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Hopf algebras of finite Gelfand-Kirillov dimension

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Abstract

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We study Hopf algebras of finite Gelfand-Kirillov dimension. By analyzing the free pointed Hopf algebra $F(t)$, we show the existence of certain Hopf subalgebras of pointed Hopf domains H with $2 \leq \text{GKdim } H < \infty$. Using Takeuchi's construction of free Hopf algebras, we study the GK-dimension of a pointed Hopf algebra H and its associated graded Hopf algebra $\text{gr } H$. We also get some lower bounds of GK-dimension of Hopf algebras in terms of certain invariants of skew primitive elements. With the help of these results, we classify connected Hopf algebras of GK-dimension 3. We introduce the notion of coassociative Lie algebras, which generalize both the notions of Lie algebras and coalgebras (without counits). It turns out to be very useful in our classification of connected Hopf algebras of GK-dimension 4, which is accomplished in the last part of this paper.

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DEDICATION

to my dear wife

Chapter 0

INTRODUCTION

The Gelfand-Kirillov dimension (GK-dimension for short), which measures the growth of an algebra, is a very useful and powerful tool for investigating noncommutative algebras. Since the popularization of quantum groups around 1980's, a great number of noncommutative Hopf algebras have been introduced and GK-dimension naturally becomes a good invariant for them. In the past few years, Hopf algebras of finite GK-dimension have been studied intensively [AA, AS1, AS2, Br1, Br2, BG1, BG2, BZ, GZ2, LWZ, WuZ1, WuZ2].

Since the structure of a Hopf algebra is very rigid by its definition, it is quite possible to classify certain Hopf algebras with finite GK-dimensions. For example, Andruskiewitsch and Angiono classified pointed Hopf domains with finitely generated abelian group $G(H)$, and generic infinitesimal braiding [AA, Theorem 1.1].

In [LWZ], Lu, Wu and Zhang introduced the notion of homological integral of Hopf algebras, which is a generalization of integral of (finite-dimensional) Hopf algebras and turns out to be a very useful tool. Also in the paper, they initiated the classification of affine noetherian Hopf algebras of Gelfand-Kirillov dimension one. This was carried on by Brown and Zhang in [BZ], where they studied some classes of prime regular Hopf algebras of GK-dimension one. In [GZ2], Goodearl and Zhang moved to GK-dimension two. To be precise, they assumed the following hypothesis

- (H) H is a Hopf algebra domain which is either affine or noetherian, the base field k is algebraically closed of characteristic zero, and $\text{GKdim } H = 2$.

They classified all Hopf algebras satisfying (H) and the extra condition $\text{Ext}_H^1(Hk, Hk) \neq 0$.

In this paper, we are going to study Hopf algebras of finite GK-dimension. Here are the brief summaries of each chapter.

Chapter 1: Preliminaries

In this chapter, we recall some definitions related to Hopf algebras.

Chapter 2: Existence of Hopf subalgebras of GK-dimension two

This chapter is devoted to the study of Hopf subalgebras of a pointed Hopf algebra. The idea is simple: in order to understand a Hopf algebra of higher GK-dimensions, we look at its Hopf subalgebras, which in general have smaller GK-dimensions, and try to find some connections.

For Hopf algebras of GK-dimension 2, the easiest examples one might think of are $U(\mathfrak{g})$, where \mathfrak{g} is a 2-dimensional Lie algebra, and the group algebra $k\mathbb{Z}^2$. While the two Hopf algebras are both pointed of the same GK-dimension, they live on two extremes: $U(\mathfrak{g})$ is connected, which means that the only group-like element is the unity 1; $k\mathbb{Z}^2$, on the other hand, is spanned by its group-like elements. In [GZ2], Brown and Zhang introduce two classes of Hopf algebras of GK-dimension 2, namely $A(b, \xi)$ and $C(b+1)$ (See Example 1.4.6 and 1.4.7), which live “between” the two extremes: they are pointed, but neither connected nor group algebras.

It turns out that the four Hopf algebras are ubiquitous in the sense that if H is a pointed Hopf algebra satisfying certain assumptions, then H must have a Hopf subalgebra isomorphic to one of the four mentioned above (Theorem 2.4.10). This result suggests that pointed Hopf algebras (at least of small GK-dimension) can be constructed from some “basic” Hopf algebras. Actually, this idea is exemplified in Chapter 3 as we build connected Hopf algebras of GK-dimension three as extensions of $U(\mathfrak{g})$ where \mathfrak{g} is a 2-dimensional Lie algebra.

Chapter 3: Properties of pointed Hopf algebras of finite Gelfand-Kirillov dimension

For a pointed Hopf algebra H , the coradical filtration $\{H_n\}_{n=0}^\infty$ is a Hopf algebra filtration in the sense that the associated graded algebra $\text{gr } H$ carries a natural Hopf algebra structure. In the first part of the chapter, we study the relation between the GK-dimension of H and that of $\text{gr } H$. A well-known result [R2, Theorem 3] states that $\text{gr } H$ has a decomposition $\text{gr } H \cong R \# kG$, where R is a certain graded subalgebra of H and G is the set of group-like elements of H . We prove that if R is finitely generated, then

$$\text{GKdim } R + \text{GKdim } kG = \text{GKdim } \text{gr } H = \text{GKdim } H.$$

For the rest of the chapter, we focus on connected Hopf algebras. The word “connected” here means that the subspace $k1$ is the only simple subcoalgebra. This very restrictive assumption forces connected Hopf algebras to have a lot of nice properties (Theorem 3.5.10 and Corollary 3.5.11). For example, the GK-dimension of a connected Hopf algebra over an algebraically closed field of characteristic 0 is either infinity or a non-negative integer. In fact, the GK-dimension of every known Hopf algebra is either infinity or a non-negative integer. But we do not know whether this is true in general.

We also classify connected Hopf algebras of GK-dimension 3. The classification gives some interesting Hopf algebras (see Theorem 3.6.8) and serves as a foothold for the study of connected Hopf algebras of higher GK-dimension, as we shall see in Chapter 6.

Chapter 4: Lower bounds of growth of Hopf algebras

A seminal result of Gromov states that a finitely generated group has polynomial growth,

or equivalently, the associated group algebra has finite Gelfand-Kirillov dimension, if and only if it has a nilpotent subgroup of finite index [Gr]. Group algebras form a special class of cocommutative Hopf algebras. It is natural to ask

Question 0.0.1. What are necessary and sufficient conditions on a finitely generated Hopf algebra H such that its Gelfand-Kirillov dimension is finite?

It is clear that an affine (i.e., finitely generated) commutative Hopf algebra has a finite GK-dimension (short for Gelfand-Kirillov dimension) which equals its Krull dimension. If H is cocommutative and the base field k is of characteristic 0, by a classification result [Mo, Corollary 5.6.4 and Theorem 5.6.5], it is isomorphic to a smash product $U(\mathfrak{g})\#kG$ for some group G and some Lie algebra \mathfrak{g} . Consequently,

$$\text{GKdim } H = \text{GKdim } kG + \dim \mathfrak{g}, \quad (\text{E0.0.1.1})$$

which solves Question 0.0.1 in terms of conditions on G and \mathfrak{g} . Question 0.0.1 is also answered for several classes of noncommutative and noncocommutative Hopf algebras, including quantum groups $U_q(\mathfrak{g})$ and $\mathcal{O}_q(G)$, see [BG2, GZ1]. This chapter attempts to study Question 0.0.1 for a larger class of noncommutative and noncocommutative Hopf algebras by providing three lower bounds of GK-dimension in terms of certain invariants of skew primitive elements.

Chapter 5: Coassociative Lie algebras

In this chapter, we introduce the notion of a coassociative Lie algebra, which is an essential ingredient in the classification of connected Hopf algebras of GK-dimension 4 in Chapter 6.

Roughly speaking, a coassociative Lie algebra L is simultaneously a Lie algebra and a coalgebra (without counit) satisfying some compatibility conditions, such that $U(L)$ has an induced bialgebra structure. However, the bialgebra $U(L)$ is not always a Hopf algebra. In fact, if $U(L)$ is a Hopf algebra then it must be a connected Hopf algebra, and this happens exactly when L is locally conilpotent (Theorem 5.3.5). Therefore, by examining locally conilpotent coassociative Lie algebras, we have a systematic way to produce connected Hopf algebras.

Chapter 6: Connected Hopf algebras of Gelfand-Kirillov dimension four

This chapter is part of the effort to understand connected Hopf algebras of finite GK-dimension. In particular, with the notion of coassociative Lie algebras developed in the previous chapter, we classify all connected Hopf algebras of GK-dimension 4. Actually, a connected Hopf algebra of GK-dimension 4 must be isomorphic to one of following.

- (a) Enveloping algebra $U(\mathfrak{g})$ over a Lie algebra \mathfrak{g} of dimension 4.

- (b) Enveloping algebra $U(L)$ over an anti-cocommutative coassociative Lie algebra L of dimension 4.
- (c) Primitively-thin Hopf algebras of GK-dimension 4.

Although the “connected” assumption leaves us with less possibilities, it is still unlikely that we can list all isomorphism classes of connected Hopf algebras. So the reason for the classification is to find some invariants that might help us to understand connected Hopf algebras in general.

Chapter 1

PRELIMINARIES

Throughout this paper, k denotes a base field. All algebras, coalgebras and tensor products are taken over k unless otherwise stated.

1.1 Gelfand-Kirillov dimension

Definition 1.1.1. The *Gelfand-Kirillov dimension* (GK-dimension for short) of a k -algebra A is

$$\text{GKdim}(A) = \limsup_V d_V(n),$$

where the supremum is taken over all finite-dimensional subspaces V of A and

$$d_V(n) = \dim_k \left(\sum_{i=0}^n V^i \right).$$

The GK-dimension can be viewed as a non-commutative analogue of the Krull dimension. In fact, for a finitely generated commutative algebra A , the GK-dimension of A is equal to its Krull dimension. For more properties of the GK-dimension, a comprehensive reference is [KL].

Example 1.1.2. Let $U(\mathfrak{g})$ be the enveloping algebra of a finite-dimensional Lie algebra \mathfrak{g} . Then

$$\text{GKdim } U(\mathfrak{g}) = \dim_k \mathfrak{g}.$$

1.2 Coalgebras

Coalgebras are structures dual to algebras. A k -algebra (with unit) is basically a k -vector space satisfying associative and unit laws, which can be expressed via the following commutative diagrams.

Definition 1.2.1. A k -algebra (with unit) is a k -vector space A together with two k -linear maps, *multiplication* $m : A \otimes A \rightarrow A$ and *unit* $u : k \rightarrow A$, such that the following diagrams are commutative:

(a) associativity

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{m} & A \otimes A \\ m \downarrow & & \downarrow m \otimes id \\ A \otimes A & \xrightarrow{id \otimes m} & A \end{array}$$

(b) unit

$$\begin{array}{ccccc}
k \otimes A & \xrightarrow{u \otimes id} & A \otimes A & \xleftarrow{id \otimes u} & A \otimes k \\
= \downarrow & & \downarrow m & & \downarrow = \\
k \otimes A & \longrightarrow & A & \longleftarrow & A \otimes k
\end{array}$$

Now we may dualize the notion of algebras by reversing all the arrows in the diagrams.

Definition 1.2.2. A k -coalgebra (with counit) is a k -vector space C together with two k -linear maps, *comultiplication* $\Delta : C \rightarrow C \otimes C$ and *counit* $\epsilon : C \rightarrow k$, such that the following diagrams are commutative:

(a) coassociativity

$$\begin{array}{ccc}
C & \xrightarrow{\Delta} & C \otimes C \\
\Delta \downarrow & & \downarrow \Delta \otimes id \\
C \otimes C & \xrightarrow{id \otimes \Delta} & C \otimes C \otimes C
\end{array}$$

(b) counit

$$\begin{array}{ccccc}
k \otimes C & \xleftarrow{1 \otimes id} & C & \xrightarrow{id \otimes 1} & C \otimes k \\
= \downarrow & & \downarrow \Delta & & \downarrow = \\
k \otimes C & \xleftarrow{\epsilon \otimes id} & C \otimes C & \xrightarrow{id \otimes \epsilon} & C \otimes k
\end{array}$$

Next we recall some basic definitions related to coalgebras. Let C be a coalgebra. An element $g \in C$ is called **group-like** if $\Delta(g) = g \otimes g$. The set of group-like elements in C is denoted by $G(C)$. An element $y \in C$ is called (g, h) -**primitive** if $g, h \in G(C)$ and $\Delta(y) = y \otimes g + h \otimes y$. We use $P_{g,h}(C)$ to denote the set of (g, h) -primitive elements in C [Mo, p.67]. If $C = H$ is a Hopf algebra (see Definition 1.4.3), we simply write $P(H)$ for $P_{1,1}(H)$, the set of primitive elements in H .

The **coradical** C_0 of a coalgebra C is defined to be the sum of all simple subcoalgebras of C . The coalgebra C is called **cosemisimple** if $C = C_0$, **pointed** if $C_0 = kG(C)$, and **connected** if C_0 is one-dimensional.

We now recall the definitions of graded coalgebras and coalgebra filtrations.

Definition 1.2.3. Let C be a coalgebra. If

- (1) $C = \bigoplus_{i=0}^{\infty} C(i)$ as k -vector spaces,
- (2) $\Delta(C(n)) \subset \bigoplus_{i=0}^n C(i) \otimes C(n-i)$ for any $n \geq 0$,
- (3) $\epsilon(C(n)) = 0$ for any $n \geq 1$,

then $C = \bigoplus_{i=0}^{\infty} C(i)$ is called a **graded coalgebra**.

Definition 1.2.4. Let C be a coalgebra and $\{F_n\}_{n \geq 0}$ a family of subcoalgebras of C . Then $\{F_n\}_{n \geq 0}$ is called a **coalgebra filtration** if

$$(1) F_n \subset F_{n+1} \text{ and } C = \bigcup_{n=0}^{\infty} F_n,$$

$$(2) \Delta F_n \subset \sum_{i=0}^n F_i \otimes F_{n-i}.$$

The associated graded coalgebra of C with respect to the filtration $\{F_n\}_{n \geq 0}$ is defined to be $\text{gr}_F C = \bigoplus_{n \geq 0} F_n/F_{n-1}$, where $F_{-1} = \{0\}$.

Starting from C_0 , one can build up a canonical coalgebra filtration $\{C_n\}$ of C , which is defined inductively by

$$C_n = \Delta^{-1}(C \otimes C_{n-1} + C_0 \otimes C).$$

This filtration is called the **coradical filtration** on C [Mo]. We simply use $\text{gr } C$ to denote the associated coalgebra of C with respect to the coradical filtration. Also, we use $\text{gr } C(n)$ to denote the n -th homogeneous component of $\text{gr } C$ (i.e. $\text{gr } C(n) = C_n/C_{n-1}$).

Definition 1.2.5. Let $C = \bigoplus_{i=0}^{\infty} C(i)$ be a graded coalgebra with coradical filtration $\{C_n\}$. Then C is called **coradically graded** if $C_0 = C(0)$ and $C_1 = C(0) \oplus C(1)$. A graded Hopf algebra H is coradically graded if it is coradically graded as a graded coalgebra.

In fact, for any coalgebra C , $\text{gr } C$ is coradically graded (see [AS2, Def 1.13]). The terminology is justified by the following lemma.

Lemma 1.2.6. [CM, Lemma 2.2] *If C is coradically graded, then*

$$C_m = \bigoplus_{i \leq m} C(i).$$

The following lemma is actually in the proof of [Mo, Theorem 5.3.1].

Lemma 1.2.7. *Let D be a coalgebra with coradical filtration $\{D_n\}$ and let I be a non-zero coideal of D . Then $I \cap D_1 \neq \{0\}$. Moreover, if D is pointed, then I contains a non-zero (g, h) -primitive element for some $g, h \in G(D)$.*

Proof. Consider the coalgebra map $\pi : D \rightarrow D/I$. Then the kernel of $\pi|_{D_1}$ is $I \cap D_1$. Suppose $I \cap D_1 = \{0\}$, then $\pi|_{D_1}$ is injective. By [Mo, Theorem 5.3.1], π is injective. But this implies $I = \{0\}$, which contradicts the choice of I .

Suppose further that D is pointed. The kernel of $\pi|_{P_{g,h}(D)}$ is $I \cap P_{g,h}(D)$. Suppose $I \cap P_{g,h}(D) = \{0\}$ for any $g, h \in G(D)$, then $\pi|_{P_{g,h}(D)}$ is injective for any $g, h \in G(D)$. By [Mo, Corollary 5.4.7], π is injective, which is a contradiction. \square

1.3 Cobar constructions

Let C be a coaugmented coalgebra in the sense that there is a coalgebra map from the trivial coalgebra k to C . (If C is a bialgebra defined in the next section, we just choose the unit $u : k \rightarrow C$ to be the coaugmentation). Let $J = C^+$, the kernel of the counit, and one defines the reduced comultiplication on J by

$$\delta(c) = \Delta(c) - (1 \otimes c + c \otimes 1),$$

where 1 is the image of $1 \in k$ under the coaugmentation. Then the **cobar construction** ΩC on C is the differential graded algebra defined as follows:

- As a graded algebra, ΩC is the tensor algebra $T(J)$,
- The differential in ΩC is given by

$$\partial_C^n = \sum_{i=0}^{n-1} (-1)^{i+1} Id^{\otimes i} \otimes \delta \otimes Id^{\otimes (n-i-1)}.$$

Dually, given an augmented algebra A , one can construct a differential graded coalgebra BA , which is called the **bar construction** of A . See [FHT, §19] for basic properties of cobar and bar constructions.

1.4 Bialgebras and Hopf algebras

Definition 1.4.1. A k -space B is a *bialgebra* if (B, m, u) is an algebra, (B, Δ, ϵ) is a coalgebra, and either of the following (equivalent) conditions holds:

- (a) Δ and ϵ are algebra morphisms,
- (b) m and u are coalgebra morphisms.

To introduce the notion of Hopf algebras, we still need the following definition.

Definition 1.4.2. Let C be a coalgebra and A an algebra. Then $\text{Hom}_k(C, A)$ becomes an algebra under the *convolution product*

$$(f * g)(c) = m \circ (f \otimes g)(\Delta c)$$

for any $f, g \in \text{Hom}_k(C, A)$ and $c \in C$. The unit element in $\text{Hom}_k(C, A)$ is $u \epsilon$.

Definition 1.4.3. Let $(H, m, u, \Delta, \epsilon)$ be a bialgebra. Then H is a *Hopf algebra* if there exists an element $S \in \text{Hom}_k(C, A)$ which is an inverse to id_H under convolution $*$. The element S is called an *antipode* for H .

Sometimes we have to consider Hopf algebras with gradings.

Definition 1.4.4. Let H be a Hopf algebra with antipode S . If

$$(1) \ H = \bigoplus_{i=0}^{\infty} H(i) \text{ is a graded algebra,}$$

$$(2) \ H = \bigoplus_{i=0}^{\infty} H(i) \text{ is a graded coalgebra,}$$

$$(3) \ S(H(n)) \subset H(n) \text{ for any } n \geq 0,$$

then $H = \bigoplus_{i=0}^{\infty} H(i)$ is called a **graded Hopf algebra**. If in addition,

$$(4) \ H = \bigoplus_{i=0}^{\infty} H(i) \text{ is a coradically graded coalgebra,}$$

then H is called a **coradically graded Hopf algebra**.

Remark 1.4.5. For a Hopf algebra H , if the coradical H_0 is a Hopf subalgebra of H , then $\text{gr } H$ is naturally a Hopf algebra [Mo, Lemma 5.2.8]. Moreover, as mentioned in [AS2, Definition 1.13], $\text{gr } H$ is a coradically graded coalgebra and therefore a coradically graded Hopf algebra.

A Hopf algebra H is called **pointed** if it is pointed as a coalgebra, i.e. $H_0 = kG(H)$. In this case, $G(H)$ is indeed a group and $\text{gr } H$ is also a pointed Hopf algebra.

We conclude this section by introducing two classes of pointed Hopf algebras defined in [GZ2, Construction 1.1 and 1.4].

Example 1.4.6. Let $A = k\langle g^{\pm 1}, y \mid gy = \xi yg \rangle$, where $\xi \in k^{\times}$. Then A has a Hopf algebra structure under which g is group-like and y is $(1, g^b)$ -primitive. We denote it by $A(b, \xi)$.

Example 1.4.7. Let $C = k\langle g^{\pm 1}, y \mid gy = yg + g^{b+1} - g \rangle$. Then C also has a Hopf algebra structure under which g is group-like and y is $(1, g^b)$ -primitive. We denote it by $C(b+1)$.

Chapter 2

EXISTENCE OF HOPF SUBALGEBRAS OF GK-DIMENSION TWO

2.1 Introduction

Throughout this chapter, we assume that the base field k is algebraically closed of characteristic 0.

The study of Hopf algebras of low Gelfand-Kirillov dimension (or GK-dimension for short) provides profound insight into basic properties of general Hopf algebras. Noetherian prime regular Hopf algebras of GK-dimension one were studied in [BZ, LWZ, Li]. Recently, in their paper [GZ2], Goodearl and Zhang classified all noetherian Hopf algebras H over an algebraically closed field k of characteristic zero which are integral domains of GK-dimension two and satisfy the condition $\text{Ext}_H^1(k, k) \neq 0$. After all the works listed above, it is natural to consider Hopf algebras of GK-dimension three. One way to understand a Hopf algebra with higher GK-dimension is to find some of its Hopf subalgebras that are more familiar to us. As a small step towards this end, we will prove in this chapter that for certain pointed Hopf algebras with GK-dimension ≥ 2 , there always exists a Hopf subalgebra GK-dimension two. We hope the result would offer some help in classifying Hopf algebras of GK-dimension three.

The key step in achieving our main result is the analysis of the free pointed Hopf algebra $F(t)$ (see Example 2.2.1 for the definition). In the classification result of [GZ2], some of the Hopf algebras discussed there are Hopf quotients of $F(t)$. When $t \neq 0$, $F(t)$ is a typical example of infinite dimensional pointed non-cocommutative Hopf algebras. On the other hand, if H is a pointed non-cocommutative Hopf algebra and $y \in H$ is a $(1, g)$ -primitive element, then the Hopf subalgebra K generated by y and $g^{\pm 1}$ is a Hopf quotient of $F(1)$. So it would be interesting to know more about the Hopf structure of $F(t)$.

In Section 3, it is shown that $F(t)$, when $t \neq 0$, is coradically graded in the sense of [CM, Lemma 2.2] with respect to the y -grading. In the same section, we also determine all skew-primitive elements of $F(t)$ when $t \neq 0$. It turns out that the coalgebra structure of $F(t)$, when $t \neq 0$, is quite different from that of $F(0)$.

By understanding the Hopf algebra structure of $F(t)$, we obtain the following theorem on the existence of certain Hopf subalgebras of pointed Hopf algebras. Recall that for any Hopf algebra H , we denote the set of group-like elements in H by $G(H)$, and the coradical filtration on H by $\{H_n\}_{n \geq 0}$. Also, given a group G , we will use the standard notation $k[G]$ to denote the group algebra of G over k .

Theorem 2.1.1 (Theorem 2.4.10). *Let H be a pointed Hopf algebra over an algebraically*

closed field k of characteristic 0. If $2 \leq \text{GKdim } H < \infty$ and H is a domain, then H contains a Hopf subalgebra of GK-dimension 2. To be precise,

- (1) If $\text{GKdim } k[G] = 0$, where $G = G(H)$, then H contains a Hopf subalgebra isomorphic to $U(\mathfrak{h})$, where \mathfrak{h} is a 2-dimensional Lie algebra.
- (2) If $\text{GKdim } k[G] \geq 2$, then H contains a Hopf subalgebra isomorphic to $k\mathbb{Z}^2$.
- (3) If $H_1 \supsetneq H_0 = k[G]$ and $\text{GKdim } k[G] \geq 1$, then H contains a Hopf subalgebra isomorphic to either $A(b, \xi)$ or $C(b+1)$ (see Example 1.4.6 and 1.4.7) for some integer b and $\xi \in k^\times$.

In fact, the characteristic zero condition is only used in part (1). Also, part (1) and (2) of Theorem 2.1.2 do not require H to be a domain. Moreover, as suggested by known examples, it is reasonable to conjecture that Theorem 2.1.2 still holds if H is weakened to a prime algebra, except that in part (3) we might have to include some other classes of Hopf algebras, besides $A(b, \xi)$ and $C(b+1)$, as possible Hopf subalgebras.

The essential ingredient for proving the main result of the chapter is the following theorem.

Theorem 2.1.2 (Corollary 2.3.5). *Let K be a pointed Hopf algebra with finite GK-dimension. If $y \in K_1 \setminus K_0$ is a $(1, g^b)$ -primitive element for some group-like element $g \neq 1$ and integer b , then there are $b_0, \dots, b_m, \beta \in k$ and $\xi \in k^\times$ such that $f = b_0y + b_1gyg^{-1} + \dots + b_mg^m yg^{-m} \in K_1 \setminus K_0$ and $gf = \xi fg + \beta g(g^b - 1)$.*

In fact, Theorem 2.1.2 can be viewed as an infinite-dimensional analogue of the following theorem of D. Ştefan.

Theorem 2.1.3. [St, Theorem 2] *Let H be a finite-dimensional pointed Hopf algebra. If H is not cosemisimple, then there exist two natural numbers m and n , with $m \neq 1$ and m dividing n , an m -th primitive root ω of unity and two elements $g, x \in H$ such that*

- (1) $gx = \omega xg$;
- (2) g is a group-like element of order n ;
- (3) $x \in P_{1,g}$ and x^m is either 0 or $g^m - 1$.

2.2 Free Pointed Hopf algebra $F(t)$

In this section we introduce the free pointed Hopf algebra $F(t)$ and study some of its basic properties.

Example 2.2.1. Let $H = k\langle x^{\pm 1}, y \rangle$ be the algebra with relations $xx^{-1} = x^{-1}x = 1$, then H becomes a Hopf algebra via

$$\Delta(x) = x \otimes x, \quad \Delta(y) = y \otimes 1 + x^t \otimes y,$$

$$S(x) = x^{-1}, \quad S(y) = -x^{-t}y,$$

$$\epsilon(x) = 1, \quad \epsilon(y) = 0,$$

where t is any given integer. We denote this Hopf algebra by $F(t)$.

Proposition 2.2.2. *The Hopf algebra $F(t)$ has a k -linear basis*

$$\{x^{i_1}yx^{i_2}y \cdots yx^{i_{n+1}} \mid (i_1, \dots, i_{n+1}) \in \mathbb{Z}^{n+1}, n \in \mathbb{N}\}.$$

Proof. This is standard from the diamond lemma. □

The Hopf algebra $F(t)$ has several gradings, which turn out to be crucial in determining its coradical filtration.

Proposition 2.2.3. *There is a natural y -grading on $F(t)$ under which $F(t)$ becomes a graded coalgebra.*

Proof. The natural y -grading is given by setting $\deg_y y = 1$ and $\deg_y x^{\pm 1} = 0$. It is easy to check that $F(t)$ becomes a graded algebra by this setting, since the relations all preserve the grading. This y -grading on $F(t)$ induces a natural y -grading on $F(t) \otimes F(t)$ by defining $\deg_y(f_1 \otimes f_2) = \deg_y f_1 + \deg_y f_2$, where $f_1, f_2 \in F(t)$. Also notice that this grading on $F(t) \otimes F(t)$ is an algebra grading.

By definition of a graded coalgebra, we only have to show that if $h \in F(t)$ is homogeneous of y -degree n , then so is $\Delta(h)$. By Proposition 3.2.2, we may assume that h is of the form $x^{i_1}yx^{i_2}y \cdots yx^{i_{n+1}}$, where $(i_1, \dots, i_{n+1}) \in \mathbb{Z}^{n+1}$. Now

$$\Delta(h) = \Delta(x^{i_1})\Delta(y) \cdots \Delta(y)\Delta(x^{i_{n+1}}).$$

Notice that $\Delta(x^{i_j})$ and $\Delta(y)$ are homogeneous of y -degrees 0 and 1, respectively. The result then follows from the fact that $F(t) \otimes F(t)$ is a graded algebra with respect to the y -grading. This completes the proof. □

By analyzing the grading, we are actually able to determine the coradical of $F(t)$. Notice that by construction, the degree zero part of $F(t)$ with respect to the y -grading is just $k[x^{\pm 1}]$, the subalgebra of $F(t)$ generated by $x^{\pm 1}$.

Corollary 2.2.4. *Let $H = F(t)$. Then $H_0 = k[x^{\pm 1}]$. As a consequence, $F(t)$ is a pointed Hopf algebra.*

Proof. By the previous proposition, H is a graded coalgebra with respect to the y -grading. The coalgebra grading induces a coalgebra filtration $\{A_n\}_{n \geq 0}$ on H with $A_0 = k[x^{\pm 1}]$. By [Mo, Lemma 5.3.4], $A_0 \supset H_0$. On the other hand, A_0 is spanned by group-like elements and thus $k[x^{\pm 1}] = A_0 = H_0$. \square

There is also a natural x -grading on $F(t)$ given by setting $\deg_x x^{\pm 1} = \pm 1$ and $\deg_x y = 0$.

Remark 2.2.5. Note that $F(0)$ is a cocommutative Hopf algebra. In this case, the Hopf structure is well understood. In fact, by [Mo, Corollary 5.6.4, Theorem 5.6.5], $F(0) \cong U(\mathfrak{g}) \# k[x^{\pm 1}]$, where \mathfrak{g} is the set of primitive elements in $F(0)$. Actually, \mathfrak{g} is the free Lie algebra generated by $\{x^n y x^{-n} \mid n \in \mathbb{Z}\}$. It is easy to see that $F(0)$ is coradically graded by setting $\deg x^{\pm 1} = 0$ and $\deg f = 1$ for any $f \in \mathfrak{g}$. Moreover, it is not hard to prove that every non-zero Hopf ideal I of $F(0)$ is generated by elements in \mathfrak{g} and possibly an element of the form $x^n - 1$ for some positive integer n . So in next section, we will focus on the case $t \neq 0$.

2.3 The coradical filtration of $F(t)$ when $t \neq 0$

In the section, we study the coradical filtration of the free Hopf algebra $F(t)$ described in Example 2.2.1 with $t \neq 0$. We can further assume that $t > 0$, since it is obvious that $F(t) \cong F(-t)$ as Hopf algebras for any integer t . We will show that H is a coradically graded Hopf algebra with respect to the y -grading. This is quite different from the $t = 0$ case. In fact, if $t = 0$, then $P(F(0))$ is the free Lie algebra generated by $\{x^n y x^{-n} \mid n \in \mathbb{Z}\}$, which implies that there are primitive elements of any y -degree in $F(0)$.

Throughout this section, let H be the Hopf algebra $F(t)$ with $t \neq 0$. The following proposition is crucial in proving that H is coradically graded with respect to the y -grading. For the rest of this section, we use $H(n)$ to denote the k -linear subspace spanned by homogeneous elements of y -degree n .

Lemma 2.3.1. *Any skew-primitive element in H has y -degree ≤ 1 .*

Proof. Suppose to the contrary that f is (x^a, x^c) -primitive element of y -degree $n \geq 2$. Replacing f with $x^{-a}f$, we can assume that f is $(1, x^b)$ -primitive for some integer b , that is to say,

$$\Delta(f) = f \otimes 1 + x^b \otimes f. \quad (\text{E2.3.1.1})$$

It is also clear from Proposition 2.2.3 that we can be further assume f is homogeneous with y -degree n .

For any $\alpha = (i_1, \dots, i_{n+1}) \in \mathbb{Z}^{n+1}$, denote the monomial $x^{i_1} y x^{i_2} y \cdots y x^{i_{n+1}}$ by M_α and $\sum_{s=1}^n i_s$ by $T(\alpha)$. Then we can write

$$f = \sum_{\alpha \in \mathcal{I}} \mu_\alpha M_\alpha,$$

where \mathcal{I} is a non-empty finite subset of \mathbb{Z}^{n+1} and $\mu_\alpha \in k^\times$ for any $\alpha \in \mathcal{I}$. A direct computation shows that

$$\Delta(M_\alpha) = M_\alpha \otimes x^{T(\alpha)} + x^{T(\alpha)+nt} \otimes M_\alpha + w_\alpha,$$

where $w_\alpha \in \sum_{\substack{i,j < n \\ i+j=n}} H(i) \otimes H(j)$. Hence

$$\Delta(f) = \sum_{\alpha \in \mathcal{I}} \mu_\alpha (M_\alpha \otimes x^{T(\alpha)} + x^{T(\alpha)+nt} \otimes M_\alpha) + w, \quad (\text{E2.3.1.2})$$

where $w \in \sum_{\substack{i,j < n \\ i+j=n}} H(i) \otimes H(j)$. By comparing (E2.3.1.1) and (E2.3.1.2) we see that $T(\alpha) = 0$ for any $\alpha \in \mathcal{I}$, and $b = nt$.

Define two bijective maps σ and τ from \mathbb{Z}^{n+1} to \mathbb{Z}^{n+1} by

$$\sigma(i_1, \dots, i_{n+1}) = (-i_{n+1} - t, -i_n - t, \dots, -i_2 - t, -i_1),$$

$$\tau(i_1, \dots, i_{n+1}) = (i_1 - b, i_2, \dots, i_{n+1}).$$

Since the antipode S is an anti-algebra map, it is easy to check that $S(f) = \sum_{\alpha \in \mathcal{I}} (-1)^n \mu_\alpha M_{\sigma\alpha}$.

On the other hand, since S is an inverse to Id_H under convolution, we have $S(f) = -x^{-b}f$, which implies that $S(f) = -\sum_{\alpha \in \mathcal{I}} \mu_\alpha M_{\tau\alpha}$. It then follows that

$$\sum_{\alpha \in \mathcal{I}} ((-1)^n \mu_\alpha M_{\sigma\alpha} + M_{\tau\alpha}) = 0. \quad (\text{E2.3.1.3})$$

Choose $\beta = (\ell_1, \dots, \ell_{n+1}) \in \mathcal{I}$. From (E2.3.1.3) we see that there is $\beta' \in \mathcal{I}$ such that $\tau\beta' = \sigma\beta$, i.e. $(\tau^{-1}\sigma)\beta \in \mathcal{I}$. By induction, the set $\mathcal{A} := \{(\tau^{-1}\sigma)^i\beta \mid i \geq 0\}$ is a subset of \mathcal{I} . But it is easy to show that

$$(\tau^{-1}\sigma)^{2s}\beta = (\ell_1 + s(n-1)t, \ell_2, \dots, \ell_n, \ell_{n+1} - s(n-1)t)$$

for any $s \geq 0$. Since $(n-1)t > 0$, by comparing the first entry, it is clear that $(\tau^{-1}\sigma)^{2s}\beta \neq (\tau^{-1}\sigma)^{2s'}\beta$ whenever $s \neq s'$. Hence the set \mathcal{A} is infinite, which is a contradiction. This completes the proof. \square

Proposition 2.3.2. *H is a coradically graded Hopf algebra with respect to the y -grading.*

Proof. By Corollary 2.2.4, $H_0 = H(0) = k[x^{\pm 1}]$. Also it is easy to check that $H_1 \supset H(0) \oplus H(1)$. By [Mo, Theorem 5.4.1], H_1 is spanned by group-like and skew-primitive elements. By Lemma 2.3.1, any skew-primitive element in H has y -degree ≤ 1 . Hence $H_1 \subset H(0) \oplus H(1)$, which completes the proof. \square

With this proposition, we can give a full description of all skew-primitive elements in H . For any $h \in H$, we define $\deg h$ to be the pair $(\deg_x h, \deg_y h)$. Notice that this $(\mathbb{Z} \times \mathbb{N})$ -grading on H induces a $(\mathbb{Z} \times \mathbb{N} \times \mathbb{Z} \times \mathbb{N})$ -grading on $H \otimes H$.

Proposition 2.3.3. *Let h be a skew-primitive element in H . Then $\deg_y h \leq 1$. If $\deg_y h = 0$, i.e. h is a polynomial in $x^{\pm 1}$, then h is of the form $\lambda x^a(x^m - 1)$, where $\lambda \in k^\times$, $a \in \mathbb{Z}$ and m is a positive integer. If $\deg_y h = 1$, then h is of the form $f + \lambda x^a(x^t - 1)$, where $\lambda \in k$ and f is homogeneous with $\deg f = (a, 1)$.*

Proof. By the definition of coradical filtration, $h \in H_1$. By Proposition 2.3.2, $H_1 = H(0) \oplus H(1)$. So $\deg_y h \leq 1$.

If $\deg_y h = 0$, then h is a polynomial in $x^{\pm 1}$. Let $h = \sum \alpha_i x^i$ and suppose that h is an (x^a, x^b) -primitive for some $a, b \in \mathbb{Z}$. Then

$$\Delta(h) = \Delta\left(\sum \alpha_i x^i\right) = \sum \alpha_i x^i \otimes x^i. \quad (\text{E2.3.3.1})$$

On the other hand, since h is (x^a, x^b) -primitive, it yields

$$\Delta(h) = h \otimes x^a + x^b \otimes h. \quad (\text{E2.3.3.2})$$

By comparing (E2.3.3.1) and (E2.3.3.2), we see that $\alpha_i = 0$ if $i \neq a$ and $i \neq b$. Therefore, $h = \alpha_a x^a + \alpha_b x^b$ and

$$\begin{aligned} \Delta(h) &= \Delta(\alpha_a x^a + \alpha_b x^b) & (\text{E2.3.3.3}) \\ &= (\alpha_a x^a + \alpha_b x^b) \otimes x^a + x^b \otimes (\alpha_a x^a + \alpha_b x^b) - \alpha_b x^b \otimes x^a - \alpha_a x^b \otimes x^a \\ &= h \otimes x^a + x^b \otimes h - (\alpha_a + \alpha_b) x^b \otimes x^a. \end{aligned}$$

It then follows that $\alpha_a = -\alpha_b \neq 0$. Hence h has the claimed form.

If $\deg_y h = 1$, we can write $h(x, y) = f_1 + \cdots + f_s + g$, where f_i is homogeneous with $\deg f_i = (a_i, 1)$, and g is a polynomial in $x^{\pm 1}$. First, we need to show $s = 1$. If not, we can rearrange f_i so that $a_1 > a_2 > \cdots > a_s$. Suppose h is an (x^m, x^n) -primitive element, that is,

$$\begin{aligned} \Delta(h) &= h \otimes x^m + x^n \otimes h & (\text{E2.3.3.4}) \\ &= f_1 \otimes x^m + x^n \otimes f_1 + \cdots + f_s \otimes x^m + x^n \otimes f_s + \cdots \end{aligned}$$

On the other hand, it is easy to check that f_i is an (x^{a_i}, x^{a_i+t}) -primitive element, so

$$\Delta(h) = f_1 \otimes x^{a_1} + x^{a_1+t} \otimes f_1 + \cdots + f_s \otimes x^{a_s} + x^{a_s+t} \otimes f_s + \cdots \quad (\text{E2.3.3.5})$$

Since $s > 1$, there is some j such that $a_j \neq m$. From (E2.3.3.5), we see that $f_j \otimes x^{a_j}$ is the non-zero component of $\Delta(h)$ in degree $(a_j, 1; a_j, 0)$. But (E2.3.3.4) implies that $\Delta(h)$ has no component in degree $(a_j, 1; a_j, 0)$. This is a contradiction.

Now we can write $h = f + g$, where f is homogeneous with $\deg f = (a, 1)$. If $g = 0$, we are done. So assume $g \neq 0$ and it suffices to show that g is of the form $\lambda x^a(x^t - 1)$ for some $\lambda \in k^\times$. Note that f is a (x^a, x^{a+t}) -primitive element, so similar to the argument above, h must be a (x^a, x^{a+t}) -primitive element. Hence g is (x^a, x^{a+t}) -primitive. Then it follows from the previous case where $\deg_y h = 0$ that g is of the form $\lambda x^a(x^t - 1)$, as desired. \square

Corollary 2.3.4. *Suppose that I is a non-zero Hopf ideal of H . Then I contains an element f of the form $x^m - 1$ for some integer m or $a_0y + a_1xyx^{-1} + \cdots + a_nx^nyx^{-n} + \lambda(x^t - 1)$, where $a_0, \dots, a_n, \lambda \in k$ and $a_0, a_n \neq 0$.*

Proof. Just combine Lemma 1.2.7 and Proposition 2.3.3. \square

Corollary 2.3.5. *Let K be a pointed Hopf algebra with finite GK-dimension. If $y \in K_1 \setminus K_0$ is a $(1, g^b)$ -primitive element for some group-like element $g \neq 1$ and integer b , then there are $b_0, \dots, b_m, \beta \in k$ and $\xi \in k^\times$ such that $f = b_0y + b_1gyg^{-1} + \cdots + b_mg^myg^{-m} \in K_1 \setminus K_0$ and $gf = \xi fg + \beta g(g^b - 1)$.*

Proof. First we claim that there are $a_0, \dots, a_n, \lambda \in k$ with $a_0 \neq 0, n \geq 1$ such that $a_0y + a_1gyg^{-1} + \cdots + a_ny^n + \lambda(g^b - 1) = 0$. If y is primitive, i.e. $g^b = 1$, then the k -linear space V spanned by $\{g^i yg^{-i} \mid i \in \mathbb{Z}\}$ is a subspace of \mathfrak{g} , the k -linear space spanned by all primitive elements in K . By the universal property of $U(\mathfrak{g})$, the inclusion $\mathfrak{g} \hookrightarrow K$ extends to an algebra map $\tau : U(\mathfrak{g}) \rightarrow K$. A direct calculation by using the PBW basis of $U(\mathfrak{g})$ shows that τ is also a coalgebra map. By [Mo, Theorem 5.3.1, Proposition 5.5.3], τ is injective. It then follows from [KL, Lemma 3.1] that $\text{GKdim } U(\mathfrak{g}) \leq \text{GKdim } K < \infty$. Now we have $\dim_k V \leq \dim_k \mathfrak{g} = \text{GKdim } U(\mathfrak{g}) < \infty$. Hence $a_0y + a_1gyg^{-1} + \cdots + a_ny^n + \lambda(g^b - 1) = 0$, for some $a_0, \dots, a_n \in k$ with $a_0 \neq 0$ and $n > 0$. If g is torsion with the order $n \geq 2$, then $y - g^nyg^{-n} = 0$. If g is torsion-free and $b \neq 0$, the Hopf subalgebra generated by y and $g^{\pm 1}$ is a Hopf quotient of $F(b)$ where $b \neq 0$. Now the statement follows from Corollary 2.3.4.

Choose n to be minimal. Since $y \in K_1 \setminus K_0$, we have $n \geq 1$. Let ξ be a root of the polynomial $a_0 + a_1s + \cdots + a_ns^n$, and set $b_0 + b_1s + \cdots + b_{n-1}s^{n-1} = (a_0 + a_1s + \cdots + a_ns^n)/(s - \xi)$. Let $f = b_0y + b_1gyg^{-1} + \cdots + b_{n-1}g^{n-1}yg^{-(n-1)}$. If $f \in K_0$, then $f \in P_{1,g^b}(K) \cap K_0 = k(g^b - 1)$. The equality $P_{1,g^b}(K) \cap K_0 = k(g^b - 1)$ can be verified by straight calculation since K_0 is spanned by group-like elements. Hence $f = \gamma(g^b - 1)$ for some $\gamma \in k$, which contradicts the minimality of n . As a consequence, $f \in K_1 \setminus K_0$. It is easy to check that $gfg^{-1} - \xi f + \beta(g^b - 1) = 0$ for some $\beta \in k$. This completes the proof. \square

2.4 Main Theorem

This section is devoted to the proof of the main theorem (Theorem 2.4.10). We also explain why the theorem fails if some of the conditions are weakened.

Lemma 2.4.1. *Let H be a noetherian prime Hopf algebra of GK-dimension 2. Suppose that $\pi : H \rightarrow K$ is a Hopf algebra projection to a domain K . If K is neither cosemisimple nor connected, then π is an isomorphism.*

Proof. If π is not an isomorphism, then $I := \ker \pi \neq 0$. By Goldie's Theorem, the ideal I contains a regular element. By [KL, Proposition 3.15], $\text{GKdim } K \leq \text{GKdim } H - 1 = 1$. Since there is no algebra with GK-dimension strictly between 0 and 1, $\text{GKdim } K = 0$ or 1.

By assumption, K is not finite-dimensional, so $\text{GKdim } K = 1$. By [GZ2, Lemma 4.5], K is commutative. Hence K is a commutative Hopf domain with $\text{GKdim } K = 1$. But by [GZ2, Proposition 2.1], K is either a group algebra or an enveloping algebra. Hence K is either cosemisimple or connected, which contradicts the assumption. \square

The next three lemmas are known. However, I am not able to locate them in literature and thus the proofs are included.

Lemma 2.4.2. *Let G be a finitely generated group which acts on a finitely generated k -algebra A . Suppose that there is a finite-dimensional generating k -subspace W of A such that W contains 1 and is stable under the action of G . Then*

$$\text{GKdim } A * G = \text{GKdim } A + \text{GKdim } k[G],$$

where $A * G$ is the skew group algebra.

Proof. It suffices to assume that both $\text{GKdim } A$ and $\text{GKdim } k[G]$ are finite. By [KL, Theorem 11.1], G has a nilpotent normal subgroup N with finite index. By Schreier's lemma, N is finitely generated. By [KL, Proposition 5.5], $\text{GKdim } A * G = \text{GKdim } A * N$ and $\text{GKdim } k[G] = \text{GKdim } k[N]$. Hence by replacing G with N , we can assume G is finitely generated nilpotent.

Suppose that the group G is generated by the set $S = \{g_1, g_2, \dots, g_m\}$. Let $g_S(n)$ be the number of distinct group elements that can be expressed as words of length $\leq n$ in the specified generators and their inverses. Without loss of generality, we can assume that S is closed under inversion. By the definition of GK-dimension and that of group algebras, $\text{GKdim } k[G] = \limsup_{n \rightarrow \infty} \log_n g_S(n)$. Moreover, if $\text{GKdim } k[G] < \infty$, then $\text{GKdim } k[G] = \lim_{n \rightarrow \infty} \log_n g_S(n)$ (see [KL, Theorem 11.14]). Let $V = W + Wg_1 + \dots + Wg_m$. It is clear that V is finite-dimensional and generates $A * G$.

First, we prove that $\text{GKdim } A * G \geq \text{GKdim } A + \text{GKdim } k[G]$. Notice that

$$\sum_i W^n f_i \subset (W + Wg_1 + \dots + Wg_m)^{2n} = V^{2n},$$

where f_i runs through all words of length $\leq n$. So $g_S(n) \cdot \dim W^n \leq \dim V^{2n}$. It then follows that

$$\begin{aligned} \text{GKdim } A + \text{GKdim } k[G] &= \limsup_{n \rightarrow \infty} \log_n W^n + \lim_{n \rightarrow \infty} \log_n g_S(n) \\ &= \limsup_{n \rightarrow \infty} \log_n (g_S(n) \cdot \dim W^n) \\ &\leq \limsup_{n \rightarrow \infty} \log_n \dim V^{2n} \leq \text{GKdim } A * G. \end{aligned}$$

For the other direction, notice that

$$(W + Wg_1 + \dots + Wg_m)^n \subset \sum_i W^n f_i,$$

where f_i runs through all words of length $\leq n$. Here we use the fact that $g_j W = W g_j$ for any j . As a consequence, $\dim V^n \leq g_S(n) \cdot \dim W^n$. Therefore,

$$\begin{aligned} \text{GKdim } A * G &= \limsup_{n \rightarrow \infty} \log_n \dim V^n \leq \limsup_{n \rightarrow \infty} \log_n (g_S(n) \cdot \dim W^n) \\ &\leq \limsup_{n \rightarrow \infty} \log_n W^n + \lim_{n \rightarrow \infty} \log_n g_S(n) \\ &= \text{GKdim } A + \text{GKdim } k[G]. \end{aligned}$$

This completes the proof. \square

Lemma 2.4.3. *Let G be a group and N a finite normal subgroup of G . Then*

$$\text{GKdim } k[G] = \text{GKdim } k[G/N].$$

Proof. Notice that there is a surjective algebra map $k[G] \rightarrow k[G/N]$. By [KL, Lemma 3.1], $\text{GKdim } k[G] \geq \text{GKdim } k[G/N]$. Now we prove the other direction. Denote G/N by \overline{G} and $k[N]$ by A . By [Mo, Example 7.1.6],

$$k[G] \cong A \#_{\sigma} k[\overline{G}],$$

where $\sigma : k[\overline{G}] \otimes k[\overline{G}] \rightarrow A$ is a cocycle. The multiplication on $A \#_{\sigma} k[\overline{G}]$ is given by

$$(a \# g)(b \# h) = a(g.b)\sigma(g, h) \# gh, \quad (\text{E2.4.3.1})$$

where $a, b \in A$ and $g, h \in \overline{G}$. Here $g.b$ is the image of $g \otimes b$ under a certain k -linear map $k[\overline{G}] \otimes A \rightarrow A$ (see [Mo, §7.1] for details). For $a \in A$ and $g \in \overline{G}$, we can identify them in $A \#_{\sigma} k[\overline{G}]$ with $a \# 1$ and $1 \# g$, respectively.

Let V be any finite-dimensional subspace of $A \#_{\sigma} k[\overline{G}]$. Then it is easy to see that $V \subset A + Ag_1 + \cdots + Ag_m$ for some $g_1, \dots, g_m \in \overline{G}$. Denote the set $\{g_1, g_2, \dots, g_m\}$ by S and let $g_S(n)$ be as defined in the previous lemma. Notice that $g_i A \subset Ag_i$ by (E2.4.3.1). As a consequence,

$$V^n \subset (A + Ag_1 + \cdots + Ag_m)^n \subset \sum_i A f_i,$$

where f_i runs through all words of length $\leq n$ in \overline{G} . Therefore,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \log_n \dim V^n &\leq \limsup_{n \rightarrow \infty} \log_n (g_S(n) \cdot \dim A) \\ &= \limsup_{n \rightarrow \infty} \log_n g_S(n) \\ &\leq \text{GKdim } k[\overline{G}]. \end{aligned}$$

Hence $\text{GKdim } k[G] \leq \text{GKdim } k[\overline{G}]$, as desired. \square

It seems that Lemma 2.4.3 is a corollary to Lemma 2.4.2. However, there is some slight difference since we need to take the cocycle σ into consideration. In fact, as we can see in the proof, the cocycle does not contribute to the GK-dimension because N is finite. One might ask whether $\text{GKdim } k[G] = \text{GKdim } k[N] + \text{GKdim } k[G/N]$ if N is not finite. The answer is negative (see Example 2.4.14).

Lemma 2.4.4. *Let G be a totally ordered group acting on a prime ring A . Then the skew group ring $B = A * G$ is also prime.*

Proof. For any non-zero elements $r, s \in B$, we need to show $rBs \neq 0$. Note that B is G -graded. Let ag and cf be the leading terms of r and s respectively, where $a, c \in A$ and $g, f \in G$. Since A is prime, there exists $b \in A$ such that $abc \neq 0$. Then

$$(ag)(b^{g^{-1}}g^{-1})(cf) = abc \neq 0.$$

Hence $r(b^{g^{-1}}g^{-1})s \neq 0$, which completes the proof. \square

Now we are ready to prove the main theorem of this chapter. We start with the case when H is pointed and cosemisimple, i.e. $H \cong k[G]$ for some group G . Before proving the proposition, we list some well-known facts about groups below.

For a group G , we have the upper central series

$$\{1\} = Z_0(G) \subset Z_1(G) \subset Z_2(G) \cdots \subset Z_i(G) \subset \cdots,$$

where $Z_{i+1}(G)/Z_i(G)$ is the center of $G/Z_i(G)$. If $Z_{r-1}(G) \subsetneq Z_r(G) = G$ for some r , then G is called a nilpotent group of nilpotency class r .

Lemma 2.4.5. *Let G be a nilpotent group. Then the following statements are true.*

- (1) *Every subgroup of G is nilpotent.*
- (2) *The elements of finite order form a normal subgroup T such that G/T is torsion-free.*
- (3) *If G is finitely generated, then every subgroup of G is also finitely generated.*
- (4) *The group G is finite if it is generated by a finite number of elements each having finite order.*
- (5) *If G is torsion-free, then $G/Z_i(G)$ is torsion-free for each $i \geq 0$.*

Proof. The first statement is easy. Statements (2), (3), (4) and (5) are [Ma, Theorem 9.16], [Ma, Corollary 9.18], [Ma, Theorem 9.17] and [E, Lemma 8.2.3] respectively. \square

Proposition 2.4.6. *Let G be a group. If $2 \leq \text{GKdim } k[G] < \infty$, then G has a subgroup isomorphic to \mathbb{Z}^2 .*

Proof. By the definition of GK-dimension,

$$\text{GKdim } k[G] = \sup_B \text{GKdim } B,$$

where B runs through all finitely generated subalgebra of $k[G]$. It is easy to see that every finitely generated subalgebra B of $k[G]$ is contained in some $k[L]$ where L is a finitely generated subgroup of G . Hence

$$\text{GKdim } k[G] = \sup_L \text{GKdim } k[L],$$

where L runs through all finitely generated subgroups of G . By [KL, Theorem 11.1, 11.14], $\text{GKdim } k[L]$ must be an integer. So there is a finitely generated subgroup N of G such that $\text{GKdim } k[N] = \text{GKdim } k[G]$. Now by [KL, Theorem 11.1], N is nilpotent by finite, which means that N has a nilpotent subgroup P with $[N : P] < \infty$. Then by [KL, Proposition 5.5], $\text{GKdim } k[P] = \text{GKdim } k[N]$. Also, by the same argument above, we can assume that P is finitely generated. Therefore by replacing G with P , we can assume that G is finitely generated nilpotent.

First, we deal with the case that G is torsion-free. If the nilpotency class $r = 1$, i.e. G is abelian, then G must be a torsion-free abelian group of rank ≥ 2 . Let N be a subgroup of G with rank 2, then $N \cong \mathbb{Z}^2$. If G has nilpotency class $r \geq 2$. Choose $h \in G \setminus Z_1(G)$. By Lemma 2.4.5 part (5), $\langle h \rangle \cap Z_1(G) = \{1\}$. Let $g \in Z_1(G) \setminus \{1\}$. Then by the choice of h , the subgroup N generated by g and h is isomorphic to \mathbb{Z}^2 .

For the general case, let T be the set of torsion elements in G . By Lemma 2.4.5, T is a finite normal subgroup of G and G/T is torsion-free. By Lemma 2.4.3, we see that $\text{GKdim } k[G/T] = \text{GKdim } k[G]$. Now by the torsion-free case, there is a subgroup B of G such that $B/T \cong \mathbb{Z}^2$. By [E, Lemma 8.2.4], B has an abelian subgroup N such that $[B : N] < \infty$. Then $\text{GKdim } k[N] = \text{GKdim } k[B] = \text{GKdim } k[B/T] = 2$, by [KL, Proposition 5.5] and Lemma 2.4.2. Hence N is an abelian group of rank 2, which means that it has a subgroup isomorphic to \mathbb{Z}^2 . This completes the proof. \square

Next we will deal with the case when H is a connected Hopf algebra. Before that, we need a lemma on the existence of 2-dimensional Lie subalgebra of a finite-dimensional Lie algebra.

Lemma 2.4.7. *Let \mathfrak{g} be a finite-dimensional Lie algebra. If $\dim_k \mathfrak{g} \geq 2$, then \mathfrak{g} has a Lie subalgebra of dimension 2.*

Proof. If \mathfrak{g} is nilpotent, by [H, 3.2, Proposition], the center of \mathfrak{g} is non-zero. Choose a non-zero element x in the center and another element y which is not a scalar multiple of x . Then the subspace of \mathfrak{g} spanned by x and y is a 2-dimensional Lie subalgebra.

If \mathfrak{g} is not nilpotent, by Engel's Theorem, there exists an element x which is not ad-nilpotent. Hence the linear map $\text{ad } x$ has a non-zero eigenvalue λ with eigenvector y . Then the k -linear span of x and y is the desired Lie subalgebra. \square

Proposition 2.4.8. *Let H be a connected Hopf algebra. If $2 \leq \text{GKdim } H < \infty$, then H contains a Hopf subalgebra isomorphic to $U(\mathfrak{h})$, where \mathfrak{h} is a 2-dimensional Lie algebra.*

Proof. Let K be the Hopf subalgebra generated by $P(H)$, the set of primitive elements in H . By [Mo, Theorem 5.6.5], $K \cong U(\mathfrak{g})$, where $\mathfrak{g} = P(H)$. First we claim that $\dim_k \mathfrak{g} \geq 2$.

If $\dim_k \mathfrak{g} = 1$, then $K \cong k[y]$, the polynomial ring of one-variable, as Hopf algebras. Obviously, $K \subsetneq H$ since they have different GK-dimensions. Let m be the smallest number such that $K_m \subsetneq H_m$. By the construction of K , we have $m \geq 2$. Choose $z \in H_m \setminus K_m$. By [Mo, Lemma 5.3.2] and its proof, we can assume that

$$\Delta(z) = 1 \otimes z + z \otimes 1 + w,$$

where $w \in \sum_{i=1}^{m-1} H_{m-i} \otimes H_i = \sum_{i=1}^{m-1} K_{m-i} \otimes K_i$. Moreover, by replacing z with $z - \epsilon(z) \cdot 1$, we can assume $\epsilon(z) = 0$. Notice that $K \cong k[y]$ is coradically graded with the usual grading and therefore K_n is spanned by all polynomials in y of degree $\leq n$. Also, the grading on K induces a grading on $K^{\otimes \ell}$ by setting $\deg f_1 \otimes f_2 \otimes \cdots \otimes f_\ell = \deg f_1 + \cdots + \deg f_\ell$. Hence we can write

$$w = \sum_{i+j \leq m} \alpha_{ij} y^i \otimes y^j.$$

By the counit axiom,

$$z = (\text{Id} \otimes \epsilon) \Delta(z) = z + \sum_{i \leq m} \alpha_{i0} y^i,$$

$$z = (\epsilon \otimes \text{Id}) \Delta(z) = z + \sum_{j \leq m} \alpha_{0j} y^j.$$

Hence $\alpha_{i0} = 0$ and $\alpha_{0j} = 0$ for any $i, j \leq m$.

If $m = 2$, then

$$\Delta(z) = 1 \otimes z + z \otimes 1 + \alpha_{11} y \otimes y,$$

Let $z' = z - \frac{\alpha_{11}}{2} y^2$. Then $z' \in H_2 \setminus K_2$ by the choice of z . But a direct calculation shows that

$$\Delta(z') = 1 \otimes z' + z' \otimes 1.$$

So $z' \in H_1 = K_1$, which is a contradiction.

Now assume that $m \geq 3$. As defined in Section 1.3, we have

$$\partial^1 : H \rightarrow H \otimes H \quad \text{and} \quad \partial^2 : H \otimes H \rightarrow H \otimes H \otimes H$$

by $\partial^1(c) = 1 \otimes c - \Delta(c) + c \otimes 1$ and $\partial^2(c \otimes d) = 1 \otimes c \otimes d - \Delta(c) \otimes d + c \otimes \Delta(d) - c \otimes d \otimes 1$ for any $c, d \in H$. Since ΩH is a complex, $\partial^2 \circ \partial^1 = 0$. Hence $\partial^2(w) = 0$.

Let $u = \sum_{i+j=m} \alpha_{ij} y^i \otimes y^j$. Notice that the map ∂^2 , when restricted from $K \otimes K$ to $K \otimes K \otimes K$, preserves the grading. Therefore $\partial^2(u) = 0$. Also, by replacing z with $z - \beta y^m$,

where $\beta = \alpha_{m-1,1}/m$, we can assume that $\alpha_{m-1,1} = 0$. Now,

$$\begin{aligned}
\partial^2(u) &= \partial^2\left(\sum_{i=2}^{m-1} \alpha_{m-i,i} y^{m-i} \otimes y^i\right) \\
&= 1 \otimes \sum_{i=2}^{m-1} \alpha_{m-i,i} y^{m-i} \otimes y^i - \sum_{i=2}^{m-1} \alpha_{m-i,i} \Delta(y^{m-i}) \otimes y^i \\
&\quad + \sum_{i=2}^{m-1} \alpha_{m-i,i} y^{m-i} \otimes \Delta(y^i) - \sum_{i=2}^{m-1} \alpha_{m-i,i} y^{m-i} \otimes y^i \otimes 1 \\
&= - \sum_{i=2}^{m-1} \sum_{j=0}^{m-i-1} \alpha_{m-i,i} \binom{m-i}{j} y^{m-i-j} \otimes y^j \otimes y^i + \sum_{i=2}^{m-1} \sum_{j=1}^i \alpha_{m-i,i} \binom{i}{j} y^{m-i} \otimes y^{i-j} \otimes y^j.
\end{aligned}$$

Since $\partial^2(u) = 0$, by comparing the coefficients of the terms $y^{m-i} \otimes y^{i-1} \otimes y$, we see that $\alpha_{m-i,i} = 0$ for $i = 2, \dots, m-1$. It then follows that

$$\Delta(z) = 1 \otimes z + z \otimes 1 + \sum_{\substack{i,j \geq 1 \\ i+j \leq m-1}} \alpha_{ij} y^i \otimes y^j.$$

But this implies that $\Delta(z) \in H_0 \otimes H + H \otimes H_{m-2}$, i.e. $z \in H_{m-1} = K_{m-1}$, which is a contradiction.

Now $K \cong U(\mathfrak{g})$ where \mathfrak{g} is a finite-dimensional Lie algebra and $\dim_k \mathfrak{g} \geq 2$. By Lemma 2.4.7, \mathfrak{g} has a 2-dimensional Lie subalgebra \mathfrak{h} . Then $U(\mathfrak{h})$ is a Hopf subalgebra of $K \cong U(\mathfrak{g})$, and thus a Hopf subalgebra of H . This completes the proof. \square

Finally, we have to handle the case when H is neither cosemisimple nor connected.

Lemma 2.4.9. *Let H be a pointed Hopf algebra with finite GK-dimension. If H is a domain and is not connected, not cosemisimple, then H contains a Hopf subalgebra isomorphic to either $A(b, \xi)$ or $C(b+1)$ for some integer b and $\xi \in k^\times$.*

Proof. Since H is not cosemisimple, by [Mo, 5.4.1], we can find $g, y \in H$ such that g is group-like and $y \notin H_0$ is $(1, g^b)$ -primitive for some integer b . Notice that g is torsion-free since H is a domain. Moreover, we can choose $g \neq 1$ since H is not connected. By Corollary 2.3.5, there are $b_0, \dots, b_m, \beta \in k$ and $\xi \in k^\times$ such that $f = b_0 y + b_1 g y g^{-1} + \dots + b_m g^m y g^{-m} \in H_1 \setminus H_0$ and $g f = \xi f g + \beta g(g^b - 1)$.

Let K be the Hopf subalgebra generated by f and $g^{\pm 1}$. By construction, K is neither cosemisimple nor connected. If $\xi \neq 1$, by replacing f with $f' = f - \frac{\beta}{1-\xi}(g^b - 1)$, we have $g f' = \xi f' g$. Hence K is a Hopf quotient of $A(b, \xi)$. If $\xi = 1$ and $\beta \neq 0$, by replacing f with $f' = \beta^{-1} f$, we have $g f' = f' g + g(g^b - 1)$, which means that K is a Hopf quotient of $C(b+1)$. If $\beta = 0$, K is a Hopf quotient of $A(b, \xi)$. In all the above cases, we can apply Lemma 2.4.1, which yields that $K \cong A(b, \xi)$ or $K \cong C(b+1)$. \square

Combining the previous lemmas together, we have the main theorem of this chapter.

Theorem 2.4.10. *Let H be a pointed Hopf algebra over an algebraically closed field k of characteristic 0. If $2 \leq \text{GKdim } H < \infty$ and H is a domain, then H contains a Hopf subalgebra of GK-dimension 2. To be precise,*

- (1) *If $\text{GKdim } k[G] = 0$, where $G = G(H)$, then H contains a Hopf subalgebra isomorphic to $U(\mathfrak{h})$, where \mathfrak{h} is a 2-dimensional Lie algebra.*
- (2) *If $\text{GKdim } k[G] \geq 2$, then H contains a Hopf subalgebra isomorphic to $k\mathbb{Z}^2$.*
- (3) *If $H_1 \supsetneq H_0 = k[G]$ and $\text{GKdim } k[G] \geq 1$, then H contains a Hopf subalgebra isomorphic to either $A(b, \xi)$ or $C(b+1)$ for some integer b and $\xi \in k^\times$.*

Proof. Since $\text{GKdim } H < \infty$, we have $\text{GKdim } k[G] < \infty$.

If $\text{GKdim } k[G] = 0$, then by assumption, G is trivial. Hence H is connected. By Proposition 2.4.8, H contains a Hopf subalgebra isomorphic to $U(\mathfrak{h})$, where \mathfrak{h} is a 2-dimensional Lie algebra.

If $\text{GKdim } k[G] \geq 2$, then by Proposition 2.4.6, $k[G]$, and thus H , contains a Hopf subalgebra isomorphic to $k\mathbb{Z}^2$.

Part (3) follows from Lemma 2.4.9. □

Remark 2.4.11. It seems that in Theorem 2.4.10, if the condition that H is a domain is dropped, we can still find Hopf subalgebra of GK-dimension 2. By Proposition 2.4.6 and Proposition 2.4.8, this is true if H is either a group algebra or connected. However, we still do not know how to deal with the case when H is neither a group algebra nor connected.

One might ask whether Lemma 2.4.9 still holds if H is a prime ring instead of a domain. The answer is negative as shown in the following example.

Example 2.4.12. Let $E = k\langle x_0^{\pm 1}, \dots, x_n^{\pm 1}, y \mid yx_i + x_iy = 0, y^2 = x_0^2 - 1, x_ix_j - x_jx_i = 0 \rangle$. Then E has a Hopf algebra structure under which x_i is group-like for $i = 0, 1, \dots, n$ and y is $(1, x_0)$ -primitive. Denote this Hopf algebra by $E(n)$. It is easy to check that $E(n)$ is not connected, not cosemisimple. Notice that $E(n)$ has a basis consisting of elements of the form $x_0^{i_0} \cdots x_n^{i_n} y^\epsilon$, where ϵ is either 0 or 1. Since $y^\epsilon \in E(n)_1$, the element $x_0^{i_0} \cdots x_n^{i_n} y^\epsilon$ is in $E(n)_1$ as well. Hence, $E(n) = E(n)_1$. By [Mo, Lemma 5.2.12], every Hopf subalgebra H of $E(n)$ has $H = H_1$. However, for $D = A(b, \xi)$ or $C(b+1)$, it is easy to find elements in $D_2 \setminus D_1$. So $E(n)$ has no Hopf subalgebra isomorphic to either $A(b, \xi)$ or $C(b+1)$.

If $n = 0$, the statement follows from [Li, Lemma 2.3 and 2.6]. If $n \geq 1$, then there is a Hopf projection from $E(n)$ to $H = k[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \cong k[\mathbb{Z}^n]$. Hence by [Mo, Proposition 7.2.3], $E(n) \cong A \# H$, where $A = E(n)^{\text{co}H}$. In fact, A is the subalgebra of $E(n)$ generated by y and $x_0^{\pm 1}$. So A is a Hopf subalgebra isomorphic to $E(0)$, which is prime of GK-dimension 1. Then it follows from Lemma 2.4.2 and 2.4.4 that $E(n)$ is prime with $\text{GKdim } E(n) = n+1$.

The next example shows that Theorem 2.4.10 fails if the Hopf algebra H is not pointed.

Example 2.4.13. Let H be $\mathcal{O}(SL_2(k))$, the coordinate ring of the algebraic group $SL_2(k)$. It is well-known that H has a 4-dimensional simple subcoalgebra, so H is not pointed. By [BG2, Theorem I.2.10], H is a noetherian domain. Also, $\text{GKdim } H = 3$ by [LS, Corollary].

Now, H is a commutative noetherian domain of GK-dimension 3. Moreover, by [Mo, §2.4], H is cosemisimple. So any subcoalgebra of H is also cosemisimple by [Mo, Lemma 5.2.12]. If H has a Hopf subalgebra K of GK-dimension 2, then by [GZ2, Proposition 2.3], K must be isomorphic to an enveloping algebra $U(\mathfrak{g})$, where \mathfrak{g} is 2-dimensional abelian, or a group algebra $k\Gamma$, where Γ is free abelian of rank 2, or a Hopf algebra $A(b, 1)$ for some nonnegative integer b . Since $U(\mathfrak{g})$ and $A(b, 1)$ are not cosemisimple, K is of the form $k\Gamma$. But the only invertible elements in H are just non-zero scalars. Therefore, H has no Hopf subalgebras of GK-dimension 2.

Another question is whether a Hopf algebra with GK-dimension > 3 contains a Hopf subalgebra of GK-dimension three. The answer is negative. In fact, this statement is not even true for group algebras, as shown in the following example.

Example 2.4.14. Let G be the free nilpotent group of class 2 on generators x and y . Thus, if we write $z := [x, y]$, then

$$[x, z] = [y, z] = 1.$$

It is easy to check that $G_1 = Z(G) = \langle z \rangle$. By [KL, Theorem 11.14], $\text{GKdim } k[G] = 4$. By construction, $G/G_1 \cong \mathbb{Z}^2$. Hence

$$\text{GKdim } k[G_1] + \text{GKdim } k[G/G_1] = 3 < 4 = \text{GKdim } k[G],$$

which gives a negative answer to the question given after Lemma 2.4.3.

Notice that every Hopf subalgebra of $k[G]$ must be of the form $k[L]$ where L is a subgroup of G . Therefore, to show $k[G]$ has no Hopf subalgebra of GK-dimension three, we only need to show that G has no subgroup of cubic growth. Suppose to the contrary that N is a subgroup of G with cubic growth.

If N has nilpotency class 1, then N must be a finitely generated torsion-free abelian group of rank 3, i.e. $N \cong \mathbb{Z}^3$. Notice that we have an exact sequence of groups,

$$1 \rightarrow N \cap G_1 \rightarrow N \rightarrow N/N \cap G_1 \rightarrow 1.$$

If $N \cap G_1 = \{1\}$, then N embeds in G/G_1 , which is impossible since G/G_1 has only rank 2. Hence $N \cap G_1$ has rank 1, which implies that $N/N \cap G_1$ has rank 2 and $G_1/N \cap G_1 \cong (N + G_1)/N$ is finite. By considering the exact sequence

$$1 \rightarrow N/N \cap G_1 \rightarrow G/G_1 \rightarrow G/(N + G_1) \rightarrow 1,$$

we see that $G/(N+G_1)$ is also finite. It follows that G/N is finite. Then by [KL, Proposition 5.5], $\text{GKdim } k[G] = \text{GKdim } k[N]$, which is a contradiction.

Now suppose N has nilpotency class 2. Then N_1 is a non-trivial subgroup of G_1 , so it must have rank 1. By [KL, Theorem 11.14], N/N_1 has rank 1. Choose $w \in N \setminus N_1$ such that the image of w in N/N_1 is torsion-free. Denote $\langle w, N_1 \rangle$ by P , then N/P is finite. On the other hand, w commutes with N_1 . Hence by construction, P is abelian of rank 2. Now $\text{GKdim } N = \text{GKdim } P = 2$ since N/P is finite. But this again is a contradiction.

Chapter 3

**PROPERTIES OF POINTED HOPF ALGEBRAS OF FINITE
GELFAND-KIRILLOV DIMENSION**

3.1 Introduction

It is well known that every Hopf algebra H has a coradical filtration $\{H_n\}_{n=0}^\infty$. If H is pointed with group-like elements G , then the associated graded algebra of H with respect to the filtration $\{H_n\}_{n=0}^\infty$ is a graded Hopf algebra, which we denote by $\text{gr } H$. The structure of $\text{gr } H$ is relatively easier in the sense that it has a nice decomposition $\text{gr } H \cong R \# kG$, where R is a certain graded subalgebra of $\text{gr } H$ (see [R2, Theorem 3]). In the first part of this chapter, we clarify the behavior of the GK-dimension of a pointed Hopf algebra under taking associated graded algebra. In fact, we prove the following Theorem.

Theorem 3.1.1 (Theorem 3.4.4). *Retain the above notation. If R is finitely generated, then*

$$\text{GKdim } R + \text{GKdim } kG = \text{GKdim } \text{gr } H = \text{GKdim } H.$$

The first equality follows from Lemma 3.4.2, which is a generalized version of Lemma 3.4.2. The proof of the second equality depends heavily on Takeuchi's construction of free Hopf algebras, which we will review briefly in Section 3.2.

An interesting phenomenon is that the GK-dimension of every known Hopf algebra is either infinity or a non-negative integer. So it is tempting to conjecture that this is always true for any Hopf algebra. As positive evidence for this conjecture, we prove in Theorem 3.5.10 that the GK-dimension of a connected Hopf algebra over an algebraically closed field of characteristic zero is either infinity or a non-negative integer. This is basically a consequence of Theorem 3.4.4 and the following result.

Theorem 3.1.2 (Proposition 3.5.5). *Let K be a connected coradically graded Hopf algebra and assume that the base field k is algebraically closed of characteristic 0. If K is finitely generated, then K is isomorphic to the polynomial ring in ℓ variables for some $\ell \geq 0$ as algebras.*

For the definition of coradically graded Hopf algebras, one can refer to Definition 1.4.4. Notice that if H is a connected Hopf algebra, then $\text{gr } H$ is a connected coradically graded Hopf algebra (see Remark 1.4.5).

In the last two sections, with the help of the results from previous sections, we classify connected Hopf algebras of GK-dimension three over an algebraically closed field of char-

acteristic zero. The result can be stated as follows. Recall that we use $P(H)$ to denote the space of primitive elements of H .

Theorem 3.1.3 (Theorem 3.6.8). *Let H be a connected Hopf algebra of GK-dimension three (over an algebraically closed field of characteristic 0). Then H is isomorphic to one of the following:*

- (I) *The enveloping algebra $U(\mathfrak{g})$ for some three-dimensional Lie algebra \mathfrak{g} ;*
- (II) *The Hopf algebras $A(0, 0, 0)$, $A(0, 0, 1)$, $A(1, 1, 1)$ or $A(1, \lambda, 0)$ from Example 3.6.1 for some $\lambda \in k$;*
- (III) *The Hopf algebras $B(\lambda)$ from Example 3.6.2 for some $\lambda \in k$.*

3.2 Takeuchi's construction of free Hopf algebras

In [T2], Takeuchi proved that for any coalgebra C there exists a Hopf algebra $\mathcal{H}(C)$ characterized by the following universal property:

- (1) There is a coalgebra map $i : C \rightarrow \mathcal{H}(C)$,
- (2) For any Hopf algebra H and coalgebra map $f : C \rightarrow H$, there is a Hopf algebra map $f' : \mathcal{H}(C) \rightarrow H$ such that $f = f'i$.

The Hopf algebra $\mathcal{H}(C)$ is called the free Hopf algebra generated by C [T2, Definition 1]. Takeuchi showed the existence of $\mathcal{H}(C)$ by an explicit construction, which we will describe briefly.

Let $V = \bigoplus_{i=0}^{\infty} V_i$ where $V_i = C$ if i is even and $V_i = C^{cop}$ if i is odd. Notice that V has a natural coalgebra structure. Let $S : V \rightarrow V^{cop}$ be the coalgebra map sending (x_0, x_1, x_2, \dots) to $(0, x_0, x_1, x_2, \dots)$. Then S induces a bialgebra map $S : T(V) \rightarrow T(V)^{op, cop}$. Let I be the two-sided ideal of $T(V)$ generated by the set

$$\{S * Id(x) - \epsilon(x)1 \mid x \in V\} \cup \{Id * S(x) - \epsilon(x)1 \mid x \in V\},$$

where $*$ represents the convolution and 1 is the identity in $T(V)$. Moreover, I is a coideal and $S(I) \subset I$. By [T2, Lemma 1], the Hopf algebra $T(V)/I$, with the antipode induced from the map S , is the universal object $\mathcal{H}(C)$.

Takeuchi's construction generalizes the notion of a free group generated by a set in the following sense.

Proposition 3.2.1 ([T2, Lemma 34]). $\mathcal{H}(kG(C)) = k\langle G(C) \rangle$, where $\langle G(C) \rangle$ is the free group generated by the set $G(C)$.

Let C be a coalgebra with coradical C_0 and let $C = C_0 \oplus V$ be a decomposition of C as a k -space. Then $\mathcal{H}(C)$ can be realized by giving a natural Hopf structure on the algebra $\mathcal{H}(C_0) \amalg T(V)$ [T2, §6], where \amalg denotes the coproduct in the category of algebras. Now the canonical coalgebra map $i : C \rightarrow \mathcal{H}(C)$ can be identified with the map induced by maps $C_0 \rightarrow \mathcal{H}(C_0)$ and $V \rightarrow T(V)$. By this characterization, we have

Lemma 3.2.2 ([T2, Theorem 35]). *Suppose that C is a pointed coalgebra and $G(C) \cup B$ is a k -basis for C . Let $X = G(C) \cup G(C)^{-1} \cup B$ and let Y be the set of finite sequences (x_1, \dots, x_n) of elements of X such that (x_i, x_{i+1}) is not of the form $(g^{\pm 1}, g^{\mp 1})$ where $g \in G(C)$. Set*

$$\bar{x} = x_1 \cdots x_n \in \mathcal{H}(C) \text{ for } x = (x_1, \dots, x_n) \in Y,$$

where by abuse of notation we still use x_j for its image in $\mathcal{H}(C)$ under the canonical map $i : C \rightarrow \mathcal{H}(C)$. Then $\{\bar{x} \mid x \in Y\}$ forms a k -basis for $\mathcal{H}(C)$.

Now we are able to prove the following proposition, which determines the coradical of $\mathcal{H}(C)$ when C is pointed. For a given k -subspace W of an algebra R and any $n \geq 1$, let W^n denote the k -subspace of R spanned by products of $\leq n$ elements in W .

Proposition 3.2.3. *Let C be a pointed coalgebra. For any $n \geq 1$. Then the following statements are true.*

- (I) *the Hopf algebra $\mathcal{H}(C)$ is pointed and the coradical of $\mathcal{H}(C)$ is equal to the subalgebra generated by $G(C)$ and their inverses, which is isomorphic to $k\langle G(C) \rangle$.*
- (II) *for any $n \geq 1$, C^n is a subcoalgebra of $\mathcal{H}(C)$ and any element in $G(C^n)$ can be expressed as a product of $\leq n$ elements in $G(C)$.*

Proof. Denote $\mathcal{H}(C)$ by H . It is clear that the subalgebra of H generated by $G(C)$ and their inverses is contained in H_0 . By Lemma 3.2.2, this subalgebra is isomorphic to $k\langle G(C) \rangle$.

Choose a subset B of C such that $B = \bigcup_{i=1}^{\infty} B_i$ and $G(C) \cup B_1 \cup \cdots \cup B_n$ is a k -basis of C_n for any n . Let V be the k -space spanned by B . By [T2, §6], $H \cong k\langle G(C) \rangle \amalg T(V)$ as an algebra. Hence we can define a grading on H by setting $\deg g = 0$ for any $g \in G(C)$ and $\deg b_i = i$ for any $b_i \in B_i$. Under this grading, H becomes a graded algebra (but not necessarily a graded coalgebra) and C is a graded subspace of H . Let $H(n)$ be the k -subspace of H spanned by homogeneous elements of degree n .

Now, fix a basis of H described in Lemma 3.2.2 with the chosen set B . Then this is a basis consisting of homogeneous elements with respect to the grading defined above. For any $m \geq 0$, let $A_m = \sum_{i=0}^m H(i)$. It is clear that $\{A_m\}_{m \geq 0}$ is an algebra filtration on H and A_0 is exactly the subalgebra of H generated by $G(C)$ and their inverses. Moreover, by the choice of B , we have $C_m \subset A_m$ for any $m \geq 0$. Let $\bar{x} = x_1 \cdots x_n$ be as in Lemma 3.2.2 and suppose that $x_j \in B_{i_j}$ (here we set $B_0 = G(C) \cup G(C)^{-1}$). Then by definition A_m is spanned by \bar{x}

such that $\sum_j i_j \leq m$. Pick such an $\bar{x} \in A_m$. Notice that $\Delta(\bar{x}) = \Delta(x_1) \cdots \Delta(x_n)$ and for each j , $\Delta(x_j) \in \sum_t C_t \otimes C_{i_j-t} \subset \sum_t A_t \otimes A_{i_j-t}$. As a consequence,

$$\Delta(\bar{x}) \in \sum_t A_t \otimes A_{m-t}.$$

This shows that $\{A_m\}_{m \geq 0}$ is also a coalgebra filtration. By [Mo, Lemma 5.3.4], $H_0 \subset A_0$. This proves the first statement.

For the second statement, it is easy to check that C^n is a subcoalgebra of H . As mentioned before, C is a graded k -subspace of H . This means that $C = \bigoplus_{i=0}^{\infty} C(i)$ where $C(i) = C \cap H(i)$. Now $G(C^n) = C^n \cap G(H) \subset C^n \cap H(0)$. Since C is a graded k -subspace of H , $C^n \cap H(0) = C(0)^n$. Notice that $C(0)$ is spanned by $G(C)$. Hence every element in $G(C^n)$ can be expressed as a linear combination with each summand a scalar multiple of a product of $\leq n$ elements in $G(C)$. Since distinct group-like elements are k -linearly independent by [Sw, 3.2.1], every element in $G(C^n)$ is actually a product of $\leq n$ elements in $G(C)$. \square

Remark 3.2.4. Suppose that $C = \bigoplus_{i=0}^{\infty} C(i)$ is a pointed graded coalgebra such that $C(0) = C_0$. Now C has a canonical decomposition $C = C_0 \oplus V$, where $V = \bigoplus_{i=1}^{\infty} C(i)$. Then by the construction of the comultiplication and the antipode on $\mathcal{H}(C_0) \amalg T(V)$ in [T2, Lemma 26, Lemma 27], $\mathcal{H}(C) = \mathcal{H}(C_0) \amalg T(V)$ becomes a graded Hopf algebra, where elements in $\mathcal{H}(C_0)$ have degree 0 and the grading on $T(V)$ is inherited from V . It is clear that the canonical inclusion $i : C \rightarrow \mathcal{H}(C)$ becomes a graded coalgebra map. Moreover, if there is a graded coalgebra map f from C to a graded Hopf algebra H , then the lifting Hopf algebra map $f' : \mathcal{H}(C) \rightarrow H$ is also graded.

Corollary 3.2.5. *Let H be a pointed Hopf algebra. Then every finite-dimensional subspace V of H is contained in a finitely generated Hopf subalgebra. Moreover, if D is a finite-dimensional subcoalgebra of H with group-like elements $G(D)$, then elements in $G(D^n)$ can be expressed as products of $\leq n$ elements in $G(D)$.*

Proof. Since V is contained in a finite-dimensional subcoalgebra of H , we can assume $V = D$ is a finite-dimensional subcoalgebra. Let C be a copy of D as coalgebras. Then there is an injective coalgebra map $f : C \rightarrow H$ whose image is D . By the universal property of $\mathcal{H}(C)$, there is a Hopf algebra map $f' : \mathcal{H}(C) \rightarrow H$ such that $f = f'i$, where i is the inclusion $C \rightarrow \mathcal{H}(C)$. (Notice here if we do not introduce a copy C of D , then by writing D we could mean either a subcoalgebra of H or a subcoalgebra of $\mathcal{H}(D)$, which may cause confusion in the proof). By Lemma 3.2.2, $\mathcal{H}(C)$ is a finitely generated algebra. By construction, D is contained in $f'(\mathcal{H}(C))$. This proves the first claim.

Let $S = \{g_1, \dots, g_\ell\}$ be the set of group-like elements of C . Then $\{f(g_1), \dots, f(g_\ell)\}$ is the set of group-like elements of D . Notice that C^n is a subcoalgebra of $\mathcal{H}(C)$. By (II) of Proposition 3.2.3, every group-like element of C^n can be expressed as a product of $\leq n$

elements in S . Since C^n is mapped onto D^n by f' , we have $G(D^n) = f'(G(C^n))$ by [Mo, Corollary 5.3.5]. The result then follows. \square

Corollary 3.2.6. *Let H be a finitely generated pointed Hopf algebra. Then H_0 is a finitely generated algebra. In fact, if D is a finite-dimensional subcoalgebra of H that generates H as an algebra, then H_0 is generated by $G(D)$.*

Proof. Since every finite-dimensional subspace of H is contained in a finite-dimensional subcoalgebra by [Mo, 5.1.1], we can assume that H is generated as an algebra by a finite-dimensional subcoalgebra D . By assumption H_0 is spanned by $G(H)$. For any $g \in G(H)$, there exists some n such that $g \in D^n$. Notice that D^n is a subcoalgebra of H . Hence $g \in G(D^n)$. Now the result follows from the second statement of Corollary 3.2.5. \square

Remark 3.2.7. Let $H = \bigoplus_{i=0}^{\infty} H(i)$ be a graded pointed Hopf algebra such that $H(0)$ is spanned by all group-like elements of H . In this case, every finite-dimensional subspace V of H is contained in a finitely generated graded Hopf subalgebra of H . In fact, without loss of generality, we can assume that V is a finite-dimensional graded subspace of H . By a similar argument as in [Mo, Theorem 5.1.1], V is contained in a finite-dimensional graded subcoalgebra C of H . Then the result follows from Remark 3.2.4 and an argument similar to that of Corollary 3.2.5.

We conclude this section with a proposition regarding the GK-dimension of a pointed Hopf algebra, which is a direct consequence of Corollary 3.2.5.

Proposition 3.2.8. *Let H be a pointed Hopf algebra. Then*

$$\text{GKdim } H = \sup_E \text{GKdim } E,$$

where the supreme is taken over all finitely generated Hopf subalgebras of H .

3.3 Pointed Hopf algebras and their associated graded Hopf algebras

Throughout this section, let H be a pointed Hopf algebra with group-like elements G . We use $\text{gr } H$ to denote the associated graded Hopf algebra of H with respect to the coradical filtration. There is a canonical Hopf projection $\psi : \text{gr } H \rightarrow H_0$. Let $R = (\text{gr } H)^{\text{co}\psi}$, the algebra of coinvariants of ψ [AS2, 1.5]. By definition R is a graded subalgebra of $\text{gr } H$. In fact, it is well known that R is a graded braided Hopf algebra in ${}^G_G\mathcal{YD}$, the Yetter-Drinfeld category over G , and

$$\text{gr } H \cong R \# H_0, \tag{E3.3.0.1}$$

as Hopf algebras. Let H_0^+ be the k -space spanned by the elements of the form $1 - g$ where $g \in G$. Notice that HH_0^+ is a coideal of H . Denote the coalgebra H/HH_0^+ by $\theta(H)$ and the coalgebra projection $H \rightarrow \theta(H)$ by π_H .

Lemma 3.3.1. *Retain the above notation. Then H/H_n is a free (right) H_0 -module for any $n \geq 0$.*

Proof. By [R1, Lemma 1], for any $m \geq 0$, H_{m+1}/H_m is a free H_0 -module. Hence inductively we see that H_{n+i}/H_n is a free H_0 -module for any $i \geq 1$ and $H_{n+i}/H_n \cong \bigoplus_{j=0}^{i-1} H_{n+j+1}/H_{n+j}$ as H_0 -modules. As a consequence, $H/H_n \cong \bigoplus_{j \geq 0} H_{n+j+1}/H_{n+j}$ as H_0 -modules. The result then follows. \square

Lemma 3.3.2. *Retain the above notation and let $I = HH_0^+$. Then $I \cap H_n = H_n H_0^+$ for any $n \geq 0$.*

Proof. For any $n \geq 0$, we have the short exact sequence

$$0 \rightarrow H_n \rightarrow H \rightarrow H/H_n \rightarrow 0.$$

Since H/H_n is a free H_0 -module, the following sequence is exact,

$$0 \rightarrow H_n \otimes_{H_0} k \rightarrow H \otimes_{H_0} k \rightarrow (H/H_n) \otimes_{H_0} k \rightarrow 0.$$

This shows that the canonical map $H_n/H_n H_0^+ \rightarrow H/I$ is injective, which implies that $I \cap H_n = H_n H_0^+$. \square

Let F_n be $\pi_H(H_n) \subset \theta(H)$. Then $\{F_n\}_{n \geq 0}$ becomes a coalgebra filtration on $\theta(H)$. The following lemma is clear.

Lemma 3.3.3. *Suppose that $f : C \rightarrow D$ is a surjective coalgebra map and C has a coalgebra filtration $\{A_n\}_{n \geq 0}$. Let $B_n = f(A_n)$. Then $\{B_n\}_{n \geq 0}$ is a coalgebra filtration on D . Moreover, f induces a surjective graded coalgebra map $\text{gr}_A C \rightarrow \text{gr}_B D$.*

By this lemma, we see that there is a surjective graded coalgebra map $\text{gr } H \rightarrow \text{gr}_F \theta(H)$ induced by π_H .

Proposition 3.3.4. *Retain the above notation. Then $\text{gr}_F \theta(H)$ is isomorphic to $\theta(\text{gr } H)$ as graded coalgebras.*

Proof. By definition, $\theta(\text{gr } H) = \text{gr } H / (\text{gr } H) H_0^+$. So we only have to show that the kernel of the map $\text{gr } H \rightarrow \text{gr}_F \theta(H)$ induced by π_H is $(\text{gr } H) H_0^+$. It suffices to prove that for any $n \geq 0$, the canonical map $H_{n+1}/H_n \rightarrow \pi_H(H_{n+1})/\pi_H(H_n)$ has kernel $(H_{n+1}/H_n) H_0^+$. Let $I = HH_0^+$. It is easy to check that the

$$\ker(H_{n+1}/H_n \rightarrow \pi_H(H_{n+1})/\pi_H(H_n)) = \frac{H_{n+1} \cap (H_n + I)}{H_n} = \frac{H_{n+1} \cap I + H_n}{H_n}.$$

By Lemma 3.3.2, $H_{n+1} \cap I = H_{n+1} H_0^+$. Therefore,

$$\frac{H_{n+1} \cap I + H_n}{H_n} = \frac{H_{n+1} H_0^+ + H_n}{H_n} = (H_{n+1}/H_n) H_0^+.$$

This completes the proof. \square

Now we are able to determine the coradical filtration of $\theta(H)$ by using the following lemma.

Lemma 3.3.5. *Let C be a coalgebra with a coalgebra filtration $\{F_n\}$ such that $F_0 = C_0$. If the associated graded coalgebra with respect to $\{F_n\}$ is coradically graded, then $\{F_n\}$ agrees with the coradical filtration of C .*

Proof. Denote the associated graded coalgebra with respect to $\{F_n\}$ by $\text{gr}_F C$. By the definition of coalgebra filtration and the fact that $F_0 = C_0$, it is easy to see that $F_n \subset C_n$ for any $n \geq 1$. If the assertion is not true, then choose n to be minimal such that $F_n \subsetneq C_n$. Pick some element $y \in C_n \setminus F_n$. Then

$$\Delta(y) \in \sum_{i=0}^n C_i \otimes C_{n-i} = C_n \otimes F_0 + F_0 \otimes C_n + \sum_{i=1}^{n-1} F_i \otimes F_{n-i}. \quad (\text{E3.3.5.1})$$

Suppose $y \in F_m \setminus F_{m-1}$ for some $m \geq n+1$. Let \bar{y} be the corresponding non-zero element in $\text{gr}_F C(m)$. Since $\text{gr}_F C$ is coradically graded, \bar{y} is not in $(\text{gr}_F C)_{m-1}$. But by (E3.3.5.1), \bar{y} is in $(\text{gr}_F C)_1$, which is a contradiction. This completes the proof. \square

Proposition 3.3.6. *The coradical filtration of the coalgebra $\theta(H)$ is $\{\pi_H(H_n)\}_{n \geq 0}$.*

Proof. By Proposition 3.3.4, $\text{gr}_F \theta(H) \cong \theta(\text{gr } H)$ as coalgebras. By the proof of [R2, Theorem 3], $\theta(\text{gr } H) \cong R$ as graded coalgebras, where R is defined in (E3.3.0.1). By [AS2, p.15], R is coradically graded. The result now follows from Lemma 3.3.5. \square

3.4 GK-dimensions of H and $\text{gr } H$

This section is devoted to the proof of Theorem 3.4.4. Let H be a pointed Hopf algebra with group-like elements G . As mentioned in the previous section, $\text{gr } H$ has a decomposition $\text{gr } H \cong R \# H_0$. By construction, the graded algebra R is connected in the sense that $R(0) = k$. So if R is finitely generated as an algebra, then it is locally finite.

Lemma 3.4.1. *Retain the above notation. Let D be a graded subcoalgebra of $\text{gr } H$. Then*

$$D \subset \bigoplus_{i \geq 0} \bigoplus_{h \in G(D)} R(i)h.$$

Proof. Let y be a non-zero element in D . By assumption we can further assume that y is homogeneous of degree s . If $s = 0$, then $y \in D(0) = kG(D)$. If $s \geq 1$, then by the decomposition (E3.3.0.1), we can write $y = \sum_{i=1}^N y_i h_i$ where h_i 's are distinct group-like elements and $0 \neq y_i \in R(n)$. We only need to show that $h_i \in G(D)$ for any i . Let $\psi : \text{gr } H \rightarrow H_0$ be the canonical Hopf projection. Then it is obvious that ψ maps D onto $D(0) = kG(D)$. Now we have

$$(\text{Id} \otimes \psi)\Delta(y) = \sum_{i=1}^N y_i h_i \otimes h_i \in D \otimes D(0).$$

Hence $h_i \in G(D)$ by the choice of y_i and h_i . \square

The next lemma is about the GK-dimensions of skew group algebras. It can be viewed as a generalization of Lemma 2.4.2. Let Γ be a group and A an algebra with a left Γ -action. As k -spaces, the skew group algebra $A * \Gamma$ is isomorphic to $A \otimes k\Gamma$. The multiplication is given by

$$(a * g)(b * h) = a(g.b) * gh,$$

where $a, b \in A$, $g, h \in \Gamma$ and $a * g$ stands for $a \otimes g$. We will omit the $*$ in $a * g$ if there is no confusion. We say that the Γ -action on A is **locally finite** if any finite-dimensional subspace of A is contained in a finite-dimensional Γ -submodule of A .

Lemma 3.4.2. *Let A and Γ be as above and suppose that the Γ -action on A is locally finite. Then*

$$\text{GKdim } A * \Gamma = \text{GKdim } A + \text{GKdim } k\Gamma.$$

Proof. We say a subalgebra B of A is **Γ -affine** if B is generated as an algebra by a finite-dimensional Γ -submodule of A . It is easy to check by the local finiteness condition that

$$\text{GKdim } A = \sup_B \text{GKdim } B,$$

where B runs over all Γ -affine subalgebras of A . Next we claim that

$$\text{GKdim } A * \Gamma = \sup_{B,L} \text{GKdim } B * L,$$

where B runs over all Γ -affine subalgebras of A and L runs over all finitely generated subgroups of Γ . In fact, by the definition of the GK-dimension, $\text{GKdim } A * \Gamma = \sup_E \text{GKdim } E$, where the supremum is taken over all finitely generated subalgebras E of $A * \Gamma$. Let V be a finite-dimensional generating subspace of E . Then there exists $g_1, \dots, g_s \in G$ and some finite-dimensional subspace W of A such that $V \subset Wg_1 + Wg_2 + \dots + Wg_s$. By the local finiteness condition we can further assume that W is a finite-dimensional Γ -submodule of A .

Let B be the subalgebra of A generated by W and let L be the subgroup of G generated by g_1, \dots, g_s and their inverses. Then it is clear that B is Γ -affine and $E \subset B * L$. This prove the claim. Now,

$$\begin{aligned} \text{GKdim } A * \Gamma &= \sup_{B,L} \text{GKdim } B * L \\ &= \sup_{B,L} (\text{GKdim } B + \text{GKdim } kL) \\ &= \sup_B \text{GKdim } B + \sup_L \text{GKdim } kL \\ &= \text{GKdim } A + \text{GKdim } k\Gamma. \end{aligned}$$

The second equality follows from Lemma 3.4.2. For the last equality, one just notices that $\text{GKdim } k\Gamma = \sup_L \text{GKdim } kL$ by Proposition 3.2.8. \square

Recall from [Mo, §5.4] that for $g \in G$,

$$P_{1,g}(H) = \{x \in H \mid \Delta(x) = x \otimes 1 + g \otimes x\}.$$

Let $P'_{1,g}(H)$ be a subspace of $P_{1,g}(H)$ such that $P_{1,g}(H) = k(1-g) \oplus P'_{1,g}(H)$ and define

$$P'_T(H) = \bigoplus_{g \in G} P'_{1,g}(H).$$

Lemma 3.4.3. *Let H be a pointed Hopf algebra. Then the following statements are equivalent.*

- (I) $\dim_k P'_T(H) < \infty$,
- (II) $\dim_k R(1) < \infty$,
- (III) *the graded algebra R is locally finite,*
- (IV) *the coalgebra $\theta(H)$ is coradically finite.*

Proof. By [Mo, Theorem 5.4.1(1)] and the definition of R , $P'_T(H) \cong R(1)$ as k -spaces. Hence (I) and (II) are equivalent. Since R is a coradically graded coalgebra and $R(0) = k$, the equivalence of (II) and (III) follows from [MS, Lemma 2.3 (2)]. As shown in the proof of Theorem 3.4.4, $\dim_k \pi_H(H_n) = \dim_k \sum_{i=1}^n R(i)$. By Proposition 3.3.6, $\{\pi_H(H_n)\}_{n \geq 0}$ is the coradical filtration of $\theta(H)$. This shows that (III) and (IV) are equivalent. \square

Now we are ready to prove the main theorem of this section.

Theorem 3.4.4. *Retain the above notation. Suppose that $\dim_k R(1) < \infty$. Then*

$$\text{GKdim } R + \text{GKdim } kG = \text{GKdim } \text{gr } H \leq \text{GKdim } H \leq \text{GKdim } kG + \gamma, \quad (\text{E3.4.4.1})$$

where $\gamma = \overline{\lim}_{n \rightarrow \infty} \log_n \dim_k \pi_H(H_n) = \overline{\lim}_{n \rightarrow \infty} \log_n \dim_k \bigoplus_{i=0}^n R(i)$. If R is a finitely generated algebra, then

$$\text{GKdim } R + \text{GKdim } kG = \text{GKdim } \text{gr } H = \text{GKdim } H = \text{GKdim } kG + \gamma. \quad (\text{E3.4.4.2})$$

Proof. Let $V_n = \bigoplus_{i=1}^n R(i)$. By Lemma 3.4.3, V_n is finite-dimensional for any n . Also, $\{V_n\}_{n \geq 0}$ is the coradical filtration of R since R is a coradically graded coalgebra. On the other hand, $R \cong \theta(\text{gr } H)$ as graded coalgebras. It then follows from Proposition 3.3.4 and Proposition 3.3.6 that $V_n \cong \pi_H(H_n)$ as k -spaces. As a consequence, the number γ is well defined.

It is well known that $R\#kG$ is just $R * G$ as algebras. Moreover, the G -action on R preserves the grading. Since every finite-dimensional subspace V of R is contained in V_s for some $s \geq 0$, we see that the G -action on R is locally finite. Now by Lemma 3.4.2, $\text{GKdim } R + \text{GKdim } kG = \text{GKdim } \text{gr } H$. By [KL, Lemma 6.5], $\text{GKdim } \text{gr } H \leq \text{GKdim } H$. Next we are going to show that $\text{GKdim } H \leq \text{GKdim } kG + \gamma$.

Now let C be a finite-dimensional subspace of H . Without loss of generality, we can assume that C is a subcoalgebra of H . Let $S = G(C)$. By the choice of C , the set S is finite. Denote by $G_S(\ell)$ the set of elements in G that can be expressed as products of $\leq \ell$ elements in $S \cup S^{-1}$ and let $g_S(\ell) = |G_S(\ell)|$. By Corollary 3.2.5, $G(C^\ell) \subset G_S(\ell)$. Suppose that $C \subset H_N$ for some $N \geq 1$. Then $C^n \subset H_{nN}$. Let $D = \text{gr } C^n$, the associated graded coalgebra of C^n with respect to its coradical filtration. Notice that D is naturally embedded in $\text{gr } H$. Since $G(D)$ can be identified with $G(C^n)$, we have $G(D) \subset G_S(n)$.

Now by Lemma 3.4.1, we have

$$\begin{aligned} D &\subset \bigoplus_{i=0}^{nN} \bigoplus_{h \in G(D)} R(i)h \\ &= \bigoplus_{h \in G(D)} V_{nN}h \subset \bigoplus_{h \in G_S(n)} V_{nN}h. \end{aligned}$$

As a consequence,

$$\dim_k C^n = \dim_k D \leq \dim_k V_{nN} \cdot g_S(n).$$

Therefore,

$$\begin{aligned} \overline{\lim}_{n \rightarrow \infty} \log_n \dim_k C^n &\leq \overline{\lim}_{n \rightarrow \infty} \log_n \dim_k V_{nN} \cdot g_S(n) \\ &\leq \overline{\lim}_{n \rightarrow \infty} \log_n \dim_k V_{nN} + \overline{\lim}_{n \rightarrow \infty} \log_n g_S(n) \\ &\leq \overline{\lim}_{n \rightarrow \infty} \log_n \dim_k V_n + \overline{\lim}_{n \rightarrow \infty} \log_n g_S(n) \\ &\leq \gamma + \text{GKdim } kG. \end{aligned}$$

This proves (E3.4.4.1).

When R is finitely generated, by [KL, Proposition 6.6], $\text{GKdim } R = \overline{\lim}_{n \rightarrow \infty} \log_n \dim_k V_n = \gamma$. Combining this fact with (E3.4.4.1), we have (E3.4.4.2). \square

Remark 3.4.5. In Theorem 3.4.4, if we further assume that G is a finite group, then $\{H_n\}_{n \geq 0}$ is a finite filtration in the sense that $\dim_k H_n < \infty$ for any $n \geq 0$. This is true because $H_n/H_{n-1} \cong R(n) \otimes kG$ as k -spaces for all n . In this case, the result $\text{GKdim } \text{gr } H = \text{GKdim } H$ follows from [KL, Proposition 6.6].

Remark 3.4.6. It is easy to check that $\text{gr } H$ is finitely generated if and only if both R and kG are finitely generated. Hence if $\text{gr } H$ is finitely generated, then $\text{GKdim } R + \text{GKdim } kG = \text{GKdim } \text{gr } H = \text{GKdim } H$.

If H is a finitely generated pointed Hopf algebra, then G is a finitely generated group by Corollary 3.2.5. However, the finite generation of H does not imply that $\dim_k R(1) < \infty$, as shown in the following example.

Example 3.4.7. Let the base field k be \mathbb{F}_p , where p is a prime number. Let $H = k[x]$. Then H is a connected Hopf algebra where x is primitive. It is well known that $\text{gr } H \cong k[x_1, x_2, \dots]/(x_1^p, x_2^p, \dots)$ with x_i being primitive. As a consequence, $\text{GKdim } \text{gr } H = 0$ since every finitely generated subalgebra of $\text{gr } H$ is finite-dimensional. On the other hand, $\text{GKdim } H = 1$.

The above example relies heavily on the assumption that the base field k has characteristic p . In fact, based on known examples, it is conjectured that if the base field is of characteristic 0, and H is finitely generated with finite GK-dimension, then $R(1)$ is always finite-dimensional.

We conclude this section with a straightforward corollary. For the definitions and basic properties of Yetter-Drinfeld modules and Nichols algebras, a good reference is [AS2].

Corollary 3.4.8. *Let H be a pointed Hopf algebra with group-like elements G . If $\text{gr } H \cong \mathcal{B}(V) \# kG$, where V is a finite-dimensional left Yetter-Drinfeld module over G and $\mathcal{B}(V)$ is the Nichols algebra of V , then*

$$\text{GKdim } \mathcal{B}(V) + \text{GKdim } kG = \text{GKdim } \text{gr } H = \text{GKdim } H.$$

3.5 Connected Hopf algebras

This section is primarily devoted to the study of connected Hopf algebras. Let H be a connected Hopf algebra. Then its associated graded Hopf algebra $\text{gr } H$ with respect to the coradical filtration is also connected. Moreover, the natural grading on $\text{gr } H$ makes it into a coradically graded Hopf algebra as mentioned in Section 1.4. In fact, we are able to show that if the base field is algebraically closed of characteristic 0 and $\text{GKdim } H < \infty$, then $\text{GKdim } H$ must be a non-negative integer ℓ and $\text{gr } H$ is isomorphic to the polynomial ring in ℓ variables as algebras (see Proposition 3.5.5 and Theorem 3.5.10). As a consequence, we derive some ring-theoretic properties of such Hopf algebras. For instance, we show that they are always domains, which reproves an unpublished result by Le Bruyn.

By [Mo, Lemma 5.2.10], a connected bialgebra is automatically a connected Hopf algebra. Furthermore, we have the following lemma.

Lemma 3.5.1. *Let H be a connected Hopf algebra and K a sub-bialgebra of H . Then K is a Hopf subalgebra of H .*

Proof. Let S be the antipode of H . We need to show that $S(K) \subset K$. It suffices to show that $S(K_n) \subset K$ for any $n \geq 0$. When $n = 0$, the statement is true since K_0 is spanned by the unit 1. For any $n \geq 1$ and $c \in K_n$, by [Mo, Lemma 5.3.2], $\Delta(c) = 1 \otimes c + c \otimes 1 + \sum a_i \otimes b_i$,

where $a_i, b_i \in K_{n-1}$. Since S is the convolution inverse of the identity map, we have $S(c) + c + \sum a_i S(b_i) = \epsilon(c)$. By induction hypothesis, $S(c) \in K$. This completes the proof. \square

The following technical lemma will be used frequently in the rest of the chapter.

Lemma 3.5.2. *Let $f : A \rightarrow B$ be a surjective algebra map. If A is a Noetherian prime algebra and $\text{GKdim } A < \text{GKdim } B + 1 < \infty$, then f is an isomorphism.*

Proof. We only have to show that $I := \ker f$ is zero. If not, then by Goldie's theorem, the ideal I contains a regular element. Now by [KL, Proposition 3.15], $\text{GKdim } B + 1 \leq \text{GKdim } A$. But this is a contradiction. \square

Lemma 3.5.3. *Let $K = \bigoplus_{n=0}^{\infty} K(n)$ be a graded Hopf algebra with $K(0) = k$. Then the following statements are true.*

(I) *If K is generated in degree one, then K is cocommutative;*

(II) *If K is coradically graded, then K is commutative.*

Proof. It is easy to check that if a Hopf algebra is generated by elements x such that $\Delta(x) = \tau\Delta(x)$, where τ is the twisting map, then the Hopf algebra is cocommutative. Since $K(1)$ is spanned by primitive elements, the statement (I) is true.

For the second statement, by Remark 3.2.7, we can assume, without loss of generality, that K is finitely generated and thus locally finite. Let $S = \bigoplus_{n=0}^{\infty} K(n)^*$ be the graded dual of K . Then S is also a graded Hopf algebra with $S(0) = k$. By [AS2, Lemma 5.5], S is generated in degree one and thus cocommutative by (I). Hence K is commutative. \square

Notice that for any connected Hopf algebra H , $\text{gr } H$ is connected coradically graded. Hence the following proposition is clear.

Proposition 3.5.4. *Let H be a connected Hopf algebra. Then $\text{gr } H$ is commutative.*

In fact, if the base field is algebraically closed of characteristic 0, we can say more about the algebra structure of a connected coradically graded Hopf algebra.

Proposition 3.5.5. *Let $K = \bigoplus_{n=0}^{\infty} K(n)$ be a coradically graded Hopf algebra with $K(0) = k$ and assume that the base field k is algebraically closed of characteristic 0. If K is finitely generated, then K is isomorphic to the polynomial ring in ℓ variables for some $\ell \geq 0$ as algebras.*

Proof. Since K is finitely generated commutative, $K \cong \mathcal{O}(\Gamma)$, the coordinate ring of some algebraic group Γ over k . Hence K has finite global dimension. Now the result follows from [NV, III.2.5]. \square

The previous proposition leads to the following theorem, which is a result by Le Bruyn (unpublished).

Theorem 3.5.6. *Assume that the base field k is algebraically closed of characteristic 0. Let H be a connected Hopf algebra. Then H is a domain.*

Proof. We only need to show that $\text{gr } H$ is a domain. By Remark 3.2.7, every finite subset of $\text{gr } H$ is contained in a finitely generated graded Hopf subalgebra of $\text{gr } H$. By Proposition 3.5.5, such subalgebras are domains. The result then follows. \square

Remark 3.5.7. In Theorem 3.5.6, the statement fails if the base field is of characteristic p . For example, let $k = \mathbb{F}_p$ and $H = k[x]/(x^p)$. It is well known that H has a unique connected Hopf algebra structure under which x is primitive. Obviously, H is not a domain.

Lemma 3.5.8. *Assume that the base field k is algebraically closed of characteristic 0. Let K be a connected coradically graded Hopf algebra and L a finitely generated graded Hopf subalgebra of K . If $L \neq K$, then $\text{GKdim } K \geq \text{GKdim } L + 1$.*

Proof. Let N be the smallest number such that $L(N) \neq K(N)$. Pick $y \in K(N) \setminus L(N)$. By the choice of N we see that $\Delta(y) = 1 \otimes y + y \otimes 1 + w$, where $w \in \bigoplus_{i=1}^{N-1} L(i) \otimes L(N-i)$. Hence the algebra P generated by L and y is a finitely generated graded sub-bialgebra of K . By Lemma 3.5.1, P is a Hopf subalgebra of K . By replacing K with P , we may assume that K is also finitely generated. Now we have an exact sequence of Hopf algebras

$$0 \rightarrow L \rightarrow K \rightarrow H \rightarrow 0,$$

in the sense that L is a normal Hopf subalgebra of K and $H = K/L^+K = K/KL^+$. Since $L \neq K$, the connected Hopf algebra H is not isomorphic to k by [T1, Theorem 4.3] and therefore $\text{GKdim } H \geq 1$. By taking the spectrum, we get an exact sequence of algebraic groups [T1, Theorem 5.2]

$$1 \rightarrow \Gamma_1 \rightarrow \Gamma_2 \rightarrow \Gamma_3 \rightarrow 1.$$

By [H, 7.4 Proposition B], $\dim \Gamma_2 = \dim \Gamma_1 + \dim \Gamma_3$, where \dim represents the dimension of an affine variety. It is well known that for any affine variety X , $\dim X = \text{GKdim } \mathcal{O}(X)$. Consequently,

$$\text{GKdim } K = \text{GKdim } L + \text{GKdim } H.$$

Now the result follows since $\text{GKdim } H \geq 1$. \square

In next section, we will lift the result to the ungraded case in Lemma 3.6.4.

Lemma 3.5.9. *Assume that the base field k is algebraically closed of characteristic 0. Let K be a connected coradically graded Hopf algebra. Then K has finite GK-dimension if and only if K is finitely generated.*

Proof. If K is finitely generated, then by Proposition 3.5.5, K has finite GK-dimension. Now assume that K is not finitely generated. If $\dim_k K(1) = \infty$, then K has a Hopf subalgebra isomorphic to $U(\mathfrak{g})$, where $\mathfrak{g} := K(1)$ is an infinite-dimensional Lie algebra. Hence $\text{GKdim } K = \infty$. Now assume that $\dim_k K(1) < \infty$. It suffices to show that there is a chain of Hopf subalgebras $K^{(1)} \subset K^{(2)} \subset \dots$ such that $\text{GKdim } K^{(i)} + 1 \leq \text{GKdim } K^{(i+1)}$. Let $K^{(1)}$ be the subalgebra generated by $K(1)$. Then $K^{(1)}$ is a finitely generated graded Hopf subalgebra of K . Since $K^{(1)} \subsetneq K$ by assumption, there is some homogeneous element $y \in K \setminus K^{(1)}$ such that $\Delta(y) = 1 \otimes y + y \otimes 1 + w$ where $w \in (K^{(1)})^+ \otimes (K^{(1)})^+$. Let $K^{(2)}$ be the subalgebra generated by $K^{(1)}$ and y . It is obvious that $K^{(2)}$ is again a finitely generated graded Hopf subalgebra. By Lemma 3.5.8, $\text{GKdim } K^{(2)} \geq \text{GKdim } K^{(1)} + 1$. Now $K^{(2)} \subsetneq K$, so we can repeat the above process and get the desired chain of Hopf subalgebras. This completes the proof. \square

Now we are able to deliver the following theorem.

Theorem 3.5.10. *Assume that the base field k is algebraically closed of characteristic 0 and let H be a connected Hopf algebra. Then the following statements are equivalent:*

- (I) $\text{GKdim } H < \infty$;
- (II) $\text{GKdim } \text{gr } H < \infty$;
- (III) $\text{gr } H$ is finitely generated;
- (IV) $\text{gr } H$ is isomorphic to the polynomial ring of ℓ variables for some $\ell \geq 0$ as algebras.

In this case, $\text{GKdim } H = \text{GKdim } \text{gr } H$, which is a positive integer.

Proof. If $\text{GKdim } H = \infty$, we need to show that $\text{GKdim } \text{gr } H$ is also infinity. If not, by Lemma 3.5.9, $\text{gr } H$ is finitely generated. Then by Theorem 3.4.4 (or [KL, Proposition 6.6]), $\text{GKdim } H = \text{GKdim } \text{gr } H < \infty$, which is a contradiction. If $\text{GKdim } H < \infty$, by [KL, Lemma 6.5], $\text{GKdim } \text{gr } H \leq \text{GKdim } H < \infty$. Hence (I) and (II) are equivalent. The equivalence of (II) and (III) is just Lemma 3.5.9. The equivalence of (III) and (IV) follows from Proposition 3.5.5.

If one of the four conditions holds, then $\text{GKdim } H = \text{GKdim } \text{gr } H$ by Theorem 3.4.4. Moreover, in this case $\text{GKdim } \text{gr } H$ is a positive integer by (IV). This completes the proof. \square

As a consequence of Theorem 3.5.10, a connected Hopf algebra enjoys many nice ring-theoretical properties. A few of them are listed in the following corollary.

Corollary 3.5.11. *Assume that the base field k is algebraically closed of characteristic 0 and let H be a connected Hopf algebra of GK-dimension $\ell < \infty$. Then H is*

- (I) a noetherian domain of global dimension ℓ and Krull dimension $\leq \ell$;
- (II) Auslander-regular;
- (III) GK-Cohen-Macaulay, i.e., for any non-zero finitely generated H -module M ,

$$j(M) + \text{GKdim } M = \text{GKdim } H,$$

where $j(M) := \min\{n \mid \text{Ext}_H^n(M, H) \neq 0\}$.

Proof. By Theorem 3.5.10, $\text{gr } H$ is a noetherian domain of global dimension and Krull dimension ℓ . Now by [BG2, Lemma I.12.12, Theorem I.12.13] and [MR, Lemma 5.6, Corollary 6.18], H is a noetherian domain of global dimension and Krull dimension $\leq \ell$. Moreover, by taking M to be the trivial H -module k in (III), we have $j(k) = \ell$. This shows that the global dimension of H is ℓ .

Since $\text{gr } H$ is noetherian, the filtration $\{H_n\}_{n \geq 0}$ on H is Zariskian by [Bj, 2.10]. Then the statement (II) follows from [Bj, Theorem 3.9].

For the statement (III), we first choose a good filtration $\{M_n\}_{n \in \mathbb{Z}}$ of M in the sense of [LV, Definition 5.1]. It then follows from [LV, Lemma 5.4] that $\text{gr } M$ is a finitely generated $\text{gr } H$ -module. It is clear that $\text{gr } H$ is GK-Cohen-Macaulay. Hence $j_{\text{gr } H}(\text{gr } M) + \text{GKdim } \text{gr } M = \text{GKdim } \text{gr } H$. As mentioned in the proof of [Bj, Theorem 3.9], $j_{\text{gr } H}(\text{gr } M) = j(M)$. By Theorem 3.5.10, $\text{GKdim } \text{gr } H = \text{GKdim } H$ and $\text{gr } H$ is a finitely generated algebra. It then follows from [KL, Proposition 6.6] that $\text{GKdim } \text{gr } M = \text{GKdim } M$. This completes the proof. \square

3.6 Connected Hopf algebras of GK-dimension three

Throughout this section, the base field k is algebraically closed of characteristic zero. We are going to classify all connected Hopf algebras of GK-dimension three. To begin with, we introduce two classes of Hopf algebras.

Example 3.6.1. Let A be the algebra generated by elements X, Y, Z satisfying the following relations,

$$\begin{aligned} [X, Y] &= 0, \\ [Z, X] &= \lambda_1 X + \alpha Y, \\ [Z, Y] &= \lambda_2 Y, \end{aligned}$$

where $\alpha = 0$ if $\lambda_1 \neq \lambda_2$ and $\alpha = 0$ or 1 if $\lambda_1 = \lambda_2$. Then A becomes a Hopf algebra via

$$\begin{aligned} \epsilon(X) &= 0, & \Delta(X) &= 1 \otimes X + X \otimes 1, & S(X) &= -X, \\ \epsilon(Y) &= 0, & \Delta(Y) &= 1 \otimes Y + Y \otimes 1, & S(Y) &= -Y, \\ \epsilon(Z) &= 0, & \Delta(Z) &= 1 \otimes Z + X \otimes Y + Z \otimes 1, & S(Z) &= -Z + XY. \end{aligned}$$

We denote this Hopf algebra by $A(\lambda_1, \lambda_2, \alpha)$.

Example 3.6.2. Let B be the algebra generated by elements X, Y, Z satisfying the following relations,

$$\begin{aligned} [X, Y] &= Y, \\ [Z, X] &= -Z + \lambda Y, \\ [Z, Y] &= \frac{1}{2}Y^2, \end{aligned}$$

where $\lambda \in k$. Then B becomes a Hopf algebra via

$$\begin{aligned} \epsilon(X) &= 0, \quad \Delta(X) = 1 \otimes X + X \otimes 1, \quad S(X) = -X, \\ \epsilon(Y) &= 0, \quad \Delta(Y) = 1 \otimes Y + Y \otimes 1, \quad S(Y) = -Y, \\ \epsilon(Z) &= 0, \quad \Delta(Z) = 1 \otimes Z + X \otimes Y + Z \otimes 1, \quad S(Z) = -Z + XY. \end{aligned}$$

We denote this Hopf algebra by $B(\lambda)$.

Proposition 3.6.3. *The algebras $A(\lambda_1, \lambda_2, \alpha)$ and $B(\lambda)$ are connected Hopf algebras of GK-dimension three.*

Proof. We only prove the statement for $B(\lambda)$. The case of $A(\lambda_1, \lambda_2, \alpha)$ can be proved analogously.

As mentioned in [GZ2, Section 1], to check $B(\lambda)$ is a Hopf algebra, it suffices to check the Hopf algebra axioms on a set of algebra generators for $B(\lambda)$, namely, X, Y and Z . This is easy and we leave it to the readers.

By Bergman's Diamond Lemma, the algebra $B(\lambda)$ has a k -linear basis of monomials

$$\{X^{w_1}Y^{w_2}Z^{w_3}\},$$

where $w_i \in \mathbb{N}$. Define the degree of $X^{w_1}Y^{w_2}Z^{w_3}$ to be $w_1 + w_2 + 2w_3$ and let F_n be the k -space spanned by all monomials of degree $\leq n$. It is easy to check that $\{F_n\}_{n \geq 0}$ is an algebra filtration on A by the defining relations. Hence by [KL, Lemma 6.1 (b)],

$$\text{GKdim } B(\lambda) = \varliminf_{n \rightarrow \infty} \log_n \dim_k F_n = 3.$$

Next, we claim that $\{F_n\}_{n \geq 0}$ is also a coalgebra filtration on B , i.e. $\Delta(F_n) \subset \sum_{i=0}^n F_i \otimes F_{n-i}$ for any n . Let $X^{w_1}Y^{w_2}Z^{w_3}$ be a monomial such that $w_1 + w_2 + 2w_3 \leq n$. Then

$$\begin{aligned} \Delta(X^{w_1}Y^{w_2}Z^{w_3}) &= \Delta(X)^{w_1} \Delta(Y)^{w_2} \Delta(Z)^{w_3} \\ &\in (F_0 \otimes F_1 + F_1 \otimes F_0)^{w_1+w_2} \cdot \left(\sum_{i=0}^2 F_i \otimes F_{2-i} \right)^{w_3} \\ &\subset \left(\sum_{i=0}^{w_1+w_2} F_i \otimes F_{w_1+w_2-i} \right) \cdot \left(\sum_{i=0}^{2w_3} F_i \otimes F_{2w_3-i} \right) \\ &\subset \sum_{i=0}^n F_i \otimes F_{n-i}. \end{aligned}$$

For the last two inclusions, we use the fact that $\{F_n\}_{n \geq 0}$ is an algebra filtration. Then it follows from [Mo, Lemma 5.3.4] that the coradical of $B(\lambda)$ is contained in F_0 , which is one-dimensional. Hence $B(\lambda)$ is a connected coalgebra. This completes the proof. \square

Before moving on to study connected Hopf algebras of GK-dimension three, we still need a few lemmas.

Lemma 3.6.4. *Let H be a connected Hopf k -algebra of finite GK-dimension and K a Hopf subalgebra of H . If $\text{GKdim } K = \text{GKdim } H$, then $K = H$.*

Proof. By [Mo, Lemma 5.2.12], $\text{gr } K$ is naturally embedded in $\text{gr } H$ as a graded Hopf subalgebra. Also, by Theorem 3.5.10, $\text{GKdim } \text{gr } K = \text{GKdim } K = \text{GKdim } H = \text{GKdim } \text{gr } H$ and they are all finitely generated. It suffices to show that $\text{gr } K = \text{gr } H$. If not, by Lemma 3.5.8, $\text{GKdim } \text{gr } H \geq \text{GKdim } \text{gr } K + 1$, which is a contradiction. \square

The following proposition is a direct consequence of Lemma 3.6.4.

Proposition 3.6.5. *Let H be a connected Hopf algebra of finite GK-dimension. Then $\text{GKdim } H \geq \dim_k P(H)$. If $\text{GKdim } H = \dim_k P(H)$, then $H \cong U(\mathfrak{g})$ as Hopf algebras, where $\mathfrak{g} = P(H)$. If $\text{GKdim } H = 3$, then $\dim_k P(H) = 2$ or 3 .*

Proof. Let $\mathfrak{g} = P(H)$. Then the injective Lie algebra map $\mathfrak{g} \hookrightarrow H$ induces a Hopf map $U(\mathfrak{g}) \rightarrow H$. It is well known that $P(U(\mathfrak{g})) = \mathfrak{g}$ and therefore by [Mo, Corollary 5.4.7] the Hopf map $U(\mathfrak{g}) \rightarrow H$ is injective. By PBW Theorem, $\text{GKdim } U(\mathfrak{g}) = \dim_k \mathfrak{g}$. Hence $\text{GKdim } H \geq \text{GKdim } U(\mathfrak{g}) = \dim_k P(H)$. If $\text{GKdim } H = \dim_k P(H)$, then we have $\text{GKdim } H = \text{GKdim } U(\mathfrak{g})$. Hence $H = U(\mathfrak{g})$ by Lemma 3.6.4. The last statement is from Lemma 2.4.7. \square

The following proposition is an easy consequence of Proposition 3.6.5. It is also mentioned in [GZ2].

Proposition 3.6.6. *Let H be a connected Hopf algebra of GK-dimension strictly less than 3. Then $\text{GKdim } H = 0, 1$ or 2 . In fact,*

- (I) *if $\text{GKdim } H = 0$, then $H \cong k$, the trivial Hopf algebra;*
- (II) *if $\text{GKdim } H = 1$, then $H \cong k[x]$ with x being primitive;*
- (III) *if $\text{GKdim } H = 2$, then $H \cong U(\mathfrak{g})$, where \mathfrak{g} is either the 2-dimensional abelian Lie algebra or the Lie algebra with basis $\{x, y\}$ and $[x, y] = y$.*

Now we focus on connected Hopf algebras of GK-dimension three. The following theorem is the key to our main theorem of this section.

Theorem 3.6.7. *Let H be a connected Hopf algebra of GK-dimension ≥ 3 such that $\dim_k P(H) = 2$. Then for any linearly independent primitive elements x, y , there exists $z \in H$ such that $\Delta(z) = 1 \otimes z + x \otimes y + z \otimes 1$. Moreover, if in addition $\text{GKdim } H = 3$, then for any such z , the set $\{x, y, z\}$ generates H as an algebra.*

We postpone the proof to the last section. Now we are ready to deliver the main theorem of this section.

Theorem 3.6.8. *Let H be a connected Hopf algebra of GK-dimension three. Then H is isomorphic to one of the following:*

- (I) *The enveloping algebra $U(\mathfrak{g})$ for some three-dimensional Lie algebra \mathfrak{g} ;*
- (II) *The Hopf algebras $A(0, 0, 0)$, $A(0, 0, 1)$, $A(1, 1, 1)$ or $A(1, \lambda, 0)$ from Example 3.6.1 for some $\lambda \in k$;*
- (III) *The Hopf algebras $B(\lambda)$ from Example 3.6.2 for some $\lambda \in k$.*

Proof. By Proposition 3.6.5, $\dim_k P(H)$ is either 2 or 3. If $\dim_k P(H) = 3$, then by Proposition 3.6.5, $H \cong U(\mathfrak{g})$ as Hopf algebras, where $\mathfrak{g} = P(H)$. This gives the Hopf algebras in (I). Now we focus on the case $\dim_k P(H) = 2$.

Let $\mathfrak{h} = P(H)$. Then \mathfrak{h} is a two-dimensional Lie algebra. It is well known that there are two isomorphic classes of two-dimensional Lie algebras.

Case 1: The Lie algebra \mathfrak{h} is spanned by x and y with $[x, y] = 0$. By Theorem 3.6.7, there is $z \in H$ such that $\Delta(z) = 1 \otimes z + x \otimes y + z \otimes 1$ and x, y, z generate H as an algebra. It is easy to check that $[z, x]$ and $[z, y]$ are primitive elements. Therefore

$$\begin{aligned} [z, x] &= a_{11}x + a_{12}y, \\ [z, y] &= a_{21}x + a_{22}y, \end{aligned} \tag{E3.6.8.1}$$

where $a_{ij} \in k$.

Let P be a 2×2 invertible matrix such that $P^{-1}AP$ is a Jordan matrix, where $A = (a_{ij})$. We take $\det P = 1$. Let $P = (b_{ij})$ and $P^{-1} = (c_{ij})$. Then by setting $x' = b_{11}x + b_{21}y$ and $y' = b_{12}x + b_{22}y$, the relations (E3.6.8.1) become

$$\begin{aligned} [z, x'] &= \lambda_1 x' + \alpha y', \\ [z, y'] &= \lambda_2 y', \end{aligned}$$

where $\begin{pmatrix} \lambda_1 & \alpha \\ 0 & \lambda_2 \end{pmatrix}$ is a Jordan matrix. Now we have

$$\Delta(z) = 1 \otimes z + (c_{11}x' + c_{21}y') \otimes (c_{12}x' + c_{22}y') + z \otimes 1.$$

Let $z' = z - \frac{1}{2}c_{11}c_{12}x'^2 - \frac{1}{2}c_{11}c_{22}y'^2 - c_{12}c_{21}x'y'$. Then a direct calculation shows that

$$\Delta(z') = 1 \otimes z' + x' \otimes y' + z' \otimes 1, \quad (\text{E3.6.8.2})$$

and

$$\begin{aligned} [z', x'] &= \lambda_1 x' + \alpha y', \\ [z', y'] &= \lambda_2 y', \end{aligned} \quad (\text{E3.6.8.3})$$

Notice that x', y', z' generate H and $[x', y'] = 0$.

If $\lambda_1 = \lambda_2 = 0$ and $\alpha = 0$ (resp. $\alpha = 1$), then there is a surjective Hopf map from $A(0, 0, 0)$ (resp. $A(0, 0, 1)$) to H sending X, Y, Z to x', y', z' , respectively.

If $\lambda_1 = \lambda_2 \neq 0$ and $\alpha = 1$, then there is a surjective Hopf map from $A(1, 1, 1)$ to H sending X, Y, Z to $x', \frac{1}{\lambda_1}y', \frac{1}{\lambda_1}z'$, respectively.

If $\lambda_1 \neq 0$ and $\alpha = 0$, then there is a surjective Hopf map from $A(1, \lambda_2/\lambda_1, 0)$ to H sending X, Y, Z to $\frac{1}{\lambda_1}x', y', \frac{1}{\lambda_1}z'$, respectively.

If $\lambda_2 \neq 0$ and $\alpha = 0$, then there is a surjective Hopf map from $A(1, \lambda_1/\lambda_2, 0)$ to H sending X, Y, Z to $\frac{1}{\lambda_2}y', -x', \frac{1}{\lambda_1}(z' - x'y')$, respectively.

By Lemma 3.5.2, all the above surjective Hopf maps are isomorphisms. This completes the proof of (II).

Case 2: The Lie algebra \mathfrak{h} is spanned by x and y with $[x, y] = y$. Again by Theorem 3.6.7, there is $z \in H$ such that $\Delta(z) = 1 \otimes z + x \otimes y + z \otimes 1$ and x, y, z generate H as an algebra. A straight calculation shows that $[z, y] - \frac{1}{2}y^2$ and $[z, x] + z$ are primitive elements. Therefore

$$\begin{aligned} [z, x] &= -z + a_{11}x + a_{12}y, \\ [z, y] &= \frac{1}{2}y^2 + a_{21}x + a_{22}y, \end{aligned}$$

where $a_{ij} \in k$. By replacing z with $z - a_{11}x$, we can assume that $a_{11} = 0$. We claim that $a_{21} = a_{22} = 0$. Notice that the relations between x, y, z can be rewritten as

$$\begin{aligned} yx &= xy - y, \\ zx &= xz - z + a_{12}y, \\ zy &= yz + \frac{1}{2}y^2 + a_{21}x + a_{22}y. \end{aligned}$$

By these relations, we have

$$z(yx) = xyz + \frac{1}{2}xy^2 + a_{21}x^2 + a_{22}xy - 2yz + (a_{12} - 1)y^2 - 2a_{21}x - 2a_{22}y.$$

On the other hand,

$$(zy)x = xyz + \frac{1}{2}xy^2 + a_{21}x^2 + a_{22}xy - 2yz + (a_{12} - 1)y^2 - a_{22}y.$$

Since $z(yx) = (zy)x$ by associativity, we have $-2a_{21}x - 2a_{22}y = -a_{22}y$, which implies that $a_{21} = a_{22} = 0$. Now it is clear that there is a surjective Hopf algebra map from $B(a_{12})$ to H sending X, Y, Z to x, y, z , respectively. By Lemma 3.5.2, this surjective map is an isomorphism. This completes the proof of (III). \square

Remark 3.6.9. It is clear from the proof of Theorem 3.6.8 that any Hopf algebra H listed in (II) and (III) has GK-dimension 3 and $\dim_k P(H) = 2$.

In fact, by following the same lines as in case 1 and case 2 in the proof of the previous theorem, we have the following proposition.

Proposition 3.6.10. *Let H be a connected Hopf algebra of GK-dimension ≥ 3 such that $\dim_k P(H) = 2$. Then for any linearly independent primitive elements x, y , there exists $z \in H$ such that $\Delta(z) = 1 \otimes z + x \otimes y + z \otimes 1$. Moreover, the algebra C generated by x, y, z is a Hopf subalgebra of H and $\text{GKdim } C = 3$.*

For the rest of this section, we are going to look closer at the Hopf algebras listed in Theorem 3.6.8 (II) and (III).

Lemma 3.6.11. *Let H be a connected Hopf algebra of GK-dimension three such that $\dim_k P(H) = 2$ and let x, y, z be a set of generators as described in Theorem 3.6.7. Denote by K the Hopf subalgebra generated by x and y . Then H_2/K_2 is spanned by the image of z .*

Proof. It is clear that $z \in H_2 \setminus K_2$. Hence we only have to show that H_2/K_2 is one-dimensional. Notice that $H_1 = K_1$. Therefore it suffices to show that $\text{gr } H(2)/\text{gr } K(2)$ is one dimensional. By Theorem 3.6.7, $\bar{x}, \bar{y} \in \text{gr } H(1)$ and $\bar{z} \in \text{gr } H(2)$ generate $\text{gr } H$. It is also clear that $\text{gr } K$ is the Hopf subalgebra of $\text{gr } H$ generated by \bar{x} and \bar{y} . As a consequence, $\text{gr } H(2) = k\bar{z} + (\text{gr } H(1))^2 = k\bar{z} + (\text{gr } K(1))^2 = k\bar{z} + \text{gr } K(2)$. This completes the proof. \square

For any Hopf algebra H , the commutator ideal $[H, H]$ is a Hopf ideal [GZ2, Lemma 3.7]. We call $H/[H, H]$ the **abelianization** of H . For any $h \in H$, let $\text{ad}(h) \in \text{End}_k(H)$ be the linear map sending u to $[h, u]$ for any $u \in H$.

Proposition 3.6.12. *For any given $\lambda \in k$, $A(0, 0, 0)$, $A(0, 0, 1)$, $A(1, 1, 1)$ and $A(1, \lambda, 0)$ are pairwise non-isomorphic. Also, $A(1, \lambda, 0) \cong A(1, \gamma, 0)$ if and only if $\lambda = \gamma$ or $\lambda\gamma = 1$.*

Proof. We start with the first statement. It is clear from defining relations that the abelianizations of $A(0, 0, 0)$, $A(0, 0, 1)$ and $A(1, 1, 1)$ are $A(0, 0, 0)$, $k[X, Z]$ and $k[Z]$, respectively. And the abelianization of $A(1, \lambda, 0)$ is $k[Z]$ if $\lambda \neq 0$, and $k[Y, Z]$ if $\lambda = 0$. Now to prove the first statement, we only have to show $A(0, 0, 1) \not\cong A(1, 0, 0)$ and $A(1, 1, 1) \not\cong A(1, \lambda, 0)$ for $\lambda \neq 0$.

Suppose that f is a Hopf isomorphism from $H' := A(1, 1, 1)$ to $H := A(1, \lambda, 0)$. We label the canonical generators of H' by X', Y' and Z' . Let K' (resp. K) be the Hopf subalgebra of H' (resp. H) generated by X', Y' (resp. X, Y). Then f restricts to a Hopf isomorphism

from K' to K since f must map primitive elements to primitive elements. As a consequence, f induces a linear isomorphism from H'_2/K'_2 to H_2/K_2 . This indicates that $f(Z')$ is of the form $aZ + u$ for some $a \in k^\times$ and $u \in K_2$. Now consider the maps

$$\text{ad}(Z') : H'_1 \rightarrow H'_1 \text{ and } \text{ad}(f(Z')) : H_1 \rightarrow H_1.$$

Since f restricts to a linear isomorphism from H'_1 to H_1 and $f \circ \text{ad}(Z') = \text{ad}(f(Z')) \circ f$, the two maps must have the same eigenvalues and the same number of independent eigenvectors. However, from the defining relations we see that $\text{ad}(Z')$ has only one linearly independent eigenvector while $\text{ad}(f(Z')) = \text{ad}(aZ + u)$ has two. This shows that $A(1, 1, 1) \not\cong A(1, \lambda, 0)$. If we replace H' and H by $A(0, 0, 1)$ and $A(1, 0, 0)$ respectively, then the above argument shows that $A(0, 0, 1) \not\cong A(1, 0, 0)$. This completes the proof of the first statement.

Next, we proceed to prove the second statement. As mentioned before, the abelianization of $A(1, \lambda, 0)$ is $k[Z]$ if $\lambda \neq 0$, and $k[Y, Z]$ if $\lambda = 0$. Hence $A(1, 0, 0) \not\cong A(1, \lambda, 0)$ for any $\lambda \neq 0$. Now assume $A(1, \lambda, 0) \cong A(1, \gamma, 0)$ where both λ and γ are non-zero, and we have to show that either $\lambda = \gamma$ or $\lambda\gamma = 1$. Repeat the argument in the second paragraph of the proof by taking $H' = A(1, \lambda, 0)$ and $H = A(1, \gamma, 0)$. It is easy to check by defining relations that $\text{ad } Z'$ has eigenvalues $\{1, \lambda\}$ and $\text{ad } f(Z')$ has eigenvalues $\{a, a\gamma\}$. Since they have the same eigenvalues, we must have

$$\begin{cases} 1 = a \\ \lambda = a\gamma \end{cases} \quad \text{or} \quad \begin{cases} 1 = a\gamma \\ \lambda = a \end{cases}.$$

Clearly, these imply that either $\lambda = \gamma$ or $\lambda\gamma = 1$.

Conversely, we only have to show that if $\lambda\gamma = 1$, then $A(1, \lambda, 0) \cong A(1, \gamma, 0)$. Label the canonical generators of $A(1, \lambda, 0)$ by X', Y' and Z' . Then there is a surjective Hopf map from $A(1, \lambda, 0)$ to $A(1, \gamma, 0)$ sending X', Y', Z' to $Y, -\lambda X, \lambda(Z - XY)$, respectively. This map is an isomorphism by Lemma 3.5.2. \square

Proposition 3.6.13. $B(\lambda) \cong B(\gamma)$ if and only if $\lambda = \gamma$.

Proof. Label the canonical generators of $B(\lambda)$ by X', Y', Z' . Suppose that f is an isomorphism from $H' := B(\lambda)$ to $H := B(\gamma)$. By Theorem 3.6.8, $P(H')$ (resp. $P(H)$) are spanned by X', Y' (resp. X, Y). Notice that f restricts to a linear isomorphism from $P(H')$ to $P(H)$. Hence

$$\begin{aligned} f(X') &= a_{11}X + a_{12}Y, \\ f(Y') &= a_{21}X + a_{22}Y, \end{aligned}$$

for some non-degenerate matrix (a_{ij}) . Since $[f(X'), f(Y')] = f(Y')$, we have

$$[a_{11}X + a_{12}Y, a_{21}X + a_{22}Y] = a_{21}X + a_{22}Y.$$

By using the fact $[X, Y] = Y$ and comparing the coefficients, we find that $a_{11} = 1$, $a_{21} = 0$ and $a_{22} \neq 0$. Since f is a coalgebra map,

$$\Delta f(Z') = (f \otimes f)\Delta(Z') = 1 \otimes f(Z') + (X + a_{12}Y) \otimes a_{22}Y + f(Z') \otimes 1.$$

Then it is easy to check that $f(Z') - a_{22}Z - \frac{1}{2}a_{12}a_{22}Y^2 \in P(H)$. As a consequence, there exists $c, d \in k$ such that

$$f(Z') = a_{22}Z + \frac{1}{2}a_{12}a_{22}Y^2 + cX + dY.$$

Since $[f(Z'), f(Y')] = \frac{1}{2}f(Y')^2$,

$$[a_{22}Z + \frac{1}{2}a_{12}a_{22}Y^2 + cX + dY, a_{22}Y] = \frac{1}{2}a_{22}^2Y^2.$$

By comparing the coefficients, we find that $c = 0$. Now the relation $[f(Z'), f(X')] = -f(Z') + \lambda f(Y')$ gives

$$[a_{22}Z + \frac{1}{2}a_{12}a_{22}Y^2 + dY, X + a_{12}Y] = -a_{22}Z - \frac{1}{2}a_{12}a_{22}Y^2 - dY + \lambda a_{22}Y. \quad (\text{E3.6.13.1})$$

The left-hand side of (E3.6.13.1) becomes

$$-a_{22}Z + a_{22}\gamma Y + \frac{1}{2}a_{12}a_{22}Y^2 - a_{12}a_{22}Y^2 - dY.$$

Comparing this with the right-hand side of (E3.6.13.1) we have $\lambda = \gamma$. This completes the proof. \square

We conclude the section by two propositions regarding the algebra structures of the Hopf algebras $A(\lambda_1, \lambda_2, \alpha)$ and $B(\lambda)$, the first of which suggests that they can be considered as coalgebra deformations of universal enveloping algebras. However, we will not pursue this direction further.

Proposition 3.6.14. *For any choice of $(\lambda_1, \lambda_2, \alpha)$ (resp. λ), as an algebra, $A(\lambda_1, \lambda_2, \alpha)$ (resp. $B(\lambda)$) is isomorphic to the enveloping algebra of a solvable Lie algebra.*

Proof. For $A(\lambda_1, \lambda_2, \alpha)$, by the defining relations in Example 3.6.1, we have $A(\lambda_1, \lambda_2, \alpha) \cong U(\mathfrak{g})$ as algebras where \mathfrak{g} is the solvable Lie algebra spanned by X, Y and Z . For $B(\lambda)$, let $Z' := Z - \frac{1}{2}XY$, then $B(\lambda)$ is generated by X, Y and Z' with the following relations

$$\begin{aligned} [X, Y] &= Y, \\ [Z', X] &= -Z' + \lambda Y, \\ [Z', Y] &= 0. \end{aligned}$$

Now it is clear that $B(\lambda) \cong U(\mathfrak{g})$ where \mathfrak{g} is the solvable Lie algebra spanned by X, Y and Z' . \square

Since the Lie algebra $U(\mathfrak{sl}_2)$ is not solvable, we have the following corollary, which suggests that $U(\mathfrak{sl}_2)$ has no non-trivial coalgebra deformations.

Corollary 3.6.15. *For any choice of $(\lambda_1, \lambda_2, \alpha)$ (resp. λ), the Hopf algebra $A(\lambda_1, \lambda_2, \alpha)$ (resp. $B(\lambda)$) is not isomorphic to $U(\mathfrak{sl}_2)$ as an algebra.*

3.7 Proof of Theorem 3.6.7

This section is devoted to the proof of Theorem 3.6.7.

Lemma 3.7.1. *Let $C = U(\mathfrak{h})$ where \mathfrak{h} is a two-dimensional Lie algebra spanned by x and y . Then $\dim_k H^2(\Omega C) = 1$ and in fact $H^2(\Omega C) = (x \otimes y)$ where $(x \otimes y)$ is the cohomology class defined by the cocycle $x \otimes y$.*

Proof. By the PBW Theorem, the coalgebra C has a basis of the form $\{x^i y^j \mid i, j \in \mathbb{N}\}$. It is also well known that C becomes a graded coalgebra by setting $\deg x^i y^j = i + j$. Denote the n -th homogeneous part by $C(n)$. Now the kernel of the counit J can be naturally identified with $\bigoplus_{i=1}^{\infty} C(i)$. Moreover, the graded k -linear dual of the graded coalgebra C is isomorphic to $A = k[x_1, x_2]$ as graded algebras, where x_i has degree 1. By [LPWZ2, Lemma 8.6 (c)], $B^\# A \cong \Omega C$ as DG algebras, where $B^\# A$ is the graded dual of the bar construction of A . On the other hand, by [LPWZ1, Lemma 4.2], $H^\bullet(B^\# A) \cong \text{Ext}_A^\bullet(k_A, k_A)$. As a consequence, $\dim_k H^2(\Omega C) = \dim_k \text{Ext}_A^2(k_A, k_A) = 1$.

It is easy to check by definition that $x \otimes y$ is a cocycle, i.e. $\partial^2(x \otimes y) = 0$. We only have to show that $x \otimes y \notin \text{im } \partial^1$. Suppose to the contrary that there is some $w \in C$ such that $\partial^1(w) = x \otimes y \in C(1) \otimes C(1)$. Then by a degree argument, the element w is in $C(2)$, i.e. w must be of the form $ax^2 + bxy + cy^2$ for some $a, b, c \in k$. However, an easy calculation shows that $\partial^1(ax^2 + bxy + cy^2)$ is in the k -subspace V spanned by $x \otimes x$, $y \otimes y$ and $x \otimes y + y \otimes x$ and clearly $x \otimes y$ is not in V . This completes the proof. \square

Now we are ready to prove Theorem 3.6.7.

Proof of Theorem 3.6.7. Let C be the subalgebra of H generated by x and y . Then C is a Hopf subalgebra of H and C is isomorphic to $U(\mathfrak{h})$ where \mathfrak{h} is a two-dimensional Lie algebra. Notice that by construction $C_1 = H_1$. Let $N \geq 2$ be the least number such that $C_N \subsetneq H_N$. By [Mo, Lemma 5.3.2], there exists $z' \in H_N \setminus C_N$ such that $\Delta(z') = 1 \otimes z' + z' \otimes 1 + u$, where $u \in H_{N-1} \otimes H_{N-1} = C_{N-1} \otimes C_{N-1} \subset C \otimes C$. Without loss of generality, we assume that $\epsilon(z') = 0$.

Now we have two DG algebras, $(\Omega H, \partial_H)$ and $(\Omega C, \partial_C)$. In fact, $(\Omega C, \partial_C)$ can be viewed as a sub-complex of $(\Omega H, \partial_H)$. Notice that $0 = \partial_H^2 \partial_H^1(z') = \partial_H^2(u) = \partial_C^2(u)$, i.e. u is a cocycle. We claim that u represents a non-zero cohomology class in $H^2(\Omega C)$. If not, there is $w \in C$ such that $\partial_C^1(w) = 1 \otimes w - \Delta(w) + w \otimes 1 = u$. As a consequence, $\Delta(z' + w) =$

$1 \otimes (z' + w) + (z' + w) \otimes 1$, i.e. $z' + w$ is a primitive element in H . By the fact that $H_1 = C_1$, $z' + w \in C_1$. But this would imply that $z' \in C$, which contradicts the choice of z' .

By Lemma 3.7.1, the cohomology classes in $H^2(\Omega C)$ represented by u and $x \otimes y$ only differ by a non-zero scalar. Hence there exists $v \in C^+$ and $a \in k^\times$ such that $\partial^1(v) = au - x \otimes y$. Let $z = az' + v$. Then $z \notin C$ and $\Delta(z) = 1 \otimes z + x \otimes y + z \otimes 1$.

Next, assume that $\text{GKdim } H = 3$. Now we have to show that H is generated by x , y and z . Let K be the subalgebra of H generated by x , y and z . Then it is easy to check that K is a sub-bialgebra and thus a Hopf subalgebra of H by Lemma 3.5.1. By the construction of K , $C \subsetneq K$. By Lemma 3.5.8, $\text{GKdim gr } K \geq \text{GKdim gr } C + 1 = 3$. On the other hand, $\text{GKdim gr } K = \text{GKdim } K \leq \text{GKdim } H = 3$ since $K \subset H$. Hence $\text{GKdim } K = 3$. Now it follows from Lemma 3.6.4 that $K = H$. This completes the proof. \square

Chapter 4

LOWER BOUNDS OF GROWTH OF HOPF ALGEBRAS

4.1 Introduction

In this chapter, we study lower bounds of GK-dimension of Hopf algebras in terms of certain invariants of skew primitive elements.

Throughout the chapter, let H be a Hopf algebra over k . A nonzero element $y \in H$ is called $(1, g)$ -primitive (or generally skew primitive) if $\Delta(y) = y \otimes 1 + g \otimes y$ and such a g is called the *weight* of y and denoted by $\mu(y)$. Let $G(H)$ denote the group of group-like elements in H and let $C_0 = kG(H)$. Here is the first lower bound theorem.

Theorem 4.1.1 (First lower bound theorem). *Let $D \supseteq C_0$ be a Hopf subalgebra of H . Let $\{y_i\}_{i=1}^w$ be a set of skew primitive elements such that*

- (a) $\{y_i\}_{i=1}^w$ is linearly independent in H/D .
- (b) for all $i \leq j$, $y_i \mu(y_j) = \lambda_{ij} \mu(y_j) y_i$ for some $\lambda_{ij} \in k^\times$,
- (c) for each i , λ_{ii} is either 1 or not a root of unity.

Then $\text{GKdim } H \geq \text{GKdim } D + w$.

In general λ_{ij} in condition (b) may not exist. If that is the case, we have other ways of obtaining lower bounds.

Let W denote the set of weights $\mu(y)$ for all skew primitive elements $y \notin C_0$ and let $W_{\sqrt{}}$ be the subset of W consisting of weights $\mu(y)$ for all y such that y^n is also a skew primitive for some $n > 1$. (Note that in this paper the term “skew primitive” means “ $(1, g)$ -primitive”). For any subset Φ of a group, the subgroup generated by Φ is denoted by $\langle \Phi \rangle$. Here is the second lower bound theorem.

Theorem 4.1.2 (Second lower bound theorem). *Suppose $\langle W \setminus W_{\sqrt{}} \rangle$ is abelian. Then*

$$\text{GKdim } H \geq \text{GKdim } C_0 + \#(W \setminus W_{\sqrt{}}). \quad (\text{E4.1.2.1})$$

There are examples such that $W = W_{\sqrt{}}$ and $\text{GKdim } H = \text{GKdim } C_0$, but $\#(W_{\sqrt{}})$ is arbitrarily large [Example 4.3.7]. Therefore $W_{\sqrt{}}$ has to be removed from W when we estimate the GK-dimension of H .

Let y be a skew primitive element not in C_0 . If

$$\mu(y)^{-1} y \mu(y) - cy \in C_0 \quad (\text{E4.1.2.2})$$

for some $c \in k^\times$, then c is called the *commutator* of y (with its weight) and denoted by $\gamma(y)$. By Lemma 4.2.6, (E4.1.2.2) is equivalent to

$$\mu(y)^{-1}y\mu(y) - cy = \tau(\mu(y) - 1) \quad (\text{E4.1.2.3})$$

for some $\tau \in k$. Define Γ to be the set of $\gamma(y)$ for all skew primitive elements $y \notin C_0$ such that $\gamma(y)$ exists and let $\Gamma_{\sqrt{}}$ be the subset of Γ consisting of those $\gamma(y)$ which are roots of unity but not 1. If $\gamma(y)$ exists, the pair $(\mu(y), \gamma(y))$ is denoted by $\omega(y)$ and is called the *weight commutator* of y . When (E4.1.2.3) holds and if $c \neq 1$, y can be replaced by $z := y + (c - 1)^{-1}\tau(\mu(y) - 1)$, which is a skew primitive element with $\omega(z) = \omega(y)$ and satisfies the equation $\mu(z)^{-1}z\mu(z) - \gamma(z)z = 0$.

Define Ω to be the set of $\omega(y)$ for all skew primitive elements $y \notin C_0$ such that $\omega(y)$ exists and let $\Omega_{\sqrt{}}$ be the subset of Ω consisting of those $\omega(y)$ in which $\gamma(y)$ is a root of unity but not 1. Theorem 4.1.2 can be improved a little under the same hypothesis:

$$\text{GKdim } H \geq \text{GKdim } C_0 + \#(\Omega \setminus \Omega_{\sqrt{}}).$$

Let y be a skew primitive element not in C_0 with $g = \mu(y)$. Let $T_{g^{-1}}$ be the inverse conjugation by g , namely, $T_{g^{-1}} : a \rightarrow g^{-1}ag$. A scalar c is called a *commutator of y of level n* if n is the least nonnegative integer such that

$$(T_{g^{-1}} - cId_H)^n(y) \in C_0. \quad (\text{E4.1.2.4})$$

In this case we also write $\gamma(y) = c$. Let Z denote the space spanned by the identity element 1 and all skew primitive elements of H and let $Y_{\sqrt{}}$ denote the subspace of Z spanned by those y with commutator of finite level and with $\gamma(y)$ being a root of unity but not 1. Here is the third lower bound theorem. Let W_\times be the subset of W consisting of weights $\mu(y)$ such that the commutator of y (as defined in (E4.1.2.4)) exists and is either 1 or not a root of unity. Note that $W \setminus W_{\sqrt{}} \subseteq W_\times$ and these are often equal [Remark 4.4.9].

Theorem 4.1.3 (Third lower bound theorem). *Suppose $\langle W_\times \rangle$ is abelian. Then*

$$\text{GKdim } H \geq \text{GKdim } C_0 + \dim Z / (C_0 + Y_{\sqrt{}}). \quad (\text{E4.1.3.1})$$

When H is cocommutative, equality holds in Theorem 4.1.3, see (E0.0.1.1). There are examples such that $Z = Y_{\sqrt{}} + C_0$ and $\text{GKdim } H = \text{GKdim } C_0$, but $\dim Y_{\sqrt{}}$ is arbitrarily large [Examples 4.3.7 and 4.4.13]. Therefore it is sensible to consider the quotient space $Z / (C_0 + Y_{\sqrt{}})$ in the above theorem. This is analogous to removing $W_{\sqrt{}}$ in Theorem 4.1.2.

If $\langle W_\times \rangle$ is abelian, Theorem 4.1.3 is a generalization of Theorem 4.1.2 [Lemma 4.4.12]. After some analysis, Theorem 4.1.1 (when $D = C_0$) can be viewed as a consequence of Theorem 4.1.3. These lower bounds provide some evidence that the GK-dimension of H is related to some combinatorial data coming from the skew primitive elements when H is pointed.

The proof of these lower bounds is based on a version of the Poincaré-Birkhoff-Witt (PBW) theorem [Theorem 4.2.5(b)] which states that under some hypotheses the set of monomials generated by skew primitive elements is linearly independent (over the Hopf subalgebra C_0). Restricted to the universal enveloping algebra of a finite dimensional Lie algebra, Theorem 4.2.5 implies the original PBW theorem. Theorem 4.2.5 is in a similar spirit to Kharchenko's quantum analog of the PBW theorem [Kh]. One of the hypotheses in Theorem 4.2.5 is (I4.2.2.3) which assume essentially that the action of the group generated by weights on the space generated by skew primitive elements is locally finite. When $\text{GKdim } H$ is finite, this is a reasonable hypothesis indicated by Theorem 2.1.2 (see also Lemma 4.3.5).

In general we are far from answering Question 0.0.1. There are a lot of unsolved questions concerning the growth of Hopf algebras. The hypotheses in Theorems 4.1.2 and 4.1.3 could be superfluous, but we don't know how to remove them at this moment. When $\langle W \rangle$ is non-abelian, a possible better lower bound could be obtained by replacing $\#(W \setminus W_{\sqrt{\cdot}})$ in Theorem 4.1.2 by $\text{GKdim } k\langle W \rangle$, see Lemma 4.3.6(b) for details. It is expected that these lower bounds can (or should) be improved and that possible upper bounds should be found once finer invariants are introduced. The ultimate goal is to find a formula for the GK-dimension of a Hopf algebra which is analogous to Bass' theorem [KL, Theorem 11.14] in the group algebra case, and then eventually to solve Question 0.0.1.

There are further connections between the growth of Hopf algebras and W and other invariants defined by skew primitive elements. Let rank denote the torsionfree rank of an abelian group.

Proposition 4.1.4. *Suppose $\langle W \rangle$ is abelian and torsionfree. If $\text{rank}\langle \Gamma \rangle > \text{rank}\langle W \rangle = 1$, then H has exponential growth.*

Note that $\text{rank}\langle \Gamma \setminus \Gamma_{\sqrt{\cdot}} \rangle = \text{rank}\langle \Gamma \rangle$ since elements in $\Gamma_{\sqrt{\cdot}}$ have finite order. The rank of $\langle W \rangle$ and $\langle \Gamma \rangle$ should be related when $\text{GKdim } H$ is finite.

Question 4.1.5. Suppose $\text{rank}\langle \Gamma \rangle > \text{rank}\langle W \rangle$. Does then H have exponential growth?

Quite a few families of Hopf algebras of finite GK-dimension have been analyzed extensively during the last few years. But the classification of such Hopf algebras is far from complete. These lower bounds are useful for studying pointed Hopf algebras of low GK-dimension. For example, if $\text{GKdim } H = 2$, then there are only three possibilities for $\text{GKdim } C_0$, $\#(W \setminus W_{\sqrt{\cdot}})$, $\#(\Omega \setminus \Omega_{\sqrt{\cdot}})$ and $\dim Z/(C_0 + Y_{\sqrt{\cdot}})$. This is one of the initial steps in our ongoing project of classifying pointed Hopf algebra domains of GK-dimension two and three.

4.2 First Lower Bound Theorem

In this section we prove Theorem 4.1.1. We need some lemmas.

Lemma 4.2.1. *Let D be a Hopf subalgebra of H and $0 \neq F \in H$. Suppose that*

(a) *L is a subcoalgebra of H containing D ,*

(b) *L is a left D -module via the multiplication, and*

(c) *there are nonzero-divisors (regular elements) $h, g \in L$ such that $\Delta(F) - F \otimes h - g \otimes F \in L \otimes L$.*

Define $V = \{a \in D \mid aF \in L\}$. Then V is either 0 or D .

Proof. Suppose V is nonzero and let a be a nonzero element in V . Let C be the subcoalgebra of D generated by a . There is a k -linear basis

$$\{a_1, \dots, a_v, a_{v+1}, \dots, a_w\}$$

of C such that $C \cap V$ is spanned by $\{a_1, \dots, a_v\}$. This means that $a_i F \in L$ for all $i \leq v$ and that any nontrivial linear combination of $\{a_{v+1}F, \dots, a_w F\}$ is not in L . Write $\Delta(a) = \sum_{1 \leq i, j \leq w} \xi_{ij} a_i \otimes a_j$ for some $\xi_{ij} \in k$.

For simplicity, we use the symbol ldt_1 for any element in L and use ldt_2 for any element in $L \otimes L$. By the definition of V , we have $aF + ldt_1 = 0$ for some $ldt_1 \in L$ and whence

$$\begin{aligned} 0 &= \Delta(aF + ldt_1) = \Delta(a)\Delta(F) + \Delta(ldt_1) \\ &= \left(\sum_{i,j} \xi_{ij} a_i \otimes a_j \right) (F \otimes h + g \otimes F + ldt_2) + ldt_2 \\ &= \left(\sum_{i,j} \xi_{ij} a_i \otimes a_j \right) (F \otimes h) + \left(\sum_{i,j} \xi_{ij} a_i \otimes a_j \right) (g \otimes F) + ldt_2 \\ &= \left(\sum_{\substack{i > v \\ \text{all } j}} \xi_{ij} a_i \otimes a_j \right) (F \otimes h) + \left(\sum_{\substack{j > v \\ \text{all } i}} \xi_{ij} a_i \otimes a_j \right) (g \otimes F) + ldt_2 \end{aligned}$$

where the last equation uses the fact $a_i F \in L$ for all $i \leq v$. The above equation implies that

$$\left(\sum_{\substack{j > v \\ \text{all } i}} \xi_{ij} a_i \otimes a_j \right) (g \otimes F) = - \left(\sum_{\substack{i > v \\ \text{all } j}} \xi_{ij} a_i \otimes a_j \right) (F \otimes h) + ldt_2 \in H \otimes L$$

or equivalently $\sum_{i=1}^w (a_i g) \otimes (\sum_{j > v} \xi_{ij} a_j F) \in H \otimes L$. Since $\{a_i g\}_{i=1}^w$ is linearly independent, we have $\sum_{j > v} \xi_{ij} a_j F \in L$ for all i . By the definition of $\{a_{v+1}, \dots, a_w\}$, we obtain that $\xi_{ij} = 0$ for all $j > v$. Similarly, $\xi_{ij} = 0$ for all $i > v$. Thus $\Delta(a) \in V \otimes V$ and hence V is a subcoalgebra of D . Since V is a subcoalgebra, there is an element $v \in V$ such that $\epsilon(v) = 1$. Then

$$1F = \epsilon(v)F = \sum S(v_1)v_2F \in L$$

since $v_2 \in V$ and $S(v_1) \in D$. This shows that $1 \in V$. Since V is a left ideal of D , $V = D$. \square

Remark 4.2.2. Recall that $(\mathbb{N}^v, +)$, for every $v \geq 1$, is a linearly ordered semigroup with respect to the following ordering. Define $(c_1, \dots, c_v) < (d_1, \dots, d_v)$ if either $\sum_{i=1}^v c_i < \sum_{i=1}^v d_i$ or $\sum_{i=1}^v c_i = \sum_{i=1}^v d_i$ and there is a $p < v$ such that $c_i = d_i$ for all $i \leq p$ and $c_{p+1} < d_{p+1}$.

Let $T := k\langle \{x_j\}_{j \in J} \rangle$ be the free algebra generated by $\{x_j\}_{j \in J}$. Given any family $(f_j)_{j \in J}$ where $f_j \in \mathbb{N}^v$, we can define an \mathbb{N}^v -graded structure on T by setting $\deg x_j = f_j$ for all $j \in J$. Then $T = \bigoplus_{w \in \mathbb{N}^v} T_w$. Since \mathbb{N}^v is linearly ordered, T has a canonical \mathbb{N}^v -filtration defined by $F_w(T) = \sum_{w' \leq w} T_{w'}$. Let B be any factor ring of T . The \mathbb{N}^v -filtration on T induces a unique \mathbb{N}^v -filtration on B , denoted by $\{F_w(B) \mid w \in \mathbb{N}^v\}$. We say that an element $x \in B$ has *filtered multi-degree*

$$\deg x := w = (d_1, \dots, d_v)$$

and *filtered total-degree* $d = \sum d_i$ if $x \in F_w(B) \setminus \sum_{w' < w} F_{w'}(B)$. Note that the filtered total-degree induces an \mathbb{N} -filtration on B .

In applications, we usually start with an algebra A generated by $\{y_1, \dots, y_v\}$ and $G = \{g_i\}_{i \in I}$ for some index set I . By the discussion in the previous paragraph, we can define two filtrations (and corresponding filtered degrees) on A such that if $f = y_{i_1} y_{i_2} \cdots y_{i_s} \in A$, then the filtered total y -degree of f is at most s and the filtered multi- y -degree of f is at most (n_1, \dots, n_v) where n_i is the number of y_i appearing in F , and if $g \in G$, the filtered total- y -degree and the filtered multi- y -degree of g are both 0.

These two filtrations can be extended to the tensor product $A \otimes A$, namely, $F_w(A \otimes A) := \sum_{w'+w'' \leq w} F_{w'}(A) \otimes F_{w''}(A)$ for all $w \in \mathbb{N}^v$ (or $w \in \mathbb{N}$). For simplicity, the words “filtered” and “filtration” might be omitted below.

Assume that $S := \{y_i\}_{i \in I}$ is a set of skew primitive elements of H where I is either \mathbb{N} or $\{1, \dots, v\}$ for some positive integer v . Suppose that D is a Hopf subalgebra of H and that

$$(I4.2.2.1) \quad g_i := \mu(y_i) \in D \text{ for all } i \in I,$$

$$(I4.2.2.2) \quad S \text{ is linearly independent in the space } H/D,$$

$$(I4.2.2.3) \quad \text{for each pair } i \leq j, y_i g_j = \lambda_{ij} g_j y_i + b_{ij} \text{ for some } \lambda_{ij} \in k^\times \text{ and } b_{ij} \in D, \text{ and there is a subalgebra } A \subset D \text{ containing all } b_{ij} \text{ such that } y_i A \subset A y_i + A \text{ and } g_i A \subset A g_i + A \text{ for all } i.$$

In most of the applications D is the coradical C_0 of H and the commutators of the y_i exist. When $b_{ij} = 0$ for all $i \leq j$, we may take $A = k$ and then (I4.2.2.3) is automatic. For every positive integer d , define

$$S^d := \{y_1^{d_1} \cdots y_n^{d_n} \cdots \mid \sum_s d_s = d\}.$$

The following lemma is known and easy to check by a direct computation.

Lemma 4.2.3. *Suppose (I4.2.2.1)-(I4.2.2.3) hold.*

(a) *For every n ,*

$$\Delta(y_i^n) = \sum_{s=0}^n \binom{n}{s} \lambda_{ii} g_i^s y_i^{n-s} \otimes y_i^s + \sum_{s+s' < n} a_{ss'} y_i^s \otimes y_i^{s'}$$

for some $a_{ss'} \in \sum_{t \geq 0} Ag_i^t$. If $b_{ii} = 0$, then $a_{ss'} = 0$ for all s, s' .

(b) *Let $\{y_1, y_2, \dots, y_z\}$ be a finite subset of S . Then, for $n_1, \dots, n_z \geq 0$,*

$$\begin{aligned} \Delta(y_1^{n_1} \dots y_z^{n_z}) &= \sum_{s_1, \dots, s_z} \left(\prod_{t=1}^z \binom{n_t}{s_t} \right) c_{(s_t)} g_1^{s_1} \dots g_z^{s_z} y_1^{n_1-s_1} \dots y_z^{n_z-s_z} \otimes y_1^{s_1} \dots y_z^{s_z} \\ &\quad + ldt_2 \end{aligned}$$

where $c_{(s_t)} = \prod_{i < j} \lambda_{ij}^{s_j(n_i-s_i)} \in k^\times$. Here ldt_2 is a linear combination of elements of the form $f y_1^{a_1} \dots y_z^{a_z} \otimes y_1^{b_1} \dots y_z^{b_z}$ with $\sum_i (a_i + b_i) < \sum_i n_i$ where $f \in \sum_{t_1, \dots, t_z \geq 0} Ag_1^{t_1} \dots g_z^{t_z}$. If $b_{ij} = 0$ for all $i \leq j$, then $ldt_2 = 0$.

For $\alpha = (n_1, \dots, n_z, 0, \dots)$, define

$$L_\alpha = \sum_G DG \tag{E4.2.3.1}$$

where G runs through elements $y_1^{m_1} \dots y_w^{m_w}$ such that $(m_1, \dots, m_w, 0, \dots) < \alpha$.

Lemma 4.2.4. *Retain the notation as above and suppose (I4.2.2.1)-(I4.2.2.3) hold. Let $\alpha = (n_1, \dots, n_z, 0, \dots)$ and $F = y_1^{n_1} \dots y_z^{n_z}$. Define*

$$V = \{a \in D \mid aF \in L_\alpha\}.$$

Then V is either 0 or D .

Proof. Let L denote L_α in the proof. First we claim that $\Delta(L) \subset L \otimes L$. It suffices to show that $\Delta(G) \in L \otimes L$ for all $G = y_1^{m_1} \dots y_w^{m_w}$ with $(m_1, \dots, m_w, 0, \dots) < \alpha$. By Lemma 4.2.3,

$$\Delta(G) = G \otimes 1 + g_1^{m_1} \dots g_w^{m_w} \otimes G + ldt_2 \in L \otimes L.$$

Thus we proved our claim. It is easy to see that the hypotheses in Lemma 4.2.1(a,b) hold. For the hypothesis in Lemma 4.2.1(c), we note that

$$\Delta(F) = F \otimes 1 + g_1^{n_1} \dots g_z^{n_z} \otimes F + ldt'_2$$

by Lemma 4.2.3(b), where $ldt'_2 \in L \otimes L$. The assertion follows from Lemma 4.2.1. \square

Here is the main result of this section. Recall that $g_i = \mu(y_i)$ for all i .

Theorem 4.2.5. *Assume that (I4.2.2.1)-(I4.2.2.3) hold. Let λ_i denote λ_{ii} for all i .*

(a) Suppose the elements in $\bigcup_{j \geq 0} S^j$ are linearly dependent over D (on the left or on the right). Then there is some $z \in \mathbb{N}$ such that

- (i) λ_z is a primitive p_z -th root of unity for some $p_z > 1$,
- (ii) there are $a_i, b_j \in k$, $p_j \in \mathbb{N}$ such that $y_z^{p_z} + \sum_i a_i y_i + \sum_{j \neq z} b_j y_j^{p_j} \in D$,
- (iii) $g_i = g_z^{p_z}$ whenever $a_i \neq 0$ in part (ii), and
- (iv) $g_j^{p_j} = g_z^{p_z}$ and λ_j is a primitive p_j -th root of unity whenever $b_j \neq 0$ in part (ii).

(b) Suppose λ_i is either 1 or not a root of unity for every i . Then the elements in $\bigcup_{j \geq 0} S^j$ are linearly independent over D (on the left and on the right). As a consequence,

$$\text{GKdim } H \geq \text{GKdim } D + \#(S).$$

Proof. (a) Suppose that $\bigcup_{j \geq 0} S^j$ is linearly dependent over D on the left. Then there is an $F = y_1^{n_1} \cdots y_z^{n_z} \in S^d$ for some $d \geq 0$ such that

$$aF \in L_\alpha, \quad \text{for some } 0 \neq a \in D, \quad (\text{E4.2.5.1})$$

where $\alpha = (n_1, \dots, n_z, 0, \dots)$. The definition of L_α is given in (E4.2.3.1). Choose F among all (a, F) satisfying (E4.2.5.1) so that α is minimal with respect to the linear order $<$ defined in the beginning of Remark 4.2.2. For simplicity let $L = L_\alpha$ for the rest of the proof. Let $V = \{b \in D \mid bF \in L\}$. Then $0 \neq a \in V$. By Lemma 4.2.4, $1 \in V$, or equivalently, $F \in L$. So we can write $F = ldt_1$ where ldt_1 denotes any element in L . By the minimality of α , L is a free left D -module with a basis $\{y_1^{m_1} \cdots y_w^{m_w} \mid (m_1, \dots, m_w, 0, \dots) < \alpha\}$. Note that $L \otimes L$ is a free $D \otimes D$ -module with a basis

$$\{y_1^{m_1} \cdots y_w^{m_w} \otimes y_1^{l_1} \cdots y_{w'}^{l_{w'}} \mid (m_1, \dots, m_w, 0, \dots), (l_1, \dots, l_{w'}, 0, \dots) < \alpha\}.$$

We define a multi-degree on L such that, for any nonzero $a \in D$, $\deg(a) = 0$ and $\deg(ay_1^{m_1} \cdots y_w^{m_w}) = (m_1, \dots, m_w, 0, \dots)$ whenever $(m_1, \dots, m_w, 0, \dots) < \alpha$. Notice that under this definition L is a graded D -module (but not an algebra), which can be viewed as a filtered D -module obviously. Extend this multi-grading naturally to $L \otimes L$ by adding the multi-degrees of the tensor components.

Recall that $F = y_1^{n_1} \cdots y_z^{n_z}$. We may assume $n_z > 0$ (if not, delete y_z in the expression of F). Following the last paragraph, there is an $ldt_1 \in L$ such that $F = -ldt_1$, or equivalently, $y_1^{n_1} \cdots y_z^{n_z} + ldt_1 = 0$. By the choice of F , any element in L has multi-degree less than α . Let ldt_2 denote any element in $L \otimes L$ and let lmt_2 denote any element in $L \otimes L$ with multi-degree less than α . Since the multi-degree of ldt_1 is less than α , $\Delta(ldt_1)$ is an lmt_2

by Lemma 4.2.3. Then, by Lemma 4.2.3 again, we have

$$\begin{aligned}
0 &= \Delta(F + ldt_1) = \Delta(F) + lmt_2 & (\text{E4.2.5.2}) \\
&= \sum_{s_1, \dots, s_z} \left(\prod_{t=1}^z \binom{n_t}{s_t}_{\lambda_t} \right) c_{(s_t)} g_1^{s_1} \cdots g_z^{s_z} y_1^{n_1 - s_1} \cdots y_z^{n_z - s_z} \otimes y_1^{s_1} \cdots y_z^{s_z} + lmt_2 \\
&= \sum_{(s_t) \neq (0), (n_t)} \left(\prod_{t=1}^z \binom{n_t}{s_t}_{\lambda_t} \right) c_{(s_t)} g_1^{s_1} \cdots g_z^{s_z} y_1^{n_1 - s_1} \cdots y_z^{n_z - s_z} \otimes y_1^{s_1} \cdots y_z^{s_z} \\
&\quad + F \otimes 1 + g_1^{n_1} \cdots g_z^{n_z} \otimes F + lmt_2 \\
&= \sum_{(s_t) \neq (0), (n_t)} \left(\prod_{t=1}^z \binom{n_t}{s_t}_{\lambda_t} \right) c_{(s_t)} g_1^{s_1} \cdots g_z^{s_z} y_1^{n_1 - s_1} \cdots y_z^{n_z - s_z} \otimes y_1^{s_1} \cdots y_z^{s_z} + lmt_2
\end{aligned}$$

where lmt_2 represents an element in $L \otimes L$ with multi-degree less than α . The multi-degree of $g_1^{n_1} \cdots g_z^{n_z} y_1^{n_1 - s_1} \cdots y_z^{n_z - s_z} \otimes y_1^{s_1} \cdots y_z^{s_z}$ equals α for any $(s_t) \neq (0), (n_t)$. Using the fact that L is a free D -module with basis $\{y_1^{m_1} \cdots y_w^{m_w} \mid (m_1, \dots, m_w, 0, \dots) < \alpha\}$, we obtain that $\left(\prod_{t=1}^z \binom{n_t}{s_t}_{\lambda_t} \right) c_{(s_t)} = 0$ or $\prod_{t=1}^z \binom{n_t}{s_t}_{\lambda_t} = 0$ for all $(s_t) \neq (0), (n_t)$. If $n_j > 0$ for some $1 \leq j < z$, we take $(s_t) = (0, 0, \dots, 0, n_z)$, then $\prod_{t=1}^z \binom{n_t}{s_t}_{\lambda_t} = 1$, a contradiction. Therefore $n_j = 0$ for all $j < z$ which means that $F = y_z^{n_z}$.

If $n_z = 1$, we have $y_z = \sum_{i < z} b_i y_i + c$ for $c, b_i \in D$. Hence $\sum_{i < z} b_i y_i + c$ is $(1, g_z)$ -primitive. Then applying Δ we obtain that

$$\begin{aligned}
\Delta(b_i) &= b_i \otimes 1, \\
\Delta(b_i)(g_i \otimes 1) &= g_z \otimes b_i, \\
\Delta(c) &= c \otimes 1 + g_z \otimes c.
\end{aligned}$$

These imply that $b_i \in k$ and $g_i = g_z$ when $b_i \neq 0$. This contradicts (I4.2.2.2). Therefore $n_z > 1$.

By the last two paragraphs, $n_z > 1$ and $n_i = 0$ for all $i < z$ and $\binom{n_z}{s_z}_{\lambda_z} = 0$ for all $1 \leq s_z \leq n_z - 1$. This can only happen when λ_z is a primitive n_z -th root of unity [GZ2, Lemma 7.5].

Next let us re-name n_z by p_z and write $F = y_z^{p_z}$. Then $y_z^{p_z} + \sum_i b_i G_i + c_0 = 0$ where $b_i, c_0 \in D$ and the G_i are monomials with multi- y -degree less than $(0, \dots, 0, p_z, 0, \dots)$ (where p_z is in the z -th position). Repeating a computation similar to (E4.2.5.2) (and the induction on the multi- y -degree of G_i) one can show that each G_i (when $b_i \neq 0$) is of the form $y_i^{n_i}$ and each $y_i^{n_i}$ is a skew primitive. If $n_i > 1$, then λ_i is a primitive n_i -th root of unity. In summary, when λ_i is not a root of unity, then $n_i = 1$ and when λ_i is a primitive p_i -th root of unity, then n_i is either 1 or p_i . So we have

$$-y_z^{p_z} = \sum_i a_i y_i + \sum_{j \neq z} b_j y_j^{p_j} + c$$

where $0 \neq a_i, b_j \in D$ and $c \in D$. Thus $\sum_i a_i y_i + \sum_j b_j y_j^{p_j} + c$ is $(1, g_z^{p_z})$ -primitive. Since L is a free left D -module, each of the nonzero $a_i y_i$, $b_j y_j^{p_j}$ and c is $(1, g_z^{p_z})$ -primitive. The coproduct computation shows that $a_i, b_j \in k$ and $g_i = g_z^{p_z}$ and $g_j^{p_j} = g_z^{p_z}$.

(b) The first assertion is an immediate consequence of part (a). To prove the second assertion, we take W to be a finite dimensional subspace of D and let S be a finite set $\{y_1, \dots, y_z\}$. For a subspace $V \subset H$, let V^n be the linear span of all elements $v_1 \cdots v_n$ for $v_i \in V$. By the first assertion,

$$\begin{aligned} \dim(W + k1 + \sum_{i=1}^z k y_i)^{2n} &\geq \dim W^n (k1 + \sum_{i=1}^z k y_i)^n \\ &\geq (\dim W^n) \#(\bigcup_{d=0}^n S^d) \geq (\dim W^n) c n^z \end{aligned}$$

for some positive constant c . This implies that $\text{GKdim } H \geq \text{GKdim } D + \#S$. If S is infinite, let S' be any finite subset of S . Then the above argument shows that $\text{GKdim } H \geq \text{GKdim } D + \#S'$ for any S' . Thus $\text{GKdim } H = \infty = \text{GKdim } D + \#S$. \square

Proof of Theorem 4.1.1. Let $S = \{y_1, \dots, y_w\}$. Then (I4.2.2.1)-(I4.2.2.3) follow easily from (a) and (b). The hypothesis in Theorem 4.2.5(b) is the same as that in Theorem 4.1.1(c). Therefore the assertion follows from Theorem 4.2.5(b). \square

The following easy lemma will be used implicitly later.

Lemma 4.2.6. *Let P be the set of all skew primitive elements in a Hopf algebra H with weight μ . Then P is a k -subspace of H and $P \cap C_0 = k(\mu - 1)$.*

Proof. It is clear that P is a k -subspace of H . For any element $y \in P \cap C_0$, write $y = \sum_{i=1}^n c_i g_i$ for some $c_i \in k$ and $g_i \in G(H)$. Then the equation $\Delta(y) = y \otimes 1 + \mu \otimes y$ forces that $y \in k(\mu - 1)$. \square

4.3 Second Lower Bound Theorem

In this section we prove Theorem 4.1.2, which is a consequence of Theorem 4.1.1. A stronger version will be proved in the next section. Lemmas presented here are also needed for the next section, and cannot be omitted even if we skip Theorem 4.1.2. If $\text{GKdim } H = \infty$, then Theorem 4.1.2 is vacuous. So we may assume that $\text{GKdim } H < \infty$. We refer to Section 0 for the definitions of W, Ω, Γ and $W_{\surd}, \Omega_{\surd}, \Gamma_{\surd}$.

Lemma 4.3.1. *Let y be a skew primitive element not in C_0 such that $\gamma(y)$ is defined.*

(a) *If $\gamma(y) \in \Gamma \setminus \Gamma_{\surd}$, then y^n is not skew primitive for any $n > 1$.*

(b) *If $\gamma(y) \in \Gamma_{\surd}$, then $\mu(y) \in W_{\surd}$.*

Proof. (a) Take S to be the singleton $\{y\}$ and $D = C_0$. Then (I4.2.2.1)-(I4.2.2.3) hold for $A = k[\mu(y)^{\pm 1}]$. Since $\gamma(y) \in \Gamma \setminus \Gamma_{\sqrt{\cdot}}$ the hypothesis in Theorem 4.2.5(b) holds. By Theorem 4.2.5(b), $\{y^n\}_{n \geq 0}$ is linearly independent over C_0 . Since $\gamma(y)$ is not a root of unity, for any $n > 1$, $\Delta(y^n) \notin H \otimes k + C_0 \otimes H$ by Lemma 4.2.3(a). Thus y^n is not a skew primitive. The assertion follows.

(b) Suppose $\gamma(y) \in \Gamma_{\sqrt{\cdot}}$. Since $\gamma(y) \neq 1$, replacing y by $y + \alpha(\mu(y) - 1)$ for a suitable $\alpha \in k$, we have $\mu(y)^{-1}y\mu(y) = \gamma(y)y$. Since $\gamma(y)$ is a primitive n -th root of unity for some $n > 1$, Lemma 4.2.3(a) says that $\Delta(y^n) = y^n \otimes 1 + \mu(y)^n \otimes y^n$, which means that y^n is a skew primitive (could be zero). Therefore $\mu(y) \in W_{\sqrt{\cdot}}$. \square

Let $G(H)$ denote the group of all group-like elements in a Hopf algebra H . Recall that $\text{GKdim } H < \infty$ by a general assumption in this section.

Lemma 4.3.2. *Let y be a skew primitive element not in C_0 and let $x = \mu(y)$. Suppose $\gamma(y)$ exists. Assume that G_0 is a subgroup of $G(H)$ commuting with x . Let $V = k(x - 1) + \sum_{g \in G_0} k(g^{-1}yg)$.*

(a) *Every $z \in V$ is $(1, x)$ -primitive; and $\omega(z) = \omega(y)$ for all $z \in V \setminus k(x - 1)$.*

(b) *If $\gamma(y) \in \Gamma \setminus \Gamma_{\sqrt{\cdot}}$, then $\dim V \leq \text{GKdim } H - \text{GKdim } C_0 + 1$.*

(c) *Suppose that V is finite dimensional and that G_0 is abelian. Then there is $z \in V \setminus k(x - 1)$ such that, either*

(ci) *for every $g \in G_0$, $g^{-1}zg = \lambda_g z$ for some $\lambda_g \in k^\times$, or*

(cii) *for every $g \in G_0$, $g^{-1}zg = z + \tau_g(x - 1)$ for some $\tau_g \in k$.*

(d) *If $\gamma(y)$ is not a root of unity, then $\mu(y)$ has infinite order.*

Proof. (a) Since $gx = xg$ for all $g \in G_0$, $g^{-1}yg$ is a $(1, x)$ -primitive with $\omega(g^{-1}yg) = \omega(y)$.

(b) Let $S = \{g_i^{-1}yg_i\}_{i=1}^w$ be a finite subset of V which is linearly independent in the space $V/k(x - 1)$. Here $g_i \in G_0$ for all $i = 1, \dots, w$. For different i , we have $\mu(g_i^{-1}yg_i) = x$, and

$$x^{-1}(g_i^{-1}yg_i)x = g_i^{-1}(x^{-1}yx)g_i = \gamma(y)(g_i^{-1}yg_i) + \tau(x - 1)$$

where τ is the same as the one in (E4.1.2.3). Then the hypotheses (I4.2.2.1)-(I4.2.2.3) hold for $A = k[x^{\pm 1}]$ and $D = C_0$. Since $\lambda := \gamma(y)$ is either 1 or not a root of unity, Theorem 4.2.5(b) says that $\#S \leq \text{GKdim } H - \text{GKdim } C_0$. Clearly $V \cap C_0 = k(x - 1)$. Thus

$$\dim V - 1 = \dim V/(V \cap C_0) = \#S \leq \text{GKdim } H - \text{GKdim } C_0$$

since S is a basis of $V/(V \cap C_0)$.

(c) First we may assume $x \in G_0$. If not, replace G_0 by the subgroup generated by G_0 and x (replacing G_0 by this larger subgroup does not enlarge V , because of (E4.1.2.3)). Then V is a G_0 -module by conjugation action. Since G_0 is abelian and k is algebraically closed, every finite dimensional simple G_0 -module is 1-dimensional. Thus V has a 1-dimensional simple G_0 -submodule kz . If $z \notin k(x-1)$, then kz being a simple G_0 -module is equivalent to (ci). Otherwise, no element $z \in V \setminus k(x-1)$ generates a simple G_0 -submodule. Hence V has a unique simple G_0 -submodule $M_0 := k(x-1)$. Note that $g^{-1}(x-1)g = (x-1)$ for all $g \in G_0$, so M_0 is the trivial G_0 -module. Since G_0 is commutative and V has only one simple submodule, every simple sub-quotient of V must be isomorphic to the simple M_0 . Pick $z \in V \setminus k(x-1)$ so that the submodule M generated by z is 2-dimensional. Then $M/k(x-1) \cong M_0$, which says that $g^{-1}zg \equiv z$ modulo $k(x-1)$. Hence $g^{-1}zg = z + \tau_g(x-1)$ for some $\tau_g \in k$.

(d) Let $G_0 = \langle \mu(y) \rangle$. It follows from the definition that the existence of $\gamma(y)$ implies that V is finite dimensional. Applying part (ci) to the cyclic group G_0 there is a skew primitive $z \in H \setminus C_0$ such that

$$g^{-1}zg = \lambda(g)z$$

for all $g \in G_0$. It is also clear that $\lambda(\mu(y)) = \gamma(y)$. Since $\gamma(y)$ is not a root of unity, the image of $\lambda : G_0 \rightarrow k^\times$ is infinite. Consequently, G_0 is infinite and $\mu(y)$ has infinite order. \square

Lemma 4.3.3. *Let $\{z_i\}_{i=1}^w$ be a set of skew primitive elements not in C_0 such that $\gamma(z_i)$ exists for each i . If the elements $\omega(z_1), \dots, \omega(z_w)$ are distinct, then $\{z_i\}_{i=1}^w$ is linearly independent in H/C_0 .*

Proof. Suppose $\{z_i\}_{i=1}^w$ is linearly dependent in H/C_0 . Pick a minimal subset, say $\{z_j\}_{j=1}^v$, such that $\sum_{j=1}^v a_j z_j =: c \in C_0$ for some scalars $a_j \in k^\times$. Thus $v > 1$ since $z_i \notin C_0$ for any i . Applying Δ to the equation $\sum_{j=1}^v a_j z_j = c$ we have

$$\Delta(c) = \Delta\left(\sum_{j=1}^v a_j z_j\right) = \sum_{j=1}^v a_j z_j \otimes 1 + \sum_{j=1}^v a_j \mu(z_j) \otimes z_j = c \otimes 1 + \sum_{j=1}^v a_j \mu(z_j) \otimes z_j.$$

Hence $\sum_{j=1}^v \mu(z_j) \otimes a_j z_j \in C_0 \otimes C_0$. By the minimality of v , $\mu(z_j) = \mu(z_{j'})$ for all j, j' .

Set $x = \mu(z_j)$ for all $1 \leq j \leq v$. Applying the conjugation by x to the equation $\sum_{j=1}^v a_j z_j = c$, we obtain $\sum_{j=1}^v \gamma(z_j) a_j z_j = -\sum_{j=1}^v \tau_j (x-1) + x^{-1} c x \in C_0$ for some $\tau_j \in k$. Using the minimality of v , $\gamma(z_j) = \gamma(z_{j'})$ for all j, j' . Thus we obtain a contradiction. The assertion follows. \square

Theorem 4.3.4. *Let $\{y_i\}_{i=1}^w$ be a set of skew primitive elements not in C_0 such that $\omega(y_1), \dots, \omega(y_w)$ are defined and distinct elements in $\Omega \setminus \Omega_{\surd}$. If the subgroup G_0 generated by $\{\mu(y_i)\}_{i=1}^w$ is abelian, then $\text{GKdim } H \geq \text{GKdim } C_0 + w$.*

Proof. By Lemma 4.3.2(a,c), for each i , there is a z_i in $k(\mu(y_i) - 1) + \sum_{g \in G_0} k g^{-1} y_i g$ but not in C_0 such that

$$\omega(z_i) = \omega(y_i),$$

and, for every $g \in G_0$,

$$g^{-1} z_i g = \lambda_{ig} z_i + \tau_{ig} (\mu(y_i) - 1)$$

for some $\lambda_{ig} \in k^\times, \tau_{ig} \in k$. Let $A = kG_0$ and $D = C_0$. Then (I4.2.2.1) is clear and (I4.2.2.2) follows from Lemma 5.3.3 for the set $\{z_i\}_{i=1}^w$. (I4.2.2.3) is a consequence of Lemma 4.3.2(c) as we have seen already. By hypothesis each $\lambda_i := \gamma(z_i)$ is either 1 or not a root of unity. Therefore $\text{GKdim } H \geq \text{GKdim } C_0 + w$ by applying Theorem 4.2.5(b) to the set $\{z_i\}_{i=1}^w$. \square

The next lemma is a slight modification of Corollary 2.3.5. As before we assume that $\text{GKdim } H < \infty$ which is one of the hypotheses in Corollary 2.3.5.

Lemma 4.3.5. *Let y be a skew primitive element not in C_0 with $g = \mu(y)$. Then there is a skew primitive element $z = \sum_{i=0}^n b_i g^{-i} y g^i \in H \setminus C_0$, where $b_i \in k$, such that $g^{-1} z g = \lambda z + \tau(g - 1)$ for some $\lambda \in k^\times$ and $\tau \in k$. Further, if $\lambda \neq 1$, then there is $z' = z + \alpha(g - 1)$ for a suitable $\alpha \in k$ such that $g^{-1} z' g = \lambda z'$.*

Proof. In Corollary 2.3.5, H is assumed to be pointed, but the statement is valid without this hypothesis. The first assertion is equivalent to Corollary 2.3.5. If $\lambda \neq 1$, take $\alpha = (\lambda - 1)^{-1} \tau$. Then $z' = z + \alpha(g - 1)$ is a $(1, g)$ -primitive element satisfying $g^{-1} z' g = \lambda z'$. \square

Now we are ready to prove Theorem 4.1.2.

Proof of Theorem 0.3. Pick any finite subset $\{\mu(y_i)\}_{i=1}^w$ of $W \setminus W_{\sqrt{\cdot}}$ where each y_i is a skew primitive not in C_0 . By Lemma 4.3.5, for each i there is a skew primitive y'_i not in C_0 such that $g_i := \mu(y'_i) = \mu(y_i)$ and that $\gamma(y'_i)$ is defined. By Lemma 4.3.1(b), $\gamma(y'_i)$ is not a root of unity or 1. Hence $\omega(y'_i) \in \Omega \setminus \Omega_{\sqrt{\cdot}}$. The assertion follows from Theorem 4.3.4. \square

Theorem 4.3.4 shows in fact that if $\langle W \setminus W_{\sqrt{\cdot}} \rangle$ is abelian, then

$$\text{GKdim } H \geq \text{GKdim } C_0 + \#(\Omega \setminus \Omega_{\sqrt{\cdot}}).$$

There is also an inequality

$$\text{GKdim } H \geq \text{GKdim } C_0 + \#(W')$$

for any $W' \subset W \setminus W_{\sqrt{\cdot}}$ such that $\langle W' \rangle$ is abelian.

Suppose there is a surjective Hopf algebra morphism $\pi : H \rightarrow C_0$ such that the restriction to C_0 is the identity. Let A be the subalgebra of H generated by all skew primitive elements in $\ker \pi$ and let G_W be the sub-semigroup of $G(H)$ generated by $\mu(y)$ for all skew primitive elements $y \in A$. We do not assume that G_W is abelian.

Lemma 4.3.6. *Suppose there is a surjective Hopf algebra morphism $\pi : H \rightarrow C_0$ such that the restriction to C_0 is the identity. Let A be defined as above.*

(a) $H = R\#C_0$ where R is the ring of right coinvariants of π . Then A is a subalgebra of R and

$$\text{GKdim } H \geq \text{GKdim } R + \text{GKdim } C_0 \geq \text{GKdim } A + \text{GKdim } C_0.$$

(b) Assume that A is a domain. Then $\text{GKdim } A \geq \text{GKdim } kG_W$. As a consequence,

$$\text{GKdim } H \geq \text{GKdim } kG_W + \text{GKdim } C_0.$$

Proof. (a) By [Mo, Theorem 7.2.2], H is isomorphic to a crossed product $R\#_\sigma C_0$ as algebras and by [Mo, Proposition 7.2.3], σ is trivial. Hence $H = R\#C_0$ where R is the ring of right coinvariants of π . It is clear that every skew primitive element in $\ker \pi$ is in R . Therefore $A \subset R$.

Since $H = R\#C_0$, $\text{GKdim } H \geq \text{GKdim } R + \text{GKdim } C_0$. The assertion follows by the fact $A \subset R$.

(b) Define a map $\rho : A \rightarrow C_0 \otimes H$ to be the composition $(\pi \otimes Id_H) \circ \Delta$. Since $\rho(y) \in kG_W \otimes A$ for all skew primitive elements $y \in A$ and since A is generated by these y 's, the image of ρ is in $kG_W \otimes A$. Consequently, (A, ρ) is a left kG_W -comodule algebra. This means that A is a G_W -graded algebra. Let $f : A \rightarrow C_0$ be the map sending any nonzero homogeneous element $h \in A$ to its degree, for example, sending $y_1 \cdots y_n$ to $\mu(y_1) \cdots \mu(y_n)$. Since A is a domain, f is multiplicative.

Pick any finite dimensional space $V = k + \sum_{i=1}^m k\mu(y_i)$ of kG_W where the y_i are skew primitive elements in A , let $W = \{1\} \cup \{y_i\}_{i=1}^m$. Then $\dim(kW)^n \geq \#(f(W^n)) \geq \#(f(W))^n \geq \dim V^n$ for all n . Hence $\text{GKdim } A \geq \text{GKdim } kG_W$. \square

The next example shows why we need to remove W_{\surd} from W (or remove Γ_{\surd} from Γ) in the lower bound theorems.

Example 4.3.7. Let B be the Hopf algebra $B(1, 1, p_1, \dots, p_s, q)$ defined in [GZ2, Construction 1.2]. This is a finitely generated, noetherian, pointed Hopf domain of GK-dimension 2. By [GZ2, Construction 1.2] B is generated by $x, x^{-1}, y_1, \dots, y_s$ where x is a group-like element and y_i 's are skew primitive elements. Let $z = y_1^{p_1}$. Then $z = y_j^{p_j}$ for all j and it is a central skew primitive element. Let $H = B/(z, x^m - 1)$ where $m = \prod_i p_i$. Then H is a finite dimensional pointed Hopf algebra of GK-dimension 0 and $C_0 = k[x, x^{-1}]/(x^m - 1)$ has GK-dimension 0.

By [GZ2, Construction 1.2], $W = W_{\surd} = \{x^{m_i}\}_{i=1}^s$ where $m_i = m/p_i$, $\Gamma = \Gamma_{\surd} = \{q^{-m_i^2}\}_{i=1}^s$, $\Omega = \Omega_{\surd} = \{(x^{m_i}, q^{-m_i^2})\}_{i=1}^s$, and $Z = Y_{\surd} + C_0$ and $Y_{\surd} = \sum_{i=1}^s ky_i$. Thus

$$\#(W_{\surd}) = \#(\Gamma_{\surd}) = \#(\Omega_{\surd}) = \dim Y_{\surd} = s$$

which can be arbitrarily large.

4.4 Third lower bound theorem

The first half of this section concerns some preliminary analysis of Hopf algebras with exponential growth and the proof of Proposition 4.1.4. The proof of the third lower bound theorem is given at the end of the section.

Let (G_0, \times) be a multiplicative abelian group and $\Lambda := \{\lambda_1, \dots, \lambda_v\}$ be a list of 1-dimensional group representations of G_0 for some $v > 1$. Note that this list is allowed to have repetitions. When some λ_i is the trivial representation of G_0 (namely, $\lambda_i(g) = 1$ for all $g \in G_0$), then we also need a group homomorphism $\tau_i : (G_0, \times) \rightarrow (k, +)$ (which must be zero if G_0 is torsion since $\text{char } k = 0$). When λ_i is not trivial, we set $\tau_i = 0$.

Now pick a list of elements $\mu := \{\mu_1, \dots, \mu_v\}$ in G_0 (again allowing repetitions). Let $K := K(\Lambda, \mu)$ be the Hopf algebra generated as an algebra by the elements in the abelian group G_0 and a set of skew primitive elements y_1, \dots, y_v subject to the relations within G_0 and the following additional relations between G_0 and $\{y_i\}_{s=1}^v$,

$$y_i g = \lambda_i(g) g y_i + \tau_i(g) g (\mu_i - 1), \text{ for all } i \text{ and all } g \in G_0.$$

The coalgebra structure of K is determined by

$$\begin{aligned} \Delta(g) &= g \otimes g, \quad \epsilon(g) = 1, \quad \text{for all } g \in G_0, \\ \Delta(y_i) &= y_i \otimes 1 + \mu_i \otimes y_i, \quad \epsilon(y_i) = 0, \quad \text{for all } i = 1, \dots, v. \end{aligned}$$

And the antipode of K is determined by

$$\begin{aligned} S(g) &= g^{-1}, \quad \text{for all } g \in G_0, \\ S(y_i) &= -\mu_i^{-1} y_i, \quad \text{for all } i = 1, \dots, v. \end{aligned}$$

Let $\lambda_{ij} = \lambda_i(\mu_j)$ for all i, j and let Λ_M be the $v \times v$ -matrix (λ_{ij}) .

By Remark 4.2.2, the total y -degree and the multi- y -degree are defined for elements in K . For example, the (filtered) multi- y -degree of $g y_3 y_2$ is $(0, 1, 1, 0, \dots, 0) \in \mathbb{N}^v$.

Let F be a nonzero skew primitive element in K with total y -degree $z \geq 2$. Write $F = \sum c_{h, (i_s)} h y_{i_1} y_{i_2} \cdots y_{i_n}$ where $h \in G_0$ and $0 \neq c_{h, (i_s)} \in k$. A *term* of F means a nonzero monomial $c_{h, (i_s)} h y_{i_1} y_{i_2} \cdots y_{i_n}$ appearing in F .

Lemma 4.4.1. *Let $K := K(\Lambda, \mu)$ be defined as above.*

(a) *K has a k -linear basis*

$$\{g y_{i_1} y_{i_2} \cdots y_{i_s}\}$$

where $g \in G_0$, $i_1, \dots, i_s \in \{1, \dots, v\}$. As a consequence, K contains a free subalgebra $k\langle y_1, y_2 \rangle$ and has exponential growth.

(b) *The coradical of K is kG_0 .*

(c) If F is a skew primitive element of total y -degree $z \geq 2$, then for any term of F with multi- y -degree (N_1, \dots, N_v) and $\sum_i N_i = z$,

$$\prod_{i=1}^v (\lambda_{ii})^{N_i(N_i-1)} \prod_{i<j} (\lambda_{ij}\lambda_{ji})^{N_i N_j} = 1.$$

Proof. (a) The first assertion follows from Bergman's Diamond Lemma [Be, Theorem 1.2]. Consequently, K contains the free algebra of rank 2, $k\langle y_1, y_2 \rangle$. Therefore K has exponential growth.

(b) By definition, Δ is compatible with filtrations defined in Remark 4.2.2. Hence Δ is a homomorphism of filtered algebras. So every group-like element must have total y -degree 0. The assertion follows.

(c) Let $F = \sum c_{h,(i_s)} h y_{i_1} y_{i_2} \cdots y_{i_n}$ with coefficients $c_{h,(i_s)} \neq 0$. For simplicity, let ldt denote any linear combination of monomials of total y -degree less than z . Then we can write $F = \sum c_{h,(i_s)} h y_{i_1} y_{i_2} \cdots y_{i_z} + ldt$. Since $\Delta(F) = F \otimes 1 + \mu(F) \otimes F$, $h = 1$ for terms with total degree z . Pick any term of y -degree z in F , say $c_{1,(i_s)} y_{i_1} y_{i_2} \cdots y_{i_z}$, and let (N_1, \dots, N_v) be its multi- y -degree.

Since F is skew primitive, $S(F) = -\mu(F)^{-1}F$. Since $S(y_i) = -\mu_i^{-1}y_i$, we have

$$\begin{aligned} S(c_{1,(i_s)} y_{i_1} \cdots y_{i_z}) &= c_{1,(i_s)} (-\mu_{i_z}^{-1} y_{i_z}) \cdots (-\mu_{i_1}^{-1} y_{i_1}) \\ &= c_{1,(i_s)} (-1)^z \prod_{s>t} \lambda_{i_s i_t}^{-1} \mu^{-1} y_{i_z} \cdots y_{i_1} + ldt \end{aligned}$$

where $\mu = \prod_{s=1}^z \mu_{i_s}$. Since $S(F) = -\mu(F)^{-1}F$, $\mu = \mu(F)$ and F contains a nonzero term of the form $c'_{(i_s)} y_{i_z} \cdots y_{i_1}$. The same computation shows that

$$\begin{aligned} S(c'_{(i_s)} y_{i_z} \cdots y_{i_1}) &= c'_{(i_s)} (-\mu_{i_1}^{-1} y_{i_1}) \cdots (-\mu_{i_z}^{-1} y_{i_z}) \\ &= c'_{(i_s)} (-1)^z \prod_{a<b} \lambda_{i_a i_b}^{-1} \mu^{-1} y_{i_1} \cdots y_{i_z} + ldt. \end{aligned}$$

Comparing the coefficients in the terms $\mu^{-1} y_{i_z} \cdots y_{i_1}$ and $\mu^{-1} y_{i_1} \cdots y_{i_z}$ in the equation $S(F) = -\mu^{-1}F$, we have

$$-c'_{(i_s)} = c_{1,(i_s)} (-1)^z \prod_{s>t} \lambda_{i_s i_t}^{-1}, \quad -c_{1,(i_s)} = c'_{(i_s)} (-1)^z \prod_{a<b} \lambda_{i_a i_b}^{-1}.$$

Since $c_{1,(i_s)}$ and $c'_{(i_s)}$ are nonzero, the above two equations imply

$$\prod_{s>t} (\lambda_{i_s i_t} \lambda_{i_t i_s}) = 1$$

or

$$\prod_{\{s \neq t\} \subset \{1, 2, \dots, z\}} (\lambda_{i_s i_t} \lambda_{i_t i_s}) = 1. \quad (\text{E4.4.1.1})$$

We know the monomial $y_{i_1} \cdots y_{i_n}$ contains N_i copies of y_i for all $i = 1, \dots, v$. Hence equation (E4.4.1.1) is in fact

$$\prod_{i=1}^v (\lambda_{ii})^{N_i(N_i-1)} \prod_{i<j} (\lambda_{ij}\lambda_{ji})^{N_i N_j} = 1.$$

□

There is a slight modification of Lemma 4.4.1. Suppose λ_{11} is a primitive p_1 -th root of unity for some $p_1 > 1$. Recycle most of the notations before Lemma 4.4.1. Let $L := L(\Lambda, \mu, p_1)$ be the Hopf algebra generated as an algebra by the abelian group G_0 and y_1, \dots, y_v subject to the relations within G_0 and the following additional relations between G_0 and $\{y_i\}_{s=1}^v$

$$\begin{aligned} y_i g &= \lambda_i(g) g y_i + \tau_i(g) g (\mu_i - 1), \quad \text{for all } i \text{ and all } g \in G_0, \\ y_1^{p_1} &= \beta (\mu_1^{p_1} - 1), \quad \text{for some } \beta \in k. \end{aligned}$$

The coalgebra structure of L is determined by

$$\begin{aligned} \Delta(g) &= g \otimes g, \quad \epsilon(g) = 1, \quad \text{for all } g \in G_0, \\ \Delta(y_i) &= y_i \otimes 1 + \mu_i \otimes y_i, \quad \epsilon(y_i) = 0, \quad \text{for all } i = 1, \dots, v. \end{aligned}$$

And the antipode of L is determined by

$$\begin{aligned} S(g) &= g^{-1}, \quad \text{for all } g \in G_0, \\ S(y_i) &= -\mu_i^{-1} y_i, \quad \text{for all } i = 1, \dots, v. \end{aligned}$$

Define $\Lambda_M := (\lambda_{ij}) = (\lambda_i(\mu_j))$. The total y -degree and the multi- y -degree are defined as before.

Lemma 4.4.2. *Let $L := L(\Lambda, \mu, p_1)$ be defined as above. Suppose either $\beta = 0$ or $\lambda_1(g)^{p_1} = 1$ for all $g \in G_0$.*

(a) *L has a k -linear basis*

$$\{g y_{i_1} y_{i_2} \cdots y_{i_s}\}$$

where $g \in G_0$, $i_1, \dots, i_s \in \{1, \dots, v\}$ and there is no u such that $i_u = i_{u+1} = \cdots = i_{u+p_1-1} = 1$. As a consequence, L contains a free subalgebra $k\langle y_1 y_2, y_1 y_2^2 \rangle$ and has exponential growth.

(b) *The coradical of L is kG_0 .*

(c) *If F is a skew primitive element of total y -degree $z \geq 2$, then for any term of F with multi- y -degree (N_1, \dots, N_v) and $\sum_i N_i = z$,*

$$\prod_{i=1}^v (\lambda_{ii})^{N_i(N_i-1)} \prod_{i<j} (\lambda_{ij}\lambda_{ji})^{N_i N_j} = 1.$$

Proposition 4.4.3. *Let H be a Hopf algebra and y_1, y_2 be two skew primitive elements linearly independent in H/C_0 . Suppose that*

- (i) *there is a group-like element x and $d_1, d_2 \in \mathbb{Z}$ such that $\mu(y_i) = x^{d_i}$ for $i = 1, 2$, and*
- (ii) *there are two scalars $q_1, q_2 \in k^\times$ such that $y_i x = q_i x y_i$ for $i = 1, 2$.*

(a) *If x has infinite order and H does not contain a free algebra of rank 2, then*

$$q_1^{d_1(M_1(M_1-1))+d_2(M_1M_2)} q_2^{d_2(M_2(M_2-1))+d_1(M_1M_2)} = 1 \quad (\text{E4.4.3.1})$$

for some integers $M_1, M_2 \geq 0$ satisfying $M_1 + M_2 \geq 2$.

(b) *Assume that one of the following holds:*

- (1) *$q_1 = q_2$ is not a root of unity and $d_1 d_2 > 0$;*
- (2) *$q_1^{d_1} = 1$ and q_2 is not a root of unity and $d_1 d_2 > 0$;*
- (3) *$q_1^{d_1} \neq 1$ is a root of unity and q_2 is not a root of unity and $d_1 d_2 > 0$;*
- (4) *the group $\langle q_1, q_2 \rangle \subset k^\times$ is free abelian of rank 2, $d_1 d_2 \neq 0$.*

Then H contains a free subalgebra of rank 2. Consequently, H has exponential growth.

Proof. (a) Let $\mu_i = x^{d_i}$ and $\lambda_{ij} = q_i^{d_j}$. Then $y_i \mu_j = \lambda_{ij} \mu_j y_i$ and $\gamma(y_i) = \lambda_{ii}$ for all $i, j \in \{1, 2\}$.

Let H_0 be the Hopf subalgebra generated by x, x^{-1}, y_1, y_2 . Let $G_0 = \langle g \rangle \cong \mathbb{Z}$ and let $\lambda_i(g^n) = q_i^n$ for $i = 1, 2$ and all n . Let $\Lambda = \{\lambda_1, \lambda_2\}$ and $\mu = \{g, g\}$. Then there is a surjective Hopf algebra homomorphism $\phi : K := K(\Lambda, \mu) \rightarrow H_0$ sending $g \mapsto x$ and $y_i \mapsto y_i$ for $i = 1, 2$, where we choose $\tau_i = 0$. By Lemma 4.4.1(a) K contains a free algebra of rank 2. If H does not contain a free algebra of rank 2, then $K \rightarrow H_0$ is not injective. By [Mo, Theorem 5.3.1], there is a nonzero skew primitive element $F \in K$ such that $\phi(F) = 0$. Since ϕ is injective on skew primitive elements of y -degree ≤ 1 , F has total y -degree $z \geq 2$. By Lemma 4.4.1(c), for any term of F with multi- y -degree (M_1, M_2) and $M_1 + M_2 = z$, we have the following,

$$(\lambda_{11})^{M_1(M_1-1)} (\lambda_{22})^{M_2(M_2-1)} (\lambda_{12} \lambda_{21})^{M_1 M_2} = 1$$

or equivalently,

$$(q_1^{d_1})^{M_1(M_1-1)} (q_2^{d_2})^{M_2(M_2-1)} (q_1^{d_2} q_2^{d_1})^{M_1 M_2} = 1.$$

This can be simplified to

$$q_1^{d_1(M_1(M_1-1))+d_2(M_1M_2)} q_2^{d_2(M_2(M_2-1))+d_1(M_1M_2)} = 1$$

which is (E4.4.3.1).

(b) Assume H does not contain a free algebra of rank 2 and we will obtain a contradiction. If one of the hypotheses holds, then x has infinite order in $G(H)$ by Lemma 4.3.2(d). Therefore we can apply part (a).

In case (b1), (E4.4.3.1) implies that

$$d_1(M_1(M_1 - 1)) + d_2(M_1M_2) + d_2(M_2(M_2 - 1)) + d_1(M_1M_2) = 0.$$

This is impossible since $d_1d_2 > 0$ and $M_1 + M_2 \geq 2$. Therefore H contains a free algebra of rank 2.

A similar argument works for case (b4).

In case (b2), equation (E4.4.3.1) implies that $(q_2^{d_2(M_2(M_2-1))+d_1(M_1M_2)})^{d_1} = 1$, or

$$d_2(M_2(M_2 - 1)) + d_1(M_1M_2) = 0$$

because q_2 is not a root of unity. Since $d_1d_2 > 0$, the only solution is $M_2 = 0$ and $M_1 = z \geq 2$. Thus we have $F = cy_1^z + ldt$ for some $c \in k^\times$. By Lemma 4.2.3 and the fact that $\lambda_{11} = q_1^{d_1} = 1$, F cannot be skew primitive for any $z \geq 2$. So case (b2) has been taken care of.

It remains to consider case (b3). Suppose $q_1^{d_1}$ is a primitive p_1 -th root of unity. Then $y_1^{p_1}$ is a skew primitive element. If $y_1^{p_1} \notin C_0$, then $\{y_1^{p_1}, y_2\}$ is linearly independent in H/C_0 . Note that if $\alpha y_1^{p_1} + \beta y_2 \in C_0$, then $x^{-1}(\alpha y_1^{p_1} + \beta y_2)x \in C_0$, which would imply $y_1^{p_1}, y_2 \in C_0$ because $q_1^{p_1} \neq q_2$. The assertion follows from case (b2) applied to $\{y_1^{p_1}, y_2\}$. If $y_1^{p_1} \in C_0$, then $y_1^{p_1} = \beta(\mu_1^{p_1} - 1)$ for some β . Replacing K by L in the above argument, (E4.4.3.1) holds again.

Since q_1 is a root of unity, we have

$$(q_2^p)^{d_2(M_2(M_2-1))+d_1(M_1M_2)} = 1$$

for some $p > 1$. Since q_2 is not a root of unity,

$$d_2(M_2(M_2 - 1)) + d_1(M_1M_2) = 0.$$

Since $d_1d_2 > 0$, the only solution is $M_2 = 0$ and $M_1 = z \geq 2$. Now $F = cy_1^z + ldt$, where $c \in k^\times$ and $z < p_1$. By Lemma 4.2.3, F cannot be skew primitive. This is a contradiction. \square

Corollary 4.4.4. *Suppose H has subexponential growth. Let y be a skew primitive element not in C_0 such that $\gamma(y)$ is defined and is not a root of unity. Let G_0 be a finitely generated abelian subgroup of $G(H)$ containing $\mu(y)$ (which has infinite order automatically). Then $V := k(\mu(y) - 1) + \sum_{g \in G_0} k(g^{-1}yg)$ is 2-dimensional. As a consequence, there is a group representation $\lambda : G_0 \rightarrow k$ such that*

$$g^{-1}y'g = \lambda(g)y'$$

for all $g \in G_0$, where $y' = y + \alpha(\mu(y) - 1)$ for some $\alpha \in k$.

Proof. Since $\gamma(y)$ is not a root of unity, we may assume that $y\mu(y) = \gamma(y)\mu(y)y$ after replacing y by $y + \alpha(\mu(y) - 1)$ for some $\alpha \in k$. Let $g \in G_0$. Let $y_1 = y$ and $y_2 = g^{-1}yg$. Then $\gamma(y_1) = \gamma(y_2)$ and it is not a root of unity. By Proposition 5.4.3(b1), y_1 and y_2 are not linearly independent in H/C_0 . The assertion that $\dim V = 2$ follows by applying Lemma 4.2.6. The consequence follows from Lemma 4.3.2(ci). \square

Proof of Proposition 4.1.4. We prove the assertion by contradiction. So we assume that H has subexponential growth.

Pick a pair of skew primitive elements (y_1, y_2) such that the subgroup of $\langle \Gamma \rangle$ generated by $\{\gamma(y_1), \gamma(y_2)\}$ has rank 2. Let $\lambda_{ii} = \gamma(y_i)$ and $g_i = \mu(y_i)$ for $i = 1, 2$. Since $\langle W \rangle$ is abelian, the subgroup $G_0 := \langle g_1, g_2 \rangle$ is abelian. By Corollary 4.4.4, we may further assume that, for $i = 1, 2$ and $j = 1, 2$, $g_j^{-1}y_i g_j = \lambda_{ij}y_i$ for some $\lambda_{ij} \in k^\times$. Since H does not contain a free subalgebra of rank 2, the proofs of Proposition 5.4.3 and (E4.4.3.1) show that

$$(\lambda_{11})^{M_1(M_1-1)}(\lambda_{22})^{M_2(M_2-1)}(\lambda_{12}\lambda_{21})^{M_1M_2} = 1 \quad (\text{E4.4.4.1})$$

for some non-negative M_1, M_2 with $M_1 + M_2 \geq 2$.

Since G_0 is a finitely generated subgroup of $\langle W \rangle$ and since $\langle W \rangle$ is abelian and torsionfree of rank 1, G_0 is isomorphic to \mathbb{Z} . Therefore there is an $x \in G_0$ such that $g_1 = x^a$ and $g_2 = x^b$ for some nonzero integers a, b . Consequently, $g_2^a = g_1^b$. Thus the equation $g_j^{-1}y_i g_j = \lambda_{ij}y_i$ implies that $\lambda_{i2}^a = \lambda_{i1}^b$ for all $i = 1, 2$. Then (E4.4.4.1) implies that

$$(\lambda_{11})^{abM_1(M_1-1)}(\lambda_{22})^{abM_2(M_2-1)}(\lambda_{11})^{b^2M_1M_2}(\lambda_{22})^{a^2M_1M_2} = 1$$

Since the rank of $\langle \lambda_{11}, \lambda_{22} \rangle$ is 2, we have

$$abM_1(M_1 - 1) + b^2M_1M_2 = abM_2(M_2 - 1) + a^2M_1M_2 = 0$$

or

$$a(M_1 - 1) + bM_2 = b(M_2 - 1) + aM_1 = 0.$$

Since a, b are nonzero, this means that $(M_1 - 1)(M_2 - 1) - M_1M_2 = 0$. This is impossible when $M_1 + M_2 \geq 2$, which yields a contradiction. \square

The rest of this section is devoted to the proof of Theorem 4.1.3. The next definition was given in the introduction, but maybe it should be reviewed here. Let y be a $(1, g)$ -primitive element in a Hopf algebra H . Let $T_{g^{-1}}$ denote the inverse conjugation by g , namely, $T_{g^{-1}} : a \rightarrow g^{-1}ag$.

Definition 4.4.5. Let y be a $(1, g)$ -primitive element of H not in C_0 . A nonzero scalar λ is called the commutator of y of level n if $(T_{g^{-1}} - \lambda Id_H)^n(y) \in C_0$ and $(T_{g^{-1}} - \lambda Id_H)^{n-1}(y) \notin C_0$. In this case we write $\gamma(y) = \lambda$. When $n = 1$, $\gamma(y)$ is the commutator of y defined as in (E4.1.2.2) or equivalently in (E4.1.2.3).

In general the commutator of y may not exist. We also need a generalization of Definition 4.4.5. Recall that W_\times is the subset of W consisting of weights $\mu(y)$ such that the commutator of y is either 1 or not a root of unity. Throughout the rest of the section let G_0 be the subgroup $\langle W_\times \rangle$ and suppose that G_0 is abelian. A 1-dimensional representation of G_0 is equivalent to a multiplicative map $\lambda : G_0 \rightarrow k^\times$. Let G_0^* denote the set of 1-dimensional representations of G_0 , which is also called the character group of G_0 .

Definition 4.4.6. Let y be a skew primitive element in $H \setminus C_0$ and let $\lambda \in G_0^*$. We say λ is the *generalized commutator* of y of level n if there is an n such that $(T_{g^{-1}} - \lambda(g)Id_H)^n(y) \in C_0$ for all $g \in G_0$ and $(T_{g^{-1}} - \lambda(g)Id_H)^{n-1}(y) \notin C_0$ for some $g \in G_0$. If $n = 1$, λ is called the *generalized commutator* of y .

Lemma 4.4.7. *In parts (a) and (c) suppose that $\text{GKdim } H < \infty$.*

- (a) *Every skew primitive element $y \in H$ is a linear combination of skew primitives with commutator of finite level and weight $\mu(y)$.*
- (b) *Suppose V is a conjugation G_0 -stable finite dimensional subspace spanned by skew primitives with their weights in G_0 . Then every element $y \in V$ is a linear combination of skew primitives with generalized commutator of finite level.*
- (c) *Every skew primitive $y \in H$ with commutator of level 1 such that $\gamma(y) \in \Gamma \setminus \Gamma_{\surd}$ is a linear combination of skew primitives with generalized commutator of finite level. Further, each nonzero summand of the linear combination has weight commutator $\omega(y)$.*

Proof. (a) If $y \in C_0$, then $y = \alpha(\mu(y) - 1)$ for some $\alpha \in k$ [Lemma 4.2.6] and the commutator of y has level 0 by definition.

If $y \notin C_0$, then, by Lemma 4.3.5, $V := \sum_{n \in \mathbb{Z}} k(g^{-n}yg^n) + k(g - 1)$ is finite dimensional over k , where $g = \mu(y)$. Then $T_{g^{-1}}$ acts on V as an invertible linear map. Pick a basis of V so that the presentation of $T_{g^{-1}}$ with respect to the basis is in the Jordan canonical form. Then each basis element is a skew primitive element with commutator of finite level. The assertion follows.

(b) Write $V = \bigoplus_{j=1}^m V_j$ where each V_j is spanned by skew primitives with weight g_j for distinct group-like elements $g_1, \dots, g_m \in G_0$. Since G_0 is abelian, each V_j is conjugation G_0 -stable. Passing from V to V_j we may assume that each element in V is a skew primitive of weight g .

For every $h \in G_0$, $T_{h^{-1}}$ acts on V as an invertible linear map. It is clear that V is a finite dimensional kG_0 -module. Since kG_0 is commutative, every finite dimensional simple G_0 -module is 1-dimensional and $\text{Ext}_{kG_0}^1(S, S') = 0$ if S and S' are distinct simple modules over kG_0 . Then V is a finite direct sum of submodules V_i so that the support of each V_i is a single closed point of $\text{Spec } kG_0$. This closed point corresponds to a 1-dimensional

G_0 -representation λ_i . Fix any i , every element in V_i has a generalized commutator λ_i with level no more than $\dim V_i$.

(c) Let V be the vector space spanned by all skew primitive elements z with commutator of level 1 such that $\omega(z) = \omega(y)$. Pick any finite set $\{z_1, \dots, z_w\}$ which is linearly independent in $V/(V \cap k(\mu(y) - 1))$. Then (I4.2.2.1)-(I4.2.2.3) hold for $D = C_0$ and $A = k[\mu(y)^{\pm 1}]$. By Theorem 4.2.5(b), $w \leq \text{GKdim } H$. Thus V is finite dimensional. Clearly, $T_{g^{-1}}$ stabilizes V for all $g \in G_0$. The first assertion follows from part (b). The final assertion is clear since every element in V has weight commutator equal $\omega(y)$. \square

We need to introduce some conventions. Let $\mu \in W$. Define P_μ to be the k -linear space spanned by all skew primitives $y \in H$ with $\mu(y) = \mu$. Let γ be a nonzero scalar and let $P_{\mu,\gamma,n}$ be the k -linear space spanned by all skew primitives $y \in P_\mu$ with commutator γ of level no more than n . Let $P_{\mu,\gamma,*} = \sum_{n \geq 0} P_{\mu,\gamma,n}$, $P_{*,*,n} = \sum_{\mu,\gamma} P_{\mu,\gamma,n}$ and $P_{*,*,*} = \sum_{\mu,\gamma,n} P_{\mu,\gamma,n}$.

Given any $\mu \in W$ and $\lambda \in G_0^*$, let $P_{\mu,\lambda,n}$ be the k -linear space spanned by all skew primitives $y \in H$ with generalized commutator λ of level no more than n and $\mu(y) = \mu$. Let $P_{\mu,\lambda,*} = \sum_{n \geq 0} P_{\mu,\lambda,n}$, $P_{*,G^*,n} = \sum_{\mu,\lambda} P_{\mu,\lambda,n}$ and $P_{*,G^*,*} = \sum_{\mu,\lambda,n} P_{\mu,\lambda,n}$.

Lemma 5.4.7(a) says that $P_{*,*,*}$ contains all skew primitive elements and Lemma 5.4.7(c) says that $P_{\mu,\gamma,1}$ is a subspace of $P_{*,G^*,*}$ when $\gamma \in k^\times$ and γ is 1 or not a root of unity.

Lemma 4.4.8. *Retain the above notation. Suppose that H does not contain a free subalgebra of rank 2.*

(a) *If $P_{\mu,\gamma,1} \subset k(\mu - 1)$, then $P_{\mu,\gamma,*} \subset k(\mu - 1)$.*

(b) *If $P_{\mu,\gamma,1} \not\subset k(\mu - 1)$ and γ is not a root of unity, then $P_{\mu,\gamma_0,*} \subset k(\mu - 1)$ for every root of unity γ_0 .*

(c) *If γ is not a root of unity, then $P_{\mu,\gamma,1}/k(\mu - 1)$ has dimension at most 1.*

(d) *Suppose γ_1 and γ_2 are two distinct scalars neither of which is a root of unity. If $P_{\mu,\gamma_i,1} \not\subset k(\mu - 1)$ for $i = 1, 2$, then $\gamma_1^N \gamma_2^M = 1$ for some positive integers N, M . Further there is no $\gamma_3 \in k^\times \setminus \{\gamma_1, \gamma_2\}$ such that $P_{\mu,\gamma_3,1} \not\subset k(\mu - 1)$.*

(e) *Suppose that $\text{GKdim } H < \infty$. If γ is not a root of unity, then $P_{\mu,\gamma,*} = P_{\mu,\gamma,1} \subset P_{*,G^*,*}$.*

Proof. (a) This is clear by induction.

(b) First assume that γ_0 is not 1. Pick $y_2 \in P_{\mu,\gamma_0,*} \setminus C_0$. If $P_{\mu,\gamma_0,*} \not\subset k(\mu - 1)$ for a root of unity γ_0 , by part (a), there is a $y_1 \in P_{\mu,\gamma_0,1} \setminus C_0$. Since γ_0 and γ are not 1, after adding a suitable term $\alpha(\mu(y_i) - 1)$ to y_i , the hypothesis in Proposition 5.4.3(ii) holds. Then we are in the situation of Proposition 5.4.3(b2,b3) (where $d_1 = d_2 = 1$). By Proposition 5.4.3, H contains a free subalgebra of rank 2, a contradiction.

Second assume that $\gamma_0 = 1$. For the statement we only need to consider the Hopf subalgebra generated by group-like and skew primitive elements. So we may assume that H is pointed. Passing to the associated graded Hopf algebra of H with respect to its coradical filtration, one can assume the hypothesis in Proposition 5.4.3(ii) holds. So the above proof works.

(c) This follows from Proposition 5.4.3(b1).

(d) By the proof of Proposition 5.4.3, see (E4.4.3.1),

$$q_1^{d_1(M_1(M_1-1))+d_2(M_1M_2)} q_2^{d_2(M_2(M_2-1))+d_1(M_1M_2)} = 1$$

where M_1 and M_2 are nonnegative integers with $M_1 + M_2 \geq 2$. Note that $\gamma_i = q_i$ and $d_1 = d_2 = 1$ in this case. Let $N = M_1(M_1 - 1) + M_1M_2$ and $M = M_2(M_2 - 1) + M_1M_2$. Then N and M are non-negative and $N + M \geq 2$. Since γ_1 and γ_2 are not roots of unity, both N and M must be positive.

If such a γ_3 exists, by part (b) it is not a root of unity. By the above assertion we have positive integers A, B, C, D such that $\gamma_1^A \gamma_3^B = 1$ and $\gamma_2^C \gamma_3^D = 1$. Then

$$1 = (\gamma_1^N \gamma_2^M)^{AC} = \gamma_1^{ACN} \gamma_2^{CAM} = \gamma_3^{-BCN-DAM}$$

which contradicts the fact γ_3 is not a root of unity.

(e) If $P_{\mu, \gamma, * } \neq P_{\mu, \gamma, 1}$, pick a skew primitive $y_2 \in P_{\mu, \gamma, 2}$. This means that $\mu^{-1}y_2\mu - \gamma y_2 = y_1 \in P_{\mu, \gamma, 1} \setminus k(\mu - 1)$. Consider the Hopf subalgebra K generated as an algebra by y_2, y_1, μ, μ^{-1} . By possibly adding a term $\beta(\mu - 1)$ to y_1 for some $\beta \in k$ and adding a term $\alpha(\mu - 1)$ to y_2 for some $\alpha \in k$, we can assume that $y_1\mu = \gamma\mu y_1$ and $y_2\mu = \gamma\mu y_2 + \mu y_1$.

It remains to show that K contains a free subalgebra of rank 2. Define a filtration F_i of K inductively as follows:

$$F_0 = C_0(K) = k[\mu^{\pm 1}],$$

$$F_1 = F_0 + F_0 y_1 = F_0 y_1 + F_0,$$

$$F_2 = F_0 + F_0 y_1 + F_0 y_1^2 + F_0 y_2 = F_0 + y_1 F_0 + y_1^2 F_0 + y_2 F_0.$$

$$F_n = \sum_{i=1}^{n-1} F_i F_{n-i} \text{ for all } n \geq 3.$$

It is easy to check that F is an Hopf algebra filtration and $\text{gr}_F K$ is a Hopf algebra. In $\text{gr}_F K$, y_1, y_2 are linearly independent in $\text{gr}_F K/k(\mu - 1)$. Since $y_1, y_2 \in P_{\mu, \gamma, 1}(K)$, by part (c), $\text{gr}_F K$ contains a free subalgebra of rank 2. Therefore K contains a free subalgebra of rank 2. This yields a contradiction. Therefore the first assertion (namely, the first equation) follows. Finally, by Lemma 5.4.7(c), $P_{\mu, \gamma, 1} \subset P_{*, G^*, *}$. \square

Remark 4.4.9. Suppose $\text{GKdim } H < \infty$. By Lemmas 4.3.1(b) and 5.4.7(a), $W \setminus W_{\sqrt{\cdot}} \subseteq W_{\times}$. In practice it often happens that $W \setminus W_{\sqrt{\cdot}} = W_{\times}$.

Now we are ready to prove the Third Lower Bound Theorem. Let Y_* be the k -linear vector space spanned by all skew primitive elements y with commutator of finite level such that $\gamma(y) \in \Gamma \setminus \Gamma_{\sqrt{\cdot}}$.

Theorem 4.4.10. *Suppose G_0 is abelian. Then*

$$\text{GKdim } H \geq \text{GKdim } C_0 + \dim Y_*/(Y_* \cap C_0).$$

Proof. Nothing needs to be proved if $\text{GKdim } H = \infty$, so we assume $\text{GKdim } H < \infty$.

Let

$$Y_n = \sum_{\mu \in W_\times, \gamma \in \Gamma \setminus \Gamma_{\checkmark}} P_{\mu, \gamma, n} \quad \text{and} \quad Y_{G_n} = \sum_{\mu \in W_\times, \lambda \in G_0^*, \gamma := \lambda(\mu) \in \Gamma \setminus \Gamma_{\checkmark}} P_{\mu, \lambda, n}$$

for all $n \geq 1$. Then $Y_* = \sum_n Y_n$. Let $Y_{G_*} = \sum_n Y_{G_n}$. We prove the following claim by induction:

Claim A:

$$\text{GKdim } H \geq \text{GKdim } C_0 + \dim Y_{G_n}/(Y_{G_n} \cap C_0)$$

for all $n \geq 1$. When $n = 1$, let $\{y_i\}$ be a basis of $Y_{G_1}/(Y_{G_1} \cap C_0)$ such that each y_i is in $P_{\mu_i, \lambda_i, 1}$ for some μ_i, λ_i . Then we have

$$\mu_j^{-1} y_i \mu_j = \lambda_i(\mu_j) y_i + \tau_{ij}(\mu_i - 1)$$

where $\lambda_i(\mu_i) = \gamma(y_i)$ is either 1 or not a root of unity and where $\tau_{ij} \in k$. By Theorem 4.2.5 (with $D = C_0$ and $A = kG_0$),

$$\text{GKdim } H \geq \text{GKdim } C_0 + \#\{y_i\} = \text{GKdim } C_0 + \dim Y_{G_1}/(Y_{G_1} \cap C_0),$$

which proves Claim A for $n = 1$.

Now assume that Claim A holds for n . Without loss of generality we may assume that H is generated as an algebra by C_0 and all skew primitive elements of H . Define a filtration F_i of H as follows:

$$F_0 = C_0,$$

$$F_1 = F_0 + F_0 Y_{G_1} = F_0 + Y_{G_1} F_0,$$

$$F_2 = F_1^2 + F_0 Z \text{ where } Z \text{ is the } k\text{-linear span of all skew primitive elements of } H, \text{ and}$$

$$F_m = \sum_{i=1}^{m-1} F_i F_{m-i} \text{ for all } m \geq 3.$$

Then $\{F_m\}$ is a Hopf algebra filtration of H . Let K be the associated graded Hopf algebra $\text{gr}_F H$. Note that if $y \in P_{\mu, \lambda, 2} \subset Y_{G_2}(H)$, then $y \in F_2$ and the associated element in K is $\text{gr } y \in F_2/F_1$ and $g^{-1}(\text{gr } y)g = \lambda(g)(\text{gr } y)$ for all $g \in G_0$ (or, when trivially $y \in F_1$, we have $\text{gr } y \in F_1/F_0$ or $\text{gr } y \in F_0$). Thus $\text{gr } y \in Y_{G_1}(K)$ (which is easy to see when $y \in F_1$). By induction one sees that

$$Y_{G_n}(K) \supseteq \{\text{gr } y \mid y \in Y_{G_{(n+1)}}(H)\} =: \text{gr } Y_{G_{(n+1)}}(H)$$

for all n . Applying the induction hypothesis to K ,

$$\text{GKdim } K \geq \text{GKdim } C_0 + \dim Y_{G_n}(K)/(Y_{G_n}(K) \cap C_0).$$

By [KL, Lemma 6.5], $\text{GKdim } H \geq \text{GKdim } K$. It is clear that

$$\dim Y_{G(n+1)}(H)/(Y_{G(n+1)}(H) \cap C_0) = \dim \text{gr } Y_{G(n+1)}(H)/(\text{gr } Y_{G(n+1)}(H) \cap C_0).$$

Therefore

$$\begin{aligned} \text{GKdim } H &\geq \text{GKdim } K \geq \text{GKdim } C_0 + \dim Y_{Gn}(K)/(Y_{Gn}(K) \cap C_0) \\ &\geq \text{GKdim } C_0 + \dim \text{gr } Y_{G(n+1)}(H)/(\text{gr } Y_{G(n+1)}(H) \cap C_0) \\ &= \text{GKdim } C_0 + \dim Y_{G(n+1)}(H)/(Y_{G(n+1)}(H) \cap C_0) \end{aligned}$$

which finishes the induction step. Therefore we proved Claim A.

When n goes to infinity, we have

$$\text{GKdim } H \geq \text{GKdim } C_0 + \dim Y_{G^*}(H)/(Y_{G^*}(H) \cap C_0). \quad (\text{E4.4.10.1})$$

Next we prove the following claim by induction:

Claim B:

$$\text{GKdim } H \geq \text{GKdim } C_0 + \dim Y_n/(Y_n \cap C_0).$$

When $n = 1$, this follows from (E4.4.10.1) since $Y_1(H) \subset Y_{G^*}(H)$ by Lemma 5.4.7(c). Note that Y_1 is G_0 -stable. Using an argument similar to the proof of Claim A by passing to the associated graded Hopf algebra $\text{gr}_F H$ (and replacing Y_{Gn} by Y_n), one sees that Claim B holds. When n goes to infinity, we have

$$\text{GKdim } H \geq \text{GKdim } C_0 + \dim Y_*(H)/(Y_*(H) \cap C_0).$$

□

The proof of Theorem 5.4.10 also shows the following.

Corollary 4.4.11. *Suppose G_0 is abelian and $\text{GKdim } H < \infty$. Let $\mu \in W$ and suppose that $\gamma \in k^\times$ is not a root of unity. Then*

- (a) $P_{\mu,1,*}$ is finite dimensional.
- (b) $P_{\mu,1,*}$ and $P_{\mu,\gamma,*}$ are subspaces of $P_{*,G^*,*}$.
- (c) $Y_* = Y_{G^*}$.

By Lemma 5.4.8(c,e), $P_{\mu,\gamma,*}$ is finite dimensional.

Proof of Corollary 4.4.11. (a) By Theorem 5.4.10, $Y_*/(Y_* \cap C_0)$ is finite dimensional. Since $P_{\mu,1,*} \subset Y_*$ and $P_{\mu,1,*} \cap C_0 = k(\mu - 1)$, we have

$$\dim P_{\mu,1,*}/k(\mu - 1) \leq \dim Y_*/(Y_* \cap C_0) < \infty$$

which implies that $P_{\mu,1,*}$ is finite dimensional.

(b) By Lemma 5.4.8(e) $P_{\mu,\gamma,*}$ is a subspace of $P_{*,G^{*},*}$.

By part (a) $P_{\mu,1,*}$ is finite dimensional. It is clear that $P_{\mu,1,*}$ is G_0 -stable. By Lemma 5.4.7(b,c) $P_{\mu,1,*}$ is a subspace of $P_{*,G^{*},*}$.

(c) As a consequence of part (b), $P_{\mu,\lambda,*} = P_{\mu,\lambda(\mu),*}$ when $\lambda(\mu)$ is either 1 or not a root of unity. The assertion follows. \square

Theorem 4.1.3 is an immediate consequence of Theorem 5.4.10.

Proof of Theorem 4.1.3. Without loss of generality we assume that $\text{GKdim } H < \infty$. First we claim that $Y_* \cap (Y_{\sqrt{\cdot}} + C_0) \subset C_0$. Suppose $y = z + c$ is in $Y_* \cap (Y_{\sqrt{\cdot}} + C_0)$ where $y \in Y_* \setminus C_0$ and $z \in Y_{\sqrt{\cdot}}$ and $c \in C_0$. It is easily reduced to the case when $\mu(y) = \mu(z) = g$ and $c = \alpha(g - 1)$ for some $\alpha \in k$. Then $y \in Y_{\sqrt{\cdot}}$, a contradiction. Therefore the claim holds.

By Lemma 5.4.7(a), every skew primitive element is a linear combination of skew primitives with commutator of finite level. This says that $Z = Y_* + Y_{\sqrt{\cdot}} + C_0$. Since $Y_* \cap (Y_{\sqrt{\cdot}} + C_0) \subset C_0$, we have

$$Z/(C_0 + Y_{\sqrt{\cdot}}) \cong Y_*/Y_* \cap (C_0 + Y_{\sqrt{\cdot}}) = Y_*/(Y_* \cap C_0).$$

The assertion follows by Theorem 5.4.10. \square

Another way of proving Theorem 4.1.2 (with a slightly stronger hypothesis that $\langle W_{\times} \rangle$ is abelian) is using Theorem 4.1.3 and the following lemma, which is due to an anonymous referee.

Lemma 4.4.12. *Suppose $\text{GKdim } H < \infty$. Then*

$$\dim Z/(C_0 + Y_{\sqrt{\cdot}}) \geq \#(W \setminus W_{\sqrt{\cdot}}).$$

Proof. For any $w \leq \#(W \setminus W_{\sqrt{\cdot}})$, take distinct elements $x_1, \dots, x_w \in W \setminus W_{\sqrt{\cdot}}$, we need to prove that $\dim Z/(C_0 + Y_{\sqrt{\cdot}}) \geq w$.

For each i , pick a skew primitive element $z_i \in H \setminus C_0$ such that $x_i = \mu(z_i)$. Lemma 4.3.5 shows how to get a suitable z_i such that $\gamma(z_i)$ is defined. Since $\mu(z_i) = x_i \notin W_{\sqrt{\cdot}}$, Lemma 4.3.1(b) says that $\gamma(z_i)$ is either 1 or not a root of unity.

It suffices to show that z_1, \dots, z_w are linearly independent in $Z/(C_0 + Y_{\sqrt{\cdot}})$, so it is enough to show that z_1, \dots, z_w, y are linearly independent in Z/C_0 for any $y \in Y_{\sqrt{\cdot}} \setminus C_0$. We can arrange $y = y_1 + \dots + y_v$ for some skew primitive elements $y_j \notin C_0$, where y_j has a commutator $\gamma(y_j)$ of finite level which is a nontrivial root of unity, and the pairs $(\mu(y_j), \gamma(y_j))$ are distinct. The pairs $(\mu(z_i), \gamma(z_i))$ are already distinct, and $(\mu(z_i), \gamma(z_i)) \neq (\mu(y_j), \gamma(y_j))$ for all i, j because $\gamma(z_i)$ is either 1 or not a root of unity.

An improved version of Lemma 5.3.3 for skew primitive elements with commutators of finite level says that $z_1, \dots, z_w, y_1, \dots, y_v$ are linearly independent in Z/C_0 . Consequently, z_1, \dots, z_w, y are linearly independent in Z/C_0 as desired. \square

Finally we end with a simple example in which $Z = Y_{\sqrt{\cdot}} + C_0$.

Example 4.4.13. Let H be generated as an algebra by x and $\{y_i\}_{i=1}^{\infty}$ subject to the relations

$$\begin{aligned}x^2 &= 1, \\y_i^2 &= 0, \\y_i y_j + y_j y_i &= 0, \\xy_i + y_i x &= 0\end{aligned}$$

for all $i, j \in \mathbb{N}$. The coalgebra structure and the antipode of H are determined by

$$\begin{aligned}\Delta(x) &= x \otimes x, & \Delta(y_i) &= y_i \otimes 1 + x \otimes y_i, \\ \epsilon(x) &= 1, & \epsilon(y_i) &= 0, \\ S(x) &= x, & S(y_i) &= -xy_i = y_i x\end{aligned}$$

for all i . It is easy to check the following

- (1) $\text{GKdim } H = 0$,
- (2) $\Omega = \{(x, -1)\} = \Omega_{\sqrt{\cdot}}$,
- (3) $Z = Y_{\sqrt{\cdot}} + C_0$ and $\dim Y_{\sqrt{\cdot}} = \infty$,
- (4) $P_{x,-1,*} = P_{*,-1,1} = Y_{\sqrt{\cdot}}$.

Chapter 5

COASSOCIATIVE LIE ALGEBRAS

5.1 Introduction

We introduce the notion of a coassociative Lie algebra which generalizes in an obvious way both a Lie algebra and a coassociative coalgebra without counit. The enveloping algebra of a coassociative Lie algebra is a bialgebra that is a generalization of the usual universal enveloping algebra of a Lie algebra. The enveloping algebra of a coassociative Lie algebra should be considered as a coalgebraic deformation of the usual universal enveloping algebra on one hand, and potentially as an algebraic deformation of the coordinate ring (regular functions) of certain algebraic groups or semigroups on the other.

Let k be a base field that is algebraically closed and everything is over k .

Let \mathfrak{g} denote an ordinary Lie algebra and L a coassociative Lie algebra. Let $U(\mathfrak{g})$ (respectively, $U(L)$) denote the enveloping algebra of \mathfrak{g} (respectively, L). It is well-known that $U(\mathfrak{g})$ is a Hopf algebra. In contrast, $U(L)$ is not a Hopf algebra in general.

Theorem 5.1.1 (Theorem 5.3.5). *The enveloping algebra $U(L)$ of a coassociative Lie algebra L is a Hopf algebra if and only if L is locally conilpotent.*

Most of unexplained terms will be defined in Sections 1 and 2.

Starting with conilpotent coalgebras or nilpotent Lie algebras, one can construct families of nontrivial coassociative Lie algebras based on them. For instance, we give explicit examples in Example 5.5.4 (based on strictly upper triangular matrix coalgebras) and Example 5.5.3 (based on Heisenberg Lie algebras). However, it is quite unsatisfactory that there is no nontrivial Lie structure based on cosemisimple coalgebras [Proposition 5.4.6] and no nontrivial coalgebra structure based on the semisimple Lie algebra sl_2 [Theorem 5.4.9]. It would be interesting if a generalized version of coassociative Lie structure can be constructed on cosemisimple coalgebras and/or semisimple Lie algebras.

Coassociative Lie algebras are helpful in the classification of connected Hopf algebras of low GK-dimension. Recall that $P(H)$ is the subspace of all primitive elements in H . Let $p(H)$ denote the dimension of $P(H)$. If $\text{GKdim } H \leq 2$, then it is well-known that $H \cong U(\mathfrak{g})$ for a Lie algebra \mathfrak{g} of dimension $p(H)$. If $\text{GKdim } H = 3$, then it follows from Theorem 3.6.8 that H is isomorphic to either

- (i) $U(\mathfrak{g})$ for a Lie algebra \mathfrak{g} of dimension 3, or
- (ii) $U(L)$ for a coassociative Lie algebra L of dimension 3.

In Chapter 6, we will give a classification in GK-dimension 4.

Theorem 5.1.2 (Theorem 6.1.3). *Suppose that k is of characteristic zero. Let H be a connected Hopf algebra of GK-dimension four. Then one of the following occurs.*

- (a) *If $p(H) = 4$, then $H \cong U(\mathfrak{g})$ for a Lie algebra \mathfrak{g} of dimension 4.*
- (b) *If $p(H) = 3$, then $H \cong U(L)$ for an anti-cocommutative coassociative Lie algebra L of dimension 4.*
- (c) *If $p(H) = 2$, then H is not isomorphic to either $U(\mathfrak{g})$ or $U(L)$ as Hopf algebras, for any Lie algebra \mathfrak{g} or any coassociative Lie algebra L . Such a Hopf algebra is isomorphic to one of the four families of Hopf algebras explicitly constructed in Section 6.5.*

The Hopf algebras in Theorem 5.1.2(b,c) are completely classified in Chapter 6. The proof of Theorem 5.1.2 is heavily dependent on the study of coassociative Lie algebras. A result of Milnor-Moore-Cartier-Kostant [Mo, Theorem 5.6.5] states that any cocommutative connected Hopf algebra over a field of characteristic zero is isomorphic to $U(\mathfrak{g})$ for some Lie algebra \mathfrak{g} . This applies to case (a) of Theorem 5.1.2. However, the Hopf algebras in Theorem 5.1.2(b,c) are not cocommutative, hence not isomorphic to $U(\mathfrak{g})$ for a usual Lie algebra \mathfrak{g} . A generalization of Theorem 5.1.2(b) states that if $p(H) = \text{GKdim } H - 1$, then H is isomorphic to $U(L)$ for some coassociative Lie algebra L by Theorem 6.3.5.

The Hopf algebra $U(\mathfrak{g})$ is always involutory. In general $U(L)$ is not involutory. By using the Calabi-Yau property of the enveloping algebra of unimodular Lie algebras, we prove the following result.

Theorem 5.1.3 (Theorem 5.3.7). *Let \mathfrak{g} be a finite dimensional unimodular Lie algebra. Suppose that $L := (\mathfrak{g}, \delta)$ is a conilpotent coassociative Lie algebra. Then $U(L)$ is involutory.*

Coassociative Lie algebras are not understood fully, and a lot of basic questions are unsolved. For example,

Question 5.1.4. Let $n(L)$ be the nilpotency of L and $con(L)$ be the conilpotency of L . If $n(L) + con(L) < \infty$, then is there the bound for the number $n(L) + con(L) - \dim L$?

Preliminary analysis of known examples shows that if $n(L) + con(L) < \infty$, then

$$n(L) + con(L) - \dim L \leq 1.$$

As one may guess, this paper grows out of the study of connected Hopf algebras. By Theorem 5.1.2(b), certain classes of connected Hopf algebras are the enveloping algebras of coassociative Lie algebras, which are noncommutative and noncocommutative. This construction is very different from the various classical constructions of quantum groups

which we are familiar with. One way of thinking of a coassociative Lie algebra is that it has an additional coalgebra structure on a Lie algebra. Our work in Chapter 6 suggests that certain “nilpotent” quantum groups could be represented by a Lie algebra with an extra coalgebra structure. This idea might be worth pursuing further.

5.2 Definitions

In this section, we start with some definitions and give some easy examples of coassociative Lie algebras.

Definition 5.2.1. A Lie algebra $(L, [,])$ together with a coproduct $\delta : L \rightarrow L \otimes L$ is called a *coassociative Lie algebra* if

- (a) (L, δ) is a coassociative coalgebra without counit, and
- (b) two operations δ and $[,]$ satisfy the following compatibility condition

$$\delta([a, b]) = b_1 \otimes [a, b_2] + [a, b_1] \otimes b_2 + [a_1, b] \otimes a_2 + a_1 \otimes [a_2, b] + [\delta(a), \delta(b)] \quad (\text{E5.2.1.1})$$

for all $a, b \in L$.

Here $\delta(x) = x_1 \otimes x_2$, which is basically the Sweedler’s notation with the summation indicator omitted.

Definition 5.2.1 is not complete without the following remark.

Remark 5.2.2. For a general Lie algebra L , the product $L \otimes L$ is not a Lie algebra in any natural way. To make sense of $[\delta(a), \delta(b)]$ in (E5.2.1.1), we need to embed L into the usual universal enveloping algebra $U(L)$. Then $L \otimes L$ can be naturally identified with a subspace of $U(L) \otimes U(L)$. Under this identification, $[f, g]$ is defined as the commutator of f and g in the associative algebra $U(L) \otimes U(L)$ for all elements $f, g \in L \otimes L$. The equation (E5.2.1.1) implies that the element $[\delta(a), \delta(b)] \in U(L) \otimes U(L)$ is actually in the subspace $L \otimes L$, namely,

$$[\delta(a), \delta(b)] \in L \otimes L. \quad (\text{E5.2.2.1})$$

Inside $U(L) \otimes U(L)$, the equation (E5.2.1.1) can be written as

$$\delta([a, b]) = [a \otimes 1 + 1 \otimes a, \delta(b)] + [\delta(a), b \otimes 1 + 1 \otimes b] + [\delta(a), \delta(b)] \quad (\text{E5.2.2.2})$$

for all $a, b \in L$. Let L^1 denote the Lie algebra extension $L \oplus k1$ where 1 is in the center (i.e., $[1, L^1] = 0$). Define $\Delta : L^1 \rightarrow L^1 \otimes L^1$ by

$$\Delta(1) = 1 \otimes 1, \quad \Delta(a) = a \otimes 1 + 1 \otimes a + \delta(a) \quad (\text{E5.2.2.3})$$

for all $a \in L$. By embedding $L^1 = L \oplus k1$ into $U(L)$ naturally, we can define $[f, g]$ for all elements in $f, g \in L^1$. For any k -space V , let $V^{\otimes 2}$ denote $V \otimes V$.

Lemma 5.2.3. *Let L be a Lie algebra, and let $\delta : L \rightarrow L^{\otimes 2}$ be any linear map. Then L is a coassociative Lie algebra if and only if (L^1, Δ) is a counital coalgebra and the equation*

$$\Delta([a, b]) = [\Delta(a), \Delta(b)] \quad (\text{E5.2.3.1})$$

holds in $U(L)^{\otimes 2}$ for all $a, b \in L^1$.

Proof. A direct calculation shows that Δ is coassociative if and only if δ is coassociative. It is also easy to see that (E5.2.1.1) is equivalent to (E5.2.3.1). \square

It follows from (E5.2.2.2) that the kernel $\ker \delta$ is a Lie subalgebra of L . Let

$$\Phi(a, b) := \delta([a, b]) - (b_1 \otimes [a, b_2] + [a, b_1] \otimes b_2 + [a_1, b] \otimes a_2 + a_1 \otimes [a_2, b]).$$

Then (E5.2.1.1) becomes

$$\Phi(a, b) = [\delta(a), \delta(b)]. \quad (\text{E5.2.3.2})$$

Let L be a coassociative Lie algebra. If $\delta = 0$, it is an ordinary Lie algebra. If $[\ , \] = 0$, then L is simply a coassociative coalgebra without counit. Let CoLieAlg denote the category of coassociative Lie algebras, LieAlg denote the category of Lie algebras and CoAlg denote the category of coalgebras without counit. Then both LieAlg and CoAlg are full subcategories of CoLieAlg . Now we give some simple examples of coassociative Lie algebras with nontrivial Lie bracket $[\ , \]$ and nontrivial coproduct δ , in which both sides of (E5.2.1.1) are trivially zero. More complicated examples will be given in later sections.

Example 5.2.4. Let $L = kx \oplus ky$ with $[x, y] = y$ and $\delta(x) = \lambda y \otimes y$ for any $\lambda \in k$ and $\delta(y) = 0$. Then L is a coassociative Lie algebra. It is clear that $(L, [\ , \])$ is a Lie algebra and (L, δ) is a coalgebra. Note that (E5.2.1.1) is trivial when $a = b$. So it suffices to check (E5.2.1.1) for $a = x$ and $b = y$, in which case both sides of (E5.2.1.1) are zero. It can be verified that, up to isomorphism, this is the unique coassociative Lie algebra with an underlying non-abelian Lie algebra of dimension 2.

Here is a slightly different situation.

Lemma 5.2.5. *Let $(L, [\ , \])$ be a Lie algebra and (L, δ) be a coalgebra. Suppose that $\delta(L) \subset Z \otimes Z$ where Z is the center of $(L, [\ , \])$ and that $\delta([L, L]) = 0$. Then L is a coassociative Lie algebra.*

Proof. The hypotheses imply that both sides of (E5.2.1.1) are zero. \square

Example 5.2.6. Suppose that L is a Lie algebra containing a Lie ideal W such that both W and L/W are abelian. For example, L is a Lie algebra of nilpotency 2. Let $\delta : L \xrightarrow{\pi} L/W \rightarrow W^{\otimes 2}$ be a k -linear map where π is the quotient map. A direct calculation shows that $(\delta \otimes 1)\delta = (1 \otimes \delta)\delta = 0$. Thus (L, δ) is coassociative. By the definition of δ , the hypotheses of Lemma 5.2.5 hold; whence L is a coassociative Lie algebra. Below are two special cases.

1. Let L be the 3-dimensional Heisenberg Lie algebra with a basis $\{x, y, z\}$ such that $[x, y] = z$ and z is central. Define $\delta(x) = z \otimes z$ and $\delta(y) = \lambda z \otimes z$, for some $\lambda \in k$, and $\delta(z) = 0$. By the above paragraph, L is a coassociative Lie algebra.
2. Let W_1 be any Lie algebra and W be a vector space. Let $\phi : W_1 \rightarrow W$ be any k -linear map. Define a Lie bracket on $L := W_1 \oplus W$ by

$$[w_1 + w, t_1 + t] = \phi([w_1, t_1]_{W_1})$$

for all $w_1, t_1 \in W_1$ and $w, t \in W$. Then W is a Lie ideal of L such that both L/W and W are abelian. Then every k -linear map $\delta : L \xrightarrow{\pi} W_1 \rightarrow W \otimes W$ defines a coassociative Lie algebra L .

Next, we give an example of “almost coassociative Lie algebra”, which is dependent on the embedding of L into an associative algebra. If A is any associative algebra, define $[\ , \]_A$ to be the commutator of A .

Example 5.2.7. Let H be a bialgebra and let $H_+ := \ker \epsilon$ where ϵ is the counit of H . Let L be a Lie subalgebra of $(H_+, [\ , \]_H)$ (for example, $L = H_+$). Define $\delta(a) = \Delta(a) - 1 \otimes a - a \otimes 1$ for all $a \in H$. Then (H_+, δ) is a coalgebra. Suppose that $\delta(L) \subset L^{\otimes 2}$. Then (L, δ) is a coalgebra. However, $(L, [\ , \]_H, \delta)$ is generally not a coassociative Lie algebra.

Let H be the 4-dimensional Taft Hopf algebra:

$$k\langle g, x \rangle / (xg + gx = 0, g^2 = 1, x^2 = 0)$$

with $\Delta(g) = g \otimes g$, $\Delta(x) = x \otimes 1 + g \otimes x$, and $\epsilon(g) = 1$, $\epsilon(x) = 0$. Let L be $H_+ = kx \oplus k(g - 1) \oplus k(gx)$. Let $y = g - 1$ and $z = gx$. Then $[x, y]_H = -2z$, $[x, z]_H = 0$, $[y, z]_H = 2x$, $\delta(x) = y \otimes x$, $\delta(y) = y \otimes y$, and $\delta(z) = z \otimes y$. Hence

$$[\delta(x), \delta(z)]_{H^{\otimes 2}} = [y \otimes x, z \otimes y]_{H^{\otimes 2}} = yz \otimes xy - zy \otimes yx =: (*).$$

Using the facts $[x, y]_H = -2z$ and $[y, z]_H = 2x$, we have

$$(*) = yz \otimes xy - (yz - 2x) \otimes (xy + 2z) = 2yz \otimes z + 2x \otimes xy + 4x \otimes z,$$

which is not in $L \otimes L$ if we embed L into $U(L)$. Hence (E5.2.1.1) does not hold for $(L, [\ , \]_H, \delta)$, and consequently, L is not a coassociative Lie algebra. After identifying yz with $-z$, and xy with $-x - z$ in H , (E5.2.1.1) does hold in $H \otimes H$.

The enveloping algebra of a coassociative Lie algebra is defined as follows.

Definition 5.2.8. Let L be a coassociative Lie algebra. The enveloping algebra of L , denoted by $U(L)$, is defined to be a bialgebra, whose algebra structure equals that of the enveloping algebra of the underlying Lie algebra L , namely,

$$U(L) = k\langle L \rangle / (ab - ba = [a, b], \forall a, b \in L),$$

and whose coalgebra structure is determined by

$$\Delta(a) = a \otimes 1 + 1 \otimes a + \delta(a), \quad \epsilon(a) = 0$$

for all $a \in L$. By (E5.2.3.1) it is easy to see that $U(L)$ is a bialgebra. We will also use $U(L, \delta)$ to denote $U(L)$ if we want to emphasize the coproduct δ .

It is clear that the assignment $L \rightarrow U(L)$ defines a functor from CoLieAlg to BiAlg where BiAlg is the category of bialgebras.

Example 5.2.9. If $\dim L = 1$, then there are exactly two coassociative Lie algebra structures on L , up to isomorphism. One is determined by $\delta = 0$. In this case, the enveloping algebra is $U(L) = k[x]$ with x being a primitive element. Consequently, $U(L)$ is a Hopf algebra. The other is determined by $\delta(x) = x \otimes x$. Here, $U(L) = k[g]$ where $g = 1 + x$ and g is a group-like element in $U(L)$. In this case, $U(L)$ is not a Hopf algebra because the group-like element g is not invertible in $U(L)$.

Let (\mathfrak{g}, δ) be a coassociative Lie algebra with underlying Lie algebra \mathfrak{g} . Then Poincaré-Birkhoff-Witt (PBW) theorem holds for $U(\mathfrak{g}, \delta)$, since, algebraically it is the usual enveloping algebra $U(\mathfrak{g})$. The difference between $U(\mathfrak{g}, \delta)$ and $U(\mathfrak{g})$ is their coalgebra structures. For many examples of (\mathfrak{g}, δ) , one can construct explicitly a family of bialgebras $B(q)$ dependent on (\mathfrak{g}, δ) , where $q \in k$, such that $B(1) = U(\mathfrak{g}, \delta)$ and $B(0) = U(\mathfrak{g})$. Hence $U(\mathfrak{g}, \delta)$ can be considered as a coalgebraic deformation of $U(\mathfrak{g})$. However, we will not pursue this topic further.

Since $U(L)$ is generated by L as an algebra, $U(L)$ is cocommutative if and only if the underlying coalgebra L is cocommutative. Similarly, $U(L)$ is commutative if and only if the underlying Lie algebra L is abelian.

Let $\delta^n = (\delta \otimes 1^{\otimes n-1})(\delta \otimes 1^{\otimes n-2}) \cdots (\delta \otimes 1)\delta$. Here is a list of definitions.

Definition 5.2.10. Let L_1, L_2, L be coassociative Lie algebras.

- (a) We say that L_1 and L_2 are *quasi-equivalent* if $U(L_1)$ is isomorphic to $U(L_2)$ as bialgebras.
- (b) A Lie algebra \mathfrak{g} is called *rigid* if every compatible δ -structure on \mathfrak{g} is zero.
- (c) A coalgebra C is called *rigid* if every compatible Lie-structure on C is trivial.
- (d) A coalgebra (C, δ) is called *anti-cocommutative* if $\tau\delta = -\delta$, where the flip $\tau : C^{\otimes 2} \rightarrow C^{\otimes 2}$ is defined by $\tau(a \otimes b) = b \otimes a$.
- (e) The *nilpotency* of L , denoted by $n(L)$, is defined to be the nilpotency of the underlying Lie algebra L .

- (f) An element $x \in L$ is called *conilpotent* if $\delta^n(x) = 0$ for some $n > 0$. We say L is *locally conilpotent* if every element in L is conilpotent.
- (g) We call L *n-conilpotent* if $\delta^n(L) = 0$. The smallest such n , denoted by $\text{con}(L)$, is called *conilpotency* of L .

5.3 Results on enveloping algebras

In this section we study some properties of the enveloping algebras $U(L)$. Let B be a bialgebra with coproduct Δ . Define $\delta_B : B \rightarrow B^{\otimes 2}$ by

$$\delta_B(x) = \Delta(x) - x \otimes 1 - 1 \otimes x$$

for all $x \in B$.

Definition 5.3.1. A subspace V in a bialgebra B is called a δ -space of B if

- (a) V is a Lie subalgebra of $(B, [\ , \]_B)$,
- (b) $\epsilon(V) = 0$,
- (c) $\delta_B(V) \subset V^{\otimes 2}$ inside $B^{\otimes 2}$, and
- (d) B is an \mathbb{N} -filtered algebra with an exhaustive filtration defined by $F_n(B) := (k1 + V)^n$, for $n \geq 0$, such that the associated graded ring $\text{gr}_F B$ is isomorphic to the commutative polynomial ring $k[V]$.

Remark 5.3.2. If L is coassociative Lie algebra, then L is δ -space of $U(L)$. In general, a δ -space of $U(L)$ is not unique. See Corollary 5.3.6.

Lemma 5.3.3. *If V is a δ -space of B , then $(V, [\ , \]_B, \delta_B)$ is a coassociative Lie algebra and $B \cong U(V)$ as bialgebras.*

Proof. Let U be the usual enveloping algebra of the Lie algebra $(V, [\ , \]_B)$. Then there is an algebra homomorphism $\phi : U \rightarrow B$ such that $\phi|_V = \text{Id}_V$. It follows from Definition 5.3.1(d) that B is generated by V and that the set $\{v_1^{n_1} \cdots v_d^{n_d} \mid n_i \geq 0\}$ is a k -linear basis of B where $\{v_1, \dots, v_j, \dots\}$ is a k -linear basis of V . Since $\{v_1^{n_1} \cdots v_d^{n_d} \mid n_i \geq 0\}$ is also a k -linear basis of U by PBW theorem, ϕ is an isomorphism of algebras. Note that B is a bialgebra and generated by V as an algebra, one can defined a canonical bialgebra structure Δ_U on U via ϕ such that ϕ is an isomorphism of bialgebras. Let $\delta_U(v) = \Delta_U(v) - v \otimes 1 - 1 \otimes v$ for $v \in V$. Since $\delta_B(x) = \Delta_B(x) - x \otimes 1 - 1 \otimes x$, $\delta_U(v) = \delta_B(v)$ for all $v \in V$ (we are identifying the subspace $V \subset B$ with the subspace $V \subset U$ via the map $\phi|_V = \text{Id}_V$). By Definition 5.3.1(c), one sees easily that $\delta_U(v) \in V^{\otimes 2}$ for all $v \in V$. Since U is a bialgebra (via the map ϕ),

(E5.2.3.1) holds. Now by Lemma 5.2.3 and Definition 5.2.8, $(V, [\ , \]_U, \delta_U)$ is a coassociative Lie algebra with enveloping algebra U . Since $(V, [\ , \]_B, \delta_B) = (V, [\ , \]_U, \delta_U)$ by construction and $U \cong B$ as bialgebras, the results follow. \square

Let (L, δ) be a coalgebra (without counit). Let $L^1 = k1 \oplus L$, and $\Delta : L^1 \rightarrow (L^1)^{\otimes 2}$ be defined as (E5.2.2.3). Moreover, let $\epsilon : L^1 \rightarrow k$ be defined by $\epsilon(1) = 1, \epsilon(x) = 0$ for all $x \in L$. Then the assignment $(L, \delta) \rightarrow (L^1, \Delta, \epsilon)$ defines a functor from CoAlg to CouAlg , where CouAlg is the category of counital coassociative coalgebras. The following lemma is easy.

Lemma 5.3.4. *Let (L, δ) be a coalgebra. Then (L, δ) is locally conilpotent if and only if (L^1, Δ, ϵ) is a connected counital coalgebra.*

Proof. Since L is a sum of its finite dimensional subcoalgebras, we can assume without loss of generality that L is finite dimensional. Note that L can be identified with the quotient coalgebra $L^1/k1$. By taking the k -linear dual, L^* becomes a subalgebra (without a unit) of $(L^1)^*$. In fact, L^* is a maximal ideal of $(L^1)^*$ of codimension 1. Now the lemma is equivalent to the statement that L^* is a nilpotent ideal if and only if L^* is the unique maximal ideal of $(L^1)^*$, which is an easy ring-theoretical fact. \square

Now we are ready to prove Theorem 5.1.1.

Theorem 5.3.5. *Let L be a coassociative Lie algebra. Then the following are equivalent:*

- (a) L is locally conilpotent;
- (b) $U(L)$ is a connected Hopf algebra; and
- (c) $U(L)$ is a Hopf algebra.

Proof. (a) \Rightarrow (b). Since (L, δ) is locally conilpotent, (L^1, Δ, ϵ) is a connected coalgebra by Lemma 5.3.4. Since $U(L)$ is generated by L^1 as an algebra, $U(L)$ is connected as a coalgebra. It follows from [Mo, Lemma 5.2.10] that a connected bialgebra is automatically a Hopf algebra.

(b) \Rightarrow (c). This is clear.

(c) \Rightarrow (a). We proceed by contradiction. Suppose that $U(L)$ is a Hopf algebra, but (L, δ) is not locally conilpotent. Then (L^1, Δ) is not connected, whence its coradical is strictly larger than $k1$. Pick a simple subcoalgebra with counit of L^1 , say C , which not equal to $k1$. Since k is algebraically closed, C is isomorphic to a matrix coalgebra $\bigoplus_{i,j=1,\dots,n} kx_{ij}$ with

$$\Delta(x_{ij}) = \sum_{s=1}^n x_{is} \otimes x_{sj}, \quad \text{and} \quad \epsilon(x_{ij}) = \delta_{ij}$$

for all $1 \leq i, j \leq n$. Here δ_{ij} is the Kronecker delta. Let $y_{ij} = x_{ij} - \delta_{ij}$, for all i, j . Note that $L = \ker(\epsilon : L^1 \rightarrow k)$. Then $\bigoplus_{i,j=1,\dots,n} k y_{ij} \subset L$ is a simple subcoalgebra of L such that $\delta(y_{ij}) = \sum_{s=1}^n y_{is} \otimes y_{sj}$ for all $1 \leq i, j \leq n$. Let $z_{ij} = S(x_{ij})$ for all $1 \leq i, j \leq n$. Let X be the matrix $(x_{ij})_{n \times n}$ and Z be the matrix $(z_{ij})_{n \times n}$. Then the antipode axiom implies that $XZ = ZX = I_n$ where I_n is the identity $n \times n$ -matrix. Note that L is a δ -space of $U(L)$, and hence $U(L)$ has a filtration defined by $F_n = (L^1)^n$ such that $\text{gr}_F U(L)$ is isomorphic to the commutative polynomial ring $k[L]$. One can extend this filtration naturally from $U(L)$ to the matrix algebra $M_n(U(L))$ such that $\text{gr}_F(M_n(U(L))) \cong M_n(k[L])$. Let gr also denote the leading terms of elements in $\text{gr}_F(M_n(U(L)))$. Then the equation $XZ = I_n$ implies that $\text{gr}(X)\text{gr}(Z) = 0$. Note that $\text{gr}(X) = (y_{ij}) \in M_n(k[L])$ and thus $\det \text{gr}(X) = \det(y_{ij})$, which is nonzero in the commutative polynomial subring $k[y_{ij}] \subset k[L]$. Consequently, the equation $\text{gr}(X)\text{gr}(Z) = 0$ implies that $\text{gr}(Z) = 0$. Hence $Z = 0$, yielding a contradiction. The assertion follows. \square

Corollary 5.3.6. *If $\text{char } k = 0$, then every locally conilpotent cocommutative coassociative Lie algebra is quasi-equivalent to a Lie algebra.*

Proof. Let L be any locally conilpotent cocommutative coassociative Lie algebra. Then $U(L)$ is a connected cocommutative Hopf algebra by Theorem 5.3.5. By Milnor-Moore-Cartier-Kostant Theorem [Mo, Theorem 5.6.5], $U(L)$ is isomorphic to $U(\mathfrak{g})$ for some Lie algebra \mathfrak{g} . The assertion follows. \square

Recall that a Lie algebra \mathfrak{g} is called *unimodular* if $\text{ad}(x)$ has zero trace for all $x \in \mathfrak{g}$, where $\text{ad}(x) \in \text{End}_k(\mathfrak{g})$ is the k -linear map sending $y \in \mathfrak{g}$ to $[x, y]$. Combining results of Koszul [Ko] and Yekutieli [Ye, Theorem A] (also see [BZ, Proposition 6.3] and [HVZ, Theorem 5.3 and Lemma 4.1]), \mathfrak{g} is unimodular if and only if $U(\mathfrak{g})[d]$ is the rigid dualizing complex over $U(\mathfrak{g})$ if $d := \dim \mathfrak{g}$ is finite. By [BZ, Proposition 6.3] and [HVZ, Theorem 5.3], \mathfrak{g} is unimodular if and only if the Hopf algebra $U(\mathfrak{g})$ is unimodular in the sense of [LWZ], if and only if $U(\mathfrak{g})$ is Calabi-Yau, and if and only if the homological integral of $U(\mathfrak{g})$ (as defined in [LWZ]) is trivial. It is well-known that all Heisenberg Lie algebras are unimodular, and that the 2-dimensional non-abelian Lie algebra is not. Next we verify Theorem 5.1.3.

Theorem 5.3.7. *Let \mathfrak{g} be a finite dimensional unimodular Lie algebra. Suppose (\mathfrak{g}, δ) is a coassociative Lie algebra such that δ is conilpotent. Then the Hopf algebra $U(\mathfrak{g}, \delta)$ is involutory.*

Proof. Let H (respectively K) denote the Hopf algebra $U(\mathfrak{g}, \delta)$ (respectively, $U(\mathfrak{g})$). Let μ_H and μ_K denote the Nakayama automorphisms of H and K respectively. Since the Nakayama automorphism is defined uniquely up to an inner automorphism, and the units in H are those elements in k^\times , the Nakayama automorphism of H (and of K) is unique. Since $H = K$ as algebras by Definition 5.2.8, we have $\mu_H = \mu_K$. Since \mathfrak{g} is unimodular, $\mu_K = \text{Id}_K$ by [BZ, Proposition 6.3(c)]. As a consequence $\mu_H = \text{Id}_H$.

Since \mathfrak{g} is unimodular, the homological integral of K , denoted by \int_K^l , is trivial. Consequently, \int_K^l equals the trivial module $K/(\mathfrak{g})$ where (\mathfrak{g}) is the ideal of K generated by subspace \mathfrak{g} . The homological integral is only dependent on the algebra structure of the Hopf algebra, so we have that $\int_H^l = H/(\mathfrak{g})$. This implies that \int_H^l is trivial. Therefore the left winding automorphism associated to \int_H^l , denoted by $\Xi_{\int^l}^l$, is the identity map of H .

Combining the above with [BZ, Theorem 0.3], we have that

$$Id_H = \mu_H = S_H^2 \circ \Xi_{\int^l}^l = S_H^2 \circ Id_H = S_H^2$$

where S_H is the antipode of H . Hence $S_H^2 = Id_H$, and H is involutory. \square

Example 5.5.2 shows that $U(\mathfrak{g}, \delta)$ may not be involutory if \mathfrak{g} is not unimodular. This is another way of showing that $U(\mathfrak{g}, \delta)$ is not isomorphic to $U(\mathfrak{g}', \delta)$ for any Lie algebra \mathfrak{g}' . For the rest of this section we assume that $\text{char } k \neq 2$.

Lemma 5.3.8. *Let L be an anti-cocommutative coalgebra. Then*

(a) $\text{con}(L) \leq 2$, and as a consequence, L is conilpotent; and

(b) $\delta(L) \subset (\ker \delta)^{\otimes 2}$.

Proof. (a) A standard calculation by using Sweedler's notation shows that

$$(1 \otimes \delta)\tau\delta = (\tau \otimes 1)(1 \otimes \tau)(1 \otimes \delta)\delta.$$

Since $\tau\delta = -\delta$ by assumption, the left-hand side of the equation becomes $-(1 \otimes \delta)\delta$ while the right-hand side is $(1 \otimes \delta)\delta$. As a consequence, $(1 \otimes \delta)\delta = 0$.

(b) For any $x \in L$, write $\delta(x) = \sum_{i=1}^n x_i \otimes y_i$ for a minimal integer n . Then $\{x_i\}_{i=1}^n$ is linearly independent. Since $\text{con}(L) \leq 2$,

$$0 = (1 \otimes \delta)\delta(x) = \sum_{i=1}^n x_i \otimes \delta(y_i).$$

Since $\{x_i\}_{i=1}^n$ is linearly independent, $\delta(y_i) = 0$ for all i . This means that $\delta(L) \subset L \otimes \ker \delta$. Similarly, $\delta(L) \subset \ker \delta \otimes L$. The assertion follows. \square

Lemma 5.3.8 also implies a nice fact about the form of the antipode.

Proposition 5.3.9. *Suppose that L is anti-cocommutative and that $U(L)$ is involutory. Then $S(x) = -x$ for all $x \in L$.*

Proof. It follows from Definition 5.2.1 that the kernel of δ , denoted by K , is a Lie subalgebra of L . If $x \in K$, then $\Delta(x) = x \otimes 1 + 1 \otimes x$. The antipode axiom implies that $S(x) = -x$. If $x \in L \setminus K$, it follows from Lemma 5.3.8(b) that

$$\delta(x) = \sum_{i < j} a_{ij}(x_i \otimes x_j - x_j \otimes x_i),$$

for some $x_i \in K$ and some $a_{ij} \in k$. Hence

$$\Delta(x) = x \otimes 1 + 1 \otimes x + \sum_{i < j} a_{ij}(x_i \otimes x_j - x_j \otimes x_i).$$

Applying the antipode axiom, and using the fact that $S(x_i) = -x_i$, we have that

$$0 = S(x) + x + \sum_{i < j} a_{ij}(S(x_i)x_j - S(x_j)x_i) = S(x) + x + \sum_{i < j} a_{ij}(-x_i x_j + x_j x_i).$$

So $S(x) = -x - Y$ where $Y = \sum_{i < j} a_{ij}(-x_i x_j + x_j x_i) \in K$. Applying S to $S(x) = -x - Y$, and using the hypothesis that $S^2 = Id$, we have $x = -S(x) + Y$. Thus $Y = 0$, and $S(x) = -x$. \square

5.4 Elementary properties of coassociative Lie algebras

In this section some elementary properties of coassociative Lie algebras are discussed. First we need some lemmas that will simplify computations.

Lemma 5.4.1. *Let L be a coassociative Lie algebra. Let $\{x_i\}_{i \in I}$ be a totally ordered k -linear basis of L . For any $a, b \in L$, write $\delta(a) = \sum_i x_i \otimes a_i$ and $\delta(b) = \sum_i x_i \otimes b_i$. Then*

(a) $[a_i, b_i] = 0$ for all i ; and

(b) $[a_i, b_j] + [a_j, b_i] = 0$ for all i, j .

If $\delta(L) \subset L' \otimes L''$ for some subspaces L' and L'' of L , then

(c) $[\delta(a), \delta(b)] \in [L', L'] \otimes [L'', L''];$

(d) $\Phi(a, b) \in [L', L'] \otimes [L'', L''];$ and

(e) $\delta([L, L]) \subset [L', L] \otimes L'' + L' \otimes [L'', L] + [L', L'] \otimes [L'', L'']$.

Proof. We compute $[\delta(a), \delta(b)]$ in $U(L)^{\otimes 2}$ as follows

$$\begin{aligned}
[\delta(a), \delta(b)] &= \sum_{i,j} x_i x_j \otimes a_i b_j - \sum_{i,j} x_i x_j \otimes b_i a_j \\
&= \sum_{i<j} x_i x_j \otimes a_i b_j + \sum_{j>i} x_j x_i \otimes a_j b_i + \sum_i x_i^2 \otimes a_i b_i \\
&\quad - \sum_{i<j} x_i x_j \otimes b_i a_j - \sum_{j>i} x_j x_i \otimes b_j a_i - \sum_i x_i^2 \otimes b_i a_i \\
&= \sum_{i<j} x_i x_j \otimes (a_i b_j - b_i a_j) + \sum_i x_i^2 \otimes [a_i, b_i] \\
&\quad + \sum_{j>i} (x_i x_j + [x_j, x_i]) \otimes a_j b_i - \sum_{j>i} (x_i x_j + [x_j, x_i]) \otimes b_j a_i \\
&= \sum_{i<j} x_i x_j \otimes ([a_i, b_j] + [a_j, b_i]) + \sum_i x_i^2 \otimes [a_i, b_i] \\
&\quad + \sum_{i<j} [x_i, x_j] \otimes (b_j a_i - a_j b_i).
\end{aligned}$$

Since $\{x_i x_j\}_{i \leq j}$ are linearly independent in $U(L)/L$ and $[x_i, x_j] \in L$ for all i, j , we have $[a_i, b_j] + [a_j, b_i] = 0$, $[a_i, b_i] = 0$. Parts (a) and (b) follow.

Now assume that $\delta(L) \subset L' \otimes L''$. By the above computation and parts (a,b), $[\delta(a), \delta(b)] = \sum_{i<j} [x_i, x_j] \otimes (b_j a_i - a_j b_i)$, which is in $[L', L'] \otimes U(L)$, as we can assume $x_i \in L'$ whenever a_i (or b_i) is nonzero. By symmetry, $[\delta(a), \delta(b)] \in U(L) \otimes [L'', L'']$. Hence $[\delta(a), \delta(b)] \in [L', L'] \otimes [L'', L'']$. This is part (c). Part (d) follows from the equation $\Phi(a, b) = [\delta(a), \delta(b)]$.

Part (e) follows from part (c) and (E5.2.1.1). \square

If V and W are subspaces of a vector space A , let V/W denote $V/(V \cap W)$.

Definition 5.4.2. 1. A subspace V of a Lie algebra L is said to have *small centralizer* if $(\ker ad(x)) \cap V$ has dimension 1 for all $x \in V \setminus \{0\}$.

2. Let Z be a Lie ideal of L . A subspace $V \subset L$ is said to have *small centralizer modulo Z* if the quotient space V/Z in L/Z has small centralizer.

Proposition 5.4.3. *Let L be a coassociative Lie algebra and Z be a Lie ideal of L . Suppose that L' and L'' are two subspaces of L such that*

(a) $[L', Z] = [L'', Z] = 0$,

(b) L' and L'' have small centralizers modulo Z , and

(c) $\delta(L) \subset L'' \otimes L' + (Z \otimes L + L \otimes Z)$.

Then $\dim \delta(L)/(Z \otimes L + L \otimes Z) \leq 1$.

Remark 5.4.4. Since Z is a Lie ideal of L , we have that L/Z is a quotient Lie algebra, but L/Z may not be a quotient of the coassociative Lie algebra L .

Proof of Proposition 5.4.3. Let $W = (Z \otimes L + L \otimes Z) \cap \delta(L)$. Without loss of generality we may assume that $\delta(L) \neq W$. Let a and b be any two elements in L . The assertion is equivalent to

Claim: $\delta(a)$ and $\delta(b)$ are linearly dependent in $\delta(L)/W$.

Proof of the Claim: If $\delta(a)$ or $\delta(b)$ is in W , the claim is obvious, so we assume during the proof that $\delta(a)$ and $\delta(b)$ are not in W .

In the rest of the proof, we will pick a k -linear basis $\{z_j\}_{j \geq 1}$ of Z , extend it to a basis $\{x_i\}_{i \geq 1} \cup \{z_j\}_{j \geq 1}$ of $L'' + Z$ where $x_i \in L'' \setminus Z$, then extend it to a basis $\{x_i\}_{i \geq 1} \cup \{z_j\}_{j \geq 1} \cup \{l_s\}_{s \geq 1}$ of the whole space L where $l_s \in L \setminus (L'' + Z)$. For simplicity, we use integers to index the basis elements. Since $\delta(L) \subset L'' \otimes L' + (Z \otimes L + L \otimes Z)$, for any $a \in L$,

$$\delta(a) = \sum_{i \geq 1} x_i \otimes a_i + \sum_{j \geq 1} z_j \otimes a'_j + \sum_{s \geq 1} l_s \otimes a''_s$$

where $a_i \in L' + Z$, $a'_j \in L$ and $a''_s \in Z$.

Case 1: Suppose that $\delta(a), \delta(b) \in u \otimes L' + (Z \otimes L + L \otimes Z)$ for some $u \in L'' \setminus Z$. In this case we can choose $x_1 = u$. By the choice of the k -linear basis, we can write

$$\delta(a) = u \otimes a_1 + \sum_{i \geq 2} x_i \otimes a_i + \sum_{j \geq 1} z_j \otimes a'_j + \sum_{s \geq 1} l_s \otimes a''_s,$$

where $a_1 \in L' + Z$, $a_i \in Z$ for all $i \geq 2$, $a'_j \in L$ and $a''_s \in Z$ for all j, s . Similarly,

$$\delta(b) = u \otimes b_1 + \sum_{i \geq 2} x_i \otimes b_i + \sum_{j \geq 1} z_j \otimes b'_j + \sum_{k \geq 1} l_s \otimes b''_s$$

where $b_1 \in L' + Z$, $b_i \in Z$ for all $i \geq 2$, and $b'_j \in L$ and $b''_s \in Z$ for all j, s . By Lemma 5.4.1(a), $[a_1, b_1] = 0$. Since L' has small centralizer modulo Z , $b_1 \in ka_1 + Z$. Thus $\delta(a)$ and $\delta(b)$ are linearly dependent in $\delta(L)/W$.

Case 2: Suppose that $\delta(a), \delta(b) \in L'' \otimes u + (Z \otimes L + L \otimes Z)$ for some $u \in L' \setminus Z$. Case 2 is equivalent to Case 1 by symmetry. Hence the claim follows by Case 1.

Case 3: Suppose that $\delta(a) \in u \otimes L' + (Z \otimes L + L \otimes Z)$ for some $u \in L'' \setminus Z$. In this case, we can choose $x_1 = u$, and $\delta(a)$ can be written as in Case 1. In particular, $\delta(a) \in L'' \otimes a_1 + (Z \otimes L + L \otimes Z)$. Write

$$\delta(b) = u \otimes b_1 + \sum_{i \geq 2} x_i \otimes b_i + \sum_{j \geq 1} z_j \otimes b'_j + \sum_{s \geq 1} l_s \otimes b''_s$$

where $b_i \in L' + Z$ for all $i \geq 1$, and $b'_j \in L$, $b''_s \in Z$ for all j, s . By Lemma 5.4.1(a), $[a_1, b_1] = 0$. Since L' has small centralizer modulo Z , $b_1 = \lambda a_1 + z$ for some $\lambda \in k$ and $z \in Z$. Replacing b_1 by $b_1 - \lambda a_1$, we may assume that $b_1 \in Z$. By Lemma 5.4.1(b), for

every $i \geq 2$, $[a_1, b_i] = -[a_i, b_1] \in Z$ since b_1 is in Z . Since L'' has small centralizer modulo Z , $b_i = \lambda_i a_1 + z_i$, for $\lambda_i \in k$ and $z_i \in Z$ for all $i \geq 2$. Thus

$$\delta(b) = u \otimes b_1 + \sum_{i \geq 2} x_i \otimes (\lambda_i a_1 + z_i) + \sum_{j \geq 1} z_j \otimes b'_j + \sum_{s \geq 1} l_s \otimes b''_s$$

which is in $L'' \otimes a_1 + (Z \otimes L + L \otimes Z)$. Therefore both $\delta(a)$ and $\delta(b)$ are in $L'' \otimes a_1 + (Z \otimes L + L \otimes Z)$. The claim follows from Case 2.

Case 4: Suppose that either $\delta(a)$ or $\delta(b)$ is in $L'' \otimes u + (Z \otimes L + L \otimes Z)$ for some $u \in L' \setminus Z$. The claim follows by symmetry and Case 3.

Case 5 [the general case]: By the choice of k -linear basis, we can write

$$\begin{aligned} \delta(a) &= \sum_{i \geq 1} x_i \otimes a_i + \sum_{j \geq 1} z_j \otimes a'_j + \sum_{s \geq 1} l_s \otimes a''_s, \\ \delta(b) &= \sum_{i \geq 1} x_i \otimes b_i + \sum_{j \geq 1} z_j \otimes b'_j + \sum_{s \geq 1} l_s \otimes b''_s, \end{aligned}$$

where $a_i, b_i \in L' + Z$, $a'_j, b'_j \in L$ and $a''_s, b''_s \in Z$ for all i, j, s . Without loss of generality, we may assume that $a_1 \in L' \setminus Z$. By Lemma 5.4.1(a), $[a_1, b_1] = 0$. Since L' has small centralizer modulo Z , $b_1 = \lambda a_1 + z$ for some $\lambda \in k$ and where $z \in Z$. Replacing b_1 by $b_1 - \lambda a_1$, we may assume that $b_1 \in Z$. By Lemma 5.4.1(b), for every $i \geq 2$, $[a_1, b_i] = -[a_i, b_1] \in Z$ since b_1 are in the Lie ideal Z . Since L' has small centralizer modulo Z , $b_i = \lambda_i a_1 + z_i$, for $\lambda_i \in k$ and $z_i \in Z$ for all $i \geq 2$. Together with the fact $b_1 \in Z$, we have that $\delta(b) \in L'' \otimes a_1 + (Z \otimes L + L \otimes Z)$. The claim now follows from Case 4. \square

For a subset $S \subset L$, the centralizer of S in L is defined to be

$$C_S(L) = \{y \in L \mid [x, y] = 0, \forall x \in S\}.$$

Lemma 5.4.5. *If there is an element $a \in L$ such that $\delta(a) = x \otimes y \neq 0$, then $\delta(L) \subset C_{\{x\}}(L) \otimes C_{\{y\}}(L)$.*

Proof. By symmetry, it suffices to show that $\delta(L) \subset L \otimes C_{\{y\}}(L)$. Pick a basis $\{x_i\}$ of L such that $x_1 = x$. Then $\delta(a) = x_1 \otimes y$. For any $b \in L$, write $\delta(b) = \sum_i x_i \otimes b_i$. By Lemma 5.4.1(a), $[y, b_1] = 0$. For any $i \geq 2$, by Lemma 5.4.1(b), $[y, b_i] = -[0, b_1] = 0$. The assertion follows. \square

Proposition 5.4.6. *Let C be the coradical of a coassociative Lie algebra L . Then $[C, C] = 0$. As a consequence, cosemisimple coalgebras are rigid.*

Proof. Since k is algebraically closed, $C = \bigoplus_i M_{n_i}(k)$ for a set of positive integers $\{n_i\}_{i \in I}$. Let $x, y \in C$; we need to show $[x, y] = 0$. By linearity, we may assume that x and y are some basis elements in C . We need to consider two cases.

Case 1: We have that x and y are in the same matrix subcoalgebra, say $M_n(k)$. If $n = 1$, $x = y$. The assertion is trivial. Now assume that $n > 1$. Then we may assume that $x = x_{ij}$

and $y = x_{kl}$ for some i, j, k, l . Consider $\delta(x_{1j}) = \sum_s x_{1s} \otimes x_{sj}$ and $\delta(x_{2l}) = \sum_t x_{2t} \otimes x_{tl}$. By Lemma 5.4.1(b), $[x_{sj}, x_{tl}] = 0$ for all s, t . The assertion follows.

Case 2: We have that x and y are in different matrix subcoalgebras. Then we may assume that $x = x_{ij} \in M_{n_1}(k)$ and $y = y_{kl} \in M_{n_2}(k)$. Consider $\delta(x_{1j}) = \sum_s x_{1s} \otimes x_{sj}$ and $\delta(y_{1l}) = \sum_t y_{1t} \otimes y_{tl}$. By Lemma 5.4.1(b), $[x_{sj}, y_{tl}] = 0$ for all s, t . The assertion follows. \square

The following lemma is also true. The proof is omitted since it is straightforward and somewhat similar to the proof of Proposition 5.4.6.

Lemma 5.4.7. *If C_1 and C_2 are subcoalgebras of a coassociative Lie algebra such that $C_1 \cap C_2 = \{0\}$, then $[\delta(C_1), \delta(C_2)] = 0$.*

Lemma 5.4.8. *Let L be a coalgebra. Then we have the following statements.*

(a) *For every $x \in L$, write $\delta(x) = \sum_{i=1}^n w_i \otimes v_i$ for a minimal n . Then $\sum_i k\delta(v_i) \subset L \otimes V_x$ and $\sum_i k\delta(w_i) \subset W_x \otimes L$ for some subspaces $V_x \subset \sum_i kv_i$ and $W_x \subset \sum_i kw_i$ of dimension no more than $\dim \delta(L)$.*

(b) *If $\delta(L)$ is 1-dimensional and L is not 2-conilpotent, then $\delta(L)$ has a basis element of the form $T \otimes T$ for some $0 \neq T \in L$.*

Proof. (a) Let $\{y_t\}_{t=1}^m$ be a basis of $\sum_s k\delta(v_s)$ for some $m \leq \dim \delta(L)$. Then there are elements $a_1, \dots, a_m \in L$ such that

$$\sum_i \delta(w_i) \otimes v_i = \sum_i w_i \otimes \delta(v_i) = \sum_{t=1}^m a_t \otimes y_t \in \left(\sum_{t=1}^m ka_t \right) \otimes L \otimes L.$$

This implies that $\delta(w_i) \in (\sum_t ka_t) \otimes L$ for each i . Since $\{y_t\}_{t=1}^m$ is a basis of $\sum_s k\delta(v_s)$, the equation

$$\sum_i w_i \otimes \delta(v_i) = \sum_{t=1}^m a_t \otimes y_t$$

implies that $\sum_t ka_t \subset \sum_i kw_i$. Therefore the second assertion follows by taking $W_x = \sum_t ka_t$. The first assertion is similar.

(b) Pick $0 \neq \Omega \in \delta(L)$, and let $\{x_i\}$ be a finite set of linearly independent elements in L such that $\Omega = \sum_{i,j} a_{ij} x_i \otimes x_j$. Since $\delta(L)$ is 1-dimensional, $\delta(x_i) = b_i \Omega$ for some $b_i \in k$. Pick $x \in L$ such that $\delta(x) = \Omega$ and $(\delta \otimes 1)\delta(x) \neq 0$ (since L is not 2-conilpotent). Then

$$(\delta \otimes 1)\delta(x) = \sum_{i,j} a_{i,j} b_i \Omega \otimes x_j = \Omega \otimes \left(\sum_{i,j} a_{i,j} b_i x_j \right) = \Omega \otimes T$$

where $T := \sum_{i,j} a_{i,j} b_i x_j \in L$, and

$$(1 \otimes \delta)\delta(x) = \sum_{i,j} a_{i,j} x_i \otimes b_j \Omega = \left(\sum_{i,j} a_{i,j} b_j x_i \right) \otimes \Omega = S \otimes \Omega$$

where $S := \sum_{i,j} a_{ij} b_j x_i$. By coassociativity, $\Omega \otimes T = S \otimes \Omega$. This implies that Hence $\Omega = c'T \otimes T$ and $S = c''T$ for some $c', c'' \in k^\times$. Since k is algebraically closed, we can choose $c' = 1$ by a scalar change of T . \square

Now we prove the main result of this section. Let

$$\left\{ e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

be a standard k -basis of sl_2 .

Theorem 5.4.9. *The simple Lie algebra sl_2 is rigid.*

Proof. By using the standard basis of sl_2 , it is straightforward to check that sl_2 has small centralizers (details are omitted). Let $L = (sl_2, \delta)$ be a coassociative Lie algebra. We need to show that $\delta = 0$. By Proposition 5.4.3 for $L' = L'' = L = sl_2$ and $Z = 0$, $\dim \delta(L) \leq 1$. To avoid the triviality, we assume that $\dim \delta(L) = 1$ and let $\Omega \in \delta(L)$ be a nonzero element.

If L is not 2-conilpotent, then Lemma 5.4.8 says that $\Omega = T \otimes T$ where $T = t_1 e + t_2 f + t_3 h$ for some $t_1, t_2, t_3 \in k$. Since $\Omega \neq 0$, not all t_i are zero. Suppose $\delta(e) = a\Omega$, $\delta(f) = b\Omega$ and $\delta(h) = c\Omega$ for some $a, b, c \in k$. By (E5.2.1.1), we have

$$\begin{aligned} 2b\Omega &= \delta(2f) = \delta([f, h]) \\ &= [f \otimes 1 + 1 \otimes f, c\Omega] + [b\Omega, h \otimes 1 + 1 \otimes h] \\ &= \{-2bt_1 e + (2ct_3 + 2bt_2)f - ct_1 h\} \otimes T \\ &\quad + T \otimes \{-2bt_1 e + (2ct_3 + 2bt_2)f - ct_1 h\}. \end{aligned}$$

Hence $b(t_1, t_2, t_3) = (-2bt_1, 2bt_2 + 2ct_3, -ct_1)$, or

$$M \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = 0 \quad \text{where} \quad M = \begin{pmatrix} b & 0 & 0 \\ 0 & b & 2c \\ c & 0 & b \end{pmatrix}.$$

Since not all t_1, t_2, t_3 are zero, the determinant of the matrix M , which is b^3 , is zero. Hence $b = 0$. Since e and f plays a very similar role, by symmetry, $a = 0$. By (E5.2.1.1) and the fact $\delta(e) = \delta(f) = 0$, we have that $\delta(h) = \delta([e, f]) = 0$. Thus $c = 0$, whence $\delta = 0$, yielding a contradiction. Therefore δ is 2-conilpotent.

Since δ is 2-conilpotent, by Theorem 5.3.5, $U(sl_2, \delta)$ is a connected Hopf algebra of GK-dimension 3. By comparing with the list in the classification of connected Hopf algebras of GK-dimension 3 [Zh2, Theorem 1.3], $U(sl_2, \delta)$ must be isomorphic to $U(sl_2)$. Since $U(sl_2)$ is cocommutative, (sl_2, δ) must be cocommutative. Hence

$$\begin{aligned} \Omega &= a_{11}e \otimes e + a_{12}(e \otimes f + f \otimes e) + a_{13}(e \otimes h + h \otimes e) \\ &\quad + a_{22}f \otimes f + a_{23}(f \otimes h + h \otimes f) + a_{33}h \otimes h \neq 0. \end{aligned}$$

Since $\delta(L)$ is 1-dimensional, the kernel $L_0 = \ker(\delta)$ is a 2-dimensional Lie subalgebra of sl_2 . By an elementary computation, it is easy to verify that any 2-dimensional Lie subalgebra of sl_2 is either (i) $kf + kh$, or (ii) $ke + kh$, or (iii) $k(e + ah) + k(-4af + h)$ for some $a \in k^\times$. In the first case, $(\delta \otimes 1)\Omega = a_{11}\delta(e) \otimes e + a_{12}\delta(e) \otimes f + a_{13}\delta(e) \otimes h$ and without loss of generality, we can assume that $\delta(e) = \Omega$. Since (sl_2, δ) is 2-conilpotent, $(\delta \otimes 1)(\Omega) = 0$, which implies that $a_{11} = a_{12} = a_{13} = 0$. It is easy to see that

$$\begin{aligned} -\Phi(e, h) &= -\delta(-2e) + [\delta(e), h \otimes 1 + 1 \otimes h] \\ &= 2\Omega + 4a_{22}f \otimes f + 2a_{23}(f \otimes h + h \otimes f). \end{aligned}$$

The equations $\Phi(e, h) = [\delta(e), \delta(h)] = [\delta(e), 0] = 0$ imply that $a_{22} = a_{23} = a_{33} = 0$. Hence $\Omega = 0$, a contradiction. The assertion follows. The second case is similar.

The final case is when $e + ah, h - 4af \in \ker(\delta)$ for some $a \in k^\times$. Let

$$\begin{aligned} e' &= \begin{pmatrix} 1 & 0 \\ -2a & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2a & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 2a & 1 \\ -4a^2 & -2a \end{pmatrix} \\ h' &= \begin{pmatrix} 1 & 0 \\ -2a & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2a & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -4a & -1 \end{pmatrix} \\ f' &= \begin{pmatrix} 1 & 0 \\ -2a & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2a & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = f. \end{aligned}$$

Then $\{e', f', h'\}$ is a new standard basis of sl_2 . It is clear that $e' = e + 2ah - 4a^2f \in \ker(\delta)$ and $h' = h - 4af \in \ker(\delta)$. Thus it is equivalent to the second case. Combining all these cases, the assertion follows. \square

Similar to Theorem 5.4.9 we show the following.

Theorem 5.4.10. *Write $gl_2 = sl_2 \oplus kz$ where kz is the center of gl_2 . If (gl_2, δ) is a coassociative Lie algebra, then $\delta|_{sl_2} = 0$ and $\delta(z) = az \otimes z$ for some scalar $a \in k$.*

Sketch of Proof. Some tedious computations are omitted in the following proof.

First of all, the δ given in the theorem gives rise to a coassociative Lie algebra structure on gl_2 . Now we assume that (gl_2, δ) is a coassociative Lie algebra.

Applying Proposition 5.4.3 to $(L, L', L'', Z) = (gl_2, sl_2, sl_2, kz)$, we obtain that

$$\dim(\delta(gl_2)/(z \otimes gl_2 + gl_2 \otimes z)) \leq 1.$$

Therefore there is an element $\Omega \in sl_2 \otimes sl_2$ such that, for every $x \in gl_2$,

$$\delta(x) = \sigma(x) \otimes z + z \otimes \tau(x) + \lambda(x)\Omega \tag{E5.4.10.1}$$

for some $\sigma(x) \in sl_2, \tau(x) \in gl_2$ and $\lambda(x) \in k$. Both $\sigma(x)$ and $\tau(x)$ are uniquely determined by (E5.4.10.1). If $\Omega \neq 0$, then $\lambda(x)$ is also uniquely determined by (E5.4.10.1). Setting $x = z$ in (E5.4.10.1), we have

$$\delta(z) = \sigma(z) \otimes z + z \otimes \tau(z) + \lambda(z)\Omega. \tag{E5.4.10.2}$$

Equation (E5.2.2.2) for $(a, b) = (z, x)$ implies that

$$\begin{aligned} 0 = & [\sigma(z), x] \otimes z + z \otimes [\tau(z), x] + [\lambda(z)\Omega, x \otimes 1 + 1 \otimes x] \\ & + [\sigma(z), \sigma(x)] \otimes z^2 + z^2 \otimes [\tau(z), \tau(x)] \\ & + [\lambda(z)\Omega, \sigma(x) \otimes z] + [\lambda(z)\Omega, z \otimes \tau(x)] \\ & + [\sigma(z) \otimes z, \lambda(x)\Omega] + [z \otimes \tau(z), \lambda(x)\Omega]. \end{aligned}$$

Since the terms in the above equation live in different k -subspaces of $U(\mathfrak{gl}_2) \otimes U(\mathfrak{gl}_2)$, we have $[\sigma(z), x] = 0 = [\tau(z), x]$ for all $x \in \mathfrak{sl}_2$. Thus $\sigma(z) = 0$ and $\tau(z) \in kz$. In this case, (E5.2.2.2) becomes, for every $x \in \mathfrak{gl}_2$,

$$0 = \lambda(z)([\Omega, x \otimes 1 + 1 \otimes x] + [\Omega, \sigma(x) \otimes z] + [\Omega, z \otimes \tau(x)]). \quad (\text{E5.4.10.3})$$

First we claim that $\lambda(z)\Omega = 0$. If not, we may assume that $\Omega \neq 0$ and $\lambda(z) = 1$. Using the fact that terms live in different k -subspaces of $U(\mathfrak{gl}_2) \otimes U(\mathfrak{gl}_2)$, (E5.4.10.3) implies that

$$[\Omega, x \otimes 1 + 1 \otimes x] = [\Omega, \sigma(x) \otimes z] = [\Omega, z \otimes \tau(x)] = 0$$

for all $x \in \mathfrak{gl}_2$. A computation shows that the first equation implies that $\Omega = c(h \otimes h + 2(e \otimes f + f \otimes e))$ for some $0 \neq c \in k$. The second and third equations imply that $\sigma(x) = 0$ and $\tau(x) \in kz$. Going back to (E5.4.10.1), we have, for every $x \in \mathfrak{gl}_2$,

$$\delta(x) = \phi(x)z \otimes z + \lambda(x)\Omega$$

for some $\phi(x) \in k$. For any $x, y \in \mathfrak{sl}_2$, (E5.2.2.2) says that

$$\delta([x, y]) = [x \otimes 1 + 1 \otimes x, \lambda(y)\Omega] + [\lambda(x)\Omega, y \otimes 1 + 1 \otimes y] = 0.$$

Since $\mathfrak{sl}_2 = [\mathfrak{sl}_2, \mathfrak{sl}_2]$, $\delta|_{\mathfrak{sl}_2} = 0$. The coassociativity on z shows that $\Omega \otimes z = 0$, a contradiction. Therefore we proved our claim.

For the rest of the proof, we have $\lambda(z)\Omega = 0$ and $\delta(z) = \sigma(z) \otimes z + z \otimes \tau(z)$. Then kz is an ideal of the coassociative Lie algebra $(\mathfrak{gl}_2, \delta)$ and $(\mathfrak{gl}_2/kz, \bar{\delta}) \cong (\mathfrak{sl}_2, \bar{\delta})$ is a quotient coassociative Lie algebra where $\bar{\delta}$ is the induced coproduct. By Theorem 5.4.9, $\bar{\delta} = 0$. This means that $\Omega = 0$. For each $x \in \mathfrak{gl}_2$, write

$$\delta(x) = \sigma_1(x) \otimes z + z \otimes \sigma_2(x) + \nu(x)z \otimes z$$

for some $\sigma_1(x), \sigma_2(x) \in \mathfrak{sl}_2$ and some $\nu(x) \in k$. By (E5.2.2.2), we have

$$\begin{aligned} \delta([x, y]) = & [\sigma_1(x), y] \otimes z + z \otimes [\sigma_2(x), y] + [x, \sigma_1(y)] \otimes z + z \otimes [x, \sigma_2(y)] \\ & + [\sigma_1(x), \sigma_1(y)] \otimes z^2 + z^2 \otimes [\sigma_2(x), \sigma_2(y)] \end{aligned}$$

for all $x, y \in \mathfrak{gl}_2$. Setting $y = z$, we have $[x, \sigma_i(z)] = 0$ for all $x \in \mathfrak{sl}_2$. This implies that $\sigma_i(z) = 0$ for $i = 1, 2$. Setting $x, y \in \mathfrak{sl}_2$, we have that

$$[\sigma_1(x), \sigma_1(y)] = 0 = [\sigma_2(x), \sigma_2(y)]$$

and that $\delta(sl_2) \subset sl_2 \otimes z + z \otimes sl_2$. Since sl_2 has small centralizers, $\dim \sigma_1(sl_2) \leq 1$ and $\dim \sigma_2(sl_2) \leq 1$. Combining above facts, there exist $w_1, w_2 \in sl_2 \setminus \{0\}$, $c \in k$ and linear maps $\phi_1, \phi_2 : sl_2 \rightarrow k$, such that

$$\begin{aligned}\delta(z) &= cz \otimes z, \\ \delta(x) &= \phi_1(x)w_1 \otimes z + \phi_2(x)z \otimes w_2\end{aligned}$$

for all $x \in sl_2$. Let $\{e, f, h\}$ be the standard basis of sl_2 , and write $w_1 = ae + bf + ch \neq 0$ for some $a, b, c \in k$. A calculation using explicit Lie product of the elements e, f , and h shows that (E5.2.2.2) implies that $\phi_1 = 0$. By symmetry, $\phi_2 = 0$. Thus the assertion follows. \square

5.5 Examples

We present several families of coassociative Lie algebras in this section. One-dimensional ones are listed in Example 5.2.9. Here is the 2-dimensional case.

Example 5.5.1. If $\dim L = 2$, then there are two Lie algebra structures on L , up to isomorphism. Namely, L is either abelian or non-abelian.

If L is abelian, then the classification of coassociative Lie algebra structures on L is equivalent to the classification of coalgebra structures on L . It is easy to show that δ -structure in L is isomorphic to one of the following:

$$(5.1.1.1) \quad \delta = 0;$$

$$(5.1.1.2) \quad (L, \delta) \text{ is cosemisimple};$$

$$(5.1.1.3) \quad L = kx_1 \oplus kx_2 \text{ and } \delta(x_1) = x_1 \otimes x_1, \delta(x_2) = 0;$$

$$(5.1.1.4) \quad L = kx_1 \oplus kx_2 \text{ and } \delta(x_1) = x_1 \otimes x_1, \delta(x_2) = x_1 \otimes x_2 + x_2 \otimes x_1;$$

$$(5.1.1.5) \quad L = kx_1 \oplus kx_2 \text{ and } \delta(x_1) = 0, \delta(x_2) = x_1 \otimes x_1.$$

If L is non-abelian, then L has a basis $\{x_1, x_2\}$ such that $[x_1, x_2] = x_2$. We have two cases.

Case 1: $\delta(x_2) = 0$. We are only interested in nonzero δ -structures. Write $\delta(x_1) = \sum_{i,j} a_{ij}x_i \otimes x_j \neq 0$. In this case

$$\Phi(x_1, x_2) = -[\delta(x_1), x_2 \otimes 1 + 1 \otimes x_2] = -(a_{11}(x_2 \otimes x_1 + x_1 \otimes x_2) + (a_{12} + a_{21})x_2 \otimes x_2).$$

By (E5.2.3.2), $\Phi(x_1, x_2) = [\delta(x_1), \delta(x_2)] = 0$. Hence $a_{11} = 0$ and $a_{12} + a_{21} = 0$. Let $a = a_{12}$ and $b = a_{22}$. We have that $\delta(x_1) = ax_1 \otimes x_2 - ax_2 \otimes x_1 + bx_2 \otimes x_2$. Now coassociativity of δ shows that $a = 0$. Thus $\delta(x_2) = 0$ and $\delta(x_1) = bx_2 \otimes x_2$. Up to a base change, we may assume $b = 1$. Hence this is Example 5.2.4.

Case 2: $\delta(x_2) \neq 0$. Since L is 2-dimensional and non-nilpotent, it has small centralizers. By Lemma 5.4.3, $\delta(L)$ is 1-dimensional. Since $\delta(x_2) \neq 0$, by replacing x_1 by $x_1 - ax_2$ for some suitable $a \in k$, we have $\delta(x_1) = 0$. Write $\delta(x_2) = \sum_{i,j} b_{ij}x_i \otimes x_j \neq 0$. In this case,

$$\begin{aligned} 0 &= [\delta(x_1), \delta(x_2)] = \Phi(x_1, x_2) = \delta(x_2) - [x_1 \otimes 1 + 1 \otimes x_1, \delta(x_2)] \\ &= b_{11}x_1 \otimes x_1 + b_{12}x_1 \otimes x_2 + b_{21}x_2 \otimes x_1 + b_{22}x_2 \otimes x_2 \\ &\quad - (b_{12}x_1 \otimes x_2 + b_{21}x_2 \otimes x_1 + 2b_{22}x_2 \otimes x_2) \\ &= b_{11}x_1 \otimes x_1 - b_{22}x_2 \otimes x_2. \end{aligned}$$

Hence $b_{11} = b_{22} = 0$ and $\delta(x_2) = b_{12}x_1 \otimes x_2 + b_{21}x_2 \otimes x_1$. The coassociativity of δ implies that $b_{12} = b_{21} = 0$. Therefore $\delta = 0$ in this case, yielding a contradiction.

Combining these two cases, the only nonzero δ -structure on the 2-dimensional non-abelian Lie algebra is the one in Example 5.2.4, up to isomorphisms.

One nice fact in 2-dimensional case is that δ is always cocommutative. If δ is conilpotent, then L is quasi-equivalent to a Lie algebra by Corollary 5.3.6.

Next, we consider some higher-dimensional examples.

Example 5.5.2. Let \mathfrak{g} be a 3-dimensional Lie algebra with a k -linear basis $\{x, y, z\}$ such that its Lie structure is determined by

$$[x, y] = y, \quad [z, y] = 0, \quad [z, x] = -z + \lambda y,$$

for any $\lambda \in k$. Let $L = (\mathfrak{g}, \delta)$ where the coproduct δ is determined by

$$\delta(x) = \delta(y) = 0, \quad \delta(z) = x \otimes y - y \otimes x.$$

It is routine to check that L is a coassociative Lie algebra (using Definition 5.2.1). It is obvious that δ is conilpotent and anti-cocommutative. Let H be the enveloping algebra $U(L)$. It follows from the antipode axiom that $S(x) = -x$, $S(y) = -y$, and $S(z) = -z + y$. Hence $S^2(z) = z - 2y$ and H is not involutory. By Theorem 5.3.7, \mathfrak{g} is not unimodular, which can also be verified directly.

Let \mathfrak{h}_{2n+1} be the $(2n+1)$ -dimensional Heisenberg Lie algebra with a standard basis $\{x_1, \dots, x_n, y_1, \dots, y_n, z\}$. Here $[x_i, y_i] = z$ for all i , and all other brackets are zero. Let $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$ and $D = (d_{ij})$ denote $n \times n$ -matrices over k , and let $E = (e_i)$ be an n -column vector over k .

Example 5.5.3. Each of the following δ defines a coassociative coalgebra structure on \mathfrak{h}_{2n+1} such that $(\mathfrak{h}_{2n+1}, \delta)$ is a coassociative Lie algebra.

(a) For every i , $\delta(x_i) = \delta(z) = 0$, and

$$\delta(y_i) = \sum_j (a_{ij}x_j + b_{ij}y_j) \otimes z + \sum_j z \otimes (c_{ij}x_j - b_{ij}y_j) + e_i z \otimes z,$$

where the coefficient matrices $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$ and $E = (e_i)$ satisfy

- (i) $BA = B^2 = BC = 0$;
- (ii) $BE = 0$;
- (iii) $A + A^\tau + C + C^\tau = 0$;
- (iv) $AB^\tau = BA^\tau$; and
- (v) $CB^\tau = BC^\tau$.

(b) For every i , $\delta(x_i) = e_i z \otimes z$, $\delta(z) = 0$, and $\delta(y_i) = \sum b_{ij}(x_j \otimes z + z \otimes x_j)$, where the coefficient matrix $B = (b_{ij})$ satisfies $B = B^\tau$.

(c) For every i , $\delta(x_i) = 0$, $\delta(z) = z \otimes z$, and

$$\delta(y_i) = \sum_j (a_{ij}x + b_{ij}y_j) \otimes z + \sum_j z \otimes (-a_{ji}x_j + (\delta_{ij} - b_{ij})y_j),$$

where the coefficient matrices $A = (a_{ij})$ and $B = (b_{ij})$ satisfy the conditions $B^2 = B$; $BA = A$; and $BA^\tau = 0$.

Proof. (a) Easy computations show that

$$\begin{aligned} & (\delta \otimes 1)\delta(y_i) \\ &= \sum_j b_{ij}\delta(y_j) \otimes z \\ &= \sum_{j,s} (b_{ij}a_{js}x_s + b_{ij}b_{js}y_s) \otimes z \otimes z + \sum_{j,s} z \otimes (b_{ij}c_{js}x_s - b_{ij}b_{js}y_s) \otimes z \\ &\quad + \sum_j b_{ij}e_j z \otimes z \otimes z, \quad \text{and} \end{aligned}$$

$$\begin{aligned} & (1 \otimes \delta)\delta(y_i) \\ &= \sum_j z \otimes (-b_{ij})\delta(y_j) \\ &= \sum_{j,s} z \otimes (-b_{ij}a_{js}x_s - b_{ij}b_{js}y_s) \otimes z + \sum_{j,s} z \otimes z \otimes (-b_{ij}c_{js}x_s + b_{ij}b_{js}y_s) \\ &\quad - \sum_j b_{ij}e_j z \otimes z \otimes z. \end{aligned}$$

Coassociativity is equivalent to equations $BA = B^2 = BC = 0$ and $BE = 0$. To check the condition (E5.2.1.1) we note that (E5.2.1.1) is trivial when $(a, b) = (z, z)$, (x_i, z) , (y_i, z) , and (x_i, x_j) . It suffices to verify (E5.2.1.1) for $(a, b) = (x_i, y_j)$ and $(a, b) = (y_i, y_j)$ for all $1 \leq i, j \leq n$.

If $(a, b) = (x_i, y_j)$, we have that

$$\text{LHS of (E5.2.1.1)} = \delta([x_i, y_j]) = \delta(\delta_{ij}z) = 0, \quad \text{and}$$

$$\text{RHS of (E5.2.1.1)} = [x_i \otimes 1 + 1 \otimes x_i, \delta(y_j)] = b_{ji}z \otimes z - b_{ji}z \otimes z = 0.$$

Hence (E5.2.1.1) holds for $(a, b) = (x_i, y_j)$.

If $(a, b) = (y_i, y_j)$, we have that

$$\begin{aligned}
\text{LHS of (E5.2.1.1)} &= \delta([y_i, y_j]) = \delta(0) = 0, \text{ and} \\
\text{RHS of (E5.2.1.1)} &= [y_i \otimes 1, \delta(y_j)] + [1 \otimes y_i, \delta(y_j)] \\
&\quad + [\delta(y_i), y_j \otimes 1] + [\delta(y_i), 1 \otimes y_j] + [\delta(y_i), \delta(y_j)] \\
&= a_{ji}z \otimes z + c_{ji}z \otimes z \\
&\quad + a_{ij}z \otimes z + c_{ij}z \otimes z \\
&\quad + \left(\sum_s a_{is}b_{js} - b_{is}a_{js} \right) z \otimes z^2 + \left(\sum_s -c_{is}b_{js} + b_{is}c_{js} \right) z^2 \otimes z.
\end{aligned}$$

Hence (E5.2.1.1) holds if and only if $A + A^\tau + C + C^\tau = 0$, $AB^\tau = BA^\tau$ and $CB^\tau = BC^\tau$. This completes the proof of (a).

The proofs of (b) and (c) are similar and therefore omitted. \square

We consider one last example. Let U_n be the strictly upper triangular $n \times n$ -matrix coalgebra, namely, it is the coalgebra with basis $\{x_{ij}\}_{1 \leq i < j \leq n}$ such that

$$\delta(x_{ij}) = \sum_{i < s < j} x_{is} \otimes x_{sj} \quad \text{for all } 1 \leq i, j \leq n.$$

In the following proposition, let $E = (e_i)$, $F = (f_i)$ and $G = (g_i)$ be three arbitrary vectors in k^{n-1} . For $1 \leq i < j \leq n$, define

$$a_{ij} = g_i + g_{i+1} + \cdots + g_{j-1}.$$

It follows from the definition that $a_{is} + a_{sj} = a_{ij}$ for all $1 \leq i < s < j \leq n$.

Example 5.5.4. Let $n \geq 3$. Then the following anti-commutative k -bilinear map $[\ , \] : U_n^{\otimes 2} \rightarrow U_n$ defines a Lie algebra structure on U_n such that $(U_n, [\ , \])$ is a coassociative Lie algebra.

$$[x_{1n}, x_{1n}] = 0, \tag{E5.5.4.1}$$

$$[x_{st}, x_{ij}] = 0 \quad \text{if } (s, t) \neq (1, n) \text{ and } (i, j) \neq (1, n), \tag{E5.5.4.2}$$

$$[x_{1n}, x_{ij}] = a_{ij}x_{ij} \quad \text{if } (i, j) \neq (1, n), (1, n-1), (2, n), \tag{E5.5.4.3}$$

$$[x_{1n}, x_{1n-1}] = a_{1n-1}x_{1n-1} + \sum_{i=1}^{n-1} e_i x_{ii+1}, \tag{E5.5.4.4}$$

$$[x_{1n}, x_{2n}] = a_{2n}x_{2n} + \sum_{i=1}^{n-1} f_i x_{ii+1}. \tag{E5.5.4.5}$$

Proof. First we prove that $(U_n, [\ , \])$ is a Lie algebra. Let $K = \bigoplus_{(i,j) \neq (1,n)} kx_{ij} \subset L$. By definition, we have $[K, K] = 0$ and $[L, K] = [K, L] \subset K$. Since we define $[\ , \]$ to be anti-commutative, it suffices to show the Jacobi identity

$$[a, [b, c]] = [b, [a, c]] + [[a, b], c]$$

for all $a, b, c \in U_n$. If $a, b, c \in K$, then the Jacobi identity is trivially true. If $a = b = x_{1n}$ and $c \in K$, the Jacobi identity is also true since it is true for all $a = b$. The Jacobi identity is stable under permutation and thus the remaining case to consider is when $a = x_{1n}$ and $b, c \in K$. In this case,

$$\begin{aligned} [a, [b, c]] &= [a, 0] = 0, \\ [b, [a, c]] + [[a, b], c] &\in [K, K] + [K, K] = \{0\}. \end{aligned}$$

Hence the Jacobi identity holds and $(U_n, [,]) is a Lie algebra.$

To prove $(U_n, [,]) is a coassociative Lie algebra, we need to verify (E5.2.1.1). By linearity, it suffices to check (E5.2.1.1) for cases listed in (E6.5.4.1)-(E5.5.4.5).$

Case 1: If $(a, b) = (x_{1n}, x_{1n})$, (E5.2.1.1) is automatic (for any $a = b$).

Case 2: Suppose that $(a, b) = (x_{st}, x_{ij})$ for $(s, t) \neq (1, n)$ and $(i, j) \neq (1, n)$. Since $\delta(L) \subset K \otimes K$ and $[K, K] = 0$, both sides of (E5.2.1.1) are zero.

Case 3: Suppose that $(a, b) = (x_{1n}, x_{ij})$ for $(i, j) \neq (1, n)$. Using the fact $[K, K] = 0$, we have

$$\begin{aligned} \text{LHS of (E5.2.1.1)} &= \delta([x_{1n}, x_{ij}]) = a_{ij} \sum_{i < s < j} x_{is} \otimes x_{sj}, \\ \text{RHS of (E5.2.1.1)} &= [x_{1n} \otimes 1 + 1 \otimes x_{1n}, \sum_{i < s < j} x_{is} \otimes x_{sj}] \\ &= (a_{is} + a_{sj}) \sum_{i < s < j} x_{is} \otimes x_{sj} = a_{ij} \sum_{i < s < j} x_{is} \otimes x_{sj}. \end{aligned}$$

Hence (E5.2.1.1) holds. This takes care of cases in (E5.5.4.3)-(E5.5.4.5).

Combining all the above cases we have checked (E5.2.1.1). Therefore $(U_n, [,]) is a coassociative Lie algebra. $\square$$

Chapter 6

**CONNECTED HOPF ALGEBRAS OF GELFAND-KIRILLOV
DIMENSION FOUR**

6.1 Introduction

For this chapter we assume that the base field k is algebraically closed of characteristic zero.

In Chapter 3, we show that if a Hopf algebra H is connected, then the associated graded Hopf algebra $\text{gr } H$ with respect to the coradical filtration is isomorphic to a commutative Hopf algebra. If H has finite GK-dimension, then $\text{gr } H$ is isomorphic to the polynomial ring $k[x_1, \dots, x_n]$, namely, the regular functions $\mathcal{O}(G)$ on a unipotent group G , or equivalently, the graded dual $U(\mathfrak{L})^*$ of the universal enveloping algebra over a graded Lie algebra \mathfrak{L} , which is called the *lantern* of H . Since the coradical filtration is naturally associated to the given Hopf algebra, the lantern \mathfrak{L} , as well as the associated unipotent group G , are invariants of H . This observation motivates the following two related questions.

Question 6.1.1. What are the invariants of H that determine completely the Hopf algebra structure of H ?

Question 6.1.2. Can we classify all connected Hopf algebras of finite GK-dimension?

The first question was suggested by Andruskiewitsch and Brown. Also see the talk given by Brown at the Banff workshop [Br3]. The second question is a subquestion of several motivating questions for a couple of ongoing classification projects initiated by many people such as Andruskiewitsch, Schneider, Brown, Goodearl and their collaborators. Some studies of general Hopf algebras of GK-dimension 1 and 2 are given in [BZ, Li, GZ2] by using homological tools. There is little chance to list all isomorphism classes of connected Hopf algebras. A more practical question is along the line of Question 6.1.1: can we classify connected Hopf algebras in terms of their invariants?

The connected hypothesis is quite restrictive, but there are some interesting new Hopf algebras in this class even when the GK-dimension is 3, see Theorem 3.6.8. Since H is a deformation of $\text{gr } H$, it is possible to understand all H in Question 6.1.2 if the (Hopf)-cohomologies of $\text{gr } H$ can be worked out completely. But we will not consider this in the present paper.

The first goal of the paper is to provide some invariants of connected Hopf algebras that help us to understand partially the structure of the Hopf algebra. One of such is the coassociative Lie algebra that was introduced in Chapter 5. Our second goal is to study and classify all connected Hopf algebras of GK-dimension 4, which should give us a better

sense of how connected Hopf algebras of higher GK-dimension look like. Here is the main result.

Theorem 6.1.3. *Let H be a connected Hopf algebra of GK-dimension 4. Then H is isomorphic to one of following.*

- (a) *Enveloping algebra $U(\mathfrak{g})$ over a Lie algebra \mathfrak{g} of dimension 4.*
- (b) *Enveloping algebra $U(L)$ over an anti-cocommutative coassociative Lie algebra L of dimension 4.*
- (c) *Primitively-thin Hopf algebras of GK-dimension 4.*

Here are some details about Theorem 6.1.3.

Remark 6.1.4. Let H be as in Theorem 6.1.3 and parts (a,b,c) here match up with that of Theorem 6.1.3.

1. All 4-dimensional Lie algebras over the complex numbers \mathbb{C} are listed in the book [OV, Theorem 1.1(iv), page 209]. In this case, the lantern $\mathfrak{L}(H)$ is the unique graded Lie algebra of dimension 4 generated by four elements in degree 1, namely, the abelian Lie algebra of dimension 4.
2. Anti-cocommutative coassociative Lie algebras of dimension 4 are classified in Theorem 6.4.5. Therefore Hopf algebras in Theorem 6.1.3(b) are completely described. In this case, $\mathfrak{L}(H)$ is, up to isomorphism, the unique graded Lie algebra of dimension 4 generated by three elements in degree 1, namely, the Lie algebra $\mathfrak{h}_3 \oplus k$ where \mathfrak{h}_3 is the 3-dimensional Heisenberg Lie algebra.
3. There are exactly four families of primitively-thin Hopf algebras of GK-dimension 4, each of which is constructed explicitly in Section 6.5, see Theorem 6.5.23. In this case, $\mathfrak{L}(H)$ is, up to isomorphism, the unique graded Lie algebra of dimension 4 generated by two elements in degree 1.

Some ideas can be extended to connected Hopf algebras of GK-dimension 5. For example, the classification of the lantern of H is given in Remark 6.3.8. In higher GK-dimension, we have the following result. Let $p(H)$ denote the dimension of the space of all primitive elements. The term $P_2(H)$ is defined in Definition 6.3.2 (b).

Theorem 6.1.5. *Suppose H is a connected Hopf algebra. If $p(H) = \text{GKdim } H - 1 < \infty$, then H is isomorphic to the enveloping algebra over an anti-cocommutative coassociative Lie algebra $P_2(H)$. In this case H is completely determined by $P_2(H)$.*

6.2 Preliminaries

Let LieAlg be the category of Lie algebras and let HopfAlg_{cc} be the category of connected cocommutative Hopf algebras. Then the first assertion of the following is a consequence of Milnor-Moore-Cartier-Kostant Theorem [Mo, Theorem 5.6.5]. The second assertion is a well-known fact in ring theory.

Proposition 6.2.1. *Then the assignment $\mathfrak{g} \rightarrow U(\mathfrak{g})$ defines an equivalence between categories LieAlg and HopfAlg_{cc} . If $\dim \mathfrak{g} < \infty$, then*

$$\text{GKdim } U(\mathfrak{g}) = \text{gldim } U(\mathfrak{g}) = \dim \mathfrak{g}.$$

In some sense, noncocommutative connected Hopf algebras are a generalization of the universal enveloping algebra over a Lie algebra.

Let H be a connected Hopf algebra. By Proposition 3.5.4 and Theorem 3.5.6, $\text{gr } H$ is a commutative domain. Suppose $\text{gr } H$ is locally finite. Then the graded dual $(\text{gr } H)^*$ is a Hopf algebra.

Definition 6.2.2. Let H be a connected Hopf algebra.

- (a) Let $P(H)$ be the space of primitive elements in H and let $p(H)$ be the dimension of $P(H)$.
- (b) We say H is *locally finite* if $p(H) < \infty$, or equivalently, H_i in the coradical filtration of H is finite dimensional for all i .
- (c) H is called *primitively-thin*, if $p(H) = 2$.
- (d) Suppose H is locally finite. The *lantern* of H is the graded Lie algebra $\mathfrak{L}(H)$ such that $U(\mathfrak{L}(H)) \cong (\text{gr } H)^*$. In other words, $\mathfrak{L}(H) = P((\text{gr } H)^*)$.

The following lemma is easy. Part (a) of the following lemma says that $\mathfrak{L}(H)$ is a kind of the abelianization of H . Parts (e,f) justify calling H primitively-thin when $p(H) = 2$.

Lemma 6.2.3. *Let H be a locally finite connected Hopf algebra.*

- (a) *If H is the (universal) enveloping algebra $U(\mathfrak{g})$ for a finite dimensional Lie algebra \mathfrak{g} , then $\mathfrak{L}(H)$ is the abelian Lie algebra of dimension equal to $\dim \mathfrak{g}$.*
- (b) *The $\mathfrak{L}(H)$ is a positively graded Lie algebra generated in degree 1 and $\dim \mathfrak{L}(H) = \text{GKdim } H$.*
- (c) $\mathfrak{L}(H)_1 = (\text{gr } H)_1^* = (H_1/k)^* = P(H)^*$.

(d) $p(H) = 1$ if and only if $H = k[x]$.

(e) (Lemma 2.4.7) If $\text{GKdim } H \geq 2$, then $p(H) \geq 2$.

(f) H is primitively-thin if and only if $\mathfrak{L}(H)$ is a graded Lie algebra generated by two elements in degree 1.

Proof. (a) In this case $\text{gr } H = U(A)$ where A is an abelian Lie algebra with $\dim A = \dim \mathfrak{g}$. So $\text{gr } H$ is a commutative and cocommutative Hopf algebra and $(\text{gr } H)^* \cong \text{gr } H$ as Hopf algebras. Consequently, $(\text{gr } H)^* \cong U(A)$. Thus $\mathfrak{L}(H) \cong A$.

(b) Since $\text{gr } H$ is coradically graded, $(\text{gr } H)^*$ is generated in degree 1 as an algebra. Since $(\text{gr } H)^*$ is cocommutative Proposition 3.5.4 and $\text{char } k = 0$, $(\text{gr } H)^* \cong U(\mathfrak{L})$ and \mathfrak{L} is a graded Lie algebra generated in degree 1. Finally,

$$\text{GKdim } H = \text{GKdim } \text{gr } H = \text{GKdim}(\text{gr } H)^* = \dim \mathfrak{L}(H).$$

(c) It follows from definition and part (b).

(d) If $p(H) = 1$, then $\mathfrak{L}(H)_1$ is 1-dimensional. By part (b), $\mathfrak{L}(H)$ is generated by $\mathfrak{L}(H)_1$. Thus $\mathfrak{L}(H) = \mathfrak{L}(H)_1$, which is 1-dimensional.

(e,f) These follow from part (b) and definition. □

Given any graded Lie algebra finitely generated in degree 1, say \mathfrak{g} , $U(\mathfrak{g})$ is a locally finite graded Hopf algebra. Since $U(\mathfrak{g})$ is isomorphic to $(U(\mathfrak{g})^*)^*$, the lantern of the Hopf algebra $H = (U(\mathfrak{g}))^*$ is \mathfrak{g} . Therefore every graded Lie algebra finitely generated in degree 1 appears as the lantern of some connected Hopf algebra.

Lemma 6.2.4. *Let H be a connected Hopf algebra of GK-dimension 4 and let \mathfrak{L} be the lantern of H . Then \mathfrak{L} is isomorphic to one of the following:*

(a) the abelian Lie algebra of dimension 4 concentrated in degree 1.

(b) the graded Lie algebra of dimension 4 with a basis $\{a, b, c, [a, b]\}$ where a, b, c , are in degree 1 and $[a, b]$ is in degree 2, subject to the relations $[c, \mathfrak{L}] = 0 = [[a, b], \mathfrak{L}]$. This Lie algebra is isomorphic to the Lie algebra $\mathfrak{h}_3 \oplus k$ where \mathfrak{h}_3 is the 3-dimensional Heisenberg Lie algebra.

(c) the graded Lie algebra of dimension 4 with a basis $\{a, b, [a, b], [[a, b], b]\}$ where a, b are in degree 1, $[a, b]$ is in degree 2 and $[[a, b], b]$ is in degree 3, and subject to the relations $[[a, b], a] = 0 = [[[a, b], b], \mathfrak{L}] = 0$.

Proof. By Lemma 6.2.3(b), $\mathfrak{L} = \bigoplus_{i \geq 1} \mathfrak{L}_i$ is a graded Lie algebra generated in degree 1 of dimension 4. By Lemma 6.2.3(c,e), the degree 1 component \mathfrak{L}_1 has dimension either 2 or 3 or 4.

If $\dim \mathfrak{L}_1 = 4$, then $\mathfrak{L} = \mathfrak{L}_1$ which must be abelian. This is case (a).

If $\dim \mathfrak{L}_1 = 3$, pick a basis, say $\{a, b, c\}$ of \mathfrak{L}_1 . Then $\dim \mathfrak{L}_2 = 1$ and $\mathfrak{L}_i = 0$ for all $i > 2$. Let z be a basis of \mathfrak{L}_2 . By linear algebra, up to a basis change, $z = [a, b]$ and c is in the center of \mathfrak{L} . This is case (b).

If $\dim \mathfrak{L}_1 = 2$, pick a basis, say $\{a, b\}$ of \mathfrak{L}_1 . Then $\dim \mathfrak{L}_2 = 1$ with a basis $[a, b]$, $\dim \mathfrak{L}_3 = 1$ and $\dim \mathfrak{L}_i = 0$ for all $i > 3$. Then $[[a, b], a]$ and $[[a, b], b]$ are linearly dependent. Up to a base change we may assume that $[[a, b], a] = 0$. Thus $[[a, b], b]$ is the fourth basis element of \mathfrak{L} . This is case (c). \square

The three different cases in Lemma 6.2.4 will be a guideline for our classification.

6.3 Coassociative Lie algebras and their enveloping algebras

The definition and basic properties of coassociative Lie algebras can be found in Chapter 5. Recall that by Lemma 5.2.3, for a coassociative Lie algebra L , $U(L)$ is a bialgebra.

Definition 6.3.1. Let L_1 and L_2 be CLAs.

- (a) We say L_1 and L_2 are *quasi-equivalent*, and denoted by $L_1 \sim L_2$, if $U(L_1)$ is isomorphic to $U(L_2)$ as bialgebras.
- (b) A coassociative coalgebra (L, δ) is called *anti-cocommutative* if $\tau\delta = -\delta$ where the flip $\tau : L^{\otimes 2} \rightarrow L^{\otimes 2}$ is defined by $\tau(a \otimes b) = b \otimes a$.

For an anti-cocommutative CLA L , the enveloping algebra $U(L)$ is a connected Hopf algebra since L is conilpotent by Lemma 5.3.8 (b).

In rest of this section we assume that $\text{char } k \neq 2$. Let H be a general Hopf algebra and let $P(H)$ denote the k -subspace of H consisting of all primitive elements in H . It is well known that $P(H)$ is Lie algebra. The dimension of $P(H)$ is denoted by $p(H)$.

Let $\delta_H : H \rightarrow H^{\otimes 2}$ be the map defined by

$$\delta_H(h) = \Delta(h) - (h \otimes 1 + 1 \otimes h) \tag{E6.3.3.1}$$

for all $h \in H$.

Definition 6.3.2. Let H be a Hopf algebra.

- (a) An element $f \in H^{\otimes 2}$ is called *symmetric* if $\tau(f) = f$. An element $f \in H^{\otimes 2}$ is called *skew-symmetric* if $\tau(f) = -f$.
- (b) Define

$$P_2(H) = \{x \in H \mid \delta_H(x) \text{ is skew-symmetric and lies in } P(H)^{\otimes 2}\}.$$

The dimension of $P_2(H)$ is denoted by $p_2(H)$.

- (c) The anti-cocommutative-space of H is the quotient space $P_2(H)/P(H)$, denoted by $P'_2(H)$. The dimension of $P'_2(H)$ is denoted by $p'_2(H)$.

Here is a list of basic properties of $P_2(H)$. Part (a) justifies Definition 6.3.2(b,c).

Lemma 6.3.3. *Let H be a Hopf algebra. Let $\delta = \delta_H$ and $[\ , \] = [\ , \]_H$.*

- (a) *Every anti-cocommutative subcoalgebra of (H, δ) is contained in $P_2(H)$. Consequently, $(P_2(H), \delta)$ is the largest anti-cocommutative subcoalgebra of (H, δ) .*
- (b) $P_2(H) = \{x \in H \mid \delta(x) \in P_2(H)^{\otimes 2}, \tau\delta(x) = -\delta(x)\}$.
- (c) $P_2(H)$ is Lie subalgebra of H if and only if $[\delta(x), \delta(y)] = 0$ for all $x, y \in P_2(H)$.
- (d) Suppose $P(H)$ is abelian. Then $(P_2(H), [\ , \])$ is a Lie subalgebra of H and $[P_2(H), P_2(H)] \subset P(H)$.
- (e) If $P_2(H)/P(H)$ is 1-dimensional, then $P_2(H)$ is a Lie subalgebra of H .
- (f) $P_2(H)$ is a Lie module over $P(H)$.
- (g) $p'_2(H) \leq \binom{p(H)}{2}$.

Proof. Clearly $P(H) \subset P_2(H)$. By definition, $\delta(P_2(H)) \subset P(H)^{\otimes 2} \subset P_2(H)^{\otimes 2}$ and $\tau\delta(x) = -\delta(x)$ for all $x \in P_2(H)$. Hence $P_2(H)$ is an anti-cocommutative subcoalgebra of (H, δ) .

(a) By definition, $\ker \delta = P(H)$. Let C be any anti-cocommutative subcoalgebra of (H, δ) . By Lemma 5.3.8(b),

$$\delta(C) \subset (\ker \delta)^{\otimes 2} \subset P(H)^{\otimes 2}.$$

Since C is anti-cocommutative, $C \subset P_2(H)$ by definition.

(b) Let $C = \{x \in H \mid \delta(x) \in P_2(H)^{\otimes 2}, \tau\delta(x) = -\delta(x)\}$. Then, by definition, $P_2(H) \subset C$. So $\delta(C) \subset C^{\otimes 2}$ and C is an anti-cocommutative subcoalgebra of (H, δ) . The assertion now follows from part (a).

(c) For any $x, y \in H$,

$$\begin{aligned} \delta([x, y]) &= \Delta([x, y]) - [x, y] \otimes 1 - 1 \otimes [x, y] \\ &= [\Delta(x), \Delta(y)] - [x, y] \otimes 1 - 1 \otimes [x, y] \\ &= [\delta(x) + x \otimes 1 + 1 \otimes x, \delta(y) + y \otimes 1 + 1 \otimes y] - [x, y] \otimes 1 - 1 \otimes [x, y] \\ &= w(x, y) + v(x, y), \end{aligned}$$

where

$$w(x, y) = [\delta(x), y \otimes 1 + 1 \otimes y] + [x \otimes 1 + 1 \otimes x, \delta(y)]$$

and

$$v(x, y) = [\delta(x), \delta(y)].$$

Now let $x, y \in P_2(H)$. By definition, $\delta(x), \delta(y) \in P(H)^{\otimes 2}$. In this case $w(x, y) \in P(H)^{\otimes 2}$. Since $\delta(x)$ and $\delta(y)$ are skew-symmetric, so is $w(x, y)$. But $v(x, y)$ is symmetric. Hence, $\delta([x, y])$ is skew-symmetric if and only if $v(x, y) = 0$. The assertion follows.

(d) By part (c), $P_2(H)$ is a Lie subalgebra of $(H, [,])$. For $x, y \in P_2(H)$, $\delta([x, y]) = w(x, y) + v(x, y)$ where $w(x, y), v(x, y)$ are defined as in the proof of part (c). Since $P(H)$ is abelian, both $w(x, y)$ and $v(x, y)$ are 0. Hence $\delta([x, y]) = 0$ and consequently, $[x, y] \in P(H)$. The assertion follows.

(e) Since $P_2(H)/P(H)$ is 1-dimensional, $P_2(H) = kf \oplus P(H)$ for some $f \in P_2(H)$. For any $x, y \in P_2(H)$, write $x = af + x_0$ and $y = bf + y_0$ for some $a, b \in k$ and $x_0, y_0 \in P(H)$. Then $\delta(x) = a\delta(f) + \delta(x_0)$ and $\delta(y) = b\delta(f) + \delta(y_0)$. Hence $[\delta(x), \delta(y)] = [a\delta(f), b\delta(f)] = 0$. The assertion follows from part (c).

(f) Let $x \in P_2(H)$ and $y \in P(H)$. Then $\delta([x, y]) = [\delta(x), 1 \otimes y + y \otimes 1] \in P(H)^{\otimes 2}$. Since $\delta(x)$ is skew-symmetric, so is $[\delta(x), 1 \otimes y + y \otimes 1]$. Hence $[x, y] \in P_2(H)$.

(g) This follows from the fact that δ defines a k -linear injective map from $P_2(H)/P(H) \rightarrow P(H) \wedge P(H)$. \square

Lemma 6.3.4. *Let H be a Hopf algebra. Suppose that $\text{char } k = 0$. In parts (c,d,e), assume that H is a connected Hopf algebra.*

(a) $p_2(H) \leq \text{GKdim } H$.

(b) Let \mathfrak{g} be a Lie algebra. Then $P_2(U(\mathfrak{g})) = P(U(\mathfrak{g})) = \mathfrak{g}$.

(c) Let U be the Hopf subalgebra of H generated by $P(H)$. If $U \neq H$, then $P(H) \neq P_2(H)$ and $\text{GKdim } U < \text{GKdim } H$.

(d) $P(H) \cong \text{gr } H(1) = P(\text{gr } H)$.

(e) $P_2(H) \cong P_2(\text{gr } H)$ and $P_2(\text{gr } H) \oplus \text{gr } H(1)^2 = \text{gr } H(1) \oplus \text{gr } H(2)$.

Proof. (a) The subcoalgebra $P_2(H) + k1$ is connected and counital by Lemma 5.3.4. Then the subbialgebra of H generated by $P_2(H) + k1$ is connected, and whence a connected Hopf algebra [Mo, Lemma 5.2.1]. Therefore, after replacing it by the Hopf subalgebra generated by $P_2(H) + k1$, we may assume that H is a connected Hopf algebra.

By Theorem 3.5.10, $\text{gr } H \cong k[x_1, \dots, x_n]$ where $n = \text{GKdim } H$. Arranging $\{x_i\}$ so that $\deg x_i = 1$ for all $i = 1, \dots, p_1$ and $\deg x_i = 2$ for all $i = p_1 + 1, \dots, p_2$ and $\deg x_i > 2$ for all $i > p_2$. Then $\{x_1, \dots, x_{p_1}\}$ is a basis of $\text{gr } H(1) = H_1/H_0 \cong P(H)$, and $\{x_{p_1+1}, \dots, x_{p_2}\}$ is a basis of $\text{gr } H(2)/\text{gr } H(1)^2$. Since $\text{gr } H(2) = H_2/H_1$ and $\text{gr } H(1) =$

H_1/H_0 , $\text{gr } H(2)/\text{gr } H(1)^2 \cong H_2/H_1^2$. Note that $P_2(H)$ is a subspace of H_2 by the definition of $P_2(H)$. For any $x \in P_2(H) \setminus P(H)$, $\delta(x)$ is nonzero and skew-symmetric by the definition of $P_2(H)$ and $P(H)$, and for every $y \in H_1^2$, an easy calculation shows that $\delta(y)$ is symmetric. Let C be a subspace of $P_2(H)$ such that $P_2(H) = C \oplus P(H)$. Then the above discussion says that $C \subset H_2$ and $C \cap H_1^2 = \{0\}$. Therefore

$$\dim P_2(H)/P(H) = \dim C \leq \dim H_2/H_1^2 = \dim \text{gr } H(2)/\text{gr } H(1)^2 = p_2 - p_1.$$

Thus

$$p_2(H) = \dim P_2(H) = \dim C + \dim P(H) \leq (p_2 - p_1) + p_1 = p_2 \leq n = \text{GKdim } H.$$

(b) Follows by a direct computation.

(c) Since $U \neq H$, by Lemma 3.6.4, $\text{GKdim } U < \text{GKdim } H$ if $\text{GKdim } U < \infty$.

Since U is the Hopf subalgebra of H generated by $P(H)$, U is cocommutative and, whence, U is the enveloping algebra $U(\mathfrak{g})$ for some Lie algebra \mathfrak{g} . Clearly, $\mathfrak{g} = P(U) = P(H)$.

Consider the coradical filtrations $\{U_i\}_{i \in \mathbb{N}}$ and $\{H_i\}_{i \in \mathbb{N}}$ of U and H respectively. Since $U \subsetneq H$, there is a minimal n such that $U_n \subsetneq H_n$. The equality $P(U) = P(H)$ implies that $n \geq 2$. Pick any $f \in H_n \setminus U_n$, $\delta(f) \in H_{n-1}^{\otimes 2} = U_{n-1}^{\otimes 2}$. This means that $\delta(f) \in U^{\otimes 2}$ is a coalgebra 2-cocycle of U .

Since U is the enveloping algebra $U(\mathfrak{g})$, U , as a coalgebra, is isomorphic to $U(V)$ where V is the abelian Lie algebra of dimension equal to $\dim \mathfrak{g}$. So one can forget about the Lie structure of \mathfrak{g} when computing the coalgebra cohomology of U . A cohomology computation shows that any 2-cocycle is congruent to an element in $\mathfrak{g} \wedge \mathfrak{g}$ modulo some 2-coboundary. This means that there is an element $g \in U$ such that $\delta(f - g) \in \mathfrak{g} \wedge \mathfrak{g}$. Since $f \notin U$, $f - g \in H \setminus U$, or $f - g \in P_2(H)$ as $\delta(f - g) \in \mathfrak{g} \wedge \mathfrak{g}$ is skew-symmetric. The assertion follows.

(d) By the definition of coradical filtration, $H_1 = k1 \oplus P(H) = H_0 \oplus P(H)$. Hence $P(H) \cong \text{gr } H(1)$. Since $\text{gr } H$ is coradically graded (Remark 1.4.5), $P(\text{gr } H) = \text{gr } H(1)$.

(e) Let $x \in P_2(H)$ and let a be the associated element in $\text{gr } H$. If $x \in P(H)$, then $a \in P(\text{gr } H)$ by part (d) and the map $x \rightarrow a$ is an isomorphism when restricted to $P(H)$. Now suppose $x \notin P(H)$. Then $a \notin P(\text{gr } H)$ by part (d). Since $\delta(x) \in P(H)^{\otimes 2}$ is skew-symmetric, so is $\delta(a) \in P(\text{gr } H)^{\otimes 2}$. Thus we have an injective map $P_2(H) \rightarrow P_2(\text{gr } H)$. Now let $a \in P_2(\text{gr } H)$. Since $P(H) \cong \text{gr } H(1)$, we may assume that $a \in \text{gr } H(2)$. Let $\{x_i\}$ be a basis of $P(H)$ and $\{y_i\}$ be the corresponding basis of $\text{gr } H(1)$. Then $\delta(a) = \sum_{i,j} c_{ij}(y_i \otimes y_j - y_j \otimes y_i)$ for some $c_{ij} \in k$. Let $x \in H_2^+$ be a preimage of a . Then $\delta(x) = \sum_{i,j} c_{ij}(x_i \otimes x_j - x_j \otimes x_i) + 1 \otimes z_1 + z_2 \otimes 1 + c \otimes 1$ where $z_1, z_2 \in P(H)$ and $c \in k$. The counit axiom for Δ implies that $z_1 = z_2 = c = 0$. Therefore $x \in P_2(H)$. The first assertion follows.

For the second assertion we note that

$$\text{gr } H(1) = P(\text{gr } H) \subset P_2(\text{gr } H) \subset \text{gr } H(1) \oplus \text{gr } H(2)$$

and that

$$P_2(\text{gr } H) \cap \text{gr } H(1)^2 = \{0\}.$$

It remains to show that $\text{gr } H(2) \subset P_2(\text{gr } H) \oplus \text{gr } H(1)^2$. By replacing H with $\text{gr } H$, we may assume that H is coradically graded. For any $x \in H_2^+$, $\delta(x) = \sum_{i,j} c_{ij} x_i \otimes x_j + 1 \otimes z_1 + z_2 \otimes 1 + c \otimes 1$ where $x_i, z_1, z_2 \in P(H)$ and $c \in k$. The counit axiom for Δ implies that $z_1 = z_2 = c = 0$. Replacing x by $x - \sum_i \frac{1}{2} c_{ii} x_i^2 - \frac{1}{2} \sum_{i < j} (c_{ij} + c_{ji}) x_i x_j$, $\delta(x)$ becomes skew-symmetric. So $x \in P_2(H)$. The second assertion follows. \square

Here is our main result of this section.

Theorem 6.3.5. *Suppose that $\text{char } k = 0$ and H is a connected Hopf algebra. If*

$$\text{GKdim } H \leq p(H) + 1 < \infty,$$

then $H \cong U(L)$ for some anti-cocommutative CLA $L = P_2(H)$. If, further, $\text{GKdim } H = p(H)$, then $H = U(\mathfrak{g})$ where $\mathfrak{g} = P(H)$.

Proof. Let \mathfrak{g} be the Lie algebra $P(H)$ of the primitive elements of H . If $H = U(\mathfrak{g})$, the assertion is trivial as we have a natural embedding $\text{LieAlg} \rightarrow \text{CoLieAlg}$.

For the rest of the proof we assume that $H \neq U(\mathfrak{g})$. By Lemma 6.3.4(c), $P_2(H) \neq P(H)$. Since $P(H)$ is a proper subspace of $P_2(H)$, $p(H) + 1 \leq p_2(H)$. By Lemma 6.3.3(h), $p_2(H) \leq \text{GKdim } H$. By hypothesis, $\text{GKdim } H \leq p(H) + 1 < \infty$. Therefore there is only one possibility, namely, $p_2(H) = p(H) + 1 = \text{GKdim } H$. By Lemma 6.3.3(e) $P_2(H)$ is a Lie subalgebra of H . By Lemma 6.3.3(a), $(P_2(H), \delta)$ is an anti-cocommutative subcoalgebra of (H, δ) .

Let K be the subbialgebra of H generated as an algebra by the connected subcoalgebra $(P_2(H) + k1, \Delta)$. Then K is a Hopf subalgebra of H as mentioned in the proof of Lemma 6.3.4(a). By Lemma 6.3.3(a), $P_2(K) \supseteq P_2(H)$ and clearly $P_2(H) \supseteq P_2(K)$, we have $P_2(K) = P_2(H)$. By Lemma 6.3.4(b), $P_2(U(\mathfrak{g})) = \mathfrak{g}$, and by assumption in the previous paragraph and Lemma 6.3.4(c), $P_2(H) \not\supseteq \mathfrak{g}$, we conclude that $U(\mathfrak{g})$ is a proper subalgebra of K . By Lemma 3.6.4,

$$\text{GKdim } K \geq \text{GKdim } U(\mathfrak{g}) + 1 = p(H) + 1.$$

By hypothesis, $\text{GKdim } H \leq p(H) + 1$ and obviously $\text{GKdim } H \geq \text{GKdim } K$. Hence $\text{GKdim } H = \text{GKdim } K = p(H) + 1 = p_2(H)$. By Lemma 3.6.4, $H = K$.

Next we show that $L := P_2(H)$ is a CLA. By the second paragraph, L is both a Lie algebra and a coalgebra. It remains to verify (E5.2.1.1). Choose a basis of \mathfrak{g} , say, $\{x_i\}$, and an element $z \in L \setminus \mathfrak{g}$. Then $\{x_i\} \cup \{z\}$ is a basis of L . It is trivial that (E5.2.1.1) holds for $(a, b) = (z, z)$ and for $(a, b) = (x_i, x_j)$ since $\delta(x_i) = 0$. It remains to show (E5.2.1.1) for $(a, b) = (z, x_i)$ (and by symmetry for $(a, b) = (x_i, z)$). Since $\delta(x_i) = 0$, we have

$$\begin{aligned} \delta([z, x_i]) &= [\delta(z), x_i \otimes 1 + 1 \otimes x_i] \\ &= [z \otimes 1 + 1 \otimes z, \delta(x_i)] + [\delta(z), x_i \otimes 1 + 1 \otimes x_i] + [\delta(z), \delta(x_i)] \end{aligned}$$

which holds in $H^{\otimes 2}$ and hence holds in $L^{\otimes 2}$. Thus the above holds in $U(L)^{\otimes 2}$, which verifies (E5.2.1.1).

Let $U(L)$ be the enveloping algebra of the CLA L . There is a canonical Hopf algebra map $\phi : U(L) \rightarrow H$ sending $x \in L$ to x . By [Mo, Theorem 5.3.1], the map ϕ is injective since the restriction of ϕ on the space of primitive elements, namely, $\phi|_{P(H)}$, is injective. Now by Lemma 3.6.4 again this is an isomorphism since $\text{GKdim } H = \text{GKdim } U(L)$. Finally by the definition of $P_2(H)$, L is anti-cocommutative. \square

Proposition 6.3.6. *Suppose $\text{char } k = 0$. Let L be an anti-cocommutative CLA.*

(a) $P_2(U(L)) = L$.

(b) Let $P = \ker(\delta : L \rightarrow L^{\otimes 2})$. Then $\text{gr } U(L) = k[P \oplus L/P]$ where elements of P are in degree 1 and that of L/P are in degree 2.

(c) The coradical of $U(L)$ is given by $U(L)_0 = k1$, $U(L)_1 = P + k1$, $U(L)_2 = (P + k1)^2 + L$, and, for $n \geq 3$,

$$U(L)_n = \sum_{i=1}^{n-1} U(L)_i \cdot U(L)_{n-i}.$$

(d) Suppose L is finite dimensional. Then $\mathfrak{L}(U(L)) \cong P^* \oplus (L/P)^*$ where the Lie algebra structure of $P^* \oplus (L/P)^*$ is induced by the coalgebra structure of L .

Proof. For simplicity, we assume that $\dim L < \infty$ in the following proof. The assertion holds in general, but the proof requires more computation which we omit here.

(a) By construction and Lemma 6.3.3(a), L is a subspace of $P_2(U(L))$. By Lemma 6.3.4(a),

$$\dim P_2(U(L)) \leq \text{GKdim } U(L) = \dim L \leq \dim P_2(U(L)).$$

Hence $L = P_2(U(L))$ as L is finite dimensional.

(b) By the proof of Lemma 6.3.4(a), $k[P \oplus L/P]$ is a Hopf subalgebra of $\text{gr } U(L)$. Obviously they have the same GK-dimension. Now the assertion follows from Lemma 3.6.4.

(c) This follows from part (b) and the fact $\text{gr } U(L)$ is coradically graded.

(d) By part (b), $H := \text{gr } U(L) = k[P \oplus L/P]$ where $P \oplus L/P$ is the minimal generating space of H . Then $P^* \oplus (L/P)^*$ is a minimal co-generating space of the dual $K := H^* = (\text{gr } U(L))^*$. Let $\{x_i\}_i$ and $\{y_j\}_j$ be a basis of P and L/P respectively, and $\{x_i^*\}_i$ and $\{y_j^*\}_j$ be the dual basis of P^* and $(L/P)^*$ respectively. It follows from the definition and the fact that x_i and y_j are generators that both x_i^* and y_j^* are primitive elements. Therefore $P^* \oplus (L/P)^* \subset \mathfrak{L}(U(L))$. Consequently, $P^* \oplus (L/P)^* = \mathfrak{L}(U(L))$ as they have the same

k -dimension. Suppose, for any s , $\delta(y_s) = \sum_{i < j} c_s^{ij} (x_i \otimes x_j - x_j \otimes x_i)$ for some $c_s^{ij} \in k$, and define $c_s^{ji} = -c_s^{ij}$. Then

$$\Delta_H(y_s) = y_s \otimes 1 + 1 \otimes y_s + \sum_{i < j} c_s^{ij} (x_i \otimes x_j - x_j \otimes x_i).$$

By the k -linear pairing $K \times H \rightarrow k$, we have

$$\begin{aligned} [x_i^*, x_j^*](y_t) &= (x_i^* x_j^* - x_j^* x_i^*)(y_t) \\ &= (x_i^* \otimes x_j^* - x_j^* \otimes x_i^*)(\Delta_H(y_t)) \\ &= (x_i^* \otimes x_j^* - x_j^* \otimes x_i^*)(y_t \otimes 1 + 1 \otimes y_t + \sum_{i' < j'} c_t^{i'j'} (x_{i'} \otimes x_{j'} - x_{j'} \otimes x_{i'})) \\ &= 2c_t^{ij} = \sum_s c_s^{ij} (2y_s^*)(y_t). \end{aligned}$$

Hence $[x_i^*, x_j^*] = \sum_s 2c_s^{ij} y_s^*$ for all i, j . Therefore the assertion follows since the coefficients c_s^{ij} are determined by the coalgebra of L . \square

Proposition 6.3.7. *Let L be a conilpotent CLA of dimension ≤ 4 . Then it is quasi-equivalent to an anti-cocommutative CLA.*

Proof. By Lemma 5.3.4, $U(L)$ is a connected Hopf algebra. By Lemma 6.2.3(e), $P(U(L))$ has dimension at least 2. If $\dim L \leq 3$, the assertion follows from Theorem 6.3.5. For the rest we consider the case when $\dim L = 4$.

Let $P_n(L) = \ker \delta^n$. Since L is conilpotent, $P_{i-1}(L) \neq P_i(L)$ if $P_{i-1}(L) \neq L$.

If $\dim P_1(L) = 4$, then L is 4-dimensional Lie algebra with trivial δ -structure.

If $\dim P_1(L) = 1$, then $\dim P_2(L) = 2$ with an element $x_2 \in P_2(L) \setminus P_1(L)$ such that $\delta(x_2) = x_1 \otimes x_1$. For any $x_3 \in P_3(L) \setminus P_2(L)$, write $\delta(x_3) = ax_1 \otimes x_2 + bx_2 \otimes x_1 + cx_2 \otimes x_2$. The coassociativity on x_3 implies that $a = b$ and $c = 0$. Without loss of generality, we assume that $\delta(x_3) = x_1 \otimes x_2 + x_2 \otimes x_1$. Thus $x_1, x_2 - \frac{1}{2}x_1^2$ and $x_3 - x_1x_2 + \frac{1}{3}x_1^3$ are linearly independent primitive elements in $U(L)$. Thus $\dim P(U(L))$ is at least 3. Hence L is quasi-equivalent to an anti-cocommutative CLA by Theorem 6.3.5.

Suppose $\dim P_1(L) = 2$ with a basis $\{x_1, x_2\}$. If $\dim P(U(L)) \geq 3$, L is quasi-equivalent to an anti-commutative CLA by Theorem 6.3.5. So we only need to consider the case when $\dim P(U(L)) = 2$. If $\dim P_2(L) = 4$, choose two basis elements y_1, y_2 in $P_2(L) \setminus P_1(L)$. Then $\delta(y_1 - f(x_1, x_2)) = a(x_1 \otimes x_2 - x_2 \otimes x_1)$ and $\delta(y_2 - g(x_1, x_2)) = b(x_1 \otimes x_2 - x_2 \otimes x_1)$ for some polynomial $f(x_1, x_2)$ and $g(x_1, x_2)$. Since $P(U(L)) = 2$, both a and b are nonzero. Thus $a(y_2 - g) - b(y_1 - f)$ is an extra primitive element, a contradiction. Hence $\dim P_2(L) = 3$ and $\dim P_3(L) = 4$. Let x_3 be a basis element in $P_2(L)$ and x_4 be a basis element in $P_3(L)$. Write $\delta(x_4) = f \otimes x_3 + x_3 \otimes g + ax_3 \otimes x_3$ with $f, g \in P_1(L)$ and $a \in k$. Coassociativity on x_4 implies that $a = 0$, $f = g \neq 0$ and $\delta(x_3) = bf \otimes f$ for some $b \in k$. Thus L is symmetric and hence quasi-equivalent to a Lie algebra by Corollary 5.3.6.

If $\dim P_1(L) = 3$, then $\dim P(U(L))$ is at least 3 and the assertion follows from Theorem 6.3.5. \square

Before we continue we would like to make a remark about the lantern of connected Hopf algebras of GK-dimension 5, which could serve as a guideline for the classification of connected Hopf algebras of GK-dimension 5.

Remark 6.3.8. Let H be a connected Hopf algebra of GK-dimension 5.

- (a) Let $p(H)$ denote the dimension of the space of all primitive elements. Then $p(H)$ is in the range $[2, 5]$.
- (b) If $p(H) = 5$, then $\mathfrak{L}(H)$ is abelian (which is unique) and H is isomorphic to the enveloping algebra over a Lie algebra.
- (c) If $p(H) = 4$, then $\mathfrak{L}(H)$ is generated by four elements in degree 1 (there are two such graded Lie algebras up to isomorphism) and H is isomorphic to the enveloping algebra over an anti-cocommutative coassociative Lie algebra.
- (d) If $p(H) = 3$, then $\mathfrak{L}(H)$ is generated by three elements in degree 1 and there are two subcases.
 - (d1) If the degree two component of $\mathfrak{L}(H)$ has dimension 2 (there is only one such graded Lie algebra), then H is isomorphic to an enveloping algebra over an anti-cocommutative coassociative Lie algebra.
 - (d2) If the degree two component of $\mathfrak{L}(H)$ has dimension 1 (there is only one such graded Lie algebra), then H is not isomorphic to the enveloping algebra over either an ordinary Lie algebra or a coassociative Lie algebra, nor a primitively-thin Hopf algebra.
- (e) If $p(H) = 2$, then $\mathfrak{L}(H)$ is generated by two elements in degree 1 (there are two such graded Lie algebras), and H is a primitively-thin Hopf algebra.

6.4 Classification of anti-cocommutative CLAs up to dimension 4

In this section we classify all anti-cocommutative CLAs of dimension up to 4. The case of dimension 1 is trivial.

6.4.1 Dimension 2

We start with an easy observation.

Lemma 6.4.1. *Let L be an anti-cocommutative CLA of dimension two. Then $\delta = 0$ and L is an ordinary Lie algebra.*

Proof. Let $I = \ker \delta$. By Lemma 5.3.8 (b), $\delta(L) \subset I^{\otimes 2}$. If $I = 0$, then $\delta = 0$. If $\dim I = 1$, pick a nonzero element $x \in I$. Then, for every $y \in L$, $\delta(y) = \lambda x \otimes x$ for some $\lambda \in k$. By the anti-cocommutativity of δ , $\lambda = 0$. Thus $\delta = 0$. The remaining case is when $\dim I = 2$, which implies that $\delta = 0$. \square

It is well known that a 2-dimensional Lie algebra is either abelian or solvable with $[x, y] = y$ for a suitable basis $\{x, y\}$.

6.4.2 Dimension 3

The following lemma is similar to Lemma 6.4.1. The classification of 3-dimensional Lie algebra is well-known, which is called Bianchi classification when the base field is either the real numbers \mathbb{R} or the complex numbers \mathbb{C} (for instance, see [OV, p. 209, Theorem 1.1] if $k = \mathbb{C}$). We will not include the list here. Next we will consider those CLAs with nontrivial δ .

Let $a(\lambda_1, \lambda_2, \alpha)$ denote the CLA with a basis $\{x, y, z\}$ whose Lie algebra structure on $L =: B(b)$ is determined by

$$\begin{aligned} [x, y] &= 0, \\ [z, x] &= \lambda_1 x + \alpha y, \\ [z, y] &= \lambda_2 y \end{aligned}$$

for some $\lambda_1, \lambda_2, \alpha \in k$, and whose coalgebra structure is determined by $\delta(x) = \delta(y) = 0$ and $\delta(z) = x \otimes y - y \otimes x$.

Lemma 6.4.2. *Let L be an anti-cocommutative CLA of dimension 3 such that $\delta \neq 0$. Then L is isomorphic to one of the following:*

(a) $a(0, 0, 0)$.

(b) $a(1, \lambda, 0)$. And $a(1, \lambda', 0)$ is isomorphic to $a(1, \lambda, 0)$ if and only if $\lambda' = \lambda$ or λ^{-1} .

(c) $a(0, 0, 1)$.

(d) $a(1, 1, 1)$.

(e) The CLA $b(\lambda)$ with a basis $\{x, y, z\}$, whose Lie algebra structure is determined by

$$\begin{aligned} [x, y] &= y, \\ [z, x] &= -z + \lambda y, \\ [z, y] &= 0, \end{aligned}$$

where λ is in k , and whose coalgebra structure is determined by $\delta(x) = \delta(y) = 0$ and $\delta(z) = x \otimes y - y \otimes x$. And $b(\lambda')$ is isomorphic to $b(\lambda)$ if and only if $\lambda' = \lambda$.

All CLAs listed above are pairwise non-isomorphic except for the isomorphisms given in part (b).

Proof. Let $I = \ker \delta$. Repeating the first part of the proof of Lemma 6.4.1, one sees that $\dim I > 1$. Since $\dim L = 3$ and $\delta \neq 0$, we have $\dim I = 2$. Pick any basis $\{x, y\}$ of I . The only skew-symmetric elements in $I^{\otimes 2}$ are $\lambda(x \otimes y - y \otimes x)$ for some $\lambda \in k^\times := k \setminus \{0\}$. Let $z \in L \setminus I$ such that $\delta(z) = x \otimes y - y \otimes x$.

Since I is a Lie subalgebra of L (see Lemma 6.3.3(e)), we have the following two cases to consider.

Case 1: I is abelian. Write

$$\begin{aligned} [z, x] &= a_1x + a_2y + a_3z, \\ [z, y] &= b_1x + b_2y + b_3z. \end{aligned}$$

Applying (E5.2.1.1) to $(a, b) = (z, x)$, we have

$$a_3(x \otimes y - y \otimes x) = \delta([z, x]) = [\delta(z), x \otimes 1 + 1 \otimes x] = 0.$$

Hence $a_3 = 0$. By symmetry, $b_3 = 0$.

Using a linear transformation $f : x \rightarrow c_{11}x + c_{12}y, y \rightarrow c_{21}x + c_{22}y$ with $\det \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = 1$, the matrix $\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$ becomes one of the Jordan forms

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}.$$

In the first Jordan case, if $a = b = 0$, this is case (a). If $a \neq 0$ (or by symmetry if $b \neq 0$) we can assume that $a = 1$ by a change of basis $\{x, y, z\} \rightarrow \{\sqrt{a^{-1}}x, \sqrt{a^{-1}}y, a^{-1}z\}$. So this is case (b). In the second Jordan case, if $a = 0$, this is case (c). If $a \neq 0$, by a change of basis, we can assume that $a = 1$, which is case (d).

Case 2: I is not abelian. So we may assume that $[x, y] = y$ where $\{x, y\}$ is a basis of I . Let $z \in L \setminus I$ such that $\delta(z) = x \otimes y - y \otimes x$. Write

$$\begin{aligned} [z, x] &= a_1x + a_2y + a_3z, \\ [z, y] &= b_1x + b_2y + b_3z. \end{aligned}$$

Applying (E5.2.1.1) to $(a, b) = (z, x)$, we have

$$a_3(x \otimes y - y \otimes x) = \delta([z, x]) = [\delta(z), x \otimes 1 + 1 \otimes x] = -(x \otimes y - y \otimes x).$$

Hence $a_3 = -1$. A similar argument shows that $b_3 = 0$. By the Jacobi identity, we have

$$\begin{aligned} b_1x + b_2y &= [z, y] = [z, [x, y]] = [[z, x], y] + [x, [z, y]] \\ &= [a_1x + a_2y - z, y] + [x, b_1x + b_2y] \\ &= a_1y - (b_1x + b_2y) + b_2y. \end{aligned}$$

Then $b_1 = 0$ and $a_1 = b_2$. Thus we have

$$\begin{aligned} [z, x] &= ax + a_2y - z, \\ [z, y] &= ay. \end{aligned}$$

After replacing z by $z - ax$, we have $a = 0$. This is the case (e).

By the above argument, the CLAs listed are pairwise non-isomorphic except for the isomorphism given in part (b). \square

The enveloping algebra of the case (e) has an interesting property that S^2 is not the identity, see Example 5.5.2.

6.4.3 Dimension 4

This is the main subsection of Section 3. We will classify all 4-dimensional anti-cocommutative CLAs. If $\delta = 0$, L is an ordinary Lie algebra of dimension four and the classification is known [OV, p. 209 Theorem 1.1] in which the base field k is \mathbb{C} . So we assume that $\delta \neq 0$. Throughout this subsection we assume that L is a 4-dimensional anti-commutative CLA such that $\delta \neq 0$.

Lemma 6.4.3. *Let $I = \ker \delta$. Then I is a 3-dimensional Lie subalgebra of L .*

Proof. An easy calculation shows that I is a Lie subalgebra of L by using (E5.2.1.1). It remains to show that $\dim I > 2$. Let $H = U(L)$. By Lemma 6.3.3(g), $p'_2(H) = \dim L - \dim I \leq \binom{\dim I}{2}$. If $\dim I \leq 2$, then $p'_2(H) \leq 1$ and $\dim L \leq 3$, a contradiction. Therefore $\dim I > 2$. \square

Lemma 6.4.4. *There are elements x_1, x_2 in I such that $\delta(L) = k(x_1 \otimes x_2 - x_2 \otimes x_1)$.*

Proof. Let $\{x'_1, x'_2, x'_3\}$ be a basis of I . Since $\dim I = 3$, $\dim \delta(L) = 1$. Pick $z \in L \setminus I$. Then we must have

$$\delta(z) = \sum_{i,j} a_{ij}(x'_i \otimes x'_j - x'_j \otimes x'_i),$$

where $A = (a_{ij})$ is a non-zero 3×3 anti-symmetric matrix. Obviously, $\delta(L)$ is spanned by $\delta(z)$. Now A has eigenvalues $0, \lambda, -\lambda$ for some $\lambda \neq 0$. By replacing z with $z/\sqrt[3]{\lambda}$, we can assume that A has eigenvalues $0, 1, -1$. By linear algebra, there exists an invertible matrix P such that

$$PAP^T = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By setting $(x_1, x_2, x_3)^T = P^{-1}(x'_1, x'_2, x'_3)^T$, we get

$$\delta(z) = x_1 \otimes x_2 - x_2 \otimes x_1.$$

This completes the proof. \square

Here is the main result of this section. The proof is computation and some details are easy to check.

Theorem 6.4.5. *Let L be an anti-cocommutative CLA of dimension four such that $\delta \neq 0$. Then there is a basis $\{x_1, x_2, x_3, z\}$ such that the coalgebra structure is given by $\delta(x_i) = 0$ and $\delta(z) = x_1 \otimes x_2 - x_2 \otimes x_1$. The Lie algebra structure of L is given by one of the following:*

(a)

$$\begin{aligned} [x_2, x_1] &= x_2, \\ [x_3, x_1] &= [x_3, x_2] = 0, \\ [z, x_1] &= z + ax_1 + cx_2, \\ [z, x_2] &= ax_2, \\ [z, x_3] &= bx_2. \end{aligned}$$

where $(a, b) = (1, 1), (1, 0), (0, 1)$ or $(0, 0)$ and $c \in k$.

(b) $L = B((a_{ij}))$ is determined by

$$\begin{aligned} [x_2, x_1] &= [x_3, x_1] = [x_3, x_2] = 0, \\ [z, x_1] &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3, \\ [z, x_2] &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3, \\ [z, x_3] &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3. \end{aligned}$$

where (a_{ij}) is a 3×3 matrix over k . The CLA $B((a_{ij}))$ is isomorphic to $B((b_{ij}))$ if and only if the matrix (a_{ij}) is similar to $\lambda(b_{ij})$ for some $\lambda \in k^\times$.

(c)

$$\begin{aligned} [x_2, x_1] &= 0 \\ [x_3, x_1] &= x_2 \\ [x_3, x_2] &= 0 \\ [z, x_1] &= ax_1 + bx_3 \\ [z, x_2] &= x_2 \\ [z, x_3] &= cx_1 + (1 - a)x_3. \end{aligned}$$

where $a, b, c \in k$.

(d)

$$\begin{aligned}
[x_2, x_1] &= 0 \\
[x_3, x_1] &= x_2 \\
[x_3, x_2] &= 0 \\
[z, x_1] &= ax_1 + bx_3 \\
[z, x_2] &= 0 \\
[z, x_3] &= cx_1 - ax_3.
\end{aligned}$$

where $a, b, c \in k$.

(e)

$$\begin{aligned}
[x_2, x_1] &= [x_3, x_2] = 0, \\
[x_3, x_1] &= x_1, \\
[z, x_1] &= ax_1, \\
[z, x_2] &= bx_1, \\
[z, x_3] &= -z + cx_1 + ax_3
\end{aligned}$$

where $(a, b) = (1, 1), (1, 0), (0, 1)$ or $(0, 0)$ and $c \in k$.

(f)

$$\begin{aligned}
[x_2, x_1] &= 0, \\
[x_3, x_1] &= x_1 + x_2, \\
[x_3, x_2] &= x_2, \\
[z, x_1] &= [z, x_2] = 0, \\
[z, x_3] &= -2z
\end{aligned}$$

(g)

$$\begin{aligned}
[x_2, x_1] &= 0, \\
[x_3, x_1] &= x_1, \\
[x_3, x_2] &= -x_2, \\
[z, x_1] &= ax_1 + cx_2, \\
[z, x_2] &= bx_1, \\
[z, x_3] &= 0.
\end{aligned}$$

where $(a, b) = (1, 1), (1, 0), (0, 1)$ or $(0, 0)$ and $c \in k$.

(h) $L = H(\lambda, a)$ is determined by

$$\begin{aligned} [x_2, x_1] &= 0, \\ [x_3, x_1] &= x_1, \\ [x_3, x_2] &= \lambda x_2, \\ [z, x_1] &= ax_2, \\ [z, x_2] &= ax_1, \\ [z, x_3] &= (-1 - \lambda)z. \end{aligned}$$

where $\lambda \in k \setminus \{0, -1\}$ and $a \in \{0, 1\}$. The CLA $H(\lambda, a)$ is isomorphic to $H(\lambda', a')$ if and only if $a' = a$ and $\lambda' = \lambda$ or $\lambda' = \lambda^{-1}$.

The CLAs listed above are pairwise non-isomorphic except for the isomorphisms given in parts (b,h).

Proof. By the previous two lemmas, for any 4-dimensional anti-cocommutative CLA L with non-zero δ , we can choose a basis $\{x_1, x_2, x_3, z\}$ such that $I = \ker \delta$ is spanned by $\{x_1, x_2, x_3\}$ and $\delta(z) = x_1 \otimes x_2 - x_1 \otimes x_2$.

Write

$$\begin{aligned} [x_2, x_1] &= a_1x_1 + b_1x_2 + c_1x_3 \\ [x_3, x_1] &= a_2x_1 + b_2x_2 + c_2x_3 \\ [x_3, x_2] &= a_3x_1 + b_3x_2 + c_3x_3 \\ [z, x_1] &= e_1z + \phi_1 = e_1z + f_1x_1 + g_1x_2 + h_1x_3 \\ [z, x_2] &= e_2z + \phi_2 = e_2z + f_2x_1 + g_2x_2 + h_2x_3 \\ [z, x_3] &= e_3z + \phi_3 = e_3z + f_3x_1 + g_3x_2 + h_3x_3. \end{aligned}$$

Applying (E5.2.1.1) to (z, x_i) for $i = 1, 2, 3$, one obtains that $c_1 = c_2 = c_3 = 0$ and that $e_1 = b_1$, $e_2 = -a_1$, $e_3 = -a_2 - b_3$. Thus (E5.2.1.1) holds if and only if the Lie bracket satisfies

$$\begin{aligned} [x_2, x_1] &= a_1x_1 + b_1x_2 \\ [x_3, x_1] &= a_2x_1 + b_2x_2 \\ [x_3, x_2] &= a_3x_1 + b_3x_2 \\ [z, x_1] &= b_1z + f_1x_1 + g_1x_2 + h_1x_3 \\ [z, x_2] &= -a_1z + f_2x_1 + g_2x_2 + h_2x_3 \\ [z, x_3] &= (-a_2 - b_3)z + f_3x_1 + g_3x_2 + h_3x_3. \end{aligned}$$

In particular, $J := kx_1 + kx_2$ is a Lie subalgebra. So J is either abelian or solvable.

If J is solvable, we may assume that $[x_2, x_1] = x_2$. Since I is a Lie subalgebra,

$$\begin{aligned} a_3x_1 + b_3x_2 &= [x_3, x_2] = [x_3, [x_2, x_1]] \\ &= [[x_3, x_2], x_1] + [x_2, [x_3, x_1]] \\ &= [a_3x_1 + b_3x_2, x_1] + [x_2, a_2x_1 + b_2x_2] \\ &= b_3x_2 + a_2x_2. \end{aligned}$$

This implies that $a_2 = a_3 = 0$.

If $b_2 \neq 0$ or $b_3 \neq 0$, after a base change, we have $b_2 = b_3 = 0$. The Jacobi identity for other elements implies that

$$\begin{aligned} [x_2, x_1] &= x_2 \\ [x_3, x_1] &= 0 \\ [x_3, x_2] &= 0 \\ [z, x_1] &= z + f_1x_1 + g_1x_2 + h_1x_3 \\ [z, x_2] &= f_1x_2 \\ [z, x_3] &= g_3x_2 \end{aligned}$$

for $f_i, g_i, h_i \in k$. There are a few cases. First of all, we can make $h_1 = 0$ by replacing z with $z + h_1x_3$. If $f_1 \neq 0$, we can assume $f_1 = 1$ by replacing z, x_2 with $\frac{1}{f_1}z, \frac{1}{f_1}x_2$, respectively. If $g_3 \neq 0$, we can also normalize it to be 1 by replacing x_3 with $\frac{1}{g_3}x_3$. All non-isomorphic Lie algebras are now listed in part (a).

If J is abelian, $[x_2, x_1] = 0$. If I is abelian, then $[z, x_i] = \phi_i$ defines a Lie algebra and for any $\phi_i \in I$, $i = 1, 2, 3$. This is part (b). Further classification can be made by taking the Jordan form of the coefficient matrix of $\{\phi_1, \phi_2, \phi_3\}$. This is a linear algebra classification and, to save space, we will not list all possibilities.

For the rest of the proof we assume that J is abelian and I is not abelian. Up to a change of basis $\{x_1, x_2\}$, we may assume that

$$\begin{aligned} [x_3, x_1] &= ax_1 + bx_2 \\ [x_3, x_2] &= cx_2 \end{aligned}$$

where

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$$

where $\lambda \in k^\times$.

Recall that

$$\begin{aligned} [z, x_1] &= \phi_1 = f_1x_1 + g_1x_2 + h_1x_3 \\ [z, x_2] &= \phi_2 = f_2x_1 + g_2x_2 + h_2x_3 \\ [z, x_3] &= (-a - c)z + \phi_3 = (-a - c)z + f_3x_1 + g_3x_2 + h_3x_3. \end{aligned}$$

The Jacobi identity implies that

$$\begin{aligned} 0 &= [\phi_2, x_1] + [x_2, \phi_1] \\ (2a + c)\phi_1 + b\phi_2 &= [\phi_3, x_1] + [x_3, \phi_1] \\ (a + 2c)\phi_2 &= [\phi_3, x_2] + [x_3, \phi_2]. \end{aligned}$$

which completely determine the Lie algebra L .

If $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then we have

$$\begin{aligned} [x_2, x_1] &= 0 \\ [x_3, x_1] &= x_2 \\ [x_3, x_2] &= 0 \\ [z, x_1] &= f_1x_1 + g_1x_2 + h_1x_3 \\ [z, x_2] &= (h_3 + f_1)x_2 \\ [z, x_3] &= f_3x_1 + g_3x_2 + h_3x_3. \end{aligned}$$

First, we can make $g_1 = g_3 = 0$ by replacing z with $z + g_3x_1 - g_1x_3$. If $h_3 + f_1 \neq 0$, we can make $h_3 + f_1 = 1$ by replacing x_1, x_2, z with $\frac{1}{\sqrt{h_3+f_1}}x_1, \frac{1}{\sqrt{h_3+f_1}}x_2, \frac{1}{h_3+f_1}z$. After recycling the letters a, b, c , we obtain part (c). If $h_3 + f_1 = 0$, this is part (d) after recycling the letters a, b, c .

For the rest we use similar computations, so some details are omitted.

If $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, then we have

$$\begin{aligned} [x_2, x_1] &= 0 \\ [x_3, x_1] &= x_1 \\ [x_3, x_2] &= 0 \\ [z, x_1] &= f_1x_1 \\ [z, x_2] &= f_2x_1 \\ [z, x_3] &= -z + f_3x_1 + g_3x_2 + f_1x_3. \end{aligned}$$

First, g_3 can be made 0 by replacing z with $z - g_3x_2$. If $f_1 \neq 0$, we can assume $f_1 = 1$ by replacing z, x_1 with $\frac{1}{f_1}z, \frac{1}{f_1}x_1$, respectively. If $f_2 \neq 0$, we can also normalize it to be 1 by replacing x_1, x_2 with $\sqrt{f_2}x_1, \frac{1}{\sqrt{f_2}}x_2$, respectively. All cases are listed in part (e).

If $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, then we have

$$\begin{aligned} [x_2, x_1] &= 0 \\ [x_3, x_1] &= x_1 + x_2 \\ [x_3, x_2] &= x_2 \\ [z, x_1] &= f_1x_1 + f_1x_2 \\ [z, x_2] &= f_1x_2 \\ [z, x_3] &= -2z + f_3x_1 + g_3x_2 + 2f_1x_3. \end{aligned}$$

First, we can make $f_3 = g_3 = 0$ by replacing z with $z - f_3x_1 - (f_3 + g_3)x_2$. Then we can assume that $f_1 = 0$ by replacing z with $z - f_1x_3$. This part (f).

Finally, if $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$ where $\lambda \neq 0$, then we have

$$\begin{aligned} [x_2, x_1] &= 0 \\ [x_3, x_1] &= x_1 \\ [x_3, x_2] &= \lambda x_2 \\ [z, x_1] &= f_1x_1 + g_1x_2 \\ [z, x_2] &= f_2x_1 + g_2x_2 \\ [z, x_3] &= (-1 - \lambda)z + f_3x_1 + g_3x_2 + (1 + \lambda)f_1x_3 \end{aligned}$$

such that $(1 + \lambda)g_1 = (1 + \lambda)f_2 = 0$ and $\lambda(1 + \lambda)f_1 = (1 + \lambda)g_2$.

Suppose $\lambda = -1$. Then we can make $g_2 = f_3 = g_3 = 0$ by replacing z with $z + f_3x_1 - g_3x_2 + g_2x_3$. Then, if $f_1 \neq 0$, we can make $f_1 = 1$ by replacing x_2, z with $\frac{1}{f_1}x_2, \frac{1}{f_1}z$. If $f_2 \neq 0$, we can make $f_2 = 1$ by replacing x_1, x_2 with $\sqrt{f_2}x_1, \frac{1}{\sqrt{f_2}}x_2$ (Notice that f_1 will not change). This is part (g).

Suppose $\lambda \neq -1$. We can also assume that $f_3 = g_3 = 0$ by replacing z with $z - \frac{f_3}{\lambda}x_1 - g_3x_2$. Now the last three relations become

$$\begin{aligned} [z, x_1] &= f_1x_1 + g_1x_2 \\ [z, x_2] &= g_1x_1 + \lambda f_1x_2 \\ [z, x_3] &= (-1 - \lambda)z + (1 + \lambda)f_1x_3 \end{aligned}$$

Then by replacing z with $z - f_1x_3$, we can assume that $f_1 = 0$. This is part (h). We finish the proof. \square

6.5 Classification of Primitively-thin algebras of dimension at most 4

Recall that a connected Hopf algebra H is primitively-thin if $\overline{p}(H) = 2$.

If $\text{GKdim } H = 2$, then $H = U(\mathfrak{g})$ for a Lie algebra of dimension 2, which follows from Lemmas 6.2.3(e) and Lemma 6.3.4(c).

6.5.1 Primitively-thin Hopf algebras of GK-dimension 3

We recall the definition of two classes of Hopf algebras from Chapter 3, which will be used later in the classification.

Example 6.5.1. Let A be the algebra generated by elements X, Y, Z satisfying the following relations,

$$\begin{aligned} [X, Y] &= 0, \\ [Z, X] &= \lambda_1 X + \alpha Y, \\ [Z, Y] &= \lambda_2 Y, \end{aligned}$$

where $\alpha = 0$ if $\lambda_1 \neq \lambda_2$ and $\alpha = 0$ or 1 if $\lambda_1 = \lambda_2$. Then A becomes a Hopf algebra via

$$\begin{aligned} \epsilon(X) &= 0, & \Delta(X) &= 1 \otimes X + X \otimes 1, \\ \epsilon(Y) &= 0, & \Delta(Y) &= 1 \otimes Y + Y \otimes 1, \\ \epsilon(Z) &= 0, & \Delta(Z) &= 1 \otimes Z + X \otimes Y - Y \otimes X + Z \otimes 1. \end{aligned} \tag{E6.5.1.1}$$

We denote this Hopf algebra by $A(\lambda_1, \lambda_2, \alpha)$. It is easy to see that $A(\lambda_1, \lambda_2, \alpha)$ is the enveloping algebra $U(a(\lambda_1, \lambda_2, \alpha))$ where $a(\lambda_1, \lambda_2, \alpha)$ is the CLA defined before Lemma 6.4.2.

Example 6.5.2. Let B be the algebra generated by elements X, Y, Z satisfying the following relations,

$$\begin{aligned} [X, Y] &= Y, \\ [Z, X] &= -Z + \lambda Y, \\ [Z, Y] &= 0, \end{aligned}$$

where $\lambda \in k$. Then B becomes a Hopf algebra via the coalgebra structure given as in (E6.5.1.1). We denote this Hopf algebra by $B(\lambda)$. This algebra is the enveloping algebra $U(b(\lambda))$ where $b(\lambda)$ is the CLA defined in Lemma 6.4.2(e).

The following proposition is from Theorem 3.6.8. It also follows now from Lemma 6.4.2 and Theorem 6.3.5.

Proposition 6.5.3. *Let H be a connected Hopf algebra. Then H is primitively-thin of GK-dimension 3 if and only if H is isomorphic to one of the following:*

1. *The Hopf algebras $A(0, 0, 0)$, $A(0, 0, 1)$, $A(1, 1, 1)$ or $A(1, \lambda, 0)$ from Example 6.5.1 for some $\lambda \in k$;*
2. *The Hopf algebras $B(\lambda)$ from Example 6.5.2 for some $\lambda \in k$.*

6.5.2 Examples of connected Hopf algebras of GK-dimension 4

We introduce four classes of connected Hopf algebras of GK-dimension 4. Later we will show that these classes give a complete description of primitively-thin Hopf algebras of GK-dimension 4.

Example 6.5.4. Let D be the algebra generated by X, Y, Z, W satisfying the following relations,

$$\begin{aligned} [Y, X] &= [Z, X] = [Z, Y] = 0, \\ [W, X] &= a_{11}X + a_{12}Y, \\ [W, Y] &= a_{21}X + a_{22}Y, \\ [W, Z] &= (a_{11} + a_{22})Z + \xi_1X + \xi_2Y, \end{aligned}$$

where $a_{ij}, \xi_i \in k$. Then D becomes a bialgebra via

$$\begin{aligned} \epsilon(X) &= 0, \quad \Delta(X) = 1 \otimes X + X \otimes 1, \\ \epsilon(Y) &= 0, \quad \Delta(Y) = 1 \otimes Y + Y \otimes 1, \\ \epsilon(Z) &= 0, \quad \Delta(Z) = 1 \otimes Z + X \otimes Y - Y \otimes X + Z \otimes 1, \\ \epsilon(W) &= 0, \\ \Delta(W) &= 1 \otimes W + W \otimes 1 \\ &\quad + \theta_1(Z \otimes X - X \otimes Z + X \otimes XY + XY \otimes X) \\ &\quad + \theta_2(Y \otimes Z - Z \otimes Y + XY \otimes Y + Y \otimes XY), \end{aligned}$$

where $\theta_i \in k$ and at least one of them is non-zero. This bialgebra is also denoted by $D(\{\theta_i\}, \{a_{ij}\}, \{\xi_i\})$ if we want to indicate the parameters. It is easy to see that the coalgebra structure is connected. Hence the bialgebra D is automatically a Hopf algebra. Note that $P(D) = kX + kY$ and $P_2(D) = kX + kY + kZ$. Let f be a Hopf algebra isomorphism between two such Hopf algebras (or a Hopf algebra automorphism). Then f preserves the subspaces $kX + kY$ and $kX + kY + kZ$, and it is now not hard to see that f sends

$$\begin{aligned} X &\longrightarrow c_{11}X + c_{12}Y, \\ Y &\longrightarrow c_{21}X + c_{22}Y, \\ Z &\longrightarrow c_{31}X + c_{32}Y + c_{33}Z, \\ W &\longrightarrow c_{44}W + w(X, Y, Z) \end{aligned} \tag{E6.5.4.1}$$

where $c_{ij} \in k$ and $w(X, Y, Z)$ is a certain polynomial of X, Y, Z . (Note that f being a Hopf algebra isomorphism implies that c_{ij} and w satisfy some conditions which we will not give details here).

Using an isomorphism f , or equivalently, choosing a new basis $\{X, Y, Z, W\}$ properly, one can first normalize the matrix $(a_{ij})_{2 \times 2}$ so that it becomes one of the following five:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \quad (\text{E6.5.4.2})$$

where $\lambda \neq 0$ and in the last class $\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$ is equivalent to $\begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$. The Hopf algebras are pair-wise non-isomorphic if these are in different classes. Within any class, two Hopf algebras $D(\{\theta_i\}, \{a_{ij}\}, \{\xi_i\})$ s can be isomorphic for different parameters $\{\theta_i\}$ and $\{\xi_i\}$, which is determined by the base changes that fixes the matrix given in (E6.5.4.2) (or change $\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$ to $\begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ if in the last class). For example, by replacing $\{X, Y, Z, W\}$ by $\{aX, aY, a^2Z, W\}$, the parameters $\{\theta_1, \theta_2\}$ becomes $\{a^{-3}\theta_1, a^{-3}\theta_2\}$. This means that we may assume that $\{\theta_1, \theta_2\} \in \mathbb{P}^1$. Dependent on the form of the matrix (a_{ij}) listed in (E6.5.4.2), one can further decide the parameters $\{\xi_1, \xi_2\}$ such that the Hopf algebras $D(\{\theta_i\}, \{a_{ij}\}, \{\xi_i\})$ are non-isomorphic. In summary, the isomorphism classes of $D(\{\theta_i\}, \{a_{ij}\}, \{\xi_i\})$ can be completely determined by easy linear algebra.

There is another way of classifying all isomorphism classes of $D(\{\theta_i\}, \{a_{ij}\}, \{\xi_i\})$. First, by choosing the basis $\{X, Y, Z, W\}$ properly, one can assume that $\theta_1 = 0$ and $\theta_2 = 1$. So we can fix $\{\theta_1, \theta_2\} = \{0, 1\}$. In particular,

$$\Delta(W) = 1 \otimes W + W \otimes 1 + Y \otimes Z - Z \otimes Y + XY \otimes Y + Y \otimes XY.$$

Under this restriction, one can further classify the parameters $\{(a_{ij}), \{\xi_i\}\}$. Unfortunately, then we can not assume that the matrix (a_{ij}) is of one of the form given in (E6.5.4.2).

From the algebraic relations, $D(\{\theta_i\}, \{a_{ij}\}, \{\xi_i\})$ is isomorphic to a universal enveloping algebra of a Lie algebra.

Example 6.5.5. Let E be the algebra generated by X, Y, Z, W satisfying the following relations,

$$\begin{aligned} [Y, X] &= [Z, Y] = 0, \\ [Z, X] &= X, \\ [W, X] &= aX, \\ [W, Y] &= bX, \\ [W, Z] &= aZ - W + \xi X + \xi' Y. \end{aligned}$$

where $\xi, \xi' \in k$. Then E becomes a bialgebra (and then a Hopf algebra) via

$$\begin{aligned}\epsilon(X) &= 0, & \Delta(X) &= 1 \otimes X + X \otimes 1, \\ \epsilon(Y) &= 0, & \Delta(Y) &= 1 \otimes Y + Y \otimes 1, \\ \epsilon(Z) &= 0, & \Delta(Z) &= 1 \otimes Z + X \otimes Y - Y \otimes X + Z \otimes 1, \\ \epsilon(W) &= 0, \\ \Delta(W) &= 1 \otimes W + W \otimes 1 \\ &\quad + Z \otimes X - X \otimes Z + X \otimes XY + XY \otimes X.\end{aligned}$$

Up to a base change (by setting $W_{new} = W - \xi'Y$), we may assume that $\xi' = 0$. We denote this Hopf algebra by $E(a, b, \xi)$ where $a, b, \xi \in k$. Using a base change

$$\begin{aligned}X &\rightarrow cX, \\ Y &\rightarrow c^{-1}Y, \\ Z &\rightarrow Z, \\ W &\rightarrow cW,\end{aligned}$$

for some $c \in k$ we can re-scale (a, b, ξ) . The complete set of non-isomorphic classes of $E(a, b, \xi)$ is corresponding to the following cases

$$(a, b, \xi) = \begin{cases} (0, 0, \xi) & \text{if } a = b = 0, \\ (0, 1, \xi) & \text{if } a = 0 \text{ and } b \neq 0, \\ (1, b, \xi) & \text{if } a \neq 0. \end{cases}$$

To unify the presentation of $\Delta(W)$, we make a change of basis

$$\begin{aligned}X &\rightarrow Y, \\ Y &\rightarrow X, \\ Z &\rightarrow -Z, \\ W &\rightarrow W.\end{aligned}$$

Under the new basis, the coalgebra structure is same except for $\Delta(W)$, which becomes

$$\Delta(W) = 1 \otimes W + W \otimes 1 + Y \otimes Z - Z \otimes Y + XY \otimes Y + Y \otimes XY.$$

The algebraic relations change accordingly, which can be easily done.

From the algebraic relations, $E(a, b, \xi)$ is isomorphic to a universal enveloping algebra of a Lie algebra.

Example 6.5.6. Let F be the algebra generated by X, Y, Z, W satisfying the following relations,

$$\begin{aligned} [Y, X] &= [Z, Y] = 0, \\ [Z, X] &= Y, \\ [W, X] &= \beta Y, \\ [W, Y] &= \gamma Y, \\ [W, Z] &= \gamma Z - \frac{2}{3}Y^3 + \xi X + \xi' Y. \end{aligned}$$

where $\beta, \gamma, \xi, \xi' \in k$. Then F becomes a bialgebra (and then a Hopf algebra) via

$$\begin{aligned} \epsilon(X) &= 0, \quad \Delta(X) = 1 \otimes X + X \otimes 1, \\ \epsilon(Y) &= 0, \quad \Delta(Y) = 1 \otimes Y + Y \otimes 1, \\ \epsilon(Z) &= 0, \quad \Delta(Z) = 1 \otimes Z + X \otimes Y - Y \otimes X + Z \otimes 1, \\ \epsilon(W) &= 0, \\ \Delta(W) &= 1 \otimes W + W \otimes 1 \\ &\quad + Y \otimes Z - Z \otimes Y + XY \otimes Y + Y \otimes XY. \end{aligned} \tag{E6.5.6.1}$$

If W is replaced by $W_{new} := W + \xi' X$, then we can assume $\xi' = 0$. We denote the Hopf algebra by $F(\beta, \gamma, \xi)$. One can make further reduction by easy linear algebra. For example, if $\gamma \neq 0$, by replacing X by $X_{new} := X - \gamma^{-1}\beta Y$, we have $\beta = 0$. By re-scalaring, we can further assume $\gamma = 1$. If $\gamma = 0$, then, by re-scalaring, we might assume $\beta = 1$. In summary, $\{\beta, \gamma\}$ is either $\{0, 1\}$ or $\{1, 0\}$. This completely determines the isomorphism classes of the Hopf algebras $F(\beta, \gamma, \xi)$.

Let $W' = W - \frac{2}{3}XY^2$. Then the algebraic relations of F becomes

$$\begin{aligned} [Y, X] &= [Z, Y] = 0, \\ [Z, X] &= Y, \\ [W', X] &= \beta Y, \\ [W', Y] &= \gamma Y, \\ [W', Z] &= \gamma Z + \xi X. \end{aligned}$$

Therefore the subspace generated by $\{X, Y, Z, W'\}$ is a 4-dimensional Lie algebra, say \mathfrak{g} , and F is isomorphic to the enveloping algebra $U(\mathfrak{g})$ as algebras.

Example 6.5.7. Let K be the algebra generated by X, Y, Z, W satisfying the following

relations,

$$\begin{aligned} [Y, X] &= [Z, Y] = 0, \\ [Z, X] &= X, \\ [W, X] &= -Z, \\ [W, Y] &= 0, \\ [W, Z] &= W - XY^2. \end{aligned}$$

The coalgebra structure of K is given as in (E6.5.6.1). Then K becomes a Hopf algebra. Let $W' = W - \frac{1}{2}XY^2$. Then the algebraic relations become

$$\begin{aligned} [Y, X] &= [Z, Y] = 0, \\ [Z, X] &= X, \\ [W', X] &= -Z, \\ [W', Y] &= 0, \\ [W', Z] &= W'. \end{aligned}$$

Therefore the subspace generated by $\{X, Y, Z, W'\}$ is a 4-dimensional Lie algebra, say \mathfrak{g} , and K is isomorphic to the enveloping algebra $U(\mathfrak{g})$ as algebras.

Proposition 6.5.8. *The algebras H defined in Examples 6.5.4-6.5.7 have the following properties.*

- (a) H is an iterated Ore extension $k[X][Y; \delta_1][Z; \delta_2][W; \sigma_3, \delta_3]$.
- (b) H is an Auslander regular Cohen-Macaulay domain.
- (c) The global dimension and GK-dimension of H is 4.
- (d) H is a Hopf algebra and connected as a coalgebra.
- (e) $P(H) = kX + kY$. The subalgebra generated by X, Y is the enveloping algebra $U(\mathfrak{g})$ where \mathfrak{g} is the Lie algebra $P(H)$.
- (f) $P_2(H) = kX + kY + kZ$. The subalgebra generated by X, Y, Z is the enveloping algebra $U(L)$ where L is the CLA $P_2(H)$.
- (g) H is not isomorphic to an enveloping algebra of either a Lie algebra or a CLA.
- (h) The lantern $\mathfrak{L}(H)$ of H is isomorphic to the graded Lie algebra of dimension four, with a basis $\{x^*, y^*, z^*, w^*\}$, such that $z^* = [x^*, y^*]$ and $w^* = [z^*, y^*]$, subject to the relation $[z^*, x^*] = 0 = [w^*, x^*] = [w^*, y^*]$.

Sketch of the proof. (a) Follows from the definitions.

(b,c) These facts are true for any iterated Ore extension.

(d,e,f) These depend on straightforward, but not trivial, computation.

(g) This follows from parts (e,f).

(h) This was proved in Lemma 6.2.4(c).

We can also check it directly here. Noting that $\{x^*, y^*, z^*, w^*\}$ can be viewed as a dual basis of $\{X, Y, Z, W\}$ in $\text{gr } H$ and that coproducts of X, Y, Z, W given in Examples 6.5.4-6.5.7 match up with the Lie structure of the $\mathfrak{L}(H)$ given in part (h). \square

6.5.3 Primitively-thin Hopf algebra of GK-dimension 4, Part I

In this and the next two subsections we classify all primitively-thin Hopf algebra of GK-dimension 4.

Let C be a primitively-thin Hopf algebra of GK-dimension 3. By Proposition 6.5.3, C is of type A or B as in Examples 6.5.1 and 6.5.2. Let D be the Hopf algebra $A(0, 0, 0)$. Then it is easy to see that D is a coradically graded Hopf algebra by setting $\deg X = \deg Y = 1$ and $\deg Z = 2$. Since $\text{gr } C \cong D$, C is a so-called PBW deformation of D .

Let $\{x, y, z\}$ be any set of generators of C such that x, y are primitive and $\Delta(z) = 1 \otimes z + x \otimes y - y \otimes x + z \otimes 1$. Then C has a basis consists of monomials of the form

$$x^{i_1} y^{i_2} z^{i_3}.$$

Notice that C^+ is spanned by $x^{i_1} y^{i_2} z^{i_3}$ with at least one i_k non-zero. Let $\bar{x}, \bar{y}, \bar{z}$ be the homogeneous elements in $D = \text{gr } C$ corresponding to x, y, z , respectively. (In fact, an easy calculation shows that $\bar{x}, \bar{y}, \bar{z}$ can be identified with the canonical generators X, Y, Z as in the definition of $A(0, 0, 0)$). Then $\text{gr } C$ has a basis $\{\bar{x}^{i_1} \bar{y}^{i_2} \bar{z}^{i_3}\}$. Now we have a k -space isomorphism from C to $\text{gr } C$ by sending $x^{i_1} y^{i_2} z^{i_3}$ to $\bar{x}^{i_1} \bar{y}^{i_2} \bar{z}^{i_3}$. Clearly this isomorphism maps C^+ onto D^+ . From now on we identify C with $D = \text{gr } C$ as k -spaces by this isomorphism, and we will abuse the notation by dropping the bars for the generators $\bar{x}, \bar{y}, \bar{z}$ of $\text{gr } C$. Define $\deg x^{i_1} y^{i_2} z^{i_3} = i_1 + i_2 + 2i_3$. This grading agrees with the natural grading on $D = \text{gr } C$. Moreover, by the defining relations of C , it is easy to check that

$$\Delta_C(a) = \Delta_D(a) + ldt,$$

where $a \in C$ and ldt denotes terms with degrees lower than $\deg a$. As a consequence, we can think about ΩC and ΩD , the cobar constructions of C and D (see Section 1.3), as the same graded k -spaces with two differentials ∂_C and ∂_D . Moreover, ∂_D respects the grading and

$$\partial_C^n(b) = \partial_D^n(b) + ldt,$$

where $b \in (C^+)^{\otimes n}$.

Lemma 6.5.9. *Let $D = A(0, 0, 0)$. Then $\dim_k H^2(\Omega D) = 2$ and $H^2(\Omega D)$ is spanned by the classes of 2-cocycles (u) and (t) , where*

$$u = z \otimes x - x \otimes z + xy \otimes x + x \otimes xy, \quad (\text{E6.5.9.1})$$

$$t = y \otimes z - z \otimes y + xy \otimes y + y \otimes xy. \quad (\text{E6.5.9.2})$$

Proof. As mentioned above, D is a graded Hopf algebra. Let A be the graded dual of D . Since D is commutative, A is cocommutative. By Proposition 6.2.1, $A = U(\mathfrak{L})$ for some graded Lie algebra \mathfrak{L} . Since D is coradically graded as a coalgebra, A is generated in degree one. This implies that \mathfrak{L} is the 3-dimensional Heisenberg Lie algebra. Or equivalently, A is generated by two degree one elements x_1 and x_2 with relations

$$x_1^2 x_2 + x_2 x_1^2 = 2x_1 x_2 x_1, \quad x_2^2 x_1 + x_1 x_2^2 = 2x_2 x_1 x_2.$$

By [LPWZ2, Lemma 8.6 (c)], $B^\# A \cong \Omega C$ as DG algebras, where $B^\# A$ is the graded dual of the bar construction of A . On the other hand, by [LPWZ1, Lemma 4.2], $H^\bullet(B^\# A) \cong \text{Ext}_A^\bullet(k_A, k_A)$. As a consequence,

$$\dim_k H^2(\Omega D) = \dim_k \text{Ext}_A^2(k_A, k_A) = 2.$$

We introduce a \mathbb{Z}^2 -grading on D by setting $\deg_2 x = (1, 0)$, $\deg_2 y = (0, 1)$ and $\deg_2 z = (1, 1)$. Then it is clear that D is a \mathbb{Z}^2 -graded Hopf algebra and therefore the differentials of ΩD preserves the \mathbb{Z}^2 -grading. A direct computation shows that both u and t are 2-cocycles. Now if u is 2-coboundary, it must be a linear combination of $\partial^1(x^2 y) = \delta(x^2 y) = x^2 \otimes y + 2xy \otimes x + 2x \otimes xy + y \otimes x^2$ and $\partial^1(xz) = \delta(xz) = x \otimes z + z \otimes x + x^2 \otimes y + x \otimes xy$. An easy calculation shows this is impossible. Hence the class (u) is a non-zero element in $H^2(\Omega D)$. Similarly, one can show that the class (t) is also non-zero in $H^2(\Omega D)$. Moreover, they are linearly independent since they have different \mathbb{Z}^2 -degrees. This completes the proof. \square

Recall that monomials of the form $x^i y^j z^k$ in C are identified with monomials $x^i y^j z^k$ in D .

Proposition 6.5.10. *Retain the above notation. Then $H^2(\Omega C)$ is spanned by the classes of 2-cocycles (u) and (t) , where u and t are given as in (E6.5.9.1)-(E6.5.9.2), and $\dim_k H^2(\Omega C) = 2$.*

Proof. Let w be a non-zero linear combination of u and t . Then w is homogeneous of degree 3 and $\partial_C^2(w) = \partial_D^2(w) = 0$ by direct calculation (which only uses (E6.5.1.1)). Suppose that there exists $c \in C^+$ such that $w = \partial_C^1(c)$. Also, we can write $\partial_C^1(c) = \partial_D^1(c) + v$, where $v \in (C^+)^{\otimes 2}$ has degree less than the degree of c . If the degree of c is larger than 3, then $\deg \partial_D^1(c) = \deg c > 3$ as ∂_D^1 is homogeneous and $\partial_D^1(c) \neq 0$. Then

$$\deg w = \deg \partial_C^1(c) = \deg \partial_D^1(c) > 3,$$

a contradiction. Therefore $\deg c \leq 3$ and consequently, v has degree less than 3.

Since $\partial_D^2 \partial_C^1(c) = \partial_D^2(w) = 0$, we have $\partial_D^2(\partial_D^1(c) + v) = 0$. Hence $\partial_D^2(v) = 0$. But $\deg v < 3$, so by Lemma 6.5.9, there exists $c' \in C^+$ such that $v = \partial_D^1(c')$. As a consequence, $w = \partial_D^1(c) + v = \partial_D^1(c + c')$, which is a contradiction.

Now, we have shown that $H^2(\Omega C)$ is at least of dimension two and $(u), (t)$ are linearly independent in $H^2(\Omega C)$. On the other hand, by a standard spectral sequence argument [W, Theorem 5.5.1], we have $\dim_k H^2(\Omega C) \leq \dim_k H^2(\Omega D) = 2$. This completes the proof. \square

Lemma 6.5.11. *Let H be a connected coalgebra and K a proper subcoalgebra of H . Let N be the smallest number such that $K_N \subsetneq H_N$ and suppose that $N \geq 2$, then δ induces a injective k -linear map from H_N^+/K_N^+ to $H^2(\Omega K)$.*

Proof. By the choice of N , we see that, for any $g \in H_N^+$,

$$\delta(g) = \Delta(g) - (1 \otimes g + g \otimes 1) \in H_{N-1}^+ \otimes H_{N-1}^+ = K_{N-1}^+ \otimes K_{N-1}^+.$$

Hence $\partial_K^2(\delta(g)) = \partial_K^2 \partial_H^1(g) = \partial_H^2 \partial_H^1(g) = 0$, which means that $\delta(g)$ is a 2-cocycle in the complex ΩK . Hence δ defines a k -linear map from $H_N^+ \rightarrow H^2(\Omega K)$. For any element $g \in K_N^+$, $\delta(g) = \partial_K^1(g)$ is a 2-boundary in the complex ΩK , whence it is zero in $H^2(\Omega K)$. Thus δ induces a k -linear map from $H_N^+/K_N^+ \rightarrow H^2(\Omega K)$.

If $g \in H_N^+ \setminus K_N^+$, we claim that $\delta(g)$ represents a non-zero cohomology class in $H^2(\Omega K)$. If not, there is $w \in K^+$ such that $\partial_K^1(w) = \delta(w) = \delta(g)$. As a consequence, $\Delta(g - w) = 1 \otimes (g - w) + (g - w) \otimes 1$, i.e. $g - w$ is a primitive element in H . By the fact that $H_1^+ = K_1^+$, $g - w \in K_1^+$. But this would imply that $g \in K^+$, which contradicts the choice of g . Therefore the map from H_N^+/K_N^+ to $H^2(\Omega K)$ is injective. \square

Theorem 6.5.12. *Suppose that H is a primitively-thin Hopf algebra of GK-dimension 4. Then for any linearly independent primitive elements x, y , there exists $z \in H$ such that*

$$\Delta(z) = 1 \otimes z + x \otimes y - y \otimes x + z \otimes 1. \quad (\text{E6.5.12.1})$$

For any such z , the algebra C generated by $\{x, y, z\}$ is a Hopf subalgebra of GK-dimension 3. Moreover, there exists $w \in H$ such that

$$\Delta(w) = 1 \otimes w + \theta_1 u + \theta_2 t + w \otimes 1, \quad (\text{E6.5.12.2})$$

where u and t are given as in (E6.5.9.1)-(E6.5.9.2) and one of the scalars θ_1, θ_2 is non-zero. For any such w , the set $\{x, y, z, w\}$ generates H .

Proof. By Proposition 3.6.10, we can find a Hopf subalgebra C of GK-dimension 3. By Proposition 6.5.3, there is an $z \in C$ such that $\Delta(z)$ is of the form (E6.5.12.1). Since $P(C) = kx \oplus ky = P(H)$, $C_1 = H_1$.

Let $N \geq 2$ be the smallest integer such that $C_N \subsetneq H_N$. By [Mo, Lemma 5.3.2], there exists $w' \in H_N \setminus C_N$ such that $\Delta(w') = 1 \otimes w' + w' \otimes 1 + f$ where $f \in C_{N-1} \otimes C_{N-1}$. Without loss of generality, we assume that $w' \in H^+$.

By Lemma 6.5.11, f represents a non-zero cohomology class in $H^2(\Omega C)$. By Proposition 6.5.10, the cohomology classes in $H^2(\Omega C)$ represented by f is a non-zero linear combination of (u) and (t) . Hence there exists $v \in C^+$ and $\theta_1, \theta_2 \in k$ such that $f = \partial^1(v) + \theta_1 u + \theta_2 t$, where at least one of θ_i is non-zero. Let $w = w' + v$. Then $w \notin C$ and $\Delta(w) = 1 \otimes w + \theta_1 u + \theta_2 t + w \otimes 1$.

Next we have to show that H is generated by x, y, z and w . Let K be the subalgebra of H generated by x, y, z and w . Then it is easy to check that K is a sub-bialgebra and thus a Hopf subalgebra of H . By the construction of K , $C \subsetneq K$. By Lemma 3.5.8, $\text{GKdim gr } K \geq \text{GKdim gr } C + 1 = 4$. On the other hand, $\text{GKdim gr } K = \text{GKdim } K \leq \text{GKdim } H = 4$ since $K \subset H$. Hence $\text{GKdim } K = 4$. Now it follows from Lemma 3.6.4 that $K = H$. This completes the proof. \square

As a direct consequence, we have the following proposition.

Corollary 6.5.13. *Let H be a commutative, connected, primitively-thin Hopf algebra of GK-dimension 4. Then H is isomorphic to $D(\{0, 1\}, \{0\}, \{0\})$.*

Proof. By Theorem 6.5.12, there is a surjective Hopf map from $D(\{\theta_i\}, \{0\}, \{0\})$ to H sending X, Y, Z, W to x, y, z, w , respectively, for some $\{\theta_1, \theta_2\}$. The map must be an isomorphism since both D and H are domains of GK-dimension 4. By definition, $D(\{\theta_i\}, \{0\}, \{0\})$ is a graded Hopf algebra with $\deg X = \deg Y = 1$, $\deg Z = 2$ and $\deg W = 3$. Hence the graded dual H^* is a graded commutative Hopf algebra, which must be isomorphic to the enveloping algebra $U(\mathfrak{L})$ for some graded Lie algebra generated by two elements in degree 1. Such a Lie algebra is unique (up to isomorphism) and is given in Proposition 6.5.8(h). Therefore H is isomorphic to $U(\mathfrak{L})^*$, which is isomorphic to $D(\{0, 1\}, \{0\}, \{0\})$. \square

Lemma 6.5.14. *Retain the notation in Theorem 6.5.12 for parts (b,c).*

1. *The Hopf algebra $D := D(\{0, 1\}, \{0\}, \{0\})$ is coradically graded by setting $\deg X = \deg Y = 1$, $\deg Z = 2$ and $\deg W = 3$.*
2. *For any connected Hopf algebra H of GK-dimension 4 with $\dim_k P(H) = 2$, $\text{gr } H$ is isomorphic to $D(\{0, 1\}, \{0\}, \{0\})$.*
3. *Working with $\text{gr } C$ (isomorphic to $A(0, 0, 0)$) and $\text{gr } H$, we have $C_2 = H_2$ and H_3^+ / C_3^+ is one-dimensional, which is spanned by the image of w .*

Proof. Let D denote the Hopf algebra $D(\{0, 1\}, \{0\}, \{0\})$.

- (a) One can check directly that $\text{gr } D = D$. Hence D is coradically graded.

(b) By Theorem 3.5.4, $\text{gr } H$ is commutative, and it is still connected and primitively-thin. The assertion follows from Corollary 6.5.13.

(c) We may replace $\text{gr } C$ by $A(0, 0, 0)$ and $\text{gr } H$ by D . Then the assertion follows by an easy computation. \square

Proposition 6.5.15. *Retain the notation in Theorem 6.5.12. Then H has a k -basis of the form*

$$\{x^{i_1}y^{i_2}z^{i_3}w^{i_4} \mid i_1, i_2, i_3, i_4 \geq 0\}.$$

Proof. Let X, Y, Z, W be the elements in $\text{gr } H$ corresponding to elements x, y, z, w in H . Then $\{X, Y, Z, W\}$ generates $\text{gr } H$ as an algebra by Theorem 6.5.12. By Lemma 6.5.14(b), $\text{gr } H \cong D$, so X, Y, Z, W satisfy the defining relations of $D(\{0, 1\}, \{0\}, \{0\})$ given in Example 6.5.4. As a consequence, $\text{gr } H$ has a k -basis of the form

$$\{X^{i_1}Y^{i_2}Z^{i_3}W^{i_4} \mid i_1, i_2, i_3, i_4 \geq 0\}.$$

Now the result follows. \square

In Theorem 6.5.12, the Hopf subalgebra C is primitively-thin. Hence by Proposition 6.5.3, C must be isomorphic to either $A(\lambda_1, \lambda_2, \alpha)$ or $B(\lambda)$.

6.5.4 Primitively-thin Hopf algebra of GK-dimension 4, Part II

In this subsection we show that $B(\lambda)$ can not appear as a Hopf subalgebra of a primitively-thin Hopf algebra of GK-dimension 4. We start with an easy observation.

Lemma 6.5.16. *Let x and y be primitive elements. Then*

$$\delta(xy^2) = y^2 \otimes x + x \otimes y^2 + 2(xy \otimes y + y \otimes xy), \quad (\text{E6.5.16.1})$$

$$\delta(x^2y) = y \otimes x^2 + x^2 \otimes y + 2(xy \otimes x + x \otimes xy), \quad (\text{E6.5.16.2})$$

$$\delta(y^3) = 3(y \otimes y^2 + y^2 \otimes y). \quad (\text{E6.5.16.3})$$

Proposition 6.5.17. *Retain the notation in Theorem 6.5.12. Then the Hopf subalgebra C can not be isomorphic to $B(\lambda)$.*

Proof. Suppose to the contrary that C is isomorphic to $B(\lambda)$ for some $\lambda \in k$. Therefore we can assume that $x, y, z \in C$ satisfies the relations listed in Example 6.5.2.

First we assume that θ_1 is not zero. By dividing w with θ_1 we may assume that $\Delta(w) = 1 \otimes w + u + \theta_2 t + w \otimes 1$. Using (E6.5.12.2), (E6.5.16.1), (E6.5.16.3) and the relations of

$B(\lambda)$, we have

$$\begin{aligned}
\delta([w, y]) &= \Delta([w, y]) - [w, y] \otimes 1 - 1 \otimes [w, y] \\
&= [\Delta(w), \Delta(y)] - [w, y] \otimes 1 - 1 \otimes [w, y] \\
&= z \otimes [x, y] - [x, y] \otimes z \\
&\quad + [xy, y] \otimes x + x \otimes [xy, y] + xy \otimes [x, y] + [x, y] \otimes xy \\
&\quad + \theta_2([xy, y] \otimes y + y \otimes [xy, y]) \\
&= -t + \delta(xy^2) + \frac{\theta_2}{3}\delta(y^3).
\end{aligned}$$

Let $w' = xy^2 + \frac{\theta_2}{3}y^3 - [w, y]$, then $\delta(w') = t$, and, whence,

$$\Delta(w') = 1 \otimes w' + t + w' \otimes 1.$$

Now, under the map given in Lemma 6.5.11, the elements w and w' are mapped to $(u) + \theta_2(t)$ and (t) , respectively. Hence $\dim_k H_3^+/C_3^+ = 2$, which contradicts Lemma 6.5.14(c).

Next we assume that $\theta_1 = 0$ and $\theta_2 \neq 0$. By dividing w with θ_2 we may assume that $\Delta(w) = 1 \otimes w + t + w \otimes 1$. A similar calculation shows that $[w, x] + 2w \in P(H)$ and $[w, y] - \frac{1}{3}y^3 \in P(H)$. As a consequence,

$$\begin{aligned}
\delta([w, z]) &= [w, x] \otimes y - y \otimes [w, x] + x \otimes [w, y] - [w, y] \otimes x & (E6.5.17.1) \\
&\quad + [y, z] \otimes z - z \otimes [y, z] + [xy, z] \otimes y + y \otimes [xy, z] \\
&\quad + [t, x \otimes y - y \otimes x] \\
&= -2w \otimes y + 2y \otimes w + \sum_s \alpha_s f_s \otimes g_s
\end{aligned}$$

where $\alpha_s \in k$, f_s, g_s are monomials of the form $x^{i_1}y^{i_2}z^{i_3}$. Let $\text{gr } H(n)$ denote the degree n piece of the graded Hopf algebra $\text{gr } H$. Since $\text{gr } H$ is commutative, $[w, z]$ represents an element $V \in \text{gr } H(4)$. Let $X \in \text{gr } H(1)$, $Y \in \text{gr } H(1)$, $Z \in \text{gr } H(2)$ and $W \in \text{gr } H(3)$ be the homogeneous elements in $\text{gr } H$ corresponding to elements x, y, z and w in H , respectively. By (E6.5.17.1), we see that

$$\delta(V) = -2W \otimes Y + 2Y \otimes W + \sum_s \alpha'_s f'_s \otimes g'_s, \quad (E6.5.17.2)$$

where $\alpha'_s \in k$, f'_s, g'_s are monomials of the form $X^{i_1}Y^{i_2}Z^{i_3}$.

Now by Theorem 6.5.12, the set $\{X, Y, Z, W\}$ generates $\text{gr } H$. Also, $\text{gr } H$ is a \mathbb{Z}^2 -graded coalgebra by setting $\deg_2 X = (1, 0)$, $\deg_2 Y = (0, 1)$, $\deg_2 Z = (1, 1)$ and $\deg_2 W = (1, 2)$. Notice that $\deg_2 V = (1, 3)$. Therefore, V must be a linear combination of linearly independent elements XY^3, Y^2Z, YW of degree $(1, 3)$. Hence there are $\beta_i \in k$ such that

$$V = \beta_1 XY^3 + \beta_2 ZY^2 + \beta_3 YW$$

or

$$\begin{aligned}\delta(V) &= \delta(\beta_1 XY^3 + \beta_2 ZY^2 + \beta_3 YW) \\ &= \beta_3(Y \otimes W + W \otimes Y) + \sum_s \alpha_s'' f_s'' \otimes g_s''\end{aligned}\tag{E6.5.17.3}$$

where $\alpha_s'' \in k$, f_s'', g_s'' are monomials of the form $X^{i_1} Y^{i_2} Z^{i_3}$. If we compare the coefficient of $W \otimes Y$ on (E6.5.17.2) and (E6.5.17.3), we get $-2 = \beta_3$. On the other hand, if we compare the coefficient of $Y \otimes W$, we have $2 = \beta_3$, which is a contradiction. This completes the proof. \square

6.5.5 Primitively-thin Hopf algebra of GK-dimension 4, Part III

In this subsection we deal with the case when $C = A(\lambda_1, \lambda_2, \alpha)$ and finish the analysis. Throughout this subsection we assume that $C = A(\lambda_1, \lambda_2, \alpha)$ where the relations of $A(\lambda_1, \lambda_2, \alpha)$ are given in Example 6.5.1; and that $(\lambda_1, \lambda_2, \alpha)$ is either $(0, 0, 0)$, or $(0, 0, 1)$ or $(1, 1, 1)$ or $(1, \lambda, 0)$ as listed in Proposition 6.5.3(a).

Lemma 6.5.18. *Let u and t be given as in (E6.5.9.1)-(E6.5.9.2).*

$$[u, x \otimes 1 + 1 \otimes x] = \alpha(y \otimes x - x \otimes y),\tag{E6.5.18.1}$$

$$[t, x \otimes 1 + 1 \otimes x] = \lambda_1(y \otimes x - x \otimes y),\tag{E6.5.18.2}$$

$$[u, y \otimes 1 + 1 \otimes y] = \lambda_2(y \otimes x - x \otimes y),\tag{E6.5.18.3}$$

$$[t, y \otimes 1 + 1 \otimes y] = 0.\tag{E6.5.18.4}$$

Proof. We use the relations of $A(\lambda_1, \lambda_2, \alpha)$ and note that $[x, y] = 0$. By an easy computation, we have

$$[u, x \otimes 1] = [z, x] \otimes x = (\lambda_1 x + \alpha y) \otimes x,$$

$$[u, 1 \otimes x] = -x \otimes [z, x] = -x \otimes (\lambda_1 x + \alpha y).$$

Now (E6.5.18.1) follows by adding the above.

By a computation,

$$[t, x \otimes 1] = -[z, x] \otimes y = -(\lambda_1 x + \alpha y) \otimes y,$$

$$[t, 1 \otimes x] = y \otimes [z, x] = y \otimes (\lambda_1 x + \alpha y),$$

and (E6.5.18.2) is obtained by adding the above. The proof of the last two are similar. \square

Lemma 6.5.19. *Let w be as in Theorem 6.5.12.*

$$[w, x] = -(\theta_1 \alpha + \theta_2 \lambda_1)z + a_{11}x + a_{12}y\tag{E6.5.19.1}$$

$$[w, y] = -\theta_1 \lambda_2 z + a_{21}x + a_{22}y\tag{E6.5.19.2}$$

for some $a_{11}, a_{12}, a_{21}, a_{22} \in k$.

Proof. We only prove the first equation and the proof of the second equation is similar.

$$\begin{aligned}
\delta([w, x]) &= \Delta([w, x]) - [w, x] \otimes 1 - 1 \otimes [w, x] \\
&= [\Delta(w), x \otimes 1 + 1 \otimes x] - [w, x] \otimes 1 - 1 \otimes [w, x] \\
&= [w \otimes 1 + 1 \otimes w + \theta_1 u + \theta_2 t, x \otimes 1 + 1 \otimes x] - [w, x] \otimes 1 - 1 \otimes [w, x] \\
&= [\theta_1 u + \theta_2 t, x \otimes 1 + 1 \otimes x] \\
&= \theta_1 [u, x \otimes 1 + 1 \otimes x] + \theta_2 [t, x \otimes 1 + 1 \otimes x] \\
&= (\theta_1 \alpha + \theta_2 \lambda_1)(y \otimes x - x \otimes y) && \text{by (E6.5.18.1)-(E6.5.18.2)} \\
&= -(\theta_1 \alpha + \theta_2 \lambda_1) \delta(z).
\end{aligned}$$

Therefore $[w, x] + (\theta_1 \alpha + \theta_2 \lambda_1)z$ is a primitive elements, whence it is of the form $a_{11}x + a_{12}y$ for some $a_{11}, a_{12} \in k$. The assertion follows. \square

Lemma 6.5.20. *Retain the notation as above. Then $\lambda_2 = 0$. Consequently, $(\lambda_1, \lambda_2, \alpha)$ is either $(0, 0, 0)$, or $(0, 0, 1)$ or $(1, 0, 0)$.*

Proof. Since $[x, y] = 0$, using Lemma 6.5.19 we have

$$\begin{aligned}
0 &= [w, [x, y]] = [[w, x], y] + [x, [w, y]] \\
&= [-(\theta_1 \alpha + \theta_2 \lambda_1)z + a_{11}x + a_{12}y, y] + [x, -\theta_1 \lambda_2 z + a_{21}x + a_{22}y] \\
&= -(\theta_1 \alpha + \theta_2 \lambda_1)[z, y] + \theta_1 \lambda_2 [z, x] \\
&= -(\theta_1 \alpha + \theta_2 \lambda_1)(\lambda_2 y) + \theta_1 \lambda_2 (\lambda_1 x + \alpha y) \\
&= -\theta_2 \lambda_1 \lambda_2 y + \theta_1 \lambda_1 \lambda_2 x.
\end{aligned}$$

Since one of θ_i 's is nonzero, $\lambda_1 \lambda_2 = 0$. We only consider those $(\lambda_1, \lambda_2, \alpha)$'s given in Proposition 6.5.3(a), therefore, $\lambda_2 = 0$. \square

Lemma 6.5.21. *Suppose $\lambda_2 = 0$. Let u and t be given as in (E6.5.9.1)-(E6.5.9.2).*

$$[u, z \otimes 1 + 1 \otimes z] = -\lambda_1 u + \alpha t - \alpha \delta(xy^2) - \lambda_1(xy \otimes x + x \otimes xy), \quad (\text{E6.5.21.1})$$

$$[t, z \otimes 1 + 1 \otimes z] = -\lambda_1(xy \otimes y + y \otimes xy) - \alpha(y^2 \otimes y + y \otimes y^2), \quad (\text{E6.5.21.2})$$

$$[u, x \otimes y - y \otimes x] = \lambda_1(x \otimes xy + xy \otimes x) + \alpha(y \otimes xy + xy \otimes y), \quad (\text{E6.5.21.3})$$

$$[t, x \otimes y - y \otimes x] = -\lambda_1(x \otimes y^2 + y^2 \otimes x) - \alpha(y \otimes y^2 + y^2 \otimes y). \quad (\text{E6.5.21.4})$$

Proof. By direct computation, we have

$$\begin{aligned}
[u, z \otimes 1] &= (\lambda_1 x + \alpha y) \otimes z - (\lambda_1 x + \alpha y)y \otimes x - (\lambda_1 x + \alpha y) \otimes xy, \\
[u, 1 \otimes z] &= -z \otimes (\lambda_1 x + \alpha y) - xy \otimes (\lambda_1 x + \alpha y) - x \otimes (\lambda_1 x + \alpha y)y.
\end{aligned}$$

Adding up and using definition and (E6.5.16.1), we obtain (E6.5.21.1). Others are similar, by using definitions and direct computations. \square

Lemma 6.5.22. *Suppose $\lambda_2 = 0$. Then*

$$\theta_1(\theta_2\lambda_1 + \theta_1\alpha) = 0. \quad (\text{E6.5.22.1})$$

Further,

(a) *If $(\lambda_1, \lambda_2, \alpha) = (0, 0, 0)$, then*

$$[w, z] = (a_{11} + a_{22})z + \xi_1x + \xi_2y$$

for some $\xi_i \in k$.

(b) *If $(\lambda_1, \lambda_2, \alpha) = (1, 0, 0)$, then $\theta_1\theta_2 = 0$. If, moreover, $\theta_1 = 0$, then*

$$[w, z] = (a_{11} + a_{22})z + w + (-\theta_2)xy^2 + \xi_1x + \xi_2y$$

for some $\xi_i \in k$. If, moreover, $\theta_2 = 0$, and

$$[w, z] = (a_{11} + a_{22})z - w + \xi_1x + \xi_2y$$

for some $\xi_i \in k$.

(c) *If $(\lambda_1, \lambda_2, \alpha) = (0, 0, 1)$, then $\theta_1 = 0$ and*

$$[w, z] = (a_{11} + a_{22})z - \frac{2}{3}\theta_2y^3 + \xi_1x + \xi_2y$$

for some $\xi_i \in k$.

Proof.

$$\begin{aligned} \delta([w, z]) &= \Delta([w, z]) - [w, z] \otimes 1 - 1 \otimes [w, z] \\ &= [\Delta(w), \Delta(z)] - [w, z] \otimes 1 - 1 \otimes [w, z] \\ &= [w \otimes 1 + 1 \otimes w + \theta_1u + \theta_2t, z \otimes 1 + 1 \otimes z + (x \otimes y - y \otimes x)] \\ &\quad - [w, z] \otimes 1 - 1 \otimes [w, z], \\ &= [w \otimes 1 + 1 \otimes w, (x \otimes y - y \otimes x)] \\ &\quad + [\theta_1u + \theta_2t, z \otimes 1 + 1 \otimes z + (x \otimes y - y \otimes x)] \\ &= -(\theta_1\alpha + \theta_2\lambda_1)(z \otimes y - y \otimes z) + (a_{11} + a_{22})(x \otimes y - y \otimes x) \\ &\quad \text{by (E6.5.19.1)-(E6.5.19.2)} \\ &\quad + \theta_1(-\lambda_1u + \alpha t - \alpha\delta(xy^2) + \alpha(y \otimes xy + xy \otimes y)) \\ &\quad \text{by (E6.5.21.1)-(E6.5.21.3)} \\ &\quad + \theta_2(-\lambda_1\delta(xy^2) + \lambda_1(xy \otimes y + y \otimes xy) - 2\alpha(y^2 \otimes y + y \otimes y^2)) \\ &\quad \text{by (E6.5.21.2)-(E6.5.21.4)} \\ &= -\theta_1\lambda_1u + (2\theta_1\alpha + \theta_2\lambda_1)t \\ &\quad + (a_{11} + a_{22})\delta(z) + (-\theta_1\alpha - \theta_2\lambda_1)\delta(xy^2) - \frac{2}{3}\theta_2\alpha\delta(y^3). \end{aligned}$$

Since $\delta([w, z]) \in C \otimes C$, $[w, z]$ induces a cohomology class in $H^2(\Omega C)$. By Lemma 6.5.14(c), $\delta([w, z])$ is a scalar multiple of $\delta(w)$. This implies that

$$\theta_2(-\theta_1\lambda_1) - \theta_1(2\theta_1\alpha + \theta_2\lambda_1) = 0$$

or, after simplifying, we obtain (E6.5.22.1).

(a) By the above computation, we have $\delta([w, z]) = (a_{11} + a_{22})\delta(z)$. The assertion follows.

(b) When $(\lambda_1, \lambda_2, \alpha) = (1, 0, 0)$, (E6.5.22.1) becomes $\theta_1\theta_2 = 0$ and

$$\delta([w, z]) = -\theta_1u + \theta_2t + (a_{11} + a_{22})\delta(z) - \theta_2\delta(xy^2).$$

Since $\theta_1\theta_2 = 0$, $-\theta_1u + \theta_2t$ is either $\delta(w)$ or $-\delta(w)$. Hence we have

$$[w, z] = (a_{11} + a_{22})z + cw + (-\theta_2)xy^2 + \xi_1x + \xi_2y$$

for some $\xi_i \in k$ and $c = \pm 1$, which gives two cases listed in part (b).

(c) If $(\lambda_1, \lambda_2, \alpha) = (0, 0, 1)$, then (E6.5.22.1) implies that $\theta_1 = 0$ and

$$\delta([w, z]) = (a_{11} + a_{22})\delta(z) - \frac{2}{3}\theta_2\delta(y^3)$$

and therefore the assertion follows. \square

Theorem 6.5.23. *Let H be a connected Hopf algebra of GK-dimension 4 with $\dim_k P(H) = 2$. Then H must be isomorphic to one of the Hopf algebras listed in Example 6.5.4, 6.5.5, 6.5.6 and 6.5.7.*

Proof. Retain the notation in Theorem 6.5.12. By Proposition 6.5.17 and Proposition 6.5.3, C must be isomorphic to $A(\lambda_1, \lambda_2, \alpha)$, where the possible choices of $(\lambda_1, \lambda_2, \alpha)$ are listed in Proposition 6.5.3(a). By Lemma 6.5.20, $\lambda_2 = 0$. Hence C is either $A(0, 0, 0)$, or $A(1, 0, 0)$, or $A(0, 0, 1)$.

Case 1: $C = A(0, 0, 0)$. By Lemma 6.5.19 and 6.5.22(a), we have

$$\begin{aligned} [w, x] &= a_{11}x + a_{12}y \\ [w, y] &= a_{21}x + a_{22}y \\ [w, z] &= (a_{11} + a_{22})z + \xi_1x + \xi_2y \end{aligned}$$

for some $a_{11}, a_{12}, a_{21}, a_{22}, \xi_1, \xi_2 \in k$. Together with the coalgebra given in Theorem 6.5.12, this is the Hopf algebra described in Example 6.5.4.

Case 2: $C = A(1, 0, 0)$. By Lemma 6.5.22(b), there are two cases to consider. If $\theta_1 = 0$, by dividing w with θ_2 we can assume that $\theta_2 = 1$. In this setting, by Lemmas 6.5.19 and 6.5.22(b), we have

$$\begin{aligned} [w, x] &= -z + a_{11}x + a_{12}y \\ [w, y] &= a_{21}x + a_{22}y \\ [w, z] &= (a_{11} + a_{22})z + w - xy^2 + \xi_1x + \xi_2y \end{aligned}$$

for some $a_{11}, a_{12}, a_{21}, a_{22}, \xi_1, \xi_2 \in k$. Applying $[w, -]$ to $[z, y] = 0$, one sees that

$$\begin{aligned} 0 &= [w, [z, y]] = [[w, z], y] + [z, [w, y]] \\ &= [(a_{11} + a_{22})z + w - xy^2 + \xi_1x + \xi_2y, y] + [z, a_{21}x + a_{22}y] \\ &= [w, y] + a_{21}[z, x] = a_{21}x + a_{22}y + a_{21}x. \end{aligned}$$

Thus $a_{21} = a_{22} = 0$. Applying $[w, -]$ to $[z, x] = x$ and using $a_{21} = a_{22} = 0$, one sees that

$$\begin{aligned} [w, x] &= [w, [z, x]] \\ &= [[w, z], x] + [z, [w, x]] \\ &= [a_{11}z + w - xy^2 + \xi_1x + \xi_2y, x] + [z, -z + a_{11}x + a_{12}y] \\ &= a_{11}[z, x] + [w, x] + a_{11}[z, x] = [w, x] + 2a_{11}x \end{aligned}$$

which implies that $a_{11} = 0$. By setting $z_{new} = z - a_{12}y$, we can make $a_{12} = 0$. By setting $w_{new} = w + \frac{1}{2}\xi_1x + \xi_2y$, we can make $\xi_1 = \xi_2 = 0$. New variables z and w still satisfy (E6.5.12.1) and (E6.5.12.2), respectively (for $\theta_1 = 0$). Therefore this is the Hopf algebra described in Example 6.5.7.

If $\theta_2 = 0$, by dividing w with θ_1 we can assume that $\theta_1 = 1$. In this setting, by Lemmas 6.5.19 and 6.5.22(b), we have

$$\begin{aligned} [w, x] &= a_{11}x + a_{12}y \\ [w, y] &= a_{21}x + a_{22}y \\ [w, z] &= (a_{11} + a_{22})z - w + \xi_1x + \xi_2y \end{aligned}$$

for some $a_{11}, a_{12}, a_{21}, a_{22}, \xi_1, \xi_2 \in k$. Applying $[w, -]$ to $[z, y] = 0$, one sees that

$$\begin{aligned} 0 &= [w, [z, y]] = [[w, z], y] + [z, [w, y]] \\ &= [(a_{11} + a_{22})z - w + \xi_1x + \xi_2y, y] + [z, a_{21}x + a_{22}y] \\ &= -[w, y] + a_{21}[z, x] = -a_{22}y. \end{aligned}$$

Thus $a_{22} = 0$. Applying $[w, -]$ to $[z, x] = x$ and using $a_{22} = 0$, one sees that

$$\begin{aligned} [w, x] &= [w, [z, x]] \\ &= [[w, z], x] + [z, [w, x]] \\ &= [a_{11}z - w + \xi_1x + \xi_2y, x] + [z, a_{11}x + a_{12}y] \\ &= a_{11}[z, x] - [w, x] + a_{11}[z, x] = -[w, x] + 2a_{11}x \end{aligned}$$

which implies that $a_{12} = 0$. This is the Hopf algebra described in Example 6.5.5.

Case 3: $C = A(0, 0, 1)$. By Lemma 6.5.22(c), $\theta_1 = 0$. By dividing w with θ_2 we can assume that $\theta_2 = 1$. In this setting, by Lemmas 6.5.19 and 6.5.22(c), we have

$$\begin{aligned} [w, x] &= a_{11}x + a_{12}y \\ [w, y] &= a_{21}x + a_{22}y \\ [w, z] &= (a_{11} + a_{22})z - \frac{2}{3}y^3 + \xi_1x + \xi_2y \end{aligned}$$

for some $a_{11}, a_{12}, a_{21}, a_{22}, \xi_1, \xi_2 \in k$. Applying $[w, -]$ to $[z, y] = 0$, one sees that

$$\begin{aligned} 0 &= [w, [z, y]] = [[w, z], y] + [z, [w, y]] \\ &= [(a_{11} + a_{22})z - \frac{2}{3}y^3 + \xi_1x + \xi_2y, y] + [z, a_{21}x + a_{22}y] \\ &= a_{21}y. \end{aligned}$$

Hence $a_{21} = 0$. Applying $[w, -]$ to $[z, x] = y$, one sees that

$$\begin{aligned} a_{22}y &= [w, y] = [w, [z, x]] = [[w, z], x] + [z, [w, x]] \\ &= [(a_{11} + a_{22})z - \frac{2}{3}y^3 + \xi_1x + \xi_2y, x] + [z, a_{11}x + a_{12}y] \\ &= (a_{11} + a_{22})y + a_{11}y. \end{aligned}$$

Hence $a_{11} = 0$. This is the Hopf algebra described in Example 6.5.6. \square

6.5.6 Proof of the main result

With the help of the last few subsections, we are able to deliver the main theorem of the paper.

Theorem 6.5.24 (Theorem 6.1.3). *Let H be a connected Hopf algebra of GK-dimension four over an algebraically closed field of characteristic zero. Then H is isomorphic to one of following.*

- (a) *Enveloping algebra $U(\mathfrak{g})$ over a Lie algebra \mathfrak{g} of dimension 4. Note that all 4-dimensional Lie algebras over the complex numbers \mathbb{C} are listed in the book [OV, Theorem 1.1(iv), page 209].*
- (b) *Enveloping algebra $U(L)$ over an anti-cocommutative CLA L of dimension 4. All anti-cocommutative coassociative Lie algebras of dimension 4 are classified in Theorem 6.4.5.*
- (c) *Primitively-thin Hopf algebras of GK-dimension four. All Primitively-thin Hopf algebras of GK-dimension four are classified in Theorem 6.5.23.*

Proof. By Lemma 6.2.3(e), $p(H) \geq 2$. Since $U(\mathfrak{g})$ embeds in H where $\mathfrak{g} = P(H)$, we have $p(H) \leq 4$.

If $p(H) = 4$, by Theorem 6.3.5, $H \cong U(\mathfrak{g})$ where $\mathfrak{g} = P(H) = P_2(H)$. This is case (a).

If $p(H) = 3$, by Theorem 6.3.5, H is isomorphic to the enveloping algebra $U(L)$ over an anti-cocommutative CLA L of dimension 4. Anti-cocommutative CLAs of dimension 4 are classified in Theorem 6.4.5.

If $p(H) = 2$, by definition, H is a primitively-thin Hopf algebras of GK-dimension four, which are classified in Theorem 6.5.23. \square

Corollary 6.5.25. *Let H be a connected Hopf algebra of dimension at most 4. Then, as an algebra, H is isomorphic to $U(\mathfrak{g})$ for some Lie algebra \mathfrak{g} .*

Proof. This is clear if $\text{GKdim } H \leq 3$. For $\text{GKdim } H = 4$, the assertion is clear for the case when $p(H) \geq 3$. The only case left is when $p(H) = 2$, in which the assertion was checked case-by-case in Examples 6.5.4-6.5.7. \square

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