Analytic and geometric aspects of the elliptic measure on non-smooth domains

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Abstract

Analytic and Geometric Aspects of the Elliptic Measure
on Non-Smooth Domains

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Harmonic/elliptic measure arises naturally in probability and in the study of boundary value problems for elliptic operators. It has attracted the attention of many mathematicians to study the relationship between the harmonic/elliptic measure ω of a given domain and its surface measure σ, in particular, whether or not they are absolutely continuous with each other. We focus on two aspects of this subject:

1) getting an equivalent characterization of the quantitative absolute continuity between these two measures, i.e. $\omega \in A_\infty(\sigma)$, in terms of the PDE solvability of the corresponding Dirichlet problem;

2) studying what the regularity of the elliptic measure (with respect to the surface measure) can tell us about the geometric properties of the domain, such as the rectifiability of the boundary.

We combine tools from PDE, harmonic analysis and geometric measure theory to answer these two questions.

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Contents

Chapter 1

Introduction

1.1 Background

The relationship between harmonic (and elliptic) measure and the geometry of a domain where they live has been at the center of the field. Roughly speaking, we consider an object traveling by a Brownian motion within a bounded domain $\Omega \subset \mathbb{R}^n$. Almost surely the object will hit the boundary within finite time. The harmonic measure characterizes the likelihood of where in the boundary the object is to exit the domain from, i.e. for any set $E \subset \partial\Omega$, its harmonic measure is

$$\omega^X(E) = \mathbb{P}\left(\text{Brownian motion } B_t \text{ starting from } X \in \Omega \text{ exits the domain from } E\right). \tag{1.1.1}$$

Another way to interprete the harmonic measure is to study the Dirichlet boundary problem

$$\begin{cases} -\Delta u = 0, & \text{in } \Omega, \\ u = f, & \text{on } \partial\Omega \end{cases} \tag{1.1.2}$$

and the harmonic measure gives a representation of the solution by its boundary value:

$$u(X) = \int_{\partial\Omega} f d\omega^X. \tag{1.1.3}$$

It is easy to see that $\omega = \omega^X$ is a probability measure on the boundary $\partial\Omega$, i.e. $\omega(\partial\Omega) = 1$. Another natural measure on the boundary is the surface measure, or the generalized surface measure $\sigma := \mathcal{H}^{n-1}|_{\partial\Omega}$, that is the $(n-1)$-dimensional Hausdorff measure restricted to the boundary. The central question in this area of research is: what is the relationship between ω and σ, and how is that related to the geometry of the domain Ω?

In 1917 F. and M. Riesz [RR] showed that if Ω is a simply connected planar domain bounded by a Jordan curve of finite length, then the two measures in question are mutually absolute continuous $\omega \ll \sigma \ll \omega$. That is to say, for any $E \subset \partial\Omega$

$$\omega(E) = 0 \iff \sigma(E) = 0.$$

Moreover if Ω is a chord-arc planar domain, Lavrentiev [La] proved that $\omega \in A_\infty(\sigma)$, which is a quantitative version of mutually absolute continuity (see Definition (2.1.14)).

In higher dimensions the situation is different. In 1974 Ziemer [Zi] found a topological ball $\Omega \subset \mathbb{R}^3$ whose boundary is 2-rectifiable and satisfies $\mathcal{H}^2(\partial\Omega) < \infty$, and yet the harmonic measure is supported in a set of zero area. In fact, Wolff [Wo] constructed a class of snowflake-type domains Ω in $\mathbb{R}^3$, whose harmonic measure may be supported in a set with Hausdorff dimension strictly bigger or smaller than 2, the canonical dimension of the geometric boundary $\partial\Omega$. This is surprising since for a planar domain with some mild assumption, we always have $\dim_{\mathcal{H}} \omega \leq 1$ (see [JWo]). This means extra geometric assumptions on Ω are needed to obtain an analogue of the Riesz Theorem in higher dimensions. In this effort, a series of work study the properties of the harmonic measure, and in particular the relationship between ω and σ, in the setting of Lipschitz domains, NTA (non-tangentially accessible) domains, and (1-sided) NTA domains with Ahlfors regularity assumptions on the boundary, see [Da1, JK, DJ, Se, Ba, HM1].

The situation becomes more delicate if the problem is non-homogenous, that is, if we consider the Dirichlet boundary problem to elliptic operator $L := -\operatorname{div}(A(X)\nabla)$

$$\text{(D)} \qquad \begin{cases} -\operatorname{div}(A(X)\nabla u) = 0, & \text{in } \Omega, \\ u = f, & \text{on } \partial\Omega \end{cases}$$

and define the elliptic measure ω_L accordingly by the representation formula (1.1.3). Here $A(X)$ a (symmetric) variable-coefficient matrix satisfying the assumption of uniform ellipticity (E). Intuitively one can think of ω_L by the probability characterization (1.1.1), but for a Brownian motion in a non-homogenous medium. In fact even in the simple case when the domain is a unit ball B in $\mathbb{R}^2$, there exist examples of elliptic matrix $A(X) \in C(\overline{B}) \cap C^\infty(B)$ for which the elliptic measure and surface measure are singular, i.e. $\omega_L \perp \sigma$ (see [CFK, MM] and also [FJK]). These examples are due to the fact that as X approaches the boundary, the matrix $A(X)$ does not have sufficient modulus of continuity, and thus affecting the effective trajectory to exit the domain. So far the best result for variable-coefficient elliptic operator is in the case of Lipschitz domains (or domains that can be approximated by Lipschitz domains, such as chord-arc domains), which says if the matrix $A(X)$ satisfies that $|\nabla A(X)|^2\delta(X)$ is a Carleson measure, then the corresponding elliptic measure ω_L is of class A_∞. See the work of Kenig and Pipher [KP, Theorem 2.6]. We remark that being a Carleson measure quantifies how fast it decays while approaching the boundary. For the precise definition see (5.0.2).

Meanwhile, in an effort to find the minimal geometric assumptions to guarantee nice properties of the harmonic measure, mathematicians also start to look at the necessary conditions for $\omega \ll \sigma \ll \omega$ and it quantitative counterpart $\omega \in A_\infty(\sigma)$. This direction is also referred to as the (non-variational) free boundary problem, as one tries to determine the regularity of the domain, or more precisely regularity of its boundary. See for example the work of [AC, Je, KT, BET]. Recently, most notable in this line of research are the work of Hofmann et al. [HMU] and Azzam et al. [AHM3TV], both of which asserted the necessity of boundary *rectifiability*, i.e. modulo a set of measure zero, the boundary can be covered by a countable union of Lipschitz graphs.

1.2 Different approaches and tools

As mentioned above, the search for necessary and sufficient conditions of $\omega \ll \sigma \ll \omega$ is an active area of research. Under this umbrella many different approaches and tools have been used and many fruitful results have been obtained. In this section we summarize these different methods in broad terms, and the reader will be reminded of them in later chapters. Since our goal is to explain the idea behind various techniques, we do not attempt to cover the most extensive list of references. We are also certainly biased by our own view and understanding of each subject.

Direct methods and PDE solvability By the representation formula (1.1.3), one can already see the connection between the harmonic/elliptic measure and the *solvability* of the corresponding PDE. By solvability, we mean for a boundary function f living in some given space, which functional space does the solution(s) live in, and do we have some control on its norm? As an example, the Dirichlet problem (D) is solvable in $L^p(\sigma)$ if and only if $\omega_L \ll \sigma$ and their Radon-Nikodym derivative $k := d\omega_L/d\sigma$ (also referred to as the *Poisson kernel*) satisfies a reverse Hölder inequality with power q (see (2.1.18)) and $1/p + 1/q = 1$.

Let us be more precise. The differential equation $-\operatorname{div}(A(X)\nabla u) = 0$ in (D) is understood in the weak sense, i.e.

$$\int_\Omega A(X)\nabla u \cdot \nabla\varphi \, dX = 0, \qquad \text{for all } \varphi \in C_c^\infty(\Omega);$$

and in general $u = f$ on $\partial\Omega$ is understood in terms of the trace on the boundary. We say the domain Ω is Wiener regular if for any continuous boundary function $f \in C(\partial\Omega)$, the solution u is continuous all the way to the boundary (see also [LSW]). If that is the case then $u = f$ in the classical sense. A fundamental result of DeGiorgi, Nash and Moser is that any solution to $Lu = 0$ in Ω is Hölder continuous on any compact sub-domain of Ω. For the study of harmonic/elliptic measure, we are more interested in the boundary behavior of the solutions, in particular positive solutions. For example recall the property of the Green's function, for smooth domains the Poisson kernel $k = d\omega_L/d\sigma$ is exactly the normal derivative of the Green's function. (We only need a domain for which the divergence theorem holds and the normal derivative of the Green's function makes sense at the boundary.) This, combined with other PDE estimates, allowed Dahlberg to show k satisfies a reverse Hölder inequality with power 2, and thus $\omega \in A_\infty(\sigma)$, in Lipschitz domains in [Da1].

In [JK] Jerison and Kenig generalized various estimates on the solutions and harmonic measures to non-tangentially accessible domains, a.k.a. NTA domains. NTA domains are domains satisfying quantitative openness (i.e. the *interior corkscrew condition*), quantitative connectedness (i.e. the *Harnack chain condition*, whose complement also satisfies quantitative openness (i.e. the *exterior corkscrew condition*), and this notion broadens the regime of domains we can analyze beyond domains given by a graph. Later, inspired by Aikawa's study of the *capacity density condition* [A1, A2, A3], in the preparation of the book [HMT2] the authors realized that the same estimates hold if we replace NTA domains by 1-sided NTA domains (also referred to as *uniform domains*) whose boundaries are Ahlfors regular. (See Theorem 2.2.6.) We will state various PDE estimates and their implications on the elliptic measure (e.g. the doubling property) in Chapter 2 without giving proofs (proper references will be given therein). For the purpose of this thesis, these estimates are mostly used as black boxes rather than the core of analysis; but we emphasize that the direct PDE methods, in particular on relatively simple domains, are often the building blocks before handling non-smooth domains.

Harmonic analysis techniques Since the harmonic/elliptic measure is doubling, various harmonic analysis techniques become available, such as the theory of weights, tent space, the square functions and non-tangential maximal functions. To our knowledge these tools were first brought into this area of research by Fefferman, Kenig and Pipher [FKP] to study how a perturbation of operators change the kernels and A_∞ property of the elliptic measures in Lipschitz domains. (Their result extends previous perturbation result of Dahlberg [Da2] from the case of vanishing constant to finite constant.) In [MPT] Milakis, Pipher and Toro brought these tools to chord-arc domains, i.e. NTA domains with Ahlfors regular boundary. Based on a dyadic decomposition of Ahlfors regular boundary (see Definition 2.1.12 and Lemma 6.1.1), Hofmann and Martell constructed a family of sawtooth domains, which are in some sense tent spaces adapted to the dyadic structure and approximate subdomains of the same class. This framework, combined with stopping time argument or extrapolation method, turns out to be very useful to facilitate analysis on non-smooth domains, as is shown in the work of [HM1, HMU, HMM, HMT1]. (See Section 6.1 of Chapter 6 for the precise construction, albeit we use it in a slightly different setting.)

In particular, the relationship between the square functions and non-tangential maximal function is proven to be closely related to $\omega_L \in A_\infty(\sigma)$. For example in [DJK] the authors showed if $\omega_L \in A_\infty(\sigma)$, then the square function and non-tangential maximal function are equivalent. In [KKiPT] the authors established a necessary condition of $\omega_L \in A_\infty(\sigma)$ by studying the square functions given by the solution, which was then used by Kenig and Pipher [KP] to obtain a necessary condition in terms of the Carleson measure of $|\nabla A|^2\delta(X)$. This pair of functionals have also been heavily used in studying the solvability of PDEs of other types (see for example [DP, DPP]).

It had been a long-standing conjecture (David-Semmes conjecture) in harmonic analysis that the boundedness of the Riesz transform implies the (uniform) rectifiability of the boundary. When the conjecture was resolved by Nazarov, Treil and Volberg [NTV], it became possible to connect the dots and find sufficient conditions for $\omega \ll \sigma$ or $\sigma \ll \omega$ by proving the boundedness of the Riesz transform, see [HMU, MT, AHM3TV].

Geometric measure theory Given some basic assumptions on a measure, its blow-ups (or tangent objects) provide information about the local and infinitesimal structure of the measure (see [Pr]). To apply this very general framework to various different settings, many analytical tools are required to quantify the estimates across different scales. Examples of the tools used in this setting are excess-decay estimates, monotonicity formulas (e.g. [All, Alm, ACF]), dimension reduction argument (see [Fe]) and more recently, quantitative stratification (e.g. [CN, NV]). These tools have been useful in geometric analysis when studying for example the regularity of minimal surfaces or free boundaries and the structure and size of the corresponding singular sets. We remark that these quantitative tools are also used sometimes in combination with a compactness argument (instead of blow-up argument), if the problem at hand is scale-invariant.

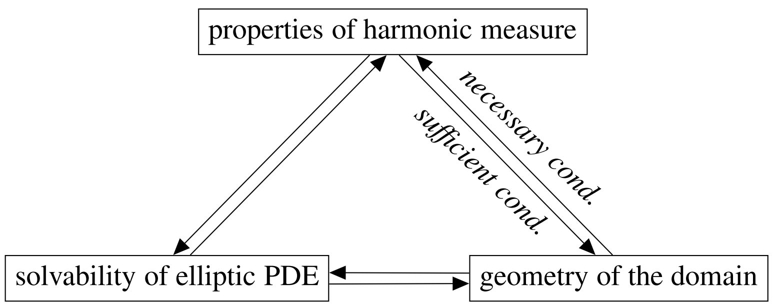
In the case of harmonic and elliptic measure, the tangent measure(s) can be realized by solving a more homogenous global problem, and thus allows one to study the structure of the original measure. For example, for the free boundary regularity problem of two phases (that is, given the relationship between the harmonic measures of a domain and its complement), the blow-up limit are domains given by harmonic polynomials (see [KP, KPT]); for elliptic operators of certain class, the tangent measures of the elliptic measure turn out to be harmonic measures, which allows one to invoke known results in harmonic measures to draw conclusion about the original elliptic measure (see [TZ, AM]). In this setting, the estimates of harmonic/elliptic measures having been obtained by direct PDE methods serve as the quantitative tools to relate and control across different scales, such as the non-degeneracy and doubling property of the elliptic measures.

Sets of higher codimensions It is well-known that sets of co-dimensions greater than two are *removable* for the Laplacian, or in plain words, harmonic measure does not see sets of higher co-dimensions. To our knowledge the only approach had been by means of the quasi-linear p-Laplacian operator and its generalizations, see [LN] and the references therein. In an effort to construct harmonic measures for low-dimensional sets, and to study the geometry of the relevant sets, David, Mayboroda and Feneuil consider for any given d-Ahlfors regular sets Γ (see Definition 2.1.12) a linear *degenerate* elliptic operator $L = -\operatorname{div}(A(X)\nabla)$, where $A(X)$ is an $n\times n$ matrix equivalent to $\delta(X)^{d-n+1}$ Id and $\delta(X) := \operatorname{dist}(X,\Gamma)$ (see (2.4.1) and (2.4.2) for the precise statement). The effect of this operator is to replace the classic Brownian motion by a Brownian motion with drift term, and the drift term attracts the movement towards Γ with magnitude in reverse proportion to $\delta(X)$. Since this operator is linear, they are able to define a measure in the same fashion as before (1.1.3), and say it is the *harmonic measure* for Γ. In [DFM1] they developed the elliptic theory for this type of operators and proved some estimates of the harmonic measure; in a follow up paper [DFM2], they proved that if Γ is a Lipschitz graph with small constant, then the harmonic measure is of class A_∞ for a special operator within this class. They conjecture that $\omega \in A_\infty(\sigma)$ should hold for a more general class of domains. This line of thoughts opens the door to many interesting questions that one hopes to answer with combined tools from PDEs, harmonic analysis, and geometric measure theory.

1.3 Summary of my work

My work is focused on two aspects of the following diagram:

- the characterization of $\omega \in A_\infty(\sigma)$ by the solvability of corresponding Dirichlet problem, in the case of co-dimension one as well as higher co-dimensions;
- the study of regularity of the domain given $\omega \in A_\infty(\sigma)$ or its qualitative counterpart for various-coefficient operators.



We state the main results in each aspect.

1.3.1 $\omega_L \in A_\infty(\sigma)$ and PDE solvability

For $1 < p < \infty$, we say the problem (D) is solvable in L^p if there exists a universal constant C such that for any continuous boundary value f and its corresponding solution u,

$$\|Nu\|_{L^p(\sigma)} \leq C\|f\|_{L^p(\sigma)}, \tag{1.3.1}$$

where $Nu(Q) = \max\{|u(X)| : X \in \Gamma(Q)\}$ is the non-tangential maximal function of u (the definition of $\Gamma(Q)$ is specified in (6.0.4)). From the representation formula (2.2.17) we know $Nu(Q)$ is comparable to the Hardy-Littlewood maximal function $M_{\omega_L} f(Q)$ with respect to ω_L. Provided that σ is doubling, the theory of weights tells us

$$\text{problem(D) is } L^p \text{ solvable, i.e. (1.3.1) holds} \iff \omega_L \in B_q(\sigma), \text{ where } \frac{1}{p} + \frac{1}{q} = 1.$$

(See Section 2.1 (2.1.18) for the definition of B_q weights.) For the Laplacian on Lipschitz domains, Dahlberg [Da1] proved the harmonic measure $\omega \in B_2(\sigma)$; therefore (D) is solvable in L^p for $2 \leq p < \infty$.

When $p = \infty$, (1.3.1) follows trivially from the maximal principle. When $L = -\Delta$ and $\Omega = \mathbb{R}^n_+$ is the upper half plane, the problem (D) is also solvable in the BMO space, that is, if $f \in BMO(\partial\mathbb{R}^n_+)$, its harmonic extension u has the property that $\mu = x_n|\nabla u|^2 dx$ is a Carleson measure on Ω (see [FS], and also Section 4.4 Theorem 3 of [St1]). In addition, the Carleson measure norm of μ is equivalent to the BMO norm of f. This BMO solvability also holds for Lipschitz domains, if μ is replaced by $\delta(x)|\nabla u|^2 dx$ and $\delta(x) = \text{dist}(x, \partial\Omega)$ (see [**?**]).

Recall that $A_\infty(\sigma) = \cup_{q>1} B_q(\sigma)$ (see the remark after Definition (2.1.14)), in other words,

$$\begin{aligned} \omega_L \in A_\infty(\sigma) &\iff \text{there exists } q_0 > 1 \text{ such that } \omega_L \in B_q(\sigma) \text{ for all } 1 < q \leq q_0 \\ &\iff \text{problem(D) is } L^p \text{ solvable for all } p \geq p_0, \text{ where } \frac{1}{p_0} + \frac{1}{q_0} = 1. \end{aligned}$$

Note that there is some ambiguity with p_0: the fact that (D) is not L^{p_0} solvable does not necessarily imply $\omega_L \notin A_\infty(\sigma)$. A natural question arises: is there a solvability criterion that directly characterizes $\omega_L \in A_\infty(\sigma)$? In 2009, Dindos, Kenig and Pipher showed that for Lipschitz domains, the elliptic measure $\omega_L \in A_\infty(\sigma)$ if and only if the problem (D) is BMO-solvable, i.e. for any continuous function $f \in C(\partial\Omega)$, the Carleson measure of $\delta(X)|\nabla u|^2 dX$ is controlled by the BMO norm of f (see [DKP]).

Definition 1.3.2. We say that the Dirichlet problem (D) is **solvable in BMO** if for any continuous boundary function $f \in C(\partial\Omega)$, the solution u to (D) given by (1.1.3) satisfies a condition that $|\nabla u|^2 \delta(X)\, dX$ is a Carleson measure with norm bounded by a constant multiple of $\|f\|^2_{BMO}$, that is,

$$\sup_{\Delta \subset \partial\Omega} \frac{1}{\sigma(\Delta)} \iint_{T(\Delta)} |\nabla u|^2 \delta(X)\, dX \leq C \|f\|^2_{BMO}. \tag{1.3.3}$$

Theorem 1.3.4. *For uniform domains with Ahlfors regular boundary, the elliptic measure $\omega \in A_\infty(\sigma)$ if and only if the Dirichlet problem* (D) *is BMO solvable.*

1.3.2 $\sigma \ll \omega$ implies boundary rectifiability for a class of variable-coefficient operators

In the joint work with Tatiana Toro that originally appeared in [TZ], our main motivation is to understand whether the elliptic measure distinguishes between a rectifiable and a purely unrectifiable boundary.

Geometrically we consider bounded uniform domains $\Omega \subset \mathbb{R}^n$ with $n \geq 3$ (see Definition 2.1.10) with Ahlfors regular boundary (see Definition 2.1.12). Analytically we consider second order divergence form elliptic and symmetric operators with $W^{1,1}(\Omega) \cap L^\infty(\Omega)$ or $C(\overline{\Omega})$ coefficients whose elliptic measure is an A_∞ weight (see Definition 2.1.14) with respect to the surface measure $\sigma = \mathcal{H}^{n-1} \llcorner \partial\Omega$. Our main goal is to understand the extent to which the regularity of the elliptic measures of these operators determines the structure of the boundary. In particular we care about whether the absolute continuity (quantitative or qualitative) of surface measure with respect to elliptic measure ensures the exterior corkscrew property of the domain or the rectifiability of its boundary. Theorems 1.3.5, 1.3.6 and 1.3.7 provide answers to these queries.

Theorem 1.3.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded uniform domain with Ahlfors regular boundary. Let $L = -\operatorname{div}(A(X)\nabla)$ with $A \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ satisfying (E). Suppose that the elliptic measure $\omega_L \in A_\infty(\sigma)$ (see Definition 2.1.14), then Ω is a set of locally finite perimeter, whose measure theoretic boundary coincides with its topological boundary $\mathcal{H}^{n-1}$-a.e. Thus $\partial\Omega$ is $(n-1)$-rectifiable.*

Theorem 1.3.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded uniform domain with Ahlfors regular boundary. Let $L = -\operatorname{div}(A(X)\nabla)$ with $A \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ satisfying (E). Suppose $X_0 \in \Omega$ is such that $\delta(X_0) \sim \operatorname{diam}\Omega$, and denote $\omega = \omega_L^{X_0}$. Then if $\sigma \ll \omega$, $\partial\Omega$ is $(n-1)$-rectifiable.*

Theorem 1.3.6 should be understood as a corollary of Theorem 1.3.5. In fact modulo a stopping time argument the proof can be reduced to applying a local version of Theorem 1.3.5. By taking this approach we would like to emphasize the fact that, in this area, quantitative results yield qualitative ones. See section 4.4.

Theorem 1.3.7. *Let $\Omega \subset \mathbb{R}^n$ be a bounded uniform domain with Ahlfors regular boundary. Let $L = -\operatorname{div}(A(X)\nabla)$ with $A \in C(\overline{\Omega})$ satisfying (E). Suppose that the elliptic measure $\omega_L \in A_\infty(\sigma)$ (see Definition 2.1.14), then there exists $r_\Omega > 0$ such that Ω satisfies the exterior corkscrew condition for all $r < r_\Omega$. In particular Ω is an NTA domain.*

Remark 1.3.8. It is important to differentiate the result in Theorem 1.3.7 and those in [HM1], [HMT1] and [HMU]. The key difference is that although one shows that the domain is NTA, the constants are not "uniform" in the sense that they do not depend only on the *allowable* constants, namely the dimension n, the ellipticity constants of A, the Ahlfors constants, and the constants that determine the uniform character of the domain. Here r_Ω is obtained via compactness and there might depend a priori on the domain Ω itself.

1.3.3 $\omega_L \in A_\infty(\sigma)$ implies uniform rectifiability for operators with small Carleson norm

As mentioned before, so far the best result of a necessary condition to guarantee the *elliptic* measure $\omega_L \in A_\infty(\sigma)$ is the following theorem by Kenig and Pipher [KP]:

Theorem 1.3.9. *Suppose Ω is a Lipschitz domain. Let $\mathcal{A}$ be an $n \times n$ matrix on Ω satisfying (E), and $\mathcal{A}$ is locally Lipschitz with $|\nabla\mathcal{A}|\delta(X) \in L^\infty(\Omega)$. Assume that $|\nabla\mathcal{A}|^2\delta(X)$ is a Carleson measure, that is,*

$$\sup_{\Delta\subset\partial\Omega} \frac{1}{\sigma(\Delta)} \iint_{T(\Delta)} |\nabla\mathcal{A}|^2 \delta(X)\, dX < \infty. \tag{1.3.10}$$

Then the elliptic measure corresponding to the operator $L = -\operatorname{div}(\mathcal{A}(X)\nabla)$ is of class A_∞.

Since chord-arc domains can be approximated by Lipschitz domains, the above theorem also hold for chord-arc domains. In [HMMTZ], I, together with S. Hofmann, J.M. Martell, S. Mayboroda and T. Toro, are able to prove a partial converse to it, which we rephrase as follows:

Theorem 1.3.11. *Suppose Ω is a uniform domain with Ahlfors regular boundary. Assume $\mathcal{A}$ is an elliptic matrix such that $|\nabla\mathcal{A}|^2\delta(X)$ is a Carleson measure with sufficiently small norm, i.e.*

$$\sup_{\Delta\subset\partial\Omega}\frac{1}{\sigma(\Delta)}\iint_{T(\Delta)}|\nabla\mathcal{A}|^2\delta(X)dX<\epsilon. \tag{1.3.12}$$

Then

$$\omega_L\in A_\infty(\sigma) \implies \Omega \text{ is a chord-arc domain and thus } \partial\Omega \text{ is uniformly rectifiable}.$$

We will state the theorem more precisely and specify what constants the smallness quantity ϵ depend on in Chapter 5, see Theorem 5.0.6 and the remarks before and after that.

1.3.4 $\omega_L\in A_\infty(\sigma)$ and PDE solvability for sets of higher co-dimensions

As mentioned in the previous section, most PDE estimates hold, or appropriate equivalent estimates exist, for the *degenerate* elliptic operators suited for the study of harmonic measure for sets of higher co-dimensions, see Section 2.4 for details. Therefore one naturally expects the PDE characterization of $\omega\in A_\infty(\sigma)$, similar to Theorem 1.3.4, holds in this scenario. This is joint work with Svitlana Mayboroda and originally appeared in [MZ].

The main difficulty is to get a bound of the square function by the non-tangential maximal function. This is often referred to as $S\leq N$ estimate in literature and is a main step in the case of co-dimension one to bound the Carleson measure of the solution (in appropriate sense) by the BMO norm of the boundary function, see [DJK, Zh] or Chapter 3.

Theorem 1.3.13. *Let Γ be a d-Ahlfors regular set in $\mathbb{R}^n$ with an integer $d\leq n-1$, and let ω be the harmonic measure of the domain $\Omega=\mathbb{R}^n\setminus\Gamma$ (see Section 2.4 for its definition). Suppose $\omega\in A_\infty(\sigma)$, then*

$$\|Su\|_{L^p(\sigma)}\leq C\|Nu\|_{L^p(\sigma)} \tag{1.3.14}$$

for any $1\leq p<\infty$ and any solution $u\in W_r(\Omega)$ to $Lu=0$ such that the right hand side is finite.

Here the constant $C>0$ depends on the allowable parameters: the dimensions d,n, Ahlfors regular constant C_0, constant of ellipticity C_1, the aperture α and the A_∞ constant(s).

Therefore combined with the PDE estimates in [DFM1], we are able to conclude:

Theorem 1.3.15. *Let Γ be a d-Ahlfors regular set in $\mathbb{R}^n$ with $d<n-1$ and $\Omega=\mathbb{R}^n\setminus\Gamma$. Consider the operator $L=-\operatorname{div}(A(X)\nabla)$ with a real, symmetric $n\times n$ matrix $A(X)$ satisfying (2.4.1) and (2.4.2). Then the harmonic measure $\omega\in A_\infty(\sigma)$ if and only if the Dirichlet problem (D) is BMO-solvable (see Definition (6.0.1)).*

Chapter 2

Preliminaries

2.1 Notations and definitions

Throughout the manuscript we always assume that Ω is a bounded domain in $\mathbb{R}^n$ ($n \geq 3$). Let L be an operator defined as $Lu := -\operatorname{div}(A(X)\nabla u)$, where $A(X) = \left(a_{ij}(X)\right)_{i,j=1}^n$ is a real $n \times n$ matrix on Ω that is bounded measurable and uniformly elliptic: that is, there exist constants $0 < \lambda \leq \Lambda$ such that

$$\lambda|\xi|^2 \leq A(X)\xi \cdot \xi \leq \Lambda|\xi|^2 \tag{E}$$

for all $\xi \in \mathbb{R}^n \setminus \{0\}$ and $X \in \Omega$.

Definition 2.1.1. An open set $\Omega \subset \mathbb{R}^n$ is said to satisfy the **(interior) corkscrew condition** (resp. the exterior corkscrew condition) with constant $M > 1$ if for every $q \in \partial\Omega$ and every $0 < r < \operatorname{diam}(\Omega)$, there exists $A = A(q,r) \in \Omega$ (resp. $A \in \Omega_{\text{ext}} := \mathbb{R}^n \setminus \overline{\Omega}$) such that

$$B\left(A, \frac{r}{M}\right) \subset B(q,r) \cap \Omega \qquad \left(\text{resp. } B\left(A, \frac{r}{M}\right) \subset B(q,r) \cap \Omega_{\text{ext}}.\right) \tag{2.1.2}$$

The point A is called a corkscrew point (or a non-tangential point) relative to $B(q,r)$ in Ω (resp. Ω_{ext}).

We define the non-tangential cone $\Gamma^\alpha(q)$ at $q \in \partial\Omega$ with aperture α as follows

$$\Gamma^\alpha(q) = \{X \in \Omega : |X - q| \leq (1+\alpha)\delta(X)\}, \tag{2.1.3}$$

and define the truncated cone $\Gamma_r^\alpha(q) = \Gamma^\alpha(q) \cap B(q,r)$. We will omit the super-index α when there is no confusion. The interior corkscrew condition in particular implies $\Gamma_r(q)$ is nonempty as long as the aperture $\alpha \geq M$. We define the non-tangential maximal function

$$Nu(q) = \sup\{|u(X)| : X \in \Gamma(q)\}, \tag{2.1.4}$$

and the square function

$$S u(q) = \left(\iint_{\Gamma(q)} |\nabla u(X)|^2 \delta(X)^{2-n} dX \right)^{1/2}. \tag{2.1.5}$$

We also consider truncated square function $S_h u(q)$, where the non-tangential cone $\Gamma(q)$ is replaced by the truncated cone $\Gamma_h(q)$.

Definition 2.1.6. An open connected set $\Omega \subset \mathbb{R}^n$ is said to satisfy the **Harnack chain condition** with constants $M, C_1 > 1$ if for every pair of points $A, A' \in \Omega$ there is a chain of balls $B_1, B_2, \dots, B_K \subset \Omega$ with $K \leq M(2 + \log_2^+ \Pi)$ that connects A to A', where

$$\Pi := \frac{|A - A'|}{\min\{\delta(A), \delta(A')\}}. \tag{2.1.7}$$

Namely, $A \in B_1$, $A' \in B_K$, $B_k \cap B_{k+1} \neq \emptyset$ and for every $1 \leq k \leq K$

$$C_1^{-1} \operatorname{diam}(B_k) \leq \operatorname{dist}(B_k, \partial\Omega) \leq C_1 \operatorname{diam}(B_k). \tag{2.1.8}$$

Remark 2.1.9. 1. If two points $A, A' \in \Omega$ do not satisfy (2.1.7), we can simply take the balls $B(A, \delta(A)/2)$ and $B(A', \delta(A')/2)$ to connect them.

2. We often want the Harnack balls to satisfy $\delta(B_j) > \operatorname{diam}(B_j)$ to be able to enlarge them. This is possible. In fact, if Ω satisfy the Harnack chain condition, then for any $C_1 > 1$, there is a constant C_2 such that Ω satisfies the above Harnack chain condition with the comparable size condition (2.1.8) replaced by

$$C_1 \operatorname{diam}(B_j) \leq \delta(B_j) \leq C_2 \operatorname{diam}(B_j). \tag{HB}$$

The number of balls M may increase, but is still of the order $\log_2 \Lambda$. Moreover, the ratio between C_2 and C_1 is fixed: $C_2/C_1 \approx C^2(C+1)^2$ Balls satisfying the condition (HB) are called *Harnack balls* with constants (C_1, C_2).

Definition 2.1.10. If Ω satisfies (1) the interior corkscrew condition and (2) the Harnack chain condition, then we say Ω is a **uniform domain**. If in addition, Ω satisfies the exterior corkscrew condition, we say it is an **NTA (non-tangential accessible) domain**.

A chord-arc domain is an NTA domain whose boundary is Ahlfors regular.

Remark 2.1.11. Uniform domain is sometimes also referred to as one-sided NTA domain in the literature, properly justified by their respective definitions.

Definition 2.1.12. Let $\Gamma \subset \mathbb{R}^n$ be a closed set and $d \leq n$ be an integer. We say Γ is d-**Ahlfors regular** if there exists a constant $C_0 \geq 1$ such that for any $q \in \Gamma$ and $r > 0$,

$$C_0^{-1} r^d \leq \mathcal{H}^d(B(q, r) \cap \Gamma) \leq C_0 r^d,$$

where $\mathcal{H}^d$ is the d-dimensional Hausdorff measure. We shall often denote $\mathcal{H}^d|_\Gamma$, that is $\mathcal{H}^d$ restricted to the set Γ, by σ and call it the *surface measure*.

There are many equivalent characterizations of a uniformly rectifiable set, see [DS2]. Since uniformly rectifiability is not the main focus of our paper, we only state one of the geometric characterizations as its definition.

Definition 2.1.13. An Ahlfors regular set $E \subset \mathbb{R}^n$ is said to be d-**uniformly rectifiable**, if it has big pieces of Lipschitz images of $\mathbb{R}^d$. That is, there exist $\theta, M > 0$ such that for each $q \in E$ and $0 < r < \operatorname{diam}(E)$, there is a Lipschitz mapping $\rho : B_d(0, r) \to \mathbb{R}^n$ such that ρ has Lipschitz norm $\leq M$ and

$$\mathcal{H}^d \left(E \cap B(q, r) \cap \rho(B_d(0, r))\right) \geq \theta r^d.$$

Here $B_d(0, r)$ denote a ball of radius r in $\mathbb{R}^d$.

In the co-dimension one case, $(n-1)$ is the canonical dimension of the boundary $\partial\Omega$ and we often assume the boundary $\partial\Omega$ is $(n-1)$-Ahlfors regular. When there is no confusion we simply drop the dimension and say $\partial\Omega$ is Ahlfors regular, $\sigma = \mathcal{H}^{n-1}|_{\partial\Omega}$ is the surface measure. For any $q \in \partial\Omega$ (or Γ) and $r > 0$, let $\Delta = \Delta(q, r)$ denote the surface ball $B_r(q) \cap \partial\Omega$, and $T(\Delta) = B_r(q) \cap \Omega$ denote the Carleson region above Δ. We always assume $r < \operatorname{diam} \partial\Omega$.

Definition 2.1.14. The elliptic measure associated with L in Ω is said to be of class A_∞ with respect to the surface measure $\sigma = \mathcal{H}^{n-1}|_{\partial\Omega}$, which we denote by $\omega_L \in A_\infty(\sigma)$, if there exist $C_0 > 1$ and $0 < \theta < \infty$ such that for any surface ball $\Delta(q, r) = B(q, r) \cap \partial\Omega$, with $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\Omega)$, any surface ball $\Delta' = B' \cap \partial\Omega$ centered at $\partial\Omega$ with $B' \subset B(q, r)$, and any Borel set $F \subset \Delta'$, the elliptic measure with pole at $A(q, r)$ (a corkscrew point relative to $\Delta(q, r)$) satisfies

$$\frac{\omega_L^{A(q,r)}(F)}{\omega_L^{A(q,r)}(\Delta')} \leq C_0 \left(\frac{\sigma(F)}{\sigma(\Delta')}\right)^\theta. \tag{2.1.15}$$

Remarks 2.1.16. (i) The above definition is symmetric: suppose $\omega \in A_\infty(\sigma)$, then we also have $\sigma \in A_\infty(\omega)$ (in a scale-invariant sense), i.e., the smallness of $\omega^A(E)/\omega^A(\Delta')$ implies the smallness of $\sigma(E)/\sigma(\Delta')$.

(ii) In particular, the assumption (2.1.15) implies that $\omega^A \ll \sigma$ when restricted to Δ. We denote the Radon-Nikodym derivative by $k^A = \frac{d\omega^A}{d\sigma}$. Since both ω^A and σ are Radon measures, we have

$$k^A(q) = \lim_{\substack{\Delta'=\Delta(q,r)\\ r\to 0}} \frac{\omega^A(\Delta')}{\sigma(\Delta')}, \qquad \text{for } \sigma\text{-a.e. } q \in \Delta. \tag{2.1.17}$$

Moreover since σ is doubling, by standard harmonic analysis techniques (see [GR] for example for the proof) (2.1.15) implies that k^A satisfies a reverse Hölder inequality: there are constants $r_0 > 1, C > 0$ such that for all $r \in (1, r_0)$,

$$\left(\fint_\Delta \left|k^A\right|^r d\sigma\right)^{\frac{1}{r}} \leq C \fint_\Delta k^A d\sigma. \tag{2.1.18}$$

The constants r_0 and C only depend on the constants characterizing the A_∞ property (2.1.15); in particular, they are independent of Δ and A.

If a kernel k satisfies the reverse Hölder inequality (2.1.18) with power r, we say $k \in RH_r(\sigma)$. In the theory of weight, k is also referred to as a B_r weight.

Definition 2.1.19. We say that the Dirichlet problem (D) is **solvable in BMO** if for any continuous boundary function $f \in C(\partial\Omega)$, the solution u to (D) given by (1.1.3) satisfies a condition that $|\nabla u|^2 \delta(X)^{d-n+2}\, dX$ is a Carleson measure with norm bounded by a constant multiple of $\|f\|_{BMO}^2$, that is,

$$\sup_{\Delta\subset\partial\Omega} \frac{1}{\sigma(\Delta)} \iint_{T(\Delta)} |\nabla u|^2 \delta(X)^{d-n+2}\, dX \le C\|f\|_{BMO}^2. \tag{2.1.20}$$

Remark 2.1.21. In the case of higher co-dimensions, we often study *unbounded* boundary sets Γ. In that case we consider boundary function $f \in C_0^0(\Gamma)$, that is, f is continuous and compactly supported in Γ.

The following lemma is relevant to Chapter 5. It establishes that if a domain satisfies the Harnack chain condition then we can modify the chain of balls so that they avoid a non-tangential balls inside:

Lemma 2.1.22. *Let $\Omega \subset \mathbb{R}^n$ be an open set satisfying the Harnack chain condition with constants $M, C_1 > 1$. Given $X_0 \in \Omega$, let $B_{X_0} = B(X_0, \delta(X_0)/2)$. For every $X, Y \in \Omega \setminus \overline{B_{X_0}}$, if we set $\Pi = |X - Y|/\min\{\delta(X), \delta(Y)\}$, then there is a chain of open Harnack balls $B_1, B_2, \dots, B_K \subset \Omega$ with $K \le 100(M + C_1^2)(2 + \log_2^+ \Pi)$ that connects X to Y. Namely, $X \in B_1$, $Y \in B_K$, $B_k \cap B_{k+1} \neq \emptyset$ for every $1 \le k \le K-1$ and for every $1 \le k \le K$*

$$(100\,C_1)^{-2}\operatorname{diam}(B_k) \le \operatorname{dist}(B_k, \partial\Omega) \le 100\,C_1^2 \operatorname{diam}(B_k). \tag{2.1.23}$$

Moreover, $B_k \cap \frac{1}{2}B_{X_0} = \emptyset$ for every $1 \le k \le K$.

Proof. Fix X, Y as in the statement and without loss of generality we assume that $\delta(X) \le \delta(Y)$. Use the Harnack chain condition for Ω to construct the chain of balls $B_1, \dots, B_K$ as in Definition 2.1.6. If none of B_k meets B_{X_0} then there is nothing to do as this original chain satisfies all the required condition. Hence we may suppose that some B_k meets B_{X_0}. The main idea is that then we can modify the chain of balls by adding some small balls that surround X_0. To be more precise, we let k_- and k_+ be respectively the first and last ball in the chain meeting B_{X_0}. Note that $1 \le k_- \le k_+ \le K$.

We pick $X_- \in B_{k_-} \setminus \overline{B_{X_0}}$: If $k_- = 1$ we let $X_- = X$ or if $k_- > 1$ we pick $X_- \in B_{k_- -1} \cap B_{k_-}$. Since B_{k_-} meets B_{X_0} then we can find $Y_- \in B_{k_-} \cap \partial B_{X_0}$ such that the open segment joining X_- and Y_- is contained in $B_{k_-} \setminus \overline{B_{X_0}}$. Analogously we can find $X_+ \in B_{k_+} \setminus \overline{B_{X_0}}$ and $Y_+ \in B_{k_+} \cap \partial B_{X_0}$ such that the open segment joining X_+ and Y_+ is contained in $B_{k_+} \setminus \overline{B_{X_0}}$.

Next set $r = \delta(X)/(16C_1)$ and let $N_\pm \ge 0$ be such that $N_\pm \le |X_\pm - Y_\pm|/r < N_\pm + 1$. For $j = 0, \dots, N_\pm$, let

$$B_\pm^j = B(X_\pm^j, r), \qquad \text{where} \quad X_\pm^j = X_\pm + jr\frac{Y_\pm - X_\pm}{|Y_\pm - X_\pm|}$$

Straightforward arguments show that $N_\pm \leq 32C_1^2$, $X_\pm \in B_\pm^0$, $Y_\pm \in B_\pm^{N_\pm}$, $B_\pm^j \cap B_\pm^{j+1} \neq \emptyset$ for every $0 \leq j \leq N_\pm - 1$, and

$$(32C_1^2)^{-1} \operatorname{diam}(B_\pm^j) \leq \operatorname{dist}(B_\pm^j, \partial\Omega) \leq 32C_1^2 \operatorname{diam}(B_\pm^j), \qquad B_\pm^j \cap \frac{1}{2}B_{X_0} = \emptyset,$$

for every $0 \leq j \leq N_\pm - 1$.

Next, since $X_\pm \in \partial B_{X_0}$ we can find a sequence of balls $B^0, \dots, B^N$ centered at ∂B_{X_0} and with radius $\delta(X)/16$ (hence $B^j \cap \frac{1}{2}B_{X_0} = \emptyset$) so that $N \leq 64$, $Y_- \in B^0$, $Y_+ \in B^N$, $B^j \cap B^{j+1} \neq \emptyset$ for $0 \leq j \leq N-1$ and $32^{-1} \leq \operatorname{dist}(B^j, \partial\Omega)/\operatorname{diam}(B^j) \leq 32$.

Finally, to form the desired Harnack chain we concatenate the sub-chains $\{B_1, \dots B_{k_- - 1}\}$, $\{B_-^0, \dots B_-^{N_-}\}$, $\{B^0, \dots B^N\}$, $\{B_+^N, \dots, B_+^0\}$, $\{B_{k_+ + 1}, \dots B_K\}$ and the resulting chain have all the desired properties. To complete the proof we just need to observe that the length of the chain is controlled by $K + N_- + N + N_+ + 3 \leq 100(M + C_1^2)(2 + \log_2^+ \Pi)$.

2.2 PDE estimates and properties of the elliptic measure

Theorem 2.2.1. *Ω is Wiener regular if and only if for any $q \in \partial\Omega$,*

$$\int_0^* \frac{\operatorname{cap}_2(B_r(q) \cap \Omega^c)}{r^{n-2}} \frac{dr}{r} = +\infty. \tag{2.2.2}$$

For any set K, the capacity is defined as follows:

$$\operatorname{cap_p}(K) = \inf \left\{ \int |\nabla\varphi|^p dx : \varphi \in C_c^\infty(\mathbb{R}^n), K \subset \operatorname{int}\{\varphi \geq 1\} \right\}. \tag{2.2.3}$$

The following condition has been explored extensively by Aikawa (the condition has been mentioned without name in the work of [An]). See [A1, A2, A3] for example.

Definition 2.2.4. A domain Ω is said to satisfy the *capacity density condition (CDC)* if there exist constants $C_0, R > 0$ such that

$$\operatorname{cap}_2(B_r(q) \cap \Omega^c) \geq C_0 r^{n-2}, \quad \text{for any } q \in \partial\Omega \text{ and any } r \in (0, R). \tag{2.2.5}$$

Clearly if the domain Ω satisfies the CDC, it satisfies (2.2.2), thus Ω is Wiener regular. What is relevant in our case is the following theorem:

Theorem 2.2.6. *If the domain Ω has Ahlfors regular boundary, it satisfies the CDC. In particular, Ω is Wiener regular.*

Remark 2.2.7. Our proof relies on geometric measure theory. See [HLMN, Lemma 3.27] for a potential analytical proof.

Proof. For any set E contained in a ball of radius r, its Hausdorff content $\mathcal{H}^{n-1}_\infty(E)$ satisfies

$$\mathcal{H}^{n-1}_\infty(E) \le C r^{n/2} \operatorname{cap}_2(E)^{1/2}, \tag{2.2.8}$$

where

$$\mathcal{H}^{n-1}_\infty(E) = \inf\left\{ \sum_i \left(\frac{\operatorname{diam} B_i}{2}\right)^{n-1} : E \subset \bigcup_i B_i, B_i\text{'s are balls in } \mathbb{R}^n \right\}. \tag{2.2.9}$$

For the proof see [EG, pp. 193-194].

We claim that for an Ahlfors regular boundary $\partial\Omega$,

$$\mathcal{H}^{n-1}_\infty(B_r(q)\cap\partial\Omega) \approx \mathcal{H}^{n-1}(B_r(q)\cap\partial\Omega) \approx r^{n-1}. \tag{2.2.10}$$

Let $E = B_r(q)\cap\partial\Omega$. It is clear from the definition (2.2.9) that $\mathcal{H}^{n-1}_\infty(E) \le \mathcal{H}^{n-1}(E)$. Let $\{B_i = B(x_i, r_i)\}$ be an arbitrary covering of E. We may assume $x_i \in \partial\Omega$; if not, there is some $x_i' \in B_i \cap E$ and we may replace B_i by $B_i' = B(x_i', 3r_i)$. By the countable sub-additivity of $\mathcal{H}^{n-1}$ and Ahlfors regularity of $\partial\Omega$, we have

$$\mathcal{H}^{n-1}(E) \le \sum_i \mathcal{H}^{n-1}(B_i\cap\partial\Omega) \lesssim \sum_i r_i^{n-1}.$$

This is true for any covering E, hence is true for the infimum. By the definition (2.2.9)

$$\mathcal{H}^{n-1}(E) \lesssim \mathcal{H}^{n-1}_\infty(E).$$

Therefore

$$\mathcal{H}^{n-1}_\infty(B_r(q)\cap\partial\Omega) \approx \mathcal{H}^{n-1}(B_r(q)\cap\partial\Omega) \approx r^{n-1}.$$

Combining (2.2.8) and (2.2.10), we get

$$\operatorname{cap}_2(B_r(q)\cap\partial\Omega) \gtrsim r^{-n}\left(\mathcal{H}^{n-1}_\infty(B_r(q)\cap\partial\Omega)\right)^2 \gtrsim r^{n-2}.$$

Therefore $\operatorname{cap}_2(B_r(q)\cap\Omega^c) \ge \operatorname{cap}_2(B_r(q)\cap\partial\Omega) \gtrsim r^{n-2}$, and the domain Ω satisfies CDC.

The work of Grüter and Widman [GW] shows the existence and some properties of a Green function in a bounded open domain. The Green function for unbounded domain was constructed using Perron's method in [HM1], and similar properties are shown to hold in [HMT2]. Moreover, if Ω is a uniform domain satisfying the CDC (in particular if $\partial\Omega$ is Ahlfors regular) and $L = -\operatorname{div}(\mathcal{A}\nabla)$ with $\mathcal{A}$ satisfying (E), the authors in [HMT2] describe the behavior of the Green function with respect to the elliptic measure. In particular the results proved in [JK] for harmonic functions on NTA domains extend to solutions of L on uniform domains with the CDC. We summarize below the results which will be used later in this paper.

Theorem 2.2.11. *Let L be a divergence form elliptic operator in a bounded open connected set $\Omega \subset \mathbb{R}^n$. There exist a unique non-negative function $G : \Omega \times \Omega \to \mathbb{R} \cup \{\infty\}$, the Green function associated with L, and a positive, finite constants C, depending only on dimension, λ, and Λ, such that the following hold:*

(2.2.12)
$$G(\cdot, Y) \in W^{1,2}(\Omega \setminus B(Y, s)) \cap W_0^{1,1}(\Omega) \cap W_0^{1,r}(\Omega), \quad \forall\, Y \in \Omega,\ \forall\, s > 0,\ \forall\, r \in \left[1, \tfrac{n}{n-1}\right);$$

$$\int \langle A(X)\nabla_X G(X, Y), \nabla\varphi(X)\rangle\, dX = \varphi(Y), \quad \text{for all } \varphi \in C_c^\infty(\Omega); \tag{2.2.13}$$

$$\|G(\cdot, Y)\|_{L^{\frac{n}{n-2},\infty}(\Omega)} + \|\nabla G(\cdot, Y)\|_{L^{\frac{n}{n-1},\infty}(\Omega)} \le C, \quad \forall\, Y \in \Omega; \tag{2.2.14}$$

$$G(X, Y) \le C|X - Y|^{2-n}; \tag{2.2.15}$$

and

$$G(X, Y) \ge C|X - Y|^{2-n}, \quad \text{if } |X - Y| \le \frac{7}{8}\delta(Y). \tag{2.2.16}$$

Furthermore, if Ω is a uniform domain satisfying the CDC, for any $\varphi \in C_c^\infty(\mathbb{R}^n)$ and for almost all $Y \in \Omega$

$$-\int_\Omega \langle \mathcal{A}(X)\nabla_X G(X, Y), \nabla\varphi(X)\rangle\, dX = \int_{\partial\Omega} \varphi\, d\omega_L^Y - \varphi(Y) \tag{2.2.17}$$

where $\{\omega_L^Y\}_{Y\in\Omega}$ is the associated elliptic measure.

We observe that (2.2.14) and Kolmogorov's inequality give that for every $1 \le r < \frac{n}{n-1}$

$$\|G(\cdot, Y)\|_{L^r(\Omega)} \le C C_3^{\frac{1}{r}} |\Omega|^{\frac{1}{r} - \frac{n-2}{n}}, \qquad \|\nabla G(\cdot, Y)\|_{L^r(\Omega)} \le C C_4^{\frac{1}{r}} |\Omega|^{\frac{1}{r} - \frac{n-1}{n}}, \tag{2.2.18}$$

where C is the constant in (2.2.14), $C_4 = (\frac{n}{(n-2)r})'$, and $C_4 = (\frac{n}{(n-1)r})'$.

Next we state some estimates for the boundary behavior of the solution and some estimates of the elliptic measure. They have been proven for various settings in the work of [CFMS, JK, Bo]. For their proof in the current setting, see [HMT2], see also [Zh, Section 3].

Lemma 2.2.19 (boundary regularity)**.** *Let Ω be a uniform domain satisfying the CDC. There exist constants $C, \beta > 0$ (depending on the allowable constants) such that for $q \in \partial\Omega$ and $0 < r <$ diam$(\partial\Omega)$, and $u \ge 0$ with $Lu = 0$ in $B(q, 2r) \cap \Omega$, if u vanishes continuously on $\Delta(q, 2r) = B(q, 2r) \cap \partial\Omega$, then*

$$u(X) \le C \left(\frac{|X - q|}{r}\right)^\beta \sup_{B(q,2r)\cap\Omega} u, \qquad \text{for any } X \in \Omega \cap B(q, r). \tag{2.2.20}$$

Lemma 2.2.21 (non-degeneracy)**.** *Let Ω be a uniform domain satisfying the CDC. There exists $m_0 \in (0,1)$ depending on the allowable constants such that for any $q \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$,*

$$\omega_L^{A(q,r)}(\Delta(q,r)) \geq m_0. \tag{2.2.22}$$

Here $A(q,r)$ denotes a non-tangential point for q at radius r.

Lemma 2.2.23 (boundary Harnack principle)**.** *Let Ω be a uniform domain satisfying the CDC. There exists a constant C (depending on the allowable constants) such that for $q \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$. If $u \geq 0$ with $Lu = 0$ in $\Omega \cap B(q,2r)$ and u vanishes continuously on $\Delta(q,2r)$, then*

$$u(X) \leq Cu(A(q,r)), \qquad \text{for any} X \in \Omega \cap B(q,r). \tag{2.2.24}$$

Lemma 2.2.25 (estimate of Green's function)**.** *Let Ω be a uniform domain satisfying the CDC. There exists $C > 0$ depending on the allowable constants such that for $q \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)/M$,*

$$C^{-1} \leq \frac{\omega_L^X(\Delta(q,r))}{r^{n-2}G(A(q,r),X)} \leq C, \qquad \text{for any } X \in \Omega \setminus B(q,4r). \tag{2.2.26}$$

Lemma 2.2.27 (doubling property)**.** *Let Ω be a uniform domain satisfying the CDC. There exists $C > 0$ depending on the allowable constants such that for any $q \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)/4$, if $X \in \Omega \setminus B(q,4r)$, then*

$$\omega_L^X(\Delta(q,2r)) \leq C\omega_L^X(\Delta(q,r)). \tag{2.2.28}$$

Remark 2.2.29. The following observation will be useful. If M denotes the corkscrew constant for Ω, it follows easily from the previous result, Lemma 2.2.22 and Harnack's inequality that

$$\omega_L^X(\Delta(q,2r)) \leq C_2\omega_L^X(\Delta(q,r)), \tag{2.2.30}$$

for every $q \in \partial\Omega$, $0 < r < \operatorname{diam}(\partial\Omega)$ and for all $X \in \Omega$ with $\delta(X) \geq r/(2M)$. Here C_2 is a constant that depends on the allowable parameters associated with Ω and the ellipticity constants of L.

Lemma 2.2.31 (Boundary comparison principle)**.** *Let u and v be non-negative solutions in $B(q,4s) \cap \Omega$ with vanishing boundary data on $\Delta(q,4s)$. Then*

$$\frac{u(X)}{v(X)} \approx \frac{u(A(q,s))}{v(A(q,s))} \quad \text{for any } X \in B(q,s) \cap \Omega. \tag{2.2.32}$$

2.3 Convergence of measures, sets and matrices

This section is in preparation for the blow-up argument and compactness argument we need in Chapters 4 and 5.

Definition 2.3.1. For any non-empty closed sets E, F in $\mathbb{R}^n$, we define their Hausdorff distance by

$$D[E, F] := \max\left\{\sup_{x\in E}\inf_{y\in F}|x-y|,\ \sup_{y\in F}\inf_{x\in E}|x-y|\right\}.$$

The following classical lemma states the compactness of non-empty closed sets in $\mathbb{R}^n$ under Hausdorff distance. Its proof can be found in [Ro, pp. 91].

Lemma 2.3.2 (Blaschke's selection theorem)**.** *Let $K \subset \mathbb{R}^n$ be a compact set. If $\{E_k\}_k$ is a sequence of non-empty closed subsets of K, then there exists a non-empty closed sets $E \subset K$ and a subsequence $\{E_{k_j}\}_j$, such that*

$$D[E_{k_j}, E] \to 0 \quad \text{as } j \to \infty.$$

We say E_{k_j} converges to E in the Hausdorff distance sense, and write $E_{k_j} \to E$.

Given a Radon measure μ on $\mathcal{R}^n$ (i.e., a non-negative Borel such that the measure of any compact set is finite) we define

$$\operatorname{spt}\mu = \overline{\left\{x \in \mathbb{R}^n : \mu(B(x,r)) > 0 \text{ for any } r > 0\right\}}.$$

Definition 2.3.3. We say that a Radon measure μ on $\mathcal{R}^n$ is Ahlfors regular with constant $C \geq 1$, if there exits a constant $C \geq 1$ such that for any $x \in E$ and $0 < r < \operatorname{diam}(E)$,

$$C^{-1}\, r^{n-1} \leq \mu(B(q,r)) \leq C\, r^{n-1}, \qquad \forall\, x \in \operatorname{spt}\mu,\ 0 < r < \operatorname{diam}(\operatorname{spt}\mu).$$

Definition 2.3.4. Let $\{\mu_j\}$ be a sequence of Radon measures on $\mathbb{R}^n$. We say μ_j converge weakly to a Radon measure μ_∞ and write $\mu_j \rightharpoonup \mu_\infty$, if

$$\int f d\mu_j \to \int f d\mu_\infty$$

for any $f \in C_c(\mathcal{R}^n)$.

We finish this section by stating a compactness type lemma for Radon measures which are uniformly doubling and "bounded below".

Lemma 2.3.5. *Let $\{\mu_j\}_j$ be a sequence of Radon measures. Let $A_1, A_2 > 0$ be fixed constants, and assume the following conditions:*

(*i*) $0 \in \operatorname{spt}\mu_j$ *and* $\mu_j(B(0,1)) \geq A_1$ *for all* j, (*ii*) *For all* $j \in \mathbb{N}$, $q \in \operatorname{spt}\mu_j$ *and* $r > 0$,

$$\mu_j(B(q,2r)) \le A_2 \mu_j(B(q,r)) \tag{2.3.6}$$

If there exists a Radon measure μ_∞ *such that* $\mu_j \rightharpoonup \mu_\infty$, *then* μ_∞ *is doubling and*

$$\operatorname{spt}\mu_j \to \operatorname{spt}\mu_\infty, \tag{2.3.7}$$

in the Hausdorff distance sense uniformly on compact sets.

Proof. Since $0 \in \operatorname{spt}\mu_j$ for all j, given any subsequence of μ_j there exists a further subsequence μ_{j_k} and a closed set Σ_∞ such that $\operatorname{spt}\mu_{j_k} \to \Sigma_\infty$ in the Hausdorff distance sense uniformly on compact sets. For $x \in \Sigma_\infty$ there exist $x_{j_k} \in \operatorname{spt}\mu_{j_k} \cap \overline{B(x,1)}$ such that $x_{j_k} \to x$. If $x \notin \operatorname{spt}\mu_\infty$ there is $r \in (0,1)$ such that $B(x,r) \cap \operatorname{spt}\mu_\infty = \emptyset$. Let $\varphi \in C_c^\infty(\mathcal{R}^n)$ be such that $\varphi \equiv 1$ on $B(x,r/2)$, and $\operatorname{spt}\varphi \subset B(x,r)$. For k large enough, we also have $\varphi \equiv 1$ on $B(x_{j_k}, r/4)$. Hence

$$\mu_{j_k}(B(x_{j_k}, r/4)) \le \int \varphi d\mu_{j_k} \to \int \varphi d\mu_\infty = 0. \tag{2.3.8}$$

Since $\{x_{j_k}\}$ is a bounded sequence in $\overline{B(x,1)}$, there is $l \in \mathbb{N}$ such that $|x_{j_k}| < 2^l$ for all j_k. Then $B(0,1) \subset B(x_{j_k}, 2^{l+1})$. Let $m \in \mathbb{Z}$ be such that $2^{-m} \le r < 2^{-m+1}$, then we have

$$\begin{aligned}\mu_{j_k}(B(x_{j_k}, r/4)) &\ge \mu_{j_k}(B(x_{j_k}, 2^{-m-2}) \ge A_2^{-(m+l+3)} \mu_{j_k}(B(x_{j_k}, 2^{l+1}) \\ &\ge A_2^{-(m+l+3)} \mu_{j_k}(B(0,1)) \ge A_1 A_2^{-(m+l+3)},\end{aligned} \tag{2.3.9}$$

which contradicts (2.3.8). Thus $\Sigma_\infty \subset \operatorname{spt}\mu_\infty$ and we have shown that any subsequential limit of $\operatorname{spt}\mu_j$ is included in $\operatorname{spt}\mu_\infty$.

On the other hand, if $y \in \operatorname{spt}\mu_\infty$, $r > 0$ and $\{j_k\}$ is the subsequence above we have

$$0 < \mu_\infty(B(y,r)) \le \liminf_{j_k \to \infty} \mu_{j_k}(B(y,r)). \tag{2.3.10}$$

Using this with $r = 1$, there exists $\tilde{j}_1$ such that if $j_k \ge \tilde{j}_1$ then $\mu_{j_k}(B(y,1)) > 0$. In particular we can pick $y_1 \in B(y,1) \cap \operatorname{spt}\mu_{\tilde{j}_1}$. Iteration guarantees that for each $k \in \mathbb{N}$ there exist $\tilde{j}_k > \tilde{j}_{k-1}$ and $y_k \in B(y, 2^{-k}) \cap \operatorname{spt}\mu_{\tilde{j}_k}$. This implies that $y_k \to y$ as $k \to \infty$ and since $\operatorname{spt}\mu_{j_k} \to \Sigma_\infty$ then $y \in \Sigma_\infty$. We have then obtained that $\operatorname{spt}\mu_\infty \subset \Sigma_\infty$ and therefore $\Sigma_\infty = \operatorname{spt}\mu_\infty$ which shows (2.3.7).

To show that μ_∞ is doubling let $x \in \operatorname{spt}\mu_\infty$ and $r > 0$. There exist $x_j \in \operatorname{spt}\mu_j$ such that $x_j \to x$. Thus for j large enough $|x_j - x| < r/4$. Since $\mu_j \rightharpoonup \mu_\infty$ we have

$$\begin{aligned}\mu_\infty(B(x,2r)) &\le \liminf_j \mu_j(B(x,2r)) \le \liminf_j \mu_j(B(x_j,3r)) \\ &\le A_2^3 \liminf_j \mu_j\left(B\left(x_j, \frac{3}{8}r\right)\right) \le A_2^3 \mu_\infty(B(x,r)).\end{aligned} \tag{2.3.11}$$

This completes the proof.

In the next lemma we describe the properties of $W^{1,1}(\Omega) \cap L^\infty(\Omega)$ which are crucial to our arguments. Recall that uniform domains are (ϵ, δ) domains in the language of Jones, see [Jo]. Thus they are extension domains and in particular, if $A \in W^{1,1}(\Omega)$ there exists $\bar{A} \in W^{1,1}(\mathbb{R}^n)$ such that $\bar{A}|_\Omega = A$ and $\|\bar{A}\|_{W^{1,1}(\mathbb{R}^n)} \leq C\|A\|_{W^{1,1}(\Omega)}$, where C depends on n and the constants describing the uniform character of Ω.

Lemma 2.3.12. *Let Ω be a uniform domain with Ahlfors regular boundary. Let $A \in W^{1,1}(\Omega)$. Then for $\mathcal{H}^{n-1}$ a.e. $q \in \partial\Omega$ there exists a symmetric constant coefficient elliptic matrix $A^*(q)$ (with constants depending on the allowable constants) such that*

$$\lim_{r\to 0} \left(\fint_{B(q,r)\cap\Omega} |A - A^*(q)|^{\frac{n}{n-1}} dX \right)^{\frac{n-1}{n}} = 0. \tag{2.3.13}$$

Proof. By the previous remark, there exists $\bar{A} \in W^{1,1}(\mathbb{R}^n)$ such that $\bar{A}|_\Omega = A$ and $\|\bar{A}\|_{W^{1,1}(\mathbb{R}^n)} \leq C\|A\|_{W^{1,1}(\Omega)}$. By Theorem 1 section 4.8 in [EG] we have that if $\bar{A} \in W^{1,1}(\mathbb{R}^n)$, then there exists a Borel set $E \subset \mathbb{R}^n$ such that $\text{cap}_1(E) = 0$ (recall the definition in (2.2.3) with $p = 1$) and

$$\lim_{r\to 0} \fint_{B(x,r)} \bar{A} = A^*(x)$$

exists for all $x \in \mathbb{R}^n \setminus E$. In addition

$$\lim_{r\to 0} \left(\fint_{B(x,r)} |\bar{A} - A^*(x)|^{\frac{n}{n-1}} dy \right)^{\frac{n-1}{n}} = 0, \quad \text{for all } x \in \mathbb{R}^n \setminus E. \tag{2.3.14}$$

Note that by Theorem 3 in section 5.6 in [EG] since $\text{cap}_1(E) = 0$ then $\mathcal{H}^{n-1}(E) = 0$. Hence (2.3.14) holds for $\mathcal{H}^{n-1}$ a.e. $q \in \partial\Omega$. Since for every $q \in \partial\Omega$ and $0 < r < \operatorname{diam}\Omega$ there exists $A(q,r) \in \Omega$ such that $B(A(q,r), r/M) \subset \Omega \cap B(q,r)$ (see (2.1.2)), we have

$$c_n \left(\frac{r}{M} \right)^n \leq |\Omega \cap B(q,r)| \leq c_n r^n,$$

where c_n denotes the volume of a unit ball in $\mathbb{R}^n$. Thus for $q \in \partial\Omega \setminus E$

$$\left(\fint_{B(q,r)\cap\Omega} |A - A^*(q)|^{\frac{n}{n-1}} dX \right)^{\frac{n-1}{n}} \leq C_{n,M} \left(\fint_{B(q,r)} |\bar{A} - A^*(q)|^{\frac{n}{n-1}} dX \right)^{\frac{n-1}{n}}$$

because $\bar{A}|_\Omega = A$. Combined with (2.3.14) we get

$$\lim_{r\to 0} \left(\fint_{B(q,r)\cap\Omega} |A - A^*(q)|^{\frac{n}{n-1}} dX \right)^{\frac{n-1}{n}} = 0,$$

and moreover, Hölder inequality gives

$$\lim_{r\to 0} \left| \fint_{B(q,r)\cap\Omega} A dX - A^*(q) \right| \leq \lim_{r\to 0} \fint_{B(q,r)\cap\Omega} |A - A^*(q)| dX$$

$$\leq \lim_{r\to 0}\left(\fint_{B(q,r)\cap\Omega}|A-A^*(q)|^{\frac{n}{n-1}}dX\right)^{\frac{n-1}{n}}=0$$

for $q \notin E$. Since (E) holds for any $\xi \in \mathbb{R}^n \setminus \{0\}$, we have

$$\lambda|\xi|^2 \leq \left\langle \left(\fint_{B(q,r)\cap\Omega} A\right)\xi, \xi\right\rangle \leq \fint_{B(q,r)\cap\Omega}\langle A\xi,\xi\rangle \leq \Lambda|\xi|^2.$$

Letting r tend to 0 we conclude that

$$\lambda|\xi|^2 \leq \langle A^*(q)\xi,\xi\rangle \leq \Lambda|\xi|^2. \tag{2.3.15}$$

Since A is symmetric, so is $\fint_{B(q,r)\cap\Omega} A$ for every q and $r > 0$. Moreover for $A \in L^\infty(\Omega)$, $\fint_{B(q,r)\cap\Omega} A$ is uniformly bounded in q and r. Thus $A^*(q)$ is a real uniformly elliptic symmetric matrix and (2.3.13) holds for $\mathcal{H}^{n-1}$ a.e. $q \in \partial\Omega$.

2.4 Construction and properties of the harmonic measure in higher co-dimensions

Let Γ be a d-Ahlfors regular set in $\mathbb{R}^n$ with $d < n-1$, and $\Omega = \mathbb{R}^n \setminus \Gamma$. Consider the degenerate elliptic operator $L = -\operatorname{div}(A(X)\nabla)$ with a real, symmetric $n \times n$ matrix $A(X)$ satisfying

$$A(X)\xi\cdot\zeta \leq C_1|\xi||\zeta|\delta(X)^{d-n+1} \text{ for } X \in \Omega \text{ and } \xi,\zeta \in \mathbb{R}^n, \tag{2.4.1}$$

$$A(X)\xi\cdot\xi \geq C_1^{-1}|\xi|^2\delta(X)^{d-n+1} \text{ for } X \in \Omega \text{ and } \xi \in \mathbb{R}^n \tag{2.4.2}$$

for some $C_1 \geq 1$.

The ground work for the elliptic theory and estimates of harmonic measures associated to L has been laid out in the work of David, Feneuil and Mayboroda, see [DFM1]. In this section we state some relevant preliminary results that were proved there and prove some corollaries that will be used later in Chapter 6. The reader will see that quite a few PDE estimates are similar to the estimates in co-dimension one from Section 2.2, such as boundary regularity, boundary Harnack, doubling property of the harmonic measure, and the estimate of the harmonic measure by the Green's function etc..

Unless specified otherwise, the constants that appear in the following lemmas would depend only on the allowable constants, namely the dimensions n, d, the Ahlfors regular constant C_0 and the ellipticity constant C_1.

We start with the following notations:

- For any $X \in \Omega$, we denote $\delta(X) = \operatorname{dist}(X,\Gamma)$, the Euclidean distance from X to Γ, and the weight $w(X) = \delta(X)^{d-n+1}$.

- We denote

$$\mathcal{A}(X) := \frac{1}{w(X)} A(X) = \delta(X)^{n-1-d} A(X).$$

By (2.4.1) and (2.4.2), $\mathcal{A}(X)$ is a uniformly elliptic matrix.

- We define a measure m on Borel sets in $\mathbb{R}^n$ by letting $m(E) = \iint_E w(X) dm(X)$. We may write $dm(X) = w(X)dX$. Since $0 < w < \infty$ a.e. in $\mathbb{R}^n$, m and the Lebesgue measure are mutually absolutely continuous.

- For any $q \in \Gamma$ and $r > 0$, we use the notation $\Delta(q, r)$, or sometimes simply Δ, to denote the surface ball $B(q, r) \cap \partial\Omega$, and $T(\Delta)$ to denote the "tent" $B(q, r) \cap \Omega$ over Δ.

- We denote the surface measure $\sigma = \mathcal{H}^d|_{\partial\Omega}$.

- If $B = B(X, r)$ is a ball and $\alpha > 0$ a constant, we use $\alpha B = B(X, \alpha r)$ to denote the concentric dilation of B. The same notation applies to surface balls $\alpha\Delta$.

Lemma 2.4.3 (Harnack chain condition, Lemma 2.1 of [DFM1])**.** *Let Γ be a d-Ahlfors regular set in $\mathbb{R}^n$ and $d < n - 1$. Then there exists a constant $c \in (0, 1)$, that depends only on d, n, C_0, such that for $\Lambda \geq 1$ and $X_1, X_2 \in \Omega$ such that $\delta(X_i) \geq s$ and $|X_1 - X_2| \leq \Lambda s$, we can find two points $Y_i \in B(X_i, s/2)$ such that $\operatorname{dist}([Y_1, Y_2], \Gamma) \geq c\Lambda^{-d/(n-1-d)} s$. That is, there is a thick tube in Ω that connects the balls $B(X_i, s/2)$.*

Remark 2.4.4. Note that

$$|Y_1 - Y_2| \leq |Y_1 - X_1| + |X_1 - X_2| + |X_2 + Y_2| < 2\Lambda s. \tag{2.4.5}$$

Let $\tau = c\Lambda^{-d/(n-1-d)} s$ and $Z_1 = Y_1$. For $2 \leq j \leq N$ let Z_j be consecutive points on the line segment $[Y_1, Y_2]$ such that $|Z_j - Z_{j-1}| = \tau/3$. Then

$$(N - 1)\frac{\tau}{3} \leq |Y_1 - Y_2| < N\frac{\tau}{3}.$$

Combined with (2.4.5) we get that the integer

$$N \sim \frac{|Y_1 - Y_2|}{\tau/3} \lesssim \Lambda^{\frac{n-1}{n-1-d}}. \tag{2.4.6}$$

Let $B_0 = B(X_1, s/2), B_j = B(Z_j, \tau/4)$ for $1 \leq j \leq N$ and $B_{N+1} = B(X_2, s/2)$. Clearly $B_j \cap B_{j+1} \neq \emptyset$ for all $0 \leq j \leq N$. Moreover $\operatorname{dist}(B_0, \Gamma), \operatorname{dist}(B_{N+1}, \Gamma) \geq s/2$ and for $1 \leq j \leq N$,

$$\operatorname{dist}(B_j, \Gamma) \geq \frac{3}{4}\tau = \frac{3}{4} c\Lambda^{-\frac{d}{n-1-d}} s, \tag{2.4.7}$$

and

$$\operatorname{dist}(B_j, \Gamma) \leq \min\{\delta(X_1), \delta(X_2)\} + \frac{s}{2} + |Y_1 - Y_2| < \min\{\delta(X_1), \delta(X_2)\} + 3\Lambda s. \tag{2.4.8}$$

Lemma 2.4.9 (estimates on the weight, Lemma 2.3 of [DFM1]).

(i) For any $\theta > 0$ *there exists* $C_\theta > 0$ *such that for any* $X \in \mathbb{R}^n$ *and* $r > 0$ *satisfying* $\delta(X) \geq (1+\theta)r$*,*

$$C_\theta^{-1} r^n w(X) \leq m(B(X,r)) = \iint_{B(X,r)} w(z)dz \leq Cr^n w(X). \tag{2.4.10}$$

(ii) There exists $C > 0$ *such that for any* $q \in \Gamma$ *and* $r > 0$*,*

$$C^{-1} r^{d+1} \leq m(B(q,r)) = \iint_{B(q,r)\cap\Omega} w(z)dz \leq Cr^{d+1}. \tag{2.4.11}$$

From the above we deduce the following estimate, which will be needed later.

Lemma 2.4.12. *Let* Γ *be d-Ahlfors regular. For any* $\alpha > -1$*, we have*

$$\iint_{T(2\Delta)} \delta(X)^\alpha dm(X) \lesssim r^{d+1+\alpha}. \tag{2.4.13}$$

Proof. The proof is a simple use of Vitali covering. For $j = 0, 1, \cdots$ let

$$T_j = T(2\Delta) \cap \{x \in \Omega : 2^{-j}r \leq \delta(X) < 2^{-j+1}r\},$$

$$T_{>j} = T(2\Delta) \cap \{x \in \Omega : \delta(X) < 2^{-j+1}r\}.$$

Then

$$\iint_{T(2\Delta)} \delta(X)^\alpha dm(X) = \sum_{j=0}^{\infty} \iint_{T_j} \delta(X)^\alpha dm(X) \leq \sum_{j=0}^{\infty} (2^{-j}r)^\alpha m(T_{>j}). \tag{2.4.14}$$

For every fixed j, we consider a covering of 4Δ by $\bigcup_{q\in 4\Delta} B(q, 2^{-j+1}r/5)$, from which one can extract a countable Vitali sub-covering $4\Delta \subset \cup_k B(q_k, 2^{-j+1}r)$, where $q_k \in 4\Delta$ and the balls $B_k = B(q_k, 2^{-j+1}r/5)$ are pairwise disjoint. The fact that $q_k \in 4\Delta = \Delta(q_0, 4r)$ implies

$$B_k := B\left(q_k, \frac{2^{-j+1}r}{5}\right) \subset B\left(q_0, 4r + \frac{2^{-j+1}r}{5}\right).$$

And the pairwise disjointness of B_k's implies that for every fixed j, there are only finitely many of them. In fact,

$$\sum_k \sigma(B_k) = \sigma\left(\bigcup_k B_k\right) \leq \sigma\left(\Delta\left(q_0, 4r + \frac{2^{-j+1}r}{5}\right)\right) \lesssim \left(4r + \frac{2r}{5}\right)^d. \tag{2.4.15}$$

Note that $\sigma(B_k) \approx \left(2^{-j+1}r/5\right)^d$ independent of k. Let N_j be the number of B_k's, by (3.1.23)

$$N_j \cdot \left(\frac{2^{-j+1}r}{5}\right)^d \leq \left(4r + \frac{2r}{5}\right)^d, \qquad \text{thus } N_j \lesssim 2^{jd}. \tag{2.4.16}$$

For any $X \in T_{>j}$, let $q_X \in \partial\Omega$ be such that $|X - q_X| = \delta(X)$. Then

$$|q_X - q_0| \leq |q_X - X| + |X - q_0| < 4r, \qquad \text{i.e. } q_X \in 4\Delta. \tag{2.4.17}$$

Hence $q_X \in B(q_k, 2^{-j+1}r)$ for some k. Moreover $T_{>j} \subset \bigcup_k B(q_k, 2 \cdot 2^{-j+1}r)$. Therefore by (3.1.24) and (2.4.11),

$$m(T_{>j}) \leq N_j \cdot \sup_k m\left(B(q_k, 2 \cdot 2^{-j+1}r)\right) \lesssim 2^{jd}\left(2^{-j}r\right)^{d+1} \sim 2^{-j}r^{d+1}.$$

Combined with (3.1.22) we get

$$\iint_{T(2\Delta)} \delta(X)^\alpha dm(X) \lesssim \sum_{j=0}^\infty (2^{-j}r)^\alpha \cdot 2^{-j}r^{d+1} = r^{d+1+\alpha}\sum_{j=0}^\infty 2^{-j(\alpha+1)} \lesssim r^{d+1+\alpha}.$$

The last sum is convergent because $\alpha + 1 > 0$.

Now we define the suitable function spaces. We denote by $C_0^0(\Gamma)$ the space of compactly supported continuous functions on Γ, that is, $f \in C_0^0(\Gamma)$ if f is defined and continuous on Γ, and there exists a surface ball Δ such that supp $f \subset \Delta$. We consider the weighted Sobolev space

$$W = \dot{W}_w^{1,2}(\Omega) = \{u \in L^1_{loc}(\Omega) : \nabla u \in L^2(\Omega, dm)\} \tag{2.4.18}$$

and set $\|u\|_W = \left(\iint_\Omega |\nabla u(X)|^2 dm(X)\right)^{\frac{1}{2}}$ for $u \in W$. In fact, it was proved in Lemma 3.3 of [DFM1] that since Γ is d-Ahlfors regular with $d < n-1$,

$$W = \{u \in L^1_{loc}(\mathbb{R}^n) : \nabla u \in L^2(\mathbb{R}^n, dm)\}. \tag{2.4.19}$$

We also define a local version of W as follows: Let $E \subset \mathbb{R}^n$ be an open set, define

$$W_r(E) = \{u \in L^1_{loc}(E) : \varphi u \in W \text{ for all } \varphi \in C_0^\infty(E)\}. \tag{2.4.20}$$

As observed in [DFM1],

$$W_r(E) = \{u \in L^1_{loc}(E) : \nabla u \in L^2_{loc}(E, dm)\}. \tag{2.4.21}$$

It is easy to see that if $E \subset F$ are open subsets of $\mathbb{R}^n$, then the function space $W_r(F) \subset W_r(E)$. We set

(2.4.22)
$$H = \dot{H}^{\frac{1}{2}}(\Gamma) = \left\{g \text{ a measurable function on } \Gamma : \int_\Gamma\int_\Gamma \frac{|g(x)-g(y)|^2}{|x-y|^{d+1}} d\sigma(x)d\sigma(y) < \infty\right\}.$$

The reader may recognize this is the homogeneous Sobolev space, a special case of the Besov spaces. The authors in [DFM1] were able to define a trace operator $T : W \to H$, see Theorem 3.13 (and Lemma 8.3 for a local version $T : W_r(E) \to L^1_{loc}(\Gamma \cap E)$) there.

Lemma 2.4.23 (interior Caccioppoli inequality, Lemma 8.26 of [DFM1]). *Let $E \subset \Omega$ be an open set, and let $u \in W_r(E)$ be a non-negative solution in E. Then for any $\phi \in C_0^\infty(E)$,*

$$\iint_\Omega \phi^2 |\nabla u|^2 dm \leq C \iint_\Omega |\nabla \phi|^2 u^2 dm, \tag{2.4.24}$$

where C depends only on n, d and C_1.

In particular, if B is a ball of radius r such that $2B \subset \Omega$ and $u \in W_r(2B)$ is a non-negative sub-solution in $2B$, then

$$\iint_B |\nabla u|^2 dm \leq C r^{-2} \iint_{2B} u^2 dm. \tag{2.4.25}$$

Lemma 2.4.26 (Harnack inequality, Lemmas 8.42 and 8.44 of [DFM1]).

1. *Let B be a ball such that $3B \subset \Omega$ and let $u \in W_r(3B)$ be a non-negative solution in $3B$. Then*

$$\sup_B u \leq C \inf_B u, \tag{2.4.27}$$

where C depends on n, d and C_1.

2. *Let K be a compact set of Ω and $u \in W_r(\Omega)$ be a non-negative solution in Ω. Then*

$$\sup_K u \leq C_K \inf_K u, \tag{2.4.28}$$

where C_K depends only on n, d, C_0, C_1, $\operatorname{dist}(K, \Gamma)$ *and* $\operatorname{diam} K$.

Lemma 2.4.29 (boundary Caccioppoli inequality, Lemma 8.47 of [DFM1]). *Let $B \subset \mathbb{R}^n$ be a ball centered on Γ of radius r, and let $u \in W_r(2B)$ be a non-negative subsolution in $2B \setminus \Gamma$ such that $Tu = 0$ a.e. on $2B$. Then for any $\phi \in C_0^\infty(2B)$,*

$$\iint_{2B} \phi^2 |\nabla u|^2 dm \leq C \iint_{2B} |\nabla \phi|^2 u^2 dm, \tag{2.4.30}$$

where C depends on n, d and C_1. In particular (2.4.30) *implies that*

$$\iint_B |\nabla u|^2 dm \leq C r^{-2} \iint_{2B} u^2 dm. \tag{2.4.31}$$

Lemma 2.4.32 (boundary Moser estimate, Lemma 8.71 of [DFM1]). *Let $p > 0$. Let B be a ball centered on Γ and $u \in W_r(2B)$ be a non-negative sub-solution in $2B \setminus \Gamma$ such that $Tu = 0$ a.e. on $2B$. Then*

$$\sup_B u \leq C_p \left(\frac{1}{m(2B)} \iint_{2B} u^p dm \right)^{\frac{1}{p}}. \tag{2.4.33}$$

Lemma 2.4.34 (boundary Hölder regularity, Lemma 8.106 of [DFM1]). *Let $B = B(q,r)$ be a ball centered on Γ and $u \in W_r(B)$ be a solution in B such that $Tu \equiv 0$ on B. There exists $\beta \in (0,1]$ such that for any $0 < s < r/2$,*

$$\operatorname*{osc}_{B(q,s)} u \leq C \left(\frac{s}{r}\right)^{\beta} \left(\frac{1}{m(B)} \iint_B |u|^2 dm\right)^{\frac{1}{2}}. \tag{2.4.35}$$

We are interested in the solution(s) of the Dirichlet problem (D).

Lemma 2.4.36 (existence and uniqueness of solution, Lemma 9.3 of [DFM1]). *For any $f \in H$, there exists a unique $u \in W$ such that*

$$\begin{cases} Lu = 0 & \text{in } \Omega \\ Tu = f & \text{a.e. on } \Gamma. \end{cases} \tag{2.4.37}$$

Moreover $\|u\|_W \leq C\|f\|_H$.

Lemma 2.4.38 (properties of solutions for $f \in C_0^0(\Gamma)$, Lemma 9.23 of [DFM1]). *There exists a bounded linear operator*

$$U : C_0^0(\Gamma) \to C(\mathbb{R}^n)$$

such that for every $f \in C_0^0(\Gamma)$

(i) the restriction of Uf to Γ is f;

(ii) $\sup_{\mathbb{R}^n} Uf = \sup_\Gamma f$ and $\int_{\mathbb{R}^n} Uf = \inf_\Gamma f$;

(iii) $Uf \in W_r(\Omega)$ and is a solution of L in Ω;

(iv) if B is a ball centered on Γ and $f \equiv 0$ on B, then Uf lies in $W_r(B)$;

(v) if $f \in C_0^0(\Gamma) \cap H$, then $Uf \in W$ and is a unique solution of (2.4.37).

Remark 2.4.39. Since $Uf \in C(\mathbb{R}^n)$, its trace $T(Uf)$ is exactly f. We also remark that $C_0^0(\Gamma) \cap H$ is dense in $C_0^0(\Gamma)$, with the supremum norm.

Lemma 2.4.40 (harmonic measure, Lemmas 9.30 and 9.33 of [DFM1]). *For any $X \in \Omega$, there exists a unique positive regular Borel measure ω^X on Γ such that*

$$Uf(X) = \int_\Gamma f d\omega^X, \quad \text{for any } f \in C_0^0(\Gamma). \tag{2.4.41}$$

Besides, for any Borel set $E \subset \Gamma$,

$$\omega^X(E) = \sup\{\omega^X(K) : E \supset K, K \text{ is compact }\} = \inf\{\omega^X(V) : E \subset V, V \text{ is open }\}. \tag{2.4.42}$$

Moreover, $\omega^X(\Gamma) = 1$.

Lemma 2.4.43 (Lemma 9.38 of [DFM1]). *Let $E \subset \Gamma$ be a Borel set and define the function u_E on Ω by $u_E(X) = \omega^X(E)$. Then*

(i) if there exists $X \in \Omega$ such that $u_E(X) = 0$, then $u_E \equiv 0$;

(ii) the function u_E lies in $W_r(\Omega)$ and is a solution in Ω;

(iii) if $B \subset \mathbb{R}^n$ is a ball such that $E \cap B = \emptyset$, then $u_E \in W_r(B)$ and $Tu_E = 0$ on $B \cap \Gamma$.

For now we are only able to write down the solution to (D) if the boundary function $f \in C_0^0(\Gamma)$, see Lemma 2.4.38. With the help of the harmonic measure, we prove the following lemma:

Lemma 2.4.44. *For any function $f \in C_0^0(\Gamma)$ and any Borel set $E \subset \Gamma$, the function*

$$u(X) := \int_E f d\omega^X \tag{2.4.45}$$

defined on Ω satisfies the following:

1. *it is continuous in Ω;*
2. *it is a solution of $Lu = 0$ in Ω and lies in $W_r(\Omega)$;*
3. *if $B \subset \mathbb{R}^n$ is an open ball such that $E \cap B = \emptyset$, then u is continuous in $B \cap \Omega$, u can be continuously extended to zero on $B \cap \Gamma$, and that $u \in W_r(B)$.*

Remark 2.4.46. We note the following:

- Compared with Lemma 2.4.40 and Lemma 2.4.38, this lemma says that $f\chi_E$ integrated against the harmonic measure gives rise to a continuous solution, for any Borel set $E \subset \Gamma$.
- If the Borel set E is bounded, then the same properties hold for any bounded continuous function $f \in C_b(\Gamma)$.

Proof. Since the definition (2.4.45) is a linear integration, we may assume without loss of generality that f is non-negative. Otherwise we just write $f = f_+ - f_-$, with $f_\pm \in C(\mathbb{R}^n)$. We first assume that E is an open set, and that $\omega^X(E) > 0$ for some $X \in \Omega$. By Lemma 2.4.43 (i) it follows that $\omega^X(E) > 0$ for all $X \in \Omega$. Fix an arbitrary $X_0 \in \Omega$. Let K_j be an increasing sequence of compact sets in E, such that $\omega^{X_0}(E \setminus K_j) < 1/j$. By Urysohn's lemma we can construct $g_j \in C_0^0(\Gamma)$ such that $\chi_{K_j} \leq g_j \leq \chi_E$, and without loss of generality we can choose the sequence g_j to be increasing. Note that $fg_j \in C_0^0(\Gamma)$, and hence by Lemma 2.4.38 we may define $u_j = U(fg_j) \in C^0(\Gamma)$. Then

$$0 \leq u(X) - u_j(X) = \int f\left(\chi_E - g_j\right) d\omega^X \leq \omega^X(E \setminus K_j)\|f\|_{L^\infty}.$$

By Lemmas 2.4.43 and 2.4.26, for any compact subset K in Ω containing X_0, we have

$$\omega^X(E \setminus K_j) \leq C_K \omega^{X_0}(E \setminus K_j)$$

holds for every $X \in K$. Here the constant C_K only depends on $n, d, C_1, \mathrm{dist}(K, \Gamma)$ and $\mathrm{diam}\, K$, and in particular it is independent of j. Therefore

$$0 \leq u(X) - u_j(X) \leq \frac{C_K \|f\|_{L^\infty}}{j},$$

namely $\{u_j\}$ converges uniformly on compact sets of Ω to u, and thus u is continuous on Ω.

Let $\phi \in C_0^\infty(\Omega)$ be arbitrary, we claim that $\{u_j\}$ has a subsequence, which we relabel, such that

$$\nabla(\phi u_j) \rightharpoonup \nabla(\phi u) \text{ in } L^2(\Omega, w). \tag{2.4.47}$$

In particular $\nabla(\phi u) \in L^2(\Omega, w)$ for all $\phi \in C_0^\infty(\Omega)$, and thus $u \in W_r(\Omega)$. Indeed, by the interior Caccioppoli inequality (2.4.24), we have

$$\iint_\Omega |\nabla(\phi u_j)|^2 dm \leq 2 \iint_\Omega \left(|\nabla \phi|^2 u_j^2 + \phi^2 |\nabla u_j|^2 \right) dm \leq C \iint_\Omega |\nabla \phi|^2 u_j^2 dm. \tag{2.4.48}$$

Recall that $u_j \to u$ uniformly on the compact set $\mathrm{supp}\, \phi$, the right hand side of (2.4.48) converges to $C \iint_\Omega |\nabla \phi|^2 u^2 dm$. As a consequence the left hand side of (2.4.48) is uniformly bounded in j. Therefore there is a subsequence (which we relabel) such that $\nabla(\phi u_j)$ converges weakly in $L^2(\Omega, w)$ to some function v. By the uniqueness of limit in the distributional sense, we conclude that $v = \nabla(\phi u)$, which finishes the proof of the claim (2.4.47).

Recall each u_j is a solution of L in Ω. Let $\varphi \in C_0^\infty(\Omega)$ be an arbitrary test function. We choose $\phi \in C_0^\infty(\Omega)$ such that $\phi \equiv 1$ on $\mathrm{supp}\, \varphi$. In particular $\nabla(\phi u) = \nabla u$, $\nabla(\phi u_j) = \nabla u_j$ on $\mathrm{supp}\, \varphi$. Thus

$$\begin{aligned}
\iint_\Omega A \nabla u \cdot \nabla \varphi dX = \iint_\Omega \mathcal{A} \nabla u \cdot \nabla \varphi dm &= \iint_\Omega \mathcal{A} \nabla(\phi u) \cdot \nabla \varphi dm \\
&= \lim_{j \to \infty} \iint_\Omega \mathcal{A} \nabla(\phi u_j) \cdot \nabla \varphi dm \\
&= \lim_{j \to \infty} \iint_\Omega \mathcal{A} \nabla u_j \cdot \nabla \varphi dm = \lim_{j \to \infty} \iint_\Omega A \nabla u_j \cdot \nabla \varphi dX = 0.
\end{aligned} \tag{2.4.49}$$

If E is not an open set, the proof is similar, and we just need to approximate E from above by open sets. We omit the details here.

Going further, if $B \subset \mathbb{R}^n$ is an open ball such that $E \cap B = \emptyset$, we first prove that u can be continuously extended to zero on $\Gamma \cap B$. Take an arbitrary $q \in \Gamma \cap B$. Choose $r > 0$ sufficiently small so that $B(q, 2r) \subset B$. Consider a function $g \in C_0^\infty(\mathbb{R}^n)$ satisfying $\chi_{B(q,r)} \leq g \leq \chi_{B(q,2r)}$. If $f \in C_0^0(\Gamma)$, then $f(1-g) \in C_0^0(\Gamma)$. If the Borel set E is bounded and f is only assumed to be bounded continuous, we let $\varphi \in C_0^\infty(\mathbb{R}^n)$ be a function such that $\varphi \equiv 1$ on a compact set containing E and $B(q, 2r)$. Then $f(1-g)\varphi \in C_0^0(\Gamma)$. Let

$$\widetilde{u}(X) := U(f(1-g)\varphi) = \int_{\partial\Omega} f(1-g)\varphi d\omega^X.$$

(For simplicity we take $\varphi \equiv 1$ for case when $f \in C_0^0(\Gamma)$.) By the positivity of the harmonic measure and the fact that $E \subset \partial\Omega \setminus B(q, 2r)$, we deduce that $0 \leq u(X) \leq \widetilde{u}(X)$ for all $X \in \Omega$. Recall by Lemma 2.4.38 that $\widetilde{u} \in C(\mathbb{R}^n)$, and as $X \to q' \in B(q, r) \cap \Gamma$, the function $\widetilde{u}(X) \to f(1-g)\varphi(q') = 0$. By the squeeze theorem u can be continuously extended to zero on $B(q, r) \cap \Gamma$, and the resulting function, still denoted as u, is continuous in $B(q, r)$.

Now we show that $u \in W_r(B)$. To this end, let $\phi \in C_0^\infty(B)$, it suffices to show that $\nabla(\phi u) \in L^2(B, w)$. From Lemma 2.4.38 (iv), Remark 2.4.39 and the boundary Caccioppoli inequality (2.4.30), we have

$$\iint_B |\nabla(\phi u_j)|^2 dM \leq 2 \iint_B \left(|\nabla\phi|^2 u_j^2 + \phi^2 |\nabla u_j|^2\right) dm \leq C \iint_B |\nabla\phi|^2 u_j^2 dm. \tag{2.4.50}$$

Recall that $u_j \to u$ pointwise on $B \setminus \Gamma$. Since u is continuous on B, $u \in L^2(\operatorname{supp}\phi, w)$. Hence by the dominated convergence theorem the right hand side of (2.4.50) converges to $C \iint_B |\nabla\phi|^2 u^2 dm$. As a consequence the left hand side is uniformly bounded, and thus passing to a subsequence $\nabla(\phi u_j)$ converges weakly in $L^2(B, w)$ to some function v. By the uniqueness of the limit we deduce $v = \nabla(\phi u)$. In particular this implies $\nabla(\phi u) \in L^2(B, w)$.

As a summary, we can write down the solution of L using the harmonic measure, for the following classes of boundary data: continuous and compactly supported functions $f \in C_0^0(\Gamma)$ (see Lemma 2.4.38), characteristic functions χ_E for Borel sets $E \subset \Gamma$ (see Lemma 2.4.43), their products $f\chi_E$ (see the above Lemma 2.4.44), or a linear combination of the above. For the third case, if the Borel set E is bounded, we only need to assume $f \in C_b(\Gamma)$.

Lemma 2.4.51 (corkscrew point, Lemma 11.46 of [DFM1]). *There exists $M > 1$ such that for any $q \in \Gamma$ and $r > 0$, there exists a point $A = A_r(q) \in \Omega$ such that*

$$|A - q| < r, \quad \delta(A) \geq \frac{r}{M}. \tag{2.4.52}$$

This point will be referred to as a corkscrew point hereafter.

Remark 2.4.53. Note that neither Lemma 2.4.3 nor Lemma 2.4.51 is automatically true if $d = n-1$. In fact in the case of co-dimension 1, people often work with domains that satisfy Harnack chain condition and the existence of corkscrew point at all scales, called uniform domains or 1-sided NTA domains in the literature.

Lemma 2.4.54 (boundary Harnack inequality, Lemma 11.50 of [DFM1]). *Let $q \in \Gamma$ and $r > 0$ be given, and let $A = A_r(q)$ be a corkscrew point as in Lemma 2.4.51. Let $u \in W_r(B(q, 2r))$ be a non-negative, non identically zero solution of $Lu = 0$ in $B(q, 2r) \cap \Omega$, such that $Tu \equiv 0$ on $\Delta(q, 2r)$. Then*

$$u(X) \leq C u(A) \quad \text{for all } X \in B(q, r). \tag{2.4.55}$$

We also recall the following "classical" Poincaré inequality for Sobolev functions.

Lemma 2.4.56 (Poincaré inequality, Lemma 4.13 of [DFM1])**.** *Let Γ be a d-Ahlfors regular set in $\mathbb{R}^n$ with $d < n-1$. For any function $v \in W$, $X \in \mathbb{R}^n$ and $r > 0$, let $B = B(X, r)$, then*

$$\left(\frac{1}{m(B)} \iint_B |v(Y) - v_B|^2 dm(Y)\right)^{\frac{1}{2}} \le Cr \left(\frac{1}{m(B)} \iint_B |\nabla v(Y)|^2 dm(Y)\right)^{\frac{1}{2}}, \tag{2.4.57}$$

where v_B denotes the average $m(B)^{-1} \int_B v dm$.

Suppose $\Delta = B(q_0, r) \cap \Gamma$ is a surface ball. For any $q \in \Delta$ and any $j \in \mathbb{N}$, let

$$\Gamma_j(q) = \Gamma(q) \cap \left(B(q, 2^{-j}r) \setminus B(q, 2^{-j-1}r)\right) \tag{2.4.58}$$

be a stripe in the cone $\Gamma(q)$ at height $2^{-j}r$, and

$$\Gamma_{j \to j+m}(q) = \bigcup_{i=j}^{j+m} \Gamma_i(q) = \Gamma(q) \cap \left(B(q, 2^{-j}r) \setminus B(q, 2^{-(j+m)-1}r)\right), \tag{2.4.59}$$

be a union of $(m+1)$ stripes. With these notations we can prove a less conventional form of Poincaré inequality, available for solutions with vanishing boundary values.

Lemma 2.4.60. *Suppose that $u \in W_r(\Omega)$ is a non-negative solution of L, $Tu = 0$ on 3Δ and $u \in W_r(B(q_0, 3r))$. There exist an aperture $\overline{\alpha} > \alpha$ and integers m_1, m_2, such that for all $q \in \Delta$,*

$$\iint_{\Gamma_j^\alpha(q)} u^2 dm(X) \le C(2^{-j}r)^2 \iint_{\Gamma^{\overline{\alpha}}_{j-m_1 \to j+m_2}(q)} |\nabla u|^2 dm(X). \tag{2.4.61}$$

The constants $m_1, m_2, \overline{\alpha}$ and C only depend on n, d, α, C_0, C_1.

Proof. Let B be a ball compactly contained in Ω. Recall that the solution $u \in W_r(\Omega)$, in particular, $\varphi u \in W$ for $\varphi \in C_0^\infty(\Omega)$ such that $\varphi \equiv 1$ on B. Apply the above Lemma 2.4.56 to φu and square both sides, we get

$$\iint_B |u(Y) - u_B|^2 dm(Y) \le Cr_B^2 \iint_B |\nabla u(Y)|^2 dm(Y), \tag{2.4.62}$$

For $j \in \mathbb{N}$, let A_j denote a corkscrew point for $B(q, 2^{-j}r)$, whose existence is guaranteed by Lemma 2.4.51. Let m be a large integer whose value is to be determined later. Take $X \in \Gamma_j^\alpha(q)$, $X' = A_{j+m}$, then

$$\delta(X) > \frac{1}{1+\alpha}|X - q| \ge \frac{2^{-j-1}r}{1+\alpha}, \quad \delta(X') \ge \frac{2^{-(j+m)}r}{M}, \tag{2.4.63}$$

$$|X - X'| \le |X - q| + |q - X'| \le 2^{-j} r + 2^{-(j+m)} r \le 2^{1-j} r.$$

Apply Lemma 2.4.3 and Remark 2.4.4 to X, X' with $s = 2^{-(j+m)} r/M$ and $\Lambda = 2^{m+1} M$, we can find balls $B_0 = B(X, s/2)$, $B_i = B(Z_i, \tau/4)$ with $\tau = c\Lambda^{-d/(n-1-d)} s$, $B_{N+1} = B(X', s/2)$ that form a Harnack chain connecting X to X', and satisfy (2.4.6), (2.4.7) and (2.4.8). Hence by Lemma 2.3 (i) of [DFM1] and (2.4.8), (2.4.7), we have

$$m(B_i) \ge C^{-1} \left(\frac{\tau}{4}\right)^n \operatorname{dist}(B_i, \Gamma)^{d-n+1} \gtrsim \tau^n (\Lambda s)^{d-n+1} \sim \Lambda^{1-n} \tau^{d+1}, \tag{2.4.64}$$

and

$$m(B_i) \le C \left(\frac{\tau}{4}\right)^n \operatorname{dist}(B_i, \Gamma)^{d-n+1} \lesssim \tau^n \tau^{d-n+1} \sim \tau^{d+1} \tag{2.4.65}$$

for all $i = 0, \cdots, N, N+1$. A simple computation shows $B_{i+1} \subset 3B_i$ for all $i = 1, \cdots N-1$, and $B_1 \subset \frac{3}{2} B_0$, $B_N \subset \frac{3}{2} B_{N+1}$, if m is sufficiently large. Therefore for each $i = 1, \cdots, N-1$,

$$\begin{aligned} |u_{B_{i+1}} - u_{3B_i}|^2 &\le \left(\frac{1}{m(B_{i+1})} \iint_{B_{i+1}} |u(X) - u_{3B_i}| dm(X)\right)^2 \\ &\le \frac{1}{m(B_{i+1})} \iint_{3B_i} |u(X) - u_{3B_i}|^2 dm(X) \\ &\lesssim \Lambda^{n-1} \tau^{1-d} \iint_{3B_i} |\nabla u(Y)|^2 dm(Y) \qquad \text{by (2.4.62), (2.4.64).} \end{aligned} \tag{2.4.66}$$

Similarly

$$|u_{B_i} - u_{3B_i}|^2 \lesssim \Lambda^{n-1} \tau^{1-d} \iint_{3B_i} |\nabla u(Y)|^2 dm(Y).$$

Hence

$$|u_{B_i} - u_{B_{i+1}}|^2 \le C \Lambda^{n-1} \tau^{1-d} \iint_{3B_i} |\nabla u(Y)|^2 dm(Y). \tag{2.4.67}$$

A similar argument shows that for the end-point case $i = 0$ or $N+1$,

$$\begin{aligned} |u_{B_i} - u_{B_{i\pm 1}}|^2 &\lesssim \max\{s^{1-d}, \Lambda^{n-1} s^2 \tau^{-1-d}\} \iint_{\frac{3}{2}B_i} |\nabla u(Y)|^2 dm(Y) \\ &\sim \Lambda^{n-1} s^2 \tau^{-1-d} \iint_{\frac{3}{2}B_i} |\nabla u(Y)|^2 dm(Y). \end{aligned} \tag{2.4.68}$$

The last line is justified since $\Lambda \gg 1$ implies $\tau \ll s$. Combining this observation, (3.2.14), (2.4.68) and (2.4.6), we get

$$\begin{aligned} \iint_{B_0} |u(X) - u_{B_{N+1}}|^2 dm(X) &\lesssim N \cdot \iint_{B_0} |u(X) - u_{B_0}|^2 dm(X) + N \cdot m(B_0) \sum_{i=0}^{N} |u_{B_i} - u_{B_{i+1}}|^2 \\ &\lesssim N \Lambda^{n-1} s^2 \left(\frac{s}{\tau}\right)^{d+1} \iint_{\frac{3}{2}B_0 \cup \left(\bigcup_{i=1}^{N} 3B_i\right) \cup \frac{3}{2}B_{N+1}} |\nabla u(Y)|^2 dm(Y) \end{aligned}$$

$$\leq C'\Lambda^{\frac{n-1+d(d+1)}{n-1-d}+n-1}s^2 \iint_{\frac{3}{2}B_0\cup\left(\bigcup_{i=1}^N 3B_i\right)\cup\frac{3}{2}B_{N+1}} |\nabla u(Y)|^2 dm(Y). \tag{2.4.69}$$

On the other hand, by Harnack inequality

$$u(X) \leq Cu(X') \quad \text{for all } X \in B_{N+1} = B(X', s/2).$$

Recall that $X' = A_{j+m}$. For any $q \in \Delta$, by the assumption we know that $u \in W_r(B(q, 2r))$ vanishes on $\Delta(q, 2r)$. By the boundary Hölder regularity (Lemma 2.4.34) and boundary Harnack principle (Lemma 2.4.54) we have

$$u(X') \leq C2^{-m\beta}u(A_j),$$

with a constant C independent of j and m. Thus

$$u_{B_{N+1}}^2 \lesssim u^2(X') \lesssim 2^{-2m\beta}u^2(A_j) \lesssim 2^{-2m\beta}\cdot\frac{1}{m(B_0)}\iint_{B_0} u^2 dm(X). \tag{2.4.70}$$

The last inequality holds because A_j is a corkscrew point and $B_0 = B(X, s/2)$ for some $X \in \Gamma_j(q)$. Combining (2.4.70) and (2.4.69) we obtain

$$\begin{aligned}
&\iint_{B_0} u^2 dm(X)\\
&\leq 2m(B_0)\left(u_{B_{N+1}}\right)^2 + 2\iint_B |u(x) - u_{B_{N+1}}|^2 dm(X)\\
&\leq A_1 2^{-2m\beta}\iint_{B_0} u^2 dm(X) + A_2\Lambda^{\frac{n-1+d(d+1)}{n-1-d}+n-1}s^2\iint_{\frac{3}{2}B_0\cup\left(\bigcup_{i=1}^N 3B_i\right)\cup\frac{3}{2}B_{N+1}} |\nabla u(Y)|^2 dm(Y).
\end{aligned} \tag{2.4.71}$$

Choose m big enough such that

$$A_1 2^{-2m\beta} \leq \frac{1}{2}, \quad \text{as well as } 2\cdot\frac{2^{-m}}{M} \leq \frac{1}{2(1+\alpha)}, \tag{2.4.72}$$

then we can absorb the first term on the right hand side of (3.2.18) to the left. Recall that $B_0 = B(X, s/2)$ for X satisfying (2.4.63). The reason for the second assumption in (2.4.72) is to guarantee the enlarged ball $\frac{3}{2}B_0$ is compactly contained in Ω. Fix the value of m from now on, thus the value of $\Lambda = 2^{m+1}/M$ is also fixed. We get

$$\iint_{B_0} u^2 dm(X) \leq Cs^2 \iint_{\frac{3}{2}B_0\cup\left(\bigcup_{i=1}^N 3B_i\right)\cup\frac{3}{2}B_{N+1}} |\nabla u(y)|^2 dy, \tag{2.4.73}$$

where $s = 2^{-(j+m)}r/M$ and the constant C depends on d, n, C_0, C_1 (Recall the values of corkscrew constant M and Harnack chain constant c only depend on d, n, C_0, C_1). Since

$B_0 = B(X, s/2)$ with center $X \in \Gamma_j^\alpha(q)$, it is a simple exercise to show that given the second assumption of (2.4.72), there exists an aperture $\alpha_1 > \alpha$ such that

$$\frac{3}{2}B_0 \subset \Gamma^{\alpha_1}_{j-1\to j+1}(q). \tag{2.4.74}$$

A similar statement holds for $\frac{3}{2}B_{N+1}$. Moreover (2.4.7) and (2.4.8) imply that for $i = 1, \cdots, N$, there exist an aperture $\alpha_2 > \alpha$ and an integer m_0 depending on the constants c, M from Lemmas 2.4.3 and 2.4.51, such that

$$3B_i \subset \Gamma^{\alpha_2}_{j-3\to j+m+m_0}(q). \tag{2.4.75}$$

Let $\overline{\alpha} = \max\{\alpha_1, \alpha_2\}$. Combining the above observations with (3.2.22) we get

$$\iint_{B_0} u^2 dm(X) \le Cs^2 \iint_{\Gamma^{\overline{\alpha}}_{j-3\to j+m+m_0}(q)} |\nabla u(y)|^2 dy. \tag{2.4.76}$$

Consider the covering

$$\Gamma_j^\alpha(q) \subset \bigcup_{X\in\Gamma_j^\alpha(q)} B\left(X, \frac{s}{10}\right). \tag{2.4.77}$$

We can extract a finite Vitali sub-covering $\{B^k = B(X_k, s/2)\}_k$ such that

$$\Gamma_j^\alpha(q) \subset \bigcup_k B^k \tag{2.4.78}$$

and $\{B^k/5 = B(X_k, s/10)\}_k$ is mutually disjoint. Moreover the number of balls B^k's is uniformly bounded by a constant $C(n, m, M)$. Note that (2.4.76) holds for all such balls B^k in place of B_0, we deduce

$$\iint_{\Gamma_j^\alpha(q)} u^2 dm(X) \le \sum_k \iint_{B^k} u^2 dm(X) \le CC(n, m, M)s^2 \iint_{\Gamma^{\overline{\alpha}}_{j-3\to j+m+m_0}(q)} |\nabla u(y)|^2 dy. \tag{2.4.79}$$

Since the value of m is fixed, we finish the proof of Lemma 3.2.5.

Lemma 2.4.80 (non-degeneracy of harmonic measure, Lemma 11.73 of [DFM1])**.** *Let $\lambda > 1$ be given. There exists a constant $C_\lambda > 1$ such that for any $q \in \Gamma$, $r > 0$, and $A = A_r(q)$, a corkscrew point from Lemma 2.4.51, we have*

$$\omega^X(B(q, r) \cap \Gamma) \ge C_\lambda^{-1} \quad \textit{for } X \in B(q, r/\lambda), \tag{2.4.81}$$

$$\omega^X(B(q, r) \cap \Gamma) \ge C_\lambda^{-1} \quad \textit{for } X \in B(A, \delta(A)/\lambda). \tag{2.4.82}$$

In [DFM1] the authors also prove the existence, uniqueness and properties of the Green function, that is, formally, a function G defined on $\Omega \times \Omega$ such that for any $Y \in \Omega$,

$$\begin{cases} LG(\cdot, Y) = \delta_Y & \text{in } \Omega \\ G(\cdot, Y) = 0 & \text{on } \Gamma \end{cases}$$

where δ_Y is the delta function.

Lemma 2.4.83 (estimates of Green function, Lemma 11.78 of [DFM1])**.** *There exists a constant $C \geq 1$, such that for any $q \in \Gamma$ and $r > 0$, $\Delta = B(q, r) \cap \Gamma$ and a corkscrew point $A = A_r(q)$, then*

$$C^{-1} r^{d-1} G(X_0, A) \leq \omega^{X_0}(\Delta) \leq C r^{d-1} G(X_0, A) \quad \text{for } X_0 \in \Omega \setminus B(q, 2r). \tag{2.4.84}$$

Lemma 2.4.85 (doubling of harmonic measure, Lemma 11.102 of [DFM1])**.** *For $q \in \Gamma$ and $r > 0$, we have*

$$\omega^X(B(q, 2r) \cap \Gamma) \leq C\omega^X(B(q, r) \cap \Gamma) \tag{2.4.86}$$

for any $X \in \Omega \setminus B(q, 4r)$.

Lemma 2.4.87 (change of poles, Lemma 11.135 of [DFM1])**.** *Let $q \in \Gamma$ and $r > 0$ be given, and let $A = A_r(q)$ be a corkscrew point as in Lemma 2.4.51. Let $E, F \subset \Delta(q, r)$ be two Borel subsets of Γ such that $\omega^A(E)$ and $\omega^A(F)$ are positive. Then*

$$\frac{\omega^X(E)}{\omega^X(F)} \sim \frac{\omega^A(E)}{\omega^A(F)}, \quad \text{for any } X \in \Omega \setminus B(q, 2r). \tag{2.4.88}$$

In particular with the choice $F = \Delta(q, r)$,

$$\frac{\omega^X(E)}{\omega^X(\Delta(q, r))} \sim \omega^A(E) \quad \text{for any } X \in \Omega \setminus B(q, 2r). \tag{2.4.89}$$

Chapter 3

$\omega_L \in A_\infty(\sigma)$ and PDE solvability

Before we start the proof, we make the following observation: the Carleson measure norm of $\delta(X)|\nabla u|^2 dX$ is in some sense equivalent to the integral of the truncated square function. Suppose $\Delta = \Delta(Q_0, r)$ is an arbitrary surface ball. For any $X \in T(\Delta)$, we define $\Delta^X = \{Q \in \partial\Omega : X \in \Gamma(Q)\}$. Let $Q_X \in \partial\Omega$ be a point such that $|X - Q_X| = \delta(X)$. Then

$$\Delta(Q_X, \alpha\delta(X)) \subset \Delta^X \subset \Delta(Q_X, (\alpha+2)\delta(X)). \tag{3.0.1}$$

Since $\partial\Omega$ is Ahlfors regular, (3.0.1) implies $\sigma(\Delta^X) \approx \delta(X)^{n-1}$. Thus

$$\begin{aligned}\iint_{T(\Delta)} |\nabla u|^2 \delta(X) dX &\approx \iint_{T(\Delta)} |\nabla u|^2 \delta(X)^{2-n} \sigma(\Delta^X) dX \\ &= \iint_{T(\Delta)} |\nabla u|^2 \delta(X)^{2-n} \int_{\Delta^X} d\sigma(Q) dX.\end{aligned} \tag{3.0.2}$$

Changing the order of integration, on one hand,

$$\begin{aligned}\iint_{T(\Delta)} |\nabla u|^2 \delta(X)^{2-n} \int_{\Delta^X} d\sigma(Q) dX &\le \int_{|Q-Q_0|<(\alpha+2)r} \iint_{\Gamma_{\alpha r}(Q)} |\nabla u|^2 \delta(X)^{2-n} dX d\sigma \\ &\le \int_{(\alpha+2)\Delta} S^2_{(\alpha+1)r}(u) d\sigma.\end{aligned} \tag{3.0.3}$$

On the other hand,

$$\begin{aligned}\iint_{T(\Delta)} |\nabla u|^2 \delta(X)^{2-n} \int_{\Delta^X} d\sigma(Q) dX &\ge \int_{|Q-Q_0|<r/2} \iint_{\Gamma_{r/2}(Q)} |\nabla u|^2 \delta(X)^{2-n} dX d\sigma \\ &\ge \int_{\Delta/2} S^2_{r/2}(u) d\sigma,\end{aligned} \tag{3.0.4}$$

where $\Delta/2 = \Delta(Q_0, r/2)$. Therefore for any $Q_0 \in \partial\Omega$,

$$\sup_{\substack{\Delta=\Delta(Q_0,s)\\ s>0}} \frac{1}{\sigma(\Delta)} \iint_{T(\Delta)} |\nabla u|^2 \delta(X) dX \approx \sup_{\substack{\Delta=\Delta(Q_0,r)\\ r>0}} \frac{1}{\sigma(\Delta)} \int_\Delta S^2_r(u) d\sigma \tag{3.0.5}$$

3.1 From $\omega_L \in A_\infty(\sigma)$ to the Carleson measure estimate

Assume $\omega \in A_\infty(\sigma)$. For any continuous function $f \in C(\partial\Omega)$, let u be the solution to the elliptic problem $Lu = 0$ with boundary data f, we want to show that $|\nabla u|^2\delta(X)$ is a Carleson measure, and in particular,

$$\sup_{\Delta\subset\partial\Omega} \frac{1}{\sigma(\Delta)} \iint_{T(\Delta)} |\nabla u|^2 \delta(X) dX \leq C\|f\|^2_{BMO(\sigma)}. \tag{3.1.1}$$

Let $\Delta = \Delta(Q_0, r)$ be any surface ball. Denote the constant $c = \max\{\alpha + 1, 27\}$ and let $\widetilde{\Delta} = c\Delta = \Delta(Q_0, cr)$ be its concentric surface ball. Let

$$f_1 = (f - f_{\widetilde{\Delta}})\chi_{\widetilde{\Delta}}, \quad f_2 = (f - f_{\widetilde{\Delta}})\chi_{\partial\Omega\setminus\widetilde{\Delta}}, \quad f_3 = f_{\widetilde{\Delta}} = \fint_{\widetilde{\Delta}} f d\sigma,$$

and let u_1, u_2, u_3 be the solutions to $Lu = 0$ with boundary data f_1, f_2, f_3 respectively. Clearly u_3 is a constant, so its Carleson measure is trivial.

We have shown in the beginning of this chapter (See (3.0.2) and (3.0.3)) that

$$\iint_{T(\Delta)} |\nabla u_1|^2 \delta(X) dX \leq C \int_{(\alpha+2)\Delta} S^2_{(\alpha+1)r}(u_1) d\sigma.$$

Since $\widetilde{\Delta} = c\Delta \supset (\alpha + 2)\Delta$, it follows that

$$\iint_{T(\Delta)} |\nabla u_1|^2 \delta(X) dX \leq C \int_{\widetilde{\Delta}} S^2_{(\alpha+1)r}(u_1) d\sigma. \tag{3.1.2}$$

By Hölder inequality, for $p > 2$

$$\int_{\widetilde{\Delta}} S^2_{(\alpha+1)r}(u_1) d\sigma \leq \sigma(\widetilde{\Delta})^{1-\frac{2}{p}} \left(\int_{\widetilde{\Delta}} S^p(u_1) d\sigma \right)^{2/p} \leq \sigma(\widetilde{\Delta})^{1-\frac{2}{p}} \|S(u_1)\|^2_{L^p(\sigma)}. \tag{3.1.3}$$

Under the assumption $\omega \in A_\infty(\sigma)$, the following theorems are at our disposal:

Theorem 3.1.4 ([Ke] Theorem 1.4.13(vii) and Lemma 1.4.2)**.** *Assume $\omega \in B_q(\sigma)$ for some $1 < q < \infty$, then the elliptic problem $Lu = 0$ is L^p−solvable with $1/p + 1/q = 1$: that is, if u is a solution with boundary value $f \in L^p(\sigma)$, then $\|Nu\|_{L^p(\sigma)} \leq C\|f\|_{L^p(\sigma)}$.*

Theorem 3.1.5 ([Ke] Theorem 1.5.10)**.** *Assume $\omega \in A_\infty(\sigma)$, then if $Lu = 0$ with boundary value f, we have $\|S(u)\|_{L^p(\sigma)} \leq C\|Nu\|_{L^p(\sigma)}$ for any $0 < p < \infty$.*

Apply Theorem 6.2.1 and Theorem 3.1.5 to u_1, and get

$$\|S(u_1)\|_{L^p(\sigma)} \leq C\|f_1\|_{L^p(\sigma)} = C \left(\int_{\widetilde{\Delta}} |f - f_{\widetilde{\Delta}}|^p d\sigma \right)^{1/p}. \tag{3.1.6}$$

Combining (3.1.2), (6.2.13) and (6.2.12), we get

$$\iint_{T(\Delta)} |\nabla u_1|^2 \delta(X) dX \leq C\sigma(\Delta) \|f\|^2_{BMO(\sigma)}. \tag{3.1.7}$$

To show similar estimate for u_2, let $\{E_m\}$ be a Whitney decomposition of $T(\Delta)$. On each Whitney cube E_m, we have the following Cacciopoli type estimate,

$$\begin{aligned}\iint_{E_m} |\nabla u_2|^2 \delta(X) dX &\lesssim \delta(E_m) \iint_{E_m} |\nabla u_2|^2 dX \\ &\lesssim \delta(E_m) \cdot \frac{1}{\delta(E_m)^2} \iint_{\frac{3}{2}E_m} |u_2(X)|^2 dX \\ &\lesssim \iint_{\frac{3}{2}E_m} \frac{|u_2(X)|^2}{\delta(X)} dX.\end{aligned}$$

Summing up, we get

$$\begin{aligned}\iint_{T(\Delta)} |\nabla u_2|^2 \delta(X) dX &\lesssim \sum_m \iint_{\frac{3}{2}E_m} \frac{|u_2(X)|^2}{\delta(X)} dX \\ &\lesssim \iint_{T(\frac{3}{2}\Delta)} \frac{|u_2(X)|^2}{\delta(X)} dX. \end{aligned} \tag{3.1.8}$$

Recall $3\Delta/2$ denotes $\Delta(Q_0, 3r/2)$, and $T(3\Delta/2)$ denotes $B(Q_0, 3r/2) \cap \Omega$.

Let $u_2^\pm$ be the solutions to $Lu = 0$ with non-negative boundary data $f_2^\pm$, then $u_2 = u_2^+ - u_2^-$ and $|u_2| = u_2^+ + u_2^-$. Let

$$v(X) = |u_2(X)| = \int_{\partial\Omega} \left(f_2^+ + f_2^-\right) d\omega^X = \int_{\partial\Omega\setminus\widetilde{\Delta}} |f - f_{\widetilde{\Delta}}| d\omega^X. \tag{3.1.9}$$

We have the following lemma:

Lemma 3.1.10. *The function v defined in (6.2.16) satisfies*

- $v(X) \leq C\|f\|_{BMO(\sigma)}$ *for all* $X \in T(9\Delta)$.
- $v(X) \leq C\left(\frac{\delta(X)}{r}\right)^\beta \|f\|_{BMO(\sigma)}$ *for all* $X \in T(3\Delta/2)$. *Here* $\beta \in (0,1)$ *is the degree of boundary Hölder regularity for non-negative solutions, and it only depends on* n *and the ellipticity of* A *(see (2.2.20)).*

Proof. By the definition (6.2.16), the function v vanishes on $\widetilde{\Delta}$. Note that $\widetilde{\Delta} \supset 27\Delta$ by the choice of $\widetilde{\Delta}$, v is a non-negative solution in $T(27\Delta)$ and vanishes on 27Δ. Let A be a corkscrew point in $T(9\Delta)$, by Lemma 2.2.23

$$v(X) \leq Cv(A), \quad \text{for all } X \in T(9\Delta).$$

Let $\bar{k} = d\omega/d\sigma$ be the Radon-Nikodym derivative of ω with respect to σ. By the assumption $\omega \in A_\infty(\sigma)$, there exists some $q > 1$ such that for any surface ball Δ',

$$\left(\fint_{\Delta'} \bar{k}^q d\sigma \right)^{1/q} \leq C \fint_{\Delta'} \bar{k} d\sigma, \tag{3.1.11}$$

Let $K(X, \cdot) = d\omega^X/d\omega$ be the Radon-Nikodym derivative of ω^X with respect to ω , i.e.

$$K(X, Q) = \lim_{\Delta' \to Q} \frac{\omega^X(\Delta')}{\omega(\Delta')}. \tag{3.1.12}$$

Then

$$\begin{aligned} v(A) = \int_{\partial\Omega \setminus \widetilde{\Delta}} |f - f_{\widetilde{\Delta}}| d\omega^A &= \int_{\partial\Omega \setminus \widetilde{\Delta}} |f - f_{\widetilde{\Delta}}| K(A, Q) \bar{k}(Q) d\sigma(Q) \\ &= \sum_{j=1}^{\infty} \int_{2^j \widetilde{\Delta} \setminus 2^{j-1} \widetilde{\Delta}} |f - f_{\widetilde{\Delta}}| K(A, Q) \bar{k}(Q) d\sigma(Q). \end{aligned} \tag{3.1.13}$$

Let Δ' be any surface ball contained in $2^j \widetilde{\Delta} \setminus 2^{j-1} \widetilde{\Delta}$, and A_j be a corkscrew point in $T(2^j \widetilde{\Delta})$. Then by Corollary 1.3.8 [Ke] (It follows easily from the boundary comparison principle (2.2.32))

$$\frac{\omega(\Delta')}{\omega(2^j \widetilde{\Delta})} \approx \omega^{A_j}(\Delta'). \tag{3.1.14}$$

On the other hand, by the boundary regularity of $\omega^X(\Delta')$ in $T(2^{j-1} \widetilde{\Delta})$, we have

$$\begin{aligned} \omega^A(\Delta') &\leq C \left(\frac{|A - Q_0|}{2^{j-1} \cdot cr} \right)^\beta \sup_{Y \in 2^{j-1} \widetilde{\Delta}} \omega^Y(\Delta') \\ &\leq C \left(\frac{9r}{2^{j-1} cr} \right)^\beta \omega^{A_j}(\Delta') \\ &\lesssim 2^{-j\beta} \omega^{A_j}(\Delta'). \end{aligned} \tag{3.1.15}$$

Combining (3.1.14) and (3.1.15) we have

$$\frac{\omega^A(\Delta')}{\omega(\Delta')} \lesssim 2^{-j\beta} \frac{\omega^{A_j}(\Delta')}{\omega(\Delta')} \approx \frac{2^{-j\beta}}{\omega(2^j \widetilde{\Delta})},$$

for any Δ' contained in $2^j \widetilde{\Delta} \setminus 2^{j-1} \widetilde{\Delta}$. Therefore by the definition (3.1.12)

$$\sup_{Q \in 2^j \widetilde{\Delta} \setminus 2^{j-1} \widetilde{\Delta}} K(A, Q) \lesssim \frac{2^{-j\beta}}{\omega(2^j \widetilde{\Delta})}. \tag{3.1.16}$$

Combining (3.1.11), (3.1.13) and (3.1.16),

$$\begin{aligned} v(A) &\lesssim \sum_j \frac{2^{-j\beta}}{\omega(2^j\widetilde{\Delta})} \left(\int_{2^j\widetilde{\Delta}} |f - f_{\widetilde{\Delta}}|^p d\sigma \right)^{1/p} \left(\int_{2^j\widetilde{\Delta}} \bar{k}^q d\sigma \right)^{1/q} \\ &\lesssim \sum_j 2^{-j\beta} \left(⨍_{2^j\widetilde{\Delta}} |f - f_{\widetilde{\Delta}}|^p d\sigma \right)^{1/p} \\ &\lesssim \|f\|_{BMO(\sigma)}. \end{aligned} \tag{3.1.17}$$

Therefore

$$v(X) \leq C\|f\|_{BMO(\sigma)} \qquad \text{for all } X \in T(9\Delta). \tag{3.1.18}$$

For any $X \in T(3\Delta/2)$, let Q_X be a boundary point such that $|X - Q_X| = \delta(X)$. Note that

$$|X - Q_X| = \delta(X) \leq |X - Q_0| < \frac{3r}{2},$$

so $X \in B(Q_X, 3r/2) \cap \Omega$. We consider the Dirichlet problem in $B(Q_X, 6r) \cap \Omega$. Note that

$$|Q_X - Q_0| \leq |Q_X - X| + |X - Q_0| < \frac{3r}{2} + \frac{3r}{2} = 3r,$$

hence $\overline{B(Q_X, 6r)} \subset B(Q_0, 9r)$. Note that $\widetilde{\Delta} \supset 9\Delta \supset \Delta(Q_X, 6r)$, v is a non-negative solution in $B(Q_X, 6r) \cap \Omega$ and vanishes on $\Delta(Q_X, 6r)$. By the boundary Hölder regularity (Proposition 2.2.19) and the first part of this lemma (6.2.17), we conclude

$$v(X) \lesssim \left(\frac{|X - Q_X|}{3r/2} \right)^{\beta} \sup_{B(Q_X, 6r)\cap\Omega} v \lesssim \left(\frac{\delta(X)}{r} \right)^{\beta} \sup_{T(9\Delta)} v \lesssim \left(\frac{\delta(X)}{r} \right)^{\beta} \|f\|_{BMO(\sigma)}.$$

Using Lemma 3.1.10 and (6.2.15), we get

$$\iint_{T(\Delta)} |\nabla u_2|^2 \delta(X) dX \lesssim \frac{\|f\|^2_{BMO(\sigma)}}{r^{2\beta}} \left(\iint_{T(\frac{3}{2}\Delta)} \delta(X)^{2\beta - 1} dX \right). \tag{3.1.19}$$

Note that $2\beta - 1 > -1$, we may use the following lemma.

Lemma 3.1.20. *For any $\alpha > -1$, we have*

$$\iint_{T(2\Delta)} \delta(X)^{\alpha} dX \lesssim r^{n+\alpha}. \tag{3.1.21}$$

Proof. If $\alpha \geq 0$, the proof is trivial. For $j = 0, 1, \cdots$ let

$$T_j = T(2\Delta) \cap \{x \in \Omega : 2^{-j}r \leq \delta(X) < 2^{-j+1}r\},$$

$$T_{<j} = T(2\Delta) \cap \{x \in \Omega : \delta(X) < 2^{-j+1} r\}.$$

Then

$$\iint_{T(2\Delta)} \delta(X)^\alpha dX = \sum_{j=0}^{\infty} \iint_{T_j} \delta(X)^\alpha dX \lesssim \sum_{j=0}^{\infty} (2^{-j} r)^\alpha m(T_{<j}). \tag{3.1.22}$$

Consider a covering of 4Δ by $4\Delta \subset \bigcup_{Q \in 4\Delta} B(Q, 2^{-j+1} r)$, from which one can extract a countable Vitali sub-covering $4\Delta \subset \cup_k B(Q_k, 2^{-j+1} r)$, where $Q_k \in 4\Delta$ and the balls $B_k = B(Q_k, 2^{-j+1} r/5)$ are pairwise disjoint. The fact that $Q_k \in 4\Delta = \Delta(Q_0, 4r)$ implies

$$B_k = B\left(Q_k, \frac{2^{-j+1} r}{5}\right) \subset B\left(Q_0, 4r + \frac{2^{-j+1} r}{5}\right).$$

And the pairwise disjointness of B_k's implies there are only finitely many of them. In fact,

$$\sum_k \sigma(B_k) = \sigma\left(\bigcup_k B_k\right) \leq \sigma\left(\Delta\left(Q_0, 4r + \frac{2^{-j+1} r}{5}\right)\right) \lesssim \left(4r + \frac{2r}{5}\right)^{n-1}. \tag{3.1.23}$$

Note that $\sigma(B_k) \approx \left(2^{-j+1} r/5\right)^{n-1}$ independent of k. Let N be the number of B_k's. By (3.1.23)

$$N \cdot \left(\frac{2^{-j+1} r}{5}\right)^{n-1} \leq \left(4r + \frac{2r}{5}\right)^{n-1}, \qquad \text{thus } N \lesssim 2^{j(n-1)}. \tag{3.1.24}$$

For any $X \in T_{<j}$, let $Q_X \in \partial\Omega$ be such that $|X - Q_X| = \delta(X)$. Then

$$|Q_X - Q_0| \leq |Q_X - X| + |X - Q_0| < 4r, \qquad \text{i.e. } Q_X \in 4\Delta. \tag{3.1.25}$$

Thus $Q_X \in B(Q_k, 2^{-j+1} r)$ for some k. Moreover $T_{<j} \subset \bigcup_k B(Q_k, 2 \cdot 2^{-j+1} r)$. Therefore

$$m(T_{<j}) \leq N \cdot \sup_k m(B(Q_k, 2 \cdot 2^{-j+1} r)) \lesssim 2^{-j} r^n.$$

Combined with (3.1.22) we get

$$\iint_{T(2\Delta)} \delta(X)^\alpha dX \lesssim \sum_{j=0}^{\infty} (2^{-j} r)^\alpha \cdot 2^{-j} r^n = r^{n+\alpha} \sum_{j=0}^{\infty} 2^{-j(\alpha+1)} \lesssim r^{n+\alpha}.$$

The last sum is convergent because $\alpha + 1 > 0$. Combining (6.2.20) and (3.1.21), we get

$$\iint_{T(\Delta)} |\nabla u_2|^2 \delta(X) dX \lesssim r^{n-1} \|f\|^2_{BMO(\sigma)} \lesssim \sigma(\Delta) \|f\|^2_{BMO(\sigma)}. \tag{3.1.26}$$

(6.2.14) and (6.2.21) together give the Carleson measure estimate (6.2.23).

3.2 From the Carleson measure estimate to $\omega_L \in A_\infty(\sigma)$

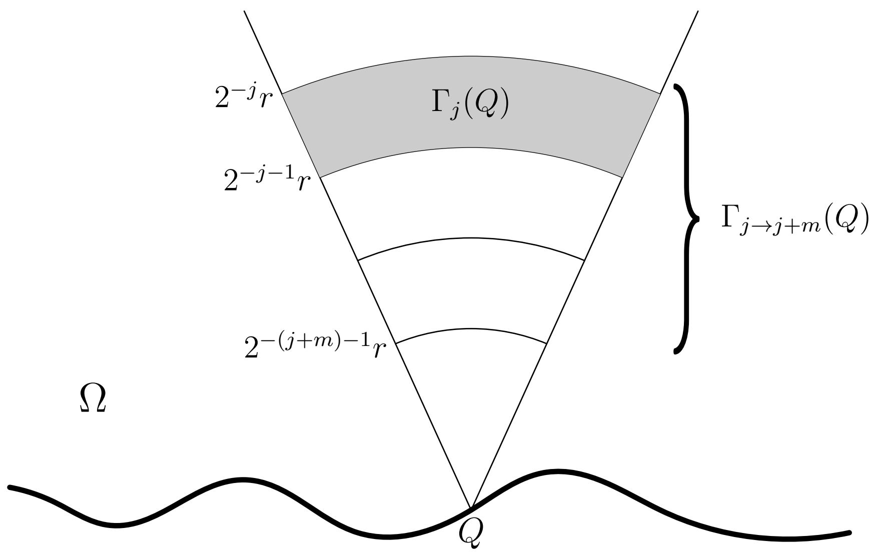
Let Δ be an arbitrary surface ball. Let f be a continuous non-negative function supported in Δ and u is the solution with boundary value f. In particular u is non-negative. Consider another surface ball Δ' of radius r that is $r-$distance away from Δ. Then by assumption,

(3.2.1) $$\iint_{T(\Delta')} |\nabla u|^2 \delta(X) dX \leq C\sigma(\Delta') \|f\|^2_{BMO(\sigma)}$$

We have shown in (3.0.4) that

(3.2.2) $$\iint_{T(\Delta')} |\nabla u|^2 \delta(X) dX \gtrsim \int_{\Delta'/2} S^2_{r/2}(u) d\sigma.$$

In order to get a lower bound of the square function $S_{r/2}(u)$, we need to decompose the non-tangential cone $\Gamma_{r/2}(Q)$ as follows.



For any $Q \in \Delta'/2$ and any $j \in \mathbb{N}$, let

(3.2.3) $$\Gamma_j(Q) = \Gamma(Q) \cap \left(B_{2^{-j}r}(Q) \setminus B_{2^{-j-1}r}(Q) \right)$$

be a stripe in the cone $\Gamma_{r/2}(Q)$ at height $2^{-j}r$, and

(3.2.4) $$\Gamma_{j\to j+m}(Q) = \bigcup_{i=j}^{j+m} \Gamma_i(Q),$$

a union of $(m+1)$ stripes. The above figure illustrates these notations, even though it over-simplifies the shape of the non-tangential cone $\Gamma(Q)$ and the relations between different radii.

We claim the following Poincaré type inequality holds, because u vanishes on Δ':

Lemma 3.2.5. *There exist an aperture $\overline{\alpha} > \alpha$, and integers m_1, m_2, such that the following Poincaré inequality holds for all $Q \in \Delta'/2$,*

$$\iint_{\Gamma_j^\alpha(Q)} u^2 dX \le C(2^{-j}r)^2 \iint_{\Gamma^{\overline{\alpha}}_{j-m_1\to j+m_2}(Q)} |\nabla u|^2 dX. \tag{3.2.6}$$

The constants m_1, m_2 and C only depend on n and α.

3.2.1 Proof of Lemma 3.2.5: Poincaré inequality

The following lemma is the standard Poincaré inequality (see (7.45) and Lemma 7.16 in [GT]).

Lemma 3.2.7. *Let B be a ball in $\mathbb{R}^n$. If the function $u \in W^{1,2}(B)$, then*

$$\|u - u_B\|_{L^2(B)} \le \left(\frac{\omega_n}{|B|}\right)^{1-1/n} \operatorname{diam}(B)^n \|\nabla u\|_{L^2(B)}. \tag{3.2.8}$$

Here $u_B = \fint_B u dx$, $|B|$ is the $n-$dimensional Lebesgue measure of B and ω_n is the volume of the unit ball in $\mathbb{R}^n$.

Let r_B denote the radius of B, then we can rewrite (3.2.8) as

$$\iint_B |u(x) - u_B|^2 dx \le 4^n r_B^2 \iint_B |\nabla u(y)|^2 dy. \tag{3.2.9}$$

Assume there is a Harnack chain $B = B_1, B_2, \cdots, B_M = B'$ from $B \subset \Gamma_j^\alpha(Q)$ to $B' \subset \Gamma_{j+m}^\alpha(Q)$, by the triangle inequality

$$\iint_B |u(x) - u_{B'}|^2 dx \le 2 \iint_B |u(x) - u_B|^2 dx + 2M|B| \cdot \sum_{j=1}^{M-1} |u_{B_j} - u_{B_{j+1}}|^2. \tag{3.2.10}$$

Assume in addition that consecutive balls $B_i = B(x_i, r_i)$ and $B_{i+1} = B(x_{i+1}, r_{i+1})$ have comparable sizes

$$c r_{i+1} \le r_i \le C r_{i+1}, \tag{3.2.11}$$

with constants $0 < c < C$. We want to estimate $|u_{B_i} - u_{B_{i+1}}|$ by the integral of ∇u. By $B_i \cap B_{i+1} \neq \emptyset$ and (3.2.11), we know

$$B_{i+1} \subset \lambda B_i \quad \text{with } \lambda = 1 + 2/c > 1. \tag{3.2.12}$$

Hence

$$|u_{B_{i+1}} - u_{\lambda B_i}|^2 \le \left(\frac{1}{|B_{i+1}|} \iint_{B_{i+1}} |u(x) - u_{\lambda B_i}| dx\right)^2$$

$$\leq \left(\frac{\lambda r_i}{r_{i+1}}\right)^{2n} \fint_{\lambda B_i} |u(x) - u_{\lambda B_i}|^2 dx$$

$$\lesssim \frac{\lambda^{n+2} C^{2n}}{|B_i|} r_i^2 \iint_{\lambda B_i} |\nabla u(y)|^2 dy \qquad \text{by (3.2.9).} \tag{3.2.13}$$

Similarly

$$|u_{B_i} - u_{\lambda B_i}|^2 \lesssim \frac{\lambda^{n+2}}{|B_i|} r_i^2 \iint_{\lambda B_i} |\nabla u(y)|^2 dy.$$

Therefore

$$|u_{B_i} - u_{B_{i+1}}|^2 \leq \frac{A(n,c,C)}{|B_i|} r_i^2 \iint_{\lambda B_i} |\nabla u(y)|^2 dy. \tag{3.2.14}$$

Plugging (3.2.14) back into (3.2.10), we get

$$\iint_B |u(x) - u_{B'}|^2 dx \leq C(n,M,c,C) r_B^2 \sum_{k=1}^{M-1} \iint_{\lambda B_k} |\nabla u(y)|^2 dy$$

$$\leq \widetilde{C}(n,M,c,C) r_B^2 \iint_{\bigcup_{k=1}^{M-1} \lambda B_k} |\nabla u(y)|^2 dy. \tag{3.2.15}$$

On the other hand, by assumption the last ball $B' \subset \Gamma^{\alpha}_{j+m}(Q)$, we have

$$u_{B'}^2 \leq \sup_{\Gamma^{\alpha}_{j+m}(Q)} u^2 \leq C \left(\frac{2^{-(j+m)} r}{2^{-j} r}\right)^{2\beta} \sup_{B(Q,2^{-j}r)\cap\Omega} u^2 \lesssim_{n,\alpha} 2^{-2\beta m} u^2(A_j). \tag{3.2.16}$$

where A_j is the corkscrew point in $B(Q, 2^{-j}r) \cap \Omega$. The second inequality is by the boundary regularity (2.2.20) and the fact that u vanishes on Δ', and the last inequality by (2.2.24). Since $B \subset \Gamma^{\alpha}_j(Q)$ is a non-tangential ball and u is non-negative, by the Harnack principle

$$u(x) \geq c_0 u(A_j) \quad \text{for all } x \in B,$$

for a constant $c_0 < 1$. Hence

$$u_{B'}^2 \lesssim 2^{-2\beta m} u(A_j)^2 \lesssim 2^{-2\beta m} \fint_B u^2 dx. \tag{3.2.17}$$

Combining (3.2.15) and (3.2.17), we obtain

$$\iint_B u^2 dx \leq 2|B|\,(u_{B'})^2 + 2 \iint_B |u(x) - u_{B'}|^2 dx$$

$$\leq A(n,\alpha,c,C) 2^{-2\beta m} \iint_B u^2 dx + \widetilde{C}(n,M,c,C) r_B^2 \iint_{\bigcup_{k=1}^{M-1} \lambda B_k} |\nabla u(y)|^2 dy. \tag{3.2.18}$$

Choose m big enough such that $A(n,\alpha,c,C)2^{-\beta m} \le 1/2$, then we can absorb the first term on the right hand side of (3.2.18) to the left, and obtain

$$\iint_B u^2 dx \lesssim_{n,\alpha,c,C} r_B^2 \iint_{\bigcup_{k=1}^{M-1} \lambda B_k} |\nabla u(y)|^2 dy. \tag{3.2.19}$$

Note that after m is fixed, by the Harnack chain condition (see Definition 2.1.6), the number of balls $M \le C(m)$ is also fixed, thus we omit the dependence on M in the above inequality. This is Poincaré inequality for non-tangential balls.

In order to prove the Poincaré inequality (3.2.6) for non-tangential cones, we just need to cover $\Gamma_j^\alpha(Q)$ by balls; we also need to choose the Harnack chain carefully so that the integration region $\cup_{k=1}^{M-1} 2\lambda B_k$ in the right hand side of (3.2.19) is also contained in a non-tangential cone, possibly of a bigger aperture and wider stripe. Let us first make the following simple observation:

Observation 1. Let B be a Harnack ball with constants (C_1, C_2) (see the definition of Harnack balls in (HB)). Assume B contains some point $X \in \Gamma^\alpha(Q)$, then

$$B \subset \Gamma^{\alpha(1+\widetilde{C}_1)}(Q), \quad \widetilde{C}_1 > 0 \text{ is a constant only depending on } C_1.$$

If in adition $X \in \Gamma_j^\alpha(Q)$, then $|X - Q| \approx 2^{-j} r$, and we can get more precise estimate:

Observation 2. Assume B contains a non-tangential point $X \in \Gamma_j^\alpha(Q)$, then

$$B \subset \Gamma_{j-1 \to j+n_0}^{\alpha(1+\widetilde{C}_1)}(Q), \quad n_0 \text{ is an integer depending only on } \alpha \text{ and } C_1.$$

Moreover, by induction:

Observation 3. If $B_1, B_2, \ldots, B_k$ are Harnack balls with constants (C_1, C_2) such that $B_j \cap B_{j+1} \neq \emptyset$, and B_1 contains some point $X \in \Gamma_j^\alpha(Q)$, then

$$\bigcup_{j=1}^k B_j \subset \Gamma_{j-k \to j+n_0+(k-1)m_0}^{\alpha(1+\widetilde{C}_1)^k}(Q),$$

where m_0 is an integer depending only on C_1 (more precisely, m_0 is such that $1/(1+\widetilde{C}_1) \ge 2^{-m_0}$).

Let A and A' be arbitrary points in $\Gamma_j^\alpha(Q)$ and $\Gamma_{j+m}^\alpha(Q)$ respectively. Then

$$\rho = \min(\delta(A), \delta(A')) \ge 2^{-(j+m)-1} r/\alpha, \qquad |A - A'| \le 2 \cdot 2^{-j} r \lesssim 2^m \rho.$$

By the Harnack chain condition, there is a chain of open Harnack balls $B_1, B_2, \cdots, B_M$ with constants (C_1, C_2) that connects A to A', and the number of balls $M \le C(m)$. By Observation 2, the balls B_1 and B_M are non-tangential balls of aperture $\alpha(1+\widetilde{C}_1)$:

$$B_1 \subset \Gamma_{j-1 \to j+n_0}^{\alpha(1+\widetilde{C}_1)}(Q), \quad B_M \subset \Gamma_{j+m-1 \to j+m+n_0}^{\alpha(1+\widetilde{C}_1)}(Q).$$

Simple computation shows that the sizes of two consecutive Harnack balls are comparable:

$$\frac{C_1}{C_2+1}\operatorname{diam}(B_j) \le \operatorname{diam}(B_{j+1}) \le \frac{C_2+1}{C_1}\operatorname{diam}(B_j). \tag{3.2.20}$$

Recall we showed in Section 3.2.1 (see (3.2.12)) that (3.2.20) implies $B_{j+1} \subset \lambda B_j$, with a constant λ depending on C_1, C_2. As we have discussed in Remark (2) after Definition 2.1.6, if Ω satisfies the Harnack chain condition with (2.1.8), we may choose C_1 (lower bound constant for the Harnack ball) appropriately such that the enlarged ball $\widetilde{B}_j = \lambda B_j$ still lies in Ω, and moreover, its distance to the boundary is still comparable to its diameter. More precisely, it is an easy exercise to show that if we choose $C_1 \approx 26C^4$, the enlarged balls $\widetilde{B}_j$'s are still Harnack balls with modified constants:

$$\frac{3}{2}\operatorname{diam}(\widetilde{B}_j) \le \delta(\widetilde{B}_j) \le C_2 \operatorname{diam}(\widetilde{B}_j). \tag{3.2.21}$$

Denote $\mathrm{IT}(A, A') = \cup_{i=1}^{M-1}\widetilde{B}_i$ (IT stands for "integration tube"). By (3.2.21) and Observation 3,

$$\mathrm{IT}(A, A') \subset \Gamma^{h(\alpha,m)}_{j-M\to j+n_0+Mm_0}(Q),$$

where $h(\alpha, m), n_0, m_0$ depend on the constants of the Harnack balls $3/2, C_2$, the number of balls $M = O(m)$ and the aperture α we start with. Thus by (3.2.19)

$$\iint_{B_1} u^2(x)dx \lesssim_{n,\alpha} r_{B_1}^2 \iint_{\mathrm{IT}(A,A')} |\nabla u(y)|^2 dy \lesssim_{n,\alpha} (2^{-j}r)^2 \iint_{\Gamma^{h(\alpha,m)}_{j-M\to j+n_0+Mm_0}(Q)} |\nabla u(y)|^2 dy.$$

To summarize, for any $A \in \Gamma^{\alpha}_j(Q)$, we can find a Harnack ball B containing A which satisfies $B \subset \Gamma^{\alpha(1+\widetilde{C}_1)}_{j-1\to j+n_0}(Q)$ and

$$\iint_B u^2(x)dx \lesssim_{n,\alpha} (2^{-j}r)^2 \iint_{\Gamma^{\overline{\alpha}}_{j-m_1\to j+m_2}(Q)} |\nabla u(y)|^2 dy, \tag{3.2.22}$$

where the aperture $\overline{\alpha} > \alpha$, and n_0, m_1, m_2 are integers depending on α. We cover $\Gamma^{\alpha}_j(Q)$ by such Harnack balls:

$$\Gamma^{\alpha}_j(Q) \subset \bigcup_{X\in\Gamma^{\alpha}_j(Q)} B^X \subset \Gamma^{\alpha(1+\widetilde{C}_1)}_{j-1\to j+n_0}(Q), \tag{3.2.23}$$

from which we can extract a Vitali sub-covering $\Gamma^{\alpha}_j(Q) \subset \cup_k B^k$, such that $\{B^k/5\}$ are pairwise disjoint. By the definition (3.2.3) and (3.2.4), the set $\Gamma^{\alpha(1+\widetilde{C}_1)}_{j-1\to j+n_0}(Q)$ is contained in an annulus with small radius $2^{-(j+n_0)-1}r$ and big radius $2^{-(j-1)}r$. By the disjointedness of $\{B^k/5\}$'s and the fact that each $B^k/5$ has radius comparable to $2^{-j}r$, we can show that the number of balls in the Vitali covering is bounded by a constant $N = N(n, \alpha)$.

Finally, by the finite overlap of Vitali covering and (3.2.22), we have

$$\begin{aligned}\iint_{\Gamma_j^\alpha(Q)} u^2(x)dx &\le C(n)\sum_k \iint_{B^k} u^2 dX \\ &\lesssim N(n,\alpha)\cdot(2^{-j}r)^2 \iint_{\Gamma^{\overline{\alpha}}_{j-m_1\to j+m_2}(Q)} |\nabla u(y)|^2 dy.\end{aligned}$$

This finishes the proof of Lemma 3.2.5.

3.2.2 From the Carleson measure estimate to estimate of the boundary value

Given $\alpha > 0$, let $\overline{\alpha} > \alpha$ and $m_1, m_2 \in \mathbb{N}$ be defined as in Lemma 3.2.5. Using the Poincaré type inequality (3.2.6), we can get a lower bound of the square function (defined in the cone $\Gamma^{\overline{\alpha}}(Q)$):

$$\begin{aligned}|S^{\overline{\alpha}}_{r/2}(u)(Q)|^2 &= \iint_{\Gamma^{\overline{\alpha}}_{r/2}(Q)} |\nabla u|^2 \delta(X)^{2-n} dX \\ &\ge \frac{1}{m_1+m_2}\sum_{j=m_1+1}^{\infty} \iint_{\Gamma^{\overline{\alpha}}_{j-m_1\to j+m_2}(Q)} |\nabla u|^2 \delta(X)^{2-n} dX \\ &\gtrsim \sum_{j=m_1}^{\infty} (2^{-j}r)^{2-n} \iint_{\Gamma^{\overline{\alpha}}_{j-m_1\to j+m_2}(Q)} |\nabla u|^2 dX \\ &\gtrsim \sum_{j=m_1}^{\infty} (2^{-j}r)^{2-n}\cdot(2^{-j}r)^{-2} \iint_{\Gamma^{\alpha}_j(Q)} u^2 dX \qquad \text{by (3.2.6)} \\ &\gtrsim \sum_{j=m_1}^{\infty} u^2(A_j),\end{aligned}$$

where $A_j \in \Gamma_j(Q)$ is a corkscrew point at the scale $2^{-j}r$. In the last inequality, we use the interior corkscrew condition, thus each stripe of cone $\Gamma_j(Q)$ contains a ball of radius comparable to $2^{-j-1}r$ (as long as α is chosen to be big, say $\alpha > 2M$, where M is the corkscrew constant). By the Harnack principle $u(A_{j+1}) \ge cu(A_j)$, where $c < 1$ is a constant independent of u and j. Thus

$$\sum_{j=m_1}^{\infty} u^2(A_j) \gtrsim u^2(A_{m_1}) \gtrsim u^2(A_1).$$

Recall for any $Q \in \Delta'/2$, the point $A_1 = A_1(Q)$ is a corkscrew point in $\Gamma_1(Q)$. Let A' be the corkscrew point in $T(\Delta'/2)$, again by the Harnack principle we have $u(A') \approx u(A_1)$. Therefore

$$|S^{\overline{\alpha}}_{r/2}(u)(Q)|^2 \gtrsim u^2(A_1) \gtrsim u^2(A'), \quad \text{for any } Q \in \Delta'/2.$$

Combining this with (6.2.26) and (6.2.27), we get

$$\sigma(\Delta')\|f\|^2_{BMO(\sigma)} \gtrsim \int_{\Delta'/2} |S^{\overline{\alpha}}_{r/2}(u)|^2 d\sigma \gtrsim \sigma(\Delta'/2)u^2(A') \gtrsim \sigma(\Delta')u^2(A'),$$

and thus

$$u(A') \lesssim \|f\|_{BMO(\sigma)}.$$

Let A be a corkscrew point in Δ. Since Δ and Δ' have the same radius r and they are $r-$distance apart, we have $u(A) \approx u(A')$. By the assumption f is supported on Δ,

$$u(A) = \int_\Delta f(Q)d\omega^A(Q) = \int_\Delta f(Q)K(A,Q)d\omega(Q) \approx \frac{1}{\omega(\Delta)}\int_\Delta f d\omega.$$

The last equality uses the estimate of $K(A,Q)$ when A is a corkscrew point in $T(\Delta)$ and $Q \in \Delta$ (see [Ke] Corollary 1.3.8). As a result, we proved the following estimate: Let f be a non-negative continuous function supported on Δ, then

$$\frac{1}{\omega(\Delta)}\int_\Delta f d\omega \leq C\|f\|_{BMO(\sigma)}. \tag{3.2.24}$$

3.2.3 Proof of $\omega_L \in A_\infty(\sigma)$

Let Δ be a surface ball with radius r. For $\epsilon > 0$ fixed, we want to find an $\eta = \eta(\epsilon)$, such that for any $E \subset \Delta$,

$$\frac{\sigma(E)}{\sigma(\Delta)} < \eta \quad \text{implies} \quad \frac{\omega(E)}{\omega(\Delta)} < \epsilon.$$

In fact, since σ and ω are Borel measures, we may assume E is an open subset of Δ.

Let $\delta > 0$ be a small constant to be determined later, we define the function

$$f(x) = \max\left\{0, 1 + \delta \log M_\sigma \chi_E(x)\right\} \tag{3.2.25}$$

where M_σ is the Hardy-Littlewood maximal function with respect to σ:

$$M_\sigma \chi_E(x) = \sup_{\widetilde{\Delta} \ni x} \frac{\sigma(\widetilde{\Delta} \cap E)}{\sigma(\widetilde{\Delta})}. \tag{3.2.26}$$

Since $(\partial\Omega, \sigma)$ is a space of homogeneous type, we can adapt the arguments in [CR] very easily and show that $\|\log M_\sigma \chi_E\|_{BMO(\sigma)}$ is bounded by some constant A (independent of the set E). Hence f is a BMO function and $\|f\|_{BMO(\sigma)} \leq A\delta$. Moreover, it is clear from the definitions (6.2.32) and (3.2.26) that $0 \leq f \leq 1$ and $f \equiv 1$ on the open set E.

Suppose $x \in \partial\Omega \setminus 2\Delta$, then $\operatorname{dist}(x, \Delta) \geq r$. Let $\widetilde{\Delta}$ be an arbitrary surface ball containing x. Since $E \subset \Delta$, in order for $\widetilde{\Delta} \cap E$ to be nonempty, the diameter of $\widetilde{\Delta}$ is at least r. Thus by Ahlfors regularity $\sigma(\widetilde{\Delta}) \gtrsim r^{n-1} \approx \sigma(\Delta)$. Therefore

$$M_\sigma \chi_E(x) = \sup_{\widetilde{\Delta} \ni x} \frac{\sigma(\widetilde{\Delta} \cap E)}{\sigma(\widetilde{\Delta})} \leq C\frac{\sigma(E)}{\sigma(\Delta)}. \tag{3.2.27}$$

This means, as long as $E \subset \Delta$ is such that

$$\frac{\sigma(E)}{\sigma(\Delta)} < \eta(\delta) = \frac{e^{-1/\delta}}{C}, \tag{3.2.28}$$

by (3.2.27) we have

$$1 + \delta \log M_\sigma \chi_E(x) < 1 + \delta \log e^{-1/\delta} = 0 \text{ outside of } 2\Delta,$$

hence $f \equiv 0$ outside of 2Δ. In other words,

$$\frac{\sigma(E)}{\sigma(\Delta)} < \eta \implies f \text{ is supported in } 2\Delta.$$

Next we want to use a mollification argument to approximate f by continuous functions, such that their BMO norms are uniformly bounded by that of f. Let φ be a radial-symmetric smooth function on $\mathbb{R}^n$ such that $\varphi = 1$ on $B_{1/2}$, $\operatorname{supp}\varphi \subset B_1$ and $0 \leq \varphi \leq 1$. Let

$$\varphi_\epsilon(z) = \frac{1}{\epsilon^{n-1}} \varphi\left(\frac{z}{\epsilon}\right), \quad f_\epsilon(x) = \frac{\int_{y \in \partial\Omega} f(y) \varphi_\epsilon(x-y) d\sigma(y)}{\int_{y \in \partial\Omega} \varphi_\epsilon(x-y) d\sigma(y)} \text{ for } x \in \partial\Omega. \tag{3.2.29}$$

The following lemma summarizes the properties of these f_ϵ's. The proof of (1) is just a standard mollification argument. However it requires more work to prove (2) and (3), since f_ϵ is a (normalized) convolution of f restricted to $\partial\Omega$, instead of all of $\mathbb{R}^n$. In particular, the proof depends on the properties of such function f defined in (6.2.32). We refer interested readers to Section 3.4 for the proof.

Lemma 3.2.30. *The following properties hold for f_ϵ's:*

1. *each f_ϵ is continuous, and is supported in 3Δ;*
2. *there is a constant C (independent of ϵ) such that $\|f_\epsilon\|_{BMO(\sigma)} \leq C\|f\|_{BMO(\sigma)}$;*
3. *$f(x) \leq \liminf_{\epsilon\to 0} f_\epsilon(x)$ for all x in their support 3Δ.*

The last property and Fatou's lemma imply

$$\int_{3\Delta} f(x) d\omega(x) \leq \int_{3\Delta} \liminf_{\epsilon\to 0} f_\epsilon(x) d\omega(x) \leq \liminf_{\epsilon\to 0} \int_{3\Delta} f_\epsilon(x) d\omega(x). \tag{3.2.31}$$

Since each f_ϵ is continuous, we can apply (6.2.25),

$$\frac{1}{\omega(3\Delta)} \int_{3\Delta} f_\epsilon(x) d\omega(x) \leq C\|f_\epsilon\|_{BMO(\sigma)} \leq C'\|f\|_{BMO(\sigma)}. \tag{3.2.32}$$

Combining (6.2.35) and (6.2.36), we get

$$\frac{1}{\omega(3\Delta)} \int_{3\Delta} f(x) d\omega(x) \leq C'\|f\|_{BMO(\sigma)} \leq C''\delta.$$

On the other hand, since $f \geq \chi_E$ and ω is a doubling measure,

$$\frac{1}{\omega(3\Delta)} \int_{3\Delta} f(x) d\omega(x) \geq \frac{\omega(E)}{\omega(3\Delta)} \gtrsim \frac{\omega(E)}{\omega(\Delta)}.$$

Therefore $\omega(E)/\omega(\Delta) \leq C\delta$ as long as the condition (6.2.33), i.e. $\sigma(E)/\sigma(\Delta) < \eta$ is satisfied. In other words, $\omega \in A_\infty(\sigma)$.

3.3 Converse to the Carleson measure estimate

Proposition 3.3.1. *Assume the elliptic measure $\omega \in A_\infty(\sigma)$. If $Lu = 0$ in Ω with boundary data $f \in C(\partial\Omega)$, then*

$$\|f\|^2_{BMO(\sigma)} \leq C \sup_{\Delta \subset \partial\Omega} \frac{1}{\sigma(\Delta)} \iint_{T(\Delta)} |\nabla u|^2 \delta(X) dX, \tag{3.3.2}$$

as long as the right hand side is finite.

Since $\omega \in A_\infty(\sigma)$, a classical result in harmonic analysis says $\|f\|_{BMO(\sigma)} \approx \|f\|_{BMO(\omega)}$. We may as well prove (3.3.2) for $\|f\|_{BMO(\omega)}$. In light of previous work [FN] and [FKN], Jerison and Kenig studied the Dirichlet problem with BMO boundary data and proved the following Theorem 3.3.3 for *the Laplacian on NTA domains* (see [JK] Theorem 9.6). The main ingredients of their proof are (2.2.26), (2.2.28) and a geometric localization theorem, which holds by the Carleson box constructed in [HM1, Section 3]. Therefore a similar proof is applicable in our case.

Theorem 3.3.3. *There exists a constant $C > 0$ such that*

$$\|f\|^2_{BMO(\omega)} \leq C \sup_{\Delta \subset \partial\Omega} \frac{1}{\omega(\Delta)} \iint_{T(\Delta)} |\nabla u|^2 G(X_0, X) dX, \tag{3.3.4}$$

on condition that the right hand side is bounded.

Proof. Let $\Delta = \Delta(Q, r)$ be an arbitrary surface ball. By [HM1] one can construct a Carleson box $\mathcal{D}$ satisfying $B(Q, 4r) \cap \Omega \subset \mathcal{D} \subset B(Q, 4Cr) \cap \Omega$, and $\mathcal{D}$ is a uniform domain with Ahlfors regular boundary. Assume r is small so that $X_0 \notin B(Q, 4Cr)$. By the interior corkscrew condition of $\mathcal{D}$, we can find a point $X_1 \in \mathcal{D} \setminus B(Q, 2r)$ and $\delta_1(X_1) := \text{dist}(X_1, \partial\mathcal{D}) \approx 2r$. For the elliptic operator L on $\mathcal{D}$, let ν be the elliptic measure and $G_{\mathcal{D}}(X_1, \cdot)$ the Green's function with pole at X_1.

We claim that for any surface ball $\Delta' \subset \Delta$.

$$\frac{\omega(\Delta')}{\omega(\Delta)} \approx \nu(\Delta'). \tag{3.3.5}$$

In fact, let Y_0 and Y' be corkscrew points with respect to Δ and Δ' respectively, and let r' be the radius of Δ'. Apply (2.2.26) to the domains Ω and $\mathcal{D}$, we get

$$\frac{\omega(\Delta')}{\nu(\Delta')} \approx \frac{G(X_0, Y')(r')^{n-2}}{G_{\mathcal{D}}(X_1, Y')(r')^{n-2}} \approx \frac{G(X_0, Y')}{G_{\mathcal{D}}(X_1, Y')}. \tag{3.3.6}$$

And similarly

$$\frac{\omega(\Delta)}{\nu(\Delta)} \approx \frac{G(X_0, Y_0)}{G_{\mathcal{D}}(X_1, Y_0)}. \tag{3.3.7}$$

Note that $X_0, X_1 \notin B(Q, 2r) \cap \Omega$, by the boundary comparison principle (2.2.32)

$$\frac{G(X_0, Y')}{G_{\mathcal{D}}(X_1, Y')} \approx \frac{G(X_0, Y_0)}{G_{\mathcal{D}}(X_1, Y_0)}. \tag{3.3.8}$$

It follows from (3.3.6), (3.3.7) and (3.3.8) that $\omega(\Delta')/\nu(\Delta') \approx \omega(\Delta)/\nu(\Delta)$. Since $\nu(4C\Delta) = \nu(\partial\mathcal{D}) = 1$ and ν is a doubling measure, we have $\nu(\Delta) \approx 1$ and thus (3.3.5).

The above estimate (3.3.5) in particular implies

$$\frac{1}{\omega(\Delta)} \int_{\Delta} |f - c_\Delta|^2 d\omega \approx \int_{\Delta} |f - c_\Delta|^2 d\nu \leq \int_{\partial\mathcal{D}} |u - c_\Delta|^2 d\nu. \tag{3.3.9}$$

Here we choose the constant $c_\Delta = \int_{\partial\mathcal{D}} u d\nu$, so that $\int_{\partial\mathcal{D}} (u - c_\Delta)\, d\nu = 0$. We have the following global estimate on $\mathcal{D}$ (see Lemma 1.5.1 in [Ke])

$$\frac{1}{2} \int_{\partial\mathcal{D}} |u - c_\Delta|^2 d\nu = \iint_{\mathcal{D}} A\nabla u \cdot \nabla u \, G_{\mathcal{D}}(X_1, Y) dY \leq \lambda \iint_{\mathcal{D}} |\nabla u|^2 G_{\mathcal{D}}(X_1, Y) dY. \tag{3.3.10}$$

Using (3.3.7) and $\nu(\Delta) \approx 1$, we have

$$G_{\mathcal{D}}(X_1, Y_0) \approx \frac{G(X_0, Y_0)}{\omega(\Delta)} \nu(\Delta) \approx \frac{G(X_0, Y_0)}{\omega(\Delta)}. \tag{3.3.11}$$

Since $X_0 \notin \mathcal{D}$, by considering Harnack chains in $\mathcal{D} \setminus B(X_1, \delta_1(X_1)/3)$ or using the boundary comparison principle (2.2.32), (3.3.11) implies

$$G_{\mathcal{D}}(X_1, Y) \approx \frac{G(X_0, Y)}{\omega(\Delta)}, \quad \text{for all } Y \in \mathcal{D} \setminus B\Big(X_1, \frac{\delta_1(X_1)}{3}\Big).$$

Thus

$$\iint_{\mathcal{D} \setminus B\left(X_1, \frac{\delta_1(X_1)}{3}\right)} |\nabla u|^2 G_{\mathcal{D}}(X_1, Y) dY \approx \frac{1}{\omega(\Delta)} \iint_{\mathcal{D}} |\nabla u|^2 G(X_0, Y) dY. \tag{3.3.12}$$

On the other hand on $B(X_1, \delta_1(X_1)/3)$, by the Harnack principle

$$|\nabla u|^2 \lesssim \frac{1}{\delta_1(X_1)^n} \iint_{B\left(X_1, \frac{\delta_1(X_1)}{3}\right)} |\nabla u|^2 dY. \tag{3.3.13}$$

By the choice of X_1 we know $\delta(X_1) \approx \delta_1(X_1) \approx 2r$ and $X_1 \in B(Q, 4Cr)$. Let $Q_{X_1} \in \partial\Omega$ satisfy $|X_1 - Q_{X_1}| = \delta(X_1)$, then by properties (2.2.26) and (2.2.28),

$$\delta_1(X_1)^{n-2} G(X_0, Y) \approx \omega(\Delta(Q_{X_1}, \delta_1(X_1)) \approx \omega(\Delta) \tag{3.3.14}$$

for any $Y \in B(X_1, \delta_1(X_1)/3)$. Plugging (3.3.14) into (3.3.13), we get

$$\begin{aligned} |\nabla u|^2 &\lesssim \frac{1}{\delta_1(X_1)^n} \frac{\delta_1(X_1)^{n-2}}{\omega(\Delta)} \iint_{B\left(X_1, \frac{\delta_1(X_1)}{3}\right)} |\nabla u|^2 G(X_0, Y) dY \\ &\lesssim \frac{1}{r^2 \omega(\Delta)} \iint_{B\left(X_1, \frac{\delta_1(X_1)}{3}\right)} |\nabla u|^2 G(X_0, Y) dY. \end{aligned} \tag{3.3.15}$$

By the maximal principle and the bound on Green's function,

$$G_{\mathcal{D}}(X_1, Y) \leq G(X_1, Y) \lesssim \frac{1}{|Y - X_1|^{n-2}}. \tag{3.3.16}$$

The last inequality is independent of $\mathcal{D}$ and X_1. Combining (3.3.15) and (3.3.16), we get

$$\begin{aligned} &\iint_{B\left(X_1, \frac{\delta_1(X_1)}{3}\right)} |\nabla u|^2 G_{\mathcal{D}}(X_1, Y) dY \\ &\qquad \lesssim \left(\frac{1}{r^2 \omega(\Delta)} \iint_{B(X_1, \frac{\delta_1(X_1)}{3})} |\nabla u|^2 G(X_0, Y) dY \right) \cdot \iint_{B\left(X_1, \frac{\delta_1(X_1)}{3}\right)} \frac{1}{|Y - X_1|^{n-2}} dY \\ &\qquad \lesssim \left(\frac{1}{r^2 \omega(\Delta)} \iint_{B(X_1, \frac{\delta_1(X_1)}{3})} |\nabla u|^2 G(X_0, Y) dY \right) \cdot \delta_1(X_1)^2 \\ &\qquad \lesssim \frac{1}{\omega(\Delta)} \iint_{\mathcal{D}} |\nabla u|^2 G(X_0, Y) dY. \end{aligned} \tag{3.3.17}$$

Summing up (3.3.12) and (3.3.17), we get

$$\iint_{\mathcal{D}} |\nabla u|^2 G_{\mathcal{D}}(X_1, Y) dY \lesssim \frac{1}{\omega(\Delta)} \iint_{\mathcal{D}} |\nabla u|^2 G(X_0, Y) dY. \tag{3.3.18}$$

Together with (3.3.9) and (3.3.10), we deduce

$$\begin{aligned} \frac{1}{\omega(\Delta)} \int_{\Delta} |f - c_\Delta|^2 d\omega &\lesssim \frac{1}{\omega(\Delta)} \iint_{\mathcal{D}} |\nabla u|^2 G(X_0, Y) dY \\ &\lesssim \frac{1}{\omega(4C\Delta)} \iint_{B(Q, 4Cr) \cap \Omega} |\nabla u|^2 G(X_0, Y) dY \\ &\lesssim \sup_{\Delta' \subset \partial\Omega} \frac{1}{\omega(\Delta')} \iint_{T(\Delta')} |\nabla u|^2 G(X_0, Y) dY. \end{aligned}$$

Recall for $X \in T(\Delta)$, the set Δ^X is defined as $\{Q \in \partial\Omega : X \in \Gamma(Q)\}$. By (3.0.1) and (2.2.26), (2.2.28), we get $G(X_0, X)\delta(X)^{n-2} \approx \omega(\Delta^X)$. Thus by changing the order of integration,

$$\begin{aligned}\iint_{T(\Delta)} |\nabla u|^2 G(X_0, X) dX &\approx \iint_{T(\Delta)} |\nabla u|^2 \delta(X)^{2-n} \omega(\Delta^X) dX \\ &\leq \int_{Q \in (\alpha+2)\Delta} \iint_{X \in \Gamma_{\alpha r}(Q)} |\nabla u|^2 \delta(X)^{2-n} dX d\omega \\ &= \int_{\widetilde{\Delta}} S^2_{(\alpha+1)r}(u) d\omega, \qquad (3.3.19)\end{aligned}$$

where $\widetilde{\Delta} = (\alpha+1)\Delta$. Since $\omega \in A_\infty(\sigma)$, there exists $q > 1$ such that the Radon-Nikodym derivative $k = d\omega/d\sigma \in B_q(\sigma)$. Let $p > 1$ be the conjugate of q, i.e. $1/p + 1/q = 1$, we have

$$\begin{aligned}\int_{\widetilde{\Delta}} S^2_{(\alpha+1)r}(u) d\omega = \int_{\widetilde{\Delta}} S^2_{(\alpha+1)r}(u) k d\sigma &\leq \left(\int_{\widetilde{\Delta}} k^q d\sigma\right)^{1/q} \left(\int_{\widetilde{\Delta}} S^{2p}_{(\alpha+1)r}(u) d\sigma\right)^{1/p} \\ &\leq C\sigma(\widetilde{\Delta})^{1/q} \left(\fint_{\widetilde{\Delta}} k d\sigma\right) \left(\int_{\widetilde{\Delta}} S^{2p}_{(\alpha+1)r}(u) d\sigma\right)^{1/p} \\ &\leq C\omega(\Delta) \sup_{\Delta_\tau \subset \partial\Omega} \left(\frac{1}{\sigma(\Delta_\tau)} \int_{\Delta_\tau} S^{2p}_\tau(u) d\sigma\right)^{1/p}. \qquad (3.3.20)\end{aligned}$$

We claim the following theorem holds for the truncated square function:

Theorem 3.3.21. *For any $2 < t < \infty$,*

$$\sup_{\substack{0<r<\operatorname{diam}\Omega \\ \Delta_r \subset \partial\Omega}} \left(\frac{1}{\sigma(\Delta_r)} \int_{\Delta_r} |S_r(u)|^t \, d\sigma\right)^{1/t} \leq C \sup_{\Delta \subset \partial\Omega} \frac{1}{\sigma(\Delta)} \iint_{T(\Delta)} |\nabla u|^2 \delta(X) dX, \qquad (3.3.22)$$

on condition that the right hand side is finite. Here Δ_r denotes any surface ball of radius r.

Assume this theorem holds, then (3.3.2) follows from combining (3.3.4), (3.3.19), (3.3.20) and (3.3.22), which concludes the proof of Proposition 3.3.1. Now we are going to prove this theorem by combining several lemmas of the truncated square function. In [MPT], the authors have proved similar lemmas for the square function (see Proposition 4.5, Lemma 4.6 and Lemma 6.2), and we are adapting their arguments to the *truncated square function*. The proof of the following two lemmas is similar to the case of non-truncated square function, so we postpone it to Appendix 3.4.2.

Lemma 3.3.23. *For any $r, \lambda > 0$, the set $\{Q \in \partial\Omega : S_r u(Q) > \lambda\}$ is open in $\partial\Omega$.*

Lemma 3.3.24. *Let $2 < t < \infty$. Assume $\overline{\alpha}$ is an aperture bigger than α and $\Delta = \Delta(Q_0, r)$ is a surface ball of radius r, then*

$$\int_\Delta |S^{\overline{\alpha}}_r u(Q)|^t d\sigma(Q) \leq \int_{2(\alpha+2)\Delta} |S_{2(\alpha+2)r} u(Q)|^t d\sigma(Q).$$

Moreover, we have the following "good-λ" inequality between $S_r u$ and the Carleson type function

$$Cu(Q) = \sup_{\Delta \ni Q} \frac{1}{\sigma(\Delta)} \iint_{\Gamma(Q)} |\nabla u|^2 \delta(X) dX. \tag{3.3.25}$$

Lemma 3.3.26. *There exist an aperture $\overline{\alpha} > \alpha$ and a constant $C > 0$, such that for any surface ball $\Delta = \Delta(Q_0, r)$ and any $\lambda, \gamma > 0$*

$$\sigma\Big(\big\{Q \in \Delta : S_r u(Q) > 2\lambda, Cu(Q) \leq \gamma\lambda\big\}\Big) \leq C\gamma^2 \sigma\Big(\big\{Q \in 4\Delta : S^{\overline{\alpha}}_{4r} u(Q) > \lambda\big\}\Big) \tag{3.3.27}$$

Proof. Let $\overline{\alpha} = \alpha + 3$. Consider the open set $O = \{Q \in 4\Delta : S^{\overline{\alpha}}_{4r} u(Q) > \lambda\}$. Similar to Lemma 3.7 and Lemma 6.2 in [MPT], let $\cup_k \Delta_k$ be a Whitney decomposition of O, such that

$$\text{for each } k, \quad \Delta\big(Q_k, \frac{1}{24} d(Q_k)\big) \subset \Delta_k \subset \Delta\big(Q_k, \frac{1}{2} d(Q_k)\big).$$

Here $Q_k \in O$, and $d(Q_k) = \operatorname{dist}(Q_k, O^c) > 0$. We claim that for all Δ_k such that $\Delta_k \cap \Delta \neq \emptyset$, we have

$$\sigma\Big(\big\{Q \in \Delta_k : S_r u(Q) > 2\lambda, Cu(Q) \leq \gamma\lambda\big\}\Big) \leq C\gamma^2 \sigma(\Delta_k). \tag{3.3.28}$$

This is clearly true if the left hand side is empty. Assume it is not empty, and

$$\text{there is some } Q'_k \in \big\{Q \in \Delta_k : S_r u(Q) > 2\lambda, Cu(Q) \leq \gamma\lambda\big\}. \tag{3.3.29}$$

Note that

$$d(Q_k) = \operatorname{dist}(Q_k, O^c) = \min\Big\{\operatorname{dist}\big(Q_k, \big\{Q \in 4\Delta : S^{\overline{\alpha}}_{4r} u(Q) \leq \lambda\big\}\big), \operatorname{dist}(Q_k, (4\Delta)^c)\Big\},$$

we need to consider two cases.

Case 1. Assume Q_k is such that

$$d(Q_k) = |Q_k - P_k|, \quad \text{for some } P_k \in 4\Delta \text{ satisfying } S^{\overline{\alpha}}_{4r} u(P_k) \leq \lambda.$$

Let $Q \in \Delta_k$ be arbitrary, recall that $\Delta_k \subset \Delta(Q_k, d(Q_k)/2)$, hence

$$|Q - P_k| \leq |Q - Q_k| + |Q_k - P_k| < \frac{1}{2} d(Q_k) + d(Q_k) = \frac{3}{2} d(Q_k).$$

For any $X \in \Gamma_r(Q)$, we define the functions

$$u_1(X) = u(X)\chi_{\{\delta(X) \geq d(Q_k)/2\}}, \text{ and } u_2(X) = u(X)\chi_{\{\delta(X) < d(Q_k)/2\}}. \tag{3.3.30}$$

Clearly $S_r u(Q) \leq S_r u_1(Q) + S_r u_2(Q)$.

If $X \in \Gamma_r(Q)$ is such that $\delta(X) \geq d(Q_k)/2$, we have

$$|X - P_k| \leq |X - Q| + |Q - P_k| < \alpha\delta(X) + \frac{3}{2}d(Q_k) \leq \overline{\alpha}\delta(X),$$

and

$$|X - P_k| \leq |X - Q| + |Q - P_k| < r + 3\delta(X) \leq 4r.$$

In other words $X \in \Gamma^{\overline{\alpha}}_{4r}(P_k)$. Hence

$$\begin{aligned} |S_r u_1(Q)|^2 &= \iint_{\Gamma_r(Q)\cap\{\delta(X)\geq d(Q_k)/2\}} |\nabla u|^2 \delta(X)^{2-n} dX \\ &\leq \iint_{\Gamma^{\overline{\alpha}}_{4r}(P_k)} |\nabla u|^2 \delta(X)^{2-n} dX \\ &= |S^{\overline{\alpha}}_{4r} u(P_k)|^2 \\ &\leq \lambda^2. \end{aligned} \tag{3.3.31}$$

If $X \in \Gamma_r(Q)$ is such that $\delta(X) < d(Q_k)/2$, recall $Q \in \Delta_k \subset \Delta(Q_k, d(Q_k)/2)$ and (3.3.29), we have

$$\begin{aligned} |X - Q_k'| &\leq |X - Q| + |Q - Q_k| + |Q_k - Q_k'| \\ &\leq (1+\alpha)\delta(X) + \frac{1}{2}d(Q_k) + \frac{1}{2}d(Q_k) \\ &< \left(\frac{\alpha}{2} + \frac{3}{2}\right) d(Q_k). \end{aligned}$$

Hence

$$\begin{aligned} \int_{\Delta_k} S_r^2 u_2(Q) d\sigma &= \int_{\Delta_k} \iint_{\Gamma_r(Q)\cap\{\delta(X)<d(Q_k)/2\}} |\nabla u|^2 \delta(X)^{2-n} dX \\ &\leq \iint_{B(Q_k', (\frac{\alpha}{2}+\frac{3}{2})d(Q_k))} |\nabla u|^2 \delta(X) dX \\ &\leq |Cu(Q_k')|^2 \cdot \sigma\left(\Delta\left(Q_k', \left(\frac{\alpha}{2} + \frac{3}{2}\right) d(Q_k)\right)\right), \end{aligned} \tag{3.3.32}$$

where the Carleson type function is defined in (3.3.25). By (3.3.29), we know that $Cu(Q_k') \leq \gamma\lambda$. In addition, σ is Ahlfors regular and $\Delta(Q_k, d(Q_k)/24) \subset \Delta_k$. Therefore it follows from (3.3.32)

$$\int_{\Delta_k} S_r^2 u_2(Q) d\sigma \leq \gamma^2\lambda^2 \cdot C_2 \left(\frac{\alpha}{2} + \frac{3}{2}\right)^{n-1} d(Q_k)^{n-1} \leq C\gamma^2\lambda^2\sigma(\Delta_k), \tag{3.3.33}$$

where the constant C only depends on the aperture α and the Ahlfors regular constants of σ. On the other hand,

$$\int_{\Delta_k} S_r^2 u_2(Q) d\sigma \geq \lambda^2 \sigma\Big(\big\{Q \in \Delta_k : S_r u_2(Q) > \lambda\big\}\Big),$$

hence $\sigma\Big(\big\{Q \in \Delta_k : S_r u_2(Q) > \lambda\big\}\Big) \leq C\gamma^2\sigma(\Delta_k)$.

Recall $S_r u_1(Q) \leq \lambda$ for all $Q \in \Delta_k$ (see (3.3.31)), therefore

$$\begin{aligned}\sigma\Big(\big\{Q \in \Delta_k : S_r u(Q) > 2\lambda, Cu(Q) \leq \gamma\lambda\big\}\Big) &\leq \sigma\Big(\big\{Q \in \Delta_k : S_r u_2(Q) > \lambda, Cu(Q) \leq \gamma\lambda\big\}\Big) \\ &\leq \sigma\Big(\big\{Q \in \Delta_k : S_r u_2(Q) > \lambda\big\}\Big) \\ &\leq C\gamma^2\sigma(\Delta_k).\end{aligned}$$

Case 2. Assume Q_k is such that $d(Q_k) = \text{dist}(Q_k, (4\Delta)^c)$. We only consider the Δ_k's such that $\Delta_k \cap \Delta \neq \emptyset$, and assume R_k is a point in the intersection. In particular $R_k \in \Delta_k \subset \Delta(Q_k, d(Q_k)/2)$. So

$$\text{dist}(R_k, (4\Delta)^c) \leq |R_k - Q_k| + \text{dist}(Q_k, (4\Delta)^c) < \frac{1}{2}d(Q_k) + d(Q_k) = \frac{3}{2}d(Q_k).$$

On the other hand, suppose $\text{dist}(R_k, (4\Delta)^c) = |R_k - R|$ for some $R \in (4\Delta)^c$, then

$$\text{dist}(R_k, (4\Delta)^c) \geq |R - Q_0| - |R_k - Q_0| > 4r - r = 3r.$$

It follows that $d(Q_k)/2 > r$. In particular, for any $Q \in \Delta_k$ and $X \in \Gamma_r(Q)$, we have $\delta(X) \leq r < d(Q_k)/2$. In other words, $u = u_2$ (see (3.3.30)). Similar to (3.3.33), one can show

$$\int_{\Delta_k} S_r^2 u(Q) d\sigma = \int_{\Delta_k} S_r^2 u_2(Q) d\sigma \leq C\gamma^2\lambda^2\sigma(\Delta_k).$$

Therefore

$$\sigma\Big(\big\{Q \in \Delta_k : S_r u(Q) > 2\lambda, Cu(Q) \leq \gamma\lambda\big\}\Big) \leq \sigma\Big(\big\{Q \in \Delta_k : S_r u(Q) > 2\lambda\big\}\Big) \leq \frac{C\gamma^2\sigma(\Delta_k)}{4}.$$

This finishes the proof of (3.3.28).

Summing up (3.3.28) for Δ_k's such that $\Delta_k \cap \Delta \neq \emptyset$, we obtain

$$\begin{aligned}&\sigma\left(\left\{Q \in \bigcup_{k:\Delta_k\cap\Delta\neq\emptyset} \Delta_k : S_r u(Q) > 2\lambda, Cu(Q) \leq \gamma\lambda\right\}\right) \\ &\leq C\gamma^2\sigma\left(\bigcup_k \Delta_k\right) \leq C\gamma^2\sigma\Big(\big\{Q \in 4\Delta : S_{4r}^{\overline{\alpha}} u(Q) > \lambda\big\}\Big). \qquad (3.3.34)\end{aligned}$$

The last inequality is because $\{\Delta_k\}$ is a Whitney decomposition of $\{Q \in 4\Delta : S_{4r}^{\overline{\alpha}} u(Q) > \lambda\}$. It also implies

$$\bigcup_{k:\Delta_k\cap\Delta\neq\emptyset} \Delta_k \supset \big\{Q \in \Delta : S_{4r}^{\overline{\alpha}} u(Q) > \lambda\big\}.$$

Therefore

$$\left\{Q \in \bigcup_{k:\Delta_k\cap\Delta\neq\emptyset} \Delta_k : S_r u(Q) > 2\lambda, Cu(Q) \le \gamma\lambda\right\}$$
$$\supset \left\{Q \in \Delta : S^{\overline{\alpha}}_{4r} u > \lambda \text{ and } S_r u(Q) > 2\lambda, Cu(Q) \le \gamma\lambda\right\}$$
$$= \left\{Q \in \Delta : S_r u(Q) > 2\lambda, Cu(Q) \le \gamma\lambda\right\} \tag{3.3.35}$$

For the last equality, we use $\overline{\alpha} > \alpha$ and thus $\{S_r u > 2\lambda\} \subset \{S^{\overline{\alpha}}_{4r} u > \lambda\}$. Combining (3.3.34) and (3.3.35), we get

$$\sigma\Big(\{Q \in \Delta : S_r u > 2\lambda, Cu \le \gamma\lambda\}\Big) \le C\gamma^2\sigma\Big(\{Q \in 4\Delta : S^{\overline{\alpha}}_{4r} u(Q) > \lambda\}\Big).$$

By Lemma 3.3.26,

$$\begin{aligned}
\int_\Delta |S_r u|^t d\sigma &= t\int_0^\infty \lambda^{t-1}\sigma\Big(\{Q \in \Delta : S_r u > \lambda, Cu \le \gamma\lambda/2\}\Big)d\lambda \\
&\qquad + t\int_0^\infty \lambda^{t-1}\sigma\Big(\{Q \in \Delta : S_r u > \lambda, Cu > \gamma\lambda/2\}\Big)d\lambda \\
&\le t\int_0^\infty \lambda^{t-1}\cdot C\gamma^2\sigma\Big(\{Q \in 4\Delta : S^{\overline{\alpha}}_{4r} u > \lambda/2\}\Big)d\lambda \\
&\qquad + t\int_0^\infty \lambda^{t-1}\sigma\Big(\{Q \in \Delta : Cu > \gamma\lambda/2\}\Big)d\lambda \\
&= C\gamma^2 2^t \int_{4\Delta} |S^{\overline{\alpha}}_{4r} u|^t d\sigma + \left(\frac{2}{\gamma}\right)^t \int_\Delta |Cu|^t d\sigma \\
&\le C'\gamma^2 \int_{4\Delta} |S^{\overline{\alpha}}_{4r} u|^t d\sigma + \left(\frac{2}{\gamma}\right)^t |\mathscr{C}(u)|^t \sigma(\Delta).
\end{aligned} \tag{3.3.36}$$

Here $\mathscr{C}(u) = \sup_{\Delta\subset\partial\Omega} \frac{1}{\sigma(\Delta)} \iint_{T(\Delta)} |\nabla u|^2 \delta(X) dX$ stands for the Carleson measure defined by the function u, and by definition $Cu(Q) \le \mathscr{C}(u)$ for all $Q \in \partial\Omega$. Apply Lemma 3.3.24 to the right hand side of (3.3.36), it becomes

$$\int_\Delta |S_r u|^t d\sigma \le C''\gamma^2 \int_{A\Delta} |S_{Ar} u|^t d\sigma + \left(\frac{2}{\gamma}\right)^t |\mathscr{C}(u)|^t \sigma(\Delta), \tag{3.3.37}$$

where $A = 8(\alpha + 1)$ is a constant and $A\Delta = \Delta(Q_0, Ar)$. If the radius r is such that $Ar < \operatorname{diam}\Omega$, we can rewrite the above inequality in the following form:

$$⨍_\Delta |S_r u|^t d\sigma \le \widetilde{C}\gamma^2 ⨍_{A\Delta} |S_{Ar} u|^t d\sigma + \left(\frac{2}{\gamma}\right)^t |\mathscr{C}(u)|^t. \tag{3.3.38}$$

Pick γ (depending on α) so that $\widetilde{C}\gamma^2 = 1/4$. Fix such γ fixed, denote $C_1 = (2/\gamma)^t$, then

$$⨍_\Delta |S_r u|^t d\sigma \le \frac{1}{4} ⨍_{A\Delta} |S_{Ar} u|^t d\sigma + C_1 |\mathscr{C}(u)|^t. \tag{3.3.39}$$

Theorem 6.1 in [MPT] states the following global estimate

$$\int_{\partial\Omega} |Su|^t d\sigma \leq C \int_{\partial\Omega} |Cu|^t d\sigma \leq C|\mathscr{C}(u)|^t \sigma(\partial\Omega). \tag{3.3.40}$$

We claim the "contraction"estimate (3.3.39), together with the global estimate (3.3.40) implies

$$\sup_{\Delta_r \subset \partial\Omega} \left(\fint_{\Delta_r} |S_r u|^t d\sigma \right)^{1/t} \leq C \cdot \mathscr{C}(u) = C \sup_{\Delta\subset\partial\Omega} \frac{1}{\sigma(\Delta)} \iint_{T(\Delta)} |\nabla u|^2 \delta(X) dX. \tag{3.3.41}$$

Firstly, for an arbitrary $r > 0$, let k be a positive integer such that $A^k r < \operatorname{diam}\Omega \leq A^{k+1} r$, then $\sigma(A^k\Delta) \approx \sigma(\partial\Omega)$. (The constants only depends on A and the Ahlfors regularity of σ. In particular they do not depend on r or Q_0.) Apply (3.3.37) to $A^k\Delta$, we get

$$\begin{aligned}
\int_{A^k\Delta} |S_{A^k r} u|^t d\sigma &\leq C''' \int_{A^{k+1}\Delta} |S_{A^{k+1} r} u|^t d\sigma + C_1 |\mathscr{C}(u)|^t \sigma(A^k\Delta) \\
&\leq C''' \int_{\partial\Omega} |Su|^t d\sigma + \widetilde{C}_1 |\mathscr{C}(u)|^t \sigma(\partial\Omega) \\
&\leq C_2 |\mathscr{C}(u)|^t \sigma(\partial\Omega). \qquad\qquad \text{by (3.3.40)}
\end{aligned}$$

Hence

$$\fint_{A^k\Delta} |S_{A^k r} u|^t d\sigma \leq C_2 |\mathscr{C}(u)|^t. \tag{3.3.42}$$

To simplify the notations, we write $a_r = \fint_\Delta |S_r u|^t d\sigma$ and $B = \max\{C_1, C_2\} \cdot |\mathscr{C}(u)|^t$, where C_1 and C_2 are the constants in (3.3.39) and (3.3.42). Hence (3.3.39) and (3.3.42) become

$$a_r \leq \frac{1}{4} a_{Ar} + B \qquad \text{if } Ar < \operatorname{diam}\Omega. \tag{3.3.43}$$

$$a_{A^k r} \leq B \quad \text{where } A^k r < \operatorname{diam}\Omega \leq A^{k+1} r. \tag{3.3.44}$$

Induction on (3.3.43), we obtain

$$a_r \leq \frac{1}{4} a_{Ar} + B \leq \frac{1}{4^k} a_{A^k r} + \left(1 + \frac{1}{4} + \cdots + \frac{1}{4^{k-1}}\right) B \leq \frac{7}{3} B. \tag{3.3.45}$$

In other words,

$$\fint_{\Delta(Q_0, r)} |S_r u|^t d\sigma \leq C |\mathscr{C}(u)|^t, \quad \text{with the constant } C = \max\{C_1, C_2\} \cdot 7/3. \tag{3.3.46}$$

This holds for arbitrary $Q_0 \in \partial\Omega$ and $r \in (0, \operatorname{diam}\Omega)$, so (3.3.41) follows. This finishes the proof of the theorem 3.3.21, hence the conclusion (3.3.2) follows.

3.4 Appendix

3.4.1 Proof of Lemma 3.2.30: Properties of f_ϵ

The function f_ϵ as in (6.2.34) is well defined because

$$\int_{y\in\partial\Omega}\varphi_\epsilon(x-y)d\sigma(y) \geq \frac{1}{\epsilon^{n-1}}\int_{y\in\Delta(x,\frac{\epsilon}{2})} d\sigma(y) \geq C_1 > 0. \tag{3.4.1}$$

We also have

$$\int_{y\in\partial\Omega}\varphi_\epsilon(x-y)d\sigma(y) \leq \frac{1}{\epsilon^{n-1}}\int_{y\in\Delta(x,\epsilon)} d\sigma(y) \leq C_2. \tag{3.4.2}$$

The constants C_1 and C_2 are independent of ϵ.

Proof of (1). For any surface ball $\Delta_0 = \Delta(x_0, r_0)$, we denote $\Delta_0^\epsilon = \Delta(x_0, r_0+\epsilon)$. Since f is supported in 2Δ, each f_ϵ is supported in $(2\Delta)^\epsilon$. Thus all f_ϵ's are supported in 3Δ if $\epsilon < r$, the radius of Δ.

Note that

$$\begin{aligned}
&\left|\int_{\partial\Omega}\varphi_\epsilon(x-y)d\sigma(y) - \int_{\partial\Omega}\varphi_\epsilon(\tilde{x}-y)d\sigma(y)\right| \\
=&\left|\int_{\partial\Omega}\int_0^1 \frac{d}{ds}\varphi_\epsilon((1-s)\tilde{x}+sx-y)dsd\sigma(y)\right| \\
\leq&\frac{|x-\tilde{x}|}{\epsilon^n}\int_0^1\int_{y\in\partial\Omega}\left|\nabla\varphi\left(\frac{(1-s)\tilde{x}+sx-y}{\epsilon}\right)\right| d\sigma(y)ds.
\end{aligned} \tag{3.4.3}$$

Since $\|\nabla\varphi\|_{L^\infty} \leq C$, for any $w \in \mathbb{R}^n$ we have

$$\int_{y\in\partial\Omega}\left|\nabla\varphi\left(\frac{w-y}{\epsilon}\right)\right| d\sigma(y) \leq C\sigma\left(B(w,\epsilon)\cap\partial\Omega\right) \leq C\epsilon^{n-1}. \tag{3.4.4}$$

Combining (3.4.3) and (3.4.4),

$$\left|\int_{\partial\Omega}\varphi_\epsilon(x-y)d\sigma(y) - \int_{\partial\Omega}\varphi_\epsilon(\tilde{x}-y)d\sigma(y)\right| \leq C\frac{|x-\tilde{x}|}{\epsilon},$$

so for any ϵ fixed, the map $x\in\partial\Omega \mapsto \int_{\partial\Omega}\varphi_\epsilon(x-y)d\sigma(y)$ is continuous. Since $0\leq f\leq 1$, we can prove similarly $\int_{\partial\Omega} f(y)\varphi_\epsilon(x-y)d\sigma(y)$ is also continuous. Thus $f_\epsilon(x)$ is continuous.

Proof of (2). Fix $\epsilon > 0$. Let $\widetilde{\Delta} = \Delta(x_0, r_0)$ be an arbitrary surface ball. Let λ be a real number to be determined later. We consider two cases.

Case 1. If $r_0 \geq \epsilon/2$, by the definition (6.2.34) and the estimates (3.4.1), (3.4.2),

$$\int_{\widetilde{\Delta}}|f_\epsilon(x)-\lambda|\,d\sigma(x) \leq \frac{1}{C_1}\int_{\widetilde{\Delta}}\left|\int_{\partial\Omega} f(y)\varphi_\epsilon(x-y)d\sigma(y) - \lambda\int_{\partial\Omega}\varphi_\epsilon(x-y)d\sigma(y)\right| d\sigma(x)$$

$$
\begin{aligned}
&\leq \widetilde{C}_1 \int_{x\in\widetilde{\Delta}} \int_{y\in\Delta(x,\epsilon)} |f(y)-\lambda|\varphi_\epsilon(x-y)d\sigma(y)d\sigma(x) \\
&\leq \widetilde{C}_1 \int_{y\in\widetilde{\Delta}^\epsilon} |f(y)-\lambda| \int_{x\in\partial\Omega} \varphi_\epsilon(x-y)d\sigma(x)d\sigma(y) \\
&\leq \widetilde{C}_1 C_2 \int_{y\in\widetilde{\Delta}^\epsilon} |f(y)-\lambda|d\sigma(y) \\
&\leq C'\sigma(\widetilde{\Delta}^\epsilon)\|f\|_{BMO(\sigma)}.
\end{aligned}
$$

The last inequality is true if we choose $\lambda = \lambda(\widetilde{\Delta}, \epsilon)$ be the constant satisfying $⨍_{\widetilde{\Delta}^\epsilon} |f(\cdot) - \lambda|d\sigma \leq \|f\|_{BMO(\sigma)}$. Thus

$$
⨍_{\widetilde{\Delta}} |f_\epsilon(x)-\lambda|\, d\sigma(x) \lesssim \frac{\sigma(\widetilde{\Delta}^\epsilon)}{\sigma(\widetilde{\Delta})}\|f\|_{BMO(\sigma)} \lesssim \left(\frac{r_0+\epsilon}{r_0}\right)^{n-1} \|f\|_{BMO(\sigma)} \lesssim \|f\|_{BMO(\sigma)}.
$$

Case 2. If $r_0 < \epsilon/2$, by the definition (6.2.34) and the estimate (3.4.1),

$$
\begin{aligned}
\int_{\widetilde{\Delta}} |f_\epsilon(x)-\lambda|\, d\sigma(x) &\leq \widetilde{C}_1 \int_{x\in\widetilde{\Delta}} \int_{y\in\Delta(x,\epsilon)} |f(y)-\lambda|\varphi_\epsilon(x-y)d\sigma(y)d\sigma(x) \\
&\leq \widetilde{C}_1 \int_{y\in\widetilde{\Delta}^\epsilon} |f(y)-\lambda| \int_{x\in\widetilde{\Delta}} \varphi_\epsilon(x-y)d\sigma(x)d\sigma(y). \qquad (3.4.5)
\end{aligned}
$$

Note

$$
\int_{x\in\widetilde{\Delta}} \varphi_\epsilon(x-y)d\sigma(x) \leq \frac{1}{\epsilon^{n-1}} \int_{x\in\widetilde{\Delta}} d\sigma(x) = \frac{\sigma(\widetilde{\Delta})}{\epsilon^{n-1}},
$$

it follows from (3.4.5) that

$$
\begin{aligned}
⨍_{\widetilde{\Delta}} |f_\epsilon(x)-\lambda|\, d\sigma(x) &\lesssim \frac{1}{\sigma(\widetilde{\Delta})}\cdot\frac{\sigma(\widetilde{\Delta})}{\epsilon^{n-1}} \int_{y\in\widetilde{\Delta}^\epsilon} |f(y)-\lambda|d\sigma(y) \\
&\lesssim \frac{\sigma(\widetilde{\Delta}+\epsilon)}{\epsilon^{n-1}}\|f\|_{BMO(\sigma)} \\
&\lesssim \|f\|_{BMO(\sigma)}.
\end{aligned}
$$

We have proved the following: for any ϵ and any surface ball $\widetilde{\Delta}$, one can find a constant $\lambda = \lambda(\widetilde{\Delta}, \epsilon)$ such that $⨍_{\widetilde{\Delta}} |f_\epsilon(x)-\lambda|\, d\sigma(x) \leq C\|f\|_{BMO(\sigma)}$. The constant C does not depend on either ϵ or $\widetilde{\Delta}$, therefore $\|f_\epsilon\|_{BMO(\sigma)} \leq C\|f\|_{BMO(\sigma)}$ for all ϵ.

Proof of (3). Fix $x \in \partial\Omega$. If $f(x) = 0$, then obviously $f(x) \leq \liminf_{\epsilon\to 0} f_\epsilon(x)$. For any arbitrary $\lambda > 0$ such that $\lambda < f(x)$, there exists $\epsilon_0 > 0$ such that $f(x) > \lambda + \epsilon_0$. It means

$$
\sup_{\Delta'\ni x} \frac{\sigma(E\cap\Delta')}{\sigma(\Delta')} = M_\sigma\chi_E(x) > e^{\frac{\lambda+\epsilon_0-1}{\delta}}.
$$

In particular, there is some surface ball $\Delta' \ni x$ such that

$$
\frac{\sigma(E\cap\Delta')}{\sigma(\Delta')} > e^{\frac{\lambda+\epsilon_0-1}{\delta}}.
$$

Then for any point $y \in \Delta'$, we also have $M_\sigma \chi_E(y) > \exp(\lambda + \epsilon_0 - 1)/\delta$ and thus $f(y) > \lambda + \epsilon_0$. Consider all f_ϵ with $\epsilon < \operatorname{dist}(x, \partial\Omega \setminus \Delta')$, we have $\Delta(x, \epsilon) \subset \Delta'$, hence

$$f_\epsilon(x) = \frac{\int_{y\in\Delta(x,\epsilon)} f(y)\varphi_\epsilon(x-y)d\sigma(y)}{\int_{y\in\Delta(x,\epsilon)} \varphi_\epsilon(x-y)d\sigma(y)} \geq (\lambda+\epsilon_0)\frac{\int_{y\in\Delta(x,\epsilon)} \varphi_\epsilon(x-y)d\sigma(y)}{\int_{y\in\Delta(x,\epsilon)} \varphi_\epsilon(x-y)d\sigma(y)} = \lambda + \epsilon_0.$$

Therefore $\liminf_{\epsilon\to 0} f_\epsilon(x) > \lambda$ for all $\lambda < f(x)$. Thus $\liminf_{\epsilon\to 0} f_\epsilon(x) \geq f(x)$. □

3.4.2 Properties of the truncated square function

Proof of Lemma 3.3.23 Assume $Q \in \partial\Omega$ satisfies $S_r^2 u(Q) = \iint_{\Gamma_r(Q)} |\nabla u|^2 \delta(X)^{2-n} dX > \lambda^2$ and is finite, then there exists $\eta < r$ such that

$$\iint_{\Gamma_r(Q)\setminus B(Q,\eta)} |\nabla u|^2 \delta(X)^{2-n} dX > \left(\frac{S_r u(Q) + \lambda}{2}\right)^2.$$

Fix η, we claim there exists $\epsilon > 0$ such that $S_r u(P) > \lambda$ for any $P \in B(Q, \epsilon\eta) \cap \partial\Omega$. In fact,

$$\left| \iint_{\Gamma_r(Q)\setminus B(Q,\eta)} |\nabla u|^2 \delta(X)^{2-n} dX - \iint_{\Gamma_r(P)\setminus B(P,\eta)} |\nabla u|^2 \delta(X)^{2-n} dX \right| \leq \iint_D |\nabla u|^2 \delta(X)^{2-n} dX \tag{3.4.6}$$

where D is the set difference between $\Gamma_r(Q) \setminus B(Q, \eta)$ and $\Gamma_r(P) \setminus B(P, \eta)$.

Assume $X \in \Gamma_r(Q) \setminus B(Q, \eta)$, then $|X - Q| \leq (1+\alpha)\delta(X)$ and $\eta \leq |X - Q| < r$. In particular $\eta \leq (1+\alpha)\delta(X)$. If in addition $X \notin \Gamma_r(P) \setminus B(P, \eta)$ for some $P \in B(Q, \epsilon\eta)$, then X falls in one of the following three categories:

- $|X-P| < \eta$, then $|X-Q| \leq |X-P|+|P-Q| < (1+\epsilon)\eta$, in particular $\eta \leq |X-Q| < (1+\epsilon)\eta$;
- $|X-P| \geq r$, then $|X-Q| \geq |X-P| - |P-Q| > r - \epsilon\eta$, in particular $r - \epsilon\eta < |X-Q| < r$;
- $|X-P| > (1+\alpha)\delta(X)$, then $|X-Q| \geq |X-P| - |P-Q| > (1-\epsilon)(1+\alpha)\delta(X)$, in particular $(1-\epsilon)(1+\alpha)\delta(X) < |X-Q| \leq (1+\alpha)\delta(X)$.

Similarly, the points in $(\Gamma_r(P) \setminus B(P, \eta)) \setminus (\Gamma_r(Q) \setminus B(Q, \eta))$ also fall in three categories, just with Q replaced by P. Therefore D, the set difference between $(\Gamma_r(Q) \setminus B(Q, \eta)) \setminus (\Gamma_r(P) \setminus B(P, \eta))$ and $(\Gamma_r(P) \setminus B(P, \eta)) \setminus (\Gamma_r(Q) \setminus B(Q, \eta))$, is contained in the union of three sets (corresponding to the above three cases):

$$V_1 = \left\{X \in \Omega : (1-\epsilon)\eta < |X-Q| < (1+2\epsilon)\eta,\ \delta(X) \geq \eta/(1+\alpha)\right\}$$

$$V_2 = \left\{X \in \Omega : r - 2\epsilon\eta < |X-Q| < r + \epsilon\eta,\ \delta(X) \geq \eta/(1+\alpha)\right\}$$

$$V_3 = \left\{X \in \Omega : (1-2\epsilon)(1+\alpha)\delta(X) < |X-Q| \leq (1+\epsilon)(1+\alpha)\delta(X),\ \delta(X) \geq \eta/(1+\alpha)\right\}.$$

Since $\delta(X) \geq \eta/(1+\alpha)$ in D,

$$\iint_D |\nabla u|^2 \delta(X)^{2-n} dX \leq \left(\frac{1+\alpha}{\eta}\right)^n \iint_{V_1 \cup V_2 \cup V_3} |\nabla u|^2 \delta(X)^2 dX. \tag{3.4.7}$$

Note that $u \in W^{1,2}(\Omega)$, we have

$$\iint_\Omega |\nabla u|^2 \delta(X)^2 dX \leq \operatorname{diam}(\Omega)^2 \iint_\Omega |\nabla u|^2 dX < \infty.$$

Hence the integral $\iint_{V_1\cup V_2 \cup V_3} |\nabla u|^2 \delta(X)^2 dX$ is small as long as the Lebesgue measures of V_1, V_2 and V_3 are small enough. Both V_1 and V_2 are contained in annuli of radius $3\epsilon\eta$, so their Lebesgue measures are small if we choose ϵ small enough (depending on η). Rewrite V_3 as

$$V_3 = \left\{ X \in \Omega : \frac{1}{(1+\epsilon)(1+\alpha)} < \frac{\delta(X)}{|X-Q|} \leq \frac{1}{(1-2\epsilon)(1+\alpha)},\ \delta(X) \geq \frac{\eta}{1+\alpha} \right\}.$$

Away from Q, say in $\Omega \setminus B(Q, \eta/2)$, the function $F(X) = \delta(X)/|X-Q|$ is Lipschitz, and $0 \leq F \leq 1$. Choose $\epsilon < 1/4$, then for any $X \in V_3$, $|X - Q| \geq (1-2\epsilon)(1+\alpha)\delta(X) \geq \eta/2$. So $V_3 \subset \Omega \setminus B(Q, \eta/2)$ and thus F is Lipschitz on V_3. By the coarea formula,

$$\mathcal{H}^n(V_3) = \int_{\frac{1}{(1+\epsilon)(1+\alpha)}}^{\frac{1}{(1-2\epsilon)(1+\alpha)}} \int_{F^{-1}(t)} \frac{1}{JF} \chi_{V_3} d\mathcal{H}^{n-1} dt. \tag{3.4.8}$$

On the other hand,

$$\int_0^1 \int_{F^{-1}(t)} \frac{1}{JF} \chi_{V_3} d\mathcal{H}^{n-1} dt \leq \int_0^1 \int_{F^{-1}(t)} \frac{1}{JF} \chi_{\Omega \setminus B(Q,\eta/2)} d\mathcal{H}^{n-1} dt = \mathcal{H}^n\big(\Omega \setminus B(Q, \eta/2)\big)$$

is finite. Therefore by (3.4.8), we may choose ϵ small enough (depending on α) such that $\mathcal{H}^n(V_3)$ is small, which in turn implies $\iint_{V_3} |\nabla u|^2 \delta(X)^2 dX$ is small. To sum up, we have shown that one can choose $\epsilon = \epsilon(\delta, \alpha, \eta, r)$ small enough such that

$$\left(\frac{\alpha}{\eta}\right)^n \iint_{V_1\cup V_2\cup V_3} |\nabla u|^2 \delta(X)^2 dX < \delta < \left(\frac{S_r u(Q) + \lambda}{2}\right)^2 - \lambda^2.$$

Therefore we conclude from (3.4.6) and (3.4.7) that

$$\begin{aligned} &\iint_{\Gamma_r(P)\setminus B(P,\eta)} |\nabla u|^2 \delta(X)^{2-n} dX \\ &\qquad \geq \iint_{\Gamma_r(Q)\setminus B(Q,\eta)} |\nabla u|^2 \delta(X)^{2-n} dX - \iint_D |\nabla u|^2 \delta(X)^{2-n} dX \\ &\qquad > \left(\frac{S_r u(Q) + \lambda}{2}\right)^2 - \delta \\ &\qquad > \lambda^2. \end{aligned}$$

Hence $S_r u(P) \geq \left(\iint_{\Gamma_r(P)\setminus B(P,\eta)} |\nabla u|^2 \delta(X)^{2-n} dX\right)^{1/2} > \lambda$, for all $P \in B(Q, \epsilon\eta) \cap \partial\Omega$. This finishes the proof that $\left\{Q \in \partial\Omega : S_r u(Q) > \lambda\right\}$ is open in $\partial\Omega$. □

Proof of Lemma 3.3.24 We prove the estimate by duality: let r be the conjugate of $q/2$, namely $1/r + 2/q = 1$, then

$$\left(\int_\Delta |S_r^{\overline{\alpha}} u(Q)|^q d\sigma(Q)\right)^{2/q} = \sup\left\{\int_\Delta |S_r^{\overline{\alpha}} u(Q)|^2 \psi(Q) d\sigma(Q) : \|\psi\|_{L^r(\Delta)} = 1\right\}. \tag{3.4.9}$$

Recall $\Delta = \Delta(Q_0, r)$. Extending ψ to all of $\partial\Omega$ by setting it to zero outside of Δ. For any X, let $Q_X \in \partial\Omega$ be a boundary point such that $|X - Q_X| = \delta(X)$. By Fubini's theorem,

$$\begin{aligned}
\int_\Delta |S_r^{\overline{\alpha}} u(Q)|^2 \psi(Q) d\sigma(Q) &= \int_\Delta \left(\iint_{\Gamma_r^{\overline{\alpha}}(Q)} |\nabla u|^2 \delta(X)^{2-n} dX\right) \psi(Q) d\sigma(Q) \\
&\leq \iint_{B(Q_0,2r)\cap\Omega} |\nabla u|^2 \delta(X)^{2-n} \int_{|Q-Q_X|\leq(\overline{\alpha}+2)\delta(X)} \psi(Q) d\sigma(Q) dX \\
&\lesssim \iint_{B(Q_0,2r)\cap\Omega} |\nabla u|^2 \delta(X) A_{(\overline{\alpha}+2)\delta(X)} \psi(Q_X) dX,
\end{aligned} \tag{3.4.10}$$

where $A_s\psi(Q)$ is defined as $A_s\psi(Q) = \frac{1}{s^{n-1}} \int_{\Delta(Q,s)} \psi(P) d\sigma(P)$. Let $\beta > 1$, simply calculations show that

$$\begin{aligned}
A_s\left(A_{\beta s}\psi\right)(Q) &= \frac{1}{s^{n-1}} \int_{\Delta(Q,s)} \left(\frac{1}{(\beta s)^{n-1}} \int_{\Delta(P,\beta s)} \psi(P') d\sigma(P')\right) d\sigma(P) \\
&\geq \frac{1}{s^{n-1}} \int_{\Delta(Q,s)} \left(\frac{1}{(\beta s)^{n-1}} \int_{\Delta(Q,(\beta-1)s)} \psi(P') d\sigma(P')\right) d\sigma(P) \\
&\gtrsim \left(\frac{(\beta-1)s}{\beta s}\right)^{n-1} A_{(\beta-1)s}\psi(Q).
\end{aligned}$$

Let $s = \alpha\delta(X)$, $\beta - 1 = (\overline{\alpha} + 2)/\alpha$, then

$$A_{(\overline{\alpha}+2)\delta(X)}\psi(Q) \lesssim_{\alpha,\overline{\alpha}} A_{\alpha\delta(X)}\left(A_{\beta s}\psi\right)(Q) \lesssim A_{\alpha\delta(X)} M\psi(Q).$$

For the last inequality, we use $|A_{\beta s}\psi(Q)| \leq C(M\psi(Q))$, where $M\psi$ is the Hardy-Littlewood maximal function of ψ with respect to σ, and the constant C only depend on the Ahlfors regularity of σ. Thus it follows from (3.4.10) that

$$\begin{aligned}
\int_\Delta |S_r^{\overline{\alpha}} u(Q)|^2 \psi(Q) d\sigma(Q) &\lesssim \iint_{B(Q_0,2r)\cap\Omega} |\nabla u|^2 \delta(X) A_{\alpha\delta(X)} M\psi(Q_X) dX \\
&\lesssim \iint_{B(Q_0,2r)\cap\Omega} |\nabla u|^2 \delta(X)^{2-n} \left(\int_{\Delta(Q_X,\alpha\delta(X))} M\psi(Q) d\sigma(Q)\right) dX.
\end{aligned} \tag{3.4.11}$$

By switching the order of integration, we can bound the right hand side by:

$$\int_\Delta |S_r^{\overline{\alpha}} u(Q)|^2 \psi(Q) d\sigma(Q) \lesssim \int_{\Delta(Q_0,2(\alpha+2)r)} M\psi(Q) \iint_{\Gamma_{2(1+\alpha)r}(Q)} |\nabla u|^2 \delta(X)^{2-n} dX d\sigma(Q)$$

$$= \int_{\Delta(Q_0,2(\alpha+2)r)} M\psi(Q)|S_{2(\alpha+1)r}u(Q)|^2 d\sigma(Q)$$

$$\leq \|M\psi\|_{L^r(\Delta(Q_0,2(\alpha+2)r))} \left(\int_{\Delta(Q_0,2(\alpha+2)r)} |S_{2(\alpha+1)r}u(Q)|^q d\sigma(Q) \right)^{2/q}. \tag{3.4.12}$$

Since $1 < r < \infty$, we have

$$\|M\psi\|_{L^r(\Delta(Q_0,2(\alpha+2)r))} \leq C\|\psi\|_{L^r(\Delta(Q_0,2(\alpha+2)r))} = C\|\psi\|_{L^r(\Delta)} = C. \tag{3.4.13}$$

By (3.4.12), (3.4.13) and the definition (3.4.9), we conclude

$$\int_\Delta |S_r^{\overline{\alpha}} u(Q)|^q d\sigma(Q) \leq C \int_{\Delta(Q_0,2(\alpha+2)r)} |S_{2(\alpha+1)r}u(Q)|^q d\sigma(Q)$$
$$\leq C \int_{\Delta(Q_0,2(\alpha+2)r)} |S_{2(\alpha+1)r}u(Q)|^q d\sigma(Q).$$

This finishes the proof of Lemma 3.3.24. □

Chapter 4

$\sigma \ll \omega$ implies boundary rectifiability for a class of variable-coefficient operators

The proof of Theorem 1.3.5 and 1.3.7 is by observing that if we blow up (i.e. zoom in and in, then take the limit) the elliptic measures near the boundary, the tangent measure we obtain corresponds to a constant-coefficient elliptic operator. Thus we can apply known results of the harmonic measure (applicable to constant coefficient operators) to the tangent measure, and draw conclusion about the original elliptic measure.

4.1 Blow-up and pseudo blow-up domains

In this section we consider tangent objects as a way to understand the fine structure of $\partial\Omega$, under the assumptions of Theorem 1.3.5 and 1.3.7. This requires looking at the tangent and pseudo-tangent domains and the corresponding functions and measures obtained via a blow-up. While tangent objects provide pointwise infinitesimal information, pseudo-tangents provide "uniform infinitesimal" information. The key point is to observe that the blow-ups or pseudo blow-ups of the operators satisfying the hypotheses of Theorems 1.3.5 and 1.3.7 lead to a constant coefficient operators. Our goal is to show that under these assumptions the tangent and pseudo-tangent objects satisfy the hypothesis of Theorem 5.0.1, which is a simple generalization of Theorem 1.23 in [HMU]. The details of its proof can also be found in [HMT1].

Theorem 4.1.1. *Let $\mathcal{D}$ be a uniform domain (bounded or unbounded) with Ahlfors regular boundary. Let L be a symmetric second order elliptic divergence form operator with constant coefficient. Assume that the elliptic measure $\omega \in A_\infty(\sigma)$ (see Definition 2.1.14), then $\partial\mathcal{D}$ is uniformly rectifiable.*

Getting to the point where we can apply this Theorem requires showing first that if Ω_∞ is a blow-up or pseudo blow-up of Ω, then Ω_∞ is an unbounded uniform domain with Ahlfors regular boundary. To accomplish this we also need to blow up the Green function. Moreover the blow-up limit of the given elliptic operator, denoted by L_∞, has constant coefficient. Once we have this, to show that $\omega_{L_\infty} \in A_\infty(\sigma_\infty)$ for the blow-up domain and the limiting operator, we need to construct the elliptic measure $\omega^Z_{L_\infty}$ for any $Z \in \Omega_\infty$ as a limiting measure compatible with the initial blow-up.

Let $X_0 \in \Omega$ and $L = -\operatorname{div}(A\nabla)$. Let $G(X_0, \cdot)$ be the Green function for L with pole X_0 and $\omega = \omega^{X_0}_L$ the corresponding elliptic measure. For $j \in \mathbb{N}$, let $q_j \in \partial\Omega$ and $r_j > 0$ such that $q_j \to q \in \partial\Omega$ and $r_j \to 0$. In some cases we assume that $q_j = q$ for all $j \in \mathbb{N}$. We now consider

$$\Omega_j = \frac{1}{r_j}(\Omega - q_j) \qquad \text{and} \qquad \partial\Omega_j = \frac{1}{r_j}(\partial\Omega - q_j). \tag{4.1.2}$$

$$u_j(Z) = \frac{r_j^{n-2} G(X_0, q_j + r_j Z)}{\omega(B(q_j, r_j))} \quad \text{for } Z \in \Omega_j \text{ and } u_j = 0 \text{ in } \Omega_j^c. \tag{4.1.3}$$

$$\sigma_j(E) = \frac{\sigma(q_j + r_j E)}{r_j^{n-1}} \qquad \text{and} \qquad \omega_j(E) = \frac{\omega(q_j + r_j E)}{\omega(B(q_j, r_j))}. \tag{4.1.4}$$

We follow the following conventions:

- For $X \in \Omega$ we denote $\delta(X) = \operatorname{dist}(X, \partial\Omega)$ and for $Z \in \Omega_j$ we denote $\delta_j(Z) = \operatorname{dist}(Z, \partial\Omega_j)$.
- For any $q \in \partial\Omega$ and $r \in (0, \operatorname{diam}\partial\Omega)$, we use $A(q, r)$ to denote a non-tangential point in Ω with respect to q at radius r, i.e.

$$|A(q, r) - q| < r, \text{ and } \delta(A(q, r)) \geq \frac{r}{M}.$$

- If $X \in \overline{\Omega_j}$ we denote by $\widetilde{X} = q_j + r_j X \in \overline{\Omega}$.
- For any $p \in \partial\Omega_j$ and $r \in (0, \operatorname{diam}\partial\Omega_j)$, we use

$$A_j(p, r) = \frac{A(\widetilde{p}, r r_j) - q_j}{r_j}$$

as a non-tangential point in Ω_j with respect to p at radius r. Here $\widetilde{p} = q_j + r_j p$.

Note that, modulo a constant, u_j is the Green function for the operator $L_j = -\operatorname{div}(A_j\nabla)$ with $A_j(Z) = A(r_j Z + q_j)$ in Ω_j with pole $X_j = (X_0 - q_j)/r_j$. Moreover ω_j is the corresponding elliptic measure with $0 \in \operatorname{spt} \omega_j$ and $\omega_j(B(0,1)) = 1$. If $p \in \partial\Omega_j$ and $r \in (0, \operatorname{diam} \partial\Omega_j)$ then $\widetilde{p} = r_j p + q_j \in \partial\Omega$,

$$\omega_j(B(p,2r)) = \frac{\omega(B(\widetilde{p}, 2rr_j))}{\omega(B(q_j, r_j))} \leq C \frac{\omega(B(\widetilde{p}, rr_j))}{\omega(B(q_j, r_j))} = C\omega_j(B(p,r)) \tag{4.1.5}$$

and

$$\frac{\sigma_j(B(p,r))}{r^{n-1}} = \frac{\sigma(B(\widetilde{p}, rr_j))}{(rr_j)^{n-1}} \sim 1 \tag{4.1.6}$$

Note also that $0 \in \operatorname{spt} \sigma_j$, $\sigma_j(B(0,1)) = \sigma(B(q_j, r_j))/r_j^{n-1} \sim 1$. Hence $\{\sigma_j\}$ and $\{\omega_j\}$ satisfy conditions i) and ii) of Lemma 2.3.5. The three theorems below describe what happens as we let j tend to infinity in the sequences defined in (4.1.2), (4.1.3) and (4.1.4).

Theorem 4.1.7. *Let $\Omega \subset \mathbb{R}^n$ be a uniform domain with Ahlfors regular boundary. Let $L_A = -\operatorname{div}(A(X)\nabla)$ be a divergence form uniformly elliptic operator in Ω, assume that $A \in L^\infty(\Omega)$. Using the notation above, modulo passing to a subsequence (which we relabel) we conclude the following*

1. *There is a function $u_\infty \in C(\mathbb{R}^n)$ such that $u_j \to u_\infty$ uniformly on compact sets. Moreover $\nabla u_j \rightharpoonup \nabla u_\infty$ in $L^2_{loc}(\mathbb{R}^n)$.*
2. *Let $\Omega_\infty = \{u_\infty > 0\}$, then $\overline{\Omega_j} \to \overline{\Omega}_\infty$ and $\partial\Omega_j \to \partial\Omega_\infty$ in the Hausdorff distance sense uniformly on compact sets.*
3. *Ω_∞ is a non-trivial unbounded uniform domain.*
4. *There is a doubling measure ω_∞ such that $\omega_j \rightharpoonup \omega_\infty$. Moreover $\operatorname{spt} \omega_\infty = \partial\Omega_\infty$.*
5. *There is an Ahlfors regular measure μ_∞ such that $\sigma_j \rightharpoonup \mu_\infty$. Moreover $\operatorname{spt} \mu_\infty = \partial\Omega_\infty$. In particular this implies that $\mu_\infty \ll \sigma_\infty := \mathcal{H}^{n-1} \llcorner \partial\Omega_\infty \ll \mu_\infty$.*

Definition 4.1.8. The domain Ω_∞ is a pseudo-tangent domain to Ω at q. The function u_∞ is a pseudo-tangent function to $G(X_0, \cdot)$ at q. The measures μ_∞ and ω_{L_∞} are pseudo-tangent measures to σ_j and ω_j at q respectively. If $q_j = q$ for all j then Ω_∞, u_∞, μ_∞ and ω_{L_∞} are called tangents at q.

Theorem 4.1.9. *Let $\Omega \subset \mathbb{R}^n$ be a uniform domain with Ahlfors regular boundary. Let $L = -\operatorname{div}(A(X)\nabla)$ be a divergence form uniformly elliptic operator in Ω. Assume that $A \in C(\overline{\Omega})$. Then using the notation in Theorem 5.2.8 we have that the function u_∞ satisfies*

$$\begin{cases} -\operatorname{div}(A(q)\nabla u_\infty) = 0 & \text{in } \Omega_\infty \\ u_\infty > 0 & \text{in } \Omega_\infty \\ u_\infty = 0 & \text{on } \partial\Omega_\infty. \end{cases} \tag{4.1.10}$$

i.e. u_∞ is a Green function in Ω_∞ for $L_\infty = \operatorname{div}(A(q)\nabla)$ with pole at ∞.

Furthermore ω_{L_∞} is the harmonic measure corresponding to u_∞, in the sense that

$$-\int_{\Omega_\infty} A(q)\nabla u_\infty \cdot \nabla\psi dZ = \int_{\partial\Omega_\infty} \psi d\omega_{L_\infty}, \quad \text{for all } \psi \in C_c^\infty(\mathbb{R}^n). \tag{4.1.11}$$

Theorem 4.1.12. *Let $\Omega \subset \mathbb{R}^n$ be a uniform domain with Ahlfors regular boundary. Let $L = -\operatorname{div}(A(X)\nabla)$ be a divergence form uniformly elliptic operator in Ω. Assume that $A \in L^\infty(\Omega) \cap W^{1,1}(\Omega)$. Then for $\mathcal{H}^{n-1}$ a.e. $q \in \partial\Omega$, using the notation in Theorem 5.2.8, under the assumption that $q_j = q$ for every j we have that the corresponding function u_∞ satisfies*

$$\begin{cases} -\operatorname{div}(A^*(q)\nabla u_\infty) = 0 & \text{in } \Omega_\infty \\ u_\infty > 0 & \text{in } \Omega_\infty \\ u_\infty = 0 & \text{on } \partial\Omega_\infty. \end{cases} \tag{4.1.13}$$

i.e. u_∞ is a Green function in Ω_∞ for $L_\infty = -\operatorname{div}(A^*(q)\nabla)$ with pole at ∞. Furthermore ω_{L_∞} is the harmonic measure corresponding to u_∞, in the sense that

$$-\int_{\Omega_\infty} A^*(q)\nabla u_\infty \cdot \nabla\psi dZ = \int_{\partial\Omega_\infty} \psi d\omega_{L_\infty}, \quad \text{for all } \psi \in C_c^\infty(\mathbb{R}^n). \tag{4.1.14}$$

Here $A^*(q)$ is obtained as in Lemma 2.3.12.

Proof of (1) in Theorem 5.2.8 Fix $R > 1$, for $j \geq j_0$ large enough we may assume $X_0 \in \Omega \setminus B(q_j, 2r_jR)$. For such j, $L_ju_j = 0$ in $B(0,2R) \cap \Omega_j$. Here $L_j = -\operatorname{div}(A_j\nabla)$ with $A_j(Z) = A(r_jZ + q_j)$. Note that $0 \in \partial\Omega_j$ and $\left(A(q_j, r_j) - q_j\right)/r_j \in \Omega_j$ is a non-tangential point for 0 at radius 1 for Ω_j, we denote it by $A_j(0,1)$. Moreover

$$u_j(A_j(0,1)) = \frac{r_j^{n-2}G(X_0, q_j + r_jA_j(0,1))}{\omega(B(q_j,r_j))} = \frac{r_j^{n-2}G(X_0, A(q_j,r_j))}{\omega(B(q_j,r_j))} \sim 1. \tag{4.1.15}$$

Let $A_j(0,R) \in \Omega_j$ denote a non-tangential point to 0 at radius R, then by Harnack's inequality we have

$$u_j(A_j(0,R)) \leq C(R)u_j(A_j(0,1)) \leq C'(R).$$

Thus for any $Z \in \Omega_j \cap B(0,R)$, using Lemma 2.2.23 we have

$$u_j(Z) \leq Cu_j(A_j(0,R)) \leq C(R).$$

Extending $u_j = 0$ on Ω_j^c we conclude that the sequence $\{u_j\}_{j\geq j_0}$ is uniformly bounded in $\overline{B(0,R)}$. Since for each j, L_j has ellipticity constants bounded below by λ and above by Λ, $\|A_j\|_{L^\infty(\Omega_j)} = \|A\|_{L^\infty(\Omega)}$ and Ω_j is uniform and satisfies the CDC (as $\partial\Omega_j$ is Ahlfors regular) with the same constants as Ω, then combining Lemma 2.2.19 with DeGiorgi-Nash-Moser we conclude that the sequence $\{u_j\}_j$ is equicontinuous on $\overline{B(0,R)}$ (in fact uniformly Hölder continuous with the same exponent). Using Arzelà-Ascoli combined with a diagonal argument applied on a sequence of balls with radii going to infinity, we produce a subsequence (which we relabel) such that $u_j \to u_\infty$ uniformly on compact sets of $\mathbb{R}^n$. Note that the boundary Cacciopoli inequality yields

$$\int_{B(0,R)} |\nabla u_j|^2 \, dZ = \int_{B(q_j,Rr_j)} \frac{r_j^{n-2}}{(\omega(B(q_j,r_j)))^2} |\nabla G(X_0,Y)|^2 \, dY$$

$$\leq C\frac{r_j^{n-4}}{(\omega(B(q_j,r_j)))^2}\int_{B(q_j,2Rr_j)} G(X_0,Y)^2 dY. \tag{4.1.16}$$

Applying the boundary Harnack principle (see (2.2.24)) and Harnack inequality to estimate $G(X_0, A(q_j, 2Rr_j))$ by $G(X_0, A(q_j, r_j))$ and noting that $A(q_j, 2Rr_j)$ can be joined to $A(q_j, r_j)$ by a chain of length independent of j, (5.2.13) becomes

$$\int_{B(0,R)} |\nabla u_j|^2\, dZ \leq C\frac{r_j^{2n-4}}{(\omega(B(q_j,r_j)))^2} G(X_0, A(q_j, 2Rr_j))^2$$

$$\leq C\frac{r_j^{2n-4}}{(\omega(B(q_j,r_j)))^2} G(X_0, A(q_j, r_j))^2. \tag{4.1.17}$$

Finally applying Lemma 2.2.25 in (4.1.17) yields

$$\sup_j \int_{B(0,R)} |\nabla u_j|^2 \leq C_R < \infty.$$

Recalling that the functions $\{u_j\}$ are uniformly bounded in $\overline{B(0,R)}$, we conclude that

$$\sup_j \|u_j\|_{W^{1,2}(B(0,R))} \leq C'_R < \infty. \tag{4.1.18}$$

Thus there exists a subsequence (which we relabel) which converges weakly in $W^{1,2}_{loc}(\mathbb{R}^n)$. A standard argument allows us to conclude that $u_j \to u_\infty$ in $L^2_{loc}(\mathbb{R}^n)$ and $\nabla u_j \rightharpoonup \nabla u_\infty$ in $L^2_{loc}(\mathbb{R}^n)$. This shows *(1)*.

Proof of (3) in Theorem 5.2.8 Let $\Omega_\infty = \{u_\infty > 0\}$. Since $0 \in \partial\Omega_j$ for all j, modulo passing to a subsequence (which we relabel) we have

$$\overline{\Omega}_j \to \Gamma_\infty \text{ and } \partial\Omega_j \to \Lambda_\infty \text{ as } j \to \infty.$$

Here Γ_∞ and Λ_∞ are closed sets, and the convergence is in the Hausdorff distance sense uniformly on compact sets.

Claim: $\Lambda_\infty = \partial\Omega_\infty$ and $\Gamma_\infty = \overline{\Omega}_\infty$.

Let $p \in \Lambda_\infty$, there is a sequence $p_j \in \partial\Omega_j$ such that $\lim_{j\to\infty} p_j = p$. Note that $u_\infty(p) = \lim_{j\to\infty} u_j(p)$. On the other hand since the u_j's are uniformly Hölder continuous on compact sets $|u_j(p)| = |u_j(p) - u_j(p_j)| \leq C|p - p_j|^\alpha$, thus $\lim_{j\to\infty} u_j(p) = \lim_{j\to\infty} u_j(p_j) = 0$, and therefore $u_\infty(p) = 0$, i.e. $p \in \Omega^c_\infty$. Assume that there exists $\epsilon \in (0, 1)$ such that $B(p, \epsilon) \subset \Omega^c_\infty$, i.e. $u_\infty \equiv 0$ on $B(p, \epsilon)$. Note that if $\widetilde{p}_j = q_j + r_j p_j$ then for j large enough

$$\left|A\left(\widetilde{p}_j, \frac{\epsilon}{2}r_j\right) - A(q_j, r_j)\right| \leq \frac{\epsilon}{2}r_j + |\widetilde{p}_j - q_j| + r_j \leq \left(\frac{\epsilon}{2} + |p_j| + 1\right) r_j \leq 2\left(|p| + 1\right) r_j$$

and

$$\delta\left(A\left(\widetilde{p}_j,\frac{\epsilon}{2}r_j\right)\right)\geq\frac{1}{M}\frac{\epsilon}{2}r_j,\quad \delta\left(A(q_j,r_j)\right)\geq\frac{r_j}{M}.$$

Applying Harnack inequality in Ω, we know there is a constant $C=C(\epsilon,|p|)$ such that

$$G\left(X_0,A\left(\widetilde{p}_j,\frac{\epsilon}{2}r_j\right)\right)\geq CG(X_0,A(q_j,r_j)).$$

Recalling that $A_j(p_j,\frac{\epsilon}{2})=\left(A\left(\widetilde{p}_j,\frac{\epsilon}{2}r_j\right)-q_j\right)/r_j$, we have

$$\begin{aligned}(4.1.19)\qquad u_j\left(A_j\left(p_j,\frac{\epsilon}{2}\right)\right)&=\frac{r_j^{n-2}G\left(X_0,A\left(\widetilde{p}_j,\frac{\epsilon}{2}r_j\right)\right)}{\omega(B(q_j,r_j))}\\ &\geq C\frac{r_j^{n-2}G\left(X_0,A(q_j,r_j)\right)}{\omega(B(q_j,r_j))}=Cu_j(A_j(0,1))\geq C'>0,\end{aligned}$$

where the constant C' is independent of j. However, since for j large enough

$$(4.1.20)\qquad A_j\left(p_j,\frac{\epsilon}{2}\right)\in B\left(p_j,\frac{3\epsilon}{4}\right)\subset B(p,\epsilon),$$

the lower bound (5.2.28) combined with (4.1.20) implies that $u_\infty\not\equiv 0$ on $B(p,\epsilon)$ which contradicts our assumption. Therefore $p\in\partial\Omega_\infty$, and $\Lambda_\infty\subset\partial\Omega_\infty$.

To show that $\partial\Omega_\infty\subset\Lambda_\infty$, we assume that $p\notin\Lambda_\infty$, thus since Λ_∞ is a closed set, there exists $\epsilon>0$ such that $B(p,2\epsilon)\cap\Lambda_\infty=\emptyset$. Since Λ_∞ is the Hausdorff limit of $\partial\Omega_j$ we have that for j large enough $B(p,\epsilon)\cap\partial\Omega_j=\emptyset$. Hence either $B(p,\epsilon)\subset\Omega_j$ or $B(p,\epsilon)\subset\operatorname{int}\Omega_j^c$. If $B(p,\epsilon)\subset\Omega_j$ then $B(q_j+pr_j,\epsilon r_j)\subset\Omega$. Hence $\delta(q_j+pr_j)>\epsilon r_j$ and $|A(q_j,r_j)-(q_j+pr_j)|\leq r_j(1+|p|)$. Thus there exists a Harnack chain joining $A(q_j,r_j)$ to $\left(q_j+pr_j\right)$ of length independent of j and depending on ϵ and $|p|$. By Harnack's inequality $G(X_0,q_j+pr_j)\sim G(X_0,A(q_j,r_j))$ which combined with (5.2.11) yields

$$(4.1.21)\qquad u_j(p)\sim u_j\left(\frac{A(q_j,r_j)-q_j}{r_j}\right)\sim u_j(A_j(0,1))\sim 1.$$

Hence for $X\in B(p,\epsilon)\subset\Omega_j$, again by Harnack inequality and (5.2.30) we have $u_j(X)\sim u_j(p)\sim 1$ with constants independent of j. Letting $j\to\infty$ we have that $u_\infty(X)\sim 1$ for $X\in B(p,\epsilon/2)$. Thus $B(p,\epsilon/2)\subset\Omega_\infty=\{u_\infty>0\}$ and $p\notin\partial\Omega_\infty$. If $B(p,\epsilon)\subset\operatorname{int}\Omega_j^c$, then $u_j(X)=0$ for all $X\in B(p,\epsilon)$. By uniform convergence of u_j in $B(p,\epsilon/2)$ we have that $u_\infty(X)=0$ for $X\in B(p,\epsilon/2)$, which implies $B(p,\epsilon/2)\subset\{u_\infty=0\}$ and $p\notin\partial\Omega_\infty$. Hence $\partial\Omega_\infty\subset\Lambda_\infty$ and we conclude $\Lambda_\infty=\partial\Omega_\infty$.

We now show that $\Gamma_\infty=\overline{\Omega}_\infty$. Note that if $Z\in\Omega_\infty$, $u_\infty(Z)>0$ and for j large enough $u_j(Z)>0$ also. Thus $Z\in\Omega_j$ for all j large enough and $Z\in\Gamma_\infty$, which yields $\Omega_\infty\subset\Gamma_\infty$. Since Γ_∞ is closed we conclude $\overline{\Omega}_\infty\subset\Gamma_\infty$. Let $X\in\Gamma_\infty$. Assume there is $\epsilon>0$ such that $\overline{B(X,2\epsilon)}\subset\Omega_\infty^c$, in particular $B(X,2\epsilon)\cap\partial\Omega_\infty=\emptyset$. Since $\partial\Omega_\infty$ is the limit of $\partial\Omega_j$'s, for j large enough $B(X,\epsilon)\cap\partial\Omega_j=\emptyset$. By the definition of Γ_∞, there is a sequence $X_j\in\overline{\Omega_j}$

converging to X. Thus for j large enough $B(X,\epsilon)$ is a neighborhood of X_j and moreover $B(X,\epsilon)\cap\Omega_j\neq\emptyset$ since $X_j\in\overline{\Omega_j}$. Since $B(X,\epsilon)\cap\partial\Omega_j=\emptyset$ we conclude that $B(X,\epsilon)\subset\Omega_j$. Using a similar argument to the one used to obtain (5.2.30) we have

$$u_j(X)\geq C(|X|,\epsilon)u_j(A_j(0,1))\geq C'>0$$

independent of j. Hence $u_\infty(X)=\lim u_j(X)\geq C'>0$ and $X\in\Omega_\infty$, contradicting the assumption that $X\in\operatorname{int}\Omega_\infty^c$. Therefore $X\in\overline{\Omega}_\infty$, that is $\Gamma_\infty\subset\overline{\Omega}_\infty$, which concludes the proof of *(2)*.

Proof of (4) in Theorem 5.2.8 Recall that since Ω is a uniform domain there is $M>1$ such that for all $q\in\partial\Omega$ and $r\in(0,\operatorname{diam}\Omega)$ there is a point $A(q,r)\in\Omega$ such that

$$B\left(A,\frac{r}{M}\right)\subset B(q,r)\cap\Omega. \tag{4.1.22}$$

Note that since each Ω_j is a dilation and translation of Ω (5.2.41) also holds for $q'\in\partial\Omega_j$ and $r\in(0,\operatorname{diam}\Omega_j)$.

Let $p\in\partial\Omega_\infty$ and $r>0$. Since $\partial\Omega_j\to\partial\Omega_\infty$, we can find $p_j\in\partial\Omega_j$ such that $p_j\to p$. For each j there exists $A_j=A_j(p_j,r/2)$ such that

$$B\left(A_j,\frac{r}{2M}\right)\subset B\left(p_j,\frac{r}{2}\right)\cap\Omega_j.$$

Note that for j large enough

$$B\left(A_j,\frac{r}{2M}\right)\subset B\left(p_j,\frac{r}{2}\right)\subset\overline{B\left(p,\frac{3r}{4}\right)}.$$

Modulo passing to a subsequence (which we relabel) we can find a point $A(p,r)$ such that $A_j\to A(p,r)$ and for j large enough

$$B\left(A(p,r),\frac{r}{3M}\right)\subset B\left(A_j,\frac{r}{2M}\right)\subset B(p,r)\cap\Omega_j. \tag{4.1.23}$$

Let $Y\in B\left(A(p,r),\frac{r}{3M}\right)$. By (5.2.45) $u_j(Y)\sim u_j(A_j)$. Since each Ω_j satisfies the Harnack chain property with the same constant as Ω we have that $u_j(A_j)\sim u_j(A_j(0,1))$ with a comparison constant that only depends on r and $|p_j|$ thus for j large enough with a comparison constant that only depends on r and $|p|$. Since $u_j(A_j(0,1))\sim 1$, we conclude that $u_j(Y)\sim 1$ with a comparison constant that only depends on r and $|p|$. Hence $u_\infty(Y)>0$ and

$$B\left(A(p,r),\frac{r}{3M}\right)\subset B(p,r)\cap\Omega_\infty, \tag{4.1.24}$$

which ensures that Ω_∞ satisfies the corkscrew condition.

Fix $X,Y\in\Omega_\infty$. Since $\partial\Omega_j\to\partial\Omega_\infty$ and $\overline{\Omega_j}\to\overline{\Omega}_\infty$, for j large enough

$$D[\partial\Omega_j,\partial\Omega_\infty]\text{ and }D[\overline{\Omega_j},\overline{\Omega_\infty}]\leq\frac{d}{2}\min\{\delta_\infty(X),\delta_\infty(Y)\}, \tag{4.1.25}$$

here $d \leq 1$ is a constant dependeing on X, Y to be determined later. Fix an j sufficiently large, we have $X, Y \in \Omega_j$ and

$$\frac{\delta_\infty(X)}{2} \leq \delta_j(X) \leq \frac{3\delta_\infty(X)}{2}, \quad \frac{\delta_\infty(Y)}{2} \leq \delta_j(Y) \leq \frac{3\delta_\infty(Y)}{2}.$$

Since Ω_j satisfies the Harnack chain property with the same constants as Ω, there are constants $c_1 < c_2 < 1$ (independent of j) and balls $B_1, \cdots, B_K$ (the choice of balls are dependent of j) connecting X to Y in Ω_j and such that

$$c_1\delta_j(B_k) \leq \operatorname{diam} B_k \leq c_2\delta_j(B_k), \tag{4.1.26}$$

for $k = 1, 2, \cdots, K$ and

$$K \leq C\left(\frac{|X-Y|}{\min\{\delta_j(X), \delta_j(Y)\}}\right) \leq C'\left(\frac{|X-Y|}{\min\{\delta_\infty(X), \delta_\infty(Y)\}}\right). \tag{4.1.27}$$

Combining (5.2.53) and (5.2.54), we know

$$\operatorname{diam} B_k \geq c(c_1, c_2, K)\min\{\delta_j(X), \delta_j(Y)\} \geq c'(c_1, c_2, K)\min\{\delta_\infty(X), \delta_\infty(Y)\}. \tag{4.1.28}$$

Combining (4.1.28) again with (5.2.53), we find a constant $d = d(c_1, c_2, K) \leq 1$ such that

$$\delta_j(B_k) \geq d\min\{\delta_\infty(X), \delta_\infty(Y)\} \quad \text{for all } k = 1, 2, \cdots, K.$$

Recall (5.2.51), we conclude

$$|\delta_j(B_k) - \delta_\infty(B_k)| \leq D[\partial\Omega_j, \partial\Omega_\infty] \leq \frac{d}{2}\min\{\delta_\infty(X), \delta_\infty(Y)\} \leq \frac{\delta_j(B_k)}{2}.$$

Thus $B_k \subset \Omega_\infty$, and moreover,

$$\frac{2\delta_\infty(B_k)}{3} \leq \delta_j(B_k) \leq 2\delta_\infty(B_k).$$

Combined with (5.2.53) we get

$$\frac{2}{3}c_1\delta_\infty(B_k) \leq \operatorname{diam} B_k \leq 2c_2\delta_\infty(B_k). \tag{4.1.29}$$

To summarize, we find balls $B_1, \cdots, B_K$ in Ω_∞ that satisfy (5.2.61) and connect X to Y, and the number of balls satisfies (5.2.54). Therefore Ω_∞ satisfies the Harnack chain condition. This combined with (5.2.47) shows that Ω_∞ is a uniform domain with constants comparable to those of Ω.

Proof of (4) and (5) in Theorem 5.2.8 As noted right after (4.1.5) and (4.1.6), $\{\sigma_j\}$ and $\{\omega_j\}$ satisfy conditions *i*) and *ii*) of Lemma 2.3.5. Moreover for $R > 0$,

$$\sup_j \sigma_j(B(0,R)) = \sup_j \frac{\sigma(B(q_j, Rr_j))}{r_j^{n-1}} \leq CR^{n-1}$$

since σ is Ahlfors regular; and

$$\sup_j \omega_j(B(0,R)) = \sup_j \frac{\omega(B(q_j, Rr_j))}{\omega(B(q_j, r_j))} \leq C(R)$$

since ω is doubling. Therefore modulo passing to a subsequence (which we relabel) we have

$$\sigma_j \rightharpoonup \mu_\infty, \quad \omega_j \rightharpoonup \omega_{L_\infty}.$$

where μ_∞ and ω_{L_∞} are Radon measures. By Lemma 2.3.5, μ_∞ and ω_∞ are doubling measures and

$$\operatorname{spt}\sigma_j \to \operatorname{spt}\mu_\infty, \quad \operatorname{spt}\omega_j \to \operatorname{spt}\omega_{L_\infty}. \tag{4.1.30}$$

Since σ_j is Ahlfors regular, it is clear that $\operatorname{spt}\sigma_j = \partial\Omega_j$; by the doubling property of ω_j and that $\omega_j(B(0,1)) = 1$ we also know $\operatorname{spt}\omega_j = \partial\Omega_j$. Recall $\partial\Omega_j \to \partial\Omega_\infty$, (4.1.30) yields

$$\operatorname{spt}\mu_\infty = \operatorname{spt}\omega_{L_\infty} = \partial\Omega_\infty.$$

To show that μ_∞ is Ahlfors regular let $q \in \partial\Omega_\infty$ and let $q_j \in \partial\Omega_j$ such that $q_j \to q$. For $r > 0$ and j sufficiently large

$$\begin{aligned} \mu_\infty(B(q,r)) &\geq \mu_\infty\left(\overline{B\left(q, \frac{r}{2}\right)}\right) \geq \limsup \sigma_j\left(\overline{B\left(q, \frac{r}{2}\right)}\right) \\ &\geq \limsup \sigma_j\left(B\left(q_j, \frac{r}{4}\right)\right) \geq Cr^{n-1}; \end{aligned} \tag{4.1.31}$$

and

$$\mu_\infty(B(q,r)) \leq \liminf \sigma_j(B(q,r)) \leq \liminf \sigma_j(B(q_j, 2r)) \leq C' r^{n-1}. \tag{4.1.32}$$

Note that (5.2.63) and (5.2.62) guarantee that μ_∞ is Ahlfors regular. Moreover by Theorem 6.9 of [Ma], there are constants C_1 and C_2 such that

$$C_1\mu_\infty \leq \mathcal{H}^{n-1} \llcorner \partial\Omega_\infty \leq C_2\mu_\infty.$$

Proof of Theorem 4.1.9 Let $\psi \in C_c^\infty(\mathbb{R}^n)$. Suppose $\operatorname{spt}\psi \subset B(0,R)$ for some large $R > 0$. Let j be large enough, so that the pole $X_0 \notin B(q_j, 4r_jR)$. Define $\varphi_j(Z) = \psi\left(\frac{Z-q_j}{r_j}\right)$ and note that $\operatorname{spt}\varphi_j \subset B(q_j, r_jR)$ thus $X_0 \notin \operatorname{spt}\varphi_j$. Using (2.2.17) as well as a change of variables we have

$$\begin{aligned} -\int_{\Omega_j} A(q_j + r_jZ)\nabla u_j \cdot \nabla\psi\, dZ &= -\int_\Omega A(X)\frac{r_j^{n-2}}{\omega(B(q_j,r_j))} r_j \nabla G(X_0, X)\cdot r_j\nabla\varphi_j r_j^{-n} dX \\ &= -\frac{1}{\omega(B(q_j,r_j))}\int_\Omega A(X)\nabla G(X_0,X)\cdot\nabla\varphi_j\, dX \end{aligned}$$

$$= \frac{1}{\omega(B(q_j, r_j))} \int_{\partial\Omega} \varphi_j d\omega^{X_0}$$
$$= \int_{\partial\Omega_j} \psi \, d\omega_j.$$

For $Z \in B(0, R)$, we have $|q - (q_j + r_j Z)| \le |q - q_j| + r_j R$, since $q_j \to q$, $r_j \to 0$ then $\lim_{j\to\infty}(q_j + r_j Z) = q$. Therefore since $A \in C(\overline{\Omega})$, we have $A(q_j + r_j Z) \to A(q_\infty)$ uniformly on $B(0, R)$. By (1), (2) and (4) in Theorem 5.2.8, $\nabla u_j \rightharpoonup \nabla u_\infty$ in $L^2_{loc}(\mathbb{R}^n)$, $\overline{\Omega_j} = \overline{\{u_j > 0\}} \to \overline{\Omega_\infty} = \overline{\{u_\infty > 0\}}$, and $\partial\Omega_j \to \partial\Omega_\infty$ in the Hausdorff distance sense. Moreover $\omega_j \rightharpoonup \omega_{L_\infty}$ with $\operatorname{spt}\omega_j \to \operatorname{spt}\omega_{L_\infty} = \partial\Omega_\infty$. Hence letting $j \to \infty$ in (5.2.82) we obtain (4.1.11).

Proof of Theorem 5.2.79 Recall we proved in Lemma 2.3.12 that for $\mathcal{H}^{n-1}$ a.e. $q \in \partial\Omega$ there exists $A^*(q)$ a symmetric constant-coefficient elliptic matrix such that (2.3.13) holds. For such q consider the blow-up given by Theorem 5.2.8 where $q_j = q$ for all j. As in the proof of Theorem 4.1.9 we have for $\psi \in C_c^\infty(\mathbb{R}^n)$ and j large enough,

$$-\int_{\Omega_j} A(q + r_j Z)\nabla u_j \cdot \nabla\psi dZ = \int_{\partial\Omega_j} \psi d\omega_j. \tag{4.1.33}$$

Note that as in the proof of Theorem 4.1.9, the right hand side

$$\int_{\partial\Omega_j} \psi d\omega_j \to \int_{\partial\Omega_\infty} \psi d\omega_{L_\infty} \quad \text{as} \quad j \to \infty. \tag{4.1.34}$$

Recall that for $R > 0$, $\sup_j \|u_j\|_{W^{1,2}(B(0,R))} \le C_R < \infty$ by (5.2.14), $u_j \to u_\infty$ in $L^2_{loc}(\mathbb{R}^n)$ and $\nabla u_j \rightharpoonup \nabla u_\infty$ in $L^2_{loc}(\mathbb{R}^n)$. Thus for j large enough since $\psi \in C_c^\infty(\mathbb{R}^n)$, by triangle inequality and Hölder inequality we have

$$\begin{aligned}
&\left| \int_{\Omega_j} \langle A(q_j + r_j Z)\nabla u_j, \nabla\psi\rangle dZ - \int_{\Omega_\infty} \langle A^*(q)\nabla u_\infty, \nabla\psi\rangle dZ \right| \\
&\le \left| \int_{\Omega_j} \left\langle \left(A(q_j + r_j Z) - A^*(q)\right) \nabla u_j, \nabla\psi \right\rangle dZ \right| \\
&\quad + \left| \int_{\Omega_j} \langle A^*(q)\nabla u_j, \nabla\psi\rangle dZ - \int_{\Omega_\infty} \langle A^*(q)\nabla u_\infty, \nabla\psi\rangle dZ \right| \\
&\le \|\nabla\psi\|_{L^\infty} \left(\int_{\Omega_j\cap B(0,R)} |A(q + r_j Z) - A^*(q)|^2 dZ \right)^{\frac{1}{2}} \left(\int_{\Omega_j\cap B(0,R)} |\nabla u_j|^2 dZ \right)^{\frac{1}{2}} \\
&\quad + \left| \int_{\Omega_j} \langle A^*(q)\nabla u_j, \nabla\psi\rangle dZ - \int_{\Omega_\infty} \langle A^*(q)\nabla u_\infty, \nabla\psi\rangle dZ \right|.
\end{aligned} \tag{4.1.35}$$

Note that since $A^*(q)$ is a constant-coefficient matrix, $\nabla u_j \rightharpoonup \nabla u_\infty$ in $L^2_{loc}(\mathbb{R}^n)$ implies $A^*(q)\nabla u_j \rightharpoonup A^*(q)\nabla u_\infty$ in $L^2_{loc}(\mathbb{R}^n)$. Thus since $\overline{\Omega_j} = \overline{\{u_j > 0\}} \to \overline{\Omega_\infty} = \overline{\{u_\infty > 0\}}$

$$\lim_{j\to\infty} \int_{\Omega_j} \langle A^*(q)\nabla u_j, \nabla\psi\rangle dZ = \int_{\Omega_\infty} \langle A^*(q)\nabla u_\infty dZ, \nabla\psi\rangle dZ. \tag{4.1.36}$$

On the other hand since $A \in L^\infty(\Omega)$, and by construction $|A^*(q)| \leq C\|A\|_{L^\infty(\Omega)}$, we have that

$$\begin{aligned}\left(\int_{\Omega_j\cap B(0,R)} |A(q+r_jZ)-A^*(q)|^2 dZ\right)^{\frac{1}{2}} &\leq \left(\frac{1}{r_j^n}\int_{\Omega\cap B(q,r_jR)} |A-A^*(q)|^2 dX\right)^{\frac{1}{2}} \\ &\leq C\|A\|_{L^\infty(\Omega)}^{2-\frac{n}{n-1}} \left(\fint_{\Omega\cap B(q,r_jR)} |A-A^*(q)|^{\frac{n}{n-1}} dX\right)^{\frac{1}{2}}. \end{aligned} \tag{4.1.37}$$

Hence by combining (2.3.13), (5.2.84), (4.1.36) and (4.1.37) we obtain

$$\begin{aligned}&\lim_{j\to\infty}\left|\int_{\Omega_j}\langle A(q_j+r_jZ)\nabla u_j, \nabla\psi\rangle dZ - \int_{\Omega_\infty}\langle A^*(q)\nabla u_\infty, \nabla\psi\rangle dZ\right| \\ &\quad\leq C\|\nabla\psi\|_{L^\infty}\sup_j\|\nabla u_j\|_{L^2(B(0,R))}\|A\|_{L^\infty(\Omega)}^{2-\frac{n}{n-1}}\cdot\lim_{j\to\infty}\left(\fint_{\Omega\cap B(q,r_jR)} |A-A^*(q)|^{\frac{n}{n-1}} dX\right)^{\frac{1}{2}} \\ &\qquad+\lim_{j\to\infty}\left|\int_{\Omega_j}\langle A^*(q)\nabla u_j, \nabla\psi\rangle dZ - \int_{\Omega_\infty}\langle A^*(q)\nabla u_\infty, \nabla\psi\rangle dZ\right| = 0.\end{aligned} \tag{4.1.38}$$

Thus combining (4.1.34) and (4.1.38), we conclude the proof of (4.1.14) and Theorem 5.2.79.

4.2 Analytic properties of the blow-up and pseudo blow-up domains

As mentioned in section 5.2, in order to apply Theorem 5.0.1 we need to study the elliptic measures of the blow-up domain with finite poles. In this section we construct these measures by a limiting procedure which is compatible with the blow-up procedure used to produce the tangent and pseudo-tangent domains.

Theorem 4.2.1. *Let $\Omega \subset \mathbb{R}^n$ be a uniform domain with Ahlfors regular boundary. Let $L = -\operatorname{div}(A\nabla)$ with $A \in C(\overline{\Omega})$ or $A \in W^{1,1}(\Omega)\cap L^\infty(\Omega)$. Suppose that the elliptic measure $\omega \in A_\infty(\sigma)$. Assume that Ω_∞ is either the pseudo-tangent or the tangent domain obtained in Theorem 5.2.8, where in the case of $A \in W^{1,1}(\Omega)\cap L^\infty(\Omega)$ we use $q_j = q$ for every j and only consider points q satisfying (2.3.13) and L_∞ is the corresponding operator as in Theorem 4.1.9 or 5.2.79, then $\omega_{L_\infty} = \omega_\infty \in A_\infty(\sigma_\infty)$ (see Definition 2.1.14).*

Proof. Our goal is to show that the elliptic measure of L_∞ with finite pole can be recovered as a limit of the elliptic measures of $L_j = -\operatorname{div}(A_j(Z)\nabla)$ with finite pole, where $A_j(Z) = A(q_j + r_jZ)$ in Ω_j, and that the A_∞ property of elliptic measures is preserved when passing to a limit.

Let $f \in C_c(\mathbb{R}^n) \cap \text{Lip}(\mathbb{R}^n) \cap W^{1,2}(\mathbb{R}^n)$, and consider the Dirichlet problem

$$\text{(4.2.2)} \qquad \begin{cases} L_j v_j = 0, & \text{in } \Omega_j \\ v_j = f, & \text{on } \partial\Omega_j \end{cases}$$

then for $Z \in \Omega_j$

$$\text{(4.2.3)} \qquad v_j(Z) = \int_{\partial\Omega_j} f(q)\, d\omega_j^Z(q).$$

Here ω_j^Z is the harmonic measure of L_j in Ω_j with pole Z. By definitions of Ω_j and L_j it is not hard to see

$$\omega_j^Z(E) = \omega^{q_j + r_j Z}(q_j + r_j E), \qquad \text{for } E \subset \partial\Omega_j.$$

By the maximum principle

$$\sup_{\Omega_j} |v_j| \leq \|f\|_{L^\infty(\partial\Omega_j)} \leq \|f\|_{L^\infty(\mathbb{R}^n)}.$$

Since the domains Ω_j have Ahlfors regular boundaries with the same constants, DeGiorgi-Nash-Moser and the assumption that the boundary data f is Lipschitz yield that the solutions $\{v_j\}$ are equicontinuous on compact sets of $\mathbb{R}^n$. Thus the sequence $\{v_j\}$ is equicontinuous and uniformly bounded. Furthermore using the variational properties of v_j we know that

$$\int_{\Omega_j} \langle A_j(Z)\nabla v_j, \nabla v_j\rangle \leq \int_{\Omega_j} \langle A_j(Z)\nabla f, \nabla f\rangle.$$

The uniform ellipticity of L_j yields

$$\lambda \int_{\Omega_j} |\nabla v_j|^2 \leq \Lambda \int_{\Omega_j} |\nabla f|^2.$$

Extending $v_j = f$ on Ω_j^c we have that

$$\sup_j \|\nabla v_j\|_{L^2(\mathbb{R}^n)} \leq \left(\frac{\Lambda}{\lambda}\right)^{\frac{1}{2}} \|\nabla f\|_{L^2(\mathbb{R}^n)}, \quad \text{and } \sup_j \|v_j\|_{L^2(B(0,R))} \leq C_R.$$

Modulo passing to a subsequence (which we relabel) we have that there is a continuous function $v \in W^{1,2}_{loc}(\mathbb{R}^n)$ with $\nabla v \in L^2(\mathbb{R}^n)$ and such that $v_j \to u$ uniformly on compact sets of $\mathbb{R}^n$ and $\nabla v_j \rightharpoonup \nabla v$ in $L^2(\mathbb{R}^n)$. Note that a priori the choice of a subsequence could depend on the boundary data f, which will be problematic. We will show later that this is not the case.

We claim that the function v solves the Dirichlet problem

$$\text{(4.2.4)} \qquad \begin{cases} L_\infty v = 0, & \text{in } \Omega_\infty \\ v = f, & \text{on } \partial\Omega_\infty. \end{cases}$$

Note that for $p \in \partial\Omega_\infty$ there exist $p_j \in \partial\Omega_j$ with $p_j \to p$. Using the continuity of v and f at p, the uniform convergence of v_j to v on compact sets (for example on $\overline{B(p,r)}$) and the fact that $v_j = f$ on $\partial\Omega_j$, we have

$$\begin{aligned} |v(p) - f(p)| &\leq |v(p) - v(p_j)| + |v(p_j) - v_j(p_j)| + |v_j(p_j) - f(p_j)| + |f(p_j) - f(p)| \\ &\leq |v(p) - v(p_j)| + \|v - v_j\|_{L^\infty(\overline{B(p,r)})} + 0 + |f(p_j) - f(p)|. \end{aligned} \tag{4.2.5}$$

Letting $j \to \infty$ in (4.2.5) yields $v(p) = f(p)$. Combined with the continuity of v in $\mathbb{R}^n$ we conclude that u tends to f continuously towards the boundary $\partial\Omega_\infty$. Let $\xi \in C_c^\infty(\Omega_\infty)$. Since Ω_j and Ω_∞ are open domains satisfying $\overline{\Omega_j} \to \overline{\Omega_\infty}$ and $\partial\Omega_j \to \partial\Omega_\infty$ uniformly on compact sets, a standard argument shows that a compact set contained in Ω_∞ is eventually contained in Ω_j. Thus $\xi \in C_c^\infty(\Omega_\infty)$ implies $\xi \in C_c^\infty(\Omega_j)$ for j sufficiently large. By (4.2.2) we have that

$$\int \langle A(q_j + r_j Z)\nabla v_j, \nabla\xi\rangle dZ = 0. \tag{4.2.6}$$

Letting $j \to \infty$ in (5.2.94) and proceeding as in the proofs of Theorem 4.1.9 and 5.2.79, we conclude that

$$\int \langle A^*(q)\nabla v, \nabla\xi\rangle dZ = 0,$$

where $A^*(q) = A(q)$ when $A \in C(\overline{\Omega})$ and $A^*(q)$ is as in Theorem 5.2.79 in the case when $A \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$. Thus in either case we have $L_\infty v = 0$ in Ω_∞. Since the tangent domain Ω_∞ is unbounded, the solution to the Dirichlet problem (5.2.88) may not be unique. It may not even satisfy the maximum principle. We need more work to show the function v we just constructed is indeed a solution we want, in particular it is indeed the solution that gives rise to the elliptic measure of Ω_∞.

Suppose that f is compactly supported in $B(0,R_0)$. Given $\epsilon > 0$ there is $j_{\epsilon,R_0} \in \mathbb{N}$ such that for $j \geq j_{\epsilon,R_0}$, the Hausdorff distance between $\partial\Omega_i \cap \overline{B(0,R_0)}$ and $\partial\Omega_\infty \cap \overline{B(0,R_0)}$ is small enough so that any $p_j \in \partial\Omega_j \cap \overline{B(0,R_0)}$, there is $p \in \partial\Omega_\infty$ such that since f is uniformly continuous on $\overline{B(0,R)}$, $|f(p) - f(p_j)| < \epsilon$. Hence

$$\sup_{\partial\Omega_j} |f| = \sup_{\partial\Omega_j \cap \overline{B(0,R)}} |f| \leq \sup_{\partial\Omega_\infty \cap \overline{B(0,2R)}} |f| + \epsilon = \sup_{\partial\Omega_\infty} |f| + \epsilon. \tag{4.2.7}$$

For $Z \in \overline{\Omega_\infty}$ there exists a sequence $Z_j \in \overline{\Omega_j}$ such that $Z_j \to Z$ and all lie in $\overline{B(0,MR_0)}$ for M large enough. Since $\sup_{\Omega_j} |v_j| \leq \sup_{\partial\Omega_j} |f|$, $v_j \to v$ and $\overline{\Omega_j} \to \overline{\Omega_\infty}$ uniformly on compact sets. For $\epsilon > 0$ there is $j'_{\epsilon,R_0,M} \in \mathbb{N}$ such that for $j \geq j'_{\epsilon,R_0,M}$, using (5.2.95) we have

$$|v(Z)| \leq |v(Z) - v(Z_j)| + |v(Z_j) - v_j(Z_j)| + |v_j(Z_j)| \leq 2\epsilon + \sup_{\partial\Omega_j} |f| \leq 3\epsilon + \sup_{\partial\Omega_\infty} |f| \tag{4.2.8}$$

Therefore (5.2.96) yields $\sup_{\Omega_\infty} |v(Z)| \leq 3\epsilon + \sup_{\partial\Omega_\infty} |f|$ for all $\epsilon > 0$, and thus $\sup_{\Omega_\infty} |v(Z)| \leq \sup_{\partial\Omega_\infty} |f|$. To summarize, for any $f \in C_c(\mathbb{R}^n) \cap W^{1,2}(\mathbb{R}^n)$ we construct a continuous function v satisfying

$$\begin{cases} L_\infty v = 0, & \text{in } \Omega_\infty \\ v = f, & \text{on } \partial\Omega_\infty \end{cases} \tag{4.2.9}$$

and satisfying the maximum principle $\sup_{\partial\Omega_\infty}|v| \le \sup_{\partial\Omega_\infty}|f|$.

We observe that even though the constructions are different, in the case when the boundary value function f is non-negative and $f \in C_c(\mathbb{R}^n) \cap \mathrm{Lip}(\mathbb{R}^n) \cap W^{1,2}(\mathbb{R}^n)$ we produce the same bounded solution v as the one constructed in [HM1, Section 3 pp.13] for the unbounded domain Ω_∞. (Note that the construction is for the Laplacian but holds for any constant coefficient operator.) We denote the solution constructed there by u. Recall that $u = \lim_{R\to\infty} u_R$, where u_R is the solution to $L_\infty u_R = 0$ in the bounded domain $\Omega_R = \Omega_\infty \cap B(0,2R)$ with boundary value $f\eta(\cdot/R)$. Here η is a smooth cut-off function such that $0 \le \eta \le 1$, $\eta = 1$ for $|Z| < 1$ and $\operatorname{spt}\eta \subset \{Z \in \mathbb{R}^n : |Z| < 2\}$. Assume f is compactly supported on $B(0,R_0)$. Then for any $R \ge R_0$, by the maximum principle $u_R \le v$ in Ω_R, thus the limit $u \le v$ on Ω_∞. Set $w = v-u$, it is a non-negative solution to $L_\infty w = 0$ in Ω_∞ with vanishing boundary value. Fix $Z \in \Omega_\infty$, since Ω_∞ is a uniform domain with Ahlfors regular boundary, by Lemma 2.2.19 for any $Z \in \Omega_\infty$ with $\delta_\infty(Z) < \frac{r}{2}$

$$w(Z) \lesssim \left(\frac{\delta_\infty(Z)}{r}\right)^\beta \sup_{\Omega_\infty} w \le 2\left(\frac{\delta_\infty(Z)}{r}\right)^\beta \sup_{\partial\Omega_\infty} f, \tag{4.2.10}$$

Letting $r \to \infty$ in (5.2.97) we get $w(Z) = 0$. Thus $v \equiv u$ in Ω_∞. Recall that at this point for $f \in C_c(\mathbb{R}^n) \cap \mathrm{Lip}(\mathbb{R}^n) \cap W^{1,2}(\mathbb{R}^n)$ we are only able to find a subsequence (possibly depending on f) converging to a continuous function v that solves (4.2.9). We claim that in the case when also $f \ge 0$, the entire sequence v_j converges to v. In fact given two arbitrary subsequences $\{v_{j_k}\}$ and $\{v_{j_k'}\}$ of $\{v_j\}$, the argument above shows that either sequence has a further subsequence that converges to a continuous function, denoted by v_1 and v_2 respectively. Both functions v_1 and v_2 satisfy the equation (4.2.9) and maximum principle. Once again by the previous argument they are both equal to u, thus $v_1 = v_2$ in Ω_∞. Therefore the entire sequence $\{v_j\}$ converges to a same continuous function $v = u$. In general if f is not necessarily non-negative, we just decompose it into two non-negative functions $f = f^+ - f^-$, with $f^\pm = \max\{0, \pm f\} \ge 0$ and $f^\pm \in C_c(\mathbb{R}^n)\cap\mathrm{Lip}(\mathbb{R}^n)\cap W^{1,2}(\mathbb{R}^n)$. The argument above yields $v^\pm$ satisfying (4.2.9) with boundary data $f^\pm$ respectively. Then $v = v^+ - v^-$ is a solution to (4.2.9) with boundary data f and satisfying the maximum principle. Hence for any $Z \in \Omega_\infty$ fixed, the operator $\Lambda_Z : C_c(\mathbb{R}^n) \cap \mathrm{Lip}(\mathbb{R}^n) \cap W^{1,2}(\mathbb{R}^n) \to \mathbb{R}$ defined by $\Lambda_Z(f) = v(Z)$ is positive bounded (with respect to the $\|\cdot\|_{L^\infty(\mathbb{R}^n)}$ norm) linear functional. Hahn-Banach theorem allows us to extend Λ_Z to a positive bounded linear functional on all of $C_c(\mathbb{R}^n)$, with the same operator norm. We still denote the functional as $\Lambda_Z : C_c(\mathbb{R}^n) \to W^{1,2}(\mathbb{R}^n)$. By Riesz representation theorem there exists a unique family of Radon measures $\{\omega_\infty^Z\}_{Z\in\Omega_\infty}$ such that

$$\Lambda_Z(f) = \int_{\partial\Omega_\infty} f(q)\,d\omega_\infty^Z(q) \quad \text{for all } f \in C_c(\mathbb{R}^n).$$

In particular for $f \in C_c(\mathbb{R}^n) \cap \mathrm{Lip}(\mathbb{R}^n) \cap W^{1,2}(\mathbb{R}^n)$, the measures $\{\omega_\infty^Z\}_{Z\in\Omega_\infty}$ satisfies

$$v(Z) = \Lambda_Z(f) = \int_{\partial\Omega_\infty} f(q)\,d\omega_\infty^Z(q). \tag{4.2.11}$$

Recall that the sequence $\{u_j\}$ converges uniformly to u in compact sets. Thus combining (5.2.89) and (4.2.11) we have that for all $f \in C_c(\mathbb{R}^n) \cap \mathrm{Lip}(\mathbb{R}^n) \cap W^{1,2}(\mathbb{R}^n)$

$$\lim_{j\to\infty} \int_{\partial\Omega_j} f(q) d\omega_j^Z(q) = \int_{\partial\Omega_\infty} f(q) d\omega_\infty^Z(q). \tag{4.2.12}$$

A standard approximation argument shows that (5.2.98) holds for all $f \in C_c(\mathbb{R}^n)$. And we conclude that $\omega_j^Z \rightharpoonup \omega_\infty^Z$ as Radon measures, for any $Z \in \Omega_\infty$.

To show that $\omega_\infty \in A_\infty(\sigma_\infty)$ (recall $\sigma_\infty = \mathcal{H}^{n-1} \llcorner \partial\Omega_\infty$ is the surface measure), let $p \in \partial\Omega_\infty$ and $r > 0$, and $\Delta' = B(m, s) \cap \partial\Omega_\infty \subset \Delta = B(p, r) \cap \partial\Omega_\infty$ with $m \in \partial\Omega_\infty$. Recall that we denote by $A(p, r)$ a non-tangential point in Ω_∞ to p at radius r; see the proof of Theorem 5.2.8 (3). Since $\partial\Omega_j \to \partial\Omega_\infty$, there exist $p_j \in \partial\Omega_j$ such that $p_j \to p$ and thus for j large enough $A(p, r)$ is also a non-tangential point to p_j in Ω_j with radius $2r$. Since $m \in \partial\Omega_\infty$, there also exist $m_j \in \partial\Omega_j$ such that $m_j \to m$. In particular for j sufficiently large

$$|m_j - m| < \frac{s}{5}. \tag{4.2.13}$$

Since the Ω_j's are uniform and satisfy the CDC with the same constant and since the operators L_j's have ellipticity constants bounded by λ and Λ, we conclude from Corollary 2.2.21 and Lemma 2.2.27 that $\omega_j^{A(p,r)}$ is doubling with a universal constant (independent of j, p, r), denoted by C. Hence by Theorem 1.24 in [Ma] we have

$$\begin{aligned}
\omega_{L_\infty}^{A(p,r)}(\Delta(m, s)) \geq \omega_{L_\infty}^{A(p,r)}\left(\overline{\Delta\left(m, \frac{4}{5}s\right)}\right) &\geq \limsup_{j\to\infty} \omega_j^{A(p,r)}\left(\overline{\Delta\left(m, \frac{4}{5}s\right)}\right) \\
&\geq \limsup_{j\to\infty} \omega_j^{A(p,r)}\left(\overline{\Delta\left(m_j, \frac{3}{5}s\right)}\right) \\
&\geq C^{-1} \limsup_{j\to\infty} \omega_j^{A(p,r)}\left(\Delta\left(m_j, \frac{6}{5}s\right)\right).
\end{aligned} \tag{4.2.14}$$

Let V be an arbitrary open set in $B(m, s)$, by (5.2.99)

$$V \subset B(m, s) \subset B\left(m_j, \frac{6}{5}s\right).$$

Again by Theorem 1.24 in [Ma] and (5.2.100) we have

$$\begin{aligned}
\frac{\omega_{L_\infty}^{A(p,r)}(V)}{\omega_{L_\infty}^{A(p,r)}(\Delta(m, s))} &\leq C \frac{\liminf_{j\to\infty} \omega_j^{A(p,r)}(V)}{\limsup_{j\to\infty} \omega_j^{A(p,r)}\left(\Delta\left(m_j, \frac{6}{5}s\right)\right)} \\
&\leq C \liminf_{j\to\infty} \left(\frac{\omega_j^{A(p,r)}(V)}{\omega_j^{A(p,r)}\left(\Delta\left(m_j, \frac{6}{5}s\right)\right)}\right).
\end{aligned} \tag{4.2.15}$$

Let $m_j = q_j + r_j m_j$ and $\widetilde{p}_j = q_j + r_j p_j$ in $\partial\Omega$, by the definition (4.1.4) of ω_j,

$$\frac{\omega_j^{A(p,r)}(V)}{\omega_j^{A(p,r)}\left(\Delta\left(m_j, \frac{6}{5}s\right)\right)} = \frac{\omega^{q_j+r_jA(p,r)}(q_j + r_j V)}{\omega^{q_j+r_jA(p,r)}\left(\Delta\left(m_j, \frac{6}{5}sr_j\right)\right)}. \tag{4.2.16}$$

The assumption $B(m,s) \subset B(p,r)$ implies $|m - p| \leq r - s$. Thus by $m_j \to m$, $p_j \to p$ we have

$$|m_j - p_j| \leq |m_j - m| + |m - p| + |p - p_j| < r - \frac{s}{5}.$$

Note $s < r$, hence

$$\Delta\left(m_j, \frac{6}{5}s\right) \subset \Delta(p_j, 2r).$$

Recall that $A(p,r)$ is a non-tangential point in Ω_j to the boundary point p_j at radius $2r$. Therefore after rescaling from Ω_j to Ω, we have that $q_j + r_j A(p,r)$ is a non-tangential point to the boundary point $\widetilde{p}_j$ at radius $2rr_j$, and that

$$q_j + r_j V \subset \Delta\left(m_j, \frac{6}{5}sr_j\right) \subset \Delta(\widetilde{p}_j, 2rr_j).$$

By the assumption that $\omega_L \in A_\infty(\sigma)$ (see Definition 2.1.14), we conclude that

$$\frac{\omega^{q_j+r_jA(p,r)}(q_j + r_j V)}{\omega^{q_j+r_jA(p,r)}\left(\Delta\left(m_j, \frac{6}{5}sr_j\right)\right)} \leq C\left(\frac{\mathcal{H}^{n-1}\left(\partial\Omega \cap \left(q_j + r_j V\right)\right)}{\mathcal{H}^{n-1}\left(\partial\Omega \cap B\left(m_j, \frac{6}{5}sr_j\right)\right)}\right)^\theta. \tag{4.2.17}$$

Combining (5.2.101), (4.2.16) and (4.2.17), using the definition (4.1.4) of σ_j, $\sigma_j \rightharpoonup \mu_\infty$ and that $\sigma = \mathcal{H}^{n-1}|_{\partial\Omega}$ and μ_∞ are Ahlfors regular with the the same constant, we get

$$\begin{aligned}
\frac{\omega_\infty^{A(p,r)}(V)}{\omega_\infty^{A(p,r)}(\Delta(m,s))} &\leq C \liminf_{j\to\infty}\left(\frac{\mathcal{H}^{n-1}\left(\partial\Omega \cap \left(q_j + r_j V\right)\right)}{\mathcal{H}^{n-1}\left(\Delta\left(m_j, \frac{6}{5}sr_j\right)\right)}\right)^\theta \\
&\lesssim \left(\liminf_{j\to\infty} \frac{\sigma(q_j + r_j V)}{(r_j s)^{n-1}}\right)^\theta \\
&\leq \left(\frac{1}{s^{n-1}} \liminf_{j\to\infty} \sigma_j(V)\right)^\theta \\
&\leq \left(\frac{\mu_\infty(\overline{V})}{\mu_\infty(\Delta(m,s))}\right)^\theta, \qquad (4.2.18)
\end{aligned}$$

Recall that μ_∞ is equivalent to the surface measure $\sigma_\infty = \mathcal{H}^{n-1} \llcorner \partial\Omega_\infty$ (see Theorem 5.2.8 (5)). Hence (4.2.18) yields that for any open set $V \subset \Delta(m,s) \subset \Delta(p,r)$ with $p, m \in \partial\Omega_\infty$

$$\frac{\omega_\infty^{A(p,r)}(V)}{\omega_\infty^{A(p,r)}(\Delta(m,s))} \leq C\left(\frac{\sigma_\infty(\overline{V})}{\sigma_\infty(\Delta(m,s))}\right)^\theta. \tag{4.2.19}$$

For any $E \subset B(m,s)$ closed, since σ_∞ is a Radon measure, given any $\epsilon > 0$ there is an open set V satisfying $E \subset V \subset B(m,s)$ and $\sigma_\infty(V \setminus E) < \epsilon$. Note that for any $x \in E$, there is $r_x > 0$ such that $B(x, 2r_x) \subset V$ and $E \subset \cup_{x\in E} B(x, r_x)$. Since E is compact we can extract a finite subcover $E \subset \cup_{i=1}^m B(x_i, r_i) = U$ and $B(x_i, 2r_i) \subset V$ for $i = 1, \cdots, m$. Note that $E \subset U \subset \overline{U} \subset V$. Thus $\sigma_\infty(\overline{U} \setminus E) < \epsilon$, and using (4.2.19) we have

$$\frac{\omega_{L_\infty}^{A(p,r)}(E)}{\omega_{L_\infty}^{A(p,r)}(\Delta(m,s))} \leq \frac{\omega_{L_\infty}^{A(p,r)}(U)}{\omega_{L_\infty}^{A(p,r)}(\Delta(m,s))} \leq C\left(\frac{\sigma_\infty(\overline{U})}{\sigma_\infty(\Delta(m,s))}\right)^\theta \leq C\left(\frac{\sigma_\infty(E)+\epsilon}{\sigma_\infty(\Delta(m,s))}\right)^\theta.$$

Letting $\epsilon \to 0$ we have that for any closed set $E \subset B(m,s)$

$$\frac{\omega_{L_\infty}^{A(p,r)}(E)}{\omega_{L_\infty}^{A(p,r)}(\Delta(m,s))} \leq C\left(\frac{\sigma_\infty(E)}{\sigma_\infty(\Delta(m,s))}\right)^\theta. \tag{4.2.20}$$

Since both $\omega_{L_\infty}^{A(p,r)}$ and σ_∞ are Radon measures, (4.2.20) holds for any Borel set $E \subset B(m,s)$, which concludes the proof of Theorem 5.2.87.

Corollary 4.2.21. *Let $\Omega \subset \mathbb{R}^n$ be a uniform domain with Ahlfors regular boundary. Let $L = -\operatorname{div}(A\nabla)$ with $A \in C(\overline{\Omega})$ (resp. $A \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$). Suppose that the elliptic measure $\omega \in A_\infty(\sigma)$. Then any pseudo-tangent domain Ω_∞ (resp. tangent domain at a point $q \in \partial\Omega$ satisfying (2.3.13)) is an NTA domain with constants depending only on the allowable constants.*

Proof. Theorem 5.2.87 combined with Theorem 5.0.1 ensures that under the hypotheses of Theorem 1.3.7 (resp. Theorem 1.3.5), all pseudo blow-ups of Ω (resp. all blow-ups of Ω at points $q \in \partial\Omega$ satisfying (2.3.13)) are uniform domains with uniformly rectifiable boundaries with constants depending on the allowable constants. By [AHMNT] we conclude that all such domains are NTA domains with constants depending only on the allowable constants.

4.3 Proof of Theorems 1.3.5 and 1.3.7

Given Corollary 4.2.21, we may assume that all pseudo-tangent domains in the case $A \in C(\overline{\Omega})$ or tangent domains at points $q \in \partial\Omega$ satisfying (2.3.13) in the case $A \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ are NTA domains with exterior corkscrew constant M_∞. That is if Ω_∞ is obtained via this blow-up procedure then for any $p \in \partial\Omega_\infty$ and $r > 0$, there exists $A_\infty^-(p,r) \subset \Omega_\infty^c \cap B(p,r)$ such that

$$B\left(A_\infty^-(p,r), \frac{r}{M_\infty}\right) \subset \Omega_\infty^c \cap B(p,r) \tag{4.3.1}$$

and in particular
$$\frac{\mathcal{H}^n(\Omega^c_\infty \cap B(p,r))}{r^n} \geq \frac{c_n}{M^n_\infty} > 0 \quad \text{for any } r > 0.$$

Proof of Theorem 1.3.7 We want to show that there exists $r_\Omega > 0$, such that Ω satisfies the exterior corkscrew condition with constant $2M_\infty$ for all $q \in \partial\Omega$ and all $r < r_\Omega$. Assume that such an r_Ω does not exist, then there are sequences $r_j \to 0$ and $q_j \in \partial\Omega$ such that we cannot find a corkscrew point in Ω^c with constant $2M_\infty$ at $q_j \in \partial\Omega$ with radius r_j. Consider $\Omega_j = (\Omega - q_j)/r_j$, then apply Theorem 5.2.8, Corollary 4.2.21 and (5.3.1) to find a point $A^-_\infty(0,1) \subset \Omega^c_\infty \cap B(0,1)$ such that $B(A^-_\infty(0,1), 1/M_\infty) \subset \Omega^c_\infty \cap B(0,1)$. Since $\overline{\Omega_j} \to \overline{\Omega_\infty}$ locally uniformly on compact sets, for j large enough
$$B\left(A^-_\infty(0,1), \frac{1}{2M_\infty}\right) \subset \Omega^c_j \cap B(0,1),$$

which implies
$$B\left(A_j, \frac{r_j}{2M_\infty}\right) \subset \Omega^c \cap B(q_j, r_j) \quad \text{with } A_j = q_j + r_j A^-_\infty(0,1). \tag{4.3.2}$$

This contradicts our assumption.

Proof of Theorem 1.3.5 Let $q \in \partial\Omega$ such that (2.3.13) holds. Recall this occurs for $\mathcal{H}^{n-1}$ a.e. $q \in \partial\Omega$ (see Lemma 2.3.12). Since Ω satisfies the interior corkscrew condition (with a constant M), for any $q \in \partial\Omega$,
$$\liminf_{r\to 0} \frac{\mathcal{H}^n(\Omega \cap B(q,r))}{r^n} \geq \frac{c_n}{M} > 0. \tag{4.3.3}$$

Let $r_j \to 0$ and $\Omega_j = (\Omega - q)/r_j$. By Theorem 5.2.8, Corollary 4.2.21, (5.3.1) and a similar argument as in (4.3.2), we have that for a subsequence (which we relabel) $B(A_j, r_j/2M_\infty) \subset \Omega^c \cap B(q, r_j)$, where $A_j = q + r_j A^-_\infty(0,1)$. Thus
$$\liminf_{j\to\infty} \frac{\mathcal{H}^n(\Omega^c \cap B(q,r_j))}{r_j^n} \geq \frac{c_n}{(2M_\infty)^n} > 0. \tag{4.3.4}$$

Combining (4.3.3) and (4.3.4), we conclude that such q belongs to the measure-theoretic boundary $\partial_*\Omega$ of $\partial\Omega$, thus $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0$. Since $\mathcal{H}^{n-1} \llcorner \partial\Omega$ is Ahlfors regular, it is in particular locally finite. Theorem 1 of Section 5.11 in [EG] ensures that Ω is a set of locally finite perimeter. Thus the reduced boundary $\partial^*\Omega$ is rectifiable. Since $\mathcal{H}^{n-1}(\partial_*\Omega \setminus \partial^*\Omega) = 0$ the fact that $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0$ implies $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial^*\Omega) = 0$. We conclude that $\partial\Omega$ is rectifiable.

4.4 Qualitative case: reduction to local quantitative case

In this section we discuss how the quantitative approach also yields information about the qualitative case. Theorem 1.3.6 is proved by reducing it to the following situation which can be seen as a local version of Theorem 1.3.5.

Theorem 4.4.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded uniform domain with Ahlfors regular boundary. Let $L = -\operatorname{div}(A(X)\nabla)$ with $A \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ satisfying (E). Suppose that $G \subset \partial\Omega$ is an open set. Assume there are uniform positive constants C_0, θ so that for any surface ball $\Delta = \Delta(q, r) \subset G$, the elliptic measure with pole at A_Δ satisfies*

$$\frac{\omega^{A_\Delta}(E)}{\omega^{A_\Delta}(\Delta')} \le C_0 \left(\frac{\sigma(E)}{\sigma(\Delta')}\right)^\theta, \tag{4.4.2}$$

where A_Δ is a non-tangential point with respect to Δ, and (4.4.2) holds for all surface balls $\Delta' \subset \Delta$ and all Borel sets $E \subset \Delta'$. Then G is $(n-1)$-rectifiable.

Remark 4.4.3. Note that the assumption (4.4.2) is a local version of $\omega \in A_\infty(\sigma)$. Recall that the proof of Theorem 1.3.5 consists of understanding the blow-ups of the domain Ω at some $q \in \partial\Omega$ and showing that the A_∞ property of the elliptic measure holds for the tangent domain Ω_∞. Since tangent objects only provide infinitesimal information at the blow-up point it is not surprising that only local assumptions are necessary to obtain rectifiablilty.

Proof. If G is empty, there is nothing to prove, so we assume $G \neq \emptyset$. Since G is an open subset of an Ahlfors regular boundary and $\sigma \ll \omega$, we have $\sigma(G) > 0$ and thus $\omega(G) > 0$. Consider

$$\widehat{G} = \left\{ q \in G : A^*(q) \text{ exists as in Lemma 2.3.12} \right\},$$

then $\sigma(G \setminus \widehat{G}) = 0$. Theorems 5.2.8 and 5.2.79 hold if we consider a geometric blow-up at $q \in \widehat{G}$. We claim that the tangent domain Ω_∞ at every $q \in \widehat{G}$ satisfies its elliptic measure ω_∞ is of class A_∞ with respect to the surface measure $\sigma_\infty = \mathcal{H}^{n-1}|_{\partial\Omega_\infty}$, i.e. Theorem 5.2.87 holds. Namely we need to show for any point $p \in \partial\Omega_\infty$, any surface ball $\Delta(m, s) = B(m, s) \cap \partial\Omega_\infty \subset B(p, r) \cap \partial\Omega_\infty$ with $m \in \partial\Omega_\infty$, $r, s > 0$ and any open subset V of $B(m, s)$,

$$\frac{\omega_\infty^{A(p,r)}(V)}{\omega_\infty^{A(p,r)}(\Delta(m, s))} \le C \left(\frac{\sigma_\infty(\overline{V})}{\sigma_\infty(\Delta(m, s))}\right)^\theta. \tag{4.4.4}$$

Note that in the proof of Theorem 5.2.87, the construction of the elliptic measure ω_∞ does not require the A_∞ property of ω. It only uses the fact that Ω is a uniform domain with Ahlfors regular boundary and that $A^*(q)$ exists. Moreover, we still have $\omega_j^Z \rightharpoonup \omega_\infty^Z$ for any $Z \in \Omega_\infty$. Recall the notations in the proof of Theorem 5.2.87 (note that in this case, we have the blow-up point $q_j = q$ for all j): there are sequences $\partial\Omega_j \ni p_j \to p \in \partial\Omega_\infty$, $\partial\Omega_j \ni m_j \to m \in \partial\Omega_\infty$, and we let $\widetilde{m}_j = q + r_j m_j$, $\widetilde{p}_j = q + r_j p_j$ on $\partial\Omega$. A close look at the proof of Theorem 5.2.87 shows that to prove (4.4.4) it is enough to show (4.2.17), which we rewrite here:

$$\frac{\omega^{q+r_jA(p,r)}(q_j+r_jV)}{\omega^{q+r_jA(p,r)}\left(\Delta\left(m_j,\frac{6}{5}sr_j\right)\right)} \le C\left(\frac{\mathcal{H}^{n-1}\left(\partial\Omega\cap\left(q+r_jV\right)\right)}{\mathcal{H}^{n-1}\left(\partial\Omega\cap B\left(m_j,\frac{6}{5}sr_j\right)\right)}\right)^{\theta}. \tag{4.4.5}$$

Moreover since G is open, for any $q\in\widehat{G}\subset G$ we can find a surface ball $\Delta_0=\Delta(q,r_0)\subset G$. Hence if j is large enough (so that r_j is small enough), we have the surface ball

$$\Delta(\widetilde{p}_j,2rr_j)\subset\Delta\left(q,2(r+|p|)r_j\right)\subset\Delta(q,r_0) \tag{4.4.6}$$

is contained in G. Therefore we may apply the assumption (4.4.2) to the surface ball

$$\Delta'=\Delta\left(m_j,\frac{6}{5}sr_j\right)=B(\widetilde{m}_j,6sr_j/5)\cap\partial\Omega\subset\Delta(\widetilde{p}_j,2rr_j),$$

with non-tangential pole $q+r_jA(p,r)$ and to the Borel set $E=q+r_jV\subset\Delta\left(m_j,\frac{6}{5}sr_j\right)$ and obtain (4.4.5). (Recall that $\sigma=\mathcal{H}^{n-1}|_{\partial\Omega}$.) By the same argument as in the proof of Theorem 5.2.87 we conclude that the tangent domain Ω_∞ satisfies $\omega_\infty\in A_\infty(\sigma_\infty)$. Hence as in the Theorem 1.3.5, we have that $\widehat{G}\subset\partial_*\Omega$, where $\partial_*\Omega$ is the measure-theoretic boundary of Ω. A local version of Theorem 1 in Section 5.11 in [EG] ensures that $\widehat{G}$ is rectifiable, and so is G.

Before reducing Theorem 1.3.6 to Theorem 4.4.1, we recall some results on uniform domains with the CDC which are needed for the proof.

Lemma 4.4.7 (Change of pole formula)**.** *Let Ω be a bounded uniform domain satisfying the CDC and $L=-\operatorname{div}(A(X)\nabla)$ be an elliptic operator satisfying (E). Let $X_0\in\Omega$ be fixed and denote the elliptic measure by $\omega=\omega^{X_0}$. Suppose $q\in\partial\Omega$ and $r<\operatorname{diam}\Omega/4$ are such that $X_0\notin B(q,4r)$, we denote $\Delta=B(q,r)\cap\partial\Omega$. Then for any surface ball $\Delta'\subset\Delta$ we have*

$$\frac{\omega(\Delta')}{\omega(\Delta)}\sim\omega^{A_\Delta}(\Delta'), \tag{4.4.8}$$

where A_Δ is a non-tangential point to surface ball Δ.

Proof. By Corollary 2.2.21, we know (4.4.8) follows directly from

$$\frac{\omega(\Delta')}{\omega(\Delta)}\sim\frac{\omega^{A_\Delta}(\Delta')}{\omega^{A_\Delta}(\Delta)},$$

i.e. the boundary comparison principle. See [Zh] for the proof of the comparison principle when Ω is a uniform domain with Ahlfors regular boundary. For the case when we only assume Ω satisfies the CDC, the proof is to appear in detail in [HMT2].

In fact (4.4.8) holds if we replace the surface ball Δ' by any Borel set $E \subset \Delta$, i.e.

$$\frac{\omega(E)}{\omega(\Delta)} \sim \omega^{A_\Delta}(E). \tag{4.4.9}$$

Suppose $V \subset \Delta$ is (relative) open in $\partial\Omega$. For any $x \in V$ let $\Delta_x = \Delta(x, r_x)$ be a surface ball satisfying $\Delta_x \subset V$ with $r_x < \frac{\delta(A_\Delta)}{16}$, then $V \subset \cup_{x\in V}\Delta_x$. By Vitali covering lemma we may extract a countable collection of pairwise disjoint balls $\{\Delta_j\}_{j\in J}$ such that

$$V \subset \bigcup_{j\in J} 5\Delta_j, \quad \text{where } 5\Delta_j := \Delta(x_j, 5r_{x_j}). \tag{4.4.10}$$

By (4.4.8), (4.4.10) and the doubling properties of ω and ω^{A_Δ}, we have

(4.4.11)
$$\frac{\omega(V)}{\omega(\Delta)} \leq \sum_{j\in J} \frac{\omega\left(5\Delta_j\right)}{\omega(\Delta)} \leq C \sum_{j\in J} \frac{\omega\left(\Delta_j\right)}{\omega(\Delta)} \leq C \sum_{j\in J} \omega^{A_\Delta}(\Delta_j) = C\omega^{A_\Delta}\left(\bigcup_{j\in J}\Delta_j\right) \leq C\omega^{A_\Delta}(V),$$

and similarly

(4.4.12)
$$\omega^{A_\Delta}(V) \leq \sum_{j\in J} \omega^{A_\Delta}(5\Delta_j) \leq C\sum_{j\in J} \omega^{A_\Delta}(\Delta_j) \leq C\sum_{j\in J} \frac{\omega(\Delta_j)}{\omega(\Delta)} = C\frac{\omega\left(\bigcup_{j\in J}\Delta_j\right)}{\omega(\Delta)} \leq C\frac{\omega(V)}{\omega(\Delta)}.$$

Now suppose E is a Borel set contained in Δ. Since ω^{A_Δ} is a Radon measure, for any $\epsilon > 0$ we can find an open set $V_\epsilon \supset E$ such that $\omega^{A_\Delta}(V_\epsilon \backslash E) < \epsilon$. We may assume $V_\epsilon \subset \Delta$ (if not, just replace V_ϵ by $V_\epsilon \cap \Delta$). Combined with (4.4.11) we get

$$\frac{\omega(E)}{\omega(\Delta)} \leq \frac{\omega(V_\epsilon)}{\omega(\Delta)} \leq C\omega^{A_\Delta}(V_\epsilon) \leq C\left(\omega^{A_\Delta}(E) + \epsilon\right). \tag{4.4.13}$$

Passing $\epsilon \to 0$ we get $\omega(E)/\omega(\Delta) \lesssim \omega^{A_\Delta}(E)$. By taking a different open set $V'_\epsilon \supset E$ satisfying $\omega(V'_\epsilon \backslash E) < \epsilon$, we can similarly use (4.4.12) to show $\omega^{A_\Delta}(E) \lesssim \omega(E)/\omega(\Delta)$. This finishes the proof of (4.4.9). □

Lemma 4.4.14 (Dyadic grids on Ahlfors regular set, see [DS1], [DS2], [Ch])**.** *Let Ω be a domain with Ahlfors regular boundary. There exist positive constants a_0, η, and C_1 depending only on n and the Ahlfors regular constants, such that for each $k \in \mathbb{Z}$ there is a collection of Borel sets ("cubes")*

$$\mathbb{D}_k := \{Q_j^k \subset \partial\Omega : j \in \mathcal{J}_k\},$$

where $\mathcal{J}_k$ denotes some (possibly finite) index set depending on k, satisfying

(i) $\partial\Omega = \cup_j Q_j^k$ *for each* $k \in \mathbb{Z}$*;*

(ii) *if* $m \geq k$*, then either* $Q_i^m \subset Q_j^k$ *or* $Q_i^m \cap Q_j^k = \emptyset$*;*

(iii) *for each* (j,k) *and each* $m < k$*, there is a unique* i *such that* $Q_j^k \subset Q_i^m$*;*

(iv) $\operatorname{diam} Q_j^k \leq C_1 2^{-k}$*;*

(v) *each* Q_j^k *contains some "surface ball"* $\Delta(x_j^k, a_0 2^{-k}) = B(x_j^k, a_0 2^{-k}) \cap \partial\Omega$*;*

(vi) $\mathcal{H}^{n-1}\left(\{x \in Q_j^k : \operatorname{dist}(x, \partial\Omega \setminus Q_j^k) \leq \tau 2^{-k}\}\right) \leq C_1 \tau^\eta \mathcal{H}^{n-1}(Q_j^k)$ *for all* k, j *and all* $\tau \in (0, a_0)$.

Proof of Theorem 1.3.6 Let $k_0 \in \mathbb{Z}$ be the smallest integer such that $C_1 2^{-k_0} \leq \operatorname{diam} \partial\Omega$. We consider a dyadic grid $\mathbb{D} = \{Q \in \mathbb{D}_k : k \geq k_0\}$ of the Ahlfors regular set $\partial\Omega$. Since $\partial\Omega$ is bounded, by property (v) of Lemma 4.4.14 there are finitely many cubes in the collection $\mathbb{D}_{k_0}$. For each $Q \in \mathbb{D}_{k_0}$ we have $\sigma(Q) \sim \left(2^{-k_0}\right)^{n-1} > 0$. Since $\sigma \ll \omega$ this implies $\omega(Q) > 0$. Now let $N_0 \in \mathbb{N}$ be the smallest integer such that

$$\frac{1}{N_0} \leq \min_{Q \in \mathbb{D}_{k_0}} \frac{\omega(Q)}{\sigma(Q)} \leq \max_{Q \in \mathbb{D}_{k_0}} \frac{\omega(Q)}{\sigma(Q)} \leq N_0.$$

We apply a stopping time argument to the descendants of each cube $Q \in \mathbb{D}_{k_0}$. Let $N \geq N_0$ be an integer and let $\mathcal{F}_N = \{B_l\} \subset \mathbb{D}$ be the collection of maximal "bad" dyadic cubes with respect to the "stopping criterion" that

$$\text{either} \qquad \frac{\omega(B_l)}{\sigma(B_l)} < \frac{1}{N} \qquad \text{or} \qquad \frac{\omega(B_l)}{\sigma(B_l)} > N.$$

In particular Q is not (a descendent of) a cube in $\mathcal{F}_N$ if it satisfies

$$\frac{1}{N} \leq \frac{\omega(Q)}{\sigma(Q)} \leq N.$$

Let

$$\Lambda_N = \partial\Omega \setminus \bigcup_{B_l \in \mathcal{F}_N} B_l. \tag{4.4.15}$$

Note that $\Lambda_N \subset \Lambda_{N+1}$ and

$$\partial\Omega = \left(\bigcap_{N \geq N_0} \bigcup_{B_l \in \mathcal{F}_N} B_l\right) \bigcup \left(\bigcup_{N \geq N_0} \Lambda_N\right) =: R_0 \bigcup \left(\bigcup_{N \geq N_0} \Lambda_N\right). \tag{4.4.16}$$

We claim that $\sigma(R_0) = 0$. In fact by the definition of R_0, each $q \in R_0$ is contained in some bad cube $B^{(N)} \in \mathcal{F}_N$, satisfying for every $N \geq N_0$

$$\text{either} \qquad \frac{\sigma(B^{(N)})}{\omega(B^{(N)})} > N \qquad \text{or} \qquad \frac{\sigma(B^{(N)})}{\omega(B^{(N)})} < \frac{1}{N}.$$

Hence every $q \in R_0$ falls into one of two cases:

- there is a sequence $N_i \to \infty$ such that $\sigma(B^{(N_i)})/\omega(B^{(N_i)}) > N_i$ for all i, in which case we say $q \in R_0^b$
- there is a sequence $N_i' \to \infty$ such that $\sigma(B^{(N_i')})/\omega(B^{(N_i')}) < 1/N_i$ for all i, in which case we say $q \in R_0^s$.

Note that both R_0^b and R_0^s are Borel sets. Since $\sigma \ll \omega$, the Radon-Nikodym derivative $h = d\sigma/d\omega$ is in $L^1(\omega)$ and is finite ω-almost everywhere. Therefore by the Lebesgue differentiation theorem,

$$h(q) = \infty \text{ for } \omega\text{-a.e. } q \in R_0^b, \tag{4.4.17}$$

and

$$h(q) = 0 \text{ for } \omega\text{-a.e. } q \in R_0^s. \tag{4.4.18}$$

Since h is finite ω-almost everywhere, (4.4.17) implies that $\omega(R_0^b) = 0$, and thus $\sigma(R_0^b) = 0$. On the other hand by (4.4.18) we have $\sigma(R_0^s) = \int_{R_0^s} h d\omega = 0$ since ω is a finite measure. We conclude that $\sigma(R_0) = \sigma(R_0^b \cup R_0^s) = 0$. Hence to show $\partial\Omega$ is rectifiable, it suffices to show Λ_N is rectifiable for all $N \geq N_0$ (see (4.4.16)).

Recalling the definition of Λ_N (see (4.4.15)) we define a collection of cubes

$$\mathcal{D}_N = \{Q \in \mathbb{D} : Q \subset \Lambda_N\} = \Big\{Q \in \mathbb{D} : Q \bigcap \bigcup_{B_l \in \mathcal{F}_N} B_l = \emptyset\Big\}. \tag{4.4.19}$$

Note that

- $\mathcal{D}_N$ is a collection of "good cubes" for N, that is,

$$\frac{1}{N} \leq \frac{\omega(Q)}{\sigma(Q)} \leq N, \quad \text{for all } Q \in \mathcal{D}_N. \tag{4.4.20}$$

- If $Q \in \mathcal{D}_N$ is a "good cube", all of its descendants are "good cubes" in $\mathcal{D}_N$.
- The set $\Lambda_N = \cup_{Q \in \mathcal{D}_N} Q$ can be decomposed into a countable union of disjoint cubes in $\mathcal{D}_N$ with diameter less then $\delta(X_0)/4$.

Let $Q_0 \in \mathcal{D}_N$ be such that $4 \operatorname{diam} Q_0 \leq \delta(X_0)$. For any descendant Q of Q_0 (thus $Q \in \mathcal{D}_N$), by (4.4.20) we have

$$\frac{1}{N^2}\frac{\omega(Q_0)}{\sigma(Q_0)} \leq \frac{\omega(Q)}{\sigma(Q)} \leq N^2 \frac{\omega(Q_0)}{\sigma(Q_0)} \qquad \text{and} \qquad \frac{1}{N^2}\frac{\sigma(Q)}{\sigma(Q_0)} \leq \frac{\omega(Q)}{\omega(Q_0)} \leq N^2 \frac{\sigma(Q)}{\sigma(Q_0)}. \tag{4.4.21}$$

To show that (4.4.2) holds, the next step is to prove that (4.4.21) holds if we replace the dyadic cube Q by any Borel set $E \subset Q_0$. The argument is similar to the one used in the proof of the change of pole formula (4.4.9), except that now we need to work with dyadic "cubes" instead of surface balls. Suppose $V \subset Q_0$ is (relatively) open. For any $x \in V$ let Q_x be a dyadic cube containing x such that

$$Q_x \subset \Delta(c_x, C_1 r_x) \subset V.$$

Here C_1, $c_x \in Q_x$ and $r_x = 2^{-k_x}$ are such that properties (iv) and (v) of Lemma 4.4.14 hold. In particular diam $Q_x \leq C_1 r_x$, and Q_x contains some surface ball $\Delta(c_x, a_0 r_x)$. Then $V \subset \cup_{x\in V} \Delta(c_x, C_1 r_x)$. By Vitali covering lemma there is a countable collection of pairwise disjoint balls $\{\Delta(c_{x_j}, C_1 r_{x_j})\}_{j\in J}$ such that

$$V \subset \bigcup_{j\in J} \Delta\left(c_{x_j}, 5C_1 r_{x_j}\right). \tag{4.4.22}$$

By (4.4.21), (4.4.22), the doubling property of ω and the fact that $\Delta(c_x, a_0 r_x) \subset Q_x$, we have

$$\begin{aligned}
\frac{\omega(V)}{\omega(Q_0)} &\leq \sum_{j\in J} \frac{\omega\left(\Delta(c_{x_j}, 5C_1 r_{x_j})\right)}{\omega(Q_0)} \leq C \sum_{j\in J} \frac{\omega\left(\Delta(c_{x_j}, a_0 r_{x_j})\right)}{\omega(Q_0)} \\
&\leq C \sum_{j\in J} \frac{\omega\left(Q_{x_j}\right)}{\omega(Q_0)} \leq CN^2 \sum_{j\in J} \frac{\sigma(Q_{x_j})}{\sigma(Q_0)} \\
&\leq CN^2 \sum_{j\in J} \frac{\sigma(\Delta(c_{x_j}, C_1 r_{x_j}))}{\sigma(Q_0)} = CN^2 \frac{\sigma\left(\bigcup_{j\in J} \Delta(c_{x_j}, C_1 r_{x_j})\right)}{\sigma(Q)} \\
&\leq CN^2 \frac{\sigma(V)}{\sigma(Q_0)}.
\end{aligned} \tag{4.4.23}$$

Since σ is a Radon measure, (4.4.23) holds if we replace open set V by any Borel set $E \subset Q_0$ (see proof of (4.4.13)). That is

$$\frac{\omega(E)}{\omega(Q_0)} \leq CN^2 \frac{\sigma(E)}{\sigma(Q_0)}, \tag{4.4.24}$$

where C only depends on C_1, a_0 and the doubling constant of ω, and which in turn only depend on the depend on n, the Ahlfors regular constant of σ and the uniform character of Ω. Since σ is Ahlfors regular, it is also a doubling Radon measure. Noting that (4.4.20) and (4.4.21) are symmetric in σ and ω. By reversing their roles in (4.4.23) and (4.4.24) we obtain that for any Borel set $E \subset Q_0$

$$C^{-1} \frac{1}{N^2} \frac{\sigma(E)}{\sigma(Q_0)} \leq \frac{\omega(E)}{\omega(Q_0)} \leq CN^2 \frac{\sigma(E)}{\sigma(Q_0)}. \tag{4.4.25}$$

Given a surface ball $\Delta' \subset Q_0$ and a Borel set $E \subset \Delta'$, combining (4.4.24) with the left hand side of (4.4.25) applied to Δ' we obtain

$$\frac{\omega(E)}{\omega(\Delta')} \leq CN^4 \frac{\sigma(E)}{\sigma(\Delta')}. \tag{4.4.26}$$

For ϵ small enough, define

$$Q_0^*(\epsilon) = \{q \in Q_0 : \mathrm{dist}(q, Q_0^c) > \tau r_{Q_0}\}.$$

Note that $Q_0^*(\epsilon)$ is open. Here $r_{Q_0} = 2^{-k}$ for some $k \in \mathbb{Z}$, $\tau = \tau(\epsilon)$ is in $(0, a_0)$ such that $C_1\tau^\eta \leq \epsilon$, and both parameters are to guarantee that properties (iv) and (v) in Lemma 4.4.14 hold. Thus we have

$$\sigma(Q_0^*(\epsilon)) \geq \sigma(Q_0) - C_1\tau^\eta\sigma(Q_0) \geq (1-\epsilon)\sigma(Q_0).$$

Therefore for any sequence $\epsilon_i \to 0$ we have $\sigma(Q_0 \setminus \cup_i Q_0^*(\epsilon_i)) = 0$. Thus in particular

$$Q_0 = \mathcal{E}_0 \cup \bigcup_i Q_0^*(\epsilon_i) \qquad \text{with} \qquad \sigma(\mathcal{E}_0) = 0. \tag{4.4.27}$$

Thus to show Q_0 is rectifiable, it suffices to show $Q_0^*(\epsilon)$ is rectifiable for ϵ small enough. We finish the proof by applying Theorem 4.4.1 to the open set $Q_0^*(\epsilon)$. Suppose $\Delta' \subset \Delta$ are surface balls in $Q_0^*(\epsilon)$, and that $E \subset \Delta' \subset \Delta$ is a Borel set. Recall that $4\,\mathrm{diam}\, Q_0 \leq \delta(X_0)$ so by the change of pole formula (4.4.8) and (4.4.9) we have

$$\omega^{A_\Delta}(\Delta') \sim \frac{\omega(\Delta')}{\omega(\Delta)}, \quad \omega^{A_\Delta}(E) \sim \frac{\omega(E)}{\omega(\Delta)}. \tag{4.4.28}$$

Combining (4.4.28) and (4.4.26) we get

$$\frac{\omega^{A_\Delta}(E)}{\omega^{A_\Delta}(\Delta')} \sim \frac{\frac{\omega(E)}{\omega(\Delta)}}{\frac{\omega(\Delta')}{\omega(\Delta)}} = \frac{\omega(E)}{\omega(\Delta')}C \leq N^4\frac{\sigma(E)}{\sigma(\Delta')}.$$

That is to say $Q_0^*(\epsilon)$ satisfies the assumption (4.4.2) of Theorem 4.4.1 with uniform constants $C_0 = CN^4$ and $\theta = 1$. Therefore we conclude that $Q_0^*(\epsilon)$ is $(n-1)$-rectifiable, and using (4.4.27) we also have that Q_0 is $(n-1)$-rectifiable. By (4.4.16) $\partial\Omega = R_0 \cup \cup_{N \geq N_0}\Lambda_N$ with $\sigma(R_0) = 0$. Since each Λ_N can be written as a countable disjoint union of cubes in $\mathcal{D}_N$ with diameter less than $\delta(X_0)/4$ (see (4.4.19)) and the properties stated thereafter) such as Q_0, we deduce that each Λ_N is $(n-1)$-rectifiable and so is $\partial\Omega$. □

Chapter 5

$\omega_L \in A_\infty(\sigma)$ implies uniform rectifiability for operators with small Carleson norm

We first recall the following theorem regarding harmonic measure:

Theorem 5.0.1. *Let $\mathcal{D} \subset \mathbb{R}^n$, $n \geq 3$, be a uniform domain (bounded or unbounded) with Ahlfors regular boundary (cf. Definitions 2.1.10 and 2.1.12) and let ω denote its associated harmonic measure. The following statements are equivalent:*

(a) *$\omega \in A_\infty(\sigma)$ (cf. Definition 2.1.14).*

(b) *$\partial\mathcal{D}$ is uniformly rectifiable. (cf. Definition 2.1.13).*

(c) *Ω satisfies the exterior corkscrew condition (cf. Definition 2.1.1), hence, in particular, it is a chord-arc domain (cf. Definition 2.1.10).*

This result in the present form appears in [AHMNT, Theorem 1.2]. That (a) implies (b) is the main result in [HMU] (see also [HM2, HLMN]); that (b) yields (c) is [AHMNT, Theorem 1.1]; and the fact that (c) implies (a) was proved in [DJ], and independently in [Se]. Even though this result was stated for the Laplacian, it is not hard to see that the proof extends to symmetric second order elliptic divergence form operators with constant coefficients.

With this and in hand, we now understand very well how the A_∞ condition of **harmonic measure** is related to the geometry of the domain Ω. Less is known when one works with the **elliptic measure** associated with an elliptic operator with variable coefficients. On the one hand, C. Kenig and J. Pipher proved in [KP] that if $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain and the elliptic matrix $\mathcal{A}$ satisfies some Carleson measure condition (later referred to as Kenig-Pipher condition), that is,

$$\sup_{\substack{q\in\partial\Omega\\0<r<\mathrm{diam}(\Omega)}} \frac{1}{r^{n-1}} \int_{B(q,r)\cap\Omega} \left(\sup_{Y\in B(X,\frac{\delta(X)}{2})} |\nabla\mathcal{A}(Y)|^2\delta(Y) \right) dX < \infty, \tag{5.0.2}$$

where here and elsewhere we write $\delta(\cdot) = \mathrm{dist}(\cdot, \partial\Omega)$, is a Carleson measure in Ω, then the corresponding elliptic measure $\omega_L \in A_\infty(\sigma)$. Combining this and the method of [DJ], one can see that the same holds in an chord-arc domain and hence (c) implies (a) holds for operators satisfying the Kenig-Pipher condition. As observed in [HMT1] one may carry through the proof in [KP] essentially unchanged with a slight reformulation of (5.0.2), namely by assuming that $|\nabla\mathcal{A}|\delta \in L^\infty(\Omega)$, along with the Carleson measure estimate

$$\sup_{\substack{q\in\partial\Omega\\0<r<\mathrm{diam}(\Omega)}} \frac{1}{r^{n-1}} \int_{B(q,r)\cap\Omega} |\nabla\mathcal{A}(X)|^2\delta(X)dX < \infty\,. \tag{5.0.3}$$

In an effort to obtain that (a) implies (b) or (c) for this class of operators, Steve Hofmann, José María Martell and Tatiana Toro have recently obtained in [HMT1] that under the same background hypothesis of Theorem 5.0.1, (a) implies (c) for elliptic operators with variable-coefficient matrices $\mathcal{A}$ satisfying $|\nabla\mathcal{A}|\delta \in L^\infty(\Omega)$ and the Carleson measure estimate

$$\sup_{\substack{q\in\partial\Omega\\0<r<\mathrm{diam}(\Omega)}} \frac{1}{r^{n-1}} \int_{B(q,r)\cap\Omega} |\nabla\mathcal{A}(X)|dX < \infty. \tag{5.0.4}$$

We note that this is stronger than the relaxed Kenig-Pipher condition mentioned above and hence Theorem 5.0.1 remains true for this new class of matrices.

The main result is quantitative in nature, and we need to fix some notations before stating it. Throughout this section and Chapter 5, and unless otherwise specified, by *allowable constants*, we mean the dimension $n \geq 3$; the constants involved in the definition of a uniform domain, that is, $M, C_1 > 1$ (see Definition 2.1.10); the Ahlfors regular constant $C_{AR} > 1$ (see Definition 2.1.12); the ellipticity constants $\Lambda \geq \lambda > 0$ (see (E)); and the A_∞ constants $C_0 > 1$ and $\theta \in (0,1)$ (see Definition 2.1.14).

Remark 5.0.5. First, we always work on $\mathbb{R}^n$, $n \geq 3$. Next, given the values of allowable constants $M, C_1, C_{AR} > 1$, $\Lambda \geq \lambda > 0$, $C_0 > 1$ and $0 < \theta < 1$, by the extension of "(a) implies (c)" in Theorem 5.0.1 to constant coefficients, or by [HMT1] applied to constant coefficient operators (in which case clearly $|\nabla\mathcal{A}|\delta \in L^\infty(\Omega)$ and the Carleson measure estimate (5.0.4) holds both with constant 0) there exists a constant $N_0 = N_0(M, C_1, C_{AR}, \Lambda/\lambda, C_0, \theta)$ such that if $\Omega \subset \mathbb{R}^n$, $n \geq 3$, is a uniform domain with constants M, C_1, such that $\partial\Omega$ is Ahlfors regular with constant C_{AR}, if $L = -\mathrm{div}(\mathcal{A}\nabla)$ is an elliptic operator with real symmetric matrix $\mathcal{A}$ satisfying (E) with ellipticity constants λ, Λ such that the corresponding elliptic measure $\omega_L \in A_\infty(\sigma)$ with constant C_0 and θ, then Ω satisfies the exterior corkscrew condition with constant N_0.

We are now ready to state the main result of this chapter:

Theorem 5.0.6. *Fix $n \geq 3$. Given the values of allowable constants $M, C_1, C_{AR} > 1$, $\Lambda \geq \lambda > 0$, $C_0 > 1$ and $0 < \theta < 1$ there exist N^* and $\epsilon > 0$ depending both on the allowable constants and n such that the following holds. Let $\Omega \subset \mathbb{R}^n$ be a bounded uniform domain with constants M, C_1 and whose boundary $\partial\Omega$ is Ahlfors regular with constant C_{AR}. Let $L = -\operatorname{div}(\mathcal{A}(X)\nabla)$ be an elliptic operator with real symmetric matrix $\mathcal{A}$ satisfying (E) with ellipticity constants λ, Λ such that the corresponding elliptic measure satisfies $\omega_L \in A_\infty(\sigma)$ with constants C_0 and θ. If $\mathcal{A}$ verifies*

$$C(\Omega, \mathcal{A}) := \sup_{X \in \Omega} \fint_{B(X,\delta(X)/2)} |\nabla \mathcal{A}(Y)| \delta(Y) dY < \epsilon, \tag{5.0.7}$$

where $\delta(\cdot) = \operatorname{dist}(\cdot, \partial\Omega)$, then Ω satisfies the exterior corkscrew condition with constant N.

Remarks 5.0.8.

(i) We note that our assumption (5.0.7) on the matrix $\mathcal{A}$ is much weaker than the smallness of the relaxed Kenig-Pipher condition (5.0.3). To see this, given $X \in \Omega$, let $q_X \in \partial\Omega$ be such that $|X - q_X| = \delta(X)$. Then by Hölder's inequality

$$\fint_{B(X,\delta(X)/2)} |\nabla \mathcal{A}(Y)| \delta(Y) dY \lesssim \left(\frac{1}{\delta(X)^{n-1}} \int_{B(q_X, 3\delta(X)/2) \cap \Omega} |\nabla \mathcal{A}(Y)|^2 \delta(Y) dY \right)^{\frac{1}{2}}. \tag{5.0.9}$$

Hence (5.0.3) with sufficiently small constant gives (5.0.7). On the other hand, it is easy to see that the latter is much weaker. Assume for instance that $|\nabla \mathcal{A}|\delta \sim \epsilon$ in Ω in which case (5.0.7) holds but (5.0.3) fails since every integral is infinity.

(ii) Our ultimate goal is to show that, under the appropriate hypothesis on the domain, the A_∞ property of the elliptic measure with respect to surface measure for operators satisfying $|\nabla A|\delta \in L^\infty(\Omega)$ and (5.0.3) ensures that the boundary is uniformly rectifiable. Preliminary work indicates that, indeed, Theorem 5.0.1 folds for the class of second order divergence form operators satisfying (5.0.3). A very different approach is needed to handle the large constant case.

(iii) As will be pointed out in the proof (see Remark 5.2.69) our condition (5.0.7) can be relaxed by assuming that

$$\operatorname{osc}(\Omega, \mathcal{A}) := \sup_{X \in \Omega} \fint_{B(X,\delta(X)/2)} |\mathcal{A}(Y) - \langle \mathcal{A} \rangle_{B(X,\delta(X)/2)}| dY < \epsilon, \tag{5.0.10}$$

where $\langle \mathcal{A} \rangle_{B(X,\delta(X)/2)}$ denotes the average of $\mathcal{A}$ on $B(X, \delta(X)/2)$.

A subtle question may arise from the statement of the theorem: fixed (among other allowable constants) the A_∞ constants C_0, θ, we wonder if we can find a matrix $\mathcal{A}$ satisfying

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*Indeed, $N = 4N_0(4M, 2C_1, 2^{5(n-1)}C_{AR}^2, \Lambda/\lambda, C_0 C_2 C_{AR}^{4\theta} 2^{8(n-1)\theta}, \theta)$ (see Remark 5.0.5) where the constant $C_2 = C_2(M, C_1, C_{AR}, \Lambda/\lambda)$ can be found in Remark 2.2.29.

(5.0.7) with small constant ϵ and whose elliptic measure has the given A_∞ constants. To answer this we consider the converse of our theorem, which is a corollary of [KP] and states that for an NTA domain with Ahlfors regular boundary, if $\mathcal{A}$ satisfies (5.0.2), then the corresponding elliptic measure $\omega_L \in A_\infty(\sigma)$, with constants depending on the *upper bound* Carleson measure constant in (5.0.2). In other words, fixed $\kappa_0 > 0$, and ellipticity constants λ, Λ, [KP] states that any $\mathcal{A}$ so that the left hand side of (5.0.2) is bounded by κ_0 has the property that $\omega_L \in A_\infty$ with constants that only depend on κ_0, ellipticity and the other allowable parameters. Hence, in particular, one could take matrices $\mathcal{A}$ so that (5.0.2) is very tiny (in particular smaller than the fixed κ_0) and still have that $\omega_L \in A_\infty(\sigma)$ with constants that do not depend on the smallness and just on κ_0. To be more specific, fix an NTA domain Ω with Ahlfors regular boundary, let $0 < \epsilon < 1$, and take the elliptic matrix $\mathcal{A}_\epsilon(X) = (1 + \epsilon\varphi(X))\,\mathrm{Id}$, $X \in \Omega$ (here Id is the identity matrix), with $0 \le \varphi \le 1$, $|\nabla\varphi| \le 1$ and $\operatorname{spt}\varphi \subset \{X \in \Omega : \delta(X) < 1\}$. Then $\mathcal{A}_\epsilon$ satisfies (*E*) with $\lambda = 1$ and $\Lambda = 2$ and it is easy to see that $\mathcal{A}_\epsilon$ satisfies the Kenig-Pipher condition with uniform constant (depending only on dimension and C_{AR} but not on ϵ), so that the corresponding elliptic measure satisfies the $A_\infty(\sigma)$ property with uniform constants that are independent of ϵ. On the other hand, $C(\Omega, \mathcal{A}_\epsilon) \le \epsilon$ which can be made as small as we want.

As a consequence we obtain the following corollary:

Corollary 5.0.11. *Under the same assumption as the Theorem 5.0.6, Ω has uniformly rectifiable boundary.*

5.1 Compactness argument

To prove Theorem 5.0.6 we will proceed by contradiction. Fixed $n \ge 3$, let us suppose that there exists a set of allowable constants $M, C_1, C_{AR} > 1$, $\Lambda \ge \lambda \ge 1$, $C_0 > 1$ and $0 < \theta < 1$, so that if we set $N = 4N_0(4M, 2C_1, 2^{5(n-1)}C_{AR}^2, \Lambda/\lambda, C_0C_2C_{AR}^{4\theta}2^{8(n-1)\theta}, \theta)$ (see Remark 5.0.5) then for every ϵ_j (with $\epsilon_j \to 0$ as $j \to \infty$), we have the following assumptions:

Assumption (a) There is a bounded domain $\Omega_j \subset \mathbb{R}^n$, which is uniform with constants M, C_1 and whose boundary is Ahlfors regular with constant C_{AR}. Also, there is an elliptic matrix $\mathcal{A}_j$ defined on Ω_j with constants λ and Λ, and we write $L_j = -\operatorname{div}(\mathcal{A}_j\nabla)$.

Assumption (b) $C(\Omega_j, \mathcal{A}_j) < \epsilon_j$ (see (5.0.7)).

Assumption (c) The elliptic measure of the operator L_j in Ω_j is of class A_∞ with respect to the surface measure $\sigma_j = \mathcal{H}^{n-1}|_{\partial\Omega_j}$ with constants C_0 and θ (see Definition 2.1.14).

Contrary to conclusion There are some $q_j \in \partial\Omega_j$ and $0 < r_j < \operatorname{diam}(\partial\Omega_j)$ such that Ω_j has no exterior corkscrew point with constant N. That is, there is no ball of radius r_j/N contained in $B(q_j, r_j) \setminus \overline{\Omega_j}$.

Our goal is to obtain a contradiction and as a consequence our Main Theorem will be proved. Without loss of generality we may assume $q_j = 0$ and $r_j = 1$ for all j, hence $\operatorname{diam}(\partial\Omega_j) > 1$. Otherwise, we just replace the domain Ω_j by $(\Omega_j - q_j)/r_j$, and replace the elliptic matrix $\mathcal{A}_j(\cdot)$ by $\mathcal{A}_j(q_j + r_j\cdot)$. Note that the new domain and matrix have the same allowable constants, in particular the corresponding A_∞ constants stay the same by the scale-invariant nature of Definition 2.1.14; moreover after rescaling, the above **Assumption (b)** is still satisfied:

$$C\left(\frac{\Omega_j - q_j}{r_j}, \mathcal{A}_j(q_j + r_j\cdot)\right) = C(\Omega_j, \mathcal{A}_j) < \epsilon_j.$$

5.2 Limiting domains

We want to use a compactness argument similar to the blow-up argument in [TZ]. Getting to the point where we can apply Theorem 5.0.1 (more precisely, its extension to the elliptic operators with constants coefficients or alternatively [HMT1] applied again to constant coefficient operators) requires showing first that if Ω_∞ is a "limiting domain" of the domains $\{\Omega_j\}$'s, then Ω_∞ is an unbounded or bounded uniform domain with Ahlfors regular boundary. To accomplish this we also need to find the limit of the Green functions. Once we have this, to show that $\omega_{L_\infty} \in A_\infty(\sigma_\infty)$ for the limiting domain Ω_∞ and the limiting operator L_∞, we need to construct the elliptic measure $\omega_{L_\infty}^Z$ for any $Z \in \Omega_\infty$ as a limiting measure compatible with the procedure. We will also show that L_∞ is an elliptic operator with constants coefficients.

Throughout the rest of paper we follow the following conventions in terms of notations:

- For any $Z \in \Omega_j$ we write $\delta_j(Z) = \operatorname{dist}(Z, \partial\Omega_j)$.
- For any $q \in \partial\Omega_j$ and $r \in (0, \operatorname{diam}(\partial\Omega_j))$, we use $A_j(q, r)$ to denote a corkscrew point in Ω_j relative to $B(q, r) \cap \partial\Omega_j$, i.e.,

$$B\left(A_j(q, r), \frac{r}{M}\right) \subset B(q, r) \cap \Omega_j. \tag{5.2.1}$$

5.2.1 Geometric limit

Since $\operatorname{diam}(\partial\Omega_j) > 1$, modulo passing to a subsequence, one of the following two scenarios occurs:

Case I: $\operatorname{diam}(\Omega_j) = \operatorname{diam}(\partial\Omega_j) \to \infty$ as $j \to \infty$.

Case II: $\operatorname{diam}(\Omega_j) = \operatorname{diam}(\partial\Omega_j) \to R_0 \in [1, \infty)$ as $j \to \infty$.

Therefore if Ω_j "converges" to a limiting domain Ω_∞, respectively **Case I** and **Case II** indicate that Ω_∞ is unbounded or bounded.

Let $X_j \in \Omega_j$ be a corkscrew point relative to $B(0, \operatorname{diam}(\Omega_j)/2) \cap \partial\Omega_j$, then

$$|X_j| \sim \delta_j(X_j) \sim \operatorname{diam}(\Omega_j), \tag{5.2.2}$$

with constants depending on the uniform constant M. Let G_j be the Green function associated with Ω_j and the operator $L_j = -\operatorname{div}(\mathcal{A}_j\nabla)$, and $\{\omega_j^X\}_{X\in\Omega_j}$ be the corresponding elliptic measure. In **Case I** we have

$$|X_j| \sim \delta_j(X_j) \sim \operatorname{diam}(\Omega_j) \to \infty, \tag{5.2.3}$$

i.e., the poles X_j tend to infinity eventually. We let

$$u_j(Z) = \frac{G_j(X_j, Z)}{\omega_j^{X_j}(B(0,1))}. \tag{5.2.4}$$

In **Case II**, we may assume that $\operatorname{diam}(\Omega_j) \sim R_0$ for all j sufficiently large. Hence, there are constants $0 < c_1 < c_2$ such that

$$c_1 R_0 \le \delta_j(X_j) \le |X_j| \le c_2 R_0 \quad \text{for all } j \text{ sufficiently large.} \tag{5.2.5}$$

Thus modulo passing to a subsequence, X_j converges to some point X_0 satisfying

$$c_1 R_0 \le |X_0| \le c_2 R_0. \tag{5.2.6}$$

Note that (5.2.5) and (5.2.6) in particular imply that for any ρ sufficiently small (depending on R_0 and c_1, c_2), the ball $B(X_0, \rho)$ is contained in Ω_j and $\operatorname{dist}(B(X_0,\rho), \partial\Omega_j) \ge c_1 R_0/2$. In this case we let

$$u_j(Z) = G_j(X_j, Z). \tag{5.2.7}$$

Our next goal is to describe what happens with the objects in question as we let $j \to \infty$. This is done in Theorems 5.2.8, 5.2.79, 5.2.87 below.

Theorem 5.2.8. *Under **Assumption (a)**, and using the notation above, we have the following properties (modulo passing to a subsequence which we relabel):*

(1) ***Case I**: there is a function $u_\infty \in C(\mathbb{R}^n)$ such that $u_j \to u_\infty$ uniformly on compact sets; moreover $\nabla u_j \rightharpoonup \nabla u_\infty$ in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$.*

(2) ***Case II**: there is a function $u_\infty \in C(\mathbb{R}^n \setminus \{X_0\})$ such that $u_j \to u_\infty$ uniformly on compact sets in $\mathbb{R}^n \setminus \{X_0\}$ and $\nabla u_j \rightharpoonup \nabla u_\infty$ in $L^2_{\mathrm{loc}}(\mathbb{R}^n \setminus \{X_0\})$.*

(3) *Let $\Omega_\infty = \{Z \in \mathbb{R}^n : u_\infty > 0\}$[†]. Then $\overline{\Omega_j} \to \overline{\Omega_\infty}$ and $\partial\Omega_j \to \partial\Omega_\infty$ locally uniformly in the Hausdorff distance sense. Moreover, Ω_∞ is an unbounded set with unbounded boundary in **Case I**, and it is bounded with diameter $R_0 \ge 1$ in **Case II**.*

[†]In **Case II**, see Remark 5.2.22 part (ii) we extend u_∞ to all of $\mathbb{R}^n$ by setting $u_\infty(X_0) = +\infty$.

(4) *Ω_∞ is a nontrivial uniform domain with constants $4M$ and $2C_1$.*

(5) *There is an Ahlfors regular measure μ_∞ with constant $2^{2(n-1)}C_{AR}$ such that $\sigma_j \rightharpoonup \mu_\infty$. Moreover,* $\operatorname{spt}\mu_\infty = \partial\Omega_\infty$. *In particular, this implies that*

$$2^{-3(n-1)}C_{AR}^{-1}\mu_\infty \le \mathcal{H}^{n-1}|_{\partial\Omega_\infty} \le 2^{3(n-1)}C_{AR}\mu_\infty. \tag{5.2.9}$$

and hence $\partial\Omega_\infty$ is Ahlfors regular with constant $2^{5(n-1)}C_{AR}^2$.

Remark 5.2.10. Note that this result is purely geometric. The proof only uses **Assumption (a)**, which states the geometric characters of domains Ω_j (i.e., they are uniform domains with Ahlfors regular boundaries) and the ellipticity of the matrix operator $\mathcal{A}_j$. The other assumptions are irrelevant for this.

Proof.[Proof of (1) in Theorem 5.2.8] Let $R > 1$ and note that for j large enough (depending on R) we have that $X_j \notin B(0, 4R)$ since by (5.2.3)

$$|X_j| = |X_j - 0| \ge \delta_j(X_j) \sim \operatorname{diam}(\Omega_j) \to \infty, \quad \text{as } j \to \infty.$$

In particular, $L_j u_j = 0$ in $B(0, 4R)\cap\Omega_j$ in the weak sense. Recall that all our domains Ω_j have Ahlfors regular boundary and hence all boundary points are Wiener regular. This in turn implies that u_j is a non-negative L-solution on $B(0, 4R) \cap \Omega_j$ which vanishes continuously on $B(0, 4R) \cap \partial\Omega_j$.

On the other hand, $0 \in \partial\Omega_j$ and, using our convention (5.2.1), $A_j(0, 1)$ is a corkscrew point relative to $B(0, 1) \cap \partial\Omega_j$ in the domain Ω_j. Thus, by Lemma 2.2.25

$$u_j(A_j(0, 1)) \sim 1. \tag{5.2.11}$$

We can then invoke Lemma 2.2.23, the fact that $A_j(0, 2R) \in \Omega_j$ is a corkscrew point relative to $B(0, 2R) \cap \partial\Omega_j$ for the domain Ω_j, Harnack's inequality, and (5.2.11) to obtain

$$\sup_{Z\in\Omega_j\cap B(0,2R)} u_j(Z) \le C u_j(A_j(0, 2R)) \le C_R u_j(A_j(0, 1)) \le C_R. \tag{5.2.12}$$

Extending u_j by 0 outside of Ω_j we conclude that the sequence $\{u_j\}_{j\ge j_0}$ is uniformly bounded in $\overline{B(0,R)}$ for some j_0 large enough. Since for each j, $\mathcal{A}_j$ has ellipticity constants bounded below by λ and above by Λ, and Ω_j is uniform and satisfies the CDC (as $\partial\Omega_j$ is Ahlfors regular) with the same constants as Ω_j, then combining Lemma 2.2.19 with the DeGiorgi-Nash-Moser estimates we conclude that the sequence $\{u_j\}_j$ is equicontinuous on $\overline{B(0,R)}$ (in fact uniformly Hölder continuous with same exponent). Using Arzela-Ascoli combined with a diagonalization argument applied on a sequence of balls with radii going to infinity, we produce $u_\infty \in C(\mathbb{R}^n)$ and a subsequence (which we relabel) such that $u_j \to u_\infty$ uniformly on compact sets of $\mathbb{R}^n$.

As observed before, u_j is a non-negative L-solution on $B(0, 4R) \cap \Omega_j$ which vanishes continuously on $B(0, 4R) \cap \partial\Omega_j$ and which has been extended by 0 outside of Ω_j. Thus it is a positive L-subsolution on $B(0,4R)$ and we can use Caccioppoli's inequality along with (5.2.12) to conclude that

$$\int_{B(0,R)} |\nabla u_j|^2 \, dZ \leq C R^{-2} \int_{B(0,2R)} |u_j|^2 \, dZ \leq C_R. \tag{5.2.13}$$

This and (5.2.12) allow us to conclude that

$$\sup_j \|u_j\|_{W^{1,2}(B(0,R))} \leq C_R < \infty. \tag{5.2.14}$$

Thus, there exists a subsequence (which we relabel) which converges weakly in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$. Since we already know that $u_j \to u_\infty$ uniformly on compact sets of $\mathbb{R}^n$, we can use again (5.2.12) to easily see that $u_\infty \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$, and $\nabla u_j \rightharpoonup \nabla u_\infty$ in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$. This completes the proof of (1) in Theorem 5.2.8.

Proof.[Proof of (2) in Theorem 5.2.8] Recall that in this case $X_j \to X_0$ as $j \to \infty$. For any $0 < \rho \leq c_1 R_0/2$ and for all j large enough we have

$$B\left(X_j, \frac{\rho}{2}\right) \subset B(X_0, \rho) \subset B(X_j, 2\rho) \subset \overline{B(X_j, 2\rho)} \subset \overline{B(X_j, \delta_j(X_j)/2)} \subset \Omega_j, \tag{5.2.15}$$

where we have used (5.2.5). Moreover, for j sufficiently large,

$$\operatorname{dist}(B(X_j, 2\rho), \partial\Omega_j) > \frac{c_1 R_0}{2}. \tag{5.2.16}$$

For any $Z \in \Omega_j \setminus B(X_j, \rho/4)$, using (5.2.7) and (2.2.15) it follows that

$$u_j(Z) \leq \frac{C}{|Z - X_j|^{n-2}} \leq \frac{4^{n-2} C}{\rho^{n-2}}. \tag{5.2.17}$$

Extending u_j by 0 outside Ω_j the previous estimate clearly holds for every $Z \in \mathbb{R}^n \setminus \Omega_j$. Thus $\sup_j \|u_j\|_{L^\infty(\mathbb{R}^n \setminus B(X_0,\rho))} \leq C(\rho)$. Moreover, as in **Case I**, the sequence is also equicontinuous (in fact uniformly Hölder continuous). Using Arzela-Ascoli theorem with a diagonalization argument, we can find $u_\infty \in C(\mathbb{R}^n \setminus \{X_0\})$ and a subsequence (which we relabel) such that $u_j \to u_\infty$ uniformly on compact sets of $\mathbb{R}^n \setminus \{X_0\}$.

Let $0 < R \leq \sup_{j \gg 1} \operatorname{diam}(\Omega_j) \sim R_0$. We claim that

$$\int_{B(0,R) \setminus B(X_0,\rho)} |\nabla u_j|^2 dZ \leq C(R, \rho) < \infty. \tag{5.2.18}$$

To prove this, we first take arbitrary $q \in \partial\Omega_j$ and s such that $0 < s \leq \delta_j(X_j)/5 \sim R_0$. In particular, if $0 < \rho < c_1 R_0/10 \leq \delta_j(X_j)/10$ it follows that $B(q, 4s) \subset \mathbb{R}^n \setminus B(X_j, 2\rho) \subset \mathbb{R}^n \setminus B(X_0, \rho)$. Thus, proceeding as in **Case I**, u_j is non-negative subsolution on $B(q, 2s)$ and we can use Caccioppoli's inequality and (5.2.17) to obtain

$$\int_{B(q,s) \setminus B(X_0,\rho)} |\nabla u_j|^2 dZ = \int_{B(q,s)} |\nabla u_j|^2 dZ \leq \frac{C}{s^2} \int_{B(q,2s)} |u_j(Z)|^2 dZ \lesssim \frac{s^{n-2}}{\rho^{2(n-2)}}. \tag{5.2.19}$$

Note that the previous estimate, with $q = 0$ and $s = R$, gives our claim (5.2.18) when $0 < R \le \delta_j(X_j)/5$.

Consider next the case $R_0 \sim \delta_j(X_j)/5 < R \le \sup_{j\gg 1} \operatorname{diam}(\Omega_j) \sim R_0$. Note first that the set $\Theta_j := \{Z \in \Omega_j : \delta_j(Z) < \delta_j(X_j)/25\}$ can be covered by a family of balls $\{B(q_i, \delta_j(X_j)/5)\}_i$ with $q_i \in \partial\Omega$ and whose cardinality is uniformly bounded (here we recall that $\delta_j(X_j) \sim \operatorname{diam}(\Omega_j)$), Thus, (5.2.19) applied to these each ball in the family yields

$$\int_{(B(0,R)\setminus B(X_0,\rho))\cap\Theta_j} |\nabla u_j|^2 dZ \le \sum_i \int_{B(q_i,\delta_j(X_j)/5)\setminus B(X_0,\rho)} |\nabla u_j|^2 dZ \le C(R,\rho) < \infty. \tag{5.2.20}$$

On the other hand, the set $\{Z \in \Omega_j \setminus B(X_j, \rho/2) : \delta_j(Z) \ge \delta_j(X_j)/25\}$ can be covered by a family of balls $\{B_i\}_i$ so that $r_{B_i} = \rho/16$, $4B_i \subset \Omega_j \setminus B(X_j, \rho/4)$. Moreover, the cardinality of the family is uniformly bounded depending on dimension and the ratio $\operatorname{diam}(\Omega_j)/\rho \sim R_0/\rho$. Using (5.2.15), Caccioppoli's inequality in each B_i since $4B_i \subset \Omega_j \setminus B(X_j, \rho/4)$, and (5.2.17) we obtain

$$\int_{(B(0,R)\setminus B(X_0,\rho))\setminus\Theta_j} |\nabla u_j|^2 dZ \le \sum_i \int_{B_i} |\nabla u_j|^2 dZ \lesssim \sum_i \frac{1}{r_{B_i}^2} \int_{2B_i} |u_j(Z)|^2 dZ \le C(R,\rho). \tag{5.2.21}$$

Combining (5.2.20) and (5.2.21) we obtain the desired estimate and hence proof of the claim (5.2.18) is complete.

Next, we combine (5.2.18) with the fact that $\sup_j \|u_j\|_{L^\infty(\mathbb{R}^n\setminus B(X_0,\rho))} \le C(\rho)$ to obtain that $\sup_j \|u_j\|_{W^{1,2}(B(0,R)\setminus B(X_0,\rho))} \le C(R,\rho) < \infty$. Thus, there exists a subsequence (which we relabel) which converges weakly in $W^{1,2}_{\text{loc}}(\mathbb{R}^n\setminus B(X_0,\rho))$. Since we already know that $u_j \to u_\infty$ uniformly on compact sets of $\mathbb{R}^n \setminus B(X_0,\rho)$, we can easily see that $u_\infty \in W^{1,2}_{\text{loc}}(\mathbb{R}^n \setminus B(X_0,\rho))$, and $\nabla u_j \rightharpoonup \nabla u_\infty$ in $L^2_{\text{loc}}(\mathbb{R}^n \setminus B(X_0,\rho))$. This completes the proof of (2) in Theorem 5.2.8.

Remark 5.2.22. In the **Case II** scenario the following remarks will become useful later. In what follows we assume that $0 < \rho \le c_1 R_0/2$ and j is large enough.

(i) Let us pick $Y \in \partial B(X_j, 3\delta_j(X_j)/4)$ and note that (5.2.5) gives $Y, A_j(0, c_1R_0/2) \in \Omega_j \setminus \overline{B(X_j, \delta_j(X_j)/2)}$, $|Y - A_j(0, c_1R_0/2)| < (c_1 + 2c_2)R_0$, and $\delta_j(Y) \ge c_1R_0/4$. Moreover, since Ω_j satisfies the interior corkscrew condition with constant M it follows that $\delta_j(A_j(0, c_1R_0/2)) \ge c_1R_0/(2M)$. All these allow us to invoke Lemma 2.1.22 to then use (2.2.16) and (5.2.5) and eventually show

$$u_j\left(A_j\left(0, \frac{c_1R_0}{2}\right)\right) \sim u_j(Y) \gtrsim \left|Y - X_j\right|^{2-n} \sim \delta_j(X_j)^{2-n} \sim R_0^{2-n}, \tag{5.2.23}$$

where the implicit constants are independent of j.

(ii) The set $\partial B(X_0, \rho)$ is compact and away from X_0, so $u_j \to u_\infty$ uniformly. Since $X_j \to X_0$, for any $Z \in \partial B(X_0,\rho)$ we have $\rho/2 < |Z - X_j| < 2\rho$ for j sufficiently large. In particular by choosing $\rho < R_0/(16M)$, we have for j large enough

$$|Z - X_j| < 2\rho < \frac{R_0}{8M} \le \frac{\operatorname{diam}(\Omega_j)}{4M} \le \frac{\delta_j(X_j)}{2}, \tag{5.2.24}$$

where the last estimate uses that $X_j \in \Omega_j$ is a corkscrew point relative to the surface ball $B(0, \operatorname{diam}(\Omega_j)/2) \cap \partial\Omega_j$ with constant M. Thus by (2.2.16) if j is large enough

$$u_j(Z) \gtrsim |Z - X_j|^{2-n} \gtrsim \rho^{2-n}, \qquad \forall\, Z \in \partial B(X_0, \rho)$$

with implicit constants which are independent of j. Therefore,

$$u_\infty(Z) = \lim_{j\to\infty} u_j(Z) \gtrsim \rho^{2-n}, \qquad \forall\, Z \in \partial B(X_0, \rho) \tag{5.2.25}$$

For this reason it is natural to extend the definition of u_∞ to all of $\mathbb{R}^n$ by simply letting $u_\infty(X_0) = +\infty$.

(iii) Since u_j is the Green function in Ω_j for L_j, and elliptic operator with uniformly elliptic constants bounded by λ, Λ, by (2.2.18) we know for any $1 < r < \frac{n}{n-1}$,

$$\|\nabla u_j\|_{L^r(\Omega_j)} \lesssim |\Omega_j|^{\frac{1}{r} - \frac{n-1}{n}} \lesssim R_0^{\frac{n}{r} - n + 1} < \infty, \tag{5.2.26}$$

provided j is large enough and where the implicit constants depend on dimension, r, λ, Λ, but are independent of j. Note that $\nabla u_j \equiv 0$ outside of Ω_j by construction. Thus, one can easily show that passing to a subsequence (and relabeling) $\nabla u_j \rightharpoonup \nabla u_\infty$ in $L^r_{\text{loc}}(\mathbb{R}^n)$ for $1 < r < n/(n-1)$.

Proof.[Proof of (3) in Theorem 5.2.8: **Case I**]

It is clear that Ω_∞ is an open set in **Case I** since $u \in C^\infty(\mathbb{R}^n)$. On the other hand, since $0 \in \partial\Omega_j$ for all j, by Lemma 2.3.2 and modulo passing to a subsequence (which we relabel) we have that there exist non-empty closed sets $\Gamma_\infty, \Lambda_\infty$ such that $\overline{\Omega_j} \to \Gamma_\infty$ and $\partial\Omega_j \to \Lambda_\infty$ as $j \to \infty$, where the convergence is in the Hausdorff distance sense uniformly on compact sets.

We are left with obtaining

$$\Lambda_\infty = \partial\Omega_\infty \qquad \text{and} \qquad \Gamma_\infty = \overline{\Omega}_\infty. \tag{5.2.27}$$

We first show that $\Lambda_\infty \subset \partial\Omega_\infty$. To that end we take $p \in \Lambda_\infty$, and there is a sequence $p_j \in \partial\Omega_j$ such that $\lim_{j\to\infty} p_j = p$. Note that $u_\infty(p) = \lim_{j\to\infty} u_j(p)$. On the other hand since the u_j's are uniformly Hölder continuous on compact sets (see the Proof of (1) in Theorem 5.2.8) and $u_j(p_j) = 0$ as $p_j \in \partial\Omega_j$ we have

$$0 \le u_\infty(p) \le |u_\infty(p) - u_j(p)| + |u_j(p) - u_j(p_j)| \lesssim |u_\infty(p) - u_j(p)| + |p - p_j|^\alpha \to 0,$$

as $j \to \infty$. Thus $u_\infty(p) = 0$, that is, $p \in \mathbb{R}^n \setminus \Omega_\infty$.

Our goal is to show that $p \in \partial\Omega_\infty$. Suppose that $p \notin \partial\Omega_\infty$, then $p \in \mathbb{R}^n \setminus \overline{\Omega_\infty}$ and there exists $\epsilon \in (0,1)$ such that $B(p,\epsilon) \subset \mathbb{R}^n \setminus \overline{\Omega_\infty}$, that is, $u_\infty \equiv 0$ on $\overline{B(p,\epsilon)}$. In Ω_j we have

$$\left|A_j\left(p_j, \frac{\epsilon}{2}\right) - A_j(0,1)\right| \leq \frac{\epsilon}{2} + |p_j| + 1 \leq 2\left(|p| + 1\right)$$

and

$$\delta_j\left(A_j\left(p_j, \frac{\epsilon}{2}\right)\right) \geq \frac{1}{M}\frac{\epsilon}{2}, \qquad \delta_j\left(A_j(0,1)\right) \geq \frac{1}{M}.$$

Note also that

$$\frac{\delta_j\left(A_j\left(p_j, \frac{\epsilon}{2}\right)\right)}{\delta_j(X_j)} + \frac{\delta_j\left(A_j(0,1)\right)}{\delta_j(X_j)} \sim \frac{1}{\operatorname{diam}(\Omega_j)} \to 0, \qquad \text{as } j \to \infty,$$

hence for j large enough, $A_j(0,1), A_j\left(p_j, \frac{\epsilon}{2}\right) \notin \overline{B(X_j, \delta_j(X_j)/2)}$.

We can then apply Lemma 2.1.22 and Harnack's inequality along the constructed chain in Ω_j to obtain

$$G_j\left(X_j, A_j\left(p_j, \frac{\epsilon}{2}\right)\right) \sim G_j(X_j, A_j(0,1)),$$

where the implicit constants depend on the allowable parameters, ϵ and $|p|$, but are uniform on j. Hence by (5.2.11),

(5.2.28)
$$u_j\left(A_j\left(p_j, \frac{\epsilon}{2}\right)\right) = \frac{G_j\left(X_j, A_j\left(p_j, \frac{\epsilon}{2}\right)\right)}{\omega_j^{X_j}(B(0,1))} \gtrsim C\frac{G_j\left(X_j, A_j(0,1)\right)}{\omega_j^{X_j}(B(0,1))} = u_j(A_j(0,1)) \geq C_0,$$

where C_0 is independent of j.

Note that since $u_j \to u_\infty$ on compact sets it follows from our assumption that for j large enough depending on C_0

$$u_j(z) = u_j(Z) - u_\infty(Z) < \frac{C_0}{2}, \qquad \forall Z \in \overline{B(p,\epsilon)}. \tag{5.2.29}$$

However, for j large enough $A_j(p_j, \epsilon/2) \in B(p_j, \epsilon/2) \subset B(p,\epsilon)$ and then (5.2.29) contradicts (5.2.28). Thus, we have shown that necessarily $p \in \partial\Omega_\infty$ and consequently $\Lambda_\infty \subset \partial\Omega_\infty$.

Let us next show that $\partial\Omega_\infty \subset \Lambda_\infty$. Assume that $p \notin \Lambda_\infty$. Since Λ_∞ is a closed set, there exists $\epsilon > 0$ such that $B(p, 2\epsilon) \cap \Lambda_\infty = \emptyset$. Since Λ_∞ is the Hausdorff limit of $\partial\Omega_j$ we have that for j large enough $B(p,\epsilon) \cap \partial\Omega_j = \emptyset$. Hence, by passing to a subsequence (and relabeling) either $B(p,\epsilon) \subset \Omega_j$ for all j large enough or $B(p,\epsilon) \subset \mathbb{R}^n \setminus \overline{\Omega_j}$ for all j large enough.

We first consider the case $B(p,\epsilon) \subset \Omega_j$. Hence, $\delta_j(p) \geq \epsilon$ and $|A_j(0,1) - p| \leq 1 + |p|$. Thus there exists a Harnack chain joining $A_j(0,1)$ and p whose length is independent of j and depends on ϵ and $|p|$. We next observe that for j large enough $|p - X_j| > \delta_j(X_j)/2$. Indeed, if we take j large enough, using that $0 \in \partial\Omega_j$ and (5.2.3) we clearly have

$$1 \leq \frac{|X_j|}{\delta_j(X_j)} \leq \frac{|X_j - p|}{\delta_j(X_j)} + \frac{|p|}{\delta_j(X_j)} < \frac{|X_j - p|}{\delta_j(X_j)} + \frac{1}{2},$$

and we just need to hide to obtain the desired estimate. Once we know that $|p - X_j| > \delta_j(X_j)/2$, we also note that $|\delta_j(A_j(0,1))| \leq 1 \ll \operatorname{diam}(\Omega_j) \sim \delta_j(X_j)$ and hence $A_j(0,1) \notin \overline{B(X_j, \delta(X_j)/2)}$ for j large enough.

We can now invoke Lemma 2.1.22 and Harnack's inequality along the constructed chain in Ω_j to obtain that $G_j(X_j, p) \sim G_j(X_j, A_j(0,1))$, which combined with (5.2.4) and (5.2.11), yields

$$u_j(p) \sim u_j(A_j(0,1)) \sim 1, \tag{5.2.30}$$

where the implicit constants depend on the allowable parameters, p and ϵ, but are uniform on j. Letting $j \to \infty$ we obtain that $u_\infty(p) \sim 1$ which implies that $p \in \Omega_\infty$, and since we have already shown that Ω_∞ is open, it follows that $p \notin \partial\Omega_\infty$.

We next consider now the case $B(p,\epsilon) \subset \mathbb{R}^n \setminus \overline{\Omega_j}$ for all j large enough which implies that by construction $u_j(X) = 0$ for all $X \in B(p,\epsilon)$. By uniform convergence of u_j in compact sets we have that $u_\infty(X) = 0$ for $X \in B(p, \epsilon/2)$, which implies $B(p,\epsilon/2) \subset \{u_\infty = 0\}$ and therefore $p \notin \partial\Omega_\infty$.

In both cases we have shown that if $p \notin \Lambda_\infty$ then $p \notin \partial\Omega_\infty$, or, equivalently, $\partial\Omega_\infty \subset \Lambda_\infty$. This together with the converse inclusion completes the proof of $\Lambda_\infty = \partial\Omega_\infty$.

Our next goal is to show that $\Gamma_\infty = \overline{\Omega}_\infty$. Note that if $Z \in \Omega_\infty$, then $u_\infty(Z) > 0$ and this implies that $u_j(Z) > 0$ for j large enough. The latter forces $Z \in \Omega_j$ for all j large enough. This implies that $Z \in \Gamma_\infty$, and we have shown that $\Omega_\infty \subset \Gamma_\infty$. Moreover since Γ_∞ is closed, we conclude that $\overline{\Omega}_\infty \subset \Gamma_\infty$.

To obtain the converse inclusion we take $X \in \Gamma_\infty$. Assume that there is $\epsilon > 0$ such that $\overline{B(X, 2\epsilon)} \subset \mathbb{R}^n \setminus \overline{\Omega_\infty}$, in particular $B(X, 2\epsilon) \cap \partial\Omega_\infty = \emptyset$. Since we have already shown that $\partial\Omega_\infty$ is the limit of $\partial\Omega_j$'s, for j large enough $B(X,\epsilon) \cap \partial\Omega_j = \emptyset$. By the definition of Γ_∞, there is a sequence $\{Y_j\} \subset \overline{\Omega_j}$ with $Y_j \to X$ as $j \to \infty$. Thus, for all j large enough $B(X,\epsilon)$ is a neighborhood of Y_j; and in particular $\Omega_j \cap B(X,\epsilon) \neq \emptyset$ since $Y_j \in \overline{\Omega_j}$. On the other hand, since $B(X,\epsilon) \cap \partial\Omega_j = \emptyset$ we conclude that $B(X,\epsilon) \subset \Omega_j$. At this point we follow a similar argument to the one used to obtain (5.2.30) replacing p by X and obtain for all j large enough

$$u_j(X) \sim u_j(A_j(0,1)) \sim 1,$$

where the implicit constants depend on the allowable parameters, $|X|$ and ϵ, but are uniform on j. Letting $j \to \infty$ it follows that $u_\infty(X) > 0$ and hence $X \in \Omega_\infty$, contradicting the assumption that there is $\epsilon > 0$ such that $\overline{B(X, 2\epsilon)} \subset \mathbb{R}^n \setminus \overline{\Omega_\infty}$. In sort, we have shown that $\overline{B(X, 2\epsilon)} \cap \overline{\Omega_\infty} \neq \emptyset$ for every $\epsilon > 0$, that is, $X \in \overline{\Omega_\infty}$. We have eventually proved that $\Gamma_\infty \subset \overline{\Omega}_\infty$ this completes the proof of (5.2.27) in the **Case I** scenario.

Since $\operatorname{diam}(\Omega_j) \to \infty$ and $0 \in \overline{\Omega_j} \to \overline{\Omega_\infty}$ uniformly on compact set, Ω_∞ is unbounded. Otherwise we would have $\overline{\Omega_\infty} \subset B(0,R)$, and for sufficiently large j one would see that $\overline{\Omega_j} \subset B(0,2R)$, which is a contradiction.

On the other hand, it is possible that $\operatorname{diam}(\partial\Omega_j) \not\to \operatorname{diam}(\partial\Omega_\infty)$, hence we do not know whether $\operatorname{diam}(\partial\Omega_\infty) = \infty$. However, under the assumption that the $\partial\Omega_j$'s are Ahlfors regular with uniform constant, we claim that $\partial\Omega_\infty$ is also unbounded. Assume not, then there is $R > 0$ such that $\partial\Omega_\infty \subset B(0,R)$. Let k be a large integer, and notice that $\partial\Omega_j \to \partial\Omega_\infty$ uniformly on the compact set $\overline{B(0,kR)}$. Thus for j sufficiently large (depending on k)

$$\partial\Omega_j \cap \overline{B(0,kR)} \subset B(0,2R). \tag{5.2.31}$$

Since $\operatorname{diam}(\partial\Omega_j) \to \infty$ we can also guarantee that $\operatorname{diam}(\partial\Omega_j) > kR$ for j sufficiently large. Recalling that $0 \in \partial\Omega_j$, we can then consider the surface ball $\Delta_j(0,kR) = B(0,kR) \cap \partial\Omega_j$. By (5.2.31) and the Ahlfors regularity of $\partial\Omega_j$,

$$C_{AR}^{-1}(kR)^{n-1} \leq \sigma_j(\Delta_j(0,kR)) \leq \sigma_j(B(0,2R) \cap \partial\Omega_j) \leq C_{AR}(2R)^{n-1}. \tag{5.2.32}$$

Letting k large readily leads to a contradiction.

Proof.[Proof of (3) in Theorem 5.2.8: **Case II**] Take $X \in \Omega_\infty$, that is, $u_\infty(X) > 0$. If $X \neq X_0$ then u_∞ is continuous at X and hence $u_\infty(Z) > 0$ for every $Z \in B(X, r_x)$ for some r_x small enough. On the other hand, if $X = X_0$, by Remark 5.2.22 part (ii) we have that $u_\infty(Z) > 0$ for all $Z \in B(X_0, \rho)$ with ρ sufficiently small (here we use the convention that $+\infty > 0$). Note that this argument show in particular that $B(X_0,\rho) \subset \Omega_\infty$.

On the other hand, since $0 \in \partial\Omega_j$ for all j, by Lemma 2.3.2 and modulo passing to a subsequence (which we relabel), there exist closed sets $\Gamma_\infty, \Lambda_\infty$ such that $\overline{\Omega_j} \to \Gamma_\infty$ and $\partial\Omega_j \to \Lambda_\infty$ as $j \to \infty$, where the convergence is in the Hausdorff distance sense uniformly on compact sets. We are going to obtain that

$$\Lambda_\infty = \partial\Omega_\infty \qquad \text{and} \qquad \Gamma_\infty = \overline{\Omega}_\infty. \tag{5.2.33}$$

Let $p \in \Lambda_\infty$, there is a sequence $\{p_j\} \subset \partial\Omega_j$ such that $p_j \to p$ as $j \to \infty$. Note that by (5.2.5)

$$c_1 R_0 \leq \delta_j(X_j) \leq |X_j - p_j| \leq |X_j - p| + |p - p_j|.$$

Thus, for j large enough, $|X_j - p| > \delta_j(X_j)/2 > c_1R_0/2$. In particular, $X_0 \neq p$ and $u_j(p) \to u_\infty(p)$ as $j \to \infty$. On the other hand since the u_j's are uniformly Hölder continuous on compact sets as observed above, $|u_j(p)| = |u_j(p) - u_j(p_j)| \leq C|p - p_j|^\alpha$, thus $u_j(p) \to 0$ as $j \to \infty$. Therefore $u_\infty(p) = 0$, that is, $p \in \mathbb{R}^n \setminus \Omega_\infty$.

Suppose now that $p \notin \partial\Omega_\infty$. Then, there exists $0 < \epsilon < \delta_j(X_j)/4$ such that $\overline{B(p,\epsilon)} \subset \mathbb{R}^n \setminus \Omega_\infty$, or, equivalently, $u_\infty \equiv 0$ on $B(p,\epsilon)$. Note that

$$\left| A_j\left(p_j, \frac{\epsilon}{2}\right) - A_j\left(0, \frac{c_1R_0}{2}\right)\right| \leq \frac{\epsilon}{2} + |p_j| + \frac{c_1R_0}{2} \leq C(\epsilon, |p|, R_0).$$

Also,

$$\frac{\epsilon}{2M} \leq \delta_j\left(A_j\left(p_j, \frac{\epsilon}{2}\right)\right) < \frac{\epsilon}{2} < \frac{\delta_j(X_j)}{2}$$

and, by (5.2.5),

$$\frac{c_1 R_0}{2M} \le \delta_j\left(A_j\left(0, \frac{c_1 R_0}{2}\right)\right) < \frac{c_1 R_0}{2} \le \frac{\delta_j(X_j)}{2}. \tag{5.2.34}$$

Notice that in particular $A_j\left(p_j, \frac{\epsilon}{2}\right), A_j\left(0, \frac{c_1 R_0}{2}\right) \notin \overline{B(X_j, \delta_j(X_j)/2)}$. We can now invoke Lemma 2.1.22, Harnack's inequality along the constructed chain in Ω_j, and (5.2.23) to see that

$$u_j\left(A_j\left(p_j, \frac{\epsilon}{2}\right)\right) \sim u_j\left(A_j\left(0, \frac{c_1 R_0}{2}\right)\right) \gtrsim 1, \tag{5.2.35}$$

with implicit constant depending on the allowable parameters, $\epsilon, |p|, R_0$ but independent of j. On the other hand, for all j large enough

$$A_j\left(p_j, \frac{\epsilon}{2}\right) \in B\left(p_j, \frac{\epsilon}{2}\right) \subset \overline{B(p,\epsilon)} \subset \mathbb{R}^n \setminus B(X_j, \delta_j(X_j)/4), \tag{5.2.36}$$

hence $u_j \to u_\infty$ uniformly on $\overline{B(p,\epsilon)}$ with $u_\infty \equiv 0$ on $\overline{B(p,\epsilon)}$. This and (5.2.36) contradict (5.2.35) and therefore we conclude that $p \in \partial\Omega_\infty$, and we have eventually obtained that $\Lambda_\infty \subset \partial\Omega_\infty$.

To show that $\partial\Omega_\infty \subset \Lambda_\infty$, we assume that $p \notin \Lambda_\infty$. If $p = X_0$, then since we observed above that $B(X_0, \rho) \subset \Omega_\infty$ (see (5.2.25)) then $X_0 \notin \partial\Omega_\infty$.

Assume next that $p \neq X_0$. Since Λ_∞ is a closed set and since $X_j \to X_0$ as $j \to \infty$, there exists $\epsilon > 0$ such that $B(p, 2\epsilon) \cap \Lambda_\infty = \emptyset$ and $X_0, X_j \notin B(p, 2\epsilon)$ for all j large enough. Moreover, since Λ_∞ is the Hausdorff limit of $\partial\Omega_j$ we have that for all j large enough $B(p,\epsilon) \cap \partial\Omega_j = \emptyset$. Hence, passing to a subsequence (and relabeling) either $B(p,\epsilon) \subset \Omega_j$ for j large enough or $B(p,\epsilon) \subset \mathbb{R}^n \setminus \overline{\Omega_j}$ for j large enough.

Assume first that $B(p,\epsilon) \subset \Omega_j$ for all j large enough. We consider two subcases. Assume first that $p \notin \overline{B(X_j, \delta_j(X_j)/2}$. Then, proceeding as before, by (5.2.34) we can apply Lemma 2.1.22 and Harnack's inequality along the constructed chain in Ω_j to get

$$u_j(p) \sim u_j\left(A_j\left(0, \frac{c_1 R_0}{2}\right)\right) \gtrsim 1, \tag{5.2.37}$$

with implicit constant depending on the allowable parameters, $\epsilon, |p|, R_0$ but independent of j. Suppose next that $p \in \overline{B(X_j, \delta_j(X_j)/2)}$. In that case we can use (5.2.7), (2.2.16), and (5.2.5) to see that for all j large enough

$$u_j(p) \gtrsim |p - X_j|^{2-n} \gtrsim \delta_j(X_j)^{2-n} \gtrsim (c_2 R_0)^{2-n}, \tag{5.2.38}$$

with implicit constants which are uniform on j. Combining the two cases together we have shown that $u_j(p) \gtrsim 1$ uniformly on j. Letting $j \to \infty$ we conclude that $u_\infty(p) \gtrsim 1$ and hence $p \in \Omega_\infty$, and since we have already shown that Ω_∞ is an open set we conclude that $p \notin \partial\Omega_\infty$

We now tackle the second case on which $B(p,\epsilon) \subset \mathbb{R}^n \setminus \overline{\Omega_j}$ for all j large enough. In this scenario $u_j(X) = 0$ for all $X \in B(p,\epsilon)$. Since $X_0 \notin B(p, 2\epsilon)$, by uniform convergence of u_j in $\overline{B(p,\epsilon/2)}$ we have that $u_\infty(X) = 0$ for $X \in B(p,\epsilon/2)$, which implies $B(p,\epsilon/2) \subset \mathbb{R}^n \setminus \Omega_\infty$ and eventually $p \notin \partial\Omega_\infty$.

In both cases we have shown that if $p \notin \Lambda_\infty$ then $p \notin \partial\Omega_\infty$, or, equivalently, $\partial\Omega_\infty \subset \Lambda_\infty$. This together with the converse inclusion completes the proof of $\Lambda_\infty = \partial\Omega_\infty$.

Our next task is to show that $\Gamma_\infty = \overline{\Omega}_\infty$. Let $Z \in \Omega_\infty$ and assume first that $Z = X_0$. By (5.2.15) and since $X_j \to X_0$ as $j \to \infty$ we have that $X_0 \in B(X_j, 2\rho) \subset \Omega_j$ for all j large enough, thus $Z = X_0 \in \Gamma_\infty$. On the other hand, if $Z \neq X_0$ since $u_\infty(Z) > 0$ we have that $u_j(Z) > 0$ for all j large enough. This forces as well that $Z \in \Omega_j$ for j all large enough and again $Z \in \Gamma_\infty$. With this we have shown that $\Omega_\infty \subset \Gamma_\infty$. Moreover, since Γ_∞ is closed we conclude as well that $\overline{\Omega}_\infty \subset \Gamma_\infty$.

Next we look at the converse inclusion and take $X \in \Gamma_\infty$. Assume that $X \in \mathbb{R}^n \setminus \overline{\Omega_\infty}$. Thus, there is $\epsilon > 0$ such that $\overline{B(X, 2\epsilon)} \subset \mathbb{R}^n \setminus \overline{\Omega_\infty}$. In particular $B(X, 2\epsilon) \cap \partial\Omega_\infty = \emptyset$ and $B(X_0, \rho) \cap B(X, 2\epsilon) = \emptyset$ (recall that we showed that $B(X_0,\rho) \subset \Omega_\infty$). Since we have already shown that $\partial\Omega_\infty$ is the limit of $\partial\Omega_j$'s, for j large enough $B(X,\epsilon) \cap \partial\Omega_j = \emptyset$. By the definition of Γ_∞, there is a sequence $\{Y_j\} \subset \overline{\Omega_j}$ so that $Y_j \to X$ as $j \to \infty$. Thus, for all j large enough $B(X,\epsilon)$ is a neighborhood of Y_j, and, in particular, $\Omega_j \cap B(X,\epsilon) \neq \emptyset$ since $Y_j \in \overline{\Omega_j}$. Besides, since $B(X,\epsilon) \cap \partial\Omega_j = \emptyset$ we conclude that $B(X,\epsilon) \subset \Omega_j$. Using a similar argument to the one used to obtain (5.2.37) and (5.2.38) we have (replacing p by X) that

$$u_j(X) \gtrsim 1$$

independently of j and with constants that depend on the allowable parameters, $\epsilon, |X|, R_0$. Since $u_j(X) \to u_\infty(X)$ we conclude that $u_\infty(X) > 0$ and thus $X \in \Omega_\infty$, contradicting the assumption that $X \in \mathbb{R}^n \setminus \overline{\Omega_\infty}$. Eventually, $X \in \overline{\Omega}_\infty$ and we have obtained that $\Gamma_\infty \subset \overline{\Omega}_\infty$.

Since $\operatorname{diam}(\Omega_j) \to R_0$ is finite and $0 \in \partial\Omega_j$, we have $\Omega_j, \Omega_\infty \subset \overline{B(0, 2R_0)}$ for j sufficiently large. Hence $\overline{\Omega_j} \to \overline{\Omega_\infty}$ uniformly, and thus $\operatorname{diam}(\Omega_\infty) = \lim_{j\to\infty} \operatorname{diam}(\Omega_j) = R_0 \geq 1$.

For later use let us remark that in the **Case II** scenario the fact that $\overline{\Omega_j} \to \overline{\Omega_\infty}$ and $\partial\Omega_j \to \partial\Omega_\infty$ in the Hausdorff distance sense uniformly on compact sets yields

$$\operatorname{diam}(\Omega_\infty) = \operatorname{diam}(\overline{\Omega_\infty}) = \lim_{j\to\infty} \operatorname{diam}(\overline{\Omega_j}) = \lim_{j\to\infty} \operatorname{diam}(\Omega_j) = R_0. \tag{5.2.39}$$

$$\operatorname{diam}(\partial\Omega_\infty) = \lim_{j\to\infty} \operatorname{diam}(\partial\Omega_j) = R_0 \tag{5.2.40}$$

Proof.[Proof of (4) in Theorem 5.2.8] Notice that Ω_∞ since $0 \in \partial\Omega_\infty$. Next we show that Ω satisfies the interior corkscrew and the Harnack chain conditions.

Interior corkscrew condition. Recall that each Ω_j is a uniform domain with constants $M, C_1 > 1$. Hence, for all $q \in \partial\Omega_j$ and $r \in (0, \operatorname{diam}(\partial\Omega_j))$ there is a point $A_j(q,r) \in \Omega_j$ such that

$$B\left(A_j(q,r), \frac{r}{M}\right) \subset B(q,r) \cap \Omega_j. \tag{5.2.41}$$

Let $p \in \partial\Omega_\infty$ and $0 < r < \operatorname{diam}(\partial\Omega_\infty)$. In **Case II**, by (5.2.40) we get that $r < \operatorname{diam}(\partial\Omega_j)$ for all j sufficiently large. In **Case I**, either $\operatorname{diam}(\partial\Omega_\infty) = \infty$ or $\operatorname{diam}(\partial\Omega_\infty) < \infty$, but we still have $r < \operatorname{diam}(\partial\Omega_j)$ for all j sufficiently large (note that in the latter case $\operatorname{diam}(\partial\Omega_j) \not\to \operatorname{diam}(\partial\Omega_\infty)$). Since $\partial\Omega_j \to \partial\Omega_\infty$, we can find $p_j \in \partial\Omega_j$ converging to p. For each j there exists $A_j(p_j, r/2)$ such that

$$B\left(A_j\left(p_j, \frac{r}{2}\right), \frac{r}{2M}\right) \subset B\left(p_j, \frac{r}{2}\right) \cap \Omega_j. \tag{5.2.42}$$

In particular we deduce that

$$\overline{B\left(A_j\left(p_j, \frac{r}{2}\right), \frac{r}{3M}\right)} \subset \Omega_j \qquad \text{and} \qquad \operatorname{dist}\left(B\left(A_j\left(p_j, \frac{r}{2}\right), \frac{r}{2M}\right), \partial\Omega_j\right) \geq \frac{r}{6M}. \tag{5.2.43}$$

Note that for j large enough

$$A_j\left(p_j, \frac{r}{2}\right) \in B\left(p_j, \frac{r}{2}\right) \subset \overline{B\left(p, \frac{3r}{4}\right)}. \tag{5.2.44}$$

Modulo passing to a subsequence (which we relabel) $A_j\left(p_j, r/2\right)$ converges to some point, which we denote by $A(p,r)$, and for all j sufficiently large (depending on r)

$$B\left(A(p,r), \frac{r}{4M}\right) \subset B\left(A_j\left(p_j, \frac{r}{2}\right), \frac{r}{3M}\right) \subset B(p,r) \cap \Omega_j. \tag{5.2.45}$$

The fact that $\overline{\Omega_j} \to \overline{\Omega_\infty}$, the first inclusion in (5.2.45), and (5.2.43) give for all j large enough

$$B\left(A(p,r), \frac{r}{4M}\right) \subset \overline{\Omega_\infty} \qquad \text{and} \qquad \operatorname{dist}\left(B\left(A(p,r), \frac{r}{4M}\right), \partial\Omega_j\right) \geq \frac{r}{6M}. \tag{5.2.46}$$

This and the fact that $\partial\Omega_j \to \partial\Omega_\infty$ yield that $\operatorname{dist}(B(A(p,r), r/4M), \partial\Omega_\infty) \geq r/6M$, hence $B(A(p,r), r/4M)$ misses $\partial\Omega_\infty$. Combining this with (5.2.46) and the second inclusion in (5.2.45), we conclude that

$$B\left(A(p,r), \frac{r}{4M}\right) \subset \Omega_\infty \cap B(p,r). \tag{5.2.47}$$

Hence, Ω_∞ satisfies the interior corkscrew condition with constant $4M$.

Harnack chain condition. Fix $X, Y \in \Omega_\infty$ and pick $q_X, q_Y \in \partial\Omega_\infty$ such that $|X - q_X| = \delta_\infty(X), |Y - q_Y| = \delta_\infty(Y)$. Without loss of generality we may assume that $\delta(X) \geq \delta(Y)$ (otherwise we switch the roles of X and Y). Let us recall that every Ω_j satisfies the Harnack chain condition with constants $M, C_1 > 1$. Set

$$\Theta := M\left(2 + \log_2^+\left(\frac{|X-Y|}{\min\{\delta_\infty(X), \delta_\infty(Y)\}}\right)\right) = M\left(2 + \log_2^+\left(\frac{|X-Y|}{\delta_\infty(Y)}\right)\right). \tag{5.2.48}$$

Choose $R \geq$ large enough (depending on X, Y) so that

$$B(q_X, \delta_\infty(X)/2), B\big(X, (2C_1^2)^{4\Theta}\delta_\infty(X)\big) \subset B(0, R) \tag{5.2.49}$$

and

$$B(q_Y, \delta_\infty(Y)/2), B\big(Y, (2C_1^2)^{4\Theta}\delta_\infty(Y)\big) \subset B(0, R) \tag{5.2.50}$$

Take also $d = 2^{-1}C_1^{-2\Theta} \leq 1$ which also depends on X, Y. Then, by (3) in Theorem 5.2.8 we can take j large enough (depending on R and d) so that

(5.2.51)

$$D\big[\partial\Omega_j \cap \overline{B(0,R)}, \partial\Omega_\infty \cap \overline{B(0,R)}\big], D\big[\overline{\Omega_j} \cap \overline{B(0,R)}, \overline{\Omega_\infty} \cap \overline{B(0,R)}\big] \leq \frac{d}{2}\delta_\infty(Y) \leq \frac{d}{2}\delta_\infty(X),$$

By (5.2.51), (5.2.49), and (5.2.50) we have that $X, Y \in \Omega_j$, and

$$\frac{\delta_\infty(X)}{2} \leq \delta_j(X) \leq \frac{3\delta_\infty(X)}{2} \qquad \text{and} \qquad \frac{\delta_\infty(Y)}{2} \leq \delta_j(Y) \leq \frac{3\delta_\infty(Y)}{2}. \tag{5.2.52}$$

Since Ω_j satisfies the Harnack chain condition with constants $M, C_1 > 1$, there exists a collection of balls $B_1, \dots, B_K$ (the choice of balls depend on the fixed j) connecting X to Y in Ω_j and such that

$$C_1^{-1}\operatorname{dist}(B_k, \partial\Omega_j) \leq \operatorname{diam}(B_k) \leq C_1 \operatorname{dist}(B_k, \partial\Omega_j), \tag{5.2.53}$$

for $k = 1, 2, \dots, K$ where

$$K \leq M\left(2 + \log_2^+\left(\frac{|X - Y|}{\min\{\delta_j(X), \delta_j(Y)\}}\right)\right) \leq 2\Theta. \tag{5.2.54}$$

Combining (5.2.53) and (5.2.54), one can see that for every $k = 1, 2, \dots, K$

$$\operatorname{dist}(B_k, \partial\Omega_j) \geq d\delta_\infty(X), \qquad \operatorname{diam}(B_k) \leq (2C_1^2)^{2\Theta}\delta_\infty(Y) \tag{5.2.55}$$

and

$$\operatorname{dist}(X, B_k) \leq 2(2C_1^2)^{2\Theta}\delta_\infty(X), \qquad \operatorname{dist}(Y, B_k) \leq 2(2C_1^2)^{2\Theta}\delta_\infty(Y). \tag{5.2.56}$$

Given an arbitrary $q_j \in \partial\Omega_j \setminus \overline{B(0,R)}$, by (5.2.49), (5.2.55), and (5.2.56) it follows that

$$\begin{aligned}(2C_1^2)^{4\Theta}\delta_\infty(X) \leq |q_j - X| &\leq \operatorname{dist}(q_j, B_k) + \operatorname{diam}(B_k) + \operatorname{dist}(X, B_k) \\ &\leq \operatorname{dist}(q_j, B_k) + 3(2C_1^2)^{2\Theta}\delta_\infty(X).\end{aligned} \tag{5.2.57}$$

Hiding the last term, using that $\Theta > 2$ and taking the infimum over the q_j as above we conclude that

$$4C_1(2C_1^2)^{2\Theta}\delta_\infty(X) < \operatorname{dist}(B_k, \partial\Omega_j \setminus \overline{B(0,R)}). \tag{5.2.58}$$

On the other hand, by (5.2.53) and (5.2.55)

$$\operatorname{dist}(B_k, \partial\Omega_j) \le C_1 \operatorname{diam}(B_k) \le C_1(2C_1^2)^{2\Theta}\delta_\infty(Y) \le C_1(2C_1^2)^{2\Theta}\delta_\infty(X),$$

which eventually leads to $\operatorname{dist}(B_k, \partial\Omega_j) = \operatorname{dist}(B_k, \partial\Omega_j \cap \overline{B(0,R)})$. Analogously, replacing q_j by $q \in \partial\Omega_\infty \setminus \overline{B(0,R)}$ in (5.2.57) we can easily obtain that (5.2.58) also holds for Ω_∞:

$$4C_1(2C_1^2)^{2\Theta}\delta_\infty(X) < \operatorname{dist}(B_k, \partial\Omega_\infty \setminus \overline{B(0,R)}). \tag{5.2.59}$$

But, (5.2.56) yields

$$\operatorname{dist}(B_k, \partial\Omega_\infty) \le \delta_\infty(X) + \operatorname{dist}(X, B_k) \le \delta_\infty(X) + 2(2C_1^2)^{2\Theta}\delta_\infty(Y) \le 3(2C_1^2)^{2\Theta}\delta_\infty(Y),$$

which eventually leads to $\operatorname{dist}(B_k, \partial\Omega_\infty) = \operatorname{dist}(B_k, \partial\Omega_\infty \cap \overline{B(0,R)})$. Using all these, (5.2.51), the triangular inequality and (5.2.51) we can obtain

$$\begin{aligned}\left|\operatorname{dist}(B_k, \partial\Omega_j) - \operatorname{dist}(B_k, \partial\Omega_\infty)\right| &= \left|\operatorname{dist}(B_k, \partial\Omega_j \cap \overline{B(0,R)}) - \operatorname{dist}(B_k, \partial\Omega_\infty \cap \overline{B(0,R)})\right| \\ &\le D\big[\partial\Omega_j \cap \overline{B(0,R)}, \partial\Omega_\infty \cap \overline{B(0,R)}\big] \le \frac{d}{2}\delta_\infty(X) \le \frac{1}{2}\operatorname{dist}(B_k, \Omega_j).\end{aligned}$$

Thus,

$$\frac{2}{3}\operatorname{dist}(B_k, \partial\Omega_\infty) \le \operatorname{dist}(B_k, \partial\Omega_j) \le 2\operatorname{dist}(B_k, \partial\Omega_\infty). \tag{5.2.60}$$

and moreover $B_k \cap \partial\Omega_\infty = \emptyset$. Note that the latter happens for all $k = 1, \dots, K$. Recall also that $X \in B_1 \cap \Omega_\infty$ and that $B_k \cap B_{k+1} \neq \emptyset$. Consequently, we necessarily have that $B_k \subset \Omega_\infty$ for all $k = 1, \dots, K$. Furthermore, (5.2.60) and (5.2.53) give

$$\frac{2}{3}C_1^{-1}\operatorname{dist}(B_k, \partial\Omega_\infty) \le \operatorname{diam}(B_k) \le 2C_1 \operatorname{dist}(B_k, \partial\Omega_\infty). \tag{5.2.61}$$

To summarize, we have found a chain of balls $B_1, \dots, B_K$, all contained in Ω_∞, which verify (5.2.61), and connect X to Y. Also, K satisfies (5.2.54) with Θ given in (5.2.48). Therefore Ω_∞ satisfies the Harnack chain condition with constants $2M$ and $2C_1$. This completes the proof of (4) in Theorem 5.2.8.

Proof.[Proof of (5) in Theorem 5.2.8] We first recall that for every j, $\sigma_j = \mathcal{H}^{n-1}|_{\partial\Omega_j}$ is an Ahlfors regular measure with constant C_{AR} and hence $\operatorname{spt}\sigma_j = \partial\Omega_j$. In particular the sequence $\{\sigma_j\}$ satisfies conditions (i) and (ii) of Lemma 2.3.5.

On the other hand, the fact that $\partial\Omega_j$ is Ahlfors regular easily yields, via a standard covering argument, that $\mathcal{H}^{n-1}(\partial\Omega_j) \le 2^{n-1}C_{AR}\operatorname{diam}(\Omega_j)^n$. Hence, using again that $\partial\Omega_j$ is Ahlfors regular we conclude that for every $R > 0$

$$\sup_j \sigma_j(B(0,R)) = \sup_j \mathcal{H}^{n-1}(\partial\Omega_j \cap B(0,R)) \le 2^{n-1}C_{AR}R^{n-1}.$$

Therefore modulo passing to a subsequence (which we relabel), there exists a Radon measure μ_∞ such that $\sigma_j \rightharpoonup \mu_\infty$ as $j \to \infty$. Using Lemma 2.3.5, $\partial\Omega_j = \operatorname{spt}\sigma_j \to \operatorname{spt}\mu_\infty$ as $j \to \infty$ in the Hausdorff distance sense uniformly on compact sets. This and (3) in Theorem 5.2.8 lead to $\operatorname{spt}\mu_\infty = \partial\Omega_\infty$.

To show that μ_∞ is Ahlfors regular take $q \in \partial\Omega_\infty$. Let $q_j \in \partial\Omega_j$ be such that $q_j \to q$ as $j \to \infty$. For any $r > 0$, using [Ma, Theorem 1.24] and that σ_j is Ahlfors regular with constant C_{AR} we conclude that

$$\mu_\infty(B(q,r)) \leq \liminf_{j\to\infty} \sigma_j(B(q,r)) \leq \liminf_{j\to\infty} \sigma_j(B(q_j, 2r)) \leq 2^{n-1} C_{AR} r^{n-1}. \tag{5.2.62}$$

On the other hand, let $0 < r < \operatorname{diam}(\partial\Omega_\infty)$. In **Case II**, by (5.2.40) we get that $r < \operatorname{diam}(\partial\Omega_j)$ for all j sufficiently large. In **Case I**, either $\operatorname{diam}(\partial\Omega_\infty) = \infty$ or $\operatorname{diam}(\partial\Omega_\infty) < \infty$, but we still have $r < \operatorname{diam}(\partial\Omega_j)$ for all j sufficiently large. Hence, using again [Ma, Theorem 1.24] and that σ_j is Ahlfors regular with constant C_{AR} we obtain

$$\begin{aligned}\mu_\infty(B(q,r)) &\geq \mu_\infty\left(\overline{B\left(q, \frac{r}{2}\right)}\right) \geq \limsup_{j\to\infty} \sigma_j\left(\overline{B\left(q, \frac{r}{2}\right)}\right) \\ &\geq \limsup_{j\to\infty} \sigma_j\left(B\left(q_j, \frac{r}{4}\right)\right) \geq 4^{-(n-1)} C_R^{-1} r^{n-1}.\end{aligned} \tag{5.2.63}$$

These estimates guarantee that μ_∞ is Ahlfors regular with constant $2^{2(n-1)} C_{AR}$. Moreover by [Ma, Theorem 6.9],

$$2^{-2(n-1)} C_{AR}^{-1} \mu_\infty \leq \mathcal{H}^{n-1}|_{\partial\Omega_\infty} \leq 2^{3(n-1)} C_{AR} \mu_\infty. \tag{5.2.64}$$

and consequently $\partial\Omega_\infty$ is Ahlfors regular with constant $2^{5(n-1)} C_{AR}^2$. This completes the proof of (5) and hence that of Theorem 5.2.8.

5.2.2 Convergence of elliptic matrices

Our next goal is to show that there exists a constant coefficient real symmetric elliptic matrix $\mathcal{A}^*$ with ellipticity constants $0 < \lambda \leq \Lambda < \infty$ (i.e., satisfying (E)) so that for any $0 < R < \operatorname{diam}(\partial\Omega_\infty)$ and for any $1 \leq p < \infty$.

$$\int_{B(0,R)\cap\Omega_j} |\mathcal{A}_j(Z) - \mathcal{A}^*|^p \, dZ \to 0, \qquad \text{as } j \to \infty. \tag{5.2.65}$$

Fix $Z_0 \in \Omega_\infty$ and set $B_0 = B(Z_0, 3\delta_\infty(Z_0)/8)$. Since $\partial\Omega_j \to \partial\Omega_\infty$ and $\overline{\Omega_j} \to \overline{\Omega_\infty}$ as $j \to \infty$, for all sufficiently large j, we can see that $Z_0 \in \Omega_j$,

$$\frac{3}{4}\delta_\infty(Z_0) \leq \delta_j(Z_0) \leq \frac{5}{4}\delta_\infty(Z_0), \tag{5.2.66}$$

and

$$B_0 \subset B\left(Z_0, \frac{\delta_j(Z_0)}{2}\right) \subset \frac{5}{3}B_0 \subset \Omega_j \quad \text{for all } j. \tag{5.2.67}$$

All these, Poincaré's inequality, and (5.0.7) yield

$$\begin{aligned} \fint_{B_0} |\mathcal{A}_j(Z) - \langle \mathcal{A}_j \rangle_{B_0}| dZ &\lesssim \delta_\infty(Z_0) \fint_{B_0} |\nabla \mathcal{A}_j(Z)| dZ \\ &\lesssim \fint_{B(Z_0, \delta_j(Z_0)/2)} |\nabla \mathcal{A}_j| \delta_j(Z) dZ \le C(\Omega_j, \mathcal{A}_j) < \epsilon_j. \end{aligned} \tag{5.2.68}$$

Remark 5.2.69. We note that if we state the Main Theorem using the oscillation assumption (5.0.10), we can easily conclude the same estimate:

$$\fint_{B_0} |\mathcal{A}_j(Z) - \langle \mathcal{A}_j \rangle_{B_0}| dZ \lesssim \fint_{B(Z_0, \delta_j(Z_0)/2)} |\mathcal{A}_j(Z) - \langle \mathcal{A}_j \rangle_{B(Z_0, \delta_j(Z_0)/2)}| dZ \le \operatorname{osc}(\Omega_j, \mathcal{A}_j) < \epsilon_j.$$

From here the proof continues the same way.

Note that all the matrices $\mathcal{A}_j$ are uniformly elliptic and bounded with the same constants $0 < \lambda \le \Lambda < \infty$ (i.e., all of them satisfy (E)), and in particular $\{\langle \mathcal{A}_j \rangle_{B_0}\}_j$ is a bounded sequence of constant real matrices. Hence, passing to a subsequence and relabeling $\langle \mathcal{A}_j \rangle_{B_0}$ converges to some constant elliptic matrix, denoted by $\mathcal{A}^*(B_0)$. Combining this with (5.2.68), the dominated convergence theorem yields

$$\fint_{B_0} |\mathcal{A}_j(Z) - \mathcal{A}^*(B_0)| dZ \to 0 \quad \text{as } j \to \infty, \tag{5.2.70}$$

that is, $\mathcal{A}_j$ converges in $L^1(B_0)$ to a constant elliptic matrix $\mathcal{A}^*(B_0)$. Moreover, passing to a further subsequence an relabeling $\mathcal{A}_j \to \mathcal{A}^*(B_0)$ almost everywhere in B_0. In particular, $\mathcal{A}^*(B_0)$ is a real symmetric elliptic matrix constants $0 < \lambda \le \Lambda < \infty$ (i.e., it satisfies (E)). It is important to highlight that all the previous subsequences and relabeling only depends on the choice of $Z_0 \in \Omega_\infty$. In any case, since $\mathcal{A}^*(B_0)$ is a constant coefficient matrix we set $\mathcal{A}^* := \mathcal{A}^*(B_0)$.

Let us pick a countable collection of points $\{Z_k\} \subset \Omega_\infty$ so that $\Omega_\infty = \cup_k B_k$ with $B_k = B(Z_k, 3\delta_\infty(Z_k)/8)$. We can repeat the previous argument with any Z_k and define $\mathcal{A}^*(B_k)$, a constant real symmetric elliptic matrix satisfying (E) so that for some subsequence depending on k, we obtain that $\mathcal{A}_j \to \mathcal{A}^*(B_k)$ in $L^1(B_k)$ and a.e in B_k as $j \to \infty$. In particular, $\mathcal{A}^*(B_{k_1}) = \mathcal{A}^*(B_{k_2})$ a.e. in $B_{k_1} \cap B_{k_2}$ (in case it is non-empty). Note that Ω_∞ is path connected (since it satisfies the Harnack chain condition), hence for any k we can find a path joining Z_k and Z_0 and cover this path with a finite collection of the previous balls to easily see that $\mathcal{A}^*(B_k) = \mathcal{A}^* = \mathcal{A}^*(B_0)$. Moreover, using a diagonalization argument, we can show that there exists a subsequence, which we relabel, so that for all k, we have that

$\mathcal{A}_j \to \mathcal{A}^*$ in $L^1(B_k)$ and a.e in B_k as $j \to \infty$. From this, and since the matrices concerned are all uniformly bounded, one can prove that for any $1 \le p < \infty$ and for all $Z \in \Omega_\infty$

$$\fint_{B_Z} |\mathcal{A}_j(Y) - \mathcal{A}^*|^p dY \to 0 \quad \text{as } j \to \infty, \tag{5.2.71}$$

where $B_Z = B(Z, \delta(Z)/2)$.

We are now ready to start proving our claim (5.2.65). Recalling that $\overline{\Omega_j} \to \overline{\Omega_\infty}$, $\partial\Omega_j \to \partial\Omega_\infty$ uniformly in $B(0,R)$ in the sense of Hausdorff distance, and that $\partial\Omega_j, \partial\Omega_\infty$ have zero Lebesgue measure since they are Ahlfors regular sets, one can see that

$$B(0,R) \cap \big(\Omega_j \triangle \Omega_\infty\big) \subset B(0,R) \cap \Big(\big(\overline{\Omega_j} \triangle \overline{\Omega_\infty}\big) \cup \big(\overline{\Omega_j} \cap \partial\Omega_\infty\big) \cup \big(\overline{\Omega_\infty} \cap \partial\Omega_j\big)\Big) \tag{5.2.72}$$

and hence the Lebesgue measure of the set on the left hand side tends to zero as $j \to \infty$. This and the fact that $\|\mathcal{A}_j\|_\infty, \|\mathcal{A}^*\|_\infty \le \Lambda$ give

$$\int_{B(0,R)\cap(\Omega_j \triangle \Omega_\infty)} |\mathcal{A}_j(Z) - \mathcal{A}^*|^p dZ \to 0, \qquad \text{as } j \to \infty. \tag{5.2.73}$$

On the other hand, let $\varrho > 0$ be arbitrarily small and let $\epsilon = \epsilon(\varrho) > 0$ be a small constant to be determined later. Set

$$\Omega_\infty^{\epsilon,1} := B(0,R) \cap \{Z \in \Omega_\infty : \delta_\infty(Z) < \epsilon\} \quad \text{and} \quad \Omega_\infty^{\epsilon,2} := B(0,R) \cap \{Z \in \Omega_\infty : \delta_\infty(Z) \ge \epsilon\}.$$

Using the notation $\Delta(q,r) := B(q,r) \cap \partial\Omega_\infty$ with $q \in \partial\Omega_\infty$ and $r > 0$, Vitali's covering lemma allows us to find a finite collection of balls $B(q_i, \epsilon)$ with $q_i \in \Delta(0, R+\epsilon)$, such that

$$\Omega_\infty^{\epsilon,1} \subset \bigcup_i B(q_i, 5\epsilon). \tag{5.2.74}$$

Calling the number of balls L_1 we get the following estimate

(5.2.75)
$$L_1 \epsilon^{n-1} \lesssim \sum_i \sigma_\infty\left(\Delta(q_i, \epsilon)\right) = \sigma_\infty\Big(\bigcup_i \Delta(q_i,\epsilon)\Big) \le \sigma_\infty\left(\Delta(0, R+2\epsilon)\right) \lesssim (R+2\epsilon)^{n-1},$$

where we have used that $\partial\Omega_\infty$ is Ahlfors regular and also that $\Delta(q_i,\epsilon) \subset \Delta(0, R+2\epsilon)$ since $q_i \in \Delta(0, R+\epsilon)$. If we assume that $0 < \epsilon < R$ we conclude that $L_1 \lesssim (R/\epsilon)^{n-1}$ and moreover by (5.2.74) we conclude that $|\Omega_\infty^{\epsilon,1}| \lesssim \epsilon$ (here the implicit constant depend on R). This and $\|\mathcal{A}_j\|_\infty, \|\mathcal{A}^*\|_\infty \le \Lambda$ give at once that for every j

$$\int_{\Omega_\infty^{\epsilon,1} \cap \Omega_j} |\mathcal{A}_j(Z) - \mathcal{A}^*|^p dZ \lesssim \Lambda^p \epsilon < \frac{\varrho}{2}, \tag{5.2.76}$$

provided ϵ is taken small enough which is fixed from now on.

On the other hand, note that $\overline{\Omega_\infty^{\epsilon,2}}$ is compact, hence we can find $Z_1, \dots, Z_{L_2} \in \overline{\Omega_\infty^{\epsilon,2}}$ so that $\Omega_\infty^{\epsilon,2} \subset \bigcup_{i=1}^{L_2} B_{Z_i}$ where L_2 depends on ϵ and R which have been fixed already. Hence, by (5.2.71)

$$\int_{\Omega_\infty^{\epsilon,2} \cap \Omega_j} |\mathcal{A}_j(Z) - \mathcal{A}^*|^p dZ \leq \sum_{i=1}^{L_2} \int_{B_{Z_i}} |\mathcal{A}_j(Z) - \mathcal{A}^*|^p dZ \to 0, \quad \text{as } j \to \infty.$$

In particular, we can find an integer $j_0 = j_0(R, \epsilon)$ such that

$$\int_{\Omega_\infty^{\epsilon,2} \cap \Omega_j} |\mathcal{A}_j(Z) - \mathcal{A}^*|^p dZ < \frac{\varrho}{2}, \quad \text{for any } j \geq j_0. \tag{5.2.77}$$

Combining (5.2.76) and (5.2.77), we conclude that

$$\int_{B(0,R) \cap (\Omega_j \cap \Omega_\infty)} |\mathcal{A}_j(Z) - \mathcal{A}^*|^p dZ < \varrho, \quad \text{for any } j \geq j_0. \tag{5.2.78}$$

This combined with (5.2.73) proves the claim (5.2.65).

5.2.3 Convergence of operator

Theorem 5.2.79. *The function u_∞ solves the Dirichlet problem*

$$\begin{cases} -\operatorname{div}(\mathcal{A}^* \nabla u_\infty) = 0 & \text{in } \Omega_\infty, \\ u_\infty > 0 & \text{in } \Omega_\infty, \\ u_\infty = 0 & \text{on } \partial\Omega_\infty, \end{cases} \tag{5.2.80}$$

*in **Case I**, and solves the Dirichlet problem*

$$\begin{cases} -\operatorname{div}(\mathcal{A}^* \nabla u_\infty) = \delta_{\{X_0\}} & \text{in } \Omega_\infty, \\ u_\infty > 0 & \text{in } \Omega_\infty, \\ u_\infty = 0 & \text{on } \partial\Omega_\infty, \end{cases} \tag{5.2.81}$$

*in **Case II**. Hence, u_∞ is a Green function in Ω_∞ for a constant-coefficient elliptic operator $L_\infty = -\operatorname{div}(\mathcal{A}^* \nabla)$ with pole at ∞ in **Case I** or at $X_0 \in \Omega_\infty$ in**Case II**.*

Proof. Let $\psi \in C_c^\infty(\Omega_\infty)$. Since $\overline{\Omega_j} \to \overline{\Omega_\infty}$ and $\partial\Omega_j \to \partial\Omega_\infty$, it follows that $\psi \in C_c^\infty(\Omega_j)$ for j sufficiently large. In **Case I**, using (5.2.4) and (**??**) we have

$$\int_{\Omega_j} \langle \mathcal{A}_j \nabla u_j, \nabla\psi \rangle dZ = \frac{1}{\omega_j^{X_j}(B(0,1))} \int_{\Omega_j} \langle \mathcal{A}_j \nabla G_j(X_j, \cdot), \nabla\psi \rangle dZ = \frac{\psi(X_j)}{\omega_j^{X_j}(B(0,1))} \to 0, \tag{5.2.82}$$

as $j \to \infty$ since $X_j \to \infty$ by (5.2.3). Analogously, in **Case II**, by (5.2.7) and (**??**) we obtain

$$\int_{\Omega_j} \langle \mathcal{A}_j \nabla u_j, \nabla\psi \rangle dZ = \int_{\Omega_j} \langle \mathcal{A}_j \nabla G_j(X_j, \cdot), \nabla\psi \rangle dZ = \psi(X_j) \to \psi(X_0). \tag{5.2.83}$$

as $j \to \infty$ since $X_j \to X_0$.

Suppose next that $\operatorname{spt}\psi \subset B(0,R)$. Let $r = 2$ for **Case I**, and pick $r \in [1, n/(n-1))$ for **Case II**. By (1) in Theorem 5.2.8 in **Case I** and (iii) in Remark 5.2.22 in **Case II** it follows that $\nabla u_j \rightharpoonup \nabla u_\infty$ in $L^r(B(0,R))$. On the other hand,

$$\begin{aligned}(5.2.84)\qquad &\left|\int_{\Omega_j} \langle \mathcal{A}_j \nabla u_j, \nabla\psi\rangle dZ - \int_{\Omega_\infty} \langle \mathcal{A}^* \nabla u_\infty, \nabla\psi\rangle dZ\right| \\ &\leq \|\nabla\psi\|_{L^\infty} \left(\int_{\Omega_j \cap B(0,R)} |\mathcal{A}_j - \mathcal{A}^*|^{r'} dZ\right)^{\frac{1}{r'}} \left(\int_{\Omega_j \cap B(0,R)} |\nabla u_j|^r\right)^{\frac{1}{r}} \\ &\quad + \left|\int_{\Omega_j \cap B(0,R)} \langle \mathcal{A}^* \nabla u_j, \nabla\psi\rangle dZ - \int_{\Omega_\infty \cap B(0,R)} \langle \mathcal{A}^* \nabla u_\infty, \nabla\psi\rangle dZ\right|.\end{aligned}$$

Using (5.2.14) in **Case I** or (5.2.26) in **Case II**, and (5.2.65) with $p = r'$, the term in the second line of (5.2.84) tends to zero as $j \to \infty$. Concerning the last term, since $\mathcal{A}^*$ is a constant-coefficient matrix, it follows that $\mathcal{A}^* \nabla u_j \rightharpoonup \mathcal{A}^* \nabla u_\infty$ in $L^r(B(0,R))$. Moreover $\overline{\Omega_j} = \overline{\{u_j > 0\}} \to \overline{\Omega_\infty} = \overline{\{u_\infty > 0\}}$ on compact sets in the sense of Hausdorff distance, thus

$$\lim_{j\to\infty} \int_{\Omega_j} \langle \mathcal{A}^* \nabla u_j, \nabla\psi\rangle = \int_{\Omega_\infty} \langle \mathcal{A}^* \nabla u_\infty, \nabla\psi\rangle.$$

Combining these with (5.2.82)–(5.2.84) we eventually conclude that

$$(5.2.85)\qquad \int_{\Omega_\infty} \mathcal{A}^* \nabla u_\infty \cdot \nabla\psi = 0 \quad \text{for all } \psi \in C_c^\infty(\Omega_\infty)$$

in **Case I**, i.e., $-\operatorname{div}(\mathcal{A}^* \nabla u_\infty) = 0$ in Ω_∞; and in **Case II**,

$$(5.2.86)\qquad \int_{\Omega_\infty} \mathcal{A}^* \nabla u_\infty \cdot \nabla\psi = \psi(X_0) \quad \text{for all } \psi \in C_c^\infty(\Omega_\infty),$$

i.e., $-\operatorname{div}(\mathcal{A}^* \nabla u_\infty) = \delta_{\{X_0\}}$ in Ω_∞.

5.2.4 Analytic properties of the limiting domains

As mentioned in Section 5.2, in order to apply Theorem 5.0.1 we need to study the elliptic measures of the limiting domain with finite poles. In this section we construct these measures by a limiting procedure which is compatible with the procedure used to produce the limiting domain Ω_∞.

Theorem 5.2.87. *Under **Assumptions (a), (b), (c)**, and using the notation from Theorems 5.2.8 and 5.2.79, the elliptic measure $\omega_{L_\infty} \in A_\infty(\sigma_\infty)$ (see Definition 2.1.14) with constants $\widetilde{C}_0 = C_2 C_{AR}^{4\theta} 2^{8(n-1)\theta}$ and $\widetilde{\theta} = \theta$, here C_2 is the constant in Remark 2.2.29.*

Proof. Our goal is to show that the elliptic measures of L_∞ with finite poles can be recovered as a limit of the elliptic measures of $L_j = -\operatorname{div}(\mathcal{A}_j(Z)\nabla)$, and the A_∞ property of elliptic measures is preserved when passing to a limit.

To set the stage we start with $0 \le f \in \operatorname{Lip}(\partial\Omega_\infty)$ with compact support. Let $R_0 > 0$ be large enough so that $\operatorname{spt} f \subset B(0, R_0/2)$. We are going to take a particular solution to the following Dirichlet problem

$$\begin{cases} L_\infty v = 0, & \text{in } \Omega_\infty \\ v = f, & \text{on } \partial\Omega_\infty, \end{cases} \tag{5.2.88}$$

In **Case II**, where the domain Ω_∞ is bounded, the Dirichlet problem (5.2.88) has a unique solution satisfying the maximum principle, then we let v_∞ be that unique solution.

In **Case I**, where Ω_∞ is unbounded, we follow the construction in [HM1, Section 3] using Perron's method (the construction is done the Laplacian but holds for any constant coefficient operator, for the general case see also [HMT2]). We denote the solution constructed there by

$$v_\infty(Z) = \int_{\partial\Omega_\infty} f(q)\, d\omega_{L_\infty}^Z(q).$$

For later use we need to sketch how it is constructed. For every $R > 4R_0$ define $f_R = f\eta(\cdot/R)$, where $\eta \in C_c^\infty(B(0, 2R)$ verifies $0 \le \eta \le 1$, $\eta = 1$ for $|Z| < 1$. Let v_R be the unique solution to $L_\infty v_R = 0$ in the bounded open set $\Omega_R = \Omega_\infty \cap B(0, 2R)$ with boundary value f_R. Then one shows that $v_R \to v_\infty$ uniformly on compacta as $R \to \infty$, and also that $v_\infty \in C(\overline{\Omega_\infty})$ satisfies the maximum principle $0 \le \max_{\Omega_\infty} v_\infty \le \max_{\partial\Omega_\infty} f$.

Once the solution v_∞ is defined we observe that since $\partial\Omega_\infty$ is Ahlfors regular we can use the Jonsson-Wallin trace/extension theory [JWo] to extend f (abusing the notation we call the extension f) so that $0 \le f \in C_c(\mathbb{R}^n) \cap W^{1,2}(\mathbb{R}^n)$ with $\operatorname{spt} f \subset B(0, R_0)$. For every j we let $h_j \in W_0^{1,2}(\Omega_j)$ be the unique Lax-Milgram solution to the problem $L_j h_j = L_j f$. Initially, h_j is only defined in Ω_j but we can clearly extend it by 0 outside so that the resulting function, which we call again h_j, belongs to $W^{1,2}(\mathbb{R}^n)$. If we next set $v_j = f - h_j \in W^{1,2}(\mathbb{R}^n)$ we obtain that $L_j v_j = 0$ in Ω_j and indeed

$$v_j(Z) = \int_{\partial\Omega_j} f\, d\omega_j^Z, \qquad Z \in \Omega_j, \tag{5.2.89}$$

see [HMT2]. Here ω_j^Z is the elliptic measure of L_j in Ω_j with pole Z and, as observed above, the fact that $\partial\Omega_j$ is Ahlfors regular implies in particular that $v_j \in C(\overline{\Omega_j})$ with $v_j|_{\partial\Omega_j} = f$. Note also that $v_j = f \in C(\mathbb{R}^n)$ on $\mathbb{R}^n \setminus \Omega_j$, hence $v_j \in C(\mathbb{R}^n)$. Moreover, by the maximum principle

$$0 \le \sup_{\Omega_j} v_j \le \|f\|_{L^\infty(\partial\Omega_j)} \le \|f\|_{L^\infty(\mathbb{R}^n)}, \tag{5.2.90}$$

thus the sequence $\{v_j\}$ is uniformly bounded.

Our next goal is to show that $\{v_j\}$ is equicontinuous. Given an arbitrary $\varrho > 0$ let $0 < \gamma < \frac{1}{32}$ to be chosen. Since $f \in C_c(\mathbb{R}^n)$, it is uniformly continuous, hence letting γ small enough (depending on f) we can guarantee that

$$|f(X) - f(Y)| < \frac{\varrho}{8}, \qquad \text{provided } |X - Y| < \gamma^{\frac{1}{4}} \tag{5.2.91}$$

Our first claim is that if γ is small enough depending on n, C_{AR}, Λ/λ , and $\|f\|_{L^\infty(\mathbb{R}^n)}$, there holds

$$|v_j(X) - v_j(Y)| < \frac{\rho}{2}, \qquad \forall\, X \in \Omega_j,\ Y \in \partial\Omega_j,\ |X - Y| < \sqrt{\gamma}. \tag{5.2.92}$$

To see this we recall that $\partial\Omega_j$ is Ahlfors regular with a uniform constant (independent of j), it satisfies the CDC with a uniform constant and [HKM, Theorem 6.18] (see also [HMT2]) yields that for some $\beta > 0$ and C depending on n, C_{AR}, and Λ/λ, but independent of j (indeed this is the same β as in Lemma 2.2.19), the following estimate holds:

$$\operatorname*{osc}_{B(Y_j,\sqrt{\gamma})\cap\Omega_j} v_j \le \operatorname*{osc}_{B(Y_j,\gamma^{1/4})\cap\partial\Omega_j} f + C\|f\|_{L^\infty(\mathbb{R}^n)}\eta^\beta < \frac{\varrho}{2},$$

where in the last estimate we have used (5.2.91) and γ has been chosen small enough so that $C\|f\|_{L^\infty(\mathbb{R}^n)}\eta^\beta < \varrho/4$.

We now fix $X, Y \in \mathbb{R}^n$ so that $|X - Y| < \gamma$ and consider several cases.

Case 1: $X, Y \in \Omega_j$ with $\max\{\delta_j(X), \delta_j(Y)\} < \sqrt{\gamma}/2$.

In this case, we take $\widehat{x} \in \partial\Omega_j$ so that $|X - \hat{x}| = \delta_j(X)$. Note that $|Y - \widehat{x}| < \sqrt{\gamma}$ and we can use (5.2.92) to obtain

$$|v_j(X) - v_j(Y)| \le |v_j(X) - v_j(\widehat{x})| + |v_j(\widehat{x}) - v_j(Y)| < \rho.$$

Case 2: $X, Y \in \Omega_j$ with $\max\{\delta_j(X), \delta_j(Y)\} \ge \sqrt{\gamma}/2$.

Assuming without loss of generality that $\delta_j(X) \ge \sqrt{\gamma}/2$, necessarily $Y \in B(X, \delta_j(X)/2) \subset \Omega_j$. Then, by the interior Hölder regularity of v_j in Ω_j (here α and C depend only on Λ/λ and are independent of j) we conclude that

$$|v_j(X) - v_j(Y)| \le C\left(\frac{|X - Y|}{\delta_j(X)}\right)^\alpha \|v_j\|_{L^\infty(\Omega_j)} \le C2^\alpha\gamma^{\frac{\alpha}{2}}\|f\|_{L^\infty(\mathbb{R}^n)} < \varrho,$$

provided ϱ is taken small enough (again independently of j).

Case 3: $X, Y \notin \Omega_j$.

Here we just need to use (5.2.91) and the fact that $v_j = f$ on $\mathbb{R}^n \setminus \Omega_j$:

$$|v_j(X) - v_j(Y)| = |f(X) - f(Y)| < \rho.$$

Case 4: $X \in \Omega_j$ and $Y \notin \Omega_j$.

Pick $Z \in \partial\Omega_j$ in the line segment joining X and Y (if $Y \in \partial\Omega_j$ we just take $Z = Y$) so that $|X - Z|, |Y - Z| \le |X - Y| < \gamma$. Using (5.2.92), the fact that $v_j = f$ on $\mathbb{R}^n \setminus \Omega_j$, and (5.2.91) we obtain

$$|v_j(X) - v_j(Y)| \le |v_j(X) - v_j(Z)| + |v_j(Z) - v_j(Y)| < \frac{\varrho}{2} + |f(Z) - f(Y)| < \varrho.$$

If we now put all the cases together we have shown that, as desired, $\{v_j\}$ is equicontinuous.

On the other hand, recalling that $h_j \in W_0^{1,2}(\Omega_j)$ satisfies $L_j h_j = L_j f$ in the weak sense in Ω_j and that $f \in W^{1,2}(\mathbb{R}^n)$ we see that

$$\lambda \|\nabla h_j\|^2_{L^2(\Omega_j)} \le \int_{\Omega_j} \langle \mathcal{A}_j \nabla h_j, \nabla h_j \rangle dX = \int_{\Omega_j} \langle \mathcal{A}_j \nabla f, \nabla h_j \rangle dX \le \Lambda \|\nabla f\|_{L^2(\Omega_j)} \|\nabla h_j\|_{L^2(\Omega_j)}.$$

We next absorb the last term, use that $v_j = f - h_j$ and that h_j has been extended as 0 outside of Ω_j:

$$\|\nabla v_j\|_{L^2(\mathbb{R}^n)} \le \|\nabla f\|_{L^2(\mathbb{R}^n)} + \|\nabla h_j\|_{L^2(\mathbb{R}^n)} = \|\nabla f\|_{L^2(\mathbb{R}^n)} + \|\nabla h_j\|_{L^2(\Omega_j)} \le (1 + \Lambda/\lambda)\|\nabla f\|_{L^2(\mathbb{R}^n)}.$$

This along with (5.2.90) yield

$$\sup_j \|\nabla v_j\|_{L^2(\mathbb{R}^n)} \le (1 + \Lambda/\lambda)\|\nabla f\|_{L^2(\mathbb{R}^n)}, \quad \text{and} \ \sup_j \|v_j\|_{L^2(B(0,R))} \le C_R. \tag{5.2.93}$$

We notice that all these estimates hold for the whole sequence and therefore, so it does for any subsequence.

Let us now fix an arbitrary subsequence $\{v_{j_k}\}_k$. By (5.2.93) there are a further subsequence and $v \in C(\mathbb{R}^n) \cap W^{1,2}_{\text{loc}}(\mathbb{R}^n)$ with $\nabla v \in L^2(\mathbb{R}^n)$, such that $v_{j_{k_l}} \to v$ uniformly on compact sets of $\mathbb{R}^n$ (hence $v \ge 0$) and $\nabla v_{j_{k_l}} \rightharpoonup \nabla v$ in $L^2(\mathbb{R}^n)$ as $l \to \infty$. Here it is important to emphasize that the choice of the subsequence may depend on the boundary data f and the fixed subsequence, and the same happens with v , and this could be problematic, later we will see that this is not the case.

To proceed we next see that v agrees with f in $\partial\Omega_\infty$. Given $p \in \partial\Omega_\infty$, there exist $p_{j_{k_l}} \in \partial\Omega_{j_{k_l}}$ with $p_{j_{k_l}} \to p$ as $l \to \infty$. Using the continuity of v and f at p, the uniform convergence of $v_{j_{k_l}}$ to v on $\overline{B(p,1)}$ and the fact that $v_{j_{k_l}} = f$ on $\partial\Omega_{j_{k_l}}$, we have

$$\begin{aligned} |v(p) - f(p)| &\le |v(p) - v(p_{j_{k_l}})| + |v(p_{j_{k_l}}) - v_{j_{k_l}}(p_{j_{k_l}})| + |f(p_{j_{k_l}}) - f(p)| \\ &\le |v(p) - v(p_{j_{k_l}})| + \|v - v_{j_{k_l}}\|_{L^\infty(\overline{B(p,1)})} + |f(p_{j_{k_l}}) - f(p)| \to 0, \quad \text{as } l \to \infty, \end{aligned}$$

thus $v(p) = f(p)$ as desired.

Next, we claim the function v solves the Dirichlet problem (5.2.88). We know that $v \in C(\mathbb{R}^n)$ with $v = f$ in $\partial\Omega_\infty$. Hence, we only need to show that $L_\infty v = 0$ in Ω_∞. To this aim, let us take $\psi \in C_c^1(\Omega_\infty)$ and let $R > 0$ be large enough so that $\operatorname{spt}\psi \subset B(0,R)$. Since $\overline{\Omega_j} \to \overline{\Omega_\infty}$, for all l large enough we have that $\psi \in C_c^1(\Omega_{j_{k_l}})$ in which case

$$\int_{\mathbb{R}^n} \langle \mathcal{A}_{j_{k_l}} \nabla v_{j_{k_l}}, \nabla\psi\rangle dZ = 0, \tag{5.2.94}$$

since $L_{j_{k_l}} v_{j_{k_l}} = 0$ in $\Omega_{j_{k_l}}$ in the weak sense. Then, by (5.2.93) and the fact that $\operatorname{spt}\psi \subset \Omega_\infty \cap \Omega_{j_{k_l}} \cap B(0,R)$,

$$\begin{aligned}
\left|\int_{\mathbb{R}^n} \langle \mathcal{A}^* \nabla v, \nabla\psi\rangle dZ\right| &= \left|\int_{\Omega_{j_k}} \langle \mathcal{A}_{j_k} \nabla v_{j_{k_l}}, \nabla\psi\rangle dZ - \int_{\Omega_\infty} \langle \mathcal{A}^* \nabla v, \nabla\psi\rangle dZ\right| \\
&\leq (1+\Lambda/\lambda)\|\nabla f\|_{L^2(\mathbb{R}^n)}\|\nabla\psi\|_{L^\infty}\left(\int_{\Omega_j\cap B(0,R)} |\mathcal{A}_{j_{k_l}} - \mathcal{A}^*|^2 dZ\right)^{\frac{1}{2}} \\
&\quad + \left|\int_{\mathbb{R}^n} \langle \mathcal{A}^* \nabla v_{j_{k_l}}, \nabla\psi\rangle dZ - \int_{\mathbb{R}^n} \langle \mathcal{A}^* \nabla v, \nabla\psi\rangle dZ\right| \to 0, \quad \text{as } l \to \infty,
\end{aligned}$$

where we have used (5.2.65) with $p = 2$ for the term in the second line, and the fact that since $\mathcal{A}^*$ is a constant-coefficient matrix, it follows that $\mathcal{A}^*\nabla v_{j_{k_l}} \rightharpoonup \mathcal{A}^*\nabla v$ in $L^2(\mathbb{R}^n)$ as $l \to \infty$. This eventually shows that $L_\infty v = 0$ in Ω_∞.

In **Case II** when the domain Ω_∞ is bounded, the Dirichlet problem (5.2.88) has a unique solution, and it satisfies the maximum principle, hence we must have that $v = v_\infty$. Therefore, we have shown that given any subsequence $\{v_{j_k}\}_k$ there is a further subsequence $\{v_{j_{k_l}}\}_l$ so that $v_{j_{k_l}} \to v_\infty$ uniformly on compact sets of $\mathbb{R}^n$ and $\nabla v_{j_{k_l}} \rightharpoonup \nabla v_\infty$ in $L^2(\mathbb{R}^n)$ as $l \to \infty$. This eventually shows that entire sequence $\{v_j\}$ satisfies $v_j \to v_\infty$ uniformly on compact sets of $\mathbb{R}^n$ and $\nabla v_j \rightharpoonup \nabla v_\infty$ in $L^2(\mathbb{R}^n)$ as $j \to \infty$.

In **Case I** where the limiting domain Ω_∞ is unbounded, we need more work to show the solution v is indeed v_∞. Recall that $f \in C_c(\mathbb{R}^n)$ with $\operatorname{spt} f \subset B(0,R_0)$. Given $\epsilon > 0$, there is an integer $j_0 = j_0(\epsilon, R_0) \in \mathbb{N}$ such that for $j \geq j_0$, the Hausdorff distance between $\partial\Omega_j \cap \overline{B(0,4R_0)}$ and $\partial\Omega_\infty \cap \overline{B(0,4R_0)}$ is small enough so that for any $p'_j \in \partial\Omega_j \cap \overline{B(0,4R_0)}$, there is $p' \in \partial\Omega_\infty$ close enough to p'_j so that $|f(p') - f(p'_j)| < \epsilon$. Consequently,

$$\sup_{\partial\Omega_j} |f| = \sup_{\partial\Omega_j \cap \overline{B(0,4R_0)}} |f| \leq \sup_{\partial\Omega_\infty \cap \overline{B(0,4R_0)}} |f| + \epsilon = \sup_{\partial\Omega_\infty} |f| + \epsilon. \tag{5.2.95}$$

For any $Z \in \Omega_\infty$ there exists a sequence $Z_j \in \Omega_j$ such that $Z_j \to Z$ and $Z_j \in \overline{B(Z, \delta_\infty(Z)/2)}$ for all j large enough. Since $v \in C(\mathbb{R}^n)$ it follows that for j large enough $|v(Z) - v(Z_j)| < \epsilon$. All these together with (5.2.90) and the fact that $v_{j_{k_l}} \to v$ uniformly on compact sets of $\mathbb{R}^n$ as $l \to \infty$ give that for all l large enough

$$0 \leq v(Z) \leq |v(Z) - v(Z_{j_{k_l}})| + |v(Z_{j_{k_l}}) - v_{j_{k_l}}(Z_{j_{k_l}})| + |v_{j_{k_l}}(Z_{j_{k_l}})| \leq 2\epsilon + \sup_{\partial\Omega_j} |f| \leq 3\epsilon + \sup_{\partial\Omega_\infty} |f|, \tag{5.2.96}$$

Letting $\epsilon \to 0$ we get $0 \leq \sup_{\Omega_\infty} v \leq \sup_{\partial\Omega_\infty} |f|$.

Let us recall that $\Omega_R = \Omega_\infty \cap B(0,2R) \subset \Omega_\infty$. Since $v \in C(\mathbb{R}^n)$ with $v|_{\partial\Omega_\infty} = f$, and since $\operatorname{spt} f \subset B(0,R_0)$, for every $R > 4R_0$ we have that $f_R|_{\partial\Omega_\infty} = f\eta(\cdot/R) \le v|_{\partial\Omega_\infty}$. Hence the maximum principle implies that $v_R \le v$ in Ω_R, and taking limits we conclude that $v_\infty \le v$ on Ω_∞. Write $0 \le \widetilde{v} = v - v_\infty \in C(\overline{\Omega_\infty})$ so that $L_\infty \widetilde{v} = 0$ in Ω_∞ and $\widetilde{v}|_{\partial\Omega_\infty} = 0$. For any $Z \in \Omega_\infty$, since Ω_∞ is a uniform domain with Ahlfors regular boundary, by Lemma 2.2.19 for any $\delta_\infty(Z) < R' < \operatorname{diam}(\partial\Omega_\infty) = \infty$ (see (3) in Theorem 5.2.8)

$$0 \le \widetilde{v}(Z) \lesssim \left(\frac{\delta_\infty(Z)}{R'}\right)^\beta \sup_{\Omega_\infty} \widetilde{v} \le 2 \left(\frac{\delta_\infty(Z)}{R'}\right)^\beta \sup_{\partial\Omega_\infty} f, \tag{5.2.97}$$

Letting $R' \to \infty$ we conclude that $\widetilde{v}(Z) = 0$ and hence $v = v_\infty$. Therefore, we have shown that given a subsequence $\{v_{j_k}\}_k$ there is a further subsequence $\{v_{j_{k_l}}\}_l$ so that $v_{j_{k_l}} \to v_\infty$ uniformly on compact sets of $\mathbb{R}^n$ and $\nabla v_{j_{k_l}} \rightharpoonup \nabla v_\infty$ in $L^2(\mathbb{R}^n)$ as $l \to \infty$. This eventually shows that entire sequence $\{v_j\}$ satisfies $v_j \to v_\infty$ uniformly on compact sets of $\mathbb{R}^n$ and $\nabla v_j \rightharpoonup \nabla v_\infty$ in $L^2(\mathbb{R}^n)$ as $j \to \infty$.

Hence, in both **Case I** and **Case II**, if $0 \le f \in \operatorname{Lip}(\partial\Omega_\infty)$ has compact support then

$$\lim_{j\to\infty} \int_{\partial\Omega_j} f(q)\,d\omega_j^Z(q) = \lim_{j\to\infty} v_j(Z) = v_\infty(Z) = \int_{\partial\Omega_\infty} f(q)\,d\omega_{L_\infty}^Z(q), \tag{5.2.98}$$

for any $Z \in \Omega_\infty$. A standard approximation argument and splitting each function on its positive and negative parts lead to shows that (5.2.98) holds for all $f \in C_c(\mathbb{R}^n)$, hence $\omega_j^Z \rightharpoonup \omega_{L_\infty}^Z$ as Radon measures for any $Z \in \Omega_\infty$.

Our next goal is to see that $\omega_{L_\infty} \in A_\infty(\sigma_\infty)$ (where $\sigma_\infty = \mathcal{H}^{n-1}|_{\partial\Omega_\infty}$). Fix $p \in \partial\Omega_\infty$ and $0 < r < \operatorname{diam}(\partial\Omega_\infty)$. Recall that whether $\operatorname{diam}(\partial\Omega_\infty)$ is finite or infinite, we always have $r < \operatorname{diam}(\Omega_j)$ for all j sufficiently large. Let $\Delta' = B(m,s) \cap \partial\Omega_\infty$ with $m \in \partial\Omega_\infty$ and $B(m,s) \subset B(p,r) \cap \partial\Omega_\infty$. Let $A(p,r) \in \Omega_\infty$ be a corkscrew point relative to $\Delta(p,r)$ (whose existence is guaranteed by (4) in Theorem 5.2.8). We can then find $p_j \in \partial\Omega_j$ such that $p_j \to p$. Thus, for all j large enough $B(p,r) \subset B(p_j,2r)$ and $\delta_j(A(p,r)) \ge r/(2M)$. Hence, $A(p,r)$ is also a corkscrew point relative to $B(p_j,2r) \cap \partial\Omega_j$ in Ω_j with constant $4M$. Since $m \in \partial\Omega_\infty$, we can also find $m_j \in \partial\Omega_j$ such that $m_j \to m$. In particular, for j sufficiently large

$$|m_j - m| < \frac{s}{5}. \tag{5.2.99}$$

Note also that since all the Ω_j's are uniform and satisfy the CDC with the same constants, and all the operators L_j's have ellipticity constants bounded below and above by λ and Λ, we can conclude from Remark 2.2.29 that there is a uniform constant C_2 depending on $M, C_1, C_{AR} > 1$, $\Lambda \ge \lambda \ge 1$ so that (2.2.30) holds for all ω_j with the appropriate changes. Using this and [Ma, Theorem 1.24] we obtain

$$\omega_{L_\infty}^{A(p,r)}(\Delta(m,s)) \ge \omega_{L_\infty}^{A(p,r)}\left(\overline{B\left(m, \frac{4}{5}s\right)}\right) \ge \limsup_{j\to\infty} \omega_j^{A(p,r)}\left(\overline{B\left(m, \frac{4}{5}s\right)}\right) \tag{5.2.100}$$

$$\geq \limsup_{j\to\infty} \omega_j^{A(p,r)}\left(\overline{B\left(m_j, \frac{3}{5}s\right)}\right) \geq C_2^{-1} \limsup_{j\to\infty} \omega_j^{A(p,r)}\left(B\left(m_j, \frac{6}{5}s\right)\right),$$

where we have used that $\delta_j(A(p,r)) \geq r/(2M) \geq \frac{3}{5}s/(2M)$.

Let V be an arbitrary open set in $B(m,s)$, and note that by (5.2.99)

$$V \subset B(m,s) \subset B\left(m_j, \frac{6}{5}s\right).$$

Using again [Ma, Theorem 1.24], we see that (5.2.100) yields

$$\frac{\omega_{L_\infty}^{A(p,r)}(V)}{\omega_{L_\infty}^{A(p,r)}(\Delta(m,s))} \leq C_2 \frac{\liminf_{j\to\infty} \omega_j^{A(p,r)}(V)}{\limsup_{j\to\infty} \omega_j^{A(p,r)}\left(B\left(m_j, \frac{6}{5}s\right)\right)} \leq C_2 \liminf_{j\to\infty}\left(\frac{\omega_j^{A(p,r)}(V)}{\omega_j^{A(p,r)}\left(B\left(m_j, \frac{6}{5}s\right)\right)}\right). \tag{5.2.101}$$

The assumption $B(m,s) \subset B(p,r)$ implies $|m-p| \leq r-s$. Using this and that $m_j \to m$, $p_j \to p$ as $j \to \infty$ one can easily see that $|m_j - p_j| < r - \frac{s}{5}$ for all j large enough and hence

$$B\left(m_j, \frac{6}{5}s\right) \cap \partial\Omega_j \subset B(p_j, 2r) \cap \partial\Omega_j. \tag{5.2.102}$$

As mentioned above $A(p,r)$ is a corkscrew point relative to $B(p_j, 2r) \cap \partial\Omega_j$ in Ω_j. This, (5.2.102) and the fact that by assumption, $\omega_j \in A_\infty(\sigma_j)$ with uniform constants C_0, θ allow us to conclude that

$$\frac{\omega_j^{A(p,r)}(V)}{\omega_j^{A(p,r)}\left(B\left(m_j, \frac{6}{5}s\right)\right)} \leq C_0\left(\frac{\sigma_j(V)}{\sigma_j\left(B\left(m_j, \frac{6}{5}s\right)\right)}\right)^\theta \leq C_0 C_{AR}^\theta \left(\frac{\sigma_j(V)}{s^{n-1}}\right)^\theta, \tag{5.2.103}$$

where in the last estimate we have used that $\partial\Omega_j$ is Ahlfors regular with constants C_{AR}. Combining (5.2.101), (5.2.103), the fact that $\sigma_j \rightharpoonup \mu_\infty$, [Ma, Theorem 1.24], and (5) in Theorem 5.2.8, we finally arrive at

$$\frac{\omega_{L_\infty}^{A(p,r)}(V)}{\omega_{L_\infty}^{A(p,r)}(\Delta(m,s))} \leq C_0 C_{AR}^\theta \left(\liminf_{j\to\infty} \frac{\sigma_j(V)}{s^{n-1}}\right)^\theta \leq C_0 C_{AR}^\theta \left(\frac{\mu_\infty(\overline{V})}{s^{n-1}}\right)^\theta \leq C_0 C_{AR}^{4\theta} 2^{8(n-1)\theta}\left(\frac{\sigma_\infty(\overline{V})}{\sigma_\infty(\Delta(m,s))}\right)^\theta.$$

and therefore we have shown that for any open set $V \subset B(m,s)$ there holds

$$\frac{\omega_{L_\infty}^{A(p,r)}(V)}{\omega_{L_\infty}^{A(p,r)}(\Delta(m,s))} \leq C_0 C_{AR}^{4\theta} 2^{8(n-1)\theta}\left(\frac{\sigma_\infty(\overline{V})}{\sigma_\infty(\Delta(m,s))}\right)^\theta. \tag{5.2.104}$$

Consider next an arbitrary Borel set $E \subset B(m,s)$. Since σ_∞ and $\omega_{L_\infty}^{A(p,r)}$ are Borel regular, given any $\epsilon > 0$ there is an open set U and a compact set F so that $F \subset E \subset U \subset B(m,s)$ and $\omega_{L_\infty}^{A(p,r)}(U \setminus F) + \sigma_\infty(U \setminus F) < \epsilon$. Note that for any $x \in F$, there is $r_x > 0$ such that $B(x, 2r_x) \subset U$. Using that F is compact we can then show there exists a finite collection of points $\{x_i\}_{i=1}^m \subset F$ such that $F \subset \bigcup_{i=1}^m B(x_i, r_i) =: V$ and $B(x_i, 2r_i) \subset U$ for $i = 1, \dots, m$. Consequently, $F \subset V \subset \overline{V} \subset U$ and $\sigma_\infty(\overline{V} \setminus F) \le \sigma_\infty(U \setminus F) < \epsilon$. We next use (5.2.104) with V to see that

$$\begin{aligned}\frac{\omega_{L_\infty}^{A(p,r)}(E)}{\omega_{L_\infty}^{A(p,r)}(\Delta(m,s))} &\le \frac{\epsilon + \omega_{L_\infty}^{A(p,r)}(F)}{\omega_{L_\infty}^{A(p,r)}(\Delta(m,s))} \le \frac{\epsilon + \omega_{L_\infty}^{A(p,r)}(V)}{\omega_{L_\infty}^{A(p,r)}(\Delta(m,s))} \\ &\le \frac{\epsilon}{\omega_{L_\infty}^{A(p,r)}(\Delta(m,s))} + C_0 C_{AR}^{4\theta} 2^{8(n-1)\theta} \left(\frac{\sigma_\infty(\overline{V})}{\sigma_\infty(\Delta(m,s))} \right)^\theta \\ &\le \frac{\epsilon}{\omega_{L_\infty}^{A(p,r)}(\Delta(m,s))} + C_0 C_{AR}^{4\theta} 2^{8(n-1)\theta} \left(\frac{\sigma_\infty(E) + \epsilon}{\sigma_\infty(\Delta(m,s))} \right)^\theta .\end{aligned}$$

Letting $\epsilon \to 0$ we obtain as desired that $\omega_{L_\infty} \in A_\infty(\sigma_\infty)$ with constants $C_0 C_{AR}^{4\theta} 2^{8(n-1)\theta}$ and θ and the proof is complete.

5.3 Proof of Theorem 5.0.6

Applying Theorem 5.2.8, we obtain that Ω_∞ is a uniform domain with constants $4M$ and $2C_1$, whose boundary is Ahlfors regular with constant $2^{5(n-1)} C_{AR}^2$. Moreover, Theorem 5.2.87 gives that $\omega_{L_\infty} \in A_\infty(\sigma_\infty)$ with constants $\widetilde{C}_0 = C_2 C_{AR}^{4\theta} 2^{8(n-1)\theta}$ and $\widetilde{\theta} = \theta$. Here $L_\infty = -\operatorname{div}(\mathcal{A}^* \nabla)$ with $\mathcal{A}^*$ a constant-coefficient real symmetric uniformly elliptic matrix with ellipticity constants $0 < \lambda \le \Lambda < \infty$ (i.e., satisfying (*E*)). We can then invoke Theorem 5.0.1, to see that Ω_∞ satisfies the exterior corkscrew condition with constant $N_0 = N_0(4M, 2C_1, 2^{5(n-1)} C_{AR}^2, \Lambda/\lambda, C_0 C_2 C_{AR}^{4\theta} 2^{8(n-1)\theta}, \theta)$ (see Remark 5.0.5). Therefore, since $0 \in \partial\Omega_\infty$, $0 < \frac{1}{2} < \operatorname{diam}(\partial\Omega_\infty)$ (recall that $\operatorname{diam}(\partial\Omega_\infty) = \infty$ in **Case I**, and $\operatorname{diam}(\partial\Omega_\infty) = \operatorname{diam}(\Omega_\infty) = R_0 \ge 1$) there exists $A_0 = A^-(0, \frac{1}{2})$ so that

$$B\left(A_0, \frac{1}{2N_0}\right) \subset B\left(0, \frac{1}{2}\right) \setminus \overline{\Omega_\infty}. \tag{5.3.1}$$

Hence

$$\operatorname{dist}\left(B\left(A_0, \frac{1}{4N_0}\right), \mathbb{R}^n \setminus \overline{\Omega_\infty}\right) \ge \frac{1}{4N_0}. \tag{5.3.2}$$

Since $\overline{\Omega_j} \to \overline{\Omega_\infty}$ in the sense of Hausdorff distance, it follows that for all j large enough

$$B\left(A_0, \frac{1}{4N_0}\right) \subset B\left(0, \frac{1}{2}\right) \setminus \overline{\Omega_j} \subset B(0,1) \setminus \overline{\Omega_j}. \tag{5.3.3}$$

Hence for all j large enough A_0 is a corkscrew point relative to $B(0,1) \cap \partial\Omega_j$ for $\mathbb{R}^n \setminus \overline{\Omega_j}$ with constant $4N_0$. This contradicts our assumption that Ω_j has no exterior corkscrew point with constant $N = 4N_0$ for the surface ball $B(0,1) \cap \partial\Omega_j$ and the proof is complete.

Chapter 6

$\omega_L \in A_\infty(\sigma)$ and PDE solvability for sets of higher co-dimensions

We first define BMO-solvability in the case of higher co-dimensions:

Definition 6.0.1. We say that the Dirichlet problem (D) is **solvable in BMO** if for any continuous boundary function $f \in C_0^0(\partial\Omega)$, the solution u to (D) given by (1.1.3) satisfies a condition that $|\nabla u|^2 \delta(X)^{d-n+2}\, dX$ is a Carleson measure with norm bounded by a constant multiple of $\|f\|_{BMO}^2$, that is,

$$\sup_{\Delta\subset\partial\Omega} \frac{1}{\sigma(\Delta)} \iint_{T(\Delta)} |\nabla u|^2 \delta(X)^{d-n+2}\, dX \leq C\|f\|_{BMO}^2. \tag{6.0.2}$$

Remark 6.0.3. We remark that this definition is almost identical to that of co-dimension one, see Definition 2.1.19. They only differ in two places

- In the case of higher co-dimensions, we often study *unbounded* boundary sets Γ. In that case we consider boundary function $f \in C_0^0(\Gamma)$, that is, f is continuous and compactly supported in Γ.
- Here the correct scaling for the Carleson measure given by the solution is $|\nabla u|^2 \delta(X)^{d-n+2}\, dX$. Note that when $d = n-1$ is of co-dimension one, the scaling is exactly $|\nabla u|^2 \delta(X)$.

We will first prove Theorem 1.3.13, i.e. the $S \leq N$ estimate, in Section 6.1, then prove the two directions of Theorem 1.3.15 in Section 6.2. We recall that in the case of co-dimension one, the $S \leq N$ estimate is achieved by a certain good-λ inequality and a local argument using the harmonic/elliptic measure of the sawtooth domains. Sawtooth domains are Carleson boxes/tents over a surface ball, and they inherit the geometric assumptions (in this case, the assumptions are uniform domain and Ahlfors regular boundary) of the original domain. In working with low-dimensional boundary sets, the Carleson box would consist of d-dimensional boundary piece, and $(n-1)$-dimensional piece in Ω, see the construction in Section 6.1. At the moment we can not make sense of harmonic/elliptic measure for domains of mixed dimensions, so we need a different argument.

The following notations will be used throughout this chapter. For any $q \in \Gamma$ and $\alpha > 0$, we define the non-tangential cone $\Gamma^\alpha(q)$ with vertex q and aperture α as

$$\Gamma^\alpha(q) = \{X \in \Omega : |X - q| < (1+\alpha)\delta(X)\}, \tag{6.0.4}$$

and a truncated cone as

$$\Gamma_r^\alpha(q) = \Gamma^\alpha(q) \cap B(q, r).$$

When there is no confusion we drop the super-index α and simply denote them by $\Gamma(q)$ and $\Gamma_r(q)$, respectively. We define the non-tangential square function

$$S u(q) = \left(\iint_{\Gamma(q)} |\nabla u|^2 \delta(X)^{1-d} dm(X) \right)^{\frac{1}{2}} \tag{6.0.5}$$

and the truncated square function

$$S_r u(q) = \left(\iint_{\Gamma_r(Q)} |\nabla u|^2 \delta(X)^{1-d} dm(X) \right)^{\frac{1}{2}}. \tag{6.0.6}$$

We also define the non-tangential maximal function and its truncated analogue

$$Nu(q) = \sup_{X \in \Gamma(q)} |u(X)|, \quad N_r u(q) = \sup_{X \in \Gamma_r(q)} |u(X)|. \tag{6.0.7}$$

Given apertures $0 < \alpha < \alpha_1 < \beta$, for simplicity we denote $Su, S'u$ as the square function on non-tangential cones of aperture α, α_1, respectively, and denote Nu the non-tangential maximal function of aperture β.

6.1 Bound of the square function by the non-tangential maximal function

The goal of this section is to prove Theorem 1.3.13. It suffices to prove (1.3.14) for non-negative harmonic functions u, because otherwise, we just split $u = u_+ - u_-$ and use the linearity of L and the triangle inequality. Before starting to prove the theorem we need to recall some notation and preliminary results.

Lemma 6.1.1 (dyadic cubes for Ahlfors regular set, [DS1, DS2, Ch]). *Let $\Gamma \subset \mathbb{R}^n$ be a d-Ahlfors regular set. Then there exist constants $a_0, A_1, \gamma > 0$, depending only on d, n and C_0, such that for each $k \in \mathbb{Z}$, there is a collection of Borel sets ("dyadic cubes")*

$$\mathbb{D}_k := \{Q_j^k \subset \Gamma : j \in \mathscr{J}_k\},$$

where $\mathscr{J}_k$ denotes some index set depending on k, satisfying the following properties.

(i) $\Gamma = \bigcup_{j\in\mathscr{J}_k} Q_j^k$ *for each* $k \in \mathbb{Z}$.

(ii) *If* $m \geq k$ *then either* $Q_i^m \subset Q_j^k$ *or* $Q_i^m \cap Q_j^k = \emptyset$.

(iii) *For each pair* (j,k) *and each* $m < k$, *there is a unique* $i \in \mathscr{J}_m$ *such that* $Q_j^k \subset Q_i^m$.

(iv) $\operatorname{diam} Q_j^k \leq A_1 2^{-k}$.

(v) *Each* Q_j^k *contains some surface ball* $\Delta(x_j^k, a_0 2^{-k}) := B(x_j^k, a_0 2^{-k}) \cap \Gamma$.

(vi) $\mathcal{H}^d\left(\left\{q \in Q_j^k : \operatorname{dist}(q, \Gamma \setminus Q_j^k) \leq \rho 2^{-k}\right\}\right) \leq A_1 \rho^\gamma \mathcal{H}^d(Q_j^k)$, *for all* (j,k) *and all* $\rho \in (0, a_0)$.

We shall denote by $\mathbb{D} = \mathbb{D}(\Gamma)$ *the collection of all relevant* Q_j^k, *i.e.*

$$\mathbb{D} = \bigcup_k \mathbb{D}_k.$$

Remark 6.1.2. i) For a dyadic cube $Q \in \mathbb{D}$, we let $k(Q)$ denote the "dyadic generation" to which Q belongs, i.e. we set $k(Q) = k$ if $Q \in \mathbb{D}_k$. We also set its "length" $\ell(Q) = 2^{-k(Q)}$. Thus $\ell(Q) = 2^{-k(Q)} \sim \operatorname{diam} Q$.

ii) Properties (iv) and (v) imply that for each cube $Q \in \mathbb{D}$, there is a point $x_Q \in \Gamma$ such that

$$\Delta(x_Q, r_Q) \subset Q \subset \Delta(x_Q, C_2 r_Q), \tag{6.1.3}$$

where $r_Q = a_0 2^{-k(Q)} \sim \operatorname{diam} Q$ and $C_2 = A_1/a_0$.

Now we define sawtooth domains following the definitions of Hofmann and Martell, see for example [HM1], [HMM] and [HMT1]. Since Ω is an open set, it has a Whitney decomposition, that is, a collection of closed "Whitney" boxes in Ω, denoted by $\mathcal{W} = \mathcal{W}(\Omega)$, which form a covering of Ω with pairwise non-overlapping interiors and satisfy

$$4 \operatorname{diam} I \leq \operatorname{dist}(4I, \partial\Omega) \leq \operatorname{dist}(I, \partial\Omega) \leq 40 \operatorname{diam} I, \quad \text{for any } I \in \mathcal{W}, \tag{6.1.4}$$

and also

$$\frac{1}{4} \operatorname{diam} I_1 \leq \operatorname{diam} I_2 \leq 4 \operatorname{diam} I_1 \tag{6.1.5}$$

whenever I_1 and I_2 in $\mathcal{W}$ touch. (See [St2] for reference.) Let X_I denote the center of I and $\ell(I)$ the side length of I, then $\operatorname{diam} I \sim \ell(I)$. We also write $k(I) = k$ if $\ell(I) = 2^{-k}$.

Let $\mathbb{D}$ be a collection of dyadic cubes for the Ahlfors regular set Γ, as in Lemma 6.1.1. For any dyadic cube $Q \in \mathbb{D}$, pick two parameters $\eta \ll 1$ and $K \gg 1$, and define

$$\mathcal{W}_Q^0 := \{I \in \mathcal{W} : \eta^{\frac{1}{4}} \ell(Q) \leq \ell(I) \leq K^{\frac{1}{2}} \ell(Q), \operatorname{dist}(I, Q) \leq K^{\frac{1}{2}} \ell(Q)\}. \tag{6.1.6}$$

Let X_Q denote a corkscrew point for the surface ball $\Delta(x_Q, r_Q/2)$. We can guarantee that X_Q is in some $I \in \mathcal{W}_Q^0$ provided we choose η small enough and K large enough. For each $I \in \mathcal{W}_Q^0$, by Lemma 2.4.3 and the discussions after that, there is a Harnack chain connecting X_I to X_Q, we call it $\mathcal{H}_I$. By the definition of $\mathcal{W}_Q^0$ we may construct this Harnack chain so that it consists of a bounded number of balls (depending on the values of η, K), and stays a distance at least $c\eta^{\frac{n-1}{4(n-1-d)}}\ell(Q)$ away from $\partial\Omega$ (see (2.4.7)). We let $\mathcal{W}_Q$ denote the set of all $J \in \mathcal{W}$ which meet at least one of the Harnack chains $\mathcal{H}_I$, with $I \in \mathcal{W}_Q^0$, i.e.

$$\mathcal{W}_Q := \{J \in \mathcal{W} : \text{there exists } I \in \mathcal{W}_Q^0 \text{ for which } \mathcal{H}_I \cap J \neq \emptyset\}. \tag{6.1.7}$$

Clearly $\mathcal{W}_Q^0 \subset \mathcal{W}_Q$. Besides, it follows from the construction of the augmented collections $\mathcal{W}_Q$ and the properties of the Harnack chains (in particular (2.4.7) and (2.4.8)) that there are uniform constants c and C such that

$$c\eta^{\frac{n-1}{4(n-1-d)}}\ell(Q) \leq \ell(I) \leq CK^{\frac{1}{2}}\ell(Q), \quad \operatorname{dist}(I, Q) \leq CK^{\frac{1}{2}}\ell(Q) \tag{6.1.8}$$

for any $I \in \mathcal{W}_Q$. In particular once η, K is fixed, for any $Q \in \mathbb{D}$ the cardinality of $\mathcal{W}_Q$ is uniformly bounded, which we denote by N_0.

Next we choose a small parameter $\theta \in (0, 1)$ so that for any $I \in \mathcal{W}$, the concentric dilation $I^* = (1 + \theta)I$ still satisfies the Whitney property

$$\operatorname{diam} I \sim \operatorname{diam} I^* \sim \operatorname{dist}(I^*, \partial\Omega) \sim \operatorname{dist}(I, \partial\Omega). \tag{6.1.9}$$

Moreover by taking θ small enough we can guarantee that $\operatorname{dist}(I^*, J^*) \sim \operatorname{dist}(I, J)$ for every $I, J \in \mathcal{W}$, I^* meets J^* if and only if ∂I meets ∂J and that $\frac{1}{2}J \cap I^* = \emptyset$ for any distinct $I, J \in \mathcal{W}$. In what follows we will need to work with further dilations $I^{**} = (1 + 2\theta)I$ or $I^{***} = (1 + 4\theta)I$ etc.. (We may need to take θ even smaller to make sure the above properties also hold for I^{**}, I^{***} etc..) Given an arbitrary $Q \in \mathbb{D}$, we may define an associated Whitney region U_Q, U_Q^* as follows

$$U_Q := \bigcup_{I \in \mathcal{W}_Q} I^*, \quad U_Q^* := \bigcup_{I \in \mathcal{W}_Q} I^{**}. \tag{6.1.10}$$

Let $\mathbb{D}_Q = \{Q' \in \mathbb{D} : Q' \subset Q\}$. For any $Q \in \mathbb{D}$ and any family $\mathcal{F} = \{Q_j\}$ of disjoint cubes in $\mathbb{D}_Q \setminus \{Q\}$, we define the local discretized sawtooth relative to $\mathcal{F}$ by

$$\mathbb{D}_{\mathcal{F},Q} := \mathbb{D}_Q \setminus \bigcup_{Q_j \in \mathcal{F}} \mathbb{D}_{Q_j}. \tag{6.1.11}$$

We also define the local sawtooth domain relative to $\mathcal{F}$ by

$$\Omega_{\mathcal{F},Q} := \operatorname{int}\left(\bigcup_{Q' \in \mathbb{D}_{\mathcal{F},Q}} U_{Q'}\right), \quad \Omega^*_{\mathcal{F},Q} := \operatorname{int}\left(\bigcup_{Q' \in \mathbb{D}_{\mathcal{F},Q}} U^*_{Q'}\right) \tag{6.1.12}$$

For convenience we set

$$\mathcal{W}_{\mathcal{F},Q} := \bigcup_{Q' \in \mathbb{D}_{\mathcal{F},Q}} \mathcal{W}_{Q'}, \tag{6.1.13}$$

so that in particular, we may write

$$\Omega_{\mathcal{F},Q} = \operatorname{int}\left(\bigcup_{I \in \mathcal{W}_{\mathcal{F},Q}} I^* \right), \quad \Omega^*_{\mathcal{F},Q} = \operatorname{int}\left(\bigcup_{I \in \mathcal{W}_{\mathcal{F},Q}} I^{**} \right). \tag{6.1.14}$$

We will need further fattened sawtooth domain $\Omega^{**}_{\mathcal{F},Q}$ etc. whose definitions follow the same lines as above. We remark that by (6.1.8), there is a constant C_3 depending on K, θ such that

$$\Omega_{\mathcal{F},Q} \subset B(x_Q, C_3\ell(Q)) \cap \Omega \tag{6.1.15}$$

for any $Q \in \mathbb{D}$ and collection of maximal cubes $\mathcal{F}$, where x_Q is the "center" of Q as in (6.1.3).

Finally, to work with sawtooth domains, it is more natural to use a discrete dyadic version of the approach region rather than the standard non-tangential cone defined in (6.0.4): for every $q \in \partial\Omega$, we define the dyadic non-tangential cones as

$$\Gamma_d(q) = \bigcup_{Q \in \mathbb{D}: Q \ni q} U_Q, \quad \widehat{\Gamma}_d(q) = \bigcup_{Q \in \mathbb{D}: Q \ni q} U_Q^{***} \tag{6.1.16}$$

where we use $\widehat{\Gamma}_d$ to denote a cone with bigger "aperture" or fattened region; we also define the local dyadic non-tangential cones as

$$\Gamma_d^Q(q) = \bigcup_{Q' \in \mathbb{D}_Q: Q' \ni q} U_{Q'}, \quad \widehat{\Gamma}_d^Q(q) = \bigcup_{Q' \in \mathbb{D}_Q: Q' \ni q} U_{Q'}^{***}. \tag{6.1.17}$$

We claim that given an aperture $\alpha > 0$, there exists K (in the definition (6.1.6)) sufficiently large such that the standard non-tangential cone $\Gamma^\alpha(q) \subset \Gamma_d(q)$ for all $q \in \partial\Omega$; and vice versa, for fixed values of η, K and the dilation constant θ, there exists $\alpha_1 > 0$ such that the dyadic cone $\Gamma_d(q) \subset \Gamma^{\alpha_1}(q)$ for all $q \in \partial\Omega$. For any $X \in \Gamma^\alpha(q)$, let I be a Whitney box such that $X \in I^*$. By (6.1.4) we know $\ell(I) \sim \delta(X)$. Let Q be a cube containing q with length $\ell(Q) = \ell(I)$. Then

$$\operatorname{dist}(I, Q) \le |X - q| < (1+\alpha)\delta(X) \le C(1+\alpha)\ell(I) = C(1+\alpha)\ell(Q). \tag{6.1.18}$$

If K is sufficiently large so that $K^{\frac{1}{2}} \ge C(1+\alpha)$, then (6.1.18) and $\ell(I) = \ell(Q)$ implies that $I \in \mathcal{W}_Q^0$. By the definition (6.1.16) it follows that $X \in \Gamma_d(q)$. In particular, since $\Gamma^\alpha(q)$ is open, we also have $\Gamma^\alpha(q) \subset \operatorname{int}\Gamma_d(q)$. On the other hand, suppose $X \in \Gamma_d(q)$, by definition (6.1.16) X is contained in some $I^* = (1+\theta)I$ for a Whitney box $I \in \mathcal{W}_Q$ and dyadic cube Q containing q. Then by (6.1.8),

$$|X - q| \le \operatorname{diam} I^* + \operatorname{dist}(I, Q) + \operatorname{diam} Q \le C(K, \theta)\ell(Q),$$

$$\delta(X) \sim \ell(I) \geq C(\eta)\ell(Q).$$

Therefore there exists α_1 sufficiently large, depending on the values of η, K, θ, such that

$$|X - q| < (1 + \alpha_1)\delta(X),$$

i.e. $X \in \Gamma^{\alpha_1}(q)$. We summarize that now we have

$$\Gamma^{\alpha}(q) \subset \operatorname{int}\Gamma_d(q) \subset \Gamma_d(q) \subset \Gamma^{\alpha_1}(q), \quad \text{for all } q \in \partial\Omega. \tag{6.1.19}$$

Clearly $\alpha_1 > \alpha$. Moreover, there exists $\beta > \alpha_1$ depending on η, K, θ such that the fattened dyadic non-tangential cone

$$\widehat{\Gamma}_d(q) \subset \Gamma^{\beta}(q) \quad \text{for all } q \in \partial\Omega. \tag{6.1.20}$$

From now on we fix the values of η, K, θ and $\beta > \alpha_1 > \alpha > 0$.

Let $F = Q \setminus \bigcup_{Q_j \in \mathcal{F}} Q_j$ and suppose it is not empty. We claim that

$$\operatorname{int}\left(\bigcup_{q\in F} \Gamma_d^Q(q)\right) \subset \Omega_{\mathcal{F},Q} \subset \overline{\Omega_{\mathcal{F},Q}} \subset \Omega_{\mathcal{F},Q}^{***} \subset \bigcup_{q\in F} \widehat{\Gamma}_d^Q(q). \tag{6.1.21}$$

In fact, for any $q \in F$, it is clear that q is in some $Q' \in \mathbb{D}_{\mathcal{F},Q}$; and by (6.1.12), the definition of $\Omega_{\mathcal{F},Q}$, we have the first inclduement. On the other hand any $X \in \Omega_{\mathcal{F},Q}^{***}$ belongs to some $U_{Q'}^{***}$ with $Q' \in \mathbb{D}_{\mathcal{F},Q}$, and thus $X \in \widehat{\Gamma}_d^Q(q)$ for arbitrary $q \in Q'$. By the definition of $\mathbb{D}_{\mathcal{F},Q}$, we know $Q' \cap F \neq \emptyset$, so by taking $q \in Q' \cap F$ we get $X \in \bigcup_{q\in F} \widehat{\Gamma}_d^Q(q)$.

For N sufficiently large, we augment the collection of maximal cubes $\mathcal{F}$ by adding all dyadic cubes in $\mathbb{D}$ of size smaller than or equal to $2^{-N}\ell(Q)$, and we denote by $\mathcal{F}^N$ a collection consisting of all maximal cubes of the above augmented collection. In particular $Q' \in \mathbb{D}_{\mathcal{F}^N,Q}$ if and only if $Q' \in \mathbb{D}_{\mathcal{F},Q}$ and $\ell(Q') > 2^{-N}\ell(Q)$. By doing this we guarantee that the sawtooth domain $\Omega_{\mathcal{F}^N,Q}$ is compactly contained in Ω (roughly speaking $\operatorname{dist}(\Omega_{\mathcal{F}^N,Q}, \Omega^c) \sim 2^{-N}\ell(Q)$). Similar to Lemma 4.44 of [HMT1] we can construct a smooth cutoff function of $\Omega_{\mathcal{F}^N,Q}$:

Lemma 6.1.22 (cut-off function of sawtooth domain)**.** *There exists $\psi_N \in C_0^\infty(\mathbb{R}^n)$ such that*

(i) $\chi_{\Omega^*_{\mathcal{F}^N,Q}} \lesssim \psi_N \leq \chi_{\Omega^{**}_{\mathcal{F}^N,Q}}$*;*

(ii) $\sup_{X\in\Omega} |\nabla\psi_N(X)|\delta(X) \lesssim 1$*;*

(iii) We abbreviate $\mathcal{W}_{\mathcal{F}^N,Q}$ as $\mathcal{W}_N$ and set $\Sigma = \partial\Omega^*_{\mathcal{F}^N,Q}$,

$$\mathcal{W}_N^\Sigma = \{I \in \mathcal{W}_N : \text{there exists } J \in \mathcal{W} \setminus \mathcal{W}_N \text{ with } \partial I \cap \partial J \neq \emptyset\}.$$

Then

$$\nabla\psi_N \equiv 0 \text{ in} \bigcup_{I\in\mathcal{W}_N\setminus\mathcal{W}_N^\Sigma} I^{***}. \tag{6.1.23}$$

(iv) *For each $I \in \mathcal{W}_{\mathcal{F}^N,Q}$, let Q_I denote a cube in $\mathbb{D}_{\mathcal{F}^N,Q}$ such that $I \in \mathcal{W}_{Q_I}$. Suppose ω is the harmonic measure with pole X_0 and X_0 satisfies* $\operatorname{dist}(X_0, \Omega^{***}_{\mathcal{F}^N,Q}) \gtrsim \ell(Q)$. *Then*

$$\sum_{I \in \mathcal{W}_N^\Sigma} \omega(Q_I) \lesssim \omega(Q) \tag{6.1.24}$$

with a constant depending on η, K, a_0, C_1, d and the Ahlfors regular constant of Γ.

Remark 6.1.25. 1. We remark that the construction of ψ_N and the proof of its properties (i), (ii), (iii) are higher codimensional analogues of Lemma 4.44 of [HMT1]. However we prove (iv) instead of the second estimate in their (4.46), because we will need to prove a good-λ inequality for the harmonic measure, instead of the surface measure. Since harmonic measure could have much worse decay properties than the surface measure, not to mention that $\partial\Omega$ and $\partial\Omega_{\mathcal{F}^N,Q}$ are objects of different dimensions, proving (iv) requires a different argument.

2. Note that in (iv), the choice of Q_I may not be unique. Suppose both $Q_I, \widetilde{Q}_I$ are cubes in $\mathbb{D}_{\mathcal{F}^N,Q}$ such that $I \in \mathcal{W}_{Q_I}$ and $I \in \mathcal{W}_{\widetilde{Q}_I}$. By the construction of $\mathcal{W}_Q$'s and in particular (6.1.8), we know

$$\ell(Q_I) \sim \ell(I) \sim \ell(\widetilde{Q}_I), \quad \operatorname{dist}(Q_I, \widetilde{Q}_I) \lesssim \ell(Q_I) \tag{6.1.26}$$

with constants depending on η, K. Since harmonic measure is doubling, we have

$$C_1\omega(Q_I) \leq \omega(\widetilde{Q}_I) \leq C_2\omega(Q_I) \tag{6.1.27}$$

with constants only depending on the doubling constant and η, K. That is to say, for different choices of Q_I the left hand side of (6.1.24) differs at most by a constant multiple. But once we associate a cube Q_I to I, the choice will be fixed.

Proof. The proof of (i) is a modification of the proof from [HMT1] in higher codimensions. We recall that given I, any closed dyadic cube in $\mathbb{R}^n$, we set $I^{**} = (1+2\theta)I$ and $I^{***} = (1+4\theta)I$. Let us introduce $\widetilde{I^{**}} = (1+3\theta)I$ so that

$$I^{**} \subsetneq \operatorname{int}\widetilde{I^{**}} \subsetneq \widetilde{I^{**}} \subset \operatorname{int} I^{***}. \tag{6.1.28}$$

Given $I_0 = [-\frac{1}{2}, \frac{1}{2}]^n \subset \mathbb{R}^n$, we fix $\phi_0 \in C_0^\infty(\mathbb{R}^n)$ such that $\chi_{I_0^{**}} \leq \phi_0 \leq \chi_{\widetilde{I_0^{**}}}$ and $|\nabla\phi_0| \lesssim 1$, with the implicit constant depending on θ. For every $I \in \mathcal{W}$ we set $\phi_I = \phi_0((\cdot - X_I)/\ell(I))$ where X_I is the center of I, so that $\phi_I \in C_0^\infty(\mathbb{R}^n)$, $\chi_{I^{**}} \leq \phi_I \leq \chi_{\widetilde{I^{**}}}$ and $|\nabla\phi_I| \lesssim 1/\ell(I)$. Let $\Phi(X) := \sum_{I \in \mathcal{W}} \phi_I(X)$ for every $X \in \Omega$. Since for each compact subset of Ω, the previous sum has finitely many non-vanishing terms, we have $\Phi \in C^\infty_{loc}(\Omega)$. Also $0 \leq \Phi(X) \lesssim C_\theta$ since the family $\{\widetilde{I^{**}}\}_{I \in \mathcal{W}}$ has bounded overlap. Hence we can set $\Phi_I = \phi_I/\Phi$ and one can easily see that $\Phi_I \in C_0^\infty(\mathbb{R}^n)$, $C_\theta^{-1}\chi_{I^{**}} \leq \Phi_I \leq \chi_{\widetilde{I^{**}}}$ and $|\nabla\Phi_I| \lesssim 1/\ell(I)$. Recall the definition of $\mathcal{W}_N = \mathcal{W}_{\mathcal{F}^N,Q}$ in (6.1.13), we set

$$\psi_N(X) = \sum_{I \in \mathcal{W}_N} \Phi_I(X) = \frac{\sum_{I \in \mathcal{W}_N} \phi_I(X)}{\sum_{I \in \mathcal{W}} \phi_I(X)}, \quad X \in \Omega. \tag{6.1.29}$$

We first note that the number of terms in the sum defining ψ_N is bounded depending on N. Indeed if $Q' \in \mathbb{D}_{\mathcal{F}^N,Q}$ then $Q' \in \mathbb{D}_Q$ and $2^{-N}\ell(Q) < \ell(Q') \le \ell(Q)$, which implies $\mathbb{D}_{\mathcal{F}^N,Q}$ has finite cardinality with bounded depending only the Alhfors regular constant and N. Also by construction $\mathcal{W}_Q$ has cardinality depending only in the allowable parameters η, K. Hence $\#\mathcal{W}_N \le C_N < \infty$. This and the fact that each $\Phi_I \in C_0^\infty(\mathbb{R}^n)$ yield that $\psi_N \in C_0^\infty(\mathbb{R}^n)$. Moreover

$$\operatorname{supp}\psi_N \subset \bigcup_{I\in\mathcal{W}_N} \widetilde{I^{**}} = \bigcup_{Q'\in\mathbb{D}_{\mathcal{F}^N,Q}} \bigcup_{I\in\mathcal{W}_Q} \widetilde{I^{**}} \subset \operatorname{int}\left(\bigcup_{Q'\in\mathbb{D}_{\mathcal{F}^N,Q}} U_{Q'}^{**}\right) = \Omega_{\mathcal{F}^N,Q}^{**}. \tag{6.1.30}$$

This and the definition of ψ_N immediately gives $\psi_N \le \chi_{\Omega^{**}_{\mathcal{F}^N,Q}}$. On the other hand if $X \in \Omega^*_{\mathcal{F}^N,Q}$ then there exists $I \in \mathcal{W}_N$ such that $X \in I^{**}$, in which case $\psi_N(X) \ge \Phi_I(X) \ge C_\theta^{-1}$. This completes the proof of (i).

To obtain (ii) we note that for every $X \in \Omega$

$$|\nabla\psi_N(X)| \le \sum_{I\in\mathcal{W}_N} |\nabla\Phi_I(X)| \lesssim \sum_{I\in\mathcal{W}} \frac{1}{\ell(I)}\chi_{\widetilde{I^{**}}}(X) \lesssim \frac{1}{\delta(X)}, \tag{6.1.31}$$

where we have used that if $X \in \widetilde{I^{**}}$ then $\ell(I) \sim \delta(I)$ and also that the family $\{\widetilde{I^{**}}\}_{I\in\mathcal{W}}$ has bounded overlap.

Now we turn to (iii). Fix $I \in \mathcal{W}_N \setminus \mathcal{W}_N^\Sigma$ and $X \in I^{***}$, and set $\mathcal{W}_X = \{J \in \mathcal{W} : \phi_J(X) \neq 0\}$. We first note that $\mathcal{W}_X \subset \mathcal{W}_N$. Indeed if $\phi_J(X) \neq 0$ then $X \in \widetilde{J^{**}}$. Hence $X \in I^{***} \cap J^{***}$ and our choice of θ gives that ∂I meets ∂J, this in turn implies that $J \in \mathcal{W}_N$ since $I \notin \mathcal{W}_N^\Sigma$. All these imply

$$\psi_N(X) = \frac{\sum_{J\in\mathcal{W}_N}\phi_J(X)}{\sum_{J\in\mathcal{W}}\phi_J(X)} = \frac{\sum_{J\in\mathcal{W}_N\cap\mathcal{W}_X}\phi_J(X)}{\sum_{J\in\mathcal{W}\cap\mathcal{W}_X}\phi_J(X)} = \frac{\sum_{J\in\mathcal{W}_N\cap\mathcal{W}_X}\phi_J(X)}{\sum_{J\in\mathcal{W}_N\cap\mathcal{W}_X}\phi_J(X)} = 1. \tag{6.1.32}$$

Hence $\psi_N|_{I^{***}} \equiv 1$ for every $I \in \mathcal{W}_N \setminus \mathcal{W}_N^\Sigma$. This and the bounded overlap of the family $\{I^{***}\}_{I\in\mathcal{W}_N}$ immediately give that $\nabla\psi_N \equiv 0$ in $\bigcup_{I\in\mathcal{W}_N\setminus\mathcal{W}_N^\Sigma} I^{***}$.

Finally, it remains to prove the most difficult property, (iv). For any $I \in \mathcal{W}_N^\Sigma$, by definition there exists some $J_I \in \mathcal{W} \setminus \mathcal{W}_N$ such that $\partial I \cap \partial J_I \neq \emptyset$. Roughly speaking, this is to say that I is a Whitney box living in the "boundary" of $\Omega^*_{\mathcal{F}^N,Q}$. Thus pick any $Q_I' \in \mathbb{D}$ such that $\mathcal{W}_{Q_I'}$ contains J_I, we know $Q_I' \notin \mathbb{D}_{\mathcal{F}^N,Q}$, that is, either $Q_I' \in \mathbb{D}_{Q_j}$ for some $Q_j \in \mathcal{F}^N$, or $Q_I' \notin \mathbb{D}_Q$. We classify $I \in \mathcal{W}_N^\Sigma$ based on which category its associated cube Q_I' lives in: We denote

$$\Sigma_j = \{I \in \mathcal{W}_N^\Sigma : Q_I' \in \mathbb{D}_{Q_j}\} \text{ for any } Q_j \in \mathcal{F}^N,$$

and

$$\Sigma_0 = \{I \in \mathcal{W}_N^\Sigma : Q_I' \notin \mathbb{D}_Q\}.$$

(Note that for each $I \in \mathcal{W}_N^\Sigma$, we associate it to a unique Q_I', even though the choice itself is not unique.) Recall (6.1.5) we have $\ell(I) \sim \ell(J_I)$. Moreover by the definition of $\mathcal{W}_Q$ and

(6.1.8),

$$\ell(Q'_I) \sim \ell(J_I) \sim \ell(I) \sim \ell(Q_I) \tag{6.1.33}$$

and

$$\operatorname{dist}(Q_I, Q'_I) \leq \operatorname{dist}(Q_I, I) + \operatorname{dist}(I, J_I) + \operatorname{dist}(J_I, Q'_I) \lesssim \ell(Q_I) + \ell(Q'_I) \lesssim \ell(Q'_I). \tag{6.1.34}$$

By similar argument as in remark 6.1.25 (2) and the doubling property of harmonic measure, we have $\omega(Q_I) \sim \omega(Q'_I)$ for any $I \in \mathcal{W}_N^\Sigma$, with a uniform constant depending on η, K. Therefore to prove (6.1.24) it suffices to show

$$\sum_{I \in \mathcal{W}_N^\Sigma} \omega(Q'_I) \lesssim \omega(Q).$$

We claim that for any $Q_j \in \mathcal{F}^N$,

$$\sum_{I \in \Sigma_j} \omega(Q'_I) \lesssim \omega(Q_j). \tag{6.1.35}$$

Recall that all such Q'_I's live in $\mathbb{D}_{Q_j}$. For each $k \in \mathbb{N}$ we denote $\Sigma_j^k = \{I \in \Sigma_j : \ell(Q'_I) = 2^{-k}\ell(Q_j)\}$. Since $Q_I \in \mathbb{D}_{\mathcal{F}^N, Q}$, $Q_j \in \mathcal{F}^N$, we always have $Q_j \cap Q_I = \emptyset$, so by (6.1.34)

$$\operatorname{dist}\left(Q'_I, \left(Q_j\right)^c\right) \leq \operatorname{dist}(Q'_I, Q_I) \lesssim \ell(Q'_I) = 2^{-k}\ell(Q_j). \tag{6.1.36}$$

That is, the smaller Q'_I is, the closer it is to the "boundary" of Q_j. The Q'_I's of different generations are very far from being disjoint, however we will sum up the $\omega(Q'_I)$'s by swapping them for the harmonic measure of mutually disjoint cubes. By (6.1.36), for ρ sufficiently small there is an integer $k_1 = k_1(\rho)$ such that for any integer $k \geq k_1$,

$$\bigcup_{k' \geq k} \bigcup_{I \in \Sigma_j^{k'}} Q'_I \subset \left\{q \in Q_j : \operatorname{dist}\left(q, (Q_j)^c\right) \leq \frac{\rho}{2}\ell(Q_j)\right\}. \tag{6.1.37}$$

In fact by choosing k_1 slightly bigger, we can even guarantee that for any integer $k \geq k_1$,

$$\bigcup_{k' \geq k} \bigcup_{I \in \Sigma_j^{k'}} Q'_I \subset \bigcup_{i \in \mathcal{I}_k} Q_j^i \subset \left\{q \in Q_j : \operatorname{dist}\left(q, (Q_j)^c\right) \leq \frac{\rho}{2}\ell(Q_j)\right\}, \tag{6.1.38}$$

where $\{Q_j^i\}_{i \in \mathcal{I}_k}$ is the collection of all dyadic cubes in $\mathbb{D}_{Q_j}$ of length $2^{-k}\ell(Q_j)$ such that $Q_j^i \subset \{q \in Q_j : \operatorname{dist}(q, (Q_j)^c) \leq \rho\ell(Q_j)/2\}$. By Lemma 6.1.1 (v) (vi) the index set $\mathcal{I}_k$ has finite cardinality and $\#\mathcal{I}_k \leq C2^{kd}$. (A priori the set $\mathcal{I}_k$ could be empty, in which case (6.1.38) just means there is no Q'_I corresponding to any $I \in \bigcup_{k' \geq k} \Sigma_j^{k'}$. This case is easy to deal with.)

On the other hand by Lemma 6.1.1, as long as we fix $\rho \in (0, a_0)$ satisfying $A_1\rho^\gamma < 1$, the set $\left\{q \in Q_j : \operatorname{dist}(q, (Q_j)^c) > \frac{\rho}{2}\ell(Q_j)\right\}$ is not empty; moreover, there is an integer k_2

sufficiently large such that for each $k \geq k_2$ we can find a cube $\widehat{Q}_j$ such that $\ell(\widehat{Q}_j) = 2^{-k}\ell(Q_j)$ and

$$\widehat{Q}_j \subset \left\{ q \in Q_j : \operatorname{dist}(q, (Q_j)^c) > \frac{\rho}{2}\ell(Q_j) \right\}. \tag{6.1.39}$$

We may think of $\widehat{Q}_j$ as sitting in the "center" of Q_j, and all Q'_I's in a $\rho/2$-boundary layer of Q_j. Let $k_0 = \max\{k_1, k_2\}$, and let N_1 denote the (maximal) number of Q'_I's with $\ell(Q'_I) = 2^{-k_0}\ell(Q_j)$. By (6.1.37) and Lemma 6.1.1 (vi), N_1 is uniformly bounded by a constant depending on a_0, A_1, ρ, k_0 and d. Moreover by the doubling property of ω, each such Q'_I satisfies

$$\omega(Q'_I) \leq \omega(Q_j) \leq C(k_0)\omega(\widehat{Q}_j), \tag{6.1.40}$$

with the constant $C(k_0)$ depending on k_0 as well as the doubling constant of ω. Recall that for each Q'_I, the number of all possible I's corresponding to it is uniformly bounded by $C(N_0)$. Therefore

$$\sum_{I \in \Sigma_j^{k_0}} \omega(Q'_I) \leq C(N_0) \sum_{Q'_I : \ell(Q'_I) = 2^{-k_0}\ell(Q_j)} \omega(Q'_I) \leq C(N_0) N_1 C(k_0) \omega(\widehat{Q}_j). \tag{6.1.41}$$

Now for any $I \in \Sigma_j^k$ with $k = 1, \cdots, k_0 - 1$, again by the doubling property of harmonic measure we have $\omega(Q'_I) \leq C(k_0)\omega(\widehat{Q}_j)$. By Lemma 6.1.1 (iv) (v), the total number of Q'_I's in $\mathbb{D}_{Q_j}$ such that $\ell(Q'_I) = 2^{-k}\ell(Q_j)$ with $k = 1, \cdots, k_0 - 1$ is uniformly bounded by a constant depending only on k_0, a_0, C_1, d and the Ahlfors regular constant of Γ. Thus the total number of I's in Σ_j^k with $k = 1, \cdots, k_0 - 1$ is also uniformly bounded. Therefore combining with (6.1.41), we get

$$\sum_{k=1}^{k_0} \sum_{I \in \Sigma_j^k} \omega(Q'_I) \lesssim \omega(\widehat{Q}_j). \tag{estimate-k_0}$$

For future generations, we recall (6.1.38), which says all the Q'_I's corresponding to some $I \in \Sigma_j^k$ with $k \geq k_0$ are contained in $\bigcup_{i \in \mathcal{I}_{k_0}} Q_j^i$. The following proof is illustrated in the (idealized) Figure 6.1, where each label denotes the cube near it enclosed or shaded by the same color. Consider any cube $Q' = Q_j^i$ for an arbitrary $i \in \mathcal{I}_{k_0}$. Apply the above argument to Q' in place of Q_j, we can find a cube $\widehat{Q}' = \widehat{Q}_j^i \in \mathbb{D}_{Q'}$ with length $\ell(\widehat{Q}') = 2^{-k_0}\ell(Q') = 2^{-2k_0}\ell(Q_j)$ sitting in the "center" of Q', in the sense that

$$\widehat{Q}_j^i \subset \left\{ q \in Q' : \operatorname{dist}(q, (Q')^c) > \frac{\rho}{2}\ell(Q') \right\}; \tag{6.1.42}$$

and all future generations satisfy

$$\bigcup_{k \geq 2k_0} \bigcup_{\substack{I \in \Sigma_j^k \\ Q'_I \in \mathbb{D}_{Q'}}} Q'_I \subset \bigcup_{i_2 \in \mathcal{I}_{k_0}} Q_j^{ii_2} \subset \left\{ q \in Q' : \operatorname{dist}(q, (Q')^c) \leq \frac{\rho}{2}\ell(Q') \right\}, \tag{6.1.43}$$

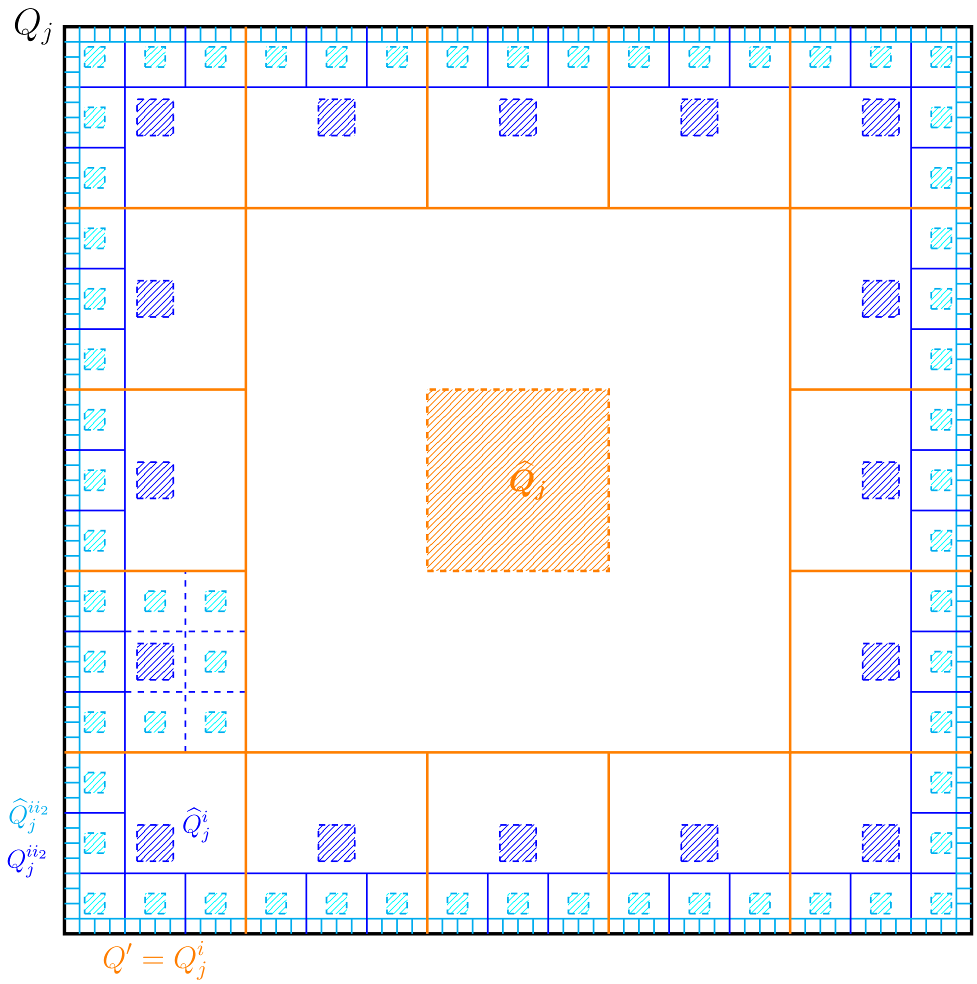

Figure 6.1: Illustration of the swap of cubes in iteration

where $\{Q_j^{ii_2}\}_{i_2 \in \mathcal{I}_{k_0}}$ is the collection of all dyadic cubes of length $2^{-k_0}\ell(Q') = 2^{-2k_0}\ell(Q_j)$ that is completely contained in $\{q \in Q' = Q_j^i : \operatorname{dist}(q, (Q')^c) \leq \rho\ell(Q')/2\}$. (The index set for i_2 may not be the same as the index set for i, but their cardinalities are uniformly bounded by $C2^{k_0 d}$, so we abuse the notation here and simply assume they are the same.) Moreover we can get an analogous estimate of (estimate-k_0):

$$\sum_{k=k_0+1}^{2k_0} \sum_{\substack{I \in \Sigma_j^k \\ Q_I' \in \mathbb{D}_{Q'}}} \omega(Q_I') \lesssim \omega(\widehat{Q}_j^i). \tag{6.1.44}$$

Summing up (6.1.44) over all cubes $Q' \in \{Q_j^i\}_{i\in\mathcal{I}_{k_0}}$, recall (6.1.38) we get

$$\sum_{k=k_0+1}^{2k_0}\sum_{I\in\Sigma_j^k}\omega(Q_I') \lesssim \sum_{i\in\mathcal{I}_{k_0}}\omega(\widehat{Q}_j^i). \tag{6.1.45}$$

Since $\{Q_j^i\}_{i\in\mathcal{I}_{k_0}}$ is a collection of cubes in the same generation, they are mutually disjoint, and their sub-cubes $\{\widehat{Q}_j^i\}_{i\in\mathcal{I}_{k_0}}$ are also mutually disjoint. Hence

$$\sum_{k=k_0+1}^{2k_0}\sum_{I\in\Sigma_j^k}\omega(Q_I') \lesssim \sum_{i\in\mathcal{I}_{k_0}}\omega(\widehat{Q}_j^i) = \omega\left(\bigsqcup_{i\in\mathcal{I}_{k_0}}\widehat{Q}_j^i\right). \tag{estimate-$2k_0$}$$

Moreover, recall the second inclument of (6.1.38) and (6.1.39), each $\widehat{Q}_j^i$ is disjoint from $\widehat{Q}_j$, so we can add up (estimate-k_0) and (estimate-$2k_0$) with ease. We can repeat this argument iteratively: for any $l\in\mathbb{N}$ we apply the argument to cube $Q' = Q_j^{i_1 i_2\cdots i_l}$ with $i_1,\cdots,i_l\in\mathcal{I}_{k_0}$ to get an analogous estimate of (6.1.44), then we sum up over the index sets and get

$$\sum_{k=lk_0+1}^{(l+1)k_0}\sum_{I\in\Sigma_j^k}\omega(Q_I') \lesssim \sum_{i_1,\cdots,i_l\in\mathcal{I}_{k_0}}\omega\left(\widehat{Q}_j^{i_1\cdots i_l}\right) = \omega\left(\bigsqcup_{i_1,\cdots,i_l\in\mathcal{I}_{k_0}}\widehat{Q}_j^{i_1\cdots i_l}\right). \tag{estimate-$(l+1)k_0$}$$

Most significantly for us, for each $l\in\mathbb{N}$ the union of cubes on the right hand side of (estimate-$(l+1)k_0$) is disjoint from all the cubes from all previous summations. Therefore we conclude that

$$\sum_{k=1}^{\infty}\sum_{I\in\Sigma_j^k}\omega(Q_I') \lesssim \omega\left(\bigsqcup_{l\in\mathbb{N}}\left(\bigsqcup_{i_1,\cdots,i_l\in\mathcal{I}_{k_0}}\widehat{Q}_j^{i_1\cdots i_l}\right)\right) \leq \omega(Q_j). \tag{6.1.46}$$

It is trivial to see $\sum_{I\in\Sigma_j^0}\omega(Q_I')\lesssim\omega(Q_j)$, so

$$\sum_{I\in\Sigma_j}\omega(Q_I') = \sum_{k\in\mathbb{N}}\sum_{I\in\Sigma_j^k}\omega(Q_I') \lesssim \omega(Q_j). \tag{6.1.47}$$

Since the maximal cubes Q_j in $\mathcal{F}^N$ are mutually disjoint and contained in Q, we have

$$\sum_{Q_j\in\mathcal{F}^N}\sum_{I\in\Sigma_j}\omega(Q_I') \lesssim \sum_{Q_j\in\mathcal{F}^N}\omega(Q_j) \leq \omega(Q). \tag{6.1.48}$$

Now we consider $I\in\Sigma_0$, which by definition means $Q_I'\notin\mathbb{D}_Q$. Recall (6.1.33) and (6.1.34), and that $\ell(I)\leq C\ell(Q)$ for all $I\in\mathcal{W}_N=\mathcal{W}_{\mathcal{F}^N,Q}$, we have

$$\ell(Q_I')\sim\ell(I)\leq C\ell(Q), \quad \operatorname{dist}(Q_I,Q_I')\lesssim\ell(Q_I')\leq C\ell(Q). \tag{6.1.49}$$

In particular since $Q_I \in \mathbb{D}_Q$, we have

$$\operatorname{dist}(Q_I', Q) \le \operatorname{dist}(Q_I', Q_I) \lesssim \ell(Q_I') \le C\ell(Q). \tag{6.1.50}$$

If $\ell(Q_I') \ge \ell(Q)$, then

$$\ell(Q_I') \sim \ell(Q), \quad \operatorname{dist}(Q_I', Q) \le C\ell(Q). \tag{6.1.51}$$

There are finitely many such Q_I''s and by the doubling property of harmonic measure $\omega(Q_I') \sim \omega(Q)$. If $\ell(Q_I') < \ell(Q)$, let $Q_0 \in \mathbb{D}$ be the cube containing Q_I' with length $\ell(Q_0) = \ell(Q)$. By the assumption $Q_I' \notin \mathbb{D}_Q$, we know Q_0 is disjoint from Q. On the other hand (6.1.50) implies

$$\operatorname{dist}(Q_0, Q) \le \operatorname{dist}(Q_I', Q) \le C\ell(Q), \tag{6.1.52}$$

that is, Q_0 is a sibling (i.e. of the same generation) of Q in a $C\ell(Q)$-neighborhood of Q. There are finitely many such Q_0's. Moreover

$$\operatorname{dist}(Q_I', (Q_0)^c) \le \operatorname{dist}(Q_I', Q) \lesssim \ell(Q_I'). \tag{6.1.53}$$

So if $\ell(Q_I') \ll \ell(Q)$, we can guarantee that Q_I' lies in the $\rho/2$-boundary layer of Q_0: $Q_I' \subset \{q \in Q_0 : \operatorname{dist}(q, (Q_0)^c) \le \rho\ell(Q_0)/2\}$. Apply the same argument to Q_0 in place of Q_j, we get

$$\sum_{\substack{I \in \Sigma_0 \\ Q_I' \in \mathbb{D}_{Q_0}}} \omega(Q_I') \lesssim \omega(Q_0) \sim \omega(Q). \tag{6.1.54}$$

Summing up (6.1.54) over all (finitely many) Q_0's satisfying (6.1.52), we get

$$\sum_{I \in \Sigma_0} \omega(Q_I') \lesssim \omega(Q). \tag{6.1.55}$$

Finally we combine (6.1.48) and (6.1.55) and conclude that

$$\sum_{I \in \mathcal{W}_N^\Sigma} \omega(Q_I') = \sum_{Q_j \in \mathcal{F}^N} \sum_{I \in \Sigma_j} \omega(Q_I') + \sum_{I \in \Sigma_0} \omega(Q_I') \lesssim \omega(Q). \tag{6.1.56}$$

Therefore

$$\sum_{I \in \mathcal{W}_N^\Sigma} \omega(Q_I') \lesssim \omega(Q). \tag{6.1.57}$$

Now that all the preparatory work has been done, we proceed to sketch the basic idea in the proof of Theorem 1.3.13. It is well known in harmonic analysis that the proof of $\|Su\|_{L^p(\sigma)} \le C\|Nu\|_{L^p(\sigma)}$ can be reduced to the proof of a certain good-λ inequality measured by σ. We first prove Proposition 6.1.58, which is a good-λ inequality measured by ω; then we use the assumption $\omega \in A_\infty(\sigma)$ to obtain the desired good-λ inequality for σ.

Recall that we use $Su, S'u, S''u$ to denote the square functions on standard non-tangential cones of aperture α, α_1, β, respectively, and Nu non-tangential maximal function on cones of aperture β, where $\beta > \alpha_1 > \alpha$ are fixed apertures (see the discussion before Lemma 6.1.22). Also recall from (6.1.15) that for any collection $\mathcal{F}$ of dyadic cubes, the sawtooth domain $\Omega_{\mathcal{F},Q} \subset B(x_Q, C_3\ell(Q)) \cap \Omega$. In fact, by choosing a slightly bigger constant C_3 we can also guarantee $\Omega^{***}_{\mathcal{F},Q} \subset B(x_Q, C_3\ell(Q)) \cap \Omega$. We prove the following good-λ inequality:

Proposition 6.1.58 (good-λ inequality for ω)**.** *Suppose Γ is a d-Ahlfors regular set in $\mathbb{R}^n$ with $d < n-1$, $\Omega = \mathbb{R}^n \setminus \Gamma$ and $\mathbb{D}$ is a collection of dyadic cubes for Γ, see Lemma 6.1.1 for the detail. Let $u \in W_r(\Omega)$ be a non-negative solution of $Lu = 0$ such that for some dyadic cube $Q \in \mathbb{D}$ and $\lambda > 0$ there exists $q_1 \in \Gamma$ with*

$$S'u(q_1) \leq \lambda \quad \text{and} \quad |q_1 - q| \leq C_2 \operatorname{diam} Q \text{ for all } q \in Q.$$

Then for any $X_Q \notin B(x_Q, 2C_3\ell(Q))$ and δ sufficiently small, we have

$$\omega^{X_Q}(\{q \in Q : Su(q) > 2\lambda, Nu(q) \leq \delta\lambda\}) \leq C\delta^2 \omega^{X_Q}(Q) \tag{6.1.59}$$

Here $x_Q, \ell(Q)$ are the "center" and "size" of Q, see Lemma 6.1.1. The constant $C > 0$ depends on the allowable parameters d, n, C_0, C_1, the apertures α, α_1, β, and the given constants C_2, C_3.

Proof. For simplicity we denote $\omega = \omega^{X_Q}$. Let $E = \{q \in Q : Su(q) > 2\lambda, Nu(q) \leq \delta\lambda\}$ and $F = \{q \in Q : Nu(q) \leq \delta\lambda\}$. If F is empty, then the left hand side of (6.1.59) is zero, and there is nothing to prove. So we assume $F \neq \emptyset$. Note that $Nu(q)$ is a continuous function, so $Q \setminus F = \{q \in Q : Nu(q) > \delta\lambda\}$ is relatively open in Q. We run a stopping time procedure for the descendants of Q, and stop at $Q' \in \mathbb{D}_Q$ whenever $Nu(q) > \delta\lambda$ for all $q \in Q'$. We denote the collection of all maximal cubes by $\mathcal{F}_2 = \{Q_j\} \subset \mathbb{D}_Q \setminus \{Q\}$. We claim that they form a partition:

$$Q \setminus F = \{q \in Q : Nu(q) > \delta\lambda\} = \bigcup_{Q_j \in \mathcal{F}_2} Q_j. \tag{6.1.60}$$

Clearly by construction $\cup_{Q_j \in \mathcal{F}_2} Q_j$ is contained in the set on the left. For any $q_0 \in Q$ such that $Nu(q_0) > \delta\lambda$, since the set $\{q \in \partial\Omega : Nu(q) > \delta\lambda\}$ is open, $Q \setminus F \neq Q$ and the cubes in $\mathbb{D}$ are nested, there exists a small cube $Q' \in \mathbb{D}_Q \setminus \{Q\}$ containing q_0, such that $Nu(q) > \delta\lambda$ for all $q \in Q'$. By the stopping time procedure, either $Q' \in \mathcal{F}_2$, or Q' is contained in some cube $Q_j \in \mathcal{F}_2$. Hence $q_0 \in Q' \subset \bigcup_{Q_j \in \mathcal{F}_2} Q_j$, and we prove the claim (6.1.60). Recall (6.1.21), which we rewrite here:

$$\operatorname{int}\left(\bigcup_{q \in F} \Gamma_d^Q(q)\right) \subset \Omega_{\mathcal{F}_2,Q} \subset \overline{\Omega_{\mathcal{F}_2,Q}} \subset \Omega^{***}_{\mathcal{F}_2,Q} \subset \bigcup_{q \in F} \widehat{\Gamma}_d^Q(q). \tag{6.1.61}$$

We claim that $|u(X)| \leq \delta\lambda$ for all $X \in \Omega^{***}_{\mathcal{F}_2,Q}$. In fact, by (6.1.20) and (6.1.61) we know that every $X \in \Omega^{***}_{\mathcal{F}_2,Q}$ is contained in some $\widehat{\Gamma}_d^Q(q) \subset \Gamma^\beta(q)$ for some $q \in F$. Since $Nu(q) = \sup_{X \in \Gamma^\beta(q)} |u(X)| \leq \delta\lambda$ for $q \in F$, we get $|u(X)| \leq \delta\lambda$.

Step 1. Recall the assumption that $S'u(q_1) \leq \lambda$ for some q_1 satisfying $|q_1 - q| \leq C_2 \operatorname{diam} Q$ for all $q \in Q$. Denote $r = \operatorname{diam} Q$. We claim that for any $\tau > 0$ there exists $\delta > 0$ sufficiently small such that the truncated square function $S_{\tau r}u(q) > \lambda$ for any $q \in E$.

Fix $q \in E$. Recall that $Su(q) > 2\lambda$ for $q \in E$. We denote $U = \Gamma^\alpha(q) \setminus B(q, \tau r)$, then we aim to show

$$\iint_U |\nabla u(X)|^2 \delta(X)^{1-d} dm(X) \leq 3\lambda^2. \tag{6.1.62}$$

Let $U_1 = \Gamma^\alpha(q) \setminus B(q, tr)$ for a constant $t > \tau$ to be chosen later, and $U_2 = \Gamma^\alpha(q) \cap (B(q, tr) \setminus B(q, \tau r))$. Then $U = U_1 \cup U_2$. A simple computation shows that

$$U_1 = \Gamma^\alpha(q) \setminus B(q, tr) \subset \Gamma^{\alpha_1}(q_1) \tag{6.1.63}$$

if the apertures satisfy

$$(1+\alpha)\left(1 + \frac{C_2}{t}\right) \leq 1 + \alpha_1,$$

that is, if t is sufficiently large such that

$$\alpha + \frac{C_2(1+\alpha)}{t} \leq \alpha_1.$$

Therefore

$$\iint_{U_1} |\nabla u(X)|^2 \delta(X)^{1-d} dm(X) \leq \iint_{\Gamma^{\alpha_1}(q_1)} |\nabla u(X)|^2 \delta(X)^{1-d} dm(X) = S'u(q_1)^2 \leq \lambda^2. \tag{6.1.64}$$

Let $\Gamma_j(q) = \Gamma^\alpha(q) \cap \left(B(q, 2^j \tau r) \setminus B(q, 2^{j-1}\tau r)\right)$ for $j = 1, 2, \cdots$, then

$$U_2 \subset \bigcup_{j: 2^{j-1}\tau r < tr} \Gamma_j(q).$$

Each $\Gamma_j(q)$ can be covered by a finite union (depending on n) of balls $B_{j,k}$ with radius $r_{j,k} \sim_\alpha 2^j \tau r$. Let $B^*_{j,k}$ denote a slight fattening of $B_{j,k}$ such that we still have $B^*_{j,k} \subset \Gamma^\beta(q)$, then by Lemma 2.4.9 (i) $m(B^*_{j,k}) \sim r_{j,k}^{d+1} \sim (2^j \tau r)^{d+1}$. Then

$$\begin{aligned}
\iint_{U_2} |\nabla u(X)|^2 \delta(X)^{1-d} dm(X) &= \sum_{2^{j-1}\tau r < tr} \iint_{\Gamma_j(q)} |\nabla u(X)|^2 \delta(X)^{1-d} dm(X) \\
&\sim_{\alpha,\beta} \sum_{2^{j-1}\tau r < tr} (2^j \tau r)^{1-d} \sum_{1 \leq k \leq C(n)} \iint_{B_{j,k}} |\nabla u(X)|^2 dm(X) \\
&\lesssim \sum_{\substack{2^{j-1}\tau r < tr \\ 1 \leq k \leq C(n)}} (2^j \tau r)^{-1-d} \iint_{B^*_{j,k}} |u(X)|^2 dm(X) \\
&\lesssim (\delta\lambda)^2 \sum_{2^{j-1}\tau r < tr} (2^j \tau r)^{-1-d} m(B^*_{j,k})
\end{aligned}$$

$$\lesssim (\delta\lambda)^2 \log_2\left(\frac{t}{\tau}\right)$$
$$< 2\lambda^2, \tag{6.1.65}$$

if δ is sufficiently small depending on the values of t, τ and α, β. Therefore (6.1.62) holds, and thus for any $q \in E$,

$$\begin{aligned} |S_{\tau r}u(q)|^2 &= \iint_{\Gamma^\alpha(q)\cap B(q,\tau r)} |\nabla u(X)|^2 \delta(X)^{1-d} dm(X) \\ &= \iint_{\Gamma^\alpha(q)\setminus U} |\nabla u(X)|^2 \delta(X)^{1-d} dm(X) \\ &> \lambda^2. \end{aligned} \tag{6.1.66}$$

Step 2. Combining (6.1.66) with $E \subset F$ we get

$$\lambda^2 \omega(E) \le \int_E |S_{\tau r}u(q)|^2 d\omega(q) \le \int_F \iint_{\Gamma^\alpha_{\tau r}(q)} |\nabla u(X)|^2 \delta(X)^{1-d} dm(X) d\omega(q). \tag{6.1.67}$$

By (6.1.19) we have

$$\Gamma^\alpha_{\tau r}(q) \subset \operatorname{int}\Gamma_d(q) \subset \Gamma_d(q) \tag{6.1.68}$$

for any $q \in Q$. In particular if X belongs to the left hand side of (6.1.68), then $X \in U_{Q'}$ for some dyadic cube Q' containing q. Moreover

$$\delta(X) \le |X - q| < \tau r = \tau \operatorname{diam} Q \sim \tau \ell(Q). \tag{6.1.69}$$

By the definition of $U_{Q'}$ and (6.1.8), we have

$$\delta(X) \gtrsim c\eta^{\frac{n-1}{4(n-1-d)}} \ell(Q'). \tag{6.1.70}$$

By combining (6.1.69), (6.1.70) and choosing τ small enough depending on η, we can guarantee that $\ell(Q') < 2\ell(Q)$. Since $Q' \cap Q \ni q$, by property (ii) of Lemma 6.1.1 we know $Q' \in \mathbb{D}_Q$. Hence $\Gamma^\alpha_{\tau r}(q) \subset \Gamma^Q_d(q)$. Again since $\Gamma^\alpha_{\tau r}(q)$ is an open set, we also have $\Gamma^\alpha_{\tau r}(q) \subset \operatorname{int}\Gamma^Q_d(q)$. Therefore

$$\bigcup_{q\in F} \Gamma^\alpha_{\tau r}(q) \subset \bigcup_{q\in F}\left(\operatorname{int}\Gamma^Q_d(q)\right) \subset \operatorname{int}\left(\bigcup_{q\in F}\Gamma^Q_d(q)\right) \tag{6.1.71}$$

Applying Fubini's theorem to the right hand side of (6.1.67), we conclude that it is bounded by

$$\iint_{\operatorname{int}\left(\bigcup_{p\in F}\Gamma^Q_d(p)\right)} |\nabla u(X)|^2 \delta(X)^{1-d} \omega\left(\{q \in F : X \in \Gamma^Q_d(q)\}\right) dm(X). \tag{6.1.72}$$

For any $p \in F$ and any $X \in \Gamma_d^Q(p)$, we have $X \in I \in \mathcal{W}_{Q'}$ for a cube Q' containing p and in $\mathbb{D}_{\mathcal{F}_1,Q}$. Thus $|X-q| \sim \ell(Q') \sim \ell(I) \sim \delta(X)$. Since the family $\{I^*\}_{I\in\mathcal{W}}$ has bounded overlap and harmonic measure ω has pole at X_Q, we conclude by Lemma 2.4.83 that

$$\omega\left(\left\{q \in F : X \in \Gamma_d^Q(q)\right\}\right) \sim \omega\left(\bigcup_{\substack{Q'\in\mathbb{D}_Q \\ \ell(Q')\sim\delta(X)\sim\operatorname{dist}(X,Q')}} Q'\right) \sim G(X_Q,X)\delta(X)^{d-1}. \tag{6.1.73}$$

Combining (6.1.67), (6.1.72), (6.1.73) and (6.1.61) and using (2.4.2), we get

$$\begin{aligned}\lambda^2\omega(E) &\lesssim \iint_{\Omega_{\mathcal{F}_2,Q}} |\nabla u(X)|^2 G(X_Q,X)dm(X) = \iint_{\Omega_{\mathcal{F}_2,Q}} |\nabla u(X)|^2 G(X_Q,X)w(X)dX \\ &\lesssim \iint_{\Omega_{\mathcal{F}_2,Q}} A\nabla u\cdot\nabla u G dX.\end{aligned} \tag{6.1.74}$$

Here we abbreviate $G(X) = G(X_Q,X)$ when there is no ambiguity as to what the pole is. Recall that $\overline{\Omega^{***}_{\mathcal{F}^2,Q}}$ and our choice of the pole $X_Q \notin B(x_Q, 2C_3\ell(Q))$. They guarantee that $X_Q \notin \overline{\Omega^{***}_{\mathcal{F}_2,Q}}$, and moreover $\operatorname{dist}(X_Q, \overline{\Omega^{***}_{\mathcal{F}_2,Q}}) \gtrsim \ell(Q)$. Hence $G(X)$ is harmonic in the fat sawtooth domain $\Omega^{***}_{\mathcal{F}_2,Q}$.

Step 3. Next we are going to prove

$$\iint_{\Omega_{\mathcal{F}_2,Q}} A\nabla u\cdot\nabla u G dX \lesssim (\delta\lambda)^2\omega(Q). \tag{6.1.75}$$

Recall the discussion before Lemma 6.1.22, we can augment $\mathcal{F}_2$ by adding all dyadic cubes of lengths smaller or equal to $2^{-N}\ell(Q)$, and denote by $\mathcal{F}_2^N$ the collection of maximal cubes giving rise to the aforementioned augmented collection. We claim that

$$\iint_{\Omega_{\mathcal{F}_2^N,Q}} A\nabla u\cdot\nabla u G dX \lesssim (\delta\lambda)^2\omega(Q) \tag{6.1.76}$$

with a constant independent of N. Thus by passing $N\to\infty$ we obtain (6.1.75).

Recall that in Lemma 6.1.22, we construct a smooth cut-off function ψ_N such that $\chi_{\Omega^*_{\mathcal{F}_N,Q}} \lesssim \psi_N \leq \chi_{\Omega^{**}_{\mathcal{F}_N,Q}}$. Hence

$$\iint_{\Omega_{\mathcal{F}_N,Q}} A\nabla u\cdot\nabla u G dX \leq \iint_{\mathbb{R}^n} A\nabla u\cdot\nabla u G\psi_N dX \tag{6.1.77}$$

Since $u, G \in W_r(\Omega^{**}_{\mathcal{F}_N,Q}) \cap L^\infty(\Omega^{**}_{\mathcal{F}_N,Q})$, we have $uG\psi_N, u^2\psi_N \in W_0^{1,2}(\Omega^{**}_{\mathcal{F}_N,Q})$. In particular they can be approximated by smooth functions in $C_0^\infty(\Omega^{**}_{\mathcal{F}_N,Q}) \subset C_0^\infty(\Omega)$. In the sawtooth region $\Omega^{**}_{\mathcal{F}_N,Q}$ we have $-\operatorname{div}(A\nabla u) = -\operatorname{div}(A\nabla G) = 0$, thus

$$\iint_{\mathbb{R}^n} A\nabla u\cdot\nabla u G\psi_N dX$$

$$
\begin{aligned}
&= \iint_{\mathbb{R}^n} A\nabla u \cdot \nabla (uG\psi_N) - \frac{1}{2} A\nabla(u^2)\cdot\nabla(G\psi_N)dX \\
&= 0 - \frac{1}{2}\iint_{\mathbb{R}^n} A\nabla(G\psi_N)\cdot\nabla(u^2)dX \\
&= -\frac{1}{2}\left(\iint_{\mathbb{R}^n} \psi_N A\nabla G\cdot\nabla(u^2) + GA\nabla\psi_N\cdot\nabla(u^2)dX\right) \\
&= -\frac{1}{2}\left(\iint_{\mathbb{R}^n} A\nabla G\cdot\nabla(u^2\psi_N) - u^2 A\nabla G\cdot\nabla\psi_N + 2uGA\nabla u\cdot\nabla\psi_N dX\right) \\
&= \frac{1}{2}\iint_{\mathbb{R}^n} u^2 A\nabla G\cdot\nabla\psi_N dX - \iint_{\mathbb{R}^n} uGA\nabla u\cdot\nabla\psi_N dX \\
&=: \frac{1}{2} I - II, \qquad (6.1.78)
\end{aligned}
$$

where we use the symmetry of A and the equation $-\operatorname{div}(A\nabla u) = 0$ in the second equality, and $-\operatorname{div}(A\nabla G) = 0$ in the second to last equality. We first estimate the second term. By (6.1.23), the contribution to the integral II only comes from Whitney boxes $I \in \mathcal{W}_N^\Sigma$. Recall the harmonic function u is non-negative and we use X_I to denote the center of Whitney box I. By Lemma 6.1.22 (ii), Hölder inequality, estimate of the weight (2.4.10), interior Caccioploi inequality (2.4.25), Harnack inequality (2.4.27) and (2.4.84), we have

$$
\begin{aligned}
|II| &\le \sum_{I\in\mathcal{W}_N^\Sigma} \frac{u(X_I)G(X_I)}{\ell(I)} \iint_{I^{***}} |\nabla u| dm \\
&\le \sum_{I\in\mathcal{W}_N^\Sigma} \frac{u(X_I)G(X_I)}{\ell(I)} \cdot m(I^{***}) \left(\fint\!\!\!\fint_{I^{***}} |\nabla u|^2 dm \right)^{\frac{1}{2}} \\
&\lesssim \sum_{I\in\mathcal{W}_N^\Sigma} u(X_I)G(X_I)\ell(I)^{d-1} \left(\fint\!\!\!\fint_{I^{****}} |u|^2 dm \right)^{\frac{1}{2}} \\
&\lesssim \sum_{I\in\mathcal{W}_N^\Sigma} u(X_I)^2 G(X_I)\ell(I)^{d-1} \\
&\sim \sum_{I\in\mathcal{W}_N^\Sigma} u(X_I)^2 \omega(Q_I), \qquad (6.1.79)
\end{aligned}
$$

where Q_I is defined as in Lemma 6.1.22 (iv). Using the estimate $|u(X)| \le \delta\lambda$ for all $X \in \Omega^{***}_{\mathcal{F}_N,Q}$ and (6.1.24), we have

$$
|II| \lesssim \sum_{I\in\mathcal{W}_N^\Sigma} u(X_I)^2 \omega(Q_I) \lesssim (\delta\lambda)^2\, \omega(Q). \qquad (6.1.80)
$$

Similarly,

$$
|I| \le \sum_{I\in\mathcal{W}_N^\Sigma} \frac{u(X_I)^2}{\ell(I)} \iint_{I^{***}} |\nabla G| dm \lesssim \sum_{I\in\mathcal{W}_N^\Sigma} u(X_I)^2 \omega(Q_I) \lesssim (\delta\lambda)^2\, \omega(Q). \qquad (6.1.81)
$$

We finish the proof of (6.1.75) by combining (6.1.78), (6.1.80) and (6.1.81).

Finally we combine (6.1.67) and (6.1.75), and get

$$\lambda^2 \omega(E) \lesssim (\delta\lambda)^2\, \omega(Q). \tag{6.1.82}$$

And thus

$$\omega(E) \le C\delta^2 \omega(Q). \tag{6.1.83}$$

This finishes the proof of the good-λ inequality for ω.

We will also need the following auxiliary fact:

Lemma 6.1.84. *For any apertures $0 < \alpha < \alpha'$ and any function $u \in W_r(\Omega)$, let $S u$ and $\widetilde{S} u$ denote the square function with aperture α and α' respectively. Suppose $\widetilde{S} u < \infty$ for σ-almost every $q \in \partial\Omega$, then the set $\{q \in \partial\Omega : S u(q) > \lambda\}$ is open for every $\lambda > 0$.*

The proof is similar in spirit to that of Lemma 4.6 in [MPT].

Proof. If $q \in \partial\Omega$ is such that $S' u(q) > \lambda$, then there exists $\eta > 0$ so that

$$\iint_{\Gamma^\alpha(q)\setminus B(q,\eta)} |\nabla u|^2 \delta(X)^{1-d} dm(X) > \left(\frac{S u(q) + \lambda}{2}\right)^2.$$

We claim that there exists $\epsilon > 0$ such that for any $p \in \Delta(q, \epsilon\eta)$, we have

$$\iint_{\Gamma^\alpha(p)\setminus B(p,\eta)} |\nabla u|^2 \delta(X)^{1-d} dm(X) > \lambda^2, \tag{6.1.85}$$

and therefore $S u(p) > \lambda$.

We observe that

$$\begin{aligned}&\left| \iint_{\Gamma^\alpha(q)\setminus B(q,\eta)} |\nabla u|^2 \delta(X)^{1-d} dm(X) - \iint_{\Gamma^\alpha(p)\setminus B(p,\eta)} |\nabla u|^2 \delta(X)^{1-d} dm(X) \right| \\ &\qquad \le \iint_D |\nabla u|^2 \delta(X)^{1-d} dm(X),\end{aligned} \tag{6.1.86}$$

where $D = (\Gamma^\alpha(q) \setminus B(q,\eta)) \triangle (\Gamma^\alpha(p) \setminus B(p,\eta))$ is the set difference. It suffices to show that the integral $\iint_D |\nabla u|^2 \delta(X)^{1-d} dm(X)$ is sufficiently small, if we choose ϵ sufficiently small.

Suppose that $X \in \Gamma^\alpha(q) \setminus B(q,\eta)$, then $|X - q| < (1+\alpha)\delta(X)$ and $|X - q| \ge \eta$. Thus $\delta(X) > \frac{\eta}{1+\alpha}$. If moreover $X \notin \Gamma^\alpha(p)\setminus B(p,\eta)$ and $p \in B(q, \epsilon\eta)$, then $|X-q| > (1+\alpha)(1-\epsilon)\delta(X)$. By symmetry, we need to study sets of the form

$$V_q = \{X \in \Omega : |X - q| \ge \eta,\ (1+\alpha)(1-\epsilon)\delta(X) < |X - q| < (1+\alpha)\delta(X)\},$$

$$V_p = \{X \in \Omega : |X - p| \ge \eta,\ (1+\alpha)(1-\epsilon)\delta(X) < |X - p| < (1+\alpha)\delta(X)\}.$$

Without loss of generality we may assume $S'u(q) < \infty$. If not, by the assumption that $S'u < \infty$ almost everywhere, we can always find $q' \in \Delta(q, \epsilon\eta/2)$ such that $S'u(q') < \infty$, and in particular $p \in \Delta(q, \epsilon\eta) \subset \Delta(q', 2\epsilon\eta)$. In this case we just replace q by q', and ϵ by 2ϵ. Moreover, if $\epsilon < 1/4$, we have that

$$V_q \cup V_p \subset V_\epsilon := \{X \in \Omega : |X - q| \geq \frac{\eta}{2},\ (1+\alpha)(1-\epsilon)^2\delta(X) < |X - q| < (1+\alpha)\frac{1-\epsilon}{1-2\epsilon}\delta(X)\}.$$

Note that for given $\alpha' > \alpha$, by choosing ϵ sufficiently small we can guarantee $(1+\alpha)\frac{1-\epsilon}{1-2\epsilon} \leq 1 + \alpha'$. Thus $V_\epsilon \subset \Gamma^{\alpha'}(q) \setminus B\left(q, \frac{\eta}{2}\right) =: V_0$, and as ϵ tends to zero, the set V_ϵ decreases to an empty set. Moreover,

$$\iint_{V_0} |\nabla u|^2 \delta(X)^{1-d} dm(X) \leq \iint_{\Gamma^{\alpha'}(q)} |\nabla u|^2 \delta(X)^{1-d} dm(X) = |S'u(q)|^2 < \infty,$$

hence by the continuity of measure from above, we deduce that

$$\iint_{V_\epsilon} |\nabla u|^2 \delta(X)^{1-d} dm(X) \searrow 0.$$

In particular, by choosing ϵ sufficiently small, we can guarantee

$$\iint_D |\nabla u|^2 \delta(X)^{1-d} dm(X) \leq \iint_{V_\epsilon} |\nabla u|^2 \delta(X)^{1-d} dm(X) < \left(\frac{Su(q) + \lambda}{2}\right)^2 - \lambda^2 \tag{6.1.87}$$

Combining (6.1.87) with (6.1.86), we conclude the proof of the claim (6.1.85).

Now we set out to complete the

Proof of Theorem 1.3.13 We first prove the theorem assuming that $\|S'u\|_{L^p(\sigma)}$ is finite. Under this assumption, we have that $\|S''u\|_{L^p(\sigma)} \sim \|S'u\|_{L^p(\sigma)} \sim \|Su\|_{L^p(\sigma)}$. For reference, see Proposition 4 of [CMS]. (The stated proof in [CMS] is for the upper half plane, but the argument goes through for Ahlfors regular sets of higher codimension.) Therefore by a standard argument, the proof of (1.3.14) can be reduced to the following good-λ inequality: For any $\epsilon > 0$ sufficiently small, we can find $\delta = \delta(\epsilon) > 0$ such that for all $\lambda > 0$,

$$\sigma\left(\{q \in \partial\Omega : Su(q) > 2\lambda, Nu(q) \leq \delta\lambda\}\right) \leq \epsilon\sigma\left(\left\{q \in \partial\Omega : S'u(q) > \lambda\right\}\right), \tag{6.1.88}$$

and $\delta \to 0$ as $\epsilon \to 0$. If $\{q \in \partial\Omega : S'u(q) > \lambda\}$ is empty, (6.1.88) is trivial, so we assume the set is not empty. We apply Lemma 6.1.84 with apertures $0 < \alpha_1 < \beta$. Since $\|S''u\|_{L^p(\sigma)} \sim \|S'u\|_{L^p(\sigma)} < \infty$, in particular $S''u(q) < \infty$ almost everywhere. Therefore $\{q \in \partial\Omega : S'u(q) > \lambda\}$ is open. We also remark that the set $\{q \in \partial\Omega : S'u(q) > \lambda\}$ has finite σ-measure, and moreover

$$\sigma\left(\{q \in \partial\Omega : S'u(q) > \lambda\}\right) \leq \frac{1}{\lambda^p}\int_{S'u(q)>\lambda} |S'u|^p d\sigma \leq \frac{\|S'u\|^p_{L^p(\sigma)}}{\lambda^p} < \infty. \tag{6.1.89}$$

In particular, for any dyadic cube $Q \in \mathbb{D}$ completely contained in $\{q \in \partial\Omega : S'u(q) > \lambda\}$

$$\ell(Q)^d \sim \sigma(Q) \leq \sigma\left(\{q \in \partial\Omega : S'u(q) > \lambda\}\right) \leq \frac{\|S'u\|_{L^p(\sigma)}}{\lambda^p}, \tag{6.1.90}$$

so its length has a uniform upper bound (albeit depending on the value of λ). Recall that $\ell(Q) \sim 2^{-k(Q)}$, and suppose $k_0 \in \mathbb{Z}$ is such that

$$2^{-k_0 d} \gtrsim \frac{\|S'u\|_{L^p(\sigma)}}{\lambda^p}, \tag{6.1.91}$$

with a sufficiently large implicit constant. Then by (6.1.90), any cube Q_0 in $\mathbb{D}_{k_0}$ can not be completely contained in $\{q \in \partial\Omega : S'u(q) > \lambda\}$.

We run a stopping time procedure as follows: For each $Q_0 \in \mathbb{D}_{k_0}$, we traverse all its descendants, and stop whenever we find a cube $Q \in \mathbb{D}_{Q_0}$ such that $S'u(q) > \lambda$ for all $q \in Q$. Let $\mathcal{F}_1 = \{Q_l\}$ be the collection of all stopping cubes in $\bigcup_{Q_0 \in \mathbb{D}_{k_0}} \mathbb{D}_{Q_0}$. Similar to the proof of (6.1.60), we can show that they form a partition:

$$\{q \in \partial\Omega : S'u(q) > \lambda\} = \bigcup_{Q_l \in \mathcal{F}_1} Q_l. \tag{6.1.92}$$

Note that the assumption $Su(q) > 2\lambda$ clearly implies $S'u(q) > \lambda$, namely

$$\{q \in \partial\Omega : Su(q) > 2\lambda\} \subset \{q \in \partial\Omega : S'u(q) > \lambda\} = \bigcup_{Q_l \in \mathcal{F}_1} Q_l.$$

Therefore to prove (6.1.88), it suffices to localize and show that

$$\sigma\left(\{q \in Q : Su(q) > 2\lambda, Nu(q) \leq \delta\lambda\}\right) \leq \epsilon\sigma(Q) \qquad \text{for any } Q = Q_l \in \mathcal{F}_1. \tag{6.1.93}$$

Recall that by (6.1.3), every $Q \in \mathbb{D}$ is contained in a surface ball $\Delta(x_Q, C_2 r_Q)$. Let X'_Q denote a corkscrew point for $B(x_Q, C_2 r_Q)$. Recall Definition 2.1.14 of $\omega \in A_\infty(\sigma)$ and Remark 2.1.16 (ii) right afterwards. Assuming $\omega \in A_\infty(\sigma)$, then to prove (6.1.93) it suffices to show

$$\omega^{X'_Q}\left(\{q \in Q : Su(q) > 2\lambda, Nu(q) \leq \delta\lambda\}\right) \leq C(\delta)\omega^{X'_Q}(Q), \tag{6.1.94}$$

with a constant $C(\delta)$ independent of Q and λ, and that $C(\delta) \to 0$ as $\delta \to 0$. Recall that for any collection $\mathcal{F}$ of dyadic cubes, there is a constant C_3 such that $\Omega^{***}_{\mathcal{F},Q} \subset B(x_Q, C_3\ell(Q)) \cap \Omega$. Let X_Q be a corkscrew point for $B(x_Q, 2C_3 M\ell(Q))$, then

$$|X_Q - x_Q| \geq \delta(X_Q) \geq 2C_3\ell(Q). \tag{6.1.95}$$

Thus $X_Q \notin B(x_Q, 2C_3\ell(Q))$, and in particular $X_Q \notin \overline{\Omega^{***}_{\mathcal{F},Q}}$. Moreover, there is a Harnack chain of finite length (depending only on M, C_2 and C_3) connecting X_Q to X'_Q; in particular the harmonic measures $\omega^{X_Q}(E) \sim \omega^{X'_Q}(E)$ for any Borel set $E \subset Q$. Therefore the proof of (6.1.94) is equivalent to the proof of

$$\omega^{X_Q}\left(\{q \in Q : Su(q) > 2\lambda, Nu(q) \leq \delta\lambda\}\right) \leq C(\delta)\omega^{X_Q}(Q)\,. \tag{6.1.96}$$

Recall that $Q = Q_l \in \mathcal{F}_1$ is a maximal cube with respect to the stopping criterion $\{S'u(q) > \lambda\}$. By maximality the parent of Q, denoted by $\widetilde{Q}$, contains at least one point $q_1 \notin \{q \in \partial\Omega : S'u(q) > \lambda\}$, that is, $S'u(q_1) \leq \lambda$. For any $q \in Q$ we have

$$|q_1 - q| \leq \operatorname{diam} \widetilde{Q} \leq A_1 2^{-k(\widetilde{Q})} = A_1 2^{-(k(Q)-1)} \leq \frac{A_1}{a_0} \operatorname{diam} Q. \tag{6.1.97}$$

Therefore for any maximal cube, we may use Proposition 6.1.58, with constant $C_2 = A_1/a_0$, to conclude the desired estimate (6.1.96).

All the above arguments show that if we know a priori $\|S'u\|_{L^p(\sigma)}$ is finite, we can prove $\|Su\|_{L^p(\sigma)} \lesssim \|Nu\|_{L^p(\sigma)}$. If we do not have this a priori information, then for κ sufficiently small we let

$$\mathbb{D}_\kappa = \{Q \in \mathbb{D} : \kappa \leq \ell(Q) \leq 1/\kappa\}\,, \tag{6.1.98}$$

$$\Omega_\kappa = \bigcup_{Q \in \mathbb{D}_\kappa} U_Q, \quad \Omega_\kappa^* = \bigcup_{Q \in \mathbb{D}_\kappa} U_Q^*, \quad \Omega_\kappa^{**} = \bigcup_{Q \in \mathbb{D}_\kappa} U_Q^{**} \quad \text{etc.} \tag{6.1.99}$$

and define the κ-approximate non-tangential cones as

$$\Gamma_\kappa^\alpha(q) = \Gamma^\alpha(q) \cap \Omega_\kappa, \quad \Gamma_\kappa^{\alpha_1} = \Gamma^{\alpha_1}(q) \cap \Omega_\kappa, \quad \Gamma_\kappa^\beta(q) = \Gamma^\beta(q) \cap \Omega_\kappa^{***},$$

define the κ-approximate *dyadic* non-tangential cones as

$$\Gamma_{d,\kappa}(q) = \Gamma_d(q) \cap \Omega_\kappa = \bigcup_{Q \in \mathbb{D}^\kappa : Q \ni q} U_Q, \quad \widehat{\Gamma}_{d,\kappa}(q) = \widehat{\Gamma}_d(q) \cap \Omega_\kappa^{***}.$$

In this regime we have the following inclusions analogous to (6.1.19) and (6.1.20):

$$\Gamma_\kappa^\alpha(q) \subset \Gamma_{d,\kappa}(q) \subset \Gamma_\kappa^{\alpha_1}(q), \quad \widehat{\Gamma}_{d,\kappa}(q) \subset \Gamma_\kappa^\beta(q). \tag{6.1.100}$$

Moreover, the κ-approximate local non-tangential cones

$$\Gamma_{d,\kappa}^Q(q) = \Gamma_d^Q(q) \cap \Omega_\kappa = \bigcup_{Q' \in \mathbb{D}_Q \cap \mathbb{D}^\kappa : Q' \ni q} U_{Q'}, \quad \widehat{\Gamma}_{d,\kappa}^Q(q) = \widehat{\Gamma}_d^Q \cap \Omega_\kappa^{***}$$

satisfy the following inclusions analogous to (6.1.21):

$$\bigcup_{q \in F} \Gamma_{d,\kappa}^Q(q) \subset \Omega_{\mathcal{F},Q} \cap \Omega_\kappa \subset \overline{\Omega_{\mathcal{F},Q} \cap \Omega_\kappa} \subset \Omega_{\mathcal{F},Q}^{***} \cap \Omega_\kappa^{***} \subset \bigcup_{q \in F} \widehat{\Gamma}_{d,\kappa}^Q(q),$$

for any dyadic cube Q and collection of maximal cubes $\Gamma \subset \mathbb{D}_Q \setminus \{Q\}$, under the assumption that $F = Q \setminus \bigcup_{Q_j \in \mathcal{F}} Q_j$ is not empty. We then define the κ-approximate square functions $S_\kappa u, S'_\kappa u$ and non-tangential maximal function $N_\kappa u$ accordingly, as integrals defined on the κ-approximate non-tangential cones instead of standard non-tangential cones. Since $N_\kappa u(q) \leq Nu(q)$ for all $q \in \partial\Omega$, we have $\|N_\kappa u\|_{L^p(\sigma)} \leq \|Nu\|_{L^p(\sigma)} < \infty$. By the interior Cacciopoli inequality (2.4.25) and that $\beta > \alpha_1 > \alpha$, we have

$$S_\kappa u(q) \leq S'_\kappa u(q) \lesssim C(\kappa) N_\kappa u(q),$$

and thus

$$\|S'_\kappa u\|_{L^p(\sigma)} \lesssim C(\kappa)\|N_\kappa u\|_{L^p(\sigma)} \leq C(\kappa)\|Nu\|_{L^p(\sigma)} < \infty. \tag{6.1.101}$$

We can not let κ go to zero in (6.1.101) since the upper bound in the right hand side depends on κ (in fact $C(\kappa) \to \infty$ as $\kappa \to 0$). However, since $\|S'_\kappa u\|_{L^p(\sigma)}$ is finite, we can apply the previous arguments and prove that $\|S_\kappa u\|_{L^p(\sigma)} \lesssim \|N_\kappa u\|_{L^p(\sigma)}$, with a constant independent of κ. Hence

$$\|S_\kappa u\|_{L^p(\sigma)} \lesssim \|N_\kappa u\|_{L^p(\sigma)} \leq C\|Nu\|_{L^p(\sigma)}$$

with a constant C independent of κ. Therefore we can safely let κ go to zero and conclude that

$$\|Su\|_{L^p(\sigma)} = \limsup_{\kappa\to 0} \|S_\kappa u\|_{L^p(\sigma)} \leq C\|Nu\|_{L^p(\sigma)}.$$

This finishes the proof of Theorem 1.3.13.

6.2 $\omega \in A_\infty(\sigma)$ is equivalent to BMO-solvability

6.2.1 From $\omega \in A_\infty(\sigma)$ to L^p-solvability

Theorem 6.2.1. *Assume $\omega \in A_\infty(\sigma)$, then there exist some $p_0 \in (1, \infty)$ such that the elliptic problem (D) is L^p−solvable for all $p \in (p_0, \infty)$, in the sense that there exists a universal constant $C > 0$ such that for any $f \in C_0^0(\Gamma)$ and any Borel set $E \subset \Gamma$, the solution $u(X) = \int_E f d\omega^X$ satisfies the estimate $\|Nu\|_{L^p(\sigma)} \leq C\|f\chi_E\|_{L^p(\sigma)}$.*

Remark 6.2.2. For a bounded set E, it suffices to assume that $f \in C_b(\Gamma)$.

Proof. We first treat the case when $E = \Gamma$. Let $q \in \partial\Omega$ and denote for any $p > 1$

$$\mathcal{M}_p f(q) = \sup_{\Delta \ni q} \left(\fint_\Delta |f|^p d\sigma \right)^{\frac{1}{p}} < \infty. \tag{6.2.3}$$

We claim

$$|u(X)| \leq C\mathcal{M}_p f(q) \quad \text{for any } X \in \Gamma(q). \tag{6.2.4}$$

Hence $Nu(q) \leq C\mathcal{M}_p f(q)$, and thus by the L^p-boundedness ($p > 1$) of Hardy-Littlewood maximal function (see [CW] for spaces of homogeneous type and [St1])

$$\|Nu\|_{L^p(\sigma)} \leq C\|\mathcal{M}f\|_{L^p(\sigma)} \lesssim \|f\|_{L^p(\sigma)}.$$

In fact, let $X \in \Gamma(q)$ be fixed and $\Delta = \Delta(q, (1+\alpha)\delta(X))$. For $j \in \mathbb{N}$ let $\Delta_j = 2^j\Delta$, and set $\Delta_{-1} = \emptyset$. We have

$$u(X) = \int f d\omega^X = \sum_{j=0}^{\infty} \int_{\Delta_j \setminus \Delta_{j-1}} f d\omega^X. \tag{6.2.5}$$

For each $j \in \mathbb{N}$ let A_j denote a corkscrew point for Δ_j. Recall Definition 2.1.14 of $\omega \in A_\infty(\sigma)$ and the discussion after that, in particular (2.1.17) and (2.1.18). We have that for each j, the Radon-Nikodym derivative

$$k^{A_j}(q') = \frac{d\omega^{A_j}}{d\sigma}(q') = \lim_{\Delta' \to q'} \frac{\omega^{A_j}(\Delta')}{\sigma(\Delta')}$$

satisfies a reverse Hölder inequality

$$\left(⨍_{\Delta_j} \left|k^{A_j}\right|^r d\sigma \right)^{\frac{1}{r}} \leq C ⨍_{\Delta_j} k^{A_j} d\sigma \tag{6.2.6}$$

for all $r \in (1, r_0)$, with uniform constants $r_0 > 1$ and $C > 0$. For any $j \geq 2$ and any surface ball $\Delta' \subset \Delta_j \setminus \Delta_{j-1}$, by the Hölder regularity of solutions near the boundary (see Lemma 2.4.34), we have

$$\omega^X(\Delta') \lesssim 2^{-j\beta}\omega^{A_{j-2}}(\Delta') \sim 2^{-j\beta}\omega^{A_j}(\Delta'). \tag{6.2.7}$$

Hence for any $q' \in \Delta_j \setminus \Delta_{j-1}$,

$$k^X(q') = \lim_{\Delta' \to q'} \frac{\omega^X(\Delta')}{\sigma(\Delta')} = \lim_{\substack{\Delta' \to q' \\ \Delta' \subset \Delta_j \setminus \Delta_{j-1}}} \frac{\omega^X(\Delta')}{\sigma(\Delta')} \lesssim 2^{-j\beta} \lim_{\substack{\Delta' \ni q' \\ \Delta' \subset \Delta_j \setminus \Delta_{j-1}}} \frac{\omega^{A_j}(\Delta')}{\sigma(\Delta')} = 2^{-j\beta} k^{A_j}(q'). \tag{6.2.8}$$

Therefore by (6.2.6), (6.2.8), and Hölder inequality for conjugates $1/p + 1/r = 1$ with $r \in (1, r_0)$, we obtain

$$\begin{aligned} |u(X)| &\leq \sum_{j=0}^{\infty} \int_{\Delta_j \setminus \Delta_{j-1}} |f k^X| d\sigma \lesssim \sum_{j=0}^{\infty} 2^{-j\beta} \int_{\Delta_j} |f| k^{A_j} d\sigma \\ &\leq \sum_{j=0}^{\infty} 2^{-j\beta} \sigma(\Delta_j) \left(⨍_{\Delta_j} |f|^p d\sigma \right)^{\frac{1}{p}} \left(⨍_{\Delta_j} |k^{A_j}|^r d\sigma \right)^{\frac{1}{r}} \\ &\lesssim \sum_{j=0}^{\infty} 2^{-j\beta} \sigma(\Delta_j) \left(⨍_{\Delta_j} |f|^p d\sigma \right)^{\frac{1}{p}} \left(⨍_{\Delta_j} k^{A_j} d\sigma \right) \end{aligned}$$

$$\leq \sum_{j=0}^{\infty} 2^{-j\beta} \mathcal{M}_p f(q) \omega^{A_j}(\Delta_j)$$

$$\lesssim \mathcal{M}_p f(q), \tag{6.2.9}$$

thus we finish proving the claim (6.2.4) for any $p \in (p_0, \infty)$, where p_0 is the conjugate of r_0. Note that we never use the continuity or compact support of f, and replacing f by $f\chi_E$ we can repeat the same argument with no change. The assumption that E is bounded or f has compact support guarantees we still have a priori finite integrability in (6.2.3).

6.2.2 Proof of the BMO-solvability

Theorem 6.2.10. *Assume that $\omega \in A_\infty(\sigma)$. For any $f \in C_0^0(\Gamma)$, let $u = Uf \in W_r(\Omega)$ be a solution to $Lu = 0$ given by Lemmas 2.4.38 and 2.4.40. Then $|\nabla u|^2 \delta(X) dm(X)$ is a Carleson measure, and moreover*

$$\sup_{\Delta \subset \partial\Omega} \frac{1}{\sigma(\Delta)} \iint_{T(\Delta)} |\nabla u|^2 \delta(X) dm(X) \leq C \|f\|_{BMO(\sigma)}^2. \tag{6.2.11}$$

Proof. Fix an arbitrary surface ball $\Delta = \Delta(q_0, r)$. Let $\alpha > 0$. Denote the constant $c = \max\{\alpha + 2, 12\}$ and let $\widetilde{\Delta} = c\Delta = \Delta(q_0, cr)$ be a concentric dilation. We denote the average $f_{\widetilde{\Delta}} = \fint_{\widetilde{\Delta}} f d\sigma$. Let

$$f_1 = (f - f_{\widetilde{\Delta}})\chi_{\widetilde{\Delta}}, \quad f_2 = (f - f_{\widetilde{\Delta}})\chi_{\partial\Omega \setminus \widetilde{\Delta}}, \quad f_3 = f_{\widetilde{\Delta}},$$

and for any $X \in \Omega$ let

$$u_1(X) = \int_{\partial\Omega} f_1 d\omega^X = \int_{\widetilde{\Delta}} \left(f - f_{\widetilde{\Delta}}\right) d\omega^X,$$

$$u_2(X) = \int_{\partial\Omega} f_2 d\omega^X = \int_{\partial\Omega \setminus \widetilde{\Delta}} \left(f - f_{\widetilde{\Delta}}\right) d\omega^X = \int_{\partial\Omega \setminus \widetilde{\Delta}} f d\omega^X - f_{\widetilde{\Delta}} \omega^X(\partial\Omega \setminus \widetilde{\Delta}),$$

$$u_3 \equiv f_{\widetilde{\Delta}}.$$

By Lemmas 2.4.38, 2.4.40, 2.4.43 and 2.4.44, they are solutions to L, and u_1, u_2 can be continuously extended to $\partial\Omega \setminus \widetilde{\Delta}$ and $\widetilde{\Delta}$, respectively. Moreover

$$(u_1 + u_2 + u_3)(X) = \int_{\partial\Omega} f d\omega^X = Uf(X) = u(X).$$

Clearly the Carleson measure of the constant function u_3 is trivial.

Apply Theorem 6.2.1 to f_1 and u_1 we get $\|Nu_1\|_{L^p(\sigma)} \leq C\|f_1\|_{L^p(\sigma)} < \infty$. Combined with Theorem 1.3.13, we get

$$\|Su_1\|_{L^p(\sigma)} \lesssim \|Nu_1\|_{L^p(\sigma)} \lesssim \|f_1\|_{L^p(\sigma)} = \left(\int_{\widetilde{\Delta}} |f - f_{\widetilde{\Delta}}|^p d\sigma \right)^{1/p} \tag{6.2.12}$$

for any $p \in (p_0, \infty)$. By (3.0.2) and (3.0.3)

$$\iint_{T(\Delta)} |\nabla u_1|^2 \delta(X) dm(X) \leq C \int_{(\alpha+2)\Delta} |S_{(\alpha+1)r} u_1|^2 d\sigma$$

Recall that $\widetilde{\Delta} = c\Delta \supset (\alpha + 2)\Delta$, thus

$$\begin{aligned}
\iint_{T(\Delta)} |\nabla u_1|^2 \delta(X) dm(X) &\leq C \int_{\widetilde{\Delta}} |S_{(\alpha+1)r} u_1|^2 d\sigma \\
&\leq C\sigma(\widetilde{\Delta})^{1-\frac{2}{p}} \left(\int_{\widetilde{\Delta}} |S u_1|^p d\sigma \right)^{\frac{2}{p}} \\
&\leq C\sigma(\widetilde{\Delta})^{1-\frac{2}{p}} \|S u_1\|^2_{L^p(\sigma)}, \qquad (6.2.13)
\end{aligned}$$

for any $p > \max\{2, p_0\}$. Combining (6.2.13) and (6.2.12) we get

$$\iint_{T(\Delta)} |\nabla u_1|^2 \delta(X) dm(X) \leq C\sigma(\Delta) \|f\|^2_{BMO(\sigma)} < \infty. \qquad (6.2.14)$$

Turning to the estimate for u_2, let $\{I_k\} \subset \mathcal{W}$ be a collection of dyadic Whitney boxes that intersect of $T(\Delta)$ (recall the properties of Whitney decomposition $\mathcal{W}$ in (6.1.4)). On each Whitney box I_k, we have by the interior Cacciopoli inequality (2.4.25)

$$\begin{aligned}
\iint_{I_k} |\nabla u_2|^2 \delta(X) dm(X) &\lesssim \ell(I_k) \iint_{I_k} |\nabla u_2|^2 dm(X) \\
&\lesssim \ell(I_k) \cdot \frac{1}{\ell(I_k)^2} \iint_{I_k^*} |u_2(X)|^2 dm(X) \\
&\lesssim \iint_{I_k^*} \frac{|u_2(X)|^2}{\delta(X)} dm(X),
\end{aligned}$$

Recall $I_k^* = (1 + \theta)I_k$ is the dilation of I_k satisfying (6.1.9). Then summing up we get

$$\begin{aligned}
\iint_{T(\Delta)} |\nabla u_2|^2 \delta(X) dm(X) &\lesssim \sum_k \iint_{I_k^*} \frac{|u_2(X)|^2}{\delta(X)} dm(X) \\
&\lesssim \iint_{T(\frac{3}{2}\Delta)} \frac{|u_2(X)|^2}{\delta(X)} dm(X). \qquad (6.2.15)
\end{aligned}$$

In the last line we use the finite overlap of $\{I_k^*\}$, and the fact that by taking θ sufficiently small, we can ensure that $I_k^* \subset T(\frac{3}{2}\Delta)$ for all I_k intersects $T(\Delta)$. Recall that $\frac{3}{2}\Delta = \Delta(q_0, \frac{3}{2}r)$ and $T(\frac{3}{2}\Delta)$ denotes $B(q_0, \frac{3}{2}r) \cap \Omega$.

Let $f_2^{\pm}$ denote the positive and negative part of f_2, and let $u_2^{\pm} = \int_{\partial\Omega \setminus \widetilde{\Delta}} f_2^{\pm} d\omega^X \geq 0$. There is a technical issue that $f_2^{\pm} \notin C_0^0(\Gamma)$, however by splitting $u_2^{\pm}$ as follows,

$$u_2^+(X) = \int_{\{f \geq f_{\widetilde{\Delta}}\} \setminus \widetilde{\Delta}} f d\omega^X - f_{\widetilde{\Delta}} \omega^X \left(\{f \geq f_{\widetilde{\Delta}}\} \setminus \widetilde{\Delta} \right),$$

$$u_2^-(X) = -\int_{\{f<f_{\widetilde{\Delta}}\}\setminus\widetilde{\Delta}} f d\omega^X + f_{\widetilde{\Delta}}\omega^X\left(\{f<f_{\widetilde{\Delta}}\}\setminus\widetilde{\Delta}\right),$$

we can confirm by combining Lemmas 2.4.43 and 2.4.44 that $u_2^\pm \in W_r(\Omega)$ are indeed legitimate solutions of L, and they can be continuously extended to $\widetilde{\Delta}$ by zero. By the linearity of integration, we have $u_2 = \int_{\partial\Omega} f_2 d\omega^X = u_2^+ - u_2^-$. Let $v(X) := u_2^+(X) + u_2^-(X)$, again by linearity we have

$$v(X) = \int_{\partial\Omega} |f_2| d\omega^X = \int_{\partial\Omega\setminus\widetilde{\Delta}} |f - f_{\widetilde{\Delta}}| d\omega^X. \tag{6.2.16}$$

Thus $|u_2(X)| \le v(X)$ for all $X \in \Omega$. Moreover by the properties of $u_2^\pm$, we know that $v \in W_r(\Omega)$ is a solution of L, $Tv = 0$ on $\widetilde{\Delta}$ and that $v \in W_r(B(q_0, cr))$. (Recall that $\widetilde{\Delta} = c\Delta = B(q_0, cr) \cap \partial\Omega$.) We claim that

$$v(X) \le C\|f\|_{BMO(\sigma)} \quad \text{for all } X \in T(6\Delta). \tag{6.2.17}$$

By the definition (6.2.16), the function v vanishes on $\widetilde{\Delta}$. Note that $\widetilde{\Delta} \supset 12\Delta$ by the choice of $\widetilde{\Delta}$, $v \in W_r(B(q_0, 12r))$ is a non-negative solution in $T(12\Delta)$ and $Tv \equiv 0$ on 12Δ. Let A be a corkscrew point for $T(12\Delta)$, by the boundary Harnack inequality (2.4.55)

$$v(X) \le Cv(A), \quad \text{for all } X \in T(6\Delta).$$

For any $j \in \mathbb{N}$, let A_j be a corkscrew point for the surface ball $2^j\widetilde{\Delta}$. Similar to (6.2.9), we get

$$\begin{aligned} v(A) &\lesssim \sum_{j=1}^\infty 2^{-j\beta} \int_{2^j\widetilde{\Delta}\setminus 2^{j-1}\widetilde{\Delta}} |f - f_{\widetilde{\Delta}}| k^{A_j} d\sigma \\ &\le \sum_{j=1}^\infty 2^{-j\beta} \left(\fint_{2^j\widetilde{\Delta}} |f - f_{\widetilde{\Delta}}|^p d\sigma\right)^{\frac{1}{p}} \left(\fint_{2^j\widetilde{\Delta}} \left|k^{A_j}\right|^r d\sigma\right)^{\frac{1}{r}} \sigma(2^j\widetilde{\Delta}) \\ &\lesssim \sum_{j=1}^\infty 2^{-j\beta} \|f\|_{BMO(\sigma)} \omega^{A_j}(2^j\widetilde{\Delta}) \\ &\lesssim \|f\|_{BMO(\sigma)}. \end{aligned} \tag{6.2.18}$$

Here p is a conjugate to r. We conclude the proof of (6.2.17).

Next, we show a finer estimate based off (6.2.17), which is

$$v(X) \le C\left(\frac{\delta(X)}{r}\right)^\beta \|f\|_{BMO(\sigma)} \quad \text{for all } X \in T\left(\frac{3}{2}\Delta\right), \tag{6.2.19}$$

where $\beta \in (0, 1]$ is the exponent from Lemma 2.4.34. To this end, for any $X \in T(\frac{3}{2}\Delta)$, let q_X be a boundary point such that $|X - q_X| = \delta(X)$. Note that

$$|X - q_X| = \delta(X) \le |X - q_0| < \frac{3}{2}r,$$

i.e. $X \in B(q_X, 3r/2) \cap \Omega$. Note also

$$|q_X - q_0| \le |q_X - X| + |X - q_0| < \frac{3r}{2} + \frac{3r}{2} = 3r,$$

so $\overline{B(q_X, 3r)} \subset B(q_0, 6r)$. Since $\widetilde{\Delta} \supset 6\Delta \supset \Delta(q_X, 3r)$, $v \in W_r(B(q_X, 3r))$ is a non-negative solution in $B(q_X, 3r) \cap \Omega$ and $Tv \equiv 0$ on $\Delta(q_X, 3r)$. By the boundary Hölder regularity (2.4.35) and the first part of this lemma (6.2.17), we conclude

$$\begin{aligned} v(X) &\lesssim \left(\frac{|X - q_X|}{3r}\right)^{\beta} \left(\frac{1}{m(B(q_X, 3r))} \iint_{B(q_X,3r)\cap\Omega} |v|^2 dm\right)^{\frac{1}{2}} \\ &\lesssim \left(\frac{\delta(X)}{r}\right)^{\beta} \sup_{T(6\Delta)} v \lesssim \left(\frac{\delta(X)}{r}\right)^{\beta} \|f\|_{BMO(\sigma)}. \end{aligned}$$

Combining (6.2.19) and (6.2.15), we get

$$\iint_{T(\Delta)} |\nabla u_2|^2 \delta(X) dm(X) \lesssim \frac{\|f\|^2_{BMO(\sigma)}}{r^{2\beta}} \left(\iint_{T(\frac{3}{2}\Delta)} \delta(X)^{2\beta-1} dm(X)\right). \tag{6.2.20}$$

Since $2\beta - 1 > -1$, we can use Lemma 3.1.20 with exponent $\alpha = 2\beta - 1$ to get

$$\iint_{T(\Delta)} |\nabla u_2|^2 \delta(X) dm(X) \lesssim r^d \|f\|^2_{BMO(\sigma)} \lesssim \sigma(\Delta)\|f\|^2_{BMO(\sigma)}. \tag{6.2.21}$$

Combining (6.2.14) and (6.2.21) finishes the proof.

6.2.3 From BMO-solvability to $\omega \in A_\infty(\sigma)$

In this subsection, we prove the other half of Theorem 1.3.15:

Theorem 6.2.22. *Assume that for any $f \in C_0^0(\Gamma)$, the solution $u = Uf \in W_r(\Omega)$ given by Lemmas 2.4.38 and 2.4.40 satisfies the property that $|\nabla u|^2 \delta(X) dm(X)$ is a Carleson measure with*

$$\sup_{\Delta \subset \partial\Omega} \frac{1}{\sigma(\Delta)} \iint_{T(\Delta)} |\nabla u|^2 \delta(X) dm(X) \le C\|f\|^2_{BMO(\sigma)}. \tag{6.2.23}$$

Then $\omega \in A_\infty(\sigma)$, with the implicit constant depending on d, n, C_0, C_1 and the above constant C.

Let us start with proving the following Lemma.

Lemma 6.2.24. *Suppose the Dirichlet problem (D) is BMO-solvable. Then any non-negative function $f \in C_0^0(\Gamma)$ whose supporte is contained in a surface ball Δ satisfies*

$$\int_{\Delta} f d\omega^A \le C\|f\|_{BMO(\sigma)}. \tag{6.2.25}$$

Here A is a corkscrew point for Δ.

Proof.

Since $f \in C_0^0(\Gamma)$ is a non-negative function, by Lemma 2.4.38 $u = Uf \in W_r(\Omega)$ is a non-negative solution of L. Suppose Δ has radius r. Consider another surface ball $\Delta' = B(q', r) \cap \Gamma$ of the same radius r and which is $2r-$distance away from Δ. Thus in particular, $Tu = 0$ on $3\Delta'$ and that $u \in W_r(B(q', 3r))$, by Lemma 2.4.38 (i) and (iv). Applying the BMO-solvability assumption to $u = Uf$ and the surface ball Δ', we have

$$\iint_{T(\Delta')} |\nabla u|^2 \delta(X) dm(X) \leq C\sigma(\Delta') \|f\|^2_{BMO(\sigma)} \tag{6.2.26}$$

We have shown in (3.0.4) that

$$\iint_{T(\Delta')} |\nabla u|^2 \delta(X) dm(X) \gtrsim \int_{\Delta'/2} |S_{r/2}u|^2 d\sigma, \tag{6.2.27}$$

where $S_{r/2}u$ is the truncated square function of aperture $\overline{\alpha} > \alpha$, whose value is determined in Lemma 3.2.5 and only depends on n, d, C_0, C_1 and α. In order to get a lower bound of the square function $S_{r/2}u$, we decompose the non-tangential cone $\Gamma_{r/2}(q)$ into stripes as in (3.2.3) and use the Poincaré-type inequality proved in Lemma 3.2.5 for surface ball Δ'. Let m_1, m_2 be integers determined in Lemma 3.2.5. We obtain

$$\begin{aligned}
|S_{r/2}u|^2(q) &= \iint_{\Gamma^{\overline{\alpha}}_{r/2}(q)} |\nabla u|^2 \delta(X)^{1-d} dm(X) \\
&\geq \frac{1}{m_1 + m_2} \sum_{j=m_1+1}^{\infty} \iint_{\Gamma^{\overline{\alpha}}_{j-m_1 \to j+m_2}(q)} |\nabla u|^2 \delta(X)^{1-d} dm(X) \\
&\gtrsim \sum_{j=m_1+1}^{\infty} (2^{-j}r)^{1-d} \iint_{\Gamma^{\overline{\alpha}}_{j-m_1 \to j+m_2}(q)} |\nabla u|^2 dm(X) \\
&\gtrsim \sum_{j=m_1+1}^{\infty} (2^{-j}r)^{1-d} \cdot (2^{-j}r)^{-2} \iint_{\Gamma^{\alpha}_{j}(q)} u^2 dm(X) \\
&\gtrsim \sum_{j=m_1+1}^{\infty} u^2(A_j),
\end{aligned}$$

where $A_j \in \Gamma_j(q)$ is a corkscrew point at the scale $2^{-j}r$. In the last inequality, we use the interior corkscrew condition, as each stripe of cone $\Gamma_j(q)$ contains a ball of radius comparable to $2^{-j-1}r$ (as long as α is chosen to be big, say $\alpha > 2M$, where M is the corkscrew constant). Moreover,

$$\sum_{j=m_1+1}^{\infty} u^2(A_j) \geq u^2(A_{m_1}) \gtrsim u^2(A_1). \tag{6.2.28}$$

Recall for any $q \in \Delta'$, the point $A_1 = A_1(q)$ is a corkscrew point of $B(q, 2^{-1}r)$. Let A' be the corkscrew point for $T(\Delta'/2)$, by Lemma 2.4.3 and Harnack inequality, $u(A') \approx u(A_1)$.

Therefore
$$|S_{r/2}u|^2(q) \gtrsim u^2(A_1) \gtrsim u^2(A'), \quad \text{for any } q \in \Delta'.$$
Combining this with (6.2.26) and (6.2.27), we get
$$\sigma(\Delta')\|f\|^2_{BMO(\sigma)} \gtrsim \int_{\Delta'/2} |S_{r/2}u|^2 d\sigma \gtrsim \sigma(\Delta'/2)u^2(A') \gtrsim \sigma(\Delta')u^2(A'),$$
and thus
$$u(A') \lesssim \|f\|_{BMO(\sigma)}. \tag{6.2.29}$$
Let A be a corkscrew point for Δ. Since Δ and Δ' have the same radius r and they are $2r-$distance apart, we have $u(A) \sim u(A')$. By assumption f is supported on Δ, hence
$$u(A) = \int_\Delta f d\omega^A. \tag{6.2.30}$$
The lemma follows by combining (6.2.29) and (6.2.30).

With that at hand, we pass to the

Proof.[Proof of Theorem 6.2.22]

By the change of pole formula in Lemma 2.4.87 and Harnack inequality, to prove $\omega \in A_\infty(\sigma)$ and in particular (2.1.15), it suffices to show: For any $\epsilon > 0$ fixed, we can find $\eta = \eta(\epsilon)$, such that for any Borel set $E \subset \Delta$,
$$\frac{\sigma(E)}{\sigma(\Delta)} < \eta \quad \text{implies} \quad \frac{\omega^A(E)}{\omega^A(\Delta)} < \epsilon. \tag{6.2.31}$$
Here Δ is a surface ball and A is a corkscrew point for Δ. In fact, since σ and ω are regular Borel measures, we may assume E is an open subset of Δ.

Recall from Lemma 2.4.80 that
$$\omega^A(\Delta) \geq C^{-1}$$
for some $C > 1$. Thus to show $\omega^A(E)/\omega^A(\Delta) < \epsilon$ it suffices to show $\omega^A(E) < C^{-1}\epsilon$. Let $\delta > 0$ be a small constant to be determined later, we define a function
$$f(x) = \max\left\{0, 1 + \delta \log M_\sigma \chi_E(x)\right\} \tag{6.2.32}$$
where M_σ is the Hardy-Littlewood maximal function with respect to σ. Similar to Section 5.3 of [Zh], f satisfies

- $0 \leq f \leq 1$, and $f \equiv 1$ on the open set E;
- $\|f\|_{BMO(\sigma)} \leq A\delta$, where A is a constant independent of E;

- If

$$\text{(6.2.33)} \qquad \frac{\sigma(E)}{\sigma(\Delta)} < \eta(\delta) \sim e^{-1/\delta},$$

  then f is supported in 2Δ.

Next we use a mollification argument to approximate f by continuous functions. Let φ be a radially symmetric smooth function on $\mathbb{R}^n$ such that $\varphi = 1$ on $B_{1/2}$, $\operatorname{supp}\varphi \subset B_1$ and $0 \leq \varphi \leq 1$. Let

$$\text{(6.2.34)} \qquad \varphi_\epsilon(z) = \frac{1}{\epsilon^d}\varphi\left(\frac{z}{\epsilon}\right), \quad f_\epsilon(x) = \frac{\int_{y\in\partial\Omega} f(y)\varphi_\epsilon(x-y)d\sigma(y)}{\int_{y\in\partial\Omega}\varphi_\epsilon(x-y)d\sigma(y)} \text{ for } x \in \partial\Omega.$$

Then these f_ϵ's satisfy the following properties:

- each f_ϵ is continuous, and is supported in 3Δ;
- there is a constant C (independent of ϵ) such that $\|f_\epsilon\|_{BMO(\sigma)} \leq C\|f\|_{BMO(\sigma)}$;
- $f(x) \leq \liminf_{\epsilon\to 0} f_\epsilon(x)$ for all x in their support 3Δ.

The proof of the above properties is a slight modification of Appendix A of [Zh]: here the mollifier $\{\varphi_\epsilon\}$ is an approximation of identity of dimension d, instead of dimension $n-1$. The proof uses standard mollification arguments and the Ahlfors regularity of $\partial\Omega$. Moreover, the proof of the last property also uses the precise definition of f in (6.2.32).

Let A' be a corkscrew point with respect to 3Δ. The last property and Fatou's lemma imply

$$\text{(6.2.35)} \qquad \int_{3\Delta} f(x)d\omega^{A'}(x) \leq \int_{3\Delta} \liminf_{\epsilon\to 0} f_\epsilon(x)d\omega^{A'}(x) \leq \liminf_{\epsilon\to 0}\int_{3\Delta} f_\epsilon(x)d\omega^{A'}(x).$$

Since each f_ϵ is non-negative, continuous and supported on 3Δ, we apply Lemma 6.2.24 and get

$$\text{(6.2.36)} \qquad \int_{3\Delta} f_\epsilon(x)d\omega^{A'}(x) \leq C\|f_\epsilon\|_{BMO(\sigma)} \leq C'\|f\|_{BMO(\sigma)}.$$

Combining (6.2.35) and (6.2.36), we get

$$\int_{3\Delta} f(x)d\omega^{A'}(x) \leq C'\|f\|_{BMO(\sigma)} \leq C''\delta.$$

On the other hand, since $f \geq \chi_E$

$$\int_{3\Delta} f(x)d\omega^{A'}(x) \geq \omega^{A'}(E) \gtrsim \omega^A(E).$$

The last inequality follows from the Harnack inequality and the fact that A, A' are corkscrew points to surface balls $\Delta, 3\Delta$ respectively. Therefore $\omega^A(E) \leq C\delta$ as long as the condition (6.2.33), i.e. $\sigma(E)/\sigma(\Delta) < \eta$ is satisfied. In other words, $\omega \in A_\infty(\sigma)$.

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