{Con}_U(K)$ since it's connected. Thus U is an open neighborhood of L such that $\mathrm{Con}_U(L)$ has empty interior. □

Corollary 1.2.35. *If K is a filament subcontinuum of X, there is an $\epsilon_K > 0$ such that any subcontinuum L containing K satisfying $H(K, L) < \epsilon_K$ is filament.*

Proof. If U is as before then eventually $L \subseteq U$ and thus $L \subseteq \mathrm{Con}_U(K)$. □

Proposition 1.2.36. *X is not cik at any point of a filament subcontinuum.*

Proof. If $x \in X$ is contained in a filament subcontinuum K, with connected neighborhood U such that $\mathrm{Con}_U(K)$ has empty interior, then any connected neighborhood of x intersects multiple components of U, so X is not cik at x. □

Corollary 1.2.37. *Every subcontinuum of a Peano continuum is ample.* □

1.3 Hyperspaces

Definition 1.3.1. *Let (X, d) be a compact metric space. Define*

(1) $2^X = \{A \subseteq X \mid A \text{ is closed and non-empty}\}$, *and*

(2) $C(X) = \{A \subseteq X \mid A \text{ is closed, connected and non-empty}\} \subseteq 2^X$

These spaces are simply referred to as *hyperspaces*. They tend to be big; it is known that 2^X for any non-degenerate continuum contains a copy of the Hilbert Cube. As it is a standard exercise, we assume known the definition of the Hausdorff distance, and that it forms a metric on 2^X, as well as the $\limsup$-$\liminf$ characterization of convergence in 2^X. It is called the Hausdorff metric; we denote it by H_d. It is a fact that the induced topology on 2^X depends only on the topology of X when X is a continuum [15] *(p. 54)*, rather than the metric. When we write 2^X we will mean this topology, and when we write $C(X)$ we will mean the topology induced by 2^X. Note that $C(X)$ contains an isometric copy of X, since the Hausdorff distance between two singletons is just the metric distance. Hyperspaces are one of the most heavily studied objects in continuum theory; for a thorough treatment, see [8].

Since the topology on 2^X only depends on the underlying topology, there should be a basis that reflects this. It is as follows.

Definition 1.3.2. ***(Vietoris Topology)*** *For open sets $U, U_1, U_2, \ldots, U_n \subseteq X$, write*

(1) $\Lambda(U) = \{A \in 2^X \mid U \cap A \neq \varnothing\}$

(2) $\langle U_1, U_2, \ldots, U_n\rangle = \{A \in 2^X \mid A \subseteq \bigcup U_i \text{ and } A \cap U_i \neq \varnothing \text{ for all } i\}$

Then $\mathcal{B} = \{\langle U_1, U_2, \ldots, U_n\rangle \mid U_i \text{ open in } X\}$ is a basis for a topology on 2^X called the Vietoris Topology, and it coincides with the topology induced by H_d when X is compact. As well, $\mathcal{C} = \{\langle U\rangle \mid U \text{ open in } X\} \cup \{\Lambda(U) \mid U \text{ open in } X\}$ is a sub-basis.

The Vietoris topology has been studied in many settings; for some results, see [14]. Our first goal is to prove that these sets are compact for a continuum X; more interesting is that they are path-connected.

Theorem 1.3.3. *If X is a compact metric space, then 2^X is compact.*

Proof. Let $B_1, B_2, \ldots$ be a basis for X and let (A_n) be a sequence of closed subsets. Inductively define sequences (A_n^i) as follows: $(A_n^0) = (A_n)$, if $(\liminf A_n^k) \cap B_{k+1} \neq \varnothing$ then $(A_n^{k+1}) = (A_n^k)$, while if $(\liminf A_n^k) \cap B_{k+1} = \varnothing$ then take a subsequence (A_n^{k+1}) all of whose members are disjoint from B_{k+1}. Consider (A_n^n), a subsequence of the original sequence. Then if $x \in \limsup A_n^n$ then $A_n^n \cap B_j \neq 0$ for any basis element B_j containing x and for every $n > j$ by construction, and thus $x \in \liminf(A_n^n)$. Thus $\liminf A_n^n = \limsup A_n^n$, i.e. (A_n^n) converges. Thus 2^X is limit point compact, and as a metric space is therefore compact. □

Corollary 1.3.4. *C(X) is compact.*

Proof. If $A_n \in C(X)$, let E be a limit point of $\bigcup\{A_n\}$ and let A_n' (which exists since 2^X is compact) be a subsequence converging to E, in other words $\liminf A_n' = \limsup A_n' = E$. Then by **1.1.1(f)** E is a continuum. Thus $C(X)$ is closed in 2^X and therefore compact, since 2^X is. □

We note that theorem **1.3.3** is true more generally if X is replaced by any compact set (thought the corollary is not). A relatively short proof can be based on the Alexander Sub-Basis Lemma.

Definition 1.3.5. ***(Nest)*** *A nest $\mathfrak{N}$ is a collection of sets A_α indexed by $\mathcal{J}$ such that for any $\alpha, \beta \in \mathcal{J}$ either $A_\alpha \subset A_\beta$ or $A_\beta \subset A_\alpha$. If for all $N \in \mathfrak{N}$ we have $A \subseteq N \subseteq B$ with $A, B \in \mathfrak{N}$, we say that $\mathfrak{N}$ is a nest from A to B.*

Definition 1.3.6. ***(Whitney Map)*** *A continuous function $w : 2^X \to \mathbb{R}$ is a Whitney map if $w(\{x\}) = 0$ for all $x \in X$ and if $w(A) < w(B)$ for all pairs $A \subsetneq B$.*

After a renormalization we may assume that $w : 2^X \to [0,1]$ with $w(X) = 1$. We call $w(A)$ the *size* of A (with respect to w). Whitney maps remain an area of active investigation. They appear similar to uniformizations, to (non-atomic) measures and to the diameter function, but there are no implications. For example, two-point sets have positive size. Considering this, one might be led to believe that size functions are hard to come by. Thankfully not [18] *(p. 78)*.

Theorem 1.3.7. *If X is a compact metric space, there is a Whitney map on 2^X.*

Proof. Let U_n be a countable open basis for X such that $U_n \neq \varnothing$ and $\overline{U_n} \neq X$ for all n. For each pair m, n such that $\overline{U_m} \subseteq U_n$ we may, by the Urysohn Lemma, pick a continuous function $f_{m,n} : X \to I$ with $f_{m,n} \equiv 0$ on $\overline{U_m}$ and $f_{m,n} \equiv 1$ on $X \setminus U_n$. Reindex this collection of functions as f_k, and for each k define $g_k(A) = \operatorname{diam}(f_k(A))$ whenever $A \in 2^X$, each continuous since f_k and $\operatorname{diam}(\cdot)$ are. Now define

$$w(A) = \sum_k \frac{g_k(A)}{2^k}$$

As a uniform limit of continuous functions w is continuous and clearly $w(\{x\}) = 0$ for all $x \in X$. So suppose $A \subseteq B$ are closed in X. If $f_k = f_{m,n}$ then since $A \setminus \overline{U_m} \subseteq B \setminus \overline{U_m}$ we have $f_{m,n}(A) \subseteq f_{m,n}(B)$, i.e. $g_k(A) \leq g_k(B)$ and thus $w(A) \leq w(B)$. The inequality will be strict as long as there is at least one U_n so that $A \setminus \overline{U_m} \neq B \setminus \overline{U_m}$, and picking a point $x \in B \setminus A$ we can select a neighborhood U_x of x not intersecting A and thus providing the desired set. $\square$

It is relatively painless to show that $A_n \to A$ if and only if given an $\epsilon > 0$ it is the case that A_n is eventually in A_ϵ (the open ϵ-neighborhood) and the size of A_n converges to the size of A with respect to any Whitney map. The following sequence of easy observations will be the crux in proving that our hyperspaces are path-connected.

Observation 1.3.8. *The following hold:*

(a) If $\mathfrak{N} \subseteq 2^X$ is a compact nest, $\mathfrak{N} \simeq w(\mathfrak{N})$ (it's injective and closed).

(b) Maximal nests in $C(X)$ are infinite by repeated use of the Extension Theorem, while maximal nests in 2^X are infinite by normality.

*(c) A maximal nest $\mathfrak{N}$ in $C(X)$ or 2^X is closed, for if $A_n \to A$ with $A_n \in \mathfrak{N}$, then A is closed (and a continuum if the A_n's are by **1.1.1(f)**). If $N \in \mathfrak{N}$ is contained in infinitely many A_n then $N \subseteq A$, while in the converse situation clearly $A \subseteq N$. Thus $\mathfrak{N} \cup \{A\}$ is a nest, so by maximality, $A \in \mathfrak{N}$.*

(d) By the above, maximal nests are compact since 2^X and $C(X)$ are.

(e) If $\mathfrak{M} \subset \mathfrak{N}$ and $\mathfrak{N}$ is maximal, then $\bigcup \mathfrak{M}$ and $\bigcap \mathfrak{M}$ are contained in $\mathfrak{N}$.

A path in 2^X that is a maximal nest is called an *order arc*.

Lemma 1.3.9. *Maximal nests in $C(X)$ are connected.*

Proof. Let $\mathfrak{N}$ be a maximal nest in $C(X)$ from A to B and let w be a Whitney map on 2^X. Since $w(\mathfrak{N})$ is closed by the above observations, suppose there is an $r \in [w(A), w(B)] \setminus w(\mathfrak{N})$ with component U in $[0,1] \setminus w(\mathfrak{N})$. Then ∂U consists of two points $s < t$ corresponding to continua $w^{-1}(s) \subsetneq w^{-1}(t)$. By the Extension Theorem there is a continuum C with $w^{-1}(s) \subsetneq C \subsetneq w^{-1}(t)$ not in $\mathfrak{N}$. But then $\mathfrak{N} \cup \{C\}$ would also be a nest in $C(X)$, contradicting maximality. □

Remark 1.3.10. *Maximal nests in 2^X are connected by normality.* □

Since the only compact, connected subsets of $[0,1]$ are arcs and will be homeomorphic to maximal nests between points in 2^X, in order to prove that these hyperspaces are path-connected it's sufficient to show the existence of maximal nests in 2^X and $C(X)$ from an arbitrary A to a base point - the point $\{X\}$ is convenient.

Lemma 1.3.11. *If $A \subseteq B \in C(X)$ there is a maximal nest from A to B in $C(X)$. If $A \subseteq B \in 2^X$ there is a maximal nest from A to B in 2^X.*

Proof. Apply the Hausdorff Maximal Principle to the nest $\{A, B\}$. □

Corollary 1.3.12. 2^X *and* $C(X)$ *are path-connected if* X *is a continuum.*

Proof. Set $B = X$ in the above lemma. □

The notion of order arc was first developed in [22] *(p. 71)*. The Order Arc Theorem was proven in [4] *(p. 149)*. One corollary is that there are continuous exhaustions by subcontinua from any point x to X. We should also make a clarification, at this point:

Definition 1.3.13. ***(Arc-Connected)*** *A space* X *is* arc-connected *if for every pair of distinct* $x, y \in X$ *there is a 1-1 embedding* $f : I \to X$ *such that* $x, y \in f(I)$.

Corollary 1.3.14. 2^X *and* $C(X)$ *are arc-connected if* X *is a continuum.* □

Corollary 1.3.15. *An order arc* $\mathfrak{N}$ *from* A *to* B *with* $A \in C(X)$ *satisfies* $\mathfrak{N} \subseteq C(X)$ *[14] (1.25).*

Proof. Let C be a component of B. If $C \cap A = \varnothing$, then both are closed in B so by the Cut Wire Theorem there is a separation of B by closed sets E, F, with $A \subseteq E$. Let $\mathcal{B} = \{N_\alpha \in \mathfrak{N} \mid N_\alpha \subseteq E\}$. Then $\mathfrak{N} = \mathcal{B} \cup \mathcal{B}^c$. Then $Z = \bigcup N_\alpha$ taken over $\mathcal{B}$ must be contained in E as the Hausdorff limit of an ascending sequence in a compact set, while $Z = \bigcap N_\beta$ taken over $\mathcal{B}^c$ must conversely be contained in F for the same reason, impossible. Thus every component of B intersects A non-trivially.

So let $D \in \mathfrak{N}$. Since D contains the connected set A and every component C_γ of D intersects A non-trivially, $D = \bigcup(C_\gamma \cup A)$ is connected, i.e. $D \in C(X)$. □

Corollary 1.3.16. 2^X *and* $C(X)$ *are locally path-connected at the point* $\{X\}$.

Proof. If $H_d(A, X), H_d(B, X) < \epsilon$ there are order arcs a, b from A, B to X. They're contained in $C(X)$ if A and B are by the above. Then concatenation $a \ominus b$ is a path from A to B all of whose elements N_α satisfy $H_d(N_\alpha, X) < \epsilon$ since all its elements contain either A or B and are thus closer to X. □

Certain subsets of 2^X appear in many proofs and applications. Similar concatenation arguments show that the following sets are path-connected, and that they are closed (i.e. subcontinua) are easy applications of the basic properties of H_d.

Remark 1.3.17. *If $A \subseteq X$ are continua, the following are path-connected continua:*

(a) $2^X(A) = \{B \in 2^X \mid A \cap B \neq \varnothing\}$

(b) $2^{X,A} = \{B \in C(X) \mid A \subseteq B\}$

(c) $2^X_A = \{B \in 2^X \mid B \subseteq A\}$

(d) $C(X, A) = \{B \in C(X) \mid A \cap B \neq \varnothing\}$

(e) $C^A(X) = \{B \in C(X) \mid A \subseteq B\}$

(f) $C_A(X) = \{B \in C(X) \mid B \subseteq A\}$

(g) $\mathfrak{F}_n(X) = \{A \in 2^X \mid A \textit{ has at most } n \textit{ components}\}$

(h) $\mathfrak{F}_{\aleph}(X) = \{A \in 2^X \mid A \textit{ has finitely many components}\}$

(i) $\mathfrak{F}_{\omega}(X) = \{A \in 2^X \mid A \textit{ has at most countably many components}\}$

See **1.3.20** for *(g)-(i)*. The following spaces are called the *symmetric product* spaces and are an area of interest in their own right, but more importantly they also appear from time to time in proofs.

Definition 1.3.18. $F_n(X) = \{A \in 2^X \mid A \textit{ contains at most } n \textit{ points}\}$.

Proposition 1.3.19. *[12] (p. 167) $2^X \setminus C(X)$ is connected.*

Proof. Let $\Delta_n = \{(x_1, \ldots, x_n) \in X^n \mid x_i = x_j \ \forall\ i, j\}$ and

$$f_n : X^n \setminus \Delta_n \to F_n(X) \setminus F_1(X), \quad f_n(x_1, \ldots, x_n) = \{x_1, \ldots, x_n\}$$

each clearly continuous. For $n \geq 3$ we have $X^n \setminus \Delta_n$ is connected and thus so is $F_n(X) \setminus F_1(X)$ for each $n \geq 3$. Write

$$Y = \cup_{n=3}^{\infty} \big(F_n(X) \setminus F_1(X)\big)$$

Y is an increasing union of connected sets and is thus connected, and $Y \subseteq 2^X \setminus C(X)$. Since X is separable Y is dense and thus $2^X \setminus C(X)$ is connected. □

We conclude this section with one last result. It is elementary but prohibitively long, and gives a converse to what we proved in **1.3.15**.

Theorem 1.3.20. ***Order Arc Theorem*** *[8] (p. 120) If $A \subsetneq B \in 2^X$ there is an order arc from A to B if and only if each component of B intersects A.* □

1.4 The Pseudo-Arc

While the concept of path-connectedness is presumably familiar, the theory of path-connected (pc) continua is specialized and we will encounter many striking - and unexpected - results. In particular, it will be important to have topological characterizations of the arc. Denote the standard unit interval $[0, 1] = I$.

Definition 1.4.1. ***(Path-Connected)*** *A Hausdorff space X is path-connected iff:*

(a) For $x, y \in X$ there exists a continuous map $f : I \to X$ with $x, y \in f(I)$, or

(b) For $x, y \in X$ distinct there is an $A \subseteq X$ containing x to y with $A \simeq I$.

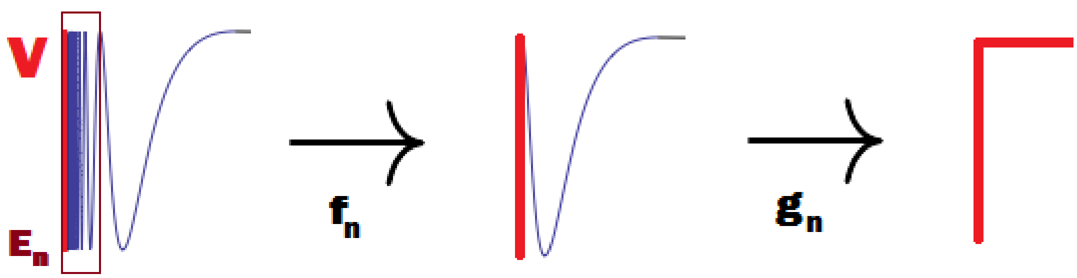
Although we have used the latter definition previously, we make a note due to the fact that the above equivalence is well-known (or well-believed) but *highly* non-trivial [15] *(8.14)*. Before characterizing the arc, we should understand the pathologies that might arise in spaces 'like the arc.' The following concepts are classical.

Definition 1.4.2. ***(ϵ-map)*** *A continuous function $f : X \to Y$ from a metric space to a topological space is called an ϵ-map if diam$(f^{-1}(y)) < \epsilon$ for all $y \in Y$.*

Definition 1.4.3. ***(X-like, $\mathfrak{X}$-like)*** *A continuum Y is X-like if for all $\epsilon > 0$, there is a surjective ϵ-map from Y onto X. If $\mathfrak{X}$ is a collection of topological spaces (e.g. trees, graphs, metric spaces etc.) then Y is $\mathfrak{X}$-like if for all $\epsilon > 0$ there is an $X_\epsilon \in \mathfrak{X}$ and a surjective ϵ-map from Y onto X_ϵ.*

Note that the definition of $\mathfrak{X}$-like does not imply that Y is X_ϵ-like for any $X_\epsilon \in \mathfrak{X}$, and this can often be the case.

Example 1.4.4. The sine continuum X is arc-like. To see this, let E_n be the closed $^1/_n$-neighborhood of the vertical bar $V \subseteq X$ and let f_n be the horizontal projection of E_n onto V. Then f_n is an ϵ-map of E_n onto V, an arc. By shrinking E_n if necessary we may assume that the furthest point in E_n from V has y-coordinate 1. Then by the gluing lemma we may map the rest of X homeomorphically to an arc via a g_n and obtain a $^1/_n$-map from X to an arc for any n. In particular, X is arc-like.



Example 1.4.5. The Warsaw Circle W is defined as the adjunction of the sine continuum X with I where $0, 1 \in I$ are identified with the right-most point of X and the point $(0, -1)$ on X respectively. It is circle-like by an identical procedure.

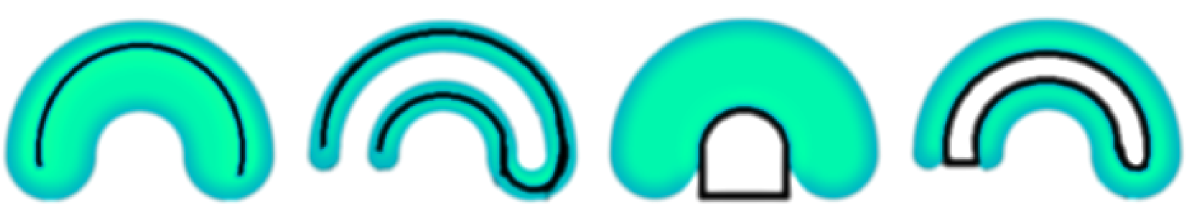
Example 1.4.6. Let T_n be the quotient of n copies of I, with only $0 \in I$ identified in each copy (i.e. an n-pointed asterisk). Then by insisting that the furthest point in E_n from V has y-coordinate 0 instead of 1, by the same procedure we see that the sine continuum is also T_3-like. Of course, T_3 is not homeomorphic to the arc.

The space T_n is called the *simple n-od*. When $n = 3$ we call it the *simple triod*.

Example 1.4.7. We now produce a continuum which is both arc-like and circle-like. It is called the Buckethandle Continuum, or sometimes the B-J-K Continuum, for Brouwer, Janiszewski and Knaster. Continua which are both arc-like and circle-like are discussed in [5] *(p. 653)*. It is constructed as a nested intersection of planar continua, and by **1.1.1(e)** will therefore be a continuum. It is prohibitively tedious to write the continuum in coordinates, but this is the only difficulty.

Let D be the closed upper half of the closed unit disc centered at $(^1\!/_2, 1)$, and let $R = [0,1] \times [-^1\!/_6, 0]$. Their union $E_0 = D \cup R$ is a continuum. Let U_n be the collection of 2^{n-1} intervals in the complement of the standard Cantor ternary set C in $[0,1]$ with lengths $^1\!/_{3^n}$. Each U_n comes in pairs of open intervals symmetric across $x = {}^1\!/_2$, each such collection of pairs U_n corresponding to open upper halves of circular annuli centered at $(^1\!/_2, 0)$ whose intersection with $[0,1]$ is U_n. We remove these successively from nested continua E_n via a tubular neighborhood of an arc, as well as removing the complements of the left-over half-discs, again as shown. The intersection B is a continuum.

The arc-likeness and circle-likeness are illustrated below. Since the continuum is a nested intersection, any ϵ-map on one of the E_n restricts to an ϵ-map on B. The relevant map for the arc case is clear; in the circle case, use the normal projection on the tube except at the 'end' of the 'tunnel', mapping the two remaining half-discs onto their bottom vertical radius by horizontal projection and then onto the straight edge of the bottom of the circle, their bottom-most points meeting in its midpoint.



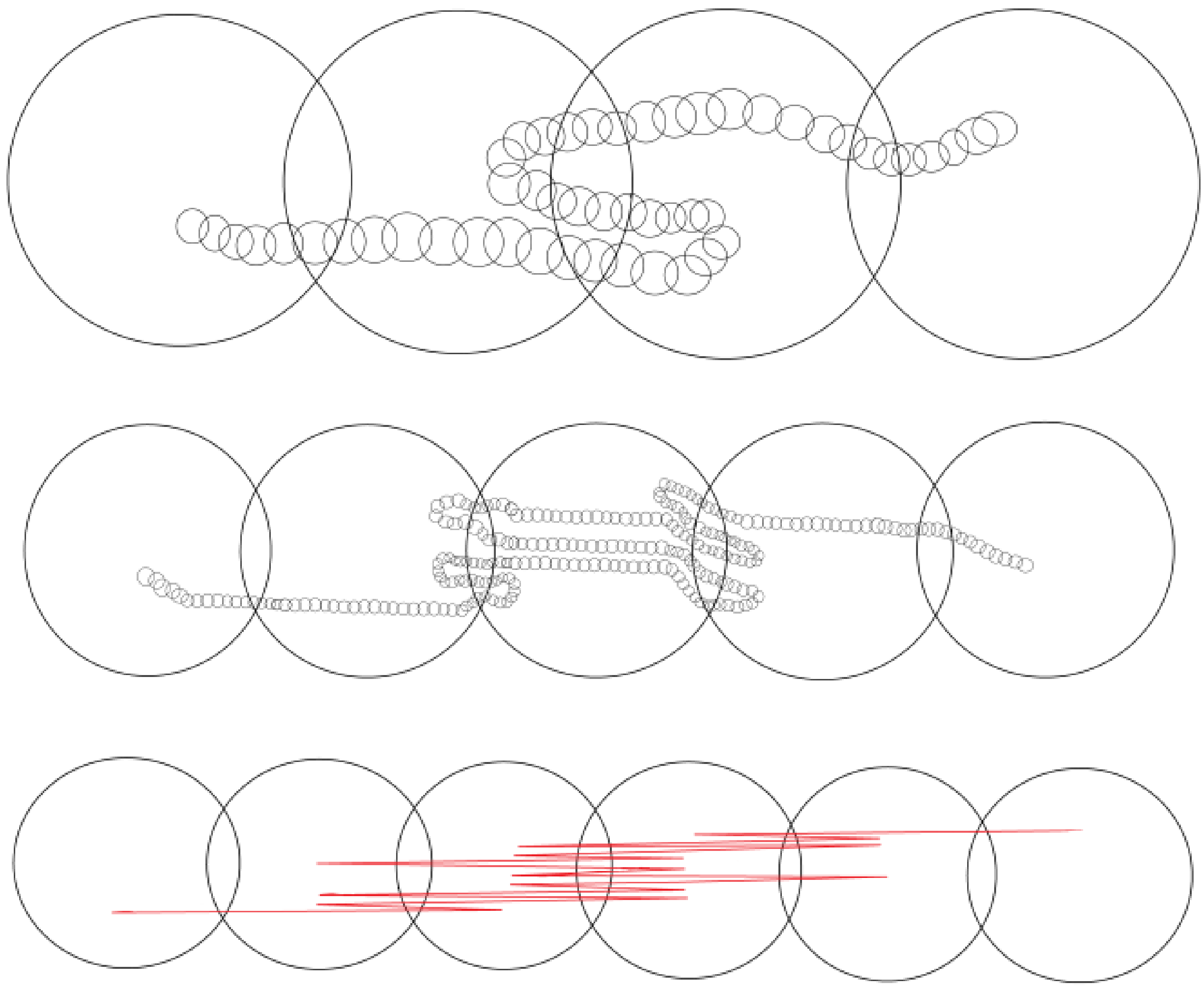
Definition 1.4.8. ***(Chain)*** *A chain in a metric space is a non-empty, finite, indexed collection* $\{B_1, \dots, B_n\}$ *of non-trivial, compact (metric) balls so that*

(a) $\mathrm{int}(B_i) \cap \mathrm{int}(B_j) \neq \varnothing$ *iff* $|i - j| \leq 1$,

(b) $B_i \cap B_j = \varnothing$ *iff* $|i-j| \geq 2$*, and*

(c) no B_j *contains another.*

Each B_j *is called a* ***link****. A* ***subchain*** $\mathcal{U}$ *of a chain* $\mathcal{V}$ *is a subset of* $\mathcal{V}$ *that is a chain. A* ***chain refinement*** $\mathcal{B}$ *of a chain* $\mathcal{A}$ *is a chain such that for every* $B \in \mathcal{B}$ *there is an* $A \in \mathcal{A}$ *with* $B \subseteq \text{int}(A)$*. A chain with* $x \in B_1$ *and* $y \in B_n$ *is said to be a chain* ***from*** x ***to*** y*. If for all* $A \in \mathcal{A}$ *we have* $\text{diam}(A) < \epsilon$ *we say* $\mathcal{A}$ *is an* ϵ*-****chain****.*



Definition 1.4.9. ***(Crooked)*** *[2] A chain refinement* $\mathcal{B}$ *of a chain* $\mathcal{A} = \{A_1, \ldots, A_n\}$ *is said to be crooked in* $\mathcal{A}$ *if for every pair of indices* $i, j \leq n$ *with* $|i-j| > 2$ *(i.e. there are at least two links between* A_i *and* A_j*) and for* $B_m \subseteq A_i, B_n \subseteq A_j$ *there are indices* k, l *with* $m < k < l < n$ *and* $B_l \subseteq A_{i+1}, B_k \subseteq A_{j-1}$ *assuming* $i < j$*, or assuming* $i > j$ *there are*

indices k, l with $m > l > k > n$ and $B_l \subseteq A_{i-1}, B_k \subseteq A_{j+1}$.

Shown above are chains of four, five and six discs in the plane and crooked chain refinements. The concept is due to Bing; it says that to get between two of the larger links which have at least two links between them, you must zig-zag back and forth, first. The issue concerning $i < j$ vs. $i > j$ simply expresses that the enforced zig-zagging is symmetric with respect to which 'direction' (in terms of the ordering) one is moving in the larger chain. In the case of only four links, it is possible to simply 'hang out' in the intersection of the middle two links and avoid the 'zig-zag' pathology, but this never causes issues in practice.

It is clear that given any two points x, y in the plane there exists a chain from x to y with at least four links, and that any such chain contains a crooked chain refinement that is a chain from x to y, since topologically it itself is just a disc. The union of the elements in such a chain is bounded, closed and connected and so is a continuum. The following was independently discovered by Knaster, Moise and Bing in different guises.

Definition 1.4.10. ***(Pseudo-Arc)*** *A pseudo-arc is a countable nested intersection of crooked chain refinements (at least one, and thus infinitely many, containing at least five links) $\mathcal{A}_n$ from fixed points x to y satisfying $A \in \mathcal{A}_n \implies \operatorname{diam}(A) < 1/n$ [2] [17] [25].*

Proposition 1.4.11. *Non-degenerate subcontinua of arc-like continua are arc-like.*

Proof. Let X be arc-like and let $Y \subseteq X$ be a non-degenerate subcontinuum. Let $\epsilon_n \to 0$ be positive numbers and let f_n be associated ϵ-maps mapping X to I. If $\epsilon_n < \operatorname{diam}(Y)$, then $f_n|Y$ is not trivial by definition of an ϵ-map and so, as a compact, connected subset of X, its image in I is compact, connected and not a point, and is thus an arc. Thus for $\epsilon_n < \operatorname{diam}(Y)$, the maps $f_n|Y$ are ϵ-maps of Y onto the arc and thus Y is arc-like. □

Corollary 1.4.12. *Every subcontinuum of the pseudo-arc is arc-like.*

Proof. The pseudo-arc is clearly arc-like by construction. Apply the above. □

Recall that a continuum is *indecomposable* if it cannot be written as the union of two proper subcontinua. It is *hereditarily indecomposable* if every one of its subcontinua is also indecomposable.

Proposition 1.4.13. *The pseudo-arc is hereditarily indecomposable.*

Proof. We follow the proof outlined in [15] *(p. 14)*. Let Y be a non-degenerate subcontinuum of the pseudo-arc, X. Suppose by contradiction that $Y = A \cup B$ where A and B are proper subcontinua of Y. Since A and B are both proper subcontinua, we can pick points $a \in A \setminus B$ and $b \in B \setminus A$. Let $m = \min\{d(a, B), d(b, A)\}$, and pick n so that $1/n < m$. Pick a $k > 2n$.

Then since $\mathcal{A}_k$ is a cover of X and thus of Y, there is an $A_r \in \mathcal{A}_k$ containing a, and since $\operatorname{diam}(A_r) < m/2$, we have that $A_r \cap Y$ is contained in $A \setminus B$. In fact (up to reversing the order) so is $A_{r+1} \cap Y$, by the definition of a chain. Similarly, there is an s with $A_s \cap Y, A_{s-1} \cap Y \subseteq B \setminus A$ with $b \in A_s$.

Then since $Y \subseteq \cup \mathcal{A}_{k+1}$, by the crookedness requirement there will be $C_r, C_s, C_t \in \mathcal{A}_{k+1}$ with (again, up to a reversal of the order) $a \in C_r \cap Y$, $a_2 \in C_t \cap Y, b_2 \in C_s \cap Y$ with $r < s < t$ and for some $a_2 \in (A_{r+1} \setminus A_r) \cap Y$, $b_2 \in (A_{s-1} \setminus A_s) \cap Y$, since to enter A_s the links in $\mathcal{A}_{k+1}$ will first have to travel back to A_{r+1} from A_{s-1}. Since the diameters of elements in $\mathcal{A}_{k+1}$ are even smaller, $C_s \cap Y \subseteq B \setminus A$. Thus $A \subseteq (\bigcup_{p<s} C_p) \cup (\bigcup_{s<p} C_p)$. That is to say the links with indices not equal to s. By the chain property and a simple induction, it is clear that $E = \bigcup_{p<s} C_p$ and $F = \bigcup_{s<p} C_p$ are connected. Since $C_x \cap C_y \neq \varnothing$ iff $|x - y| \leq 1$, E and F are separated by the Cut Wire Theorem. As well, $a \in E$ and $a_2 \in F$, so $(A \cap E, A \cap F)$ is a separation of A, contradicting its connectedness. □

Corollary 1.4.14. *The pseudo-arc contains no arc.*

Proof. The arc is decomposable. □

We will now show that there is indeed only one 'pseudo-arc' up to homeomorphism. First, a definition. If $\mathcal{A}$ is a chain, let $\mathcal{A}^*$ denote the union of its elements.

Definition 1.4.15. ***(Chainable)*** *A continuum X is chainable if there is a sequence $\mathcal{A}_n$ of ϵ_n-chains in X with $\epsilon_n \to 0$, $\mathcal{A}_{n+1}$ is a refinement of $\mathcal{A}_n$ for all n and $X = \cap \mathcal{A}_n^*$.*

The pseudo-arc is chainable by construction. The next few theorems, by Bing [2] [3], show that up to homeomorphism there is only one hereditarily indecomposable, chainable continuum. We say that a link A_s in a chain $\mathcal{A}$ is *between A_r and A_t* if either $r < s < t$ or $r > s > t$.

Lemma 1.4.16. *If $X = \cap \mathcal{A}_n^*$ is hereditarily indecomposable and each $\mathcal{A}_n$ is a $1/n$-chain, then for any j there is an $n(j)$ with $\mathcal{A}_k$ crooked in $\mathcal{A}_j$ for all $k > n(j)$.*

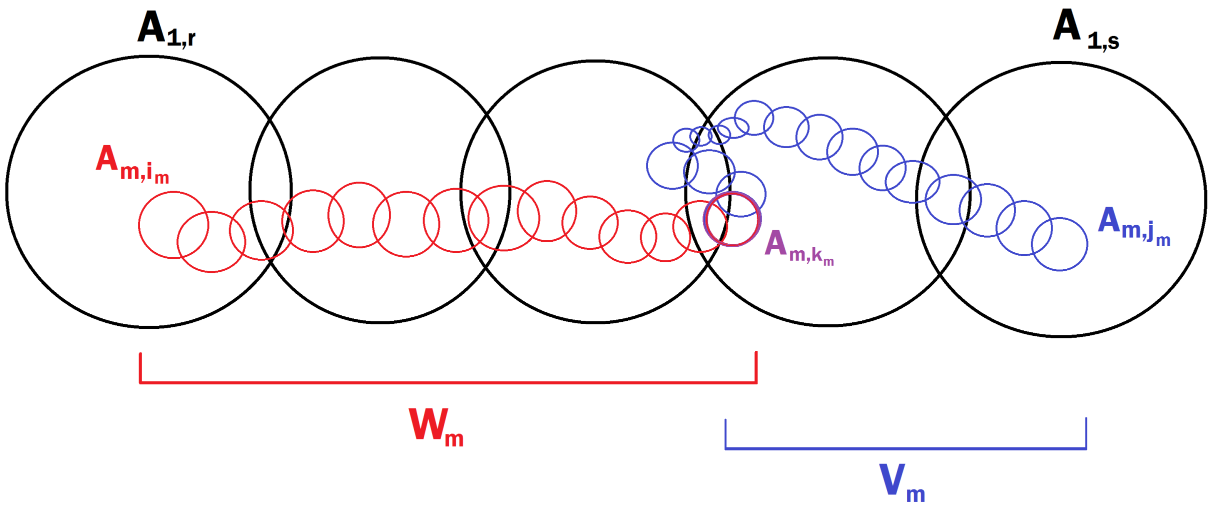
Proof. Write $\mathcal{A}_n = \{A_{n,1}, \ldots, A_{n,t_n}\}$. It's enough to prove the case $j = 1$. Assume the lemma is false; then we may pick $A_{1,r}, A_{1,s}$ with $|s - r| > 2$ and infinitely many $\mathcal{A}_m$ each with links A_{m,i_m} and A_{m,j_m} satisfying:

(a) $A_{m,i_m} \subseteq A_{1,r}$ and $A_{m,j_m} \subseteq A_{1,s}$,
(b) if $A_{m,k}$ is between A_{m,i_m} and A_{m,j_m} and contained in $A_{1,s-1}$ there is no link of $\mathcal{A}_m$ contained in $A_{1,r+1}$ between $A_{m,k}$ and A_{m,j_m}.

This is because there are only finitely many ordered pairs of elements from $\mathcal{A}_1$, so if there are infinitely many chains $\mathcal{A}_m$ that fail to be crooked in $\mathcal{A}_1$ there must be some pair of elements where the crookedness condition fails infinitely many times by the pigeonhole principle (though here we have assumed $r < s$, $i_m < j_m$, which we may do without loss of generality by reversing orderings if necessary). We may assume it happens 'at the ends' $A_{1,r+1}$ and $A_{1,s-1}$ by taking a subchain of $\mathcal{A}_1$ if needed. Write

$$W_m = \bigcup_{i_m \leq x \leq k_m} A_{m,x}, \quad V_m = \bigcup_{k_m \leq x \leq j_m} A_{m,x}$$

We pick the k_m that is the smallest index between i_m and j_m such that $A_{m,k_m} \subseteq A_{1,s-1}$.



Since $\{V_m\}, \{W_m\}$ are infinite subsets of C^X and C^X is compact, after taking subsequences we may assume they converge to continua V, W respectively, and since $X = \cap \mathcal{A}_n^*$ they are contained in X. Neither is contained in the other since W intersects $A_{1,r}$ non-trivially while V does not and vice-versa. However, $V \cup W$ is also a nested intersection of continua and thus a subcontinuum of X containing V, W and has decomposition (V, W), contradicting hereditary indecomposability of X. □

The following theorem will be the engine for displaying a homeomorphism between any two hereditarily indecomposable, chainable continua.

Lemma 1.4.17. *Let X_1, X_2 be compact metric spaces, and let $\epsilon_n \to 0$ be positive with $\sum \epsilon_n < \infty$. Let $\mathcal{A}_1, \mathcal{A}_2, \ldots$ and $\mathcal{B}_1, \mathcal{B}_2, \ldots$ be ordered finite open covers by non-empty, connected subsets of X_1, X_2 respectively such that $Y \in \mathcal{A}_n$ or $Y \in \mathcal{B}_n$ implies diam$(Y) < \epsilon_n$.*

Further, assume that whenever the jth element of $\mathcal{A}_{i+1}$ intersects the kth element of $\mathcal{A}_i$, the distance between the jth element of $\mathcal{B}_{i+1}$ and the kth element of $\mathcal{B}_i$ is less than ϵ_i. Vice-versa assume that whenever the jth element of $\mathcal{B}_{i+1}$ intersects the kth element of $\mathcal{B}_i$, the distance between the jth element of $\mathcal{A}_{i+1}$ and the kth element of $\mathcal{A}_i$ is less than ϵ_i. Then there is a homeomorphism between X_1 and X_2.

Proof. Write $\mathcal{A}_i = \{A_i^1, \ldots, A_i^{m(i)}\}$. Write $\mathcal{B}_i = \{B_i^1, \ldots, B_i^{n(i)}\}$ similarly. As well, define Q_i^k to be the $\epsilon_i + 2(\epsilon_{i+1} + \epsilon_{i+2} + \cdots)$ neighborhood of A_i^k. Define R_i^k similarly for B_i^k. Finally, let Y_i be the sequence $(Q_i^1, Q_i^2, \ldots, Q_i^{m(i)})$ of these 'fattened' elements from $\mathcal{A}_i$, and let $Z_i = (R_i^1, R_i^2, \ldots, R_i^{n(i)})$ similarly.

Note the following observation: If $A_i^k \cap A_{i+1}^j \neq \varnothing$, then since $\mathrm{d}(B_i^k, B_{i+1}^j) < \epsilon_i$ and the fact that $\mathrm{diam}(B_{i+1}^j) < \epsilon_{i+1}$, the closure of R_{i+1}^j is contained in R_i^k by the main assumption in the lemma and the definition of R_i^k. Repeating this argument inductively, we see that if $i < l$ and $A_i^k \cap A_l^j \neq \varnothing$, the closure of R_l^j is contained in R_i^k. Similarly, the observation holds replacing A with B and Q with R.

Since $X_1 = \cup \mathcal{A}_i^*$ (where we define $\mathcal{A}_i^*$ as for chains), every point $p \in X$ is contained in some nested intersection of sets $A_1^{p(1)}, A_2^{p(2)}, \ldots$, and since the diameters of these sets tend to zero p is the unique point in their intersection. Define a relation f, ostensibly a function from $X_1 \to X_2$, by $f(p) = \cap B_i^{p(i)}$. Since $\mathrm{diam}(B_i^{p(i)})$ tends to zero, p is related to at most a single point. Since the *closure* of R_{i+1}^k, a compact set, is contained in R_i^l whenever A_{i+1}^k intersects A_i^l, by the nested intersection property of compact sets in X_2 there is a point in X_2 related to p under f, and by the same token $f(p)$ is well-defined in the sense that it does not matter which sequence of $A_i^{p(i)}$ whose intersection is p we take in X_1 since the elements of one such sequence will intersect the elements of the other.

Thus $f : X_1 \to X_2$ is a well-defined function. It is continuous since if $f(p) \in U$ we may pick i large enough so that any element of Z_i (open) containing $f(p)$ is contained in U, and if A_i^r (open) contains p then by definition of f, any $x \in A_i^r$ is mapped into $R_i^r \subseteq U$. Thus f is continuous. Since X_1 is compact, to show that f is surjective it's sufficient to show that $f(X_1)$ is dense, since its image is already closed. Let U be open in X_2 and let $q \in U$. Let j be such that every element of Z_j containing q is contained in U. Let B_j^r be an element of $\mathcal{B}_j$

containing q. Then $f(A_j^r) \subseteq R_j^r \subseteq U$, as needed.

Assume there are distinct points p_1, p_2 with $f(p_1) = f(p_2) = q$ and $d(p_1, p_2) > 5\epsilon_j$. Without loss of generality assume $e_{k+1} + \epsilon_{k+2} + \cdots < \epsilon_k$ for all k. Let $A_j^{p_1(r)}$ and $A_j^{p_2(r)}$ contain p_1, p_2 respectively, and let A_{j+1}^s contain p_1; note that $d(A_{j+1}^s, A_j^{p_2(r)}) > \epsilon_j$. Their images contain q and so $d(B_j^{p_1(r)}, B_{j+1}^s) < \epsilon_j$ by diameter considerations. But then by assumption $d(A_j^{p_2(r)}, A_{j+1}^s) < \epsilon_j$, impossible. Thus f is a continuous bijection between two compact spaces and is therefore a homeomorphism. □

Definition 1.4.18. ***(Consolidation)*** *A consolidation of a chain $\mathcal{B}$ is a chain $\mathcal{A}$ such that every link in $\mathcal{A}$ is a union of links in $\mathcal{B}$ and $\mathcal{A}^* = \mathcal{B}^*$.*

Definition 1.4.19. ***(Pattern)*** *If $\mathcal{B}$ is a refinement of $\mathcal{A}$ with $B_{x_n} \subseteq A_{y_n}$ then $\mathcal{B}$ follows the pattern $P = ((x_1, y_1), \ldots, (x_n, y_n))$ in $\mathcal{A}$.*

We will use the following two lemmas [2] which are visually clear but extraordinarily tedious, so we state them without proof. By $\mathcal{A}(r, s)$ we mean the subchain of $\mathcal{A}$ composed of those links between A_r and A_s inclusively.

Lemma 1.4.20. *If $\mathcal{B}$ is crooked in $\mathcal{A} = \{A_1, \ldots, A_n\}$ and $\mathcal{B}(r, s)^* \cap A_1 \neq \varnothing$, $\mathcal{B}(r, s)^* \cap A_n \neq \varnothing$, then there is a consolidation $\mathcal{C}$ of $\mathcal{B}$ such that each element of $\mathcal{C}$ intersects at most two elements of $\mathcal{A}$, D_r is contained only in the first link of $\mathcal{C}$ and D_s is contained only in the last.* □

Lemma 1.4.21. *Let $P = ((1, y_1), (2, y_2), \ldots, (n, y_n))$ with $1 = y_1 \leq y_i \leq y_n$ and $|y_i - y_{i+1}| \leq 1$ for all $i \leq n-1$. Let $p, q \in X$ and let $\mathcal{A}_k$ be a sequence of $1/k$-chains in X from p to q such that $\mathcal{A}_1$ has y_n links and for all k we have that $\mathcal{A}_{k+1}$ is a crooked refinement of $\mathcal{A}_k$. Then there is an integer j and a chain $\mathcal{B}$ from p to q such that $\mathcal{B}$ is a consolidation of $\mathcal{A}_j$ and $\mathcal{B}$ follows the pattern P in $\mathcal{A}_1$.* □

Corollary 1.4.22. *Let X_1, X_2 be continua with pairs of distinct points $p_i, q_i \in X_i$. Let $\epsilon_n \to 0$ and let $\mathcal{A}_k$ be a sequence of ϵ_k-chains in X_1 from p_1 to q_1 such that $\mathcal{A}_{k+1}$ is a crooked refinement of $\mathcal{A}_k$ with $X_1 = \mathcal{A}_k^*$ for all k. Let $\mathcal{B}_k$ be defined similarly in X_2. Then there is a homeomorphism from X_1 to X_2 sending p_1 to p_2 and q_1 to q_2.*

Proof. Without loss of generality assume $\mathcal{A}_n$ and $\mathcal{B}_n$ are 2^{-n}-chains. Set $\mathcal{C}_1 = \mathcal{A}_1$, let $\mathcal{D}_1$ be a chain from p_2 to q_2 with the same number of links as $\mathcal{C}_1$ that is a consolidation of some $\mathcal{B}_k$, and let $\mathcal{D}_2 = \mathcal{B}_{k+1}$. By lemma **1.4.21** there is an integer j and a chain $\mathcal{C}_2$ from p_1 to q_1 consolidating $\mathcal{C}_1$ and following the pattern in $\mathcal{C}_1$ that $\mathcal{D}_2$ follows in $\mathcal{D}_1$. Let $\mathcal{C}_3 = \mathcal{A}_2 \subseteq \mathcal{C}_2$; then by **1.4.21** there is an integer j and a chain $\mathcal{D}_3$ from p_2 to q_2 consolidating $\mathcal{B}_2$ following the pattern in $\mathcal{D}_2$ that $\mathcal{C}_3$ follows in $\mathcal{C}_2$.

Repeating back and forth, by induction we obtain a sequence $\mathcal{C}_n$ of 2^{-n}-chains from p_1 to q_1 covering X_1 and a sequence $\mathcal{D}_n$ of 2^{-n}-chains from p_2 to q_2 covering X_2 such that $\mathcal{C}_{i+1}$ follows the same pattern in $\mathcal{C}_i$ that $\mathcal{D}_{i+1}$ follows in $\mathcal{D}_i$ for all i. Then by definition the sequences $(\mathcal{C}_i), (\mathcal{D}_i)$ satisfy the requirements of lemma **1.4.17** and so $X_1 \simeq X_2$. If f is as in the proof of that lemma, then $f(p_1) = \cap \mathcal{D}_i^1 = p_2$, and similarly $f(q_1) = q_2$. □

Thus the pseudo-arcs as previously constructed in the plane using intersections of crooked chain refinements are all homeomorphic. Before proving our main theorem, we need a few classical results of general interest.

Lemma 1.4.23. *No subcontinuum separates an indecomposable continuum.*

Proof. Suppose X is indecomposable and C is a proper subcontinuum. If $X \setminus C = A \cup B$ were a separation, then by the Boundary Bumping Theorem $A \cup C$ and $B \cup C$ would be connected with $X = (A \cup C) \cup (B \cup C)$. Then $\overline{C \cup A} = C \cup \overline{A} = (C \cup \overline{A}) \cap (C \cup A \cup B) = C \cup A$ since $A \cap B = \varnothing$. Thus $C \cup A$ is closed; similarly, $C \cup B$ is closed, and therefore $X = (A \cup C) \cup (B \cup C)$ is a decomposition of the indecomposable X, a contradiction. Thus no subcontinuum separates X. □

Corollary 1.4.24. *Indecomposable continua have no cut points.* □

Lemma 1.4.25. *A subcontinuum of an indecomposable continuum is nowhere-dense.*

Proof. If C is proper then by the above $X \setminus C$ is connected, and thus so is $\overline{X \setminus C}$. If C is not nowhere-dense, then as a closed set it contains a set open in X, and so $\overline{X \setminus C} \neq X$. Thus $C \cup \overline{X \setminus C} = X$ is a decomposition, impossible. □

Corollary 1.4.26. *Indecomposable continua have uncountably many composants.*

Proof. Combine **1.4.25**, **1.1.24** and Baire. □

Corollary 1.4.27. *An indecomposable continuum X is irreducible.* □

Now we prove our main theorem [3]. The first part is as promised; the second part is probably surprising. Recall that a space is *homogeneous* if for every $p, q \in X$ there is a homeomorphism of X with itself that sends p to q.

Theorem 1.4.28. *If X_1, X_2 are non-degenerate, hereditarily indecomposable and chainable, then $X_1 \simeq X_2$ and each is homogeneous.*

Proof. By **1.4.16** there is a sequence of $1/2n$-chains $\mathcal{C}_n$ each a crooked refinement of the previous so that $X_1 = \cap \mathcal{C}_n^*$. let $p_1, q_1 \in X_1$ belong to different composants, which exist by **1.4.27**. *Claim*: for each j, there is a $k > j$ such that the subchain $\mathcal{D}_k$ of $\mathcal{C}_k$ from p_1 to q_1 intersects both the first and last links of $\mathcal{C}_j$.

Proof of claim: Let W_i be the union of the links of the subchain $\mathcal{C}_i$ from p_1 to q_1. Then by compactness of $C(X)$ the set $\{W_i\}$ has a limit point in $C(X)$ which would be a subcontinuum of X containing both p_1 and q_1 and must thus be all of X, since they are in different composants. Thus for any open set, some W_k with $k > j$ intersects it, in particular the interiors of the first and last links of $\mathcal{C}_j$, which is what was to be shown.

Now, by **1.4.20** there is a chain $\mathcal{E}_j$ from p_1 to q_1 that consolidates $\mathcal{C}_j$ with each link of

$\mathcal{E}_j$ intersecting at most two links of $\mathcal{C}_j$, and in particular $\mathcal{E}_j$ is a $1/n$-chain. By another application of **1.4.16** we may assume that each $\mathcal{E}_j$ is a crooked refinement of the previous, and each is a chain from p_1 to q_1. Similarly we can write $X_2 = \cap\mathcal{F}_n^*$ where each $\mathcal{F}_j$ is an $1/n$-chain from p_2 to q_2 for some points p_2, q_2 in different composants of X_2 and each is a crooked refinement of the previous. Thus we are in the setting of **1.4.22**.

Therefore there is a homeomorphism of X_1 and X_2 sending p_1 to p_2, q_1 to q_2. Setting $X_1 = X_2$ then we see that each is homogeneous. □

Thus the pseudo-arc is homogeneous; it doesn't have special end points like the arc. It also has no cut points and contains no arc. It can be shown without undue difficulty that being chainable and being arc-like are equivalent properties for a continuum (both directions are elementary) [15] *(p. 235)*. Thus the pseudo-arc is not an arc, but does have the following properties: It is an arc-like, irreducible planar continuum homeomorphic to each of its non-degenerate subcontinua.

1.5 Regular Continua

In the next section we will obtain the following well-known characterizations of the arc [26] [15]. This section will develop some of the necessary tools that we will need throughout the text.

Proposition 1.5.1. *A continuum X is an arc if and only if either:*

(a) X has precisely two non-cut points.

(b) X is arc-like and path-connected.

If X is irreducible between p and q we write $X = \mathrm{Irr}(p, q)$ and say that p is a *point of irreducibility* for X. Generalizing this definition, if C is a subcontinuum of X and $A, B \subseteq X$ we write $C = \mathrm{Irr}(A, B)$ if no proper subcontinuum of C intersects both A and B. If A and B are closed and $C_n \cap A \neq \varnothing, C_n \cap B \neq \varnothing$ for all n then $\cap C_n$ also intersects A and B, and thus since the nested intersection of continua is a continuum, by the Brouwer Reduction Theorem [37] *(p. 17)* the collection of continua which intersect both A and B contains a minimal member.

If C is a subcontinuum of X and A is a closed, we say C is *irreducible about* A if no proper subcontinuum of C contains A. In this case, we write $C = \text{Irr}(A)$. Again by the Brouwer Reduction Theorem, if A is closed there is a subcontinuum of X which is irreducible about it. Thus:

Proposition 1.5.2. *If A and B are closed subsets of X then there is a subcontinuum C of X such that $C = Irr(A, B)$. If A is closed, there is a subcontinuum D irreducible about it. Thus if $p, q \in X$ there is a subcontinuum irreducible between p and q.* □

Remark 1.5.3. *By definition, if $C = \text{Irr}(A, B)$ with A, B closed and $p \in A, q \in B$, then $C = \text{Irr}(p, q)$.*

Another quick observation is the following: If $X = \text{Irr}(a, b) = \text{Irr}(c, d)$ then either $X = \text{Irr}(a, d)$ or $X = \text{Irr}(b, d)$. For otherwise there would be proper subcontinua A, B of X with $a, d \in A$ and $b, d \in B$ whose union would then be a subcontinuum of X. Since $X = \text{Irr}(a, b)$ it would be all of X. Thus one of them contains c, but also contains d by definition, contradicting $X = \text{Irr}(c, d)$.

Proposition 1.5.4. *If X is a continuum and $X = \text{Irr}(a, b) = \text{Irr}(c, d)$ then either $X = \text{Irr}(a, d)$ or $X = \text{Irr}(b, d)$.* □

The reader should have in mind the sine continuum, which is irreducible between any point on the vertical segment and the rightmost point. Thus we see that non-degenerate continua are replete with irreducible subcontinua, and this concept has important interplay between separations and decompositions. Irreducibility between p and q also has some similarities with being an arc between p and q, for example there is the following.

Proposition 1.5.5. *If $X = \text{Irr}(p, q)$ then p and q are not cut points.*

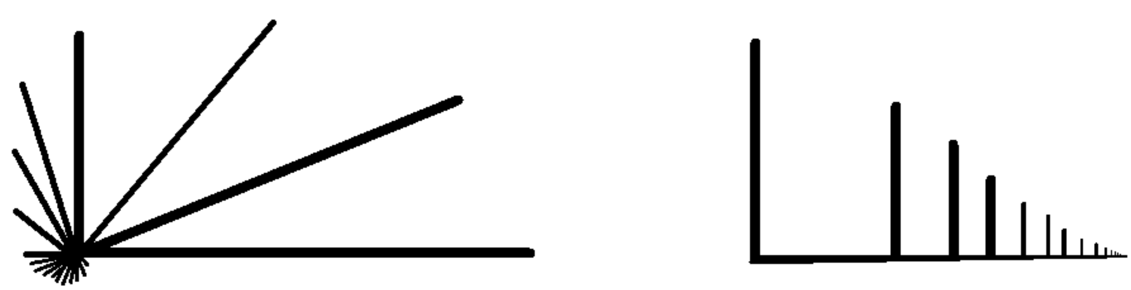
Proof. Let $X = \text{Irr}(p, q)$ and suppose p is a cut point with $X \setminus p = Q|R$ with $q \in Q$. Then $\{p\} \cup Q$ is a subcontinuum of X as shown in the proof of **1.4.23**, and since by assumption $R \neq \varnothing$, it is proper, contradicting that $X = \text{Irr}(p, q)$. $\square$

It turns out that more is true: They are also the only possible 'end points' of X.

Definition 1.5.6. ***(Order)*** *Let $A \subseteq X$. Then the order of A in X, written* $\text{ord}_X(A)$ *(or* $\text{ord}(A)$ *when X is understood), is the minimal cardinal number α (which exists by well-ordering) such that for any open neighborhood U of A there is another open neighborhood V of A with $V \subseteq U$ such that $|\partial(V)| \leq \alpha$.*

A point $p \in X$ is an end point if $\text{ord}_X(p) = 1$*, an edge point if* $\text{ord}_X(p) = 2$ *and a ramification point if* $\text{ord}_X(p) \geq 3$.

A continuum X is finite if for every $p \in X$, there is a local basis $\mathcal{B}_p = \{B_1, B_2, \dots\}$ at p such that there is an integer n_p with $|\partial(B_j)| \leq n_p$ for all j. We say X is regular if for every p there is a local basis at p all of whose elements have finite boundary. We say X is rational if for every p there is a local basis at p all of whose elements have countable boundary.



The continuum on the left, which is composed of countably many line segments L_n with length $1/n$ respectively emanating from the origin is called the *hairy point continuum.* It is an example of a regular, but not finite, continuum. The continuum on the right, composed of $[0, 1] \times \{0\}$ along with countably many line segments with lengths $1/n$ situated atop a sequence converging to 1, is a finite continuum with infinitely many end points called the *null comb*. The cone over the Cantor Set is clearly not rational. The sine continuum is rational but not regular. The concept is due to Menger.

Note that in the above definitions we need not assume the neighborhoods U or V are open. To see this, note that if X is any connected space and $A \subseteq X$ is non-empty and not all of X, then it has non-empty boundary. Since $\partial(\text{int}(A)) \subseteq \partial(A)$, if we started with generic neighborhoods we could obtain open neighborhoods the cardinality of whose boundaries would also be 1, or finite, or countable as needed. For edge points and ramification points, simply note that open local bases are, of course, also local neighborhood bases.

Observation 1.5.7. *The definitions in **1.5.6** are equivalent if 'open neighborhood' is replaced with 'neighborhood' and 'local basis' with 'local neighborhood basis'.*

Proposition 1.5.8. *If X is regular then X is Peano.*

Proof. Let U be a neighborhood of $p \in X$ with $|\partial(U)| \leq n$ and $\text{diam}(U) < \epsilon$. Then by the Boundary Bumping Theorem, the number of components of $\overline{U}$ is dominated by $|\partial(U)|$, so letting C be the component of $\overline{U}$ containing p we find that it is a neighborhood of p in $\overline{U}$ and thus in X since it's separated from the other finitely many components, and since $C \subseteq \overline{U}$, we have $\text{diam}(C) < \epsilon$. Thus X is cik at p. But p was arbitrary, so X is cik. Thus by **1.1.3** X is Peano. □

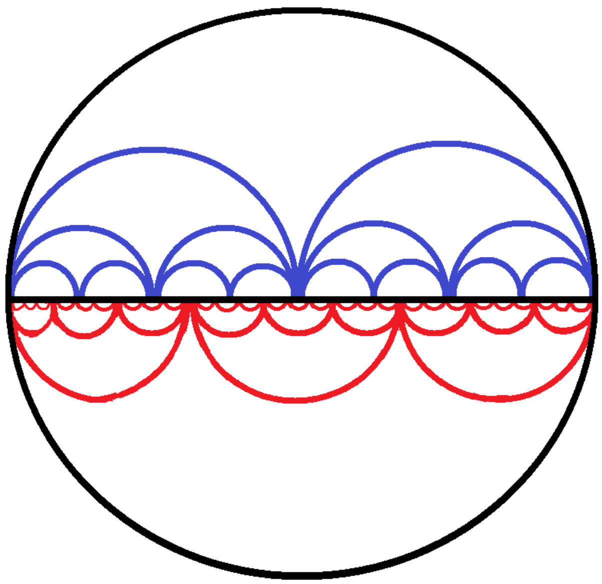
Corollary 1.5.9. *Every subcontinuum of a regular continuum is Peano.*

Proof. Every subcontinuum of a regular continuum is regular. □

A continuum all of whose subcontinua are Peano is called *hereditarily locally connected* or hlc. For a proof of the following, see [37] *(p. 94)*.

Theorem 1.5.10. *Every hlc continuum is rational.* □

Thus regular $\implies$ hlc $\implies$ rational. The sine continuum is rational but not hlc. A rational Peano continuum *is* hlc, however, since the characterization in [15] *(p. 167)* shows that hlc is equivalent to not containing a convergence continuum. Thus if a subcontinuum of X contained a convergence continuum it would be a convergence continuum of X as well. As shown below, there are hlc continua which are not regular. While regular continua are exceptionally nice, it is also an example showing that the union of two regular continua need not be regular.



This continuum M was first exhibited by Menger [19] *(p. 284)*. The patterns on top and bottom of $1/2^n$ and $1/3^n$ circles continue for all n. Clearly it's closed, bounded and connected, so it's a continuum. It is locally connected at every point off the diameter since those points have neighborhoods homeomorphic to arcs. If p is on the diameter, B is a metric ball centered at p and if C is a half-circle intersecting B, then by curvature considerations either $B \cap C$ is an arc or all of C, either of which is connected and intersects the diameter. Thus M is locally connected; since it's clearly rational, M is hlc.

The top half of M is regular: If p is off the diameter then $\text{ord}(p) = 2$, if p is a dyadic then the balls of radius $1/2^n$ intersect M only finitely many times by a quick check, and otherwise p lies under infinitely many half-circles which induce open discs in the plane having them in their boundary, and clearly each intersects M in only two points and contain p. Similarly, the bottom half of M is regular.

But M itself is not regular: If U is an open set (sufficiently small) containing the left-most point then for its boundary to avoid intersecting infinitely many of the upper half-circles (and thus have infinite boundary), it must intersect the diameter at a dyadic. But then in the bottom half, by the same reasoning, it will intersect infinitely many half-circles away from the diameter and thus $|\partial(U)| = \aleph_0$.

Returning to the notion of irreducibility, we show how it relates with end points.

Lemma 1.5.11. *No end point is a cut point.*

Proof. If p is an end point and $X\backslash p = A|B$ then pick a neighborhood V of p so that $|\partial(V)| = 1$ which does not contain either A or B. Without loss of generality assume $\partial(V) \in A$. If $V \cap B = \varnothing$ then $X = (A \cup V)|B$ is not connected (impossible), and thus $V \cap B \neq \varnothing$. Since it's not all of B it has non-empty boundary in B and thus a second boundary point in X, a contradiction. □

Note that we actually showed $X \setminus V$ is connected since its boundary must be in, say, A so that $A \cup V$ will be closed and disjoint from B, also closed, separating X.

Corollary 1.5.12. *If p is an end point of X then p has arbitrarily small open neighborhoods U with connected complements.* □

Proposition 1.5.13. *If $X = \mathrm{Irr}(p, q)$ then p and q are its only possible end points.*

Proof. If p, q, r were end points and $X = \mathrm{Irr}(p, q)$ then by **1.5.12** r has an open neighborhood not containing either p or q whose complement C is a continuum. C is proper and contains p and q, a contradiction. □

Corollary 1.5.14. *The pseudo-arc has no end points.*

Proof. The pseudo-arc is indecomposable, and thus irreducible by **1.4.28**. Thus it has at most two end points. However, by **1.4.29** it is homogeneous. Since a homeomorphism clearly preserves being an end point and the pseudo-arc contains points which are not end points, none of its points are end points. □

We can strengthen **1.5.11**. Denote the set of end points of X by $X^{[1]}$.

Proposition 1.5.15. *If X is a continuum and $A \subseteq X^{[1]}$ then $X \setminus A$ is connected.*

Proof. Fix $x \in X \setminus A$ and let $p \in X \setminus A$. By **1.5.2** there is a subcontinuum C_p of X irreducible between p and x. Suppose that $q \in C_p \cap X^{[1]}$. Note that for an open neighborhood U of q away from x and p we have $|C_p \cap \partial(U)| \leq |X \cap \partial(U)|$ so q is an end point of C_p as well. Then by **1.5.12**, we may assume that U is such that $C_p \setminus U$ is a proper subcontinuum of C_p containing p and x, impossible since $C_p = \mathrm{Irr}(x,p)$. Thus $C_p \cap X^{[1]} = \varnothing$. Thus $X \setminus A = \cup C_p$ for $p \in X \setminus A$ and since each contains x this union is connected. □

If a set C separates a space X, we call C a *cutting* of X and we write $X \setminus C = A|B$ to mean that (A, B) is a separation of $X \setminus C$ (esp. A and B are non-empty).

Corollary 1.5.16. *Non-degenerate continua have uncountably many non-end points.*

Proof. Fix $p \in X$ and suppose $d(p, X) > \epsilon$. Let C be a Cantor set on the interval $[0, \epsilon]$ and for $c \in C$ let B_c be the open ball of radius c centered at p. Then each is properly contained in X; let $D_c = \partial(B_c)$. Then for distinct $r, s \in C$ we have $D_r \cap D_s = \varnothing$ and this is thus an uncountable collection of mutually disjoint closed sets, each satisfying $X \setminus D_c = B_c|(X \setminus \overline{B_c})$. Thus X contains an uncountable family of mutually disjoint cuttings, none of which are contained in $X^{[1]}$ by **1.5.15**, proving the theorem. □

Compare **1.5.12** with the following. These are given as theorems 11.6-11.9 in [15]. The proofs are fairly direct so we omit them.

Proposition 1.5.17. *Let $X = \mathrm{Irr}(p, q)$ and let A, B be subcontinua.*

(a) If A contains p, then $X \setminus A$ is connected.

(b) If $p \in A$ *and* $q \in B$ *then* $X \setminus (A \cup B)$ *is connected.*

(c) If $X \setminus A = P|Q$ *then* $X \setminus A$ *has two components, containing* p *and* q *respectively.*

(d) $\mathrm{Int}(A)$ *is connected.* $\square$

Thus the notions of end point, non-cut point and point of irreducibility share many similarities. Note, however, that a continuum need not be irreducible about $X^{[1]}$, e.g. take a disc with two arcs sticking off it. Now, we saw that the pseudo-arc is irreducible but has no cut points; is there a continuum which *only* has cut points?

Theorem 1.5.18. *A non-degenerate continuum contains at least two non-cut points. As well, if* $X \setminus c = U|V$ *then each of* U *and* V *contain a non-cut point of* X*.*

Proof. See [19] *(p. 177)*. $\square$

Corollary 1.5.19. *No proper connected subset of* X *contains all its non-cut points.*

Proof. Following [15] *(p. 90)* let N be the set of non-cut points and suppose Z is connected and contains N. Suppose $c \in X \setminus Z$. Thus c is a cut point so $X \setminus x = U|V$ with, without loss of generality, $N \subseteq Z \subseteq U$, contradicting **1.5.18**. $\square$

We would be remiss not to present the following theorem, which gives a nice characterization of irreducible continua. First, a quick lemma.

Lemma 1.5.20. *Suppose* X *is a connected space and* $X = A \cup B$ *where* A *and* B *are both closed. If* $A \cap B$ *is connected then so are* A *and* B*.*

Proof. Suppose $A = U|V$. Then since $A \cap B$ is connected, without loss of generality assume $A \cap B \subseteq U$. Then $X = (B \cup U) \cup V$; each is non-empty, and we claim $(B \cup U)|V$ is a separation. $\overline{B \cup U} = B \cup \overline{U}$ doesn't intersect V since $A \cap B \subseteq U$ and U, V are separated in A, and thus in X since A is closed. Similarly, $\overline{V} \cap (B \cup U) = \varnothing$ since U, V are separated and $A \cap B \subseteq U$. Thus $X = (B \cup U)|V$, impossible. $\square$

Theorem 1.5.21. ***(Kuratowski's Theorem)*** *Let X be a continuum and let $p \in X$. Then p is a point of irreducibility of X if and only if X is not the union of two proper subcontinua both of which contain p.*

Proof. If p were contained in two such subcontinua A, B (by the Extension Theorem one of which may be taken to be non-degenerate) then for any $q \in X$ either $q \in A$ or $q \in B$ would be non-degenerate proper subcontinua, and so $X \neq \mathrm{Irr}(p, q)$ for any q. Thus p is not a point of irreducibility of X.

If p is not a point of irreducibility then $\kappa(p) = X$ and, by **1.4.26**, $\kappa(p) = \cup A_n = X$ for some proper subcontinua A_n of X containing p. By taking finite unions, we can assume that A_n is an increasing sequence of sets. If by contradiction we assume that X is not the union of two proper subcontinua containing p, then by taking unions it follows that X is not the union of finitely many proper subcontinua containing p, and thus the sequence A_n is infinite, so we may take it to be *strictly* increasing.

Pick $x_n \in A_{n+1} \setminus A_n$ and let x be a limit point of this set. Since $X = \cup A_n$, there is some k with $x \in A_k$. Since x_n escapes every A_j we have that $x \in \overline{X \setminus A_j}$ for all j, and thus $(\overline{X \setminus A_j}) \cap A_k \neq \varnothing$ for all j. Since $A_j \setminus A_k$ is open in A_j (and thus X) for every j, then since continua are perfectly normal each is an F_σ set. Since $X = \cup_j (A_j \setminus A_k)$, by the Baire Category Theorem there is an N such that $A_N \setminus A_k$ has non-empty interior. In particular, $(\overline{X \setminus A_N}) \cup A_k \neq X$.

Note that $X = A_N \cup [A_k \cup (\overline{X \setminus A_N})]$ and both contain p, so it's enough to show that the latter is a continuum, i.e. that it's connected since it's clearly closed. Since by assumption $A_k \cap (\overline{X \setminus A_N}) \neq \varnothing$ and A_k is a continuum it's enough to show that $(\overline{X \setminus A_N})$ is connected.

Suppose to the contrary that $(\overline{X \setminus A_N}) = C|D$. Then C and D are closed in $(\overline{X \setminus A_N})$

and thus in X, as is A_N, and $X = (A_N \cup C) \cup (A_N \cup D)$ while $(A_N \cup C) \cap (A_N \cup D) = A_N$. Thus by **1.5.20** $(A_N \cup C)$ and $(A_N \cup D)$ are connected, and in fact continua. Both contain p, so by the main assumption either $X = A_N \cup C$ or $X = A_N \cup D$.

Assuming without loss of generality that $X = A_N \cup C$, then since $C \cap D = \varnothing$ we see that $D \subseteq A_N$. Thus it must be in the boundary of A_N since $D \subseteq (\overline{X \setminus A_N})$, which is impossible because $\varnothing \neq \text{int}(A_N) \subseteq C$ and thus $D \subseteq \overline{C} = C$ which is separated from D. Thus $(\overline{X \setminus A_N})$ is connected, proving the theorem. □

Proposition 1.5.22. *Suppose X is a Peano continuum.*

(a) X is path-connected.

(b) X is locally path-connected.

(c) Open subsets of X are locally path-connected.

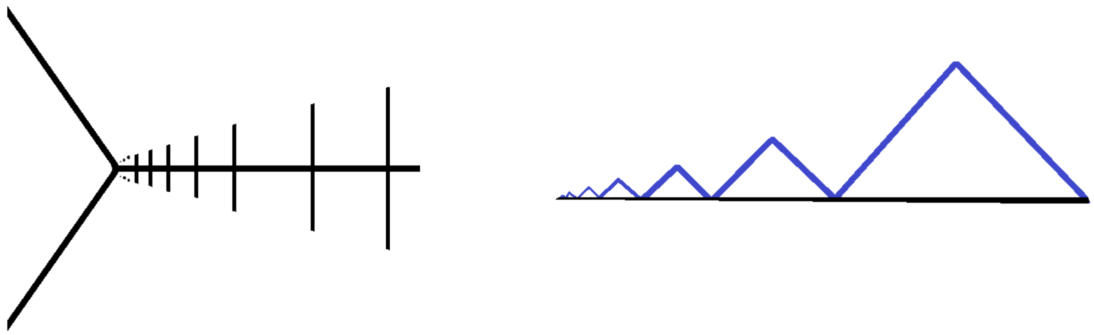
(d) Thus connected open subsets of X are path-connected.

(e) Any Peano continuum is uniformly locally path-connected *(ulac), that is for every $\epsilon > 0$ there is a $\delta > 0$ such that if $d(p, q) < \delta$ there is an arc from p to q whose diameter is less than ϵ [15] ch. 8.* □

Theorem 1.5.23. ***(n-Beinsatz Theorem)*** *Let X be the union of $n \geq 2$ arcs containing p that are pairwise-disjoint except for $\{p\}$. Then $\text{ord}_X(p) = n$.*

Conversely, if X is a Peano continuum and $\text{ord}_X(p) \geq n \geq 2$ there is a copy of T_n contained in X centered at p [19] (p. 177). □

The point p need not have neighborhoods homeomorphic to T_n (*left figure*).



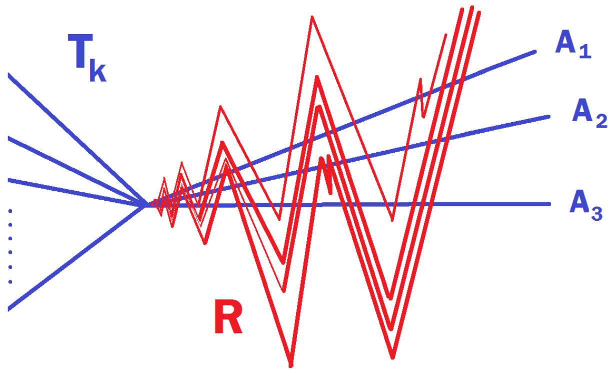
Definition 1.5.24. *If A and B are arcs with a common end point p we say that B is* dense along A at p *if p is a limit point of $(A \cap B) \setminus \{p\}$ (right figure). If T_n is a simple n-od centered at p and, as per usual, is the union of n arcs A_k that are pairwise-disjoint except for p, we call the arcs A_k the* chords *of T_n.*

The following is proven in Menger's *Kurventheorie* in the context of Peano continua and called the ∞-*beinsatz*; we will state the result in the same capacity, though the proof used here will work just as well for arbitrary topological spaces upon replacing the metric local basis with a general one.

Corollary 1.5.25. ***(Hairy Point Theorem)*** *[29] If X is a Peano continuum and $\text{ord}_X(p) \geq \aleph_0$ then X contains a copy of the hairy point (see **1.5.6**) centered at p.*

Proof. By the n-Beinsatz Theorem there are simple n-ods $T_n \subseteq X$ centered at p. Fix a k; we claim that there is a simple $(k+1)$-od $S \subseteq X$ centered at p such that $S = T_n$ outside the $1/k+1$-neighborhood of p.

(Case 1): First assume there is an n so that some chord B of T_n is not dense at p along any chord in T_k, whose chords we denote by $A_1, A_2, \dots, A_k$. Then by taking a (non-degenerate) sub-arc B' of B if necessary, we may assume that B' is contained in the $1/k+1$-neighborhood of p and $B' \cap A_j = \{p\}$ for all j. Thus $B' \cup T_k = S$ is a simple $(k+1)$-od centered at p, $T_k \subseteq S$ and $S \setminus T_k$ is an arc contained in the $1/k+1$-neighborhood of p, as desired.

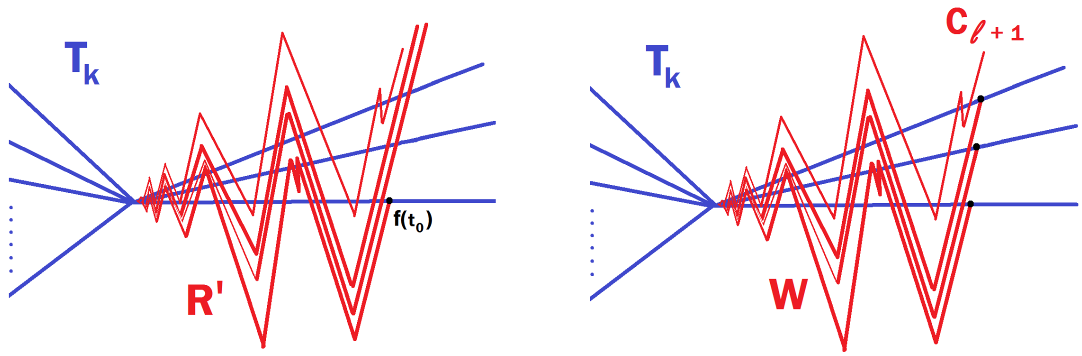


(Case 2): Now assume that for every T_n with $n > k$ we have that every chord of T_n is dense along at least one of $A_1, \ldots, A_k$ at p (such a chord may be dense along more than one A_j at p, for example by spiraling). As before, for each T_n we may replace it by a simple n-od T'_n so that every arc B of T'_n is contained in the $1/k+1$-neighborhood of p and only intersects those A_j along which it is dense at p.

Noting that 2^k is the number of subsets of $\{1, 2, \ldots, k\}$, by the pigeonhole principle if $n = (k+1) \cdot 2^k$ then for at least one subset $\{m_1, \ldots, m_\ell\}$ of $\{1, 2, \ldots, k\}$ the simple n-od T'_n contains some $k+1$ chords $C_1, \ldots, C_{k+1}$ each dense along all A_{m_j} at p. In particular, there are at least $\ell + 1$ of them $C_1, \ldots, C_{\ell+1}$. Without loss of generality assume $m_j = j$ for $1 \leq j \leq \ell$, and let $R = C_1 \cup \cdots \cup C_{\ell+1}$ be the simple $(l+1)$-od centered at p. Recall that its chords only intersect the chords of T_k along which it is dense at p, i.e. $A_1, \ldots, A_\ell$.

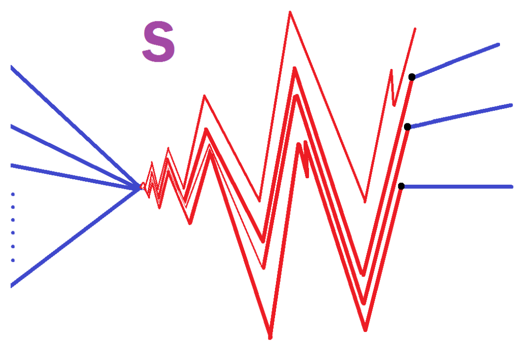
Assume each A_j is ordered so that its initial point is p. Fix an A_j. Among the first ℓ chords $C_1, \ldots, C_\ell$ of R, let i be the index such that $A_j \cap C_i$ is the *last* point of $(C_1 \cup \cdots \cup C_\ell) \cap A_j$ with respect to the order on A_j, and denote it by $c(A_j)$. This index exists since the C_i's are closed and disjoint. Consider the set $\{c(A_1), \ldots, c(A_\ell)\}$.

Assume there is an index h, for $1 \leq h \leq \ell$, such that $h \notin \{c(A_1), \ldots, c(A_\ell)\}$. Then some collection of distinct chords $A_{j_1}, \ldots, A_{j_h}$ of T_k are last intersected by the same C_i amongst $C_1, \ldots, C_\ell$. Assume it is i; let $f_i : I \to C_i$ be a homeomorphism with $f_i(0) = p$, and consider the sub-arcs $f_i([0,t])$. If C_i *last* intersects $A_{j_1}, \ldots, A_{j_h}$ in that order, by closedness of C_i there is a maximal $t_0 \in (0,1)$ such that $f_i(t_0) \in A_{j_1}$.



By continuity of f_i, for $s > 0$ there are only finitely many times the set $f_i([s,1])$ changes which A_{j_r} it last intersects (lest those times have an accumulation point, impossible since the A_j's are mutually separated outside any neighborhood of p). If all of $A_{j_1}, \ldots, A_{j_h}$ are still last intersected by $f_i([0,t_0])$ (as opposed to some C_j with $j \neq i$), then since C_h is dense along each A_{j_m} there is an $x \in C_h \cap A_{j_m}$ and a $y \in C_i \cap A_{j_m}$ such that $y < x$ with respect to the ordering on A_{j_m}. Letting $s = f_i^{-1}(x) > 0$ we have, for any j_m, that after a finite number of excisions of terminal sub-arcs from C_i we obtain $c(A_{j_m}) \neq i$. Thus after finitely many such excisions we may assume that $c(A_{j_m}) = i$ for precisely one m.

Replacing C_i as above we obtain a new simple $(\ell+1)$-od $R' \subseteq R$ so that i appears in $\{c_{R'}(A_1), \ldots, c_{R'}(A_\ell)\}$ precisely once. Note that this replacement does not decrease the number of distinct elements of $\{c(A_1), \ldots, c(A_\ell)\}$. If still $h \notin \{c(A_1), \ldots, c(A_\ell)\}$ we may repeat this same procedure on some other C_j's from R', obtaining an R'' and proceeding inductively. Since for any j there are only finitely many times C_j changes which A_i it last intersects, after finitely many iterations some A_i will be last intersected by C_h. Thus after a finite number of steps we obtain a simple $(\ell+1)$-od W satisfying $\{c(A_1), \ldots, c(A_\ell)\} = \{1, 2, \ldots, \ell\}$. After taking a sub-arc, assume $C_{\ell+1}$ doesn't intersect any A_{j_m} before one of the others does.



Order the first ℓ chords of W, which we also call $C_1, \ldots, C_\ell$, so that they're in bijection with the chords $A_1, \ldots, A_\ell$ such that C_j is the chord of W that last intersects A_j. For each pair (A_j, C_j) let S_j be the arc from p to the other end point of A_j that is the concatenation of C_j up to the last point of $A_j \cap C_j$, followed by the remaining terminal sub-arc of A_j as shown above. Let S be the union of $C_{\ell+1}$, S_j for $1 \leq j \leq \ell$ and A_i for $i > \ell$.

We show that S is a simple $(k+1)$-od centered at p, which will prove the claim. By construction it is the union of $k+1$ arcs each having p as an end point; we must show this is the only point where any pair of intersect. If $i > \ell$ then A_j doesn't intersect any other (alleged) chord of S except at p by construction of T'_n and certainly no other A_j does, since T_k is a simple k-od. Since $C_{\ell+1}$ doesn't intersect any A_i for $i > \ell$ except at p by construction of T'_n, and doesn't intersect any S_j by construction of W, it only remains to show that $S_i \cap S_j = \{p\}$ if $i \neq j$.

Writing $S_h = D_h \oplus B_h$, where D_h is the portion of the chord C_h and B_h is the portion of the chord of A_h comprising S_h, for $x \in S_i \cap S_j$, if $x \in D_i \cap D_j$ then $x = p$ since W is a simple $(\ell+1)$-od, and since $B_i \cap B_j = \varnothing$ we may assume $x \in D_i \cap B_j$, without loss of generality. But then $C_i \cap A_j$ has a point of intersection after $C_j \cap A_j$, impossible by construction of W. Thus $S_i \cap S_j = \{p\}$ for $i \neq j$, proving the claim.

Returning to the main problem, by taking sub-arcs of the chords of T'_n if necessary, assume T'_n doesn't contain any end points (besides p) of the chords of T_k (since they are bounded away from p). Thus we may assume each S_j contains a non-trivial sub-arc of A_j $(*)$. Similarly, we may assume that any A_i is only intersected by a C_j at a time (with respect to the order on A_i) before it first intersects the complement of the $1/k+1$-neighborhood B of p (†) (the left figure is permitted; the right is not).

Thus there is a sequence of simple n-ods T_n centered at p such that, by (†), $T_{n+1} = T_n$ outside the $1/n+1$-neighborhood of p - we showed more, namely that by $(*)$ we may assume all but one chord of T_{n+1} is a concatenation of a non-trivial arc and a non-trivial terminal sub-arc $B_i^{(n)}$ of a unique chord of T_n (in *Case 1* we simply subdivide each chord of T_n into two non-trivial arcs). By taking sub-arcs if necessary, we may assume that $T_{n+2} \cap B_i = \varnothing$ for all i such that $1 \leq i \leq n$(††).

We claim $\lim_n T_n = G$ is a hairy point centered at p. As $T_{n+1} = T_n$ outside the $1/n+1$-neighborhood of p, this limit exists since it's eventually constant outside every neighborhood of p, and $p \in T_n$ for all n. Thus G is a continuum containing p by **1.1.1**(f). Setting notation, let H be the hairy point with 'chords' $H_1, H_2, \ldots$ where H_n is the segment of length $1/n$ emanating from p. If h_j is the linear homeomorphism of I with H_j with $h_j(0) = (0,0)$ then by the dyadics 2^{-n} in H_j we mean the images under h_j of the sequence 2^{-n} in I.

Let $f_0 : T_2 \to H$ map T_2 homeomorphically to $H_1 \cup H_2$ with $f_0(p) = (0,0)$ and $f_0(A_j) = H_j$ for its chords A_1, A_2. By assumption, two chords C_1, C_2 of T_3 terminate in non-trivial sub-arcs $B_1^{(2)}, B_2^{(2)}$ of the chords A_1, A_2 of T_2 respectively.

Define $f_1 : T_3 \to H$ so that $f_1(C_j) = H_j$ for $j = 1, 2, 3$, while $f_1(B_j) = [2^{-1}, 1]$ in H_j as per the notation defined above for $j = 1, 2$. We may inductively define $f_n : T_{n+2} \to H$ so that it sends its chord C_{n+2} homeomorphically onto H_{n+2} with $f_n(p) = (0, 0)$ and sends the corresponding $B_j^{(n+1)}$ of T_{n+1} to $[2^{-n}, 1]$ in H_j. By (††) we have that $B_j^{(n)} \subseteq B_j^{(n+1)}$, since T_{n+2} doesn't intersect $B_j^{(n)}$ for any j with $1 \le j \le n$. Thus we may assume $f_n \equiv f_k$ on $B_j^{(k)}$ for all j and for all $n \ge k + 2$.

If $x \in G = \lim_n T_n$ with $x \neq p$ then x is contained in the (closed) complement E of some $1/n$-neighborhood of p. Since $T_k \cap E$ is a sequence of closed sets that is eventually constant, $G \cap E = T_N \cap E$ for some N, which thus contains x. Thus x is contained in $B_j^{(N+2)}$ for some j, on which the sequence f_n is eventually constant, so $\lim_n f_n(x)$ is well-defined. If $x = p$ then since $f_n(p) \equiv 0$ we have $\lim_n f(p) = 0$, so $\lim_n f_n = f : G \to H$ is a well-defined function on G.

We wish to show that f is a homeomorphism. If $x \in G$ with $x \neq p$ then as before $x \in B_j^{(N+2)}$ for some j and for some N sufficiently large; since the f_n's are eventually constant, each a homeomorphism on $B_j^{(N+2)}$, we have that $f = \lim_n f_n$ is continuous at x. If $x_k \to p$ with $x_k \in G$ then eventually x_k is contained in the $1/n+2$-neighborhood of p and therefore mapped to $[0, 2^{-n}]$ in some $H_{j(k)}$ by definition of f_n. Thus f is continuous at p, i.e. f is continuous.

By construction, $f(G)$ contains the end point of H_j for every j; since it also contains p and is the continuous image of a connected set, $f(G)$ contains H_j for all j. Since $H = H_1 \cup H_2 \cup \cdots$, we have that f is surjective. If $x, y \in G$ are two distinct points other than p, then for N sufficiently large we have $x \in B^{(N+2)}_{j(x)}$ and $y \in B^{(N+2)}_{j(y)}$. Since $f(B^{(N+2)}_{j(x)}) \cap f(B^{(N+2)}_{j(y)}) = \varnothing$ if $j(x) \neq j(y)$ and f is a homeomorphism on any given $B^{(N+2)}_{j(x)}$ we have that $f(x) \neq f(y)$. Similarly, since $(0,0) \notin f(B^{(N+2)}_{j(x)})$ if $x = p$ and $y \neq p$ we have $f(x) \neq f(y)$.

Thus f is a continuous bijection of the compact G and the Hausdorff H, and is therefore a homeomorphism. □

Definition 1.5.26. *If X is decomposable and (A, B) is a decomposition of X, we write $X = A \oplus B$.*

Definition 1.5.27. ***(Essential Sum)*** *A continuum X is the essential sum $X = X_1 \oplus \cdots \oplus X_n$ of subcontinua X_j if $X = \bigcup X_j$ and $X_j \not\subset \bigcup_{i \neq j} X_i$ for all j.*

In particular, if $X = X_1 \oplus \cdots \oplus X_n$ each X_j is proper and non-degenerate. Thus this definition generalizes the notation $X = A \oplus B$ in **1.5.25**. Attaching two copies of the (indecomposable) pseudo-arc at a point shows that even if X is decomposable there may not be subcontinua X_1, X_2, X_3 with $X = X_1 \oplus X_2 \oplus X_3$. However, if $X = X_1 \oplus X_2 \oplus X_3$ then they can't all be mutually separated, so some pair of them (say X_1, X_2) are such that $X_1 \cup X_2$ is a continuum. It doesn't contain X_3, so $X = (X_1 \cup X_2) \oplus X_3$ is a decomposition of X.

Definition 1.5.28. ***(Triod)*** *If $X = X_1 \oplus X_2 \oplus X_3$ with $\cap X_i = C$ a non-empty continuum and $X_i \setminus C$ are pairwise-separated we say X is a* triod.

Definition 1.5.29. ***(Unicoherent)*** *A continuum X is unicoherent if whenever $X = X_1 \oplus X_2$ we have that $X_1 \cap X_2$ is a continuum.*

We will devote an entire section to unicoherence, but for now we just point out that the arc is unicoherent. The circle, for example, is not. We now state two theorems, both of which are sometimes called 'Sorgenfrey's Theorem.' The second half can be deduced from the first half, although historically they were discovered in the other order [33] [32] [1].

Theorem 1.5.30. ***(Sorgenfrey's Theorem)*** *X is irreducible about some set of n points if and only if whenever $X = X_1 \oplus \cdots \oplus X_{n+1}$ some union of n of them is not connected.*

Every non-degenerate unicoherent continuum which is not a triod is irreducible. □

In this regard we mention two other characterizations that were later discovered. The first is by Maćkowiak, and the second by Ryden [21] [28].

Theorem 1.5.31. *X is irreducible about some n point set if and only if:*

(a) X is not the countable union of any increasing sequence of subcontinua, or

(b) every pairwise-disjoint collection of non-separating open sets is finite. □

We mention the following three classical theorems which will be less useful for our purposes, but which help square visual intuition with the definitions.

Theorem 1.5.32. ***(Cut Point Order Theorem)*** *If a point p separates one of its open neighborhoods, call it a* local separating point. *If X is a continuum then all but at most countably many locally separating points have order 2. Thus all but at most countably many cut points have order two [37] (p. 49).* □

Theorem 1.5.33. *If X is a continuum, then X is regular (resp. rational) if and only if for every pair of distinct points $p, q \in X$ there is a separation of X between p and q by a finite (resp. countable) set [37] (p. 99).* □

Theorem 1.5.34. ***(n-arc Connectedness Theorem)*** *If X is a Peano continuum and $x, y \in X$ are such that no collection of n points separates them, then there are n arcs from x to y that are pairwise-disjoint except at x and y [36] (p. 452).* □

1.6 The Arc

Now we give several characterizations of the arc.

Theorem 1.6.1. *A non-degenerate Peano continuum is an arc if and only if it contains no simple triod and no simple closed curve.*

Proof. Clearly an arc doesn't contain a simple triod (since it has no points of order 3) or a simple closed curve. Now suppose X is a non-degenerate Peano continuum and is not an arc. By the Non-Cut Point Existence Theorem there are points $p, q \in X$ that are non-cut points, and by **1.5.22**(a) there is an arc A in X from p to q. By **1.5.22**(d) the sets $P = X \setminus \{p\}$ and $Q = X \setminus \{q\}$ are path-connected. Since $X \neq A$, let s be a point in $X \setminus A$.

Let $r \in A \setminus \{p, q\}$. Then there is an arc $B \subseteq P$ oriented from r to s. First suppose that $q \notin B$. With respect to the order on B, let $x \in B \cap A$ be the last time B intersects A. Since $q \notin B$ we have that $[a, x] \cup [x, q] \cup [x, s]$ is the union of three arcs in X that are pairwise-disjoint except at x. Thus it is a simple triod centered at x.

Thus assume that every arc $B \subseteq P$ from r to s contains q, for every $r \in A \setminus \{p, q\}$. In particular, every arc from q to s in P is disjoint from $A \setminus \{q\}$ lest it intersect at some $t \neq q$; but then there would be a path from t to s not containing q, impossible by assumption. Fix an arc Y from q to s in P, which by the previous comment doesn't intersect A except at q.

Similarly, we may fix an arc Z from p to s in Q that doesn't intersect A except at p. Let z be the first point that Y and Z intersect, assuming they're ordered from q to s and from p to s respectively. Let Y' be the sub-arc of Y from q to z, and let Z' be the sub-arc of Z from p to z. Then the concatenation $A \oplus Y' \ominus Z'$ is a simple closed curve running from p to q to x to p. □

It's straight-forward to show that a simple triod and a simple closed curve are not arc-like. Since non-degenerate subcontinua of arc-like continua are arc-like by **1.4.11**, it can be deduced from the above that a Peano continuum is arc-like if and only if it's an arc. However, there is a stronger result [15] *(p. 230)*.

Lemma 1.6.2. *An arc-like continuum X is unicoherent.*

Proof. Suppose by contradiction that $X = A \oplus B$ and $A \cap B = E|F$ is separated. Let U, V be open sets containing E, F respectively with disjoint closures; since A and B are connected and $U \cup V$ is not, neither is contained in $U \cup V$. If $d(C, D) = \min\{d(c, d) \mid c \in C, d \in D\}$ then let ϵ be the minimum of $d(\overline{U}, \overline{V})$, $d(A \setminus (U \cup V), B)$ and $d(B \setminus (U \cup V), A)$. Since each is taken over a pair of disjoint compact sets, $\epsilon > 0$.

Let f be an ϵ-map from X onto I, and let $G = f(A) \cap f(B)$, which as the intersection of two connected subsets of I is an interval. By a basic continuity argument, the sets $H = \{t \in G \mid f^{-1}(t) \subseteq U\}$ and $J = \{t \in G \mid f^{-1}(t) \subseteq V\}$ are open in G and disjoint since $U \cap V = \varnothing$; we claim they are a separation of G, a contradiction.

First, H is non-empty: If $e \in E \subseteq U$ and $f(e) = t$, then since f is an ϵ-map any point x of $f^{-1}(t)$ is in the ϵ-neighborhood of e. By definition of ϵ we have $x \notin \overline{V}$ and since $x \in A \cap B$ we have $x \notin A \setminus (U \cup V)$ nor $B \setminus (U \cup V)$ and thus not in $(A \cup B) \setminus (U \cup V) = X \setminus (U \cup V)$. Thus $x \in U$ since $x \notin V$. Similarly, J is non-empty.

If $t \in G \setminus (H \cup J)$ there is an $x \in f^{-1}(t)$ with $x \in X \setminus (U \cup V)$. Without loss of generality, assume $x \in A \setminus (U \cup V)$. But since $G = f(A) \cap f(B)$ there is a $y \in B$ with $f(y) = t = f(x)$, impossible by definition of ϵ. Thus $H \cup J$ is a separation of G, a contradiction. □

A continuum is *hereditarily unicoherent* if all its subcontinua are unicoherent. Since non-degenerate subcontinua of arc-like continua are arc-like, we have:

Corollary 1.6.3. *Arc-like continua are hereditarily unicoherent.* □

Corollary 1.6.4. *An arc-like continuum does not contain a simple closed curve.* □

Recall **1.5.28**. A continuum X is *atriodic* if it doesn't contain a triod.

Lemma 1.6.5. *An arc-like continuum X is atriodic.*

Proof. Since non-degenerate subcontinua of arc-like continua are arc-like, it's sufficient to show that X isn't a triod. Suppose by contradiction that $X = X_1 \oplus X_2 \oplus X_3$ with $\cap X_i = C$

a continuum and with the sets $X_i \setminus C$ pairwise separated; let U_i be pairwise-disjoint open neighborhoods of $X_i \backslash C$ in X, which we may assume are each disjoint from C since C is closed.

By **1.1.12** each of $C \cup U_i$ is a continuum. Letting $x_i \in U_i$ and $d(x_i, X \backslash U_i) = \min\{d(x_i, p) \mid p \in X \backslash U_i\}$ if $\epsilon = \min_i d(x_i, X \backslash U_i)$ then $\epsilon > 0$, and since X is arc-like there is an ϵ-map $f : X \to I$. If $G_i = f(U_i \cup C)$ then since $U_i \cup C$ is a continuum, G_i is an interval and $I = \cup G_i$. If some G_i contains both 0 and 1 then $G_i = I$; otherwise, since each contains the image of C which in turn contains some point $y \in I$, and at least one contains 0 while another contains 1, the union of these two contains the intervals $[0, x]$ and $[x, 1]$.

Thus I is covered by some two of G_1, G_2, G_3, and without loss of generality we may assume $I = G_1 \cup G_2$. Thus $G_3 \subseteq (G_1 \cup G_2)$. But this is impossible, since f is an ϵ-map and $x_3 \in U_3$, whereas $f^{-1}(x_3)$ non-trivially intersects $C \cup (U_1 \cup U_2)$ since $f(x_3) \in G_1 \cup G_2$, a contradiction. Thus X is not a triod. □

Corollary 1.6.6. *An arc-like continuum X is irreducible.*

Proof. By the previous two theorems, X is unicoherent and is not a triod. Thus by Sorgenfrey's Theorem, X is irreducible. □

Now we obtain another characterization of the arc [15] *(p. 231)*.

Theorem 1.6.7. *A continuum X is an arc iff it's path-connected and arc-like.*

Proof. An arc is path-connected and arc-like. Now assume that X is path-connected and arc-like. By **1.6.6** X is irreducible, say between p and q. Since X is path-connected there is an arc A from p to q in X. But then $A = X$ since X is irreducible between p and q. □

Corollary 1.6.8. *A Peano continuum X is arc-like iff X is an arc.*

Proof. Peano continua are path-connected. □

By a similar argument to that used in **1.6.7**, we obtain the following.

Proposition 1.6.9. *If X is a path-connected continuum, then X is decomposable.*

Proof. Suppose by contradiction that X is indecomposable. Then by **1.4.28** X is irreducible, say between p and q. Since X is path-connected, there is an arc A from p to q in X which, by $X = \mathrm{Irr}(p, q)$ must satisfy $X = A$. Thus X is an arc, which is decomposable, a contradiction. $\square$

Definition 1.6.10. ***(Graph)*** *A continuum is a graph if it is the union of n arcs which pairwise-intersect at only finitely many points.*

By taking subdivisions, clearly it is equivalent to assume that X is the union of finitely many arcs that are pairwise-disjoint except possibly at their end points. Much of the basic theory of graphs is showing that they are topologically equivalent to the 'usual' definition of a graph, i.e. that this topological definition is sufficient to eliminate any 'pathological' behavior. Our judicious plucking of the n-Beinsatz Theorem trivializes much of this work, as follows.

Suppose $X = A_1 \cup \cdots \cup A_n$ that are arcs which are pairwise-disjoint except possibly at their end points. Then if $x \in A_j$ is not an intersection point with any other A_i, which are closed and thus bounded away from x, it has a neighborhood basis of arcs. If x is an end point of A_j and is not intersected by any other A_i then it is an end point of A_j and thus also has a neighborhood basis of arcs all of which it is an end point. If x is intersected by some other arcs A_i, then since there are only finitely many such possible intersection points it is bounded away from the rest and since $A_i \cap A_j$ is empty except possibly for some of these intersection points, x has a neighborhood that is a simple k-od.

Thus X is locally connected. Since neighborhoods in an arc or simple k-od are arcs and simple l-ods, subcontinua of graphs are graphs. Turning this around, by the n-Beinsatz theorem, if X is Peano and $x \in X$ with $\mathrm{ord}(x) = n$ then X is contained in a simple n-od T_n centered at x, and if the set of points with order greater than 2 are discrete then we may take T_n to be a neighborhood of x. In particular, if $\mathrm{ord}(x)$ is finite for every $x \in X$ and is equal to 2 for all but finitely many points, then X is a graph. Especially, we obtain the following characterization.

Proposition 1.6.11. *If X is a continuum, then $ord_X(x) \leqq 2$ iff $X \simeq I$ or $X \simeq \mathbb{S}^1$.*

Proof. By the above observation, X is a graph and thus Peano. Suppose it contains a simple closed curve S; if $X \neq S$ then let $x \in X \setminus S$. Since X is path-connected there is an arc A from x to S, and after taking a sub-arc we may assume $A \cap S = \{y\}$. Then y has a neighborhood C in S that is a closed arc, and $A \cup C$ is a simple triod centered at y. But then $\text{ord}(y) \geq 3$ by the n-Beinsatz Theorem, a contradiction.

Thus assume that X doesn't contain a simple closed curve. Since it doesn't contain a simple triod by the above argument, by **1.6.1** X is an arc. □

Corollary 1.6.12. *$X \simeq \mathbb{S}^1$ iff* $\text{ord}_X(x) \equiv 2$. □

The following surprising fact comprises our interest in graphs and will facilitate our next characterization of the arc. Our exposition differs from, but is similar to, that in [15].

Theorem 1.6.13. *A Peano continuum X is a graph if and only if each of its subcontinua has only finitely many end points.*

Proof. One direction is trivial, so assume X is such that all of its subcontinua have at most finitely many end points. By the Hairy Point Theorem, X contains no point with infinite order since H has infinitely many end points. Thus it's sufficient to show that X doesn't contain infinitely many point with order greater than two.

By contradiction, assume there is a sequence of distinct x_j with $\text{ord}(x_i) \geq 3$. After taking a subsequence if necessary, assume $x_j \to x$, and let U_j be pairwise-disjoint connected open neighborhoods of x_j with diameters bounded by $1/j$ none of which contains x. First assume there is an arc A containing infinitely many x_j; after taking a further subsequence we can assume $x_j \in A$ for all j. Since $\text{ord}(x_j) \geq 3$ for all j there is a $p_j \in U_j \setminus A$ for all j, and since U_j is connected there is an arc $A_j \subseteq U_j$ from x_j to p_j for all j.

We claim that $Y = A \cup (\cup A_j)$ is the null comb (see **1.5.6**) N, which clearly has infinitely many end points. Let $f : A \to I \times \{0\}$ be a homeomorphism with $f(x) = (1, 0)$. Extend f to each A_j by mapping it homeomorphically to the vertical segment of length $1/j$ with $f(p_j) = (1 - 1/j, 1/j)$. Then this is a well-defined bijection between Y and N which, by definition, is continuous at every point other than x since the U_j's are mutually separated, and thus the A_j's are. Thus it's sufficient to show continuity at x.

Since $\mathrm{diam}(U_j) \to 0$ any sequence of points $z_j \in U_j$ converges to x since $x_i \to x$. Since each U_j is bounded away from x, every sequence converging to x escapes each U_j. As f is a homeomorphism on A, if $z_n \subseteq A$ with $z_n \to x$ then $f(z_n) \to f(x)$. Thus given an arbitrary sequence $y_n \to x$ we may assume $y_n \notin A$ for all n, and thus $y_n \in A_{n(i)} \subseteq U_{n(i)}$ for all n. Thus $f(y_n) \to f(x)$ since the vertical segments converge to $f(x)$ and y_n escapes every U_i, i.e. $f(y_n)$ escapes each vertical segment.

Now assume that each arc in X contains only finitely many x_i. Since Peano continua are locally path-connected, after taking a subsequence we may assume there are arcs A_j from x_j to x with $\mathrm{diam}(A_j) < 1/n$ and since no such arc contains infinitely many x_i, after iteratively taking subsequences we may assume that $x_i \notin A_j$ for $i \neq j$. Let $Y = \cup A_j$. Y is connected since it's the union of arcs with a common point x, and it's closed since $\lim A_n = \{x\} \subseteq Y$. Letting U_i be as before and assuming that $\mathrm{diam}(U_i) < \frac{1}{n} d(x_i, x)$ then since $\mathrm{diam}(A_i) \to 0$ each U_i is intersected by only finitely many of the A_i. Since none contain x_i there is a neighborhood of x_i away from $\cup_{j \neq i} A_j$, i.e. a neighborhood which is sub-arc of A_i for which x_i is an end point. Thus Y has infinitely many end points, completing the proof. □

Definition 1.6.14. ***(Tree)*** *A tree is a graph containing no simple closed curve.*

It's a standard application of the Seifert-Van Kampen Theorem to show that, equivalently, trees are the simply connected graphs, equivalently the contractible graphs. In the same vein as the theorem above, we have the following.

Theorem 1.6.15. *If X is a tree, its non-cut points are precisely its end points.*

Proof. By **1.5.11** end points are not cut points. So assume x is a non-cut point and, by contradiction, that $\mathrm{ord}(x) \geq 2$. Let $Y = X \setminus \{x\}$, which is open and connected since x is not a cut point, and is in fact path-connected since X is a graph and thus Peano. Since $\mathrm{ord}(x) = n \geq 2$ there is a simple n-od T_n centered at x by the n-Beinsatz Theorem, and since X is a graph we can assume that T_n is a neighborhood of x in X. Since Y is path-connected, we may pick distinct chords C_1, C_2 of T_n and a path P in Y from a point $p \in C_1$ to a point $q \in C_2$.

Let $[x, p] \subseteq C_1$ be the arc from x to p in C_1 and define $[x, q]$ similarly. Since P is bounded away from x and since C_1 and C_2 are mutually separated outside any neighborhood of x, we may assume that p is the last time P intersects C_1 with respect to the order on P, and that q is the first time it intersects C_2. Then the concatenation $[x, p] \oplus P \ominus [x, q]$ is a simple closed curve, a contradiction. □

This property is shared by the arc, and will the crux of our argument. To utilize it, we need the following concept, discussed in [13].

Definition 1.6.16. ***(Disconnection Number)*** *Suppose there is a minimal cardinal $\alpha \leq \aleph_0$ such that every countable subset A of X with cardinality at least α separates X. Then we say that $\alpha = D(X)$ is the disconnection number of X.*

Thus the only possible disconnection numbers for a continuum are $1, 2, \ldots, \aleph_0$. In fact, by the Non-Cut Point Existence Theorem we have $2 \leq D(X) \leq \aleph_0$. The following is intuitive.

Proposition 1.6.17. *If α is a disconnection number for a continuum X and $\alpha < \beta \leq \aleph_0$, then any subset of cardinality β also separates X.*

Proof. Let $|B| = \beta$ and let A be a subset of B with $|A| = \alpha$. Then $X \setminus A$ is not connected, and by the Boundary Bumping Theorem the closure of each component C_r of $X \setminus A$ intersects

A. Picking two components C_1, C_2 then since they are non-degenerate, connected metric spaces they're uncountable, and thus neither is contained in B since $|B| = \beta \leq \aleph_0$. Hence $C_1 \setminus B$ and $C_2 \setminus B$ are non-empty and mutually separated in $X \setminus B$, and thus B also separates X. □

The following is similar, but the result is less intuitive.

Lemma 1.6.18. *If X is a continuum and $D(X) \leq \aleph_0$, then for any proper subcontinuum A of X the number of components of $X \setminus A$ is finite.*

Proof. Suppose by contradiction that $X \setminus A$ has infinitely many components C_i. By **1.1.15** $C_i \cup A = E_i$ is a continuum, so by **1.5.19** each E_i has a non-cut point p_i not in A, since A is connected. Let $F_i = E_i \setminus \{p_i\}$, which is connected. Let A' be the union of A and the components of $X \setminus A$ other than the C_i's, which is again connected by **1.1.15**. Since each F_i intersects A, letting $F = \cup F_i$ we have that $F \cup A'$ is connected. But if $P = \{p_1, p_2, \dots\}$ then $F \cup A' = P^c$ is connected, whereas P is infinite. By the previous proposition, P separates X, a contradiction. □

Lemma 1.6.19. *If K is a convergence continuum of the continuum X then it contains at most countably many cut points of X.*

Proof. Suppose $\mathfrak{A}$ is an uncountable collection of cut points of X. Writing $X \setminus \{p_\alpha\} = U_\alpha | V_\alpha$, we first show that there is a p_α so that both U_α and V_α contain an element of $\mathfrak{A}$. If not, then assume $\mathfrak{A} \setminus \{p_\alpha\} \subseteq U_\alpha$ for all α. Let $p_\beta, p_\gamma \in \mathfrak{A}$ be distinct. Then since $p_\gamma \in U_\beta$ and vice-versa we have $X = (U_\beta \cup U_\gamma) \cup (V_\beta \cap V_\gamma)$, with the latter being separated from the former and thus empty since X is connected. But then $\{V_\alpha\}$ is an uncountable collection of pairwise-disjoint open sets in the second-countable X, impossible.

Thus if $\mathfrak{A}$ is an uncountable collection of cut points of X contained in K, there is a $p_\alpha \in K \cap \mathfrak{A}$ with $\mathfrak{A} \cap U_\alpha \neq \varnothing, \mathfrak{A} \cap V_\alpha \neq \varnothing$ with U_α, V_α separated in X. In particular, each contains a point of K. As a convergence continuum, K is the limit of a sequence of subcontinua

A_j of X disjoint from K, which must thus each be contained in either U_α or V_α. Since $U_\alpha \cap K \neq \varnothing$ the sequence A_j is not eventually contained in V_α, and vice-versa it's not eventually contained in U_α. But then the only set which A_j can converge to is p_α, impossible since $\lim A_j = K \neq \{p\}$. □

By replacing the cut points with pairwise-disjoint closed cuttings and the requirement that there is a p_α so that both U_α and V_α contain an element of $\mathfrak{A}$ with the requirement that there is a C_α so that both U_α and V_α intersect $\cup\mathfrak{A}$, the proof of the following proceeds almost ver batim:

Corollary 1.6.20. *If K is a convergence continuum of the continuum X then it contains at most countably many mutually disjoint closed cuttings.* □

Corollary 1.6.21. *If a continuum X has a disconnection number then X is Peano.*

Proof. Suppose X is not Peano. By the Continuum of Convergence Theorem it contains a convergence continuum K. Since K is uncountable it contains uncountably many countably infinite subsets A_α. Since K is compact, each A_α contains a closed, countably infinite B_α, e.g. a convergent sequence. By the above corollary, at least one of B_α doesn't separate X, but this contradicts that $D(X) \leq \aleph_0$. □

Lemma 1.6.22. *If C is a non-degenerate subcontinuum of the continuum X and $D(X) \leq \aleph_0$, then $D(C) \leq \aleph_0$.*

Proof. By **1.6.16** $X \setminus C$ has finitely many components $C_1, \ldots, C_n$, and by the Boundary Bumping Theorem we may pick points $c_i \in \overline{C_i} \cap C$. Then $D_i = C_i \cup \{c_i\} \subseteq \overline{C}$ is connected. Let $A \subseteq C$ be countably infinite and suppose by contradiction that $C \setminus A$ is connected. Then similarly $E = (C \setminus A) \cup \{c_1, \ldots, c_n\}$ is connected and thus $F = D_1 \cup \cdots \cup D_n \cup E$ is connected since each D_i contains c_i and is connected.

Then by construction, $F^c = A \setminus \{c_1, \ldots, c_n\}$. But $A \setminus \{c_1, \ldots, c_n\}$ is countably infinite and F is connected. Thus it doesn't separate X, contradicting $D(X) \leq \aleph_0$. □

Lemma 1.6.23. *If C is a subcontinuum of the continuum X and $D(X) \leq \aleph_0$, then C has only finitely many end points.*

Proof. If C is a single point then it has no end points, so assume otherwise. Then by the above, $D(C) \leq \aleph_0$. If $C^{[1]}$ is infinite it contains a countable set A with $X \setminus A$ connected, by **1.5.15**. But this contradicts **1.6.17**. $\square$

Corollary 1.6.24. *If X is a continuum and $D(x)$ is finite, then X is a graph.*

Proof. By **1.6.21** X is Peano. By **1.6.23** every subcontinuum of X has only finitely many end points. By **1.6.13**, X is a graph. $\square$

Corollary 1.6.25. *A continuum X with only finitely many non-cut points is a tree.*

Proof. Suppose X has precisely n non-cut points, and let A contain $n + 1$ points, one of which p must be a cut point. Then if $X \setminus \{p\} = U|V$ we have that U and V are uncountable, so $X \setminus A$ is also separated between $U \setminus A$ and $V \setminus A$. Thus $D(X) \leq n + 1$ since A was arbitrary, and so by **1.6.23** X is a graph.

If X contains a simple closed curve S, then since X is a graph only finitely many points of S don't have neighborhoods in X homeomorphic to arcs. Pick a point x that does and let B be an arc in S which is a neighborhood of x; if X is written as the finite union of arcs $A_1, \ldots, A_k$, since B is infinite it contains infinitely many points x_j that are not end points of any A_i.

We claim that each x_j is a non-cut point, contradicting the initial assumption. Fix an x_j and assume $x_j \in A_j$. By assumption x_j is contained in no other A_i. After taking a subdivision of A_j if necessary we may assume $A_j \subseteq S$, and after taking a further subdivision may assume that it intersects only two other A_i, each of which is an adjacent arc contained in S.

Since X is path-connected, by construction every path from x_j to any point outside of

A_j passes through one of its end points. Thus $X \setminus \text{Int}(A_j)$ is path-connected since $S \setminus \text{Int}(A_j)$ is a path between the end points of A_j and every point in X has a path to at least one of them contained outside of $\text{Int}(A_j)$. Recalling that x_j is not an end point of A_j, we have that x separates the arc A_j into two half-open intervals, each of which is path-connected and contains one of the end points of A_j. Thus $X \setminus x_j$ is path-connected, and in particular connected, a contradiction since x_j was arbitrary among an infinite collection of points. □

Now we can prove the following classic theorem:

Theorem 1.6.26. *A continuum X is an arc iff it has exactly two non-cut points.*

Proof. Clearly an arc has exactly two non-cut points. So assume X is a continuum with exactly two non-cut points. By the corollary above, X is a tree. Thus X is path-connected, as a Peano continuum. Let the two non-cut points be p and q, and let A be an arc from p to q. Then A is a connected subset of X containing all its non-cut points, so by **1.5.19** $X = A$. □

The meat of this section is generally due to Nadler [13]. We mention that there is a universal arc-like continuum, constructed by Schori [30], in the sense that every arc-like continuum can be topologically embedded in it. By standard applications of dimension theory and inverse limits it is also true that every arc-like continuum is embeddable in $\mathbb{R}^2$.

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