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Investigating and Improving Student Understanding of Key Ideas in Quantum Mechanics throughout Instruction

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Abstract

Investigating and Improving Student Understanding of Key Ideas in Quantum Mechanics throughout Instruction

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This dissertation describes research on student understanding of quantum mechanics across multiple levels of instruction. The primary focus has been to identify patterns in student reasoning related to key concepts in quantum mechanics. The specific topics include quantum measurements, time dependence, vector spaces, and angular momentum. The research has spanned a variety of different quantum courses intended for introductory physics students, upper-division physics majors, and graduate students in physics. The results of this research have been used to develop a set of curriculum, *Tutorials in Physics: Quantum Mechanics*, for addressing the most persistent student difficulties. We document both the development of this curriculum and how it has impacted and improved student understanding of quantum mechanics.

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DEDICATION

for Heather, who is always there

Chapter 1
Introduction and Overview

"The lecture was good for getting the mathematical side of QM but the tutorial section was awesome for getting a conceptual understanding of QM."

—Junior-level quantum mechanics student, end-of-quarter survey

Quantum mechanics is a central part of what most physics students are expected to learn throughout both undergraduate and graduate studies. This dissertation describes an investigation of student understanding of multiple topics in quantum mechanics, with a focus on how student understanding changes throughout multiple levels of instruction.

As with most fields in physics, each concept in quantum mechanics builds on previous ones to construct a powerful and descriptive model of real-world phenomena. When examining student understanding of quantum mechanics, it is thus important to identify the extent to which the fundamental ideas are understood. In this dissertation, we examine this question by probing the ability of students to apply concepts when they are first introduced and later when they are applied to more complicated systems. This dissertation shows that many students have difficulty with concepts that are foundational to quantum mechanics, and that such difficulty persists throughout instruction, in some cases even among graduate students.

This chapter provides an introduction and overview to the research on student understanding of quantum mechanics at the University of Washington over the last several years (2011-2015). Section 1.1 gives a brief overview of the prior research that is relevant to the research discussed throughout this dissertation. Section 1.2 describes in detail the context for this research at the

University of Washington, including the student populations and courses. Then, section 1.3 discusses the methodology associated with this investigation. Lastly, section 1.4 provides an overview of the structure of this dissertation as a whole.

1.1 – Prior Research and Curriculum Development

Since the 1970s, there has been a steady growth in research on the teaching and learning of physics, much of it taking place within physics departments.[1] Overall, this research has contributed to a substantial body of knowledge regarding both student understanding of physics and instructional tools and methods that are effective at improving student understanding. This has allowed for the development of curriculum that is well-suited both to addressing the particular difficulties that students have with learning physics and to the ways in which most students are best able to learn.[1] In more recent years, there has been a concerted effort to extend such research and instructional practices to courses beyond the introductory university level and into upper-division courses, including quantum mechanics.

A. Research on Student Understanding of Quantum Mechanics

Systematic research on student understanding of quantum mechanics is relatively new. Before 1999, only the anecdotal evidence of experienced instructors had been published.[2] Much of the work since then has focused on investigations of student understanding at various levels, including high-school, undergraduate, and graduate students.[3-6] Most commonly, each investigation has used long-answer and multiple-choice questions to explore student ideas about a specific concept or set of related concepts.[7-11] Often, individual demonstration interviews have

been used to probe student understanding in detail, as has been done at the introductory level.[12] Several conceptual surveys have been designed and tested to provide both instructors and researchers with standardized assessment tools for quantum mechanics.[13-16] These surveys have helped contribute to a body of knowledge about student understanding of a variety of different fundamental quantum concepts.

The topics that have been touched on by prior research include probability,[7] wave properties,[8] quantum tunneling,[9] time dependence,[10] and energy measurements,[11] along with many others. The specific conclusions of these prior studies are not discussed in this section. Instead, each subsequent chapter of this dissertation summarizes the prior research on those particular topics that are relevant to that chapter. However, some important overarching findings are discussed here.

Several investigations have identified that students (including incoming graduate students in physics) often have conceptual difficulties after traditional lectures on quantum mechanics.[4-6] For example, many students are unable to make qualitative predictions about measurements and time dependence, even for simple quantum systems.[10] It is common for students to give incorrect expressions for the time dependence of quantities such as the wave function, the probability density, and the probabilities of various outcomes of energy measurements.[4-5] Many students also do not appear to understand the effect of measurement on a quantum system, including the time evolution of such a system after a measurement has been made.[11] Overall, these results suggest that traditional undergraduate instruction often leaves students with gaps in their conceptual knowledge of quantum mechanics.

B. Development of Curriculum to Address Student Difficulties with Quantum Mechanics

There have been several efforts to design curriculum using the results of the research mentioned in the previous section.[17-21] These curricula also tend to make use of the large body of research supporting the use of interactive engagement in promoting student learning.[1] Often, computer-based visualization tools have been incorporated into the curriculum. Several prominent examples are discussed in more detail below.

Visual Quantum Mechanics, developed at Kansas State University, uses a set of interactive computer programs to introduce quantum mechanical ideas to students in introductory physics courses, or even in high school physics courses.[17] In *Visual Quantum Mechanics*, students work through guided exercises to explore quantum phenomena such as atomic spectroscopy, wave functions, and quantum tunneling. Although this curriculum is no longer being actively researched, it is one of the earliest examples of a set of interactive instructional activities specifically aimed at quantum ideas.

The University of Colorado – Boulder has developed an extensive set of open-source, online simulations (the PhET simulations) that cover a broad spectrum of topics in science, including quantum mechanics.[18] These simulations are appropriate to students at a wide variety of instructional levels. While the PhET simulations do not necessarily constitute a formal quantum curriculum, they are one example of a large number of different online tools that are freely available both to instructors and to students, such as Open Source Physics[22] and the Quantum Mechanics Visualisation Project.[23]

The QuILTs (Quantum Interactive Learning Tutorials) from the University of Pittsburgh are an example of curriculum intended as supplementary material to traditional lecture content for advanced undergraduate students.[20] They make use of both interactivity and computer-based resources to help improve student understanding of a wide variety of content, including measurement, time dependence, spin, and angular momentum. Preliminary assessments of the QuILTs have suggested that they can be effective.[20,24-26] However, these assessments have not yet demonstrated how broadly such curriculum is capable of affecting student understanding.

The Physics Education Group at UW has been developing and assessing tutorial curriculum for improving student understanding of quantum mechanics for almost two decades.[27] The evolution of the tutorials has followed the iterative process of research, curriculum development, and instruction characteristic of *Tutorials in Introductory Physics*.[28] Through this process, the tutorials have been continuously modified over a period of many years based on exam-based assessments of student understanding after completion of the tutorials.

The initial development of the quantum tutorials is documented in the PhD dissertations of Bradley Ambrose and Andrew Crouse.[29,30] This dissertation describes further development of the tutorials. This development has included three different focuses, each conducted on the basis of ongoing research: (1) continuous modifications to existing tutorials, (2) the development of new tutorials, and (3) the adaptation of tutorials for populations at different levels of instruction. We also conduct an extensive assessment of the curriculum at both UW and at other institutions that have served as pilot sites for the tutorials. The courses that have been involved in this effort are discussed in more detail in the following section.

1.2 – Context for Research: Instruction and Curriculum Development

Much of the research discussed in this dissertation has been carried out over many years (1996-2015) within the context of the physics program at UW. Details of the primary classes that have participated in this investigation are discussed in section 1.2A. In section 1.2B, we describe the research-based curriculum that has been developed to supplement instruction in the various quantum courses at UW. Particular attention is paid to the incorporation and evolution of this curriculum, along with changes to the structure of the courses over time and how these changes have impacted this work. Some data has also been gathered from other universities, and any similarities or differences between these classes and those at UW are discussed in section 1.2C.

A. Quantum Mechanics at the University of Washington

The Physics Department at UW offers three courses for undergraduates that are primarily focused on quantum mechanics: PHYS 225, 324, and 325. Most of the work represented in this dissertation has been conducted within the context of one or more of these three classes. Some research has also been conducted in the calculus-based introductory physics course at UW (PHYS 121, 122, and 123). The classes are discussed below in order by instructional level: introductory physics (section 1.2A1), sophomore-level introductory quantum mechanics (section 1.2A2), and junior-level quantum mechanics (section 1.2A3). In 2013, the sophomore-level course was modified substantially on the basis of the research discussed throughout this dissertation. This new course is discussed in section 1.2A4. Section 1.2A5 discusses several courses at UW that are also somewhat related to our investigation.

1. Introductory physics

The introductory calculus-based physics sequence at UW consists of three quarters: PHYS 121 (mechanics), PHYS 122 (electromagnetism), and PHYS 123 (waves, optics, and selected topics). The course is predominantly taken by students in engineering or science majors, with physics majors comprising less than 5% of the total enrollment. Quantum mechanics is covered for about 1-2 weeks near the end of PHYS 123 for most students.

There is also an honors version of PHYS 123. The honors version spends about five weeks on quantum mechanics.[31] These students therefore cover more topics than those in the non-honors version of the course. These students also typically complete between one and three tutorials (see section 1.2B) related to quantum concepts.

In this dissertation, all results related to quantum mechanics from students at the introductory level are taken from the honors version of the class. Some chapters also discuss student understanding of concepts that are *not* directly related to quantum concepts. In these cases, results are drawn from the non-honors version of the introductory physics sequence.

2. Sophomore-level modern physics/introduction to quantum mechanics

PHYS 225 is intended for sophomores, and is required of all students who graduate with a major or minor in physics. The exact syllabus has varied from year to year and instructor to instructor, but the current version has tended to serve as either an introduction to quantum mechanics or a survey of topics in modern physics. Prerequisites for this course include a full year of introductory physics, two quarters of introductory calculus, one quarter of either differential equations or linear algebra, and one quarter of mathematical methods in physics.

The course consists of three hours of lecture per week in a large lecture hall with class sizes of one hundred students or more. The exact style of the lectures has ranged from traditional to interactive, depending on the instructor. The textbook used for this course (prior to the redesign discussed in section 1.2A4) was by Scherrer.[32]

3. Junior-level quantum mechanics

Together, PHYS 324 and 325 cover the entirety of the widely used textbook by Griffiths.[33] This sequence of courses is usually taken by students in their third or fourth year of study. The students in this course have typically completed PHYS 225 and its prerequisites, along with one additional quarter of differential equations or linear algebra and one additional quarter of mathematical methods in physics. Almost all students in the undergraduate physics program complete PHYS 324. PHYS 325 is not required of any students, but is one of several optional courses. Both courses consist of three hours of (traditional) lecture each week, and the class size has varied between 50 and 90 students in PHYS 324, about half of whom continue to PHYS 325.

Each course also includes a weekly small-group section. In some cases, the small-group section has been focused on problem-solving; in others, research-based curriculum has been administered to the students. This curriculum is discussed in detail in section 1.2B1.

4. Other courses at UW

In a few cases, we have conducted research within the context of other courses at UW. The purpose of this research has typically been to compare student performance on specific questions

across courses with different focuses. The courses that have been involved in this way include a junior-level course in electromagnetism (PHYS 321) and a senior-level course in nuclear/particle physics (PHYS 422).

We have also collected data from graduate students throughout the course of their first-year quantum mechanics course. This has allowed us to assess how incoming graduate students think about quantum physics and to compare the lines of reasoning that they use to those that are most common among undergraduates.

B. Tutorial Curriculum for Quantum Mechanics

The Physics Education Group at UW has been developing curriculum to address known student difficulties with quantum mechanics for many years. This curriculum is known as *Tutorials in Physics: Quantum Mechanics*,[27] and is modeled after the tutorials developed by our group for use in introductory physics.[28] In section 1.2B1, we describe the format and the origin of the tutorials within the context of the junior-level quantum course at UW. Then, in section 1.2B2 we discuss how the sophomore-level course at UW has been changed over the years to incorporate the results of research on student understanding of quantum mechanics, including the incorporation of worksheets based on the tutorials. Lastly, section 1.2B3 assesses student perception of the tutorials.

1. Tutorial curriculum in junior-level quantum mechanics (PHYS 324 and 325)

In addition to the lectures, the PHYS 324 and 325 classes each include small-group discussion sections that meet once per week for 50 minutes. We have developed and

implemented a research-based curriculum, *Tutorials in Physics: Quantum Mechanics*,[27] for use in these discussion sections. These are modeled after the tutorials developed by our group for use in introductory physics.[28]

Classical probability	Addition of angular momentum
Relating classical and quantum mechanics	Identical particles
Treating functions as vectors	**Prelude to perturbation theory**
Representations of wave functions	Time-independent perturbation theory
Probability amplitude	Degenerate perturbation theory
Superposition in quantum mechanics	**The WKB approximation**
Time dependence in quantum mechanics	Time-independent perturbation theory part I: Foundations
Energy measurements	**Time-independent perturbation theory part II: Formalism**
Position, momentum, and energy measurements	**Scattering in quantum mechanics**
Degenerate states	
Angular momentum in quantum mechanics	

Figure 1.1: List of tutorial topics for a junior-level quantum course

The tutorials listed have been included in the first edition of *Tutorials in Physics: Quantum Mechanics*. Bolded topics are those that were first developed as part of the effort documented in this dissertation (all topics listed have been substantially modified as a result of this effort). Three additional topics—*Uncertainty: Classical and quantum*, *Probability current*, and *Reflection and transmission of matter*—were part of the original development of the tutorials, but have not undergone further development or assessment.

During the tutorials, groups of 3-5 students work through 4-6 page worksheets that contain carefully structured sequences of questions to help them build an understanding of physics concepts and address common difficulties. The students are assisted by teaching assistances (TAs) whose role is not to provide students with the correct answers, but rather to ask the students questions in order to give them a chance to build the knowledge for themselves. The exercises are often modeled on certain instructional strategies, such as elicit-confront-resolve,

that past research has shown can improve student understanding of physics concepts.[34] The students also complete homework problems outside of class that have been designed to complement the in-class worksheets. Figure 1.1 gives a list of the topics for which tutorials have been used in both PHYS 324 and PHYS 325. The figure also notes how they have been modified over the years.

Attendance/participation in the tutorial sections contributes to the final course grade, and content from the tutorials is included on each exam. The degree to which the tutorials have been used has varied from quarter to quarter and instructor to instructor. As a result, during some quarters, the discussion sections have used tutorials during some weeks, while during others they consisted of more traditional, TA-led problem-solving sessions. For a description of the origin of the discussion sections and the initial development of the tutorials, see the PhD dissertation of Andrew Crouse.[30] More detailed descriptions and assessments of the tutorials may be found in chapters 5, 8, and 10 of this dissertation.

2. Tutorial curriculum in sophomore-level physics (PHYS 225)

In 2013, the Physics Education Group participated in a redesign of PHYS 225 with the primary goal of better preparing the students for the more difficult courses that they would take during the following year. Part of the reason for this redesign was the fact that many of the students in PHYS 324 and PHYS 325 were unable to apply concepts that they had already covered in PHYS 225. We found that even though the students learned this material in PHYS 225, they had to relearn it in the context of the more advanced course.

The decision was made to teach PHYS 225 from a "spins-first" perspective. The order of topics is the same as that of the first four chapters of McIntyre's *Quantum Mechanics: A Paradigms Approach*.[35] The topics covered in the course include: the Stern-Gerlach experiment and spin-1/2 systems, an overview of quantum computing, Hamiltonians and time dependence for two-state systems, and an introduction to position space in terms of square well and free particle systems. This focus on two-state systems allows the students to encounter many of the properties of quantum systems within a relatively straightforward context, before moving on to more abstract, infinite-dimensional systems. This approach to teaching quantum mechanics is not uncommon, and can be found in several textbooks appropriate to students at different levels.[36-37]

The new structure of PHYS 225 was designed to take advantage of the body of research on student understanding of quantum mechanics, as well as to utilize the active engagement techniques that have been shown to be effective within the constraints of a large lecture-hall setting. The instructors of the course rely heavily on clicker questions as a means of engaging the students during lectures. In addition, we have designed and adapted several worksheets from *Tutorials in Physics: Quantum Mechanics* to administer as interactive tutorial-lectures throughout the course (see Fig. 1.4 for a list of the topics for which tutorials have been used in PHYS 225).

Blackbody radiation

Spin

Dirac notation

Two-state time dependence

Quantum key distribution

Superposition in quantum mechanics

Figure 1.4: List of tutorial topics included in the first edition of *Tutorials in Physics: Quantum Mechanics* that have been used in the sophomore-level introduction to quantum mechanics course at UW. The bolded topic was also included on the list of tutorials appropriate for junior-level students (see Fig. 1.1). All the tutorials listed here were developed as part of the work described in this dissertation.

The format of the tutorial-lectures is similar to that of the tutorials for small-group sections: the students are expected to work with their neighbors, to give thorough explanations, and to discuss their answers with an instructor or a TA. After most students have worked through a section of the worksheet, they are typically brought together and asked to answer a clicker question and to share their answers. This is done, in part, to prevent some groups of students from falling behind, but also to compensate for the fact that it is often difficult for instructors to have in-depth discussions with each group of students. In our experience, it takes the students slightly longer to complete any particular tutorial in this setting than in small-group sections. To address this finding, we have shortened some of the worksheets, or allotted more than one class period for a single tutorial topic. In some cases, the instructor also gives a targeted lecture (usually for about 10 minutes) before or partway through a tutorial, to supply students with necessary information that is beyond the scope of the worksheet itself. This has also facilitated the dividing of the

tutorial-lectures between different class periods, so that they can be integrated with lectures more smoothly.

In addition to the in-class techniques described above, we have developed a system of "pre-lectures" that the students are expected to complete online prior to each class period. These pre-lectures typically take the form of a sequence of questions on a topic related to the upcoming lecture, and have often included links to and questions about online simulations.[38] These pre-lectures have several goals, similar to the tutorial pretests described in the previous section. One goal is to help prepare the students for what they will discuss during the class period. Another is to give both researchers and instructors an idea of the topics with which the students are struggling. As with the tutorial pretests (see section 1.3B), the students are not graded on the correctness of their answers, but instead receive credit for completion. In some instances, we have also used the same online tools to create online homeworks for the students, which work very similarly. These are often more likely than the pre-lectures to include links to online simulations, and have occasionally been graded in order to provide students with feedback.

3. Student perception of the tutorials in the junior-level quantum course

At the end of each quarter of instruction, we have asked students to give their opinion on the tutorials. Figure 1.2 shows two questions administered to students who had completed PHYS 324 during a quarter that tutorials were used. Each question asks the students to agree or disagree with a statement on a Likert scale. The answer option "Neutral" was not included during most quarters—when it was included, very few students selected it.

PHYS 324 end-of-quarter survey

Question 1

The tutorials were helpful to my learning.

Strongly agree Agree Somewhat agree Neutral Somewhat disagree Disagree Strongly disagree

Question 2

I would like to see tutorials in other advanced topics in physics for which there are no tutorials (such as electromagnetism or classical mechanics).

Strongly agree Agree Somewhat agree Neutral Somewhat disagree Disagree Strongly disagree

Figure 1.2: End of quarter survey

Two questions administered on an end-of-quarter survey to students in PHYS 324. Not all quarters included the answer option "Neutral", though very few students selected this option even when it was offered.

The results of the survey are shown in Fig. 1.3. The most common answer chosen by the students was "Strongly agree" in response to both statements, and very few students disagreed with either statement. It is thus clear that the students view the tutorials as a positive contribution to the course. Not only do they consider the tutorials helpful, but the majority were strongly interested in seeing tutorials in other upper-division classes. In fact, this response has encouraged the development of tutorials for the junior-level electromagnetism course at UW, based initially on curriculum developed at the University of Colorado – Boulder.[39-40]

The results of the questions on this survey are examined in more detail in Ref. 1.19. In particular, we divided the students from one quarter into three groups based on the average grade that they received on the course exams. We found that all three groups of students found the tutorials helpful and were interested in tutorials for other upper-division courses, regardless of the grade. This result indicates that the tutorials are not exclusively helpful to only one group of

students (*i.e.*, either the top students in the class or the bottom students in the class), but instead are helpful to all students.

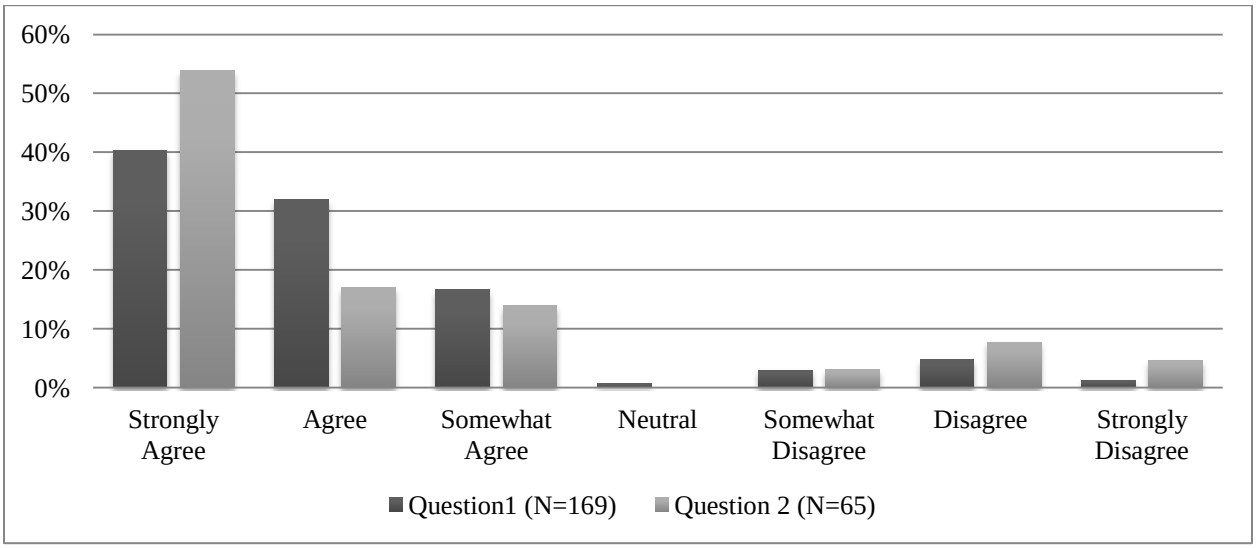

Figure 1.3: Survey results from PHYS 324

The percentage of the students who selected each option is shown.

C. Quantum Mechanics at Other Universities

As part of our effort to develop, assess, and validate *Tutorials in Physics: Quantum Mechanics*, we have partnered with other universities in order to collect data from a wider variety of students than just those at UW. For example, from 2009 to 2011 the University of Colorado – Boulder (UCB) administered many of our research tasks (see section 1.3B) to a class that is roughly analogous to the junior-level course at UW (PHYS 324/325), covering a similar amount of material and using the same textbook. They also made use of an early version of our curriculum. However, while the lectures at UW have usually been "traditional", those at UCB involved a greater use of interactive techniques, such as clicker questions. It should be noted that

UCB modified the tutorials somewhat for their use, as well as writing their own tutorials on some topics.

In 2013, a faculty member from Mount Allison University administered several tutorials, along with their associated pre-tutorial research tasks. The size of the class was very small (fewer than ten students), and it only covered the beginning of Griffiths' textbook. The data that we received has been helpful, as it contributes to the picture of student difficulties that our research has identified at UW.

In 2015, three universities (Texas State University, California Poly – Pomona, and the University of St. Andrews) each made use of several tutorials in their quantum mechanics courses. Each university also administered several exam questions similar to those asked at UW, which has contributed to our assessment of the impact of the tutorials. The assistance of the instructors and researchers at these various universities has been invaluable to our efforts to ensure that the tutorials are not only research-based but also research-validated.[41]

1.3 – Research Methodology

The Physics Education Group at UW has used a wide variety of research techniques to conduct research on the teaching and learning of physics at the university level. The most common method of gathering data is to ask students sequences of questions that have been carefully designed to probe their understanding of physics concepts. These questions typically require the students to explain their answers in detail. However, our group has also administered multiple-choice questions, both with and without accompanying explanations, due to constraints on time or resources. The questions are asked in two different settings: as tutorial pretests, which

are often administered online, and as a part of course exams. In some cases, we have also conducted individual student interviews in order to probe student ideas about physics more deeply.

In this investigation, we have made use of many of the same tools that have proved useful to our group at the introductory level. However, there are substantial differences between introductory and upper-division physics courses that have resulted in challenges to the use of these tools. These challenges, and the changes that we have made in response to them, are discussed below. We also include an overview of how the data is analyzed throughout the rest of this dissertation.

A. Tutorial Pre- and Post-tests

Much of the research that we have done has been structured around the tutorial curriculum. Each individual tutorial is typically associated with both a pretest and a post-test. The pretests are given before students complete the tutorial but are usually given after lecture instruction. They are composed of qualitative questions designed to probe student thinking about one or more physics concepts, typically those concepts associated with the corresponding tutorial. The post-test questions are often similar to but more challenging than the corresponding pretest questions. They also tend to be qualitative questions focusing on specific physics concepts.

Pretests: The pretests serve several purposes. The questions on the pretest are often very similar to those asked near the beginning of the tutorial, so they offer the students an opportunity to see and think about some of the ideas that they will encounter on the tutorial worksheet. The results of these pretests are therefore a valuable means for tutorial instructors to identify what

ideas their students have prior to that week's tutorial (both correct and incorrect). Lastly, the results provide insight into student understanding of different topics that can be used to improve curriculum in the future.

The pretest questions are often changed slightly from quarter to quarter. The changes are made both to test different ideas related to the tutorial and also to identify whether or not small changes (*e.g.,* wording, the presence or absence of figures, or the exact sequence of questions) have an impact on the kinds of answers that the students provide. When there are substantial differences in how students respond to slight variations of the same question, this can provide insight into the factors that affect student understanding of particular topics. Often, the results do not vary from version to version, which indicates that the ideas expressed by students are indicative of their beliefs. In cases where the results do not vary beyond what would be expected statistically, we typically group together students who completed each variation on the same topic.

The pretests are now administered almost entirely online. In the past they were given on paper. The online format has several advantages and some disadvantages. First of all, the students can complete the pretest on their own schedule, which is often more convenient and can lead to a higher participation rate. The online pretests do not require the use of class time. The data is also easier to access, analyze, and store. Additionally, the pretests can make use of online tools and visualizations, and can even be designed to offer the students feedback depending on the exact form of their answers. However, students cannot currently make sketches to support their answer when the pretests are given online, and it can be more difficult for them to use equations or formulas in their answers.

We have made substantial use of the tutorial pretests in the various quantum mechanics courses at UW. As at the introductory level, an online pretest is administered each week, and the students receive credit for completing the pretest, regardless of whether or not their answers are correct. Because they are typically administered following lecture instruction, they provide us with an opportunity to document student understanding of quantum mechanics after traditional instruction but before tutorial instruction. In our experience, at both the introductory and upper-division levels, most students take these pretests seriously, and their responses are an accurate reflection of their understanding at this stage of instruction.

Post-tests: Every exam in each course that utilizes the tutorials includes one or more questions that focus on material covered on the tutorial worksheets. These exam questions are typically long-answer, and the students are made aware of the particular topics that might be tested on each exam. In many cases, the exam questions are modified versions of the research tasks given on the pretests, although the context and the phrasing are often different in order to serve as a better assessment of the tutorials. Not only do the exam questions provide us with an opportunity to assess student performance after the tutorials, but they also indicate to the students that the material covered during the tutorial sessions is a valuable part of the course that they should treat as seriously as other components, such as the lectures and the homework.

In some cases, the pretests for tutorials administered later in class may serve as post-tests for earlier tutorials. For example, much of our curriculum involves sequences of tutorials for related topics that build on each other. Pretests near the end of such a sequence sometimes ask questions very similar to those asked near the beginning, and therefore can serve as post-tests of the earlier tutorials. In other cases, we intentionally design and administer post-test questions on tutorial

pretest, often as a means for exploring different aspects of student understanding than we do on the course exams. Throughout this dissertation, we clearly identify whether each question was given as part of the online tutorial pretest or on one of the course exams.

B. Individual Student Interviews

Individual student interviews have been an important part of the research conducted by the Physics Education Group for four decades. These interviews have helped our group to probe different lines of student reasoning on a wide variety of topics. For many topics, simple demonstrations often provide a basis for the context of the interview. This, however, is more difficult to achieve for phenomena associated with quantum mechanics than with introductory physics. As a result, the interviews that various members of the group have conducted to investigate student understanding of quantum mechanics have often started by presenting the students with a simple quantum mechanical situation and then asking them a sequence of questions intended to elicit different ideas related to that context. The interviews typically take anywhere from 30-60 minutes.

C. Methods of Analysis

The responses to both the pre- and post-test questions have been analyzed in several different ways over the course of this investigation. The responses to each individual question were analyzed in order to identify the most common lines of reasoning, both correct and incorrect, used by the students. These categories were refined as a result of discussions with various members of the Physics Education Group and by comparing them to previously published

research on student understanding of quantum mechanics. Each student response was then classified into one or more of these categories depending on both the answer and the explanation given by the student. When many students give the same answer with the same incorrect line of reasoning, it is interpreted as a difficulty. In some situations, a student's explanations for multiple questions on the same topic are used to categorize their responses, especially in cases where an incorrect line of reasoning can give a correct answer for one or more questions. It should be noted that the existence of a difficulty does not necessarily indicate that the students believe the associated ideas, but rather that they give answers that are consistent with the associated line of reasoning whether or not they consciously believe it.

In order to assess the effectiveness of the tutorials, the responses to post-test questions were analyzed using the same framework as was used for the pretests. The analysis of the post-tutorial questions rarely resulted in additional categorizations of student reasoning not already identified by analysis of the pretest responses. In the chapters of this dissertation concerned with curriculum assessment, we have used a variety of different methods to compare the pre- and post-test results for a particular topic. Not all of these methods are necessary in every case, so some of them are not discussed with respect to every post-test question. However, many of these methods are necessary to overcome some of the challenges associated research in most upper-division courses: namely, that such courses are offered less frequently and to smaller populations of students than introductory physics courses.

The most straightforward method of assessment is to compare the percentage of students who answer correctly both before and after the tutorial curriculum is administered. Because the primary goal of the tutorials is to improve student understanding of quantum mechanics, we

usually compare only those students who answer correctly *with correct reasoning*, not including those students who provide incorrect or incomplete explanations. The criteria for correct reasoning are described in each instance, and usually consist of correctly applying the appropriate quantum concept(s) to the question. When only two proportions are being compared, we use a one-tailed t-test to evaluate whether or not a particular difference is statistically significant.[42]

In order to use the student's t-test, it is assumed that one can reasonably approximate the data as a normal distribution. In some cases, the class size or the number of correct answers is too small for this assumption to be valid. In these situations, the beta distribution, rather than the normal distribution, can be used to make the comparison more exact.[43] To simplify the calculations for the beta distribution, we simply replace the standard error for the normal distribution, $\sigma_{\text{normal}} = \sqrt{p(1-p)/n}$, with the standard deviation for the beta distribution, $\sigma_{\text{beta}} = \sqrt{p(1-p)/(n+1)}$, where p is the proportion and n is the size of the sample. Note that these standard deviations are essentially the same for large values of n, so in those cases either expression may be used.

There are some instances in which we are interested in comparing more than two proportions (*e.g.*, in cases where different versions of the same tutorial were administered in different quarters). In these cases, we use a one-way ANOVA rather than a t-test, followed by Tukey's range test to identify which differences, if any, are statistically significant. As with the t-test, the ANOVA assumes that the data can be approximated by the normal distribution, so we use σ_{beta}

for this test as well. These statistical methods are used in all circumstances that require a comparison between two or more populations, unless noted otherwise.

In some cases, we have also compared the prevalence of the most common student difficulties among the pre- and post-test responses. This comparison is valuable because many of the tutorials are written in such a way that they attempt to address particular difficulties directly, which we would expect to be less prevalent after the students have completed the tutorial. In most cases, however, we will not evaluate whether or not these differences are statistically significant, primarily for two reasons. First, carrying out additional statistical tests on the same data would increase the probability of making a Type-I error (*i.e.*, identifying an effect where none exists). Second, the tutorials are modified slightly from year to year, so the results from multiple courses cannot always be combined. This increases the number of comparisons that must be made and decreases the size of each sample, both of which make it more difficult to identify statistically significant results. However, comparing the prevalence of particular difficulties is often still valuable because it gives a sense for the strengths and weaknesses of the tutorials, which in turn will allow us to make future modifications to further improve their ability to improve student understanding.

1.4 – Overview

This dissertation focuses on student understanding of several key ideas in quantum mechanics. In particular, we examine student understanding of various topics throughout undergraduate instruction (in some cases, through graduate instruction as well). Each of this dissertation's three parts describes our investigation into student understanding of a different set

of quantum topics: time dependence and measurements (part I), vector spaces (part II), and angular momentum (part III).

In each part, we describe the most common student difficulties that have been identified. We first describe the conceptual, qualitative questions that were used to probe student understanding. We then examine and classify student responses to these questions. In many cases, we present questions that span across a wide variety of contexts in order to probe the persistence of student ideas.

After identifying common student difficulties and the contexts in which they occur, each part then describes our effort to develop and assess curriculum to improve student understanding. This effort typically involves designing curriculum, then using tutorial post-tests to assess the curriculum, and finally revising the curriculum based on the results. This process is repeated several times in a continuous cycle of development and assessment.

The three different aspects of our investigation (identification of student difficulties, development of curriculum, and assessment of curriculum) are highly interrelated. For example, the results from assessments of the tutorials were often used not only to make further modifications to the curriculum, but also to devise new conceptual survey questions to explore different facets of student understanding. In addition, interactions in the classroom between researchers, instructors, and students often provided insight into student thinking.

There is substantial overlap between the investigations discussed in each of the three parts. For example, even though part I and part II focus on different student understanding of different key ideas, concepts such as superposition and inner products (part II) are often needed when answering questions about time dependence and measurement (part I), and *vice versa*.

Furthermore, concepts discussed in part I and part II are relevant for the topic of angular momentum (part III). Thus, part III serves not only to investigate student understanding of a challenging topic (angular momentum), but also to explore how student understanding of more fundamental concepts (*e.g.*, measurement and inner products) manifests in more advanced physical contexts.

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[40] For more information about upper-division electromagnetism tutorials at UW, see R. L. C. Hazelton, "Identifying and Addressing Student Difficulties with Electrostatics and Magnetostatics in Introductory and Advanced Courses," PhD dissertation, University of Washington (2015).
[35] This textbook is the result of the Paradigms in Physics program at Oregon State University: D. McIntyre, *Quantum Mechanics: A Paradigms Approach*, Addison-Wesley (2012).
[36] At the undergraduate level, see, for example, J. S. Townsend, *A Modern Approach to Quantum Mechanics*, University Science Books (2000).
[37] At the graduate level, see, for example, J. J. Sakurai, *Modern Quantum Mechanics*, Addison-Wesley (1994).
[38] Examples of simulations that we have used include the PhET on blackbody radiation (see Ref. 18 and http://phet.colorado.edu/en/simulation/blackbody-spectrum) and the QuVis from the University of St. Andrews on the difference between superposition states and mixed states (see Ref. 21 and http://www.st-andrews.ac.uk/physics/quvis/embed_item_3.php?anim_id=5&file_sys=index).
[41] See, for example, N. D. Finkelstein and S. J. Pollock, "Replicating and Understanding Successful Innovations: Implementing Tutorials in Introductory Physics," *Physical Review Special Topics – Physics Education Research* **1**, 010101 (2005)
[42] D. S. Moore and F. P. McCabe, *Introduction to the Practice of Statistics*, W. H. Freeman and Company (2006).
[43] See, for example, L. D. Brown, T. T. Cai, and A. DasGupta, "Interval Estimation for a Binomial Proportion," *Statistical Science* **16** (2001).

Part I
Student Understanding of Time Dependence and Measurements in Quantum Mechanics

This dissertation begins with investigations of two of the most fundamental concepts in quantum mechanics: time dependence and measurements. The manifestation of each concept is substantially different in quantum mechanics than in classical mechanics, yet they are no less important. Without measurements, physical quantities could not be determined, and experimental results could not be obtained. Without time dependence, all quantities would be static and unchanging. Together, these two concepts constitute much of the foundation on which quantum mechanics is built, and they occur again and again throughout its study.

Despite the importance these ideas, past research has shown that many students struggle with both of them. Some students do not understand how the wave function and the associated probabilities evolve in time (see Ref. 2-5 and 14-17 in chapter 2). Others have difficulty identifying the possible outcomes of measurements of basic quantities such as energy (see Ref. 2-13 in chapter 3) or position (see Ref. 1-4 in chapter 4). These difficulties occur even after lecture instruction, and some studies have shown that even some incoming graduate students are unable to answer basic questions about time dependence and measurements correctly. These findings indicate the need for specialized curriculum that directly targets the most common errors made by students.

The Physics Education Group at UW has conducted extensive research over the past two decades on student understanding of both time evolution and measurements in quantum mechanics. This effort has included both the identification of student difficulties and the

development and assessment of curriculum to address these difficulties. Much of this work began with the efforts of Brad Ambrose and Andrew Crouse, and is based on research that Crouse's PhD dissertation documents.

This part of the dissertation is divided into chapters that each focus on our initial identification of student difficulties with a particular subject. The first of these, chapter 2, is dedicated to student difficulties with time dependence. The second, chapter 3, describes student difficulties with energy measurements. Each chapter is reproduced from one of a pair of papers that we published in 2015 in close collaboration with Gina Passante and Peter Shaffer. The object of the papers is to document common student difficulties with time dependence or energy measurements after lecture instruction. Both works draw heavily on data collected in the junior-level quantum mechanics sequence at UW, but also rely on supporting data drawn from sophomores and graduate students. In this way, we examine not only the difficulties that students encounter when they are introduced to the material, but also at several different points of instruction and for a variety of physical contexts that range from two-state systems to time-dependent perturbation theory. Chapter 4 discusses a similar investigation on student difficulties with sequences of measurements, including position measurements, that has not yet been published elsewhere. The student populations for this study were similar to those in chapters 2 and 3.

Lastly, chapter 5 describes our efforts to develop and assess curriculum to improve student understanding of quantum time evolution and measurement theory. In particular, we focus on the changes that have been made since Andrew Crouse's preliminary assessment of this curriculum in his dissertation. Most of the curriculum that we have developed has been

primarily intended for junior-level students. However, we have had some success adapting this curriculum for students in a sophomore-level introduction to quantum mechanics. We also address the challenges associated with evaluating curriculum for upper-division physics students, and the methods that we used to overcome these challenges.

Chapter 2
Student Difficulties with Time Dependence

"It is called a stationary state for a reason. If the potential doesn't change with time then the probability shouldn't either."

—Junior-level quantum mechanics student, pretest

This chapter has been reproduced from Ref. 1, which was written in close collaboration with Gina Passante and Peter Shaffer. A few additional comments that are not included in the published paper have been added where relevant, and some figures have been added. It has also been edited to match the format of this dissertation.

2.1 – Introduction

There is an increasing body of research on the teaching and learning of quantum mechanics, a topic that is abstract, mathematical, and often counter-intuitive.[2-5] The findings indicate that student conceptual and reasoning difficulties are widespread, and independent of textbook, population, and instructor, a situation similar to that at the introductory level.[6] Efforts have been made to design instructional strategies to address some of the issues that have been identified,[7-13] but much more needs to be done.

A fundamental concept in quantum mechanics is the time evolution of quantum states. It is necessary for interpreting results from laboratory techniques such as nuclear magnetic resonance (NMR), for explaining new physics like neutrino oscillations, and for understanding the relationship between classical and quantum mechanics. Yet, there is evidence that many

students fail to recognize the importance of time dependence in quantum mechanics.[14] Even for simple systems, students often have difficulty in identifying the correct time dependence not only for wave functions, but also for probabilities and probability densities.[15] Many are unable to determine the source of the time dependence, and some believe that probabilities or expectation values never depend on time.

Although some student difficulties related to time dependence have been previously identified,[3,16,17] this paper details a systematic and in-depth investigation into student ideas about time evolution in quantum mechanics. We describe some of the overall problems that students have in determining and generalizing the time dependence of quantum states, together with the prevalence in various contexts at different points during instruction.[18] In particular, we focus on patterns in student reasoning after lecture instruction on time dependence and related topics.

In this investigation, we use the framework for research and curriculum development that characterizes the work of the Physics Education Group at the University of Washington (UW).[19] Student responses to carefully designed questions (both in written form and administered during interviews) are examined to identify common incorrect answers or explanations that might suggest incorrect beliefs or lines of reasoning. Multiple versions of each question are used to assess the extent to which the ideas are strongly held or persist throughout instruction. Strategies for improving instruction are then developed and tested. They are deemed effective if, afterward, students can apply the appropriate concepts properly in contexts that differ from those they have previously seen. The investigation into student understanding, the development of curriculum, and the testing of instructional materials are done together in an iterative process

with each component informing the other. This article focuses on the first part of this process—identification of common difficulties and of ways to elicit them.

2.2 – Context for Research

This investigation has been carried out over a period of many years (2003-2015). The data come primarily from a two-quarter, junior-level quantum mechanics sequence at UW. Supporting results are also reported from sophomore- and graduate-level courses. In the junior-level sequence, between 50 and 90 students are typically enrolled in the first-quarter course each year. About half of the students continue to the second quarter. The textbook used is Griffiths' *Introduction to Quantum Mechanics*.[20]

In addition to data from UW, some of our results are drawn from junior-level courses at other universities: the University of Colorado – Boulder (2009-2011) and Mount Allison University (2013). The textbook and content have been essentially the same across all the courses in this study, but the courses have been taught by different instructors and using different lecturing styles. Some have included tools for interactive engagement, such as clickers. The results have been similar for all of the courses included in this study.[21]

Each course also utilized *Tutorials in Physics: Quantum Mechanics.*[22] These are modeled after the tutorials that our group has developed to supplement instruction in introductory physics.[23] Each tutorial is preceded by a tutorial "pretest". These pretests are given before tutorial instruction, but after lecture instruction on the relevant material. Each pretest consists of one or more tasks involving a series of questions. Students are asked to explain their answers to each question. The pretests are timed and have been given both on paper during class time and

online outside of class. Students receive credit for completing the pretests; credit is not based on correctness. In our experience at the introductory level, most students take the pretests seriously and they provide a good gauge of student understanding. We have found the results to be similar when a given question is asked either on a tutorial pretest or on a course exam when the corresponding tutorial is not used. The concepts and reasoning addressed in the tutorials are assessed through tutorial post-test questions on course examinations.

Since the focus of this paper is on student understanding after lecture instruction, most of the findings discussed in this paper are based on student responses to tasks given as tutorial pretests. In those places where data are drawn from exams administered after relevant tutorial instruction, it is explicitly noted. We also report results from individual student interviews conducted to probe various aspects of student reasoning in greater detail.

2.3 – Tasks Used to Probe Student Reasoning

A variety of questions have been used in this investigation. Four representative tasks are discussed in this paper. The first two are based on physical contexts normally introduced near the beginning of instruction on quantum mechanics. They require students to reason about time dependence, in particular for spatial probability densities and probabilities of energy measurements for a variety of initial states. The other two questions probe similar ideas for more complicated systems. One of these involves degenerate energy eigenvalues; the other, a time-dependent Hamiltonian. The four tasks are shown in Figures 2.1a-d.

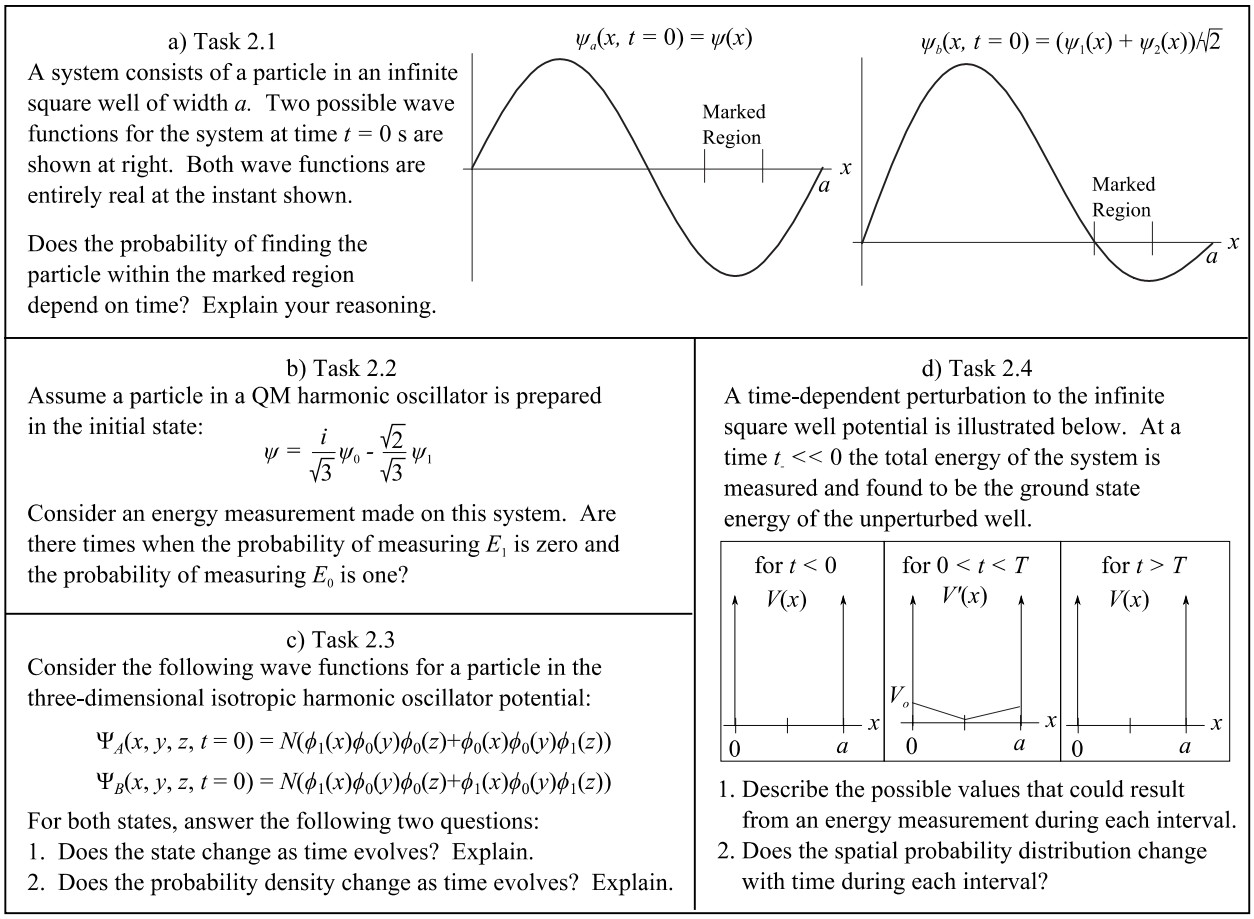

Figure 2.1: Four time dependence tasks

Example versions of four tasks that were administered over the course of many years in a junior-level quantum mechanics course. (a) Task 2.1 and (b) Task 2.2 were each administered more than 400 students. (c) Task 2.3 was given to more than 250 students, and (d) Task 2.4 was given to more than 200 students. This figure also appeared in Ref. 1.

All of the tasks are intended to probe the specific approaches that students use when answering questions about time dependence. In each task, students are given one or more quantum states for a single instant in time (*e.g.,* for $t = 0$). The form of the wave function for other times is not given explicitly. Multiple versions of each task have been used. All versions of the four tasks from which we have drawn student responses may be found in the appendix.

They have been continually modified to probe different aspects of student understanding and to determine whether or not the results are due to a particular phrasing, representation, or physical context. The versions discussed below are representative, as are the student responses used to illustrate common lines of reasoning. Some of the tasks have been given to more than 400 students in junior-level courses at multiple universities. In addition, we discuss results from a version of Task 2.1 given in a sophomore-level course and from versions of Tasks 2.1-2.3 given in a graduate-level course. These versions are presented in the appropriate subsection.

Table 2.1: Results from Tasks 2.1-2.4

The percentage of the junior-level students who gave correct answers (with correct reasoning) is shown for each of the four research tasks. (See Fig. 2.1). Results are shown for the individual parts of each task and for each task as a whole. Data are aggregated across all courses in which each task was administered after relevant lecture instruction, but before tutorial instruction.

Question	Task 2.1 (N=416)			Task 2.2 (N=439)	Task 2.3 (N=285)			Task 2.4 (N=215)			
	ψ_a	ψ_b	Both		Ψ_A	Ψ_B	Both	Before	During	After	All
Correct w/ reasoning	35%	20%	20%	5%	20%	50%	20%	50%	20%	20%	10%

Table 2.1 shows the percentage of junior-level students who answered correctly and provided correct explanations for each task. The criteria used for an explanation to be considered correct are discussed below with the description of the corresponding task. The results are aggregated over several different years, instructors, and universities, as well as over different versions of the research tasks. (A total of 115, 131, and 91 students from the University of Colorado, Boulder are included in the results for Tasks 2.1, 2.2, and 2.3, respectively. A total of 5 and 9 students from Mount Allison University are included in the results for Tasks 2.1 and 2.2.) During this study, the percentage of the students who have answered a given task correctly has typically been

similar. (In most cases, the percentages are what would be expected statistically given the sizes of the different classes involved.) The table includes only results from courses in which the research tasks were administered after lecture instruction and before students had worked through any tutorials on the relevant material.

Task 2.1: On Task 2.1 (Fig. 2.1a), students consider two different initial states for a particle in the infinite square well potential: an energy eigenstate and a superposition of two energy eigenstates. In each case, they are asked whether or not the probability of finding the particle within a specified region of the well depends on time. In some versions, students are asked to rank the probabilities of finding the particle within this region at specific times.

One way to answer this question is to determine whether or not the probability density, which is given by the modulus (or complex) square of the wave function, depends on time. Since students are given the wave function at $t = 0$, one step in the reasoning is to recognize that each term in the wave function has a time-dependent phase that depends on the associated energy eigenvalue. This phase cancels for the particle in the energy eigenstate, resulting in a constant probability density, which is the hallmark of a *stationary state.* The components of the wave function for the superposition state, however, have unequal phases. Thus, for this state, there are time-dependent cross-terms in the square of the modulus. Students were considered to have correct reasoning for this task if they identified a connection between time dependence and energy eigenstates (*e.g.*, if students simply stated that the first state is a stationary state and the second is not).

Task 2.1 has been given to 416 students at three institutions since 2007. The questions have proved challenging for most students. The percentage of correct answers (with correct

reasoning) has been about 35% for ψ_a (the energy eigenstate) and about 20% for ψ_b (the superposition state), with about 20% correct for both.

A version of Task 2.1 involving only the superposition state has also been asked in a graduate course on quantum mechanics at UW. It was given at the start of the course before any instruction. About two-thirds of the students (N=31) answered correctly with correct reasoning.

Task 2.2: On Task 2.2 (Fig. 2.1b), students are presented with an initial state written as a superposition of two energy eigenstates in a harmonic oscillator potential. They are asked if there are times when the probability of measuring the energy associated with one of the eigenstates is zero. Some variants of this task directly asked whether or not the probabilities of the energy measurements depend on time.

Since the potential is constant in time and the state is expressed in the energy eigenbasis, the probability of measuring a given energy depends only on the square of the coefficient of the corresponding energy eigenstate. The time-dependent phase of each coefficient (not given to the students) squares to 1, so the probability does not depend on time. Answers were considered as having correct reasoning if students related the coefficients in the wave function to the probability for measuring the energy.

About half of the students answered correctly (N=439). However, most of these students simply asserted that the result did not depend on time. Only about 5% of the students also gave correct explanations.[24] Incoming graduate students at UW performed better: about 70% answered correctly, although only 15% provided correct explanations (N=31).

Task 2.3: On Task 2.3 (Fig. 2.1c), students are given two initial states for a three-dimensional harmonic oscillator, which has degenerate energy eigenstates. The two states (Ψ_A

and Ψ_B) are each superpositions of two eigenstates that have the same or different energies, respectively. Students are also given the functional form of the potential, the energy eigenvalues, and the energy eigenfunctions (not shown). Students are asked whether or not each state changes with time and whether or not each probability density changes with time.

The criteria we used to assess reasoning on this task are very similar to the criteria for Task 2.1. Each term in each state has a time-dependent phase (not given to the students) that depends on the corresponding energy eigenvalue of that term; therefore, each state has time dependence. However, the degenerate energy eigenvalues for Ψ_A result in a single time-dependent phase for the entire wave function. Thus, the probability density has no time dependence. The degeneracy therefore results in a time-independent probability density for the first state, even though it is written as a superposition of energy eigenstates—in fact, this state is also an energy eigenstate (or stationary state) because of this. The same is not true for Ψ_B, which has distinct energy eigenvalues, so the phases do not cancel and the probability density depends on time.

In the junior-level course (N=285), about 20% of the students answered the entire task correctly (with correct reasoning). Relatively few students gave correct answers without correct reasoning. About 20% answered correctly for Ψ_A alone and about 50% answered correctly for Ψ_B alone.[25] Students who did not answer correctly for either state typically gave a time evolution that was incorrect for both the state and its associated probability density. Overall, these results suggest that even near the end of the course, many students struggle with the underlying ideas of time dependence. Even graduate students have difficulty with this task. When a version of Task 2.3 was given at the start of the third quarter of a graduate course on

quantum mechanics, only about 60% of the students (N=19) answered the entire task correctly with correct reasoning.

We have asked similar tasks involving other physical systems that have degenerate states (*e.g.*, systems with angular momentum or identical particles). The results have been similar across the different contexts.

Task 2.4: The context for Task 2.4 (Fig. 2.1d) is time-dependent perturbation theory. Students consider a particle that is initially in the ground state of the infinite square well. A perturbation is applied to the potential and then removed. (Both changes are made instantaneously.) Students are asked whether or not the probability density depends on time for three separate time-intervals: *before*, *during*, and *after* the applied perturbation. On some versions of this question, students are also asked whether or not the probabilities of energy measurements depend on time for the same three intervals.

The probability density for the first interval (before the perturbation) can be determined in exactly the same manner as Task 2.1: the particle is in an energy eigenstate, so the probability density will not depend on time. The wave function itself is unchanged immediately after the potential changes. Therefore, it does not correspond to an energy eigenfunction of the perturbed system. As a result, the probability density evolves with time during the interval in which the potential is perturbed. Finally, the wave function is also continuous in time at the instant the perturbation is removed, but it has changed, so it no longer corresponds to an energy eigenstate of the unperturbed potential. Thus, the probability density also changes in time after the perturbation has been removed.

A total of 215 juniors near the end of their second quarter of introductory quantum mechanics at UW have responded to this task. About 50%, 20%, and 20% of the students correctly identified and explained whether or not probability density would depend on time before, during, and after the perturbation, respectively. About 10% of the students answered correctly for all three intervals.[26] As with the other tasks discussed above, different variants of Task 2.4 have been asked in different years.[27]

It should be noted that both Task 2.3 and Task 2.4, which involve advanced contexts typically not discussed at the beginning of a quantum mechanics course, were administered after students had completed tutorial curriculum specifically designed to address student difficulties in simpler contexts. This suggests that time evolution is a particularly challenging topic, and that many students are unable to transfer what they have learned from one context to another, often more difficult context. Despite the fact that the tutorials have been shown to be effective at improving student understanding of time dependence in quantum mechanics (see Ref. 10), more work needs to be done. For a more in-depth discussion of this curriculum and its effectiveness, see chapter 5.

2.4 – Specific Difficulties Elicited by Research Tasks

Student responses to each task were examined by several members of our group to attempt to identify each student's underlying reasoning. The questions on each task were analyzed both individually and together with all other questions on the same task. When many responses were compared, the patterns that emerged suggested some common approaches to the tasks as a whole. In many cases, interactions with students in class or in individual interviews provided

supporting evidence for the interpretations. In what follows, incorrect lines of reasoning that were common among many students are termed *difficulties.* The most common and persistent of these are described in the numbered subsections below, together with representative student statements. Some of these difficulties have been documented elsewhere.[28] They are included below if we have identified them as arising in response to different kinds of questions than previously asked or in a broader variety of contexts than has been previously reported.

The specific difficulties have been grouped into four categories (given by the lettered headings below) that can be considered to represent similar types of errors. Note that some responses do not fit into a single category. While other categorization schemes are possible, we have found these categories to be useful in communicating the specific difficulties to other instructors teaching quantum mechanics and for guiding the design of curriculum to address the issues.[22]

We also report the percentage of student responses that are consistent with each difficulty (aggregated across all courses) and the tasks on which they have been identified. The data in this section are taken mostly from tasks administered after lecture instruction in junior-level courses (other sources of data will be noted explicitly). In some cases, which are noted when relevant, we found that the percentages changed somewhat from year to year or version to version. These variations may be due to differences in the lectures or a result of small changes to the questions. We regard the exact percentages reported below as less important than the presence and persistence of the individual student difficulties.

Some of the difficulties we have identified occurred at only the 5% level on individual questions. They are included here if they occurred consistently across multiple contexts.

Moreover, we have observed that in some cases students made multiple errors on a given question or task. Thus, some difficulties may be masked by others. In addition, some student answers (ranging between 5%-20% on various questions) were too brief for us to be able to characterize their reasoning. As a result, the given percentages represent a lower bound for the prevalence of specific difficulties.

A. Tendency to Confuse the Time Dependence of Different Quantum Mechanical Quantities

In many of their responses, students attributed the time dependence of one quantity to that of another. In some cases, they treated the presence (or absence) of time dependence for one quantity as implying the presence (or absence) of time dependence for another quantity. Often, they seemed to believe that the time evolution of both quantities must be exactly the same. This tendency was evident for quantities such as the wave function, the probability density, the probabilities of individual energy measurements, and the potential. It was especially common in student responses for the more advanced contexts of Tasks 2.3 and 2.4, which suggests that even students who have completed most of their instruction in quantum mechanics may confuse the time dependence of various quantities.

1. Confusion between the time dependence of wave functions and probability densities

On Task 2.1 (N=416), about half of the students correctly stated that the probability of finding the particle in a given region does not depend on time for an energy eigenstate.

However, between 5% and 20% of the students seemed to assign the time dependence of the wave function to the probability density (*i.e.*, without referencing the modulus square).

> *"The wave function is time independent. Thus, its probability density does not change. If the wave function is time dependent, then [its] probability density would change in time too."* (Task 2.1)

This student has clearly connected the time dependence of the two quantities. Many students obtained the correct answer because they incorrectly treated the wave function as being time-independent. (See also section 2.4B1.) Students often demonstrated reasoning of this sort even when they were able to express the relationships mathematically (*e.g.*, by writing that the probability density is the modulus square of the wave function).

About 5% of the students stated explicitly that the wave function itself does not depend on time.

> *"This is a stationary state so the wave function will not evolve with time."* (Task 2.1)

The reasoning above suggests that this student thinks the term *stationary state* means that the wave function, rather than the probability density, has no time dependence. This error was more common for energy eigenstates than for superpositions of energy eigenstates. Some students also seemed to believe that everything about stationary states is time independent.

On Task 2.3 (N=285), about 25% of the students seemed to be using the time dependence of the probability density to determine whether or not the state itself is time-dependent. On one version of Task 2.4 (N=34), about 25% of the students stated that the probability density would (or would not) depend on time because the wave function did (or did not) depend on time.[29] The

presence of these errors in the more-advanced contexts of Tasks 2.3 and 2.4 suggests that they are particularly resistant to instruction, since they seem to persist throughout all of undergraduate quantum mechanics.

2. Confusion between the time dependence of probabilities of energy measurements and other quantities

Some students incorrectly stated that the probability of measuring a given energy depends on time. They related the time dependence of the probability to the time dependence of other quantities. About 10% of the responses to Task 2.2 (N=439) involved this line of reasoning.

> *"It [the energy probability] would change because the wave function would change."* (Task 2.2)

> *"It [the energy probability] depends on the probability density. If it's time independent then no, if time dependent then yes."* (Task 2.2)

The first student infers a time dependence for the probability of an energy measurement based on the fact that the wave function changes in time. The second student associates the probability of measuring a given energy to the time dependence of the *probability density*. However, the probabilities for energy measurements do not depend on time, regardless of whether or not the wave function or the probability density has a time dependence. (Only an energy eigenstate with zero energy would have no time dependence.) In some cases (*e.g.*, for stationary states) this incorrect line of reasoning led students to give the correct answer.

Confusion about the time dependence of the probabilities for energy measurements was also common among graduate students. Even after two quarters of graduate instruction, 40% (N=19) gave answers to a version of Task 2.3 consistent with the line of reasoning discussed above.

The following response illustrates another way in which students applied the time dependence of some quantity inappropriately to that of energy measurements.

> *"A linear combo of stationary states is not stationary. The system will oscillate around E_0 and E_1."* (Task 2.2)

This student appears to believe that the probability of measuring a given energy changes with time, alternating between being greater for one of the eigenstates in the wave function and then, at later times, greater for the other. This behavior could correctly describe the real (or imaginary) part of the wave function, which oscillates in the real and imaginary planes, or the probability density, which has a peak that moves back and forth. However, it is incorrect when applied to energy probabilities, which are constant in time.

3. Confusion between the time dependence of the potential and other quantities

On Task 2.4 (N=215), some students used the time dependence of the potential to argue about the time dependence of other quantities. On this task, students are asked to consider the effect of a perturbation on the time dependence of the probability density for a particle.

> *"If the perturbation is time dependent, then yes [the probability density depends on time]."* (Task 2.4)

Although this student arrived at the correct answer, their reasoning is not based on the relevant aspects of the quantum formalism. Approximately 20-40% of students gave answers to Task 2.4 consistent with this response. This type of reasoning was evident on other questions as well, such as Task 2.3, although only at about the 5% level.

B. Failure to Ascribe the Correct Time-dependent Phases to the Wave Function

Many students had difficulty in determining, applying, or interpreting the phases that arise from the time-dependent Schrödinger equation. Each term in a wave function that is written in the energy eigenbasis is associated with a phase that depends both on time and on the energy of the associated eigenstate. Students often failed to apply this formalism to the wave function. Some treated the wave function as not having a time-dependent phase; others included only a single time-dependent phase for the entire wave function. Still others incorrectly associated phases with *distinct* time dependence to terms that have the *same* energy (*e.g.,* when there was degeneracy).

1. Belief that the wave function is time-independent

In the research tasks, the wave functions were usually given to students, either mathematically or graphically, for a particular instant in time. The functional form of the time dependence was not given. On some versions of Tasks 2.1 and 2.2 (not shown) we asked students explicitly about the time dependence of the wave function prior to asking about probabilities of energy or position measurements. In these cases, some students responded that

there is no time dependence or that they had not been given enough information to find it. The following response is illustrative:

> *"The graph does not provide any information about the time dependence of the wave function."* (Task 2.1)

This student did not recognize that, in the absence of external factors, the wave function at an initial time determines the wave function for all times. On each of the four tasks, about 5% gave similar responses, regardless of whether the question asked about the wave function, the probability density, or both. These students tended to focus on the fact that no explicit time dependence was given for the wave function. For example, one student answered Task 2.2 by saying "No, there is no time in the equation". Although this student correctly stated that the probability for an energy measurement does not change with time, the reasoning is incorrect.

Some students attempted to give a mathematical basis for why they believed that the wave function does not depend on time.

> *"[Both wave functions] satisfy the time independent Schrödinger equation so* ψ_1 *and* ψ_2 *do not have time dependence."* (Task 2.2)

This student is correct that the solutions to the time-independent Schrödinger equation do not depend on time. However, the solutions appear as terms in the wave function with a time-dependent phase. Like the students above, this student does not seem to recognize that the wave function at an instant does not provide a complete description of the state. When reasoning about the time dependence of non-stationary states, this belief has led some students to think that no states have time dependence.

Some of the student responses discussed in section 2.4A also reflect the idea that a wave function consisting of a single eigenstate is time-independent. These students often argued about the time evolution of the state based on the fact that the probability density does not depend on time (see section 2.4A1).

We have probed student thinking about the time evolution of states in greater detail during interviews. The results suggest that the errors are not superficial. For example, we conducted a series of interviews in the context of perturbation theory (similar to Task 2.4).[5] Even those students who recognized that there is a time dependence associated with an energy eigenstate of the infinite square well often did not do so for an eigenstate of an unspecified potential. This finding suggests that many students do not have a general understanding of the time evolution of quantum states.

2. Tendency to treat all wave functions as having a single phase

Perhaps the most prominent error documented by prior research on student understanding of quantum mechanics is a tendency of students to associate a *single* time-dependent phase with the entire wave function, rather than to associate an individual phase with each term.[15] The equation below is a student's response to the superposition state question from Task 2.1. Note that the student wrote a wave function, even though the question asked about the probability density.

$$\sqrt{1/2}e^{-iEt/\hbar}\left(\psi_1+\psi_2\right) \quad \text{(Task 2.1)}$$

The time dependence expressed by this student would be correct for an energy eigenfunction, but not for a superposition of energy eigenfunctions with different energies. Moreover, the student

has not identified which energy should be used, but simply written a generic "E". Other students wrote "E_n" without specifying the value of n that should be used. (The tendency for students to give an answer in terms of a generic energy (*e.g.*, E_n) has been identified previously in, for example, Ref. 14.)

Some students correctly identified the different phase factors for each term in the wave function, but failed to recognize how the different phases impact the probability density.

> *"While it is true that the general wavefunction is of the form* $\sqrt{\tfrac{1}{2}}\varphi_1 e^{-iE_1 t/\hbar} + \sqrt{\tfrac{1}{2}}\varphi_2 e^{-iE_2 t/\hbar}$, *again the function we're interested in is* $P(x) = |\varphi|^2$ *which loses its t-dependence."* (Task 2.1)

Although the wave function is correct, the student claims that the overall phase vanishes for the probability density. This statement is consistent with what we have observed when working with students in class. Many simply state that "time drops out" or "probability is squared and [the] time component won't matter", independent of whether or not the initial state was an energy eigenstate or not. Between 10% and 20% of the responses to Task 2.1 at the junior level were similar (N=416).

The tendency to treat the time dependence of all wave functions as consistent with the presence of only a single phase has been very persistent. We have observed this difficulty in response to questions asked on exams in multiple courses. Even after students had worked through tutorials designed to address this difficulty, the problem still arose when the context of the exam question was different from that of the tutorials.

For example, on a final exam in a sophomore-level quantum mechanics course, students were given a version of Task 2.1 that explicitly gave the time-dependent form of the wave function.

(In Task 2.1, we typically gave the wave function for only a single instant in time.) This version is shown in Fig. 2.2. The students were asked about the time dependence of the probability of a particular outcome of a position measurement. About 25% of the students (N=223) gave answers consistent with treating the wave function as having a single time-dependent phase. This finding suggests that the underlying problem goes beyond a failure to recognize that the wave function has time dependence. Some students appear to have a strongly held belief that the time dependence vanishes in the probability density, even when they are given the information necessary to perform a calculation that contradicts this belief.

Consider a particle in the quantum mechanical infinite square well potential. Let the eigenfunctions of this potential be given by $\phi_n(x)$. The wave function is given by $\psi(x,t) = \frac{1}{\sqrt{2}}\left(\phi_3(x)e^{-iE_3t/\hbar} + \phi_2(x)e^{-iE_2t/\hbar}\right)$.

Suppose you were to measure the position of this particle at some time $t > 0$. Assume that no other measurements have been made on this particle.

Is there a time $t > 0$ when the probability that this particle is found to be on the left half of the well is *exactly the same* as the probability that it was found to be on the left half of the well at $t = 0$? If so, identify at least one such time in terms of the given variables. Explain your reasoning.

Figure 2.2: A variant of Task 2.1

This variant was administered to 223 students on the final exam of an introductory quantum mechanics course for sophomores.

We have also found that the tendency to treat the wave function as having a single phase (whether or not students explicitly write it with a single phase) is particularly prevalent when the initial state is not given in terms of the energy eigenbasis. For example, on several exams in the junior-level course we have asked students about the time dependence of both wave functions and probabilities for an initial state that was not written explicitly as a sum of energy eigenstates. The contexts for these questions have included the infinite square well, the harmonic oscillator,

and spin (see Fig. 2.3a-c). The number of students in these classes has ranged from 40 to 70. Between 10% and 25% of the responses to each exam question were consistent with treating the wave function as having a single, time-dependent phase.

(a) The infinite square well

At $t = 0$, the wave function is given by $\Psi(x, t=0) = \sqrt{\frac{30}{a^5}}\left(x^2 - ax\right)$ for $0 < x < a$ and 0 elsewhere.

Does the probability that the position of this particle is given by $x < a/3$ depend on the time at which the measurement is made? Explain.

(b) The harmonic oscillator

At $t = 0$, the system is prepared in an initial state $\Psi(x,0) = \frac{1}{\sqrt{2}}\left(\psi_1(x) + \psi_2(x)\right)$ for $0 < x < a$ and 0 elsewhere.

Are $\langle x \rangle$ and $\langle p \rangle$ evolving in time? Explain.

(c) Spin

Two non-identical spin-1/2 particles interact with a magnetic field, with different magnetic moments:

$$H_1 = \frac{1}{\hbar}\vec{B} \cdot \left(\mu_1 \vec{S}_1 + \mu_2 \vec{S}_2\right)$$

If the system of both particle is initially known to have total spin $s = 0$, does the probability $P_1(t)$ that at time t the two particles will be found in a state with total spin $s = 1$ change with time? Explain your reasoning.

Figure 2.3: Three exam questions on time dependence

The questions that asked the students to identify whether or not the probabilities associated with some measurable quantity would depend on time. The contexts for these questions were (a) the infinite square well, (b) the harmonic oscillator, and (c) spin for a two-particle system.

3. Tendency to treat every superposition as having multiple distinct phases

In responding to the questions in the research tasks about the time dependence of a wavefunction or another quantity, even students who answered correctly did not usually discuss

the phases of the individual terms. Many appeared to use a rule that any superposition state has a time-dependent probability density. This reasoning gives the correct answer for cases with distinct energy eigenvalues. However, it does not work for Task 2.3, which involves a superposition of states with degenerate energies. On this task, about 30% of the students (N=285) gave reasoning consistent with this idea.

"It is a linear combo so time dependence does not cancel out." (Task 2.3)

This student responded in the same way to *both* of the questions on Task 2.3, regardless of whether or not the energies were degenerate. Many students gave identical responses for both states.

Consider the four hydrogen atom wave functions given below:

$$\psi_1(r,\theta,\phi) = \tfrac{1}{\sqrt{2}}\left(R_{21}(r)Y_1^1(\theta,\phi) + R_{21}(r)Y_1^{-1}(\theta,\phi)\right)$$

$$\psi_2(r,\theta,\phi) = \tfrac{1}{\sqrt{2}}\left(R_{21}(r)Y_1^1(\theta,\phi) + R_{31}(r)Y_1^1(\theta,\phi)\right)$$

$$\psi_3(r,\theta,\phi) = \tfrac{1}{\sqrt{2}}\left(R_{21}(r)Y_1^0(\theta,\phi) + R_{20}(r)Y_0^0(\theta,\phi)\right)$$

$$\psi_4(r,\theta,\phi) = \tfrac{1}{\sqrt{2}}\left(R_{21}(r)Y_1^{-1}(\theta,\phi) + R_{10}(r)Y_0^0(\theta,\phi)\right)$$

where $R_{nl}(r)$ are the radial wave functions and $Y_l^{m_l}(\theta,\phi)$ are the spherical harmonics.

Suppose the position of each particle is measured. For which, if any, of the wave functions above does the probability that the particle is found near $r = a$ (the Bohr radius) depend on time?

Figure 2.4: An alternate version of Task 2.3

In this version, students were asked which of four wave functions, each written as a superposition of energy eigenstates, is associated with a position probability that depends on time. Versions like the one shown above, which uses the eigenstates of the hydrogen atom potential, have been asked for contexts such as spin, angular momentum, and identical particles.

The tendency to treat all superpositions as having distinct phases has arisen in student answers to several variants of Task 2.3. For example, on some versions, students were shown

four different superpositions of two eigenstates of the hydrogen atom potential. This version is shown in Fig. 2.4. Two of the superpositions had degenerate energies and two had non-degenerate energies. About 30% of junior-level students (N=42) answered that the probability density for all four states depends on time. The result was the same on similar tasks in the context of identical particles, spin, or angular momentum. About 10% of graduate students (N=19) also made similar errors, even after almost a full year of advanced instruction on quantum mechanics.

C. Tendency to Misinterpret the Mathematical Formalism used for Time Dependence in Quantum Mechanics

In this investigation, we have found that many students seem to have difficulty in determining the effect that the time-dependent phases have on the evolution of different physical quantities. Even students who are able to determine the correct time dependence of a state often misinterpret the mathematics in one or more ways. Some confuse the behavior of traveling waves and standing waves; others have difficulty in distinguishing between real and imaginary exponentials or between coordinate space and Hilbert space. We have grouped these problems together because they have in common a strong mathematical component.

1. Tendency to treat wave functions for bound systems as travelling waves

Students in quantum mechanics classes have typically studied waves in a variety of contexts (*e.g.*, in introductory and upper-division courses on classical mechanics and electromagnetism).

We have found that some students seem to confuse the properties of two common kinds of waves: traveling waves and standing waves.

The explanation below was given in response to a version of Task 2.1. The student was shown a graph of the wave function at an initial time, $t = 0$, and asked how the graph would change in time for a single point labeled x_1.

> *"The graph of [the] wavefunction shows that it is a function of sin(t) and at t=0, $x_1=3\pi/2$. So when the wavefunction move[s] to the right $\pi/2$ second later, the zero point on graph hit[s] x_1."* (Task 2.1)

This student appears to be treating the wave function as a traveling wave that moves from left to right, rather than as a standing wave. On Task 2.1 (N=416), about 10% of the students gave similar responses. This tendency has been most prominent for tasks that include a visual representation of the wave function.

2. Tendency to treat time-dependent phase factors as decaying exponentials

The time-dependent phase factors in a superposition of energy eigenstates are imaginary exponentials. One of their properties is that the overall magnitude does not change; it is always equal to 1. Previous research has shown that many students treat these factors as rising or falling exponentials.[28] This tendency appears in about 10% of the responses to Task 2.1 (N=416), which asked students to determine and describe the time dependence of probability densities.

> *"Since the wave equation will gain a* $e^{-E_2 t/\hbar}$ *term to represent its evolution as time goes on, the probability of finding the particle in the marked area will decrease […] since the square of its wave equation will decrease as well."* (Task 2.1)

This student has omitted the imaginary unit in the exponent, which is not uncommon.[14] However, even students who wrote the term correctly often gave answers consistent with treating the term as a rising or falling exponential.

It should be noted that falling exponentials are sometimes described as *decaying* exponentials. In addition to the mathematical error in the response above, some students may interpret "decay" as referring to a change in which the energy of a particle decreases. (See section 2.4D2.)

3. Tendency to misinterpret the real and imaginary components of the wave function

Complex numbers play a critical role in quantum mechanics, especially with respect to time dependence. Student responses on many of the tasks presented in this paper revealed fundamental errors in their reasoning about the real and imaginary parts. Figure 2.5 (see next page) shows part of a hand-written response to a version of Task 2.1. In this version, students were told to consider a particle in the first excited state of an infinite square well and asked about the probability of finding the particle in a given region at several times. This student correctly identified the time dependence of the wave function and recognized that the overall phase is given by 1, i, and –1 at times t_0, t_1 and t_2 respectively. However, the student appears to associate the imaginary wave function with a probability of zero, and is apparently uncertain about the probability when the wave function is negative.

The response below articulates the same idea for a student who is thinking about time evolution graphically:

"When the wave function rotates to being entirely in the imaginary axis the wave function will be equal to zero." (Task 2.1)

In both of these responses, the students correctly identify a rotation of the wave function in the complex plane. However, both students seem to treat the imaginary part of the wave function as if it does not contribute to the probability density. This line of reasoning was evident in about 10% of the responses to both Tasks 2.1 and 2.2.

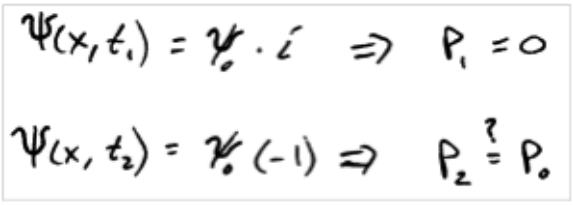

Figure 2.5: A hand-written student response

This response was given to a version of Task 2.1 that illustrates a common student difficulty in relating imaginary numbers to probabilities in quantum mechanics. The subscripts 0, 1, and 2 refer to instants in time.

In response to some versions of Task 2.3, students made a different error involving imaginary numbers. In these versions, students were asked to consider several superposition states, with either degenerate or non-degenerate energies, some of which include a term that has an imaginary coefficient. Some students asserted that the wave function, the probability density, or the energy probabilities only change with time for the states that include an imaginary term.

"ψ^2 depends on time because it has a complex component." (Task 2.3)

"The imaginary component rotates as time passes." (Task 2.3)

These students seem to be associating a time dependence with the *presence* of imaginary numbers. We have also identified similar errors on questions about wave functions and complex numbers that do not ask about time dependence. See chapter 3 for more details.

4. Confusion between coordinate and Hilbert spaces

On Task 2.3 (N=285), students are asked to consider a particle in an isotropic three-dimensional harmonic oscillator. Often, this is one of the first times that they have considered a potential that depends on more than one spatial dimension. One of the questions asked about the time evolution of a superposition state consisting of two eigenstates with degenerate energies. About 5% of the students seemed to treat the particle as if it were moving alternately in different spatial directions.

"[The state] will oscillate between x and z with y component constant." (Task 2.3)

Students who give answers like this seem to be confusing the three spatial directions with the real and imaginary axes through which the wave function rotates in time. Similar difficulties involving student confusion of physical spaces and Hilbert spaces have been previously documented.[30,31] It is also possible that this student views the particle as switching from one state to another, similar to the difficulty discussed in section 2.4A2.

D. Tendency to Apply Ideas about Time Evolution that Lie Outside the Model for QM

A number of errors that students made on the research tasks suggest that they were using ideas that are inconsistent with the basic model and formalism that are being presented in the

course. (These difficulties were documented in detail in the PhD thesis of Andrew Crouse, with which our results are consistent over hundreds of responses.[4]) In some cases, these alternative ideas might be considered as arising from mathematical difficulties, but many seem to be conceptual in nature. We have grouped these together since they often appeared to be an attempt by students to incorporate outside knowledge (*e.g.*, electron transitions in an atom) into the model for quantum mechanics that they are being taught. We have also found this interpretation useful in guiding the design of curriculum to address the underlying problems.

1. Belief that the wave function will spread out over time

One of the systems typically described near the beginning of an introductory quantum mechanics course is that of a free particle. The general form of the wave function for a free particle is often termed a "wave packet", and the time dependence is frequently described as "spreading" from a narrow to a broad distribution in position space. Some students use similar language to describe the time dependence for other quantum systems.

For example, the student below describes how the probability density for a finite region in an infinite square well would change for the first excited state of the particle.

> *"Over time the function would need to spread out and become symmetric."* (Task 2.1)

This response is consistent with the behavior of a wave packet. The example below expresses the same idea in a different way.

> *"All regions have equal possibility because there is enough time for the wave to become undefined."* (Task 2.1)

Between 5% and 15% of the students gave responses to Task 2.1 (N=416) consistent with the idea that an energy eigenfunction would spread out over time.

In a variant of Task 2.2, students were asked whether or not the probability of measuring the ground state energy would change with time for a state initially in an unequal superposition of the ground and first excited states. The following response to this task illustrates how reasoning about the spreading of the wave function has been applied to the probabilities of energy measurements.

"Still [could measure] either E_1 or E_2 but [the probabilities] will disperse." (Task 2.2)

This student seems to believe that the probabilities of each possible outcome, which were initially unequal, will change. Other students have stated that as time progresses it becomes possible to measure other energies, until all allowed energies are both possible and equally probable. (The response of the prior student, who stated that all positions eventually become equally probable, is consistent with the idea that the state eventually includes terms corresponding to all energy eigenvalues.) We have also seen evidence of this line of reasoning in other contexts, such as angular momentum.

2. Belief that the wave function will return to its initial state

On Task 2.4 (N=215), students are asked to consider the time dependence of a wave function in a potential that is perturbed. They are asked about several different intervals: *before, during,* and *after* the perturbation. The following response is for the interval after the perturbation is removed.

"Now it will time evolve back to [the] original wave function." (Task 2.4)

The idea that the wave function reverts to the form it had before the perturbation ("revival" of the wave function) is very common on this task. It is also common on questions that deal with successive measurements of a system or when students were asked about the long-term time dependence of a system, for example, on versions of Task 2.2 that include questions about an energy measurement after an initial measurement of energy or position (see chapter 3). About 20% of student responses to all versions of Task 2.4 are consistent with this idea (the percentage for individual quarters varied from as little as 5% to as high as 45% of the students).

In some cases, we have asked a version of Task 2.4 in which students consider a particle that is initially in an excited state, instead of being initially in the ground state. While some students still give answers consistent with revival, others explain that the particle will "decay" to the ground state. It is possible that student knowledge of atomic or radioactive decay may influence these answers, even though this is not consistent with the formalism they have learned. It is also possible that this idea is reinforced by the terminology of a decaying exponential discussed in section 2.4C2.[4]

2.5 – Summary

This paper documents results from an investigation into student understanding of time dependence in quantum mechanics. The focus is on the extent to which students are able to recognize which quantities depend on time, how they depend on time, and the impact of the time dependence on real-world phenomena. The findings suggest that many students have not understood key aspects of the formalism that have been taught in the course. The persistence of

specific difficulties throughout many different courses—taught by different professors, using a wide variety of lecture styles, over many years, and at several different institutions—indicates the extent to which they are strongly held and resistant to instruction. The identification of these difficulties is an important first step toward developing curriculum specifically tailored to address the most common student difficulties.

Some of the problems that we have identified are related to the mathematics underlying the physical concepts, for example, in interpreting the role of imaginary numbers. However, many are related directly to the physics, such as the finding that many students incorrectly relate the time dependence of different quantities. For example, some treat quantum states as if they do not depend on time and base this belief on the fact that the probabilities for stationary states are time independent. Others reason in reverse, assigning the time dependence of a quantum state to that of the associated probability density.

Perhaps most troubling is the failure of many students to recognize which phenomena can and cannot be accounted for by the model for quantum mechanics that they are studying. When asked about the time evolution of a system, many give an answer that is not prescribed by the Hamiltonian. Some treat quantum states as if they will "decay" to states with lower energy or as if the corresponding probability distributions will "spread out" over time. They seem not to understand critical elements of the model that is being presented (*e.g.*, the basic assumptions that are included in the model), as well as how to use it to make predictions about the behavior of a system. This is particularly apparent in the relatively difficult context of time-dependent perturbation theory. In this application, we found that many students were unable to describe the

time dependence of the wave function, the probability density, and the probabilities of energy measurements.

Most of the difficulties were more common among students in the junior-level course, but some occurred at all levels of instruction and persisted to the graduate level. In some cases, the graduate students were more likely to give correct answers; however, their explanations were often incomplete and contained conceptual and reasoning difficulties similar to those of students at the junior- or sophomore-levels. This finding suggests that the ideas are not easy, and not addressed simply by additional instruction.[3,14,10,32]

The findings from this research have guided our group in the design of *Tutorials* on quantum mechanics. We have been developing and testing instructional strategies for helping students deepen their understanding of this topic while attempting to address the most common incorrect ideas. Overall, the results have been promising,[33-34] but in the process, we have found that certain errors seem to be very persistent. Small changes to the phrasing of a question or a change in context can elicit problems that had previously seemed to be addressed. Moreover, the underlying problems may appear to be resolved early in the course, but resurface when students study more complicated phenomena. The persistence of certain ideas suggests that the problems are deeply rooted and that more research needs to be done. There is a need for ongoing efforts that describe not only the specific problems that students encounter but that document instructional strategies that have proved effective at improving the learning and teaching of this difficult and abstract topic. See chapter 5 for a more detailed discussion of the curriculum that we have developed based on this research.

2.6 – Notes and References

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[18] A related investigation into student reasoning about energy measurements can be found in chapter 3 of this dissertation, and in a companion paper: G. Passante, P. J. Emigh, and P. S. Shaffer, "Examining Student Ideas about Energy Measurements on Quantum States across the Sophomore, Junior, and Graduate Levels," *Physical Review Special Topics – Physics Education Research* **11**, 020111 (2014).
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[21] We have found that the percentage of the students who give a correct answer to qualitative questions (similar to those asked in this paper) does not vary substantially from one class to another. For a quantitative discussion of this finding at the introductory level, see: P. Heron, "Student Performance on Conceptual Questions: Does Instruction Matter?" presented at the *Physics Education Research Conference 2012*, Philadelphia, PA (2012).
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[24] On Task 2.2 a substantial fraction of the students who responded correctly did not give complete reasoning. This was not the case on the other tasks. Most of these students did not give incorrect reasoning, but rather their explanations did not seem to be directly related to the time-dependent phases. Thus, it was difficult to determine whether or not they understood the underlying ideas.

[25] Note that the percentage of students who answered correctly with correct reasoning for a superposition state on Task 2.3 (~50%) is greater than the percentage who correctly answered the corresponding part of Task 2.1 (~20%). There is evidence that this difference might be a result of tutorial curriculum administered between the two tasks.

[26] Results from the administration of Task 2.4, including a description of several follow-up interviews, are discussed in more detail in Ref. 10. However, the percentages in this paper differ slightly due to the inclusion of additional data obtained in subsequent years.

[27] Task 2.4 has been given on paper and online. The online versions have tended to be multiple-choice with explanations. We have noticed somewhat different patterns in the incorrect responses when the task was given online rather than hand-written.

[28] See, for example, Ref. 15 and 7.

[29] This was only present for the interval before the perturbation was turned on, and was not seen in the paper version of the task. We suspect this is because in the paper version, students are aware that a perturbation will be added to the system. The students who took the online version did not receive this information until after they had answered the question for the "before" interval.

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Chapter 3
Student Difficulties with Energy Measurements

"If we measure real things (and I assume we measure real things) at t = 0 ψ_1 is entirely imaginary and therefore will not be measured. Maybe?"

—Junior-level quantum mechanics student, pretest

This chapter has been reproduced from Ref. 1, which was written in close collaboration with Gina Passante and Peter Shaffer. A few additional comments that are not included in the published paper have been added in square brackets where relevant. It has also been edited to match the format of this dissertation.

3.1 – Introduction

Experienced instructors recognize that quantum mechanics is a difficult subject. There is mounting evidence that many students complete a physics degree without having a firm grasp of some fundamental concepts.[2-5] Even majors who continue to graduate school in physics have similar difficulties.[6-8] However, a working knowledge of quantum mechanics is becoming increasingly important as the market for quantum technologies continues to grow. Thus, there is a need for ongoing research to document the problems that students encounter and to provide the kind of information needed to be able to develop and assess instructional strategies.

The energy eigenbasis defines the time dependence of the quantum state and is thus arguably the most important basis in quantum mechanics. Full understanding of a physical system is not

complete without knowledge of the energy eigenvalues. In a typical course on quantum mechanics, students are asked to apply their knowledge of energy eigenvalues and eigenstates to answer questions about energy measurements. For example, they are often asked to determine the possible values that could result from an energy measurement and the probabilities of obtaining each value. Implicit in many questions is an understanding of the role of the Hamiltonian. Previous research studies have demonstrated that students struggle in answering these (and other) questions. The errors suggest a broad lack of understanding of the underlying quantum formalism that can affect student ability to apply these principles to more complex and physically realistic contexts.[9-15]

In this work we analyze student responses to a set of open-ended questions. The goal is to elicit common errors and to probe the lines of reasoning that many students use. This detailed information not only informs the research base on student learning but also can guide the design of instructional materials. The results have implications for teaching both energy measurements and related topics, such as the WKB approximation, position and momentum measurements, and time evolution.

3.2 – Context for Research

The research discussed in this paper has taken place in a sequence of courses at the University of Washington (UW). An undergraduate physics degree at UW requires between one and three courses on quantum mechanics. A sophomore-level course is the first formal introduction and is the only required course on quantum mechanics for physics minors and for one of the tracks for a physics major in our Department. The course uses a spins-first approach

that covers the first five chapters of McIntyre's *Quantum Mechanics* textbook.[16] It ends with solutions to the Schrödinger equation in position-space for simple potentials. The second and third courses are at the junior level and together they cover the entire Griffiths' textbook (*Introduction to Quantum Mechanics*).[17] Most of the data in this paper is drawn from these three courses. However, we also present data collected during the first week of instruction in a graduate course on quantum mechanics. This latter data is used to document the understanding of strong physics undergraduates after typical instruction on quantum mechanics.

In all cases, the data come from qualitative questions administered after relevant instruction. These questions have been designed to try to elicit explanations that provide insight into student conceptions and common chains of reasoning. Supporting evidence also comes from individual student interviews. Both the written questions and the interviews have probed student ideas related to energy measurements in a wide range of contexts from the infinite square well potential to perturbation theory.

The sophomore and junior-level courses involved in this study have used preliminary versions of a set of tutorials for quantum mechanics that is being developed by our group.[18] Typically, the questions presented in this paper were asked *before* relevant tutorial instruction, but after lecture instruction. However, in a few cases, they were asked after early versions of the tutorials that may have somewhat influenced the results. It is for this reason, in part, that we do not focus on the exact percentages of students who have specific difficulties. Percentages that are reported should be taken as a general measure of the prevalence of particular difficulties; the percentages obtained at other institutions may differ. However the underlying difficulties reported in this paper have been present across all years of our investigation and have proved

particularly resistant to instruction. Thus, we believe they are common among similar groups of students.

3.3 – Research Perspective

The perspective taken in this paper toward research on learning and teaching has grown out of the experience that the UW Physics Education Group has had in examining student understanding and in developing research-validated curriculum. (See Ref. 19 for a more detailed discussion.) For this project, we began by trying to identify what students could and could not do through discussions with students in the classroom and analysis of written responses to questions on homework and course examinations. Our goal was to try to gain insight into the approaches that students use in solving a wide variety of problems. In the process we began to develop and modify questions that allowed us to probe incorrect lines of reasoning that seemed to be applied repeatedly by multiple students and that arose across a variety of questions and contexts.

The research involved members of our group independently interpreting many student statements as to the possible underlying approach taken by the students. Some responses did not include much detail, but were noted to be similar to more complete statements given by other students. In the process we came to identify certain approaches that we interpret as being consistent with one or more underlying conceptual or reasoning difficulties. We use the term *difficulty* to describe incorrect or inappropriate ideas or awed patterns of reasoning that students use in answering one or more of the questions.

The difficulties that we discuss in this paper stood out during the analysis, as they occurred on a wide variety of questions in different contexts and often at several different levels of instruction. However, different formulations of each question often elicited different aspects of student thinking. Individual difficulties did not necessarily arise in equal percentages on all variations of the questions. Moreover, individual students did not always respond in the same way when asked different, related questions in the same course. The results suggest that students do not necessarily have a firmly held model that is driving their responses. Nonetheless, we are finding the identification of conceptual and reasoning difficulties to be fruitful in guiding the design of curriculum.[18,20-22]

In this paper, we articulate five broad and overlapping sets of difficulties relating to energy measurements and discuss how these manifest at different levels of instruction. Those that are presented were chosen for several reasons: they are prevalent in different contexts (*e.g.*, the infinite square well and perturbation theory); they occur at different levels of instruction (*e.g.*, sophomore and junior); and they have been identified in every year of our study across multiple instructors. We include a discussion of some difficulties even if certain aspects have been documented previously in order to extend or contrast the results, or for completeness. Previous results from research are cited where appropriate.

3.4 – Sample Questions from the Investigation

A functional understanding of energy measurements in quantum mechanics requires that students be able to interpret and apply steps in reasoning to contexts they have not previously encountered. In this study, we have focused on the ability of students to recognize and apply the

following ideas: (1) a given quantum mechanical system has a set of allowed energy values and that these depend on the eigenvalues of the Hamiltonian, and thus, upon the potential; (2) a wave function defines the state of the system and provides information about the probability of measuring each energy; and (3) how time evolution and measurement can affect the possible measurement outcomes and the associated probabilities.

We have administered a wide variety of questions to probe student ideas about these aspects of energy measurements. In this paper, we illustrate three tasks, each of which elicits student ideas about several ideas mentioned above. Multiple versions of each task have been asked in various quarters to ensure that the difficulties encountered are generalizable and not an artifact of the phrasing of the question or the context in which it is asked. All variants of these three tasks are shown in the appendix.

Task 3.1: Task 3.1, shown in Fig 3.1, asks students to find the probability of an energy measurement at $t = 0$ (question 3.1a) and $t > 0$ (question 3.1b). Several versions have been given at the sophomore, junior, and graduate level. The version in Fig. 3.1 was given on a final examination in a single sophomore quantum mechanics course at UW (N=104). It indirectly probes student reasoning about the probability of measuring various outcomes for the energy of a single quantum state by asking about a large number of measurements made on many identical systems.

A correct response to question 3.1a requires noting that the absolute square of the coefficients for both terms in the wave function are equal at the given time ($t = 0$). Thus, the number of particles measured to have energy E_1 is expected to be about the same as that for energy E_2. For question 3.1b, in which the time dependence of the wave function is explicitly

shown, the reasoning is similar. The time components vanish when the absolute square is taken, so there is no time when the number of particles measured to have energy E_1 is expected to be greater than the number measured to have energy E_2.

Task 3.1

Recall that the eigenfunction of the infinite square well potential as given by $\phi_n(x)$. Collection A contains a large number of identical particles, each of which is described by the initial wave function:

$$\Psi_A(x, t=0) = \frac{1}{\sqrt{2}}\left(\phi_1(x) - \phi_2(x)\right).$$

Question 3.1a

Suppose you were to measure the energy of a very large number of particles at $t = 0$. Would the number of particles with energy E_1 be greater than, less than, or approximately equal to the number of particles with energy E_2? Explain your reasoning.

Question 3.1b

At some later time, the wave function for the particles is given by $\Psi_A(x,t) = \frac{1}{\sqrt{2}}\left(\phi_1(x)e^{-iE_1t/\hbar} - \phi_2(x)e^{-iE_2t/\hbar}\right)$.

Suppose you were to measure the energy of a very large number of particles at a later time. Is there a time $t > 0$ when the number of particles with energy E_1 is greater than the number of particles with energy E_2? If so, determine at least one such time. If not, state why not.

Figure 3.1: First sequence of questions on energy measurements

Task 3.1 was given after all instruction in the sophomore quantum mechanics course. Similar questions have been given to students at the junior and graduate level.

Versions of this task have been given to juniors and graduate students (see, for example question 3.2a in Fig. 3.2), but usually with two differences. First, the questions have involved a single particle (rather than an ensemble) and students have been directly asked about the probabilities of various outcomes. Second, we have given students the state at an instant and have not provided the time dependence (as shown in question 3.1b in Fig. 3.1). This omission adds another step to the reasoning, since students need to identify the time dependence of each term in the wave function. These versions have been asked of more than 300 students at the

junior level (after relevant lecture instruction) and of 31 students at the graduate level (before any graduate instruction).

Task 3.2: The question shown in Fig. 3.2 begins by providing students with a wave function at time $t = 0$ written as a superposition of two energy eigenstates of the infinite square well. Variations of Task 3.2 have been given to students in the junior-level course beginning in 1996.

Task 3.2

A particle in an infinite square well, with energy eigenstates $\Psi_n(x,t)$, is prepared so that its wave function at time $t = 0$ is: $\Psi(x,0) = 0.6\Psi_1(x,0) + 0.8i\Psi_2(x,0)$.

Question 3.2a

Suppose you were to measure the energy of this particle at time $t_1 > 0$. What value or values would a measurement of the energy yield?

Question 3.2b

Suppose that at a later time t_2 you measured the energy of the particle again. (Assume that the particle remains isolated from its surroundings during the time interval $t_1 < t < t_2$.) Which of the following statements would best describe the result of your second energy measurement?

i) *Definitely* the same result as the first.

ii) *Possibly* the same result as the first.

iii) *Definitely not* the same result as the first.

Question 3.2c

A particle is prepared in the ground state of the infinite square well. Its position is then measured and found to be $x = L/3$ (where L is the length of the well). Suppose you now measured the energy of the particle. Which of the following statements best describes the result of your second energy measurement?

i) *Definitely* the ground state energy.

ii) *Possibly* the ground state energy.

iii) *Definitely not* the ground state energy.

Figure 3.2: Second sequence of questions on energy measurements

Task 3.2 was given after approximately one month of instruction in the junior-level quantum mechanics course at UW. Students are also asked to explain their reasoning after each question.

Question 3.2a asks students to identify the values that could result from an energy measurement at time $t > 0$. The correct answer is that both E_1 and E_2 are possible energy measurements since they correspond to the two eigenstates represented in the wave function. Question 3.2b asks students about possible results from an energy measurement made after the first energy measurement. To answer, students could recognize that after the first measurement the wave function of the particle corresponds to an energy eigenstate. Thus, a second measurement must yield the same result as the first.

In question 3.2c, students are told that a position measurement has been made and are asked about the possible results of a subsequent energy measurement. A correct solution is to recognize that after the position measurement, the wave function is in an eigenstate of position (*e.g.*, a delta function in position space), which can be written as a sum of many energy eigenstates. Therefore, a subsequent energy measurement could yield almost any energy eigenvalue for the potential. Task 3.2 has been given to more than 200 students at the junior level.

Note that in this question, students are required to think about the state of a quantum system after a measurement has been made. In undergraduate and graduate textbooks, students are typically told that the possible outcomes of a measurement are the eigenvalues of the operator associated with the observable and that after a measurement the system is in an eigenstate corresponding to the observed eigenvalue (the "measurement postulate").[23] Sometimes the phrase "collapse of the wave function" is used. There is some disagreement among instructors as to whether or not this postulate should be taught in undergraduate courses.[24] The arguments often focus on the difficulty (or, in some cases, impossibility) of performing such measurements.

Nonetheless, many undergraduate courses discuss the state of a system after a measurement is made and we have found this context provides an excellent opportunity to probe student understanding of energy measurements.

Task 3.3: Task 3.3, shown in Fig. 3.3, involves a particle that is measured to be in the ground state of an infinite square well potential. At a later time, a perturbation is applied for a finite amount of time, after which it is removed. Students are told to treat the perturbation as beginning and ending instantaneously. They are asked to determine the values that could result from an energy measurement that is made before (question 3.3a), during (question 3.3b), or after (question 3.3c) the perturbation is applied. Versions of this question have involved several different perturbations (*e.g.,* a delta function or a small rectangular perturbation at the center of the well).

Task 3.3

A particle is measured to be in the ground state of the infinite square well at time $t < 0$. At a later time, $t_{on} > 0$, a small perturbation is added to the center of the well. At a later time, t_{off}, the perturbation is removed.

Question 3.3a

Before the perturbation, $0 < t < t_{on}$, what possible values could an energy measurement yield?

Question 3.3b

During the perturbation, $t_{on} < t < t_{off}$, what possible values could an energy measurement yield?

Question 3.3c

After the perturbation has been removed, $t_{off} < t$, what possible values could an energy measurement yield? Assume that no measurements where made while the perturbation was present.

Figure 3.3: Third sequence of questions on energy measurements.

Task 3.3 has been given after lecture instruction on perturbation theory at the junior level. Questions testing similar ideas have been asked at different points in the junior year and at the graduate level.

Before the perturbation has been applied, only the ground state energy of the infinite square well can be measured, since the particle was initially in the corresponding eigenstate. For the instant at which the perturbation is applied, students should assume that the wave function is continuous. Therefore the mathematical form just after the perturbation is applied still corresponds to that of the lowest energy eigenstate of the infinite square well. However, during the period that the perturbation exists, the new Hamiltonian determines the energy eigenstates of the wave function, and only the corresponding energies can be measured. At the start of the perturbation, the wave function in general will not be in an eigenstate of the new potential and its time evolution is determined by the new energies. Thus, the wave function will evolve out of the initial unperturbed eigenstate. Finally, at the instant the perturbation is removed, the wave function can again be assumed to be continuous. The energies that can be measured after the perturbation is removed are once again those corresponding to the square well. However, at this point the wave function corresponds to a superposition of many energy eigenstates of the infinite square well and a measurement of the energy is no longer restricted to the ground state energy. More than 140 students at the junior level have answered versions of this question. The results are similar when it is given before or after lecture instruction on time-dependent perturbation theory.

3.5 – Student Understanding of Energy Measurements

In responding to the questions described in the previous section, students have made many, various errors. We have identified five broad categories of conceptual and reasoning difficulties that appear to underlie many of their answers and that seem to persist throughout instruction. In

this section we provide examples of student responses when they help to illustrate or substantiate specific difficulties. It is important to recognize that individual student statements may fit into several categories and that no single statement completely encapsulates a particular difficulty. The categories presented have been developed on the basis of examining a large number of student responses to multiple different questions and draw on data obtained from individual student interviews. Less formal discussions with students in class or in office hours have also contributed to the analysis. The percentage of students who made a particular error is included for those cases in which the responses did not appear to depend strongly on the version of the question that was asked.

A. Failure to Understand the Relationship between the Wave Function and Possible Energy Measurements

In their responses to the questions above, many students revealed a lack of understanding of the relationship between the wave function and the possible energy values and their corresponding probabilities. Some of the questions probed student ability to find these quantities from the wave function; others examined student ability to reason in reverse (*e.g.*, by asking students to find the possible results of an energy measurement after a first measurement has been made.) The latter reasoning is arguably more difficult, since students must first determine the wave function after an energy measurement and then use it to obtain information about a future measurement.

1. Finding the probability of an energy measurement from the wave function

Research by others has demonstrated that students often think that an energy measurement can yield the expectation value of energy[10] or they calculate the probability density in order to find the probability of an energy measurement.[9] In our research we have also identified these errors at the junior-level, however, we find that these errors are less prevalent among graduate students, and they did not appear at all in sophomore students responses to Task 3.1.

The errors made by students in the sophomore-level course often seemed to be at a more basic level than those mentioned above. A common error was to relate the probability of making a specific energy measurement directly to the coefficient of the corresponding term in the wave function—without performing the absolute square (15%, N=104). Some students did this explicitly. For others this error was more implicit. See, for example, the following response to question 3.1b (Fig. 3.1).

> *"These [energy eigenstates] are rotating around according to the Schrödinger time evolution as given by* $e^{iEn/\hbar}$*. The value for* $iE_n/\hbar$ *is not the same so they are rotating at different speeds so at some point the probability of being in* E_1 *will be greater than* E_2*."* (sophomore, question 3.1b)

This student appears to have a good understanding of the time evolution of this wave function. He describes the rotation of the real and imaginary components of the wave function in the complex plane. However, the student does not correctly associate the square of the coefficients in the wave function to the probabilities of the energy measurements. Like many of the students on this question, he seems to be treating the real part of the coefficient as the only important component.

The tendency to focus on the real part of the wave function was not limited to students in the sophomore course. One of the variations of question 3.1b asked in the junior-level course presented students with the wave function $\Psi_A(x,0) = i\sqrt{\frac{1}{3}}\psi_0 - \sqrt{\frac{2}{3}}\psi_1$ and asked whether there is a time when the probability of measuring E_0 is equal to 1.

> *"Yes, ψ_1 also rotates through the complex axis in time, meaning there are times at which it is completely imaginary, and thus unobservable."* (junior, variation of question 3.1b)

This student explicitly states that only real parts of the wave function can be measured. Approximately 10% (N=123) of the junior-level students gave answers consistent with treating an imaginary coefficient for a term in the wave function as meaning that the corresponding energy value is not measurable.

Some of the students who did not properly square the coefficient also treated the complex exponential as if it were instead a decaying exponential. This tendency is discussed in greater detail in section 3.5D. Even some students at the graduate level struggled with versions of question 3.1b. For example, when asked if the probability of measuring a particular energy at a later time is the same as it is at t = 0, many students explained that the time dependence of energy measurements depends on the time dependence of the state. Statements like the one below were common.

> *"It [the probability of measuring E_n at a later time] should be the same [at all times]. $|\psi\rangle$ is not relevant with t. So $\langle\psi|\hat{H}|\psi\rangle$ doesn't change."* (graduate student, variation of question 3.1b)

This student obtained the correct answer, but the underlying reasoning is flawed, since $|\psi\rangle$ does in fact depend on time. The student also seems to treat $\langle\psi|\hat{H}|\psi\rangle$ as representing the probability of an energy measurement, rather than the expectation value. Similar tendencies have been reported in Ref. 9.

2. Determining the outcomes of a subsequent energy measurement

Many of the responses to the questions suggested that students were having difficulty in determining the wave function after an energy measurement is made. To answer Tasks 3.2B and 3.3A, for example, students must recognize that after the energy measurement, the wave function corresponds to an energy eigenstate. Thus, there is only one possible outcome for a future energy measurement. Although these questions do not explicitly ask students to write the wave function after the first measurement, the explanations indicate that many failed to recognize or apply this key idea. On question 3.2b, for example, 45% of junior-level students (N>200) incorrectly answered that the result of a second measurement would possibly, but not necessarily, be the same as the first. The student statement below is exemplary.

> *"The particle is described by a wave function with elements in both eigenstates. Although a measurement of energy collapses it to one, the possibility of the other still exists, so a second measurement could get the other E."* (junior, question 3.2b)

This student states that there is still a non-zero probability associated with the other energy value, even though he has knowledge of a "collapse" of the wave function. These results corroborate findings by Zhu and Singh.[9]

We have asked questions similar to the one above but differ in that students are not given the initial wave function, but rather are told the result of the first energy measurement, *e.g.*, that the measurement yielded the ground state energy. (See, for example, question 3.3a.) Lacking information about the state before the measurement, students often answered that any of the energy eigenvalues of the Hamiltonian could be obtained.

The failure to recognize how a measurement affects the state of a system does not seem to be limited to measurements of energy. In the sophomore course, we have asked similar questions about measurements of spin. Many of the same types of difficulties have emerged. For a more detailed discussion of student difficulties with sequences of measurements, see chapter 4.

B. Tendency to Associate the Time Dependence of Energy Measurements with Properties of Stationary States

Some previous studies have demonstrated that students often have difficulty in identifying stationary states or in answering questions that ask explicitly about properties of stationary states.[9] We have found that even on questions that do not involve stationary states, students spontaneously invoke the term, often incorrectly, in justifying their ideas about how the energy of a state does or does not change with time. (In response to question 3.1b, about 10% of the sophomore students (N=104) explained their answer in terms of stationary states.)

Stationary states are energy eigenstates and as such, their probability densities do not change with time. We have found that students at all levels have a tendency to extend this property both to the wave function and to the probability of energy measurements. For the scope of this paper we restrict ourselves to discussing the latter.[25]

Consider the following explanation to Task 3.1B regarding the state $\Psi_A(x,t) = \frac{1}{\sqrt{2}}\left(\phi_1(x)e^{-iE_1t/\hbar} - \phi_2(x)e^{-iE_2t/\hbar}\right)$.

> *"*$\Psi_A(x,t)$ *is a stationary state, the probabilities [of energy measurements] will not change."* (sophomore, question 3.1b)

The student has provided the correct answer, but has made two important errors. First, this student has incorrectly identified the wave function as being a stationary state. (In our study, we found many students treated the sum of stationary states as also being a stationary state. This error is common and has been documented in several prior studies.) Second, this student has indicated that because it is a stationary state the probability of energy measurement outcomes will not change in time. Although the probabilities do not depend on time, this is true for all quantum states in a time-independent potential, not just stationary states.

A similar question was asked on a final examination in the junior-level course. Students were given a three-dimensional wave function $\psi(r,\theta,\phi,t=0) = N\left(\psi_{321} + 2\psi_{311} + 2\psi_{210}\right)$ where ψ_{nlm} are the normalized energy eigenfunctions with energy eigenvalues E_{nl}. They were asked to find the probability that an energy measurement at $t = 0$ would result in E_{21} and asked whether or not their answer would change if instead the energy were measured at a later time $t > 0$. Almost all students (approximately 80%, N=44) correctly answered that the probability would not change if it were measured at a later time; however, only 25% gave correct explanations. Many reasoned using stationary states, as demonstrated below.

> *"No because all [three terms] are energy eigenstates which do not change with time. They are stationary."* (junior, variation of question 3.1b)

> *"It [the probability of an energy measurement] would not change. The eigenvectors of energy are stationary states, so their coefficients will not be changing in time."* (junior, variation of question 3.1b)

The first student correctly identifies the ψ_{nlm} as stationary states, but then states, incorrectly, that they do not depend on time. The student then uses that as the basis for stating that the energy probabilities do not depend on time. The second student also gives a correct answer, but explicitly states that since each term in the wave function corresponds to a stationary state, none of the coefficients have a time-dependent phase. Although each of these students arrives at the correct answer, they do not correctly relate the time dependence of quantum states to the time-independence of the probability of energy measurements. We have found that some graduate students make similar errors, although less frequently.

C. Difficulties Related to Outside Knowledge

On all of the questions we have asked, some students have answered based on ideas that lie outside of the model for quantum mechanics that has been developed in class. The specific errors and the types of reasoning have varied depending on the level of instruction.

On Task 3.1 (Fig. 3.1), a consistent fraction of the students (about 10%, N=104) in the sophomore course stated that lower energy values are more likely to be measured than higher values. (Recall that the particle is in an equal superposition of two energy eigenstates.)

> *"[The probability of finding the particle in the ground state is] greater than [the first excited state]. It requires more energy for particles to stay in E_2 than E_1. So particles are more likely to be in a more stable state E_1."* (sophomore, question 3.1a)

This student does not seem to recognize that the probabilities for the energy measurements are determined by the corresponding coefficients in the wave function. (See the discussion in section 3.5A.) A possible explanation for this response is that the student is using reasoning from classical physics about particles being more likely to have the lowest available energy. Another explanation is that the student recalls information about spontaneous emission. In either case, the student is neglecting to apply the model that has been developed in class. We found that reasoning based on outside knowledge was more prevalent at the junior level when the questions involved subsequent measurements. For example, on question 3.2b (Fig. 3.2), about 10% (N=141) of students used conservation of energy to explain the possible outcomes of future energy measurements. The following response is typical of these students.

> *"Since no energy was added or subtracted [during the measurement], there MUST be the same E due to the law of conservation of energy."* (junior, question 3.2b)

This student arrived at the correct answer, but did so by assuming that the prior measurement did not affect the energy of the system. The same reasoning yields an incorrect answer when it is applied to energy measurements performed after a position measurement (or a measurement of any other incompatible observable). For example, on Task 3.2C, some students stated that the position measurement did not add energy to the system, so the energy distribution will be the same as before the position measurement. Other students, like the one following, stated that the position measurement would *add* energy to the system, so a subsequent energy measurement would result in an increased value.

> *"The particle's position was measured which means some energy must have been [added] into the system—which would excite the particle above the ground state."* (junior, question 3.2c)

We have also seen instances of this chain of reasoning from graduate students. The explanation below was given in response to a question similar to Task 3.3 (Fig. 3.3). In this version, the potential was instantly changed from the infinite square well to the simple harmonic oscillator and students were asked about the possible results of an energy measurement some time after the change.

> *"The energy will be unchanged. I'm guessing this because I don't want to violate conservation of energy."* (graduate student, variation of question 3.3b)

Some students used similar reasoning in their responses to question 3.3b, in which a small perturbation is added to an infinite square well. In many versions of the question, the perturbation is non-zero over only a small portion of the well (*e.g.*, a small constant positive or negative correction applied to a finite region near the center of the well). Some students reasoned that the possible energy measurements are found by adding the value of the perturbation to the energy eigenvalues for the infinite well. In some cases, students thought that the energy that could be measured would depend on where inside the well it was measured. For example, the possible energies would be those of the unperturbed infinite square well if the measurement were made where the perturbation is zero.

Related to these errors is a tendency of students to treat the energy of any system as being well-defined (*i.e.*, there is only one possible outcome from an energy measurement). Perhaps students who indicate that the expectation value of the energy could result from a single energy

measurement (see section 3.5A) are using this reasoning. Interactions with students during tutorials and office hours provide us with additional support that this may be affecting some of their answers.

D. Failure to Recognize that the Time Evolution of an Isolated System Is Determined by the Schrödinger Equation

On some of the questions that we have asked, student responses have reflected a tendency to ascribe a time dependence to isolated systems that goes beyond that described by the Schrödinger equation. Students have often predicted that a particle will either decay to the ground state, return to its initial state (revival), or evolve to include all possible values (diffusion). (These have all been previously identified in the Ph.D. Dissertation of Andrew Crouse,[2] and the revival evolution has also been discussed by Zhu and Singh.[9,11]) In some cases, students are using knowledge from outside the course as was discussed in the previous section. The difficulties below are treated separately since they all seem to be related to the dynamics of quantum systems.

Decay reasoning: In some versions of the questions that we have administered, students are asked about the possible energies that can be measured shortly after an energy measurement and a long time afterward. On these versions, many students respond that the particle will eventually decay into the ground state. We also see evidence of this in the time-dependent perturbation theory question in Fig. 3.3, for which some students have stated that the particle will eventually end up in the ground state. This reasoning is most common when students are told that the particle begins in an excited state (as opposed to the ground state).

> *"As time goes on the particle is more likely to settle into the ground state whatever that may be."* (junior, question 3.3b)

Many students fail to provide a detailed explanation for why they think the particle will evolve to be in the ground state. However, the reasoning given by some suggests that they are recalling ideas they have studied earlier, such as radioactive decay and spontaneous or stimulated emission.

Revival reasoning: On questions that ask about the possible energy values that could be measured after a prior energy measurement, students often indicate that the particle will return to the initial state. Others do not explicitly indicate that it will return to that state, but they give probabilities for measuring the energy that are the same as they were before the measurement. These types of answers were given by about 20% (N=141) of the students on question 3.2b (Fig. 3.2). A sample response is given below.

> *"The time interval, minute, is long enough for the wave to return back to normal, so the option for both [energy measurements] is there like before."* (junior, question 3.2b)

It is not clear what this student means by the term normal, however the student clearly states that both measurement options are possible, even after the first energy measurement has been made.

We also have found that many students express the idea that a quantum system has a "memory" in time-dependent perturbation theory. For example, in question 3.3c (Fig. 3.3) students often predict that the particle will return to its initial state once the perturbation is removed. This idea is not restricted to the context of energy measurements. On questions involving successive spin measurements, students often express similar ideas.

Diffusion reasoning: In response to many of the questions we have asked, some students have responded that after a measurement has been made, the state will eventually consist of an equal superposition of all available eigenstates. The following student statement is in response to a version of question 3.2b (Fig. 3.2) in which a particle was first measured to have energy E_1 and the question asks what are the possible results of a subsequent energy measurement.

> *"If the energy were measured immediately afterward, the probability is likely to remain E_1 because the wave function has not yet spread out into its full linear combination. However, if the energy is measured a long time later, then the probability of measuring E_1 decreases, because the particle is now in a superposition of states whose energies are clearly not all E_1."* (junior, variation of question 3.2b)

We have also observed this line of reasoning in other contexts, such as angular momentum. For example, if a particle is known to have the angular momentum quantum numbers $l = 1$ and $m_l = 1$ (where m_l is the quantum number for the z-component of angular momentum), then the possible quantum numbers for angular momentum in the x-direction are –1, 0, and 1. Many students incorrectly state that each option must have an equal probability of 1/3. This particular result is discussed more in section 9.3.

E. Failure to Recognize the Role of the Hamiltonian in Determining the Possible Energy Values

We have found that many students have difficulty in relating the Hamiltonian to the possible energy values for a system. Two types of errors were common in response to the questions we asked.

The first type of error was evident on versions of question 3.1b, all of which involve a superposition of two energy eigenstates. The questions probe student understanding that the probability of energy measurements for such a system does not change in time. Many students answered correctly, and correctly stated that the absolute squares of the coefficients are not time dependent. However, some based their reasoning on that the fact that the potential is not changing. (It is correct in that, if the potential did depend on time, then the probability of the energy measurements would change. However, in that case, the measurable energy values themselves would also vary.) These responses are consistent with the belief that the probabilities don't have time dependence if the Hamiltonian is time-independent. We have also paired this question with one that asks about the time dependence of the wave function, and have found that students have answered in the same way for both questions. This suggests that the difficulty is more general and reflects a misunderstanding of the role of the Hamiltonian in quantum systems.[26]

The second type of error was demonstrated in responses to Task 3.3 (Fig. 3.3), which asks about possible energy measurements for a particle in the ground state a system, before, during, and after a perturbation. Almost 30% (N=140) of the (junior-level) students incorrectly indicated that the original, infinite square well energies could be measured while the perturbation is present. These students seem to be using the basis in which the wave function is written, and not that for the current Hamiltonian, to determine the possible results of an energy measurement. This error also occurred for the period after the perturbation has been removed. About 40% answered correctly that any of the energy values associated with the original, unperturbed potential could be measured; however, about half of them have stated that it takes time for the

particle to "feel the effects" of the change to the potential. They do not seem to be arguing that the potential itself takes time to change, but rather that the wave function needs time to adapt. The explanations suggest that students are treating the wave function, and not the Hamiltonian, as determining the possible outcomes of an energy measurement.[27]

Graduate students gave very similar responses on a variation of Task 3.3. This version began with a particle in the first excited energy eigenstate of an infinite square well. Students were told that the potential then instantaneously changes to that for a harmonic oscillator and were asked to list the possible results of an energy measurement made immediately after the change. Only 25% (N=31) of the graduate students answered correctly. About 40% incorrectly stated that the original infinite square well energy could still be measured, as seen below.

> *"It should still be E_2—the wave will take time to conform."* (graduate student, variation of question 3.3b)

Many, like the student above, reasoned that it would take time for the particle to feel the effects of the change. About half of the students who gave this response predicted that some time later the harmonic oscillator energies would be measured.

3.6 – Summary

In this paper, we have documented some of the difficulties that students have in reasoning about the results of energy measurements on quantum systems. We have probed student thinking about which energy values can be measured for a given physical system, how to calculate the probabilities of obtaining each energy value, and how both time evolution and measurement can affect these probabilities. The results provide insight into some of the

conceptual and reasoning difficulties that students encounter in the study of quantum mechanics. We have found it useful to group the underlying problems into five related categories. These are not limited to sophomore or junior-level students, most also arise at the graduate level.

At the sophomore level, many of the errors seem to be relatively basic in nature, such as a failure to use the absolute square of the coefficients in a wave function to find the probabilities for the measurements of energy. However, the errors often seemed to have conceptual underpinnings. For example, some responses are consistent with a belief that only the real part of the wave function contributes to the energies that can be measured and their associated probabilities. This and other ideas seem to persist to the junior level and beyond. In the sophomore level course, for example, we found evidence of students treating the term *stationary state* as implying that the wave function itself (not just the probability density) has no time dependence. The students then used this reasoning as the basis for arguing that the probability of energy measurements do not depend on time. This incorrect chain of reasoning proved to be persistent and we saw evidence for it at both the junior and graduate levels.

One of the most interesting errors we found at the sophomore level was evident in only a small fraction of the students (~5%), but the reasoning they used provided insight into errors made by students at more advanced levels. Some of the sophomores seemed to be reasoning that the probability of a given energy measurement depends on time only if the potential has a time dependence. Although this chain of reasoning was not common among sophomores, it was relatively common at the junior level in the context of perturbation theory. We had seen this reasoning previously among these students and thought it was an isolated and question-dependent error. However, the underlying ideas appear to be much deeper, and suggest that

many students fail to understand the relationship between the time-dependent and the time-independent Schrödinger equations.

At the junior level we also asked questions about subsequent measurements made on a system. Chapter 4 of this dissertation explores student understanding of sequences of measurements in more detail. The results suggest that there is considerable confusion about the effect of a first measurement on the possible outcomes of later energy measurements. Students often gave answers that suggested they thought that the particle eventually returns to the initial state or would evolve to include all possible energies. Some provided explanations that indicated that the energy of a system does not change when a measurement is made.

Finally, at both the junior and graduate level we found that when faced with an abrupt change to the potential, a large fraction of students claim that the energy eigenvalues of the former potential can still be measured after the change. They reason it would "take time for the particle to feel the effects of the change." These students seem to be aware that the potential determines the possible energy values, as many of them articulate that, some time after the change, the energy eigenvalues of the new potential can be measured. However, instead of using knowledge of the instant the measurement is made, they appear to be incorporating their knowledge of the system moments earlier.

We have found that a critical step in the design of effective curriculum is the identification of ideas that underlie the errors that students make on given topic. This process is guiding the design of curriculum for the junior-level quantum course (*Tutorials in Physics: Quantum Mechanics*).[18] At this stage, there is evidence that some of these tutorials have a direct impact on student conceptual understanding.[28] However, significant work still needs to be done. Some of

the most common difficulties seem to be persistent and resurface when students are asked questions in different ways or in new, more difficult contexts. The discussion in this paper has focused on some of the areas that we have found relevant for most topics in quantum mechanics, and in recent versions of the tutorials we have made an effort to continually reinforce many ideas relating to energy measurements throughout the undergraduate course content.

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Chapter 4
Student Difficulties with Sequences of Measurements

"Once you measure the position it should not change, like other eigenvalues/states.

I can measure the position of a gnat flying around ... but it has momentum ... so if I measure it again it may be in a new position .... And since I measure position, it has all momentums, so why would I expect that it hasn't moved?"

—Junior-level quantum mechanics student, pretest

This chapter presents results from research into student understanding of the impact of measurements on quantum states. In a typical junior-level quantum mechanics course, students are taught that a measurement can alter the state of a particle, and thus that a measurement can influence the results of future measurements. (See section 3.4 for a discussion of our perspective on why it is valuable to examine student understanding of ideas such as the "measurement postulate" and the "collapse" of the wave function.) The effect that measurement can have on the state of a system is one of the defining features of quantum mechanics that sets it apart from classical physics. Indeed, the consequences of quantum measurements seem to be particularly challenging concepts for many students. In chapter 3 we demonstrated that students often give incorrect answers and explanations when they are asked to determine the possible outcomes of a measurement of a particle's energy after a previous measurement of that particle's position has already been made (see, for example, question 3.2C).

Many incorrect student beliefs about time dependence and measurement in quantum mechanics, such as those discussed throughout chapters 2 and 3, arise when students reason

about measurements that happen sequentially.[1-4] This chapter focuses on those difficulties that appear to be related directly to changes in quantum states resulting from measurements and also related to the behavior of subsequent measurements. Where there is overlap between the findings in this chapter and previous chapters, we give a brief description of the related difficulties and reference the relevant sections of previous chapters.

We describe the context for this research in section 4.1. Then, section 4.2 discusses student difficulties with sequences of energy and position measurements, including a more in-depth analysis of student responses to question 3.2C. Section 4.3 focuses on a similar set of questions that involve successive measurements of spin in different directions. The goal of these questions is not to identify new difficulties, but to probe the extent to which the difficulties identified in section 4.2 persist in questions using alternate contexts.

4.1 – Context for Research

Most of the research discussed in this section involves students in the junior-level quantum mechanics course at UW. In this course, students are introduced to the effect of measurement on a quantum system after about 4-5 weeks of instruction.[5] All responses from junior-level students come from online surveys administered after students have covered the effects of quantum measurements in class. In some cases, students also completed tutorial instruction related to this topic; when this is the case, it is noted.

Most of the junior-level students have also been introduced to the effects of quantum measurement in a prerequisite quantum course at the sophomore level. During the years included in this investigation, the exact content of the prerequisite course has varied. In some

cases, it was taught spins-first; in others, it was a more traditional introductory quantum course focusing on wave functions and position space (for more information about the changes to the sophomore-level course, see section 1.2A). In each case, students learned about the effects of quantum measurements. All responses from sophomore-level students discussed in this chapter were gathered during quarters when the class was taught spins-first. These responses also come from online surveys given after lecture instruction on the relevant topics. In the sophomore-level class, students had no tutorial instruction focusing on sequences of measurements.

4.2 – Sequences of Energy and Position Measurements

In this section, we focus on identifying student difficulties with quantum measurements made *after* a previous measurement has been made. Section 4.2A describes the questions that were used to probe student understanding. Section 4.2B discusses the common difficulties that we identified in the responses to each question. Lastly, section 4.2C summarizes the results, and relates them to prior research on student understanding of the effects of quantum measurements.

A. Tasks for Identifying Student Difficulties

Figure 4.1 shows the two questions (Task 4.1) that were given together as part of a tutorial pretest in the junior-level quantum course at UW. Prior to the pretest, students had completed tutorial instruction on time dependence and energy measurements (see section 5.1C), but had not yet completed tutorial instruction on position measurements. The completed tutorial instruction included exercises intended to help students determine how the wave function is changed when a measurement of energy is made, and the time dependence of the wave function afterward.

Task 4.1

Question 4.1a.

A particle is in the infinite square well potential from $x = 0$ to $x = a$. At time $t = 0$, the energy of the system is measured and found to be E_2, the first excited-state energy.

At time $t_1 > 0$ you measure the *position* of the particle and find that it is located near $x = a/3$.

Suppose a short time later, at a time t_2, you measured the energy of the particle. (Assume that no other measurements are made between t_1 and t_2.)

For each of the possible energies below, select the statement that best describes the probability that the energy measurement will result in this value. Explain your reasoning.

Ground state energy First excited-state energy Second excited-state energy

i) I will definitely measure this value

ii) It is possible (but not certain) that I will measure this value

iii) I will definitely not measure this value

iv) I do not have enough information to tell

Question 4.1b.

A particle is in the infinite square well potential from $x = 0$ to $x = a$. At time $t = 0$, the energy of the system is measured and found to be E_1, the ground state energy.

At time $t_1 > 0$ you measure the *position* of the particle and find that it is located near $x = 2a/3$.

Suppose a short time later, at a time t_2, you measured the *position* of the particle again. (Assume that no other measurements are made between t_1 and t_2.)

For each of the possible positions below, select the statement that best describes the probability that the energy measurement will result in this value. Explain your reasoning.

Near $x = a/3$ Near $x = a/2$ Near $x = 2a/3$

i) I will definitely measure this value

ii) It is possible (but not certain) that I will measure this value

iii) I will definitely not measure this value

iv) I do not have enough information to tell

Figure 4.1: Questions about sequences of measurements

Task 4.1 asks students about sequences of energy and position measurements. It was administered on a tutorial pretest to students in the junior-level quantum mechanics course at UW. Question 4.1a involves a measurement of position following by a measurement of energy on a particle initially in an energy eigenstate. Question 4.1b instead involves two separate measurements of position on a particle initially in an energy eigenstate. A version of question 4.1a also appeared in chapter 3 as Task 3.2C.

Each question in Task 4.1 describes a sequence of two measurements made on a particle that is initially known to have a well-defined energy. In question 4.1a, the particle is initially in the first excited state of the infinite square well before its position is determined to be near $x = a/3$ at t_1. (The word "near" is used instead of "at" because the probability of measuring a particle at any particular position is zero.) The students are asked about the likelihood of several possible results of a subsequent measurement of the *energy* at time t_2. The students are told that time t_2 is a short time after t_1 (other questions, not discussed in this chapter, have been used to probe student understanding of measurements after a *long time* has elapsed). Question 4.1a was also discussed in chapter 3 (as question 3.2C) as part of our effort to identify student difficulties with energy measurements.

Question 4.1a can be answered correctly by recognizing that the particle can be thought of as being in an eigenstate of the position operator (*i.e.*, a Dirac delta function)[6] immediately after the measurement of position. Since the eigenstates of position are *superpositions* of energy eigenstates, in general any energy eigenvalue could be measured at t_2 after an arbitrary position measurement. If the measurement is at certain locations (*e.g.*, $x = a/3$), the energies associated with eigenfunctions that have nodes at those locations cannot be measured, since those terms would not contribute to this superposition. In this case, $x = a/3$, thus, for the ground state energy and the first excited-state energy the correct answer is *ii* (it is possible to measure those values), but for the second excited-state energy the correct answer is *iii* (it is not possible to measure that value). For the purpose of this discussion, students were considered correct if they chose *ii* for the second excited-state energy but gave explanations that were otherwise correct.

Question 4.1b is nearly identical to question 4.1a. In question 4.1b, we ask students about the possible results of a second position measurement at t_2 (question 4.1a instead asks about an energy measurement). The particle in question 4.1b also begins in an energy eigenstate (in this case, the ground state instead of the first excited state) and the position is still measured at t_1 (near position $x = 2a/3$ rather than $x = a/3$). Like question 4.1a, question 4.1b can be answered by noting that the particle is in a superposition of energy eigenstates at t_1, after the first measurement of position. Since each energy eigenstate in the superposition changes in time at a different rate, the probability density changes with time after t_1. Thus, it is possible to measure the particle at t_2 to be near essentially any value of x that lies within the bounds of the infinite square well. For all three positions listed in question 4.1b, the correct answer is *ii* (it is possible to measure those values of position).

It should be noted that question 4.1b is a more complicated form of Task 2.1 (see section 2.3), in which students are asked whether or not the probability density depends on time for a superposition of two energy eigenfunctions. However, question 4.1b requires the additional step of recognizing that the state of the particle after the measurement at t_1 is given by a superposition of energy eigenstates. It is the impact of this additional step that is the focus of our investigation in this chapter.

The responses to questions 4.1a and 4.1b were individually analyzed by examining both the answers that each student chose and the explanation that they gave. We then grouped the students according to commonalities in the underlying reasoning that they used to answer. Table 4.1 shows the percentage of the students who gave answers consistent with each of the most

common types of reasoning identified by this analysis (the bolded rows indicate correct reasoning). Each of these lines of reasoning is discussed in more detail below.

On question 4.1a, more than half of the students gave the correct answer. Only 18% gave explanations that were judged to be correct; two lines of reasoning were treated as correct. Some students (12%) argued that a position eigenstate can be specifically rewritten as a superposition of energy eigenstates. Another 6% based their answer on the uncertainty principle. These students tended to note that the momentum of the particle would be uncertain after a measurement of position, and since the total energy depends on the momentum, the energy would also be uncertain. Most of the other students who gave the correct answer (34%) simply asserted that many energy values would be possible, without giving any further reasoning.

Table 4.1: Results from Task 4.1

The top row gives the percentage of the students who gave correct answers. (The percentage in parentheses includes only students who also gave correct reasoning.) Each subsequent row indicates a common incorrect line of reasoning used by students. The results for both questions are aggregated across the three quarters for which the version shown in Fig. 4.1 was administered (N=145).

		Question 4.1a	Question 4.1b
Correct answer (correct with correct explanations)		58% (18%)	23% (3%)
Incorrect lines of reasoning	The state of the particle is unaffected by the measurement at t_1	35%	10%
	The position of the particle is fixed after the measurement at t_1	N/A	44%
	Classical reasoning	6%	10%

On question 4.1b, 23% of the students gave the correct answer (only 3% also gave correct reasoning). About half the students who gave the correct answer gave incorrect explanations

(see below), while the others did not give any further justification. Only 15% of the students answered both questions correctly.

B. Specific Student Difficulties

Students expressed several common incorrect ideas in response to the questions on Task 4.1. Each is discussed below. Some incorrect answers were related to incorrect lines of reasoning related to time evolution, as identified in chapter 2. Representative student quotes are used to illustrate the predominant reasoning associated with each difficulty.

1. Tendency to think that measurement does not affect the state of a particle

On both questions on Task 4.1, many students gave answers for the state that are not consistent with the fact that the measurement of position made at t_1 changes the state of the particle. Below is an example from a response to question 4.1a.

"Once the energy is defined it cannot change." (Question 4.1a)

This student appears to believe that the energy of a particle cannot change once it is determined. The student does not give any further reason for this answer, which was very common among students who stated that the energy measured after a measurement of position would be the same as the energy of the initial state. Responses consistent with the position measurement having no effect on the energy of a particle were more common on question 4.1a (given by 35% of the students) than on question 4.1b (given by10% of the students).

An example of this kind of response for question 4.1b is shown below.

> *"For the ground state energy, it is possible for the particle to be found anywhere in the well, although it is more likely to be found near the center."* (Question 4.1b)

This student clearly begins with the assumption that the particle is still described by the ground state energy, despite the intervening position measurement. The student then gives an answer consistent with using the ground-state wave function to determine which positions are allowed. Other students who gave this kind of explanation also described a probability distribution based on the ground state.

Some students (about 5%) who answered that only the original energy could be measured justified their answer using the mathematics associated with quantum mechanics, such as the example below.

> *"Position and energy operators commute. Therefore the measurement of position does not affect energy. So the wave function will still be collapsed in the first excited state."* (Question 4.1a)

This student's answer is based on the (incorrect) belief that the position operator and the energy operator (*i.e.*, the Hamiltonian) commute. The student uses commutation as a basis for why measuring position does not affect the energy of the particle. This reasoning would be correct if the operators did commute. However, some students noted that the operators do *not* commute, but came to the same conclusion (that the energy would be the same before and after the position measurement). It is likely that students who attempt to use commutation relations in this way are confused about how commutators relate to the uncertainty principle, and how to use the uncertainty principle to make claims about measurements.

2. Belief that a position measurement results in a fixed position

The most common answer to question 4.2B was that the second measurement of position will definitely result in the same value of x as was found by the first measurement of position. The response below demonstrates some of the reasoning given in support of this answer.

> *"By measuring the position, we have made it well-defined. The infinite square well is not time dependent, so there is no reason for the particle to move from that position. The wave function is set as being at that position for all t."* (Question 4.1b)

This student begins by noting that the measurement causes the position to become well-defined, which is correct. Many other students also used this fact in their explanations, frequently citing the "collapse" of the wave function to justify it. However, the student above then notes that the particle's position will not change, because the wave function is fixed at that position, and because the potential is time-independent. The language that the student uses is similar to what has been discussed in chapter 2 for students who have difficulty with the time evolution of general quantum states.

Almost half of the students (44%) responded to question 4.1b using similar reasoning. These students concluded that $x = 2a/3$ would definitely be the result of the second position measurement. In most cases the students clearly seemed to be thinking that the position would remain fixed (*i.e.*, it would not change in time). However, about one-third of these students explicitly noted that their answer was for "a short time later" as specified in the problem. These students did not describe how their answer would change if more time were to elapse. Although we have asked how some students' answers would change if the measurement were made after a very long time, such questions bring up additional difficulties related to the behavior of the wave

function after a long time. Examples of this kind of response, which are discussed in more detail in section 4.2A4 and in Ref. 4, include the belief that the wave function returns to the shape it had before the measurement, or decays to the ground state. As a result, it is difficult to determine whether or not all students who gave this answer believe that position is independent of time after a measurement of position.

3. Tendency to use classical reasoning

In answering Task 4.1, many students seemed to reason based on knowledge from classical physics rather than from the quantum mechanical formalism that they have learned. (This finding is also discussed in section 3.5C.) Often, students apply concepts from classical physics incorrect to quantum mechanical objects. For example, on question 4.1a many students stated that energy should be conserved, and used this to reason that the energy of the particle is always the same, regardless of the position measurement.

Some students gave responses to question 4.1b (for a successive measurement of position) that suggested the use of classical reasoning.

"After measuring the position, the particle can still move around." (Question 4.1b)

This student answered that it is possible to find the particle at any position, because the particle is still capable of motion. Many other students gave responses as simple as "the particle will move". Ten percent of the students answered question 4.1b in a way that is consistent with incorrectly applying classical reasoning. While these students often gave the correct answer, this

kind of reasoning is incorrect because it is not related to the quantum model that the students have been taught.

4. Additional, less common, student difficulties

In addition to the difficulties listed above, there were some less common patterns in student reasoning. Many of these have also been observed in the responses to other tasks we have administered. For example, about 5% of the responses to questions 4.1A and 4.1B state that the wave function will "spread out" or "diffuse" after the measurement of position. We consider responses of this nature to be incomplete because they do not explain why the wave function undergoes this behavior. Some students who use this language frequently describe the final wave function as being uniform across all space, which is incorrect. A handful of students also specifically stated that the wave function would evolve back to the original state from before the first measurement. Each of these difficulties are discussed in chapters 2 and 3, and Ref. 4 documents their prevalence in much greater detail.

C. Summary

In this section, we identified several difficulties that arise when students are asked about the results of measurements on quantum states that are made after prior measurements have already been made. In some cases, students do not appear to recognize that a quantum measurement causes the state of a system to change. For example, we found that students often believe that a measurement of one quantity (*e.g.*, energy) will not affect the possible outcomes of future measurements of a different quantity (*e.g.*, position), and that some believe a measurement will

not affect the possible outcomes of *any* future measurements. This is a difficulty that was also identified in Ref. 3.

Other students recognize that a quantum measurement alters the state of a system, but do not appear to understand how the state changes in time after such a measurement. When we told students that a measurement of position had been made on a quantum particle, we found that many gave answers consistent with treating the particle as "fixed" at that position. This difficulty is also discussed briefly in Ref. 3. Other students ascribed classical behavior to the particle after the measurement: that is, they described the particle as "moving around" rather than using the wave formalism they had been taught.

The difficulties described above illustrate that traditional instruction on quantum measurements may not be sufficient for students to make correct predictions about the behavior of quantum systems after measurements. Section 5.1 describes a sequence of tutorial curriculum intended to help improve student understanding of measurements and time dependence, including how to determine the state of a system after a measurement and its subsequent time evolution.

4.3 – Sequences of Spin Measurements

We have encountered incorrect lines of reasoning like those described in the previous section in the responses to questions about sequences of measurements for other quantities. This finding suggests that student difficulties often persist throughout all of quantum mechanics. Chapters 2 and 3 discuss the persistence of student difficulties for time dependence and energy measurements, respectively. It is therefore not surprising to find that student difficulties with

sequences of energy and position measurements occur when they are asked about sequences of measurements involving other quantities, regardless of whether or not they have previously studied other sequences of measurements, such as energy and position.

We asked many students about sequences of spin measurements for different directions (*e.g.*, S_z and S_x). Since the S_z and S_x operators do not commute, the eigenstates of S_z are *superpositions* of the eigenstates of S_x (and vice versa). The operators S_z and S_x are thus like the operators associated with energy and position, which also do not commute. This analogy prompted us to use questions about spin to probe student understanding of sequences of measurements.[7] (Difficulties that students have with spin that are not related to sequences of measurement are discussed in more detail in part III of this dissertation; in this section, we focus on those difficulties that appear to be related directly to sequences of measurement, rather than to the context of spin.

We have asked questions about sequences of spin measurements at many levels throughout instruction on quantum mechanics. In section 4.3A, we discuss the results from a task that focuses on determining possible outcomes after a sequence of spin measurements. In section 4.3B, we discuss the results from questions that ask about the time dependence of spin probabilities after a measurement of spin. In both sections, we focus on whether or not certain difficulties were present in the responses to questions about sequences of spin measurements. The spin questions are somewhat different from the questions about sequences of energy and position measurements, so we do not attempt to compare the percentages of students who gave responses consistent with the same difficulty across the two types of question.

A. Difficulties with Sequences of Spin Measurements – Determining Possible Outcomes

Task 4.2 (see Fig. 4.2) was given to students near the end of the first quarter of the junior-level quantum mechanics course at UW. It was part of a pretest after lecture instruction on spin and orbital angular momentum. The pretest followed both lecture and tutorial instruction on time dependence and measurements. (See section 5.1 for a description of the tutorial curriculum designed to address some of the difficulties described in section 4.2.) The pretest was given before any tutorial instruction for spin and angular momentum.

Task 4.2

Suppose the initial spin state of a particle is given by: $\left|s, m_s\right\rangle_z = \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle_z$.

Suppose you were to measure the ***y*-component** of the spin angular momentum for this particle, and the result is $+\hbar/2$.

After this measurement, you measure the ***z*-component** of the spin angular momentum for this particle. Determine the possible outcomes of this second measurement, and the probability associated with each outcome. Explain your reasoning.

Figure 4.2: A question on successive spin measurements

In Task 4.2, students are told the initial spin state of a spin-1/2 particle is $\left|s, m_s\right\rangle_z = \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle_z$. They are then told that the result of a measurement in the *y*-direction is $+\hbar/2$. Finally, they are asked about the possible outcomes of a measurement in the *z*-direction. Task 4.2 can be answered by writing the state of the particle as the following superposition of the eigenstates of S_z after the measurement in the *y*-direction: $\left|\chi\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle_z + i\left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle_z\right)$. Thus, the possible outcomes are $\pm\hbar/2$, each with probability 50%. We consider this question analogous to question 4.1a, in which measurements of energy, position, and then energy again were made, because the

same pattern of measurements is made here. In the discussion below, we focus on the outcomes given by the students, rather than the probabilities they gave.

We found that 39% of the students gave the correct answer to Task 4.2, 16% with correct reasoning. The correct lines of reasoning were very similar to those given in response to Task 4.1 (see section 4.2). On this task, we also found that some students made errors similar to those made when we asked students about sequences of energy and position measurements. Specifically, 7% of the students (N=74) indicated that the state of the particle would not be affected by the measurement of S_y. The reasoning given by these students was often very similar to what has been discussed for Task 4.1. An example response is included below.

"S_y measurement doesn't affect S_z." (Task 4.2)

This student states that a measurement of the spin in one direction does not affect the spin in a different direction. This response is very similar to those seen in section 4.2B1 when the students were asked about the results of an energy measurement after a position measurement had been made. Other incorrect lines of reasoning associated with spin are not discussed in this chapter (instead, see chapter 9).

B. Difficulties with Sequences of Spin Measurements – Time Dependence

Task 4.3 (see Fig. 4.3) was given to students in a sophomore-level introduction to quantum mechanics that was taught spins-first. The focus of this task is on whether or not the probability of measuring the spin of an electron depends on time *after* a prior measurement of the spin has

been made. The task was given as part of a pretest after time dependence and the Schrödinger equation was covered in class.

Task 4.3

Question 4.3a.

An electron's spin is measured and determined to be up in **the *z*-direction**. At $t = 0$, a constant magnetic field B_o is turned on in the *z*-direction.

You decide to conduct another measurement of the spin in **the *z*-direction**. Does the probability that the electron is measured to have spin up again depend on time? Explain your reasoning.

Question 4.3b.

A different electron's spin is measured and determined to be up in **the *x*-direction**. At $t = 0$, a constant magnetic field B_o is turned on in the *z*-direction.

You decide to conduct another measurement of the spin in **the *x*-direction**. Does the probability that the electron is measured to have spin up again depend on time? Explain your reasoning.

Figure 4.3: A sequence of questions on spin measurements

Students are first told the outcome of a measurement of one component of an electron's spin. Students are then told that a magnetic field is turned on in the *z*-direction. Each question then asks whether or not the probability that a second measurement of the same component of the electron's spin depends on time.

Task 4.3 has two questions. Question 4.3a tells students that an electron has been measured to be spin-up in the *z*-direction. They are then told that a constant magnetic field, which is also in the *z*-direction, is turned on at $t = 0$. They are then asked whether or not the state of the electron depends on time (not shown in Fig. 4.3), and whether or not the probability of again measuring spin-up in the *z*-direction depends on time. Students are asked to explain their reasoning.

The correct answer to question 4.3a is that the state is an energy eigenstate, so it depends on time via a single, time-dependent phase factor. Since only the overall phase changes with time, the probability that the particle is measured to have spin-up in the *z*-direction is independent of time.

Question 4.3b is the same as question 4.3a, except that both of the measurements are made in the *x*-direction (the magnetic field remains in the *z*-direction). Students are again asked to explain their reasoning. In this case, the particle is in a superposition of energy eigenstates, each of which evolves in time with a different phase factor. The probability of measuring spin-up in the *x*-direction depends on time because the two phases interfere with each other.

Task 4.3 asks about the time dependence of probabilities, rather than about the possible outcomes of a subsequent measurement, so it is somewhat different from Task 4.1. Nevertheless, the answers and the explanations that students give can still be used to identify incorrect ideas that are related to sequences of measurements. Table 4.2 shows the percentage of the students who gave answers or explanations consistent with the difficulties identified in section 4.2 that have to do with sequences of measurements. Many students also made errors related to time dependence (discussed in chapter 2) or to spin (discussed in chapter 9), which are not shown in the table.

Table 4.2 Results from Task 4.3

The percentage of the students who answered correctly is shown in the first row (the percentage who gave correct reasoning is bolded). The percentage of the students who gave incorrect explanations related to sequences of measurements is shown in the second row (other incorrect explanations not shown). The results are aggregated across three quarters (N=141).

		Question 4.3a	Question 4.3b
Correct answer (correct reasoning)		92% (**21%**)	55% (**17%**)
Incorrect lines of reasoning	The result of the second measurement is always the same as the result of the first measurement	9%	9%
	Classical reasoning	13%	30%

Most of the students answered question 4.3a correctly, though only 21% gave correct reasoning. About half (55%) answered question 4.3b, 17% with correct reasoning. Many students gave incorrect explanations that did not seem to be related to sequences of measurements. These explanations were often consistent with other difficulties discussed in chapter 2.

Many students gave explanations to question 4.3a (13%) and question 4.3b (30%) that indicated classical reasoning. Most of these students indicated that the spin of the particle would align with or precess around the magnetic field. Some explicitly stated that the magnetic field would exert a torque on the electron (this answer would be correct for a classical object with a magnetic moment). Although the context is very different from Task 5.1, which asked about sequences of energy and position measurements for a particle in the infinite square well, many students answered both tasks based on what would happen to an object in the analogous classical system.

On each question, 9% of the students specifically indicated that the second measurement should always give the same result as the first measurement did. The response below illustrates the reasoning used by these students.

> *"The probability that the spin will be measured as up again does not depend on time. Once it is measured the spin is 'stuck' in the measured state."* (Question 4.2a)

This student states that the first measurement of spin fixes the state of the particle, which cannot then be measured to have a different initial spin. This kind of response was also common when we asked questions about sequences of energy and position measurements (see section 4.2B2).

While this reasoning gives the correct answer to question 4.2a, it gives an incorrect answer to question 4.2b.

4.4 – Summary

The results discussed in this section indicate that many students have difficulty in reasoning about the impact of measurements on the state of a system. We have asked questions that probe their ideas when two successive measurements are of the same quantity or of different (non-commuting) quantities. Examples of the sequence of measurements that we used are an initial energy measurement, followed by a measurement of position, before finally measuring energy again (*i.e.*, question 4.1a).

Several common beliefs emerged from our investigation. One is that quantum states are not changed by measurement, or that measurement does not change any quantity except the one that is measured (*e.g.*, a measurement of position affects position but not energy). Thus, students often fail to recognize how the state of a particle changes after a measurement has been made. In some cases, this failure seems to be related to the student difficulties with time evolution, as discussed in chapter 2. For example, relatively few students wrote the state that results from a measurement (*e.g.*, of position) in terms of the energy eigenstates of the system, which is a crucial step in determining time dependence.

We found that some difficulties are not unique to sequences of measurements involving position and energy. Questions about successive measurements of spin in various directions elicit the same ideas. Together, the results from multiple student populations for questions in more than one context support the idea that the impact of measurements on a system is

challenging for students. They also seem to be challenging in ways that transcend context, which we have found to be true for most of the student difficulties described in this dissertation. Further questions about quantities such as momentum could be used to identify whether or not our results can be generalized beyond the contexts we have analyzed in this chapter.

The identification of student difficulties discussed in this chapter has proved valuable in the design of tutorial curriculum for measurements in quantum mechanics. This curriculum is discussed in more detail in chapter 5.

4.4 – Notes and References

[1] C. Singh, "Student Understanding of Quantum Mechanics," *American Journal of Physics* **69**, 885 (2001).
[2] G. Zhu and C. Singh, "Surveying Students' Understanding of Quantum Mechanics in One Spatial Dimension," *American Journal of Physics* **80**, 252 (2012).
[3] G. Zhu and C. Singh, "Improving Students' Understanding of Quantum Measurement I: Investigation of Difficulties," *Physical Review Special Topics – Physics Education Research* **8** (1), 010117 (2012).
[4] A. Crouse, "Research on Student Understanding of Quantum Mechanics as a Guide for Improving Instruction," PhD dissertation, University of Washington (2007).
[5] In the commonly used textbook by Griffiths, the fact that the state of a particle immediately after a measurement is given by an eigenstate corresponding to the measured observable is covered in Section 3.4 – Generalized Statistical Interpretation.
[6] The Dirac delta function is the wave function that would result from an idealized measurement. This function can be thought of as a Gaussian in the limit that the width is taken to zero.
[7] Analogous questions about sequences of Stern-Gerlach experiments have been discussed in some previously published work. However, these publications do not discuss specific student difficulties with sequences of measurements in this context. See, for example, G. Zhu and C. Singh, "Improving Students' Understanding of Quantum Mechanics via the Stern–Gerlach Experiment," *American Journal of Physics* **79**, 499 (2011).

Chapter 5
Development and Assessment of Tutorial Curriculum for Time Dependence and Measurements

"Yes, $\rho(x,t)$ changes with time. As explained before, we know that $\psi(x,t)$ is a superposition of energy eigenstates that all rotate (if you will) at different rates. When you take $|\psi(x,t)|^2$, you will have ugly cross terms and the exponents with the time dependence $e^{-iE_n t/\hbar}$ won't cancel out!"

—Junior-level quantum mechanics student, midterm exam

The results discussed in chapters 2-4 indicate that many students leave a junior-level quantum mechanics course with some serious conceptual and reasoning difficulties related to quantum measurements and time dependence. For example, students frequently fail to relate the possible outcomes of a measurement and the associated probabilities to the wave function for a system. Some students are unable to determine the correct time evolution for even simple quantum states. Even when they are given the correct time dependence, many are unable to use it to determine how measurements and probabilities will evolve in time. We have also found that student understanding of time dependence and measurements often appear to be related (*e.g.*, when they are asked to reason about sequences of measurements), and that difficulties with one often lead to difficulties with the other.

The lecture instruction of typical undergraduate courses in quantum mechanics does not seem to be enough to help students develop the tools necessary to answer questions about time dependence and measurements. This result is consistent with the findings on a variety of topics across introductory physics courses.[1] At the introductory level, a variety of techniques (*e.g.*,

active engagement) has been used to address the difficulties that students have after lecture instruction. Some have been shown to be effective.[2] Some techniques can be implemented during lectures (*e.g.*, using in-class clicker questions, lecture demonstrations, or flipped classrooms)[3] or in small-group sections (*e.g.*, tutorials or other supplementary curriculum).[4] These methods have begun to see more use in upper-division courses, where they have also proved successful.[5-6]

The Physics Education Group at UW has been developing *Tutorials in Physics: Quantum Mechanics* to supplement lecture instruction.[7] (For more detail about the tutorials as a whole, see section 1.2A.) This chapter discusses the parts of this curriculum that have focused on time evolution and quantum measurements. In section 5.1, we discuss tutorials designed to address student difficulties in basic contexts (*i.e.*, those typically encountered near the beginning of instruction on quantum mechanics). In section 5.2, tutorials that involve the same ideas in more advanced contexts are discussed. Lastly, section 5.3 discusses an effort to adapt the curriculum for sophomore-level students studying two-state systems (*e.g.*, spin). In each section, we first describe the curriculum itself before presenting our assessment of the impact of this curriculum on student performance. We also discuss how the results of our assessments have informed changes to the tutorials, challenges to assessment in upper-division courses, and the techniques that we have used to overcome these challenges.

It should be noted that there are several other examples of existing curriculum intended to improve student understanding of time dependence and measurements in quantum mechanics. For example, there are multiple research-based online simulations that address quantum measurement and/or time evolution. Examples include the PhET simulations from the

University of Colorado – Boulder,[8-9] the QuVIS from the University of St. Andrews,[10-11] and several downloadable programs from Open Source Physics.[12] Although these simulations are based on an extensive body of research on student understanding of time dependence and measurement, there have been few publications specifically assessing their effectiveness.

The University of Pittsburgh has briefly assessed a Quantum Interactive Learning Tutorial (QuILT) on measurement that incorporates the Open Source Physics simulations.[13] The QuILT includes exercises designed to help students determine the outcomes of quantum measurements and their associated probabilities, both for an initial superposition of energy eigenstates and for states after measurements have been made. Their preliminary evaluation (N~30) determined that students receive a better average score on a set of post-test questions than on analogous questions given after only lecture instruction (but before the QuILT). Thus, there is some evidence that interactive curriculum that targets specific student difficulties can be effective at the upper-division level.

5.1 – Tutorial Curriculum for Time Dependence and Measurements in Basic Contexts

As discussed in chapter 1, much of the early development of the quantum tutorials was done by Bradley Ambrose and Andrew Crouse. This work is documented in the PhD dissertation of Crouse.[14] In this section, we begin by describing the original tutorial curriculum that focused on time dependence and measurement in basic quantum contexts (*e.g.*, the infinite square well or harmonic oscillator potentials). We also summarize the preliminary assessment of this curriculum. Then, we document several changes that have been made to this sequence of tutorials as a result of the research documented in chapters 2-4. This is followed by an

assessment of the modified tutorials that focuses on comparing student performance on pre- and post-tutorial questions. Then, we discuss some alternative methods of assessment that we have used. Finally, we examine results from students at other universities that have served as pilot sites for the tutorials in an effort to determine if the results at UW can be duplicated at different institutions.

A. Description of Original Tutorial Curriculum

The original tutorial curriculum for time dependence and measurement was developed, tested, and modified from 2002-2007. The primary setting for this effort was the junior-level quantum mechanics sequence at UW. A sequence of three tutorials focuses on these topics.[15] Each tutorial was designed to improve student understanding by targeting known student difficulties, such as those identified in chapters 2-4. Each of the three original tutorial worksheets in this sequence is discussed below. The homework set assigned by the lecturer for each week included one problem related to the tutorial.

1. Original tutorial: *Time dependence in quantum mechanics*

The first tutorial in the original sequence on time dependence and measurement focuses on helping students recognize how the wave function and its corresponding probability density depend on time, both for energy eigenstates and for superpositions of energy eigenstates. The first section asks students to sketch the first two energy eigenfunctions of the infinite square well potential. Students are then asked to consider whether they need a two- or a three-dimensional graph to represent these wave functions, and they are guided to discuss how to represent the real

and imaginary parts. At this point, students are provided with a visualization tool that allows them to represent the wave functions three-dimensionally. This tool consists of a set of real and imaginary axes printed on transparencies that fit together to form a three-dimensional coordinate system (a copy of this tool, along with instructions for using it, may be found in the appendix). Next, students are asked whether or not the wave functions depend on time. They check their answer using the time-dependent Schrödinger equation, and many students are surprised to find that time dependence is required in order for a wave function to satisfy this equation.[14] The transparencies can be used to demonstrate the time evolution spatially.

The tutorial then asks students to consider the probability density corresponding to each of these energy eigenfunctions. They find that the probability density is constant in time, and relate this to the fact that the absolute value of the wave function does not change. Students also think about the amount of time required for each wave function to return to its original phase: these times are also used in section III of the original tutorial.

In section II, students are given the graph of a wave function that is initially entirely real and asked several questions that guide them to recognize that this wave function can be written as a sum of the two energy eigenfunctions that they considered in the previous section. At this point, they consider several student statements designed to address specific common difficulties: (1) that a sum of eigenstates is an eigenstate (see Ref. 14); (2) that the wave function is associated with a definite energy because it is a solution to the Schrödinger equation (see Ref. 14); and (3) that the wave function goes back and forth between the ground state and the first excited state (see chapter 3).

In the last section, students sketch the probability density for the superposition state at several different instants in time. As a result of this exercise, students are expected to recognize that the probability density for this state depends on time. They are also provided with a handout demonstrating the time evolution as a sequence of graphs, so that they can check their answer. Finally, they are asked what will happen to the wave function after a long time. This line of questioning is meant to address the common belief of students that the wave function eventually decays to the ground state or that it eventually spreads out so that it is equal across all positions (see section 2.4D).

2. Original tutorial: *Energy measurements*

The tutorial that focuses on energy measurements was the second in the original sequence. The first part focuses on the difference between superposition states and mixed states. It does so by giving students a superposition state and then asking them to agree or disagree with the following statement.

> *"The wave function given by represents a lack of knowledge about the state of the system. The system is definitely in either the ground state or the first excited state. The wave function simply tells you that the probability is 1/2 that the system is really in the ground state and 1/2 that it is really in the first excited state."* (Text from original *Energy measurements* tutorial)

Then students are asked about the probability of measuring the position of particles in each kind of system. They find for themselves that there is an experimental difference between the two. They then revisit their initial response to the incorrect student statement shown above.

In section II of the original tutorial, students are asked to consider a particle in the harmonic oscillator potential that has a wave function that is equal to the ground-state wave function for the *infinite square well*. They are then asked to determine the time dependence of the wave function and the probability density by considering whether or not this state is an energy eigenstate for this potential. This section is intended to address the common incorrect idea that the eigenfunctions of the infinite square well are eigenfunctions of any potential (see, for example, Ref. 14). The goal of this section is for students to recognize that the energy eigenfunctions of the infinite square well are not energy eigenfunctions of the harmonic oscillator, but that they are still valid for describing a particle in either potential.

The third section of the original tutorial focuses on energy measurements themselves. First, students are asked about the possible outcomes of an energy measurement on a superposition state that is written in the energy basis. They are also asked about the associated probabilities. Then, they are asked to determine whether or not the probabilities depend on time, and consider how this is consistent with the time dependence of the wave function itself. They are also asked whether or not the energy is a well-defined quantity. Lastly, students are told the result of an energy measurement, and consider the outcome of a second energy measurement at a later time. They consider this case only briefly, before moving on questions that ask them to summarize their ideas by considering a similar set of questions for a generic wave function.

3. Original tutorial: *Position, momentum, and energy measurements*

In the last of the original tutorials, *Position, momentum, and energy measurements*, students are first asked to write down the eigenvalue equations for energy, position, and momentum.

Throughout the tutorial, they are expected to note that the energy eigenvalue equation depends on the system under consideration (*i.e.*, the potential). They then solve these equations for position and momentum, resulting in a Dirac delta function and an imaginary exponential, respectively. They are asked to consider whether the eigenvalues are discrete or continuous, but otherwise they are not given any guiding questions. We have found that these exercises are very challenging for most students.

After students have determined the eigenfunctions, they are asked a series of questions about measurements of position, momentum, and energy. Students are asked each question about all three kinds of measurement before they move on to the next question. First, they are asked to describe the state after a measurement of each quantity. They also determine whether or not the action of an operator on the initial state is equal to the state after measurement (a common error discussed in Ref. 14). Then, students are asked whether or not the probability density would depend on time after the measurement is made. The students are not guided to consider how each wave function may be written in terms of energy eigenfunctions at this point. Lastly, they are asked to consider the result of a second measurement of the same type made both immediately and a few minutes after the first measurement.

The last section of the original tutorial presents students with a specific example of one of the exercises from the previous section: a particle in an energy eigenstate of the infinite square well. The position of this particle is measured, and then students are asked about the results of subsequent measurements of both energy and position. They are explicitly guided to consider whether or not the particle is in an energy eigenstate or a superposition of energy eigenstates after the measurement of position. This example is used to distinguish between measurements

that are "compatible" and "incompatible", an idea that is then extended to momentum measurements at the end of the original tutorial.

B. Preliminary Assessment of Original Tutorial Curriculum

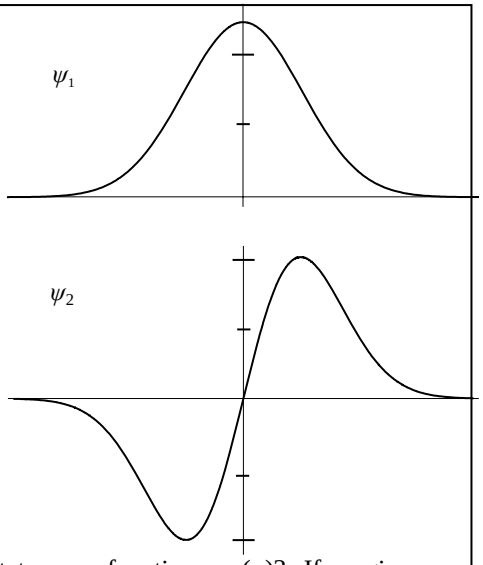
Crouse's PhD thesis discusses several different methods that were used to assess the original sequence of tutorials.[14] The first method involved comparing the performance of two groups of students on exam questions: (1) students in courses with tutorials at UW and (2) students in courses without tutorials at other universities.[16] On each question, students who had completed the tutorials performed significantly better than those at other universities, including, in some cases, incoming graduate students. The second method compared the performance of two separate populations at UW: those in a course with tutorials and those in a different course without tutorials. The students with tutorials performed significantly better according to this comparison as well. The final method of comparison involved pre- and post-tutorial performance for the same students. There was a substantial increase in the percentage of the students who gave correct responses to questions after tutorial instruction.

Each of the comparisons discussed above suggests that the original tutorials are effective at helping the students. However, these comparisons included only about 40-50 students from classes with tutorials. In order to give a more complete assessment of the effect that the tutorials have, a larger sample of students is necessary. Below, we discuss the results to exam questions administered to students in the junior-level quantum sequence at UW (N=130 over two academic quarters). The exam took place after the full original sequence of tutorials described in the previous section.

Task 5.1 (see Fig. 5.1) presents students with graphs of the wave functions for the ground state and the first excited state of an unfamiliar potential. Students are then told that a system has an initial state that is a superposition of these two states. Question 5.1a asks the students if the state of the system is a stationary state. Students should recognize that it is not a stationary state because it is a superposition of energy eigenstates with different energy eigenvalues. Thus, the state has two terms, each with a different time-dependent phase, so the probability density depends on time. Question 5.1b asks if there is a time when the wave function is exactly equal to the ground-state wave function. Students should recognize that there is no such time because the time-dependent phase factor for each term in the state never goes to zero. Lastly, question 5.1c asks if there is a time when the probability density has the same shape as a new graph corresponding to a superposition of the same two states with a different relative phase. This question can be answered by computing the probability density as a function of time. It can also be answered by making a qualitative argument about the relative phase as the state evolves through the complex plane. For the probability density shown in Fig. 5.1, the relative phase between the energy eigenfunctions must be $\pi/2$, which would occur, for example, when ψ_2 is entirely real and ψ_1 is entirely imaginary.

Task 5.1

The symmetric ground-state wave function, $\psi_1(x)$, and the antisymmetric first excited-state wave function, $\psi_1(x)$, for a system are shown at right. Assume that the energy of the first excited state, E_1, is twice the energy of the ground state, E_o.



The system is initially in a state given by:

$$\Psi(x, t=0) = \frac{\left(\psi_1(x) - \psi_2(x)\right)}{\sqrt{2}}.$$

Question 5.1a

Is this state a stationary state? Explain.

Question 5.1b

Is there a time when the wave function is exactly equal to the ground state wave function, $\psi_1(x)$? If so, give a time when this is true. Give your answers in terms of the variables given or describe a limiting case. Explain.

Question 5.1c

Is there a time when the probability distribution looks like the graph below? If so, give a time when this is true? Give your answer in terms of the variables given or describe a limiting case. Explain.

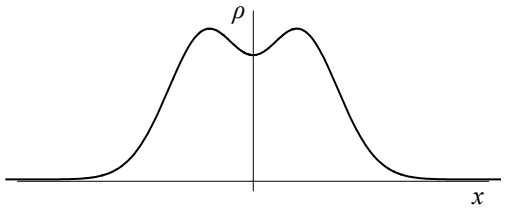


Figure 5.1: Sequence of exam questions on time dependence

Task 5.1 was administered to students in a junior-level quantum mechanics course. The exam was administered after the students had completed various amounts of tutorial instruction. The exact wave functions given to the students varied from quarter to quarter, but always consisted of a symmetric ground state and an antisymmetric first excited state. The exact questions also differed somewhat from quarter to quarter: see the appendix for an exact copy of each question.

Task 5.1 has been asked with different sets of functions, initial states, and possible probability densities (for question 5.1c) in several different quarters. However, all versions focused on similar aspects of time dependence for wave functions or probability densities, so we consider them together. All variants may be found in the appendix.

Each question on Task 5.1 was designed to elicit particular student difficulties identified in chapter 2. Question 5.1a is intended to assess whether or not students understand how the term "stationary state" is used in quantum mechanics (see section 2.4A). Question 5.1b is aimed at assessing whether or not students understand how to consider both the real and imaginary parts of the wave function (see section 2.4C3). Question 5.1c focuses on student ideas about how the probability density depends on time (see sections 2.4A and 2.4B).

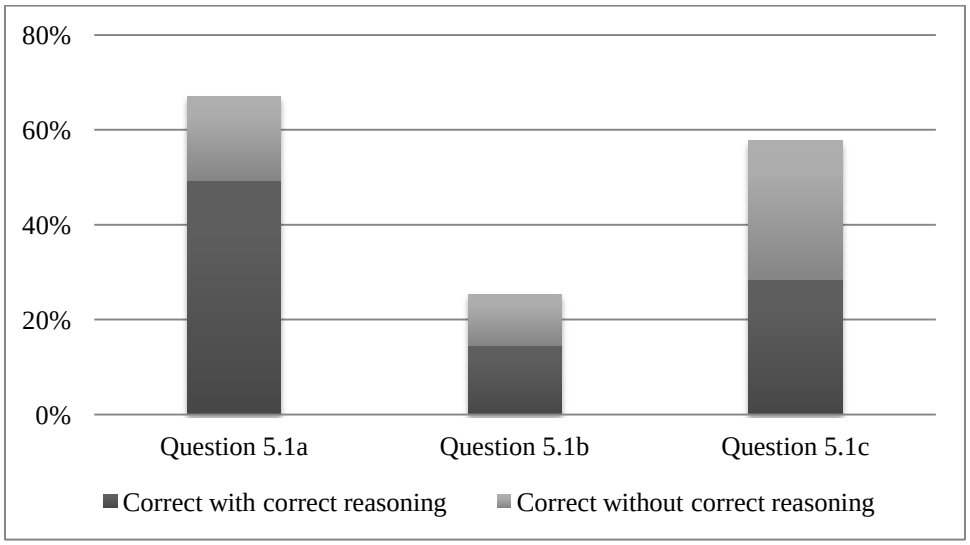


Figure 5.2: Results from Task 5.1

The task was administered to 130 junior-level quantum students at UW. The results are aggregated across two different quarters.

Figure 5.2 shows the percentage of the students who answered each question on Task 5.1 correctly, both with and without correct reasoning. (These questions were not asked before the tutorials were administered to students.) On question 5.1a, 67% of the students correctly answered that the given state is not a stationary state. Most students (49%) also gave correct reasoning: they either noted that the state is a superposition of energy eigenstates or that the

probability density depends on time. Only one-quarter of the students answered question 5.1b correctly, 15% with correct reasoning. Most students incorrectly stated that the wave function would have the same shape as the ground state when the other term is imaginary. Lastly, on question 5.1c, 58% of the students correctly stated that there is a time when the probability density has the shape given on the graph. About half of those students (28% of all students) gave correct reasoning.

Some of the results on this question are encouraging. About 60% of the students answered questions 5.1a and 5.1c correctly. However, there is still room for improvement. Many students did not give correct reasoning to the questions on Task 5.1, even if they answered correctly. Additionally, three-quarters of the students answered question 5.1b incorrectly. Thus, even though there is evidence that students in courses with tutorials perform better on exams than students in courses without tutorials, the tutorials can still better address quantum measurement and time dependence. Section 5.1C discusses how we have modified the tutorials to take this research into account. Section 5.1D details our effort to assess this modified sequence of tutorials.

C. Modified Tutorial Curriculum

The results from exam questions given after administration of the original tutorial curriculum (see section 5.1B) suggested the need for modifications to the tutorials. We drew on these results, as well as on observations of student work during the tutorial sessions to help us identify particular tutorial exercises that need improvement. In some cases we realized that students did not fully understand the questions they were asked. In other cases, students required too much

time to answer them. In addition, we realized that many students failed to understand a major goal of this sequence of tutorials: to construct a model for quantum measurement and time evolution that can be applied to any quantum system, not just to those covered in the tutorials.

Starting in 2011, the tutorials have been continually modified as a result of ongoing research at UW. Based on the points discussed above, the overall revision was undertaken to accomplish the following objectives: each tutorial should not address too many topics, but should instead focus on a small set of particular ideas; most groups of students should be able to complete each tutorial within a single 50-minute period; and the individual questions should be free of ambiguity and easy to interpret. We also decided to develop a set of tutorial-style homework problems (based on the homework problems from *Tutorials in Introductory Physics*)[4] for each tutorial topic. Very few homework questions existed for the original tutorial curriculum discussed in section 5.1A. Some exercises were moved from the original in-class worksheets to the new homeworks. Additionally, a new tutorial, *Superposition in quantum mechanics*, was created based on exercises that were removed from the original sequence of curriculum.

In this section, we discuss the changes that were made to the tutorials for each of the three topics discussed in the previous section (5.1A). Then, the new tutorial is discussed. The assessment of this curriculum, both by individual topic and together as a sequence, is discussed in the following section (5.1C). Copies of all the modified and new tutorial worksheets and homeworks may be found in the appendix.

1. Modified tutorial: *Time dependence in quantum mechanics*

Observations of students in tutorial sections indicated that relatively few groups were able to complete the original time dependence tutorial within a single, 50-minute period. Most students spend a substantial amount of time on the first two pages of the original tutorial, determining the time dependence of the wave function for energy eigenstates. Since responses to pre-tutorial questions indicate that students often do not recognize that wave functions change in time, we decided to keep the section since it is important that students have an opportunity to consider this time dependence in detail. However, in order to help students complete this section more quickly, we added several guiding questions to emphasize two specific ideas about time dependence: the difference between the rate of change of the wave function for the ground state and the first excited state, and the fact that a probability density that is independent of time is typically associated with a wave function that is not independent of time (as is the case for an energy eigenfunction).

Even with the changes discussed above, we have found that some students still take as much as 30-40 minutes to complete the first two pages of the tutorial. As a result, we restructured the last section of the tutorial so that it focuses on what we consider the most important idea: how the probability density for a superposition of energy eigenfunctions changes with time. This decision necessitated the removal of several exercises from the original tutorial. The students are no longer asked to check their answer using the time-dependent Schrödinger equation. The two sets of student statements, which address common student difficulties with eigenstates, have been removed from the modified tutorial. Instead, we have included several of these statements

in one of the homework exercises for the modified *Energy measurements* tutorial (see section 5.1C2).

Other than these differences, the overall structure of this tutorial has not changed substantially. Students are still asked to consider the time dependence for wave functions of energy eigenstates and their associated probabilities, and then to repeat this exercise for a superposition of energy eigenstates. Several summary questions are no longer included, since few students completed the rest of the tutorial and reached these ideas. However, some of the homework questions for this topic help students summarize their ideas in similar ways.

The homework for the modified time dependence tutorial consists of six problems, each of which focuses on a different aspect of time dependence. Only a few of these problems are typically assigned when the tutorials are administered at UW. This is done to keep the total homework for the course at a reasonable length, since the lecturer also assigns weekly homework problem sets. The modified tutorial homework covers: the relationship between the time dependence for classical and quantum particles; practice with finding probability densities at different times; the common student difficulty that the wave function "spreads out"; how the time dependence of the probability density is related to energy *differences* rather than directly to energy; and how degenerate energies impact time dependence.

2. Modified tutorial: *Energy measurements*

The original version of the *Energy measurements* tutorial is also one that many students did not complete in the allotted time. As with the time dependence tutorial, part of the reason for this seemed to be the large number of different ideas that the students were asked to consider.

We decided to prioritize questions about three topics: (1) the possible outcomes of energy measurements and their associated probabilities; (2) the time dependence of these probabilities; and (3) how measuring the energy affects the state of a particle and future measurements. The first section of the original tutorial, which dealt with the difference between superpositions and mixed states, was removed altogether. It now forms the basis for a new tutorial (see section 5.1C4). The second section, which focused on the fact that energy eigenfunctions for the infinite square well are not energy eigenfunctions for the harmonic oscillator, was moved from the beginning to the end of the tutorial.

The remainder of the tutorial was expanded to emphasize questions about the time dependence of energy probabilities, after both short and long intervals of time have elapsed. The set of questions focusing on what happens after the energy of the particle is measured has also been expanded. In one exercise, students are presented with three incorrect statements, each of which represents a common reasoning difficulty, about what happens to the state after a long time has elapsed. The final page of the tutorial, which asks students to summarize their results, has not been changed substantially. Although not all students reach this page, it is designed so that they can complete it on their own time, and almost all students now work through most of the exercises in the tutorial.

The homework associated with the modified tutorial on energy measurements includes three problems. The first problem provides practice with determining the probabilities of energy measurements and the state after a measurement has been made. The second problem asks students to verify that both an energy eigenstate and a superposition of energy eigenstates satisfy the time-dependent Schrödinger equation, and then to consider several incorrect statements that

target specific common difficulties. The exercises in this problem are based primarily on questions that were removed from the original version of the tutorial *Time dependence in quantum mechanics*. The last homework problem asks students a sequence of questions about a system whose potential changes from an infinite square well of width a to an infinite square well of width $2a$. It helps them to recognize that the energy eigenfunctions for one potential are not necessarily energy eigenfunctions for a different potential.

3. Modified tutorial: *Position, momentum, and energy measurements*

The original version of the tutorial *Position, momentum, and energy measurements* was too long for students to complete in the allotted time. It contained several questions that many students found difficult to answer. The tutorial also included a number of exercises that were more mathematical than conceptual. As a result, we substantially redesigned this tutorial to serve both as an opportunity for students to explore the effect of position measurements and as a synthesis of the ideas in the previous tutorials. In order to reduce the number of ideas covered in the tutorial, momentum measurements are now only discussed in the homework.

The first section of the modified tutorial asks students to describe the state of a particle after its position has been measured and to sketch a wave function for such a state. As in the tutorial *Energy measurements*, students consider an incorrect statement that expresses the idea that making a measurement corresponds to acting the corresponding operator on the initial state of the particle. Then, they are asked about a second position measurement made both immediately after and a short time after the initial measurement.

The second section of the modified tutorial is based on the section of the original tutorial in which the students consider a specific sequence of measurements for the infinite square well. In the modified version, students begin by sketching the wave function for the third excited state and identifying whether or not the probability density and the energy probabilities will depend on time. They are encouraged to generalize their results from the specific case that they considered to any energy eigenstate. For students who completed the previous tutorial, this should be review, but we have found that it is important to strengthen their understanding by revisiting it in this tutorial. Students are then told the result of a position measurement and asked the same sequence of questions—in particular, they are asked whether the wave function is an energy eigenstate or a superposition of energy eigenstates. Lastly, students fill out tables to summarize and bring together their results.

Due to time constraints, the exercises from the original tutorial that asked students to find the eigenvalues and the eigenfunctions of the position and momentum operators have been deemphasized. The tutorial worksheet now includes a supplementary page at the end that focuses on the position operator. Several questions have been added to help guide students to the correct answer, since many do not complete this section during class. One of the problems on the homework follows the same sequence of questions for the momentum operator. Another question on the homework asks students to expand the summary tables from the tutorial to include momentum. We have found that this is still challenging for students, and it should be noted that there is relatively little research on student understanding of momentum in quantum mechanics. This topic is thus suitable for future research and curriculum development.

4. New tutorial: *Superposition in quantum mechanics*

Many of the ideas removed from the original sequence of tutorials are important, and we have tried to include them in other places, such as the tutorial homeworks. One of the most important topics from the original curriculum that is not addressed in the modified curriculum discussed above is the difference between superposition states and mixed states. We used the exercises from the original tutorial as the basis for a new tutorial, *Superposition in quantum mechanics*. Several questions were added to guide the students in sketching wave functions and probability densities for both superpositions and mixed states, and in identifying that while energy measurements cannot be used to distinguish them, position measurements can. This topic forms the second section of the new tutorial.

The first section asks students about a superposition of energy eigenstates in the infinite square well. Students confront several common incorrect ideas, such as the idea that sums of eigenstates are always also eigenstates and that only eigenstates are valid states. Then they are shown the graphs of two possible wave functions and asked to identify which could represent a particle. Students should recognize that they can use the boundary conditions for the infinite square well to answer. The homework problems offer students practice with the ideas in this section.

It should be noted that this tutorial is appropriate for students at a variety of different points throughout instruction. Students are not required to use ideas related either to time dependence or to the effects of measurement on a quantum state in order to work through this tutorial. Thus, *Superposition in quantum mechanics* can be administered at any point before, during, or after the

sequence of tutorials on time dependence and measurements, as long as the students have studied the time-independent Schrödinger equation, boundary conditions, and the infinite square well potential. At UW, we have administered this tutorial near the end of the sophomore-level introduction to quantum mechanics course, rather than in the junior-level course. This decision was primarily made for two reasons: there is not enough time in the first quarter of the junior-level course to complete this tutorial and the sophomore-level students are taught the difference between superpositions and mixed states using the context of spin-1/2 states.[17]

D. Assessment of the Modified Sequence of Tutorials for Quantum Measurements and Time Dependence

Preliminary assessments of the original sequence of tutorials on quantum measurements and time dependence (see section 5.1B) were encouraging. The results indicated that these tutorials can be used to improve student understanding of many ideas related to time dependence and measurement. Students who completed the original tutorials performed better on several exam questions that focus on these topics than a variety of populations of students who did not have tutorials.[14]

In this section, we discuss an analysis of how students who completed the *revised* tutorials performed on exam questions. In particular, we compare student performance on post-tutorial questions to performance on similar pretest questions administered before the tutorial. For those post-test questions for which there is no suitably matching pretest question, the results are only compared qualitatively. Pretest results are drawn from the tutorial pretests, which are given after

lecture instruction on the relevant topics, but before the tutorials. The results from many of the pretest questions are discussed in chapters 2-4.

The responses to pre- and post-test questions were each analyzed to determine the type of reasoning used by each student. We used the same criteria to determine whether or not a student's reasoning was correct on the pre- and post-test questions. In the sections below, we typically compare the percentage of the students who answered correctly with correct reasoning before and after the tutorials. We also examine how the prevalence of certain incorrect lines of reasoning changes after the tutorials. Not all common incorrect lines of reasoning were present in the responses to each question; thus, only the most prevalent are discussed.

Each of the numbered sections below focuses on the exam questions for a particular topic (*e.g.*, question about energy measurements). Many exam questions were administered after students had completed only one or two of the sequence of three tutorials. In what follows, we group students by the number of tutorials administered prior to the exam: group 1 completed only *Time dependence in quantum mechanics*; group 2 completed this tutorial along with *Energy measurements*; and group 3 completed all three tutorials. Some questions were also asked of students who completed the *original* versions of the tutorials; they are referred to as group A. Students in group A had completed the entire sequence of original tutorials (see section 5.1B). "Pretest" is used to refer to students who had not completed any tutorial curriculum, but who had lecture instruction on these topics.

The ideas in all three tutorials are interrelated. Although each tutorial focuses on different ideas, the later tutorials provide practice with some of the ideas from the beginning of the sequence. On the other hand, the new tutorial, *Superposition of quantum mechanics*, focuses on

ideas that are only somewhat related to time dependence and measurements. This tutorial was not given as part of the sequence on quantum measurement and time dependence to the junior-level students at UW. However, most students in the junior-level course had completed the superposition tutorial in their sophomore-level introduction to quantum mechanics course, typically 2-6 months prior to the start of the junior-level course.

1. Questions about the time dependence of wave functions and probability densities

We used variants of Task 5.1 (see Fig. 5.1 in section 5.1B) to assess the effectiveness of the modified sequence of tutorials. In Task 5.1, students were shown two energy eigenfunctions—one symmetric and one antisymmetric—and asked three questions related to the time evolution of a superposition of these eigenfunctions. Question 5.1a asked if the state is stationary; question 5.1b asked if there is an instant in time when the wave function is exactly equal to the ground state; and question 5.1c asks if there is an instant in time when the probability density has a given shape. The only differences between the versions given during each quarter were the shapes of the energy eigenfunctions, the initial superposition state of the system, and the possible probability density graph in question 5.1c. All variants may be found in the appendix.

Question 5.1a has been administered to students from group A (after all original tutorial curriculum), group 2 (after two revised tutorials), and group 3 (after all three revised tutorials). Figure 5.3 shows the percentage of the students who answered question 5.1a correctly, both with and without correct reasoning. When comparing the percentage of the students in each group who gave correct answers with correct reasoning, only the difference between group A and group 2 is statistically significant ($p<0.05$).

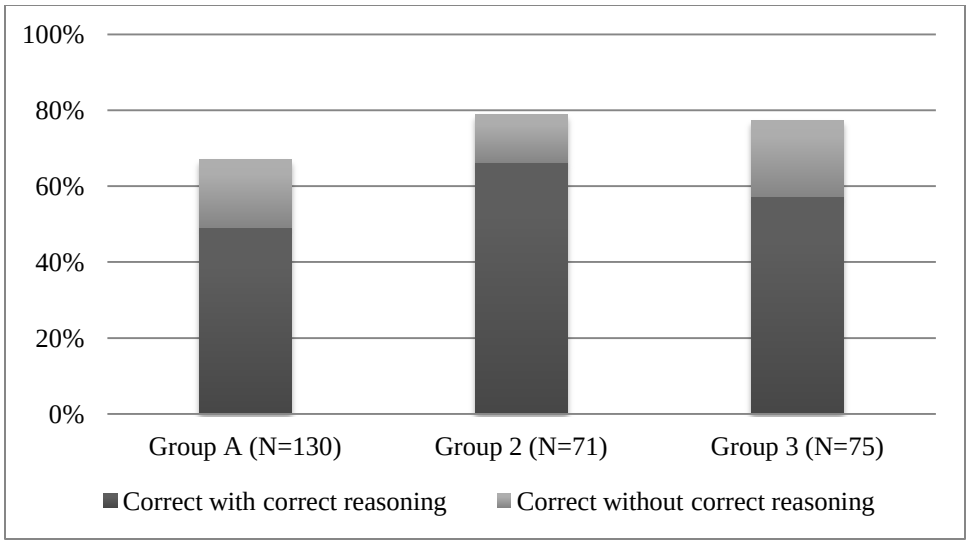


Figure 5.3: Results for question 5.1a

The percentage of correct responses, both with and without correct reasoning, is shown.

As discussed in chapter 2, students often misuse the term "stationary state" after lecture instruction. They often give answers that are not consistent with the definition of "stationary state" as a state for which the probability density is independent of time (*i.e.*, an energy eigenstate). For example, many students believe that the wave function itself does not depend on time for a stationary state (see section 2.4A1). After tutorial instruction, this kind of response was extremely uncommon (about 5% or less on question 5.1a). This suggests that the tutorials are effective at helping students understand what is meant by a "stationary state" in quantum mechanics.

A version of question 5.1b, which asks if there is an instant when the wave function has the same shape as the ground state, has been asked after the students have completed one, two, and three of the tutorials in the modified sequence. (No students saw the question more than once).

Figure 5.4 shows the results for question 5.1b. The results for group 3 are statistically different from the results for all other groups ($p<0.01$), while the difference between group 1 and group A is also statistically significant ($p<0.05$). The other differences are not statistically significant.

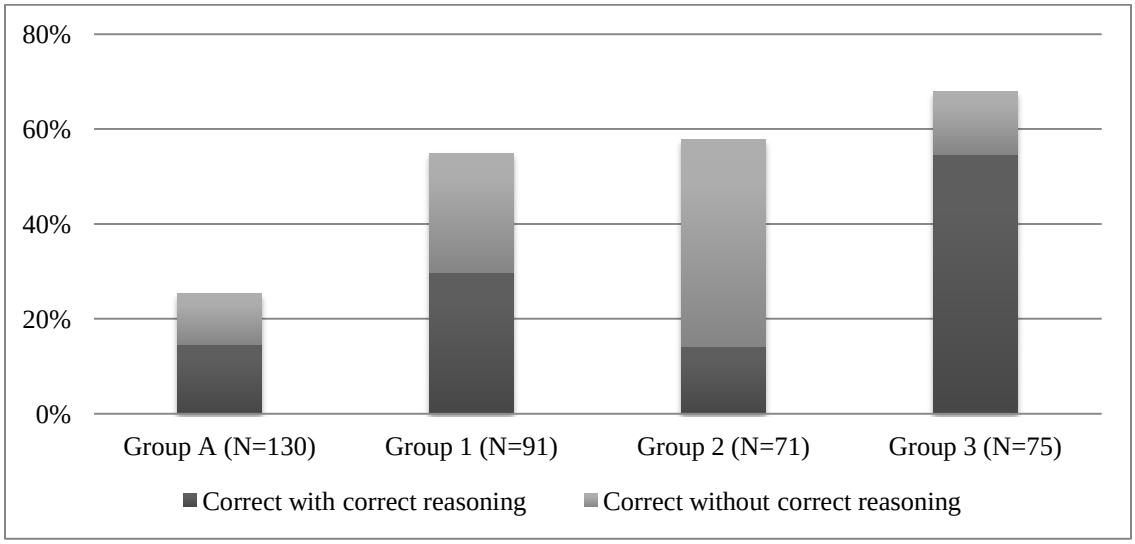


Figure 5.4: Results for question 5.1b

The percentage of correct responses, both with and without correct reasoning, is shown.

One of the common incorrect answers identified on pre-tutorial questions similar to Task 5.1 is students stating that the wave function is equal to the ground state when the phase of the first excited-state term was entirely imaginary (see section 2.4C3 for a more detailed discussion of this difficulty). This difficulty was also prevalent in the responses to question 5.1b. About one-third of the students from group A, group 1, and group 2 gave answers consistent with this difficulty. However, only 4% of the students in group 3 gave this answer. None of the other common incorrect lines of reasoning identified in chapter 2 were prevalent in the responses to question 5.1b.

The results above suggest that the tutorials help students answer this question. The modified tutorials in particular appear to be more helpful than the original sequence of tutorials. Students also tended to be more likely to give correct reasoning after more tutorials were administered. The performance of group 2 does not follow this trend, however. Much of the difference between the post-test performance of groups 1 and 2 can be accounted for by noting that 17% of the students in group 2 answered question 5.1b correctly but gave no explanation. In each of the other groups, less than 5% of the students answered correctly without offering any explanation.

Question 5.1c asks if there is an instant in time when the probability density has a particular shape. The results are shown in Fig. 5.4. This question has been administered to students from group A, group 1, and group 2. A version of this question was also given on a pretest (see Task 2.1). The pretest asks whether or not the probability density changes with time, whereas the exam question asks whether or not there is a time when the probability density has a specific shape. We consider the pretest question to be less difficult than the post-test question because the students are not required to reason about a particular instant in time on the pretest. The percentage of the students who answered correctly with correct reasoning is statistically greater for all three post-test groups ($p<0.05$) than for the pretest group. The differences between the results on the three post-test questions are not significant.

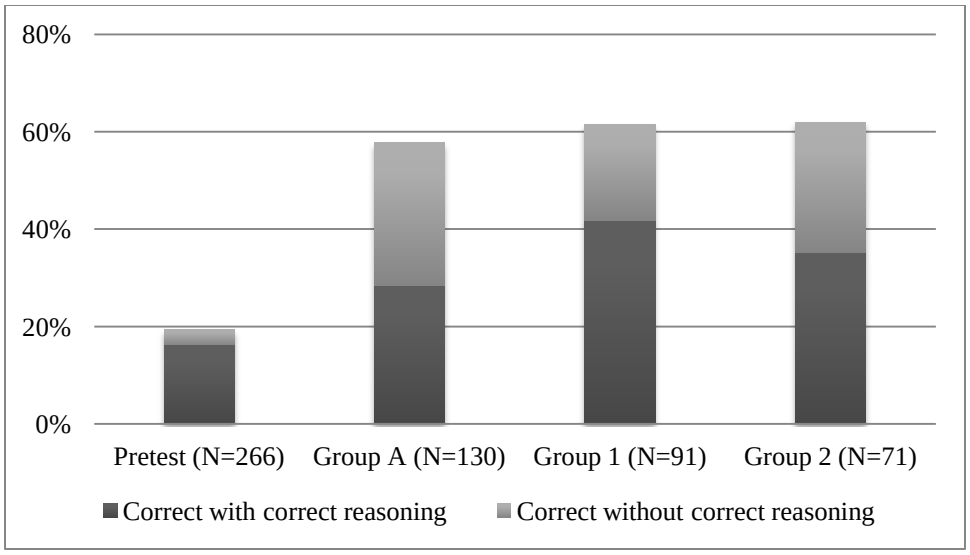


Figure 5.5: Results for question 5.1c

The percentage of correct responses, both with and without correct reasoning, is shown.

In summary, the sequence of modified tutorials described in section 5.1C appears to help students answer questions about the time dependence of wave functions and probability densities in quantum mechanics. As documented in chapter 2, students who have completed only traditional lecture instruction often have difficulty with time dependence. We found, for example, that many students believe that the probability density for a superposition of states spreads out or does not evolve in time at all. Responses like these were very uncommon when question 5.1c was given after tutorial instruction. Overall, students did much better on the post-tutorial questions than they did on pre-tutorial questions.

However, there still appears to be room for improvement. After tutorial instruction, many students made errors related to combining the graphs of the ground-state and first excited-state wave function, or used symmetry incorrectly. We did not encounter difficulties of this nature on

the pretests discussed in chapter 2, and they do not seem to be directly related to time dependence. The prevalence of errors related to student ability to add wave functions on this question, and other similar questions, suggested that students were having difficulty unrelated to quantum mechanics. We were encouraged to investigate student ability to add and subtract wave functions beyond the context of time dependence. The results of this investigation are discussed in chapter 6.

2. Questions about energy measurements

Much of the sequence of tutorials described in section 5.1C is intended to help improve student understanding of energy measurements. We designed the task shown in Fig. 5.6 (Task 5.2) to assess whether or not the tutorials have accomplished this goal. Task 5.2 was given to students after the first two tutorials in the sequence were administered. The task provides students with an initial state consisting of a superposition of energy eigenstates. Question 5.2a asks students to determine the probability of measuring the ground state energy at $t = 0$, the time for which the initial state is specified. Students are then given a specific instant in time and asked about the same probability at that instant in time (question 5.2b). The correct answer to question 5.2a is 1/2, and can be determined by taking the complex square of the coefficient. The correct answer to question 5.2b is that the probability is the same at any later time. Since each term in the wave function changes only according to a phase factor, the complex square of the wave function, and thus the probability, does not change in time. Question 5.2b is specifically designed to assess whether or not students think that the probability of an outcome of an energy measurement depends on time.

Task 5.2

Suppose that a system is initially in a state given by: $\Psi(x,t=0)=\frac{\psi_1-\psi_2}{\sqrt{2}}$.

Question 5.2a

Suppose you were to measure the energy of the system at $t = 0$. What is the probability that the result of this measurement is E_1? Explain.

Question 5.2b

Suppose instead that you were to measure the energy of the system at time $t_1 = \hbar/E_2$ (no measurement was made at $t = 0$). Is the probability that the result of this measurement is E_1 *greater than, less than,* or *equal to* your answer to the previous question? Explain.

Figure 5.6: Exam questions related to energy measurements

Task 5.2 was administered to 71 students from Population 2.

Almost all students (89%) answered question 5.2a correctly after tutorial instruction. Most of them (77%) gave correct reasoning. The results for question 5.2b are shown in Fig. 5.7. The figure includes the results from a similar pretest question (see Task 2.2) that gives students a superposition of energy eigenfunctions and asks them if there is an instant at which the probability of measuring the ground-state energy is zero.

Relatively few students provided correct reasoning on this question, either when it was asked as a pretest or as a post-test. Most of the students who gave the correct answer simply stated that energy probabilities do not depend on time, but did not provide any reasoning beyond this statement. While this statement is not incorrect, it is incomplete. The difference between the percentage who answered correctly on the pretest and the post-test (both with and without correct reasoning) is statistically significant ($p<0.01$).

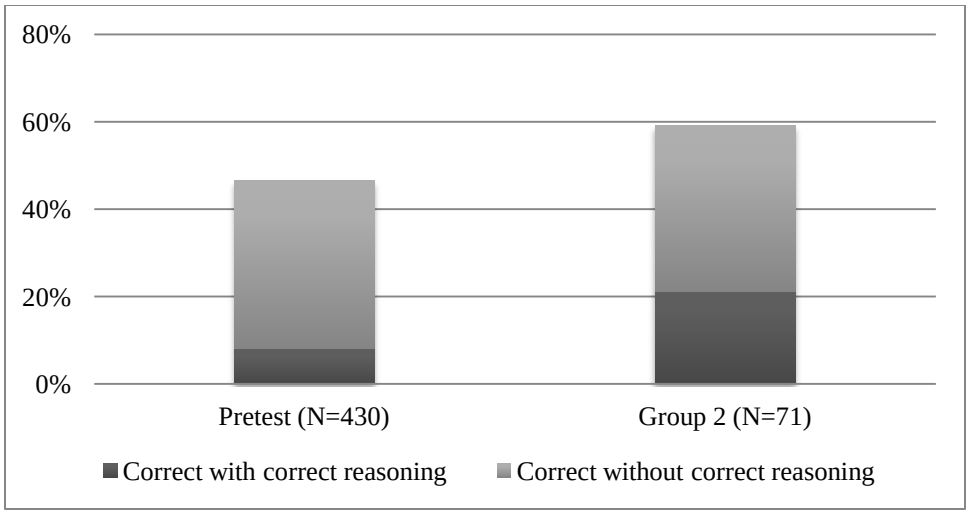


Figure 5.7: Results for question 5.2b

The percentage of correct responses, both with and without correct reasoning, is shown.

For students who gave the incorrect answer, there were several common incorrect lines of reasoning. Eight percent of the students on the exam stated that the probability would be zero when the phase factor is imaginary. About the same percentage (11%) used this reasoning on the pretest (see section 3.5A1). This kind of explanation was also common on question 5.1b (see previous section). On the post-test, an additional 8% of the students stated that the probability would decrease as time increases because of the exponential phase factor. These students appear to be treating the phase factor as a *real* rather than as an imaginary exponential. This difficulty has also been identified at about the 10% level in the responses to some pretest questions (see section 2.4C2). Some students (8%) made mathematical errors on question 5.2b that led to an incorrect answer, while 13% of the students left the question blank.

The results from Task 5.2 suggest that the modified sequence of tutorials is somewhat effective at improving student understanding of energy measurements. However, many students

continue to answer incorrectly after tutorial instruction, and more than half of the students who answer correctly do not give reasoning that is fully correct. These findings indicate that further modifications, particularly to the tutorial *Energy measurements*, may be useful.

3. Questions about sequences of measurements

We have used a variety of different questions to assess student understanding of sequences of quantum measurements. Since the last tutorial in the sequence on time dependence and measurement (*Position, momentum, and energy measurements*) focuses primarily on sequences of measurement, the questions are typically asked after the full sequence of tutorials has been administered. In this section, we discuss the results to two different tasks (Tasks 5.3 and 5.4) that focus on the time dependence of probabilities after measurements of different types have been made.

Task 5.3: On Task 5.3 (see Fig. 5.8) students are presented with a superposition of energy eigenstates for the hydrogen atom potential. They are asked about the time dependence of measurements of energy, position, and momentum both before (question 5.3a) and after (question 5.3b) a measurement of energy is made. Students should recognize that energy probabilities do not depend on time either before or after the measurement, since the time dependence in each coefficient does not affect the corresponding magnitude. However, since the particle is initially in a superposition of energy eigenstates, the probabilities associated with position and momentum do depend on time. After the measurement, the particle is in an energy eigenstate, so no probabilities depend on time. Since the pretest questions about sequences of

measurement did not ask about momentum measurements (see Task 4.1), the analysis of the student responses in this section focuses on the energy and position measurements.

Task 5.3

Consider an electron in the hydrogen atom potential. The particle's initial state is given by the expression $|\Psi(t=0)\rangle = \frac{|2,1,-1\rangle - |3,2,0\rangle}{\sqrt{2}}$, where $|n,l,m_l\rangle$ are the energy eigenstates of the system.

Question 5.3a

For the particle defined above, would the probability of measuring a particular value for each of the following quantities depend on time: 1) *energy,* 2) *position,* or 3) *momentum?* Explain your reasoning for full credit. If you do not have enough information to tell, state so explicitly.

Question 5.3b

Suppose that the energy of the particle initially defined above is measured and determined to be equal to the energy of the second excited state (E_3).

After the measurement described above, would the probability of measuring a particular value for each of the following quantities depend on time: 1) *energy,* 2) *position,* or 3) *momentum?* Explain your reasoning for full credit. If you do not have enough information to tell, state so explicitly.

Figure 5.8: Exam questions about sequences of measurements

Task 5.3 asks students about the time dependence of various measurement probabilities both before and after a measurement of energy is made on a superposition of energy eigenstates. This sequence of questions was administered to 87 students in Population 3.

Task 5.3 was administered to 87 students from group 3. Most students (78%) answered question 5.3a correctly, almost all (72%) with correct reasoning. This is consistent with the high performance of students from group 3 on other questions about time dependence (see section 5.1D1). We do not discuss the results to question 5.3a in more detail because it does not focus on sequences of measurements; that is, it does not ask students about probabilities or time dependence *after* a previous measurement has been made.

The percentage of the students who gave the correct answer to question 5.3b, which asks about the time dependence of probabilities after an energy measurement has been made, is

shown in Fig. 5.9. The figure also shows the results for question 3.2B (given before tutorial but after lecture instruction; see section 3.4), which asks whether an energy measurement will *definitely*, *possibly*, or *definitely not* give the same value as an energy measurement made a short time earlier. Since the pretest question does not ask about time dependence or about subsequent position measurements, we regard it as less difficult. The difference between the pre- and post-test results (percentage who gave correct answer with correct reasoning) is statistically significant ($p<0.001$).

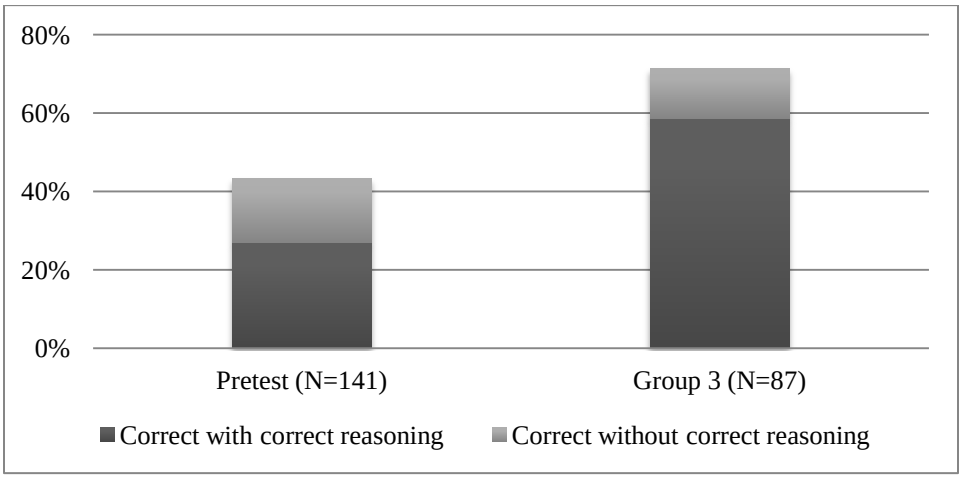


Figure 5.9: Results for question 5.3b

The percentage of correct responses, both with and without correct reasoning, is shown.

The most common incorrect answer on the pretest (given by about 45% of the students) was that the initial energy measurement would not affect the state of the particle. They predicted that a second measurement could result in the same set of possible values as the first measurement. An additional 20% of the students on the pretest answered that the wave function would return to

its initial state after some amount of time. Lastly, about 10% of the students based their answer to the pretest classical ideas, such as conservation of energy.

On the post-test (question 5.3b), almost no students gave explanations consistent with any of these three lines of reasoning. This suggests that the sequence of three tutorials was highly effective at helping students to understand the effects of energy measurements on quantum systems.

Task 5.4: This task (see Fig. 5.10) includes two questions about whether certain probabilities are time-dependent or time-independent after a measurement of position has been made. (In contrast, the questions on Task 5.3 only ask about what happens after an energy measurement.) The first four questions on Task 5.4 are designed to assess other ideas that the sequence of tutorials on time dependence and measurement is intended to address.

Task 5.4

Consider a particle in the quantum mechanical infinite square well of width a. At $t = 0$, the normalized wave function for the particle is given by $\Psi(x, t=0) = \sqrt{\frac{30}{a^5}}\left(x^2 - ax\right)$ for $0 < x < a$ and 0 elsewhere.

You may receive full credit on this problem without evaluating any integrals; however, writing down an integral without explaining where it comes from will not result in credit.

Question 5.4a

Suppose you were to measure the *energy* of this particle at $t = 0$. Determine the probability that the energy is equal to E_1, the energy of the ground state. Explain your reasoning.

Question 5.4b

Does the probability that the energy is equal to E_1 depend on the time at which the measurement is made? Explain.

Question 5.4c

Suppose you were to measure the *position* of this particle at $t = 0$ (no other measurements have been made). Determine the probability that the position of the particle is given by $x < a/3$. Explain your reasoning.

Question 5.4d

Does the probability that the position of this particle is given by $x < a/3$ depend on the time at which the measurement is made? Explain.

Question 5.4e

Now suppose that the position of the particle is measured to be at $x = a/3$ at time t_1, where $t_1 > 0$.

Suppose you were to measure the position of this particle at t_2 ,where $t_2 > t_1$ (assume no measurements have been made between t_1 and t_2). Does the probability that the position of this particle is given by $x < a/3$ depend on the time at which the measurement is made? Explain your reasoning.

Question 5.4f

Suppose instead you were to measure the energy of this particle at t_2 ,where $t_2 > t_1$ (assume no measurements have been made between t_1 and t_2). Does the probability that the energy of this particle is E_1 (the ground state energy) depend on the time at which the measurement is made? Explain your reasoning.

Figure 5.10: Another sequence of exam questions

The questions were given after all three tutorials on time dependence and measurement

At the beginning of Task 5.4, students are given an initial wave function that is not explicitly written in terms of energy eigenfunctions. They are asked to determine the probability of measuring the ground state energy (question 5.4a) and the probability of measuring the position

to be $x < a/3$ (question 5.4c) at $t = 0$. Each probability can be determined by taking a suitable inner product and then converting each inner product into an integral, as shown below.

$$P(E_1) = \left|\langle \psi_1 | \Psi \rangle\right|^2 = \left|\int_0^a \psi_1^*(x)\Psi(x,t=0)dx\right|^2 \qquad \text{(Question 5.4a)}$$

$$P(x < a/3) = \int_0^{a/3} \left|\langle x | \Psi \rangle\right|^2 dx = \int_0^{a/3} \left|\Psi(x,t=0)\right|^2 dx \qquad \text{(Question 5.4c)}$$

The students are also asked whether or not each of these probabilities depends on time (questions 5.4b and 5.4d, respectively). The energy probability (question 5.4b) does not depend on time because the inner product between the state of the particle and an energy eigenstate leaves only a single time-dependent phase, which cancels upon taking the complex square. The position probability (question 5.4d) does depend on time because the particle is in a superposition of energy eigenfunctions, so there is interference between the time-dependent phase factors of each term. The responses to questions 5.4a-d, which are not related to sequences of measurements, are not discussed in this section—they are instead discussed in section 5.1F.

Students are then told the result of a position measurement. They are again asked whether or not the probabilities associated with position (question 5.4e) and energy (question 5.4f) depend on time. Since the state of the particle after the measurement may still be written as a (different) superposition of energy eigenfunctions, the position probability does depend on time while the energy probability do not. Figure 5.11 shows the percentage of the students who answered questions 5.4e and 5.4f correctly. Results from pretest questions about sequences of measurements (see section 4.2A) are also included in the figure. Since the pretest questions only asked whether or not it is possible to measure various outcomes after a measurement of position, we regard them as easier than the post-test questions.

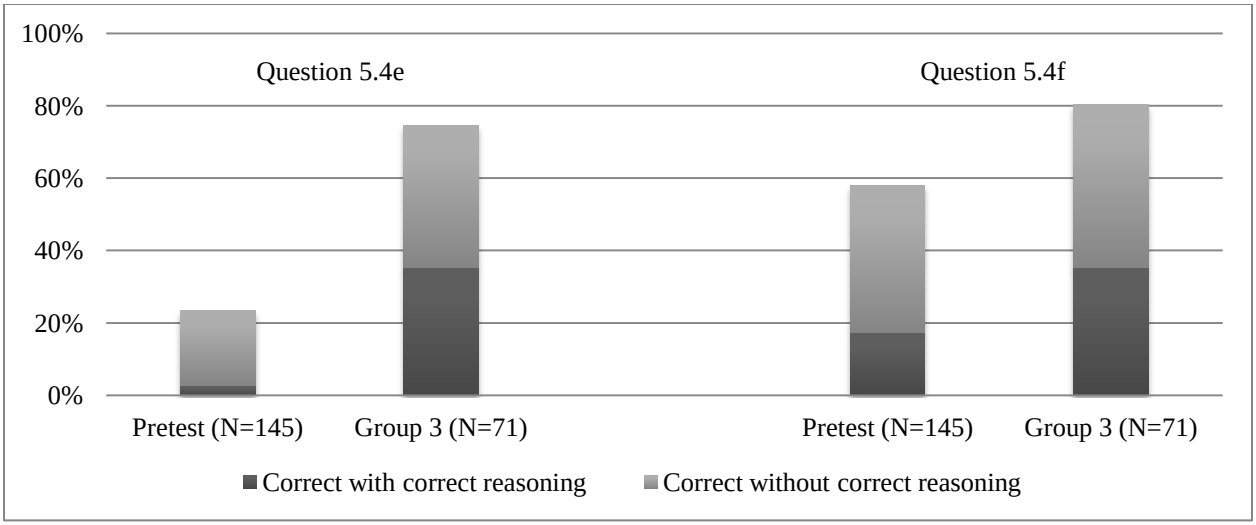


Figure 5.11: Results from Task 5.4

The percentage of correct responses to question 5.4e (left) and 5.4f (right), both with and without correct reasoning, are shown.

The performance of the students who had completed the tutorials was significantly better than the pretest performance for both question 5.4e ($p<0.001$) and question 5.4f ($p<0.01$). While more than three-fourths of the students answered each post-test question correctly, only about 35% gave correct reasoning. Some common incorrect lines of reasoning gave the correct answer. For example, 17% of the students stated that the wave function would "spread out" over time after the measurement. Thirteen percent of the students answered that the wave function would return to a previous form (*i.e.*, from before the measurement). Both of these errors are also common on questions asked before tutorial instruction (see sections 2.4D, 3.5D, and 4.1A). In addition, 15% of the students gave the incorrect answer that the state would remain "collapsed" after the measurement of position at time t_1.

In summary, student performance on questions about sequences of measurements is improved after working through the sequence of tutorials described in section 5.1C. However, it

appears that there is still room for the tutorials to be improved such that they might better address ideas related to measurements made after a position measurement. In particular, the *Position, momentum, and energy measurements* tutorial has been modified to take into account the results discussed in this section. The modifications to this tutorial are discussed in section 5.1G.

E. Alternative Method of Assessment

Direct comparison of pre- and post-tutorial results is not the only method that we have used to assess the effectiveness of the tutorials. In 2013, we used an alternative method for making a comparison between students who did and did not complete the tutorials. In the junior-level quantum course at UW, not all students attended every tutorial session (between 55% and 75% of the students attended each week during 2013). Thus, we were able to group the students based on whether or not they attended each tutorial. Then, for each exam question that was directly related to one or more of the tutorials, we compared the performance of those students who attended the relevant tutorial to the performance of those students who did not attend. Lastly, we aggregated the results from all of the tutorial-related exam questions, including several questions that were related to tutorials on angular momentum, which are described in chapter 10 of this dissertation.

The result of this analysis was that a greater percentage of the students who attended each tutorial gave correct answers on the exam questions related to that tutorial than those students who did not attend the corresponding tutorial. We attempted to control for the fact that the students who attend the tutorials are self-selected by dividing the students into three groups based on their average exam score for non-tutorial questions across the entire quarter. We found

that students in all three groups performed better when they had attended the relevant tutorials. In other words, students benefited from the tutorials regardless of how they performed on exam questions written by the lecturer. This assessment of the tutorials is described in more detail in Ref. 6.

F. Assessment of Student Performance at a Pilot Site for Tutorial Curriculum

An important part of validating the effectiveness of the sequence of three tutorials for time dependence and measurement is assessing student performance at other universities. Over the past two years, several different universities have served as pilot sites for preliminary versions of *Tutorials in Physics: Quantum Mechanics* (see chapter 1 for more information about the pilot sites). One university, Texas State University (TSU), administered Task 5.4 (from Fig. 5.10) to 24 students who had completed the entire sequence of modified tutorials for time dependence and measurements. Another university, California State Polytechnic University – Pomona (CPSU), administered Task 5.4 to 16 students who had completed the modified version of *Position, momentum, and energy measurements*. The students at CPSU also completed a version of the tutorial *Energy measurements* developed at the University of Colorado – Boulder. They did not complete any tutorial analogous to *Time dependence in quantum mechanics*.

Table 5.1 shows the results to Task 5.4 at both UW and at the pilot sites. The number of students at each pilot site is too small for statistical comparisons to highlight potential differences between the results at each university. (In general, the standard error for the given samples, which roughly corresponds to one standard deviation, is about 20-25%.) In addition, we did not ask any pre-tutorial questions at either pilot sites, and it is possible that the student populations

are different. Instead of making a statistical comparison, we focus on the general results: student performance does not appear to be substantially different at either pilot site than at UW.

Table 5.1: Results from Task 5.4

The percentage of correct answers, both with and without correct reasoning, is shown. The percentage with correct reasoning is shown in parentheses. Task 5.4 was administered on an exam after tutorial instruction at UW. It was also given to students at two pilot sites for the tutorials (TSU and CPSU). TSU completed the entire sequence of three modified tutorials, while CPSU completed only some of these tutorials.

University	Question 5.4a	Question 5.4b	Question 5.4c	Question 5.4d	Question 5.4e	Question 5.4f
UW (N=71)	70% (59%)	85% (32%)	73% (63%)	66% (44%)	75% (35%)	80% (35%)
TSU (N=24)	54% (46%)	75% (13%)	54% (54%)	63% (38%)	46% (21%)	67% (16%)
CPSU (N=16)	50% (50%)	94% (38%)	94% (94%)	19% (13%)	75% (56%)	81% (25%)

Only the results on question 5.4d seem remarkably different between the three universities. This question asks students whether or not a probability associated with position depends on time. Students at CPSU perform much worse (19%, 13% with correct reasoning) than students at UW (66%, 44% with correct reasoning) and students at TSU (63%, 38% with correct reasoning). The differences in performance on these two questions could be due to the fact that students at CPSU did not complete the tutorial *Time dependence in quantum mechanics*, which focuses on how wave functions and spatial probability densities change in time.

Overall, the results on Task 5.4 from the two pilot sites suggest that the tutorials can be just as effective when given at universities other than UW. Although these results are preliminary and the sample sizes are small, the results are promising. However, some results indicate that this sequence of tutorials may be less effective when certain tutorials are omitted from the

sequence. This suggests that it may be important for the entire sequence to be used. Further research is necessary to explore student performance at pilot sites in more detail.

G. Discussion

The objective of the sequence of tutorials described in this section is to improve student understanding of quantum measurements and time dependence. We used a variety of qualitative exam questions focusing on these topics to assess how effective the tutorials are at accomplishing this objective. We looked not just at whether or not students could answer the exam questions correctly, but at whether or not they could give correct explanations for those correct answers based on the model for quantum mechanics that they have been taught. The performance of the students on the (post-tutorial) exam questions was compared to the performance on similar questions that were given before tutorial but after lecture instruction. We interpret gains in the percentage of the students who answer correctly with correct explanations as evidence that student understanding has improved as a result of the tutorials.

The results from our assessment overwhelming indicate that the sequence of tutorials on time dependence and measurements has improved student understanding of these topics. Students performed universally better on post-test questions than they did on similar (and often less difficult) pretest questions. Not only did a larger percentage of the students give the correct answer, but more students gave correct reasoning as well. A closer examination of the results from one year of the junior-level quantum course at UW showed that the tutorials are helpful to students in the top, middle, and bottom thirds of overall exam performance in that course. Results from exam questions given to students at other universities where the tutorials have been

administered indicate that the effectiveness of the tutorials extends beyond just the structure of quantum mechanics at UW.

During the last several years, we have modified the sequence of tutorials on time dependence and measurements based on the results of much of the research described in chapters 2-5 of this dissertation. The revised sequence of tutorials still consists of three in-class worksheets; some are relatively similar to the original worksheets, while others are dramatically different. A fourth worksheet on a related topic was developed, but has not been used as part of this sequence. Tutorial-style homework problems were also developed for each of the three tutorial topics, replacing a single problem included on the weekly homework problem set. Comparisons of student performance on several exam questions administered after the revised and original tutorials demonstrated that the modified sequence of tutorials is, in general, more effective than the original sequence.

However, there is still room for improvement. On some post-test questions, many students answer correctly, but give reasoning that is incomplete or even incorrect. For example, about 60% of the students gave a correct answer when asked about the time dependence of an energy probability, which is independent of time. However, very few students (about 20%) correctly explained *why* the probability is independent of time by using the definition of the probability and the time dependence of the wave function. When we asked students about the time dependence of various probabilities after a measurement of position, only about 35% of the students correctly explained how the wave function changed as a result of the measurement, and how the wave function evolved in time afterward.

We have continued to modify the tutorials in an attempt to improve student understanding of time dependence and measurements further. For example, several changes have been made to *Position, momentum, and energy measurements* as a result of our assessment. In the most recent version of this tutorial, students begin by considering several possible descriptions of how a wave function evolves in time after a position measurement. Later in the tutorial, students are given a handout showing the time evolution of the probability density that allows them to check their answer. Future research will assess this newest version of the tutorial, and possibly result in additional changes to its structure or content. A similar process of revision and assessment may also be necessary for the other tutorials in this sequence of curriculum.

5.2 – Tutorial Curriculum for Time Dependence and Measurements in Advanced Contexts

The results discussed in the previous section indicate that the tutorial curriculum helps improve student understanding of quantum measurements and time dependence in basic quantum mechanical contexts. In addition to questions in basic contexts, such as the infinite square well or harmonic oscillator potentials, we have asked some questions, still focusing on time dependence and measurements, in more advanced quantum contexts. These contexts include two- and three-dimensional harmonic oscillator potentials, spin and angular momentum, identical particles, and perturbation theory. The questions were asked after lecture instruction on the relevant context, and in many cases after students had worked through the sequence of tutorials on time dependence and measurements, as well.

The results from many of these questions, which are discussed in more detail in chapters 2 and 3, indicate that students often give responses representative of the difficulties similar to those

that are common in more basic contexts before students have worked through the tutorials. The persistence of these difficulties, despite tutorial instruction, suggests that there is a need for curriculum to help students transfer their knowledge about quantum measurements and time dependence from basic to more challenging contexts. In this section, we discuss two example contexts—degenerate states (section 5.2A) and time-dependent perturbation theory (section 5.2B)—for which we have developed tutorial curriculum. The format of sections 5.2A and 5.2B are the same as in section 5.1: first, we describe an original version of the curriculum; second, we discuss how and why the curriculum has been modified as a result of ongoing research; and third, we provide an assessment of the revised curriculum. Since the tutorials for advanced contexts are at an earlier stage in their development than the sequence of tutorials for basic contexts, the assessments of these tutorials are more preliminary than what was discussed in section 5.1.

A. Tutorial Curriculum for a System with Degenerate Energy Eigenvalues

During the first half of the first-quarter junior-level quantum course at UW, students primarily consider systems that do not have degenerate energy eigenvalues. When all energy eigenvalues are distinct, any superposition of two or more energy eigenstates has a time-dependent probability density (*i.e.*, it is not a stationary state). For systems with degeneracy (*e.g.*, the two- or three-dimensional isotropic harmonic oscillator), superpositions of energy eigenstates with the same energy eigenvalue can be constructed. Such superpositions would *not* have time-dependent probability densities, since they would also be energy eigenstate and hence stationary states.

We have asked several questions about the time dependence of superpositions of energy eigenstates either with or without degenerate energies. These questions (see Task 2.3) were administered after lecture instruction on systems with degenerate energy eigenvalues, and also after the sequence of tutorials on quantum measurements and time dependence described in section 5.1. We found (see chapter 2) that many students treated superpositions of energy eigenstates with the same energy eigenvalue as if they were superpositions with different energy eigenvalues: that is, students often stated that the probability density depends on time for each superposition. Based on these results, a new tutorial based on a system with degenerate energies was created. The following sections discuss the original development and assessment of this tutorial, which was done by Andrew Crouse.[14] Then, we describe how we have revised the tutorial in recent years. Finally, we discuss how we have assessed the modified tutorial.

1. Description of original tutorial curriculum: *Degenerate states*

The tutorial *Degenerate states* uses the context of the isotropic two-dimensional harmonic oscillator to help students think about how the presence of degeneracy can impact the time dependence of wave functions and probability densities. The original tutorial begins by reminding students of the time-dependent Schrödinger equation and its solutions (in particular, the exponential phase factor that carries the time dependence). Then, students are asked to determine the wave functions for the ground state and the first excited state of a two-dimensional potential. They are expected to recognize that more than one wave function can be associated with the first excited-state energy level.

Students then consider several different wave functions for the two-dimensional system, some of which are associated with the same energy eigenvalue. They are asked whether or not each state and each probability density changes in time. Then, they are asked whether or not a superposition of stationary states can also be a stationary state. The objective of these questions is for students to generalize their knowledge about time dependence for simple systems to a system that has degenerate energy eigenvalues.

In the next section of the original tutorial, students consider the same potential using polar coordinates instead of Cartesian coordinates. They are given two initial states and are asked the same questions about time dependence as in the previous section for wave functions expressed in Cartesian coordinates. In the last section, students are asked the same questions about a state that is expressed using mixed coordinates (*i.e.*, as a superposition of one polar and one Cartesian state).

2. Assessment of original tutorial curriculum

This tutorial was still in early development at the time Crouse published his dissertation, so he does not discuss any assessment of the original tutorial.[14] Very few exam questions about quantum measurements and time dependence for systems with degenerate energy eigenvalues have been asked after administration of the original tutorial curriculum. On those that were asked, student performance tended to be poor. For example, some students were not able to identify the correct outcomes of an energy measurement on a two-dimensional system in a superposition of energy eigenstates. Many students also failed to determine the correct time

dependence for a wave function written as a superposition of energy eigenstates with *the same* energy eigenvalue.

The results from these questions suggest that the original tutorial is not sufficient to address student difficulties with time dependence and measurements for systems with degenerate energy eigenvalues. In addition, the original tutorial worksheet suffers from some of the same problems as the original sequence of tutorials described in section 5.1A1: it is too long for many students to finish in one class period, the topics it covers are too broad, and students have difficulty in understanding some questions. The following section discusses how we revised the tutorial based on these concerns.

3. Description of modified tutorial curriculum: *Degenerate states*

Of the ideas covered in the original tutorial, we decided to emphasize how to determine when the probability density for a superposition of eigenstates depends on time and when it does not. We made this choice because a large percentage of the students were unable to identify which probability densities depend on time when they were given Task 2.3 as a tutorial pretest (see chapter 2). The revised tutorial can thus serve as a review or summary of many of the ideas in the sequence of tutorials described in section 5.1A and 5.1B, in addition to extending students' model for time dependence to include cases with degenerate energy levels. The revised tutorial now begins by reminding students of the connection between a "stationary state" and a state with a well-defined energy. Students then find the three lowest energy eigenvalues of the isotropic two-dimensional harmonic oscillator. They are also given a definition for degeneracy.

In the second section of the revised tutorial, students are asked to consider two states that are written as superpositions of two energy eigenstates. One superposition has terms associated with different energies, while the other has terms associated with the same energy. Students are asked if the state of either particle depends on time, and whether or not either state is a stationary state.

The remainder of the revised tutorial guides students through a graphical exercise designed to help them recognize that some of the degeneracy in a system such as the two-dimensional harmonic oscillator can be accounted for by the rotational degree of freedom in the potential. Students sketch approximations of several wave functions and see that they are related by rotation. They are asked to identify how choosing a different set of initial coordinates would impact their answers. In the original tutorial, students also considered how a change from Cartesian to polar coordinates would affect their answer—this exercise is now in the homework. The homework also has students sketch approximate wave functions for the two-dimensional infinite square well, and to identify similarities and differences between the two potentials.

4. Assessment of modified tutorial curriculum

The revised tutorial has been administered to students in the fourth week of the second quarter of the junior-level quantum course at UW. In contrast, the original tutorial was given to students near the end of the first quarter of the same course. There are two reasons for this change: first, there was not enough time available to give this tutorial in the first quarter of the course; and second, another tutorial (*Degenerate perturbation theory*; see section 8.3) uses the same context as the tutorial *Degenerate states*, and we have found that doing these tutorials in successive weeks is useful for the students. We believe that the new version of the tutorial is

appropriate for groups of students at any point in a junior-level quantum course after they have studied time dependence and systems with degenerate energy eigenvalues.

Task 5.5

Consider a single particle in a three-dimensional infinite square well of side length *a* (that is, the potential is zero from $x = 0$ to a, $y = 0$ to a, and $z = 0$ to a).

In Cartesian coordinates the energy eigenfunctions are given in terms of the solutions to the one-dimensional infinite square well by:

$$\psi(x,y,z) = \varphi_{n_x}(x)\varphi_{n_y}(y)\varphi_{n_z}(z) \qquad E_{n_x n_y n_z} = E_1(n_x^2 + n_y^2 + n_z^2)$$

Several possible wave functions for this particle are given below:

$$\psi_A(x,y,z) = \frac{1}{\sqrt{2}}\left(\varphi_1(x)\varphi_1(y)\varphi_1(z) + \varphi_2(x)\varphi_2(y)\varphi_2(z)\right)$$

$$\psi_B(x,y,z) = \frac{1}{\sqrt{2}}\left(\varphi_1(x)\varphi_2(y)\varphi_2(z) - \varphi_2(x)\varphi_1(y)\varphi_2(z)\right)$$

$$\psi_C(x,y,z) = \frac{1}{\sqrt{2}}\left(\varphi_1(x)\varphi_2(y)\varphi_2(z) - i\varphi_1(x)\varphi_1(y)\varphi_3(z)\right)$$

$$\psi_D(x,y,z) = \frac{1}{\sqrt{2}}\left(\varphi_4(x)\varphi_4(y)\varphi_1(z) + \varphi_5(x)\varphi_2(y)\varphi_2(z)\right)$$

Question 5.5a

Which, if any, of the four wave functions listed above would change with time? Explain.

Question 5.5b

Which, if any, of the four wave functions listed above have a *corresponding probability density* that would change with time? Explain.

Question 5.5c

For which, if any, of the four wave functions listed above would the probability of measuring a particular energy value depend on time? Explain.

Figure 5.12: Sequence of exam questions involving degenerate states

Task 5.5 was administered as an online pretest *after* students had worked through the tutorial *Degenerate states*.

In assessing the revised version of the tutorial, we administered several questions (Task 5.5; see Fig. 5.12) after students had completed the new tutorial worksheet. These questions were given as an online survey rather than as part of a midterm exam. Since the modified tutorial

focuses on determining whether or not probability densities depend on time, this was the focus of the assessment questions.

In Task 5.5, students are first given four wave functions for a system consisting of a three-dimensional infinite square well. Each wave function is written as a superposition of two energy eigenfunctions. Then, question 5.5a asks them which, if any, of the four wave functions depend on time. Question 5.5b asks which, if any, have corresponding probability densities that depend on time. Finally, question 5.5c asks which, if any, have energy probabilities that depend on time. Very similar questions were asked on the pretest before the students worked through the tutorial, but for only two superpositions (see Task 2.3). The post-test questions are somewhat more difficult than the pretest questions because students are asked about more superpositions.

The correct answer to question 5.5a is that all wave functions depend on time. Students were not required to give further justification to be considered correct. For question 5.5b, only those wave functions that are superpositions of energy eigenfunctions with *different* (non-degenerate) energy eigenvalues (A and C) give time-dependent probability densities. It should be noted that many students who gave otherwise correct explanations failed to recognize that both terms in wave function D are associated with the same energy. Since our objective is to assess whether or not the tutorial is effective at helping students with time dependence, students who made this error were still counted as having given correct answers with correct reasoning. The correct answer to question 5.5c is that none of these wave functions are associated with time-dependent energy probabilities. For an explanation to be considered correct, it was sufficient for the students to note that energy probabilities never depend on time. There is no pretest question analogous to question 5.5c.

The percentage of the students who answered each question from Task 5.5 correctly with correct reasoning is shown in Fig. 5.13. Results from pretest questions analogous to questions 5.5a and 5.5b are shown in Fig. 5.13. Results from the pre- and the post-test questions include data from the second quarter of the junior-level quantum course at UW during two different years. On question 5.5a and on question 5.5b, students performed statistically better ($p<0.05$) afte the tutorial than before the tutorial. Very few students answered question 5.5c correctly or gave correct reasoning.

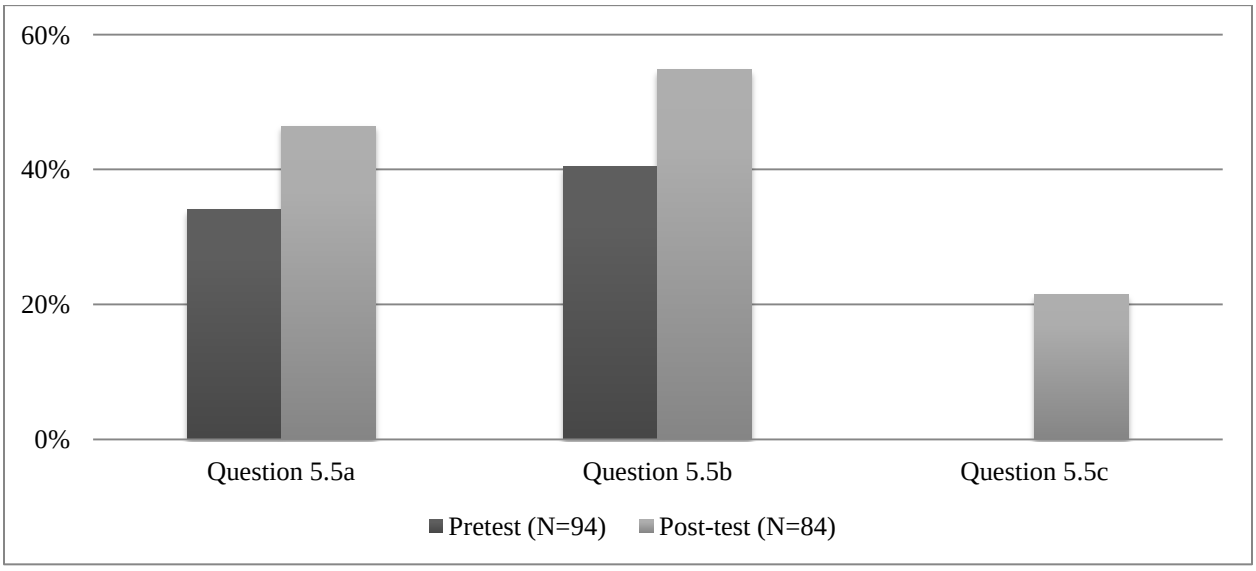


Figure 5.13: Results from Task 5.5

The percentage of the students who gave correct responses with correct reasoning is shown. Results on analogous pretest questions (see chapter 2) are also included for questions 5.5a and 5.5b.

The most common incorrect answers on the post-test were very similar to those on the pretest, but were less common. Some students (18%) answered question 5.5a by stating that the wave function depends on time only for superposition states (27% on pretest). On question 5.5b, 20% stated that all four probability densities depend on time; this error was made by 29% of the

students who completed the pretest. See section 2.4A for a more detailed discussion of difficulties related to student confusion between the time dependence of wave functions and the time dependence of probability densities.

Two difficulties were prominent on question 5.5c. Some students (23%) stated that energy probabilities depend on time for states that have time-dependent probability densities. Others (26%) stated that only superpositions of energy eigenstates have time-dependent energy probabilities (many did not correctly identify which states are superpositions and which are energy eigenstates). Since this question has not yet been asked on a pretest, further research is needed to determine the extent to which the revised tutorial helps students answer questions about the time dependence of probabilities of energy measurements.

In summary, the post-test results suggest that the modified version of the tutorial is effective at helping students answer questions about measurements and time dependence for systems with degenerate energies. However, there is still substantial room for improvement, especially in terms of students' ability to identify that energy probabilities are independent of time. In fact, the percentage of the students who answered question 5.5c correctly with correct reasoning is no different than when similar questions have been asked for systems that do not have degenerate energies (see section 5.1C2). The findings suggest that the current version of the tutorial, which does not specifically address this idea, should be modified to target this difficulty.

B. Tutorial Curriculum for Time-dependent Perturbation Theory

Another advanced context in which we have asked questions about time dependence and measurements is time-dependent perturbation theory. This is one of the final topics covered in

the second quarter of the junior-level quantum mechanics course at UW. Thus, the students have completed virtually all undergraduate instruction in quantum mechanics when they first learn time-dependent perturbation theory. This includes lecture and tutorial instruction on quantum measurement and time dependence, not just for simple contexts but also for systems with degenerate energy eigenvalues.

We have administered several questions in the context of time-dependent perturbation theory: Task 2.4, which focuses on the time dependence of energy and position probabilities, and Task 3.3, which focuses on the possible outcomes of energy measurements. Many of the student responses to each of these tasks (discussed in chapters 2 and 3, respectively) are consistent with the same difficulties that were common in the responses to questions about time dependence and measurements in basic contexts like the infinite square well. Some examples of common student difficulties include confusion between the time dependence of quantities (such as the wave function, the probability density, the energy probabilities, and the potential) and a belief that the wave function will spread out, decay, or return to a previous form after a long time has elapsed. This section describes the development and assessment of tutorial curriculum aimed at addressing these difficulties.

1. Original tutorial: *Time-dependent perturbation theory*

An initial version of a tutorial worksheet designed to address student difficulties with time-dependent perturbation theory is described very briefly in Andrew Crouse's dissertation.[14] The context of the tutorial is a particle in the ground state of an infinite square well. A small perturbation is added to the system at $t = 0$. The perturbation is then removed at a later time,

$t = T$, so that it is the infinite square well again. The tutorial asks students to consider whether or not it is possible to write the wave function for $0 < t < T$ as a superposition of the *unperturbed* energy eigenfunctions, even though these are not the energy eigenfunctions of the system during this time interval. The students then use equations derived from the time-dependent Schrödinger equation to determine the behavior of the wave function and its associated probabilities for $t > T$. Though some qualitative questions about measurements and time evolution appear throughout the tutorial, students are expected to do a substantial amount of mathematical manipulations. The original tutorial was often administered over two weeks to give students enough time to complete it.

2. Assessment of original tutorial curriculum

The original tutorial was not formally assessed by Crouse (see Ref. 14). However, his dissertation includes informal discussion of student performance indicating some student difficulties with the ideas associated with time-dependent perturbation theory even after completing the tutorial. (Some student difficulties with this topic are discussed in chapters 2 and 3.) The most prevalent difficulties after the original tutorial was administered were related to the behavior of the wave function after the perturbation is turned off: specifically, students often state that the wave function will decay to the ground state or return to the original state from before the perturbation. These results suggest that the curriculum requires additional development along with more formal assessment. The following section describes how we changed the original curriculum. Section 5.2B4 discusses how we have assessed the revised curriculum.

3. New sequence of two tutorials

Time-dependent perturbation theory involves challenging and advanced mathematics. After one administration of the tutorial on this topic, we identified several places where we thought the worksheet could be modified in order to focus on the conceptual ideas that underlie the mathematics. During the process of modifying the tutorial, we decided to divide the ideas related to time-dependent perturbation theory into two separate topics. *Time-dependent perturbation theory Part I: Foundations* is more qualitative. In the tutorial, students are asked about the possible outcomes of measurements and general time dependence at several different instants in time throughout the perturbation. *Time-dependent perturbation theory Part II: Formalism* is more quantitative. It guides students to determine the specific form of the wave function at any instant after the perturbation has been turned off.

The first tutorial in the new sequence covers much of the material discussed in the original tutorial, but with a somewhat different emphasis. In the first section, students consider a particle in the first excited state of the infinite square well. They are asked basic questions about the possible outcomes of an energy measurement, and about time dependence. This section is meant as a review of material that has been covered previously in both lectures and tutorials.

The second section introduces a small perturbation to the potential. Students are asked to predict whether or not the probability density depends on time, as well as the possible results of a measurement of the energy. They are then guided to think about how they might construct the time dependence of the wave function. They check their results and their predictions with an instructor before moving on. Then, students are asked to explain why the wave function can still

be written in terms of the energy eigenfunctions of the *unperturbed* potential. They are then asked why the coefficients in this expression are not related to the probabilities of the outcomes of an energy measurement.

The last section in the first new tutorial asks students to relate their wave function from section II with the wave function after the perturbation has been turned off. In particular, students think about the fact that the wave function is now in a superposition of energy eigenstates, and how this change impacts various probabilities of energy and position. They summarize their answers in a table and consider several incorrect statements that express two common student difficulties: (1) that the state returns to its initial state and (2) that the state decays to the ground state after the perturbation has been turned off.

A short homework assignment for the first tutorial in the new sequence has students practice the ideas from the in-class worksheet. First, they are asked about a particle that begins in the tenth excited state of the infinite square well; second, they are asked about a superposition of the first two energy eigenstates. For each initial state, students are asked about the possible results of an energy measurement made during or after the perturbation. They are also asked whether or not the probability density depends on time. Another student dialogue helps address the most persistent difficulties identified in chapters 2 and 3 (these are the same difficulties addressed at the end of the tutorial).

The second tutorial in the sequence addresses some of the more mathematical ideas related to time-dependent perturbation theory, while also helping students to solidify their understanding of the model they are using for this topic and the consequences of that model. The first section provides a review of what students completed during the first tutorial. It also asks them to

predict whether or not an energy probability depends on T, the length of time for which the perturbation is turned on. The correct answer is that the probability does depend on T, since the wave function is a different superposition of unperturbed energy eigenstates at each instant in time. Many students initially answer that the probability does not depend on time.

In the second section, students revisit the wave function during the perturbation, which is written in terms of unperturbed energy eigenfunctions. They are shown how the coefficients in this expression can be expanded, and given an equation (derived in lectures and in the textbook from the time-dependent Schrödinger equation) that describes how each coefficient changes in time. In the original version of the tutorial, students were asked to derive this equation on their own. In the modified version, students are guided to apply the correct initial conditions to the system, and then to use integrals to calculate the coefficients. These coefficients are used to determine probabilities, and students are asked several conceptual questions about these probabilities. In particular, they are asked how the probabilities depend on T, how they depend on energy, and whether or not they are normalized.

The homework for this tutorial offers students both practice and an opportunity to expand what they investigate in the in-class worksheet. The tutorial begins with a particle in the ground state, so the homework asks students to repeat their calculations for a particle that is instead initially in the j^{th} state. Another homework question asks students to compare probabilities for transitions from different energy levels. The rest of the homework guides students to calculate probabilities for a perturbation that is sinusoidal in time between $t = 0$ and $t = T$, rather than constant during this interval. They explore the differences between the two cases, and consider how their answers are related to ideas such as absorption and emission of radiation.

Together, this sequence of tutorials attempts to guide students to a more complete conceptual understanding of time-dependent perturbation theory than the original tutorial. Although the emphasis on mathematics is less than in the previous version of the tutorial, they are explicitly asked to consider several important mathematical steps. We feel that this is appropriate for an advanced tutorial that is meant to be given near the end of a junior-level quantum mechanics course. However, the qualitative ideas related to measurements and time dependence are just as important, as evidenced by our research has indicated that students struggle with such fundamental ideas even after both lecture and tutorial instruction.

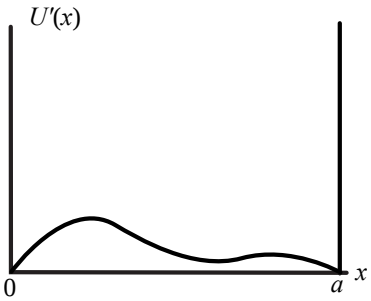
4. Assessment of modified tutorial curriculum

The new sequence of tutorials for time-dependent perturbation theory has been administered at the end of the second quarter of the junior-level quantum course at UW during two different years. (The original version of the tutorial was given to students at the same point in instruction.) In order to assess the effectiveness of the new curriculum, we asked a sequence of questions (Task 5.6, see Fig. 5.14) on the final exam at the end of the course. The questions focus on the probabilities associated with measuring the energy both during and after a time-dependent perturbation. The pretest questions for this topic (see chapter 2) have a somewhat different focus from Task 5.6: the pretests tended to ask whether or not various probabilities depend on time before, during, and after the perturbation. As a result, we do not attempt to make a direct comparison between the pre- and post-tutorial results. Instead, we look for general patterns in the lines of reasoning that students used to answer the pre- and post-test questions, and at how the prevalence of each line of reasoning may have changed as a result of the tutorials.

Task 5.6

Consider a particle in an infinite square well of width a. Recall that the energy eigenvalues of this system are given by $E_n = n^2 E_1$, where E_1 is the ground-state energy. Suppose that this particle is initially in the **second excited-state (n = 3)**.

At t = 0, the small perturbation $U'(x)$ shown below is turned on suddenly.



Question 5.6a

Suppose you were to measure the energy of the particle at some time $t > 0$. Rank the probabilities, P_n, that the energy of the particle is determined to be E_n (the n^{th} **unperturbed** energy) for the first four values of n (*i.e.*, n = 1, 2, 3, 4). If any probabilities are equal to 1 or 0, state so explicitly. Explain your reasoning.

Question 5.6b

Recall that this particle is initially in the **second excited-state (n = 3)** of the unperturbed potential. Assume that no measurements have been made on this system. Suppose that T is chosen such that the following equation is true for n = 1, 2, and 4: $\sin^2\left(\frac{E_n - E_3}{2\hbar}T\right) = 1$.

Suppose you were to measure the energy of the particle at some time $t > T$. Rank the probabilities, P_n, that the energy of the particle is determined to be E_n (the n^{th} **unperturbed** energy) for the first four values of n (*i.e.*, n = 1, 2, 3, 4). If any probabilities are equal to 1 or 0, state so explicitly. Explain your reasoning.

Figure 5.14: Exam questions for time-dependent perturbation theory

Task 5.6 was administered on the final exam of the second-quarter junior-level quantum mechanics course at UW.

Question 5.6a asks students to rank the probabilities of measuring the four lowest energy eigenvalues of the unperturbed potential during the time that the perturbation is on. The correct answer is that they are all equal to zero, since only the perturbed energy eigenvalues can be measured. Question 5.6b asks students to rank the probabilities of measuring the four lowest energy eigenvalues of the unperturbed potential *after* the perturbation has been turned off. Since

these probabilities are proportional to $(E_3 - E_n)^{-2}$, the probability is greatest for the values of n closest to 3, as follows: $P_3 > P_2 > P_4 > P_1$.

Figure 5.15 shows the percentage of the students who answered each question correctly, both with and without correct reasoning. (Results from the pretest are not shown in the figure.) Very few students answered either question correctly without also giving a correct explanation. On question 5.6a, most of the students who were correct simply noted that the eigenvalues of the perturbed and unperturbed potentials are different. The most common incorrect response (given by 26% of the students) was to rank the probabilities using a formula or idea only appropriate for *after* the perturbation has been turned off. The remaining students tended to base their answer on the shape of the perturbation (*e.g.*, on the fact that the perturbation is entirely positive).

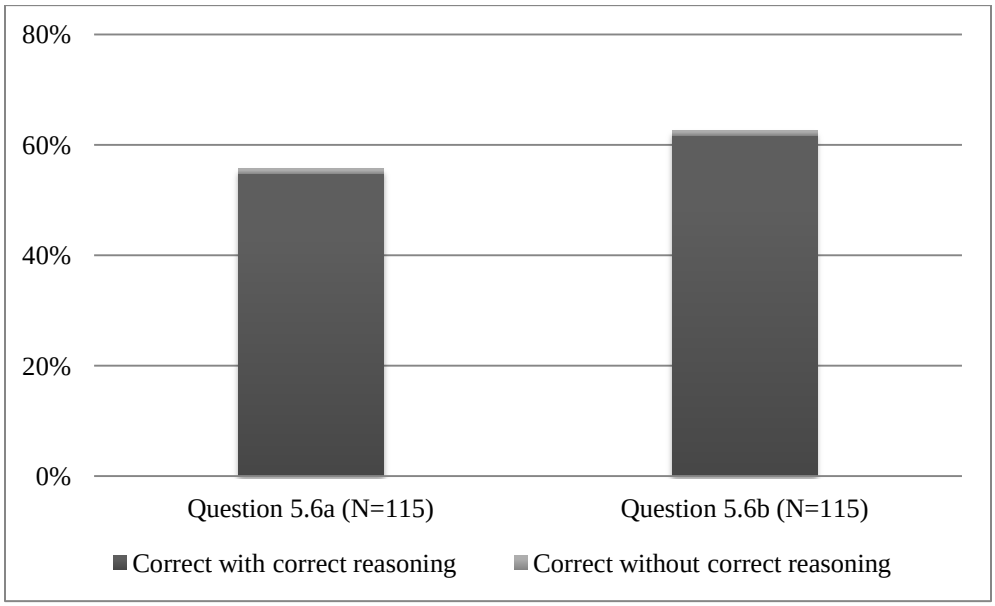


Figure 5.15: Results from Task 5.6

The percentage of the students who gave the correct answer to question 5.6a (left) and question 5.6b (right) is shown. Both questions were given after the revised sequence of tutorial curriculum.

On question 5.6b, two different methods of answering were deemed correct. Many students (42%) based their ranking on the fact that the probabilities are proportional to $\left(E_3 - E_n\right)^{-2}$ after the perturbation has been turned off. Others (20%) described how to write the wave function after $t = T$ and related the probabilities to the coefficients in their expression. These students often did not tend to give a final ranking for the probabilities. (It was not uncommon for these students to state that the probabilities would all be equal; see section 9.3 for further discussion of this error.) We consider explanations of this kind to be correct because the students described how to determine the probabilities, even if they lacked the appropriate formula to reach a final answer.

Chapter 2 discusses the most common incorrect errors on several pretest questions related to time-dependent perturbation theory. On these pretest questions, many students (about 20%) indicate that after the perturbation has been removed, the wave function will revert to the form it had before the perturbation. This was only present in 6% of the responses to the post-test (question 5.6b).

In summary, the results to Task 5.6 indicate that most students have a good understanding of some of the basic concepts related to time-dependent perturbation theory after completing the sequence of tutorials described in this section. Performance on pretest questions, however was generally poor.[18] Although the post-test questions we have asked are not similar enough to the pretest questions to make a direct comparison, the results are still suggestive that the tutorials are helpful. Future exam questions are being designed to serve as a better basis for comparing pre- and post-test results.

C. Discussion

In this section (5.2), we discussed the development of tutorial curriculum to address student difficulties with quantum measurements and time dependence in several advanced contexts. The two discussed in this section are the two-dimensional isotropic harmonic oscillator and time-dependent perturbation theory. Preliminary assessments of the curriculum for each of these two topics suggest that the tutorials help improve student understanding of time dependence and measurements in advanced contexts. However, some of the results from this section indicate that additional modification and assessment is necessary.

5.3 – New Tutorial Curriculum: *Two-state time dependence*

In this section, we discuss an effort to adapt parts of the curriculum for quantum measurements and time dependence for a sophomore-level introduction to quantum mechanics. As discussed in chapter 1, the Physics Education Group at UW was involved in a department effort to redesign the sophomore-level introduction to quantum mechanics course. This course now uses the context of spin-1/2 to introduce students to quantum ideas. Previously, this course covered the same material as the first quarter of the junior-level quantum course at UW. One of the topics covered for spin-1/2 systems is time dependence. We decided to adapt certain pieces of the sequence of tutorials on time dependence and measurements (described in section 5.1) for this population and context. This resulted in the creation of a new tutorial, titled *Two-state time dependence*.

The sophomore-level course does not have small-group sections and is taught in a large lecture hall with an enrollment of 100 students or more during some quarters. As a result, the

new tutorial was designed to work primarily as an interactive tutorial-lecture. In an interactive tutorial-lecture, students work together with their neighbors while several instructors (the lecturer and a few teaching assistants) help students. At several points the lecturer may bring the class together to discuss important results, often by using clicker questions.

In this section, we describe the iterative process of development and assessment for the tutorial *Two-state time dependence*. In section 5.3A we describe the tutorial that was originally adapted from the curriculum described above. Then, we provide an assessment of this tutorial in section 5.3B. The results of the assessment resulted in minor modifications to the tutorial (section 5.3C), which was then tested again (section 5.3D). Finally, the tutorial was restructured substantially as a result of this assessment (section 5.3E). This final version has not yet been assessed.

A. Original Adapted Tutorial: *Two-state time dependence*

When the sophomore-level quantum course was first redesigned, we decided to emphasize the topic of time dependence. This topic was chosen for several reasons: it is fundamental to much of quantum mechanics, there is a substantial body of research on the associated student difficulties, and it is conceptually challenging to students at all levels of instruction. In the original development of this tutorial, we were guided primarily by the tutorials and the methods described in section 5.1, which have proved effective at the junior level. The results to pretest questions focusing on two-state time dependence, some of which were described in chapters 2 and 4, also helped guide our adaptation of the junior-level curriculum.

The tutorial begins with students being told that an electron's spin has been measured to be spin-up in the *z*-direction, after which a magnetic field is turned on in the positive *z*-direction. Students are asked to write a column vector for the initial state of the particle and to sketch each element of this vector on a set of axes representing the complex plane. Then, students determine the functional form of the time dependence of the state, and are guided to graph the state at several different times. This exercise is designed to help students recognize how the state of the particle evolves in time as a rotation in the complex plane. This exercise is similar to the start of the tutorial *Time dependence in quantum mechanics* (see section 5.1B1). When the tutorial is used as an interactive tutorial-lecture, the class is typically asked a clicker question about the time evolution of the state after completing this sequence of questions.

Next, students consider whether or not the probability of a second spin measurement giving spin-up or spin-down would depend on time. They are encouraged to consider the graphical property that is directly related to the constant probability (*i.e.*, the absolute value). They are told that this is an example of a "stationary state" and asked to consider why that name is applicable. At this point, another class discussion is usually held.

The second section of the tutorial guides students through the same sequence of steps that is outlined above for a particle initially measured to have spin up in the *x*-direction. Students are expected to recognize that the two coefficients for this state evolve in different ways (one clockwise in the complex plane, the other counterclockwise) and that this causes some probabilities, but not all, to depend on time. Students are asked to find the probability that a second measurement of spin in the *x*-direction yields spin up as a function of time. They also graph this function. Students then contrast this state with the stationary state from the first

section. A brief homework assignment provides practice with time dependence for a different superposition, and students receive feedback on their responses to this assignment.

B. Assessment of the Original Adapted Tutorial

Task 5.7 (see Fig. 5.16) was administered to 102 students after the completion of the interactive tutorial-lecture. This task gives students an initial state along with the energy eigenstates and eigenvalues of the system. Question 5.7a asks the students if there is a time when the probability of measuring spin up in the *z*-direction is equal to zero. Question 5.7b asks the same question for spin up in the *x*-direction. Students are asked to explain their reasoning on both questions. Each question can be answered by first determining the state of the particle as a function of time, computing the correct inner product, and squaring it to get a probability. Students can also answer qualitatively; for example, by determining the relative phase between the two coefficients that is necessary to give the desired probability. In this case, there is no time at which the probability of measuring spin up in the *z*-direction is equal to zero (question 5.7a) because the magnitudes of the coefficients do not change in time. There are many instants at which the probability of measuring spin up in the *x*-direction is equal to zero (question 5.7b): any time at which the relative phase between the two coefficients is zero.

The results to Task 5.7 are compared to a similar set of questions asked before the tutorial was administered (but after lecture material on time dependence had been covered). The pretest (see Task 4.3 in section 4.3B) also had two questions about an electron in a magnetic field oriented in the *z*-direction. In the first question, analogous to question 5.7a, students were told that the electron is initially spin-up in the *z*-direction. They were then asked whether or not the

probability of measuring spin-up in the *z*-direction depends on time. In the second question, analogous to question 5.7b, students were told that the electron is spin-up in the *x*-direction, and asked whether or not the probability of measuring spin-up in the *x*-direction depends on time.

Task 5.7

An electron is initially in the state $\psi(t=0) = \frac{1}{\sqrt{8}}\begin{pmatrix} -2i \\ 2 \end{pmatrix}$. At $t = 0$, a constant magnetic field is turned on in the *z*-direction, such that the energy of the state $|+\rangle_z$ is $+\omega_p\hbar$ and the energy of the state $|-\rangle_z$ is $-\omega_p\hbar$.

Question 5.7a

Is there a time $t_1 > 0$ when the probability of measuring **spin up** in the *z*-direction is *equal to zero?* If so, determine a time when this occurs and explain your reasoning. If not, explain why not.

Question 5.7b

Is there a time $t_1 > 0$ when the probability of measuring **spin up** in the *x*-direction is *equal to zero?* If so, determine a time when this occurs and explain your reasoning. If not, explain why not.

Figure 5.16: A pair of exam questions on time dependence

Task 5.7 was administered to students after completion of the interactive tutorial-lecture *Two-state time dependence*.

Figure 5.17 shows the percentage of the students who answered each question on Task 5.7 correctly with correct reasoning (44% for question 5.7a and 31% for question 5.7b). Very few students answered either post-test question correctly without correct reasoning. The figure also shows results for the analogous pretest questions (Task 4.3). The pretest results are aggregated across three different quarters. Because the pretest question uses different initial states, a large number of students gave the correct answer with incorrect reasoning (not shown in Fig 5.17).

More students gave the correct answer with correct reasoning on the post-test than on the pretest for both question 5.7a and question 5.7b. This result was highly statistically significant

($p<0.001$) on each question. However, less than half of the students answered each question correctly. Thus, there still appears to be substantial room for improvement.

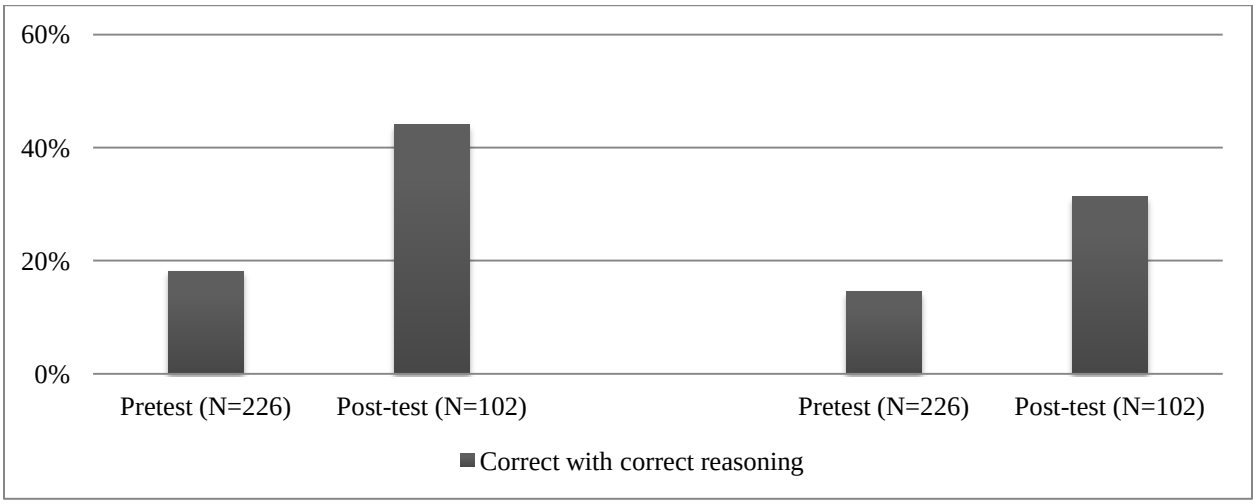


Figure 5.17: Results from Task 5.7

The percentage of the students who answered questions 5.7a (left) and 5.7b (right) correctly with correct reasoning before and after completing the tutorial *Two-state time dependence* is shown.

One of the most common incorrect lines of reasoning on the post-test questions was for students to associate a single time-dependent phase with the state, rather than a separate phase to each term (see section 2.4B2). Nine percent of the students gave responses to question 5.7b consistent with this line of reasoning. On question 5.7a, 8% of the students stated that the probabilities would not depend on time because $|+\rangle_z$ is a stationary state. While it is true that this state would be stationary, this is not correct reasoning for why the probability of measuring spin in the *z*-direction is independent of time. This kind of response demonstrates a misuse of the term "stationary state" that we have also seen in response to other questions about time dependence (see, for example, section 2.4A1).

No other incorrect response was prevalent, although many of the difficulties identified in chapter 2 appeared in about 5% of the responses to this question. However, many students (24% for question 5.7a and 36% for question 5.7b) did not answer the question at all, or provided an unclear explanation. This suggests that students found Task 5.7 very difficult.

C. First Modification to the Tutorial

Although the original tutorial was only three pages long, some students were not able to complete the last exercises during a 50-minute class period. Classroom observations revealed that students had difficulty with the sections that involved graphing and manipulating complex numbers. Student responses to pre-tutorial questions (not shown) confirm that many students could not sketch complex numbers written in exponential form. Thus, we modified the tutorial by providing Euler's formula, which we believed would help students connect the graph of a complex number with more familiar sines and cosines. We removed a question asking students to identify the Hamiltonian because we found that it distracted some students from the focus of the tutorial (*i.e.*, concepts related directly to the time dependence of a quantum state vector).

The rest of the tutorial was not changed substantially—the overall structure of the original tutorial remained intact. However, we developed an additional resource for the students: a pair of videos demonstrating how the state in each section of the tutorial evolves in time, along with the probabilities of measuring spin in the *z*- and *x*-directions. A screenshot of the videos is shown in Fig. 5.18. At the end of each section, after most students had completed all exercises, we asked students a clicker question about the time dependence of a probability and then showed the appropriate video. In this way, students not only constructed the time dependence for

themselves, but also had an opportunity to see the correct answer visually. Students were still assigned a brief homework assignment for additional practice with time dependence.

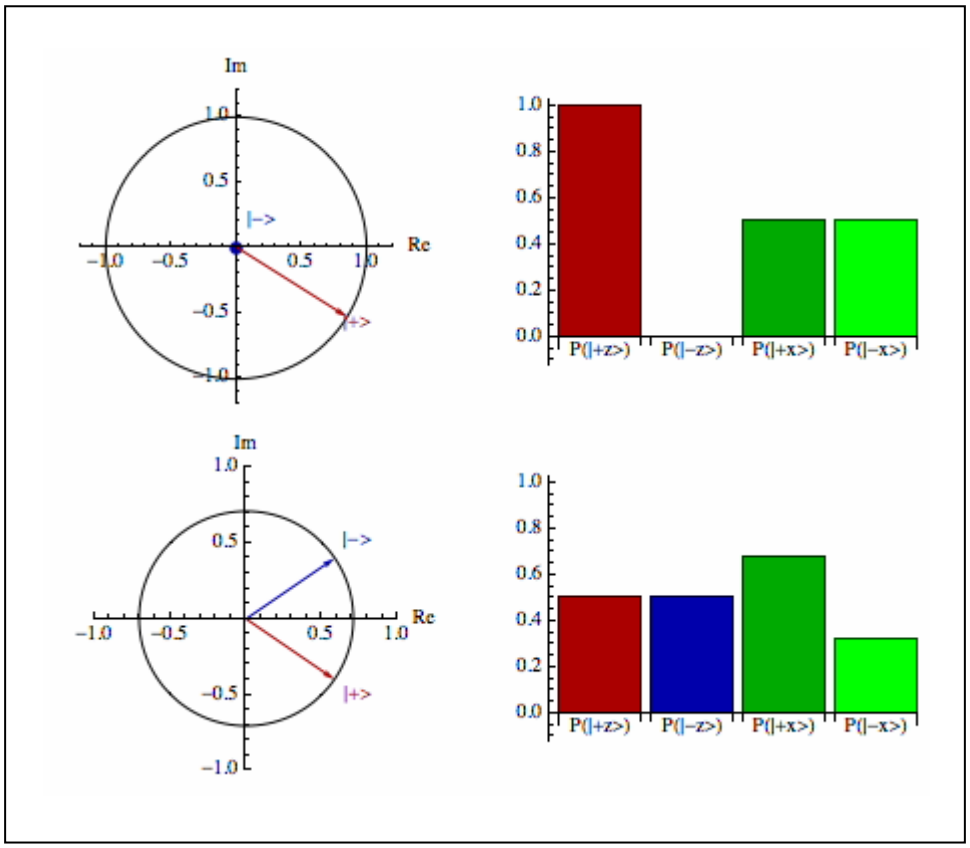


Figure 5.18: Videos used in a tutorial for time dependence

The images above are taken from two videos shown to students during the tutorial *Two-state time dependence*. Each video shows how each term in a column vector changes in time on the left, and how each probability in the *z*- and *x*-direction changes in time on the right. The top video is for a particle that is initially spin-up in the *z*-direction, while the bottom video is for a particle that is initially spin-up in the *x*-direction.

D. Assessment of the First Modification to the Tutorial

Two versions of Task 5.7 were administered to students in two separate quarters who had completed the modified version of the tutorial. Most of the students (about 75%) saw a version

that differed from the version shown in Fig. 5.19 only by using the initial state $\psi(t=0)=\frac{1}{\sqrt{5}}\begin{pmatrix} i \\ -2 \end{pmatrix}$ instead of $\psi(t=0)=\frac{1}{\sqrt{8}}\begin{pmatrix} -2i \\ 2 \end{pmatrix}$. The remaining students were asked slightly different questions (see the appendix) that focused on the same ideas: do the probabilities of spin measurements in the *z*- and *x*-directions depend on time? The same procedure was used to analyze and categorize the responses to both versions, and the results were similar, so they are aggregated below. Figure 5.19 shows the results of Post-test 1 and Post-test 2 from students who completed the original and the modified versions of the tutorial, respectively. The pretest results, aggregated across all quarters, are also shown.

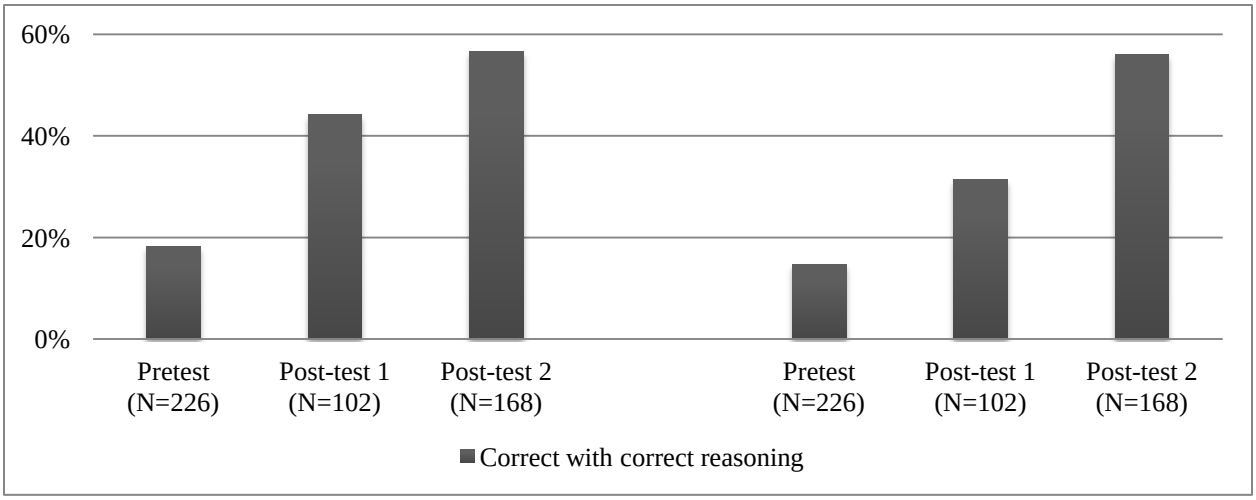


Figure 5.19: Additional results from Task 5.7

The percentage of the students who answered questions 5.7a (left) and 5.7b (right) correctly after completing two different versions of the tutorial *Two-state time dependence* is shown. Only students who gave correct reasoning are included.

The difference between the percentage of the students who answered correctly with correct reasoning after each version of the tutorial is statistically significant on question 5.7a ($p<0.05$) and on question 5.7b ($p<0.001$). Students who completed the modified tutorial were more likely

to make mathematical errors on question 5.7b than those who completed the original tutorial. This difference is likely due to the fact that Task 5.7 involved a different initial state during each quarter, and some states require more algebra to answer correctly. Because of this, students who made algebraic errors on any version of question 5.7b, but whose explanations were otherwise correct, were counted as having given correct answers with correct reasoning. These errors tended to involve squaring an inner product incorrectly, either by failing to take the complex conjugate or by making a mistake with Euler's formula when rewriting the result in terms of trigonometric functions. A few students also wrote the correct expression for the probability as a function of time, but misinterpreted it when answering whether or not there was an instant at which the probability is zero. Although these errors are not directly related to time dependence, it is worth noting that students may need additional practice with squaring complex numbers.

The percentage of the students who did not answer Task 5.7, or who gave explanations that could not be categorized, dropped substantially, from 24% to 10% on question 5.7a and from 36% to 20% on question 5.7b. The prevalence of the most common student error (treating the state as if it had only a single time-dependent phase) increased slightly from 9% to 14%. No other error was made by more than 5% of the students.

E. Second Modification to the Tutorial

The results from both the original and the modified tutorial, while superior to performance on the pretest, still leave room for improvement. In particular, some students appear to be confused about the meaning of the term "stationary state", while others treat a superposition state as if had only a single time-dependent phase. We decided to modify the tutorial so that students consider

a superposition state *before* considering a stationary state, which is opposite the order from the original tutorial, in the first modification to the tutorial, and in the tutorial *Time dependence in quantum mechanics*. This choice was made to emphasize that most quantum states and their associated probabilities depend on time (*i.e.*, that stationary states are not the norm).

The tutorial now begins by asking students to plot (1) the complex number $1 - i$ and (2) the complex function of time $f(t) = e^{-2it}$ on a set of coordinate axes. We intend this exercise to be somewhat easier than the similar exercise in the previous versions of this tutorial, which instead asked students to graph the state of a particle at various instants in time. In section II, students consider a particle measured to have spin up in the *x*-direction. They are given the functional form of the time dependence for a general state with a magnetic field in the *z*-direction. Students graph each element of the column vector at several instants in time. Then, they use this answer to determine whether or not various probabilities depend on time and construct a summarizing table, similar to the table from the tutorial *Position, momentum, and energy measurements*.

In the last section of the tutorial, students are asked a similar sequence of questions for a particle that is initially known to have spin up in the *z*-direction. They are guided to recognize that no probabilities depend on time for this state. Students are introduced to the term "stationary state" and are asked to justify using it to describe a state that depends on time.

F. Discussion

In this section, we discussed an adaptation of tutorial curriculum, initially developed for a junior-level quantum course, for a new context and population. The adapted tutorial, *Two-state time dependence*, brings together many of the ideas in the sequence of tutorials described in

section 5.1: namely, how do quantum states and probabilities depend on time? Since two-state systems are often used to introduce students to quantum mechanics, this tutorial was written for sophomore-level students who have not previously learned about quantum time dependence.

The development of the new tutorial was initially guided by several different factors: (1) other curriculum that has been shown to be effective at improving student understanding of time dependence; (2) research on student understanding of complex numbers and of time dependence, especially in the context of two-state systems; (3) the structure of the course at UW for which the curriculum is intended; and (4) the results on midterm exams designed to assess the tutorial. After each version of the tutorial was administered, it was then revised based on the in-class experience of the instructors and the exam results.

Over the course of several quarters, we found that the original structure of the tutorial, which was heavily based on the structure of the junior-level tutorials from which it was adapted, was not as well-suited to addressing student difficulties with time dependence in the context of two-state systems. In particular, the tendency of many students to overgeneralize the time dependence of stationary states to all states suggested that a different emphasis might be needed at this level. This resulted in our decision to reverse the order of the exercises in the tutorial. In addition, students proved less capable of graphing complex numbers than we had initially believed, and a new section was added to the beginning of the tutorial to help guide students to recognize how to graph complex numbers and functions of time in a non-quantum context before they are required to make similar graphs of quantum states.

The second modification to the tutorial (discussed in section 5.3E) incorporates all of the changes discussed in the previous paragraph. This version has not yet been formally assessed.

However, preliminary results—both in terms of the experience of the instructors and in terms of exam results—suggest that the newest version may be even more effective than the first two versions of the tutorial. Future work will evaluate and assess the impact of this version on student understanding of time dependence for two-state systems. Another area for future research is the development of homework exercises to reinforce the ideas from the tutorial, and to target some specific student difficulties in more detail than is possible with the in-class activities.

5.4 – Summary

In this chapter, we described the development and assessment of tutorial curriculum on quantum measurements and time dependence. This work is the result of many years of research and curriculum development by the Physics Education Group at the University of Washington. In particular, we discussed tutorial curriculum that is appropriate for students in both sophomore- and junior-level courses. The curriculum grew out of the research discussed in chapters 2-4, in which we tried to identify specific student difficulties with time dependence, energy measurements, and sequences of measurements. The results from our assessment of post-tutorial responses overwhelming indicate that the tutorials help improve student understanding of time dependence and measurements in quantum mechanics.

The initial development of the tutorials was predominantly the work of Andrew Crouse, and is documented in his PhD dissertation.[14] Although he demonstrated that the original versions of the tutorials had some effect on student understanding, there were several aspects that needed improvement. For example, students often did not complete the tutorial worksheets during the

allotted class time. Additionally, there are some topics related to time dependence and measurements that students struggle with even after completing the original tutorial curriculum.

This chapter described the changes that we made to the tutorials over the last several years. In particular, our effort has been devoted to honing and focusing the tutorials so that: (1) each tutorial is dedicated to address specific student difficulties; (2) each tutorial can completed by most students during class; and (3) students clearly understand each exercise.

This chapter also discussed how we assessed the effectiveness of the tutorials. Crouse's dissertation provides only a brief, preliminary assessment of these original tutorials.[14] We demonstrated that more students give correct answers with correct reasoning to questions about quantum measurements and time dependence after they have worked through the tutorials than before. Student performance on post-tutorial exam questions was also better for students in classes with the modified versions of the tutorials than it was for students in classes with the original versions of the tutorials. We also showed that the percentage of the students who gave responses consistent with the most prevalent student difficulties tended to decrease after the tutorials had been administered.

The effectiveness of the tutorials has not been limited to basic quantum contexts in the junior-level course at UW. We discussed tutorial curriculum for advanced quantum contexts, such as systems with degenerate energy eigenvalues and time-dependent perturbation theory. The assessment also showed improved student understanding of time dependence and measurements. We also described the cycle of development and assessment used to adapt parts of the curriculum for students in a sophomore-level introduction to quantum mechanics. In addition, we discussed student performance at other universities where the tutorials have been

administered. The results from other universities, which have helped guide further modifications to the tutorials, indicate that the tutorials can be just as effective in other settings as they have been at UW.

Despite the success of these tutorials, there is still more work to be done. The results of the assessments discussed in this chapter indicate some areas where the tutorials focusing on quantum measurements and time dependence can be revised and improved. In particular, students often answer exam questions correctly while giving incomplete, or in some cases incorrect, explanations. Modifications to the tutorials that focus on guiding students to improving the quality of their explanations is one possible avenue for future research in this area.

There are many other topics in quantum mechanics that are conceptually challenging for students. Since the tutorial model has proved successful for addressing student difficulties with time dependence and measurements, we believe that these other topics would benefit from tutorial curriculum. The remainder of this dissertation focuses on two different areas in quantum mechanics for which we have investigated student difficulties and begun to develop tutorial curriculum: vector spaces (Part II) and angular momentum (Part III).

5.5 – Notes and References

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[8] S. B. McKagan, K. K. Perkins, M. Dubson, C. Malley, S. Reid, R. LeMaster, & C. E. Wieman, "Developing and Researching PhET Simulations for Teaching Quantum Mechanics," *American Journal of Physics* **76**, 406 (2008).
[9] For the PhET simulations that deal with quantum phenomena, see the following website: https://phet.colorado.edu/en/simulations/category/physics/quantum-phenomena.
[10] A. Kohnle *et al.*, "A New Multimedia Resource for Teaching Quantum Mechanics Concepts," *American Journal of Physics* **80**, 148 (2012).
[11] For the QuVIS, see the following website: http://www.st-andrews.ac.uk/physics/quvis/.
[12] For the Open Source Physics simulations related to time dependence and quantum measurement, see the following website: http://www.compadre.org/OSP/items/detail.cfm?ID=6814.
[13] G. Zhu and C. Singh, "Improving Students' Understanding of Quantum Measurement II: Development of Research-based Learning Tools," *Physical Review Special Topics – Physics Education Research* **8** (1), 010118 (2012).
[14] A. Crouse, "Research on Student Understanding of Quantum Mechanics as a Guide for Improving Instruction," PhD dissertation, University of Washington (2007).
[15] Descriptions of this curriculum may also be found in chapters 6 and 8 of Ref. 14, and copies of each worksheet may be found in Appendix A of Ref. 14.
[16] For a discussion of the results from question at the University of Pittsburgh, see C. Singh, "Assessing and Improving Student Understanding of Quantum Mechanics," presented at the *Physics Education Research Conference 2005*, Salt Lake City, UT (2005).
[17] A detailed discussion of our investigation into student understanding of the difference between superpositions and mixed states may be found in: G. Passante, P. J. Emigh, and P. S. Shaffer, "Student Ability to Distinguish between Superposition States and Mixed States in Quantum Mechanics," *Physical Review Special Topics – Physics Education Research* (Under review). This paper also documents our assessment of the new tutorial *Superposition in quantum mechanics*. The findings indicate that students performed substantially better after completing this tutorial than after only lecture instruction and that this improvement persisted through an additional quarter of instruction on quantum mechanics. Student difficulties with identifying which states are valid may be found in Ref. 14.
[18] For further discussion of pretest results for time-dependent perturbation theory, see G. Passante, P. J. Emigh, and P. S. Shaffer, "Investigating Student Understanding of Basic Quantum Mechanics in the Context of Time-dependent Perturbation Theory," presented at the *Physics Education Research Conference 2013*, Portland, OR (2013).

Part II
Student Understanding of Vector Spaces in Quantum Mechanics

The study of quantum mechanics involves the use of mathematics that is often substantially more advanced than what is required of introductory physics students. Undergraduate physics students who are learning quantum mechanics for the first time are called upon to use concepts from topics in mathematics such as complex analysis, linear algebra, and partial differential equations, among others. In addition, a unique and abstract notation (*i.e.*, Dirac notation) is used to describe mathematical objects in quantum mechanics. Many students are introduced to this new notation side-by-side with unintuitive physical concepts. Thus, students may have difficulties with the mathematics that go beyond those associated with physics concepts alone.

During the course of investigating student understanding of quantum mechanics at UW, we have identified many student difficulties that seem to have their roots in mathematics, even though this was not the principal objective of our original study. Chapters 6, 7, and 8 of this dissertation discuss student understanding of a specific mathematical subject within the context of quantum mechanics: vector spaces. There are many other mathematical topics that have implications for quantum mechanics, but we have focused on vector spaces in part because they occur across multiple contexts and throughout multiple years of instruction in quantum mechanics.

Vectors are ubiquitous in the study of physics, from introductory courses to advanced courses like quantum mechanics and electromagnetism. They are versatile tools that can be used to represent both physical and abstract quantities. Even though much of introductory quantum

mechanics focuses on solving problems in one spatial dimension, the mathematics of vectors and vector spaces is critical. This part of this dissertation focuses on student understanding of fundamental vector properties across a variety of different physics courses and some of the different contexts studied in those courses. In particular, we highlight similarities between the incorrect lines of reasoning that students use on different problems and at different points in their undergraduate instruction. We also identify some unique difficulties associated with questions about vector spaces that seem to arise due to specific contexts, especially those in quantum courses.

This part of the dissertation is divided into chapters that focus on different aspects of student understanding of vector spaces. Chapter 6 discusses student difficulties with superposition in the contexts of vectors, waves, and quantum wave functions. This chapter includes data from introductory physics courses as a means of tracing student ideas through instruction in prior courses. Chapter 7 describes how students answer questions about inner products and changes of basis in a variety of quantum representations. Chapter 8 discusses a preliminary effort to develop and assess curriculum that addresses the student difficulties with vector spaces identified in chapters 6 and 7.

Chapter 6
Student Difficulties with Addition of Waves and Vectors

"The real part of the wave function is the same so that the probability is the same."

—Junior-level quantum mechanics student, pretest

The idea of superposition is crucial to recognizing not only how quantum mechanical objects behave, but also how the behavior of quantum objects is different from the behavior of classical objects. Students are typically asked to use the principle of superposition, in the form of addition and subtraction of vectors, starting at the beginning of introductory physics. However, we and others have found that even in upper-division courses, many students have difficulty adding and subtracting vectors.

Since quantum states are frequently written as wave functions (in the position representation), students are often called upon to add together two or more wave functions using superposition at each point. Wave functions can be complex, so adding wave functions is equivalent to adding two-dimensional vectors for each individual value of position. In many cases, the quantity of interest is the probability density, which is given by the square of the wave function's magnitude. Thus, both the real and imaginary parts of the wave function are in general of interest. For a discussion of related student difficulties with imaginary numbers, see, for example, section 2.4C3. Over the course of many years of teaching and research in quantum mechanics courses, members of the Physics Education Group have observed that adding wave functions can prove very challenging for students.

In this chapter, we discuss the results from several questions that have been designed to probe student understanding of superposition at different levels and in different contexts. In section 6.1, we describe the results of research that probes student ability to add and subtract vectors in introductory physics. In section 6.2, we discuss student difficulties with superposition of waves, also at the introductory level. In section 6.3, we describe difficulties that students have when similar questions are asked of sophomore- and junior-level students while studying quantum mechanics. Throughout this chapter, we focus on identifying the similarities and differences in the reasoning used by students on the different questions at different levels of instruction.

6.1 – Student Difficulties with Addition of Vectors in Introductory Physics

In this section, we provide an overview of student difficulties with addition of vectors in introductory physics. Section 6.1A describes the prior research that has been done on this topic. Section 6.1B presents a small portion of a larger study at UW focused on investigating student difficulties with vectors.

A. Prior Research on Student Understanding of Addition of Vectors

In recent years, there have been several investigations of student understanding of vectors in introductory physics courses, both at UW and at other universities. Some of these studies have focused on vectors solely as mathematical objects, without assigning them any physical meaning.[1-2] Others have asked questions that involve physical quantities (*e.g.*, velocities or forces).[3-4] There have also been several efforts to compare student performance on isomorphic

questions across different contexts.[5-10] Many of these studies have examined student understanding of inner products. These are described in chapter 7. In this chapter we only discuss those results related to addition of vectors.

Nguyen and Meltzer administered seven basic questions about generic vectors (with no physical context) to several populations of students in both algebra- and calculus-based introductory physics courses.[2] They found that students in a calculus-based course outperformed those in an algebra-based course on each question, and that students performed only slightly better after one semester of physics instruction than before. Their results for students at the beginning of the first-semester calculus-based course (N=608) are most relevant to this study. More of their students correctly answered questions that required addition or subtraction of vectors in one dimension (about 85%) than in two dimensions (about 55%).

Nguyen and Meltzer also discuss the most common incorrect answers for each question. In particular, they report that the most common error on questions about vector addition is for students to "split the difference" between the given vectors (such that the sum is about halfway between them). When asked to subtract vectors, many students attempted to find a vector that they could "average" with the initial vector in order to get the given resultant. On both questions, they report that many students attempted to use a correct vector algorithm, but made a mistake in applying it to this situation. For example, some students used an algorithm appropriate for addition on problems about subtraction, and *vice versa*. These student errors have been confirmed by subsequent studies.[6,7,9]

B. Identifying Student Difficulties with Addition and Subtraction of Vectors

We have been investigating student understanding of addition and subtraction of vectors at UW throughout introductory physics. In particular, we have focused on asking questions that require students to make qualitative or semi-quantitative comparisons of the magnitudes of different vector sums. The specific questions that we asked were often chosen in order to complement our research on student understanding of quantum mechanics. In many cases, our findings are consistent with those from other published studies that have administered similar questions to similar populations.[2,3]

In this section, we discuss three different vector tasks. The first task includes two questions taken directly from Ref. 2, and is designed to assess student ability to add or subtract two vectors. The second task is based on a question developed by Steve Kanim, and asks students to compare the magnitudes of sums of different pairs of vectors.[3] The last task was designed to probe student understanding of vector addition in a more general and abstract context rather than by asking about the sums of specific vectors.

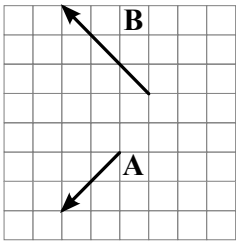
Figure 6.1 shows Task 6.1, which consists of two questions about addition and subtraction of vectors. Task 6.1 was administered to students at the start of the first quarter of calculus-based introductory physics at UW (N=130). The questions are based on two of those described by Nguyen and Meltzer.[2] Because the questions are given online at UW, the students were provided with multiple-choice answers, shown in Figure 6.2. We found that a greater percentage of students answered correctly (85% and 77% on the addition and subtraction questions, respectively) than Nguyen and Meltzer reported (58% and 54%). The incorrect answers chosen

by the students were spread among the different options such that no single option was chosen by a substantial portion of the students. It is possible that UW students performed better on this task than students in the Nguyen and Meltzer study because the UW students were given a set of answers to choose from, rather than being asked to sketch the vectors as in Ref. 2.

Task 6.1

Question 6.1a

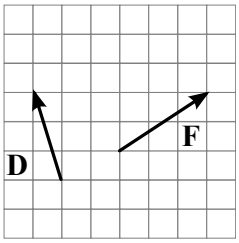
Two vectors, **A** and **B**, are drawn to scale on the grid below.



Draw a vector **C** that is the sum of the two.

Question 6.1b

Two different vectors, **D** and **F**, are drawn to scale on the grid below.



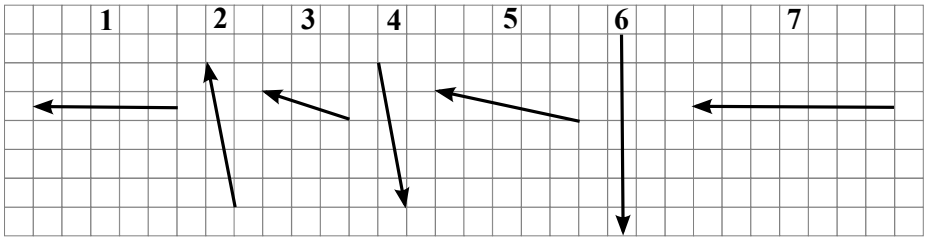
Draw a vector **E** that must be added to **D** in order to obtain **F**.

Figure 6.1: Two questions on vectors

The questions are based on two of the questions in Ref. 2. The questions focus on (a) addition and (b) subtraction of vectors.

We have also administered Task 6.1 to 333 students in sophomore-level courses over several years. The questions were given near the beginning of instruction on quantum mechanics. For the most part, the results were similar to the results from introductory courses at UW: 88% and 83% of the students gave the correct answer on question 6.1a and 6.1b, respectively. The distribution of the incorrect answers selected was again relatively even.

(a)



(b)

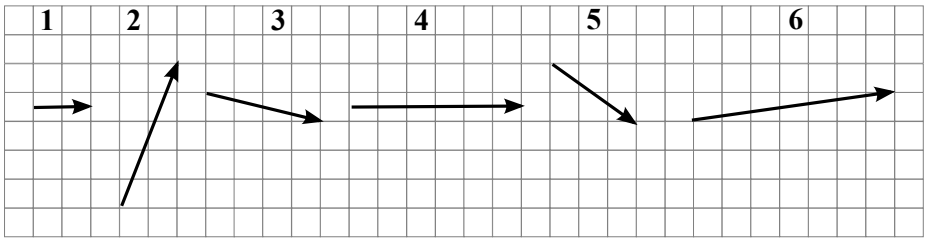


Figure 6.2: Answer choices for Task 6.1

The answers shown are for (a) question 6.1a and (b) question 6.1b.

In addition to Task 6.1, we have used a more qualitative question to probe student understanding of vector addition. The more qualitative question was developed at UW by Steve Kanim and has been used elsewhere.[2,3,9] We have developed several versions, collectively referred to as the "magnitude comparison task". Each magnitude comparison task presents students with two pairs of vectors and asks them to identify whether one sum is greater than, less than, or equal to the other sum. Two such questions (Task 6.2), asked together on the same

tutorial pretest, are shown in Fig. 6.3. Question 6.2a asks about two pairs of vectors that have opening angles that are relatively close to each other, while in question 6.2b the pairs of vectors have very different opening angles. Some of the students were shown the pairs of vectors with no grid, as in the figure, while others were shown a background grid for scale.

Task 6.2

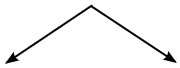
Question 6.2a

Consider the vectors shown below. Each vector shown has the same length.

Is the magnitude of the sum of the vectors in Pair A *greater than, less than,* or *equal to* the magnitude of the sum of the vectors in Pair B? Explain your reasoning.

Pair A

Pair B

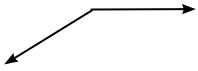


Question 6.2b

Consider the vectors shown below. Each vector shown has the same length.

Is the magnitude of the sum of the vectors in Pair C *greater than, less than,* or *equal to* the magnitude of the sum of the vectors in Pair D? Explain your reasoning.

Pair C



Pair D

Figure 6.3: Two different versions of the magnitude comparison task

In each task, students are asked to compare the magnitudes of sums of different pairs of vectors. Both question 6.2a and question 6.2b use generic vectors. In some cases a grid was provided for scale.

The magnitude comparison task can be answered by noting that each individual vector has the same magnitude, so there is less cancellation between the vectors when they are closer to being parallel. Thus, the sum of the vectors in Pair A is greater than the sum of the vectors in Pair B, while the sum of the vectors in Pair C is less than the sum of the vectors in Pair D. In cases where a grid or an angle is given, it is also possible to calculate the horizontal and vertical

components of each vector and then add them to determine the resultant. We considered answers by students using either of these methods to be fully correct.

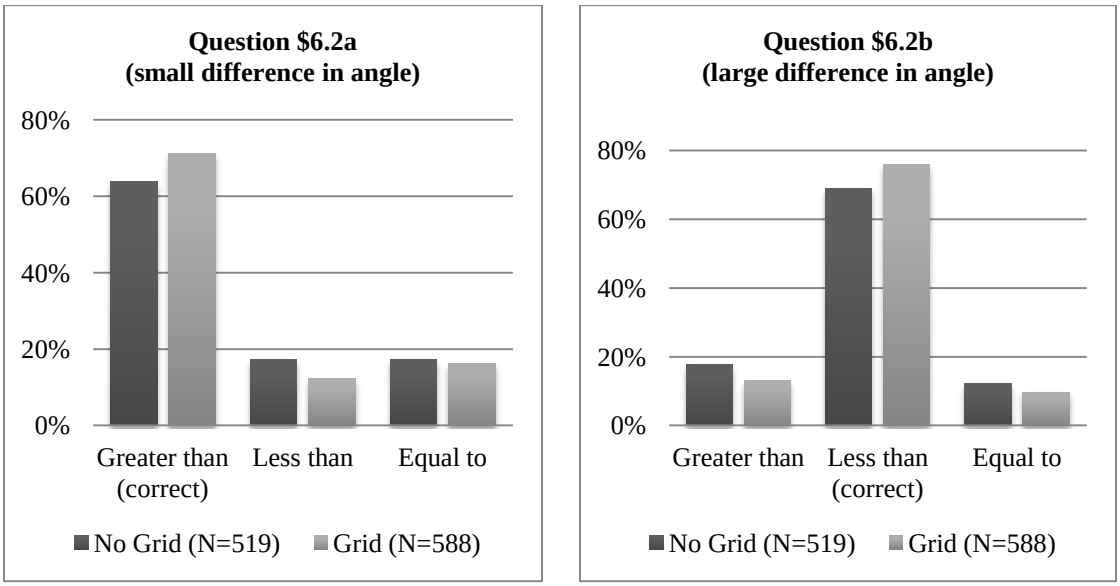


Figure 6.4: Results from Task 6.2

The percentage of the students who selected each answer choice on the magnitude comparison task (Task 6.2) is shown. Results are shown for question 6.2a (small difference between the angles in the two cases) and question 6.2b (large difference between the angles in the two cases). Results are included for grid and no-grid versions of each task. Task 6.2 was given during the first week of instruction in an introductory calculus-based physics course at UW.

Over a period of several years, a total of 1107 students answered Task 6.2. The task was always administered during the first week of introductory calculus-based physics. Some students (N=588) were shown a grid, while others (N=519) were not shown a grid. Figure 6.4 shows the percentage of the students who selected each answer choice ("greater than", "less than", or "equal to") for both questions and both populations. The correct answer was given by 76% and 69% of the students when question 6.2b (large difference in angle) was asked with and without a grid, respectively. Fewer students answered question 6.2a (small difference in angle between the two cases) correctly: 71% and 64% for the grid and no-grid versions, respectively. Overall, 68%

of the students answered both questions correctly when they were shown on grids, while 62% of the students who were not given grids answered both questions correctly. Although the students do only slightly better on the grid version than the no-grid version of each question, the result is statistically significant for each question ($p<0.05$ for question 6.2a and $p<0.1$ for question 6.2b). Overall, the results are similar to what has been reported in prior studies.[2]

Neither incorrect answer was strongly preferred on any of the questions. About 15% of the students chose an answer for each question that is opposite the correct answer (*e.g.*, that Pair B's sum is greater than Pair A's sum, rather than smaller). Some of the students who chose this answer stated that a larger angle implies a larger sum. Others seemed to believe that the sum is given by a vector that points from the tip of one vector to the tip of the other. While this would be a correct way of determining the *difference* between two vectors (a common student error identified in Ref. 2), the explanations given at UW are not sufficient to determine if this was the line of reasoning used by these students.

The remaining students (about 15% for question 6.2a and about 10% for question 6.2b) answered that the sums are equal. Many students justified this answer by stating that each of the vectors given in the problem has the same magnitude. This kind of reasoning is indicative of adding vectors as if they were scalars. A few students who stated that the sums are equal based their answer on geometric properties (*e.g.,* the area enclosed by the vectors). Some students who answered incorrectly gave explanations indicating that they attempted to use an algorithm for vector addition, such as placing the vectors tip-to-tail. Most of the students (about 85%) were consistent between the two different versions of Task 6.2 (*e.g.,* they stated that the sums are equal in both cases, or that the sum corresponding to the wider angle is always smaller).

In order to investigate the different forms of incorrect reasoning about vector sums further, we designed Task 6.3 (shown in Fig. 6.5). This question asks students to identify which of the following pieces of information are sufficient, in general, to determine the magnitude of the sum of two vectors: the magnitude of the first vector, the magnitude of the second vector, and the angle between the two vectors. Some variants of this question used slightly different wording than what is shown in the figure. The results did not vary substantially from version to version, and are combined in the discussion below.

Task 6.3

Suppose you are given two vectors, $\vec{A}$ and $\vec{B}$, that lie in the *xy*-plane.

Let $\vec{C}$ be given by the sum of these two vectors: $\vec{C} = \vec{A} + \vec{B}$.

Suppose you wanted to determine the magnitude of vector $\vec{C}$. (You are not concerned about its direction.) Which of the following pieces of information would be sufficient? (Check all that apply.) Explain your reasoning.

1. The magnitude of vector A
2. The magnitude of vector B
3. The angle between A and B
4. None of the above are necessary

Figure 6.5: A question about basic vector knowledge

This question was administered to students at the start of introductory calculus-based physics.

The correct answer to Task 6.3 is that all three pieces of information are needed: either different magnitudes or different angles would result in a different amount of cancellation between the two vectors. The objective of this question was to determine if there are any differences between students' knowledge about vectors in general and the ways in which they answer questions that involve actual vector manipulation. In particular, we were interested in the

explanations to Task 6.3 given by students who stated that only the magnitude of each vector, but not the angle between them, is necessary to determine their sum.

Task 6.3 has been asked of 651 students during the first week of introductory calculus-based physics (the same point in instruction as Task 6.2). About half of the students (49%) correctly selected all three pieces of information. Thirty percent of the students chose only the two magnitudes, but not the angle between the two vectors. Below are two examples of the reasoning used by students who gave this answer.

> *"Magnitude is the absolute value so in order to find the magnitude of vector C you have to have the magnitude of vectors A and B."* (Task 6.3)
>
> *"You do not need the angle since the direction is not concerned with this question."* (Task 6.3)

The first student relates the magnitude of a vector to the absolute value, and concludes that only the magnitudes of the other vectors are needed to determine this. The second student specifically states that the angle is not needed because the question does not ask about the angle of the resultant. Some other students invoked the Pythagorean Theorem as justification for why the angle does not matter (this line of reasoning is also identified in Ref. 2). Responses like those above suggest that students often do not recognize on their own the importance of direction for vector quantities, and that they seem to treat vectors more like scalars in many situations.

A few students (9%) answered that they only needed to know the angle between the vectors in order to determine the sum. Some of these students seemed to believe that this angle could be used with an appropriate formula from trigonometry (*e.g.*, the Law of Cosines) to answer the question. However, many of their explanations indicate that they would also require the

magnitudes of the two vectors. Some of the students who indicated that none of the given pieces of information were necessary also gave explanations that were mostly correct. Thus, it is possible that the 49% of the students who chose the correct answer underestimates the percentage who are able to recognize which pieces of information are important for determining a vector sum. However, there still appear to be a substantial number of students at the start of introductory physics who do not think the angle between two vectors is necessary to determine their sum. This is consistent with the results from question 6.2, in which 10-15% of the students stated that two vector sums would be equal despite the fact that the opening angles are different.

6.2 – Student Difficulties with Addition of Waves in Introductory Physics

Superposition of wave functions is regularly studied in most courses on quantum mechanics. Therefore, we decided to investigate what students do and do not understand about superposition of waves, a topic studied in introductory physics. In this section, we discuss student difficulties with addition of waves in the third quarter of introductory physics course at UW. The context is waves on a spring. Section 6.2A describes prior research on this topic. Section 6.2B discusses the results from a question, administered at UW, that was specifically designed to be compared to the similar questions about spatial vectors discussed in section 6.1.

A. Prior Research on Student Difficulties with Waves

There has been some research showing that students in introductory physics often have difficulty with questions that involve superposition of waves.[11-16] Some of this research has focused on student difficulties associated with *traveling* waves, or with the applications of

superposition to phenomena such as interference and diffraction. The discussion below focuses on one aspect of superposition that is relevant for quantum mechanics: how the total displacement for more than one wave can be determined by summing the displacements of each individual wave at every point.

Wittmann presented students with two pulses traveling in opposite directions on the same spring, and asked them to sketch the shape of the spring at times of various degrees of overlap.[11] The exact shape of the pulses varied from question to question. He reported that many students did not add the two pulses together at all unless their peaks overlapped, or added together only the amplitudes of the pulses. In both cases, the students failed to recognize that the question could be answered by summing the displacement of each pulse at each point along the spring. Any difficulties the students might have with carrying out the actual addition of the displacements were often hidden by the presence of other difficulties. It is possible that other common student difficulties with superposition mask students' ability to add and subtract waves. For example, Grayson reported that many students confused the transverse velocity of the wave with its displacement.[16]

B. Identifying Student Difficulties with Addition of Waves

We designed a pair of questions to probe student understanding of superposition of waves based on the questions discussed in section 6.1 for vectors. Figure 6.8 shows the two questions (6.4a and 6.4b) that were administered to students near the beginning of their third quarter of introductory physics, which primarily covers waves and optics. The questions have been given to 1420 students over several quarters.

Under the assumption that all displacements are substantially smaller than the size of the spring, both questions can be answered using linear superposition. The correct answer to question 6.4a is that the displacement at point O is 3 cm, equal to the sum of the displacements of the two pulses at the given point and at the specified time. Note that the displacement of each pulse is upward. Question 6.4b can be answered by noting that the displacement at point P is 1cm, because one of the pulses has a downward displacement at the specified location and instant. The total displacement is thus less than the displacement at point O.

Task 6.4

Consider a long string held between two students (Student A and Student B). Two points on the string are marked (point O and Point P).

Student A creates a pulse that moves to the right along the string. One second after creating the pulse, a photograph of the string is taken. In this photograph, Point O is displaced upward by 1 cm and point P is displaced downward by 1 cm.

Student B creates a pulse that moves to the left along the string. One second after creating the pulse, a photograph of the string is taken. In this photograph, both Point O and point P are both displaced upward by 2 cm.

Now, Student A and Student B create the same pulses as before on the string. However, they create their pulses at the same time. One second later, a photograph of the string is taken.

Question 6.4a

What is the displacement of the string at Point O in the photograph of both pulses? (If you cannot give an exact value for the displacement, give a range of values.) Explain your reasoning.

Question 6.4b

Is the displacement of the string at Point P in the photograph of both pulses *greater than, less than,* or *equal to* the displacement of the string at Point O in the photograph of both pulses? Explain your reasoning.

Figure 6.8: Two questions about superposition of wave pulses

The questions were given near the beginning of instruction on waves and optics. They were administered as part of a tutorial pretest. Note that the waves are only described with words; no diagrams or photographs were given to the students on Task 6.4.

The net displacement at point O was correctly identified by 67% of the students. The most common incorrect answer was 1 cm. The explanations of students who gave this incorrect answer were typically consistent with subtracting the displacements of the pulses rather than adding them. Some students answered 1.5 cm, the average displacement due to each pulse. Other students gave a range of values, often between 1 cm and 2 cm. Some students indicated that they could not identify the exact displacement because the given displacements only represented maximum values. There were also a large number of answers that were difficult to categorize, often with little or no accompanying explanation.

In answering question 6.4b, about 60% of the students correctly identified that the displacement at point P is smaller than the displacement at point O. The distribution of incorrect answers chosen by the students was similar to that of the magnitude comparison task (see section 6.1): about 20% of the students chose "equal to" and about 15% of the students chose "greater than". The reasoning provided by the students who answered incorrectly was often vague or circular, and frequently referenced various wave quantities such as frequency, amplitude, and wave speed. However, there were some students who clearly indicated that the displacements at the two points are equal because only the magnitude of the displacement of each pulse matters. This is very similar to the reasoning that many students use when answering about vectors (*i.e.*, that only the magnitude of each vector, not their relative direction, is relevant). Additionally, it should be noted that about 55% of the students gave the correct answer for both questions, so the students who answered question 6.4b incorrectly were often comparing the displacement at point P to an incorrect displacement at point O. Future versions of this question could be modified to address this issue.

The results to Task 6.4 suggest that some of the common difficulties that students have with adding vectors are also present when students are asked to add waves. In both contexts, students often fail to recognize that two quantities with the same magnitude (or absolute value) can be summed in different ways such that the result has a different magnitude (or absolute value). The fact that the same difficulty is predominant in each context indicates that this difficulty may also arise in response to questions with similar features in other contexts, such as those in the next section.

6.3 – Student Difficulties with Superposition of Wave Functions in Quantum Mechanics

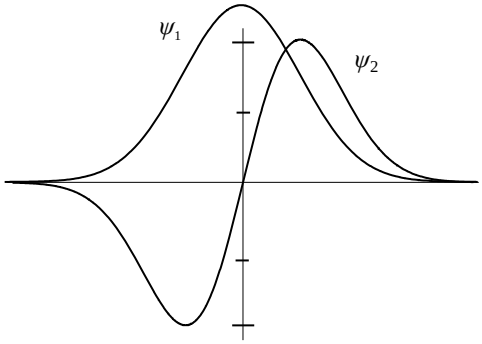
In sections 6.1 and 6.2, we examined student thinking about superposition in introductory physics courses. In this section, we describe the results from tasks asked of students in introductory-, sophomore-, and junior-level courses based on quantum mechanics. The tasks were designed to probe students' ability to sum two wave functions with a variety of relative phases. We asked students about the probability density (or a related quantity) for three different kinds of superpositions of two energy eigenfunctions: when the eigenfunctions are added, when they are subtracted, and when they are summed with a relative imaginary phase. The purpose of these questions was to explore whether or not students recognize that adding two wave functions with different relative phases produces different outcomes that are physically measurable. Since variants of this question have been asked at the various levels, we also discuss the similarities and differences in the responses given in each course.

Task 6.5 (see Fig. 6.9) shows one version of the task that was designed to probe student understanding of superposition. The task was given in a number of classes at various levels on

midterm exams. It was always given in free-response format. In the task, students are presented with a symmetric ground-state wave function and an anti-symmetric first excited-state wave function for an unknown potential. During some quarters, a different pair of functions than what is shown in Fig. 6.9 was used (see the appendix for all figures used for this task). The students are told to consider a particle in the state $\Psi = (\psi_1 + i\psi_2)/\sqrt{2}$, which is a superposition of the given states. The students are then asked whether the probability of measuring this particle to be to the left of the origin is *greater than*, *less than*, or *equal to* the probability of measuring it to be to the right of the origin. They are also asked to explain their reasoning.

Task 6.5

Consider a system with a symmetric ground state wave function, ψ_1, and an anti-symmetric first excited state wave function, ψ_2, shown below. Both wave functions are entirely real.



Consider a particle with the initial state $\Psi(t=0) = (\psi_1 + i\psi_2)/\sqrt{2}$.

Is the probability that this particle is measured to be to the left of the origin at $t = 0$ *greater than, less than,* or *equal to* the probability that it is measured to be to the right of the origin at $t = 0$? Explain your reasoning.

Figure 6.9: A question about superposition of wave functions

Some students saw a different pair of functions than the ones shown above. However, all pairs of functions included one symmetric and one antisymmetric function. See the appendix for all pairs of functions that were used.

One way to answer Task 6.5 is to write an expression for (or make a sketch of) the probability density. The expression for the probability density in this case is given by:

$$\rho(x) = |\Psi(x)|^2 = (\psi_1(x) + i\psi_2(x))(\psi_1(x) - i\psi_2(x))/2 = (\psi_1^2(x) + \psi_2^2(x))/2$$

Since the square of each energy eigenfunction is symmetric about the origin, the probability density for the full wave function is also symmetric. As a result, the probability of measuring the particle to be on the left is equal to the probability of measuring the particle to be on the right. Students were considered to have correct reasoning if (1) they identified this formula for the probability density or (2) if they discussed how to add and square the wave functions graphically or in words, including how to account for the imaginary unit in the definition of the state.

In section 6.3A, we discuss the results to Task 6.5 among junior- and introductory-level students. In particular, this section describes all of the common student errors on questions of this type. Section 6.3B focuses on the results to a similar task, given to sophomores and introductory students, that does not include an imaginary term. Section 6.3C discusses the results when sophomore- and junior-level students are asked about a superposition involving subtraction. Lastly, section 6.3D discusses the overall results from all variants of Task 6.5. Note that we do not make an effort to compare student performance across the different versions of Task 6.5 that have been asked. We focus on the specific difficulties that students have, rather than the exact percentage of the students whose responses were consistent with each difficulty.

A. Student Difficulties with Superpositions that Include Imaginary Terms

Task 6.5 has been asked of students at UW in both the junior-level quantum mechanics course and in the honors section of introductory physics. In each course, the task was

administered as part of a midterm exam. In this section, we focus first on the responses given by juniors in two different quarters (N=129). Then, we discuss the responses given by students in the introductory course.

A majority of the junior-level students (78%) answered Task 6.5 correctly. However, only 26% of the students also gave correct reasoning. Some students (36%) identified that the probability density for both the ground state and the first excited state is symmetric, but did not justify why this implies that the probability density is also symmetric. This reasoning would only be valid in situations where the relative phase between the two eigenfunctions is $\pi/2$—in all other cases there is interference between the two wave functions. As a result, it is difficult to determine the level of understanding of those students whose explanations are based entirely on symmetry from only the responses to this question.

About 20% of the students incorrectly stated that the probability would be greater on the side where both wave functions are positive. Almost all of these students seemed to be summing the two eigenfunctions directly, with no recognition of the impact of the relative phase. This behavior is similar to the errors that students make when they are asked to add or subtract vectors with different orientations. Many students add vectors as if they were scalars, ignoring the relative angle between the vectors (see section 6.1).

There were also some students who answered correctly, but gave explanations that are incorrect, rather than incomplete. For example, 5% of the students stated that ψ_2 does not contribute to the probability because it is imaginary. This difficulty has also been discussed in section 2.4C3. In addition, 7% of the students seem to believe that the result of a position measurement depends on whether the state is *actually* ψ_1 or ψ_2. That is, the students seemed to

be thinking of the state as a mixed state,[17] which in quantum mechanics behaves very different from a superposition state.

Task 6.5a has also been administered to students in the honors section of the introductory physics sequence, which includes several weeks of instruction on quantum mechanics. Only 50% of these students (N=72) gave the correct answer, about 21% with correct reasoning. Note that this is about the same percentage (25%) as for the students in the junior-level course who gave correct reasoning. The distribution of incorrect answers is different, however. The introductory students were more likely to add the wave functions with no relative phase (32% *vs.* 15%), and were less likely to answer based solely on symmetry (12% *vs.* 36%). About the same percentage of the introductory- and the junior-level students treated the imaginary part of the wave function as irrelevant (9% *vs.* 5%) or treated the state as a mixed state (0% *vs.* 7%). This suggests that while the particular difficulties change between these two different points of instruction, the percentage of the students who are able to give both the correct answer and a correct explanation is not substantially different on the same question about superposition of wave functions.

B. Student Difficulties with Superpositions that Involve only Addition

The task discussed in the previous section, in which two eigenstates were added with a relative imaginary phase, can also be asked using superpositions with other phase differences. One variant of Task 6.5 asks about a superposition of states that involves only real, positive coefficients: $\Psi = (\psi_3 + \psi_2)/\sqrt{2}$. A total of 123 students in a sophomore-level introduction to quantum mechanics were given this task on their final exam.

The correct answer was given by 61% of the students. Almost all of these students (59%) also gave correct reasoning: they summed the wave function at each point graphically, squared it to determine the probability density, and compared the area under the curve on each half of the infinite square well. Some students (11%) sketched only the wave function, not the probability density; they were not considered incorrect if they drew a qualitatively correct wave function.

A similar variant was also asked of 38 students in honors introductory physics. They were asked about the state $\Psi = (\psi_1 + \psi_2)/\sqrt{2}$. Most of these students (79%) answered correctly, almost all (68%) with correct reasoning. Although a small number of students drew graphs to support their answer, most of the students who gave correct explanations did so by describing in words where the waves would interference both constructively and destructively.

Most of the students who answered incorrectly stated that the probability for finding the particle on each side would be equal (21% of introductory students and 36% of sophomores). Almost all of these students seemed to believe that, because the probability density for each energy eigenfunction is symmetric, the overall probability density must also be symmetric. This is identical to the reasoning that many students gave when one of the two terms was imaginary (see section 6.3A). This suggests that many students do not recognize when they should add wave functions before squaring and when they should square before adding, even if they are capable of both.

C. Student Difficulties with Superpositions that Involve Subtraction

As described at the beginning of the previous section, we decided to ask students about superpositions of wave functions with a variety of different relative phases. A variant of Task

6.5 different from those in the two previous sections was administered to 102 students in the sophomore class. The task was given as part of a midterm exam. It asked about a superposition of wave functions with a relative minus sign: $\Psi = (\psi_1 - \psi_2)/\sqrt{2}$. An additional 41 students were asked about a state with coefficients of unequal magnitude: $\Psi = \left(-\psi_1 + \sqrt{3}\psi_2\right)/2$. We consider both groups of students together because the underlying reasoning necessary to answer the question is the same.

About half of the students (52%) correctly identified that the probability on the left half of the well is less than the probability on the right half of the well (48% with correct reasoning). As with question 6.1b, almost all of these students used sketches of the wave function and/or the probability density to answer. This result is only slightly lower than the percentage of sophomores who answered correctly when the superposition was entirely real and positive (61%, N=123), as described in section 6.3B. The difference is not statistically significant (p=0.16).

However, the distribution of incorrect answers for the variant involving subtraction is substantially different from the distribution for the variant involving addition. While virtually none of the students incorrectly subtracted wave functions when asked to add them, 16% of the students incorrectly *added* wave functions instead of *subtracting* them. Also, there were still 28% of the students who believed that the probability density would be symmetric, although the reasoning that these students used was highly variable.

When we asked students in the junior-level quantum course (N=71) to sketch the probability density for the state $\Psi = (\psi_1 - \psi_2)/\sqrt{2}$, we found that the graphs drawn by the students split almost evenly into three groups. Some students (35%) drew the correct graph, others (34%)

sketched a graph that was symmetric about the origin, and the rest (30%) drew a graph consistent with adding the wave functions instead of subtracting them.

D. Discussion of Results

The results from the various administrations of Task 6.5 suggest that students often have difficulty adding together wave functions with different relative phases. Because each individual variant of the task was given in a different form to somewhat different populations of students, we do not directly compare the percentages of the students who answered correctly across all versions. However, it is worth nothing that the same errors were present, to a not-insubstantial degree, in the responses to nearly every version of Task 6.5 that we have asked.

The errors that students made on the tasks presented in this section are similar to those students made with addition and subtraction of vectors (see section 6.1). For example, when asked to subtract vectors, many students incorrectly add them instead. This kind of error has also been observed in response to questions about superposition of classical waves (see section 6.2). Other students use a formula such as the Pythagorean Theorem that is appropriate only when two vectors are perpendicular. On the other hand, when students are asked to sum vectors with some nontrivial angle between them, some instead treat those vectors as if they were in the same direction or as if they added like scalars. While it is clear that there are also other difficulties that the students have with superposition of wave functions, such as treating them like mixtures or ignoring imaginary components altogether, it also appears that many students struggle simply to add wave functions while correctly accounting for their relative phase.

6.4 – Summary

This chapter has focused on student understanding of addition and subtraction of vectors, waves, and wave functions. Although each of these quantities is slightly different, the mathematical and graphical tools required to add and subtract them are similar. The results span populations ranging from introductory physics courses to a junior-level course on quantum mechanics.

With regard to vectors, the simplest context, the results indicate that many students are not able to give the correct answer when asked to add vectors with different orientations. For example, many students appear to believe that the sum of two vectors does not depend on the angle between them. Other students do not appear to have a procedure for comparing vector sums when the pairs of vectors differ only by the angle between them. In many cases, the difficulties that students have when they are asked to use superposition for waves are very similar to the difficulties that students have when they are asked to add vectors.

Students in quantum mechanics courses are often asked questions that call upon them to sum wave functions. Summing wave functions requires many of the same tools as adding together vectors and waves. We found that when quantum students are asked to add wave functions, they often make mistakes that are very similar to the mistakes made by students answering questions about superposition of vectors and waves. While many students in quantum courses seem to be capable of adding vectors, it is not uncommon for students to overlook one or more of the steps necessary to correctly solve questions about superposition of wave functions. Note that we do not necessarily have evidence for whether the students are entirely unaware of those steps that

they overlook, but it is clear that many students fail to make use of one or more of them in various situations. It seems that students, even at the junior level, may benefit from help in recognizing how some mathematical tools they have already learned, such as addition and subtraction of vectors, may be used to help them answer questions about superposition quantum mechanics.

6.5 – Notes and References

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Chapter 7
Student Difficulties with Inner Products and Changes of Basis

"Honestly, I just kind of looked at [the vectors] and decided those two were the closest to opposite. I know there is a mathematical justification, but I can't for the life of me remember it."

—Junior-level quantum mechanics student, pretest

Inner products are ubiquitous in quantum mechanics. They are particularly important for the computation of probabilities, because probability amplitudes are given by inner products between eigenstates and general quantum states. Inner products are also important when changing from one basis to another (*e.g.*, when writing a wave function in terms of energy eigenstates) and for calculating expectation values.

In this chapter, we describe preliminary results from an investigation into student understanding of inner products in quantum mechanics. This chapter builds upon the research in chapter 6, in which we describe student difficulties related to superposition. Here, we focus on student difficulties with multiplying vectors (*i.e.*, inner products) rather than with adding vectors. Furthermore, the investigation discussed throughout this chapter is still in its beginning. In many cases, the questions that we have used to probe student thinking have not yet been modified and refined over the course of several years, as is typical of most questions discussed elsewhere in this dissertation. It is hoped that the results discussed here may be used to inform future research on this topic that is more detailed.

Section 7.1 discusses the results from several questions designed to probe student understanding of inner products. This section includes results from both sophomore- and junior-level students' responses to questions about inner products in three different but related contexts: generic vectors, quantum mechanical state vectors, and wave functions represented graphically. Section 7.2 focuses on student difficulties with changes of basis (*e.g.*, determining the probabilities of measurement outcomes associated with a basis different from the basis in which the state of the particle is written). This topic is included here because it requires students not only to recognize that inner products are necessary to determine probabilities, but also to calculate the corresponding inner products. Lastly, section 7.3 examines student understanding of time-independent perturbation theory, for which the calculation of inner products is critical. Although some student difficulties with perturbation theory are unique to the particular context, others appear rooted in the errors that students make when they are asked about inner products, particularly inner products between graphically-represented functions.

7.1 – Student Difficulties with Inner Products

In the course of our investigation, we have administered various questions about inner products for a wide variety of situations to students in sophomore- and junior-level quantum courses. Section 7.1A provides a brief summary of prior research on student understanding of inner products in introductory physics courses. Sections 7.1B-D discuss the results of tasks for several different contexts that can each be answered by calculating inner products. In each section, we describe the student difficulties that are particular to that context, as well as identifying which difficulties are common across multiple contexts. Section 7.1B focuses on

inner products between spatial vectors, section 7.1C focuses on quantum states written using Dirac notation, and section 7.1D focuses on position-space wave functions represented graphically. We have found that examining student responses to similar questions in a variety of contexts helps us to identify incorrect patterns in student reasoning and to design curriculum to address the most common difficulties.

The questions discussed in sections 7.1B-D have tended to focus on the calculation of the inner products. We have also asked questions with two more complicated contexts that involve similar reasoning: energy probabilities and time-independent perturbation theory. Student responses to questions in these two topics are often similar,[9] and are discussed in sections 7.2 and 7.3, respectively. In this section, we focus specifically on student difficulties with the calculation of inner products, not with other quantum concepts that are related to inner products. In addition, it should be noted that the results described throughout this section are preliminary, and more research needs to be done to explore student understanding of inner products at the upper-division level.

A. Prior Research on Student Difficulties with Inner Products

There have been several published studies of student understanding of inner (or dot) products at the introductory level.[1-6] Some of these investigations compared student performance on questions about inner products for vectors with no physical context to performance on analogous questions for contexts such as work. Other studies have focused on only one context or the other. While there has been some research on the ability of students in upper-division courses to answer questions that involve dot products (*e.g.,* for Gauss's Law or Ampère's Law),[7-8] prior

research has tended to focus on the overarching physics concepts rather than the actual calculation of dot products. However, the ways in which introductory students struggle may help to identify how students in quantum mechanics courses are likely to have difficulty.

Zavala and Barniol asked students at the end of introductory physics several questions about both the interpretation and the calculation of dot products for spatial vectors.[2] They reported that many students gave answers consistent with an interpretation of the dot product as a vector rather than as a scalar. In particular, some students answered that the dot product is equal to the sum of two vectors, or that it represents the magnitude of the sum. When they asked students to *calculate* a dot product, about 65% answered correctly. The most common error that they identified was a tendency for the students to use the sine of the angle between the vectors rather than the cosine. A few students simply multiplied the magnitudes of the two vectors together, with no dependence on the angle between them. Zavala and Barniol found that each of these errors was also common when the students were asked about work or electric flux rather than about generic vectors, although the distribution of answers was not the same across each context.[3,4]

B. Student Difficulties with Determining Inner Products for Spatial Vectors

To investigate student understanding of inner products, we have asked questions on tutorial-style pretests in both sophomore- and junior-level quantum courses. The students in each of these courses have completed a full year of introductory physics, a full year of calculus, and several additional math courses taken from the Math and Physics Departments at UW. (For

more information about these classes, see section 1.2.) The questions discussed in this section were administered during the first weeks of each class.

Section 7.1B1 discusses the responses to questions about inner products for spatial vectors from students in a sophomore-level introduction to quantum mechanics. Then, section 7.1B2 discusses the responses to a different set of questions given to junior-level students. In both sections, we attempt to identify the tools and knowledge related to inner products that students bring to upper-division courses.

1. Results from sophomore-level quantum mechanics

We administered Task 7.1 (see Fig. 7.1) to students in a sophomore-level introduction to quantum mechanics (N=333 over five quarters). The questions were in long-answer format and were administered as part of a pre-lecture assignment near the beginning of the course. Question 7.1a asks students to compute the dot product of two vectors that have been represented graphically. Note that the vectors are not drawn tail-to-tail. Question 7.1b asks students to give a general interpretation of the information provided by the dot product. Questions 7.1c and 7.1d each ask students about the dot product between a vector and itself is zero and the relationship between two vectors whose dot product is zero, respectively.

Task 7.1

The grid below shows two vectors (**R** and **S**) drawn to scale.

Question 7.1a

Determine the dot product between **R** and **S**.

Question 7.1b

Give a short explanation of what the dot product tells you about the vectors involved. You may find it useful to refer to the previous question.

Question 7.1c

What is the dot product of a vector with itself? Explain your reasoning.

Question 7.1d

What must always be true of two vectors if their dot product is zero? Explain your reasoning.

Figure 7.1: Four related questions about vectors

Task 7.1 includes questions that ask students to (a) calculate and (b)-(d) interpret the dot product for spatial vectors. The questions were administered on a single pre-lecture near the beginning of a sophomore-level quantum course.

Question 7.1a can be answered by multiplying the *x*- and *y*-components of the two vectors, which can be read off the graph, and then summing the results, as shown below.

$$\vec{R}\cdot\vec{S} = R_x S_x + R_y S_y = (2)(4) + (-2)(1) = 8 - 2 = 6$$

Most students (73%) answered this question correctly. Many simply wrote the answer, while others showed a calculation similar to the one above. In addition, 2% responded with the (correct) formula $\vec{R}\cdot\vec{S} = \left|\vec{R}\right|\left|\vec{S}\right|\cos(\theta)$, but did not give a number. Most of the remaining students (17%) gave an incorrect number. Many showed no work; others gave answers that suggest they may have been using a correct procedure. For example, some students gave answers consistent with reversing the sign of one or more of the components or miscounting the

number of boxes spanned by one of the vectors (though many did not show enough work in order to determine a specific mistake). These errors seemed to be relatively minor, and did not appear to be representative of any deeper difficulties with calculating dot products.

A small fraction of the students (5%) gave an answer consistent with treating the dot product as a vector rather than a scalar. Many of these students gave answers such as "$8x-2y$" (or versions of this with different numerical coefficients). Answers of this type appear to come from treating the inner product as a vector in which each component is given by multiplying the corresponding components of the multiplied vectors. One student, who gave the correct magnitude (6) for the dot product, also provided an angle of 45 degrees. This suggests that the student is also treating the dot product as a vector, possessing both magnitude and direction.

This finding is consistent with that of Zavala and Barniol (discussed in section 7.1A), who found that some students in introductory physics classes treat the dot product as a vector rather than a scalar.[2] It appears that this confusion is still present among a few sophomore-level physics students. However, it is interesting to note that most of the students who gave such responses went on to give interpretations of the dot product that are *not* consistent with treating it like a vector, which suggests that they are unable to recognize the inconsistency between their answers to various questions about vector products.

Question 7.1b asks the students to explain what the dot product tells them about the vectors involved. There are many correct answers to this question. Most of the students gave an answer consistent with one or more possible (correct) interpretations of the dot product. For example, nearly half of the students (47%) identified that the dot product could be used to determine the angle between two vectors, or the relative direction of the vectors. The level of detail provided

by these students in their explanations was highly variable: some gave little supporting reasoning, while others described exactly how to determine the angle given the dot product and the magnitude of each vector. Some students (22%) explained that the dot product gave the projection of one vector onto another—most of them used the word "projection" in their answer. Other students (16%) gave a definition of the dot product in words that was equivalent to either $\vec{R}\cdot\vec{S} = R_x S_x + R_y S_y$ or $\vec{R}\cdot\vec{S} = \left|\vec{R}\right|\left|\vec{S}\right|\cos(\theta)$. Although some of these explanations are not as detailed as we would prefer, they all demonstrate at least one adequate interpretation of the dot product.

As with question 7.1a, a small number of the students (7%) gave answers that seemed to treat the dot product as a vector, as in the response below.

"It [the dot product] gives the magnitude of the resultant vector." (Question 7.1b)

This student uses the term "resultant", which is typically used to refer to the sum of two vectors.

As reported by Zavala and Barniol, associating the dot product with the vector sum, or the magnitude of the vector sum, was also common among introductory students.[2] Overall, 11% of the sophomore-level students gave responses to either question 7.1a or question 7.1b consistent with mistaking the dot product for a vector or the magnitude of a vector. Only 3 students (less than 1%) made this mistake for both questions.

The students performed relatively well on both of the remaining question. Question 7.1c asks students to interpret the result of the dot product of a vector with itself. The correct answer, that it is equal to the square of the magnitude (or the length) of the vector, was given by 73% of the students. A few students (9%) instead stated that the dot product of a vector with itself is

equal to the magnitude, rather than the square of the magnitude. An additional 2% of the students stated that it is equal to one, which would be correct only for a normalized vector. Some students (4%) stated that the dot product is equal to the "vector squared" without specifying either length or magnitude. In general, most students gave correct or almost correct answers, though not all of them gave complete explanations. A few students (2%) stated that the dot product is equal to zero; they seemed to be confusing this case with when the vectors are orthogonal.

Question 7.1d asks the students to identify what is true of two vectors whose dot product is zero. The students performed very well on this question: 94% identified that the vectors would be orthogonal (some used the terms "perpendicular" or "linearly independent"). The students who answered incorrectly tended to state that two vectors with a dot product of zero were either in the same direction (3%) or in opposite directions (1%). One student indicated that a dot product of zero implied that the vectors are identical.

2. Results from junior-level quantum mechanics

A different sequence of questions about spatial vectors, Task 7.2 (see Fig. 7.2) was administered near the beginning of the junior-level quantum mechanics course at UW (N=156 over two quarters). The questions were given online as part of a tutorial pretest. Task 7.2 defines three vectors and asks students which vector has the largest magnitude. Students are then shown a new vector and asked which of the three original vectors has the greatest overlap with the new vector. Finally, students are asked which pair of the first three vectors is closest to perpendicular. Each question provided the students with multiple choice options, and also asked

the students to explain their reasoning. Unlike Task 7.1, which directly asks students to compute and interpret inner products, Task 7.2 does not specifically tell students to use inner products—they are expected to recognize that inner products are appropriate and then to use them to answer the questions.

Task 7.2

Consider the vectors shown below.

$\vec{a} = 2\vec{x} - \vec{y} + 2\vec{z}$

$\vec{b} = 3\vec{x} - \vec{y} - \vec{z}$

$\vec{c} = -\vec{y} - 3\vec{z}$

Question 7.2a

Which of the vectors above has the largest magnitude? Explain.

Question 7.2b

Consider the vector shown below.

$\vec{d} = -\vec{x} + 4\vec{y} - 5\vec{z}$

Which of the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ has the greatest overlap with this vector? Explain.

Question 7.2c

Of the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$, which pair is closest to perpendicular? Explain.

Figure 7.2: Three questions about vectors

Task 7.2 was administered on a tutorial pretest. The questions were given to students near the beginning of a junior-level quantum course at UW. Each question was multiple choice, but also asked the students to explain their reasoning.

On question 7.2a, most of the students (86%) correctly identified that $\vec{b}$ has the largest magnitude. Virtually all of these students explained their answer, either in words or equations,

by noting that the magnitude of a vector is given by the formula $\left|\vec{b}\right| = \sqrt{b_x^2 + b_y^2 + b_z^2}$. (Only two students correctly related this formula to the dot product of a vector with itself.) Seven percent of the students answered that $\vec{a}$ has the largest magnitude, while 4% stated that $\vec{a}$ and $\vec{b}$ have the same magnitude. Most of the students who selected an incorrect answer gave explanations similar to those provided by the students who answered correctly, which suggests that they may have made a mistake in their calculations. However, a few students seemed to believe that the magnitude of a vector is determined (or can be inferred) by the sum of the *magnitudes* of the components, rather than by the sum of the *squares* of the magnitudes of the components.

Question 7.2b asks students to identify which of the three vectors from question 7.2a has the greatest "overlap" with a new vector, $\vec{d}$. In this context, "overlap" is meant to refer to the inner product. The values of the three inner products that the students are asked to compare are:

$$\vec{a} \cdot \vec{d} = -16 \qquad \vec{b} \cdot \vec{d} = -2 \qquad \vec{c} \cdot \vec{d} = 11$$

Although this question was designed to evaluate students' ability to use inner products, we found that many students did not correctly interpret "overlap" to mean inner product. As a result, we do not discuss the responses to this question in detail. Future versions of this question should be made clearer to avoid this confusion.

Question 7.2c asks students to determine which pair of vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ is the closest to perpendicular. This question can be answered by calculating the dot product for each pair of vectors, then dividing the result by the magnitude of each vector. Whichever of these three numbers is closest to zero corresponds to the pair of vectors that is closest to perpendicular. The dot product for each pair of vectors is shown below.

$$\vec{a} \cdot \vec{b} = 5 \qquad \vec{a} \cdot \vec{c} = -5 \qquad \vec{b} \cdot \vec{c} = 4$$

Since $\vec{b} \cdot \vec{c}$ has the dot product with the smallest absolute value (and the magnitudes of the three vectors are not substantially different), $\vec{b}$ and $\vec{c}$ are closest to perpendicular. About half of the students (47%) selected the correct answer, typically by comparing the dot product for each case. However, as with question 7.2b, this is not the only correct method of solution (for example, the angle between each pair of vectors could be determined from a graph). Additionally, some of the students who gave correct explanations made mistakes in their calculations. As a result, we again focus on the specific kind of reasoning that the students used, rather than the answer that they chose.

While about half of the students gave explanations that involved the dot product, 6% of the students invoked the cross product instead. Although the cross product can also be used to determine whether or not two vectors are perpendicular, some of these students appeared to be confusing the two kinds of vector product. For example, one student's explanation for selecting that $\vec{b}$ and $\vec{c}$ are orthogonal (the correct answer) was "because their cross product is closest to zero". Some students (13%) instead based their answer on the angle between each pair of vectors, typically determined by graphing the vectors. All of these explanations have correct elements, though about half of the students who gave them still failed to identify the correct answer.

Some students (15%) answered based on the similarity (or lack thereof) between the components of each vector. These students tended to believe that two vectors were close to perpendicular if they had components that were directly opposite (*e.g.*, one vector with a positive

z-component and the other with a negative *z*-component of roughly the same magnitude). The responses below are indicative of this kind of reasoning.

> *"$\vec{a}$ and $\vec{b}$ have similar x-components, identical y-components, and nearly opposite z-components. They come closest to laying almost orthogonal in the same plane."* (Question 7.2c)

> *"Based on how each coordinate is oriented [for $\vec{a}$ and $\vec{b}$], the only opposites are their z-coordinates, [while] the rest are relatively close."* (Question 7.2c)

Both of these students identify the same two features of $\vec{a}$ and $\vec{b}$: that the *z*-components are mostly opposite, and that the other components are essentially the same. This reasoning is incorrect, as the relationship between the components in all three directions is necessary to determine how close to perpendicular a pair of vectors is. Only 38% of the 24 students who gave similar explanations also chose the correct answer—most (46%) chose $\vec{a}$ and $\vec{b}$ instead.

3. Discussion of results

Overall, the results from the tasks administered to sophomore- and junior-level quantum students suggest that most upper-division physics students are able to calculate inner products for spatial vectors. Even those who answered incorrectly seemed only to be making basic mathematical mistakes (*e.g*, sign errors). The questions about the interpretation of the dot product proved to be more difficult for both groups. For example, some students incorrectly associated the dot product with the sum of the two vectors. Additionally, only about half of the junior-level students (or less) used the dot product to determine whether or not pairs of vectors were orthogonal. While the dot product is not the only way to answer this question, it is one of

the most common and most efficient. Students are typically taught to use the dot product in this way in math classes, as well as in courses on mathematical methods in physics. Since calculating and understanding inner products is crucial for most quantum mechanics classes, it may be important to strengthen students' conceptual understanding of inner products in the simpler case of spatial vectors before introducing them to more abstract inner products involving Dirac notation or wave functions.

C. Student Difficulties with Determining Inner Products for States Written in Dirac Notation

Although the notation typically used for quantum states is very different from the notation commonly used for spatial vectors, the method for calculating inner products is very similar. One important difference between the two cases is the fact that the components of quantum states can be complex numbers. Thus, an inner product between quantum states involves both a regular state vector (denoted by a "ket") and the Hermitian conjugate of a state vector (denoted by a "bra"). Another distinction is that quantum states can span more than three dimensions. Otherwise, the procedure for calculating inner products is the same for the two cases.

In this section, we discuss the results from a task designed to explore student understanding of inner products between state vectors written in Dirac notation. The two specific features of student understanding in which we are interested are: (1) what student difficulties with inner products are similar between the contexts of spatial vectors and state vectors, and (2) what student difficulties are unique to Dirac notation? The sequence of questions (Task 7.3) is shown in Fig. 7.3. It was administered to students in a sophomore-level introduction to quantum

mechanics (N=220 across three quarters). Variants of this question have not been asked in the junior-level quantum course because the sophomore-level course introduces Dirac notation at the beginning of the quarter while the junior-level course waits until later the quarter. See section 7.1D for a discussion of questions about the inner products between wave functions given to junior-level students.

Task 7.3 version α

The basis vectors $|\chi_i\rangle$ are orthonormal in a 4-dimensional Hilbert space. Consider the three state vectors defined below.

$$|A\rangle = |\chi_1\rangle + 2|\chi_3\rangle + i|\chi_4\rangle$$
$$|B\rangle = 2i|\chi_1\rangle + |\chi_2\rangle + |\chi_4\rangle$$
$$|C\rangle = i|\chi_2\rangle - i|\chi_3\rangle - 2|\chi_4\rangle$$

Question 7.3a

Rank these three state vectors by magnitude from greatest to smallest. Explain your reasoning.

Question 7.3b

Evaluate the expression $\langle A|B\rangle$. Show your work.

Question 7.3c

Determine which of the three state vectors are orthogonal to each other. Explain your reasoning.

Task 7.3 version β

The basis vectors $|\chi_i\rangle$ are orthonormal in a 4-dimensional Hilbert space. Consider the three state vectors defined below.

$$|A\rangle = |\chi_1\rangle + 2|\chi_3\rangle + i|\chi_4\rangle$$
$$|B\rangle = 2i|\chi_1\rangle + |\chi_2\rangle + |\chi_4\rangle$$
$$|C\rangle = i|\chi_2\rangle - i|\chi_3\rangle + \gamma|\chi_4\rangle$$

Question 7.3a

Is the magnitude of $|A\rangle$ *greater than, less than,* or *equal to* the magnitude of $|B\rangle$? Explain your reasoning.

Question 7.3b

Evaluate the expression $\langle A|B\rangle$. Show your work.

Question 7.3c

Determine the value of γ such that $|C\rangle$ is orthogonal to $|A\rangle$. Explain how you arrived at your answer.

Figure 7.3: A sequence of questions on inner products
Task 7.3 was designed to evaluate student understanding of inner products administered on a midterm exam in a sophomore-level introduction to quantum mechanics. Version α was asked of 89 students during one quarter, while version β was asked of 131 students during a different quarter.

Task 7.3 was administered on a midterm exam after both traditional and tutorial instruction on Dirac notation and inner products. Two different versions are shown in the figure (version α

and version β). The questions in each version are intended to be parallel to some of the questions in Tasks 7.1 and 7.2: question 7.3a asks students about the magnitudes of state vectors (similar to questions 7.1c and 7.2a); question 7.3b asks students to compute an inner product (similar to questions 7.1a and 7.2b); and question 7.3c asks students about orthogonality (similar to questions 7.1d and 7.2c).

Versions α and β of question 7.3a ask the students to compare the magnitudes of either three or two quantum states in a four-dimensional Hilbert space. Since the magnitude of each state is given by the square root of the inner product of the state with itself, the correct answer is that all of the states have the same magnitude: $\sqrt{6}$. About half of the students (56%) gave the correct answer. Some students (20%) gave explanations that included inner products. Most students (33%) did not explicitly relate the magnitudes of the states to inner products. Rather, they took the complex square of the coefficients of each state vector and then added them. However, both of these methods are essentially correct.

Many students (33%) failed to take the *complex* square of each coefficient. Instead, they simply squared each number, which makes the contribution of imaginary coefficients negative rather than positive. While some of these students took the absolute value at the end, others indicated that the magnitude of some states would be negative—or even imaginary—even though this is not consistent with how the term "magnitude" is typically used in physics. The example below shows how one student ranked imaginary values as smaller than real values.

> *"Magnitude of $|B\rangle$ is imaginary hence compared to real numbers it is last in ranking."*
> (Question 1 – version a)

Many students gave a ranking consistent with this explanation, in which the "imaginary magnitudes" are smallest. This suggests that some students may have difficulty interpreting how imaginary numbers are being used in quantum mechanics. In particular, some were not able to recognize that imaginary numbers should not appear in the magnitudes of quantum states. In addition to this error, a few students (4%) simply summed the absolute values of the coefficients without squaring them.

Question 7.3b asks students to calculate the inner product of two state vectors. It is important to note that students are told the Hilbert space is four dimensional, but that each state vector includes terms for only three basis vectors, and each of the given states includes a different set of basis vectors with nonzero coefficients. Because the basis states are orthonormal, the correct answer can be determined by multiplying the corresponding coefficients of each basis state and then summing over the results (this can also be done using column vectors as shorthand). The result of this procedure for $|A\rangle$ and $|B\rangle$ is i. About two-thirds of the students (66%) gave the correct answer.

Some of the students (15%) did not recognize the need to take the Hermitian conjugate of one of the state vectors when computing the inner product. This error is very similar to the one described above; however, substantially fewer students made this error when they were directly asked about an inner product than when they were asked about the magnitudes of state vectors. Additionally, not all of these students made this error on both questions. Thus, most students seem to be aware that finding a "bra" from a "ket" involves taking the complex conjugate of each coefficient, but they do not apply this rule consistently. Moreover, there are some contexts in which they are less likely to recognize that it is necessary to take the complex conjugate.

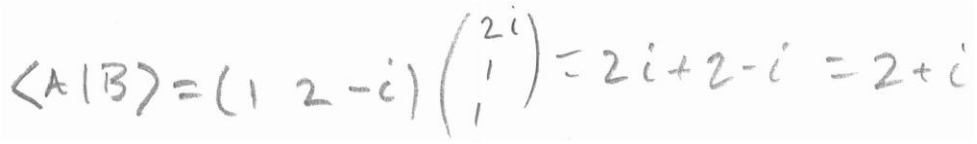


Figure 7.4: Sample student response from a midterm exam in a sophomore-level introduction to quantum mechanics. The question (question 7.3b – version α) asks students to determine the inner product of two state vectors in a *four-dimensional* Hilbert space. This student has omitted the fourth element from their column vector, which results in multiplying the wrong components of the respective state vectors.

As mentioned above, the Hilbert space for this system is four-dimensional. Some students (8%) represented the state vectors with three-dimensional column vectors, as shown in the sample response in Fig. 7.4. In this response, the student incorrectly omits the component of each state vector for which the coefficient is zero.

A few students (5%) answered that the inner product of two state vectors is itself a state vector. Figure 7.5 shows an example of this kind of response. Note that the coefficient of each basis state in this student's response is equal to the products of the corresponding coefficients of the states for which the inner product is being calculated. We have also seen evidence of students treating the inner product between two vectors as a vector in spatial contexts. For further discussion of this difficulty in the context of spatial vectors, see section 7.1B.

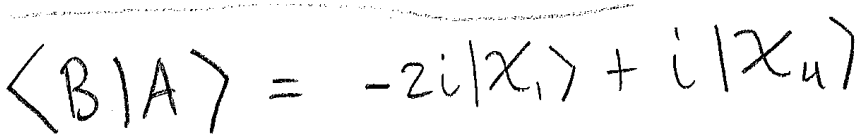


Figure 7.5: Sample student response from a midterm exam

The exam was given in a sophomore-level quantum course (question 7.3b – version β). The student has indicated that the inner product between two state vectors is another state vector, with components given by the product of the components of the original state vectors.

Finally, in question 7.3c, the students are asked about orthogonality. In version α, they are asked to identify which, if any, of the three quantum states are orthogonal to each other. In version β, one of the components of $|C\rangle$ is a variable (γ). Students are asked to determine the value of γ such that the state vector would be orthogonal to one of the other state vectors. Both versions require students to recognize that they can use the inner product to answer. Almost all of the students, whether they answered correctly or incorrectly, evaluated an inner product and then comparing it to zero. About half of the students answered correctly: 53% aggregated across both versions. An additional 10% of the students seemed to have made minor mathematical errors, such as adding or multiplying incorrectly. The distribution of incorrect answers was otherwise very similar to that of question 7.3b, which asked students to calculate an inner product explicitly. However, students often did not make the same error on both questions. Fifteen percent of the students did not take the complex conjugate of the state vector represented by the "bra". Relatively few students made this error on more than one of the questions from Task 7.3, while 43% of the students failed to take the complex conjugate on at least one question. In addition, 9% of the students treated the Hilbert space as three- instead of four-dimensional. Almost all of the students who used this incorrect method used it for more than one question.

The example below is from the response of a student who did not use an inner product to determine whether or not $|A\rangle$ and $|C\rangle$ are orthogonal.

> *"$|C\rangle$ has a component in $|\chi_2\rangle$ while $|A\rangle$ does not. $|A\rangle$ has a component in $|\chi_1\rangle$ while $|C\rangle$ does not. Vectors are orthogonal regardless of γ [the undetermined coefficient of the $|\chi_4\rangle$ component of $|C\rangle$]."* (Question 7.3c – version β)

This student appears to believe that these quantum states are orthogonal simply because each state possesses at least one component that the other state does not. Other students stated that it is not possible for $|A\rangle$ and $|C\rangle$ to be orthogonal because they do not share the same set of basis vectors. Responses of this nature indicate that some students may not understand what it means for two states to be orthogonal. It should be noted that this is not a difficulty that we have observed for spatial vectors, which could be due to differences in how students view quantum states, or because the spatial vectors we used did not lead students to give responses of this nature. Future versions of Task 7.3 could be designed to explore this line of reasoning further.

Overall, the results of Task 7.3 indicate that as many as half of the students in sophomore-level quantum courses have difficulty in calculating inner products for state vectors written in Dirac notation. Only a small number of students (about 5%) seemed to believe that the result of an inner product between two vectors is another vector (rather than a scalar). Most of the students who answered Task 7.3 incorrectly had difficulty with the properties of quantum state vectors that spatial vectors lack, such as having imaginary coefficients and having more than three components. Since most students are able to determine inner products for spatial vectors, and the most common difficulties are with features that are new to quantum mechanics, instruction that highlights the similarities and differences between the two contexts could prove useful (see section 8.2 for a discussion of tutorial curriculum for addressing these difficulties).

D. Student Difficulties with Determining Inner Products for Functions

While inner products for quantum states are sometimes calculated using Dirac notation, it is also common in the study of quantum mechanics to write each state as a wave function in

position space and to evaluate the inner product as an integral. It is reasonable to expect that students might have difficulty with this method for determining inner products because it is substantially different from the methods typically used for spatial vectors. As discussed in the previous sections, many students have more difficulty with interpreting the dot product than they have with calculating it. We therefore decided to evaluate how students interpret inner products of wave functions, which are used at least as often as, if not more often than, Dirac notation to calculate inner products.

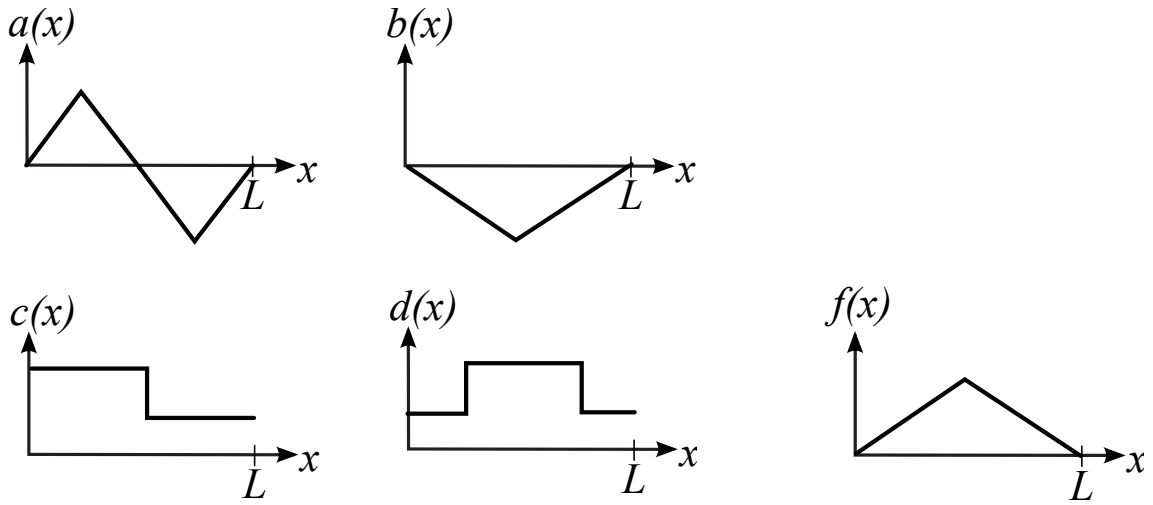
In this section, we discuss the responses to a task that asks about inner products for graphical functions. We focus on graphs of functions, rather than functions represented algebraically, because we are interested in investigating conceptual understanding of integrals in position space. In particular, we have found that, while students are sometimes able to determine the results of integrals of functions, this does not necessarily indicate that they understand the underlying concepts. The task discussed in this section was administered to students in a junior-level quantum course. These questions have not been asked in a sophomore-level course because those students do not cover position space until the end of the quarter, and that course emphasizes spin rather than position-space wave functions. See section 7.1C for a description of inner product questions asked in the sophomore-level course.

Task 7.4, shown in Fig. 7.6, is designed to probe student understanding of inner products for wave functions. This task was given as part of a tutorial pretest administered near the beginning of the junior-level quantum course at UW (N=77 in one quarter). An alternate version, which asks about the "overlap" between similar functions, has been given to 53 junior-level quantum students. Since many of these students did not appear to understand what was meant by the term

"overlap", these results are not included here. We also found that some sophomore-level students did not understand what was meant by this term in the responses to question 7.2b, which focuses on inner products for spatial vectors (see section 7.1B2).

Task 7.4

Consider the function *f(x) shown below right*, and the four other functions shown below left. All functions are zero except as indicated.



Recall that the inner product between two functions is defined to be the integral over all space of the product of the functions.

Question 7.4a

Is the inner product of *a(x)* with *f(x) greater than, less than, or equal to* the inner product of *b(x)* with *f(x)? Explain your reasoning.*

Question 7.4b

Is the inner product of *c(x)* with *f(x) greater than, less than,* or *equal to* the inner product of *d(x)* with *f(x)?* Explain your reasoning.

Question 7.4c

Do any of the functions shown above (*a-d*) have **zero** inner product with the function *f(x)?* Select all that apply. Explain your reasoning.

Figure 7.6: A sequence of questions on inner products between functions

Task 7.4 was administered near the start of a junior-level quantum mechanics course. The questions focus on student ability to compare the values of inner products between graphically-represented functions.

Most of the students who completed this pretest had previously used integrals to calculate inner products in several courses. However, the students did not have any prior tutorial instruction devoted to inner products or to graphs of wave functions. On Task 7.4, students are shown graphs of five functions defined between $x = 0$ and $x = L$. They are asked to make two comparisons between different pairs of inner products between these functions. Then, they are asked whether or not any of the first four functions has an inner product of zero with the fifth function. The questions were multiple-choice, and students were asked to explain their reasoning for each question. The goal of Task 7.4 is to probe student ability to calculate inner products, rather than to recognize when inner products are necessary or useful. Thus, students were given the definition of the inner product at the beginning of the task.

The correct answer to question 7.4a is that the inner product between $a(x)$ and $f(x)$ is zero, while the inner product between $b(x)$ and $f(x)$ is negative. Thus, the inner product between $a(x)$ and $f(x)$ is greater. About two-thirds of the students (66%) answered question 7.4a correctly. On question 7.4b, 60% of the students gave the correct answer: the inner product between $d(x)$ and $f(x)$ is larger than the inner product between $c(x)$ and $f(x)$ because $d(x)$ has a larger value than $c(x)$ at the points where $f(x)$ is also large, which results in a larger contribution to the integral. Only about 40% of the students answered both questions correctly. Most of the students who answered correctly also gave correct reasoning.

Most of the students who answered question 7.5a incorrectly (25%) stated that the inner product between $b(x)$ and $f(x)$ is larger than the inner product between $a(x)$ and $f(x)$. On question 7.5b, the most common incorrect answer (given by 26% of the students) was that the inner product between $c(x)$ and $f(x)$ is equal to the inner product between $d(x)$ and $f(x)$. The students

who gave these answers often gave vague answers based on similarities between the shapes of the functions. An example of this is shown below.

> *"d(x) has a shape that more closely resembles f(x)."* (Question 7.4b)

About one-third of the students gave an explanation that appeared to be based on the general similarity of the functions rather than based on the definition of an inner product that was given at the beginning of the task. In addition, 11% of the students gave explanations that were blank or unclear.

Question 7.5c asks students if any of the inner products between $f(x)$ and the other four functions is equal to zero. Only one of the functions shown, $a(x)$, has an inner product with $f(x)$ that is equal to zero. About half of the students (49%) gave the correct answer. Most of the remaining students (35%) explicitly noted that the inner product of $b(x)$ and $f(x)$ is zero, as in the example below.

> *"The areas of b(x) and f(x) directly cancel each other out, resulting in zero."* (Question 7.4c)

This student seems to believe that the inner product is zero when the functions are exactly opposite; *i.e.*, when they "cancel out". (Although some students gave this reasoning in response to question 7.5a, many did not apply this idea consistently on question 7.5c.) This represents a fundamental misunderstanding of what an inner product of zero means: that the two functions are orthogonal. In fact, $b(x)$ and $f(x)$ have a large (and negative) inner product, because $b(x) = -f(x)$.

Overall, the results from Task 7.5 suggest that determining inner products for graphical functions is challenging. Only about 40% of junior-level physics students were able to answer

the pair of qualitative questions in Fig. 7.4 about integrals of products correctly. Many students do not seem to have a correct method for evaluating the inner product between two functions graphically, even when the definition of the inner product is given. Some students are unable to identify cases in which an inner product is equal to zero. It seems likely that students who have difficulty calculating inner products may also have difficulty determining probabilities of measurement outcomes that require inner products. Further research is necessary to explore student difficulties with inner products at the upper-division level in more detail.

7.2 – Student Difficulties with Changes of Basis in Quantum Mechanics

In section 7.1 we examined student difficulties with inner products. In section 7.2, we probe student difficulties a topic that requires the use of inner products: changes of basis. Rewriting a state that is written in one basis (*e.g.,* as a position-space wave function) in terms of a different basis (*e.g.,* the energy eigenbasis) is a fundamental mathematical tool for studying quantum mechanics. The ability to change from one basis to another is one area of quantum mechanics in which inner products are used frequently. Changes of basis are especially important for evaluating the time dependence of a quantum state and for predicting the probabilities of measurement outcomes. While there have been significant efforts to investigate student difficulties specifically with time dependence and measurement (see, for example, chapters 2-4 of this dissertation), there has been no prior research directly focused on student difficulties with the changes of basis and the associated inner products that underlie these concepts.

In this section, we examine the responses to questions designed to probe student understanding of changes of basis. Throughout this section, we focus primarily on two different

topics: (1) students' ability to *calculate* inner products when a change of basis is required, and (2) students' ability to *recognize* that such a change of basis is required. In section 7.2A, we describe student difficulties with determining inner products between state vectors written in Dirac notation in terms of different basis states. In section 7.2B, we discuss the responses to an open-ended task in which students were asked whether or not it is possible to determine various measurement probabilities for quantum states defined using different representations. It should be noted that this section is meant to describe a preliminary effort to investigate student understanding of changes of basis, and that further research is needed.

A. Student Difficulties with Inner Products between Two States Written in Different Bases

When two quantum states are written in Dirac notation using the same basis states, the procedure for determining the inner product between them is relatively straightforward. Since quantum bases are almost always orthonormal, the inner products between the basis states result in either zero or one. Thus, the inner product between general states is given by multiplying each coefficient from one state vector with the corresponding coefficient from the other state vector, and then summing over all basis states. As discussed in section 7.1C, about two-thirds of the students in an introduction to quantum mechanics for sophomores were able to carry out this kind of calculation. The most prominent difficulty was a failure to take the complex conjugate of one of the state vectors.

However, it is often necessary in quantum mechanics to determine the inner product between states that are written in *different* quantum bases (*e.g.*, position and energy). In this section, we discuss the results from two tasks in which students are asked to determine inner products

between states that are written in different bases. Both tasks were given as long-answer questions on course exams in the sophomore-level quantum course at UW. They have not been given in the junior-level course because that course introduces Dirac notation much later. The first task (section 7.2A1) involves state vectors for a spin-1/2 system, while the second task (section 7.2A2) instead uses state vectors for a two-state system different from those introduced during the class.

1. Questions about inner products between spin-1/2 states

Students at UW first learn Dirac notation in a sophomore-level introduction to quantum mechanics course. This course introduces Dirac notation in the context of spin-1/2 systems, which exist in a Hilbert space with only two dimensions. Students are taught how to represent quantum states in different bases, such as the *x*-basis and the *z*-basis, and how to change from one basis to another. During the period of this study, students would also complete an interactive tutorial-lecture (see section 8.2A1) that focuses on helping students recognize the similarities and differences between spatial (Cartesian) vectors and state vectors in a Hilbert space.

Several questions on the first midterm exam in the sophomore-level course included questions about states written in different bases. One such question, Task 7.5, is shown in Fig. 7.7. Task 7.5 has been administered to 323 students over a period of two years. The task presents students with one state (for a spin-1/2 system) written in the *z*-basis and two states written in the *x*-basis. They are then asked to compute an inner product. The inner product that the students were asked to compute varied from quarter to quarter, but always included the state written in the *z*-basis.

Task 7.5

Spin-1/2 systems are prepared in these states (note that not all states are expressed in the same basis):

$$|\alpha\rangle = \tfrac{1}{\sqrt{2}}|\uparrow\rangle - \tfrac{1}{\sqrt{2}}|\downarrow\rangle, \quad |\beta\rangle = \sqrt{\tfrac{3}{5}}|+\rangle_x - \sqrt{\tfrac{2}{5}}|-\rangle_x, \quad |\gamma\rangle = \tfrac{1}{\sqrt{2}}|+\rangle_x - \tfrac{i}{\sqrt{2}}|-\rangle_x$$

Compute $\langle\alpha|\gamma\rangle$. Show your work and/or explain your reasoning.

Figure 7.7: An exam question requiring a change of basis

Task 7.5 was asked on the first midterm exam of a sophomore-level introduction to quantum mechanics. During some quarters, the states $|\uparrow\rangle$ and $|\downarrow\rangle$ were written as $|+\rangle$ and $|-\rangle$, respectively (students were told to treat all states with no subscript letter as referring to the *z*-basis). For these quarters, students were instead asked to compute $\langle\alpha|\beta\rangle$.

The inner product in Task 7.5 may be computed in several different ways, all of which involve a change of basis. For example, $|\gamma\rangle$ can be written in terms of the *z*-basis using the relationship $|\pm\rangle_x = \tfrac{1}{\sqrt{2}}|+\rangle \pm \tfrac{1}{\sqrt{2}}|-\rangle$. Then, the inner product can be evaluated by using the fact that the basis kets for the *z*-basis are orthonormal:

$$\langle\alpha|\gamma\rangle = \left(\tfrac{1}{\sqrt{2}}\langle\uparrow| - \tfrac{1}{\sqrt{2}}\langle\downarrow|\right)\left(\tfrac{1-i}{2}|\uparrow\rangle + \tfrac{1+i}{2}|\downarrow\rangle\right) = \tfrac{1-i}{2\sqrt{2}} - \tfrac{1+i}{2\sqrt{2}} = -\tfrac{i}{\sqrt{2}}$$

The same result can be achieved by rewriting $|\alpha\rangle$ in terms of the *x*-basis. In this basis, $|\alpha\rangle$ is equal to $|-\rangle_x$. Either method can also be executed using column vectors instead of using Dirac notation. Seventy percent of the students used one of these methods. However, not all of them gave the correct answer. Most of the errors made by these students appeared to be algebraic in nature, rather than being related to the underlying concepts associated with finding inner products.

A few students (6%) gave an answer to Task 7.5 that is a state vector rather than a scalar. The error of giving a vector as the result of an inner product is also discussed in section 7.1B in the context of spatial vectors.

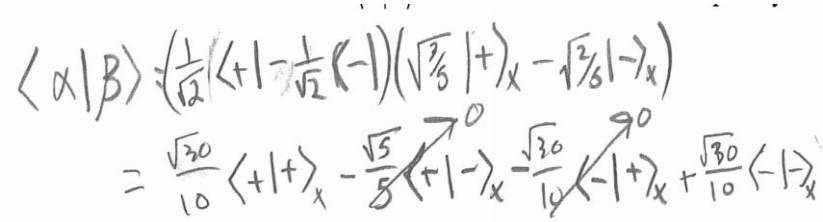


Figure 7.8: Sample student response

The response was given on Task 7.5 (see Fig. 7.7).

Other students (18%) gave answers that did not reflect any attempt to change basis. These students failed to distinguish between the different basis states used in the task. For example, one student's response (see Fig. 7.8) explicitly indicates that $|+\rangle$ is orthogonal to $|-\rangle_x$, despite the fact that they are basis states for *different* bases. Some students were less explicit about the assumptions they made about the orthogonality of various states (*i.e.*, they did not note that $|+\rangle$ and $|-\rangle_x$ were orthogonal, but gave an answer consistent with treating them as orthogonal). Others performed a calculation in terms of row and column vectors *without* changing basis. While a few of these students may simply have failed to notice that the states are written in different bases (even though it was explicitly pointed out in most versions of the question), the responses indicate that this was not usually the case. Instead, many students did not seem to recognize that it is necessary to rewrite one of the states in a different basis in order to answer Task 7.5. This indicates that some students may not understand *when* it is necessary to change basis, even though they may otherwise be able to calculate inner projects.

A few students (3%) stated that the answer to Task 7.5 is zero. Most of these students specifically noted that they believed that $|\alpha\rangle$ and $|\beta\rangle$ were orthogonal. A few of these students stated that they believed them to be orthogonal because the states were written in different bases. This kind of response has not proved to be common, either for Task 7.5 or for other inner-product tasks that we have administered.

2. Questions about inner products for a non-spin two-state system

On the final exam in the sophomore-level quantum course at UW, we asked another sequence of questions about inner products that require changing basis. The context of these questions (Task 7.6, see Fig. 7.9) was changed to something other than spin, which students use throughout the bulk of the sophomore course. The reason for this modification was to investigate whether students can transfer their knowledge about inner products and changes of basis to an unfamiliar quantum system.

The context used for Task 7.6 is a "quantum mechanical mouse", which is described as a two-state system with operators for "weight" and for "mood". (This context was originally developed by the University of Colorado – Boulder for instructional use.) On Task 7.6, students are told that the eigenstates of each operator are orthonormal (*i.e.*, "light" is orthogonal to "heavy", while "happy" is orthogonal to "sad"). They are then given a superposition state $|\varphi\rangle$ in the "mood" basis and told that it corresponds to a system that is always measured to be $|light\rangle$. The task shown in Fig. 7.9 was given on the final exam to 123 students in a sophomore-level quantum course.

Task 7.6

Consider a quantum mechanical mouse. You can observe the *weight* of the mouse (using the operator $\hat{W}$) and the *mood* of the mouse (using the operator $\hat{M}$). There are only two possibilities for the weight of the mouse and two possibilities for the mood of the mouse.

The eigenequations for weight are: $\hat{W}|light\rangle = 1|light\rangle$ $\quad \hat{W}|heavy\rangle = 5|heavy\rangle$.

The eigenequations for mood are: $\hat{M}|happy\rangle = 1|happy\rangle$ $\quad \hat{W}|sad\rangle = -1|sad\rangle$.

Assume that the eigenstates of $\hat{M}$ are orthonormal, and that the eigenstates of $\hat{W}$ are also orthonormal.

Question 7.6a

Consider many identical copies of the mouse described by the state $|\varphi\rangle = \frac{3}{5}|happy\rangle - \frac{4i}{5}|sad\rangle$.

Suppose you measured the weight of each mouse and determine that every mouse had a *weight* of 1 (*i.e.,* "light").

Determine the value of the inner product between a "light" mouse and a "happy" mouse. Explain how you arrived at your answer.

Question 7.6b

Now consider a mouse that is known to have a *mood* of –1 (*i.e.,* "sad").

If you were to measure the weight of this mouse, what is the probability that this mouse has *weight* = 5 (*i.e.,* "heavy")? Explain how you arrived at your answer.

Figure 7.9: Exam questions about a "quantum mouse"

Task 7.6 was administered on the final exam (N=123) in a sophomore-level introduction to quantum mechanics. A variant of question 7.6b, which asked for an inner product rather than a probability, was asked during a different quarter (N=102). This variant may be found in the appendix.

Question 7.6a asks students to determine the inner product between a "light" mouse and a "happy" mouse. This question can be answered by observing that $|\varphi\rangle$ must be equal to $|light\rangle$. This is true because each quantum mouse known to be in this state was measured to have a weight of 1. It is then possible to take the inner product between $|\varphi\rangle$ and $|happy\rangle$. Since $|happy\rangle$ and $|sad\rangle$ are orthonormal basis states, the inner product between $|\varphi\rangle$ and $|happy\rangle$ is equal to 3/5.

Question 7.6b asks students to determine the probability that a mouse known to be "sad" is then measured to be "heavy". This question can be answered by using the fact that $|heavy\rangle$ and $|light\rangle$ are orthogonal to determine how to write $|heavy\rangle$ in terms of $|sad\rangle$ and $|happy\rangle$. Then the probability can be determined by squaring the inner product between $|sad\rangle$ and $|heavy\rangle$.

An alternate version of question 7.6b has also been used. In that version, students are asked to determine an inner product instead of a probability. The alternate version was given to 102 students on the final exam during a different quarter than the version shown in Fig. 7.9. Overall, the results on each version were similar.

Question 7.6a was answered correctly by 58% of the students, most with correct reasoning. The most common incorrect answers were that the inner product in this question is equal to one (11%) or zero (7%). Most students who stated that the inner product is equal to one seemed to misinterpret the question: their responses indicated that they were finding the inner product between $|light\rangle$ and $|\varphi\rangle$ rather than between $|light\rangle$ and $|happy\rangle$. The students who stated that the inner product is equal to zero tended to state that all the states are orthogonal, even those that are basis states for different bases. Another 11% of the students attempted to find an inner product between a pair of states different from the one indicated in the text of the question.

Across both versions of question 7.6b, only 21% of the students gave the correct answer (none of the results differed by more than about 10%). Most of the students (20% of the class) also gave correct explanations. Some students (12%) stated that more information—namely, the form of the unknown state—is necessary in order to answer. These students failed to recognize that this information could be determined using the orthogonality between $|light\rangle$ and $|heavy\rangle$.

Another 8% of the students answered that the probability (or inner product) is zero, or that it is not possible to determine, because the states are written in different bases.

A large fraction of the students recognized that it was necessary to use orthogonality to determine an expression for $|heavy\rangle$ in terms of $|sad\rangle$ and $|happy\rangle$, but then did not use orthogonality correctly. The following response is an example of one of the most common incorrect expressions (given by 12% of the students).

> *"From above, we have that* $|light\rangle = \frac{3}{5}|happy\rangle - \frac{4i}{5}|sad\rangle$. *Furthermore, since* $|light\rangle$ *and* $|heavy\rangle$ *are orthonormal,* $|heavy\rangle = \frac{3}{5}|happy\rangle + \frac{4i}{5}|sad\rangle$. *"* (Question 7.6b)

This student simply changed the sign of the term associated with $|sad\rangle$, which does not result in a state that is orthogonal to $|light\rangle$. In order for this procedure to result in the correct expression, the magnitudes of the coefficients of $|sad\rangle$ and $|happy\rangle$ must also be switched. In the spin-1/2 case that the students covered extensively in class, the magnitudes *are* equal, so it would be sufficient to make one of the terms negative in order to find an orthogonal state. Although this is not the only case that students covered in the course, it was used almost exclusively. The results to this question suggest that some students failed to generalize correctly to the case where the magnitudes of the coefficients are unequal.

Another common incorrect expression given by students for $|heavy\rangle$ is shown in Fig. 7.10. This student appears to use the eigenvalues of the weight operator to determine the coefficients of the respective eigenstates. The student also indicates that "the mood doesn't effect [sic] the weight", which suggests that the answer does not depend on the mood state that they are

considering. This kind of answer suggests that some students may not recognize how to use eigenvalues and eigenvectors to write a general state vector in terms of a new set of basis states. A few students wrote a state similar to the one shown in the figure, but did not square the coefficients when normalizing, so they arrived at an answer of 5/6 for the probability, rather than 25/26. Answers with this form were given by 12% of the students who answered this version of question 7.6b (N=123), though some did not explicitly indicate all of their algebra.

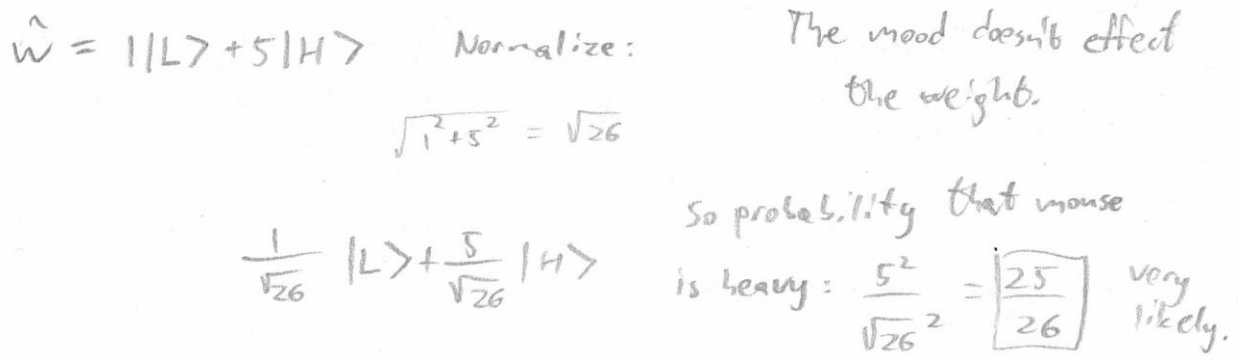


Figure 7.10: A sample response to question 7.6b

In the other version of question 7.6b (see the appendix), the students are given that $|happy\rangle = \frac{\sqrt{3}}{2}|light\rangle - \frac{i}{2}|heavy\rangle$ and asked to determine the inner product $\langle light|sad\rangle$. As with the version shown in Fig. 7.9, students can determine that $|sad\rangle = \frac{1}{2}|light\rangle - \frac{i\sqrt{3}}{2}|heavy\rangle$ by noting that $|happy\rangle$ and $|sad\rangle$ are orthogonal.

Figure 7.11 shows one student's response to this question. As with the response shown in Fig. 7.10 above, the student appears to use the *eigenvalue* of one of the operators to determine what the state of the particle should be, rather than using the orthogonality between $|happy\rangle$ and $|sad\rangle$. In this case, because the eigenvalue of $|sad\rangle$ is –1, the student multiplies the entire state

$|happy\rangle$ by –1 to determine $|sad\rangle$. A total of 17% of the students who answered this variant of question 7.6b (N=102) gave this answer.

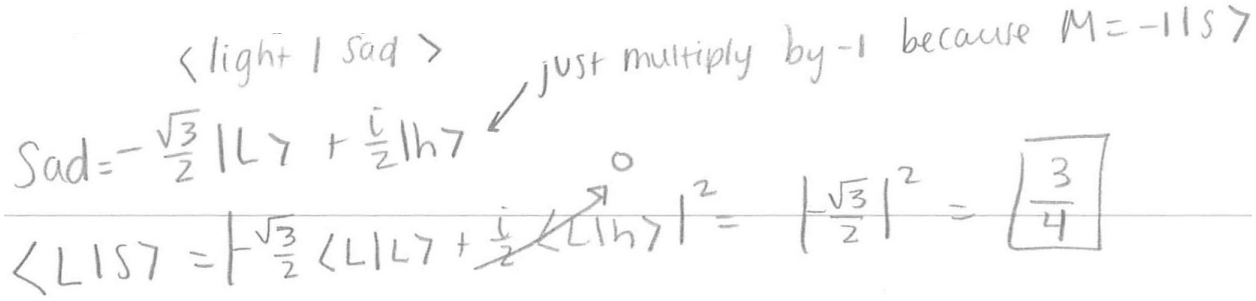


Figure 7.11: A sample response to a variant of question 7.6b
See the appendix for the specific version of the question asked.

3. Discussion of results

Overall, the results from Task 7.5 and 7.6 suggest that many students struggle to determine inner products for states that are not written in the same basis. Some students did not recognize when a change of basis was necessary, while others did not seem to know which states should be orthogonal. Furthermore, many of the errors that students make are similar to those that have been identified on questions that ask about inner products between states or vectors that are written in the same basis. The fact that some basic errors become more common when students are asked more challenging questions suggests that students' conceptual understanding of inner products may not be as robust as their answers to simpler questions (such as those discussed in section 7.1B for spatial vectors) would indicate.

The results to Task 7.6 (the quantum mouse question) provide further evidence that students often lack the conceptual understanding necessary for changing basis. When the students were asked to consider that unfamiliar system, they found it even more difficult to answer questions that required a change from one basis to another. For example, the large number of students who failed to identify the correct expression for $|heavy\rangle$ suggests that students do not find it easy to work with more than one basis. In particular, some students do not appear to understand the procedure for determining an expression for a basis vector in terms of the states for a different basis. This appears to be especially true when the bases are less familiar (or more complicated) than the bases formed by the eigenstates of S_z, S_x, and S_y, which the students in the sophomore class use most of the time. The fact that fewer students can determine inner products between states written in different bases when the system is unfamiliar suggests that some students may not understand the fundamental properties of orthonormal bases.

B. Student Difficulties with Determining Probabilities of Energy, Position, and Momentum for Different Representations

In section 7.2A, we described student difficulties with questions that asked them about specific inner products and probabilities requiring a change of basis. In this section, we instead discuss student responses to questions that ask about whether it is *possible* to determine probabilities for various quantities, including those that would require a change of basis.

The first several weeks of the junior-level quantum course at UW covers quantum mechanics for one-dimensional systems (*e.g.*, the infinite square well or the simple harmonic oscillator). Students at this level are often expected to use multiple quantum bases; for example, they might

be asked to use a position-space wave function to answer questions about the probabilities of possible energy values. Almost all of the students in the course have also completed one quarter (10 weeks) of sophomore-level instruction on quantum mechanics that covered two-state systems and basic one-dimensional systems such as the free particle, the infinite square well, and the finite square well.

Figure 7.12 (see next page) shows a sequence of questions (Task 7.7) that we have designed to probe student understanding of the process of changing from one basis to another. Task 7.7 was administered to 88 students on a tutorial pretest after about 2 weeks of junior-level instruction.

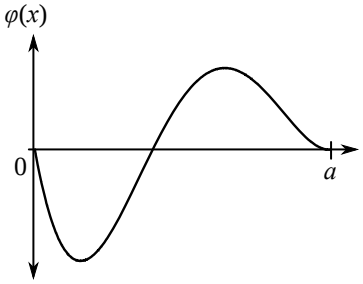
Task 7.7 asks students to consider three different cases corresponding to a different representation of a quantum state: (1) a state written in terms of energy eigenstates using Dirac notation, (2) a graph of a position-space wave function, and (3) an algebraic expression for a position-space wave function. When given to students, each case was on a different page, and once students had completed a page they were not permitted to revise their answers on previous pages. On each page, students were asked to determine whether or not it is possible to determine three quantities of interest, corresponding to probabilities for measurements of energy, position, and momentum.

The answer to these questions is that any quantity can be determined for any representation by computing the appropriate inner products. The fact that probabilities are given by the complex square of inner products is one of the postulates of quantum mechanics.[11]

Task 7.7

Case 1: Consider a particle in the infinite square well. Suppose this particle is described by the following expression in Dirac notation: $|\psi\rangle = \frac{3}{5}|\psi_2\rangle + \frac{4}{5}|\psi_3\rangle$.

Case 2: Consider a different particle in the same infinite square well. This particle is known to have the wave function shown below.



Case 3: Consider a different particle in the infinite square well. This particle is known to have the wave function given by (where N is a constant): $\psi_A(x) = N\left(\sin(\pi x/a) + \sin^3(\pi x/a)\right)$.

For each of the following quantities, state whether or not it is possible to determine the quantity of interest given the above representation of the state.

Quantity 1: The probability that an energy measurement will result in E_2, the first excited-state energy.

Quantity 2: The probability that a position measurement will find the particle on the left half of the well.

Quantity 3: The probability that a momentum measurement will find the particle moving to the left.

Please explain your reasoning for ***all*** of the above cases.

Figure 7.12: A sequence of questions on representing wave functions

Task 7.7 was administered on a tutorial pretest to 88 students in a junior-level quantum course. Each of the three cases was presented on a different page, and students were not allowed to backtrack.

For each of the three cases, about 20-30% of the students gave answers and explanations that we considered correct. Students were counted as correct either if they identified that all three quantities of interest could be determined, or if they stated that the quantities could only be determined if they also knew the energy eigenfunctions. Students did not have to explain exactly how they would determine the quantities in order to be considered correct. Few students gave the correct answer without also giving a correct explanation.

The most common incorrect answer to each case was that only one quantity could be determined: the quantity in terms of which the state was given (*i.e.*, energy for case 1 and position for cases 2 and 3). This answer was most common in case 2 when the wave function was given graphically (30% of the students) and least common in case 3 when it was given as a general function (15%). In case 1, 23% of the students gave this answer when the state was represented in Dirac notation in terms of energy eigenstates. The following explanation, given in response to case 2, when the wave function was represented graphically, is representative of this answer.

> *"What we have above is the state written in terms of position space. Thus, the only information we can get from it is about its position. The other two quantities we can't find given just the wave function."* (Task 7.7, case 2)

This student seems to think that only position can be determined from a position-space wave function. The student gave a very similar response to case 3. Interestingly, the answer that the student gave for case 1 was correct, and it was not uncommon for students to respond to the three cases differently. The fact that very few students gave answers that were consistent across all three cases suggests that some students may not view all representations as equivalent, in addition to viewing some representations as lacking certain information.

Some students noted specific information that could *not* be determined from the representation that was given. The example below, given in response to case 1 (for a particle represented in the energy basis using Dirac notation) is illustrative.

> *"It is possible to determine the probability of position, since we are given the position function. However we are given no information about momentum, therefore we cannot determine anything about this."* (Task 7.7, case 1)

This student states that the wave function was given in terms of position, even though the state was given in Dirac notation in terms of energy eigenstates. The response below shows a different explanation used to support the answer that position probabilities can be determined but momentum probabilities cannot.

> *"We can find the energy from the formula in Dirac notation. We can find the probability of the location of the particle. However, we cannot find the probability of a momentum measurement because we cannot measure both position and velocity simultaneously."* (Task 7.7, case 1)

This student's answer seems to be based on ideas related to the uncertainty principle, though that term is not used in the response. It is true that the uncertainty principle states that position and momentum cannot be well-defined at the same instant in time. However, the uncertainty principle does not restrict the determination of position or momentum outcomes or probabilities. For case 1, 28% of the students gave explanations similar to the two examples above, with 19% and 16% giving similar answers for cases 2 and 3, respectively.

Some students stated that both position and momentum could be determined, but not energy. This was more common for case 2 (13%) than for case 1 (6%) or case 3 (8%). Most of the students who gave this answer indicated that the energy could be determined if the energy eigenstates were given, though a few noted that it was not possible to determine the energy because of the way in which the state was represented.

C. Discussion of Results

The results discussed in sections 7.2A and 7.2B suggest that many students have difficulty with several concepts related to changes of basis in quantum mechanics. In particular, students often fail to recognize how probabilities can be determined from quantum states represented in different ways. For example, the results from Task 7.7 indicate that many students do not have a fundamental understanding of how different quantities such as energy, position, and momentum are related to each other in quantum mechanics. The results from Tasks 7.5 and 7.6 revealed that students also have difficulty with the mechanics of calculating the inner products necessary to change from one quantum basis to another.

Together, the responses to the three tasks described in section 7.2 indicate that there may be gaps in some students' understanding of quantum mechanics that go beyond common difficulties with specific quantum contexts (*e.g.*, time dependence and measurements). When students are asked to determine the possible outcomes of measurements and their associated probabilities, especially when a quantum state is not represented in the same basis as the measurement, some students make errors that appear to be related directly to the necessary change of basis. Other students do not seem to recognize that changes of basis are needed at all. The preliminary results discussed in this section suggest that it is crucial to conduct further research to investigate how students approach questions that require changes of basis, and to study how student difficulties with changes of basis may be related to other ideas, such as measurements and time dependence. Moreover, we have found that students have difficulty with changes of basis for some of the

more advanced topics in quantum mechanics, including the subject of chapters 9 and 10 of this dissertation—angular momentum.

7.3 – Student Difficulties with Time-independent Perturbation Theory

In this section, we focus on student understanding of a topic that requires the extensive use of inner products: time-independent perturbation theory. Sections 7.1 and 7.2 have been predominantly focused on student difficulties with inner products related to the calculation of probabilities. This chapter includes a discussion of perturbation theory because it is a context that requires the use of inner products in a different way than those we have examined previously in this chapter. Thus, perturbation theory provides an opportunity to investigate whether or not students can transfer their knowledge about inner products from one quantum context to another. In addition, the difficulties with inner products that have already been identified, especially with inner products between functions represented graphically (see section 7.1D), can help inform our investigation into student understanding of perturbation theory.

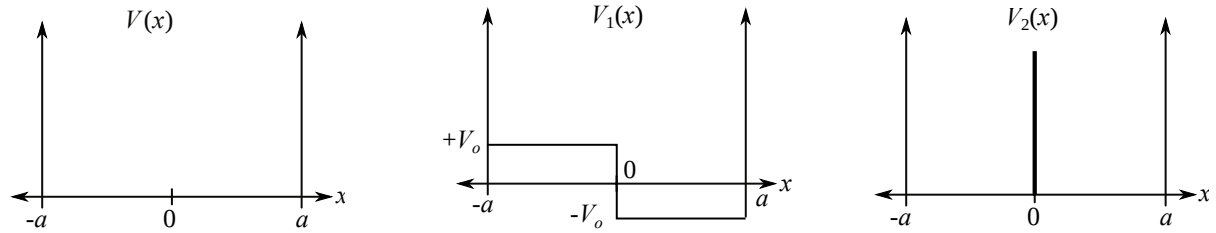
Perturbation theory is an advanced topic that students at UW do not encounter until the second quarter of the junior-level quantum mechanics course. By the time they begin studying perturbation theory, students have completed one quarter each of sophomore- and junior-level instruction in quantum mechanics. Thus, they have studied inner products in a wide variety of other contexts before learning perturbation theory. In this section, we also describe some student difficulties particular to the context of perturbation theory, which do not appear to be related directly to inner products.

In his dissertation, Andrew Crouse describes the results of a preliminary investigation into student understanding of perturbation theory.[11] He focuses primarily on the errors made in response to questions that asked students to determine the sign of first-order corrections to the energy for perturbations presented graphically. We review and add to his discussion of these errors in section 7.3A below. Section 7.3B discusses some preliminary research on student difficulties with *degenerate* perturbation theory. Curriculum designed to improve student understanding of perturbation theory is discussed in section 8.3. Much of the work in this section was done in collaboration with Gina Passante and Tong Wan, and some of the results are discussed in Ref. 9.

A. Student Difficulties with Non-degenerate Perturbation Theory

We have administered a variety of question to probe student understanding of perturbation theory. One sequence of questions (Task 7.8) is shown in Fig. 7.13. Task 7.8 was administered as a hand-written tutorial pretest to 152 students from 2003-2008. (The results from the pretest during those years are discussed briefly in Ref. 11.) Task 7.8 has also been administered as an online pretest to 95 students (in two separate quarters from 2014-2015). All students had some lecture instruction on time-independent perturbation theory prior to completing the pretest. The goal of this pretest is to determine if students can make qualitative predictions about the signs and magnitudes of several different first-order corrections to the energy, given the general expression for the correction in Dirac notation, which was derived in class and in the textbook.

Task 7.8



Shown above is an infinite square well of width $2a$ and two possible perturbations, $V_1(x)$ and $V_2(x)$. For what follows, recall that $E_n^{(1)} = \left\langle \psi_n^{(0)} \middle| \hat{H}' \middle| \psi_n^{(0)} \right\rangle$.

Question 7.8a

Is the first-order correction to the energy of the ground state, $E_1^{(1)}$, *positive*, *negative*, or *zero* for $V_1(x)$? Explain.

Question 7.8b

Is the first-order correction to the energy of the first excited state, $E_2^{(1)}$, *positive*, *negative*, or *zero* for $V_1(x)$? Explain.

Question 7.8c (online only)

Is the magnitude of $E_1^{(1)}$ *greater than*, *less than*, or *equal to* the magnitude $E_2^{(1)}$ for $V_1(x)$? Explain.

Question 7.8d

Is the first-order correction to the energy of the ground state, $E_1^{(1)}$, *positive*, *negative*, or *zero* for $V_2(x)$? Explain.

Question 7.8e

Is the first-order correction to the energy of the first excited state, $E_2^{(1)}$, *positive*, *negative*, or *zero* for $V_2(x)$? Explain.

Question 7.8f (online only)

Is the first-order correction to the energy of the second excited state, $E_3^{(1)}$, *positive*, *negative*, or *zero* for $V_2(x)$? Explain.

Question 7.8g (online only)

Is the magnitude of $E_1^{(1)}$ *greater than*, *less than*, or *equal to* the magnitude $E_2^{(1)}$ for $V_2(x)$? Explain.

Question 7.8h (online only)

Is the magnitude of $E_1^{(1)}$ *greater than*, *less than*, or *equal to* the magnitude $E_3^{(1)}$ for $V_2(x)$? Explain.

Figure 7.13: A sequence of questions about time-independent perturbation theory

Task 7.8 was administered on a tutorial pretest in the second quarter of a junior-level quantum course. On the hand-written version, the delta function was located on the right half of the well and the order of the perturbations was reversed. In addition, the students on the hand-written version were asked to write an expression for the functional form of the second perturbation, and were not asked questions 7.8c and 7.8f-h.

Task 7.8 shows a graphical representation of two different small perturbations to the infinite square well potential. The first perturbation $V_1(x)$ consists of a positive constant on the left and a negative constant of equal magnitude on the right. The second perturbation $V_2(x)$ is a delta

function at the center of the well. The students are also given the definition (in Dirac notation) of the first-order correction to the energy of the n^{th} energy eigenstate: $E_n^{(1)} = \left\langle \psi_n^{(0)} \middle| \hat{H}' \middle| \psi_n^{(0)} \right\rangle$. Students are then asked to determine the sign of the first-order correction to the energy of several states (questions 7.8a-b and 7.8d-f). They are also asked to compare the magnitudes of several first-order corrections to the energy (questions 7.8c and 7.8g-h). The hand-written version (see Ref. 11) differed slightly from the online version: (1) the delta function perturbation was located on the right half of the well, (2) the perturbations were presented in opposite order, and (3) the students were not asked to compare the magnitudes of the various first-order corrections.

Each question can be answered by rewriting the first-order correction to the energy as an integral: $E_n^{(1)} = \int \psi_n^{*(0)} \hat{H}' \psi_n^{(0)} dx = \int H'(x) \left| \psi_n^{(0)} \right|^2 dx$. Note that $V_1(x)$ is antisymmetric with respect to the center of the well, while the *probability density* (the complex square of the wave function) for every energy eigenfunction of the unperturbed infinite square well is symmetric with respect to the center. Thus, the first-order correction to the energy for $V_1(x)$ is thus equal to the integral of an antisymmetric function, and is therefore zero.

Table 7.3: Results from Task 7.8

The percentage of the students who gave the correct answer to each question is shown. The gray columns include only responses from the online administrations (N=95). The numbers in parentheses in the last two columns indicate the percentage of the students who identified the sign of each correction to the energy in addition to correctly comparing the magnitudes of the indicated corrections to the energy.

N=247	$E_1^{(1)}$	$E_2^{(1)}$	$E_3^{(1)}$	All signs correct	Correct reasoning for all signs	$E_1^{(1)}$ *vs.* $E_2^{(1)}$	$E_1^{(1)}$ *vs.* $E_3^{(1)}$
$V_1(x)$	57%	38%	N/A	32%	16%	38% (26%)	N/A
$V_2(x)$	68%	52%	46%	34%	9%	51% (29%)	45% (24%)

The second perturbation $V_2(x)$ is a delta function, so the integral above gives the value of the probability density at the position where the delta function is nonzero. This is positive unless the delta function is located at a node of the wave function, which is only true for the first excited state when the delta function is located at the origin, as in the online administrations of the question.

The percentage of the students who answered each question correctly is shown in Table 7.3. The table also shows the percentage of the students who were able to determine the sign of the first-order correction to all energies for a given perturbation, and the percentage who gave correct reasoning for the entire sequence of questions. Students were considered to have correct reasoning if they wrote the first-order correction as an integral and used it correctly. Students who based their answer on the functional forms of the perturbed potentials and the probability density were considered correct even if they did not reference an integral. Students who made a conceptual error while attempting to calculate an integral, even if the integral was correct, were not counted as having correct reasoning. The results in the table have been aggregated over all quarters. The variation from quarter to quarter has been within the range that would be expected statistically (less than ±20% for quarters with about 20-30 responses).

About half of the students were able to determine the correct sign of most of the individual corrections to the energy. However, only about one-third of the students determined the correct sign of both (or all three) corrections. Even fewer students gave correct reasoning in support of the correct answer. The same pattern was true for comparing the magnitudes of the corrections: about half gave the correct answer, but a substantial proportion of these students identified incorrect signs for the corrections to the energy. In addition, 35% of the students did not give

any explanation, gave an unclear explanation, or specifically noted that they guessed or did not know how to answer. Overall, the results suggest that answering qualitative questions about perturbation is challenging for many students. Below, we discuss the most common errors that the students tended to make.

4. Belief that the sign of the correction is the same as the sign of the perturbation

In their responses, some students indicated that the first-order correction to the energy for the delta function perturbation was positive because the perturbation itself is positive, as below.

> *"The spike is positive, so the energy correction should be positive as well."* (Question 7.8d)

> *"The perturbation is to the whole system, so the energy correction should occur to all the states."* (Question 7.8f)

This pair of explanations was given by a single student, who stated that the first-order correction to the energy would be positive for all states. The student seems to believe that the sign of the perturbation determines the sign of the correction, even though the energy eigenfunctions are different. This reasoning gives the correct answer in many cases, such as when the perturbation is positive everywhere in space. However, it does not give the correct answer unless the perturbation is positive (or negative) everywhere, or when a delta function perturbation is located at the node of a wave function. Some students gave similar explanations stating that the delta function potential would "cause an increase in energy" or "add energy" to the system. Twelve percent of the students gave responses to questions 7.8d-h (the delta function perturbation) consistent with this error, while only 5% gave such responses to questions 7.8a-c.

1. Tendency to estimate the correction based on the wave function instead of the probability density

When a perturbation to a Hamiltonian depends only on position, the first-order correction to the energy may be written as an integral of the product of the perturbed potential and the probability density. Many students, however, answer as if the product in this integral is between the perturbed potential and the *wave function*, as in the example below.

"Negative—the delta function is in the region where ψ_2 is less than zero." (Question 2B)

This student states that a negative value for the wave function results in a negative value for the correction. Many of the students who gave similar explanations sketched the wave function for the first excited state, showing the region where it is negative. (In contrast, many of those who answered correctly sketched the probability density, either instead of or in addition to sketching the wave function.) This suggests that some students are failing to recognize that the wave function must be squared to calculate the first-order correction to the energy.

In the questions about the delta function perturbation (questions 7.8d-h), 13% of the students gave answers that were consistent with this error. On the questions about the step-function perturbation (questions 7.8a-c), 14% of the students argued that $E_2^{(1)}$ is positive because both the wave function and the perturbation are antisymmetric about the center of the well. This use of symmetry to evaluate an integral is productive, but incorrect because it fails to account for the second factor of the wave function in the expression for the correction. About 5% of the students

made this error on *both* the questions about the delta function perturbation and the step function perturbation.

2. Belief that the correction is given by the average of the perturbation

The shape of the first perturbation (questions 7.8a-c) is uniformly positive on the left and uniformly negative on the right. Many students gave the correct answer—that the first-order correction to the energy is zero for both the ground state and the first excited state—while giving *incorrect* reasoning such as the response below.

> *"The first-order correction to ground state is 0, since the perturbation averages to 0."* (Question 1A)

This student's argument seems to be based on the *average* of the perturbation: because the size and width of the positive part of the perturbation to the potential and the negative part of the perturbation to the potential are the same, their combined effect averages to zero. Although zero is the correct answer, the explanation is incorrect because it does not take into account the shape of the probability density for the ground state. If the perturbation had instead been uniformly positive near the center of the well and uniformly negative near the edges, the first-order correction to the energy would no longer be zero (it would be positive) even though the perturbation would still average to zero.

Some students gave responses similar to the one above, but instead argued that the perturbation would affect the energy differently in the two regions, and that these effects would cancel out because they are equal. The example below illustrates this line of reasoning.

> *"Energy goes up in +V region and down in –V region so energy correction equals 0."* (Question 1A-B)

This student states that the energy changes in each region, and concludes that the correction is zero. The student does not appear to recognize that the energy should be constant across the entire well; that is, the student appears to believe that the energy depends on position in the well. The student also fails to use the expression for the first-order correction to the energy given at the top of the page. Overall, 34% of the students responded to questions 7.8a-c in a way that was consistent with the effects of the perturbation "averaging" or "cancelling" between the two regions. Some students gave the same answer and explanation for both the ground state and the first excited state. Others instead gave explanations of this type for question 7.8a (about $E_1^{(1)}$), but gave explanations for question 7.8b (about $E_2^{(1)}$) consistent with the error discussed in section 7.3A2.

3. Less common student errors

A few students (7%) based their answers to questions 7.8a-c on how the shape of the wave function is related to the shape of the potential, as in the example below.

> *"The ground state would fit in the well of length a on the right and thus be lower. The first excited state would be lower if it is still in the small well or the same if it were out of it."* (Question 7.8a-b)

This student appears to believe that the correction to the energy depends on whether or not the wave function "fits" within the new well created by the perturbation. It was not uncommon for

students to assume that the wave function would be confined to this region, and then to answer that the energy would have to increase because the width or wavelength had decreased.

A small percentage of the students (2%) gave similar explanations to questions 7.8d-h. These students tended to treat the delta function potential as something that decreased the width of the well, and therefore increased the energy.

There were also a handful of errors made by very small percentages of the students (2% or less). Some of these errors are discussed in more detail in Ref. 11. For example, some students seemed to think about the perturbations using language associated with reflection and transmission (*i.e.*, as if the wave function were moving from one side of the potential well to the other). Others based their arguments on classical ideas, such as assuming that the total energy is constant, or that the total energy would have to decrease in order to counter the effect of an increase in potential energy. However, none of these other errors was prevalent.

B. Student Difficulties with Degenerate Perturbation Theory

We have asked students in the junior-level quantum course about *degenerate* perturbation theory after lecture instruction on this topic. (For a more detail discussion of student understanding of systems with degenerate energies, see section 5.2.) Task 7.9, shown in Fig. 7.14, was given as part of a tutorial pretest in the junior-level quantum course, a few weeks after the questions discussed in the previous section (Task 7.8). The context for Task 7.9 is the two-dimensional infinite square well potential, which has a first excited state with a degeneracy of two (*i.e.*, there are two linearly independent energy eigenstates with energy $E_{21} = E_{12}$). Students are provided with a perturbation to this potential consisting of a delta function. They are asked

to determine whether the first-order correction to the energy is positive, negative, or zero for several states, each of which is an energy eigenstate of the unperturbed system with energy E_{21}. Students are explicitly told to note if any correction is not well defined.

Task 7.9

Consider a single particle in a two-dimensional quantum mechanical infinite square well (*i.e.*, the potential is zero from $x = 0$ to $x = a$ and from $y = 0$ to $y = a$, and infinite everywhere else). Let the energy eigenfunctions and energy eigenvalues of this system be given by $|\psi\rangle = |n_x, n_y\rangle$ and $E_{n_x n_y} = \left(n_x^2 + n_y^2\right)E_{11}$, respectively.

Suppose the small perturbation below is added to the system, where α is a small, positive constant:

$$V'(x, y) = \alpha\delta\left(x - 3a/4\right)\delta\left(y - 3a/4\right).$$

For each of the following states, determine whether the first-order correction to the energy is *positive*, *negative*, or *zero*. If any correction is not well defined, note that explicitly. Explain.

$$|A\rangle = |1,2\rangle \qquad |B\rangle = |2,1\rangle \qquad |C\rangle = \left(|1,2\rangle + |2,1\rangle\right)/\sqrt{2} \qquad |D\rangle = \left(|1,2\rangle - |2,1\rangle\right)/\sqrt{2}$$

Figure 7.14: A question about degenerate perturbation theory

The question was administered as part of a tutorial pretest in the second quarter of a junior-level quantum course.

The correct answer to this question is that the corrections to states C and D are well defined, while the corrections to states A and B are not well defined. The reason for this is that ordinary perturbation theory is only valid for states that are not degenerate. When more than one of the unperturbed energy eigenstates has the same eigenvalue, a different procedure must be used.[12] The result of degenerate perturbation theory is that the first-order correction to the energy is only well defined for states that are eigenvectors of the following matrix:

$$\begin{pmatrix} \langle\psi_{12}^{(0)}|H'|\psi_{12}^{(0)}\rangle & \langle\psi_{12}^{(0)}|H'|\psi_{21}^{(0)}\rangle \\ \langle\psi_{21}^{(0)}|H'|\psi_{12}^{(0)}\rangle & \langle\psi_{21}^{(0)}|H'|\psi_{21}^{(0)}\rangle \end{pmatrix}$$

For the perturbation used in Task 7.9, states C and D are the eigenvectors of this matrix, so only they have well-defined first-order corrections to the energy. The corrections are given by the corresponding eigenvalues: $4\alpha/a^2$ (positive) for state C and 0 for state D.

No students (N=84) gave the correct answer for all four states. Only two students gave explanations indicating that they had attempted to account for the fact that this system has degenerate energies. For each state, most students stated that the first-order correction to the energy is positive. Very few students gave an explanation more detailed than referencing the formula for the first-order correction to the energy derived using non-degenerate perturbation theory.

The fact that virtually no students were able to answer Task 7.9 correctly, even though it was administered *after* students had covered degenerate perturbation theory in lecture, suggests that students do not understand when degenerate perturbation theory should be used. Interactions with students in the classroom have shown that students often struggle to carry out the calculations necessary for degenerate perturbation theory. Chapter 8 discusses our preliminary effort to develop tutorial curriculum for improving student understanding of this challenging topic.

7.4 – Summary

In this chapter, we discuss preliminary results from an investigation focusing on student understanding of inner products and changes of basis. This investigation was carried out in both a sophomore-level introduction to quantum mechanics and a junior-level course in quantum mechanics at UW. The overall goal of this chapter was to identify student difficulties with inner

products and changes of basis in a variety of different contexts. In particular, we have focused on identifying patterns of student reasoning that are similar across these different contexts.

In section 7.1, we examined student responses to question about inner products in several different contexts, including spatial vectors, Dirac notation, and functions represented graphically. We found that most (but not all) upper-division physics students are able to calculate dot products for spatial vectors, such as those used throughout introductory physics. These students also tended to be able to interpret the dot product correctly for spatial vectors.

However, we identified several difficulties that the same students have when asked about inner products in more difficult contexts (Dirac notation and functions). In particular, many students give answers consistent with failing to take the complex conjugate of a state vector with imaginary components. Some students fail to use a correct method for determining inner products for vector spaces that have more than three dimensions. These findings suggest that some upper-division physics students may have difficulty with the concepts needed to determine inner products, even if they are able to answer basic questions about inner products for spatial vectors.

Those tasks that asked students to find inner products between functions represented graphically were the most challenging. Large percentages of the students in junior-level quantum courses were not able to evaluate and compare such inner products qualitatively. Even though they were given a definition for the inner product as an integral, most students did not give correct answers or correct explanations. The errors that many students tended to make suggest that they lack a method for evaluating inner products graphically. We have begun to

develop tutorial curriculum (see chapter 8) to help improve student ability to determine inner products.

Our investigation of student understanding of inner products has also examined student understanding of a related topic in quantum mechanics: changes of basis. We have asked a variety of questions that provide students with a state in one quantum basis and require them to rewrite it in a different quantum basis. We found that students often fail to recognize when the question they are answering requires them to changes from one basis to another. We also found that many students failed to recognize that all aspects of a state are encapsulated in the wave function regardless of the basis used to represent the wave function. This is manifested in students saying that it is not possible to determine probabilities for one observable (*e.g.*, momentum) given only a state written in a distinct quantum basis (*e.g.*, as a position-space wave function). We also found that some students fail to distinguish between different quantum bases, and that they did not believe it was necessary to change basis. It is possible that some student difficulties with inner products and changes of basis underlie some of the common difficulties (discussed in chapters 2-4) that students have with quantum measurements and time dependence.

The last part of the investigation discussed in this chapter focuses on student difficulties with time-independent perturbation theory. Perturbation theory is an advanced topic that is typically taught near the end of undergraduate instruction in quantum mechanics. It is also a topic that involves calculating inner products, often involving wave functions and potentials that may be represented graphically. We found that some of the difficulties that students have with finding inner products are also common when they are asked qualitative questions about corrections to

the energy in perturbation theory. In particular, some students did not appear to recognize that the correction to the energy could be determined using an appropriate integral.

Our findings throughout this chapter suggests that even students near the end of undergraduate instruction in quantum mechanics struggle with concepts that are typically taught near the beginning of quantum instruction. Though the study we have described is only in its beginnings, it may have strong implications for the teaching of quantum mechanics. There is a clear need for additional research into student understanding of topics in quantum mechanics that require inner products or changes of basis. Part of this research includes the development and assessment of targeted curriculum to improve student understanding of vector spaces throughout quantum mechanics. The development of a preliminary set of such curriculum for this topic is discussed in chapter 8.

7.5 – Notes and References

[1] R. Knight, "The Vector Knowledge of Beginning Physics Students," *The Physics Teacher* **33** (2), 74 (1995).
[2] G. Zavala and P. Barniol, "Students' Understanding of the Concepts of Vector Components and Vector Products," presented at the *Physics Education Research Conference 2010*, Portland, OR (2010).
[3] G. Zavala and P. Barniol, "Students' Understanding of Dot Product as a Projection in No-Context, Work and Electric Flux Problems," presented at the *Physics Education Research Conference 2012*, Philadelphia, PA (2012).
[4] P. Barniol and G. Zavala, "Force, Velocity, and Work: The Effects of Different Contexts on Students' Understanding of Vector Concepts Using Isomorphic Problems," *Physical Review Special Topics – Physics Education Research* 10 (2014).
[5] J. Van Deventer, "Comparing Student Performance on Isomorphic Math and Physics Vector Representations," Master's Thesis, University of Maine (2008).
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[7] Colin S. Wallace and Stephanie V. Chasteen, "Upper-division students' difficulties with Ampère's law," Phys. Rev. ST Phys. Educ. Res. 6, 020115 (2010).
[8] Rachel E. Pepper, Stephanie V. Chasteen, Steven J. Pollock, and Katherine K. Perkins, "Observations on student difficulties with mathematics in upper-division electricity and magnetism," *Physical Review Special Topics – Physics Education Research*, 8, 010111 (2012).
[9] See, for example, G. Passante, T. Wan, P. J. Emigh, and P. S. Shaffer, "Investigating Student Qualitative Understanding of Perturbation Theory and the Inner Products of Functions," presented at the *Physics Education Research Conference 2015*, College Park, MD (2016).

[11] See, for example, chapter 3 of N. Zettili, *Quantum Mechanics: Concepts and Applications* (John Wiley & Sons, Chichchester, UK, 2001).

[11] A. Crouse, "Research on Student Understanding of Quantum Mechanics as a Guide for Improving Instruction," PhD dissertation, University of Washington (2007).

[12] For more details about degenerate perturbation theory, see, for example, section 6.2 of D. J. Griffiths, *Introduction to Quantum Mechanics*, Pearson Prentice Hall (2005).

Chapter 8
Development and Assessment of Tutorial Curriculum for Vector Spaces in Quantum Mechanics

" $P(E) = \left| \int_{-\infty}^{\infty} \psi_E^*(x) \psi_{A,B,C}(x) dx \right|^2$ *. The amount that the graphs match up (essentially) gives an estimate of relative probability.* ψ_A *would produce an integrand nearly odd about x = a/2, so the integral would be nearly zero. Similarly,* ψ_B *would create equal regions of positive and negative area, so the integral would be zero.* ψ_C *would produce an asymmetric integrand with* ψ_E *and the complex square would yield something nonzero. So particle C is most likely to be found with energy E."*

—Junior-level quantum mechanics student, midterm exam

The results from chapters 6 and 7 suggest that students often have difficulty with many aspects of vector spaces that are important to quantum mechanics. There is evidence that student difficulties with vector spaces might underlie some difficulties with other topics in quantum mechanics, such as measurements and time dependence (see chapters 2-5). For example, we found that students may not relate probabilities in quantum mechanics to the appropriate inner product (see section 7.2). We have also found that many students fail to calculate such inner products properly and lack a conceptual understanding of the inner product. It is likely that student difficulties with inner products have contributed to some students' failure to make correct predictions about probabilities in quantum mechanics, even after tutorial instruction for quantum measurements and time dependence (described in chapter 5).

This chapter describes an effort by the Physics Education Group at UW[1] to develop tutorials that address common student difficulties with topics such as superposition and inner products.

Section 8.2 discusses the development of four tutorials that focus on vector spaces, especially inner products and changes of basis. They were designed to address the problems identified in chapters 6 and 7. This section also provides a preliminary assessment of this curriculum. Section 8.3 discusses the development and assessment of tutorial curriculum for time-independent perturbation theory, which has also been informed by the research on student difficulties with vector spaces described in chapters 6 and 7.

8.1 – Prior Research

In chapter 5, we describe tutorials that focus on concepts such as quantum measurements and time dependence. These are topics for which there have been significant previous efforts to design curriculum, both at UW and at other universities.[2-6] However, very little prior development and assessment has been done for curriculum designed to address common student difficulties with vector spaces in quantum mechanics.

In his PhD dissertation, Andrew Crouse briefly describes some research on student difficulties with specific elements of Dirac notation.[2] Based on this research, he developed a tutorial worksheet (*Treating functions as vectors*; see section 8.2A2). This research and curriculum development was very preliminary, however. There is no description of the tutorial curriculum in his dissertation, and little assessment of effectiveness.

There have been no substantive efforts at other universities to design curriculum for this subject. In some cases, other quantum curricula include exercises related to vector spaces as a part of curriculum for some other topic. However, none of this material appears to focus directly on vector spaces in quantum mechanics.

8.2 – Tutorial Curriculum for Vector Spaces in Quantum Mechanics

In section 8.2A, we describe four tutorials that have been developed with the goal of improving student understanding of vector spaces in quantum mechanics. In general, these tutorials have been designed for use in small-group sections to supplement lecture instruction at the junior level (for more detail, see section 1.2A). In some cases, individual tutorials have been used in other ways (*e.g.*, as interactive tutorial lectures) or for different populations (*e.g.*, sophomore-level quantum courses). When the usage of the tutorials was not typical, it is explicitly noted. In section 8.2B, we provide a preliminary assessment of some of the tutorials. This assessment focuses only on those tutorials that have been given in the junior-level quantum course, and only assesses a few of the goals of this sequence of curriculum. Lastly, section 8.2C discusses the overall results from this effort, and indicates possible lines of future research.

A. Description of Tutorial Curriculum for Vector Spaces

The sequence of tutorials described below grew out of the ongoing research conducted in the sophomore-, junior-, and graduate-level quantum courses at UW. The sequence as a whole is structured to guide students toward the development of a coherent model for vector spaces in quantum mechanics. Each individual tutorial addresses specific student difficulties with particular aspects of quantum vector spaces (*i.e.*, Hilbert spaces), as identified in chapters 6 and 7. The overarching goal is for students to be able to use inner products to change from one quantum basis to another, and to be able to calculate such inner products in a variety of different forms (*e.g.*, for state vectors written in Dirac notation or for wave functions represented mathematically or graphically). The tutorials were developed over a period of several years in an

iterative process of instructional use, assessment, and modification. For more detail about this process, see chapter 1. Below, the important features of the final versions of each tutorial are described, along with any significant modifications over the course of their development. The discussion also makes clear the reasons for certain key modifications. Copies of each tutorial may be found in the appendix. Lastly, it should be noted that the tutorials are still in development, and future research (see section 8.2C) may result in additional modifications to the versions discussed below.

1. Tutorial curriculum: *Dirac Notation*

Beginning in 2013, the Physics Education Group participated in a redesign of the sophomore-level modern physics course at UW (for more information, see section 1.2A4). This course was changed from using a non-spins-first textbook by Scherrer[7] to using the first four chapters of the textbook by McIntyre, which cover spin and two-state systems prior to introducing wave functions and position space.[8] This structure required introducing Dirac notation during the first few weeks of the course. Since Dirac notation can be difficult even for more advanced students,[9] we developed a tutorial to guide students to recognize the similarities between vector spaces described by Dirac notation and Cartesian vectors, which are familiar to students from introductory math and physics courses. It should be noted that this tutorial was initially designed as an interactive tutorial-lecture; however, it has been written so that it can also serve as a more typical tutorial in a small-group section.

The beginning of this tutorial asks students to recall some of the basic properties of Cartesian vectors. Students are asked to write mathematical expressions for sample vectors that are shown

on a grid. They are then asked to calculate and interpret the inner product in several cases. Lastly, they are asked to generalize their answers to the case of *n*-dimensional vectors. In particular, students are asked to determine the inner product between basis (or unit) vectors.

The second section of the tutorial focuses on drawing a comparison between the features of state vectors written in Dirac notation and Cartesian vectors. The features that the tutorial focuses on are: the inner product between basis vectors, the inner product between a vector and itself, and the inner product between two non-basis vectors. In the first version of this tutorial (see the appendix), this comparison was done explicitly by having students carry out each operation side-by-side for both cases. However, we found that this exercise was time-consuming, and that many students continued to have difficulty after completing the tutorial. In particular, many students failed to conjugate the dual vector when finding inner products on midterm exams (see section 7.1C). As a result, this section was rewritten so that students are primarily asked about a vector space written in Dirac notation, and then asked to compare their answers to the case of Cartesian vectors. In addition, students are given a pair of incorrect calculations for the dot product in which the dual vector is not conjugate, and asked to identify the error in each calculation.

The original tutorial also had a third section that focused on operators. This section is now a part of the homework; see below. A different section that focuses on changes of basis was added to the tutorial. In this new section, students are given an initial state vector for a spin-1/2 particle, and are asked to find several inner products. They are guided to recognize that certain inner products are easier to compute in a particular basis, while others are more difficult, and they are asked to rewrite the initial state in a different basis to make other inner products easier to

compute. A complimentary exercise using a classical context has been administered online prior to the in-class tutorial. In this exercise, students are reminded of a familiar problem from introductory physics: calculating the acceleration of a block on a ramp. Students are asked to describe the set of coordinates that would be most useful for solving such a problem. The goal of this exercise is to help students recognize other problems in their study of physics for which there is a need to change basis, and why this change of basis is useful.

The homework for the tutorial *Dirac notation* focuses on some of the general properties of operators. Two important properties that students investigate are (1) how operators act on and change vectors, and (2) how acting a pair of operators on a vector in a different order can produce different results. On the homework, students are first asked to normalize two vectors written in Dirac notation, and then to represent them as column vectors. Students are then given an operator (written as a two-by-two matrix) and asked to graph the result of acting this operator on each vector on the same set of axes as the original vectors. They are then asked to identify a general rule for the graphical effect of the operator. Students then repeat this exercise with a second operator. Lastly, students consider acting both operators on the same vector, and whether or not the result depends on the order in which the two operators act.

It should be noted that the sequence of exercises on operators was originally given as part of the in-class worksheet during the first few years that this tutorial was administered. Since then, it has been moved to the homework to make room for the sequence of questions on changes of basis. This change was made in part because of the research, discussed in section 7.2.

2. Tutorial curriculum: *Treating functions as vectors*

Only one tutorial on vector spaces, titled *Treating functions as vectors*, was administered by Andrew Crouse during the initial development of the tutorials. The object of the original tutorial is to help students recognize that some of the properties of vector spaces, such as orthogonality and linear independence, can also be applied to certain sets of functions. In the original tutorial, students are asked to discretize the four lowest energy eigenfunctions of the infinite square well by dividing the *x*-axis into several equal-sized regions and making a bar graph of each function. Students are asked to check that the wave functions are orthogonal using the definition of the inner product in terms of an integral over all space. They are then guided to recognize that dividing the *x*-axis into more and more regions provides a better and better approximation to any function. Lastly, students are asked to identify the basis functions associated with the discrete approximation used throughout the tutorial, and to describe how these basis functions change as the number of regions into which the *x*-axis is divided becomes very large. At the conclusion of the tutorial, students are expected to identify that the wave function $\psi(x)$ is equivalent to the inner product $\langle x|\psi\rangle$, which is related to the probability of finding the particle near position *x*.

The original version of this tutorial was administered near the end of the first quarter of the junior-level quantum course. It was only used from 2003-2004, so it had not been assessed and modified over a period of many years. In addition, the original version of the tutorial had been given only *after* students had worked through a set of tutorials that focused on quantum measurements and time dependence (see chapter 5). Thus, students had been asked to use their knowledge of concepts related to vector spaces in the context of time dependence and

measurements prior to completing the *Treating functions as vectors* tutorial. We decided to modify the tutorial so that students could consider these ideas *before* working through the sequence of tutorials for time dependence and measurements.

The new version of the tutorial begins with a short exercise reviewing basis vectors and inner products for Cartesian vectors. The exercise is similar to the first section of the tutorial *Dirac Notation* described in section 8.2A1. Students are then given the graphs of three wave functions: the ground state and the second excited state of the infinite square well, and a superposition of these two functions. They are then asked to divide the *x*-axis into three regions, to approximate each function as a constant throughout each region, and to graph these approximations. Students then told to identify the basis states for this new, approximated representation. They are then asked to consider the consequences of increasing the number of regions into which the *x*-axis is divided. They are expected to recognize that since each basis state corresponds to a single region along the *x*-axis, and since the number of regions is increasing, the graphs of the basis states must become narrower.

We found that the original tutorial was challenging and time consuming. The changes seem to have helped clarify the intent for students. Furthermore, students now typically require much less time to complete the section of the tutorial described above, and they have fewer questions about each individual exercise.

The last section of the new tutorial focuses on inner products. In particular, students are asked to use their answers from the previous section to approximate the value for an integral between two functions. We expect students to divide the *x*-axis into three regions, multiply the two functions together, multiply by the width of each region, and then add up the result for each

region. Students are directed to describe their procedure in detail, especially how they approximated the infinitesimal width from the integral, *dx*. Students are then told that this integral is equivalent to an inner product between the functions, and asked to compare it to an inner product between two orthogonal Cartesian vectors. Lastly, students use their method to determine two additional inner products.

Two homework problems help students extend two different ideas from the tutorial. The first problem asks students to determine the inner products from the tutorial using a representation of the same functions using the energy eigenbasis. They then compare their results to the inner products they determined during the tutorial using a discretized position space. Students are expected to recognize that inner products should be the same regardless of the basis used to calculate them.

The second problem gives students three different functions and asks them to rank the inner products between each pair of functions by absolute value, and to determine whether each inner product is positive, negative, or zero. The last part of the second problem gives students two statements that are based on lines of incorrect reasoning we have found common among students prior to tutorial. The lines of incorrect reasoning that we have chosen to focus on are those in which students relate the value of an inner product to similarities between the "shapes" of the functions (see, for example, sections 7.1D and 7.3A).

3. Tutorial curriculum: *Representations of wave functions*

In section 7.2B, we presented evidence that shows some students seem to believe that when the state of a quantum system is represented in a particular way (*e.g.*, as a position-space wave

function), probabilities associated with *other* quantities (*e.g.*, momentum or energy) cannot be determined. That is, students do not seem to recognize that the wave function is a complete representation of a quantum state, regardless of the basis in which it is written. We have also found that many students have difficulty *calculating* probabilities, even when they believe it is possible to determine them from the wave function. As a result of these findings, we developed a tutorial to help students recognize that any representation of a quantum state provides information about *all* quantities of interest. The tutorial is also intended to help students learn how to change from one basis to another. A particular focus is on writing a state in the energy basis when it is given in the position basis (as a wave function).

The tutorial begins by giving students a state vector written as a superposition of energy eigenstates for the infinite square well in Dirac notation. Students are asked to determine the possible outcomes of an energy measurement, and the corresponding probabilities. Then, they are given a graphical representation of the same information, and guided to recognize that each representation contains the same information. They are also asked about the effect on energy probabilities of making one of the coefficients negative. In the second section of the tutorial, students are given the position-space wave function corresponding to the system from the first section. They are asked sequence of questions about position probabilities similar to the questions in the first section. Then, they are asked to identify the similarities and differences between the two representations they have seen.

The third section of the tutorial focuses on the effect of interference on quantum probabilities. Students are given three wave functions that only differ by the phase of one term. They are asked to predict whether the graphs of these three wave functions and their associated

probability densities will be the same as or different from each other. After they have discussed their predictions with fellow students and instructors, they are shown the graphs, which are all distinct from each other. Many students predict that they will be the same. They are then given a chance to revisit their predictions and to recognize the important role that interference places in quantum mechanics.

The final section of the tutorial focuses on changes of basis. Students are given a formula for a new position-space wave function, along with a graph, and asked to predict the relative size of several energy probabilities. They are given several examples of incorrect reasoning that are based on common incorrect answers to pretest questions. Students are asked to identify any errors in these statements. Students are then guided to determine a method for evaluating the probabilities, similar to the method for evaluating inner products that students considered in the prior tutorial *Treating functions as vectors*. Lastly, they are given the values for each inner product and asked to reconcile these values with their previous answers.

The first homework problem extends the tutorial by asking students to consider momentum space. Students are given a momentum-space wave function and asked several questions about momentum probabilities. They are also asked to determine expressions for the energy and position representations based on the momentum-space wave function. The second homework problem focuses on interference and inner products. Students are given the graph of a wave function that is entirely imaginary, and asked to determine several inner products and the associated probabilities. They are then given two additional wave functions in which the phase of part of the wave function has been changed, and asked to compare the probabilities. The goal of this exercise is for students to consider a case of interference distinct from what is in the

tutorial, and also to practice their method for finding inner products from wave functions represented graphically.

4. Tutorial curriculum: *Probability amplitude*

Interactions with students over the course of many years have indicated that students often do not consider the wave function as a probability amplitude. We consider the interpretation of a wave function as a probability amplitude to be an essential part of the model for quantum mechanics that the tutorials are helping students to construct. The original version of the tutorial *Treating functions as vectors* attempted to help students regard the wave function as a probability amplitude. However, the modified version of that tutorial was too long for this topic to be included. Additionally, some post-test results (see section 8.2B) suggested that students would benefit from the opportunity to explore probability amplitude in the context of energy and position in further detail. Thus, we developed a new tutorial to help students combine their understanding of concepts such as inner products, probability amplitudes, probabilities, and changes of basis into a coherent model.

The first part of the tutorial asks students to write a general expression for the state of a particle in the infinite square well. Students are then asked to determine the relationship between the probability of an energy measurement and a corresponding inner product (*e.g.*, $P(E_n) = \left|\langle \psi_n | \psi \rangle\right|^2$). Students are told that the inner product in this relationship is an example of a probability amplitude. They are asked to determine the impact of changing the sign of this probability amplitude on the corresponding probability.

In the second section of the tutorial, students are given a position-space wave function and asked to use it to determine a position probability. They are then told that the wave function can be written as an inner product: $\psi(x) = \langle x|\psi\rangle$. Students are then asked if the term probability amplitude can be used to describe this inner product. They are then directed to compare the relationship between probability amplitude and probability for the cases of energy and position. They are expected to recognize that the procedure for determining a probability is essentially the same: first, determine the probability amplitude (*i.e.*, the inner product); second, square the probability amplitude. Lastly, students are asked to write a general expression for the state in terms of the eigenstates of *position*. The goal of this exercise is to help students recognize how writing a state in terms of position basis states is similar to writing it in terms of the basis states for energy (or for any other operator).

On the homework, students are introduced to the projection operator, $|\psi_n\rangle\langle\psi_n|$, and are asked to describe its effect on a general quantum state. They then consider the identity operator written as a sum of projection operators over all values of *n*. They repeat this exercise using the basis states for position, momentum, and for a general operator with either discrete or continuous eigenvalues. The goal of this question is to help students build a model for changing basis in quantum mechanics: that is, helping them to recognize that a quantum state can be rewritten in a new basis by determining inner products between the state and the basis states.

A second problem on the homework revisits the idea of interference introduced in the tutorial *Representations of wave functions*. Students are given a wave function that looks like the first excited state of the infinite square well, but with the sign of the wave function reversed on the

right half of the well. Students are asked to predict position and energy probabilities, and are then guided through a process for calculating them. An optional exercise asks students to repeat this procedure for momentum.

B. Preliminary Assessment of Tutorial Curriculum for Vector Spaces

In this section, we provide a preliminary assessment of the last three tutorials described in the previous section. We focus here on one task (Task 8.1) that has been given to three classes after varying amounts of tutorial instruction. All results in this section are from administrations of Task 8.1 in the junior-level quantum course at UW. This section does not attempt to assess the impact of the tutorial *Dirac notation*. The reason for this is that the implementation of this tutorial in the sophomore-level course has differed substantially from the implementation of the sequence of tutorials given to the juniors. *Dirac notation* has been administered as an interactive tutorial-lecture, and has tended to be integrated with lecture material on this topic. This has made it impossible for us to ask true "pretest" questions that are given both before the tutorial and after lecture instruction.

Populations: The sequence of tutorial curriculum given at the junior level has been continuously modified over the last several years. Thus, both the number of tutorials administered and the exact format of the tutorials has varied substantially. Table 8.1 shows which tutorials in the sequence were administered during three separate academic quarters.

Class α consists of students who had only the first tutorial (*Treating functions as vectors*); students in class β completed this tutorial and then completed *Representations of wave functions*; and students in class γ completed all three tutorials, including *Probability amplitude*. The

tutorial used in class α consisted of an interactive tutorial-lecture, while in all other classes the tutorials were run in small-group sections. Furthermore, the tutorial *Treating functions as vectors* was modified substantially after it was administered to class β. Thus, the students in class γ not only completed more tutorials, but also completed tutorials that had undergone an additional iteration of revision based on the results of much of the research described in chapters 6 and 7 along with the in-class experience of the tutorial instructors.

Table 8.1: Tutorial curriculum for vector spaces

The specific set of tutorials run in the junior-level quantum course at UW during three academic quarters is shown.

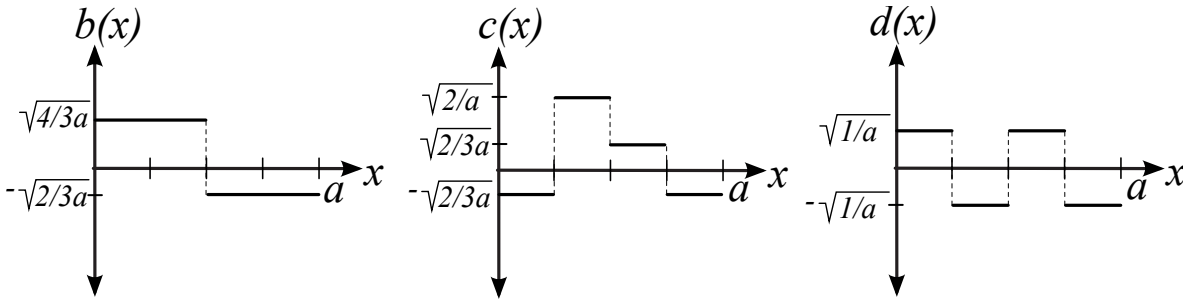
Class	List of Tutorials
Class α	*Treating functions as vectors*
Class β	*Treating functions as vectors* *Representations of wave functions*
Class γ	*Treating functions as vectors* *Representations of wave functions* *Probability amplitude*

Task 8.1: Task 8.1 (shown in Fig. 8.1) was designed because we are interested in students' ability (1) to associate a probability with the square of an inner product, and (2) to calculate inner products for graphically represented functions. This task is similar to a sequence of questions given on a tutorial pretest (see Task 7.4 in section 7.1D). In the pretest, students were shown graphs of several functions and asked several questions about various inner products between the functions. The pretest did not ask students to relate the inner products to probabilities.

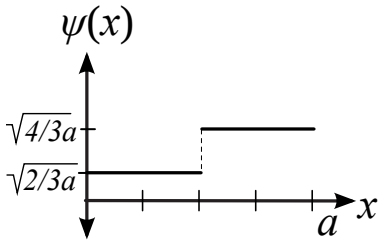
Task 8.1 shows students three general wave functions—$b(x)$, $c(x)$, and $d(x)$—along with a fourth wave function, $\psi(x)$, representing an energy eigenfunction for this potential with energy E. They are asked to determine which of the three general wave functions is most (question 8.1a) and least (question 8.1b) likely to be measured to have energy E, and to explain their reasoning.

Task 8.1

The three functions shown below each represent the wave function for a particle in an unknown potential well. The wave functions are zero outside of the region shown. Each function is drawn to the same scale.



The wave function below represents an energy eigenstate of the system with energy eigenvalue E. It is zero outside the region shown.



Question 8.1a

If the energy of each particle were measured, which of the three particles above is most likely to be found to have energy E? Explain your reasoning.

Question 8.1b

If the energy of each particle were measured, which of the three particles above is least likely to be found to have energy E? Explain your reasoning.

Figure 8.1: A pair of questions on inner products and probabilities

Task 8.1 was administered online after various amounts of tutorial instruction on vector spaces.

Task 8.1 may be answered by recognizing that the probability of measuring the given energy is equal to the complex square of the inner product between the particle's wave function and the given energy eigenfunction. The inner product between $b(x)$ and $\psi(x)$ is zero, since the area under the curve for the product of the functions is positive on the left half and negative (with the same absolute value) on the right half. Similarly, the inner product between $d(x)$ and $\psi(x)$ is zero, since $d(x)$ is an antisymmetric function with respect to the center of each half of the well, while $\psi(x)$ is symmetric over the same intervals. Thus, each of these particles is least likely to be found with energy E. The inner product between $c(x)$ and $\psi(x)$ is nonzero, so the associated probability is the greatest of $b(x)$, $c(x)$, and $d(x)$.

It should be noted that the shapes of the wave functions $b(x)$, $c(x)$, and $d(x)$ were chosen such that students using the most common incorrect lines of reasoning would give incorrect answers to the task as a whole. In particular, students often answered the pretest (Task 7.4) by relating inner products to similarities between the "shapes" of the wave functions, often with no reference to any mathematical definition for the inner product. Function $b(x)$ was chosen to have a shape that is very similar to $\psi(x)$, even though the inner product between these functions is zero. Function $c(x)$ was chosen to have a nonzero inner product with $\psi(x)$ despite the fact that they do not have similar shapes, areas, or values. Lastly, function $d(x)$ was chosen so that students would need to think about all four regions marked along the x-axis to determine that the inner product with $\psi(x)$ is zero.

Results: The results from questions 8.1a and 8.1b for each of the three classes (α, β, and γ) are shown in Table 8.2. The first two columns in the table show the percentage of the students who chose the correct answer on each question. The last column indicates the students who

answered both questions correctly, and the percentage in parentheses includes only students whose explanations on the entire task were judged to be fully correct. The results from the pretest (Task 7.4) are not shown because the pretest only asked students about inner products, but did not ask about probabilities.

Table 8.2: Results from Task 8.1

The percentage of the students who answered each question correctly is shown. The results are shown for each of the three classes described above in Table 8.1. The last column shows the percentage of the students who answered both questions on Task 8.1 correctly (students who gave correct reasoning on Task 8.1 as a whole are shown in parentheses in the last column).

Class	Question 8.1a	Question 8.1b	Task 8.1 as a whole (Correct reasoning)
Class α (N=58)	17%	22%	10% (9%)
Class β (N=66)	26%	17%	12% (8%)
Class γ (N=51)	55%	37%	33% (27%)

In general, student performance on Task 8.1 is poor. At most one-quarter of the students from classes α and β answered either question 8.1a or question 8.1b correctly. Less than 10% of each class answered the entirety of Task 8.1 correctly with correct reasoning. Class γ, which completed all three tutorials in the sequence, performed somewhat better. However, only 27% of class γ gave correct answers with correct reasoning for the whole task. Although the difference between the performance of class γ and classes α and β is statistically significant ($p<0.05$), it is clear that there is still substantial room for this sequence of tutorials to be improved.

In order to assess the impact of the tutorials in greater detail, we examine the explanations given by students who answered incorrectly. Table 8.3 shows the percentage of the students from each class who used various lines of reasoning that were incomplete or incorrect. The table

includes two sections: the columns on the left correspond to lines of reasoning that have correct elements, while the columns on the right correspond to lines of reasoning that are entirely incorrect.

Table 8.3: Additional results from Task 8.1

The percentage of the students who answered Task 8.1 incorrectly is shown. The results are shown for each of the three classes described above in Table 8.1. Note that some students may fall into more than one category. The columns on the left indicate lines of reasoning that have correct elements, while the columns on the right are entirely incorrect. Individual lines of reasoning that were relatively rare (given by less than 5% of the students in each class) are not shown.

Class	Lines of reasoning with correct elements		Incorrect lines of reasoning				
	Incorrect answer w/ correct expl.	Superposition	Similarity of "shape"	Comparison of "squares"	Overlap	Not enough information	No expl.
Class α (N=58)	0%	14%	35%	17%	5%	5%	24%
Class β (N=66)	8%	5%	44%	5%	12%	12%	14%
Class γ (N=51)	27%	4%	31%	0%	4%	2%	4%

The most obvious difference in the table between class γ and classes α and β is in the first column. Students were only included in this category if they gave a correct explanation that made reference to an inner product or an integral over all space, but then answered one or both of the questions on Task 8.1 incorrectly. Thus, this column represents students who recognized that the energy probabilities could be determined by appropriate inner products, but who were not able to calculate all of those inner products correctly given the graphs of each function.

Students were included in the second column if they described expanding each wave function in terms of energy eigenfunctions as part of their answer. Although these students did not explicitly describe inner products as a means for determining this expansion, they were still

describing elements of the correct procedure. However, students in this category often went on to use entirely incorrect lines of reasoning as their basis for determining how to expand each wave function.

The most common lines of reasoning that are entirely incorrect were those in which students based their answers on some amount of general similarity between the wave functions. Often, students whose responses were placed in this category stated that the probability was largest for the wave function whose "shape" was most similar to $\psi(x)$. Many of these students did not explicitly describe what they meant by the word "shape". Others noted specific features of the wave function that were similar, including the absolute values of the functions in each region and the number of "nodes" or "oscillations" that each wave function undergoes. Some students explicitly stated that they should square each wave function before making this kind of comparison (they are included in a separate column in the table). Others stated that the probability is related to the "overlap" between the functions (also in a separate column), without giving a definition for this term. Responses that relied on a general similarity between the shapes of different wave functions were also very common on pretest questions about inner products given before any tutorial instruction on this topic (Task 7.4; see section 7.1D).

Some students stated that there was not enough information to determine the probabilities in Task 7.3. A few specifically noted that the functions were not represented in the correct basis to determine energy probabilities. This is a different kind of error than students basing their answer on the general similarities between wave functions. Similar student responses have been identified on the pretest discussed in section 7.2B.

C. Discussion and Future Research

The results from Task 8.1 suggest that inner products and changes of basis can be very challenging for students, and that they persist. Even after targeted tutorial instruction students continued to have difficulty with these ideas. In particular, early versions of the tutorials (administered to classes α and β) do not seem to have been very effective. When the most recent sequence of tutorials, which was revised based on the results from classes α and β, was administered to class γ, the results were promising. However, there is substantial room for the sequence of tutorials for vector spaces in quantum mechanics (described in section 8.2A) to be improved. In particular, more than half of the students from class γ failed to recognize that inner products are necessary to determine probabilities. One-quarter of that class correctly related the probabilities to inner products, but then calculated those inner products incorrectly. Additional modifications to the tutorials are essential to address these two aspects of student understanding.

It should be noted that Task 8.1 only explores student understanding of inner products and changes of basis in one particular context and for one particular course. The sequence of tutorials described in section 8.2A also addresses other important topics. Superposition of wave functions, which is discussed in section 6.3, is one of these topics. Further assessment of the tutorials involving questions with different focuses is therefore crucial. Additionally, we have not yet performed a formal assessment of the tutorial *Dirac notation*, which is intended for students in a spins-first, sophomore-level introduction to quantum mechanics. Future research on the topics introduced in this tutorial could provide further insight into student understanding

of vector spaces by allowing comparisons of student difficulties across the sophomore and junior levels.

8.2 – Tutorial Curriculum for Time-independent Perturbation Theory

In this section, we describe the development and assessment of tutorial curriculum for an advanced topic in quantum mechanics: time-independent perturbation theory. This topic requires students to apply many of the fundamental concepts that they have learned throughout a quantum mechanics course. At UW, perturbation theory is taught in the second quarter of the junior-level quantum course.[10] Since much of perturbation theory involves the use of Dirac notation and integrals of products of functions, research on student understanding of inner products is particularly relevant for the design and evaluation of curriculum aimed at improving instruction of this difficult subject.

In section 8.3A, we describe tutorial curriculum for both degenerate and non-degenerate perturbation theory.[1] These tutorials were initially developed by Andrew Crouse but have been substantially modified in recent years based on the results presented in section 7.3 of this dissertation. Section 8.3B provides a preliminary assessment of the modified tutorials.

A. Description of Tutorial Curriculum for Time-independent Perturbation Theory

The University of Washington has been developing tutorial curriculum for perturbation theory for a period of many years.[11] The PhD dissertation of Andrew Crouse includes a copy of the original version of the tutorial worksheets, but he does not explicitly discuss their development.[2] Since that time, the tutorials have undergone substantial modifications based on

the results of research at UW (see section 7.3). Below, we describe the structure of the most recent version of each tutorial, along with any major modifications that have been made, and the research basis for those modifications.

1. Tutorial curriculum: *Time-independent Perturbation Theory*

The tutorial for non-degenerate, time-independent perturbation theory focuses on helping students to build a functional and conceptual understanding of the first-order corrections to the energy for various simple perturbations to the potential. It does so by directly addressing two common student difficulties (see section 7.3A). The first difficulty is a failure to use integrals to determine first-order corrections to the energy. The second difficulty is a tendency to include only a single factor of the wave function (rather than two factors of the wave function) in the integral described above. The tutorial addresses each of these difficulties by guiding students to use and explain each part of the integral expression for the formula of the first-order correction to the energy. A secondary goal of the tutorial is to help students better understand the first-order corrections to the wave function.

The worksheet begins by asking students to sketch the wave functions for the ground state and for the first excited state of the unperturbed infinite square well. Although tutorials for other topics have typically utilized a square well defined between $x = 0$ and a, this tutorial asks the students to consider the square well defined between $x = -a$ and a. This decision was made to take advantage of symmetry about the origin. The students are also shown the graph of a small perturbation to the infinite square well. The original version of this tutorial used a delta-function potential: the perturbation was changed to have non-infinitesimal width because we observed

that some students had difficulty with the delta function potential that seemed to get in the way of their understanding of the goals of this tutorial.

The students are then asked to compare the signs and the magnitudes of the first-order correction to the energy of the two lowest-energy eigenstates. They are given an expression in Dirac notation for the correction. Students are expected to recognize that they can determine which correction is larger by comparing the values of the two wave functions in the region where the perturbation is nonzero. The tutorials also confront the common student belief that the correction is negative if the wave function is negative in the region where the perturbation is positive (see section 7.5A2). Students then consider several different perturbations to the potential (one symmetric and two antisymmetric) and identify patterns in the first-order corrections to the energy.

The remainder of the tutorial focuses on the first-order correction to the wave function. For this exercise, they consider a step-function potential that is slightly larger than zero on the left and slightly smaller than zero on the right. Students are first asked to predict the overall shape of the ground-state wave function using the general properties of the time-independent Schrödinger equation (*e.g.*, by noting that the wavelength should be larger when the potential is larger). Then, students are given an expression for the first-order correction to the wave function and are guided to use it to determine the ground-state wave function of the perturbed potential. Lastly, a handout is provided for students to check their answer and their initial prediction. We have found that the sequence of questions on the first-order correction to the wave function is challenging for students.

A short homework question (based on exercises that were removed from the original version of the tutorial), asks students to make sketches of potential wells whose corrections to the energy satisfy various conditions (*e.g.,* for which the first-order correction to the energy is zero for all energy levels). The remainder of the homework consists of an exercise in which students determine the first-order correction to the wave function for the first excited state of the potential used throughout the last section of the tutorial.

2. Tutorial curriculum: *Degenerate Perturbation Theory*

We have observed that degenerate perturbation theory is very challenging for students in the junior-level class. The original version of the tutorial required students to use several sophisticated mathematical concepts: integrals, symmetry, eigenvectors, and commutators. This early version of the tutorial had very few intermediate questions linking one idea to the next, and provided little guidance for students to connect the ideas to physics concepts. Based on experience in the classroom, we modified the tutorial to deemphasize much of the mathematics in favor of the concepts.

The tutorial begins by asking students to recall the eigenvalues and eigenfunctions for the two-dimensional isotropic harmonic oscillator, which they also considered in the tutorial *Degenerate States* (see section 5.2A). In particular, students are asked if either of the first two energy levels are degenerate, and to determine the degeneracy. Students also use perturbation theory to determine the first-order correction to the energy of the ground state, which is not degenerate.

Students are then guided through a process for determining the first-order corrections to the energy of the (doubly-degenerate) first excited state. The tutorial asks students how many terms must be calculated to find these corrections, and to arrange these terms in a matrix. They are specifically asked why a matrix is the correct mathematical object in this case. Students are then given the value of each term and asked to find the eigenvalues and eigenvectors of the resulting two-by-two matrix. In the original version of this tutorial, students were asked to calculate the terms, but we found that this exercise was time-consuming and did not contribute substantially to the students' conceptual understanding of degenerate perturbation theory.

Students are then asked to relate the eigenvalues and eigenvectors that they calculated to the first-order corrections to the first excited state. This step is particularly challenging for students, who often do not recognize that each eigenvalue represents a *different* correction that can only be associated with one particular linear combination of the unperturbed energy eigenstates. Students are then asked to read several incorrect statements and to identify the errors. This is intended to help them recognize and avoid common errors.

The homework for this topic contains many of the more mathematical exercises that had been part of the original version of the tutorial. First, students are asked to calculate all of the terms that they used throughout the tutorial. These require students to evaluate two-dimensional integrals. Then, students solve the perturbed potential from the tutorial exactly, and compare the result to the approximate values from perturbation theory. Lastly, students consider how operators and commutators could be used to determine the first-order corrections to the energy and the associated "good" eigenstates without determining eigenvectors and eigenvalues as they did in the tutorial.

3. Tutorial curriculum: *Prelude to Perturbation Theory*

During the course of developing the tutorials described in the previous two sections (and the tutorials on time-dependent perturbation theory discussed in section 5.2B), we found that some students struggled with certain elements of perturbation theory that went beyond calculating corrections to the energy or the wave function. In particular, students often failed to recognize that the unperturbed energy eigenvalues are in general no longer possible results of a measurement of the energy of the perturbed system. We developed additional curriculum, first as supplementary homework and then as an in-class tutorial, to address ideas related to the different bases used in both time-independent and time-dependent perturbation theory. Although we do not regard this topic as critical for students to consider, we have found that students who complete this tutorial are more prepared for future tutorials on perturbation theory.

This tutorial entitled *Prelude to perturbation theory* begins by asking students why it is possible to write a general wave function as a superposition of the system's energy eigenfunctions. The goal of this question is to get them to recognize that the energy eigenfunctions form a complete set of functions in terms of which other functions can be written. They are also asked to write a mathematical expression for this superposition, including how the coefficients may be determined. A perturbation is then added to the system, and students are asked the same questions regarding the *perturbed* energy eigenfunctions. They are asked whether the coefficients in this superposition are different, and whether the coefficients in the original superposition would be different (at the same instant in time) due to the added perturbation. As a check on their reasoning, students are then told that the particle is known to

be in the ground state of the perturbed Hamiltonian, and asked about the possible results of a measurement of the energy. Lastly, students are given an incorrect statement. This statement associates the probability of measuring the energy of the *unperturbed* ground state with the square of the corresponding coefficient in one of the superpositions they have written down. Students are asked to determine the flaws in the reasoning included in the incorrect statement.

The homework for this tutorial gives students a wave function corresponding to an energy eigenfunction of the infinite square well. They are then told that the width of the well is changed and asked a sequence of questions about the wave function. This affords students an opportunity to explore the ideas introduced in the tutorial, and also to practice changing from one quantum basis to another.

B. Preliminary Assessment of Tutorial Curriculum for Time-independent Perturbation Theory

The three tutorials described in the previous section are all intended to improve student understanding of perturbation theory. Each focuses on very different aspects of this difficult and advanced topic. In this section, we assess the two tutorials for non-degenerate perturbation theory separate from the single tutorial for degenerate perturbation theory. In each case, we have analyzed the responses to long-answer questions administered on midterm exams, and compared the results to the results from similar questions asked on tutorial pretests (see section 7.3). Section 8.3B1 focuses on questions about non-degenerate perturbation theory asked at the University of Washington. Section 8.3B2 discusses the results from a pilot site for the tutorials

on the same questions. Lastly, section 8.3B3 describes the results from a question on degenerate perturbation theory given at UW.

1. Assessment for non-degenerate perturbation theory

In order to assess the tutorials that focus on non-degenerate perturbation theory, we have designed a variety of questions that focus on (1) determining the sign of first-order corrections to the energy for simple perturbations and (2) comparing the magnitudes of these corrections. In this section, we present the results from two such questions asked on a midterm exam in the second quarter of a junior-level quantum course. Although we have occasionally asked about more advanced applications of what is done in the tutorial—namely, about the first-order correction to the wave function or about the second-order correction to the energy—the questions have not been refined sufficiently or administered often enough for us to use them in assessing the curriculum. Figure 8.2 shows a pair of questions (Task 8.2) asked together on the same exam during three separate classes.

Table 8.4: Tutorial curriculum for perturbation theory

The specific set of tutorials run in the junior-level quantum course at UW during three academic quarters is shown.

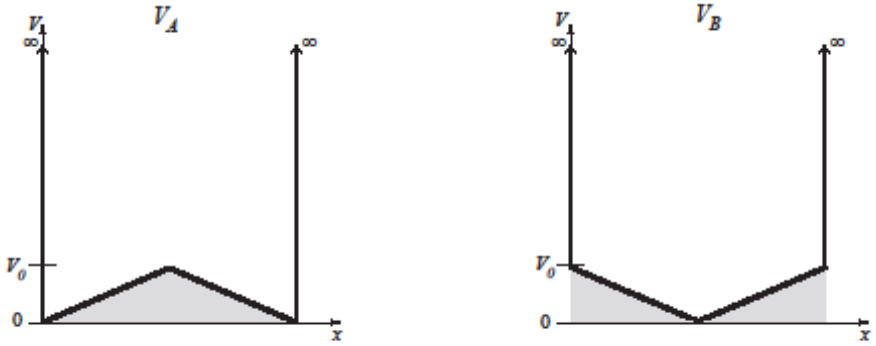
Class	List of Tutorials
Class α	*Time-independent perturbation theory* (preliminary)
Class β	*Prelude to perturbation theory* (preliminary, HW) *Time-independent perturbation theory*
Class γ	*Prelude to perturbation theory* *Time-independent perturbation theory*

Task 8.2 has been asked of three different classes, described in Table 8.4. In class α, the students had completed a preliminary version of the tutorial *Time-independent perturbation theory* prior to the exam. In class β and class γ the students had completed the final version of the tutorial, as discussed section 8.3A1. Students in class β had also been given a preliminary version of *Prelude to perturbation theory* as part of their tutorial homework, and class γ had completed the final version of this curriculum as an in-class tutorial (see section 8.3A3).

Question 8.2a asks the students to compare the first-order correction to the energy of the ground state and to the energy of the first excited state for a symmetric perturbation that is peaked near the center of the infinite square well (V_A). Question 8.2b asks the students to compare the first-order correction to the energy of the ground state for two different potentials: V_A and a similar symmetric perturbation that is peaked near the edges of the infinite square well, V_B. We have modified the graphs of V_A and V_B somewhat in later administrations because it seemed that a small number of students misinterpreted the representation. For example, some students believed that V_0 represents the level of the unperturbed potential, and thus that the perturbations were negative. When students who made these errors gave reasoning that was otherwise correct, they were counted as having given correct answers with correct explanations. The final versions of the graphs are shown in Fig. 8.2. A few other changes were made to improve the clarity of the questions, and additional questions about higher-order perturbation theory were included (not shown).

Task 8.2

In this question you will consider two different potentials that can be considered perturbations to the infinite square well. In each diagram, the dark black line represents the potential and the shaded portion indicates the perturbation to the infinite square well. The value V_0 is the same in each diagram.



Question 8.2a

For the potential V_A, is the first-order correction to the energy for the ground state (ψ_1) *greater than, less than,* or *equal to* the first-order correction to the energy for the first excited state (ψ_2)? Explain your reasoning for full credit.

Question 8.2b

Consider a different small perturbation (V_B) to the infinite square well (shown above). For the ground state, is the first-order correction to the energy for potential V_B *greater than, less than,* or *equal to* the first-order correction to the energy for potential V_A? Explain your reasoning for full credit.

Figure 8.2: Two exam questions on perturbation theory

Task 8.2 was administered to students on a midterm exam in the second quarter of a quantum mechanics course for juniors. These questions were asked in three separate years; in each year, the description of the perturbations changed in order to improve readability, but they were essentially the same.

Question 8.2a can be answered by noting that the correction to the energy is given by the integral of the product of the perturbed potential and the probability density. This integral is larger for the ground state because the wave function for the ground state is larger than the wave function for the first excited state in the region where the perturbation is also large. It should be noted that the expression for the first-order correction to the energy was not given to the students on the exam. Question 8.2b can be answered in the same way: V_B is smaller than V_A in the

region where the ground-state wave function is largest, so the first-order correction to the energy will also be smaller. The results from each administration of Task 8.2 are shown in Table 8.5.

Table 8.5: results from Task 8.2

The percentage of the students who gave each answer on Task 8.2 is shown. The correct answers are bolded. The percentages in parentheses include only students who also gave correct reasoning.

Question	Answer (N)	Class α (N=50)	Class β (N=58)	Class γ (N=61)	All classes (N=169)
Question 8.2a $E_1^{(1)}$ for V_A *vs.* $E_2^{(1)}$ for V_A	**Greater**	**70% (26%)**	**83% (64%)**	**87% (56%)**	**80% (50%)**
	Less	18%	12%	5%	11%
	Equal	10%	5%	7%	7%
Question 8.2b $E_1^{(1)}$ for V_B *vs.* $E_1^{(1)}$ for V_A	Greater	12%	16%	5%	11%
	Less	**56% (34%)**	**59% (57%)**	**85% (64%)**	**67% (53%)**
	Equal	30%	26%	7%	20%
Questions 8.2a and 8.2b	**Correct on both**	**50% (20%)**	**53% (50%)**	**80% (48%)**	**62% (40%)**

A majority of students gave the correct answer to each question, with a greater percentage of the students answering correctly for question 8.2b than question 8.2a. Almost all of the students who answered question 8.2a correctly had also given the correct answer for question 8.2b. The percentage of the students who gave the correct reasoning along with the correct answer is also indicated in the table. Students who gave an incorrect answer because they misinterpreted question 8.2b, but who still gave fully correct reasoning, are also included in this percentage (this is only about 5% of the students from each quarter).

These results suggest that the tutorial curriculum can be effective at improving student ability to determine first-order corrections to the energy graphically and qualitatively. On the pretest, which was given after lecture instruction on perturbation theory, only 24-29% of the students

were able to give correct answers with correct reasoning to questions that asked them to compare the magnitudes of the corrections for different states (see section 7.3A). Even fewer students gave correct explanations on the pretest when they were asked to determine the sign of these corrections. On the post-test questions, however, substantially more students answered correctly. This is especially true in class β and class γ, in which the students completed the final version of the tutorial *Time-independent perturbation theory*. In these two classes, 56-64% of the students were able to answer each post-test question correctly with correct reasoning, with about half the students giving correct reasoning for both questions. In addition, few students from class β or class γ gave an incorrect answer, and none of the common incorrect answers discussed in section 7.3A were prevalent in the explanations given to the post-test questions.

2. Assessment for non-degenerate perturbation theory at a pilot site

The tutorial *Time-independent perturbation theory* has also been administered at a pilot site for *Tutorials in Physics: Quantum Mechanics*.[11] The materials were used in a junior-level quantum mechanics course for physics majors at the University of St. Andrews in 2015. The students had already completed lecture material and homework assignments on the subject of perturbation theory prior to the administration of any tutorial material. Several of the pretest questions for this topic (see section 8.3A) were given to the students as in-class clicker questions, with no discussion between students allowed. On each pretest question (N=63), the correct answer was given by 16-52% of the students. This range is the same as or less than the performance of the students at UW on the same questions. Although the clicker question format does not allow us to examine the reasoning of the students at the pilot site, the fact that their

performance on pre-tutorial, post-lecture questions was similar to the results at UW suggests that many of them seem to be struggling with perturbation theory.

The students at the pilot site completed the tutorial in a lecture-hall setting. They worked in groups of 2-4, with only one instructor in the room. Most of the instructor's time was spent helping individual groups, with some time devoted to class discussions. In the 50-minute session, most students completed the first three pages of the tutorial (which focus on the first-order correction to the energy). Although the remaining pages of the tutorial were made available to the students, almost none completed it in class, and the answers to these pages were not discussed.

Task 8.2 (see Fig. 8.2 in the previous section) was administered to the students at the pilot site as a pair of clicker questions at the start of the lecture that followed the completion of the tutorial. As for the pretest, the students were not permitted to discuss the questions with each other, and answers were not discussed until after the students had completed both questions. A total of 62% of the students (N=63) gave the correct answer to each question, with 44% answering both questions correctly. Only about 15% of the students answered all of the corresponding pretest questions correctly, so there appears to be substantial improvement after the tutorial.

It is difficult to compare the performance of the students at the pilot site to the performance of students at UW, since the questions were given in different formats. However, in each case there was improvement from before the tutorial (but after lecture instruction) to after the tutorial. This suggests that the tutorial for this topic can be effective when modified for use in a lecture-hall setting, and that the results are generalizable beyond the University of Washington.

3. Assessment for degenerate perturbation theory

We have found degenerate perturbation theory to be one of the most difficult quantum topics for which we have attempted to design curriculum. It is abstract and mathematical, and students often do not see the purpose or the applicability of the procedure that is used. We have also found it to be a challenging topic to assess. The focus of the tutorial for degenerate perturbation theory (see section 7.3B) is on helping students recognize the *process* used to determine corrections to the energy when there are degenerate energies. Figure 8.3 shows a pair of questions (Task 8.3) that we developed to attempt to assess student understanding of this process. This task was given to 58 students on a midterm exam during the second quarter of a junior-level quantum mechanics course, after both lecture and tutorial instruction on degenerate perturbation theory. These students had also completed the tutorial *Time-independent perturbation theory*, which focuses on the non-degenerate case (see section 8.3A1).

In question 8.3a, students are asked how to calculate the first-order corrections to the energy for the first excited state of the two-dimensional infinite square well. A different question on the same exam (not shown) asked them to determine the degeneracy of this state. About 80% of the students correctly identified that the state is doubly degenerate. Question 8.3b then asks students how to determine the unperturbed states corresponding to these corrections. In neither question are students asked to determine values for the corrections—instead, the focus is on having students articulate the process that would be necessary to determine these values. It should be noted that the pretest used for degenerate perturbation theory (see Task 7.9 in section 7.3B) is

relatively different from Task 8.3. As a result of this difference, we do not directly compare the pre- and post-test results for this topic.

Task 8.3

Consider a particle in a two-dimensional infinite square well for which the potential is infinite except from $x = 0$ to $x = a$ and from $y = 0$ to $y = a$. the energy eigenstates and eigenvalues of the one-dimensional system are given below, where E_1 is the ground-state energy of the one-dimensional system

$$\varphi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) \qquad E_n = n^2 E_1 \qquad n = 1, 2, 3...$$

A perturbation given by the form $V'(x) = \frac{a^2 E_1}{10} \delta(x - 2a/3)\delta(y - a/3)$ is added to this system.

Question 8.3a

Explain how you would calculate the first-order correction(s) to the energy for the **first excited-state** of this system. For full credit, you must fully explain each step but you do not need to perform the calculation.

Question 8.3b

Explain how you would determine the **unperturbed** states that correspond to the first-order correction(s) from the previous question.

Figure 8.3: Two exam questions on degenerate perturbation theory

Task 8.3 was administered to students on a midterm exam in the second quarter of a quantum mechanics course for juniors.

The correct answer to question 8.3a is that the first-order corrections to the energy of the first excited state are given by the eigenvalues of the following matrix:

$$\begin{pmatrix} \langle \psi_{12}^{(0)} | H' | \psi_{12}^{(0)} \rangle & \langle \psi_{12}^{(0)} | H' | \psi_{21}^{(0)} \rangle \\ \langle \psi_{21}^{(0)} | H' | \psi_{12}^{(0)} \rangle & \langle \psi_{21}^{(0)} | H' | \psi_{21}^{(0)} \rangle \end{pmatrix}$$

More than half the students (56%) answered this question correctly. Some of these students explicitly described the process they would use, while others made reference to the "W-matrix", the term used by the course textbook for the matrix above. The only common error made by

students (19%) was to use non-degenerate perturbation theory to answer the question. There were no other common incorrect patterns of reasoning.

The correct answer to question 8.3b is that the eigenvectors of the matrix above represent the unperturbed states associated with the first-order corrections to the energy. A correct answer referencing these eigenvectors was given by 42% of the students. Again, the most common incorrect answer (given by 16% of the students) was to use non-degenerate perturbation theory. A large percentage of the students gave confusing answers (12%) or essentially left the question blank (16%).

The results from this question suggest that after the set of three tutorials, about half of the students are able to describe at least part of the process for determining first-order corrections to the energy using degenerate perturbation theory. However, many students seemed to struggle with Task 8.3, even after they had worked through the tutorial on degenerate perturbation theory. This suggests that there is still room for the tutorial to be improved. In particular, improving the questions that are intended to help students to recognize a *need* for degenerate perturbation theory could prove fruitful, especially if students can be guided to make connections between this need and their final answers. In addition, future questions could be developed to provide a better assessment of the ideas covered in the tutorial.

8.3 – Summary

In this chapter, we discussed results from a preliminary effort to develop and assess tutorial curriculum for topics related to vector spaces in quantum mechanics. This effort has been part of a larger investigation on student understanding of vector spaces in quantum mechanics discussed

throughout part II of this dissertation (chapters 6, 7, and 8). The first two chapters each described the identification of student difficulties with different aspects of vector spaces. In chapter 6, we focused on student difficulties with superposition in three different contexts: vectors, waves, and wave functions. In chapter 7, we discussed student difficulties with inner products, changes of basis, and perturbation theory. Together, the findings from these two chapters have indicated that students often have difficulty adding vectors and finding inner products, in both quantum and non-quantum contexts.

The development of the tutorials chronicled in this chapter (chapter 8) has focused on two different areas: (1) tutorials intended to be used near the beginning of a quantum mechanics course to improve student understanding of mathematical topics such as inner products (section 8.2) and changes of basis (section 8.3), and (2) tutorials for time-independent perturbation theory, a more advanced topic that requires the use of mathematical tools like inner products. These two sequences of tutorials have gone through an iterative process of research, testing, and revision over the course of several years. While some of the results are promising, additional cycles of research on student understanding, including further development, modification, and assessment of curriculum, are necessary.

One of the principal findings of part II has been that students often fail to recognize when inner products are necessary to solve a quantum problem. We have seen evidence of this difficulty in a variety of contexts, but it has been especially common in response to questions that require a change of basis (see section 7.2). Even after targeted tutorial instruction (see section 8.2B), as many as half of junior-level students fail to use inner products to determine probabilities when a change of basis is required.

We have also found that some students have difficulty *calculating* inner products, even when they recognize that inner products are necessary to solve a problem. In particular, many students do not appear to have a clear method for evaluating inner products between functions represented graphically (see section 7.1D). There is evidence from questions about perturbation theory (see section 7.3A) that students have difficulty computing graphical inner products even after more than a full quarter of junior-level quantum instruction. Tutorial curriculum has been developed to improve student ability to determine graphical inner products both in basic contexts (see section 8.2) and in the advanced context of perturbation theory (section 8.3). Results from post-test questions suggest that certain recent modifications to the tutorials may be helpful. However, there is still substantial room for improvement. Further research—especially individual student interviews—should be conducted to probe student understanding of vector spaces in more detail. Student interviews could also be used to investigate how specific exercises in the tutorials do or do not help change student thinking about vector spaces.

8.4 – Notes and References

[1] Much of the work described in this chapter was done in collaboration with Gina Passante and Tong Wan, who made invaluable contributions to both the design and the assessment of this curriculum.
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[5] A. Kohnle *et al.*, "A New Multimedia Resource for Teaching Quantum Mechanics Concepts," *American Journal of Physics* **80**, 148 (2012).
[6] G. Zhu and C. Singh, "Improving Student Understanding of Addition of Angular Momentum in Quantum Mechanics," *Physical Review Special Topics – Physics Education Research* **9**, 010101 (2013).
[7] R. Scherrer, *Quantum Mechanics: An Accessible Introduction*, Addison-Wesley (2006).
[8] D. H. McIntyre, *Quantum Mechanics: A Paradigms Approach*, Addison-Wesley (2012).
[9] C. Singh, "Student Understanding of Quantum Mechanics at the Beginning of Graduate Instruction," *American Journal of Physics* **76**, 277 (2008).
[10] In the widely used textbook by Griffiths, time-independent perturbation theory is covered in chapter 6.

[11] L. C. McDermott, P. S. Shaffer, P. R. L. Heron, and the Physics Education Group, *Tutorials in Physics: Quantum Mechanics* (in preparation).

Part III
Student Understanding of Angular Momentum in Quantum Mechanics

Angular momentum is an important but difficult topic in quantum mechanics. An understanding of angular momentum, including spin, is particularly useful for studying atomic, nuclear, and particle physics. Previous research, including this dissertation, has shown that students have difficulty with many concepts—such as time dependence (chapter 2), measurement (chapters 3 and 4), and changes of basis (chapter 7)—that are necessary for the study of angular momentum. However, there has been almost no published research on what students do or do not understand about this fundamental topic.

In this part of this dissertation, we present results from an investigation into student understanding of angular momentum. Chapter 9 examines student responses to several research tasks related to orbital, spin, and total angular momentum. Chapter 10 details the evolution and assessment of tutorial curriculum to address student difficulties with these topics. Much of the work in these chapters was inspired by the PhD work of Andrew Crouse, which includes a brief examination of student understanding of angular momentum and development of a preliminary tutorial worksheet on addition of angular momentum.

Chapter 9
Student Difficulties with Angular Momentum

"I don't know, orbital angular momentum confuses me :("

—Junior-level quantum mechanics student, pretest

There is relatively little published research on student understanding of angular momentum in quantum mechanics. In one of the earliest papers that touches on this topic, Styer notes that some students believe that the three components of orbital angular momentum are well-defined but rapidly changing in time.[1] Singh and Zhu have documented student difficulties with spin and the Stern-Gerlach experiment,[2,3] and with some mathematical aspects of the addition of angular momentum.[4,5] Andrew Crouse gave a brief description of student difficulties with both orbital and addition of angular momentum in his dissertation.[6] There has also been some discussion from of chemistry education research on student understanding of atomic orbitals;[7] however, the instructional context for these students was very different from that of the introductory quantum mechanics courses offered by a physics department.

In this chapter, we describe the results of an investigation into student understanding of angular momentum in quantum mechanics. Section 9.1 describes the methodology that we have used throughout the investigation. Section 9.2 discusses difficulties students have in determining the possible results of measurements of angular momentum. The focus is primarily on orbital angular momentum (L), with some discussion of spin angular momentum (S). Section 9.3 examines student understanding of angular momentum, probability, and uncertainty. Section 9.4

discusses student difficulties with total angular momentum (J) resulting from the addition of spin and orbital angular momentum.

9.1 – Methodology

We have designed several research tasks to probe student understanding of various aspects of angular momentum. They have been administered mostly as tutorial pretests in a junior-level quantum course at UW (recall that the pretests are typically given after relevant lecture instruction). The responses to the questions on each task were analyzed to identify common student errors by considering both the answers and the explanations given by the students. Commonalities between the various errors and the accompanying student explanations were then used to determine the underlying lines of reasoning that students seem to be using when answering conceptual questions about angular momentum. In particular, we attempted to identify lines of reasoning common in responses to a variety of different questions. As discussed in section 1.3C, we term these incorrect lines of reasoning *difficulties*.

In each section below, we first describe a sequence of questions, the correct answers and possible explanations, and the percentage of students who answered correctly (both with and without correct reasoning). Then, we discuss the common errors that students make, along with representative examples of explanations given in support of each error. Finally, we describe the most prominent student difficulties that we believe underlie many of the errors that students make. In some places, we include supporting results from individual student interviews conducted after both lecture and tutorial instruction on angular momentum.

9.2 – Student Difficulties with Measurements of Orbital Angular Momentum

Task 9.1 (see Fig. 9.1) was designed to probe student ideas about measurements of orbital angular momentum in a very simple case. On this task, students consider the wave function $\psi(r,\theta,\phi) = f(r)Y_2^1(\theta,\phi)$, where $f(r)$ is an unspecified function of r and $Y_2^1(\theta,\phi)$ is one of the spherical harmonics. They are asked to determine the possible results of measurements of: (a) the orbital angular momentum squared (L^2), (b) the z-component of the orbital angular momentum (L_z), and (c) the x-component of the orbital angular momentum (L_x).

Task 9.1

Suppose a particle in a three-dimensional quantum system has a wave function given by $\psi(\vec{r}) = f(r)Y_2^1(\theta,\varphi)$, where $f(r)$ is an arbitrary function of r and $Y_2^1(\theta,\varphi)$ is one of the spherical harmonics.

Question 9.1a

If the orbital angular momentum squared (L^2) of this particle were measured, which of the following would be possible results? Explain your reasoning. (Circle all that apply.)

0 $\hbar^2$ $2\hbar^2$ $4\hbar^2$ $6\hbar^2$ any value other (please specify)

Question 9.1b

If the z-component of this particle's orbital angular momentum were measured, which of the following would be possible results? Explain your reasoning. (Circle all that apply.)

$-2\hbar$ $-\hbar$ $-\hbar/2$ 0 $\hbar/2$ $\hbar$ $2\hbar$ any value other (please specify)

Question 9.1c

If the x-component of this particle's orbital angular momentum were measured, which of the following would be possible results? Explain your reasoning. (Circle all that apply.)

$-2\hbar$ $-\hbar$ $-\hbar/2$ 0 $\hbar/2$ $\hbar$ $2\hbar$ any value other (please specify)

Figure 9.1: Three questions on orbital angular momentum

Task 9.1 was administered on a tutorial pretest near the end of the first quarter of a junior-level quantum mechanics course. Questions 9.1a-c asked the students to determine the possible results of a measurement of L^2, L_z, and L_x, respectively, for a particle with a wave function associated with a single spherical harmonic. In each case, the students were asked to explain their reasoning.

This task was given online following lecture instruction on the hydrogen atom potential and orbital angular momentum. The multiple choice answers are based on responses to previously administered free response questions, and students were asked to explain their reasoning.

The correct answers to the first two questions can be determined by noting that the spherical harmonic given is an eigenstate of both L^2 and L_z, so there can be only one possible result for a measurement of either of these two quantities. The eigenvalue equations $\hat{L}^2 Y_l^{m_l} = \hbar^2 l(l+1) Y_l^{m_l}$ and $\hat{L}_z Y_l^{m_l} = \hbar m_l Y_l^{m_l}$ give values of $6\hbar^2$ and $+\hbar$ for L^2 and L_z, respectively. Note that we use m_l, rather than the commonly used m, in order to distinguish it from quantum numbers such as m_s and m_j.

A more detailed analysis is needed to determine the possible results of a measurement of L_x. At this level of instruction, we do not necessarily expect the students to be able to rewrite the given spherical harmonic in terms of the eigenfunctions of L_x. Instead, we expect them to identify the allowed eigenvalues of the operator, which range from $-l$ to l in integer steps, times $\hbar$. (See, for example, section 4.3 of Griffiths.)[8] For the given particle, this results in the values $-2\hbar$, $-\hbar$, 0, $+\hbar$, and $+2\hbar$, since $l = 2$. This answer was considered correct. If a student were to work out the full algebra from the matrix form of the operator for L_x, they would find that the probability for the value of zero vanishes for the given initial state.

This task has been administered near the end of the first quarter of junior-level quantum mechanics in three different quarters. (A version of question 9.1b has been asked of a slightly different population of students, with similar results.)[6] The percentage of the students from each quarter who gave the correct answer is shown in Table 9.1. The percentage in parentheses

includes only students who also gave correct reasoning. The results are fairly consistent from quarter to quarter. Even though angular momentum is introduced in lecture prior to students completing the pretest, relatively few answer each question correctly, and even fewer give correct reasoning. Only 3 students gave correct answers and explanations for all three questions.

Table 9.1: Results from Task 9.1

The percentage correct (correct reasoning in parentheses) is shown for questions about orbital angular momentum given to students at the end of first-quarter junior-level quantum mechanics.

Population	Question 9.1a (L^2)	Question 9.1b (L_z)	Question 9.1c (L_x)
Autumn 2013 (N=58)	38% (36%)	22% (14%)	21% (5%)
Summer 2014 (N=16)	19% (19%)	19% (19%)	6% (0%)
Autumn 2014 (N=74)	24% (22%)	18% (12%)	19% (0%)

A. Identification of Student Errors

In this section, we discuss the most common errors in the responses to Task 9.1. The percentages below are aggregated across all three quarters, and should be considered representative of students in upper-division quantum mechanics who have only recently learned this material.

1. Confusion between the eigenvalue and the quantum number for L^2

Many students answered question 9.1a (about the possible outcomes of a measurement of L^2) in a way that does not seem to distinguish between the value of the orbital angular momentum squared (the eigenvalue of L^2) and the quantum number l that is used to index the spherical

harmonics (the eigenfunctions of L^2). Few students provided a detailed explanation to this question. Shown below are examples of quantitative and qualitative reasoning, respectively, given in support of this answer.

"$L = l\hbar$ and $l = 2$, so $L^2 = l^2\hbar^2$." (Question 9.1a)

"The 2 in Y_2^1 is the [angular] momentum and this gets squared." (Question 9.1a)

Both students identify the correct quantum number, but they use it incorrectly. They seem to be treating l as equal to the magnitude of the angular momentum vector, which they square in order to answer the question. Many students also gave answers such as l or $l\hbar^2$. In all, 22% of the students gave an answer or an explanation to question 9.1a consistent with this error.

2. Confusion between the quantum numbers *l* and m_l

The spherical harmonic given in Task 9.1, $Y_2^1(\theta,\phi)$, specifies two quantum numbers, $l = 2$ and $m_l = 1$. Many students confused or reversed these numbers; that is, they used m_l to answer about L^2 and they used l to answer about L_z.

"The 1 next to the Y_2^1 corresponds to the angular momentum." (Question 9.1a)

"The l superscript on Y_2^1 is 1, so that means the orbital angular momentum is 1." (Question 9.1a)

Both of these students reverse the location of l and m_l; the second student even identifies the correct letter, but still reverses them. We have also found this error to be prevalent when l and m_l are explicitly identified for the students, when the initial state is given in Dirac notation rather than as a spherical harmonic, and in more complicated circumstances, such as those that involve

total angular momentum (see section 9.4). In addition, this error is in conflict with the condition that $m_l \leq l$, revealing that some students are unable to check whether or not their answer is reasonable.

Eleven percent of the students answered question 9.1a in a way that is consistent with switching these two quantum numbers. Some students (about 6%) made a similar mistake on question 9.1b (which asks about L_z instead of L^2). The fact that this error persists across multiple questions, together with similar instances of confusion between a variety of quantities related to angular momentum, suggests that students may have difficulty with the notation associated with angular momentum.

3. Confusion between spin and orbital angular momentum

Spin and orbital angular momentum are very similar in quantum mechanics. However, some students seemed to confuse spin and orbital angular momentum on Task 9.1, which does not involve spin, as in the response below.

> *"I believe orbital angular momentum corresponds to the particle's spin, and only half spins and zero spins are possible."* (Question 9.1b)

This student appears to believe that orbital angular momentum is synonymous with spin: they answered that the possible outcomes of a measurement of L_z are $-\hbar/2$, 0, and $+\hbar/2$.

Many other students answered that half integer values are possible outcomes of a measurement of L_z without any specific reference to spin in their explanations, as in the following example.

> *"Since [the] eigenvalue of L_z is m where m = –l, –l + 1, …, l – 1, l, and l is either integer or half integer."* (Question 9.1b)

While this student uses the correct quantization condition for m_l, they explicitly state that l can take either integer or half-integer values. The student selected all of the possible answer choices: $-2\hbar$, $-\hbar$, $-\hbar/2$, 0, $\hbar/2$, $\hbar$, and $2\hbar$. On question 9.1b, many other students gave similar ranges of values that included both integer *and* half-integer values, often with little or no accompanying explanation. It seems as though these students were treating orbital angular momentum as if it includes spin or spin-like properties. However, some students did not mention spin in their explanations, and it is possible that these students have instead failed to recognize that the derivation of the eigenvalues of L^2 and L_z excludes half-integer values.

Some students, in response to both question 9.1b and question 9.1c, gave only two answers with equal magnitude: one positive and one negative. The values $\pm\hbar/2$ were the most common, but some students appeared to base their answers on one of the other quantum numbers given on Task 9.1, as in the example below.

> *"It's in the second harmonic, so [the answer is] 2ħ/2 and plus or minus as it could occupy spin up or down states."* (Question 9.1b)

This student indicates that the particle could be either "spin up" or "spin down", language that is commonly used to characterize the states of spin-1/2 particles. The student answers in terms of the "second harmonic", which they seem to relate to the given value for l (*i.e.*, 2). Some students gave similar reasoning in support of the answer $\pm m_l\hbar$. Note that the prior student divides this answer ($2\hbar$) by two, which could also be due to confusion with spin.

About 20-25% of the students gave answers to questions 9.1b and 9.1c consistent with confusion between spin and orbital angular momentum (*i.e.,* either they referenced spin directly in their explanation or they included half-integer values in their answer). Less than 5% of the students gave responses of this type to question 9.1a (which asks about L^2). However, some students give responses consistent with confusion between L and S even after all undergraduate instruction in quantum mechanics.

4. Tendency to associate m_l with L_x

On question 9.1c, the students were asked to identify the possible outcomes of a measurement of L_x for a particle associated with the spherical harmonic $Y_2^1(\theta,\phi)$. Because the spherical harmonics are defined in terms of the z-axis, this function is associated with a single value of L_z ($+\hbar$), but it is not associated with a single value of L_x. The following explanation was given by a student who stated that both L_z and L_x are equal to $+\hbar$ for this particle.

"Eigenvalues of L_x are $\hbar m$." (Question 9.1c)

This student had given a virtually identical response to question 9.1b, and appears to believe that there should not be a difference between the z- and x-directions at all. While L_z and L_x do have the same eigenvalues in general, the associated eigenfunctions are different because the operators do not commute. Below is a response from a student who used different reasoning to support the same answers ($+\hbar$ for both L_z and L_x).

"No orientation should be preferred." (Question 9.1c)

This student, who provided a correct explanation for question 9.1b (which asks about L_z), seems to believe that each direction should be equivalent, and ignores the fact that spherical harmonics are only eigenfunctions of L_z, not of L_x. In addition, the answer given by each of these students is inconsistent with the uncertainty principle, since it is not possible for two components of L to be known precisely. (See section 9.3 for further discussion of student difficulty with the consequences of the uncertainty principle for angular momentum.)

Although only about 5% of the students appeared to associate m_l with L_x, one-quarter of the students who gave the correct answer to question 9.1b (about L_z) made this error on question 9.1c. A related error, wherein students gave different answers in which they seemed to equate the *x*- and *z*-directions, is discussed in the next section (9.2A5).

5. Failure to recognize when L^2 or L_z is well-defined

By the time students completed the pretest, they had been exposed to the derivation of how angular momentum is quantized. The ideas covered included the following: the eigenvalues of L_z are quantized in integer units of $\hbar$; only those integers with absolute values less than or equal to l (the quantum number associated with L^2) are allowed; and each of the spherical harmonics is an eigenfunction of L_z and is associated with a single eigenvalue, $m_l\hbar$.

On the pretest, students often failed to apply these ideas. The following explanation is representative of many responses given when the students were asked to identify the possible outcomes of a measurement of L_z for a particle described by the spherical harmonic $Y_2^1(\theta,\phi)$.

> *"The eigenvalues of the orbital angular momentum is the magnetic quantum number m, which runs from –l to l over the integers. Therefore, since l = 2, m can be any integer from –2 to 2."* (Question 9.1b)

This student correctly identifies the possible eigenvalues of L_z for a state known to have $l = 2$. However, they fail to use the fact that m_l is also specified, so that the *z*-component of angular momentum is equal to $+\hbar$. The student gave exactly the same answer when asked to identify the possible outcomes of a measurement of L_x.

At this point in the course, the students have learned two rules that follow a similar pattern: (1) that m_l can take any value from $-l$ to l and (2) that l can take any value from 0 to $n - 1$ (for the hydrogen atom potential only). The example below shows a student who has applied such a rule incorrectly.

> *"$-l < m < l$ and the z-component must be less than or equal to m."* (Question 9.1b)

The student appears to think that while l provides a bound on the allowed values of m_l, the quantum number m_l provides a bound on the possible outcomes of a measurement of L_z. This reasoning suggests that some students may be confused about the circumstances under which they can use a rule of this form (*i.e.,* that some quantity is restricted to be less than some related quantity).

Approximately 35% of the responses to question 9.1b were consistent with the reasoning discussed above. These students tended to answer that L_z can be any integer between some maximum and minimum value, and they often gave the same answer for question 9.1c (about L_x). Many students also included half-integer values (see section 9.2A3) or excluded zero from their list of values, though they did not often justify these decisions.

In addition to not recognizing that L_z is well-defined for the particle on Task 9.1, 13% of the students gave responses to question 9.1a indicative of a failure to recognize that L^2 is also well defined. These students typically claimed that l could take any integer value for this state, or any integer value up to 2 (the given value of l), and they often did not find the correct values of L^2 to which the quantum number l corresponds (similar to the error discussed in section 9.1A1).

6. Failure to treat angular momentum as discrete rather than continuous

The eigenvalues associated with angular momentum are discrete. The response below shows an example of a student whose response is inconsistent with this requirement.

> *"L only gives us a value for the total angular momentum, so the value of the z-component can be almost any value, as long [as] the total [angular] momentum is $6\hbar^2$."* (Question 9.1b)

This student states that the *z*-component of a vector is less than or equal to its total magnitude, which would be correct for a vector associated with a continuous spectrum of eigenvalues (*e.g.*, linear momentum) or for a classical angular momentum vector. However, this is not compatible with the requirement that the component of the angular momentum along any direction be an integer multiple of $\hbar$. It should also be noted that this student also fails to recognize that L_z is fully specified for the particle they are asked to consider (similar to the error discussed in section 9.2A5).

Some students used similar reasoning to determine L_x. The student whose response is shown in the previous paragraph, for example, used exactly the same reasoning for L_x that they used to

determine L_z. The two students below, who answered correctly that L_z is well-defined (including correct explanation), both give a continuous range of values for L_x.

> *"$L^2 = 6\hbar^2$, and $L_z^2 = \hbar^2$, and $L^2 = L_x^2 + L_y^2 + L_z^2$, so L_x^2 might be anything within –5 and 5ħ."* (Question 9.1c)

> *"The x-component is not well-defined: if we already know [the] z-component, then [the] x- and y-components are uncertain.* $-\sqrt{5}\hbar < L_x < \sqrt{5}\hbar$ *."* (Question 9.1c)

Both students answer explicitly that L_x could be any continuous value, contingent on the values of L^2 and L_z. The first student's argument is not quantum mechanical: they use the Pythagorean relationship between the components of a vector to calculate the missing component. The second student seems to use this relationship implicitly, but also answers in familiar quantum terms: "not well-defined" and "uncertain". The ideas associated with these terms can be productive, although the student applies them to this system incorrectly. (See section 9.3B for further discussion of how some students interpret terms like "uncertain".)

Five percent of the students treated angular momentum as continuous when determining the possible values of L_z. Ten percent made this error when answering about L_x. Additional students (up to 10%) answered that any value is possible for either measurement, but they did not always clearly indicate whether this should be taken to mean that the measurement could be any real number, any real number between some defined limits, or any integer or quantized value between incorrect limits.

B. Underlying Student Difficulty: Belief that L_x and L_z are equivalent

When answering question 9.1c (about L_x), 43% of the students gave answers or explanations identical to the answers or the explanations that they used for question 9.1b (about L_z). It should be noted that the task always asks about L_z before it asks about L_x. Some students used exactly the same language in their explanations to both question. The failure of students to recognize the component of L specified by m_l (section 9.2A4) and the failure to recognize that m_l does specify L_z (section 9.2A5) are each associated with students who used reasoning that is identical for both questions.

The response below is representative of the explanations that many students gave in support of a wide variety of answers, both correct and incorrect, to question 9.1c.

"Spherically symmetric, so L_x should equal L_z." (Question 9.1c)

This student appears to use symmetry as a basis for arguing that the possible values for each component should be the same. This reasoning would be appropriate in the absence of a particular choice of direction, or for comparing the *x*- and *y*-directions when only the *z*-direction is specified. Symmetry is both a useful and a powerful concept in physics, but in this case many students do not use it correctly or consistently. Many students gave similar explanations, often using the word "symmetry", citing the lack of a "special" direction, or noting that the choice of axis is "arbitrary".

Most quantum textbooks are careful to note that the *z*-axis is not special when angular momentum is first introduced.[9] While identifying how the *z*-axis is chosen is important, some students may use such a statement to justify a belief that the directions should always be

equivalent. See chapter 10 for a discussion of how we have attempted to address this difficulty in tutorial curriculum on angular momentum.

9.3 – Student difficulties with angular momentum, probability, and uncertainty

Task 9.1 asks about the possible values of a measurement of L_x; it does not ask about the probabilities associated with those values. Task 9.2 (see Fig. 9.2) was specifically designed to elicit student ideas about the probabilities of different values for L_x for a state initially written in the L_z basis. On this task, students consider a particle that is in an eigenstate of the L_z operator with eigenvalue $+\hbar$. They are told to consider only orbital angular momentum. Question 9.2a asks the students to determine the possible values of a measurement of L_x, similar to question 9.1c discussed in section 9.2. Question 9.2b asks them to determine the probability that the result of this measurement of L_x is equal to $+\hbar$. The probability of this result, given by the expression $\left|{}_x\langle 1,1|1,1\rangle_z\right|^2$, is equal to 1/4.[11]

The second pair of questions on Task 9.2 is very similar to the first pair. Students consider an initial state that is a superposition of eigenstates of L_z. We chose to use the specific combination of states that is exactly equal to an eigenstate of L_x with eigenvalue 0. Thus, only one possible value (0) can be measured and the probability of measuring $+\hbar$ is equal to zero.

This question was administered to two populations of students, α (N=73) and β (N=39), from the two-quarter sequence of junior-level quantum mechanics at UW. There were several differences between the two populations. Population β consists of students midway through the second quarter of the junior-level course, and population α consists of students at the end of the

first quarter. The students in population β had completed more overall instruction, but both populations had covered orbital angular momentum in class.

Task 9.2

Consider a spinless particle in the orbital angular momentum state: $|\psi_A\rangle = |l, m_l\rangle_z = |1,1\rangle_z$.

Question 9.2a

Suppose the *x*-**component** of the orbital angular momentum of this particle is measured. Which of the following values are possible results of this measurement?? Explain your reasoning. (Circle all that apply.)

$-2\hbar$ $-\sqrt{2}\hbar$ $-\hbar$ $-\hbar/2$ 0 $\hbar/2$ $\hbar$ $\sqrt{2}\hbar$ $2\hbar$

Question 9.2b

Which of the following represents the probability that the result of the measurement of the *x*-component of orbital angular momentum from the previous question is $+\hbar$? Explain your reasoning.

0 1/5 1/4 1/3 1/2 1

The same particle is now put in the orbital angular momentum state: $|\psi_B\rangle = |l, m_l\rangle_z = \frac{1}{\sqrt{2}}\left(|1,1\rangle_z - |1,-1\rangle_z\right)$

Question 9.2c

Suppose the *x*-**component** of the orbital angular momentum of this particle is measured. Which of the following values are possible results of this measurement? Explain your reasoning. (Circle all that apply.)

$-2\hbar$ $-\sqrt{2}\hbar$ $-\hbar$ $-\hbar/2$ 0 $\hbar/2$ $\hbar$ $\sqrt{2}\hbar$ $2\hbar$

Question 9.2d

Which of the following represents the probability that the result of the measurement of the *x*-component of orbital angular momentum from the previous question is $+\hbar$? Explain your reasoning.

0 1/5 1/4 1/3 1/2 1

Figure 9.2: Sequence of questions on angular momentum probabilities

Task 9.2 was designed to probe student understanding of the probabilities associated with measurements of angular momentum for different initial states.

The background of the students varied in another way. The content of a prerequisite course taken by the students was also different, because the populations are drawn from different years. For most of the students in population α (61 out of 73), this prerequisite course focused heavily on spin and two-state systems. Most of the students in population β (30 out of 39) instead took a

version of the same course that only covered spin briefly. Very few students, if any, had not previously been introduced to spin.

Table 9.2: Results from Task 9.2

The percentage correct or nearly correct on questions 9.2b and 9.2d is shown. The raw number of correct or nearly correct responses is shown in parentheses. Students were considered nearly correct if they gave an incorrect answer that was justified with correct reasoning.

Population	Question 9.2b	Question 9.2d	Notes
β (N=39)	8% (3)	8% (3)	• Undergraduates midway through 2nd quarter of QM • Most students' modern physics course covered spin briefly
α (N=73)	1% (1)	21% (15)	• Undergraduates at the end of 1st quarter of QM • Most students' modern physics course covered spin in detail

Table 9.2 shows the percentage of the students who gave responses to questions 9.2b and 9.2d that were correct or nearly correct (questions 9.2a and 9.2c are not included in the table, because student responses to those questions are discussed in section 9.2). Virtually no students gave a fully correct explanation in response to either question. On question 9.2b (about a particle in an eigenstate of L_z), students were considered nearly correct if they identified each of the possible outcomes of the measurement and described a method for calculating either coefficients or probabilities. On question 9.2d, students were considered nearly correct if they were able to identify that the state should be written in a different basis or that the state was an eigenstate of L_x, even if they misidentified the corresponding eigenvalue. Students who guessed the correct answer, or gave entirely incorrect reasoning in support of the correct answer, were not considered correct for either question.

In addition to Task 9.2, we have also conducted several individual student interviews after all lecture and tutorial instruction on angular momentum. Each interview lasted between 30 and 60

minutes. In each interview, the students were told to consider a system consisting of two particles, one with spin-3/2 and the other with spin-1, and were asked several questions about the total spin and the spin of the individual particles for a variety of initial states. These interviews were conducted using this context in order to investigate ideas beyond just orbital angular momentum; however, many of the ideas expressed by the students are similar to those that appear in the student responses to the questions on Task 9.2.

A. Student Error: Tendency to treat the probabilities for all allowed values as equal

Each of the responses below was given by a student who correctly identified the three possible outcomes of a measurement of L_x for a particle in the state $|l, m_l\rangle_z = |1,1\rangle_z$ and then answered that the probability of this measurement resulting in $+\hbar$ is equal to 1/3.

> *"All options are equally probable, and there are three options."* (Question 9.2b, population α)
>
> *"[1/3] since there [are] only 3 possible measurements."* (Question 9.2b, population β)

The first student clearly states that the reason for the probability being equal to 1/3 is because all three of the possible outcomes are equally probable. The second student's reasoning is less clear, but they seem to be connecting the number of options with the probability in the same way. Many students also claimed that the probabilities are equal, but incorrectly identified the measurement outcomes. The two most common examples of this were to exclude zero (two possibilities with a probability of 1/2 each) and to include $\pm\hbar/2$ (a total of five possibilities with a

probability of 1/5 each). About 60-65% of the students in each population gave answers to question 9.2b consistent with all allowed values being equally probable.

Many students also gave this answer when asked about the superposition state $|\psi_B\rangle = |l, m_l\rangle_z = \frac{1}{\sqrt{2}}\left(|1,1\rangle_z - |1,-1\rangle_z\right)$, which happens to be an eigenstate of L_x with eigenvalue 0. Most responses simply noted the similarity between the two initial setups. While 59% of the students in population β stated that the probabilities were equal for question 9.2d, only 30% of the students in population α gave this answer. Many of the students in population α noted that this state was an eigenstate of L_x. Most did not identify the correct eigenvalue, perhaps due to the resemblance of $|\psi_B\rangle$ to the spin-1/2 state $|{+}\tfrac{1}{2}\rangle_x = \frac{1}{\sqrt{2}}\left(|{+}\tfrac{1}{2}\rangle_z - |{-}\tfrac{1}{2}\rangle_z\right)$. The discrepancy between the two populations may be due to the fact that a prerequisite course dealt heavily with spin for the students in population α, while the same course for population β discussed spin only briefly. This result is also interesting because we might otherwise expect the population β students to be more advanced or sophisticated in their reasoning due to the fact that they had several additional weeks of junior-level instruction.

There has been research on student understanding of probability indicating that many students tend to believe that probabilities for different outcomes are always equal.[11] We have seen evidence of this result in quantum mechanics in the responses to a wide variety of questions. About 10% of the students who answered the "quantum mouse" question discussed in section 7.2A2 stated that the probabilities of two outcomes would each be 50%, even though they were given a quantum state with coefficients of unequal magnitude. We have also noticed a common belief of students that wave functions evolve in time until all values (of either position

or energy) have become equally probable (see section 2.4D1). While this reasoning is not present in response to all questions, it has been common in cases where information about the measurement quantity is not obvious; *e.g.*, when the state is written in a basis other than that of the measurement. These conditions are met for the questions on Task 9.2, so it is not surprising that some students would rely on this reasoning.

B. Description of Underlying Student Difficulties

1. Tendency to treat all eigenstates of L_z as equal superpositions of the eigenstates of L_x

As discussed earlier, most of the students (in both populations) had studied spin-1/2 systems prior to completing Task 9.2. The response below indicates that some of the students tended to overgeneralize from the spin-1/2 case. They seem to think that since the probabilities are almost always equal for spin-1/2, the probabilities must always be equal in other cases.

> *"$|1,1\rangle_z$ is an equal superposition of the 3 [L_x] eigenstates, so the probability is 1/3."*
> (Question 9.2b, population α)

This student describes writing the given eigenstate of L_z in terms of the eigenstates of L_x, which is an important step in answering correctly. However, the student explicitly identifies this state as an "equal superposition" of the three eigenstates of L_x: $|1,1\rangle_x$, $|1,0\rangle_x$, and $|1,-1\rangle_x$. This is language that some instructors use to characterize states with coefficients that are the same, or have the same magnitude, which is common for spin-1/2 systems. One student, who also answered that the three probabilities would be equal, gave the following justification: "intuition

from the spin-1/2 states." Although most responses do not directly relate this idea to spin, it seems very likely that this is a possible source of confusion.

The following excerpt from an interview shows a student (from population α) generalizing what they have learned about the spin-1/2 case, in which the probabilities tend to be equal, to the case of a spin-3/2 particle, in which they are not. The student determined that there would be four possible outcomes if S_z were measured immediately prior to what is quoted.

Interviewer: *Would those four values be equally probable or would there be some other distribution?*

Student: *I think that they would be equally probable. I don't see why they wouldn't be.*

Interviewer: *Would their probabilities depend on the z-component of the total spin?*

Student: *Wait I want to say that they are equally probable because I know that when we have spin-1/2 that we could have 50% chance it would be spin up or down because we haven't specified anything about the system that it's in, so it could be one or the other, so it has to be equal probability.*

This student initially believes that the four possible values for spin-3/2 would all be equally probable, but is uncertain as to the reason. However, they recall the result for spin-1/2 and use it as justification for their earlier answer. Since the spin-1/2 case is both common and easy to remember, it is likely that many students are using it either as a basis for their answers about more complicated systems or as a *post hoc* justification for an answer that matches the result from the simpler case. This finding suggests that greater care should be taken to present non-equal probability cases for spin-1/2, although further research is needed to determine whether or not this would impact student understanding at higher levels of instruction.

2. Confusion between uncertainty and lack of knowledge

It is very common, even for students who are not studying quantum mechanics, to assume that all the possible values of a certain result are equally probable.[11] However, some responses suggest that, in the context of angular momentum in particular, more than one incorrect idea may be contributing to this error, as in the examples below.

> *"The angular momentum in [the] z-direction is well-defined, so angular momentum in [the] x cannot be, so [the] probability to measure any given value in x is just 1/(number of possible values) = 1/3."* (Question 9.2b, population β)

This student starts with a correct statement: that the x-component of angular momentum is not well-defined because the z-component is well-defined. However, they appear to use this fact as justification for why the three values of L_x are equally probable. This student may be misinterpreting the term "well-defined". In quantum mechanics, a quantity is said to be well-defined if its uncertainty is equal to zero; *i.e.,* if there is only one possible outcome of a measurement of that quantity. However, this student seems to think that when L_x is uncertain, each possible value must have the same probability, perhaps because they do not understand what is meant in this situation by the term *uncertain* (*i.e.,* that L_x is not well-defined, and thus that more than one outcome is possible if it were measured). The response below demonstrates this line of reasoning even more clearly.

> *"Possible values of L_x are integer values of $\hbar$, and knowing L_z doesn't tell us anything about L_x."* (Question 9.2b, population α)

This student states that knowledge of one quantity (L_z) conveys *no information* about a different quantity (L_x)! They give this response despite their answer to question 9.2b (that the probability of measuring $+\hbar$ is 1/3), which is information about L_x. Furthermore, the student has failed to recognize that the state vector for a particle specifies all the information that can be known about the particle, namely the probabilities of measurement outcomes for any and all quantities of interest.[12] The student appears to be confusing uncertainty, which indicates that measuring L_x could result in more than one possible outcome, with an absence of knowledge about the quantity, and then associating that absence of knowledge with an equal probability of all outcomes. Similar results have been discussed in chapters 2-4 of this dissertation for quantities such as energy, position, and momentum.

About 5% of the students from each population gave reasoning that was directly indicative of this difficulty. While some of these students also claimed that the probabilities were equal, a few answered that it was not possible to determine the probabilities, or even that it was not possible to measure the given quantities. Although this percentage is small, it only includes the students who specifically made reference to some lack of knowledge. Many other student responses are suggestive of similar lines of reasoning. The interview excerpt below gives some insight into how students might be thinking. The student (from population β) was asked about the probabilities of the various outcomes of a measurement of the *x*-component of the total spin for a system with spin-5/2.

Interviewer: *You said I'd have six possible values, would those values be equally probable?*

Student: *No. Most definitely not equally probable because I just remember from working out the math on systems like this that you get different probabilities for the values.*

Interviewer: *Okay. Would the distribution of probabilities depend on which state we started in?*

Student: *No, as long as the total [spin] was not changed.*

Interviewer: *Okay, so if we had a total [spin] of 5/2 but we had a different [z-component] like 3/2 or 5/2, we'd expect the same distribution of probabilities if we measured in the x-direction?*

Student: *Yes. […]*

Interviewer: *Why is that?*

Student: *Again if you measure a non-commuting operator you no longer have that information about the z-component of your wave function: it's just not a factor I guess.*

This student correctly identifies that the probabilities are not equal. However, when they are asked whether changing the initial state will result in a different distribution of probabilities, they answer incorrectly that the exact form of the initial state does not impact the probabilities. They support this argument by noting that the information about the *z*-component is not a factor. Because they cite non-commuting operators, it seems likely that they are conflating the loss of certainty about the *z*-component of spin after the measurement to a lack of information prior to the measurement. Of the eleven interview subjects, seven students gave answers that referenced either equal probabilities or some lack of information. Even students who stated that the probabilities would not be equal to each other tended to have difficulty describing exactly how they would determine these probabilities.

9.4 – Student Difficulties with Measurements of Total Angular Momentum

Addition of angular momentum is one of the most challenging topics in quantum mechanics. Zhu and Singh have identified several student difficulties with this topic.[4,5] In particular, they report that students struggle to understand the Hilbert space(s) associated with angular momentum, such as incorrectly calculating the dimension of a Hilbert space and confusing various Hilbert spaces with position space. They also indicate that some students are unable to determine the outcomes of a measurement of total angular momentum (or the corresponding probabilities). We are interested in determining the most common incorrect lines of reasoning associated with the errors that students make. Although some of the student responses that we discuss below are consistent with errors identified in this previous work, we focus on certain conceptual and reasoning difficulties that have not been reported elsewhere.

Task 9.3 (see Fig. 9.3) is an extension of Task 9.1 that asks about total angular momentum. On this task, students are given the following superposition of three states written as a tensor product of different spin and orbital angular momentum states.

$$|l, m_l\rangle \otimes |s, m_s\rangle = \tfrac{1}{\sqrt{6}}\left(|3,0\rangle \otimes |\tfrac{1}{2}, \tfrac{1}{2}\rangle + \sqrt{2}\,|3,1\rangle \otimes |\tfrac{1}{2}, -\tfrac{1}{2}\rangle - \sqrt{3}\,|2,0\rangle \otimes |\tfrac{1}{2}, \tfrac{1}{2}\rangle\right)$$

They are then asked to determine the possible results of measurements of two quantities: J^2, the total angular momentum squared (question 9.3a), and J_z, the z-component of the total angular momentum (question 9.3b). Both questions are free response, and the students are asked to explain their reasoning.

Task 9.3

Quantum states on this page are written in Dirac notation as $|\psi\rangle = |l, m_l\rangle \otimes |s, m_s\rangle$, where the first ket refers to the *orbital* angular momentum states, and the second ket refers to the *spin* angular momentum states.

Assume the electron is in the state:

$$|\psi_1\rangle = \frac{1}{\sqrt{6}}\left(|3,0\rangle \otimes |\tfrac{1}{2}, \tfrac{1}{2}\rangle + \sqrt{2}|3,1\rangle \otimes |\tfrac{1}{2}, -\tfrac{1}{2}\rangle - \sqrt{3}|2,0\rangle \otimes |\tfrac{1}{2}, \tfrac{1}{2}\rangle\right)$$

Let $\vec{J} = \vec{L} + \vec{S}$ be the total angular momentum of the electron.

Question 9.3a

Suppose you measured the total angular momentum squared (J^2). Determine all the possible values of this measurement. Explain your reasoning.

Question 9.3b

Suppose you measured the *z*-component of the total angular momentum (J_z). Determine all the possible values of this measurement. Explain your reasoning.

Figure 9.3: Two questions on addition of angular momentum

Task 9.3 was administered on a tutorial pretest near the end of the first quarter of a junior-level quantum mechanics course. Questions 9.3a-b ask the students to determine the possible results of a measurement of J^2 and J_z, respectively, for a particle in a superposition state.

In order to answer correctly, students can make use of the relationship between total, orbital, and spin angular momentum stated in the problem: $\vec{J} = \vec{L} + \vec{S}$. For J_z, this results in the equation $J_z = L_z + S_z$ (or equivalently, $m_j = m_l + m_s$). Since all three terms have *z*-components that sum to give the value $m_j = 1/2$, there is only one possible outcome for a measurement of J_z for this particle. The quantum number *j*, however, can take multiple values for each term, ranging from $|l - s|$ to $l + s$ in integer steps. For the first two terms, this gives 7/2 and 5/2; for the last term, 5/2 and 3/2. Finally, the eigenvalues of J^2, given by $j(j+1)\hbar^2$, are $15\hbar^2/4$, $35\hbar^2/4$, and $63\hbar^2/4$.

Table 9.3 shows the percentage of the students who answered correctly. (The percentage shown in parentheses includes only students who also provided correct explanations.) The

results for both questions are very low. Less than 10% of the students answered question 9.3a correctly (only 4 out of all 194 students provided correct reasoning). Anywhere from as few as 2% to as many as 45% answered question 9.3b correctly; only about half gave correct reasoning. The larger percentage of the students from Autumn 2014 who answered question 9.3b correctly may be due to the use of tutorial curriculum on the topic of angular momentum in quantum mechanics given prior to the administration of Task 9.3 (see chapter 10 for more details). Only four students across all quarters answered both questions correctly—none of these students gave fully correct reasoning for both questions.

Table 9.3: Results from Task 9.3

The percentage correct on addition of angular momentum questions (correct reasoning in parentheses) is shown.

Population	Question 9.3a (J^2)	Question 9.3b (J_z)	Notes
Autumn 2012 (N=57)	11% (2%)	2% (2%)	• Online and multiple-choice (with explanation) • Given to students at the end of 1st quarter of QM • Given before students had completed all lecture material on addition of angular momentum • **Bold: given after one tutorial on angular momentum**
Autumn 2013 (N=59)	**4% (2%)**	**20% (12%)**	
Summer 2014 (N=15)	**7% (7%)**	**20% (13%)**	
Autumn 2014 (N=73)	**8% (1%)**	**45% (25%)**	

This topic is clearly difficult for students. Although some students had not completed all lecture instruction on addition of angular momentum prior to completing the pretest, the results have been similar when administered after all lecture instruction.

A. Identification of Student Errors

Below, we discuss the most common errors given in response to Task 9.3. As in section 9.2A, the percentages reported are aggregated across all quarters.

1. Confusion between J^2, J_z, and other angular momentum quantities

Many of the errors identified in section 9.1 for orbital angular momentum were also common when students were asked about total angular momentum. Since J is a form of angular momentum, and it is often presented in the same manner as L and S, it is not surprising to find similar kinds of reasoning.

For example, about 5% of the students confused the quantum numbers j and m_j, which was also very common for l and m_l (see section 9.1A2). Another 15% of the students claimed that any value from $-j$ to $+j$ can be measured for J_z, either in integer or in half-integer steps (see section 9.1A5). This reasoning would be correct for J_x, but not for J_z, which is well-defined for the state given on Task 9.3.

The two cases above illustrate ways in which the students failed to distinguish between the different quantities associated with angular momentum. Many students also failed to distinguish between the different *forms* of angular momentum, such as spin and orbital (see section 9.1A3). About 5-10% of the students also confused J with L or with S, as in the example below, which was given in response to question 9.3b, which asked about J_z for a particle in the state $\frac{1}{\sqrt{6}}\left(|3,0\rangle \otimes |½,½\rangle + \sqrt{2}\,|3,1\rangle \otimes |½,-½\rangle - \sqrt{3}\,|2,0\rangle \otimes |½,½\rangle\right)$.

"0ħ and 1ħ, since the eigenvalues are still mħ." (Question 9.3b)

This student appears to base their answer on the quantum number m_l for orbital angular momentum, with no influence at all from the particle's spin. Many student responses were less clear than this, consisting only of equations or numbers that included only orbital or only spin angular momentum.

About 30% of the responses to question 9.3b (about J_z) and about 20% of the responses to question 9.3a (about J^2) included at least one of the errors discussed in this section. In most cases, the reasoning used to support these answers was very similar to that described in section 9.2, so additional examples are not included here.

2. Adding values of m_l and m_s from different terms to determine m_j

Many of the students who recognized that the *z*-components of both orbital and spin angular momentum are necessary to determine J_z did not use these components correctly. Some added together all permutations of L_z and S_z from the three states in the superposition given to the students, as illustrated below.

"If you measure L_z, you can get 0 and ħ, if you measure S_z, you can get ħ/2 and –ħ/2, so J_z can be the sum of them, which can be ħ/2, –ħ/2, and 3/2ħ." (Question 9.3b)

This student has identified the correct possible values for both L_z and S_z, but combines all possible values of each to determine the possible values of J_z. The student has failed to recognize that, each value of L_z is associated only with the value of S_z corresponding to the same term in the overall superposition, even though two different values can be measured for each.

About 5% of the students gave answers or explanations to question 9.3b consistent with combining values from different term to determine J_z. A few students (3%) gave similar reasoning in response to question 9.3a (which asks about the possible outcomes of a measurement of J^2). Although this error was not predominant on Task 9.3, it is included here because it was much more common (16%) when a similar question about the possible outcomes of J_z was asked on an exam. See section 10.3 for further discussion of this result.

3. Adding vectors as if they were scalars

On Task 9.3, the total angular momentum is defined to be $\vec{J} = \vec{L} + \vec{S}$. Many students, when asked to determine the possible values of J^2, treated *J*, *L*, and *S* as scalar quantities rather than vector quantities, as in the example below.

> *"For all the possibilities* $s = \sqrt{3/4}\hbar$ *and L depends on l so* $J^2 = (L + \sqrt{3/4}\hbar)^2$*."* (Question 9.3a)

This student has simply squared each side of the equation that relates *J*, *L*, and *S*, ignoring the fact that it is a vector equation by adding only the *magnitudes* of the *L* and *S* vectors (which they calculated correctly). This process would be correct for two parallel vectors (*i.e.*, two vectors that have exactly the same direction); however, angular momentum does not have a well-defined direction in quantum mechanics. In the example below, a student uses the same equation in a slightly different way.

> *"$J^2 = (L + S)^2$ and for this particle l = 0, 1, 2, 3 and s = ½."* (Question 9.3a)

This student references the same equation as the previous student, but uses the quantum numbers associated with L^2 and S^2 rather than their magnitudes, which is reminiscent of previously discussed difficulties (see section 9.2A1). Both of the students fail to take into account the fact that angular momentum is a vector. About 5% of the responses to question 9.3a were consistent with this error.

4. Adding vectors as if they were perpendicular

The response below shows a student using a different equation for J^2 than the one discussed in the previous section.

> *"$J^2 = L^2 + S^2$ as L and S are orthogonal (they contain no information about each other)."* (Question 9.3a)

This student articulates that L and S are perpendicular, rather than perfectly aligned. Many students used the same equation in their responses, but gave no further justification (*i.e.*, they did not discuss the relative orientation of the vectors at all), as in the example below.

> *"$J^2 = L^2 + S^2$ and $S^2 = 3/4\ \hbar^2$. L^2 applied to this state will return either 0, $2\hbar^2$, or $6\hbar^2$. Thus, these three results [$3/4\ \hbar^2$, $11/4\ \hbar^2$, and $27/4\ \hbar^2$] are the only possible results."* (Question 9.3a)

This student simply adds together the various possible values of L^2 and S^2 (which they determined correctly). The student makes no claim about whether or not the vectors are orthogonal, despite using a formula only valid for that case. Thus, it is possible that the student does not view L and S as vectors at all, but instead is thinking of them as scalars, while applying an incorrect formula for determining the square of their sum.

The response below shows a different justification for the use of the same formula.

> *"As we're dealing with linear operators, we can simply add the possible eigenvalues of L^2 and S^2 for this state (corresponding to l(l+1) and s(s+1))."* (Question 9.3a)

This student misuses the term "linear operator", which refers to an operator that satisfies the distributive property when acting on sums of state vectors, to imply that J^2 can be written as a linear sum of L^2 and S^2. Again, this reasoning does not correctly make use of the definition of $\vec{J} = \vec{L} + \vec{S}$ as a vector sum, which was given to the students on Task 9.3. Twenty percent of the students gave explanations to question 9.3a consistent with the use of the formula $J^2 = L^2 + S^2$. Students in introductory physics who are asked about vector addition and subtraction have also been observed to add vectors as if they were perpendicular, regardless of the particular orientation of the vectors.[13]

5. Using the equation *j = l + s*

The errors described in the previous two sections involved students who attempted to calculate J^2 directly. Some students instead found the quantum number j and then used it to determine J^2. The example below is representative of the explanations given by these students.

> *"First add the l and s values for each term to obtain j, then the eigenvalues of J^2 are j(j+1)."* (Question 9.3a)

This student directly adds the quantum numbers for the associated operators, rather than attempting to add the corresponding magnitudes of the L and S vectors in some way, as discussed in sections 9.4A4 and 9.4A5. Many students simply stated the formula $j = l + s$ with no further

justification, while others only provided numbers consistent with this equation. It is possible that students who gave such responses are misusing the correct formula, since the value $l + s$ is one of the allowed values of j. About 16% of the students gave answers consistent with this error.

B. Underlying Difficulty: Failure to account for the fact that the directions of *L* and *S* are not well-defined

Vectors are an important part of physics at many levels, and there has been a large amount of research on student understanding of vectors in introductory physics.[14-16] These and other studies have identified the difficulties that students have with addition and subtraction of vectors. For example, many students fail to recognize that the angle between two vectors is relevant when finding their sum.[16]

A large fraction of the students answered question 9.3a in way that was consistent with the use of either $J^2 = (L + S)^2$ or $J^2 = L^2 + S^2$. The students who used these equations—which would be correct for vectors that are either parallel or perpendicular, respectively—failed to account for the fact that J, L, and S are vectors whose directions are not well-defined. Very few students gave responses indicating that they had consciously considered the vector nature of angular momentum prior to answering. The students who made use of the equation $j = l + s$ also overlooked the fact that the sum of two vectors can have a range of possible magnitudes depending on the relative orientation of the vectors that are being summed. In all, about 40% of the students gave answers to question 9.3a that did not correctly account for the indeterminacy of the directions of the spin and orbital angular momentum vectors.

The results to several alternate questions help illustrate the ways in which students treat vectors and indeterminacy, particularly with regards to angular momentum. These questions have been asked of students at many different levels of instruction, including introductory physics students and students in a sophomore-level introduction to quantum mechanics. The results of each question are given in the three numbered subsections below.

1. What are the allowed values of *j*?

Students (N=81) in a sophomore-level introduction to quantum mechanics were asked the following question on their final exam: "What are the possible values of j for an electron in the $n = 3$, $l = 2$ orbit of the hydrogen atom?" When the question was phrased in this way, asking for the possible values of the quantum number j rather than the possible outcomes of a measurement of J^2, 95% of the students were able to give the correct answer (that $j = 3/2$ or $5/2$). Most of the students (69%) gave correct explanations (that j ranges from $|l - s|$ to $l + s$ in integer steps). When Task 9.3 was administered to the same population of students in the junior-level quantum course (eight months later), only about 10% could identify the possible outcomes of a measurement of J^2. While students may be able to use a formula to answer some questions about addition of angular momentum, they do not always seem to recognize the applicability of that formula when asked to answer slightly different questions.

2. What are the allowed values of $\vec{\mathbf{S}} \cdot \vec{\mathbf{S}}'$?

Figure 9.4 shows the text of a question that was administered on the final exam of the second quarter of a junior-level quantum course—after essentially all undergraduate instruction on

quantum mechanics. The question asked the students (N=59) to determine the hyperfine splitting for a nitrogen atom. They had previously been asked to determine the total spin of the electrons (S = 3/2, although some students answered as if the total spin were S = 1/2). As a hint, the students were given the relationship between the dot product of two spins and the squares of the magnitudes of those spins. Thus, in order to answer, the students simply needed to identify the possible values of the quantity $\left(\vec{S}+\vec{S}'\right)^2$, which is very similar to question 9.3a. In this case, since S = 3/2 and $S'=1$, the only possible values are those associated with the quantum number S_{total} = 5/2, 3/2, and 1/2.

> Consider a nitrogen atom with electron configuration $1s^2 2s^2 2p^3$.
> The hyperfine splitting is due to the interaction H_{hf} between the electron and nuclear spins magnetic moments, which you may approximate as $A\vec{S}\cdot\vec{S}'$, where A is a constant. Assuming a total nuclear spin of $S'=1$, determine the possible values of $A\vec{S}\cdot\vec{S}'$ and the maximum hyperfine splitting ΔE_{hf}. Hint: use the identity $\vec{S}\cdot\vec{S}'=\left[\left(\vec{S}+\vec{S}'\right)^2-\vec{S}^2-\vec{S}'^2\right]/2$.

Figure 9.4: An exam question requiring addition of angular momentum

The question was administered on the final exam of the second quarter of a junior-level quantum mechanics course. Prior to this question, the students were asked to determine that the total spin of the electrons was S = 3/2, although some students answered as if the total spin were S = 1/2 instead.

Very few students (8%) were able to answer this question correctly. About 22% of the students stated that there would be two possible cases, corresponding to the two spin vectors being either "aligned" or "anti-aligned". Twenty percent of the students gave only one possible value for this quantity, corresponding to the sum of the quantum numbers for S and S'. About 15% of the students explicitly used the relation $\left(\vec{S}+\vec{S}'\right)^2=S^2+S'^2+2SS'$. This distribution of

errors is very similar to what was discussed in section 9.4A in response to question 9.3a. These results continue to suggest that most students lack a fundamental understanding of angular momentum in quantum mechanics, and that they struggle when asked to apply the concepts they have learned to new contexts.

3. What is the magnitude of the total angular momentum for a planet?

Figure 9.5 shows a "classical" version of question 9.3a that we administered to students at four different points during the course of undergraduate instruction. On this question, the students consider a planet with angular momentum both due to its motion around the sun, the direction of which is known, and due to rotation about its own axis, the direction of which is *not* known. The students are then asked to determine the possible value(s) of the total angular momentum of the planet, which is defined to be the vector sum of the planet's orbital and spin angular momentum. The question was administered to four different populations: students who had completed a quarter of introductory mechanics (denoted by PHYS 121), a full year of introductory physics (PHYS 123), a one-quarter introductory quantum course for sophomores (PHYS 225), and a two-quarter quantum mechanics course for juniors (PHYS 325). The students in PHYS 225 had also completed a full year of introductory physics, while the 325 students had completed both introductory physics and the PHYS 225 course.

Imagine that you are an astronomer who has just discovered a new planet in the Solar system! You have made several measurements of this planet and determined the following information:

The planet is orbiting the Sun in the same plane as the Earth. The magnitude of the angular momentum due to this motion is $\left|\vec{L}_1\right| = 2.5\times10^{30}\,\mathrm{J\cdot s}$.

The planet is also spinning about an axis that points in an *unknown* direction. The magnitude of the angular momentum due to this motion is $\left|\vec{L}_2\right| = 1.2\times10^{30}\,\mathrm{J\cdot s}$.

You are interested in determining the *total* angular momentum of this planet, which is defined to be $\vec{L}_{\mathrm{tot}} = \vec{L}_1 + \vec{L}_2$.

Determine the value of the *magnitude* of the total angular momentum of this planet. If more than one value is possible, please list all possible values. Explain your reasoning.

Figure 9.5: A "classical" version of question 9.3a

In this question, students are asked to determine the possible values of the magnitude of the total angular momentum of a planet whose spin angular momentum has an unknown direction. The question was administered to students after one quarter and after three quarters of introductory physics, as well as to students at the end of both sophomore- and junior-level quantum courses.

The correct answer is that the total angular momentum of the planet could be any (continuous) value between $1.3\times10^{30}\,\mathrm{J\cdot s}$ and $3.7\times10^{30}\,\mathrm{J\cdot s}$. This is because the direction of the planet's spin axis is not known, and so the two angular momentum vectors could add with any orientation. It is for this reason that the question can be considered a classical equivalent of question 9.3a, which also asks the students to determine values associated with the total angular momentum when the directions of the individual angular momenta are not known. Very few students at the introductory and sophomore levels answered this question correctly, although those who did also gave correct reasoning, and performance was slightly better at each successive level of instruction. Even at the end of junior-level quantum mechanics, a course that

included tutorial instruction on addition of angular momentum in quantum mechanics, only 57% gave the correct answer. The percentage from each population who gave each of the most common answers is shown in Table 9.4.

Table 9.4: Results from classical addition of angular momentum

The percentage of the students from four different populations who gave each of the most common answers to the classical version of question 9.3a is shown. The row corresponding to the correct answer is bolded.

Answer		PHYS 121 (N=753)	PHYS 123 (N=274)	PHYS 225 (N=82)	PHYS 325 (N=47)
Correct:	$1.3\times10^{30} \le \left\vert\vec{L}\right\vert \le 3.7\times10^{30}$	**9%**	**18%**	**33%**	**57%**
Sum or difference:	$\left\vert\vec{L}\right\vert = 1.3\times10^{30}$ or 3.7×10^{30}	49%	32%	23%	15%
Sum only:	$\left\vert\vec{L}\right\vert = 3.7\times10^{30}$	21%	38%	15%	9%
Perpendicular only:	$\left\vert\vec{L}\right\vert = 2.8\times10^{30}$	5%	6%	15%	9%

The most common incorrect answer was to give two values: $1.3\times10^{30}\,\mathrm{J\cdot s}$ and $3.7\times10^{30}\,\mathrm{J\cdot s}$. The following response is an example of the kind of reasoning given in support of this answer.

> *"Because the direction of the second angular momentum is unknown it could be either the same or opposite direction of L_1. You would add L_2 to L_1 if the direction is the same and you would subtract L_2 to L_1 if the directions are opposite. This is because [angular] momentum is a vector so direction matters."* (PHYS 121)

This student notes that the direction is important because angular momentum is a vector; however, they incorrectly assume that the directions of L_1 and L_2 must be either parallel or anti-parallel, when in fact they could have any relative orientation. Most of the students who gave this answer provided explanations consistent with the example above, although few of them explicitly described angular momentum as a vector. A few of these students also stated that one

or both of the values could be either positive or negative. This error was present in about 5% of the responses given to question 9.3a for the quantum case.

Another common incorrect answer was $3.7\times10^{30}\,\mathrm{J\cdot s}$, which is equal to the sum of the planet's two forms of angular momentum. In fact, most of the students who gave this answer did not give any explanation other than indicating that they had added together the two numbers given in the problem. Some students explicitly referenced the equation given to them in the text of the question, $\vec{L}_{\mathrm{tot}} = \vec{L}_1 + \vec{L}_2$. This error is consistent with treating angular momentum solely as a scalar rather than a vector, and appears very similar to the errors discussed in sections 9.4A3 and 9.4A5 for the quantum case. The error discussed in section 9.4A4 (treating *L* and *S* as if they are perpendicular) was present in about 5-15% of the responses to the classical question, which was more common in response to question 9.3a (about 20%).

The fact that angular momentum does not have a well-defined direction is a defining feature of quantum mechanics. We have found that physics students at many levels find it difficult to add two vectors with unknown direction, even in classical physics. In addition, the context of angular momentum seems to be particularly difficult, as when we asked a very similar question about the sum of two *position* vectors for which the relative direction was unknown, substantially more students were able to answer correctly (about 64% of 470 students in the first week of introductory physics). Because students struggle with this question even in the context of classical angular momentum, it is not surprising that they struggle when they are asked a similar question for a quantum mechanical system. The evidence indicates that many students do not understand how the indeterminacy associated with angular momentum in quantum

mechanics impacts the possible values of the total angular momentum, and that this lack of understanding goes beyond a simple failure to add vectors.

9.5 – Summary and Discussion

The results discussed in this chapter suggest that angular momentum in quantum mechanics is a conceptually challenging subject. The presence of such a wide variety of errors involving so many different quantities on tasks administered after lecture instruction suggests that many students do not understand the model for angular momentum that they have been taught. While few students correctly identified the possible outcomes of angular momentum measurements, there were relatively few incorrect lines of reasoning used consistently across multiple questions. This indicates not only that the students lack an understanding of the correct model, but also that they do not seem to have a coherent alternative model. Instead, the students appear to be using pieces of what they have been taught to answer Tasks 9.1 and 9.3, as identified in sections 9.2A and 9.4A, respectively.[10]

It should be noted that this lack of a model is unlike many of the difficulties that have been discussed in previous chapters (*e.g.*, the tendency to confuse the time dependence of different quantum mechanical quantities, section 2.4A). Most of those difficulties represent incorrect lines of reasoning that many students apply somewhat consistently to a variety of questions. However, the evidence suggests that many students do not have a cohesive model, correct or incorrect, for angular momentum in quantum mechanics. In particular, many students appeared to have difficulty distinguishing between different quantities: for example, the quantum numbers l and m_l (see section 9.2A2), the possible outcomes of L_z and L_x (see section 9.2B), spin and

orbital angular momentum (see section 9.2A3), and total angular momentum with all of these (see section 9.4A1). Furthermore, many students made multiple errors across Tasks 9.1, 9.2, and 9.3, many made more than one error on the same question, and 10-20% either stated that they did not know how to answer, gave responses that were unclear, or gave no explanation at all.

There is evidence that some students are aware of their own lack of understanding of angular momentum. The following quote was given by a student during an interview conducted after all instruction on angular momentum in quantum mechanics (both lecture and tutorial).

> *"I honestly found this angular momentum thing to be like impossible to understand. We just memorize things that we learn so every single time I want to answer a question I have to go look it up again because there seems to be at least no intuition."*

This student, after also discussing the impact of the textbook, the lectures, and the tutorials on their understanding of angular momentum, went on to say:

> *"I think all the pieces were there but they never really told you how to fit it all together."*

The student also noted that while the tutorials were somewhat useful, they were still left with an incomplete picture of how the various aspects of angular momentum fit together. These quotes indicate that some students are able to recognize that they have not developed a conceptually robust model for angular momentum in quantum mechanics.

Many of the other difficulties discussed in this chapter are similar to difficulties identified in other areas of quantum mechanics. For example, some student reasoning about angular momentum evokes ideas from classical physics, which is also common for position, momentum, and energy (see, for example, section 4.2B3). Some angular momentum questions require

students to change from one basis to another, which is not easy for students at any level of quantum mechanics, as discussed in section 7.2. In particular, many students do not correctly interpret how indeterminacy relates to probabilities for components of angular momentum in different directions. While some of these difficulties are present in other quantum mechanical contexts, the fact that angular momentum brings so many of them together makes it a particularly challenging topic for both students and instructors.

9.6 – Notes and References

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[9] See, for example, page 272 of N. Zettili *Quantum Mechanics: Concepts and Applications*, John Wiley & Sons (2001) or page 4 of D. McIntyre, *Quantum Mechanics: A Paradigms Approach*, Addison-Wesley (2012).
[10] These results are consistent with a model of student thinking in terms of phenomelogical primitives. For further discussion of this framework, see: D. Hammer, A. Elby, R. E. Scherr, and E. F. Redish, "Resources, Framing, and Transfer," in *Transfer of Learning: Research and Perspectives* (2004); E. F. Redish, "A Theoretical Framework for Physics Education Research: Modeling Student Thinking," in Proceedings of the International School of Physics, "Enrico Fermi" Course CLVI (2004); L. Bao and E. F. Redish, "Model Analysis: Representing and Assessing the Dynamics of Student Learning," *Physical Review Special Topics – Physics Education Research* **2**, 010103 (2006); and M. Sabella and E. F. Redish, "Knowledge Organization and Activation in Physics Problem Solving," *American Journal of Physics* **75**, 1017 (2007).
[11] See, for example: J. H. Albert, "College Students' Conceptions of Probability," *The American Statistician* **57** (1), 37 (2003).
[12] This is one of the postulates of quantum mechanics. See, for example, N. Zettili *Quantum Mechanics: Concepts and Applications*, John Wiley & Sons (2001).
[13] For more detail about student difficulties with addition of vectors, see chapter 6 in this dissertation.
[14] P P. Barniol and G. Zavala, "Force, Velocity, and Work: The Effects of Different Contexts on Students' Understanding of Vector Concepts Using Isomorphic Problems," *Physical Review Special Topics – Physics Education Research* 10 (2014).

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Chapter 10
Development and Assessment of Tutorial Curriculum for Angular Momentum

"We can measure one of the directions precisely (x, y, z) but not the other two."

—Junior-level quantum mechanics student, final exam

This chapter discusses the development of curriculum at UW to address the student difficulties with spin, orbital, and addition of angular momentum described in the previous chapter. There have been several variations of this curriculum since it was first administered in 2003. We also assess the effectiveness of this curriculum by discussing the results from several questions administered on course exams.

There have been a few efforts to design material for some aspects of angular momentum in quantum mechanics. For example, m simulations are available to aid students in visualizing the Stern-Gerlach experiment,[1] and some textbooks and curriculum have been developed specifically for the treatment of spin and two-state systems.[2] The University of Pittsburgh (UP) has developed and tested Quantum Interactive Learning Tutorials (QuILTs) on both the Stern-Gerlach experiment and addition of angular momentum.[3-5] In the QuILT on the Stern-Gerlach experiment, students use the SPINS program[1] to explore different sequences of measurement using simulated Stern-Gerlach apparatus. Students also compare these experiments to applying successive polarizers to beams of photons. Pre- and post-tests by UP have tended to ask about the probabilities resulting from various Stern-Gerlach experiments for different initial states.

Their preliminary evaluation (N=35) indicates that this activity helps students to answer these questions correctly.

The QuILT for addition of angular momentum guides students to build vectors and matrices in Hilbert spaces appropriate for a variety of spin systems, including the "coupled representation" for two spin-1/2 particles. The UP pre- and post-test questions for this topic ask students to choose a set of basis vectors and write two Hamiltonians in that basis (as matrices with the appropriate dimensions). The pretest asks about two spin-1/2 particles, while the post-test asks about a system consisting of a spin-1 particle and a spin-1/2 particle. Their preliminary evaluation (N=25) suggests that the curriculum helps students to answer these questions correctly. However, they note that there is still room for improvement.

The pre- and post-test questions asked at the University of Pittsburgh differ substantially from those that have been asked at UW. We have focused on observables rather than on mathematics by asking students to determine the possible outcomes of various measurements and their associated probabilities. Our questions therefore probe different aspects of addition of angular momentum than has been previously reported. Additionally, no prior publications have discussed student ideas related to orbital angular momentum, or described and assessed curriculum specifically targeted to improve student understanding of this topic.

10.1 – Development and Assessment of Original Tutorial Curriculum: *Addition of angular momentum*

In his PhD dissertation, Crouse briefly describes a tutorial worksheet developed to help students with addition of angular momentum.[6] This worksheet can be found in the appendix of

Ref. 6. This worksheet asked students to consider the representation of angular momentum states as cones swept out by vectors with only one well-defined component. This representation is common in many quantum mechanics textbooks.[8] The worksheet took this approach because previous interactions with students had indicated that this representation was confusing and difficult for them to interpret. The worksheet tries to help students understand the representation and helps them to see both its advantages and its limitations.

In this worksheet, the students construct the "cone" representation for (1) the spin state $|s, m_s\rangle = |½, ½\rangle$ and (2) the spherical harmonic $Y_2^0(\theta, \phi)$. In the first case, they are asked whether or not each of the quantities S_x, S_y, and S_z is well-defined. The students then repeat this exercise for the total angular momentum of a particle associated with *both* the spherical harmonic $Y_2^0(\theta, \phi)$ and the spin state $|s, m_s\rangle = |½, ½\rangle$. The students are asked if it is possible to draw more than one cone consistent with the known values for *l* and *s*, but it does not have any questions to help guide the students to recognize that there are two possible cones before providing the correct formula for the quantum number *j*.

From 2005-2008, students in the second quarter of introductory quantum mechanics worked through this tutorial after covering both spin and addition of angular momentum during lecture. Andrew Crouse, in his dissertation, describes a preliminary effort to assess the tutorial.[6] He found that students who had completed the tutorial described above did not perform well on exam questions about addition of angular momentum. In particular, only about 40% of the students who completed the tutorial correctly identified the outcomes of a measurement of J^2, with only about half providing correct reasoning. Crouse concludes his discussion of angular

momentum by stating that "a sequence of tutorials spanning several weeks [may] be necessary to see real improvement in student understanding."[6]

10.2 – Development of New and Revised Tutorial Curriculum for Angular Momentum

Since the development of the original tutorial, there have been some changes to the introductory quantum mechanics sequence at UW. Several topics, including spin and addition of angular momentum, are now taught near the end of the first quarter rather than at the beginning of the second quarter of the junior-level quantum course. Classroom interactions with students in the changed course suggested that the students required more preparation for addition of angular momentum. The results discussed in the previous section also suggest that there is still substantial room to improve the tutorial curriculum for angular momentum.

The work described in this section is the result of an effort to modify the tutorial worksheet discussed in section 10.1 so that it better addresses common student errors and difficulties with angular momentum (see chapter 9). In particular, our goal has been to help students understand the model for angular momentum in quantum mechanics in a complete and cohesive way. This project has been guided by several important factors: the results of research tasks and interviews; the methods that have proved effective at improving student understanding in both introductory physics and quantum mechanics;[10,11] and the experience of various members of the Physics Education Group at UW who have been involved in the teaching of angular momentum in quantum mechanics over the years. The curriculum currently consists of a sequence of two tutorials: (1) a new tutorial titled *Angular momentum in quantum mechanics*, and (2) a heavily

revised version of the original tutorial, still titled *Addition of angular momentum*. Both worksheets can be found in the appendix.

The original tutorial on addition of angular momentum covered a substantial amount of content for a worksheet intended to be finished in about 50 minutes. Students were asked to consider all of the following: (1) both spin and orbital angular momentum; (2) how the uncertainty principle relates to angular momentum; (3) the previously described "cone" representation; and (4) addition of angular momentum. Few students were able to complete the entire worksheet within a single class period. Thus, we created a new tutorial so that students would have more time to consider the ideas listed above. The new tutorial focuses on the first two concepts identified above: (1) the similarities and differences between L and S, and (2) the uncertainty principle. One of the goals of this new tutorial is to help students improve their understanding of the quantum model for angular momentum by explicitly considering the similarities and differences between various related but distinct quantities. (See chapter 9, especially sections 9.2 and 9.4, for a discussion of difficulties of this nature.)

A. Description of New Tutorial Curriculum: *Angular momentum in quantum mechanics*

In the first section of the new tutorial, students are guided to determine the possible results of a measurement of L_z for a particle with $l = 3$. The students first make an analogy to classical vectors. We have found that some students attempt to use classical reasoning on their own (see section 9.2A6), and we believe that it can be very productive to consider classical angular momentum first, both as a tool for making sense of quantum mechanical results and as a way of contrasting the quantum and classical cases. The students are asked to use this analogy to

identify the maximum and minimum values that m_l can take. They are also guided to recognize that the allowed values for any direction are the same as the allowed values for the *z*-direction.

In section II, the students consider the spin of an electron. Throughout this section, students compare the properties of spin to the properties of orbital angular momentum from section I. The students are asked to identify which quantities are well-defined for the state $|\chi\rangle = |½, -½\rangle_z$, and they are explicitly asked whether or not S_x is well-defined for this state.[12] They are then told that S_x is measured and given a possible result. They are then asked about the possible results of re-measuring S_z. The students are guided to recognize that measurements can change the spin state of a particle, and are asked to explain how to determine the possible outcomes of such a measurement, along with the corresponding probabilities.

The final section focuses on uncertainty. The students identify which quantities are and are not well-defined for the initial state $|\psi\rangle = |l, m_l\rangle_z = |2,1\rangle_z$. They are then asked to consider two statements that identify L_x, L_y, and L_z as being simultaneously well-defined. They are told that both statements are incorrect and asked to explain why.

Lastly, the students are asked to predict the ranking of the probabilities of a measurement of L_z after L_x has been determined to be $-2\hbar$. Many students predict that all of the possible results will be equally probable (see section 9.3A). They are given the means to resolve any inconsistencies in their prediction, and are asked to explain several important features of the probability distribution.

Due to the overall length of the tutorial, some students do not complete section III during a single, 50-minute period. In some instances, we have given students two class periods to complete the entire worksheet.

We designed several homework questions to compliment different aspects of the tutorial worksheet. These homework questions cover a wide variety of topics, including: time dependence and conservation; the "cone" representation for a spin-1/2 particle; the uncertainty principle; and how to calculate the coefficients associated with changing from the L_x basis to the L_z basis. Final versions of these homework questions may be found in the appendix.

B. Description of Revised Tutorial Curriculum: *Addition of angular momentum*

The division of the original tutorial into two separate worksheets resulted in an opportunity for students to investigate total angular momentum in greater detail. The revised version of *Addition of angular momentum* assumes that students are already familiar with the basic properties of quantum mechanical angular momentum, especially how the states are represented in Dirac notation and the fact that two components may not be simultaneously well-defined. These are all ideas covered in the new tutorial described in the previous section. Although the tutorial worksheets no longer utilize the "cone" representation (which does, however, appear in some homework problems), some students still tend to use this representation because it has been introduced in class or in the textbook.

Section I begins by asking students to identify the quantities that are well-defined for the initial state $\left|l, m_l; s, m_s\right\rangle = \left|2, 0; \tfrac{1}{2}, \tfrac{1}{2}\right\rangle$. They are then asked to identify which components of J (the total angular momentum) are well-defined, and to check that their answer is still consistent

with the uncertainty principle. Since students have already discussed the uncertainty principle for spin and orbital angular momentum in the previous tutorial, this exercise requires less time than it did in the original version of the worksheet. Students are also asked to determine the value of the quantum number m_j (corresponding to J_z) and to predict whether or not the magnitude of the total angular momentum vector will be well-defined.

The second section consists of an exercise that uses a classical argument, like the one used in the previous tutorial, to understand why J^2 is not well-defined. The students are asked to determine the largest and smallest values of J^2 treating L and S as classical vectors whose directions were not well-defined. Then they use their knowledge of how angular momentum is quantized to determine the allowed values of j (and J^2) for this system.[13] The students are also presented with a student dialogue of the most common incorrect patterns of reasoning (see section 9.4) and asked to explain why they are not correct.

Section III states the correct quantum mechanical formula for addition of angular momentum. The students are then guided to rewrite the initial state in the $\left|l,s;j,m_j\right\rangle$ basis by identifying what terms should be included and then by using the Clebsch-Gordon table to evaluate these terms. The students are also told that the same formalism can be applied to sums of angular momenta other than $L + S$. In our experience, not all students reach section III in a 50-minute period, so the questions that require the most help from an instructor are located before this section.

As with the first tutorial in this sequence, we have devised several homework questions that are relevant to this tutorial. Some are intended to give the students practice applying the ideas from the tutorial, while others ask the students to consider new situations, such as a system

initially defined in the $\left|l,s;j,m_j\right\rangle$ basis. There is also a question that asks about how the "cone" representation may be used for addition of angular momenta, and another that asks the students to compare the "coupled" and "uncoupled" bases and to calculate the Clebsch-Gordon coefficients that connect them. A final version of the entire homework may be found in the appendix.

10.3 – Assessment of New and Revised Tutorial Curriculum

We have used a variety of questions on course exams to assess the effectiveness of this curriculum. All exam questions were free-response, and students were asked to explain their reasoning for every question. Not all questions were given during every quarter. Those that were given during more than one quarter varied slightly from quarter to quarter. Figure 10.1 shows a version of each exam question discussed in this section.[14]

The first three questions are based on the pretest questions from Task 9.1; questions 10.1d and 10.1e are based on the pretest questions from Task 9.3. Each question is described in more detail in chapter 9. Question 10.1f is based on a question about sequences of spin measurements (Task 4.3), and is discussed in section 4.2B. The responses to these tasks are used for the pre-tutorial results included throughout this section. The number of students who answered each pretest question varied between 74 and 194.[15] Figure 10.2 shows the results from both the pre- and post-tutorial administrations of each question. The figure shows only the percentage of the students who answered each question correctly with correct reasoning. The criteria for correct reasoning on each question are discussed in the individual subsections of section 10.3B.

Task 10.1

Consider the state of a particle given by: $|\psi\rangle = |l, m_l; s, m_s\rangle = |1, -1; 3/2, 1/2\rangle$, which is represented using the $|l, m_l; s, m_s\rangle$ basis. Let total angular momentum be defined by $\vec{J} = \vec{L} + \vec{S}$.

Determine the possible results of a measurement of each of the following quantities on the particle in the state above. In each case, explain your reasoning.

Question 10.1a: L^2

Question 10.1b: L_z

Question 10.1c: L_x

Question 10.1d: J_z

Question 10.1e: J^2

Suppose that L_x is measured to be $+\hbar$.

Question 10.1f

After this measurement, determine the possible results of a measurement of L_z.

Figure 10.1: Several exam questions on angular momentum

Task 10.1 was administered on exams in the junior-level quantum mechanics course. Each question asks the students to determine the possible results of a measurement. In many cases, the students were also asked to determine the corresponding probabilities; however, our analysis does not focus on this aspect of the question. The number of students who answered each question varied between 68 and 172.

On every question except for question 10.1f, performance on the post-test question was statistically greater than performance on the analogous pretest question. This suggests that the tutorials are effective at improving student understanding of angular momentum. However, the overall percentage of the students who gave correct answers with correct reasoning is still less than 40% on three of the questions, which indicates that there is still substantial room for improvement in this sequence of tutorials.

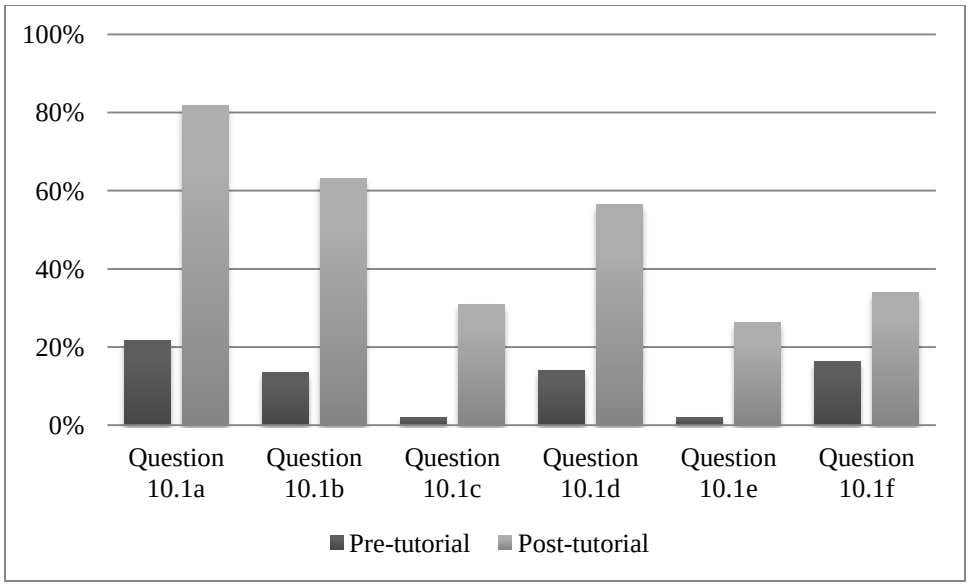

Figure 10.2: Pre- and post-test results from Task 10.1

The questions in Fig. 10.1 were also administered to students who had only completed part of the sequence of tutorials (*e.g.*, only the new tutorial *Angular momentum in quantum mechanics*). Below, we describe the various situations under which the questions have been given. Then we compare the percentage of the students from each administration who answered each exam question correctly with correct reasoning. All statistical comparisons in this chapter are done using a one-way ANOVA, followed by a *post hoc* Tukey's test if a significant difference is detected (see chapter 1 for more details about these statistical tests).

We also compare the percentage of the responses to each question that are consistent with the most common incorrect patterns of reasoning both before and after tutorial instruction. Student reasoning on each post-test question was evaluated using the same criteria as we used for the pretest questions discussed in chapter 9.

Last of all, section 10.3C discusses the results from a pilot site for the tutorials on the same set of questions.

A. Description of Student Populations and Groups

The new curriculum has been administered to students in five different quarters. In quarters two of these quarters, only the second tutorial in the sequence was used, while both tutorials were used in the remaining three quarters. Most students were all taught by the same instructor during the autumn quarters, which is when most physics majors at UW take introductory quantum mechanics. Two quarters, however, were taught during the summer to smaller populations of students, each by a different instructor. The results from tasks administered after lecture instruction in both the summer and autumn courses indicate that there is not a substantial difference between these populations before tutorial instruction.

As indicated above, most post-test questions were administered on the final exam. However, during one quarter, we re-administered Task 9.1 (corresponding to questions 10.1a-c; see section 9.1) a few days after students had completed the first two sections of the first tutorial worksheet. The task was given online rather than on an exam, and the only difference was in the initial state that we asked students to consider. We also asked one question on a midterm exam, after these students had completed the remainder of the first tutorial worksheet (but before they had completed the second worksheet).

We have grouped students across all quarters by the total amount of tutorial curriculum completed prior to answering each question. Group α consists of students who completed only the first two sections of the first tutorial (*Angular momentum in quantum mechanics*), while

group β consists of students who completed all of the first tutorial, but group had completed the second tutorial worksheet. Additionally, students in groups α and β may or may not have completed the tutorial homework associated with the first tutorial. The students in group γ had completed only the second tutorial (*Addition of angular momentum*) and its associated homework, but did not work through the first tutorial. Lastly, group ω includes only students who had completed both tutorials, and the associated tutorial homeworks, prior to answering the post-test questions. Figure 10.3 gives a visual representation of when each group responded to the questions relative to the curriculum they had completed.

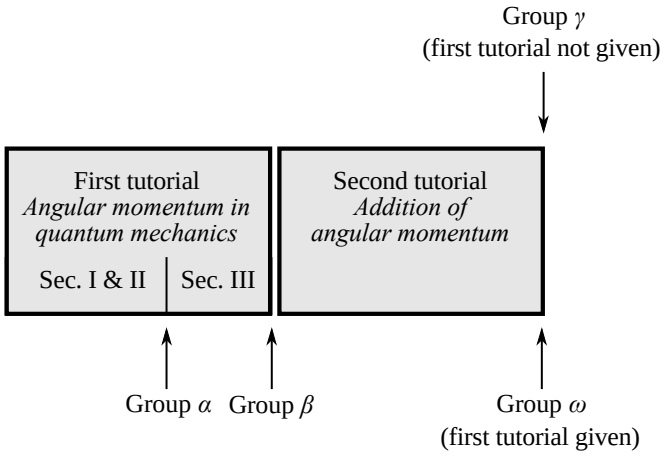

Figure 10.3: Visual representation of angular momentum tutorials

The figure shows the overall sequence of tutorials, as well as where the questions on Task 10.1 were administered to each of the four groups of students. Group γ, shown at the top of the figure, was given after the second tutorial during a quarter in which the first tutorial was not used.

It should be noted that some students are included in more than one group, since some questions were administered at different points within the same quarter. All four groups also include some students who did not attend one or more of the tutorial sessions, although a few of these students completed the curriculum on their own time.

B. Results from Pre- and Post-tests

The results for each of the questions from Task 10.1 are discussed in the following subsections. We present the percentage of the students who answered each question correctly, with and without correct reasoning, along with a discussion of the most prevalent incorrect lines of reasoning. In each case, the results from different groups of students, including results from the pretest, are compared.

1. Questions about determining the outcomes of a measurement of L^2

Figure 10.4 shows the percentage of the students who answered question 10.1a (about L^2) correctly each time it was asked, both with and without correct reasoning. Only responses that referenced the correct quantum number (l) or the corresponding eigenvalue equation for L^2 were considered to be representative of correct reasoning. On this question, relatively few students gave the correct answer without also giving correct reasoning.

The difference between the percentage who answered correctly with correct reasoning for the first two administrations of this question is statistically significant ($p<0.01$). The first tutorial on angular momentum is therefore somewhat effective at helping students answer this question. However, only about half answered correctly at this point during instruction, while 83% answered correctly after working through both tutorials. The difference between these administrations is also statistically significant ($p<0.001$). Thus, while either tutorial on its own may be helpful to the students, together they are very successful, as the percentage of the students who were able to answer correctly went from 24% to 83% and those who gave correct explanations went from 22% to 83%.

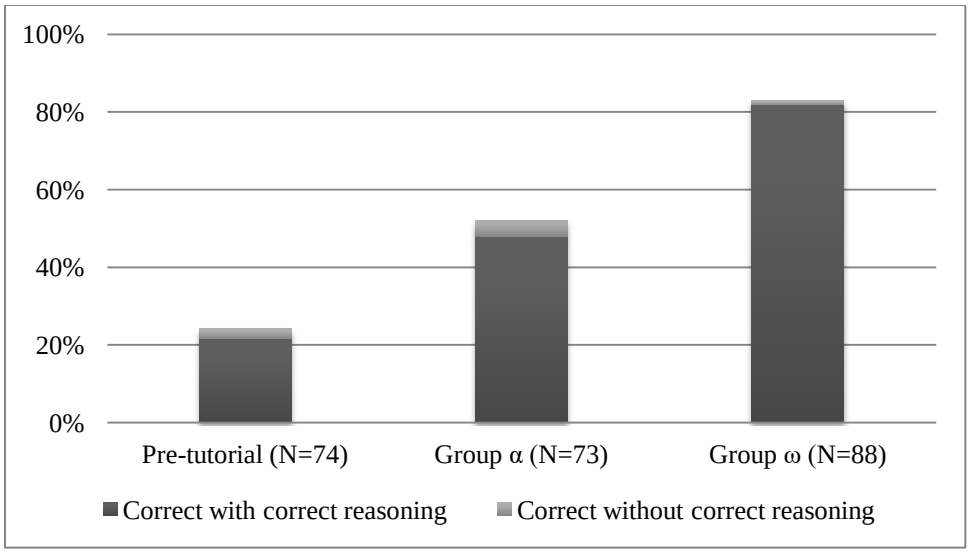

Figure 10.4: Results for question 10.1a

The percentage of correct answers to questions about L^2, both with and without correct reasoning, is shown. The results are shown for students with different amounts of tutorial instruction.

Table 10.1 shows the percentage of the students who gave the most common incorrect answers for each population, along with the section in chapter 9 where each was discussed. The error that decreases the most after only the first tutorial is confusion between the quantum numbers l and m_l. However, the first tutorial appears to have relatively little impact on the percentage of the students who failed to distinguish between the quantum number l and its eigenvalue, or who stated that there is a range of possible values for this quantity. This suggests that the tutorial could be improved by addressing these errors more directly, though these errors become substantially less prevalent after students have completed both tutorials. After each tutorial, there is also a decrease in the percentage of the students who leave this question blank or give reasoning that is unclear, which indicates that fewer students are confused about how to answer the question after working through the tutorials.

Table 10.1: Results for question 10.1a

Shown in the table is the percentage of the students who gave common incorrect answers on pre- and post-test questions that asked them to identify the possible values corresponding to a measurement of L^2. The second column indicates the section in chapter 9 where each common incorrect answer is discussed in more detail.

Incorrect answer	Section	Pre-tutorial (N=205)	Group α (N=73)	Group ω (N=87)
Using l instead of $l(l+1)$	9.2A1	16%	22%	6%
Using m_l instead of l	9.2A2	11%	3%	2%
Range of quantized values	9.2A5	10%	18%	8%
Blank or unclear reasoning	N/A	20%	8%	3%

2. Questions about determining the outcomes of a measurement of L_z

Figure 10.5 shows the percentage of the students who answered this question correctly (both with and without correct reasoning) for several different group. Only students who identified that the eigenvalues of the L_z operator (*i.e.*, m_l) correspond to the possible results of this measurement were considered to have correct reasoning. The difference between the percentage of the students who answered correctly with correct reasoning on the pretest and on the post-test during each individual quarter is statistically significant ($p<0.001$ in each case). The only other statistically significant difference ($p<0.05$) is between group α and group ω. This implies that while the first tutorial is helpful, the full sequence of tutorials is more helpful.

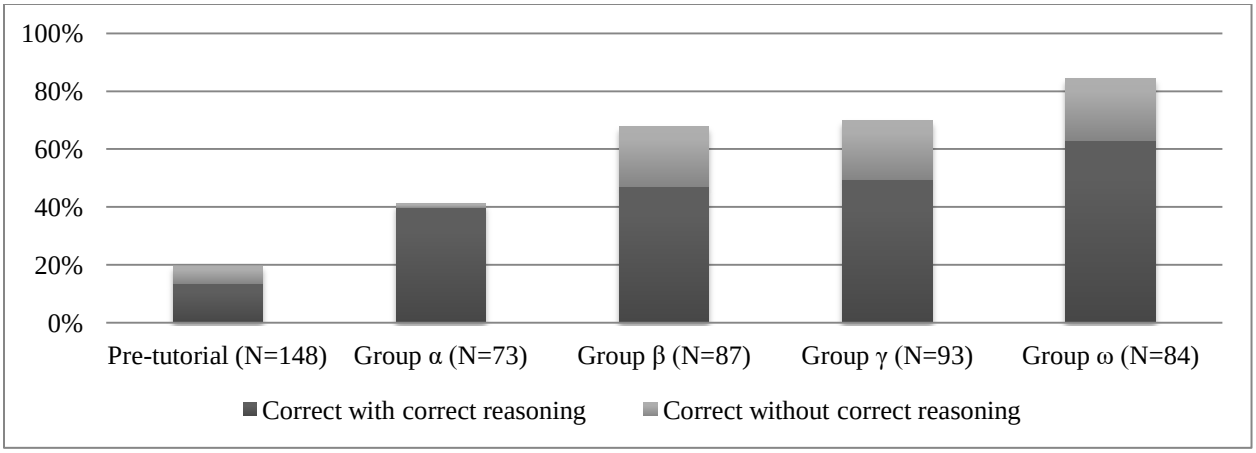

Figure 10.5: Results for question 10.1b

The percentage of correct answers to questions about L_z, both with and without correct reasoning, is shown. The results are shown for students with different amounts of tutorial instruction.

Table 10.2 shows the distribution of several common incorrect answers on both the pre- and post-test questions. The most common incorrect answer on the pretest (35%) was that any integer value between $-l$ and l times $\hbar$ can be measured for L_z. Answers consistent with confusion between spin and orbital angular momentum were the second most common answer (18%). A small fraction of the students on the pretest also answered that L_z is given by $l\hbar$ instead of $m_l\hbar$ (10%). Each of these three types of responses was relatively uncommon when this question was asked on the final exam after all tutorial instruction.

The results from populations α and β provide some insight into which incorrect ideas may be helped the most by various parts of the tutorials. For example, very few of the students in population α (after completing the first two sections of the tutorial *Angular momentum in quantum mechanics*) gave answers consistent with confusing L and S, while the other common incorrect responses from the pretest were just as prevalent after the tutorial. By the time the students had completed the last section of the tutorial (population β), very few students used the

quantum number l to answer questions about L_x, and the percentage who stated that any quantized value is possible for L_z decreased from 40% to 25%. As noted in section 10.3B1, this response was also prevalent on questions about L^2 at this stage of instruction (before the students had worked through the tutorial *Addition of angular momentum*).

Table 10.2: Results for question 10.2b

Shown in the table below is the percentage of students who gave common incorrect answers on pre- and post-test questions asking students to identify the possible values corresponding to a measurement of L_z. The second column indicates the section in chapter 9 where each common incorrect answer is discussed in more detail.

Incorrect answer	Section	Pre-tutorial (N=148)	Group α (N=73)	Group β (N=87)	Group γ (N=74)	Group ω (N=68)
Range of quantized values	9.2A5	35%	40%	25%	3%	2%
Confusion with spin	9.2A3	18%	4%	5%	4%	2%
Using l instead of m_l	9.2A2	10%	18%	1%	10%	4%
Blank or unclear reasoning	N/A	11%	1%	5%	11%	5%

3. Questions about determining the outcomes of a measurement of L_x

The question from Task 9.1 that proved to be most difficult asked the students to identify the possible values of a measurement of L_x. Only about 20% of the students answered this question correctly before tutorial instruction, almost none of them with correct reasoning. Because each initial state is an eigenstate of L_z, the correct answer is that any of the allowed eigenvalues of L_x, which run in integer steps from $-l$ to l times $\hbar$, can be measured. When this question was given on exams, it asked about J_x rather than about L_x, but was otherwise identical. Students were considered to have given correct reasoning if they explained that this result is a consequence of the fact that L_z is well-defined, or if they correctly cited a relevant concept such as the

uncertainty principle. The results, including the pretest results aggregated over all quarters, are shown in Figure 10.6 below.

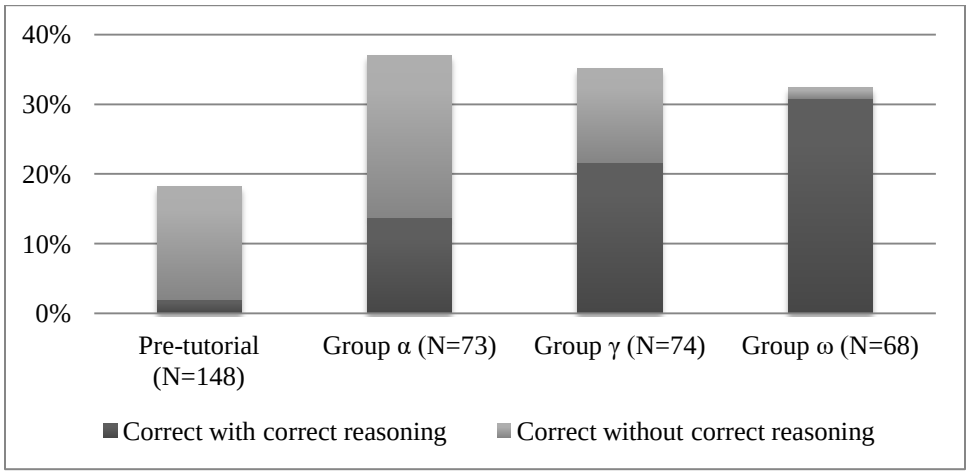

Figure 10.6: Results for question 10.1c

The percentage of correct answers to questions about L_x, both with and without correct reasoning, is shown. The results are shown for students with different amounts of tutorial instruction.

About 35% of the students in each group gave the correct answer on the post-test, which is about double the percentage of the students who answered the pretest correctly. As with questions about L^2 and L_z, however, more of the students who completed the full sequence of tutorial curriculum gave correct reasoning than the students who worked through only one tutorial worksheet or the other. Although the percentage of the students from each quarter who gave correct answers with correct explanations is statistically different from the pretest results ($p<0.05$ for population α, $p<0.001$ for the other cases), there is still substantial room for improvement. It should also be noted that the students in group α had not completed the section of the tutorial *Angular momentum in quantum mechanics* that deals specifically with measurements of L_x when L_z is well-defined.

Table 10.3 shows the most prevalent patterns of incorrect reasoning. None of these was predominant. Many students gave answers consistent with confusion between spin and orbital angular momentum (*e.g.*, including half-integer values in their answers). Others gave answers that are consistent with classical reasoning, typically by stating that any (continuous) value could be measured, often bounded by $\pm l$ or by $\pm\sqrt{l(l+1)}$. Both of these answers were relatively uncommon on the questions given on the final exam, but they were just as common on the post-test for population α as they were on the pretest.

Table 10.3: Results for question 10.1c

Shown in the table below is the percentage of the students who gave common incorrect answers on both pre- and post-test questions asking students to identify the possible values corresponding to a measurement of L_x. The second column indicates the section in chapter 9 where each common incorrect answer is discussed in more detail.

Incorrect answer	Section	Pre-tutorial (N=148)	Group α (N=73)	Group γ (N=74)	Group ω (N=68)
Confusion with spin	9.2A3	17%	19%	0%	3%
Classical reasoning	9.2A6	19%	15%	4%	3%
Using m_l for L_x	9.2A4	8%	7%	16%	13%
Blank or unclear reasoning	N/A	18%	12%	20%	16%

The percentage of the students who stated that the quantum number m_l determines L_x is relatively small on the pretest, but it does not appear to be substantially different after the students have worked through the tutorials. This result suggests that the tutorials could be improved to address this idea more directly. The number of students who did not answer the question at all, or gave unclear explanations, also did not change, which also indicates that the tutorials should focus more on this topic.

4. Questions about determining the outcomes of a measurement of J_z

The correct answer to question 10.1d is that the possible results of a measurement of J_z are given by $\hbar$ times the values of m_j for each term in the superposition, where $m_j = m_l + m_s$. Students were only considered to have correct reasoning if they stated and used this equation; they were not required to identify the correct probabilities, although most students who answered correctly were able to determine these probabilities. This question was typically asked along with a question about L_z (see section 10.3A2) and a question about J^2 (see section 10.3A6).

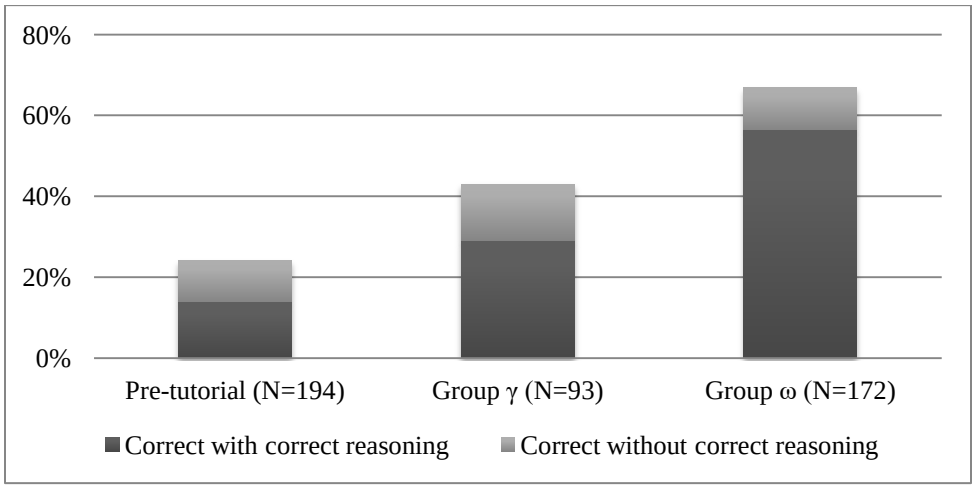

Figure 10.7: Results for question 10.1d

The percentage of correct answers to questions about J_z, both with and without correct reasoning, is shown. The results are shown for students with different amounts of tutorial instruction.

Figure 10.7 shows the percentage of the students who answered this question correctly, with and without correct reasoning. Although the percentage of the students who answered correctly with correct reasoning for each case is statistically different from the pretest results ($p<0.05$ for

group γ, $p<0.001$ for group ω), the students who completed both tutorials also performed significantly better than the students who completed only the second tutorial ($p<0.001$).

Table 10.4: Results for question 10.1d

Shown in the table below are the results for pre- and post-test questions asking students to identify the possible values corresponding to a measurement of J_z. The second column indicates the section in chapter 9 where each common incorrect answer is discussed in more detail.

Incorrect answer	Section	Pre-tutorial (N=194)	Group γ (N=93)	Group ω (N=172)
Any quantized value	9.4A1	14%	4%	10%
Confusion with J^2	9.4A1	7%	8%	4%
Confusion with S or L	9.4A1	7%	2%	1%
Combining m_l and m_s for different states	9.4A2	5%	18%	6%
Blank or unclear reasoning	N/A	32%	18%	5%

Table 10.4 shows the distribution of common incorrect lines of reasoning on this question. Because many students leave this question blank on the pretest, it is difficult to identify which incorrect ideas may be the most prevalent. As with several other quantities related to angular momentum, many students stated that it is possible to measure any value for J_z in integer steps times $\hbar$ within some defined range. Some students used the formula for the allowed values of j to determine the allowed values of m_j, while others gave answers that involve either the spin or the orbital angular momentum, instead of both. When the initial state is a superposition of many values of m_l and m_s, some students combined these in all possible permutations, rather than associating each value of m_l with the corresponding value of m_s for the same term in the superposition. Only this last error was somewhat prevalent when this question was asked after tutorial instruction. The most obvious difference between the pre- and post-test results is in the

percentage of the students who leave the question blank or provide unclear reasoning, especially for population ω.

5. Questions about determining the outcomes of a measurement of J^2

As with questions about J_z, we have also asked about J^2 on the final exam during every quarter. These questions also tended to be similar to those on Task 9.3, asking the students to identify the possible outcomes of a measurement of J^2 on a specified initial state (usually a superposition of states in the $|l, m_l; s, m_s\rangle$ basis). The correct answer to question 10.1e is that the possible outcomes of a measurement of J^2 are given by $j(j+1)\hbar^2$, where the quantum number j ranges from $|l - s|$ to $l + s$ in integer steps for each term in the superposition. Several different kinds of explanations were considered as correct reasoning for this question, such as: citing the relevant equation, expressing the initial state in terms of the $|l, s; j, m_j\rangle$ basis, and using the reasoning from the tutorials in which quantization is applied to the classical range of values determined from vector addition. Students who used the formula $j = l \pm s$, which gives the correct answer for $s = ½$ but not for larger values of s, were never considered to have correct reasoning, but were always considered to be correct in order to be consistent from quarter to quarter. The students were not typically asked to determine the probabilities corresponding to each allowed value of J^2 because it would be infeasible for them to calculate them during a timed exam.

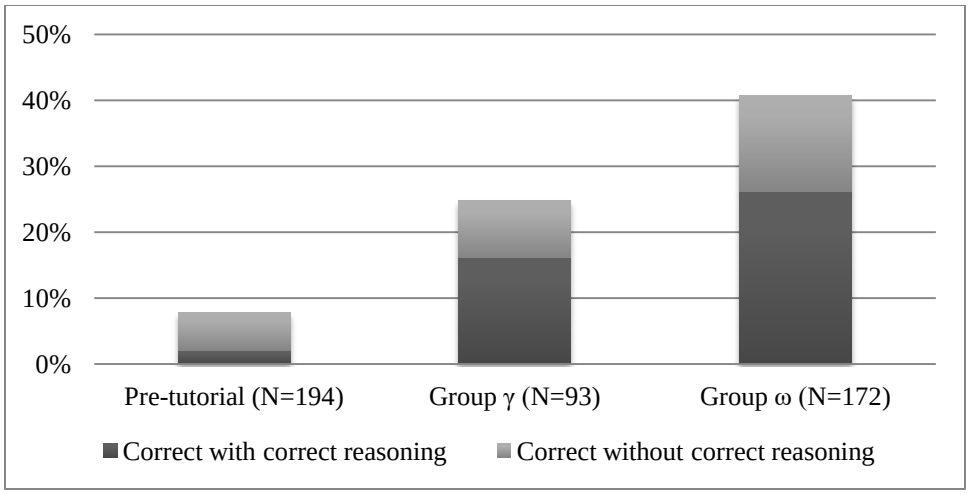

Figure 10.8: Results for question 10.1e

The percentage of correct answers to questions about J^2, both with and without correct reasoning, is shown. The results are shown for students with different amounts of tutorial instruction.

The percentage of the students from each population who answered this question correctly is shown in Figure 10.8. The difference between the pre- and post-test results (correct answer and correct explanation) is statistically significant for both populations ($p<0.01$). As with the questions about J_z, the students who had worked through both new tutorials performed better on the J^2 question than those who had only worked through the second tutorial, although this difference is not statistically significant. There is clearly room for improvement, as more than half of the students answer this question incorrectly, and only about one quarter provide correct reasoning.

Table 10.5 shows the distribution of common incorrect answers in response to questions about J^2. Note that a large number of the students left this question blank on the pretest, so these errors may be more prevalent before tutorial instruction than the table indicates. Fewer students left this question blank, or provided unclear reasoning, on the exam, especially for population ω.

The most common incorrect answers on the pretest involved students using a classical vector formula such as $J^2 = (L+S)^2$ or $J^2 = L^2 + S^2$, or using the formula $j = l + s$. These answers were also common on the post-test questions. This result suggests that the tutorials can still be improved to address these ideas better.

Table 10.5: Results for question 10.1e

Shown in the table below are the results for pre- and post-test questions asking students to identify the possible values corresponding to a measurement of J^2. The second column indicates the section in chapter 9 where each common incorrect answer is discussed in more detail.

Incorrect answer or explanation	Section	Pre-tutorial (N=194)	Population γ (N=93)	Population ω (N=172)
$j = l \pm s$ (correct w/o correct reasoning)	N/A	3%	2%	4%
Classical vector formula	9.4A3-4	24%	18%	24%
Direct sum ($j = l + s$)	9.4A5	16%	17%	15%
Blank or unclear reasoning	N/A	32%	19%	9%

6. Questions about sequences of measurements

Figure 2.9 shows the results to question 10.1f, which asks about L_z after a measurement of L_x has been made. The correct answer is that the possible outcomes of a measurement of L_z are now given by $m_l\hbar$, for integer values of m_l between $-l$ and l. (This answer is the same as the answer to question 10.1c, which asks about L_x.) Figure 10.6 shows the percentage of the students from each administration of this question who answered correctly (with and without correct reasoning). An ANOVA shows no statistically significant difference between the percentages of the students from each population who answered this question correctly with correct reasoning.

This result indicates that the tutorials do not appear to impact student performance on this question.

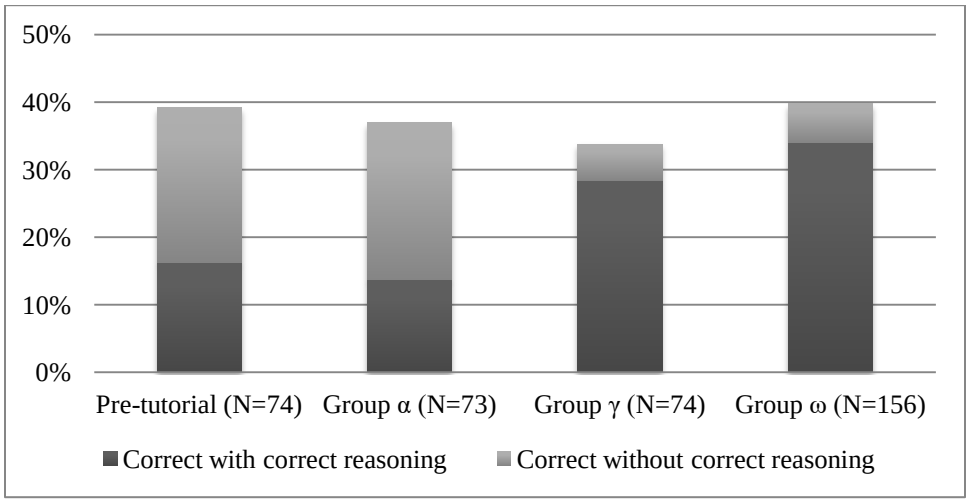

Figure 10.9: Results from question 10.1f

The percentage of correct answers to questions about a measurement of L_z after a measurement of L_x has been made, both with and without correct reasoning, is shown. The results are shown for students with different amounts of tutorial instruction.

Many students answered the pretest correctly without giving correct reasoning, which was relatively uncommon when this question was asked on post-tests. This could be due to a difference in context: spin-1/2 for the pretest *vs.* total angular momentum for the post-test. Future administrations of the pretest question could be modified to see if students answer differently for different contexts.

There were few incorrect answers given by a substantial percentage of the students. Most of the incorrect answers were similar to the incorrect answers given in response to questions about L_x with no intervening measurement, and some students made the same error on both questions. The distribution of incorrect answers also varied more from quarter to quarter than the responses to the other questions discussed in this section. As a result, we believe that more research is

necessary to draw substantive conclusions about student understanding of this topic, both before and after the administration of the tutorials.

C. Assessment of Student Performance at a Pilot Site for Tutorial Curriculum

All of the results discussed in section 10.3B were drawn from responses given by students in the junior-level quantum mechanics course at UW. The sequence of tutorial curriculum on angular momentum and several of the related exam questions were also administered to 13 students in a similar class at California State Polytechnic University – Pomona (CSPU). As with the class at UW, they used the Griffiths textbook and had lectures on angular momentum prior to working through the tutorials and the associated tutorial homework questions. The instructor of the course consulted with experienced tutorial instructors at UW on the intended implementation of the tutorials. Post-test questions were administered on the final exam for the course. These students were not given the pretests.[16]

Table 10.6: Results for question 10.1f

Shown in the table below is the percentage of correct answers with correct reasoning given by students at a pilot site for the tutorials and for analogous exam questions given at UW. Only results from students at UW who completed both tutorial worksheets (population ω) are included in this table.

University	L^2	L_z	L_x	J_z	J^2	L_z after L_x
UW (N varies)	82% (N=88)	63% (N=84)	31% (N=68)	56% (N=172)	26% (N=172)	34% (N=156)
CPSU (N=13)	100%	77%	38%	69%	31%	31%

Table 10.6 shows the percentage of the students at CPSU who answered each of the questions administered at that site correctly with correct reasoning. The table also shows the

percentages for the students at UW who answered an analogous question after all tutorial instruction.

On each question, the students at CPSU seemed to perform at about the same level as the students at UW, although there are not enough students at the pilot site to perform a useful statistical test to compare the percentages. The results are encouraging and suggest that the tutorials can be just as effective when given at another university. This result has been observed at the introductory level[17] and noted in section 5.1F for other quantum concepts.

10.4 – Discussion of Results

Overall, the results from a series of post-test questions administered after the new sequence of tutorials on angular momentum in quantum mechanics were significantly better than the pre-tutorial results on nearly every question. The results suggest that the tutorials help improve student ability to identify the possible outcomes of measurements of most quantities related to angular momentum. The students perform particularly well (55% or greater answered correctly with correct reasoning) on post-test questions that asked about L^2, L_z, and J_z. They perform less well (about 20-30% correct with correct reasoning) on questions that ask about L_x and J^2. However, the latter results are still superior to performance on pretest questions that ask about the same quantities. Only on questions that ask about a measurement of L_z after an intervening measurement of L_x was there no significant difference between the pre- and post-test results. It is clear that there is still substantial room for improvement in some areas of the tutorials, and we have made some revisions to the tutorials to incorporate these findings, though they have not yet been tested.

Furthermore, students who had completed only part of the tutorial sequence prior to the administration of the post-test tended to perform less well than students who had completed the entire sequence of two tutorials. In particular, results were poor when only the modified version of the tutorial *Addition of angular momentum* was given to the students, whereas the inclusion of the tutorial *Angular momentum in quantum mechanics* helped students perform better, even on questions about J_z and J^2, quantities only discussed on the second tutorial. Interestingly, the reverse was true, as well: students who completed both tutorials performed better on questions about quantities such as L_z and L^2 than students who completed only one tutorial or the other. Additionally, the difference is apparent in more than just the percentage of students who are able to give the correct answer: on many questions, there were clear improvements in the quality of the explanations given by the students, as evidenced by the larger percentages of correct answers with correct reasoning.

Together, all of these results suggest that the sequence of tutorials as a whole is guiding the students to build a coherent framework that can help improve their understanding of how different various angular momentum concepts fit together in quantum mechanics. While particular sections of individual tutorials may help target certain common student errors, extensive and consistent exercises in a variety of different contexts are required for students to develop a robust understanding of the model for angular momentum in quantum mechanics. Since our findings in chapter 9 showed that many students appear to lack such a cohesive model for angular momentum, the progress that the tutorials have made at furthering student understanding is promising. However, there remains work to be done, as many students still struggle to apply what they have learned to certain situations.

10.5 – Notes and References

[1] Examples of simulations for the Stern-Gerlach experiment include PhET (https://phet.colorado.edu/), QM SPINS (http://www.opensourcephysics.org/items/detail.cfm?ID=7011), and QuVis (http://www.st-andrews.ac.uk/physics/quvis/index.php).
[2] D. McIntyre, *Quantum Mechanics: A Paradigms Approach*, Addison-Wesley (2012).
[3] C. Singh, "Interactive Learning Tutorials on Quantum Mechanics," American Journal of Physics 76, 400 (2008).
[4] G. Zhu and C. Singh, "Improving Students' Understanding of Quantum Mechanics via the Stern–Gerlach Experiment," *American Journal of Physics* **79**, 499 (2011).
[5] G. Zhu and C. Singh, "Improving Student Understanding of Addition of Angular Momentum in Quantum Mechanics," *Physical Review Special Topics – Physics Education Research* **9**, 010101 (2013).
[6] A. Crouse, "Research on Student Understanding of Quantum Mechanics as a Guide for Improving Instruction," PhD dissertation, University of Washington (2007).
[8] See, for example, D. J. Griffiths *Introduction to Quantum Mechanics*, 2nd edition, page 165 (figure 4.9).
[9] These results are also reported in Ref. 6; some numbers may be slightly different due to differences in how certain responses were classified.
[10] See, for example, P. S. Shaffer and L. C. McDermott, "Research as Guide for Curriculum Development: An Example from Introductory Electricity. Part II: Design of Instructional Strategies," *American Journal of Physics* **60**, 1003 (1992).
[11] See, for example, chapter 5 and chapter 8 in this dissertation, and G. Passante, P. J. Emigh, and P. S. Shaffer, "Testing Tutorials in Upper-division: An Example from Quantum Mechanics," presented at the *Physics Education Research Conference 2014*, Minneapolis, MN (2014).
[12] This section of the tutorial has been specifically tailored to match the instruction on spin-1/2 that students at UW have in a prior course. The students are expected to base their answers on the results of Stern-Gerlach experiments encountered in that course.
[13] This process is not meant to be a rigorous argument for the quantum case, but rather is an analogy that can make the quantum result more understandable. Most undergraduate textbooks, including Griffiths, do not present the full quantum calculation, but instead present only the result.
[14] It would be natural to compare student performance after the new sequence of tutorials to the performance of students after the original tutorial worksheet (see section 10.1). However, there are many differences between the populations that make this comparison difficult. For example, the population in 2005 included only students enrolled in the second quarter of junior-level quantum mechanics, who tended to be those who performed well during the first quarter, whereas the other populations (see below) included all students from the first quarter. The students from 2005 had several additional weeks of instruction, and there have been substantial changes to the organization of the quantum mechanics course since 2005. The instructors for the courses were also different, and no pretest data was gathered for 2005. Because there are so many differences between the populations, we do not compare results between them, but rather focus our analysis on populations that we know to be similar.
[15] The pre-tutorial results are aggregated across all five quarters in which each question has been asked. These results tend to be similar from quarter to quarter. Results from other quarters, especially when the pretest was administered paper rather than online, are not included in this chapter. See chapter 9 for a more complete description of the pretest results.
[16] Results from other pilot sites on different pretests have not tended to vary across multiple universities. See, for example, P. J. Emigh, Gina Passante, and P. S. Shaffer, "Student Understanding of Time Dependence in Quantum Mechanics," *Physical Review Special Topics – Physics Education Research* **11**, 020112 (2015).
[17] See, for example, N. D. Finkelstein and S. J. Pollock, "Replicating and Understanding Successful Innovations: Implementing Tutorials in Introductory Physics," *Physical Review Special Topics – Physics Education Research* **1**, 010101 (2005).

Chapter 11
Conclusion

"I felt that I did better than I might have in this course because of the ideas brought to me through the tutorial sections."

—Junior-level quantum mechanics student, end-of-quarter survey

This dissertation presents results from a long-term investigation of student understanding of quantum mechanics. A major focus has been to improve the teaching and learning of this challenging and important topic in various courses at the sophomore, junior, and graduate levels. The procedures we have used are similar to those that have proved effective for improving student understanding of other topics in physics. First, we conduct research to identify what students do and do not appear to understand after lecture instruction. Then, we design curriculum based on this research to address common conceptual and reasoning difficulties. Lastly, we assess the impact of the curriculum on student learning, and revise, modify, and test the curriculum in a continuous cycle until a significant and reproducible impact on student performance has been achieved.

The research has involved the design of many written questions to examine student thinking about important concepts and principles. In many cases, we have used multiple questions to probe similar and related ideas in a wide range of contexts, from the infinite square well potential to spin and angular momentum. An important finding, discussed throughout this dissertation, is that students often express similar conceptual and reasoning difficulties in their study of quantum mechanics, and that these difficulties often persist despite instruction. Even when the known

problems are targeted explicitly by instruction, many remain or resurface when the physical context is changed. The specific difficulties do not appear to be superficial, but rather reflect students' failure to interpret the formalism of quantum mechanics, and their lack of understanding of the fundamental postulates.

The findings presented in this dissertation have guided the development of a research-based and research-validated curriculum: *Tutorials in Physics: Quantum Mechanics*.[1] An initial version of these materials has been developed on the basis of research conducted prior to the investigation presented in this dissertation.[2] These had been used and assessed to a limited degree in a two-quarter, junior-level sequence of quantum mechanics courses at UW. The research and classroom experience discussed here has since contributed to substantial revisions, rewriting, and expansions of the tutorials.

We have made an effort to assess the impact of the tutorials in two different ways: (1) do the tutorials help students answer conceptual, qualitative questions correctly with correct reasoning, and (2) does the incidence of the most common lines of incorrect reasoning decrease after the tutorials are administered? The results have been promising: the post-test questions are typically answered correctly by substantially more students than the pretest questions—in many cases, by 50% or greater—while there is a corresponding decrease in the number of students using incorrect or unproductive lines of reasoning. In some cases, we have demonstrated similar impacts at other universities that have served as pilot sites for the curriculum.

This chapter briefly summarizes the findings from each of this dissertation's three parts (section 11.1). This section also presents an overview of the most prevalent patterns in student

reasoning that we have observed during the course of the investigation chronicled in this dissertation. Lastly, section 11.2 discusses several avenues for future research.

11.1 – Summary of Results

Each of the three parts of this dissertation focus on exploring different interrelated aspects of student understanding of quantum mechanics: time dependence and measurements (part I); vector spaces (part II); and angular momentum (part III). Some of the research discussed in any individual part is interrelated with the research discussed in each of the others. For example, we have observed that some student difficulties appear to transcend context, and arise in student responses across the different contexts examined in this dissertation. Furthermore, the investigations discussed in each part were carried out essentially simultaneously, and thus the results have influenced and contributed to each other. In this section, we summarize the research that was directly addressed in each of the three parts, and also draw attention to the commonalities that stretch across them.

Part I: The focus of part I (chapters 2-5) is on student understanding of quantum measurements and time dependence. The overarching goal of this investigation has been to examine the ideas that students use to make predictions about the outcomes of various measurements, and the corresponding probabilities of each outcome. Specific concepts that we have probed include the time evolution of wave functions and probabilities and the effects of quantum measurements on the system. Although there have been prior studies on these topics,[2-6] we examine patterns in student reasoning across many different contexts spanning multiple years of instruction on quantum mechanics.

In chapter 2, we discuss several common student difficulties with time dependence in quantum mechanics. The most prevalent and persistent of these is a tendency of students to associate a single time-dependent phase with any wave function. That is, when students are given an initial state, they often determine the state at a later time by multiplying the state by one time-dependent phase, even when more than one phase would be appropriate (such as for a superposition of energy eigenstates with distinct eigenvalues). This tendency is especially common for wave functions that are not represented, or are not easily represented, in terms of energy eigenfunctions. This finding suggests that students do not correctly associate time dependence with the expression of a quantum state in terms of the energy eigenstates of the system.

We also demonstrate that many students appear to confuse the time dependence of various quantities. One prominent example is that students often ascribe the time dependence of the wave function to the probability density, and *vice versa*. For example, some students appear to believe that if the wave function depends on time then the probability density also depends on time—in some cases, students explicitly state that the time dependence of these two quantities is *exactly* the same. Furthermore, it is not uncommon for students to exhibit similar confusion between the time dependences of other quantities. Another example we have observed is that many students, even students who have completed two quarters of graduate instruction in quantum mechanics, seem to believe that all states with probability densities that depend on time also have energy probabilities that depend on time.

In chapters 3 and 4, we identify numerous student difficulties with quantum measurements. One of the most common is the belief that the state of a system does not change after a

measurement has been made. In particular, many students believe that measurements of one quantity (*e.g.*, position) will not have an effect on the possible outcomes of a measurement of a different quantity (*e.g.*, energy) or on the corresponding probabilities. We also find that students often fail to determine how the state of a system changes in time after a measurement. Some appear to believe that wave functions do not depend on time after a measurement (*e.g.*, once the position of a system has been measured, all subsequent measurements of position must give the same result as the first measurement). Others describe the evolution of the system after a measurement using ideas that are outside the model for quantum mechanics that they have learned: in particular, students often appear to believe (1) that the wave function spreads out until the probability distribution is uniform, (2) that the system eventually returns to the state of the system before the measurement, or (3) that the system decays to the ground state of the system after enough time has passed.

We describe in chapter 5 a sequence of specialized tutorial curriculum[7,8] based on the research presented in chapters 2-4. We illustrate various strategies that we use to try to help students understand the time dependence of wave functions and various probabilities. We demonstrate that this curriculum can be effective at improving student understanding of quantum measurements and time dependence. Since many student difficulties appear to persist across multiple contexts, we also describe the development of tutorial curriculum for advanced topics that build on the exercises in the sequence of tutorials for more basic topics. A preliminary assessment suggests that the advanced tutorials are also effective.

Overall, we find that student difficulties with time dependence and measurements often persist from context to context. The incorrect lines of reasoning that students use near the end of

instruction are often the same as or very similar to those that students use near the beginning of instruction in quantum mechanics. Some difficulties are even prevalent after students have completed tutorial instruction targeted at addressing the most common difficulties. Some difficulties are so persistent that they are present in the responses of physics students at the introductory, sophomore, junior, and even graduate levels.

Part II: In part II of this dissertation (chapters 6-8), we describe preliminary research into student understanding of vector spaces in quantum mechanics. Vector spaces and their properties are useful and necessary throughout all of quantum mechanics. They constitute a mathematical means for specifying and extracting information about quantum systems. There are many elements of vector spaces that prove challenging to students, especially because students are also asked to learn a new notation alongside quantum concepts that are already difficult and abstract. Furthermore, the skills necessary for students to answer questions about vector spaces often overlap substantially with those necessary for other quantum topics, such as time dependence and measurements. Thus, the results that we discuss in part II have implications for much of the research from part I. The overall structure of part II is therefore very similar to the structure of part I: first, we identify common student difficulties with vector spaces; then, we discuss tutorial curriculum designed to address those difficulties.

As part of our investigation of student understanding of vector spaces, we decided to examine student ability to add Cartesian vectors at many different instructional levels, from introductory through upper-division courses in physics (chapter 6). We began by looking at students in introductory physics courses. We found that students at this level often fail to recognize that the magnitude of a vector sum depends on the relative direction of the vectors that

are being added. Similar difficulties arise when introductory students are asked to add waves. Even junior-level students make related errors (*e.g.*, when adding wave functions with different relative phases). In chapter 6, we show that some of the incorrect lines of reasoning associated with superposition of wave functions are very similar to lines of reasoning given by introductory students in response to questions about addition of vectors and superposition of waves.

Another important aspect of vector spaces is the inner (or dot) product. In our investigation, we found that most students in sophomore- and junior-level quantum courses are able to determine the inner products for spatial vectors (chapter 7). However, many are unable to determine inner products in quantum contexts, such as for wave functions or spin-1/2 state vectors. For example, we designed a question that asks about inner products between functions represented graphically. These proved to be particularly challenging for students. Even when students are given the definition of the inner product in terms of an integral, many base their answers on whether or not the "shapes" of the functions are similar. We also describe results from questions that focus on finding inner products or probabilities that require a change of basis. The results indicate that these questions are particularly challenging for students. Furthermore, we find that students often fail to recognize that inner products are necessary to determine quantum probabilities, especially when changes of basis are required.

As with other topics, we used the research on student understanding of vector spaces to design tutorial curriculum aimed at improving student understanding. Chapter 8 describes the initial development of curriculum for topics such as superposition, inner products, and changes of basis in quantum mechanics. We also provide a preliminary assessment of this tutorial curriculum. While the results are promising, there appear to be many opportunities for further

improvements to the curriculum. In fact, since student understanding of vector spaces has not been researched in as much detail as it has for measurements and time dependence, there is much room for additional research on this topic.

Part III: The third part of this dissertation (chapters 9 and 10) focuses on student understanding of angular momentum. Research on this topic is highlighted in this dissertation because it involves many of the concepts that fall within other parts of our investigation, especially changes of basis and quantum measurement. Not only does angular momentum serve as an instance in which our research from other areas of quantum mechanics proves relevant, but it is also an area where there has been little overall prior research.

Over the course of many years in the classroom, we have found that angular momentum is one of the most challenging quantum topics for students. Not only are students asked to make use of other concepts that they have learned (*e.g.*, superposition, inner products, and measurement), but they are also expected to learn new notation and properties that are unique to the context. We have found that students make a variety of errors on relatively straightforward questions, such as when they are asked to identify the possible outcomes of angular momentum measurements (chapter 9). For example, students often use their knowledge of spin to answer questions about orbital angular momentum, and *vice versa*. We find that some topics related to angular momentum, especially addition of angular momentum, are so difficult that fewer than 10% of junior-level students give correct answers after lecture instruction. By considering the nature of the various errors that students make, we are led to conclude that many students lack a cohesive model for angular momentum in quantum mechanics.

The tutorial curriculum for angular momentum has undergone substantial evolution over a period of many years. Chapter 10 describes our effort to modify an existing tutorial that students found very challenging, and that prior research suggested could be more effective. This effort included two primary focuses: (1) developing a new tutorial designed to address the apparent lack of model identified in chapter 9, and (2) revising the original tutorial to target known student problems more directly. We now find that student performance after the tutorials is substantially improved, but that, as in part II, there is still room for improvement.

Difficulties that transcend context: Throughout this dissertation, we have identified a number of student difficulties with quantum concepts that seem to appear again and again in response to a variety of different questions in new contexts. Below, we highlight one particular difficulty that has been encountered in the research discussed across all three of this dissertation's parts.

In section 3.5C, we show that students often treat energy probabilities as being related to the *energy eigenvalues*, rather than to the corresponding coefficients or inner products. In section 7.2B, we show that many students fail to recognize that probabilities for measurements can be determined from a quantum state regardless of the basis in which the state is represented. Lastly, in chapter 9, we show that very few students are able to determine the probabilities associated with a measurement of angular momentum when the state of the system is not represented in terms of the quantity that is being measured (sections 9.3 and 9.4). These difficulties appear to be related. Each demonstrates that some students do not seem to have a reliable produced for relating probabilities of measurement outcomes to inner products.

Overview: Throughout the research presented in this dissertation, we have found that some specific student difficulties are both prevalent and persistent. They are seen early in the course, when students are first introduced to the relevant concepts, and also observed later, when students are asked questions across a range of more difficult contexts. Many persist despite targeted tutorial instruction. Even for cases in which the current versions of the tutorials are effective at lowering the prevalence of a specific difficulty, some reappear in student responses to similar questions in new contexts or those that differ substantially from those used in the tutorial. These results suggest that in these cases it is not easy for students to transfer the relevant concepts from one context to a new, unfamiliar, and often more advanced context.

One way to interpret the persistence of specific difficulties is that curriculum needs to focus strongly on helping students build a robust understanding of these concepts, especially those that are fundamental to quantum mechanics. We have evidence that the tutorial curriculum for such fundamental topics seems to be most effective when students are asked to consider the same ideas repeatedly over the course of many tutorials in a variety of different contexts. We have begun to incorporate the overall findings of this dissertation into much of the curriculum that we have been developing. In each of this dissertation's three parts, we describe a different sequence of tutorials designed to address the most prevalent and persistent student difficulties for three different topics: time dependence and measurements (section 5.2), vector spaces (section 8.2), and angular momentum (section 10.1). Each sequence guides students through exercises that reinforce each other to help students build understanding of the relevant concepts. In addition, we have developed tutorials for several advanced topics that focus on connections between the concepts necessary for different quantum contexts. These topics include systems with

degenerate energy eigenvalues (section 5.3A), time-independent perturbation theory (section 8.3), and time-dependent perturbation theory (section 5.3B). Our preliminary results indicate that the tutorials for the advanced contexts can be just as effective as those for more fundamental contexts.

11.2 – Future Research

The investigation discussed in this dissertation focuses primarily on examining broad patterns in student understanding of quantum mechanics. The identification of these patterns has been critical to our efforts at designing curriculum for improving student understanding. However, deeper examinations of the lines of reasoning that underlie the most prevalent student difficulties could contribute substantially both to our understanding of student thinking and to our ability to design and modify tutorial curriculum. In this section, we discuss possible areas for future research identified during the course of this investigation. Some chapters in this dissertation also explicitly discuss possibilities for future research. This future research can be divided into two areas, each of which can provide valuable insights for the other: (1) further investigation into student conceptual and reasoning difficulties with quantum mechanics, and (2) further development and assessment of curriculum.

In part I of this dissertation, we focus on student understanding of quantum measurements and time dependence. This is a topic that has also been the subject of several other investigations.[2-6] However, as shown in this dissertation, there is still work to be done, especially with respect to student understanding of time dependence and measurements in advanced contexts. The findings indicate that many student difficulties transcend context.

Future research could examine student responses in more detail with the goal of identifying those concepts for which transfer is most difficult. Contexts that are already known to be difficult for students, such as time-dependent perturbation theory and angular momentum, could prove to be valuable for this kind of study.

Further assessment and revision of the tutorials for time dependence and measurements (as described in chapter 5) is also necessary. Although the tutorials have proved successful at improving student understanding of specific contexts, both at UW and at other universities, there are some difficulties that appear to be particularly resistant to instruction. Furthermore, tutorials focusing on more advanced contexts have only undergone limited iterations of testing and modification. Additional testing—and, if necessary, additional revision—is still needed. The findings in this dissertation suggest that it is important for tutorial curriculum to focus on strengthening student ability to use concepts developed in one context when answering questions in new, unfamiliar, or more advanced contexts.

The second part of this dissertation describes an investigation into student understanding of vector spaces in quantum mechanics. Because these results are preliminary, there is substantial room for future research. For example, in chapter 6 we discuss student responses to questions that ask about vector addition. Only questions involving generic vectors (*i.e.*, vectors with no physical context) are included in this chapter. However, very similar questions about vector addition have been asked throughout the introductory physics sequence at UW over a period of many years. The questions have been asked in different forms (*e.g.*, as online tutorial pretests or course exams) and for a wide variety of physics contexts (*e.g.*, velocities, forces, and momenta). Thus, it would be possible to determine how changing the physical context of a vector addition

problem affects student reasoning. It would also be possible to assess the effect of different amounts of instruction (both lecture and tutorial). This kind of study could provide valuable insights into the patterns of reasoning that students appear to use when adding wave functions in quantum mechanics, and would also have more general implications for instruction on vectors and superposition at the introductory level.

The results presented in chapter 7 suggest that many students have difficulty calculating inner products in quantum mechanics. Some students do not seem to recognize when it is necessary to compute inner products (*e.g.*, when students are given a position-space wave function and asked about energy probabilities). These results are only preliminary, but they indicate that some students appear to lack the fundamental understanding of how quantum mechanics may be used to make predictions about measureable quantities. A much deeper investigation, involving in-depth techniques such as student interviews, will likely be necessary to probe student understanding in greater detail.

We have developed an initial sequence of tutorials (described in chapter 8) to address student difficulties with various aspects of vector spaces, including superposition, inner products, and changes of basis. Although these tutorials appear promising, there is still substantial room for them to be improved based on further research. The results may also have implications for tutorials for other topics, such as angular momentum (see chapter 10), that require students to change basis.

In the process of conducting the research presented in this dissertation, we observed numerous other aspects of student understanding of quantum mechanics that are worthwhile of investigation but that are not directly related to the research documented in this dissertation. One

example is students' use (or misuse) of terminology. This is especially the case for terms that have specific, precise definitions in quantum mechanics, that are very different from their use in non-quantum contexts. Examples of such terminology include: "state" (*e.g.*, state vector), "uncertain", and "representation". Research on students' use of quantum-specific terminology could provide insight into some student difficulties, and could also help inform instruction by calling attention to differences in the usage of certain terms between students and instructors.

11.3 – Notes and References

[1] L. C. McDermott, P. S. Shaffer, P. R. L. Heron, and the Physics Education Group, *Tutorials in Physics: Quantum Mechanics* (in preparation).
[3] C. Singh, "Student Understanding of Quantum Mechanics," *American Journal of Physics* **69**, 885 (2001).
[2] A. Crouse, "Research on Student Understanding of Quantum Mechanics as a Guide for Improving Instruction," PhD dissertation, University of Washington (2007).
[4] G. Zhu and C. Singh, "Surveying Students' Understanding of Quantum Mechanics in One Spatial Dimension," *American Journal of Physics* **80**, 252 (2012).
[5] C. Singh, "Student Understanding of Quantum Mechanics at the Beginning of Graduate Instruction," *American Journal of Physics* **76**, 277 (2008).
[6] G. Zhu and C. Singh, "Improving Students' Understanding of Quantum Measurement I: Investigation of Difficulties," *Physical Review Special Topics – Physics Education Research* **8** (1), 010117 (2012).
[7] L. C. McDermott, P. S. Shaffer, and the Physics Education Group, *Tutorials in Introductory Physics*, Prentice Hall College Div (2001).
[8] L. C. McDermott, P. S. Shaffer, P. R. L. Heron, and the Physics Education Group, *Tutorials in Physics: Quantum Mechanics* (in preparation).

Appendices

This dissertation includes four appendices with supplementary material. Appendix A contains the tutorial pretests that were given to the students online. The tutorial worksheets and homework assignments make up appendix B and appendix C, respectively. Lastly, appendix D includes copies of all exam questions discussed in this dissertation. The contents of each appendix are made up of exact replicas of what was administered to students. Each appendix also gives some additional description of the included material.

In general, each item in the appendices is associated with one (or more) of the tutorials. For ease of reference, each tutorial is assigned a code based on the name of its topic. Table A.1 gives the name of each tutorial, the corresponding code, and the section(s) of the dissertation where that element of the curriculum is described. The order of topics in the table corresponds to the order in which the curriculum is typically administered to students, not to the order in which they appear in the body or the appendices of this dissertation. Note that some tutorials in the table are not discussed in detail in this dissertation; they are included here for completeness.

Table A.1: Codes associated with the tutorials referenced throughout the appendices

Code	Name of tutorial	Section
BBR	*Blackbody radiation*	N/A
SPN	*Spin*	N/A
DIR	*Dirac notation*	8.2A1
2ST	*Two-state time dependence*	5.3
CPR	*Classical probability*	N/A
RCM	*Relating classical and quantum mechanics*	N/A

Table A.1 continued.

Code	Name of tutorial	Section
FAV	*Treating functions as vectors*	8.2A2
RWF	*Representations of wave functions*	8.2A3
PAI	*Probability amplitude*	8.2A4
SQM	*Superposition in quantum mechanics*	5.1C4
TDQ	*Time dependence in quantum mechanics*	5.1C1
EME	*Energy measurements*	5.1C2
PME	*Position, momentum, and energy measurements*	5.1C3
2DH	*Degenerate states*	5.2A
LQM	*Angular momentum in quantum mechanics*	10.2A
AAM	*Addition of angular momentum*	10.2B
IDP	*Identical particles*	N/A
PPT	*Prelude to perturbation theory*	8.3A3
TIP	*Time-independent perturbation theory*	8.3A1
DPT	*Degenerate perturbation theory*	8.3A2
WKB	*The WKB approximation*	N/A
TDF	*Time-dependent perturbation theory Part I: Foundations*	5.2B1
TDP	*Time-dependent perturbation theory Part II: Formalism*	5.2B2
SCT	*Scattering in quantum mechanics*	N/A

Appendix A
Pretests

"Go Cougs!"

—Author

This appendix includes copies of all pretests discussed throughout this dissertation. Every variant of a pretest for which the results are discussed is included separately. Almost every pretest was given online. The pretests included in this appendix are taken directly from the online system used to administer them to students. For more information about the administration of the pretests, see section 1.3B.

Each sub-section of this appendix (A.I, A.II, and A.III) includes those pretests that are discussed in the corresponding part of this dissertation: part I (quantum measurements and time dependence), part II (vector spaces), and part III (angular momentum). The top left of each page gives the three-letter code for the tutorial associated with each pretest.

A.I – Pretests for Part I

This sub-section includes copies of all pretests discussed in part I of this dissertation. The first pretests are surveys designed for students to provide us with feedback regarding the tutorial portion of the course. The results from these surveys are discussed in chapter 1. The results from all other pretests in this section are discussed in chapters 2-4.

Print view of '143 _324 Survey'

Print this page

The survey below is designed for you to give us feedback regarding the tutorials so that we can improve future teaching of quantum mechanics. We appreciate your thoughts and any information you can give us will be extremely helpful.

Question 1.
When did you take the modern physics course, PHYS 225?

- ○ I took PHYS 225 this winter (2014)
- ○ I took PHYS 225 last summer (2013)
- ○ I took PHYS 225 last winter (2013)
- ○ I took PHYS 225 before winter 2013
- ○ I did not take PHYS 225, but I have taken a modern physics course
- ○ I did not take PHYS 225, and I have never taken a modern physics course
- ○ Other:

Question 3.
Which of the following physics courses have you taken (or are currently taking)?

- ☐ PHYS 224 (Thermal Physics)
- ☐ PHYS 226 (Particles and Symmetries)
- ☐ PHYS 227 (Math Methods I)
- ☐ PHYS 228 (Math Methods II)
- ☐ PHYS 321 (Electromagnetism I)
- ☐ PHYS 322 (Electromagnetism II)

Question 4.
The tutorials were helpful to my learning.

- ○ Strongly Agree
- ○ Agree
- ○ Somewhat Agree
- ○ Neutral
- ○ Somewhat Disagree
- ○ Disagree
- ○ Strongly Disagree

Question 5.
I would have benefitted from having worked through more tutorials in this course.

- ○ Strongly Agree
- ○ Agree
- ○ Somewhat Agree
- ○ Neutral
- ○ Somewhat Disagree
- ○ Disagree
- ○ Strongly Disagree

Question 6.
I would like to see tutorials in other advanced topics in physics (such as electromagnetism or classical mechanics.

- ○ Strongly Agree
- ○ Agree
- ○ Somewhat Agree
- ○ Neutral
- ○ Somewhat Disagree

- ○ Disagree
- ○ Strongly Disagree

Question 7.
Are there any topics for which you think a tutorial would have been of benefit to you? (e.g. topics that you found particularly difficult, or that you found particularly interesting and would have liked to have studied in greater depth)

Question 8.
In the space below, please give any additional comments on the tutorial portion of the course.

Questions or comments?
Contact us or email catalysthelp@uw.edu

Print view of '(TDQ)U1a'

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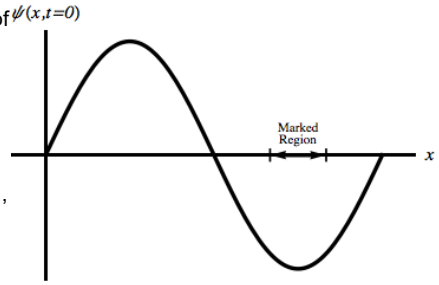
Q1: Suppose a system consists of a particle in an infinite square well potential of width a. The wave function for the system at time t = 0 is entirely real and is shown at right. This state satisfies the eigenvalue equation $\hat{H}\psi = E_2\psi$, where $\mathrm{E}_2 = \frac{4\pi^2\hbar^2}{ma^2}$.

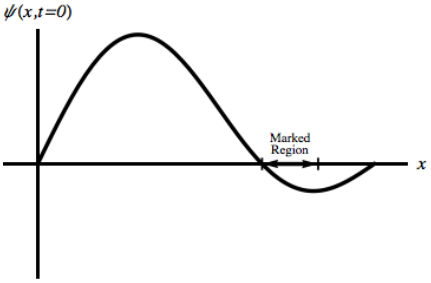


Consider the marked region along the x-axis.

Rank from greatest to least the probabilities of finding the particle within the marked region at the following four times: $\mathrm{t}_0 = 0$, $\mathrm{t}_1 = \frac{\pi\hbar}{2E_2}$, $\mathrm{t}_2 = \frac{\pi\hbar}{E_2}$, and $t_3 = \lim_{t\to\infty} t$. Use $\mathrm{P}_0, P_1, P_2, \text{and } P_3$ as symbols for your probabilities.

Explain your ranking.

Q2: Another system consisting of a particle in an infinite square well potential of width a is initially prepared such that its wave function at time t = 0 is



$\Psi(x, t=0) = \sqrt{\frac{1}{2}}\left(\psi_1(x) + \psi_2(x)\right)$ as shown at right. ψ_1 and ψ_2 satisfy the eigenvalue equation $\hat{H}\psi_n = E_n\psi_n$. Here $\mathrm{E}_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$.

Consider the marked region along the x-axis.

Rank the probabilities of finding the particle within the marked region at the following four times: $\mathrm{t}_0 = 0$, $\mathrm{t}_1 = \frac{2\hbar\pi}{E_2}$, $\mathrm{t}_2 = \frac{4\hbar\pi}{E_2}$, and $\mathrm{t}_3 = \frac{8\hbar\pi}{E_2}$. Use $\mathrm{P}_0, P_1, P_2, \text{and } P_3$ as symbols for your probabilities.

Explain your ranking.

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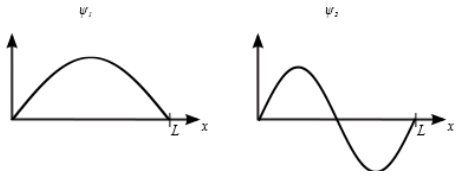
Part I
A particle is in a quantum mechanical infinite square well of width *L*. The wave function of the particle at *t* = 0 is given below.



$$\Psi(x, t=0) = \frac{1}{\sqrt{2}}(\psi_1 + \psi_2)$$

where ψ_1 and ψ_2 (pictured at right) represent the ground state and first excited state wave functions.

Question 1.
At *t* = 0, is the probability of finding the particle in the left half of the well *greater than, less than,* or *equal to* the probability of finding the particle in the right half of the well?

- Greater than
- Less than
- Equal to
- Not enough information

Question 2.
Explain your reasoning.

Question 3.
A second particle is in an identical infinite square well of width *L*. The wave function of this particle at *t* = 0 is given below.

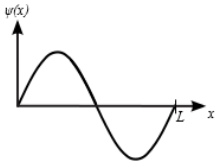
$$\Psi = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2)$$

At *t* = 0, is the probability of finding the particle in the left half of the well *greater than, less than,* or *equal to* the probability of finding the particle in the right half of the well?

- Greater than
- Less than
- Equal to
- Not enough information

Question 4.
Explain your reasoning.

Part II
A particle is prepared in the first excited state of the quantum mechanical infinite square well of width *L*. The wave function at *t = 0* is shown at right.



Question 5.
Does the wave function for this particle change in time?

- Yes
- No
- Not enough information

Question 6.
Explain your reasoning.

Question 7.
The *probability density* at $t = 0$ is shown at right for the same particle.

Does the probability density for this particle change in time?

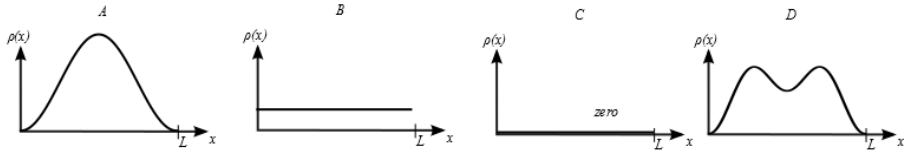
	Logic destinations
○ Yes	Don't skip (default)
○ No	Question 11: [IMAGE]You answered that th...
○ Not enough information	Question 12: [IMAGE]You answered that th...
No response	Question 12: [IMAGE]You answered that th...

Question 8.

You answered that the probability density shown at right does change in time.

Explain the reasoning you used to answer the above question. If possible, give an explicit form for the time dependence of the probability density.

Question 9.
Which, if any, of the following graphs could represent the probability density for this system at the indicated times?



Rows
a short time later (check all that apply)
a very long time later (check all that apply)

- ☐ A
- ☐ B
- ☐ C
- ☐ D

Question 10.
Explain your reasoning.

Logic destination
[IMAGE]Part III A particle ...

Question 11.
You answered that the probability density shown at right does not change in time.

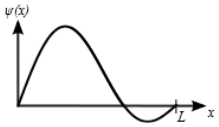
Explain your reasoning. If possible, identify a state for which the probability density would change in time.

Logic destination
[IMAGE]Part III A particle ...

Question 12.
You answered that there is not enough information to determine whether the probability density shown at right is changing in time.

Explain your reasoning. Specify what additional information would allow you to answer this question.

Part III
A particle is prepared in a superposition of the first two eigenstates of the quantum mechanical infinite square well of width *L*. The wave function at $t = 0$ is shown at right, and the functional form of the wave function at $t = 0$ is given below.



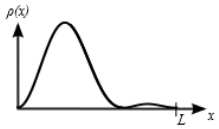
$$\psi(x, t = 0) = \frac{1}{\sqrt{2}}(\psi_1 + \psi_2)$$

Question 13.
Does the wave function for this particle change in time?

- Yes
- No
- Not enough information

Question 14.
Explain your reasoning.

Question 15.
Question 5
The probability density for this particle at $t = 0$ is shown at right.

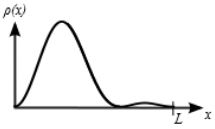


Does the probability density for this particle change in time?

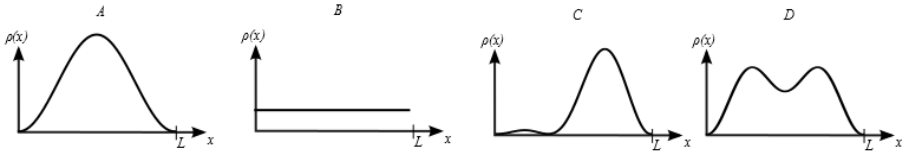
	Logic destinations
Yes	Don't skip (default)
No	Question 19: [IMAGE]You answered that th...
Not enough information	Question 22: [IMAGE]You answered that th...
No response	Don't skip (default)

Question 16.
You answered that the probability density shown at right does change in time.

Explain the reasoning you used to answer the above question. If possible, give an explicit form for the time dependence of the probability density.



Question 17.
Which, if any, of the following graphs could represent the probability density for this system at the indicated times?



Rows
a short time later (check all that apply)
a very long time later (check all that apply)

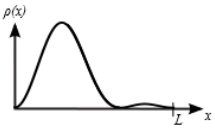
- ☐ A
- ☐ B
- ☐ C
- ☐ D

Question 18.
Explain your reasoning.

Logic destination
End of Survey

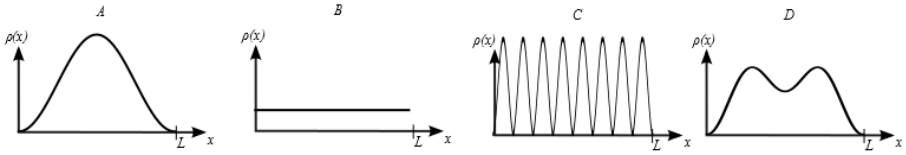
Question 19.

You answered that the probability density shown at right does not change in time.



Explain the reasoning you used to answer the above question.

Question 20.
Which, if any, of the following graphs represent a state of the infinite square well for which the probability density would change in time? Check all that apply.



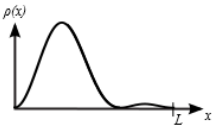
- ☐ A
- ☐ B

☐ C
☐ D

Question 21.
Explain your reasoning.

Logic destination
End of Survey

Question 22.
You answered that there is not enough information to determine whether the probability density shown at right is changing in time.



Explain your reasoning. Specify what additional information would allow you to answer this question.

Logic destination
End of Survey

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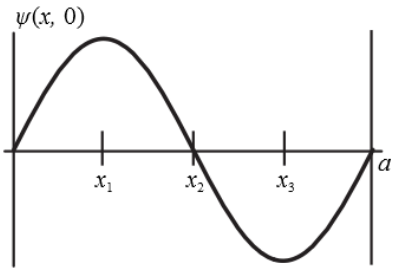
Suppose a system consists of a particle in an infinite square well potential of width *a*. The wave function for this system at time *t* = 0 is entirely real (shown at right). This state is an energy eigenstate with the energy given below.

$$E_2 = \frac{2\pi^2\hbar^2}{ma^2}$$



Question 1.
At time *t* = 0, near which of the marked points (x_1, x_2, or x_3) is the particle most likely to be found?

- The particle is most likely to be found near x1
- The particle is most likely to be found near x2
- The particle is most likely to be found near x3
- The particle is equally likely to be found near x1 or x3
- The particle is equally likely to be found near x1, x2, or x3
- There is not enough information to answer

Question 2.
Explain your reasoning.

Question 3.
Is the probability that the particle is found to be near x_3 at time $t_2 = \pi\hbar/2E_2$, *greater than, less than,* or *equal to* the probability that the particle is found to be near x_3 at *t* = 0?

- Greater than
- Less than
- Equal to
- Not enough information

Question 4.
Explain your reasoning.

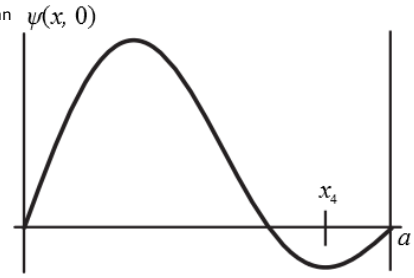
Question 5.
Is the probability that the particle is found to be near x_3 at time $t_3 = \pi\hbar/E_2$, *greater than, less than,* or *equal to* the probability that the particle is found to be near x_3 at *t* = 0?

- Greater than
- Less than
- Equal to
- Not enough information

Question 6.
Explain your reasoning.

Question 7.
Describe the wave function for this particle as *t* approaches infinity.

Another system consisting of a particle in an infinite square well potential of width *a* is initially prepared with the wave function given below.



$$\Psi(x, t = 0) = \sqrt{\tfrac{1}{2}}\left(\psi_1(x) + \psi_2(x)\right)$$

$$\hat{H}\psi_n = E_1\psi_n$$

$$\mathrm{E}_n = \tfrac{n^2\pi^2\hbar^2}{2ma^2}$$

Question 8.
Is this particle in a stationary state?

- Yes
- No
- Not enough information

Question 9.
Explain your reasoning.

Question 10.

Is the probability that the particle is found to be near x_4 at time $\mathrm{t}_2 = 2\pi\hbar/E_2$, *greater than, less than,* or *equal to* the probability that the particle is found to be near x_4 at $t = 0$?

- Greater than
- Less than
- Equal to
- Not enough information

Question 11.
Explain your reasoning.

Question 12.

Is the probability that the particle is found to be near x_4 at time $\mathrm{t}_3 = 4\pi\hbar/E_2$, *greater than, less than,* or *equal to* the probability that the particle is found to be near x_4 at $t = 0$?

- Greater than
- Less than
- Equal to
- Not enough information

Question 13.
Explain your reasoning.

Question 14.
Describe the wave function for this particle as *t* approaches infinity.

Question 15.

Which of the following are necessary to determine the probability density for a quantum mechanical system?

- ○ Only the real part of the wave function
- ○ Only the imaginary part of the wave function
- ○ Both the real and imaginary parts of the wave function
- ○ None of the above are necessasry

Question 16.
Explain your reasoning.

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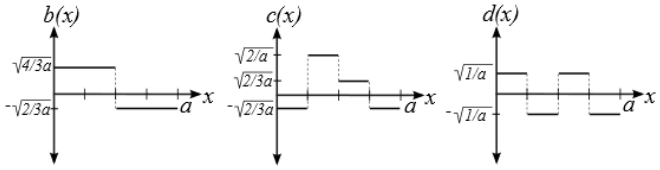
The three functions shown below each represent the wave function for a particle in an unknown potential well. The wave functions are zero outside of the region shown. Each function is drawn to the same scale.



Question 1.
Which of the three particles above is ***most likely*** to be found between $x = a/4$ and $x = 3a/4$ (the two central regions of the well)?

- b
- c
- d
- More than one of the above (please specify)
- I need more information to answer (please specify)

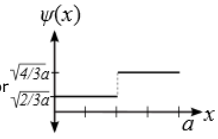
Question 2.
Explain your reasoning.

Question 3.
Which of the three particles above is ***least likely*** to be found between $x = a/4$ and $x = 3a/4$ (the two central regions of the well)?

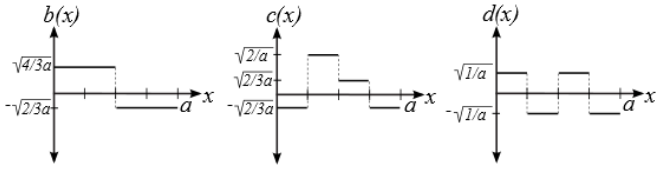
- b
- c
- d
- More than one of the above (please specify)
- I need more information to answer (please specify)

Question 4.
Explain your reasoning.

The wave function at right represents an energy eigenstate of this system with energy eigenvalue *E*. It is zero outside the region shown.



The wave functions from the previous page are reproduced below for your convenience.



Question 5.
If the energy of each particle were measured, which of the three particles above is ***most likely*** to be found to have energy *E?*

- b
- c
- d
- More than one of the above (please specify)
- I need more information to answer (please specify)

Question 6.
Explain your reasoning.

Question 7.
If the energy of each particle were measured, which of the three particles above is ***least likely*** to be found to have energy *E?*

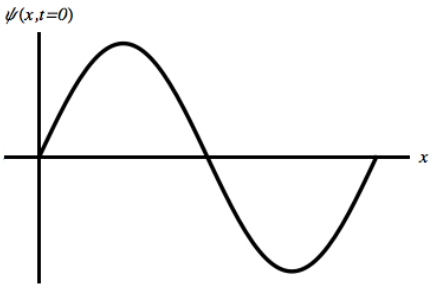
- b
- c
- d
- More than one of the above (please specify)
- I need more information to answer (please specify)

Question 8.
Explain your reasoning.

Suppose a system consists of a particle in an infinite square well potential of width *a*. The wave function for this system at time *t* = 0 is entirely real (shown at right). This state satisfies the eigenvalue equation below with the energy given.

$$\hat{H}\psi = E_2\psi$$

$$\mathrm{E}_2 = \frac{2\pi^2\hbar^2}{ma^2}$$



Question 9.
At time *t* = 0, is the probability that the particle is found to be on the left half of the well *greater than, less than,* or *equal to* the probability that the particle is found to be on the right half of the well?

- Greater than
- Less than
- Equal to
- Not enough information

Question 10.
Explain your reasoning.

Question 11.
Is the probability that the particle is found to be on the left half of the well at time $\mathrm{t}_2 = \pi\hbar/2E_2$, *greater than, less than,* or *equal to* the probability that the particle is found to be on the left half of the well at *t* = 0?

- Greater than
- Less than

- Equal to
- Not enough information

Question 12.
Explain your reasoning.

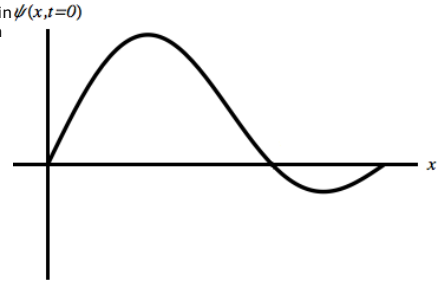
Question 13.
Is the probability that the particle is found to be on the left half of the well at time $t_3 = \pi\hbar/E_2$, *greater than, less than,* or *equal to* the probability that the particle is found to be on the left half of the well at *t* = 0?

- Greater than
- Less than
- Equal to
- Not enough information

Question 14.
Explain your reasoning.

Question 15.
Describe the wave function for this particle as *t* approaches infinity.

Another system consisting of a particle in an infinite square well potential of width *a* is initiall prepared with the wave function given below.



$$\Psi(x, t=0) = \sqrt{\tfrac{1}{2}}\left(\psi_1(x) + \psi_2(x)\right)$$

$$\hat{H}\psi_n = E_1\psi_n$$

$$\mathrm{E}_n = \tfrac{n^2\pi^2\hbar^2}{2ma^2}$$

Question 16.
At time *t* = 0, is the probability that the particle is found to be on the left half of the well *greater than, less than,* or *equal to* the probability that the particle is found to be on the right half of the well?

- Greater than
- Less than
- Equal to
- Not enough information

Question 17.
Explain your reasoning.

Question 18.
Is the probability that the particle is found to be on the left half of the well at time $t_2 = 2\pi\hbar/E_2$, *greater than, less than,* or *equal to* the probability that the particle is found to be on the left half of the well at *t* = 0?

- ○ Greater than
- ○ Less than
- ○ Equal to
- ○ Not enough information

Question 19.
Explain your reasoning.

Question 20.
Is the probability that the particle is found to be on the left half of the well at time $t_3 = 4\pi\hbar/E_2$, *greater than, less than,* or *equal to* the probability that the particle is found to be on the left half of the well at *t* = 0?

- ○ Greater than
- ○ Less than
- ○ Equal to
- ○ Not enough information

Question 21.
Explain your reasoning.

Question 22.
Describe the wave function for this particle as *t* approaches infinity.

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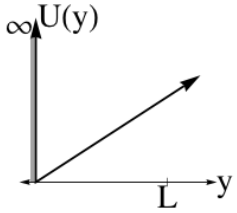
Consider a quantum mechanical system with the potential shown at right. The system is prepared so that at t = 0 it is in a state described by a wave function identical to that of the ground state of the infinite square well. That is,

$$\Psi(y,0) = \sqrt{\tfrac{2}{L}}\sin\left(\tfrac{\pi y}{L}\right) \quad 0 < y < L$$
$$\Psi(y,0) = 0 \quad y \le 0, y \ge L$$



Q1:
a) Does the wave function associated with this state depend on time? If so, briefly describe how you would write down its time dependence. If not, explain why not.

b) At *t* = 0, what is the probability density for this particle at y = L/2?

c) At *t* = 0, what is the probability density for this particle at *y* = 3L/2?

d) At some later time t_1 > 0, is the probability density for this particle at y = 3L/2 *larger, smaller,* or *exactly the same* as it was at *t* = 0? Explain your reasoning.

Q2:
a) Does the probability density associated with this state depend on time?

- (•) Select one...
- ( ) Definitely yes
- ( ) It depends (specify below)
- ( ) No, the probability density is time independent

b) If so, describe its time dependence. If not, explain why not.

Assume that a quantum mechanical harmonic oscillator is prepared so that its initial state is given by:

$$\Psi_i = \Psi(x,0) = i\sqrt{\tfrac{1}{3}}\psi_0 - \sqrt{\tfrac{2}{3}}\psi_1$$

where ψ_0 and ψ_1 are the ground state (n=0) and first excited state (n=1) which satisfy the time independent Schrödinger equation:

$$\hat{H}\psi_0 = E_0\psi_0$$

and

$$\hat{H}\psi_1 = E_1\psi_1$$

Q3:
a) Consider an energy measurement made on this system. Are there times when the probability of measuring E_1 is zero and the probability of measuring E_0 is one?

- (•) Select one...
- ( ) Definitely yes
- ( ) It depends (specify below)
- ( ) No, there is no such time

b) If so, give the first such time. If not, explain why not.

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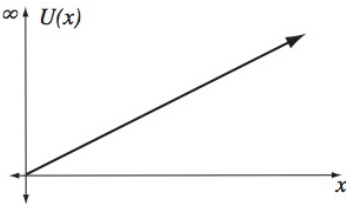
Part A

Consider a quantum mechanical system with the potential shown at right. The system is prepared so that at t = 0 it is in a state described by a wave function identical to that of the ground state of the infinite squae well. That is,



$$\Psi(x,0) = \begin{cases} 0 & \text{for } x \leq 0 \\ \sqrt{\frac{2}{L}}\sin(\frac{\pi x}{L}) & \text{for } 0 < x < L \\ 0 & \text{for } x > L \end{cases}$$

Note that: $U(x \leq 0) = \infty$

1.) Does the wave function associated with this state depend on time?

- Select one...
- Yes, the wave function is definitely time dependent
- No, the wave function is not time dependent
- Need more information

Explain your reasoning.

2.) At t = 0, what is the probability density for this particle at x = 3L/2?

3.) At some later time t_1 > 0, is the probability density for this particle at x = 3L/2 greater than, less than, or equal to what it was at t = 0?

- Select one...
- Greater than
- Less than
- Equal to
- Need more information

Explain your reasoning.

4.) Does the probability density associated with this state depend on time?

- Select one...
- Yes, the probability density is definitely time dependent
- No, the probability density is not time dependent
- Need more information

Explain your reasoning.

Part B

Assume that a quantum mechanical harmonic oscillator is prepared so that its initial state is

given by:

$$\Psi_i = \Psi(x,0) = i\sqrt{\tfrac{1}{3}}\psi_0 - \sqrt{\tfrac{2}{3}}\psi_1$$

where ψ_n is the n^{th} energy eigenstate of the harmonic oscillator potential, with energy E_n.

1.) Does the probability density associated with this state change in time?

- Yes
- No
- Not enough information

2.) Consider an energy measurement made on this system. Are there times when the probability of measuring E_0 is greater than the probability of measuring E_1?

- Yes
- No
- Not enough information

Explain your reasoning.

3.) An energy measurement is made on the system and the energy is found to be E_1.
a) Suppose that a second energy measurement were made a short time later. What are the possible outcomes of that measurement? Explain your reasoning.

b) Suppose instead, that the second energy measurement were made a long time later? What are the possible outcomes of that measurement? Explain your reasoning.

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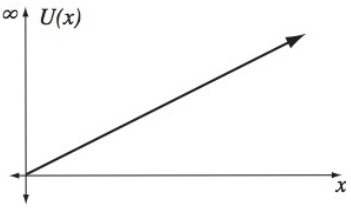
Part A

Consider a quantum mechanical system with the potential shown at right. The system is prepared so that at t = 0 it is in a state described by a wave function identical to that of the ground state of the infinite squae well. That is,



$$\Psi(x,0) = \begin{cases} 0 & \text{for } x \leq 0 \\ \sqrt{\frac{2}{L}}\sin(\frac{\pi x}{L}) & \text{for } 0 < x < L \\ 0 & \text{for } x > L \end{cases}$$

Note that: $U(x \leq 0) = \infty$

1.) Does the wave function associated with this state depend on time?

- Select one...
- Yes, the wave function is definitely time dependent
- No, the wave function is not time dependent
- Need more information

Explain your reasoning.

2.) At t = 0, what is the probability density for this particle at x = 3L/2?

3.) At some later time t_1 > 0, is the probability density for this particle at x = 3L/2 greater than, less than, or equal to what it was at t = 0?

- Select one...
- Greater than
- Less than
- Equal to
- Need more information

Explain your reasoning.

4.) Does the probability density associated with this state depend on time?

- Select one...
- Yes, the probability density is definitely time dependent
- No, the probability density is not time dependent
- Need more information

Explain your reasoning.

Part B

Assume that a quantum mechanical harmonic oscillator is prepared so that its initial state is

given by:

$$\Psi_i = \Psi(x,0) = i\sqrt{\tfrac{1}{3}}\psi_0 - \sqrt{\tfrac{2}{3}}\psi_1$$

where ψ_n is the n^{th} energy eigenstate of the harmonic oscillator potential, with energy E_n.

1.) Does the probability density associated with this state change in time?

- Yes
- No
- Not enough information

2.) Consider an energy measurement made on this system. Are there times when the probability of measuring E_0 is greater than the probability of measuring E_1?

- Yes
- No
- Not enough information

Explain your reasoning.

Part C

Now assume that a quantum mechanical harmonic oscillator is prepared so that its initial state is given by:

$$\Psi(x, t=0) = \psi_2,$$

where ψ_n is the n^{th} energy eigenstate of the harmonic oscillator potential, with energy E_n.

Is this state an eigenstate of position? Explain your reasoning.

If you were to measure the momentum of the particle, are the values that you could measure discrete or continuous? Explain your reasoning.

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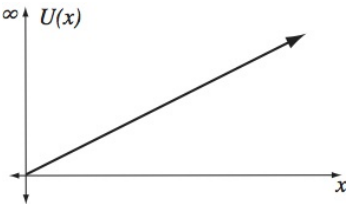
Part I

Consider a quantum mechanical system with the potential shown at right. The system is prepared so that at t = 0 it is in a state described by a wave function identical to that of the ground state of the infinite square well. That is,



$$\Psi(x,0) = \begin{cases} 0 & \text{for } x \leq 0 \\ \sqrt{\frac{2}{a}}\sin(\frac{\pi x}{a}) & \text{for } 0 < x < a \\ 0 & \text{for } x > a \end{cases}$$

Note that: $U(x \leq 0) = \infty$

Question 1.
Does the wave function associated with this state depend on time?

- Yes, the wave function is definitely time dependent
- No, the wave function is not time dependent
- I need more information to answer this question (please specify)

Question 2.
Explain your reasoning.

Question 3.
At *t* = 0, is the probability density for this particle at *x* = 3*a*/2 *greater than, less than,* or *equal to* the probability density at *x* = *a*/2?

- Greater than
- Less than
- Equal to
- There is not enough information to answer

Question 4.
Explain your reasoning.

Question 5.
At some later time $t_1 > 0$, is the probability density for this particle at *x* = *a*/2 *greater than, less than,* or *equal to* the probability density at the same point at *t* = 0?

- Greater than
- Less than
- Equal to
- There is not enough information to answer this question

Question 6.
Explain your reasoning.

Question 7.
At $t = t_1$, is the probability density for this particle at *x* = 3*a*/2 *greater than, less than,* or *equal to* the probability density at the same point at *t* = 0?

- Greater than
- Less than
- Equal to
- There is not enough information to answer this question

Question 8.
Explain your reasoning.

Question 9.
Does the probability density associated with this state depend on time?

- Yes, the probability density is definitely time dependent
- No, the probability density is not time dependent
- I need more information to answer this question (please specify)

Question 10.
Explain your reasoning.

Part II

Assume that a quantum mechanical harmonic oscillator is prepared so that its initial state is given by:

$$\Psi_i = \Psi(x,0) = i\sqrt{\tfrac{1}{3}}\psi_0 - \sqrt{\tfrac{2}{3}}\psi_1$$

where ψ_n is the n^{th} energy eigenstate of the harmonic oscillator potential, with energy E_n.

Question 11.
Does the probability density associated with this state change in time?

- Yes
- No
- Not enough information

Question 12.
Explain your reasoning.

Question 13.
Consider an energy measurement made on this system at $t = 0$. Is the probability of measuring E_1 *greater than, less than,* or *equal to* the probability of measuring E_0?

- Greater than
- Less than
- Equal to
- There is not enough information to answer

Question 14.
Explain your reasoning.

Question 15.
Assume that no measurements have been made on the state shown above. Are there times when the probability of measuring E_1 is zero? If so, give such a time in your explanation.

- Yes, there are times when this probability is zero
- Yes, but only after waiting a very long time
- No, there are no times when this probability is zero
- There is not enough information to answer

Question 16.

Explain your reasoning.

An energy measurement is made on the system and the energy is found to be E_1.

Question 17.
Suppose that a second energy measurement were made a few minutes later. What are the possible outcomes of this second measurement? Explain your reasoning.

Question 18.
Suppose instead that the second energy measurement were made *a very long time* later? What are the possible outcomes of this second measurement? Explain your reasoning.

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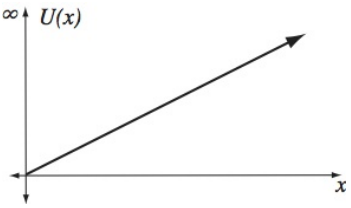
Part I

Consider a quantum mechanical system with the potential shown at right. The system is prepared so that at t = 0 it is in a state described by a wave function identical to that of the ground state of the infinite square well. That is,



$$\Psi(x,0) = \begin{cases} 0 & \text{for } x \leq 0 \\ \sqrt{\frac{2}{a}}\sin(\frac{\pi x}{a}) & \text{for } 0 < x < a \\ 0 & \text{for } x > a \end{cases}$$

Note that: $U(x \leq 0) = \infty$

Question 1.
Does the wave function associated with this state depend on time?

- Yes, the wave function is definitely time dependent
- No, the wave function is not time dependent
- I need more information to answer this question (please specify)

Question 2.
Explain your reasoning.

Question 3.
At *t* = 0, is the probability density for this particle at *x* = 3*a*/2 *greater than, less than,* or *equal to* the probability density at *x* = *a*/2?

- Greater than
- Less than
- Equal to
- There is not enough information to answer

Question 4.
Explain your reasoning.

Question 5.
At some later time $t_1 > 0$, is the probability density for this particle at *x* = *a*/2 *greater than, less than,* or *equal to* the probability density at the same point at *t* = 0?

- Greater than
- Less than
- Equal to
- There is not enough information to answer this question

Question 6.
Explain your reasoning.

Question 7.
At $t = t_1$, is the probability density for this particle at *x* = 3*a*/2 *greater than, less than,* or *equal to* the probability density at the same point at *t* = 0?

- Greater than
- Less than
- Equal to
- There is not enough information to answer this question

Question 8.
Explain your reasoning.

Question 9.
Does the probability density associated with this state depend on time?

- Yes, the probability density is definitely time dependent
- No, the probability density is not time dependent
- I need more information to answer this question (please specify)

Question 10.
Explain your reasoning.

Part II

Assume that a quantum mechanical harmonic oscillator is prepared so that its initial state is given by:

$$\Psi_i = \Psi(x,0) = i\sqrt{\tfrac{1}{3}}\psi_0 - \sqrt{\tfrac{2}{3}}\psi_1$$

where ψ_n is the n^{th} energy eigenstate of the harmonic oscillator potential, with energy E_n.

Question 11.
Is this state an eigenstate of the Hamiltonian for this system (the simple harmonic oscillator)?

- Yes
- No
- Not enough information

Question 12.
Explain your reasoning.

Question 13.
Consider an energy measurement made on this system at t = 0. Is the probability of measuring E_1 *greater than, less than,* or *equal to* the probability of measuring E_0?

- Greater than
- Less than
- Equal to
- There is not enough information to answer

Question 14.
Explain your reasoning.

Question 15.
Assume that no measurements have been made on the state shown above. Are there times when the probability of measuring E_1 is zero? If so, give such a time in your explanation.

- Yes, there are times when this probability is zero
- Yes, but only after waiting a very long time
- No, there are no times when this probability is zero
- There is not enough information to answer

Question 16.

Explain your reasoning.

An energy measurement is made on the system and the energy is found to be E_1.

Question 17.
Suppose that a second energy measurement were made a few minutes later. What are the possible outcomes of this second measurement? Explain your reasoning.

Question 18.
Suppose instead that the second energy measurement were made *a very long time* later? What are the possible outcomes of this second measurement? Explain your reasoning.

Question 19.
Does the probability density associated with this state change in time?

- ○ Yes
- ○ No
- ○ Not enough information

Question 20.
Explain your reasoning.

Question 21.
Which of the following statements about **stationary states** in quantum mechanics do you agree with? (Select all that apply.)

- ☐ The wave function for a stationary state does not depend on time
- ☐ The probability of an energy measurement for a stationary state does not depend on time
- ☐ The probability density for a stationary state does not depend on time

Question 22.
Explain your reasoning.

Question 23.
Which of the following statements about **energy eigenstates** in quantum mechanics do you agree with? (Select all that apply.)

- ☐ All energy eigenstates are stationary states
- ☐ If a state can be written as a superposition of energy eigenstates, then it is a stationary state
- ☐ If a state can be written as a superposition of energy eigenstates, then it is NOT a stationary state

Question 24.
Explain your reasoning.

Question 25.
Which of the following statements about **time dependence** in quantum mechanics do you agree with? (Select all that apply.)

- ☐ If the wave function for a particle does not depend on time, then it is in a stationary state
- ☐ If the probability of an energy measurement for a particle does not depend on time, then it is in a stationary state
- ☐ If the probability density for a particle does not depend on time, then it is in a stationary state

Question 26.
Explain your reasoning.

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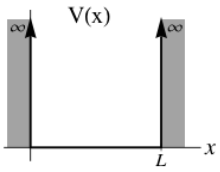

At time t = 0, the energy of the system is measured and found to be E_2, the first excited state energy.

Immediately after the measurement, is the system in an eigenstate of position?

- Select one...
- Yes
- No
- It depends

Explain your reasoning.

At time $t_1 > 0$ you measure the position of the particle and find that it is located at x = 0.3L.

Suppose a few minutes later, at a time t_2 you measured the energy of the particle. (Assume that, during the time interval $t_1 < t < t_2$, the particle remains isolated from the environment.)

Which of the following statements would best describe the result of your energy measurement?

- Select one...
- The value that you measure would definitely be the ground state energy.
- The value that you measure would definitely be the first excited state energy.
- The value that you measure could possibly (but not necessarily) be the first excited state energy.
- The value that you measure would definitely not be the first excited state energy.

Explain your reasoning:

Suppose that instead of measuring the energy at time t_2, you measure the energy a very long time later. (Assume that, during the time interval between position and energy measurements, the particle remains isolated from the environment.)

Which of the following statements would best describe the result of your energy measurement?

- Select one...
- The value that you measure would definitely be the ground state energy.
- The value that you measure would definitely be the first excited state energy.
- The value that you measure could possibly (but not necessarily) be the first excited state energy.
- The value that you measure would definitely not be the first excited state energy.

Explain your reasoning:

Suppose the orbital angular momentum squared, L^2, is measured for an electron in a spherically symmetric potential and is found to be $12\hbar^2$.

Which of the following represent possible wave functions for the electron just before the measurement is made? Here $f(r)$ and $g(r)$ are normalized functions of the radial coordinate r only. The $Y_l^m(\theta,\varphi)$ are the normalized spherical harmonics and the χ_i are the eigenspinors of S_z.

$$\text{A: } \Psi(r,\theta,\phi) = f(r)\left[\sqrt{\tfrac{1}{6}}Y_3^0(\theta,\phi)\chi_+ + \sqrt{\tfrac{2}{6}}Y_3^1(\theta,\phi)\chi_-\right] + g(r)\sqrt{\tfrac{3}{6}}Y_2^0(\theta,\phi)\chi_+$$
$$B: \Psi(r,\theta,\phi) = f(r)\left[\sqrt{\tfrac{1}{3}}Y_3^0(\theta,\phi)\chi_+ + \sqrt{\tfrac{2}{3}}Y_3^1(\theta,\phi)\chi_-\right]$$
$$C: \Psi(r,\theta,\phi) = f(r)Y_3^0\chi_+$$
$$D: \Psi(r,\theta,\phi) = g(r)Y_2^0\chi_+$$

- ☐ A
- ☐ B
- ☐ C
- ☐ D

Explain your choice(s).

Which of the following represent possible wave functions for the electron just after the measurement is made? Here $f(r)$ and $g(r)$ are normalized functions of the radial coordinate r only. The $Y_l^m(\theta,\varphi)$ are the normalized spherical harmonics and the χ_i are the eigenspinors of S_z.

$$\text{A: } \Psi(r,\theta,\phi) = f(r)\left[\sqrt{\tfrac{1}{6}}Y_3^0(\theta,\phi)\chi_+ + \sqrt{\tfrac{2}{6}}Y_3^1(\theta,\phi)\chi_-\right] + g(r)\sqrt{\tfrac{3}{6}}Y_2^0(\theta,\phi)\chi_+$$
$$B: \Psi(r,\theta,\phi) = f(r)\left[\sqrt{\tfrac{1}{3}}Y_3^0(\theta,\phi)\chi_+ + \sqrt{\tfrac{2}{3}}Y_3^1(\theta,\phi)\chi_-\right]$$
$$C: \Psi(r,\theta,\phi) = f(r)Y_3^0\chi_+$$
$$D: \Psi(r,\theta,\phi) = g(r)Y_2^0\chi_+$$

- ☐ A
- ☐ B
- ☐ C
- ☐ D

Explain your choice(s).

Assume that the electron has the wave function

$$\Psi(r,\theta,\phi) = f(r)\left[\sqrt{\tfrac{1}{6}}Y_3^0(\theta,\phi)\chi_+ + \sqrt{\tfrac{2}{6}}Y_3^1(\theta,\phi)\chi_-\right] + g(r)\sqrt{\tfrac{3}{6}}Y_2^0(\theta,\phi)\chi_+$$

Suppose you measured the z-component of the orbital angular momentum, L_z. What values might you get, and what is the probability of each? Explain.

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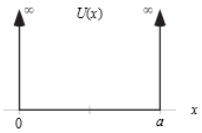


At time t = 0, the energy of the system is measured and found to be E_2, the first excited state energy.

Question 1.
Immediately after the measurement, is the system in an eigenstate of position?

- Select one...
- Yes
- No
- It depends

Question 2.
Explain your reasoning.

Question 3.
At time t_1 > 0 you measure the *position* of the particle and find that it is located at x = a/3.

Suppose a short time later, at a time t_2 you measured the energy of the particle. (Assume that no other measurements are made between t_1 and t_2.)

For each of the possible energies below, select the statement that best describes the probability that the energy measurement will result in this value.

Rows
Ground state energy
First excited state energy
Second excited state energy

- I will definitely measure this value
- It is possible (but not certain) that I will measure this value
- I will definitely not measure this value
- I do not have enough information to tell

Question 4.
Explain your reasoning:

Question 5.
At which of the times below is the *position* of the particle well-defined? (Check all that apply)

- ☐ Just AFTER the measurement at t = 0
- ☐ Just BEFORE the measurement at t1
- ☐ Just AFTER the measurement at t1
- ☐ Just BEFORE the measurement at t2

Question 6.
Explain your reasoning:

Question 7.
At which of the times below is the *energy* of the particle well-defined? (Check all that apply)

- ☐ Just AFTER the measurement at t = 0
- ☐ Just BEFORE the measurement at t1
- ☐ Just AFTER the measurement at t1
- ☐ Just BEFORE the measurement at t2

Question 8.
Explain your reasoning:

Question 9.
Suppose that instead of measuring the energy at time t_2, you measure the energy a *very long time* later. (Assume that no other measurements of the particle are made after t_1).

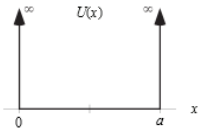
For each of the possible energies below, select the statement that best describes the probability that the energy measurement will result in this value.

Rows
Ground state energy
First excited state energy
Second excited state energy

- ○ I will definitely measure this value
- ○ It is possible (but not certain) that I will measure this value
- ○ I will definitely not measure this value
- ○ I do not have enough information to tell

Question 10.
Explain your reasoning:

A different particle is in an identical infinite square well potential.



At time t = 0, the energy of the system is measured and found to be E_1, the ground state energy.

Question 11.
At time $t_1 > 0$ you measure the *position* of the particle and find that it is located near x = 2*a*/3.

Suppose a short time later, at a time t_2 you measured the *position* of the particle again. (Assume that no other measurements are made between t_1 and t_2.)

For each of the possible positions below, select the statement that best describes the probability that the position measurement will result in this value.

Rows
Near x = a/3
Near x = a/2
Near x = 2a/3

- ○ I will definitely measure this value
- ○ It is possible (but not certain) that I will measure this value
- ○ I will definitely not measure this value
- ○ I do not have enough information to tell

Question 12.
Explain your reasoning:

Question 13.
At which of the times below is the *position* of the particle well-defined? (Check all that apply)

- ☐ Just AFTER the measurement at t = 0
- ☐ Just BEFORE the measurement at t1
- ☐ Just AFTER the measurement at t1
- ☐ Just BEFORE the measurement at t2

Question 14.
Explain your reasoning:

Question 15.
At which of the times below is the *energy* of the particle well-defined? (Check all that apply)

- ☐ Just AFTER the measurement at t = 0
- ☐ Just BEFORE the measurement at t1
- ☐ Just AFTER the measurement at t1
- ☐ Just BEFORE the measurement at t2

Question 16.
Explain your reasoning:

Question 17.
Suppose that instead of measuring the position at time t_2, you measure the position again a *very long time* later. (Assume that no other measurements of the particle are made after t_1.)
For each of the possible positions below, select the statement that best describes the probability that the position measurement will result in this value.

Rows
Near x = a/3
Near x = a/2
Near x = 2a/3

- ○ I will definitely measure this value
- ○ It is possible (but not certain) that I will measure this value
- ○ I will definitely not measure this value
- ○ I do not have enough information to tell

Question 18.
Explain your reasoning:

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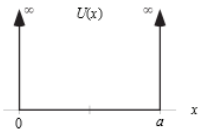


At time t = 0, the energy of the system is measured and found to be E_2, the first excited state energy.

Question 1.
Immediately after the measurement, is the system in an eigenstate of position?

- Select one...
- Yes
- No
- It depends

Question 2.
Immediately after the measurement, is the system in an eigenstate of energy?

- Select one...
- Yes
- No
- It depends

Question 3.
Explain your reasoning to the previous two questions.

Question 4.
At time $t_1 > 0$ you measure the *position* of the particle and find that it is located at $x = a/3$.

Suppose a short time later, at a time t_2 you measured the energy of the particle. (Assume that no other measurements are made between t_1 and t_2.)

For each of the possible energies below, select the statement that best describes the probability that the energy measurement will result in this value.

Rows
Ground state energy
First excited state energy
Second excited state energy

- I will definitely measure this value
- It is possible (but not certain) that I will measure this value
- I will definitely not measure this value
- I do not have enough information to tell

Question 5.
Explain your reasoning:

Question 6.
At which of the times below is the *position* of the particle well-defined? (Check all that apply)

- ☐ Just AFTER the measurement at t = 0

- ☐ Just BEFORE the measurement at t1
- ☐ Just AFTER the measurement at t1
- ☐ Just BEFORE the measurement at t2

Question 7.
Explain your reasoning:

Question 8.
At which of the times below is the *energy* of the particle well-defined? (Check all that apply)

- ☐ Just AFTER the measurement at t = 0
- ☐ Just BEFORE the measurement at t1
- ☐ Just AFTER the measurement at t1
- ☐ Just BEFORE the measurement at t2

Question 9.
Explain your reasoning:

Question 10.
Suppose that instead of measuring the energy at time t_2, you measure the energy a *very long time* later. (Assume that no other measurements of the particle are made after t_1).

For each of the possible energies below, select the statement that best describes the probability that the energy measurement will result in this value.

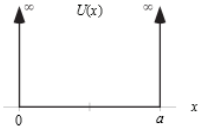
Rows
Ground state energy
First excited state energy
Second excited state energy

- ○ I will definitely measure this value
- ○ It is possible (but not certain) that I will measure this value
- ○ I will definitely not measure this value
- ○ I do not have enough information to tell

Question 11.
Explain your reasoning:

A different particle is in an identical infinite square well potential.

At time t = 0, the energy of the system is measured and found to be E_1, the ground state energy.



Question 12.
At time $t_1 > 0$ you measure the *position* of the particle and find that it is located near x = 2*a*/3.

Suppose a short time later, at a time t_2 you measured the *position* of the particle again. (Assume that no other measurements are made between t_1 and t_2.)

For each of the possible positions below, select the statement that best describes the probability that the position measurement will result in this value.

Rows
Near x = a/3
Near x = a/2

Near x = 2a/3

- I will definitely measure this value
- It is possible (but not certain) that I will measure this value
- I will definitely not measure this value
- I do not have enough information to tell

Question 13.
Explain your reasoning:

Question 14.
At which of the times below is the *position* of the particle well-defined? (Check all that apply)

- ☐ Just AFTER the measurement at t = 0
- ☐ Just BEFORE the measurement at t1
- ☐ Just AFTER the measurement at t1
- ☐ Just BEFORE the measurement at t2

Question 15.
Explain your reasoning:

Question 16.
At which of the times below is the *energy* of the particle well-defined? (Check all that apply)

- ☐ Just AFTER the measurement at t = 0
- ☐ Just BEFORE the measurement at t1
- ☐ Just AFTER the measurement at t1
- ☐ Just BEFORE the measurement at t2

Question 17.
Explain your reasoning:

Question 18.
Suppose that instead of measuring the position at time t_2, you measure the position again a *very long time* later. (Assume that no other measurements of the particle are made after t_1.)
For each of the possible positions below, select the statement that best describes the probability that the position measurement will result in this value.

Rows
Near x = a/3
Near x = a/2
Near x = 2a/3

- I will definitely measure this value
- It is possible (but not certain) that I will measure this value
- I will definitely not measure this value
- I do not have enough information to tell

Question 19.
Explain your reasoning:

Question 20.
For each of the states indicated below, determine whether the state is also a *stationary state?*

Rows
An eigenstate of the Hamiltonian operator
An eigenstate of the position operator
An eigenstate of the momentum operator

- Yes, this is always a stationary state
- No, this is never a stationary state
- The answer depends on other information (please specify)

Question 21.
Explain your reasoning:

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Print view of '(2DH)U1b'

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Consider a three-dimensional harmonic oscillator described by the Hamiltonian:

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\omega^2 r^2$$

In Cartesian coordinates the energy eigenfunctions are given by:

$$\psi(x,y,z) = N\phi_{n_x}(x)\phi_{n_y}(y)\phi_{n_z}(z)$$
$$\mathrm{E}_{n_x n_y n_z} = \hbar\omega(n_x + n_y + n_z + 3/2)$$

In spherical polar coordinates the energy eigenfunctions are given by:

$$\psi(r,\theta,\phi) = NR_{nl}(r)Y_l^m(\theta,\phi)$$
$$\mathrm{E}_{nl} = \hbar\omega(2n + l + 3/2)$$

Assume that *N* is the proper normalization.

Question 1.
Consider the state:
$$\Psi_a(\vec{r}, t=0) = N\left(\phi_1(x)\phi_0(y)\phi_0(z) + \phi_0(x)\phi_0(y)\phi_1(z)\right)$$

Does the state change as time evolves?

- Yes
- No
- Not enough information (please specify)

Question 2.
Please explain your reasoning to the previous question.

Question 3.
Consider the same state:
$$\Psi_a(\vec{r}, t=0) = N\left(\phi_1(x)\phi_0(y)\phi_0(z) + \phi_0(x)\phi_0(y)\phi_1(z)\right)$$

Does the probability density change as time evolves?

- Yes
- No
- Not enough information (please specify)

Question 4.
Please explain your reasoning to the previous question.

Question 5.
Consider the different state:
$$\Psi_b(\vec{r}, t=0) = N\left(\phi_1(x)\phi_0(y)\phi_0(z) + \phi_1(x)\phi_0(y)\phi_1(z)\right)$$

Does the state change as time evolves?

- Yes
- No
- Not enough information (please specify)

Question 6.
Please explain your reasoning to the previous question.

Question 7.
Consider the same state:
$\Psi_b(\vec{r}, t=0) = N\left(\phi_1(x)\phi_0(y)\phi_0(z) + \phi_1(x)\phi_0(y)\phi_1(z)\right)$

Does the probability density change as time evolves?

- Yes
- No
- Not enough information (please specify)

Question 8.
Please explain your reasoning to the previous question.

Question 9.
Consider the different state:
$\Psi_c(\vec{r}, t=0) = N\left(R_{11}(r)Y_1^0(\theta,\varphi) + R_{21}(r)Y_1^{-1}(\theta,\varphi)\right)$

Does the state change as time evolves?

- Yes
- No
- Not enough information (please specify)

Question 10.
Please explain your reasoning to the previous question.

Question 11.
Consider the same state:
$\Psi_c(\vec{r}, t=0) = N\left(R_{11}(r)Y_1^0(\theta,\varphi) + R_{21}(r)Y_1^{-1}(\theta,\varphi)\right)$

Does the probability density change as time evolves?

- Yes
- No
- Not enough information (please specify)

Question 12.
Please explain your reasoning to the previous question.

Question 13.
Consider the different state:
$\Psi_d(\vec{r}, t=0) = N\left(\phi_1(x)\phi_1(y)\phi_0(z) + R_{11}(r)Y_1^1(\theta,\varphi)\right)$

Does the state change as time evolves?

- Yes
- No
- Not enough information (please specify)

Question 14.
Please explain your reasoning to the previous question.

Question 15.
Consider the same state:
$\Psi_d(\vec{r}, t=0) = N\left(\phi_1(x)\phi_1(y)\phi_0(z) + R_{11}(r)Y_1^1(\theta,\varphi)\right)$

Does the probability density change as time evolves?

- Yes
- No

- Not enough information (please specify)

Question 16.
Please explain your reasoning to the previous question.

Question 17.
Consider the different state:
$\Psi_e(\vec{r}, t=0) = N\left(\phi_2(x)\phi_1(y)\phi_0(z) + R_{11}(r)Y_1^{-1}(\theta,\varphi)\right)$

Does the state change as time evolves?

- Yes
- No
- Not enough information (please specify)

Question 18.
Please explain your reasoning to the previous question.

Question 19.
Consider the same state:
$\Psi_e(\vec{r}, t=0) = N\left(\phi_2(x)\phi_1(y)\phi_0(z) + R_{11}(r)Y_1^{-1}(\theta,\varphi)\right)$

Does the probability density change as time evolves?

- Yes
- No
- Not enough information (please specify)

Question 20.
Please explain your reasoning to the previous question.

Suppose the system now consists of two identical (non-interacting) fermions with spin-1/2 in the same, three-dimensional harmonic.

The one-particle wave functions and energies are repeated below:

$\psi(x,y,z) = N\phi_{n_x}(x)\phi_{n_y}(y)\phi_{n_z}(z)$
$\mathrm{E}_{n_x n_y n_z} = \hbar\omega(n_x + n_y + n_z + 3/2)$

Question 21.
What is the degeneracy of the ground state of the entire two-particle system?

Question 22.
Please explain your reasoning to the previous question.

Question 23.
Suppose the entire two-particle system is known to have its ground-state energy. Which of the following best describes the result of a measurement of the position of both particles?

- A: Both particles are likely to be found near the same position
- B: Both particles are unlikely to be found near the same position
- Either A or B, depending on the exact form of the spatial state of both particles
- Either A or B, depending on the exact form of the spin state of both particles
- There is not enough information to answer (please specify)
- Other (please specify):

Question 24.
Please explain your reasoning to the previous question.

Question 25.
What is the degeneracy of the first excited-state of the entire two-particle system?

Question 26.
Please explain your reasoning to the previous question.

Question 27.
Suppose the entire two-particle system is known to have its first excited-state energy. Which of the following best describes the result of a measurement of the position of both particles?

- A: Both particles are likely to be found near the same position
- B: Both particles are unlikely to be found near the same position
- Either A or B, depending on the exact form of the spatial state of both particles
- Either A or B, depending on the exact form of the spin state of both particles
- There is not enough information to answer (please specify)
- Other (please specify):

Question 28.
Please explain your reasoning to the previous question.

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Part I
Suppose a particle in a three-dimensional quantum system has a wave function given by

$$\psi(r,\theta,\phi) = f(r)Y_2^1(\theta,\phi)$$

where $f(r)$ is any function of the radial coordinate, r, and $Y_2^1(\theta, \varphi)$ is one of the spherical harmonics.

Question 1.
Suppose the *orbital* angular momentum squared, L^2, of this particle is measured.

Which of the following would be possible results of this measurement? (Check all that apply)

- ☐ 0
- ☐ 1 hbar^2
- ☐ 2 hbar^2
- ☐ 4 hbar^2
- ☐ 6 hbar^2
- ☐ Any value is possible
- ☐ Other:

Explain your reasoning.

Question 2.
Suppose the *z*-component of the *orbital* angular momentum, L_z, of this particle is measured.

Which of the following would be possible results of this measurement? (Check all that apply)

- ☐ -2 hbar
- ☐ -1 hbar
- ☐ -1/2 hbar
- ☐ 0
- ☐ 1/2 hbar
- ☐ 1 hbar
- ☐ 2 hbar
- ☐ Any value is possible
- ☐ Other:

Explain your reasoning.

Question 3.
Suppose the ***x*-component** of the *orbital* angular momentum, L_x, of this particle is measured.

Which of the following would be possible results of this measurement? (Check all that apply)

- ☐ -2 hbar
- ☐ -1 hbar
- ☐ -1/2 hbar
- ☐ 0
- ☐ 1/2 hbar
- ☐ 1 hbar
- ☐ 2 hbar
- ☐ Any value is possible
- ☐ Other:

Explain your reasoning.

Part II

States will be represented on this page using the following notation: $\psi = |n, l, m_l\rangle$.

Each of the following represents the state of an electron in identical hydrogen atom potentials.

$\psi_1 = |2,1,1\rangle$

$\psi_2 = \frac{|2,1,1\rangle + i|2,1,0\rangle}{\sqrt{2}}$

$\psi_3 = \frac{|2,1,1\rangle - |3,1,1\rangle}{\sqrt{2}}$

$\psi_4 = \frac{|2,0,0\rangle + |2,1,0\rangle}{\sqrt{2}}$

Question 4.
For which of the initial states above will the corresponding *wave function* depend on time? (Check all that apply.)

- ☐ ψ1
- ☐ ψ2
- ☐ ψ3
- ☐ ψ4

Explain your reasoning.

Question 5.
For which of the initial states above will the corresponding *probability density* depend on time? (Check all that apply.)

- ☐ ψ1
- ☐ ψ2
- ☐ ψ3
- ☐ ψ4

Explain your reasoning.

Question 6.
For which of the initial states above will the *expectation value of the energy* depend on time? (Check all that apply.)

- ☐ ψ1
- ☐ ψ2
- ☐ ψ3
- ☐ ψ4

Explain your reasoning.

Question 7.
For which of the initial states above will the *probability that the energy is measured to be* E_2 depend on time? (Check all that apply.)

- ☐ ψ1
- ☐ ψ2
- ☐ ψ3
- ☐ ψ4

Explain your reasoning.

Suppose the position of electron 1 is measured at t_1. A short time later, at t_2, another measurement is made on this system.

Question 8.
For each of the different kinds of measurement below, indicate which measurements could result in *more than one* possible value at t_2. (Check all that apply.)

- ☐ Position (x)
- ☐ Momentum (p)
- ☐ Energy (E)
- ☐ Angular momentum squared (L^2)
- ☐ The z-component of angular momentum (Lz)
- ☐ The x-component of angular momentum (Lx)

Explain your reasoning.

Question 9.
For each of the different kinds of measurement below, indicate which measurements have probabilities that *depend on the time,* t_2, at which the measurement is made. (Check all that apply.)

- ☐ Position (x)
- ☐ Momentum (p)
- ☐ Energy (E)
- ☐ Angular momentum squared (L^2)
- ☐ The z-component of angular momentum (Lz)
- ☐ The x-component of angular momentum (Lx)

Explain your reasoning.

Part III

Consider a particle with total spin-1/2. The particle is in a system with energy eigenvalues +ε corresponding to spin-up in the *z*-direction and -ε corresponding to spin-down in the *z*-direction.

Suppose you measure this particle to be spin-up in the *z*-direction at $t = 0$. A short time later, another measurement is made at $t_1 > 0$.

Question 10.
For each of the different kinds of measurement below, indicate which measurements could result in *more than one* possible value at t_1. (Check all that apply.)

- ☐ Energy (E)
- ☐ Total spin (|S|)
- ☐ The z-component of spin (Sz)
- ☐ The x-component of spin (Sx)

Explain your reasoning.

Question 11.
For each of the different kinds of measurement below, indicate which measurements have probabilities that *depend on the time,* t_1, at which the measurement is made. (Check all that apply.)

- ☐ Energy (E)
- ☐ Total spin (|S|)

- ☐ The z-component of spin (Sz)
- ☐ The x-component of spin (Sx)

Explain your reasoning.

Suppose you measure this particle to be spin-up in the ***x*-direction** at t_1. A short time later, another measurement is made at $t_2 > t_1$.

Question 12.
For each of the different kinds of measurement below, indicate which measurements could result in *more than one* possible value at t_2. (Check all that apply.)

- ☐ Energy (E)
- ☐ Total spin (|S|)
- ☐ The z-component of spin (Sz)
- ☐ The x-component of spin (Sx)

Explain your reasoning.

Question 13.
For each of the different kinds of measurement below, indicate which measurements have probabilities that *depend on the time, t_2,* at which the measurement is made. (Check all that apply.)

- ☐ Energy (E)
- ☐ Total spin (|S|)
- ☐ The z-component of spin (Sz)
- ☐ The x-component of spin (Sx)

Explain your reasoning.

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Part I
Consider the two hydrogen atom wave functions given below:

$$\psi_1(r,\theta,\phi) = \frac{1}{\sqrt{2}}\left(R_{21}(r)Y_1^1(\theta,\phi) + R_{21}(r)Y_1^{-1}(\theta,\phi)\right)$$
$$\psi_2(r,\theta,\phi) = \frac{1}{\sqrt{3}}\left(R_{21}(r)Y_1^1(\theta,\phi) + R_{32}(r)Y_2^1(\theta,\phi) + R_{31}(r)Y_1^0(\theta,\phi)\right)$$

where $R_{nl}(r)$ are the radial wave functions and $Y_l^m(\theta, \varphi)$ are the spherical harmonics.

Question 1.
Suppose the orbital angular momentum squared, L^2, of each particle is measured.

Which of the following would be possible results of this measurement for each particle?
(Check all that apply)

Rows
ψ1
ψ2

- ☐ 0
- ☐ 1 hbar^2
- ☐ 2 hbar^2
- ☐ 4 hbar^2
- ☐ 6 hbar^2
- ☐ Any value greater than zero is possible

Question 2.
Explain your reasoning.

Question 3.
Suppose the *z*-component of the *orbital* angular momentum, L_z, of each particle is measured.

Which of the following would be possible results of this measurement for each particle?
(Check all that apply)

Rows
ψ1
ψ2

- ☐ -2 hbar
- ☐ -1 hbar
- ☐ -1/2 hbar
- ☐ 0
- ☐ 1/2 hbar
- ☐ 1 hbar
- ☐ 2 hbar
- ☐ Other values are possible (please specify below)

Question 4.
Explain your reasoning.

Question 5.
Suppose the ***x*-component** of the *orbital* angular momentum, L_x, of each particle is measured.

Which of the following would be possible results of this measurement for each particle?
(Check all that apply)

Rows

ψ1
ψ2
- ☐ -2 hbar
- ☐ -1 hbar
- ☐ -1/2 hbar
- ☐ 0
- ☐ 1/2 hbar
- ☐ 1 hbar
- ☐ 2 hbar
- ☐ Other values are possible (please specify below)

Question 6.
Explain your reasoning.

Part II
Consider the four hydrogen atom wave functions given below:

$$\psi_1(r,\theta,\phi) = \frac{1}{\sqrt{2}}\left(R_{21}(r)Y_1^1(\theta,\phi) + R_{21}(r)Y_1^{-1}(\theta,\phi)\right)$$
$$\psi_2(r,\theta,\phi) = \frac{1}{\sqrt{2}}\left(R_{21}(r)Y_1^1(\theta,\phi) + R_{31}(r)Y_1^1(\theta,\phi)\right)$$
$$\psi_3(r,\theta,\phi) = \frac{1}{\sqrt{2}}\left(R_{21}(r)Y_1^0(\theta,\phi) + R_{20}(r)Y_0^0(\theta,\phi)\right)$$
$$\psi_4(r,\theta,\phi) = \frac{1}{\sqrt{2}}\left(R_{21}(r)Y_1^{-1}(\theta,\phi) + R_{10}(r)Y_0^0(\theta,\phi)\right)$$

where $R_{nl}(r)$ are the radial wave functions and $Y_l^m(\theta, \varphi)$ are the spherical harmonics.

Question 7.
Suppose the position of each particle is measured.

For which, if any, of the wave functions above does the probability that the particle is found near $r = a$ (the Bohr radius) depend on time? (Check all that apply.)

- ☐ ψ1
- ☐ ψ2
- ☐ ψ3
- ☐ ψ4

Question 8.
Explain your reasoning.

Question 9.
Suppose the orbital angular momentum squared (L^2) of each particle is measured. (Assume no other measurements have been made.)

For which, if any, of the wave functions above does the probability that the particle is found to have $L^2 = 2\hbar^2$ depend on time? (Check all that apply.)

- ☐ ψ1
- ☐ ψ2
- ☐ ψ3
- ☐ ψ4

Question 10.
Explain your reasoning.

Question 11.
Suppose the z-component of the orbital angular momentum (L_z) of each particle is measured. (Assume no other measurements have been made.)

For which, if any, of the wave functions above does the probability that the particle is found to

have $L_z = \hbar$ depend on time? (Check all that apply.)

- ☐ ψ1
- ☐ ψ2
- ☐ ψ3
- ☐ ψ4

Question 12.
Explain your reasoning.

Part III

Consider a particle with total spin-1. The particle is in a system with energy eigenvalues {9ε, 4ε, ε} corresponding to the states with an ***x*-component** of spin angular momentum equal to {$+\hbar$, 0, $-\hbar$}, respectively (*i.e.*, $[H, L_x] = 0$).

At $t = 0$, suppose the ***x*-component** of this particle's spin angular momentum is measured to be equal to 0. A short time later, another measurement is made at $t_1 > 0$.

Question 13.
For each of the different kinds of measurement below, indicate which measurements have probabilities that *depend on the time,* t_1, at which the measurement is made. (Check all that apply.)

- ☐ Energy (E)
- ☐ Total spin (|S|)
- ☐ The z-component of spin (Sz)
- ☐ The x-component of spin (Sx)

Question 14.
Explain your reasoning.

At t_1, suppose the ***z*-component** of this particle's spin angular momentum is measured to be equal to 0. A short time later, another measurement is made at $t_2 > t_1$.

Question 15.
For each of the different kinds of measurement below, indicate which measurements could result in *more than one* possible value at t_2. (Check all that apply.)

- ☐ Energy (E)
- ☐ Total spin (|S|)
- ☐ The z-component of spin (Sz)
- ☐ The x-component of spin (Sx)

Question 16.
Explain your reasoning.

Question 17.
For each of the different kinds of measurement below, indicate which measurements have probabilities that *depend on the time,* t_2, at which the measurement is made. (Check all that apply.)

- ☐ Energy (E)
- ☐ Total spin (|S|)
- ☐ The z-component of spin (Sz)
- ☐ The x-component of spin (Sx)

Question 18.
Explain your reasoning.

Question 19.
What is the energy of the particle at time t between the measurements at t_1 and t_2 (*i.e.*, after the measurement of S_z but before any other measurement)?

- ○ E = ε
- ○ E = 4ε
- ○ E = 13ε/3
- ○ E = 9ε
- ○ E = <ψ|H|ψ>
- ○ E is different at different times
- ○ E is not well defined
- ○ Other:

Question 20.
Explain your reasoning.

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Part I
Suppose that you are given a red die weighted such that the probability of rolling an "i" on the red die is given by $P_r(i)$. (*i.e.,* the probability that rolling a red die would result in a 1 is $P_r(1)$.)

Do not assume that $P_r(i) = 1/6$ (that it is a fair die).

Question 1.
Write an expression for the probability that the roll would result in a "1" *or* a "2".

Question 2.
Explain the reasoning you used to answer the previous question.

You are now given a blue die weighted differently such that the probability of rolling an "j" on the blue die is given by $P_b(j)$.

Question 3.
Write an expression for the probability that rolling both dice would result in a blue "1" *and* a red "6".

Question 4.
Explain the reasoning you used to answer the previous question.

Question 5.
Write an expression for the probability that rolling both dice would result in a "1" and a "6", where it does not matter which number appears on which colored die.

Question 6.
Explain the reasoning you used to answer the previous question.

Question 7.
If you have made any assumptions in answering the previous questions, please list them here.

Part II
Assume you have two particles in the same infinite square well. The particles are identical, and there is no interaction between them.

Suppose the probability that both particles are measured to be in ***the same half of the well*** (*i.e.,* both particles on the right *or* both particles on the left) is greater than the probability that the particles will be found in ***different halves of the well*** (*i.e.,* one on the right and one on the left).

Question 8.
Is it possible that these particles are identical bosons?

- ○ Yes
- ○ No
- ○ There is not enough information to answer

Question 9.
Explain the reasoning you used to answer the previous question.

Question 10.
If you answered "yes" please give at least one example of the full wave function for this system that satisfies the condition above.

Question 11.
Is it possible that these particles are identical fermions?

- Yes
- No
- There is not enough information to answer

Question 12.
Explain the reasoning you used to answer the previous question.

Question 13.
If you answered "yes" please give at least one example of the full wave function for this system that satisfies the condition above.

Part III
Consider the following spatial wave functions for a system of two particles (labeled a and b) in an infinite square well potential, where $\psi_n(x)$ is the nth excited-state wave function. Assume that the spin wave function is symmetric in each case.

$$\psi_A(x_a, x_b) = \psi_1(x_a)\psi_2(x_b)$$

$$\psi_B(x_a, x_b) = \frac{1}{\sqrt{2}}(\psi_1(x_a)\psi_2(x_b) + \psi_2(x_a)\psi_1(x_b))$$

$$\psi_C(x_a, x_b) = \frac{1}{\sqrt{2}}(\psi_1(x_a)\psi_2(x_b) + \psi_2(x_a)\psi_3(x_b))$$

$$\psi_D(x_a, x_b) = \frac{1}{\sqrt{2}}(\psi_1(x_a)\psi_2(x_b) - \psi_2(x_a)\psi_1(x_b))$$

$$\psi_E(x_a, x_b) = \frac{1}{\sqrt{2}}(\psi_2(x_a)\psi_2(x_b) - \psi_1(x_a)\psi_1(x_b))$$

Question 14.
Which of the wave functions above could represent a system consisting of identical bosons? (Check all that apply.)

- ☐ ψA
- ☐ ψB
- ☐ ψC
- ☐ ψD
- ☐ ψE

Question 15.
Explain the reasoning you used to answer the previous question.

Question 16.
Which of the wave functions above could represent a system consisting of identical fermions? (Check all that apply.)

- ☐ ψA
- ☐ ψB
- ☐ ψC
- ☐ ψD
- ☐ ψE

Question 17.
Explain the reasoning you used to answer the previous question.

Question 18.
For which of the wave functions above does the corresponding probability density depend on time? (Check all that apply.)

- ☐ ψA
- ☐ ψB
- ☐ ψC
- ☐ ψD
- ☐ ψE

Question 19.
Explain the reasoning you used to answer the previous question.

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For this question, hydrogen atom states will be written in Dirac notation as follows:
$\psi = |n, l, m_l; s, m_s\rangle$

In this expression, the first three quantum numbers refer to spatial coordinates, while the last two quantum numbers refer to *spin* angular momentum.

Consider the set of four wave functions shown below.

$\psi_1 = |3, 2, -1; 1/2, 1/2\rangle$

$\psi_2 = \frac{1}{5}(4|3, 2, -1; 1/2, 1/2\rangle - 3|2, 1, -1; 1/2, -1/2\rangle)$

$\psi_3 = \frac{1}{3}(i|3, 2, -1; 1/2, 1/2\rangle - 2|3, 1, -1; 1/2, -1/2\rangle + 2|3, 0, 0; 1/2, 1/2\rangle)$

$\psi_4 = \frac{1}{\sqrt{2}}(1 - i)|2, 1, 1; 1/2, -1/2\rangle$

Question 1.
Suppose the *orbital* angular momentum squared, L^2, is measured for an electron in the hydrogen atom and is found to be $6\hbar^2$.

Which of the above represent possible states for the electron *just before* the measurement is made? (Select all that apply)

- ☐ ψ1
- ☐ ψ2
- ☐ ψ3
- ☐ ψ4

Explain your reasoning.

Question 2.
Which of the above represent possible states for the electron *just after* the measurement is made? (Select all that apply)

- ☐ ψ1
- ☐ ψ2
- ☐ ψ3
- ☐ ψ4

Explain your reasoning.

Consider again the set of four wave functions from the previous page (shown below).

$\psi_1 = |3, 2, -1; 1/2, 1/2\rangle$

$\psi_2 = \frac{1}{5}(4|3, 2, -1; 1/2, 1/2\rangle - 3|2, 1, -1; 1/2, -1/2\rangle)$

$\psi_3 = \frac{1}{3}(i|3, 2, -1; 1/2, 1/2\rangle - 2|3, 1, -1; 1/2, -1/2\rangle + 2|3, 0, 0; 1/2, 1/2\rangle)$

$\psi_4 = \frac{1}{\sqrt{2}}(1 - i)|2, 1, 1; 1/2, -1/2\rangle$

Recall that these states are written in Dirac notation as follows: $\psi = |n, l, m_l; s, m_s\rangle$

Question 3.
Suppose the orbital angular momentum squared (L^2) of each particle is measured. (Assume no other measurements have been made.)

For which, if any, of the wave functions above does the probability that the particle is found to have $L^2 = 6\hbar^2$ depend on time? (Check all that apply.)

- ☐ ψ1
- ☐ ψ2
- ☐ ψ3
- ☐ ψ4

Explain your reasoning.

Question 4.
Suppose the *z*-component of the *orbital* angular momentum (L_z) of each particle is measured. (Assume no other measurements have been made.)

For which, if any, of the wave functions above does the probability that the particle is found to have $L_z = \hbar$ depend on time? (Check all that apply.)

- ☐ ψ1
- ☐ ψ2
- ☐ ψ3
- ☐ ψ4

Explain your reasoning.

Question 5.
Suppose the *z*-component of the *spin* angular momentum (S_z) of each particle is measured. (Assume no other measurements have been made.)

For which, if any, of the wave functions above does the probability that the particle is found to have $S_z = \hbar/2$ depend on time? (Check all that apply.)

- ☐ ψ1
- ☐ ψ2
- ☐ ψ3
- ☐ ψ4

Explain your reasoning.

Question 6.
Suppose the position of each particle is measured. (Assume no other measurements have been made.)

For which, if any, of the wave functions above does the probability that the particle is found near r = 1.4 nm depend on time? (Check all that apply.)

- ☐ ψ1
- ☐ ψ2
- ☐ ψ3
- ☐ ψ4

Explain your reasoning.

For the remaining questions, assume the electron is in the state:

$$\psi_3 = \tfrac{1}{3}(i|3,2,-1;1/2,1/2\rangle - 2|3,1,-1;1/2,-1/2\rangle + 2|3,0,0;1/2,1/2\rangle)$$

Question 7.
Suppose you measured the *z*-component of the *orbital* angular momentum (L_z). What values might you get, and what is the probability of each? Explain your reasoning.

Question 8.
Let **J** = **L** + **S** be the total angular momentum of the electron.

Suppose you measured the total angular momentum squared (J^2). Determine all the possible values of this measurement. Explain.

Question 9.
Suppose you measured the *z*-component of the total angular momentum (J_z). Determine all the possible values of this measurement. Explain your reasoning.

Question 10.

What is the probability that this measurement will result in $\hbar/2$? Explain your reasoning.

Question 11.
Would this probability depend on the time at which the measurement is made? Explain your reasoning.

Two non-identical spin-1/2 particles interact with a uniform magnetic field, B_o, directed in the *z*-direction. The particles have different magnetic moments; the magnetic moment of particle 1 (μ_1) is three times the magnetic moment of particle 2 (μ_2). The Hamiltonian for this system is given by

$$\hat{H} = \vec{B} \cdot \left(\mu_1 \vec{S_1} + \mu_2 \vec{S_2} \right)$$

where S_1 and S_2 are the spin operators for particles 1 and 2, respectively.

The states for this system will be written using the following notation:

$$\chi = |m_1, m_2\rangle$$

where m_1 is the *z*-component of the spin for particle 1 and m_2 is the *z*-component of the spin for particle 2.

Question 12.
Consider the state given below:

$$\chi_2 = |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

Is this state an energy eigenstate of the Hamiltonian for this system? If so, calculate the energy eigenvalue in terms of the quantities B_o and μ_2. If not, determine the most probable value(s) of the energy in terms of the same quantities.

Question 13.
Consider the state given below:

$$\chi_3 = |-\tfrac{1}{2}, \tfrac{1}{2}\rangle$$

Is this state an energy eigenstate of the Hamiltonian for this system? If so, calculate the energy eigenvalue in terms of the quantities B_o and μ_2. If not, determine the most probable value(s) of the energy in terms of the same quantities.

Explain how you determined your answers.

Question 14.
Suppose that at time t = 0 the two particles in this system are in a state with total spin s = 0 (otherwise known as the *singlet* state):

$$\chi(t=0) = \frac{1}{\sqrt{2}}\left(|\tfrac{1}{2}, -\tfrac{1}{2}\rangle - |-\tfrac{1}{2}, \tfrac{1}{2}\rangle\right)$$

Give an expression for the wave function $\chi(t)$ at later times.

Explain how you determined your answer.

Question 15.
Suppose you were to measure the *total spin* of this system a short time later at time t > 0. Does the probability that this measurement results in s = 1 change in time?

- Yes, the probability of measuring s = 1 changes with time
- No, the probability of measuring s = 1 is constant in time (but nonzero)
- No, the probability of measuring s = 1 is constant in time and equal to zero
- There is not enough information to answer this question

Explain how you determined your answer.

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Print view of '(TDP)U1'

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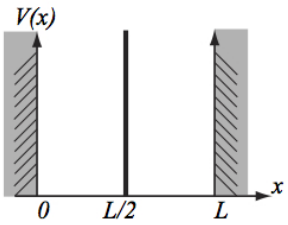
A particle is placed in the infinite square well, which has energy eigenvalues of E_1, E_2, E_3… etc. The energy of the particle is measured to be equal to E_1, the ground state energy.

Question 1.
Does the spatial probability density of this particle depend on time? Explain.

Question 2.
If the energy of the particle were measured again, what is the probability you would obtain E_1? Does this probability depend on time?

At t = 0, a small delta-function perturbation is turned on in the center of the well, as shown at right.



Question 3.
At some later time t > 0, does the spatial probability density of this particle depend on time? Explain.

Question 4.
If the energy of the particle were measured at some later time t > 0, which of the unperturbed infinite square well energies (E_1, E_2, E_3…) could you obtain? If none of the listed energies could be obtained, state so explicitly. Explain.

At t = t_1, the perturbation is turned off, such that the potential is again described by the unperturbed infinite square well. Assume that no measurements were made while the perturbation was on.

Question 5.
At some later time t > t_1, does the spatial probability density of this particle depend on time? Explain.

Question 6.
If the energy of the particle were measured at some later time t > t_1, which of the unperturbed infinite square well energies (E_1, E_2, E_3…) could you obtain? If none of the listed energies could be obtained, state so explicitly. Explain.

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A particle is placed in the infinite square well, which has energy eigenvalues of E_1, E_2, E_3... etc. The energy of the particle is measured to be equal to E_1, the ground state energy.

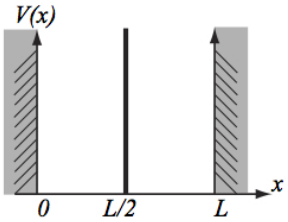
Question 1.
Does the spatial probability density of this particle depend on time?

- Yes
- No
- Not enough information

Question 2.
Explain the reasoning you used to answer the previous question.

Question 3.
If the energy of the particle were measured again, what is the probability you would obtain E_1? Does this probability depend on time?

At t = 0, a small delta-function perturbation is turned on in the center of the well, as shown at right.



Question 4.
At some later time t > 0, does the wave function for this particle depend on time?

- Yes
- No
- Not enough information

Question 5.
Explain the reasoning you used to answer the previous question.

Question 6.
At some later time t > 0, does the spatial probability density of this particle depend on time?

- Yes
- No
- Not enough information

Question 7.
Explain the reasoning you used to answer the previous question.

Question 8.
If the energy of the particle were measured at some later time t > 0, which of the following **unperturbed** infinite square well energies could you obtain? (Select all that apply.)

- ☐ E1

- ☐ E2
- ☐ E3
- ☐ E4
- ☐ E5

Question 9.
Explain the reasoning you used to answer the previous question.

Question 10.
Does the probability of getting any particular value of energy depend on time? Explain your reasoning.

At $t = t_1$, the perturbation is turned off, such that the potential is again described by the unperturbed infinite square well. Assume that no measurements were made while the perturbation was on.

Question 11.
At some later time $t > t_1$, does the wave function for this particle depend on time?

- ○ Yes
- ○ No
- ○ Not enough information

Question 12.
Explain the reasoning you used to answer the previous question.

Question 13.
At some later time $t > t_1$, does the spatial probability density of this particle depend on time?

- ○ Yes
- ○ No
- ○ Not enough information

Question 14.
Explain the reasoning you used to answer the previous question.

Question 15.
If the energy of the particle were measured at some later time $t > t_1$, which of the following **unperturbed** infinite square well energies could you obtain? (Select all that apply.)

- ☐ E1
- ☐ E2
- ☐ E3
- ☐ E4
- ☐ E5

Question 16.
Explain the reasoning you used to answer the previous question.

Question 17.
Does the probability of getting any particular value of energy depend on time? Explain your reasoning.

Questions or comments?

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Part I
An electron's spin is measured and determined to be up in the *z*-direction. At $t = 0$, a constant magnetic field B_o is turned on in the *z*-direction.

Question 1.
Does the state of this electron depend on time? Explain your reasoning.

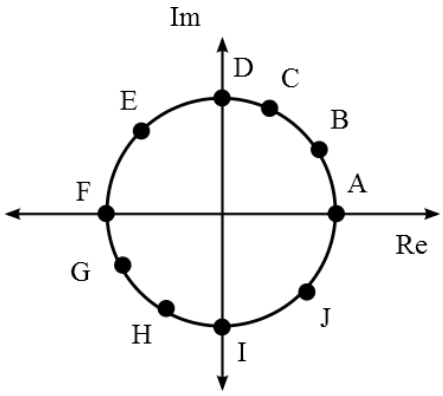
Question 2.
You decide to conduct another measurement of the spin in the *z*-direction. Does the probability that the electron is measured to have spin up again depend on time? Explain your reasoning.

Part II
A different electron's spin is measured and determined to be up in the ***x*-direction**. At $t = 0$, a constant magnetic field B_o is turned on in the *z*-direction.

Question 3.
Does the state of this electron depend on time? Explain your reasoning.

Question 4.
You decide to conduct another measurement of the spin in the ***x*-direction**. Does the probability that the electron is measured to have spin up again depend on time? Explain your reasoning.

Part III
For each of the following questions, choose the point from the graph below that best represents the indicated point (ignore the relative magnitudes of the points).



Question 5.

Point 1: $i - 1$ Point 2: $e^{i\pi}$ Point 3: $-e^{-1}$ Point 4: i^3 Point 5: $e^{i\frac{\pi}{2}} - e^{-i\frac{\pi}{2}}$

Point 6: sin(30°) Point 7: $e^{i\frac{\pi}{3}}$ Point 8: $e^{-i\frac{5\pi}{6}}$

Rows
Point 1

Point 2
Point 3
Point 4
Point 5
Point 6
Point 7
Point 8

- ○ A
- ○ B
- ○ C
- ○ D
- ○ E
- ○ F
- ○ G
- ○ H
- ○ I
- ○ J

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A.II – Pretests for Part II

This sub-section includes copies of all pretests discussed in part II of this dissertation.

Print view of 'PL_DIR_U1a'

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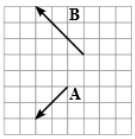
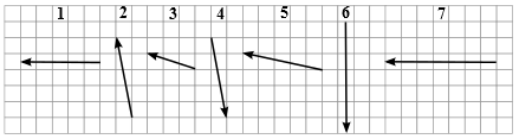
Part I
Two vectors, A and B, are drawn to scale on the grid at right.



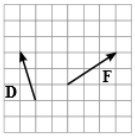
Question 1.
Which of the following represents the vector C = A + B? (Each vector is drawn to scale)



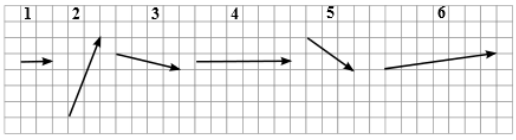
- Select one...
- 1
- 2
- 3
- 4
- 5
- 6
- 7
- None of the above

Question 2.
Explain the reasoning you used to answer the previous question.

Two different vectors, D and F, are drawn to scale on the grid at right.



Question 3.
Which of the following represents the vector E that must be added to the vector D in order to obtain the vector F? (Each vector is drawn to scale)



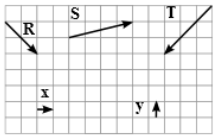
- Select one...
- 1
- 2
- 3
- 4
- 5

- 6
- None of the above

Question 4.
Explain the reasoning you used to answer the previous question.

Part II
The grid at right shows three vectors (R, S, and T) and two basis vectors (x and y) drawn to scale. Use these vectors to answer the questions below.

If it is not possible to answer any of the questions below, state so explicitly.



Question 5.
Write an expression for the vector R in terms of the basis vectors given above (for example, R = 6x - 9y).

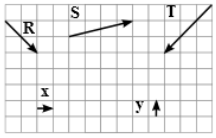
Question 6.
Write an expression for the vector S in terms of the basis vectors given above.

Question 7.
Write an expression for the vector T in terms of the basis vectors given above.

Question 8.
Write an expression for the vector x in terms of **only** the vectors R and T.

Question 9.
Write an expression for the vector y in terms of **only** the vectors R and S.

Part III
The vectors from the previous page are reproduced at right.



Question 10.
Determine the dot product between R and S.

Question 11.
Give a short explanation of what the dot product tells you about the vectors involved. You may find it useful to refer to the previous question.

Question 12.
What is the dot product of a vector with itself? Explain your reasoning.

Question 13.
What must always be true of two vectors if their dot product is zero? Explain your reasoning. If possible, give one or two examples of pairs of vectors for which this is true.

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Part I
Consider the vectors given below:

$\vec{a} = 2\vec{x} - \vec{y} + 2\vec{z}$
$\vec{b} = 3\vec{x} - \vec{y} - \vec{z}$
$\vec{c} = -\vec{y} - 3\vec{z}$

Question 1.
Which of the vectors above has the largest magnitude?

- a
- b
- c
- a and b
- b and c
- a and c
- all have the same magnitude

Question 2.
Explain how you answered the previous question.

Question 3.
Consider the vector shown below:

$\vec{d} = -\vec{x} + 4\vec{y} - 5\vec{z}$

Which of the vectors a, b, and c has the greatest overlap with this vector?

- a
- b
- c
- a and b
- b and c
- a and c
- all have the same overlap with this vector

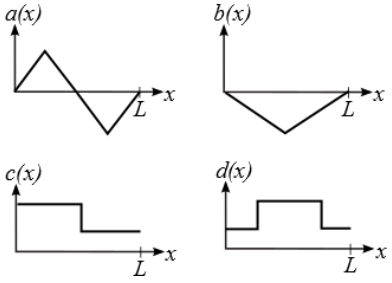
Question 4.
Explain how you answered the previous question.

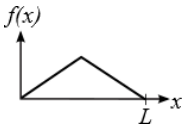
Question 5.
Of the vectors a, b, and c, which pair of vectors are closest to perpendicular?

- a and b
- b and c
- a and c
- all three

Question 6.
Explain how you answered the previous question.

Part II
Consider the function *f(x)* shown at right, and the four functions shown below.
All functions are zero except as indicated.





Recall that the inner product between two functions is defined to be the integral over all space of the product of the functions.

Question 7.
Is the inner product of *a(x)* with *f(x) greater than, less than, or equal to* the inner product of *b(x)* with *f(x)?*

- Greater than
- Less than
- Equal to
- There is not enough information to answer

Question 8.
Explain the reasoning you used to answer the previous question.

Question 9.
Is the inner product of *c(x)* with *f(x) greater than, less than,* or *equal to* the inner product of *d(x)* with *f(x)?*

- Greater than
- Less than
- Equal to
- There is not enough information to answer

Question 10.
Explain the reasoning you used to answer the previous question.

Question 11.
Do any of the functions shown above (*a-d*) have **zero** inner product with the function *f(x)? (Check all that apply)*

- a
- b
- c
- d
- none of the above

Question 12.
Explain the reasoning you used to answer the previous question.

Question 13.
In your own words, please give a definition for how to determine whether two vectors are *linearly independent.*

Print view of '(RWF)U1a'

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Consider a particle in the infinite square well. Suppose this particle is described by the following expression in Dirac notation:

$$|\psi\rangle = \frac{3}{5}|\psi_2\rangle + \frac{4}{5}|\psi_3\rangle$$

Question 1.
For each of the following quantities, state whether or not it is possible to determine the quantity of interest given the above representation of the state.

Rows

The probability that an energy measurement will result in E2, the first excited-state energy

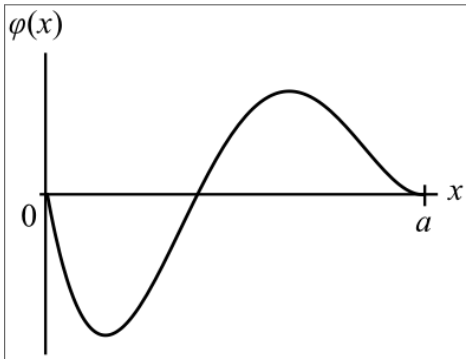
The probability that a position measurement will find the particle on the left half of the well

The probability that a momentum measurement will find the particle moving to the left

- ○ Yes, it is possible to determine this
- ○ No, it is not possible to determine this

Question 2.
Please explain your reasoning for ***all*** of the above cases.

Consider a different particle in the same infinite square well. This particle is known to have the wave function shown at right.



Question 3.
For each of the following quantities, state whether or not it is possible to determine the quantity of interest given the above representation of the state.

Rows

The probability that an energy measurement will result in E2, the first excited-state energy

The probability that a position measurement will find the particle on the left half of the well

The probability that a momentum measurement will find the particle moving to the left

- ○ Yes, it is possible to determine this
- ○ No, it is not possible to determine this

Question 4.
Please explain your reasoning for ***all*** of the above cases.

Consider a different particle in the infinite square well. This particle is known to have the wave function given below (where *N* is a constant):

$$\psi_A(x) = N(sin(\pi x/a) + sin^3(\pi x/a))$$

Question 5.
For each of the following quantities, state whether or not it is possible to determine the quantity of interest given the above representation of the state.

Rows

The probability that an energy measurement will result in E2, the first excited-state energy

The probability that a position measurement will find the particle on the left half of the well

The probability that a momentum measurement will find the particle moving to the left

- ○ Yes, it is possible to determine this
- ○ No, it is not possible to determine this

Question 6.
Please explain your reasoning for ***all*** of the above cases.

Consider the two wave functions given below for the infinite square well (where *N* is a constant):

$$\psi_A(x) = N(sin(\pi x/a) + sin^3(\pi x/a))$$

$$\psi_B(x) = N(sin(\pi x/a) + isin^3(\pi x/a))$$

Question 7.
Would the (position space) probability densities for these two particles be *the same* or *different?*

- ○ The same
- ○ Different
- ○ Not enough information to tell

Question 8.
Please explain your reasoning to the previous question.

Question 9.
Would the **(momentum space)** probability densities for these two particles be *the same* or *different?*

- ○ The same
- ○ Different
- ○ Not enough information to tell

Question 10.
Please explain your reasoning to the previous question.

Question 11.
Would the probabilities for each particle to be found in the ground state be *the same* or *different?*

- ○ The same
- ○ Different
- ○ Not enough information to tell

Question 12.
Please explain your reasoning to the previous question.

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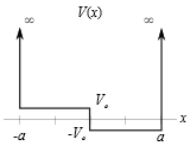
Part I
The potential well shown at right may be considered as an infinite square well of width 2*a* with a small perturbation of size V_o.



Recall that in perturbation theory, the first-order correction to the energy is given by the expression below:

$$\langle \psi_n | \hat{V}' | \psi_n \rangle$$

Question 1.
Is the first-order correction to the energy of the **ground state** of this system *positive, negative,* or *zero?*

- Positive
- Negative
- Zero
- There is not enough information to answer

Question 2.
Explain your reasoning.

Question 3.
Is the first-order correction to the energy of the **first excited-state** of this system *positive, negative,* or *zero?*

- Positive
- Negative
- Zero
- There is not enough information to answer

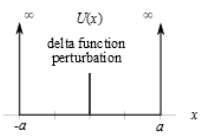
Question 4.
Explain your reasoning.

Question 5.
Is the magnitude of the first-order correction to the energy of the **ground state** of this system *greater than, less than,* or *equal to* the magnitude of the first-order correction to the energy of the **first excited-state** of this system?

- Greater than
- Less than
- Equal to
- There is not enough information to answer

Question 6.
Explain your reasoning.

Part II
The potential well shown at right may be considered as an infinite square well of width 2*a* with a small perturbation delta function perturbation centered at the origin.



Recall that in perturbation theory, the first-order correction to the energy is given by the expression below:

$$\langle \psi_n | \hat{V}' | \psi_n \rangle$$

Question 7.
Is the first-order correction to the energy of the **ground state** of this system *positive, negative,* or *zero?*

- Positive
- Negative
- Zero
- There is not enough information to answer

Question 8.
Explain your reasoning.

Question 9.
Is the first-order correction to the energy of the **first excited-state** of this system *positive, negative,* or *zero?*

- Positive
- Negative
- Zero
- There is not enough information to answer

Question 10.
Explain your reasoning.

Question 11.
Is the first-order correction to the energy of the **second excited-state** of this system *positive, negative,* or *zero?*

- Positive
- Negative
- Zero
- There is not enough information to answer

Question 12.
Explain your reasoning.

Question 13.
Is the magnitude of the first-order correction to the energy of the **ground state** of this system *greater than, less than,* or *equal to* the magnitude of the first-order correction to the energy of the **first excited-state** of this system?

- Greater than
- Less than
- Equal to
- There is not enough information to answer

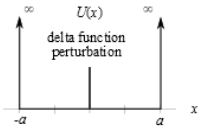
Question 14.
Explain your reasoning.

Question 15.
Is the magnitude of the first-order correction to the energy of the **ground state** of this system *greater than, less than,* or *equal to* the magnitude of the first-order correction to the energy of the **second excited-state** of this system?

- Greater than
- Less than
- Equal to
- There is not enough information to answer

Question 16.
Explain your reasoning.

Part III
Consider again the delta-function perturbation shown at right.



Recall that in perturbation theory, the second-order correction to the energy is given by the expression below:

$$\sum_{m \neq n} \frac{|\langle \psi_n | \hat{V}' | \psi_n \rangle|^2}{E_n - E_m}$$

Question 17.
Is the second-order correction to the energy of the **ground state** of this system *positive, negative,* or *zero?*

- Positive
- Negative
- Zero
- There is not enough information to answer

Question 18.
Explain your reasoning.

Question 19.
Is the second-order correction to the energy of the **first excited-state** of this system *positive, negative,* or *zero?*

- Positive
- Negative
- Zero
- There is not enough information to answer

Question 20.
Explain your reasoning.

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Print view of '(DPT)U2a'

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Consider a single particle in a two-dimensional quantum mechanical infinite square well (*i.e.,* the potential is zero from $x = 0$ to $x = a$ and from $y = 0$ to $y = a$, and infinite everywhere else). Let the energy eigenfunctions and energy eigenvalues of this system be given by $|\psi\rangle = |n_x, n_y\rangle$ and $\mathrm{E}_{n_x n_y} = (n_x^2 + n_y^2)E_{11}$, respectively.

Question 1.
Which, if any, of the following states are valid first excited-states for this system? (Check all that apply.)

A: $|12\rangle$

B: $|22\rangle$

C: $(|12\rangle - |21\rangle)/\sqrt{2}$

D: $(|12\rangle + i|21\rangle)/\sqrt{2}$

- ☐ A
- ☐ B
- ☐ C
- ☐ D

Question 2.
Please explain your reasoning to the previous question.

Suppose the small perturbation below is added to the system, where α is a small, positive constant:

$$\mathrm{V}' = \alpha\delta(x - 3a/4)\delta(y - 3a/4)$$

Question 3.
Is the first-order correction to the ground state *positive, negative,* or *zero?*

- ○ Positive
- ○ Negative
- ○ Zero
- ○ There is not enough information to answer

Question 4.
Please explain your reasoning to the previous question.

Question 5.
For each of the following states, determine whether the first-order correction to the energy is *positive, negative,* or *zero.* If any correction is not well defined, note that explicitly.

A: $|12\rangle$

B: $|21\rangle$

C: $(|12\rangle + |21\rangle)/\sqrt{2}$

D: $(|12\rangle - |21\rangle)/\sqrt{2}$

Rows

A

B

C

D

- Positive
- Negative
- Zero
- Not well defined

Question 6.
Please explain your reasoning to the previous question.

Consider a single particle in a three-dimensional infinite square well of side length *a* (that is, the potential is zero from *x* = 0 to *a*, *y* = 0 to *a*, and *z* = 0 to *a*).

In Cartesian coordinates the energy eigenfunctions are given in terms of the solutions to the one-dimensional infinite square well by:

$$\psi(x,y,z) = \phi_{n_x}(x)\phi_{n_y}(y)\phi_{n_z}(z)$$
$$\mathrm{E}_{n_x n_y n_z} = \frac{\pi^2\hbar^2}{2ma^2}(n_x^2 + n_y^2 + n_z^2)$$

Several possible wave functions for this particle are given below:

$$\psi_A(x,y,z) = \frac{1}{\sqrt{2}}(\phi_1(x)\phi_1(y)\phi_1(z) + \phi_2(x)\phi_2(y)\phi_2(z))$$

$$\psi_B(x,y,z) = \frac{1}{\sqrt{2}}(\phi_1(x)\phi_2(y)\phi_2(z) - \phi_2(x)\phi_1(y)\phi_2(z))$$

$$\psi_C(x,y,z) = \frac{1}{\sqrt{2}}(\phi_1(x)\phi_2(y)\phi_2(z) - i\phi_1(x)\phi_1(y)\phi_3(z))$$

$$\psi_D(x,y,z) = \frac{1}{\sqrt{2}}(\phi_4(x)\phi_4(y)\phi_1(z) + i\phi_5(x)\phi_2(y)\phi_2(z))$$

Question 7.
Which, if any, of the four wave functions listed above change with time? (Check all that apply.)

- ☐ ψA
- ☐ ψB
- ☐ ψC
- ☐ ψD

Question 8.
Please explain your reasoning to the previous question.

Question 9.
Which, if any, of the four wave functions listed above have a *corresponding probability density* that changes with time? (Check all that apply.)

- ☐ ψA
- ☐ ψB
- ☐ ψC
- ☐ ψD

Question 10.
Please explain your reasoning to the previous question.

Question 11.
For which, if any, of the four wave functions listed above would the probability of measuring a particular energy value depend on time? (Check all that apply.)

- ☐ ψA
- ☐ ψB
- ☐ ψC
- ☐ ψD

Question 12.
Please explain your reasoning to the previous question.

Consider the following wave function for the three-dimensional infinite square well potential discussed on the previous page:

$$\psi(x,y,z) = \frac{1}{\sqrt{2}}(\phi_1(x)\phi_2(y)\phi_3(z) + \phi_\alpha(x)\phi_\beta(y)\phi_\gamma(z))$$

Question 13.
Are there any values of *α, β,* and *γ* such that the state above is a stationary state?

- ○ Yes, any set of values work
- ○ Yes, several sets of values work
- ○ Yes, but only one set of values works
- ○ No set of values works
- ○ There is not enough information to answer
- ○ Other:

Question 14.
Please explain your reasoning to the previous question. If you answered yes, list the values of *α, β,* and *γ* that you think will work.

Recall the wave function from the previous page given by:

$$\psi_C(x,y,z) = \frac{1}{\sqrt{2}}(\phi_1(x)\phi_2(y)\phi_2(z) - i\phi_1(x)\phi_1(y)\phi_3(z))$$

Question 15.
If you were to measure the total energy of this system with this wave function, determine the *number* of distinct possible energy values that could result from such a measurement.

Question 16.
Please explain your reasoning to the previous question.

Recall the different wave function from the previous page given by:

$$\psi_D(x,y,z) = \frac{1}{\sqrt{2}}(\phi_4(x)\phi_4(y)\phi_1(z) + i\phi_5(x)\phi_2(y)\phi_2(z))$$

Question 17.
If you were to measure the total energy of the system with this wave function, determine the *number* of distinct possible energy values that could result from such a measurement.

Question 18.
Please explain your reasoning to the previous question.

A.III – Pretests for Part III

This sub-section includes copies of all pretests discussed in part III of this dissertation.

Print view of '(LQM)U2a'

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Part I
Suppose a particle in a three-dimensional quantum system has a wave function given by

$$\psi(r,\theta,\phi) = f(r)Y_2^1(\theta,\phi)$$

where $f(r)$ is any function of the radial coordinate, r, and $Y_2^1(\theta, \varphi)$ is one of the spherical harmonics.

Question 1.
Suppose the *orbital* angular momentum squared, L^2, of this particle is measured.

Which of the following would be possible results of this measurement? (Check all that apply)

- ☐ 0
- ☐ 1 hbar^2
- ☐ 2 hbar^2
- ☐ 4 hbar^2
- ☐ 6 hbar^2
- ☐ Any value is possible
- ☐ Other:

Explain your reasoning.

Question 2.
Suppose the *z*-component of the *orbital* angular momentum, L_z, of this particle is measured.

Which of the following would be possible results of this measurement? (Check all that apply)

- ☐ -2 hbar
- ☐ -1 hbar
- ☐ -1/2 hbar
- ☐ 0
- ☐ 1/2 hbar
- ☐ 1 hbar
- ☐ 2 hbar
- ☐ Any value is possible
- ☐ Other:

Explain your reasoning.

Question 3.
Suppose the ***x*-component** of the *orbital* angular momentum, L_x, of this particle is measured.

Which of the following would be possible results of this measurement? (Check all that apply)

- ☐ -2 hbar
- ☐ -1 hbar
- ☐ -1/2 hbar
- ☐ 0
- ☐ 1/2 hbar
- ☐ 1 hbar
- ☐ 2 hbar
- ☐ Any value is possible
- ☐ Other:

Explain your reasoning.

Part II

States will be represented on this page using the following notation: $\psi = |n, l, m_l\rangle$.

Each of the following represents the state of an electron in identical hydrogen atom potentials.

$\psi_1 = |2,1,1\rangle$

$\psi_2 = \frac{|2,1,1\rangle + i|2,1,0\rangle}{\sqrt{2}}$

$\psi_3 = \frac{|2,1,1\rangle - |3,1,1\rangle}{\sqrt{2}}$

$\psi_4 = \frac{|2,0,0\rangle + |2,1,0\rangle}{\sqrt{2}}$

Question 4.
For which of the initial states above will the corresponding *wave function* depend on time? (Check all that apply.)

- ☐ ψ1
- ☐ ψ2
- ☐ ψ3
- ☐ ψ4

Explain your reasoning.

Question 5.
For which of the initial states above will the corresponding *probability density* depend on time? (Check all that apply.)

- ☐ ψ1
- ☐ ψ2
- ☐ ψ3
- ☐ ψ4

Explain your reasoning.

Question 6.
For which of the initial states above will the *expectation value of the energy* depend on time? (Check all that apply.)

- ☐ ψ1
- ☐ ψ2
- ☐ ψ3
- ☐ ψ4

Explain your reasoning.

Question 7.
For which of the initial states above will the *probability that the energy is measured to be E_2* depend on time? (Check all that apply.)

- ☐ ψ1
- ☐ ψ2
- ☐ ψ3
- ☐ ψ4

Explain your reasoning.

Suppose the position of electron 1 is measured at t_1. A short time later, at t_2, another measurement is made on this system.

Question 8.
For each of the different kinds of measurement below, indicate which measurements could result in *more than one* possible value at t_2. (Check all that apply.)

- ☐ Position (x)
- ☐ Momentum (p)
- ☐ Energy (E)
- ☐ Angular momentum squared (L^2)
- ☐ The z-component of angular momentum (Lz)
- ☐ The x-component of angular momentum (Lx)

Explain your reasoning.

Question 9.
For each of the different kinds of measurement below, indicate which measurements have probabilities that *depend on the time,* t_2, at which the measurement is made. (Check all that apply.)

- ☐ Position (x)
- ☐ Momentum (p)
- ☐ Energy (E)
- ☐ Angular momentum squared (L^2)
- ☐ The z-component of angular momentum (Lz)
- ☐ The x-component of angular momentum (Lx)

Explain your reasoning.

Part III

Consider a particle with total spin-1/2. The particle is in a system with energy eigenvalues +ε corresponding to spin-up in the *z*-direction and -ε corresponding to spin-down in the *z*-direction.

Suppose you measure this particle to be spin-up in the *z*-direction at $t = 0$. A short time later, another measurement is made at $t_1 > 0$.

Question 10.
For each of the different kinds of measurement below, indicate which measurements could result in *more than one* possible value at t_1. (Check all that apply.)

- ☐ Energy (E)
- ☐ Total spin (|S|)
- ☐ The z-component of spin (Sz)
- ☐ The x-component of spin (Sx)

Explain your reasoning.

Question 11.
For each of the different kinds of measurement below, indicate which measurements have probabilities that *depend on the time,* t_1, at which the measurement is made. (Check all that apply.)

- ☐ Energy (E)
- ☐ Total spin (|S|)

- ☐ The z-component of spin (Sz)
- ☐ The x-component of spin (Sx)

Explain your reasoning.

Suppose you measure this particle to be spin-up in the ***x*-direction** at t_1. A short time later, another measurement is made at $t_2 > t_1$.

Question 12.
For each of the different kinds of measurement below, indicate which measurements could result in *more than one* possible value at t_2. (Check all that apply.)

- ☐ Energy (E)
- ☐ Total spin (|S|)
- ☐ The z-component of spin (Sz)
- ☐ The x-component of spin (Sx)

Explain your reasoning.

Question 13.
For each of the different kinds of measurement below, indicate which measurements have probabilities that *depend on the time, t_2,* at which the measurement is made. (Check all that apply.)

- ☐ Energy (E)
- ☐ Total spin (|S|)
- ☐ The z-component of spin (Sz)
- ☐ The x-component of spin (Sx)

Explain your reasoning.

Questions or comments?
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Print view of '(LQM)U2b'

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Part I
Suppose a particle in a three-dimensional quantum system has a wave function given by

$$\psi(r,\theta,\phi) = f(r)Y_2^1(\theta,\phi)$$

where $f(r)$ is any function of the radial coordinate, r, and $Y_2^1(\theta, \varphi)$ is one of the spherical harmonics.

Question 1.
Suppose the *orbital* angular momentum squared, L^2, of this particle is measured.

Which of the following would be possible results of this measurement? (Check all that apply)

- ☐ 0
- ☐ 1 hbar^2
- ☐ 2 hbar^2
- ☐ 4 hbar^2
- ☐ 6 hbar^2
- ☐ Any value is possible
- ☐ Other:

Explain your reasoning.

Question 2.
Suppose the *z*-component of the *orbital* angular momentum, L_z, of this particle is measured.

Which of the following would be possible results of this measurement? (Check all that apply)

- ☐ -2 hbar
- ☐ -1 hbar
- ☐ -1/2 hbar
- ☐ 0
- ☐ 1/2 hbar
- ☐ 1 hbar
- ☐ 2 hbar
- ☐ Any value is possible
- ☐ Other:

Explain your reasoning.

Question 3.
Suppose the ***x*-component** of the *orbital* angular momentum, L_x, of this particle is measured.

Which of the following would be possible results of this measurement? (Check all that apply)

- ☐ -2 hbar
- ☐ -1 hbar
- ☐ -1/2 hbar
- ☐ 0
- ☐ 1/2 hbar
- ☐ 1 hbar
- ☐ 2 hbar
- ☐ Any value is possible
- ☐ Other:

Explain your reasoning.

Part II

States will be represented on this page using the following notation: $\psi = |n, l, m_l\rangle$.

Each of the following represents the state of an electron in identical hydrogen atom potentials.

$\psi_1 = |2,1,1\rangle$

$\psi_2 = \frac{|2,1,1\rangle + i|2,1,0\rangle}{\sqrt{2}}$

$\psi_3 = \frac{|2,1,1\rangle - |3,1,1\rangle}{\sqrt{2}}$

$\psi_4 = \frac{|2,0,0\rangle + |2,1,0\rangle}{\sqrt{2}}$

Question 4.
For which of the initial states above will the corresponding *wave function* depend on time? (Check all that apply.)

- ☐ ψ1
- ☐ ψ2
- ☐ ψ3
- ☐ ψ4

Explain your reasoning.

Question 5.
For which of the initial states above will the corresponding *probability density* depend on time? (Check all that apply.)

- ☐ ψ1
- ☐ ψ2
- ☐ ψ3
- ☐ ψ4

Explain your reasoning.

Question 6.
For which of the initial states above will the *expectation value of the energy* depend on time? (Check all that apply.)

- ☐ ψ1
- ☐ ψ2
- ☐ ψ3
- ☐ ψ4

Explain your reasoning.

Question 7.
For which of the initial states above will the *probability that the energy is measured to be* E_2 depend on time? (Check all that apply.)

- ☐ ψ1
- ☐ ψ2
- ☐ ψ3
- ☐ ψ4

Explain your reasoning.

Suppose the position of electron 1 is measured at t_1. A short time later, at t_2, another measurement is made on this system.

Question 8.
For each of the different kinds of measurement below, indicate which measurements could result in *more than one* possible value at t_2. (Check all that apply.)

- ☐ Position (x)
- ☐ Momentum (p)
- ☐ Energy (E)
- ☐ Angular momentum squared (L^2)
- ☐ The z-component of angular momentum (Lz)
- ☐ The x-component of angular momentum (Lx)

Explain your reasoning.

Question 9.
For each of the different kinds of measurement below, indicate which measurements have probabilities that *depend on the time, t_2,* at which the measurement is made. (Check all that apply.)

- ☐ Position (x)
- ☐ Momentum (p)
- ☐ Energy (E)
- ☐ Angular momentum squared (L^2)
- ☐ The z-component of angular momentum (Lz)
- ☐ The x-component of angular momentum (Lx)

Explain your reasoning.

Part III

Consider the following wave function for a particle in a three-dimensional quantum mechanical potential:

$$\psi_1(\vec{r}) = A_1 r^2 \tan\theta$$

Question 10
Suppose you were to measure the *z*-component of orbital angular momentum for this wave function. List all the possible result of this measurement below.

Explain your reasoning.

Also consider the following wave function for the same potential:

$$\psi_2(\vec{r}) = A_2 e^{-r^3}(\theta^2 - \pi\theta)$$

Question 11
Suppose you were to measure the *z*-component of orbital angular momentum for this wave function. List all the possible result of this measurement below.

Explain your reasoning.

Question 12
Describe how you would determine the possible results and the associated probabilities of a measurement of the orbital angular momentum squared for these wave functions.

Questions or comments?
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Part I
Suppose a particle in a three-dimensional quantum system has a wave function given by

$$\psi(r,\theta,\phi) = f(r)Y_2^1(\theta,\phi)$$

where $f(r)$ is any function of the radial coordinate, r, and $Y_2^1(\theta, \varphi)$ is one of the spherical harmonics.

Question 1.
Suppose the *orbital* angular momentum squared, L^2, of this particle is measured.

Which of the following would be possible results of this measurement? (Check all that apply)

- ☐ 0
- ☐ 1 hbar^2
- ☐ 2 hbar^2
- ☐ 4 hbar^2
- ☐ 6 hbar^2
- ☐ Any value is possible
- ☐ Other:

Explain your reasoning.

Question 2.
Suppose the *z*-component of the *orbital* angular momentum, L_z, of this particle is measured.

Which of the following would be possible results of this measurement? (Check all that apply)

- ☐ -2 hbar
- ☐ -1 hbar
- ☐ -1/2 hbar
- ☐ 0
- ☐ 1/2 hbar
- ☐ 1 hbar
- ☐ 2 hbar
- ☐ Any value is possible
- ☐ Other:

Explain your reasoning.

Question 3.
Suppose the ***x*-component** of the *orbital* angular momentum, L_x, of this particle is measured.

Which of the following would be possible results of this measurement? (Check all that apply)

- ☐ -2 hbar
- ☐ -1 hbar
- ☐ -1/2 hbar
- ☐ 0
- ☐ 1/2 hbar
- ☐ 1 hbar
- ☐ 2 hbar
- ☐ Any value is possible
- ☐ Other:

Explain your reasoning.

Part II

Spin states will be represented on this page using the following notation: $|\chi\rangle = |s, m_s\rangle_z$, where *s* is the quantum number associated with the spin angular momentum squared and m_s is the quantum number associated with the *z*-component of spin angular momentum. You may ignore any spatial degrees of freedom on this page.

Suppose the initial spin state of a particle is given by: $|\chi\rangle = |1/2, 1/2\rangle_z$.

Question 4
Suppose you were to measure the *magnitude* of the spin angular momentum for this particle.

Determine the possible outcomes of this measurement, and the probability associated with each outcome. Explain your reasoning.

Question 5
Suppose instead you were to measure the *z*-component of spin angular momentum for this particle.

Determine the possible outcomes of this measurement, and the probability associated with each outcome. Explain your reasoning.

Question 6
Suppose instead you were to measure the *x*-component of spin for this particle.

Determine the possible outcomes of this measurement, and the probability associated with each outcome. Explain your reasoning.

Question 7
Suppose you were to measure the ***y*-component** of the spin angular momentum for this particle, and the result is +ħ/2.

After this measurement, you measure the ***z*-component** of the spin angular momentum for this particle. Determine the possible outcomes of this second measurement, and the probability associated with each outcome. Explain your reasoning.

Question 8
Suppose you prepare a different particle to have the initial spin state given by: $|\chi\rangle = |1/2, -1/2\rangle_z$.

You measure the ***q*-component** of the spin angular momentum for this particle, where the *q*-axis is defined to lie in the *xz*-plane at a 45 degree angle between the *x*-axis and the *z*-axis. The result of this measurement is +ħ/2.

After this measurement, you measure the ***z*-component** of the spin angular momentum for this particle.

Which of the following correctly describes the probability that the result of this measurement is +ħ/2?

- ○ The probability of getting +ħ/2 is zero
- ○ The probability of getting +ħ/2 is less than 50%
- ○ The probability of getting +ħ/2 is equal to 50%
- ○ The probability of getting +ħ/2 is greater than 50%
- ○ The probability of getting +ħ/2 is one
- ○ There is not enough information (please specify)

Explain your reasoning.

Part III

Consider the following wave function for a particle in a three-dimensional quantum mechanical potential:

$$\psi_1(\vec{r}) = A_1 \frac{\cos\theta}{r}$$

Question 9
Suppose you were to measure the *z*-component of orbital angular momentum for this wave function. List all the possible result of this measurement below. Explain your reasoning.

Question 10
Suppose you were to measure the orbital angular momentum squared for this wave function. List all the possible result of this measurement below. Explain your reasoning.

Also consider the following wave function for the same potential:

$$\psi_2(\vec{r}) = A_2 e^{-r^3} \sin^2\theta \sin 2\phi$$

Question 11
Suppose you were to measure the *z*-component of orbital angular momentum for this wave function. List all the possible result of this measurement below. Explain your reasoning.

Question 12
Suppose you were to measure the orbital angular momentum squared for this wave function. List all the possible result of this measurement below. Explain your reasoning.

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Part I
Suppose a particle in a three-dimensional quantum system has a wave function given by

$$\psi(r,\theta,\phi) = g(r)Y_1^{-1}(\theta,\phi)$$

where $g(r)$ is a function of the radial coordinate, r, and $Y_1^{-1}(\theta, \varphi)$ is one of the spherical harmonics.

Question 1.
Suppose the *orbital* angular momentum squared, L^2, of this particle is measured.

Which of the following would be possible results of this measurement? (Check all that apply)

- ☐ 0
- ☐ 1 hbar^2
- ☐ 2 hbar^2
- ☐ 4 hbar^2
- ☐ 6 hbar^2
- ☐ Any value is possible
- ☐ Other:

Explain your reasoning.

Question 2.
Suppose the *z*-component of the *orbital* angular momentum, L_z, of this particle is measured.

Which of the following would be possible results of this measurement? (Check all that apply)

- ☐ -2 hbar
- ☐ -1 hbar
- ☐ -1/2 hbar
- ☐ 0
- ☐ 1/2 hbar
- ☐ 1 hbar
- ☐ 2 hbar
- ☐ Any value is possible
- ☐ Other:

Explain your reasoning.

Question 3.
Suppose the ***x*-component** of the *orbital* angular momentum, L_x, of this particle is measured.

Which of the following would be possible results of this measurement? (Check all that apply)

- ☐ -2 hbar
- ☐ -1 hbar
- ☐ -1/2 hbar
- ☐ 0
- ☐ 1/2 hbar
- ☐ 1 hbar
- ☐ 2 hbar
- ☐ Any value is possible
- ☐ Other:

Explain your reasoning.

Part II
Consider a particle in the orbital angular momentum state:

$$|\psi_A\rangle = |l, m_l\rangle_z = |1, 1\rangle_z$$

You may ignore any other degrees of freedom for this particle (*i.e.,* consider only orbital angular momentum).

Question 4
Suppose the ***x*-component** of the orbital angular momentum of this particle is measured. Which of the following values are possible results of this measurement? (Check all that apply.)

- ☐ 2ħ
- ☐ √2ħ
- ☐ ħ
- ☐ ħ/2
- ☐ 0
- ☐ -ħ/2
- ☐ -ħ
- ☐ -√2ħ
- ☐ -2ħ

Explain your reasoning.

Question 5
Which of the following represents the probability that the result of the measurement of the *x*-component of orbital angular momentum from the previous question is +ħ?

- ○ 0
- ○ 1/5
- ○ 1/4
- ○ 1/3
- ○ 1/2
- ○ 1

Explain your reasoning.

The same particle is now put in the orbital angular momentum state:

$$|\psi_B\rangle = |l, m_l\rangle_z = \frac{1}{\sqrt{2}}(|1, 1\rangle_z - |1, -1\rangle_z)$$

Question 6
Suppose the ***x*-component** of the orbital angular momentum of this particle is measured. Which of the following values are possible results of this measurement? (Check all that apply.)

- ☐ 2ħ
- ☐ √2ħ
- ☐ ħ
- ☐ ħ/2
- ☐ 0
- ☐ -ħ/2
- ☐ -ħ
- ☐ -√2ħ
- ☐ -2ħ

Explain your reasoning.

Question 7
Which of the following represents the probability that the result of the measurement of the *x*-component of orbital angular momentum from the previous question is +*ħ*?

- 0
- 1/5
- 1/4
- 1/3
- 1/2
- 1

Explain your reasoning.

Part III

Suppose a spin-1/2 particle is known to be in the following spin state:

$$|s, m_s\rangle_z = |1/2, +1/2\rangle_z$$

Question 8
Suppose you were to measure the *magnitude* of the spin angular momentum for this particle. List all the possible result of this measurement below, and determine the probability associated with each. Explain your reasoning.

Question 9
Suppose you were to measure the *x*-component of spin angular momentum for this particle. Which of the following best describes the probability (P) that the result of this measurement is +*ħ*/2?

- P = 0
- 0 < P < 1/2
- P = 1/2
- 1/2 < P < 1
- P = 1
- There is not enough information to answer

Explain your reasoning.

Question 10
Assume that no measurement was made in the previous question (*i.e.,* the particle is still in the spin state given by $|s, m_s\rangle_z = |1/2, +1/2\rangle_z$). Let the *q*-direction point in the *xz*-plane at a 60 degree angle to the *z*-axis.

Suppose you were to measure the *q*-component of spin angular momentum for this particle. Which of the following best describes the probability (P) that the result of this measurement is +*ħ*/2?

- P = 0
- 0 < P < 1/2
- P = 1/2
- 1/2 < P < 1
- P = 1
- There is not enough information to answer

Explain your reasoning.

Question 11

Assume that the measurement in the previous question was made, and the result was $+\hbar/2$ in the ***q*-direction**. Recall that the *q*-direction is at an angle of 60 degrees to the z-axis.

Suppose you were now to measure the *x*-component of spin angular momentum for this particle. Which of the following best describes the probability (P) that the result of this measurement is $+\hbar/2$?

- P = 0
- 0 < P < 1/2
- P = 1/2
- 1/2 < P < 1
- P = 1
- There is not enough information to answer

Explain your reasoning.

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Part I
Consider a quantum system for which the operators Â and Ô are Hermitian.

Question 1.
Suppose this system is in the state |ψ>. How would you determine the possible results of a measurement of the observable corresponding to the operator Â?

Question 2.
Can you determine whether the probability of a given measurement of the operator above depends on time?

- ○ Yes, you can always tell
- ○ Yes, if you have more information (please specify)
- ○ No, you can never tell

Question 3.
Explain your reasoning to the previous question.

Question 4.
Suppose that you have just made a measurement of the observable associated with the operator Â. How would you determine the possible outcomes of a measurement of the observable associated with the operator Ô immediately after the measurement above?

Question 5.
Describe how you would calculate the probabilities of the outcomes of the measurement from the previous question.

Part II
Consider a spinless particle in the orbital angular momentum state:

$$|\psi_A\rangle = |l, m_l\rangle_z = |1, 1\rangle_z$$

Question 6.
Suppose the ***x*-component** of the orbital angular momentum of this particle is measured. Which of the following values are possible results of this measurement? (Check all that apply.)

- ☐ 2ħ
- ☐ √2ħ
- ☐ ħ
- ☐ ħ/2
- ☐ 0
- ☐ -ħ/2
- ☐ -ħ
- ☐ -√2ħ
- ☐ -2ħ

Question 7.
Explain your reasoning.

Question 8.
Which of the following represents the probability that the result of the measurement of the *x*-component of orbital angular momentum from the previous question is +ħ?

- ○ 0
- ○ 1/5
- ○ 1/4
- ○ 1/3
- ○ 1/2
- ○ 1
- ○ Other:

Question 9.
Explain your reasoning.

The same particle is now put in the orbital angular momentum state:

$$|\psi_B\rangle = |l, m_l\rangle_z = \frac{1}{\sqrt{2}}(|1,1\rangle_z - |1,-1\rangle_z)$$

Question 10.
Suppose the ***x*-component** of the orbital angular momentum of this particle is measured. Which of the following values are possible results of this measurement? (Check all that apply.)

- ☐ 2ħ
- ☐ √2ħ
- ☐ ħ
- ☐ ħ/2
- ☐ 0
- ☐ -ħ/2
- ☐ -ħ
- ☐ -√2ħ
- ☐ -2ħ

Question 11.
Explain your reasoning.

Question 12.
Which of the following represents the probability that the result of the measurement of the *x*-component of orbital angular momentum from the previous question is $+\hbar$?

- ○ 0
- ○ 1/5
- ○ 1/4
- ○ 1/3
- ○ 1/2
- ○ 1
- ○ Other:

Question 13.
Explain your reasoning.

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In the following questions, we will use quantum states made up of the hydrogen energy eigenstates:

$$\psi_{nlm}(r,\theta,\phi) = R_{nl}(r)Y_l^m(\theta,\phi)$$

The energy of these states is given by:

$$\mathrm{E}_n = \frac{E_1}{n^2}$$

where E_1 is the energy of the ground state. For simplicity, we will leave out the coordinates and use:

$$\psi_{nlm} = R_{nl}Y_l^m$$

We will also list the spin state of the electron as

$$\chi_+ = |\tfrac{1}{2},+\tfrac{1}{2}\rangle, \quad \chi_- = |\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

Consider the electron in a hydrogen atom initially in the state:

$$\Psi(t=0) = i\sqrt{\tfrac{1}{6}}R_{10}Y_0^0\chi_+ + \sqrt{\tfrac{1}{3}}R_{21}Y_1^1\chi_- - \sqrt{\tfrac{1}{2}}R_{32}Y_2^1\chi_+$$

Question 1
If the energy of this state is measured, what is the probability of measuring the energy to be equal to E_2?

- ○ 0
- ○ 1/6
- ○ 1/3
- ○ 1/2
- ○ 5/6
- ○ 1
- ○ Other:

Explain your reasoning.

Consider the electron in a hydrogen atom to be in the same initial state.

Question 2
If the *z*-component of the orbital angular momentum of this state is measured, what is the probability that the result will be $\hbar$?

- ○ 0
- ○ 1/6
- ○ 1/3
- ○ 1/2
- ○ 5/6
- ○ 1
- ○ Other:

Explain your reasoning.

Consider the electron in the hydrogen atom to be in the same initial state.

Question 3
If the value of the spin angular momentum vector squared (S^2) is measured, which of the following are possible results? (check all that apply)

- ☐ 0
- ☐ 1/2
- ☐ 1/2 hbar
- ☐ 1/2 hbar^2
- ☐ 3/4
- ☐ 3/4 hbar
- ☐ 3/4 hbar^2
- ☐ Other:

Explain your reasoning.

Consider again the electron in a hydrogen atom initially in the state:

$$\Psi(t=0) = i\sqrt{\tfrac{1}{6}}R_{10}Y_0^0\chi_+ + \sqrt{\tfrac{1}{3}}R_{21}Y_1^1\chi_- - \sqrt{\tfrac{1}{2}}R_{32}Y_2^1\chi_+$$

Question 4
If the value of the total angular momentum vector squared (J^2) is measured, which of the following are possible results? (check all that apply)

- ☐ 0
- ☐ 3/4 hbar^2
- ☐ 2 hbar^2
- ☐ 15/4 hbar^2
- ☐ 6 hbar^2
- ☐ 35/4 hbar^2
- ☐ Other:

Explain your reasoning.

Consider the electron in a hydrogen atom to be in the same initial state.

Question 5
If the value of the *z*-component of the angular momentum vector (J_z) is measured, which of the following are possible results? (check all that apply)

- ☐ -5/2 hbar
- ☐ -2 hbar
- ☐ -3/2 hbar
- ☐ -1 hbar
- ☐ -1/2 hbar
- ☐ 0 hbar
- ☐ 1/2 hbar
- ☐ 1 hbar
- ☐ 3/2 hbar
- ☐ 2 hbar
- ☐ 5/2 hbar
- ☐ Other:

Explain your reasoning.

Question 6

What is the probability that this measurement will result in $\hbar/2$? Explain your reasoning.

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For this question, states will be written in Dirac notation as follows:
$\psi = |l, m_l\rangle \otimes |s, m_s\rangle$

In this expression, the first ket will refer to the *orbital* angular momentum states, and the second ket will refer to the *spin* angular momentum states.

Consider the set of four wave functions shown below.

$$\psi_1 = \frac{1}{\sqrt{6}}(|3,0\rangle \otimes |1/2,1/2\rangle + \sqrt{2}|3,1\rangle \otimes |1/2,-1/2\rangle - \sqrt{3}|2,0\rangle \otimes |1/2,1/2\rangle)$$

$$\psi_2 = \frac{1}{\sqrt{3}}(i|3,0\rangle \otimes |1/2,1/2\rangle + \sqrt{2}|3,1\rangle \otimes |1/2,-1/2\rangle)$$

$$\psi_3 = |3,0\rangle \otimes |1/2,1/2\rangle$$

$$\psi_4 = |2,0\rangle \otimes |1/2,1/2\rangle$$

Question 1.
Suppose the *orbital* angular momentum squared, L^2, is measured for an electron in a three-dimensional quantum mechanical potential and is found to be $12\hbar^2$.

Which of the above represent possible states for the electron *just before* the measurement is made? (Select all that apply)

- ☐ ψ1
- ☐ ψ2
- ☐ ψ3
- ☐ ψ4

Question 2.
Explain your reasoning.

Question 3.
Which of the above represent possible states for the electron *just after* the measurement is made? (Select all that apply)

- ☐ ψ1
- ☐ ψ2
- ☐ ψ3
- ☐ ψ4

Question 4.
Explain your reasoning.

For the remaining questions, assume the electron is in the state:

$$\psi_1 = \frac{1}{\sqrt{6}}(|3,0\rangle \otimes |1/2,1/2\rangle + \sqrt{2}|3,1\rangle \otimes |1/2,-1/2\rangle - \sqrt{3}|2,0\rangle \otimes |1/2,1/2\rangle)$$

Question 5.
Suppose you measured the *z*-component of the *orbital* angular momentum (L_z). What values might you get, and what is the probability of each? Explain your reasoning.

Question 6.
Let **J** = **L** + **S** be the total angular momentum of the electron.

Suppose you measured the total angular momentum squared (J^2). Determine all the possible values of this measurement. Explain.

Question 7.
Suppose you measured the *z*-component of the total angular momentum (J_z). Determine all the possible values of this measurement. Explain your reasoning.

Question 8.
What is the probability that this measurement will result in $\hbar/2$? Explain your reasoning.

Consider an electron (a spin-1/2 particle) placed in a uniform magnetic field B_o that points in the *z*-direction. The Hamiltonian for this system is:

$$\hat{H} = \mu \vec{B} \cdot \vec{S}$$

where B is the magnetic field, S is the spin of the electron, and μ is the magnitude of the magnetic moment of the electron.

At t = 0, the electron is measured and determined to have spin-up in the ***x*-direction**.

Question 9.
If a measurement is carried out a short time later, at $t_1 > 0$, which of the following have probabilities that depend on the time at which the measurement is made? (Check all that apply.)

- ☐ The probability of measuring spin-up in the z-direction
- ☐ The probability of measuring spin-up in the x-direction
- ☐ The probability of measuring spin-up in the y-direction

Question 10.
Explain your reasoning.

Recall that the particle has spin-up in the *x*-direction at t = 0 (*i.e.,* no measurements have been made on the system).

At t = 0 the direction (but not the magnitude) of the magnetic field is changed, so that it points in an unknown direction.

Question 11.
Suppose it is known that the probability of measuring spin up in the *z*-direction is known to be constant in time after t = 0. Which of the following directions could the magnetic field be pointing in for this to be true? (Check all that apply.)

- ☐ The negative z-direction
- ☐ The positive x-direction
- ☐ The positive y-direction
- ☐ Halfway between the negative z- and negative x-directions
- ☐ There is not enough information to answer this question (specify)
- ☐ None of the above directions will work

Question 12.
Explain your reasoning.

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For this question, states will be written in Dirac notation as follows:
$\psi = |l, m_l\rangle \otimes |s, m_s\rangle$

In this expression, the first ket will refer to the *orbital* angular momentum states, and the second ket will refer to the *spin* angular momentum states.

Consider the set of four wave functions shown below.

$$\psi_1 = \frac{1}{\sqrt{6}}(|3,0\rangle \otimes |1/2,1/2\rangle + \sqrt{2}|3,1\rangle \otimes |1/2,-1/2\rangle - \sqrt{3}|2,0\rangle \otimes |1/2,1/2\rangle)$$

$$\psi_2 = \frac{1}{\sqrt{3}}(i|3,0\rangle \otimes |1/2,1/2\rangle + \sqrt{2}|3,1\rangle \otimes |1/2,-1/2\rangle)$$

$$\psi_3 = |3,0\rangle \otimes |1/2,1/2\rangle$$

$$\psi_4 = |2,0\rangle \otimes |1/2,1/2\rangle$$

Question 1.
Suppose the *orbital* angular momentum squared, L^2, is measured for an electron in a three-dimensional quantum mechanical potential and is found to be $12\hbar^2$.

Which of the above represent possible states for the electron *just before* the measurement is made? (Select all that apply)

- ☐ ψ1
- ☐ ψ2
- ☐ ψ3
- ☐ ψ4

Question 2.
Explain your reasoning.

Question 3.
Which of the above represent possible states for the electron *just after* the measurement is made? (Select all that apply)

- ☐ ψ1
- ☐ ψ2
- ☐ ψ3
- ☐ ψ4

Question 4.
Explain your reasoning.

For the remaining questions, assume the electron is in the state:

$$\psi_1 = \frac{1}{\sqrt{6}}(|3,0\rangle \otimes |1/2,1/2\rangle + \sqrt{2}|3,1\rangle \otimes |1/2,-1/2\rangle - \sqrt{3}|2,0\rangle \otimes |1/2,1/2\rangle)$$

Question 5.
Suppose you measured the *z*-component of the *orbital* angular momentum (L_z). What values might you get, and what is the probability of each? Explain your reasoning.

Question 6.
Let **J** = **L** + **S** be the total angular momentum of the electron.

Suppose you measured the total angular momentum squared (J^2). Determine all the possible values of this measurement. Explain.

Question 7.
Suppose you measured the *z*-component of the total angular momentum (J_z). Determine all the possible values of this measurement. Explain your reasoning.

Question 8.
What is the probability that this measurement will result in $\hbar/2$? Explain your reasoning.

Consider a particle that has the initial orbital angular momentum state given below:

$$|l, m_l\rangle_z = |2, 0\rangle_z$$

Question 9.
Suppose you were to measure the ***x*-component** of the orbital angular momentum of this particle. What are the possible results of this measurement? Explain your reasoning.

Question 10.
Rank the probabilities of the possible values you identified in the previous question from greatest to least.

Question 11.
Explain your reasoning.

Now consider a particle that has the initial orbital angular momentum state given below:

$$|l, m_l\rangle_q = |2, 0\rangle_q$$

where the *q*-axis is defined to lie in the *xz*-plane at an angle of 45° to the *z*-axis.

Question 12.
Suppose that the *x*-component of the orbital angular momentum of this particle is also measured.

Indicate whether you believe each of the following probabilities will *increase, decrease,* or *remain the same* compared with the corresponding probability for the first particle.

Rows
The probability that Lx is 0
The probability that Lx is ħ
The probability that Lx is 2ħ

- ○ Increase
- ○ Decrease
- ○ Remain the same

Question 13.
Explain your reasoning.

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On this pretest, quantum states will be written in Dirac notation as follows:
$|\psi\rangle = |l, m_l\rangle \otimes |s, m_s\rangle$

In this expression, the first ket will refer to the *orbital* angular momentum states, and the second ket will refer to the *spin* angular momentum states.

Consider the set of four wave functions shown below.

$|\psi_1\rangle = \frac{1}{\sqrt{6}}(|3,0\rangle \otimes |1/2,1/2\rangle + \sqrt{2}|3,1\rangle \otimes |1/2,-1/2\rangle - \sqrt{3}|2,0\rangle \otimes |1/2,1/2\rangle)$

$|\psi_2\rangle = \frac{1}{\sqrt{3}}(i|3,0\rangle \otimes |1/2,1/2\rangle + \sqrt{2}|3,1\rangle \otimes |1/2,-1/2\rangle)$

$|\psi_3\rangle = |3,0\rangle \otimes |1/2,1/2\rangle$

$|\psi_4\rangle = |2,0\rangle \otimes |1/2,1/2\rangle$

Question 1.
Suppose the *orbital* angular momentum squared, L^2, is measured for an electron in a three-dimensional quantum mechanical potential and is found to be $12\hbar^2$.

Which of the above represent possible states for the electron *just before* the measurement is made? (Select all that apply)

- ☐ ψ1
- ☐ ψ2
- ☐ ψ3
- ☐ ψ4

Question 2.
Explain your reasoning.

Question 3.
Which of the above represent possible states for the electron *just after* the measurement is made? (Select all that apply)

- ☐ ψ1
- ☐ ψ2
- ☐ ψ3
- ☐ ψ4

Question 4.
Explain your reasoning.

Quantum states on this page are still written in Dirac notation as follows:
$|\psi\rangle = |l, m_l\rangle \otimes |s, m_s\rangle$

where the first ket refers to the *orbital* angular momentum states, and the second ket refers to the *spin* angular momentum states.

For the remaining questions, assume the electron is in the state:

$|\psi_1\rangle = \frac{1}{\sqrt{6}}(|3,0\rangle \otimes |1/2,1/2\rangle + \sqrt{2}|3,1\rangle \otimes |1/2,-1/2\rangle - \sqrt{3}|2,0\rangle \otimes |1/2,1/2\rangle)$

Question 5.

Suppose you measured the *z*-component of the *orbital* angular momentum (L_z). What values might you get, and what is the probability of each? Explain your reasoning.

Question 6.
Let **J** = **L** + **S** be the total angular momentum of the electron.

Suppose you measured the total angular momentum squared (J^2). Determine all the possible values of this measurement. Explain your reasoning.

Question 7.
Suppose you measured the *z*-component of the total angular momentum (J_z). Determine all the possible values of this measurement. Explain your reasoning.

Question 8.
What is the probability that this measurement (of J_z) will result in $+\hbar/2$? Explain your reasoning.

Consider a quantum particle that has spin-3/2 (*i.e.,* the quantum number *s* is given by 3/2).

Question 9.
Suppose you were to measure the **magnitude** of the spin angular momentum for this particle. List the possible outcomes of this measurement. Explain your reasoning.

Now suppose you were to consider a system of **two** identical spin-3/2 particles such as the one described above.

Let the total spin vector for this particle be given by: $\vec{S}_{total} = \vec{S}_1 + \vec{S}_2$

Question 10.
Suppose you were to measure the **magnitude** of the **total** spin angular momentum for this particle. List all the possible outcomes of this measurement. Explain your reasoning.

Now suppose you were to measure the *z*-component of spin angular momentum for both particles, and that the result of this measurement were $+\hbar/2$ for one particle and $-\hbar/2$ for the other particle.

Question 11.
Suppose you were to measure the *z*-component of the **total** spin angular momentum for this particle. List the possible outcomes of this measurement. Explain your reasoning.

Question 12.
Suppose you were to measure the ***y*-component** of the **total** spin angular momentum for this particle. List the possible outcomes of this measurement. Explain your reasoning.

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Appendix B
Tutorial Worksheets

"I really like the tutorials, and I know having taken Classical Mechanics [it] needs tutorials or something. Can't say much about Statistical Mechanics but I'd assume it's the same."

—Junior-level quantum mechanics student, end-of-quarter survey

This appendix consists of copies of the tutorial curriculum discussed in this dissertation (primarily in chapters 5, 8, and 10). Each sub-section of this appendix (B.I, B.II, and B.III) includes those tutorials described in part I (for quantum measurements and time dependence), part II (for vector spaces), and part III (for angular momentum). Section B.IV includes additional tutorials that have been developed at UW, but which are not explicitly discussed in this dissertation.

B.I – Tutorials for Part I

This sub-section includes copies of all tutorials discussed in part I of this dissertation. Each tutorial is discussed in chapter 5.

TIME DEPENDENCE IN QUANTUM MECHANICS

I. Energy eigenstates of the infinite square well

Consider the quantum mechanical infinite square well potential, $V(x)$, defined at right. The energy eigenvalues for this potential are given by $E_n = n^2 E_1$. At time $t = 0$ a particle is in the ground state; assume the wave function is entirely real at $t = 0$.

$$V(x) = \begin{cases} \infty & x < 0 \\ 0 & 0 \le x \le a \\ \infty & x > a \end{cases}$$

A. Sketch and label the wave function for this particle at $t = 0$. Explain.

B. How does the wave function for this particle change with time? Write the real and imaginary parts of the wave function as functions of time. Explain.

One way to graph one-dimensional, complex-valued functions (such as wave functions) is to plot the real part of the function, the imaginary part of the function, and the spatial dimension on three orthogonal axes. Ask an instructor for one or more tools to help you visualize this representation.

C. Sketch and label the wave function for this particle at $t_1 > 0$. Use the visualization tool provided by the instructor to help illustrate the time dependence. Explain.

D. How is the wave function at $t = t_1$ similar to and different from the wave function at $t = 0$? Explain.

Consider a second particle in the first excited state of the infinite square well. Assume its wave function is also entirely real at $t = 0$.

E. Does the wave function for the second particle change *faster*, *slower*, or *at the same rate* as the first particle? Explain. (*Hint:* Consider the energy associated with each particle.)

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F. Consider the first instant, $T > 0$, when the graph of the wave function for the *second particle* (in the first excited state) looks exactly the same as it did at $t = 0$.

1. Determine the value of T in terms of E_1, the energy of the ground state.

2. Sketch the wave function for the *first particle* (the ground state) at this new time, $t = T$.

3. Sketch both wave functions at time $t = 2T$.

4. Use your visualization tool to demonstrate the time dependence of these states.

G. Sketch the probability density for both particles at $t = 0$.

Do the probability densities for these states depend on time? If so, how?

H. Suppose you were to consider a particle for which the probability density did depend on time. How would the wave function for such a particle change in time? Explain.

✓ Check your results with a tutorial instructor.

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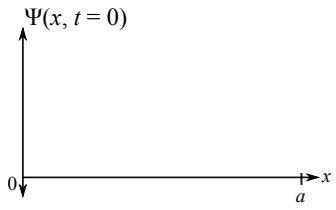
II. Linear superpositions of energy eigenstates

Consider a particle in a quantum mechanical infinite square well that is identical to the well in section I. The wave function at $t = 0$ is given by $\Psi(x,t=0) = \frac{1}{\sqrt{2}}\left(\psi_1(x) + \psi_2(x)\right)$, where $\psi_n(x)$ is the energy eigenfunction associated with the energy $E_n = n^2E_1$.

A. In the space at right, sketch the wave function for this particle at $t = 0$. Explain.



B. Does the wave function for this particle depend on time? Explain.

C. In the space at right, sketch the probability density for this particle at $t = 0$. Explain.

D. *Predict* whether the probability density for this particle depends on time. Explain.

E. Is this particle in an energy eigenstate? Explain.

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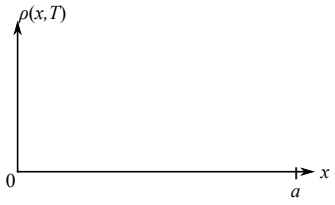
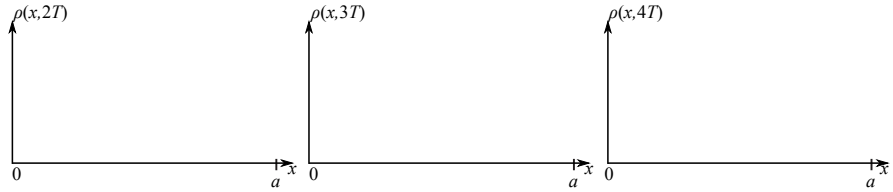
F. Recall the definition for T from page 2. In the space below, sketch the real and imaginary parts of the wave function for this particle at $t = T$. You may find your visualization tool helpful.

G. In the space at right, sketch the probability density at $t = T$. Explain.



H. Sketch the probability density at each of the following times: *2T, 3T,* and *4T*.



I. Are there times between T and $2T$ when the probability density is the same as it was at $t = 0$? Use your visualization tool to illustrate your answer.

J. Obtain a handout showing the time evolution of the probability density. Resolve any inconsistencies with your answers above.

K. What will happen to the state after a *very long time* has elapsed? Explain.

L. What must be true of a quantum mechanical state if the probability density associated with it is to change with time? Explain.

✓ Check your results with a tutorial instructor.

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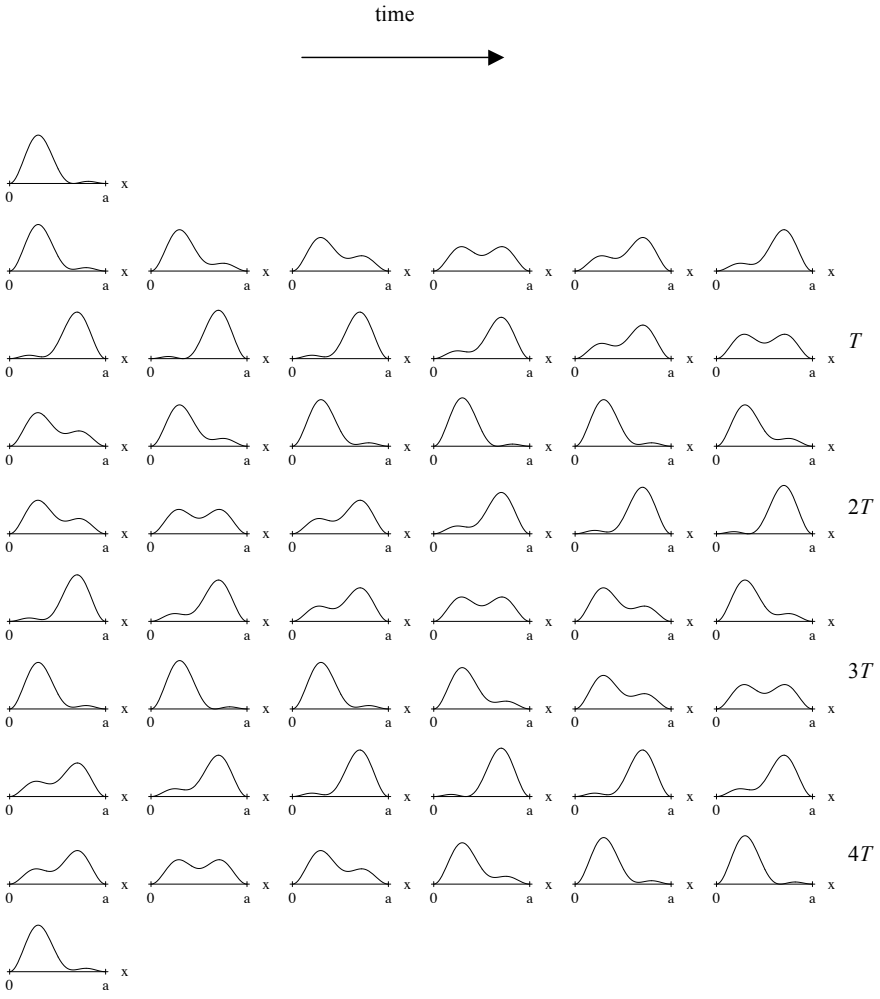
QM **HANDOUT: TIME DEPENDENCE IN QUANTUM MECHANICS**

The time evolution of the probability density for a superposition of the first two eigenstates of the infinite square well is shown below. Each picture is separated by a time equal to $\frac{\hbar\pi}{24E_1}$.



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VISUAL AID: TIME DEPENDENCE IN QM

QM

The following pages can be used to create 3-dimensional diagrams for students to use to demonstrate the time evolution of energy eigenstates and superpositions of energy eigenstates.

Steps:

1. Print the following two pages onto transparencies. You should note that there is a real axis, an imaginary axis, a ground state wave function (of the infinite square well), and a first excited state wave function.
2. Cut out each graph to roughly the same size.
3. Cut along the x-axis on the real component graph from the left side to the center (x=a/2)
4. Cut along the x-axis on the imaginary component of the graph from the right side to the center (x=a/2)
5. Slide the real and imaginary axes together along the x-axis to create a 3-dimensional graph.
6. Have your students orient an energy eigenfunction beside the 3-D axis and indicate its evolution as time progresses. (You may find it easier to cut a slit halfway through the x-axis of each of the wave function transparencies so they can slide together with the axis.)
7. Using these tools students can demonstrate the direction of rotation and the difference in speed of rotation for the two eigenfunctions.

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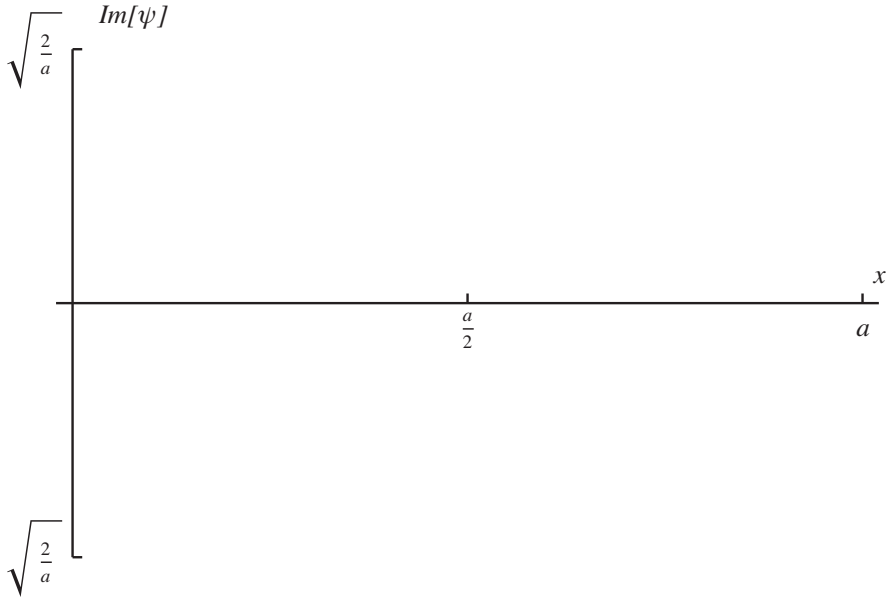
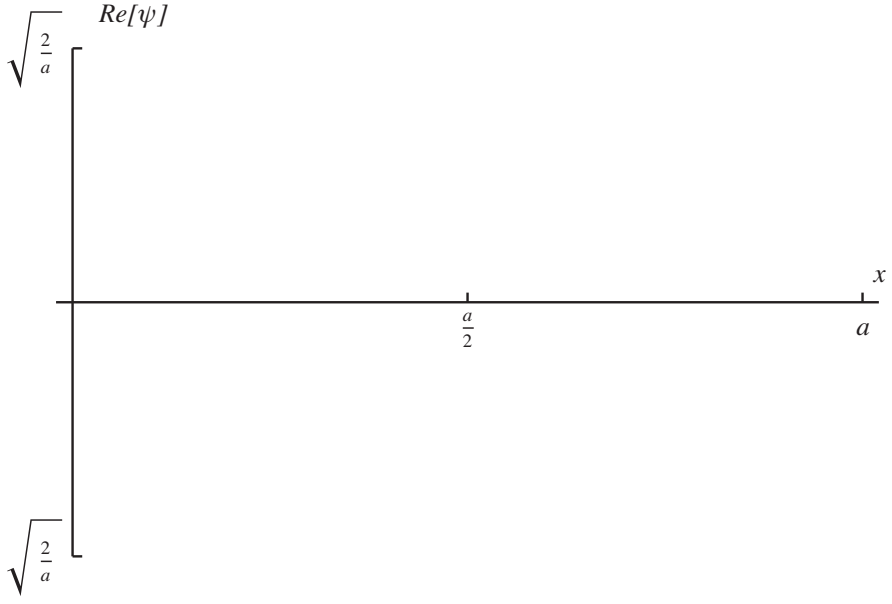
QM ***Visual Aid: Time Dependence in QM***





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ENERGY MEASUREMENTS

QM 1

I. Superpositions of energy eigenstates

Consider a particle in a quantum mechanical harmonic oscillator, where ψ_0 and ψ_1 are the first two energy eigenstates of the system. The initial state of the particle is given by:

$$|\psi\rangle = \frac{i}{\sqrt{3}}|\psi_0\rangle - \sqrt{\frac{2}{3}}|\psi_1\rangle$$

A. Is this particle in an energy eigenstate of this system? Justify your answer.

B. Suppose that at time $t = 0$ you measured the energy of the particle. What value (or values) could an energy measurement yield? Explain.

What is the probability that your energy measurement would yield the value E_0 (the ground state energy)? Show how you determined your answer from the initial state given above.

C. On the basis of your results above, would you say that "the energy of the system" for the particle in this state can be represented by a single number or is it not well-defined? Explain.

D. Determine the time dependence of the state given above and write a new expression, $|\psi(t)\rangle$, for the state at a later time, $t > 0$.

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QM 2 *Energy measurements*

E. Suppose instead that you measure the energy of the particle at a different time, $t_1 > 0$ (assume that no other measurements have been made on this particle).

1. What value (or values) could an energy measurement at t_1 yield? Explain.

2. What is the probability that your energy measurement at t_1 would yield the value E_0? Show how you determined your answer from $|\psi(t)\rangle$.

F. Suppose that you wait a *very long time* to measure the energy of the particle (assume that no earlier measurements have been made).

1. What value or values could an energy measurement yield? Explain.

2. What is the probability that this energy measurement would yield the value E_0? Show how you determined your answer from $|\psi(t)\rangle$.

G. Let $P(E_n)$ represent the probability that an energy measurement results in E_n. Does $P(E_n)$ depend on time? Explain.

H. Explain how your answers about energy measurements are consistent with the time dependence of $|\psi(t)\rangle$ that you discussed above.

✓ Discuss your answers with an instructor before continuing.

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II. Energy eigenstates

A. Suppose that an energy measurement at t_1 on the particle from the previous section yields E_1.

1. Write an expression for the state of the system immediately after the energy measurement is made. Explain how you arrived at your answer.

2. Consider the student discussion below.

Student 1: "Acting the Hamiltonian on the initial wave function corresponds to measuring the energy, because all the Hamiltonian does is pull out a factor of E."

Student 2: "But if I act on a superposition with the Hamiltonian I just multiply each part of the state by its energy. Since the state after a measurement is just a single eigenstate, that can't correspond to making an energy measurement."

With which student do you agree, if either? Explain.

Suppose you were to measure the energy of the particle again at some time $t_2 > t_1$. Assume that no other measurements are made between these times.

B. What value (or values) could this measurement yield? What is the probability associated with each possible measurement? Explain.

Would your answers change if you instead waited *a very long time* to make a second energy measurement. Explain.

C. Consider the following student dialogue about energy measurements.

Student 1: "If we wait long enough, the wave function will spread out until it is equal everywhere."

Student 2: "I disagree. We measured the particle in an excited state, but eventually it will decay back to the ground state, because that's where every particle wants to be."

Student 3: "I agree that the particle won't stay in this state forever, but I was thinking about the original superposition. Eventually it will remember the state it started in, and it will go back to that state again."

All three students are *incorrect.* Identify the flaw(s) in each student's reasoning.

✓ Discuss your answers with an instructor before continuing.

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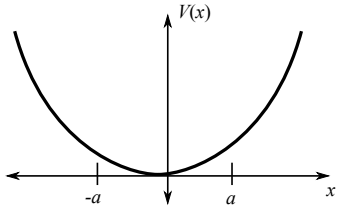
QM 4 *Energy measurements*

III. Energy eigenstates of different potentials

Consider a particle in the quantum mechanical harmonic oscillator shown at right. At $t = 0$ the state is described by a wave function identical to that of the ground state of the infinite square well: $\Psi(x,0) = \begin{cases} \frac{1}{\sqrt{a}}\cos\left(\frac{\pi x}{2a}\right) & |x| < a \\ 0 & |x| > a \end{cases}$.



A. Predict whether the probability density for this particle will depend on time? Explain.

B. Consider the prediction made by the following student.

"This state depends on time. The full wave function is $\Psi(x,t) = \frac{1}{\sqrt{a}}\cos\left(\frac{\pi x}{2a}\right)e^{-\frac{iE_1 t}{\hbar}}$. The probability density *does not* depend on time since the phase cancels out upon squaring."

Do you agree or disagree with this statement? Explain.

C. Is the initial state for this particle $\Psi(x, 0)$, associated with a single energy? Explain. (*Hint:* Is this state an energy eigenstate of the Hamiltonian for the harmonic oscillator potential?)

D. Is $\Psi(x, 0)$ an allowed initial state for this potential? Explain.

E. Describe how you would write the full, time-dependent wave function, $\Psi(x, t)$, for this particle.

Resolve any inconsistencies between your answers to questions A and E.

✓ Discuss your answers with an instructor before continuing.

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IV. Summary

Assume $\Psi(x, 0)$ is a general quantum mechanical wave function describing the initial state of an isolated system with a time-independent potential.

A. Would the possible values for energy measurements on this system change as time evolves? Explain.

Would the relative probabilities of measuring these values change as time evolves? Explain.

B. Would the possible values for position measurements on this system change as time evolves? Explain.

Would the relative probabilities of measuring these values change as time evolves? Explain.

Describe how you would find the wave function, $\Psi(x, t)$, as a function of position and time given the functional form of $\Psi(x, 0)$ and the functional form of the potential.

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POSITION, MOMENTUM, AND ENERGY MEASUREMENTS

QM 1

I. Measurements of position

Consider a particle in an unknown potential that is initially in an arbitrary and unknown superposition of energy eigenstates. At $t = 0$, the position of the particle is measured.

A. What are the possible results of this measurement? Are there any positions that are *not* possible? Explain how you would be able to tell.

B. Assume that the result of this measurement is $x = \alpha$. Describe the state and sketch the wave function of the system immediately after this measurement. Explain

C. Consider the student discussion below.

Student 1: "I think that if I act an operator on a wave function, that corresponds to measuring that operator's observable."

Student 2: "But the wave function is different after the measurement than it was before, and the operator doesn't change which eigenstates are present in the state, it just changes the factor in front of each eigenstate, which tells me what the *possible* results are."

With which student do you agree, if either? Explain.

Suppose that *immediately* after the first position measurement the position is measured again.

D. What are the possible results of this measurement? Explain.

Suppose instead that the second measurement were made at time $t > 0$.

E. What are the possible results of this measurement? Explain.

F. Does the probability density for a system change with time after a position measurement is made? Explain.

✓ Check your results with a tutorial instructor.

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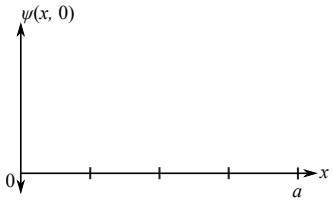
QM 2 *Position, momentum, and energy measurements*

II. Measurements of position and energy

A. At time $t = 0$, a particle in an infinite square well is in the third excited energy eigenstate ($n = 4$). Let E_4 represent the energy of this state.



1. On the axes provided at right, sketch the wave function for this particle at $t = 0$. Assume the wave function is real at $t = 0$.

2. On your graph, clearly label the locations where the particle would be (i) *most* likely and (ii) *least* likely to be found at $t = 0$. Explain.

3. Will the probability that the particle is measured in a particular region (*e.g.*, near $x = a/8$) change with time? (*Hint:* Does the probability density for this state depend on time?) Explain.

4. Suppose that an energy measurement were made on this particle at $t > 0$. List all possible results and the probability associated with each.

5. Does your answer to the previous question depend on time? Explain.

B. If a particle starts in an *energy eigenstate,* do position and/or energy measurements have probabilities that depend on time? Which measurements have probabilities that *do not* depend on time? Explain your reasoning.

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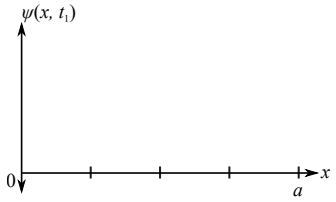
C. Suppose you measured the *position* of the particle at $t_1 > 0$ and found it at $x_1 = a/8$. (Assume no other measurements have been made.)



1. On the axes provided at right, sketch the shape of the wave function at t_1. Assume the wave function is real at t_1.
2. On your graph, clearly label the locations where the particle would be (i) *most* likely and (ii) *least* likely to be found at t_1. Explain.
3. Will the probability that the particle is measured in a particular region (*e.g.*, near $x = a/8$) change with time? (*Hint:* Does the probability density for this state depend on time?) Explain.
4. Is the wave function that you have drawn a single energy eigenstate or a superposition of many energy eigenstates? Explain how you can tell from the shape of the wave function.
5. Suppose that an energy measurement is made on this particle at a later time $t > t_1$. Would you expect only *one* possible result (*e.g.*, E_4), or *many* possible results? Explain.
6. Is the probability of finding a particular result for your energy measurement (*e.g.*, E_4) dependent on time? Explain.

   Would your answer change if you waited *a very long time?* Explain.

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D. Use the tables below to summarize your results.

1. In the table at right, enter the number of possible measurements at a later time of each type (*e.g.*, **one** or **many**) for a system that starts in a given eigenstate. (*Hint:* Examine your results on page 2.)

Table 1: # of Possible Measurements

	Energy Eigenstate	Position Eigenstate
# of possible energies		
# of possible positions		

2. In the table at right, determine whether the probability associated with a single result of a given measurement depends on the time at which the measurement is made. (*Hint:* Examine your results on page 3.)

   Would any of your entries in the second table change if you waited *a very long time* to make the measurement? Explain.

Table 2: Time-dependence of Possible Measurements

	Energy Eigenstate	Position Eigenstate
Energy measurements		
Position measurements		

E. Consider the student discussion below:

Student 1: "Eigenstates of any operator do not change in time. For example, whether a particle starts as an eigenstate of energy or position, it will remain in that eigenstate forever."

Student 2: "You are right about energy eigenstates. But those are special because they are stationary states. Eigenstates of a different operator will not necessarily stay the same, and the probability of measurements would change for them."

Student 3: "I agree, but the probability of making a certain energy measurement doesn't seem to change in time, no matter what state we start in."

With which student(s) do you agree or disagree? Explain your reasoning.

F. Suppose you are told that a given observable is known to have more than one possible measurement outcome. From only this information, can you determine whether the probabilities of those outcomes would depend on time? Explain.

✓ Check your results with a tutorial instructor.

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III. Supplement: Eigenvalues and eigenvectors of position

A. In Dirac notation, write an equation relating an eigenvector $|q_\alpha\rangle$ of the *position* operator $\hat{x}$ and its corresponding eigenvalue α. Explain.

B. Rewrite your equation in position space; that is, in terms of x. Indicate explicitly which symbols are functions of x and which are not. Explain.

Rearrange your equation such that one side is equal to zero.

C. Determine the value of $q_\alpha(x)$ for $x \neq \alpha$. Explain.

D. Sketch a graph of $q_\alpha(x)$. Label your axes, as well as the location of the eigenvalue α.

E. For which values of α does a solution to the equation above exist? Use your answer to determine all the allowed eigenvalues of the position operator. Explain.

F. Are the eigenfunctions of position also eigenfunctions of energy? Explain.

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DEGENERATE STATES

QM 1

I. Dirac notation for multiple-dimension systems

Do not use functions to represent quantum states in this section; use Dirac notation only.

Consider the solutions to the one-dimensional quantum mechanical harmonic oscillator. The energy eigenstates and eigenvalues are given by $|\psi_n\rangle \equiv |n\rangle$ and $E_n = (n+1/2)\hbar\omega$, respectively, for $n = 0, 1, 2, \ldots$.

A. Assume that a particle in this system can be associated with a single energy, E_i. Write down the state of the particle using the notation above.

Is this a stationary state? Explain how you arrived at your answer.

Now consider the *two-dimensional* harmonic oscillator with Hamiltonian $\hat{H}_{2D} = \hat{H}(x) + \hat{H}(y)$. The energy eigenstates and eigenvalues are given by $|\psi_m\rangle_x \otimes |\psi_n\rangle_y \equiv |m,n\rangle$ and $E_{mn} = E_m + E_n$, respectively, where the one-dimensional states and energies are the same as above.

B. Use the information given above to determine the three lowest allowed energy values for this system. Write down the state corresponding to each value of energy. If there is more than one state corresponding to a single value of energy, write down all such states. Explain.

The *degeneracy* of a state is the number of linearly independent eigenstates that correspond to the same energy. A state with degeneracy greater than one is said to be *degenerate*.

C. Determine the degeneracy of each of the states in question B.

Does it appear that the degeneracy for this system depends on the energy level? Explain.

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II. Time dependence for degenerate states

A. Consider a particle in the system defined in section I given by $|\varphi_A\rangle = \frac{1}{\sqrt{2}}\left(|0,0\rangle + |0,1\rangle\right)$.

1. Is it possible to associate a single value of energy with this state? Explain.

2. Does this state depend on time? If so, determine the time dependence of the state. Explain.

B. Consider a different particle in this system given by $|\varphi_B\rangle = \frac{1}{\sqrt{2}}\left(|1,0\rangle + |0,1\rangle\right)$.

1. Is it possible to associate a single value of energy with this state? Explain.

2. Does this state depend on time? If so, determine the time dependence of the state. Explain.

C. Which of the two states defined above, if either, is a stationary state? Explain.

D. Consider the student discussion below. With which student do you agree, if either? Explain.

Student 1: "Both of these are superpositions of stationary states, so neither is a stationary state."

Student 2: "That was true before because the parts of the superposition had different energy, so they had different time dependence. When they have the same energy, each term has the same time dependence, which cancels out when we square the wave function."

E. How many different states have the same energy as $|\varphi_B\rangle = \frac{1}{\sqrt{2}}\left(|1,0\rangle + |0,1\rangle\right)$? Explain how this is consistent with the definition of degeneracy given in section I.

✓ Discuss your answers with an instructor before continuing.

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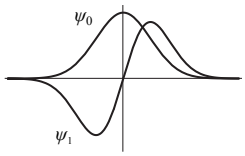
III. Position space

The first two energy eigenfunctions for the *one-dimensional* harmonic oscillator are given below and shown at right. (A_0 and A_1 are normalization constants, and a is a positive constant.)



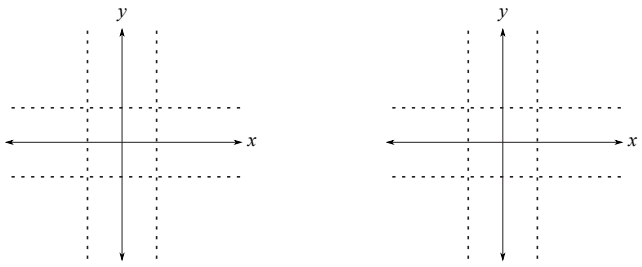
$$|0\rangle = \psi_0(x) = A_0 e^{-x^2/a^2} \qquad |1\rangle = \psi_1(x) = A_1 x e^{-x^2/a^2}$$

A. Write the wave function for each of the following states in the *two-dimensional* harmonic oscillator. (*Hint:* They should have the form $f(x)g(y)$.)

1. $|\varphi_A\rangle = |1,0\rangle$

2. $|\varphi_B\rangle = |0,1\rangle$

B. Graph these functions on the axes below by indicating whether each wave function is *positive*, *negative*, or *zero* at the center of each of the nine regions shown. (*Note:* The middle region corresponds to the origin; assume that the center of the outer regions corresponds roughly to the location of the extrema of ψ_1.)



C. How are the wave functions for $|\varphi_A\rangle$ and $|\varphi_B\rangle$ graphically related? (*Hint:* What could you do to one of them to get the other?) Explain.

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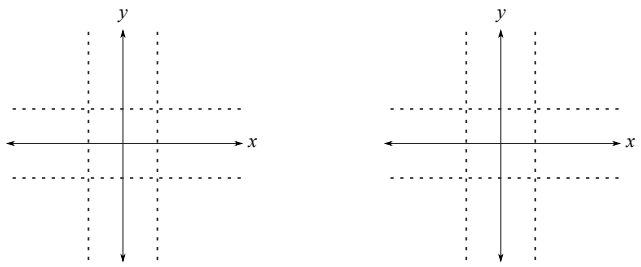
D. Use your expressions from question A to write the wave function the following states.

1. $|\varphi_C\rangle = \frac{1}{\sqrt{2}}\left(|0,1\rangle + |1,0\rangle\right)$

2. $|\varphi_D\rangle = \frac{1}{\sqrt{2}}\left(|0,1\rangle - |1,0\rangle\right)$

E. Graph these functions on the axes below by indicating whether each wave function is *positive*, *negative*, or *zero* at the center of each region. (*Note:* The middle region corresponds to the origin.)



F. Are $|\varphi_C\rangle$ and $|\varphi_D\rangle$ related to each other in the same way as $|\varphi_A\rangle$ and $|\varphi_B\rangle$? How are they related to $|\varphi_A\rangle$ and $|\varphi_B\rangle$? Explain.

G. Consider the student discussion below.

Student 1: "The superpositions are just rotations of the first wave functions. We could just as easily have chosen different x- and y-axes to begin with, and we would have started off with those states as our energy basis instead of the original ones."

Student 2: "I disagree. Once you have a set of energy eigenstates, you can't change your mind and use different ones. Only the original solutions of the Schrödinger Equation for this potential are a valid energy basis."

With which student do you agree, if either? Explain.

✓ Check your results with a tutorial instructor.

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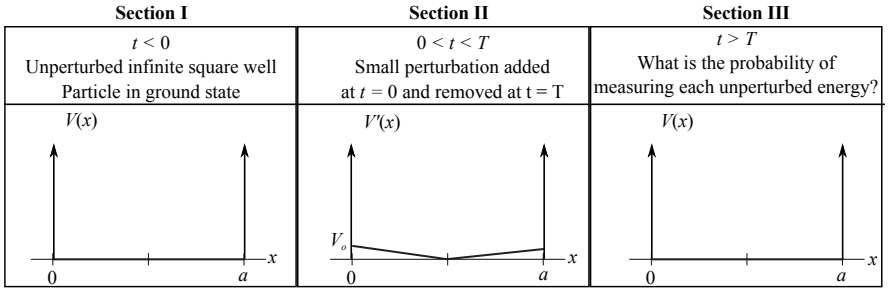
TIME-DEPENDENT PERTURBATION THEORY PART 1: FOUNDATIONS

QM 1

In this tutorial, we will consider a quantum mechanical potential that is *not constant* in time. In section I, the potential has the form of an unperturbed square well. In section II, a small (constant) perturbation is turned on. In section III, this perturbation is turned off at a later time, leaving the original, unperturbed infinite square well.



I. Before the perturbation

Consider a particle in the infinite square well whose energy eigenfunctions are given by $\varphi_m^{(0)}(x)$. The particle is in the first excited-state, $\varphi_2^{(0)}(x)$, with energy $E_2^{(0)}$.

A. Describe the time dependence, if any, of the wave function for this particle.

B. Describe the time dependence, if any, of the probability density for this particle.

C. Suppose the energy of the particle were measured. What is the probability that the measurement would result in $E_2^{(0)}$? Explain.

D. Does your answer to question C depend on the time at which the measurement is made?

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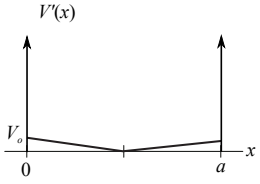
II. During the perturbation



At $t = 0$, the potential is perturbed such that it has the form shown at right, where V_o is a small, positive constant. The perturbation remains constant until it is removed at $t = T$.

A. *Predict* whether the probability density of the particle will depend on time between $t = 0$ and T (during the time when the perturbation is on). Explain.

B. Suppose the energy of this system were measured at a time between $t = 0$ and T. *Predict* the probability that the result of the measurement will be $E_2^{(0)}$. Explain.

C. Does your answer to question B depend on time? Explain.

D. Describe the form of the wave function *immediately after* $t = 0$ (the instant when the perturbation is turned on). (*Hint:* Should the wave function be continuous in time at $t = 0$?)

E. Is the wave function for this particle an energy eigenfunction of the system during the time from $t = 0$ to T? Explain your answer.

Is your answer consistent with questions A-C?

F. Do you agree or disagree with the statement below? Explain.

"At $t = 0$, the wave function is the original first excited-state, so at later times the wave function will have the same time dependence it always does, $\varphi_2^{(0)}(x)e^{-iE_2^{(0)}t/\hbar}$."

✓ Check your results with an instructor.

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At any given instant, the wave function of a particle in the perturbed potential may be written in terms of the eigenstates of the *unperturbed* potential: $\psi(x) = \sum_m c_m \varphi_m^{(0)}(x)$.

G. Explain why it is possible to write the wave function in this form, even when the potential has been perturbed.

H. Explain why the time dependence of the above wave function from $t = 0$ to T is *not* given by the expression $\psi(x,t) \neq \sum_m c_m \varphi_m^{(0)}(x) e^{-iE_m^{(0)}t/\hbar}$.

I. Explain why you can write $\psi(x,t) = \sum_m c_m(t) \varphi_m^{(0)}(x) e^{-iE_m^{(0)}t/\hbar}$, even though $\varphi_m^{(0)}(x)$ are *not* the eigenfunctions for the perturbed potential. (*Hint:* Consider your answer to question G.)

Suppose you were to make a measurement of the energy of this system between $t = 0$ and T.

J. What are the possible results of this measurement? Explain.

K. Will the probabilities associated with each of the possible energy measurements for this system depend on time? Explain.

L. The statement below is incorrect. Identify the flaw(s) in the student's reasoning. Explain.

"As usual, $|c_m(t)|^2$ represents the probability of measuring the unperturbed energy, $E_m^{(0)}$, which depends on time because the system has been perturbed."

✓ Check your results with an instructor.

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III. After the perturbation

Recall that the perturbation to the potential is removed at $t = T$, after which the potential is the infinite square well.

A. Is it possible to write the wave function for $t > T$ as $\psi(x,t) = \sum_m a_m \varphi_m^{(0)}(x) e^{-iE_m^{(0)}t/\hbar}$? Explain.

(*Note:* a_m **does not** depend on time in this expression.)

B. How is the coefficient a_m in the expression above related to the coefficient $c_m(t)$ from the previous section? *(Hint:* Should the wave function be continuous in time at $t = T$*?)*

C. Does the probability density for this particle depend on time for $t > T$ (after the perturbation has been turned off). Explain.

Suppose the energy of this system were measured at some time $t > T$.

D. Determine an expression for the probability that the result of the measurement is $E_m^{(0)}$, the energy of the m^{th} unperturbed state. Explain.

E. Do you expect there to be values of m for which the probability of measuring $E_m^{(0)}$ is equal to one? Are there any you expect to be equal to zero? Explain.

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F. Does the probability that the result of the measurement is $E_m^{(0)}$ depend on the exact time, $t > T$, at which you make the measurement? Explain.

G. Fill in the table at right to indicate whether it is possible (P), not possible (0), or certain (1) to measure each energy value during each of the three intervals considered in this tutorial. Explain.

Energy Measurements	Before	During	After
Unperturbed ground state			
Unperturbed first excited-state			
Unperturbed highly excited-states			
Perturbed states			

H. Consider the following discussion between three students about what happens after the perturbation is turned off.

Student 1: "During the perturbation, the wave function for the particle changed; it is now in a superposition of energy eigenstates. That means we could measure almost any of the unperturbed energies."

Student 2: "Maybe we evolve into a superposition, but once we turn the potential off, the particle will return to the state it started in, which was the first excited-state."

Student 3: "I agree, but if we wait long enough, then the particle will decay, and we will only be able to measure the ground state energy."

With which student(s) do you agree, if any? For each student with whom you disagree, explain the flaws in their reasoning.

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TIME-DEPENDENT PERTURBATION THEORY PART 2: FORMALISM

I. Foundations of time-dependent perturbation theory

Recall the *non-constant* potential from the previous tutorial, in which an initial system is perturbed at $t = 0$. The perturbation is removed at $t = T$, at which point the wave function may be represented by $\psi(x,t) = \sum_m a_m \varphi_m^{(0)}(x) e^{-iE_m^{(0)}t/\hbar}$.

Suppose the particle begins in the n^{th} energy eigenstate of the unperturbed system, and an energy measurement is made after the perturbation is removed (at $t > T$).

A. Is the energy of the particle well-defined after the perturbation is removed (but before any measurement is made)? Explain.

B. Write an expression for the probability that the result of the measurement is $E_m^{(0)}$, the energy of the m^{th} unperturbed state. Explain.

C. Does the probability that the result of the measurement is $E_m^{(0)}$ depend on the exact time, $t > T$, at which you make the measurement? Explain.

D. Predict whether the probability that the result of the measurement is $E_m^{(0)}$ will depend on the amount of time, T, for which the potential is perturbed? Explain.

✓ Check your results with a tutorial instructor.

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QM 2 *Time-dependent perturbation theory Part 2: Formalism*

II. Time-dependent perturbation theory

Recall the expression from the previous tutorial for the time dependence of the wave function during the time when the perturbation is active, $\psi(x,t) = \sum_m c_m(t)\varphi_m^{(0)}(x)e^{-iE_m^{(0)}t/\hbar}$. We can use perturbation theory to determine the coefficients as the expansion: $c_m(t) = c_m^{(0)}(t) + c_m^{(1)}(t) + \ldots$.

Suppose that the system is in the ground state before the perturbation is turned on.

A. Use your knowledge of the wave function at $t = 0$ to determine the zeroth-order coefficients, $c_m^{(0)}$. (*Hint:* They should not depend on time.)

Equation (1) below (which is derived from the Schrödinger Equation) is accurate to first order.

$$\frac{\partial c_n^{(1)}(t)}{\partial t} = -\frac{i}{\hbar}\sum_m c_m^{(0)}\left\langle \varphi_n^{(0)}\middle|H'(t)\middle|\varphi_m^{(0)}\right\rangle e^{i(E_n^{(0)}-E_m^{(0)})t/\hbar} \qquad (1)$$

B. The expression above contains two indices, *n* and *m*. Explain the purpose of each index, and describe how they are different from each other.

C. Rewrite equation (1) by inserting the values of $c_m^{(0)}$ that you determined in questions A on the right side of the equation. For each value of *n*, how many terms in the sum are non-zero for the system described above?

The matrix element in equation (1) above is often written as $\left\langle \varphi_n^{(0)}\middle|H'(t)\middle|\varphi_m^{(0)}\right\rangle = H'_{nm}(t)$. If the Hamiltonian is constant in time during the interval from $t = 0$ to T (*i.e.,* the only time dependence is in the turning on and off of the potential), then we can ignore the specific form of the perturbation in order to determine the general time-dependent behavior.

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D. Integrate both sides of your expression in part C to obtain $c_n^{(1)}(t)$ for $n = 1$. Be sure to include the correct limits of integration on both sides of the equation.

E. Integrate both sides of your expression in part C to obtain $c_n^{(1)}(t)$ for $n > 1$. Be sure to include the correct limits of integration on both sides of the equation.

F. Write an expression for the wave function for this particle for $t > T$. (*Hint:* Recall that the wave function should be continuous in time when the perturbation is turned off at $t = T$.)

G. For some time $t > T$, determine P_n, the probability that an energy measurement results in $E_n^{(0)}$, the unperturbed energy of the n^{th} unperturbed eigenstate. Does P_n depend on the exact time, t, at which the measurement is made? Explain.

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H. The quantity P_n is known as the transition probability. Why is this name appropriate?

I. Does the transition probability depend on the length of time the perturbation is left on, *T?* Is your prediction in question D from section I consistent with this answer? If not, resolve any inconsistency.

J. Examine your expression for P_n for $n > 1$.

1. In general, for which values of n will this expression be maximal? For which n will it be minimal? Explain.

2. Is there ever a value $T = T_n^{min}$ for which P_n is equal to zero? Develop a general rule for determining the time at which this expression is minimal as well as the time at which it is maximal (T_n^{max}). Explain.

3. Describe how you could *maximize* the transition probability to a given state, ψ_n.

K. What answer do you *expect* for the sum of all the transition probabilities? Explain.

Are your transition probabilities consistent with your answer above?

The transition probabilities you have derived contain two powers of the magnitude of the perturbation; however, they are only accurate to *first-order*. The probabilities for $n > 1$ are still approximately accurate because only the $n = 1$ probability receives a correction at second order.

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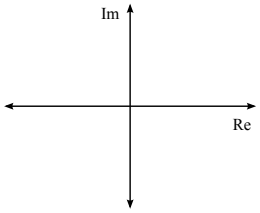
TWO-STATE TIME DEPENDENCE

I. Stationary state

An electron's spin is measured and determined to be $+\hbar/2$ in the z-direction. At time $t = 0$, a constant magnetic field B_o is turned on in the z-direction.

A. The state of a particle with spin-½ may be written in the form $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$, where α and β are complex parameters. Determine the initial state of the electron described above (*i.e.,* find α and β at $t = 0$).



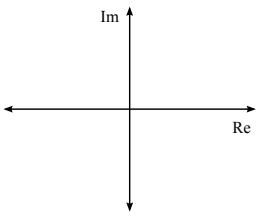
B. Plot these two numbers as points on the axes at right (be sure to label each point).

C. Determine the state of this electron as a function of time. (*Hint:* Write the state as $\begin{pmatrix}\alpha(t)\\ \beta(t)\end{pmatrix}$ using what you have learned in class.)

D. Find a time, T, such that $\alpha(T) = \alpha(0)$ (this time is known as the period). You may find the identity $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ useful.

E. Graph $\alpha(t)$ on the axes at right, following the steps below.

1. Plot $\alpha(t)$ for the points $t = 0$, $T/8$, $T/4$, $T/2$, $3T/4$, and T.
2. Connect these points with a dashed line. Be sure to indicate the direction of increasing time.
3. Describe the overall time dependence of this state in words.



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F. Suppose you were to make a second measurement of the z-component of the electron's spin at a later time $t > 0$. What is the probability that this measurement will result in $+\hbar/2$? Explain how you determined your answer.

G. Does your answer to the previous question depend on time? Explain.

H. Examine your graph on the previous page. What geometric feature of this graph can you use to answer question F?

I. The state you have examined in this part is known as a stationary state. Explain why this name is appropriate. Does the state of a particle depend on time in a stationary state?

Please wait for the class discussion.

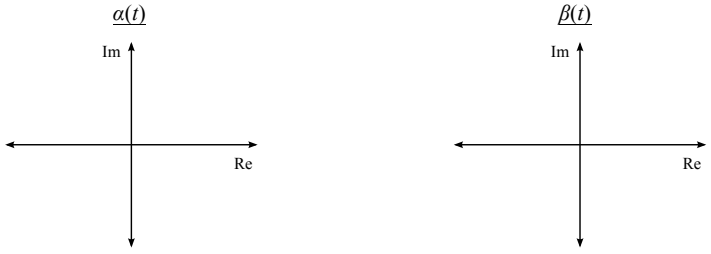
II. Superposition of stationary states

A different electron's spin is measured and determined to be $+\hbar/2$ in the ***x*-direction**. At time $t = 0$, a constant magnetic field B_o is turned on in the z-direction, as in the previous section.

A. Determine the initial state of this electron in the form $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$. Explain your reasoning.

B. Determine the time dependence of this state.

C. Graph $\alpha(t)$ and $\beta(t)$ on the axes below. You may find it useful to follow the steps from the previous section.



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D. Suppose you were to make a second measurement of the electron's spin at a later time $t > 0$, this time in the z-direction. What is the probability that this measurement will result in $+\hbar/2$? Explain how you determined your answer.

E. Does your answer to the previous question depend on time? Explain.

F. Suppose instead that your measurement at $t > 0$ were of the ***x*-component** of the electron's spin. Determine $P_{x+}(t)$, the probability that the result of this measurement is $+\hbar/2$ at a later time $t > 0$? Express your answer in terms of $\alpha(t)$ and $\beta(t)$ as well as a numeric value.

G. In the space below, sketch a plot of $P_{x+}(t)$ as a function of time.

H. Is your sketch periodic? If so, identify the period and compare it to the value of T you determined in section I.

I. Explain how the graphs in part C are related to your plot in part G. (*Hint:* Can you use the principle of superposition here?)

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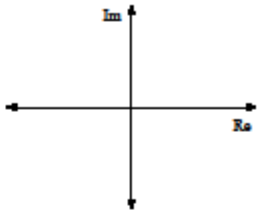
TWO-STATE TIME DEPENDENCE

QM 1

I. Complex numbers

Consider the complex number $z = 1 - i$.



A. Plot this number on the set of axes shown at right.

B. Draw a vector from the origin to the plotted number.

1. What does the length of this vector represent about the complex number z? Determine the value in this case.

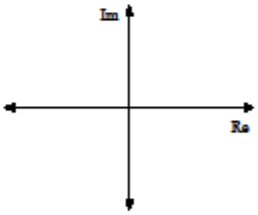
2. What does the angle between the real axis and this vector represent about the complex number z? Determine the value in this case. (*Note:* Is the angle positive or negative?)

3. Rewrite z as a complex exponential. Describe how you used the numbers you determined in the previous two questions.

Consider the complex function of time $f(t) = e^{-2it}$.

C. Find the smallest positive time, $T > 0$, such that $f(T) = f(0)$ (this time is known as the period). (*Hint:* Recall that $e^{\pm 2\pi i} = 1$)

D. Graph $f(t)$ on the axes at right, following the steps below.



1. Plot $f(t)$ for the points $t = 0$, $T/8$, $T/4$, $T/2$, $3T/4$, and T.
2. Connect these points with a dashed line. Be sure to indicate the direction of increasing t.
3. As t increases, does the length of the vector between the origin and the point described by $f(t)$ change? If so, describe how it changes.

4. As t increases, does the angle between the positive real axis and the vector from the origin to the point described by $f(t)$ change? If so, describe how it changes.

✓ Please wait for the class discussion.

QM 2 *Two-state time dependence*

II. Time dependence for a quantum state

Consider an electron in the spin state $|\chi\rangle \equiv \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. (*Note:* when no basis is given, use the z-basis.)

A. Determine the inner product ${}_x\langle + | \chi \rangle$ in terms of α and β.

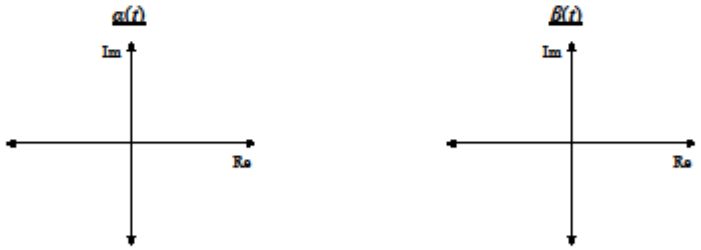
B. Is this inner product related to a probability? If so, which probability? If not, explain why not.

C. An electron's spin is measured just before $t = 0$ and determined to be $+\hbar/2$ in the **x-direction**.

1. Determine the state of this electron as a column vector at $t = 0$. Explain.

At $t = 0$, a constant magnetic field B_o is turned on in the **z-direction**. Recall that, at $t > 0$, the state of a particle in a magnetic field is given by $|\chi(t)\rangle \equiv \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix} = \begin{pmatrix} \alpha(0)e^{-i\omega_o t/2} \\ \beta(0)e^{+i\omega_o t/2} \end{pmatrix}$, where ω_o depends on the strength of both the magnetic field and the particle's magnetic moment.

2. For the electron described above, graph $\alpha(t)$ and $\beta(t)$ on the axes below, following the procedure from the previous section. Indicate the phase and the magnitude of $\alpha(t)$ and $\beta(t)$ at several instants in time (*e.g.*, $t = T/8$, $t = T/4$, *etc.*).

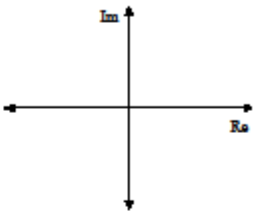


Describe how $\alpha(t)$ and $\beta(t)$ are changing in time. Discuss any similarities and differences between them.

Are $\alpha(t)$ and $\beta(t)$ changing in time at the same rate?

3. Write an expression for the inner product ${}_x\langle +|\chi(t)\rangle$ as a function of time.

4. On the graph at right, plot the inner product ${}_x\langle +|\chi(t)\rangle$ at several different instants *(e.g., $t = T/8$, $t = T/4$, etc.)* for this particle.



D. Suppose that you were to measure the ***x*-component** of the electron's spin at a later time, $t > 0$.

1. Does the probability that the result of this measurement is $+\hbar/2$ depend on time? Explain.

2. Use your graph of ${}_x\langle +|\chi\rangle$ to determine the smallest time $t > 0$ at which the probability of measuring $+\hbar/2$ in the x-direction is *equal to* the probability at $t = 0$. Explain.

How does this time compare to T, the period of $\alpha(t)$ and $\beta(t)$?

3. Write an expression for the probability of measuring $+\hbar/2$ in the x-direction in terms of $\alpha(t)$ and $\beta(t)$.

Suppose instead that you were interested in the probability that the ***z*-component** of spin is equal to $+\hbar/2$. Assume that no measurement was made in the previous question.

4. Write an expression for this probability in terms of $\alpha(t)$ and $\beta(t)$. Does this probability depend on time? Explain.

E. Use your answers from this section to fill out the first two columns of the handout.

✓ Please wait for the class discussion.

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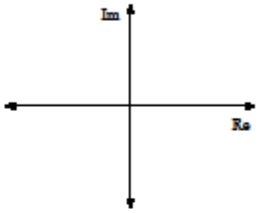
QM 4 *Two-state time dependence*

III. Time dependence for a stationary state

Now suppose that an electron's spin is measured and determined to be $+\hbar/2$ in the z-direction. As in the previous section, a constant magnetic field B_o is turned on in the z-direction at $t = 0$

A. Determine the state of this electron as a function of time in the form $\begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix}$. (*Hint:* What is $\alpha(t)$ at $t = 0$?)

B. Graph $\alpha(t)$ on the set of axes at right by following the procedure from the previous sections.



C. What is the probability that a measurement of the z-component of this electron's spin results in $+\hbar/2$? Justify your answer using your graph.

D. What is the probability that a measurement of the **x-component** of this electron's spin results in $+\hbar/2$? (Assume no measurements were made in the previous question.)

E. Which of the probabilities in this section, if any, depend on time? Explain. Use your answers to fill out the rest of the handout.

The term *stationary state* may be used to describe the state of the electron in this section.

F. Justify the use of this term for this state. For a stationary state, does the state depend on time?

G. Consider the student discussion below.

Student 1: "The probability of measuring the spin in the z-direction was independent of time for both of the states we considered in this tutorial, so they are both stationary states."

Student 2: "I disagree. The state that started as spin up in the z-direction is stationary because *all* probabilities are independent of time, even though the state does depend on time."

Student 3: "I think it depends on what we measure. Any state is stationary if we measure the spin in the z-direction, but not stationary if we were measure the spin in the x-direction."

With which student(s) do you agree, if any? For each student who is incorrect, identify the flaw(s) in that student's reasoning. Explain.

✓ Please wait for the class discussion.

B.II – Tutorials for Part II

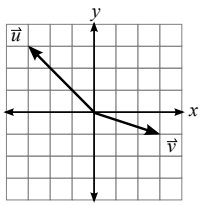
This sub-section includes copies of all tutorials discussed in part II of this dissertation. Each tutorial is discussed in chapter 8.

DIRAC NOTATION PART 1

QM 1

I. Vectors and Inner Products

Consider the vectors $\vec{u}$ and $\vec{v}$ shown at right.



A. Write mathematical expressions for these vectors in terms of the basis vectors $\hat{x}$ and $\hat{y}$.

B. Evaluate the inner product $\vec{u} \cdot \vec{v}$. Is your answer a *vector* or a *scalar?*

C. What information does $\vec{u} \cdot \vec{v}$ tell you about the relationship between the vectors $\vec{u}$ and $\vec{v}$? Explain. (*Hint:* Consider the cases where $\vec{u} \cdot \vec{v}$ is positive, negative, or zero.)

D. Suppose each of the vectors given above also had a component out of the page *(i.e.*, u_z and v_z*)*. Write an expression for $\vec{u} \cdot \vec{v}$ given this information. Does your interpretation of the inner product from question C change in three dimensions? Explain

E. Suppose instead that you had two *n*-dimensional vectors, $\vec{a}$ and $\vec{b}$ Write expressions for these vectors in terms of a set of basis vectors, $\hat{x}_i$ (where $i = 1, 2, \ldots n$).

F. Write a general expression for $\vec{a} \cdot \vec{b}$. Does your interpretation of the dot product change in *n* dimensions? Explain.

✓ Please wait for the class discussion.

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QM 2 ***Dirac notation Part 1***

II. Dirac notation

Consider the following systems: system 1 consists of two state vectors in a *6-dimensional* space; system 2 consists of two three-dimensional spatial vectors.

System 1	System 2
$\|\psi_A\rangle = 2\|\phi_1\rangle - \|\phi_2\rangle + i\|\phi_4\rangle + \|\phi_5\rangle$	$\vec{a} = \hat{x}_1 - \hat{x}_2 + \hat{x}_3$
$\|\psi_B\rangle = i\|\phi_1\rangle - \sqrt{2}\|\phi_3\rangle + i\|\phi_4\rangle + \|\phi_5\rangle$	$\vec{b} = 2\hat{x}_1 - 3\hat{x}_3$

A. How many basis vectors are necessary to fully describe an arbitrary vector in each space? Which objects in the expressions above correspond to these basis vectors? Explain.

System1 System 2

B. Evaluate the inner products shown below. For each system, determine whether your answer is consistent with question A above. Explain.

$\langle\phi_i|\phi_j\rangle$ $\hat{x}_i \cdot \hat{x}_j$

C. Evaluate the pairs of inner products shown below. For which system, if either, should the answers be equal? Explain.

$\langle\psi_A|\psi_B\rangle$ and $\langle\psi_B|\psi_A\rangle$ $\vec{a}\cdot\vec{b}$ and $\vec{b}\cdot\vec{a}$

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D. What information do the expressions below convey about their respective vectors? Explain.

$\langle \psi_A | \psi_A \rangle$ $\qquad$ $\vec{a} \cdot \vec{a}$

E. Determine the values of q_A and q_B defined by the vectors $|\psi_A'\rangle = q_A|\psi_A\rangle$ and $|\psi_B'\rangle = q_B|\psi_B\rangle$ such that $\langle \psi_A'|\psi_A'\rangle = \langle \psi_B'|\psi_B'\rangle = 1$.

F. How is $\langle \psi_A'|\psi_B'\rangle$ related to $\langle \psi_A|\psi_B\rangle$? Explain.

Multiplying a vector by a constant to ensure $\langle \varphi|\varphi\rangle = 1$ is a process known as *normalization.*

When you consider state vectors in quantum mechanics, they will usually be normalized in this fashion.

✓ Check your results with a tutorial instructor.

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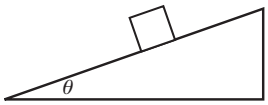
DIRAC NOTATION PART 2

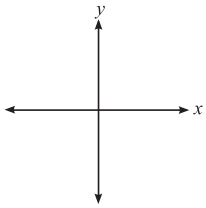
QM 1

I. Change of basis

A. A block is moving down on a frictionless ramp.



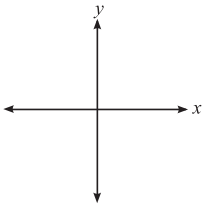
1. Draw a free body diagram for the block in the coordinate system below. Draw a dot to represent the block at the origin.



2. Write mathematical expressions for these force vectors in terms of the basis vectors $\hat{x}$ and $\hat{y}$.

3. Determine the acceleration of the block in terms of the basis vectors $\hat{x}$ and $\hat{y}$. Do you need any additional information in order to answer this question?

4. Is it possible to choose a more convenient coordinate system to answer the questions above? If so, sketch this coordinate system at right. Explain.



✓ Check your results with a tutorial instructor.

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QM 2 ***Dirac notation part 2***

B. The spin of an electron is prepared in the state $|\chi\rangle = \frac{4}{5}|+\rangle_z + \frac{3}{5}|-\rangle_z$.

1. Calculate the inner product ${}_z\langle +|\chi\rangle$. Clearly show each step in your solution.

2. Write an expression for the inner product ${}_x\langle +|\chi\rangle$. Do you need any additional information in order to evaluate this expression? Explain.

Consider another electron whose spin is prepared in the state $|\Psi\rangle = \frac{1}{2}|+\rangle_x + \frac{\sqrt{3}}{2}|-\rangle_x$.

3. Which of the following inner products is easier to calculate? Explain.

${}_z\langle +|\Psi\rangle$

${}_x\langle +|\Psi\rangle$

4. Is it possible to rewrite $|\Psi\rangle$ in terms of the basis vectors $|+\rangle_z$, and $|-\rangle_z$? If so, write the expression in this basis. Explain.

✓ Check your results with a tutorial instructor.

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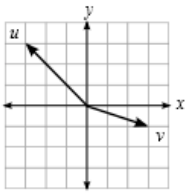
DIRAC NOTATION

I. Inner products for spatial vectors

Consider the vectors $\vec{u}$ and $\vec{v}$ shown at right.



A. Write mathematical expressions for these vectors in terms of the basis vectors $\hat{x}$ and $\hat{y}$.

B. Evaluate the inner product $\vec{u} \cdot \vec{v}$. Explain why the inner product results in a *scalar* rather than a vector.

C. In the space below, sketch a pair of vectors that have an inner product with the indicated sign. In each case, explain how you determined your answer.

1. Positive

2. Negative

3. Zero

D. Consider two vectors, $\vec{a}$ and $\vec{b}$, each of which has n components instead of two.

1. Use the symbol Σ to write expressions for $\vec{a}$ and $\vec{b}$ in terms of a set of n orthonormal basis vectors, $\hat{x}_i$ (where $i = 1, 2, \ldots, n$).

2. Determine the value of the inner product between two basis vectors: $\hat{x}_i \cdot \hat{x}_j$. (*Hint:* Consider the cases where $i = j$ and $i \neq j$ separately.)

3. Use the symbol Σ to write a general expression for $\vec{a} \cdot \vec{b}$. Explain why only one sum is necessary in this expression.

✓ Check your results with an instructor.

II. Inner products in Dirac notation

Consider a 6-dimensional Hilbert space with orthonormal basis state vectors $|\varphi_i\rangle$, where *i* is an integer between 1 and 6. Recall that the inner product between two state vectors, $|\alpha\rangle$ and $|\beta\rangle$, written in Dirac notation is given by the expression $\langle\alpha|\beta\rangle$.

A. Determine the value of the inner product between two basis state vectors: $\langle\varphi_i|\varphi_j\rangle$.

How is your answer similar to the inner product between two spatial basis vectors, $\hat{x}_i \cdot \hat{x}_j$? Explain.

B. Consider the following state vector for this system: $|\psi_A\rangle = 2|\varphi_1\rangle - |\varphi_2\rangle + i|\varphi_4\rangle + |\varphi_6\rangle$.

1. Write the dual vector, $\langle\psi_A|$, corresponding to the state vector above. Explain how you arrived at your answer.

2. Determine the inner product between this state vector and itself, $\langle\psi_A|\psi_A\rangle$.

3. How is the inner product between a state vector and itself related to the *magnitude* of that state vector? Explain.

4. How is the process for finding the inner product between two state vectors in Dirac notation different from the process for finding the inner product between two spatial vectors? Explain.

Consider a different state vector for the same system: $|\psi_B\rangle = i|\varphi_1\rangle - 2|\varphi_3\rangle + i|\varphi_4\rangle + |\varphi_6\rangle$.

C. Is the magnitude of $|\psi_B\rangle$ *greater than, less than,* or *equal to* the magnitude of $|\psi_A\rangle$? Explain.

D. Consider the dialogue between two students below.

Student 1: "Terms that are imaginary don't contribute at all to the magnitude of a state vector, so the magnitude of state B is only 5, which is smaller than the magnitude of state A."

Student 2: "You're right that B is smaller than A, but I think the imaginary terms take away from the magnitude of the state, since $i^2 = -1$. So the magnitude of B is really only 3."

Both students calculated the magnitude of $|\psi_B\rangle$ incorrectly. Identify the error(s) made by each student.

E. Determine the value of the following inner product: $\langle\psi_A|\psi_B\rangle$.

F. Do you expect the inner product $\langle\psi_B|\psi_A\rangle$ to be equal to $\langle\psi_A|\psi_B\rangle$? Explain your reasoning.

How does your answer differ from what you would expect for the inner product between two spatial vectors? Explain.

✓ Check your results with an instructor

III. Changes of basis

Recall that the state vector for a spin-1/2 particle (such as an electron) may be written in terms of the two orthonormal basis states corresponding to the *z*-direction: $|+\rangle_z$ and $|-\rangle_z$. Such a state vector may also be written in terms of the two orthonormal basis states corresponding to the *x*-direction: $|+\rangle_x$ and $|-\rangle_x$.

Suppose an electron is described by the state vector $|\chi\rangle = \frac{4}{5}|+\rangle_z + \frac{3}{5}|-\rangle_z$.

A. Calculate the inner product ${}_z\langle +|\chi\rangle$. Clearly show each step in your calculation, including any use of orthonormality.

B. Suppose you are interested in the inner product ${}_x\langle +|\chi\rangle$. What additional information is necessary in order to calculate this inner product? Explain.

C. Consider the dialogue below between three students who are attempting to determine the inner product ${}_x\langle +|\chi\rangle$.

Student 1: "This is just like the first inner product: ${}_x\langle +|-\rangle_z$ is zero because the plus and minus states are always orthogonal to each other. Therefore, the answer is 4/5."

Student 2: "The *x*- and z-directions are orthogonal to each other, so $|+\rangle_x$ is orthogonal to both $|+\rangle_z$ and $|-\rangle_z$. That means the answer is zero."

Student 3: "I don't think this inner product has a well-defined answer. The two state vectors are written in terms of different bases, so we can't find the inner product between them."

With which student(s), if any do you agree? Explain your reasoning.

D. Determine the inner product ${}_x\langle +|\chi\rangle$. Clearly show each step in your calculation.

E. Consider another electron described by the state vector $|\psi\rangle = \frac{1}{2}|+\rangle_x + \frac{\sqrt{3}}{2}|-\rangle_x$.

1. Calculate each of the following inner products.

   ${}_z\langle +|\psi\rangle$

   ${}_x\langle +|\psi\rangle$

2. Which of the two inner products above is easier to calculate? Explain.

3. Is it possible to rewrite $|\psi\rangle$ in terms of the basis state vectors $|+\rangle_z$ and $|-\rangle_z$? Explain why or why not.

   If it is possible, write $|\psi\rangle$ in terms of these state vectors. Clearly show your work.

F. Explain why it is important to be able to rewrite a state vector in a different basis. When is it most necessary to change from one basis to another?

✓ Check your results with an instructor.

TREATING FUNCTIONS AS VECTORS

QM 1

I. Vectors

Consider the vector given by the expression $\vec{v} = 3\hat{x} - 2\hat{y}$.

A. What do $\hat{x}$ and $\hat{y}$ represent in the expression above? How are they related to each other?

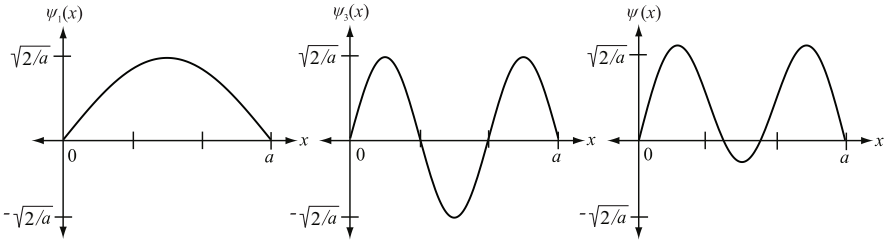
B. Explain why the inner (dot) product $\hat{x} \cdot \vec{v}$ is equal to the x-component of the vector $\vec{v}$.

C. Write $\vec{v}$ as a column vector; express each element as both an inner product and a number.

II. Position space

Consider the three functions defined from $x = 0$ to $x = a$ below.

$$\psi_1(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right) \qquad \psi_3(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{3\pi x}{a}\right) \qquad \psi(x) = \frac{1}{\sqrt{13}}\left(2\psi_1(x) + 3\psi_3(x)\right)$$



A. Divide each graph into three regions of equal width. Approximate the value of each function by its value at the center of the region and write your results in the table at right.

B. In the space below, sketch a new graph of each function by assuming it is equal to the value written in the table throughout each region (*i.e.,* each graph should be a step function with three levels).

x	$\psi_1(x)$	$\psi_3(x)$	$\psi(x)$
x_1			
x_2			
x_3			

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C. Represent each function as a column vector in which the entries are equal to the values of the function in each of the equal-width regions.

D. Describe the basis vectors for the column vector representation from the previous question.

In the space below, sketch graphs of functions (from $x = 0$ to $x = a$) to represent the basis vectors, and write the corresponding column vectors next to each sketch.

E. Describe how the column vector representation used above could be modified to provide a *better* approximation to a function.

F. Assume you were to divide your graphs into N equally-sized regions.

1. As N becomes very large, how do the column vectors that approximate each function change?

What happens to the number of basis states and to their graphs?

One way of describing a function is as a list of numbers: a single number, $\psi(x)$, associated with each value of position, x.

2. How is the column vector representation of the functions on this page similar to this definition of a function? Explain.

✓ Check your results with an instructor.

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III. Inner products

A. Consider two vectors given by $\vec{u} = 2\hat{x} + 3\hat{y}$ and $\vec{v} = 3\hat{x} - 2\hat{y}$.

1. Write each of these vectors as a column vector.

2. Use the column vector representation to compute the dot product $\vec{u} \cdot \vec{v}$.

Use your result to explain how $\vec{u}$ and $\vec{v}$ are related.

B. Consider the function given by the expression $\psi_1(x)\psi_3(x)$.

1. Use the approximation from section II to represent this function in the table at right.

2. Sketch a graph of this approximation in the space below.

x	$\psi_1(x)\psi_3(x)$
x_1	
x_2	
x_3	

3. Determine an approximate value for the integral $\int_0^a \psi_1(x)\psi_3(x)\,dx$. Describe each step of the procedure you use.

How did you approximate the infinitesimal width dx in your procedure?

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QM 4 ***Treating functions as vectors***

The inner product between two *functions, b*(*x*) and *c*(*x*), is defined to be the following integral:

$\int b^*(x)c(x)\,dx$, where $b^*(x)$ indicates the complex conjugate of $b(x)$.

C. Explain why the expression $b^*(x)c(x)$, without an integral, does *not* give the inner product of two functions.

D. Describe how to approximate the inner product between two functions of *x* using your result from part B.

How does this procedure for finding the inner product differ from the procedure you used in question A2?

What does your result for the inner product of $\psi_1(x)$ and $\psi_3(x)$ suggest about how these two functions are related to each other?

E. Determine an approximate value for the inner product between $\psi_1(x)$ and $\psi(x)$.

Compare your answer to the original definition of $\psi(x)$ in terms of $\psi_1(x)$ and $\psi_3(x)$.

F. Should the inner product between $\psi_1(x)$ and itself be *greater than, less than,* or *equal to* one? Explain.

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Use your procedure from the previous page to check your answer.

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REPRESENTATIONS OF WAVE FUNCTIONS

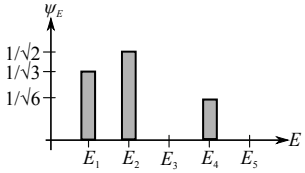
I. The energy basis

Consider a particle in the quantum mechanical infinite square well of width a described by $|\psi\rangle = \frac{1}{\sqrt{3}}|\psi_1\rangle + \frac{1}{\sqrt{2}}|\psi_2\rangle + \frac{1}{\sqrt{6}}|\psi_4\rangle$, where $|\psi_n\rangle$ is the n^{th} energy eigenstate with eigenvalue E_n.

A. List the possible results of an energy measurement on the particle described by the state above and the probability associated with each result. Explain how you determined your answers from the given representation of this quantum state.

B. The quantum state above is said to be represented in the *energy basis*. Explain why this name is appropriate for this representation.

C. The graph shown at right is a visual representation of the state given above.



1. Would it still be appropriate to say that the state is represented in the energy basis? Explain.

2. Give a physical interpretation for the vertical component of the graph for each value of E_n. Is the normalization of the state consistent with your interpretation?

3. Suppose that one of the terms in the expression at the top of the page were made negative. How would you represent this change on a graph like the one above?

   How would this change affect the physical interpretation you identified in question 2?

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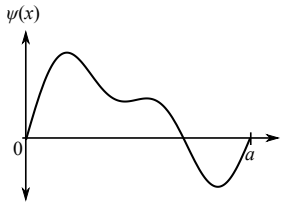
II. The position basis

A different visual representation of the state of the particle from the previous page is shown at right.



A. What probabilities can you readily infer about this state from the graph at right? Explain in detail how to determine these values using only the graph.

B. Consider the student discussion below.

Student 1: "The probability of finding the particle is largest where the wave function is positive, and lowest where the wave function is negative."

Student 2: "I disagree. The particle wants to be where the wave function is lowest, so we would be more likely to find it where $\psi(x)$ is very negative."

With which student do you agree, if either? Identify and explain the flaws in the reasoning of each student with whom you disagree.

C. The graph above is a representation of a quantum state in the *position basis*, also known as the *(position-space) wave function*. Explain why these names are appropriate.

D. What are the units on the vertical axis of the graph above? Make sure your answer is consistent with the overall normalization of the state. Explain.

E. The two graphs you have seen in this tutorial represent the same quantum state. How are the two visual representations similar to each other in terms of how they convey information? How are they different? Explain.

✓ Check your results with an instructor.

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III. Interference

Consider three particles (A, B, and C) described by the states $|\varphi_A\rangle = \frac{1}{\sqrt{3}}|\psi_1\rangle + \frac{1}{\sqrt{2}}|\psi_2\rangle + \frac{1}{\sqrt{6}}|\psi_4\rangle$, $|\varphi_B\rangle = \frac{1}{\sqrt{3}}|\psi_1\rangle - \frac{1}{\sqrt{2}}|\psi_2\rangle + \frac{1}{\sqrt{6}}|\psi_4\rangle$, and $|\varphi_C\rangle = \frac{1}{\sqrt{3}}|\psi_1\rangle + i\frac{1}{\sqrt{2}}|\psi_2\rangle + \frac{1}{\sqrt{6}}|\psi_4\rangle$.

A. *Predict* (without sketching) whether the wave functions in position space associated with each of these three states will be *the same* or *different*. Briefly explain your reasoning.

B. *Predict* (without sketching) whether the *probability densities* in position space associated with each of these three states will be *the same* or *different*. Briefly explain your reasoning.

C. Ask an instructor for a handout showing the wave function and the associated probability density for each particle.

   If your predictions were incorrect, resolve any inconsistencies between the handout and your answers on the previous page. Explain.

D. Suppose the sign (or the complex phase) of a single term in a quantum state written in the energy basis is changed. Indicate whether or not each of the following would be different. Explain.

   1. Energy probabilities

   2. The probability density

The results above indicate that in quantum mechanics, wave functions are subject to interference.

E. Compare the interference between wave functions in quantum mechanics to other examples of interference that you have seen (*e.g.*, for pulses on a spring or for light waves).

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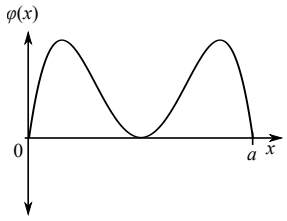
QM 4 *Representations of Wave Functions*

IV. Changes of basis

Consider the wave function for a particle in the infinite square well given by $\varphi(x) = Nx(a-x)(2x-a)^2$, where N is the appropriate normalization constant. This wave function is also graphed at right.



A. In what basis is this particle represented? Explain how you determined your answer.

Suppose an energy measurement were made on the particle with the wave function above. Let $P(E_n)$ represent the probability that this measurement results in E_n. Consider the probabilities of the three lowest energy eigenvalues: $P(E_1)$, $P(E_2)$, and $P(E_3)$.

B. *Predict* which of the three probabilities above will be *smallest* and which will be *largest*. Briefly explain (do not write any equations; base your answer on a qualitative argument).

C. Consider the student discussion below.

Student 1: "This wave function is all positive, just like the ground state, so I think that $P(E_1)$ will be the greatest."

Student 2: "I disagree. The wave function looks more like the first excited state, since they both have two bumps, so I think $P(E_2)$ will be the greatest."

Student 3: "I don't know anything about the probabilities for energy because this particle is represented in the position basis. We don't have any information about energy, so the probabilities are all equal."

With which student(s), if any, do you agree? For each statement with which you disagree, identify and explain the errors made by that student.

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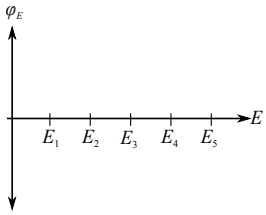
D. Write a mathematical expression for $P(E_n)$, but do not evaluate it.

E. Describe a method for evaluating your expression from question D qualitatively, without calculating any integrals mathematically.

Use your method to check your prediction in question B. Resolve any inconsistencies.

F. Ask an instructor for a handout that will help you to make a graph on the axes at right of this state in the energy basis.



Resolve any inconsistencies with your answers to questions B and E.

G. Consider the student discussion below.

Student 1: "If I know any representation for a quantum state, then I know everything there is to know about it."

Student 2: "I disagree, because if all I know is energy space, then I can't know anything about position space, and vice versa. That's the uncertainty principle."

With which, if either, of the students do you agree? Explain your reasoning.

✓ Check your results with an instructor.

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PROBABILITY AMPLITUDE

QM 1

I. Probability amplitude

Suppose that particle A is placed in the infinite square well potential. Let the state of the particle be given by $|\varphi_A\rangle$ and let the system's energy eigenstates and eigenvalues be given by $|\psi_n\rangle$ and E_n, respectively, for $n = 1, 2, 3, \ldots$.

A. Write the state of particle A as a sum of the energy eigenstates of the system.

Describe how to determine the coefficient of each term in this sum.

Ask an instructor to check your expression, and to provide a handout that includes the values of the coefficients in the sum above for a particle A. Write a final expression for the state of particle A in the space below.

B. Determine the inner product of the state with itself, $\langle\varphi_A|\varphi_A\rangle$. Show your work.

Does your answer agree with what you expect? Explain.

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C. Suppose you were to measure the energy of particle A. Which value would be the most likely outcome of this measurement? What is the probability of this outcome? Explain.

D. Explain why it would be *incorrect* to say that $\langle \psi_n | \varphi_A \rangle$ is the probability that particle A is measured to have energy E_n.

The inner product, $\langle \psi_n | \varphi_A \rangle$, between a state that represents a particle, such as $|\varphi_A\rangle$, and an eigenstate associated with an observable, such as $|\psi_n\rangle$, is called a *probability amplitude*.

E. Discuss with your group why the term probability amplitude is appropriate for this inner product.

F. Suppose that the value of $\langle \psi_2 | \varphi_A \rangle$ for particle A were changed to $-\frac{1}{\sqrt{2}}$.

1. Is the *probability amplitude* associated with $n = 2$ *the same* or *different?* Explain.

2. Is the *probability* of measuring E_2 *the same* or *different?* Explain.

✓ Discuss your answers with an instructor.

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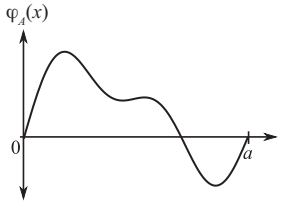
II. Wave functions

The wave function for particle A, $\varphi_A(x)$, is shown at right.



A. Describe how to use wave function to determine the probability that a particle is measured within the region between $x = a/5$ and $x = a/3$.

The wave function for particle A can be written as the following inner product: $\varphi_A(x) = \langle x|\varphi_A\rangle$, where $\langle x|$ is the basis state associated with position x.

B. Would it be appropriate to use the term *probability amplitude* to describe the wave function for particle A? Explain.

C. Use the inner product above to write an expression for the probability that a particle is measured within the region between $x = a/5$ and $x = a/3$.

D. Consider the student discussion below.

Student 1: "When I square the probability amplitude for energy, I get the probability of measuring that energy. Since the wave function is also a probability amplitude, the square of the wave function is also a probability."

Student 2: "I agree that the wave function is a probability amplitude. However, since the eigenvalues of position are continuous, squaring the wave function gives a probability density instead of a probability. That means I need to integrate the square of the probability amplitude to get probability."

With which student do you agree, if either? Explain.

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E. Compare the procedure for determining probabilities of energy measurements (section I) to the procedure for determining probabilities of position measurements (section II). How are these procedures similar? How are they different? Explain.

F. Write an expression for the state of particle A, $|\varphi_A\rangle$, in terms of the basis states associated with position, $|x\rangle$. Explain. (*Hint:* What changes do you need to make to the expression in question A of section I?)

Describe how to determine the coefficient of each term in your expression.

Can these coefficients be imaginary? Explain.

✓ Discuss your answers with an instructor.

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PERTURBED POTENTIALS AND WAVE FUNCTIONS

QM 1

Consider a quantum mechanical system with energy eigenfunctions $\psi_n^{(0)}(x)$ and corresponding energy eigenvalues $E_n^{(0)}$.

A. Suppose the wave function for this system is given by $\psi(x)$.

1. Explain why you can write this wave function as a superposition of the energy eigenfunctions for this system, $\psi_n^{(0)}(x)$.

2. Write a mathematical expression for this superposition. Determine expressions in Dirac notation for all unknown coefficients.

B. Suppose a perturbation is added to the system. Let the energy eigenfunctions and corresponding eigenvalues of the perturbed system be given by $\psi'_n(x)$ and E'_n, respectively. Assume that the wave function is the same as in question A, $\psi(x)$.

1. Is it possible to write the wave function, $\psi(x)$, for the system as a superposition of the new (perturbed) energy eigenfunctions, $\psi'_n(x)?\psi'_n(x)$? If so, write a mathematical expression for this superposition. Explain.

2. Determine expressions in Dirac notation for any unknown coefficients. Are these coefficients the same as those you determined in question A? Explain.

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3. Is it still possible to write the wave function, $\psi(x)$, for the system as a superposition of the original (unperturbed) energy eigenfunctions, $\psi_n^{(0)}(x)$? If so, write a mathematical expression for this superposition. Explain.

4. Determine expressions in Dirac notation for any unknown coefficients. Are these coefficients the same as those you determined in question A? Explain.

C. At $t = 0$, assume that the system is perturbed, and that the particle is known to be in the ground state, $\psi'_o(x)$, of the perturbed Hamiltonian.

1. Is the energy of the system well-defined at $t = 0$? Explain.

2. Suppose you were to measure the energy of the system at a later time, $t > 0$. *Predict* the probability that the result of this measurement is $E_o^{(0)}$, the ground state energy of the *unperturbed* system? Briefly explain.

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D. Determine an expression for the wave function for this particle at time $t > 0$.

In which basis is it easier to determine the time dependence of this wave function: the energy eigenfunctions of the *perturbed* or the *unperturbed* system? Explain.

E. Based on your answer to question D, determine an expression for the probability that the energy is measured to be $E_o^{(0)}$, the ground state energy of the *unperturbed* system. Explain.

Determine an expression for the probability that the energy is measured to be E'_o, the ground state energy of the *perturbed* system. Explain.

F. Compare your answers with your prediction in question C2. Resolve any inconsistency.

G. Identify the flaws in the following *incorrect* statement. Explain.

"The ground state of the perturbed system can be written in terms of the original eigenfunctions as $\psi'_o(x) = \sum_n c_n \psi_n^{(0)}(x)$, so the probability of measuring $E_o^{(0)}$, the unperturbed ground state energy, is given by the square of the coefficient: $|c_o|^2$."

✓ Check your answers with a tutorial instructor.

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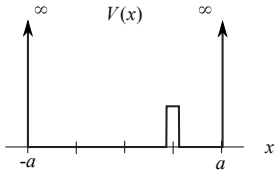
FIRST-ORDER TIME-INDEPENDENT PERTURBATION THEORY

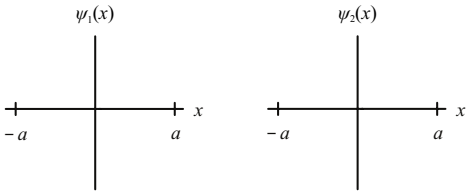
QM 1

I. First-order corrections to the energy

Consider an infinite square well of width $2a$, shown at right. A perturbation is added to this potential that is zero everywhere except within a small region, centered at $a/2$, at which it takes a (small) uniform value.



A. In the spaces below, draw the wave functions for the (unperturbed) ground state ($n = 1$) and first excited state ($n = 2$). Draw each graph using the same vertical scale.



B. As presented in lecture, an important result from time-independent perturbation theory is that the first-order correction to the n^{th} energy eigenvalue is given by $E_n^{(1)} = \left\langle \psi_n^{(0)} \middle| H' \middle| \psi_n^{(0)} \right\rangle$, where H' is the perturbation (when the perturbation is to the potential, $H' = V'$).

1. Rewrite the expression for $E_n^{(1)}$ as an integral. (Do not write out $\psi_n^{(0)}(x)$ or $V'(x)$ in terms of x.)

2. For the ground state of the system above, determine whether the first-order correction to the energy is *positive*, *negative*, or *zero*. Explain your answer.

3. For the first-excited state of the system above, determine whether the first-order correction to the energy is *positive*, *negative*, or *zero*. Explain your answer.

4. For which of the above states is the magnitude of the first-order correction to the energy larger? Explain your answer.

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C. Are there any energy eigenstates for which the first-order correction to the energy caused by this perturbation is *zero?* If so, what is the lowest energy eigenstate for which this is true? If not, explain why not.

D. Are there any energy eigenstates for which the first-order correction to the energy caused by this perturbation is *negative?* If so, what is the lowest energy eigenstate for which this is true? If not, explain why not.

E. Consider the student discussion below.

Student 1: "Only the regions where the perturbation is nonzero affect the first-order correction to the energy, because the part of the integral where the perturbation is zero will vanish."

Student 2: "I agree. If the wave function is negative in that region, then the first-order correction to the energy will be negative as well."

Student 3: "I agree that we do not need to consider regions where the perturbation is zero. However, the correction includes both $\psi^*(x)$ and $\psi(x)$, so any negative or imaginary parts of the correction will become positive"

With which student(s) do you agree, if any? Explain.

F. Consider a perturbation of the same size, but centered at the origin (instead of at $a/2$).

1. For the ground state, is the first-order correction to the energy for the perturbation centered at the origin *greater than*, *less than*, or *equal to* that produced by the perturbation centered at $a/2$? Explain your reasoning.

2. For the first excited-state, is the first-order correction to the energy for the perturbation centered at the origin *greater than*, *less than*, or *equal to* that produced by the perturbation centered at $a/2$? Explain your reasoning.

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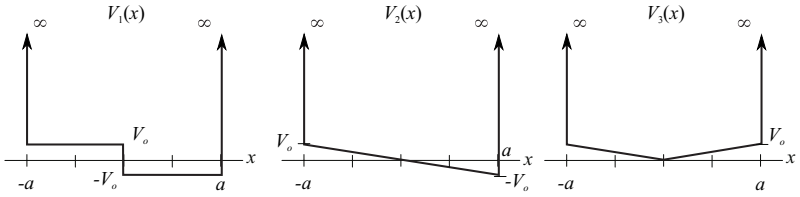
G. Consider the three potential wells shown below, each of which contains a small perturbation to an infinite square well of width $2a$.



1. For each well, determine whether the first-order correction to the **ground state energy** is *zero* or *nonzero*. Explain your reasoning in each case.

2. For each well, determine whether the first-order correction to the **first excited-state energy** is *zero* or *nonzero*. Explain your reasoning in each case.

3. For which of the wells above, if any, is the first-order correction to the energy equal to zero for all energy levels? Explain.

4. For V_3, is the first-order correction to the energy of the ground state *greater than, less than,* or *equal to* the first-order correction to the energy of the first-excited state? Explain.

H. Do you agree or disagree with the following statement? Explain.

"There are some potentials for which $E_n^{(1)} = 0$ for every eigenstate. That means that the energy of each perturbed eigenfunction is equal to the energy of the corresponding unperturbed eigenfunction."

✓ Check your results with a tutorial instructor.

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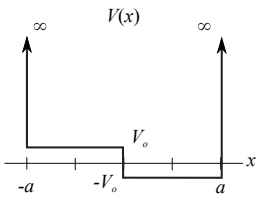
QM 4 *First-order time-independent perturbation theory*

II. First-order corrections to the wave functions

The potential well shown at right may be considered as an infinite square well of width $2a$ with a small perturbation of size V_o.

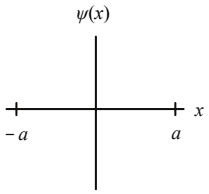
A. Consider a particle in the ground state of this well (assume that the energy of the ground state is much larger than V_o).



1. Is the wavelength of the wave function to the left of the origin *greater than*, *less than*, or *equal to* the wavelength of the wave function to the right of the origin? Explain.

2. Should the maximum value of the wave function appear *to the right of the origin*, *to the left of the origin*, or *exactly at the origin?* Explain. (*Hint:* Recall that the slope of the wave function should be continuous at the origin.)

3. Sketch the wave function for a particle in the ground state of this well on the set of axes at right (pay close attention to the height of the wave function in each half of the well).



✓ Check your results with a tutorial instructor.

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The first-order correction to the wave function of the *n*th energy level is given by the expression $\psi_n^{(1)}(x) = \sum_{m \neq n} \frac{\langle \psi_m^{(0)} | V' | \psi_n^{(0)} \rangle}{E_n^{(0)} - E_m^{(0)}} \psi_m^{(0)}(x)$, where V' is the perturbation to the potential.

B. The expression above contains two indices for labeling states, *n* and *m*. Explain the difference between these two indices.

C. The wave function correct to first-order has the form $\psi'_n(x) \approx \psi_n^{(0)}(x) + \psi_n^{(1)}(x)$. Is this wave function normalized? If not, describe how to normalize it. Explain.

D. Which terms in the sum given above will contribute most to the first-order correction to the ground state wave function? Explain.

E. Some of the terms in the sum above are zero. Determine which terms by making an argument based on symmetry. Explain.

F. Determine whether the lowest-order term in the sum above is *positive*, *negative*, or *zero* for the ground state wave function. Explain. (*Note:* Be sure to include the sign of the denominator.)

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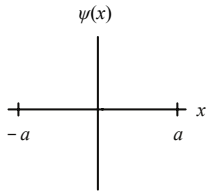
G. Explain why it is reasonable to neglect the higher-order terms in the sum above.

H. If you have time, calculate the lowest-order term in the sum (if not, ask an instructor for the value of this term so that you can proceed). You may assume that $V_o = E_1/5$.

I. Sketch the wave function correct to first-order on the axes at right.



J. Are your answers to question D consistent with your answers and graph in part A? Explain.

Ask an instructor for a graph of the wave function for the ground state of this potential and discuss your results.

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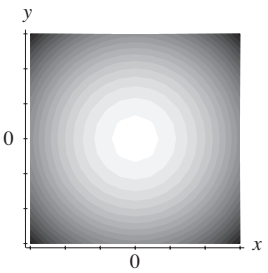
DEGENERATE TIME-INDEPENDENT PERTURBATION THEORY

I. Two-dimensional isotropic harmonic oscillator

A two–dimensional isotropic harmonic oscillator is described by the Hamiltonian $H = -\frac{\hbar^2}{2M}\nabla^2 + \frac{1}{2}M\omega^2\left(x^2 + y^2\right)$. A contour plot of the potential is shown at right. The first two energy eigenfunctions for the one–dimensional harmonic oscillator are: $\varphi_0(x)$ and $\varphi_1(x)$. The eigenvalues corresponding to these eigenfunctions are $E_0 = \frac{1}{2}\hbar\omega$ and $E_1 = \frac{3}{2}\hbar\omega$.



A. Determine the energy eigenfunction for the ground state of the two-dimensional harmonic oscillator in Cartesian coordinates, $\psi_0(x,y)$. (*Hint:* The *x* and *y* directions are independent.)

B. Determine the energy eigenvalue of this state.

C. Determine the energy eigenfunction for the first excited-state of the two-dimensional harmonic oscillator in Cartesian coordinates.

D. Determine the energy eigenvalue of this state.

E. Is either energy level degenerate? (*i.e.*, Is there more than one energy eigenfunction associated with one of the energy eigenvalues you identified above?) If so, determine the degeneracy.

✓ Check your results with a tutorial instructor.

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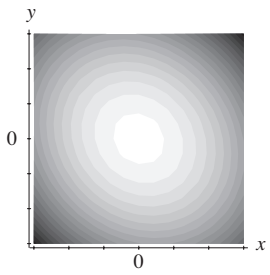
QM 2 *Degenerate time-independent perturbation theory*

II. An anisotropy

A small anisotropy is added to the potential as illustrated by the contour plot at right. The new Hamiltonian is now:

$$H = -\frac{\hbar^2}{2M}\nabla^2 + \frac{1}{2}M\omega^2\left(x^2+y^2\right) + \gamma\frac{1}{2}M\omega^2(x+y)^2,$$

where γ is small and positive.



A. The full Hamiltonian can be written as the Hamiltonian for the isotropic oscillator plus a perturbation: $H = H^{(0)} + H'$. What is H'?

B. Is the first-order correction to the energy of the ground state *positive*, *negative*, or *zero?* If you do not have enough information to tell, state so explicitly. Explain.

The first-order correction to the energy of a state is usually written as $\left\langle \psi_n^{(0)} \middle| H' \middle| \psi_n^{(0)} \right\rangle$. When the unperturbed system is degenerate, you must calculate all terms of the form $\left\langle \psi_i^{(0)} \middle| H' \middle| \psi_j^{(0)} \right\rangle$, where $\left\{\psi_i^{(0)}\right\}$ is a set of unperturbed degenerate wave functions.

C. For an energy level with degeneracy of *n*, how many correction terms of the form $\left\langle \psi_i^{(0)} \middle| H' \middle| \psi_j^{(0)} \right\rangle$ can be calculated? Explain.

D. What is the degeneracy of the first excited-state of the unperturbed potential? Write each degenerate state in terms of $\varphi_0(x)$ and $\varphi_1(x)$. Label these states $\left\{\psi_i^{(0)}\right\}$, as above.

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E. Write the first-order correction terms for the first excited-state wave functions. (Do not calculate any integrals.)

Organize these terms as a 2x2 matrix. Explain why a matrix is the correct mathematical object for terms of this form.

Ask an instructor for a handout that will let you check your answers above.

F. Find the eigenvalues and eigenvectors of the matrix.

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These eigenvectors represent linear combinations $\left\{\tilde{\psi}_i^{(0)}\right\}$ of the original (unperturbed) first excited-state wave functions $\left\{\psi_i^{(0)}\right\}$.

G. Rewrite these eigenvectors in terms of $\left\{\psi_i^{(0)}\right\}$ and also of $\varphi_0(x)$ and $\varphi_1(x)$. Label these states $\left\{\tilde{\psi}_i^{(0)}\right\}$.

H. Are the states $\left\{\tilde{\psi}_i^{(0)}\right\}$ energy eigenstates of the *unperturbed* system? If so, determine their energy eigenvalues. Explain.

I. Determine the first-order correction to the energy of the states represented by $\left\{\tilde{\psi}_i^{(0)}\right\}$. Explain. (*Hint:* What do the eigenvalues from your matrix on the previous page represent?)

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J. Consider the following student statements. For each statement, state whether you agree or disagree. Explain your reasoning.

1. "The first-order corrections to the energy calculated using degenerate perturbation theory apply to every degenerate state with the same unperturbed energy."

2. "Both $\left\{\psi_i^{(0)}\right\}$ and $\left\{\tilde{\psi}_i^{(0)}\right\}$ are valid first excited-states of the unperturbed potential, so it doesn't matter which set we pick to compute the first-order corrections to the **energy**."

The states $\left\{\tilde{\psi}_i^{(0)}\right\}$ are often called the *good* eigenstates in degenerate perturbation theory, because they are the basis in the degenerate space corresponding to the corrected energy. If these wave functions are corrected using perturbation theory, they will approach the correct energy eigenfunctions of the perturbed potential.

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B.III – Tutorials for Part III

This sub-section includes copies of both tutorials discussed in part III of this dissertation. Each tutorial is discussed in chapter 10.

ANGULAR MOMENTUM IN QUANTUM MECHANICS

I. Orbital angular momentum

A. Consider a particle in a three-dimensional potential that has quantum number $l = 3$.

1. Suppose you were to measure the magnitude of the orbital angular momentum vector for this particle. What value(s) could you get? Explain.

2. Can the magnitude of the orbital angular momentum vector for a particle be negative? Can the quantum number l for a particle be negative?

The orbital angular momentum vector satisfies the following eigenvalue equation: $\hat{L}^2 |l, m_l\rangle = l(l+1)\hbar^2 |l, m_l\rangle$ where l is a nonnegative integer.

3. Resolve any inconsistencies between this expression and your answers above.

B. Consider a *classical* vector $\vec{A}$ with magnitude 3.5. The direction of $\vec{A}$ is not known.

1. Determine the *largest* possible value for the z-component of $\vec{A}$.

2. Determine the *smallest* possible value for the z-component of $\vec{A}$. (*Hint:* Can a single component of a vector be negative?)

3. Determine the set of all possible values for the z-component of $\vec{A}$. Explain.

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C. Suppose you were to measure the magnitude of the z-component of the orbital angular momentum vector for a particle that has $l = 3$. Determine the set of all possible value(s) for this measurement. Explain. (*Hint:* Use your argument from part B, but recall the quantization condition for angular momentum in quantum mechanics.)

The z-component of the orbital angular momentum satisfies the following eigenvalue equation: $\hat{L}_z|l,m_l\rangle_z = m_l\hbar|l,m_l\rangle_z$ where m_l is an integer with $|m_l| \le l$.

D. Determine the set of all possible values that could result from a measurement of the x-component of the particle's angular momentum. Assume no other measurements have been made. Explain. (*Hint:* Is there anything unique about the z-direction?)

Would your answer change if you were to consider a measurement of the component of angular momentum along an arbitrary axis (*e.g.*, the y-axis, or halfway between the x- and z-axes)?

E. Do you agree or disagree with the following statement? Explain.

"When we measure the z-component of L, we always get something less than the total L, so angular momentum can never point in the z-direction. But if we happened to choose the direction in which L did point, then we could measure the full magnitude."

F. Consider a particle for which only the quantum number m_l is known. What are the possible values of the quantum number l for this particle? Explain.

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✓ Check your results with a tutorial instructor.

II. Spin angular momentum

In addition to orbital angular momentum, quantum mechanical particles also possess spin angular momentum, which obeys the same formalism. The quantum number associated with $\hat{S}^2$ can take both integer or half-integer values (*e.g.*, $s = ½$ for an electron).

A. If you were to measure the *z*-component of spin angular momentum for an arbitrary electron, what are the possible results?

Would your answer change if instead you had measured a different component of the spin (*e.g.*, the *x*-component)? Explain.

The spin state of a particle can be defined by the quantum numbers s and m_s, where m_s corresponds to a component of the spin angular momentum vector. The state is often written as $|\chi\rangle = |s, m_s\rangle_i$, where i specifies the direction of the given m_s component. When the state is represented in the *z*-direction, the subscript i is often omitted.

B. Assume an electron is in the initial state $|\chi\rangle = |½, -½\rangle_z$.

1. Determine two quantities related to spin that are well-defined for this electron, and the value of each. Explain.

2. Is the *x*-component of spin angular momentum for this electron (S_x) well-defined? Explain. (*Hiint:* Consider the uncertainty principle.)

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C. Suppose that the result of a measurement of S_x is $+\hbar/2$.

1. Using Dirac notation, write the spin state of the electron after the measurement.

2. If you were now to measure S_z, what would be the possible results? Explain.

The state of a particle with spin may be written in terms of the eigenstates of the spin operator for any direction. For example, the state from question C1 can be written in terms of the eigenstates of S_x instead of those of S_z as follows: $\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle_x = \frac{1}{\sqrt{2}}\left(\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle_z + \left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle_z\right)$.

3. Determine the probability associated with each of the possible results you identified in the previous question. Clearly show your work and explain your reasoning.

D. Do you agree or disagree with the following statement? Explain.

"In the initial state, we knew the *z*-component of spin exactly. Since *x* and *z* are orthogonal, measuring the *x*-component of spin doesn't affect the *z*-component at all, so when we re-measure that component we still get $-\hbar/2$."

E. Which of the following determines the possible results of a measurement in quantum mechanics: the *initial* state of the system or the state of the system *immediately before* the measurement? Explain.

Check your results with a tutorial instructor.

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III. Angular momentum and uncertainty

Consider a particle with the initial orbital angular momentum state $|\psi\rangle = |l, m_l\rangle_z = |2,1\rangle_z$.

A. Determine two quantities related to orbital angular momentum that are well-defined for this particle, and the value for each. Explain.

B. Is the *x*-component of the orbital angular momentum, L_x, well-defined for the initial state given above? Explain.

If L_x is well-defined, what is its value? If it is not well-defined, determine the possible results of a measurement of the *x*-component of the orbital angular momentum. Explain.

What additional information would you need to determine the probabilities of the possible results of this measurement? Explain.

C. Suppose instead that you were to measure the *y*-component of the orbital angular momentum, L_y. Would your answers to part B change? Explain.

Suppose that a measurement of the *y*-component of the orbital angular momentum of the initial state above resulted in $+\hbar$.

D. Consider the student discussion below.

Student 1: "We started knowing the *z*-direction was $1\hbar$, and now we know the *y*-direction is also $1\hbar$. The magnitude of angular momentum is only $2\hbar$, so the *x*-component must be 0."

Student 2: "Angular momentum squared is $l(l+1)\hbar^2$, which for this state is $6\hbar^2$. I know that $L^2 = L_x^2 + L_y^2 + L_z^2$, so if *y* and *z* are both $1\hbar$, I get that the *x*-component must be $2\hbar$."

Neither statement is completely correct. Explain where each statement is incorrect.

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QM 6 ***Angular momentum in quantum mechanics***

Suppose that a measurement in the *x*-direction resulted in –2ħ; *e.g.*, $|\psi\rangle = |l, m_l\rangle_x = |2,-2\rangle_x$.

E. Suppose you made a second measurement, this time of the *z*-component.

1. *Predict* the ranking of the probabilities of all the possible results of this measurement from most probable to least probable. Explain your reasoning.

The angular momentum state of this particle can be rewritten in terms of the *z*-basis, as follows: $|2,-2\rangle_x = \frac{1}{4}\left(|2,2\rangle_z - 2|2,1\rangle_z + \sqrt{6}|2,0\rangle_z - 2|2,-1\rangle_z + |2,-2\rangle_z\right)$.

2. Use the expression above to determine the probability associated with each value of L_z for this particle. Show your work. Resolve any inconsistencies with your prediction.

F. Which of the possible outcomes of a measurement of L_z would be impossible to measure for a classical system? Explain.

G. Justify why the probabilities are the same for positive and negative results with the same absolute value. Do you expect this to be true in general?

H. Justify why the probabilities are *not* equally distributed among all possible values. Do you expect this to be true in general?

According to the uncertainty principle, only one of the components of angular momentum in quantum mechanics may be known precisely at any one instant. If a particle is in an eigenstate of $\hat{L}_z$ then a measurement along a perpendicular axis, such as $\hat{L}_x$, can (usually) result in any allowed eigenvalue of $\hat{L}_x$. You will investigate this further in the homework.

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ADDITION OF ANGULAR MOMENTUM

QM 1

I. Total angular momentum

Recall that quantum mechanical particles can possess *both* orbital, $\hat{\vec{L}}$, and spin, $\hat{\vec{S}}$, angular momentum. The state of such a particle may be written in a basis of the eigenvectors of $\hat{L}^2$, $\hat{L}_z$, $\hat{S}^2$, and $\hat{S}_z$: $|\psi\rangle \otimes |\chi\rangle = |l, m_l\rangle \otimes |s, m_s\rangle \equiv |l, m_l; s, m_s\rangle$.

A. Consider an electron in the angular momentum state $|l, m_l; s, m_s\rangle = |2,0; \tfrac{1}{2}, \tfrac{1}{2}\rangle$.

1. Determine four quantities related to angular momentum that are well-defined for this electron, and the value of each. Explain.

2. What must be true about these four quantities if they are all well-defined?

3. For the state above, do the *x*- or *y*-components of the orbital angular momentum have well-defined values? Explain.

   Is your answer the same for the *x*- and *y*-components of the spin angular momentum?

The *total* angular momentum, $\hat{\vec{J}}$, for a particle is defined as the vector sum of all forms of angular momentum. For a single quantum mechanical particle, this takes the form $\hat{\vec{J}} = \hat{\vec{L}} + \hat{\vec{S}}$.

B. Using $\hat{\vec{J}} = \hat{\vec{L}} + \hat{\vec{S}}$, write down separate equations for each of the three components of $\hat{\vec{J}}$ (*e.g.*, $\hat{J}_z$) in terms of the components of $\hat{\vec{L}}$ and $\hat{\vec{S}}$.

1. Is the *z*-component of the total angular momentum $\hat{J}_z$ for the electron in part A well-defined? If so, give the value; if not, list all possible values. Explain.

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QM 2 ***Addition of angular momentum***

The questions on this page refer to the electron with the state $|l, m_l; s, m_s\rangle = |2,0; ½, ½\rangle$.

2. Is the *x*-component of the total angular momentum $\hat{J}_x$ for this electron well-defined? If so, give the value; if not, list all possible values. Explain.

3. Is the *y*-component of the total angular momentum $\hat{J}_y$ for this electron well-defined? If so, give the value; if not, list all possible values. Explain.

4. Check that your answers are consistent with the fact that only a single component of angular momentum can well-defined at the same time.

Consider the quantum numbers *j* and m_j associated with $\hat{J}^2$ and $\hat{J}_z$, respectively. Note that $\hat{\vec{J}}$ obeys the same formalism as other forms of angular momentum, such as $\hat{\vec{L}}$ and $\hat{\vec{S}}$.

C. Determine the value of m_j for this electron. Is this quantity well-defined? Explain.

D. For the electron in this state, will *j* be an integer or a half-integer? Explain.

E. *Predict* whether the magnitude of the total angular momentum vector, $\hat{\vec{J}}$, for this electron will be well-defined. Explain.

✓ Check your results with a tutorial instructor.

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II. Addition of angular momentum

In this section, you will make a classical analogy as a guide to determining the allowed values of the quantum number *j* associated with the magnitude of the total angular momentum.

A. Consider two classical vectors, $\vec{A}$ and $\vec{B}$ of lengths 5 and 2, respectively. There is no knowledge of the individual components or directions of the two vectors. Let $\vec{C} = \vec{A} + \vec{B}$.

1. What is the largest possible value for the magnitude of the vector $\vec{C}$? Draw an alignment of the vectors $\vec{A}$ and $\vec{B}$ that could result in this value.

2. What is the smallest possible value for the magnitude of the vector $\vec{C}$? Draw an alignment of the vectors $\vec{A}$ and $\vec{B}$ that could result in this value.

3. What are all the possible values for the magnitude of the vector $\vec{C}$? Illustrate the possible alignments of the vectors $\vec{A}$ and $\vec{B}$ that could result in these values. Explain.

B. Consider again the electron in the state $|l, m_l; s, m_s\rangle = |2,0; \tfrac{1}{2}, \tfrac{1}{2}\rangle$.

1. Determine the length of the *orbital* angular momentum vector, $\hat{\vec{L}}$. Approximate this value to one decimal in terms of $\hbar$. *(Hint:* It is *not* just $l\hbar$.*)*

2. Determine the length of the *spin* angular momentum vector, $\hat{\vec{S}}$. Approximate this value to one decimal in terms of $\hbar$.

3. Suppose $\hat{\vec{J}}$ were a *classical* vector. Determine both the largest and the smallest possible values for its magnitude. Also, find the largest and possible values for $\hat{J}^2$. Approximate these values to one decimal in terms of $\hbar$ and enter them in the table at right.

Quantity	Largest value	Smallest value
$\hat{J}$		
$\hat{J}^2$		

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QM 4 ***Addition of angular momentum***

C. For each of the given values for the quantum number j, fill in the table below for the value of $\hat{J}^2$. Approximate each value to one decimal in terms of $\hbar$.

Quantum number	j	0	1/2	1	3/2	2	5/2	3	7/2
Value of $\hat{J}^2$	$j(j+1)$								
Approximation									

Is it possible to measure values for $\hat{J}^2$ that do not appear in the table (*i.e.*, values that lie in between the values shown)? Explain.

Consider again the electron in the state $|l,m_l;s,m_s\rangle = |2,0;\tfrac{1}{2},\tfrac{1}{2}\rangle$.

D. Recall your answer to question D from section I. Can j be a half-integer for this electron? Can it be an integer? Cross out any values in the table above that are *not allowed* for this electron.

E. Cross out any values in the table above that would not be *classically allowed* for this electron. Check that your answer is consistent with the table in question B3.

F. For this electron, list all the possible values of $\hat{J}^2$ and its corresponding quantum number, j. Explain.

G. Consider the discussion between two students below.

Student 1: "Originally, we knew that $l = 2$ and $s = 1/2$. Since $J = L + S$, we just add l and s together to get j, which would be equal to 5/2 for this particle."

Student 2: "You can't do that because J, L, and S are vectors. Since L and S could point in any direction, the magnitude of J could be any number between the magnitude of $L - S$ and the magnitude of $L + S$, which for this particle would be $1.5 < J < 3.3$."

Both students are incorrect. Identify the flaws in each student's reasoning. Explain.

✓ Check your results with a tutorial instructor.

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III. The total angular momentum basis

Since angular momentum is quantized, the quantum number *j* associated with the total angular momentum vector $\hat{\vec{J}} = \hat{\vec{L}} + \hat{\vec{S}}$ can take values from $|l - s|$ to $|l + s|$ in *integer* steps.

A. Use this formula to determine the possible values of the quantum number *j* for an electron in the state $|l, m_l; s, m_s\rangle = |2,0; ½, ½\rangle$.

Is your answer consistent with your answer from the previous section?

B. Is there a single, well-defined value for this quantum number for this electron? Explain.

The state of a particle with both spin and orbital angular momentum may be written not only in the $|l, m_l; s, m_s\rangle$ basis, but also in the following different basis: $|\psi\rangle = |l, s; j, m_j\rangle$.

C. Which four quantities related to angular momentum can be well-defined for a particle written in this basis? Explain why this is possible.

The $|l, s; j, m_j\rangle$ basis is often abbreviated as $|j, m_j\rangle$ (omitting the quantum numbers *l* and *s*). This can make it easier to distinguish between this basis and the $|l, m_l; s, m_s\rangle$ basis. Note that *l* and *s* are still known and relevant even though they are omitted from the notation. In the tutorials, we will specify which form of this basis will be used to represent states.

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D. Write down a general expression (*i.e.*, with undetermined coefficients) in the $\left|l,s;j,m_j\right\rangle$ basis for the total angular momentum state of the electron defined at the top of the page. Explain. (*Hint:* How many terms should there be in your expression?)

E. Should the probabilities associated with each term in your expression be equal?

F. The undetermined coefficients in your expression are known as Clebsch-Gordon coefficients; they can be found in a table on page 188 of *Introduction to Quantum Mechanics 2nd Edition* by David Griffiths. Use this table to find the coefficients in your expression above.

Resolve any inconsistencies with your other answers.

The formalism introduced in this tutorial for the addition of angular momentum is valid for any sum of two (or more) angular momentum states. Expressions such as $\hat{\vec{L}}_1+\hat{\vec{L}}_2$, $\hat{\vec{S}}_1+\hat{\vec{S}}_2$, and $\hat{\vec{S}}_1+\hat{\vec{S}}_2+\hat{\vec{S}}_3+\ldots$ all obey this formalism.

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B.IV – Additional Tutorials

This sub-section includes copies of those tutorials that are not discussed at length in this dissertation.

BLACKBODY RADIATION

QM 1

I. Light emitted from an object with changing temperature

Ask an instructor to show you a video before answering the following questions.

A. Describe what happens to the iron nail before it breaks. In your answer, specify (*i*) what is physically happening inside the iron nail and (*ii*) how this is related to the curve displayed to the right of the video.

B. Explain why the nail breaks. (*Hint:* What physical property of iron is related to what is happening when it breaks?)

C. Describe what happens to the iron nail after it breaks. In your answer, specify (*i*) what is physically happening inside the iron nail and (*ii*) how this is related to the curve displayed on the right.

D. Light bulb filaments are made of tungsten. What property of tungsten makes this metal better for light bulbs than iron? Explain.

✓ Discuss your answers with an instructor.

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II. Incandescent light bulbs

A. Sketch the spectrum for a light bulb on the axes shown below. You may find the simulation at the website http://phet.colorado.edu/en/simulation/blackbody-spectrum helpful.

1. Discuss the labels you chose for the horizontal and vertical axes.

2. Clearly label the location of the visible part of spectrum on your graph. Specify the approximate locations of the red and blue parts of the visible spectrum. Should the peak of your curve be *to the right, to the left,* or *within* the visible spectrum? Explain.

3. Explain how to use the graph to determine the total power emitted by the light bulb.

4. What change to the graph would make this curve more efficient at emitting visible light? What physical changes might cause this result? Explain.

5. Are there any objects that are more efficient at emitting visible light than an incandescent light bulb? If so, identify any major differences between them.

✓ Discuss your answers with an instructor.

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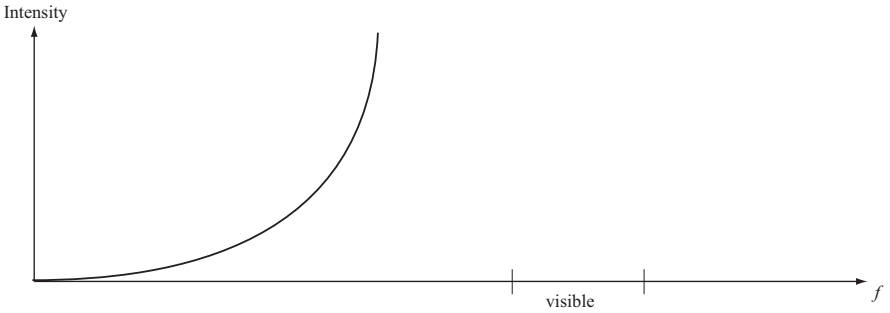
III. The ultraviolet catastrophe

Below is a sketch of a classical prediction for the spectrum of a *blackbody* (such as an incandescent light bulb) at a particular temperature made by the Rayleigh-Jeans Law, which is a consequence of the laws of thermodynamics. The Rayleigh-Jeans Law is only a good approximation at *low frequencies*.



A. The graph above shows the intensity of light emitted by a blackbody *per unit frequency*. Some graphs instead show the intensity emitted per unit wavelength.

1. How would a graph of the Rayleigh-Jeans Law per unit wavelength differ from the graph shown above? Explain.

2. Label the approximate locations of red and blue light on the horizontal axis above. How would these locations differ for a graph of wavelength? Explain.

B. Make a sketch of the *correct* spectrum for a blackbody at this temperature on the same axes.

1. Discuss any similarities or differences between the two graphs.

2. Suppose you wanted to use the Rayleigh-Jeans Law to calculate the total power emitted by this blackbody. What result would this calculation yield? Explain.

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QM 4 ***Blackbody radiation***

The fact that the Rayleigh-Jeans Law predicts that a blackbody will emit an infinite amount of total power is known as *the ultraviolet catastrophe.*

3. Explain why this name is appropriate for this result.

4. Do you agree or disagree with the following statement? Explain.

"The problem with the Rayleigh-Jeans Law is that when you heat something up to a high enough temperature, it emits more and more power until the curve diverges. This is why it was called the ultraviolet catastrophe—because it only happens at high temperatures."

The completely correct curve for the spectrum of a blackbody is described by Planck's Law. Planck's Law is calculated from the same thermodynamic principles as the Rayleigh-Jeans Law, but under the additional assumption that light can only be emitted in discrete (or quantized) packets of energy. These packets of energy are known as photons.

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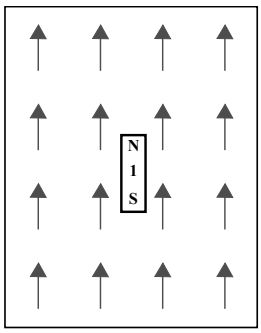
SPIN

QM 5

I. Review of magnets

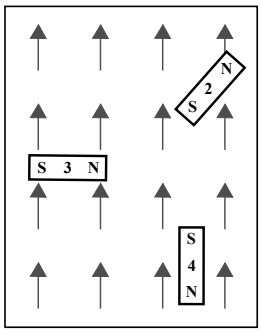
A. A bar magnet is placed in the uniform magnetic field shown at right.



1. Draw a vector indicating the direction of the magnetic dipole moment of the magnet.

2. Determine the direction of the net force exerted on the magnet by the external magnetic field. If the net force is zero, state so explicitly. Explain.

B. Three identical magnets are shown in the same magnetic field, each with a different orientation.

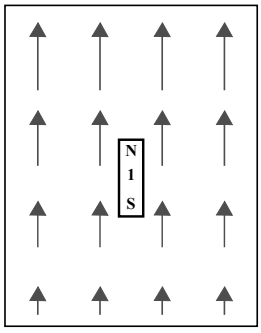


1. Draw a vector indicating the direction of the magnetic dipole moment of each magnet.

2. Does the magnetic force on a bar magnet due to a uniform magnetic field depend on the orientation of the magnet? Explain.

3. Would your answer to the previous questions change if the magnets had been moving with velocity v to the right? Explain.

C. An identical magnet is now placed in a *non-uniform* magnetic field as shown. The gradient of the magnetic field is constant.



1. Determine the direction of the net force exerted on the magnet by a *non-uniform* magnetic field. If the net force is zero, state so explicitly. Explain.

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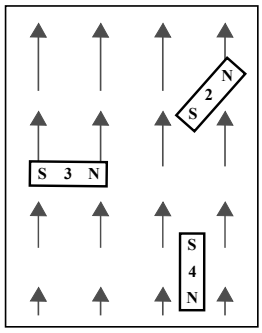
QM 6 ***Spin***

2. Three identical magnets are shown in the same non-uniform magnetic field, each with a different orientation. Determine the direction of the magnetic force on each magnet. Explain.



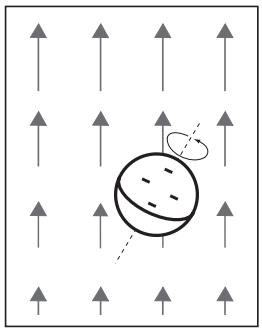
3. Rank the four magnets by the magnitude of the force exerted on each magnet by the external magnetic field. If any of the forces are zero, state so explicitly. Explain.

4. Consider the following student statement:

   The force can be different magnitudes depending on which way the magnetic moment points. If I know the force on a magnet, I can fully determine the direction of the magnetic moment.

   Do you agree or disagree with this statement? Explain.

D. A solid spinning sphere of negative charge is placed in the same non-uniform magnetic field as shown.



1. Draw a vector indicating the direction of the magnetic dipole moment of the sphere. (*Hint:* What direction is the current?)

2. Determine the direction of the net force exerted on the sphere by the external magnetic field. Explain.

3. Would your answer to the previous question change if the sphere were moving with velocity v to the right? If so, describe how to use an electric field to negate the part of the force due to the velocity.

✓ Check your results with an instructor.

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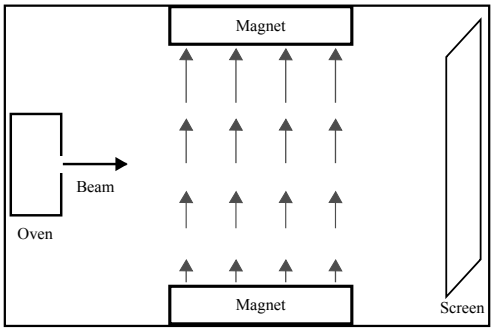
II. The Stern-Gerlach experiment (classical)



In the experiment at right, a beam of objects is sent through a non-uniform magnetic field such as the one in section I. The particles are then detected on the screen on the far right. Such a setup is known as a Stern-Gerlach experiment. The magnetic field points in the *z*-direction.

The beam is composed of identical neutral particles with nonzero magnetic moments of equal magnitude, such as the bar magnets from Section I. The particles emerge from the oven one at a time with a magnetic moment that points in a random direction.

A. The first particle strikes the screen in the center. Did this particle experience a force due to the magnetic field? If so, determine the direction of this force.

What can you say about the direction of the magnetic moment of this particle? Can you specify *all three* components of the magnetic moment?

B. The next three particles strike the screen as indicated below.

Particle 2 strikes the screen below the first particle.

Particle 3 strikes the screen below particle 2.

Particle 4 strikes the screen above the first particle, equally far from the center of the screen as particle 3.

In the table at right, indicate (1) the direction of the net force on each particle due to the magnetic field and (2) a possible direction for the magnetic moment of each particle.

#	Net Force	Magnetic Moment
1		
2		
3		
4		

C. Predict the pattern that will appear on the screen after a long time. Draw a picture of what you expect to see on the screen. Explain how you arrived at your answer.

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D. What does the location where a particle strikes the screen tell you about the magnetic moment of that particle?

E. A different kind of particle is sent through an identical apparatus in the same fashion as described above. After you have waited a very long time, you find that all of these particles have hit the center of the screen. What can you say about the magnetic moment of these particles?

✓ Check your results with an instructor.

III. The Stern-Gerlach experiment (quantum mechanical)

In the previous section, you examined the classical consequences of the Stern-Gerlach experiment. You will now consider the same situation with quantum mechanical particles.

A. A Stern-Gerlach experiment is conducted on a beam of randomly oriented electrons. Predict what will appear on the screen after a large number of electrons have passed through the magnetic field. Draw a picture of what you expect to see on the screen. Explain.

B. Ask an instructor for a handout showing the results of this experiment.

1. Are these results consistent with your prediction? Explain.

2. Are these results consistent with the classical picture from the previous section? Explain.

3. What information about the electron and its magnetic momentum can you draw from this information? Explain.

The electron has a magnetic moment due to a property known as *spin*. It is analogous to the classical case of a charged sphere spinning about an axis through its center, although there is no experimental evidence that there is something spinning inside an electron. Quantum mechanical *spin* is different from classical spinning in additional ways that will be explored in this course.

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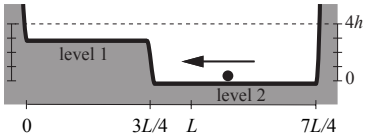
CLASSICAL PROBABILITY

QM 1

I. Probability

A ball rolls back and forth on a track with very steep sides. Two levels of unequal length are joined by a steep ramp to form the base of the track. Assume that the time spent on the steep portions is negligible, that there is no friction in the system, and that the ball rolls smoothly, without bouncing, forever. Level 1 has a length of $3L/4$, and level 2 has a length of L. Assume the ball was dropped from a height of $4h$.



A. Is the speed of the ball on level 1 *greater than, less than,* or *equal to* its speed on level 2? Explain.

Assume the speed of the ball on level 1 is v. Calculate the speed of the ball on level 2.

B. Is the amount of time that the ball spends on level 1 *greater than, less than,* or *equal to* the amount of time it spends on level 2? Explain.

C. Suppose a single photograph were taken at a random time. On the basis of your results above, is the probability of the photograph showing the ball on level 1 *greater than, less than,* or *equal to* that of the photograph showing the ball on level 2? Explain.

D. Determine the probability of finding the ball on each level. Explain.

✓ Check your results with an instructor.

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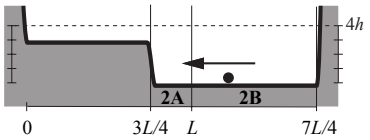
QM 2 *Classical Probability*

II. Probability density



Imagine splitting level 2 into two unequal segments: segment "2A" from $x = 0.75L$ to $x = L$, and segment "2B" from $x = L$ to $x = 1.75L$.

A. Find the probability that the ball is found in the indicated region (out of anywhere in the system). Explain how you got your answer.

1. Segment 2A

2. Segment 2B

Consider the following ratio for each segment: the probability of finding the ball within that segment divided by the length of that segment.

B. Is the above ratio for segment 2A *greater than*, *less than*, or *equal to* the above ratio for segment 2B? Explain.

The ratio defined above is called *probability density*.

C. In your study of physics, you may have encountered different kinds of density. For example, (linear) charge density may be interpreted as the amount of charge in a region of unit length.

1. How are charge density and probability density similar?

2. Consider the student discussion below.

Student 1: "If the probability in one region is larger than another region, then the probability density must also be larger."

Student 2: "I disagree. We are dividing by length, so a smaller region will always have a larger probability density."

Student 3: "Both of you are partially right. I can only really compare the probability density in two regions if I know both the probability and the length of the two regions."

With which student do you agree, if any? Explain.

✓ Check your results with an instructor.

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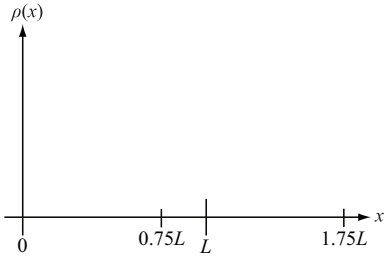
III. Probability density graphs

A. In the space at right, carefully draw a graph of *probability density*, $\rho(x)$, versus *position* from $x = 0$ to $x = 1.75L$. Label relevant values on the vertical axis.



B. How would you determine the probability of finding the ball between $x = L/2$ and $x = L$? Explain.

C. What is the probability of finding the ball *exactly* at $x = L$? Explain.

D. What answer would you expect for the probability of finding the ball between $x = 0$ and $x = 1.75L$? Show that your graph of probability density gives you the answer you expect.

E. Suppose you were given an arbitrary probability density function $\rho(x)$ (*i.e.*, one that does not have a shape as simple as the one above). Write a mathematical expression for the probability of finding the ball between $x = x_1$ and $x = x_2$.

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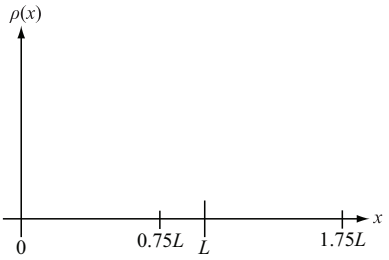
F. Do you agree or disagree with the statement below? Explain.

"The probability densities have to add up to one, just like the probabilities, because it has to be somewhere."

✓ Check your results with an instructor.

IV. Time-dependent probability densities

A. Suppose that at time t_1 the ball is located at $x = L$ and moving to the right. Sketch the probability density for the ball at time t_1. *(Hint:* What is the probability of finding the ball at $x = L$*?)*



B. On the same graph, sketch the probability density for the ball a short time later. Explain.

C. Explain how the graph of the probability density above changes with time.

When a graph such as the one above gives the probability density for different values of time, this is said to be a *time-dependent probability density*, $\rho(x, t)$.

D. How is $\rho(x, t)$ related to the graph you drew in section III? Explain.

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RELATING CLASSICAL MECHANICS TO QUANTUM MECHANICS

I. The Schrödinger equation and the shape of the wave function

The Schrödinger equation in one dimension is given by $i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi$. Its solutions, in general, can be written as $\Psi(x,t) = \sum c_n\psi_n(x)e^{-iE_nt/\hbar} \equiv \sum c_n\Psi_n(x,t)$. In this tutorial we consider only solutions in the form $\Psi(x,t) = \Psi_n(x,t)$, where the functions $\psi_n(x)$ satisfy the time independent Schrödinger equation, $\frac{d^2\psi(x)}{dx^2} = -\frac{2m}{\hbar^2}\left[E - U(x)\right]\psi(x)$.

A. What can be said about the form of the function, $\psi(x)$, if the total energy in a given region is:

1. greater than the potential energy in that region?

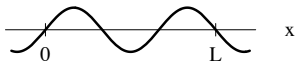
2. less than the potential energy in that region?

3. equal to the potential energy in that region?

B. In those region(s) in which the function $\psi(x)$ is sinusoidal, it may be expressed mathematically in the form, $\psi(x) = A\cos kx + B\sin kx$, where k is a constant.

1. In describing periodic waves, k is often called the *wave number*. Give an interpretation of the wave number. (*Hint:* How is the wave number related to the *wavelength* λ of a sinusoidal wave?)

2. Find the wave number, k, in terms of L for the following wave.

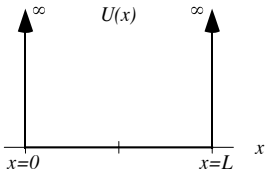


3. How is the wave number related to quantities that appear in the Schrödinger equation?

✓ Check your results with a tutorial instructor.

II. The infinite square well potential

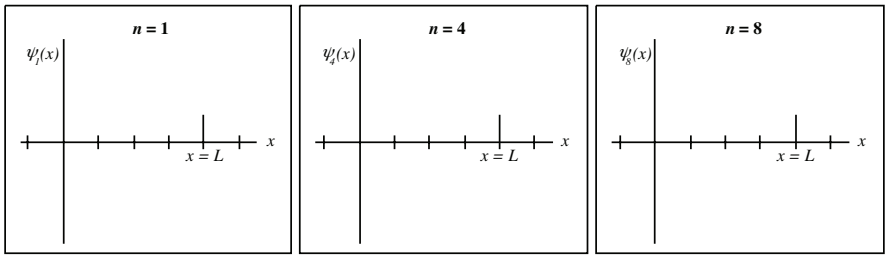
Consider a particle with total energy E in an infinite square well potential of width L. (As shown in the graph, the potential energy rises abruptly from zero to a very large value, essentially infinite, at $x = 0$ and at $x = L$.)



The allowed energy states correspond to the functions, $\psi_n(x)$, and energy eigenvalues, E_n, below:

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right); \qquad E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$$

A. Sketch the function, $\psi_n(x)$, that corresponds to a particle in each of the energy states below. Label the vertical axis with the appropriate values for each case.



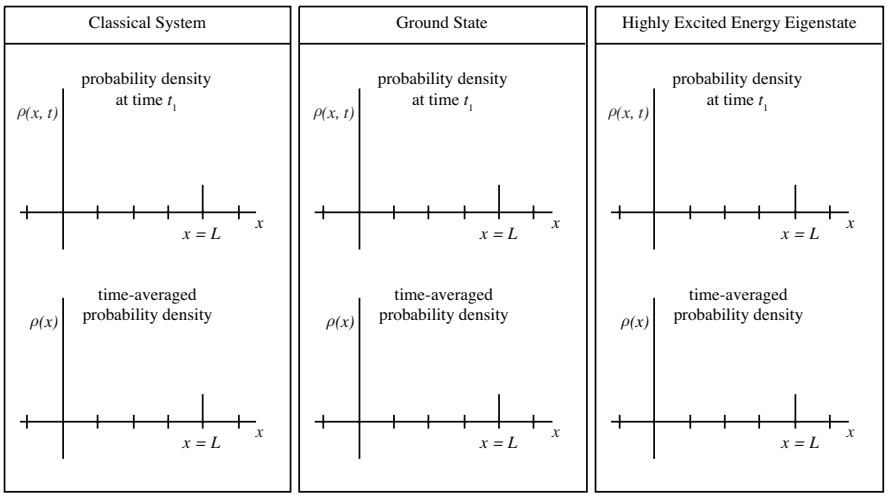
B. What happens to the *wave number* of the function as the energy increases?

C. Write the full wave functions, $\Psi(x,t)$, for each of the states above (*e.g.*, $\Psi_1(x,t), \Psi_4(x,t)$, …).

D. Find the probability density as a function of position and time, $\rho(x,t)$, for each of the states above.

III. Classical and quantum systems compared

A. In this section we consider the quantum mechanical infinite square well, as well as its classical analog having the same potential. For each of the systems below complete the graphs of (1) probability density at some arbitrary time, t_1, and (2) the time-averaged probability density. Explain your reasoning.



B. The special states that we have been considering (*i.e.*, those that can be written as $\Psi_n(x,t)$, rather than $\Psi(x,t) = \sum c_n \Psi_n(x,t)$) are often given the name *stationary states*. Why do you think this is appropriate?

C. Suppose that you have been given a large ensemble of identically prepared systems all corresponding to a single one of the three possible cases above. Imagine that you also have a large number of detectors that you could use to measure position.

1. If, for each system, you made a measurement of position at the same time t_1, could you decide to which case your ensemble corresponds? Explain.

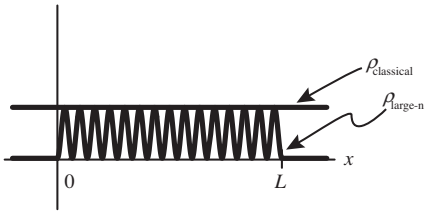
2. If, instead, you made measurements of position at random times, could decide to which case your ensemble corresponds? Explain.

3. Would your answers to parts 1 and 2 change if the resolution of your detectors changed (*i.e.*, was made better or worse)?

Quantum mechanics was developed in order to account for phenomena that classical mechanics could not. If it is to be regarded as a complete theory, it must also be able to account for systems that classical mechanics can describe. This idea underlies the *correspondence principle.* A consequence of the correspondence principle is that the time average probability density for a highly excited system should be experimentally indistinguishable from that of the analogous classical system.

D. Consider the following discussion between two physics students:

Student 1: "For a classical particle in a box of length L, the time-averaged probability density is uniform along the length of the box and is equal to (1/L). I have drawn a time-averaged quantum mechanical probability distribution corresponding to large n. As you can see this probability distribution approaches the classical one just as the correspondence principle says it should.



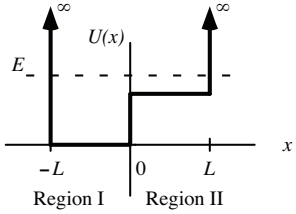
Student 2: "No, the area under the quantum mechanical distribution that you have drawn is half that of the classical probability distribution. That violates the correspondence principle.

Which student, if either, is correct? Identify any errors made by either student.

✓ Check your results with a tutorial instructor.

IV. Using classical mechanics to reason about highly excited energy eigenstates

A particle of energy E is in a bound state of the idealized potential well shown at right.



A. First, treat this problem from a purely ***classical*** standpoint.

1. Give an example of a real physical situation that corresponds to this potential well.

2. In which region of the well would the particle have greater kinetic energy? Explain.

3. In which region of the well would the particle spend more time? Explain.

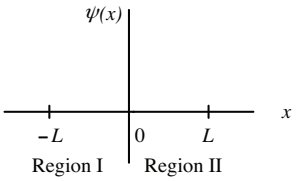
B. Now treat the problem from a quantum mechanical point of view. Consider specifically the case in which the particle occupies a *highly excited* energy eigenstate of the well for which the function $\psi_n(x)$ has many nodes (*i.e.*, zero points) between $x = -L$ and $x = L$.

1. What general shape do you expect for the function $\psi_n(x)$ in each region of the well? Base your answers on part IA.

2. How would you expect the *amplitude* of the function $\psi_n(x)$ to compare in the two regions of the potential well? Base your answers on part IV A and the consequences of the correspondence principle. Explain.

3. How would you expect the *wave number* of the function $\psi_n(x)$ to compare in the two regions of the potential well? Explain.

4. Sketch a graph of this highly excited eigenstate.



SUPERPOSITION IN QUANTUM MECHANICS

I. Mixed states and superpositions

A. Consider the following statement about a quantum mechanical infinite square well where $\psi_1(x)$ and $\psi_2(x)$ are the ground state ($n = 1$) and first excited state ($n = 2$) wave functions.

"The wave function given by $\frac{1}{\sqrt{2}}\left(\psi_1(x)+\psi_2(x)\right)$ represents a lack of knowledge about the state of the system. The system is definitely in either the ground state or the first excited state. The wave function simply tells you that the probability is 1/2 that the system is really in the ground state and 1/2 that it is really in the first excited state."

Do you agree or disagree with this statement?

B. Consider an ensemble of particles in identical quantum mechanical infinite square wells. Half the particles in this ensemble are in the ground state; the other half are in the first excited state. We will call this the *mixed state ensemble*.

1. Sketch the possible wave functions for a particle in this ensemble.

Suppose you were to select a large number of particles from this ensemble and measure the position of each particle.

2. Will the number of particles you find in the left half of the well be *greater than*, *less than*, or *equal to* the number of particles you find in the right half of the well? Explain.

Now suppose you were to select *a single particle* at random from this ensemble and measure its position.

3. Is the probability that this particle is found in the left half of the well *greater than*, *less than*, or *equal to* the probability that it is found in the right half of the well? Explain.

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QM 2 ***Superposition in quantum mechanics***

C. Consider a different ensemble of particles in identical quantum mechanical infinite square wells. The wave function for each of the particles in this ensemble is given by $\frac{1}{\sqrt{2}}\left(\psi_1(x)+\psi_2(x)\right)$. We will call this the *superposition ensemble*.

1. Sketch the wave function for a particle in this ensemble.

Suppose you were to select a large number of particles from this ensemble and measure the position of each particle.

2. Will the number of particles you find in the left half of the well be *greater than, less than,* or *equal to* the number of particles you find in the right half of the well? Explain.

Now suppose you were to select *a single particle* at random from this ensemble and measure its position.

3. Is the probability that this particle is found in the left half of the well *greater than, less than,* or *equal to* the probability that it is found in the right half of the well? Explain.

D. If you were to select a single particle from one of the ensembles above at random, would you be able to determine whether the particle came from the *mixed state ensemble* or the *superposition ensemble?* Explain.

E. If instead you were to select a hundred particles from one of the ensembles above at random, would you be able to determine whether the particles came from the *mixed state ensemble* or the *superposition ensemble?* Explain.

F. Consider the statement at the start of section I. Do you agree or disagree with this student? Make sure your answer is consistent with your results in this section.

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II. Linear superpositions of eigenstates

Let the energy eigenfunctions and eigenvalues for a system be $\psi_n(x)$ and E_n, respectively. These functions satisfy the eigenvalue equation $\hat{H}\,\psi_n(x) = E_n\,\psi_n(x)$ where $\hat{H}$ is the Hamiltonian.

A. Consider the wave function given by $\frac{1}{\sqrt{2}}\left(\psi_1(x) + \psi_2(x)\right)$.

1. Is this state an eigenstate of the infinite square well? Explain. (*Hint:* Does it satisfy the eigenvalue equation given above?)

2. Could this state describe a particle in the infinite square well? Explain.

3. Consider the following student discussion.

   Student 1: "This state is a sum of eigenstates which makes it an eigenstate."

   Student 2: "No, it does not satisfy the eigenvalue equation, so it is not an eigenstate. That means it is not a valid state for a particle in this system."

   Student 3: "You're right that it is not an eigenstate, but it could still represent a particle. Any state that can be written as a sum of eigenstates is valid."

   With which, if any, of the students do you agree? Explain your reasoning. For each student who is incorrect, identify the flaws in that student's reasoning.

4. Write a general expression for an allowed wave function $\psi(x)$ in terms of the energy eigenfunctions of the system, $\psi_n(x)$. Explain.

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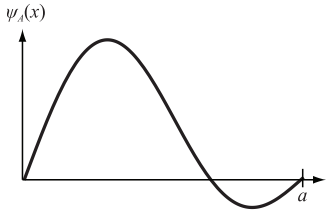
QM 4 ***Superposition in quantum mechanics***

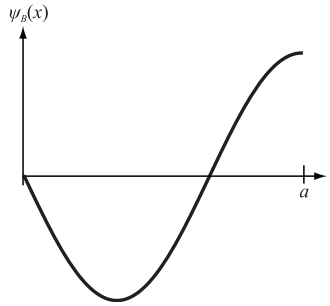
B. Consider the graph shown at right.



1. Could this graph be a possible wave function for a particle in the infinite square well? Explain.

2. Could this graph represent an *energy eigenstate* of the infinite square well? If so, determine which eigenstate. If not, could it be a sum of two or more eigenstates? Explain.

C. Consider a different graph shown at right.



1. Could this graph be a possible wave function for a particle in the infinite square well? Explain.

2. Could this graph represent an *energy eigenstate* of the infinite square well? If so, determine which eigenstate. If not, could it be a sum of two or more eigenstates? Explain.

3. Consider the following discussion between two students.

   Student 1: "$\psi_B(x)$ is an allowed wave function because it is made up of just a single sine function, which means it satisfies the Schrödinger Equation."

   Student 2: "But the boundary conditions for the square well demand that the wave function be zero at the boundaries, and $\psi_B(x)$ does not satisfy that. $\psi_A(x)$ satisfies the boundary condition, so it is a valid state."

   With which student, if either, do you agree? Explain your reasoning.

✓ Check your results with an instructor.

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IDENTICAL PARTICLES

QM 1

I. Two-particle wave functions

Consider a system of two non-interacting, distinguishable (*i.e.*, non-identical) particles in the same quantum mechanical infinite square well of width a. The wave function for this particle is given by $\Psi_A(x_a, x_b) = \psi_1(x_a)\psi_2(x_b)$, where x_a and x_b refer to the first and second particles, respectively, and $\psi_n(x)$ is a single-particle energy eigenfunction with energy eigenvalue E_n.

A. Can you determine which single-particle state is occupied by each particle? Explain.

B. Interpret the quantity $\left|\Psi_A(x_a, x_b)\right|^2 dx_a dx_b$ in your own words.

Consider the wave function $\Psi_B(x_a, x_b) = \psi_1(x_b)\psi_2(x_a)$.

C. How would your answer to question A change if you were to use this wave function instead? Explain.

How would your answer to question B change? Explain.

Consider the *improper* wave function $\Psi(x_a, x_b) = \frac{1}{\sqrt{2}}\left(\psi_1(x_a) + \psi_2(x_b)\right)$.

D. Explain why this wave function could not describe a two-particle system.

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II. Indistinguishable particles

Now consider the case in which particles *a* and *b* from section I are *identical, indistinguishable bosons*. In this case, the wave function $\Psi(x_a, x_b)$ must have the property that exchanging x_a and x_b does not change the wave function. A wave function with this property is called *symmetric under exchange*.

A. Consider the two wave functions from the previous section, $\psi_1(x_a)\psi_2(x_b)$ and $\psi_1(x_b)\psi_2(x_a)$. Is it possible for either of these wave functions to correctly describe indistinguishable bosons (*i.e.*, is either wave function symmetric under exchange)? Explain.

B. Determine a wave function $\Psi(x_a, x_b)$ that would be valid to describe a system of two bosons. (*Hint:* Consider how to combine the two wave functions from question A in a symmetric fashion.) Explain.

Now suppose the particles are *identical, indistinguishable fermions*. The wave function for fermions must have the property that exchanging x_a and x_b changes the wave function by introducing an overall minus sign (*i.e.*, $\Psi(x_a, x_b) = -\Psi(x_b, x_a)$). A wave function with this property is called *antisymmetric under exchange*.

C. Determine a wave function $\Psi(x_a, x_b)$ that would be valid to describe a system of two fermions. (*Hint:* What can you change about your wave function for identical bosons to go from symmetric to antisymmetric under exchange?) Explain.

D. Could a system of two identical particles have both particles in the same state (*e.g.*, both particles in the ground state of the one-particle system)? If so, determine the form of the wave function, and whether they could be bosons or fermions. Explain.

✓ Check your results with a tutorial instructor.

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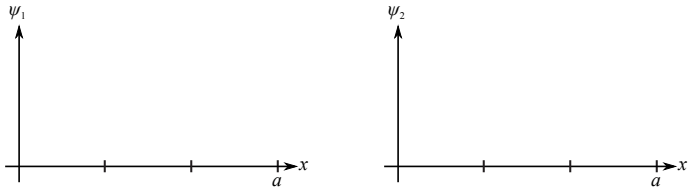
III. Properties of identical particles

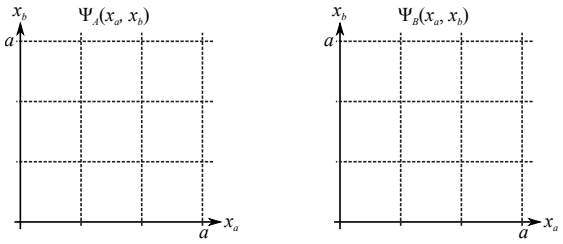
Consider two particles in a one dimensional infinite square well of width a. Let ψ_1 be the ground state of this well, and let ψ_2 be the first excited state.

A. Sketch $\psi_1(x)$ and $\psi_2(x)$ below.



B. Consider again the two-particle wave functions from the previous sections:
$\Psi_A(x_a, x_b) = \psi_1(x_a)\psi_2(x_b)$ and $\Psi_B(x_a, x_b) = \psi_1(x_b)\psi_2(x_a)$.

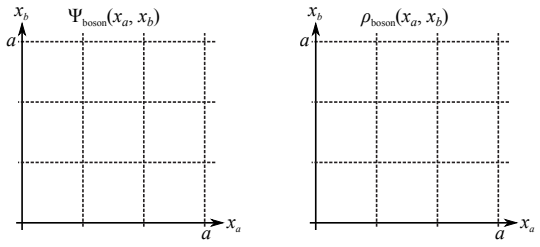
1. Represent both of these functions below by marking "+", "–", or "0" in each region to indicate whether the wave function at the center of each box is positive, negative, or zero.



2. Mark any lines of symmetry on the above plots.

C. Now consider the case of two indistinguishable bosons (see question B from section II).

1. Represent the spatial part of the two-particle wave function for bosons on the graph below as above. Do the same for the spatial part of the probability distribution.



2. Mark any lines of symmetry on the above plots.

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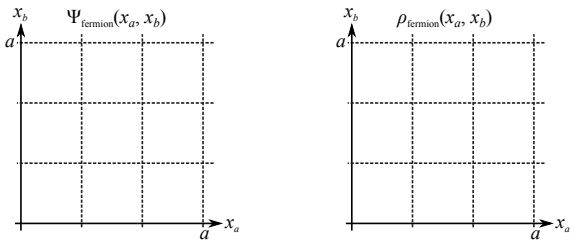
D. Now consider the case of two indistinguishable fermions (see question C from section II).

1. Represent the spatial part of the two-particle wave function for fermions on the graph below as above. Do the same for the spatial part of the probability distribution.



2. Mark any lines of symmetry on the above plots.

E. From your plots, are bosons or fermions more likely to be found in the same region of space? Explain.

In the previous questions, you considered only the spatial part of the wave function for a two-particle system. However, the symmetry condition on identical bosons and fermions applies to the entire wave function, which is often written as the product of a spatial wave function (such as those above) and a spin wave function.

F. Suppose that the fermions in the system above had an *anti-symmetric* spin state.

1. Determine the form of the spatial part of the wave function.

2. The following statement is incorrect. Identify the error(s) made by the student.

"It is never likely to find fermions in the same region of space; their wave function has to be anti-symmetric, so you will almost always find them far apart."

3. Resolve any inconsistency between your answer and part E above.

4. Would it be possible for these two fermions to both occupy the ground state of the one-particle infinite square well potential? Explain.

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THE WKB APPROXIMATION

I. The shape of the wave function

In this tutorial, you will only consider wave functions which are energy eigenstates. For a state with energy E_n in a system with potential $U(x)$, the wave function, $\psi_n(x)$, will satisfy the equation

$$\frac{\partial^2 \psi_n}{\partial x^2} = -\frac{2m}{\hbar^2}\left[E_n - U(x)\right]\psi_n(x)$$

A. Consider a region where the total energy, E_n, is *less than* the potential energy, $U(x)$. What is the general shape of the function, $\psi_n(x)$, in such a region? *(i.e.,* is the wave function linear, exponential, sinusoidal, quadratic, *etc.?)* Explain.

B. Now consider a region where the total energy is *equal to* the potential energy. What is the general shape of the function, $\psi_n(x)$, in such a region? Explain.

C. Now consider a region where the total energy is *greater than* the potential energy. What is the general shape of the function, $\psi_n(x)$, in such a region? Explain.

D. In those region(s) in which the wave function is sinusoidal, it may be expressed in the form, $\psi_n(x) = A \cos kx + B \sin kx$. The constant k is often called the *wave number*.

1. How is k related to the quantities in the equation at the top of the page? (*Hint:* Insert the form given for $\psi_n(x)$ into the equation at the top of the page, and solve for k.)

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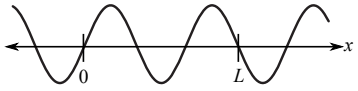
2. Find k in terms of L for the wave at right.
Sketch waves with wave numbers of $2k$ and $k/2$.



3. Give an interpretation of the wave number. (*Hint:* How is k related to *wavelength?*)

E. Consider the student discussion below.

Student 1: "The wave number is proportional to energy. The more oscillations per unit length, the more energy the wave has."

Student 2: "I think k is proportional to the difference between total energy and potential energy. So higher wave number means higher *kinetic energy,* not just energy."

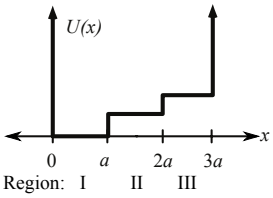
With which student do you agree, if either? Explain.

✓ Check your results with a tutorial instructor.

II. Sketching wave functions

A particle is in a *highly-excited* energy eigenstate of the potential well shown at right.

A. Sketch a line representing the total energy of this particle on the graph at right. Explain how you arrived at your answer.



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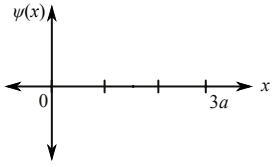
B. What general shape do you expect for the function $\psi_n(x)$ in each region of the well? (*i.e.*, Is the wave function linear, exponential, sinusoidal, *etc.)* Explain.

C. In which region is the *wave number* of the function $\psi_n(x)$ largest? In which region is it smallest? Explain.

D. In which region is the *amplitude* of the function $\psi_n(x)$ largest? In which region is it smallest? Explain.

E. Sketch a graph of this highly-excited eigenstate on the figure at right.



In the WKB approximation, the potential energy as a function of position may be represented by dividing space into small regions of uniform width within which the potential is nearly flat. The wave function for a highly-excited energy eigenstate may be represented as sinusoidal within each small region, $\psi(x) \approx \frac{A}{\sqrt{|p(x)|}} e^{\pm i \int p(x)dx/\hbar}$, with amplitude and wave number determined by the form of the (spatially-dependent) momentum, $p(x) = \sqrt{\frac{2m(E - V(x))}{\hbar^2}}$.

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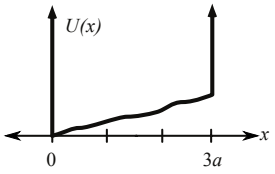
QM 4 *The WKB Approximation*

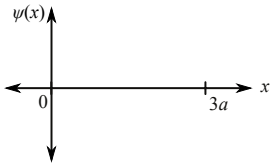
III. Approximate wave functions

Consider a particle in a *highly-excited* energy eigenstate of the different potential well shown at right.

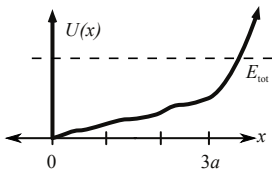


A. Explain why it is reasonable to use the WKB approximation for this system.

B. Sketch an approximate wave function for this particle on the figure at right. Explain your reasoning.



Suppose the infinite wall on the right is replaced by an increasing function, as shown in the graph at right. The dashed line indicates the total energy of the particle.



C. The intersection of the total energy with the potential energy curve is known as the *classical turning point*. Explain why this name is appropriate for this position.

D. What general shape do you expect for the wave function for values of x significantly greater than the classical turning point? Explain.

E. Using the WKB approximation, what happens to the amplitude of the wave function near the classical turning point?

Explain why the WKB approximation cannot be used to determine the wave function for regions of space near the classical turning points.

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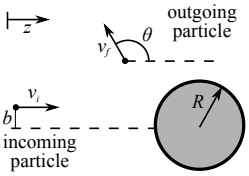
SCATTERING IN QUANTUM MECHANICS

I. Classical scattering theory

Consider a classical point particle incident on a large, hard sphere of radius R as shown at right. Suppose that the collision with the sphere is elastic, and the mass of the sphere is much greater than that of the incoming particle.



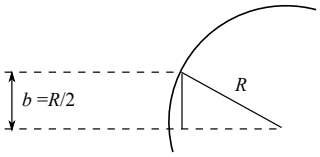
A. Is the momentum of the system of the sphere and the particle conserved in this collision? Is the energy of this system conserved? Explain how you were able to determine your answer.

B. Is the outgoing speed of the particle *greater than*, *less than*, or *equal to* the initial velocity of the particle? Explain. (*Hint:* Does the large sphere gain significant energy during this collision?)

Is the time it takes the incoming particle to move a distance d toward the sphere *greater than*, *less than*, or *equal to* the time it takes the outgoing particle to move the same distance d away from the sphere.

The quantities b and θ are known as the *impact parameter* and *scattering angle,* respectively.

C. Determine the scattering angle assuming that $b = R/2$ (the diagram at right may be useful).



Is there more than one answer, or is θ well defined for this case?

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D. Suppose that, for a different particle incident on the same hard sphere, you knew only that the scattering angle was within a small range of values represented by $d\theta$. Could you use this information to determine the range of possible impact parameters, db, for this particle? Explain.

A full theory of scattering typically requires more than just the one dimension represented by quantities such as db and $d\theta$. The quantities $d\sigma$ and $d\Omega$ are two-dimensional generalizations of db and $d\theta$, respectively. That is, a particle that is incident within the small area $d\sigma$ will be scattered into a corresponding small solid angle, $d\Omega$. In scattering theory, we typically combine these quantities to define the *differential cross section*, $\frac{d\sigma}{d\Omega}$.

E. In your own words, give a physical interpretation of the differential cross section. (*Note:* You may find it useful to consider a large number of identical particles with different impact parameters scattered from the same potential.)

II. Quantum mechanical scattering

Suppose that a quantum mechanical particle with the wave function $\psi_{\text{in}}(r,\theta,\phi) = Ne^{ikz}$ were incident on a spherically symmetric potential.

A. Is the momentum for this particle well-defined? If so, determine both the magnitude and the direction of the initial momentum. Explain.

B. Is the impact parameter b for this particle well-defined? If so, determine its value. Explain.

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C. Suppose you were to measure the position of the particle while it is incident on the potential.

1. Determine an expression for the probability that the incident particle is measured to be within an infinitesimal volume element, dV_{in}. Explain.

2. In the space below, make a sketch of dV_{in} and label it in terms of dz and $d\sigma$, the infinitesimals parallel and perpendicular to the direction of travel of the wave, respectively.

Use your sketch to rewrite dV_{in} in terms of dz and $d\sigma$.

Does the probability that the particle is found within this volume depend on where $d\sigma$ lies in the plane perpendicular to the direction of travel? Explain.

Assuming that no previous measurements have been made, the wave function for this particle after it has been scattered from the potential may be written $\psi_{sc}(r,\theta,\phi) = Nf(\theta)\frac{e^{ikr}}{r}$, for large values of the radial distance from the scattering center, r.

D. Is the scattering angle, θ, for this particle well-defined? If so, determine its value. Explain.

E. Is the *magnitude* of the momentum for the scattered wave function *greater than*, *less than*, or *equal to* the magnitude of the momentum for the incident wave function? Explain.

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F. Suppose you were to measure the position of the particle after it has scattered from the potential.

1. Determine an expression for the probability that the *scattered* particle is measured to be within an infinitesimal volume element, dV_{sc}. Explain.

2. In the space below, make a sketch of dV_{sc} and label it in terms of dr and $d\Omega$, the infinitesimals parallel and perpendicular to the direction of travel of the wave, respectively. (*Hint:* Should your coordinate system be the same as the one you used for the incident wave?)

Use your sketch to rewrite dV_{sc} in terms of dr and $d\Omega$.

Does the probability that the particle is found within this volume depend on the angle θ at which $d\Omega$ lies? Explain.

3. How is dr in your expression for dV_{sc} related to dz in your expression for dV_{in}? Explain. (*Hint:* Consider your answer to question E.)

Consider two small volume elements, dV_{in} and dV_{sc}, for which the probabilities are equal.

G. Use this assumption and your answers to the previous questions to determine an expression for the differential cross section, $\frac{d\sigma}{d\Omega}$, in quantum mechanics.

✓ Check your results with a tutorial instructor.

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Appendix C
Tutorial Homeworks

"The tutorial section was great and very helpful. I would recommend it for other physics courses as well."

—Junior-level quantum mechanics student, end-of-quarter survey

This appendix consists of copies of the homework associated with the tutorial curriculum discussed in this dissertation (primarily in chapters 5, 8, and 10). Each sub-section of this appendix (C.I, C.II, and C.III) includes homeworks for those topics described in part I (for quantum measurements and time dependence), part II (for vector spaces), and part III (for angular momentum). Section C.IV includes homework assignments associated with additional tutorials that have been developed at UW, but which are not explicitly discussed in this dissertation.

C.I – Homeworks for Part I

This sub-section includes copies of all tutorial homeworks discussed in part I of this dissertation. Each homework is discussed in chapter 5.

HOMEWORK: TIME DEPENDENCE IN QUANTUM MECHANICS

Name _____ QM HW-1

1. Consider a particle in the quantum mechanical infinite square well. At $t = 0$, the particle is in the state $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\psi_1\rangle - i|\psi_2\rangle\right)$.

   a. Sketch both the real and imaginary parts of the wave function for this particle at $t = 0$. You may use two separate graphs or a single graph with three axes.

   b. Sketch the probability density for this particle at $t = 0$. Explain.

   c. Recall the definition of T from the tutorial (the period of the first excited state). Sketch the probability density for this particle at $t = T$, $2T$, $3T$, and $4T$. Explain.

   d. What effect does the phase of the coefficients in a superposition of energy eigenstates have on the overall behavior of the time dependence of the *probability density?* Explain.

QM HW-2 *Homework: Time dependence in quantum mechanics*

2. Consider a particle of mass m in a quantum mechanical infinite square well of width a. At $t = 0$, the particle is in the state $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\psi_2\rangle + |\psi_3\rangle\right)$, where $|\psi_n\rangle$ and $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$ are the energy eigenstates and eigenvalues, respectively.

   a. Verify that the time dependence of the probability density for this state has the form:

$$\rho(x,t) = \frac{\psi_2^2(x) + \psi_3^2(x)}{2} + \psi_2(x)\psi_3(x)\cos\left(\frac{(E_3 - E_2)t}{\hbar}\right)$$

   b. Determine the first time when the probability density for this particle has exactly the same form as it does at $t = 0$. Explain.

Note that $\rho(x, t)$ is a periodic function.

   c. Determine the angular frequency, ω_ρ, for the probability density of a superposition of two energy eigenstates. How is ω_ρ related to the angular frequency associated with an individual eigenstate? Explain.

d. In each of the following questions, a **single change** is made to the situation above. In each case, determine whether ω_ρ will be *greater than*, *less than*, or *equal to* the value of ω_ρ for the original state (part b)? Explain your reasoning for each case.

i. The mass of the particle is increased to $2m$.

ii. The width of the square well is decreased to $a/2$.

iii. The initial state of the particle is given by $|\psi\rangle = \frac{1}{\sqrt{5}}\left(2|\psi_2\rangle + |\psi_3\rangle\right)$.

iv. The initial state of the particle is given by $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\psi_3\rangle + |\psi_4\rangle\right)$.

3. Consider the following statement:

"The time dependence we determined for an eigenstate doesn't make sense. No matter what state a particle starts in, the wave function has to spread out so that it fills all the available space, due to the uncertainty principle. Eventually, that means that all positions will be equally probable."

Do you agree or disagree with this statement? Explain your reasoning. You may wish to include examples that support or refute different aspects of this statement.

4. Consider a quantum mechanical harmonic oscillator. Let the energy eigenstates and eigenvalues of this system be given by $|\psi_n\rangle$ and $E_n = (n + ½)\,\hbar\omega$, respectively.

   a. At $t = 0$, the state of a particle in this potential is given by: $|\psi_a\rangle = \frac{1}{\sqrt{2}}\left(|\psi_0\rangle + |\psi_1\rangle\right)$.

   Determine ω_ρ (see problem 3) for $|\psi_a\rangle$. How is it related to ω, the frequency of the harmonic oscillator potential? Explain.

   b. At $t = 0$, the state of an identical particle is given by: $|\psi_b\rangle = \frac{1}{\sqrt{2}}\left(|\psi_0\rangle + |\psi_2\rangle\right)$. Is the value of ω_ρ for $|\psi_b\rangle$ *greater than*, *less than*, or *equal to* the value of ω_ρ for $|\psi_a\rangle$? Explain.

   c. At $t = 0$, the state of an identical particle is given by: $|\psi_c\rangle = \frac{1}{\sqrt{2}}\left(|\psi_{17}\rangle + |\psi_{18}\rangle\right)$. Is the value of ω_ρ for $|\psi_c\rangle$ *greater than*, *less than*, or *equal to* the value of ω_ρ for $|\psi_a\rangle$? Explain.

   d. Consider the following student dialogue:

   Student 1: "Higher excited-states have higher energies, so the wave functions rotate faster. This implies that the probability density will also change faster."

   Student 2: "I disagree. We cannot actually observe the wave function, so it doesn't matter how fast they are rotating. All that matters is the difference between the energies. A probability density will only move faster if the energy *difference* is larger."

   With which student do you agree, if either? Explain.

Recall that in classical mechanics it is sufficient to know differences in energy in order to determine changes in dynamical quantities.

   e. Which of the following quantities is more useful in determining how the probability density changes in time: *energies* or *differences in energy?* Explain.

HOMEWORK: ENERGY MEASUREMENTS Name _____ QM HW-1

1. Consider a particle in a quantum mechanical infinite square well. The wave function of this particle at $t_o = 0$ is given below, along with the allowed energies of this potential.

$$|\psi\rangle = \frac{1}{\sqrt{6}}\left(\sqrt{2}|\psi_1\rangle - \sqrt{3}|\psi_2\rangle + i|\psi_3\rangle\right) \qquad E_n = n^2 E_1$$

   a. At time t_o, the energy of this particle is measured. What is the most likely measurement? What is the probability of measuring this value? Explain.

   b. What is the expectation value of this measurement in terms of E_1, the ground state energy?

      Is it possible for an energy measurement on this particle to result in this value? If so, give the associated probability. Explain.

   c. Assume that the measurement results in the value E_3. Write down the form of the wave function immediately after the measurement.

   d. Consider the statements given by the students below. State whether you agree or disagree with each student. Explain your reasoning.

      Student 1: "The energy of the particle was always E_3 because the particle was really in the state ψ_3 the whole time."

      Student 2: "Before the measurement, the energy of the particle is equal to the expectation value. Measuring changes the energy, so now the expectation value is E_3."

      Student 3: "Initially the energy is not well-defined; a measurement could yield any of the first three energy eigenvalues. After the measurement the energy is well-defined, because the particle is now in an energy eigenstate."

QM
HW-2

Homework: Energy measurements

2. Consider a particle in the *ground state* of the quantum mechanical infinite square well of width a. (*Note:* Recall that this wave function is equal to 0 for $x > a$.)

   At time t_1 you increase the size of the square well so that it is now $2a$. It is increased in such a way that the wave function of the particle just after t_1 is **the same** as the wave function of the particle just before t_1. The energy levels of the new potential are given by:

$$E_n' = n^2 E_1' = n^2 E_1 / 4$$

   a. Is the particle in an energy eigenstate *immediately before* time t_1? If so, specify which state. If not, describe the state of the particle. Explain.

   b. Is the particle in an energy eigenstate *immediately after* time t_1? If so, specify which state. If not, describe the state of the particle. Explain.

   c. Suppose you wait until time $t_2 > t_1$, making no other changes to the system. Is the particle in an energy eigenstate at time t_2? If so, specify which state. If not, describe the state of the particle.

   d. Several students discuss the possibility of measuring the energy of this particle again. State whether you agree or disagree with each student. Explain your reasoning.

      Student 1: "If the energy is measured again at a later time, it will continue to yield E_1, because that is still an allowed energy."

      Student 2: "I agree, but the energy of the ground state changed, so we should measure the new ground state energy, since that's what the wave function looks like now."

      Student 3: "I disagree. The particle is in a new potential, so we have to use the eigenstates of the new system. We could measure any of the new energy eigenvalues."

   e. Suppose you were to measure the particle's energy at t_2. Determine the possible results of this measurement, and describe how you would determine the probability associated with each measurement. Explain your reasoning.

Homework: Energy measurements Name _____ QM HW-3

3. Consider a particle in a quantum mechanical infinite square well. At $t = 0$, the particle is in the ground state, with a wave function given by $\Psi(x,t) = \psi_1(x)e^{\frac{-iE_1t}{\hbar}}$.

   a. Show explicitly that this wave function is an eigenstate of the operator, $i\hbar\frac{\partial}{\partial t}$, and find its eigenvalue.

Now consider a particle given by the wave function $\Psi(x,t) = \frac{1}{\sqrt{2}}\left(\psi_1(x)e^{\frac{-iE_1t}{\hbar}} + \psi_2(x)e^{\frac{-iE_2t}{\hbar}}\right)$.

   b. Show explicitly that this wave function is *not* an eigenstate of the same operator.

   c. Consider the statements about this second wave function given below.

   Student 1: "The wave function is a solution of the Schrödinger Equation. So there must be a definite energy associated with it."

   Student 2: "At times the wave function is like the ground state, and at other times it is like the first excited state. So, sometimes it has energy E_1, and at other times E_2."

   Student 3: "When written as a sum of eigenstates, the coefficients are always the same. So the probabilities of measuring E_1 and E_2 don't change, even though the wave function looks like ψ_2 when $e^{\frac{-iE_1t}{\hbar}} = 0$ and like ψ_1 when $e^{\frac{-iE_2t}{\hbar}} = 0$."

   Each student is incorrect. Identify the flaw in each student's reasoning. Explain.

HOMEWORK: POSITION, MOMENTUM, AND ENERGY MEASUREMENTS Name _____ QM HW-1

1. The following questions focus on the momentum operator in quantum mechanics.

   a. In Dirac notation, write an equation relating an eigenvector $|k_i\rangle$ of the *momentum* operator $\hat{p}$ and its corresponding eigenvalue λ_i. Explain.

   b. Rewrite your equation in terms of x and derivatives with respect to x. Indicate explicitly which symbols are functions of x and which are not. Explain.

   c. Solve this equation for the eigenfunction corresponding to the eigenvalue λ_i, and graph at least one example. Show your work.

   d. For which values of λ_i does a solution to the equation above exist? Use your answer to determine all the allowed eigenvalues of the momentum operator. Explain.

   e. Are the eigenfunctions of momentum also eigenfunctions of *position?* If you need additional information to answer, describe that information. Explain.

   f. Are the eigenfunctions of momentum also eigenfunctions of *energy?* If you need additional information to answer, describe that information. Explain.

QM HW-2 *Homework: Position, momentum, and energy measurements*

2. In this problem, you will add a discussion of momentum to your summary from the tutorial.

a. Consider the operators for position, $\hat{x}$, momentum, $\hat{p}$, and energy, $\hat{H}$. Determine which pairs of operators commute with each other. If any of your results depend on the system under consideration, state so explicitly. Explain.

b. Fill in the table below using your results from the tutorial. Use the commutation relations you found in part a to answer the questions about momentum. If any of your results depend on the system under consideration, state so explicitly. Explain.

	Table 1: Number of Possible Measurements (at a later time)			**Table 2: Time-dependence of Possible Measurements**		
	Energy Eigenstate	Position Eigenstate	Momentum Eigenstate	Energy Eigenstate	Position Eigenstate	Momentum Eigenstate
Energy Measurements						
Position Measurements						
Momentum Measurements						

c. Consider a hypothetical operator, $\hat{O}$, which commutes with the Hamiltonian.

i. If you started in an eigenstate of $\hat{O}$, which observables would have multiple possible results? Explain.

ii. If you started in an eigenstate of $\hat{O}$, which observables would have probabilities that depend on time? Explain.

Homework: Position, momentum, and energy measurements Name _____ QM HW-3

3. Consider a particle that is initially in the ground state of the infinite square well (with energy eigenvalues E_i). The initial wave function for this particle is given by

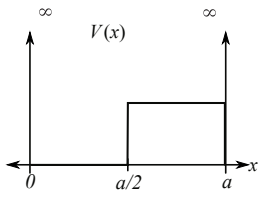
$$\psi(x,0) = \sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right) \text{ for } 0 < x < a.$$

a. At $t = 0$, is the probability that the particle is found on the left half of the well *greater than, less than,* or *equal to* the probability it is found on the right half of the well? Explain.

b. At a later time, $t_1 > 0$, is the probability that the particle is found on the left half of the well *greater than, less than,* or *equal to* the probability it is found on the right half of the well? Explain.

c. Assume you measure the energy of this particle at time t_1 (no previous measurements have been made). What is the probability that the result of this measurement will be E_1? Explain.

Consider a particle in the potential shown at right, which consists of a step added to the right half of an infinite square well. The initial wave function of this particle is the same as for the particle above: $\psi(x,0) = \sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)$ for $0 < x < a$.

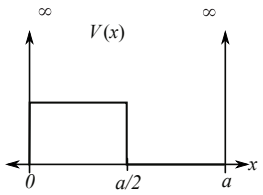


d. At $t = 0$, is the probability that this particle is found on the left half of the well *greater than, less than,* or *equal to* the probability it is found on the right half of the well? Explain.

e. Assume you measure the energy of this particle at time t_1 (no other measurements have been made). What is the probability that the result of this measurement will be E_1? Explain.

QM HW-4 **Homework: Position, momentum, and energy measurements**

Now consider a particle in the potential at right, which consists of a step of the same height added to the *left* half of an infinite square well. The wave function of this particle is the same as the particles above: $\psi(x,0) = \sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)$ for $0 < x < a$.



f. At $t = 0$, is the probability of measuring the position to be $a/4$ in this system *greater than*, *less than*, or *equal to* the probability of measuring the position to be $a/4$ in the previous system (with the step on the right half of the well)? Explain.

g. At $t = 0$, is the probability of measuring the momentum to be to the left with magnitude $2\pi\hbar/a$ in the system with the new potential *greater than*, *less than*, or *equal to* the probability of measuring the momentum to be to the left with magnitude $2\pi\hbar/a$ in the system with the previous potential? Explain.

h. Consider the following *incorrect* statement.

"The potential for a system tells me everything I need to know about it. It can tell me the allowed energies, the functional form and time dependence of the wave function, and the probability that I get different energies, positions, and momenta."

Identify which portions of the statement are incorrect. Explain your reasoning.

HOMEWORK: DEGENERATE STATES

Name _____ QM HW-1

1. Recall the first two energy eigenfunctions for the one-dimensional harmonic oscillator (where A_0 and A_1 are normalization constants): $|0\rangle \equiv \psi_0(x) = A_0 e^{-x^2/a^2}$ $|1\rangle \equiv \psi_1(x) = A_1 x e^{-x^2/a^2}$

   The wave function for a *two-dimensional* harmonic oscillator is given by: $\psi(r,\phi) = Are^{-r^2/a^2} e^{i\phi}$.

   a. Show that this state is an energy eigenstate of this system and find its eigenvalue. Explain. (*Hint:* Write the state in terms of the eigenfunctions you determined in the tutorial.)

   b. A different particle has the wave function: $\psi(r,\phi) = Are^{-r^2/a^2} e^{i\varphi} + Bye^{-\left(x^2+y^2\right)/a^2}$.

      i. Is it possible for a particle to have a wave function with this form? Explain.

      ii. Is this state an energy eigenstate of the system? Explain how you can tell.

      iii. Does the wave function for this state depend on time? If so, determine the time dependence of the wave function. Explain.

      iv. Is this state a stationary state? Explain.

   c. Consider the student discussion below.

      Student 1: "You cannot mix notations between polar and Cartesian coordinates. This state is not even allowed for a particle in the 2D harmonic oscillator."

      Student 2: "I disagree. There's nothing wrong with this state, since it's still square normalizable. But it can't be a stationary state because it's a sum of two other states."

      Student 3: "You're right that this is a sum of states, but these two states both have the same energy. It's even more obvious if you write the whole thing in one coordinate system, like Cartesian. That means this state is also an energy eigenstate."

      With which student do you agree, if any? Explain.

QM
HW-2

Homework: Degenerate states

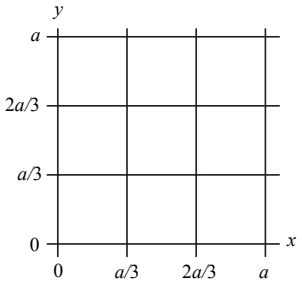
2. Consider the solutions to the two-dimensional quantum mechanical **infinite square well** given by the general form $\psi_{mn}(x,y) = \psi_m(x)\psi_n(y)$, where the $\psi_n(x)$ are the solutions to the one-dimensional infinite square well of width a with energy eigenvalues $E_n = n^2E_1$, for $n = 1, 2, \ldots$.

   a. What is the degeneracy of states where $m = n$? Explain.

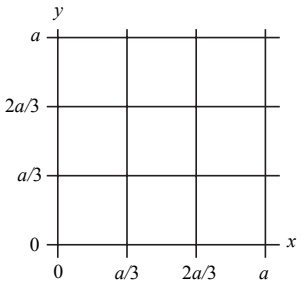
   b. What is the degeneracy of states where $m \neq n$? Explain.

   c. For each of the states below, divide each axis into three equal regions, so that the xy-plane is divided into nine equal areas. Approximate each wave function by its value at the center of the region, and enter this number in the appropriate area on the graph at right.

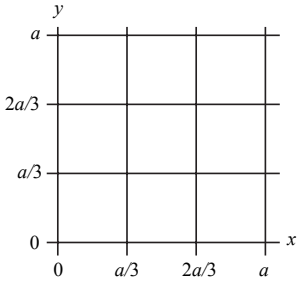
      i. $\psi_{11}(x,y)$



      ii. $\psi_{21}(x,y)$

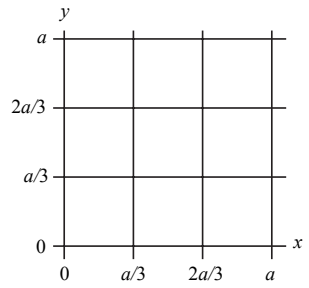


      iii. $\psi_{12}(x,y)$



Homework: Degenerate states Name _____ QM HW-3

iv. $\psi_{22}(x,y)$



d. Identify any lines of symmetry in your graphs above.

e. Which of the four states that you graphed are degenerate?

Are there any similarities between the graphs of the wave functions for the degenerate states?

f. Compare degenerate eigenstates of the infinite square well to the degenerate eigenstates of the harmonic oscillator you studied in the tutorial. What similarities can you find between them? What differences can you find? Explain.

g. Extra Credit: Graph the states above using a computer program, and compare the graphs to your approximation. Also graph additional pairs of higher-energy degenerate eigenstates, and determine if your findings extend to these states as well.

QM
HW-4

Homework: Degenerate states

3. Recall that the Hamiltonian for the two-dimensional quantum mechanical harmonic oscillator is given by $\hat{H}_{2D} = \hat{H}(x) + \hat{H}(y)$. The energy eigenstates are $|\psi_m\rangle_x \otimes |\psi_n\rangle_y \equiv |m,n\rangle$.

   a. Show that the energy of the state $|m,n\rangle$ is given by the sum of the energies of the one-dimensional states, $E_{mn} = E_m + E_n$.

Consider the operator for the angular momentum of a particle confined to move in two dimensions, $\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x = i\hbar\left(\hat{a}_{y+}\hat{a}_{x-} - \hat{a}_{y-}\hat{a}_{x+}\right)$, where the raising and lowering operators in one dimension are defined by: $\hat{a}_+|n\rangle = \sqrt{n+1}\,|n+1\rangle$ and $\hat{a}_-|n\rangle = \sqrt{n}\,|n-1\rangle$. Note that this operator commutes with the Hamiltonian.

   b. Use these relationships to show that the energy eigenstate given by $|m,n\rangle$ is *not* an eigenstate of the operator $\hat{L}_z$.

   c. Show that the states given by $\frac{1}{\sqrt{2}}\left(|1,0\rangle \pm i|0,1\rangle\right)$ are eigenstates of the operator $\hat{L}_z$, and determine the eigenvalue for each.

   Are these two states also energy eigenstates? Explain.

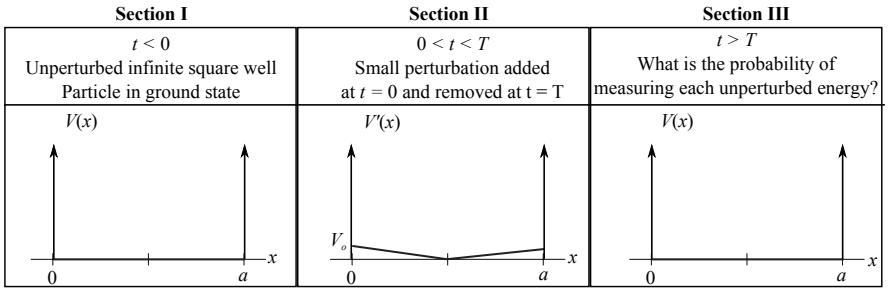
   d. The statement below is incorrect. Explain the flaws in the student's reasoning.

   "The original energy eigenfunctions we found are not eigenfunctions of angular momentum, so the eigenfunctions of angular momentum can't be energy eigenfunctions."

HOMEWORK: TIME-DEPENDENT PERTURBATION THEORY PART 1

Name _____ QM HW-1

1. Consider the time dependent system from the tutorial represented below (the sections in the figure refer to the sections of the tutorial during which you considered that potential). Let the energy eigenfunctions and eigenvalues of the unperturbed system be given by $\varphi_m^{(0)}(x)$ and $E_m^{(0)}$, and those of the perturbed system be given by $\varphi_m^{(p)}(x)$ and $E_m^{(p)}$, respectively.



a. Suppose the system is initially in the **tenth** excited-state of the unperturbed potential.

   i. What are the possible results of an energy measurement of this system at $t = T/2$? Explain how you would calculate the probability of each result. Indicate whether or not your probabilities should depend on time.

   ii. What are the possible results of an energy measurement of this system at $t = 3T/2$? Explain how you would calculate the probability of each result. Indicate whether or not your probabilities should depend on time.

   iii. What are the possible results of an energy measurement of this system as t approaches infinity? Explain how you would calculate the probability of each result.

   iv. Does the probability density for the system change with time during the interval from $t = 0$ to T? Does it change with time when $t > T$? Explain.

b. Suppose instead the system is initially in the state given by $\psi(x) = \frac{1}{5}\left(4\varphi_1^{(0)}(x) + 3\varphi_2^{(0)}(x)\right)$.

i. What are the possible results of an energy measurement of this system at $t = T/2$? Explain how you would calculate the probability of each result. Indicate whether or not your probabilities should depend on time.

ii. What are the possible results of an energy measurement of this system at $t = 3T/2$? Explain how you would calculate the probability of each result. Indicate whether or not your probabilities should depend on time.

iii. What are the possible results of an energy measurement of this system as t approaches infinity? Explain how you would calculate the probability of each result.

iv. Does the probability density for the system change with time during the interval from $t = 0$ to T? Does it change with time when $t > T$? Explain.

c. Consider the discussion between three students below.

Student 1: "In both cases, the wave function after the perturbation turns off should look just like it did before the perturbation. If we were in an eigenstate, we should be in the same eigenstate; if we were in a superposition we should be in the same superposition."

Student 2: "I disagree. If it started in an eigenstate it will stay there, but if we start in a superposition then the perturbation will cause it to drop into an eigenstate."

Student 3: "Yes, and if we wait long enough, we will always end up in the ground state no matter what we started with, because the particle will decay."

With which student do you agree, if any? For each student with whom you disagree, explain the flaws in their reasoning.

HOMEWORK: TIME-DEPENDENT PERTURBATION THEORY PART 2

Name _____ QM HW-1

1. Suppose that the particle in the time-dependent potential from the tutorial begins in the j^{th} excited-state instead of the ground state.

   a. Show that equation 1 reduces to the following expression:

$$\frac{\partial c_n^{(1)}(t)}{\partial t} = -\frac{i}{\hbar}\sum_m c_m^{(0)} H'_{nm} e^{i(E_n^{(0)}-E_m^{(0)})t/\hbar} = -\frac{i}{\hbar} c_j^{(0)} H'_{nj} e^{i(E_n^{(0)}-E_j^{(0)})t/\hbar}$$

   b. Integrate both sides of your expression above to obtain $c_n^{(1)}(t)$ for $n = j$. Be sure to include the correct limits of integration on both sides of the equation.

   c. Integrate both sides of your expression above to obtain $c_n^{(1)}(t)$ for $n \neq j$. Be sure to include the correct limits of integration on both sides of the equation.

   d. Write an expression for the wave function for this particle for $t > T$ (after the perturbation has been turned off).

   e. For some time $t > T$, determine the transition probability, P_{nj}, for a particle to be measured in the n^{th} excited-state given that it began in the j^{th} excited-state.

f. Compare your expression to the transition probability you determined in the tutorial. Identify any similarities. Explain.

2. Consider two particles in the following (identical) time-dependent potentials: the infinite square well for $t < 0$, a (constant) perturbed system for $0 < t < T$, and a return to the infinite square well for $t > T$. Recall that the energies of the infinite square well are given by n^2E_1 ($n = 1, 2, 3, \ldots$).

   Particle 1 is initially in the state with $n = 2$; particle 2 is initially in the state with $n = 4$.

   a. If the energy of both particles is measured at some time $t > T$, which particle is more likely to be found with the ground state energy (E_1)? Explain.

   b. Which particle is more likely to be found with the second excited-state energy (E_3)? Explain.

   c. Suppose that you could choose the length of time, T, for which the perturbation is active.

      i. Is there a value of T that maximizes the probability that both particles are found with the ground state energy (E_1). If so, determine at least one such value. Explain.

      ii. Is there a value of T that maximizes the probability that both particles are found with the second excited-state energy (E_3). If so, determine at least one such value. Explain.

Homework: Time-dependent perturbation theory Part 2 Name _____ QM HW-3

3. Consider a time-dependent perturbation given by $H'(x,t) = V'(x)\cos(\omega t)$ for $0 < t < T$. Note that the simplification made at the bottom of page 2 of the tutorial is no longer applicable. In this case, equation 1 from the tutorial may be rewritten as: $\frac{\partial c_n^{(1)}(t)}{\partial t} = -\frac{i}{\hbar}\sum_m c_m^{(0)} V'_{nm} \cos(\omega t) e^{i\omega_{nm} t}$, where $\omega_{nm} = \left(E_n^{(0)} - E_m^{(0)}\right)/\hbar$. This angular frequency differs from the quantity ω in the definition of the perturbation, which is known as the *external frequency*.

   Suppose that the system is in the ground state before the perturbation is turned on.

   a. Show that the first-order coefficients are given by the following formula:

$$c_n^{(1)}(t) = -\frac{V'_{n1}}{2\hbar}\left[\frac{e^{i(\omega_{n1}+\omega)t}-1}{\omega_{n1}+\omega} + \frac{e^{i(\omega_{n1}-\omega)t}-1}{\omega_{n1}-\omega}\right]$$

   b. Which of the two terms in the expression will have a larger contribution to $c_n^{(1)}(t)$? Explain.

   c. Assuming you can neglect the smaller of the two terms, determine the transition probability, P_n, for this system. Explain how you arrived at your answer.

d. For which values of ω (the external frequency) will P_n be largest? Explain the physical significance of this value.

e. Is there a value $T = T_n^{min}$ for which P_n is equal to zero? How does it compare with the value you determined in the tutorial? Explain.

f. Suppose that the particle began in the j^{th} excited-state instead of the ground state.

i. How would this change the expression given in question a? Explain.

ii. Consider the term in this expression that you neglected in question b. Determine the value of ω (the external frequency) that will maximize the magnitude of this term. Explain.

iii. What is the physical significant of this value of ω? How does this value differ from the value you determined in question d? Explain.

HOMEWORK: TWO-STATE TIME DEPENDENCE

Name _____ QM HW-1

1. An electron is placed in a region of uniform magnetic field B_o in the z-direction. The electron is initially known to be in the state $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}$.

   a. Determine the probability of measuring the particle to have spin-up in the x-direction, the y-direction, and the z-direction at $t = 0$? Explain.

   b. Determine the state of the particle for time $t > 0$. Explain how you arrived at your answer.

   c. At what time (or times) is the probability of measuring spin up in the x-direction the same as it was at $t = 0$? At these times, is the probability of measuring spin-up in the y-direction also the same as it was at $t = 0$? Explain.

   d. Consider the statements given below.

      Student 1: "The spin-down part of this state is imaginary, so the probability of measuring spin-down in the z-direction must be zero."

      Student 2: "I agree, but only for $t = 0$. At later times, the time dependence makes both terms oscillate between real and imaginary, and only the real part of the state affects probabilities."

      With which student, if either, do you agree? Explain your reasoning.

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2. A different electron is placed in a region of uniform magnetic field B_o in the z-direction. This electron is initially known to be in the state $\frac{1}{2}\begin{pmatrix}\sqrt{3}\\ 1\end{pmatrix}$.

   a. Show that this state is an eigenstate of the operator $\frac{1}{2}\hat{S}_z + \frac{\sqrt{3}}{2}\hat{S}_x$ and determine its eigenvalue. Show your work.

   b. Calculate the expectation value of the x-component, y-component, and z-component of spin as a function of time. Show your work.

   c. Give a physical interpretation for the time dependence of the expectation values you identified in the previous question.

   d. Consider the statements given below.

      Student 1: "Stationary states do not depend on time. The state we were given is not a stationary state, so it does depend on time."

      Student 2: "I agree. Once we know that the state depends on time, we know that all probabilities must also depend on time."

      Both students are incorrect. Use an example to identify the flaws in each student's reasoning.

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C.II – Homeworks for Part II

This sub-section includes copies of all tutorial homeworks discussed in part II of this dissertation. Each homework is discussed in chapter 8.

HOMEWORK: DIRAC NOTATION

Name _____ QM HW-1

1. Consider the two-dimensional spatial vectors written at right in Dirac notation.

$$|u\rangle = -3|x\rangle + 3|y\rangle$$
$$|v\rangle = 3|x\rangle - |y\rangle$$

   a. Are these vectors normalized? If not, calculate the normalized vectors, $|u'\rangle$ and $|v'\rangle$. What important property does a normalized vector have?

   b. Represent the normalized $|u'\rangle$ and $|v'\rangle$ as column vectors. Express each element as both an inner product and a number

   c. Consider the operator $\hat{P}$ defined by the matrix at right.

$$\hat{P} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

      i. Graph $|u'\rangle$ and $\hat{P}|u'\rangle$ on the same set of axes in the space below.

      ii. Graph $|v'\rangle$ and $\hat{P}|v'\rangle$ on the same set of axes in the space below.

      iii. Determine a general rule for the action of the operator $\hat{P}$. (*Hint:* Consider the graphical result of the operator's action on each vector.)

d. Now consider the operator $\hat{Q}$ defined by the matrix at right. $\hat{Q} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

i. Graph $|u'\rangle$ and $\hat{Q}|u'\rangle$ on the same set of axes in the space below.

ii. Graph $|v'\rangle$ and $\hat{Q}|v'\rangle$ on the same set of axes in the space below.

iii. Determine a general rule for the action of the operator $\hat{Q}$.

e. Suppose you were to act both operators $\hat{P}$ and $\hat{Q}$ on one of the vectors above. Would the result depend on the order in which you chose to act the two operators? Explain using both the mathematical result and the graphical interpretation of each operator.

HOMEWORK: TREATING FUNCTIONS AS VECTORS

Name _____ QM HW-1

1. Recall the functions from the tutorial: $\psi_1(x)$, $\psi_3(x)$, and $\psi(x) = \frac{1}{\sqrt{13}}\left(2\psi_1(x) + 3\psi_3(x)\right)$. Let the energy eigenfunctions of the infinite square well be given by $\psi_n(x)$, for $n = 1, 2, 3, \ldots$.

   a. Write the function $\psi(x)$ as a column vector in the energy basis *(i.e.,* such that each entry corresponds to a different value of *n)*.

   How many entries should a column vector in this basis have? Explain.

   b. Write the functions $\psi_1(x)$ and $\psi_3(x)$ as column vectors in the energy basis.

   Explain how you can tell that these functions correspond to basis vectors in this basis.

   c. Use the energy basis to calculate the following inner products.

      i. $\psi_1(x)$ and $\psi_3(x)$

      ii. $\psi_1(x)$ and $\psi(x)$

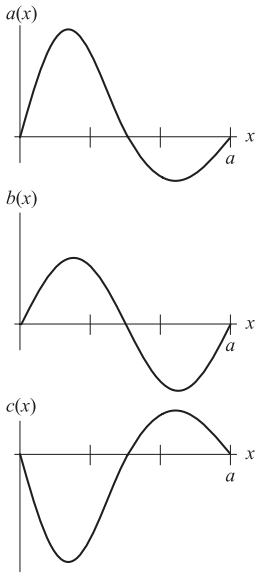
      iii. $\psi_1(x)$ and $\psi_1(x)$

   Are your answers the same as when you calculated these inner products using the approximated column vectors from the tutorial? Explain.

QM
HW-2

Homework: Treating functions as vectors

2. Shown at right are the graphs of three functions.



   a. Determine whether each of the following inner products is *positive*, *negative*, or *zero*. In each case, explain your reasoning.

      i. $a(x)$ and $b(x)$

      ii. $a(x)$ and $c(x)$

      iii. $b(x)$ and $c(x)$

   b. Rank the three inner products by absolute value from greatest to least. Explain.

   c. Consider the student discussion below.

      Student 1: "The inner product of a(x) and b(x) has the greatest absolute value because they look very similar: they both have a positive bump on the left and a negative bump on the right."

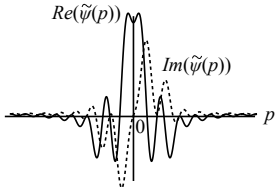
      Student 2: "The inner product of a(x) and c(x) is zero, because the functions are equal and opposite, so they will cancel each other out."

      With which student do you agree, if either? Explain.

HOMEWORK: REPRESENTATIONS OF WAVE FUNCTIONS

Name _____ QM HW-1

1. Recall the quantum state for a particle in the infinite square well, $|\psi\rangle = \frac{1}{\sqrt{3}}|\psi_1\rangle + \frac{1}{\sqrt{2}}|\psi_2\rangle + \frac{1}{\sqrt{6}}|\psi_4\rangle$, from the tutorial. A different visual representation of this state is shown at right. (*Note:* solid lines represent real parts while dashed lines represent imaginary parts.)



   a. The graph at right is a representation of a quantum state in the *momentum basis,* also known as the *(momentum-space) wave function*. Explain why these names are appropriate.

   b. What are the units for the vertical axis? Is your answer consistent with the overall normalization of the state? Explain.

      How do these units compare to the units for the position-space wave function?

   c. Suppose you were given only the momentum-space wave function for a particle, $\tilde{\psi}(p)$.

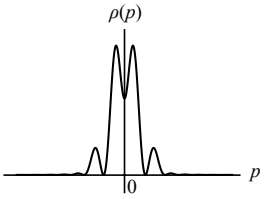
      i. Determine an expression for this particle's state in the energy representation. Explain how you arrived at your answer using both words and equations.

      ii. Determine an expression for this particle's state in the position representation (*i.e.,* the position-space wave function). Explain how you arrived at your answer using both words and equations.

d. Do you agree or disagree with the statement below? Explain your reasoning.

"After I square the wave function for momentum, I can use that to go back and find probabilities for other quantities, like energy or position. So I don't need to know the wave function at all if I already know what the probability density is."

e. The momentum-space probability density for this particle (which is in the infinite square well) is shown at right. Use this graph to answer the following questions.



i. Write an expression for the probability that the momentum is measured to be within range dp of the value $p = 0$. Explain.

ii. This graph is symmetric about the value $p = 0$. What does this observation tell you about the probabilities of different values of momentum? Explain.

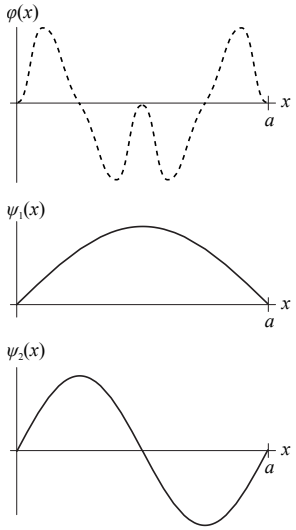
iii. Identify and label the most and least probable value(s) of momentum for this particle. Explain.

iv. What is the expectation value of momentum for this particle? Explain.

v. Should the expectation value of momentum for this particle be one of the most probable values of momentum? Explain.

Homework: Representations of wave functions Name _____ QM HW-3

2. Consider a particle in the infinite square well. Shown at right is a wave function for this particle that is entirely imaginary, $\varphi(x)$. The wave functions for the ground state, $\psi_1(x)$, and the first excited state, $\psi_2(x)$, are also shown. (Solid and dashed lines represent real and imaginary parts, respectively.)



   a. Consider the inner product between $\varphi(x)$ and $\psi_1(x)$.

      Is the real part of this inner product *positive*, *negative*, or *zero?* Is the imaginary part *positive*, *negative*, or *zero?* Explain. (*Hint:* Consider dividing the *x*-axis into several regions of equal width.)

   b. Do you agree or disagree with the following statement? Explain.

      "This inner product is imaginary, so the probability of measuring E_1 is zero. Imaginary things can't be measured."

   c. Consider the inner product between $\varphi(x)$ and $\psi_2(x)$.

      Is the real part of this inner product *positive*, *negative*, or *zero?* Is the imaginary part *positive*, *negative*, or *zero?* Explain.

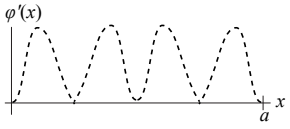
   d. Is the probability that this particle is measured to have energy E_1 *greater than*, *less than*, or *equal to* the probability that it is measured to have energy E_2? Explain.

   e. In the space below, sketch the probability density for the particle described by $\varphi(x)$. On your sketch, indicate where the particle is (1) *most likely* and (2) *least likely* to be found if the position were measured. Explain.

QM
HW-4

Homework: Representations of wave functions

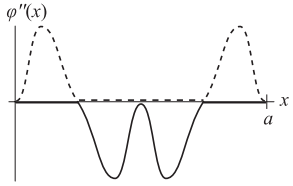
f. Consider the wave function, $\varphi'(x)$, shown at right.



Is the probability density for $\varphi'(x)$ *the same as* or *different from* the probability density for $\varphi(x)$? Explain.

Is the probability that a particle described by $\varphi'(x)$ is measured to have energy E_1 *greater than*, *less than*, or *equal to* the same probability for the particle described by $\varphi(x)$? Explain.

g. Consider the wave function, $\varphi''(x)$, shown at right. (Recall that solid lines represent the *real* part of the function.)



Is the probability density for $\varphi''(x)$ *the same as* or *different from* the probability density for $\varphi(x)$? Explain.

Is the probability that a particle described by $\varphi''(x)$ is measured to have energy E_1 *greater than*, *less than*, or *equal to* the same probability for the particle described by $\varphi(x)$? Explain.

Is the probability that a particle described by $\varphi''(x)$ is measured to have energy E_1 *greater than*, *less than*, or *equal to* the same probability for the particle described by $\varphi'(x)$? Explain.

h. With which of the following students, if either, do you agree? Explain.

Student 1: "Whether part of a wave function is positive or negative, or whether it is real or imaginary, doesn't affect probabilities because phases don't matter."

Student 2: "I agree. Even though the wave function is different, the probabilities for energy should stay the same, because energy and position are different quantities. They shouldn't be able to affect each other."

HOMEWORK: PROBABILITY AMPLITUDE

Name _____ QM HW-1

1. Consider a particle in the quantum mechanical infinite square well from $x = 0$ to $x = a$. Let the state of the particle be given by $|\varphi_A\rangle$ and let the system's energy eigenstates and eigenvalues be given by $|\psi_n\rangle$ and E_n, respectively, for $n = 1, 2, 3, \ldots$. Let the system's position eigenstate and eigenvalue be given by $|x\rangle$ and x, respectively, where $0 \le x \le a$.

   a. Consider the operator given by the expression $|\psi_n\rangle\langle\psi_n|$.

      i. Write an expression for this operator acting on the state, $|\varphi_A\rangle$?

      ii. Give a physical interpretation for the inner product that results from this operator.

   b. Consider the operator given by the expression $\sum_n |\psi_n\rangle\langle\psi_n|$.

      i. Write an expression for this operator acting on the state, $|\varphi_A\rangle$?

      ii. Compare the result with the expression you wrote in question A of section I of the tutorial. Is the state that results from the action of the operator *the same as* or *different from* the initial state $|\varphi_A\rangle$? Explain.

   c. Consider the operator given by the expression $\int_0^a dx|x\rangle\langle x|$.

      i. Write an expression for this operator acting on the state $|\varphi_A\rangle$? Explain.

      ii. Compare the result with the expression you wrote in question F of section II of the tutorial. Is the state that results from the action of the operator *the same as* or *different from* the initial state $|\varphi_A\rangle$? Explain.

d. Let $|p\rangle$ and p represent the eigenstates and eigenvalues, respectively, of the momentum operator.

i. Describe how to find the probability that a measurement of momentum results in a value between $p = k_1$ and $p = k_2$.

ii. Write an expression for the state $|\varphi_A\rangle$, in terms of the eigenstates of the momentum operator. Explain.

e. Consider a hypothetical operator, $\hat{Y}$.

i. Suppose the eigenvalues of $\hat{Y}$ are discrete. Let the eigenstates and eigenvalues be given by $|y_n\rangle$ and y_n, respectively, for $n = 1, 2, 3, \ldots$. Describe how to find the probability of measuring y_n.

ii. Now suppose the eigenvalues of operator $\hat{Y}$ are continuous. Let the eigenstates and eigenvalues be given by $|y\rangle$ and y, respectively, where y is a continuous variable. Describe how to find the probability of measuring eigenvalue y in the region between $y = b$ and $y = c$.

2. The infinite square well of width a has energy eigenfunctions $\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$.

Homework: probability amplitude Name _____ QM HW-3

Suppose that particle B has the initial wave function defined at right. (*Note:* The wave function is zero outside of the well.)

$$\varphi(x,t=0)=\sqrt{\frac{2}{a}}\begin{cases}\sin\left(\dfrac{2\pi x}{a}\right) & 0\le x\le\dfrac{a}{2}\\ -\sin\left(\dfrac{2\pi x}{a}\right) & \dfrac{a}{2}\le x\le a\end{cases}$$

a. Sketch the wave function for this particle in the space below. (*Hint:* Is the wave function *positive* or *negative* on the right half of the well?)

b. Sketch the probability density for this particle in the space below.

c. Explain why the normalization constant for this particle's wave function is the same as the normalization constant for the first excited-state wave function.

d. Predict whether the probability that this particle is measured to have energy E_1 is *greater than, less than,* or *equal to* the probability that this particle is measured to have energy E_2. Explain your reasoning.

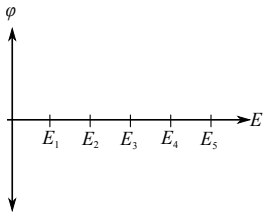
e. *Predict* whether the probability that this particle is measured to have energy E_1 is *greater than, less than,* or *equal to* the probability that this particle is measured to have energy E_3. Explain your reasoning.

f. Determine the first five coefficients in the energy basis. (You may wish to use computer software to carry out the integration.)

QM
HW-4

Homework: probability amplitude

g. Graph this state on the axes below using the energy representation.



h. Optional (extra credit): Determine the momentum-space wave function for this particle and sketch a graph of this function in the space below or using a computer program.

HOMEWORK: PERTURBED POTENTIALS AND WAVE FUNCTIONS

Name _____ QM HW-1

1. Consider a particle in the quantum mechanical infinite square well of width $2a$ centered on the origin. The wave function for this particle is $\psi(x) = \sqrt{\frac{1}{a}}\cos\left(\frac{\pi x}{2a}\right)$ for $-a < x < a$ and zero for all other values of x.

   Suppose that, at $t = 0$, the width of the infinite square well is doubled, such that the potential is now zero for $-2a < x < 2a$ and infinite for all other values of x. The wave function for the particle at this instant is the same as the wave function described above.

   a. In the space below, sketch both the potential and the wave function at $t = 0$. (*Note:* pay attention to the positions for which the initial wave function defined above is zero.)

   b. Determine an expression for the wave function in terms of the energy eigenfunctions of the new potential (of width $4a$). Write an integral for any coefficients in your expression, but do not evaluate any integrals yet. Explain.

   c. Write an expression for the wave function for this particle at a later time $t > 0$. Explain how you arrived at your answer.

d. Determine an expression for the probability of measuring each of the following energies. In each case, if any probability is equal to zero, state so *explicitly*. Explain.

i. The *n*th excited state of the *original* infinite square well (width 2*a*).

ii. The *n*th excited state of the *new* infinite square well (width 4*a*).

Note that the energy eigenfunctions for the new potential (of width 4*a*) can be written $\psi'_n(x) = \sqrt{\frac{1}{2a}}\cos\left(\frac{n\pi x}{4a}\right)$ for *odd* values of *n* and $\psi'_n(x) = \sqrt{\frac{1}{2a}}\sin\left(\frac{n\pi x}{4a}\right)$ for *even n*.

e. **Extra credit:** calculate the coefficients and the probabilities. Show your work. If you make use of any integration software, please state so explicitly (you may attach additional paper with your work on it).

HOMEWORK: TIME-INDEPENDENT PERTURBATION THEORY

Name _____ QM HW-1

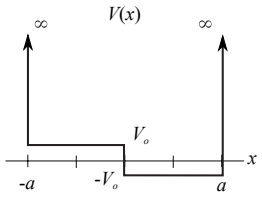
1. Sketch at least one example of a potential well that would result from a perturbation to the infinite square well such that the following will be true. Explain how you determined each sketch.

   a. The first order correction to the energy is positive for all energy levels?

   b. The first order correction to the energy is negative for all energy levels?

   c. The first order correction to the energy is zero for all energy levels? (Assume the potential is not equal to zero everywhere.)

2. Consider the perturbation to the infinite square well shown at right. Suppose that V_o is equal to $E_1/10$, where E_1 is the unperturbed energy of the ground state.



   a. Is the first-order correction to the energy for the *first excited-state* of this system *positive*, *negative*, or *zero?* Explain using the ideas developed in the tutorial; do not perform any integrals for this question.

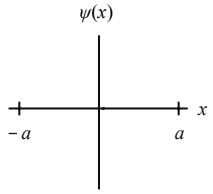
   b. Determine the wave function, correct to first order, for the first excited-state for this system using the following steps.

      i. Identify any terms that are zero due to symmetry. Explain.

ii. Identify any terms that you can neglect because they are small relative to other terms. Explain. (*Hint:* You should only have two terms remaining.)

iii. Calculate each term not identified as zero or negligible in the expression for the first-order correction to the wave function. (You may wish to make use of trigonometric identities or integration software.)

iv. Carefully graph the result in the space below by adding the corrections to the unperturbed wave function.



c. Are your answers consistent with the Schrödinger Equation? (*Hint:* Compare the wavelength on each half of the well to what you would expect from $\frac{\partial^2 \psi_n}{\partial x^2} = -\frac{2m}{\hbar^2}\left[E_n - V(x)\right]\psi_n(x).$)

Bonus question: On a separate sheet of paper, calculate the full **second-order** correction to the energy for the first-excited state for this system (do *not* neglect any terms). Show that your answer has the correct sign using the ideas developed in the tutorial.

HOMEWORK: DEGENERATE PERTURBATION THEORY

Name _____ QM HW-1

1. Recall the potential from the tutorial, $H = -\frac{\hbar^2}{2M}\nabla^2 + \frac{1}{2}M\omega^2\left(x^2+y^2\right) + \gamma\frac{1}{2}M\omega^2(x+y)^2$. You may find the integrals shown at right helpful for this problem.

   a. Show that the first-order correction to the energy of the ground state is $\gamma\hbar\omega/2$.

$$\int_{-\infty}^{\infty}|\varphi_0(x)|^2\,dx = \int_{-\infty}^{\infty}|\varphi_1(x)|^2\,dx = 1$$

$$\int_{-\infty}^{\infty}x^2|\varphi_0(x)|^2\,dx = \frac{\hbar}{2M\omega}$$

$$\int_{-\infty}^{\infty}x^2|\varphi_1(x)|^2\,dx = \frac{3\hbar}{2M\omega}$$

$$\int_{-\infty}^{\infty}x\varphi_0(x)\varphi_1(x)\,dx = \sqrt{\frac{\hbar}{2M\omega}}$$

   b. Calculate each of the terms in the matrix from the tutorial (shown below).

$$\begin{pmatrix} \iint\left(\varphi_0(x)\varphi_1(y)\right)H'(x,y)\left(\varphi_0(x)\varphi_1(y)\right)dx\,dy & \iint\left(\varphi_1(x)\varphi_0(y)\right)H'(x,y)\left(\varphi_0(x)\varphi_1(y)\right)dx\,dy \\ \iint\left(\varphi_0(x)\varphi_1(y)\right)H'(x,y)\left(\varphi_1(x)\varphi_0(y)\right)dx\,dy & \iint\left(\varphi_1(x)\varphi_0(y)\right)H'(x,y)\left(\varphi_1(x)\varphi_0(y)\right)dx\,dy \end{pmatrix}$$

QM HW-2 **Homework: Degenerate perturbation theory**

2. Consider the anisotropic harmonic oscillator potential from the tutorial, for which the Hamiltonian is $H = -\frac{\hbar^2}{2M}\nabla^2 + \frac{1}{2}M\omega^2\left(x^2+y^2\right) + \gamma\frac{1}{2}M\omega^2(x+y)^2$, where γ is small and positive. In this problem, you will solve this potential exactly and compare the energies to those calculated in the tutorial via degenerate perturbation theory.

   a. Find *u* and *v* in terms of *x* and *y* using substitution of variables such that the Hamiltonian is:

$$H = -\frac{\hbar^2}{2M}\nabla^2 + \frac{1}{2}M\omega^2\left(u^2 + (1+2\gamma)v^2\right)$$

   b. Determine the (exact) energy of the first three states of the perturbed potential and enter them in the first column of the table at right. Explain.

Perturbed state	Exact energy	Approx. energy	Perturbed energy
Ground state			
1st excited-state			
2nd excited-state			

   c. Approximate the exact energies using the relation $\sqrt{1+2\gamma} \approx 1+\gamma+\ldots$ and enter them in the middle column of the table above.

   d. Fill in the last column in the table using your results from the tutorial. Compare this to the middle column.

      i. To which of the *good* (unperturbed) degenerate first excited-states from the end of the tutorial does the **first** excited-state energy of the perturbed system correspond? Explain.

      ii. To which of the *good* (unperturbed) degenerate first excited-states does the **second** excited-state energy of the perturbed system correspond? Explain.

Homework: Degenerate perturbation theory Name _____ QM HW-3

3. Consider the operator defined by: $\hat{R}\psi(x,y) = \psi(y,x)$.

   a. Show that this operator is Hermitian.

   b. Does this operator commute with the perturbed Hamiltonian (see problem 2)? Explain.

   c. Does this operator commute with the *unperturbed* Hamiltonian (see problem 1)? Explain.

d. Show that the *good* degenerate first excited-states you found in the tutorial are eigenstates of this operator. Explain why this makes sense.

e. Determine the eigenvalue of $\hat{R}$ for each of these states. Do they have the same or different eigenvalues? Explain.

f. Explain why it would be useful to find such an operator, which commutes with both the unperturbed and perturbed Hamiltonians, *before* diagonalizing the degenerate matrix from the tutorial.

C.III – Homeworks for Part III

This sub-section includes copies of all tutorial homeworks discussed in part III of this dissertation. Each homework is discussed in chapter 10.

HOMEWORK: ANGULAR MOMENTUM IN QUANTUM MECHANICS

Name _____ QM HW-1

1. Consider an electron in an unknown three-dimensional potential, $V(r, \theta, \varphi)$.

   a. In classical mechanics, under what conditions is the angular momentum a constant of motion (*i.e.*, a conserved quantity)? Explain. (*Hint:* On which variables can the potential depend?)

   b. When can you simultaneously determine l and the total energy of the system?

      When can you simultaneously determine m_l and the energy?

   c. Do your answers to questions a and b, for the classical and quantum mechanical cases, respectively, agree? Explain.

Recall that the eigenfunctions for angular momentum are usually written in terms of the spherical harmonics, $Y_l^m(\theta,\varphi)$. For the remainder of this problem, assume that these functions are ***not*** the energy eigenfunctions of the system.

   d. At $t = 0$, let the wave function for this electron be $\psi(\vec{r}) = f(r)Y_2^{-1}(\theta,\varphi)$. Assume this wave function is normalized, and $f(r)$ is a function of the radial coordinate only.

      i. Determine the possible values of a measurement of the z-component of the orbital angular momentum for this electron at $t = 0$. If you do not have enough information to answer, state so explicitly. Explain.

      ii. Determine the possible values of a measurement of the z-component of the orbital angular momentum for this electron at $t_1 > 0$. If you do not have enough information to answer, state so explicitly. Explain.

      iii. Under what conditions will the probability of measuring $m_l = -1$ be independent of time? Explain.

QM HW-2 ***Homework:*** Angular momentum in quantum mechanics

2. Consider an electron in a hydrogen atom potential.

   a. Assume the state of the particle is given by $|\psi\rangle = |n,l,m_l\rangle = |3,2,-1\rangle$.

      i. Write the wave function for this state in terms of $R_{nl}(r)$ and $Y_l^{m_l}(\theta,\varphi)$ (you do not have to write these out in terms of *r, θ,* and *φ*).

      ii. On which quantum numbers does the energy of this electron depend? If there are any quantum numbers on which the energy *does not* depend, state so explicitly. Explain.

   b. Now assume that the wave function for this particle is given by:

   $$\psi(\vec{r}) = \frac{1}{\sqrt{14}}\left(R_{42}(r)Y_2^1(\theta,\varphi) - 3R_{31}(r)Y_1^1(\theta,\varphi) + 2iR_{32}(r)Y_2^1(\theta,\varphi)\right).$$

      i. Write the state of the particle in Dirac notation.

      ii. Assume you were to make a measurement of the energy of this particle. List all possible results of this measurement and the probability associated with each. Explain.

      iii. Assume you were to make a measurement of the orbital angular momentum squared (L^2) for this particle. List all possible results of this measurement and the probability associated with each. Explain.

      iv. Assume you were to make a measurement of the *z*-component of the orbital angular momentum *(L_z)* for this particle. List all possible results of this measurement and the probability associated with each. Explain.

      v. Which, if any, of the three quantities above are well-defined for this particle? Explain.

Homework: Angular momentum in quantum mechanics Name _____ QM HW-3

3. Recall the generalized uncertainty principle, $\sigma_A \sigma_B \geq \frac{1}{2}\left|\left\langle\psi\left|\left[\hat{A},\hat{B}\right]\right|\psi\right\rangle\right|$. Note that the quantity σ_A is known as the *uncertainty* in quantity *A* (and similarly for *B*).

   a. Suppose that quantity *A* is well-defined for a quantum mechanical particle (*i.e.*, there is only one possible outcome of a measurement of that quantity). Would the uncertainty in quantity *A*, σ_A, be *greater than*, *less than*, or *equal to* zero? Explain.

   b. Suppose that *all three* components of the angular momentum vector for a particle with $l \neq 0$ are well-defined. Show that this is inconsistent with the uncertainty principle.

   c. Suppose a quantum mechanical particle has a well-defined value of L_z only. Determine the expectation value of L_y, $\left\langle\psi\left|\hat{L}_y\right|\psi\right\rangle$, starting from the left hand side of the uncertainty principle? *(Hint:* What should you choose for quantities *A* and *B?)*

      Does your answer depend on the value of L^2 for the particle? Explain.

   d. With which of the following statements, if either, do you agree? Explain.

      Student 1: "If one component of angular momentum is well-defined, then the expectation value of the perpendicular components will be zero."

      Student 2: "I agree. That means the probability of measuring zero for those other components will always be the largest, no matter which value we started with."

QM HW-4 **Homework:** Angular momentum in quantum mechanics

4. An electron is known to be in the orbital angular momentum state $\left|l,m_l\right\rangle_x = \left|1,1\right\rangle_x$.

   a. List the possible results of a measurement of the z-component of orbital angular momentum. Explain.

   b. *Predict* the ranking of the probabilities of each of the possible results of this measurement from most probable to least probable. Explain.

Note the column vector representations: $\left|1,1\right\rangle_z \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\left|1,0\right\rangle_z \equiv \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $\left|1,-1\right\rangle_z \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

   c. Optional: Show on your own paper that the operator for L_x, written in the z-basis, is given by: $\frac{\hbar\sqrt{2}}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. (*Hint:* Recall that $\hat{L}_x = \frac{\hat{L}_+ + \hat{L}_-}{2}$ and take advantage of the following property of the raising and lowering operators: $\hat{L}_\pm\left|l,m_l\right\rangle = \hbar\sqrt{l(l+1)-m_l(m_l\pm1)}\left|l,m_l\pm1\right\rangle$.)

   d. Find the column vector in the z-basis that represents the initial orbital angular momentum state $\left|l,m_l\right\rangle_x = \left|1,1\right\rangle_x$. (*Hint:* Use an eigenvalue equation.) Show your work.

   Use your answer to find the probability of each possible value of a measurement in the z-direction. Explain how you arrived at your answer.

e. Repeat your calculation from question d to find the column vector in the z-basis that represents the initial orbital angular momentum state for a particle with the initial state $|l, m_l\rangle_x = |1, 0\rangle_x$.

Use your answer to find the probability of each possible value of a measurement in the z-direction on this different particle. Explain how you arrived at your answer.

f. Consider the following statement.

"There are three possibilities for this measurement, so the probability of each possible outcome should be one-third. That's how it was for spin-1/2: the probabilities were always 50% for the two options.."

Explain why this statement is *incorrect*.

g. Consider the following statement.

"When we know the angular momentum in the z-direction, we know nothing about the other directions—we have zero information about them. This is a consequence of the uncertainty principle."

Explain why this statement is *incorrect*.

QM
HW-6

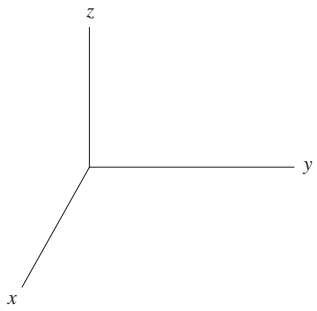
Homework: Angular momentum in quantum mechanics

5. An electron is known to be in the spin angular momentum state $|\chi\rangle = |s, m_s\rangle_z = |1/2, 1/2\rangle_z$.

   a. What is the magnitude of the spin angular momentum vector, *S*, for this electron? Explain.

   b. What is the *z*-component of the spin angular momentum, S_z, for this electron? Explain.

   c. On the diagram at right, draw the set of all possible vectors that could represent the angular momentum of a *classical* particle with an angular momentum equal to the spin of this electron. Label the total length as well as the length of each known component.

   What shape does your graph form?

   d. Is your answer consistent with your knowledge of the *x*- and *y*-components of the spin angular momentum of this electron? Explain.

   e. Do you agree or disagree with the following statement? Explain.

   "This cone is composed of all the possible vectors for the angular momentum of the electron. One of these is the actual vector, but I don't know which one it is until I measure it."

   f. How would the picture you drew above differ if instead the electron were known to be in an eigenstate of the *y*-component of the spin angular momentum, S_y? Explain.

HOMEWORK: ADDITION OF ANGULAR MOMENTUM

Name _____ QM HW-1

1. Consider an electron in a hydrogen atom potential. Let the wave function for this particle be:

$$\frac{1}{\sqrt{7}}\left(R_{54}(r)Y_4^1(\theta,\varphi)\chi_- + i\sqrt{3}R_{61}(r)Y_1^0(\theta,\varphi)\chi_+ - \sqrt{2}R_{64}(r)Y_4^0(\theta,\varphi)\chi_+ + e^iR_{21}(r)Y_1^1(\theta,\varphi)\chi_-\right)$$

   a. Write the state of the particle in Dirac notation (using the $|n,l,m_l;s,m_s\rangle$ basis).

   b. Assume you were to make a measurement on this particle. For each possible measurement, list all possible results and the probability associated with each. Explain.

Measurement	Possible Values	Probabilities
E		
L^2		
L_z		
S_z		
J_z		
J^2		

   c. Which, if any, of the quantities in parts b–g are well-defined for this particle? Explain.

QM HW-2 *Homework: Addition of angular momentum*

2. Consider an electron in a hydrogen atom potential with the state $|l,s;j,m_j\rangle = |1,½;½,-½\rangle$.
(Note that this is represented in a different basis from problem 1.)

   a. Assume you were to make a measurement of L_z for this particle. List all possible results of this measurement and the probability associated with each. Explain. (*Note:* If you use any resources, such as the textbook, be sure to note it explicitly.)

   b. Assume you were to make a measurement of S_z for this particle. List all possible results of this measurement and the probability associated with each. Explain.

   c. Suppose a measurement of L_z on this particle resulted in a value of 0. Is the quantity S_z well defined? If so, give its value. Explain.

   d. Consider the statements given by the students below. State whether you agree or disagree with each student. Explain your reasoning.

   Student 1: "l = 1 for this state, so there are three possible values, just like always: 0 or ±ℏ."

   Student 2: "I don't think you could measure +ℏ for this state, though, because that would mean the z-component of spin would have to be -3/2, and that's not possible."

   Student 3: "I agree, there are only two options. That means there must be a 50% chance of measuring 0 and a 50% chance of measuring –ℏ."

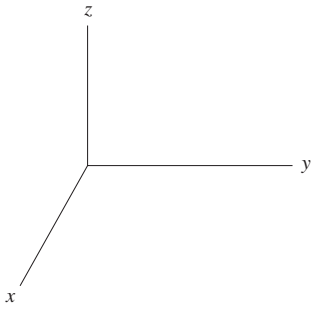
Homework: Addition of angular momentum Name _____ QM HW-3

3. Consider a particle in the state given by $|l, m_l; s, m_s\rangle = |2,1;1,1\rangle$.

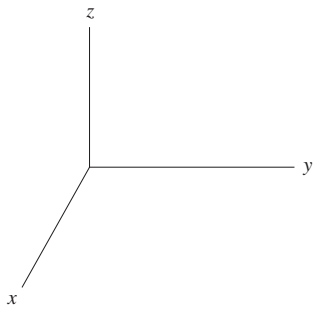
   a. What is the magnitude of the orbital angular momentum vector, *L*, for this particle? Explain.

   b. What is the *z*-component of the spin angular momentum, L_z, for this particle? Explain.

   c. On the diagram at right, use a cone to represent the orbital angular momentum of this particle. Draw relative lengths as accurately as possible, and label the total length as well as the length of each known component.



   d. On the diagram at right, use a cone to represent the *spin* angular momentum of this particle. Draw relative lengths as accurately as possible, and label the total length as well as the length of each known component.



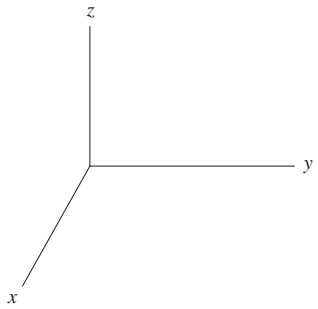
   Consider the *total* angular momentum, $\vec{J} = \vec{L} + \vec{S}$.

   e. Use the cone representation to represent several different possibilities for the total angular momentum for this particle. (*Hint:* Start by drawing the cone for *L*, then place the base of the cone for *S* at one of the points corresponding to a possible tip for *L*. Repeat this with several possible points.)

f. Is there a single, well-defined value for the magnitude of the total angular momentum, J, of this particle? Explain.

g. Determine all the possible values of the total angular momentum, J, of this particle. Explain. Make sure your answer is consistent with the quantum number m_j for this state.

h. Use cones to represent the total angular momentum of this particle. (*Hint:* Based on your previous answer, how many cones should you draw? What should each of these cones have in common?)



i. Consider the student statements below:

Student 1: "I found the total angular momentum by adding a cone to another cone, so **J** can be any of the vectors that result from adding all the possible vectors for **L** and **S**."

Student 2: "But in quantum mechanics angular momentum is quantized, so the vector **J** for our particle must also correspond to an integer or half-integer value of *j*."

Student 3: "I agree that it must be quantized, but we also cannot know all three components of **J** at the same time. So this representation can tell us something about the allowed values of *j*, but it can't actually tell us which vector corresponds to **J** because there is no such vector."

In the space below, write down whether you agree or disagree with each student. Explain.

Homework: Addition of angular momentum Name _____ QM HW-5

4. The quantum state of a system consisting of two identical spin-1 particles may be written in terms of the basis vectors given by $|s_1, m_1; s_2, m_2\rangle = |1, m_1; 1, m_2\rangle$.

   a. Write out each of the basis states for this system. What is the dimension of this basis?

   Let $\vec{J} = \vec{S}_1 + \vec{S}_2$ be the total spin angular momentum of this system. Note that we will use the $|s_1, s_2; j, m_j\rangle \equiv |j, m_j\rangle$ basis, where s_1 and s_2 have been omitted to distinguish this basis from the $|s_1, m_1; s_2, m_2\rangle$ basis.

   b. What are the possible values of the quantum number *j?* Explain.

   c. For each value of *j,* what are the allowed values of m_j*?* Use this to write down all the possible states in the $|j, m_j\rangle$ basis.

   d. What is the dimension of the $|j, m_j\rangle$ basis? Is your answer consistent with your answer to part a?

   e. There are two states in either basis that can be written in terms of exactly one of the basis states in part c. Determine these states and their corresponding form in the other basis. (*Hint:* Compare the quantum numbers for the *z*-direction.)

f. Recall the definition of the lowering operator for angular momentum, $\hat{J}_-$, which lowers the quantum number m_j by one unit: $\hat{J}_-\left|j,m_j\right\rangle = \sqrt{j(j+1)-m_j(m_j-1)}\left|j,m_j-1\right\rangle$.

i. Beginning with $\left|2,2\right\rangle = \left|1,1;1,1\right\rangle$, write $\left|2,1\right\rangle$ as a linear combination of states in the $\left|s_1,m_1;s_2,m_2\right\rangle$ basis by acting with the operator $\hat{J}_- = \hat{S}_{1-} + \hat{S}_{2-}$. (*Note:* S_1 and S_2 have the same effect as *J,* but each acts only on its respective state.)

ii. Consider the state $\left|j,m_j\right\rangle = \left|1,1\right\rangle$.

What are the possible values of m_1 and m_2 for this state?

Is this state orthogonal to the state $\left|j,m_j\right\rangle = \left|2,1\right\rangle$?

Use your answers to the previous two questions to write this state as a linear combination of states in the $\left|s_1,m_1;s_2,m_2\right\rangle$ basis.

g. Repeat the procedure from part f on the states with $m_j = 1$, $\left|2,1\right\rangle$ and $\left|1,1\right\rangle$, to find expressions in the $\left|s_1,m_1;s_2,m_2\right\rangle$ basis for each of the three states with $m_j = 0$: $\left|2,0\right\rangle$, $\left|1,0\right\rangle$ and $\left|0,0\right\rangle$. Show your work.

h. Write expressions for the two states with $m_j = -1$ in the $\left|s_1,m_1;s_2,m_2\right\rangle$ basis. (*Hint:* Take advantage of the natural symmetry with $m_j = 1$; you do not have to do any new algebra.)

C.IV – Additional Homeworks

This sub-section includes copies of tutorial homeworks not discussed in detail in this dissertation.

HOMEWORK: BLACKBODY RADIATION

Name _____ QM HW-1

You may find the simulation at http://phet.colorado.edu/en/simulation/blackbody-spectrum helpful in answering the questions on this homework.

1. Four different rooms with no external sources of light are described below. Assume that each bar of iron is identical.

   i. Room 1 contains one of iron at room temperature (300 K).

   ii. Room 2 contains one bar of iron at 900 K.

   iii. Room 3 contains four bars of iron at 900 K.

   iv. Room 4 contains one bar of iron at 1100 K.

   a. Briefly describe the appearance of each room (recall that there are no external sources of light).

   b. Rank the rooms from greatest to least by the total amount of power emitted. If the total power emitted from any of the rooms is zero, state so explicitly. Explain.

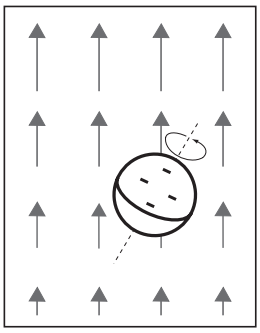
   You look inside a fifth room and see three identical bars of iron, but observe that the total power emitted from room 5 is the same as the total power emitted from room 4.

   c. Determine the temperature of room 5. If the temperatures are the same, state so explicitly. Explain.

   d. Explain how it is possible for two objects to have the same temperature but emit different amounts of total power. In particular, what physical property must differ between the two objects? What numerical quantity (related to power) would be the same for the two objects?

HOMEWORK: SPIN Name _____ QM HW-1

2. Consider a solid spinning sphere of negative charge in a *non-uniform* magnetic field at instant t_1 as shown at right (Note: you thought briefly about a similar situation during class.)



a. Indicate the direction of the magnetic moment of the sphere at instant t_1.

b. Indicate the direction of the angular momentum of the sphere at instant t_1. How is the angular momentum related to the magnetic moment?

c. Determine the direction of the net torque on the sphere at instant t_1. (*Hint:* use the relationship $\vec{\tau} = \vec{\mu} \times \vec{B}$.)

d. Compare the direction of the net torque with the direction of the angular momentum at instant t_1. How does the net torque on an object affect its angular momentum?

e. Describe the direction of the angular momentum vector for the sphere at instant t_2, a short time after t_1. Explain your reasoning.

f. Describe the direction of the net torque for the sphere at instant t_2. Compare the direction of the net torque with the direction of the angular momentum at instant t_2.

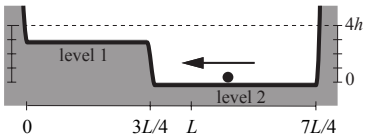
g. What motion does the sphere undergo as time evolves?

h. Would the motion you described above affect the component of the magnetic moment measured by the Stern-Gerlach experiment you studied in class? Explain your reasoning.

HOMEWORK: CLASSICAL PROBABILITY

Name _____ QM HW-1

1. Consider the situation from the tutorial, reproduced at right.



   a. What is the average position of the ball when it is on level 1? Explain.

   b. What is the average position of the ball when it is on level 2? Explain.

   c. What is the average position of the ball in the whole system? Explain how you used the answers to questions a and b.

   d. Imagine you were to divide the whole track into seven equal-sized regions. Explain how to use the average position within each small region and the probability of finding the ball within each small region to find the average position for the whole system.

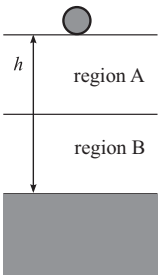
   e. Write a general expression that shows how the probability density, $\rho(x)$, can be used to calculate the average position of the ball. Explain your reasoning.

   f. Write a general expression that shows how the probability density, $\rho(x)$, can be used to calculate the average speed of the ball. Explain your reasoning. (*Hint:* Use a process similar to questions a–e above.)

QM
HW-2

Homework: Classical Probability

2. Consider a system (shown at right) consisting of a ball dropped from a height *h* above the ground. Assume the ball bounces and rises back to the same height, and repeats this motion forever.



   a. Consider the two regions of equal length shown in the figure at right. If the position of the ball is measured at a random time, is the probability that the ball is found in region A *greater than, less than,* or *equal to* the probability that the ball is found in region B? Explain.

   b. Divide region B into three equal-sized sub-regions and rank them by the probability that the ball is found in each sub-region. Explain.

   c. Divide the entire height into two new regions, region 1 and region 2 such that the probability that the ball is found in region 1 is equal to the probability that the ball is found in region 2. Determine the length of each region. Explain.

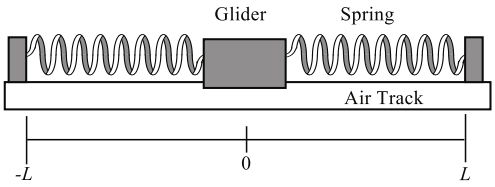
   d. Make a *qualitative* graph of the probability density for this ball, $\rho(y)$, versus the height, y, above the ground. Explain how you arrived at your graph.

   e. Do you agree or disagree with the following statement? Explain your reasoning.

   "Since the speed of the ball is different at each position, there is no way to define probability density because I can't calculate the probability at just one point."

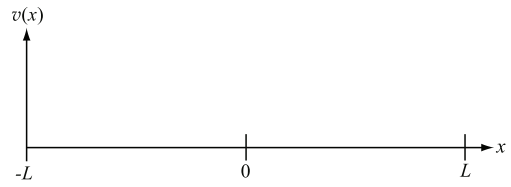
Homework: Classical Probability Name _____ QM HW-3

3. In the experiment at right, a glider on an air track is attached to two identical springs, one on each end. The glider moves without friction with amplitude L and period T.

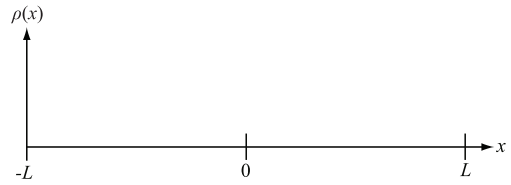


   a. Qualitative analysis

      i. In the space at right, sketch a qualitatively correct graph of the speed $v(x)$ of the glider as a function of position x along the track. Explain.



      ii. In the space at right, sketch a qualitatively correct graph for the probability density $\rho(x)$ of the glider as a function of x. Explain.



      iii. Consider the following student dialogue.

      Student 1: "As the glider moves, it is always being pulled toward the center of the track. That means you're most likely to find the glider at the center."

      Student 2: "That's right. The glider spends the same amount of time to the left of center as it does to the right of it, so it would be close to the center most of the time."

      Both students are *incorrect*. Identify the flaw(s) in each student's reasoning.

b. Consider the probability density for a glider given by $\rho(x) = \dfrac{A}{\sqrt{L^2 - x^2}}$, where A is a constant.

i. What is the probability that the glider is found between $x = -L$ and $x = L$? Explain.

ii. Determine the value of A that is consistent with your answer to the previous question. (*Hint:* The substitution $x = L \sin\theta$ might be helpful.)

iii. In the space at right, sketch a graph of $\rho(x)$ *vs.* x.

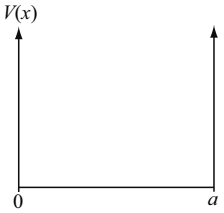
iv. Is your graph consistent with the graph you drew in part a? Resolve any inconsistencies. In particular examine the points $x = 0$ and $x = \pm L$.

v. Determine the average position of the glider. Show your work. Check that your answer is consistent with the graph you drew above.

HOMEWORK: RELATING CLASSICAL AND QUANTUM MECAHNICS

Name _____ QM HW-1

1. In the tutorial we considered the high-energy limit of a quantum mechanical system. In this problem you will consider the large mass limit.

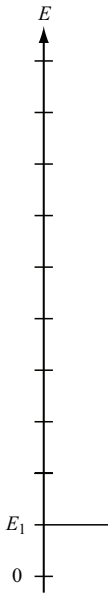


Consider the infinite square well potential shown at right.

a. Sketch the probability density for the ground state and the second excited state ($n = 3$) for a particle of mass m in the infinite square potential shown.

b. Sketch the probability density for the ground state and the second excited state ($n = 3$) for a particle of mass $10m$ in the same infinite square potential.

c. On the energy level diagram at right, the ground state of the particle of mass m is labeled E_1.



Draw the energy levels for each of the following cases.

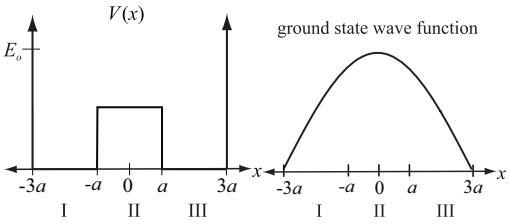
- the second excited state of the particle of mass m, E_3
- the ground state of the particle of mass $10m$, E_1'
- and the second excited state of the particle of mass $10m$, E_3'

d. Suppose a particle of mass m_1 is in a given excited state n of an infinite square well and particle m_2 (much larger than m_1) is in a different excited state n' of the same potential.

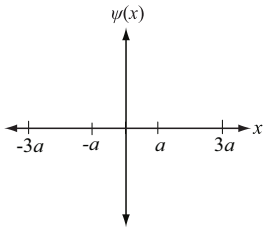
i. Is it possible for the two particles to have the same energy? If so, give a numerical example for which it is possible. In either case, explain.

ii. How do the forms of the wave functions for these two states compare to each other? Which one more closely resembles a classical particle? Explain how your answer is consistent with the correspondence principle.

2. A particle of mass m is bound in the potential well shown at right. The wave function for the ground state is also shown. It has energy E_o.



a. Sketch the wave function for a very highly-excited energy state of this potential well *(i.e.*, for a state with energy much greater than E_o*)*. Explain your reasoning.



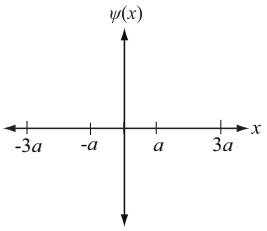
Explain how your answer is consistent with the correspondence principle.

b. Suppose a very massive particle were bound in an energy eigenstate of this same potential well with energy, E_o, equal to the ground state energy of the **original** particle of mass m.

i. In which regions will the wave function have the largest wave number? Explain.

ii. In which regions will the wave function have the largest amplitude? *(Hint:* Use the correspondence principle, and what you learned in the tutorial *Classical probability.)*

iii. Sketch the wave function for this very massive particle.



c. Compare the wave functions you drew for the highly energetic case and the highly massive case. In particular, consider how the details of the potential well are reflected in both cases.

d. Consider the student discussion below.

Student 1: "You can always tell if you can apply classical reasoning to a quantum mechanics problem by considering the energy alone."

Student 2: "No, you only approach the classical limit if the mass is very large."

Student 3: "The mass alone is not enough to tell. You must consider the particle's energy relative to the energy it has in the ground state."

With which student do you agree? For each student with whom you do not agree, identify a specific case from above which contradicts that student's statement.

3. Consider a particle in the quantum mechanical harmonic oscillator potential.

   a. In the space below, sketch a graph of the wave function and the probability density for a highly-excited energy eigenstate in this potential.

   b. On your graphs above, identify the regions where the particle is most likely and least likely to be found. Explain.

   c. On your graphs above, identify the regions where the wave number of the particle is largest and smallest. Explain.

   d. The classical probability distribution for a particle in a harmonic oscillator potential has the form $\rho(x) = \frac{1}{\pi\sqrt{L^2 - x^2}}$, where L is the maximum displacement. Graph this in the space below and compare it to your result from part A.

   e. Are your answers consistent with the correspondence principle? Explain.

   f. For the quantum particle, is the probability that the particle is found to be beyond the classical turning point equal to zero? Explain.

   If necessary, modify your graphs so that they are consistent with your answer.

HOMEWORK: IDENTICAL PARTICLES

Name _____ QM HW-1

1. Consider a system consisting of two identical particles in the same infinite square well of width a. Two possible wave functions for this particle are given below:

$$\Psi_{\pm}(x_a, x_b) = \frac{1}{\sqrt{2}}\left(\psi_1(x_a)\psi_2(x_b) \pm \psi_1(x_b)\psi_2(x_a)\right)$$

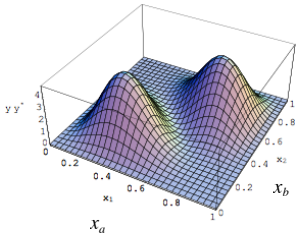
where x_a and x_b are the coordinates associated with the two particles, and $\psi_n(x)$ is the nth one-particle wave function for the infinite square well.

a. Show that the functional form of the probability density for these two wave functions is given below.

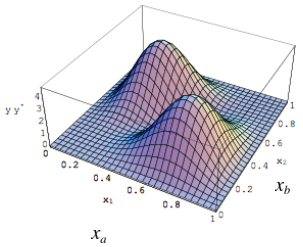
$$\rho_{\pm}(x_a, x_b) = \frac{1}{2}\left(\psi_1^2(x_a)\psi_2^2(x_b) + \psi_1^2(x_b)\psi_2^2(x_a) \pm 2\psi_1(x_a)\psi_1(x_b)\psi_2(x_b)\psi_2(x_a)\right)$$

b. Consider the set of points where x_a is approximately equal to x_b. Is the value of ρ_+ *greater than, less than,* or *equal to* the value of ρ_- at these points? Explain.

c. Consider the set of points where x_a and x_b are not approximately equal. Is the value of ρ_+ *greater than, less than,* or *equal to* the value of ρ_- at these points? Explain. (*Hint:* Use conservation of probability.)



d. Consider the two three-dimensional plots at right. Determine which plot corresponds to ρ_+ and which plot corresponds to ρ_-. Explain.



QM HW-2 *Homework: Identical particles*

2. Consider two identical spin-1/2 fermions in the same quantum mechanical harmonic oscillator potential. Recall that the energies of a single particle in this potential are given by $E_n = (n+½)\hbar\omega$.

   a. Suppose that both particles are known to be in the (single-particle) ground state. Determine all possible forms of the spin wave function. Explain.

   b. Suppose that both particles are known to have spin-up.

      i. Determine the lowest possible energy of such a configuration. Explain.

      ii. Determine the degeneracy of this energy level (*i.e.*, the total number of different combinations of spin and spatial states with this energy). Explain.

   c. Consider the second excited-state energy level for the two-particle system.

      i. Determine the value of the energy for this level (*Note:* more than one combination of spatial states will work). Explain.

      ii. Determine the degeneracy of this energy level. Explain.

      iii. Write the full wave function (spatial and spin) for each of the linearly independent states that compose this energy level. Explain.

Homework: Identical particles Name _____ QM HW-3

3. Consider two identical quantum particles in the same system with energy levels E_n that do not depend on spin. Let the one-particle energy eigenfunctions be given by $\psi_n(x)$.

   a. Suppose that the particles are known to have a spin wave function that is symmetric under exchange. For each of the three wave functions below, identify whether it could describe *identical bosons*, *identical fermions*, *both*, or *neither?* In each case, explain your reasoning.

      i. $\psi_A(x_1,x_2)=\frac{1}{\sqrt{2}}\left(\psi_2(x_1)\psi_3(x_2)+\psi_3(x_1)\psi_2(x_2)\right)$

      ii. $\psi_B(x_1,x_2)=\frac{1}{\sqrt{2}}\left(\psi_2(x_1)\psi_3(x_2)-\psi_3(x_1)\psi_2(x_2)\right)$

      iii. $\psi_C(x_1,x_2)=\frac{1}{\sqrt{2}}\left(\psi_2(x_1)\psi_2(x_2)-\psi_3(x_1)\psi_3(x_2)\right)$

   b. Suppose instead the particles are known to have a spin wave function that is **antisymmetric** under exchange. For each of the wave functions above, identify whether it could describe *identical bosons*, *identical fermions*, *both*, or *neither?* In each case, explain your reasoning.

   c. Consider the statements given by the students below. State whether you agree or disagree with each student. Explain your reasoning.

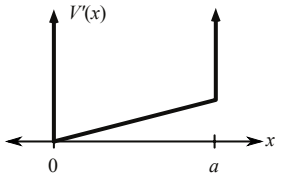
      Student 1: "The first wave function is symmetric, so it must be for a boson. The other two are antisymmetric because of the minus sign, so they must be for fermions."

      Student 2: "The third wave function has a minus sign, but it isn't symmetric or antisymmetric. All two-particle wave functions have to be one of those, so it can't describe this system."

HOMEWORK: WKB APPROXIMATION

Name _____ QM HW-1

1. Consider a particle in a highly-excited energy eigenstate of the potential shown at right, given by the functional form $V'(x) = E_1 x / a$, where E_1 is the energy of the unperturbed ground state.



   a. If you were to measure the position of this particle, would the probability that it is found on the right half of the well be *greater than, less than,* or *equal to* the probability that it is found on the left half of the well? Explain.

   b. Sketch the approximate form of this wave function in the space below. Clearly indicate the important features of your graph and how you were able to determine them.

   c. Determine $p(x)$ as a function of x for this system. Show your work.

   d. Draw an approximate phase space diagram ($p(x)$ vs. x) for this system for some fixed value of E much greater than E_1.

e. Use the following steps to determine the approximate energy levels of this well by quantizing the classical action in units of *h*.

i. Integrate the momentum as a function of x to get the classical action.

ii. Expand your result to first order by assuming that $E >> E_1$. Solve for E and compare your result with the unperturbed energies of the infinite square well.

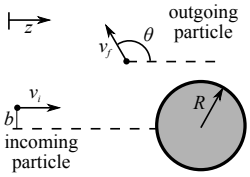
iii. Expand your result to second order by assuming that $E >> E_1$. Solve for E and compare it to the previous result.

f. Extra Credit: Treat the potential above as a perturbation to the infinite square well, and use first-order perturbation theory to graph several excited-states of this potential correct to first-order using computer software and compare it to your previous graphs.

HOMEWORK: SCATTERING IN QUANTUM MECHANICS

Name _____ QM HW-1

1. Consider a classical point particle of mass m incident on a large, hard sphere of radius R as shown in the figure at right. Suppose that the collision with the sphere is elastic, and the mass of the sphere is much greater than that of the incoming particle. The impact parameter b, the initial velocity v_i, and the positive z-direction are also indicated on the figure.



   a. Determine the magnitude and direction of the initial angular momentum of this particle in terms of the given quantities. What is the z-component of this angular momentum?

   b. Is the magnitude of the angular momentum of the outgoing particle *greater than*, *less than*, or *equal to* the magnitude of the angular momentum of the incoming particle? Explain.

   c. Determine the direction of the outgoing angular momentum of this particle. What is the z-component of this angular momentum?

   Suppose the impact parameter b were increased.

   d. Would this change the magnitude of the angular momentum of this particle?

   e. Would this change the direction of the angular momentum of this particle?

   Consider a quantum mechanical particle with the wave function $\psi_{in}(r,\theta,\phi) = Ne^{ikz}$ that is incident on a spherically symmetric potential.

   f. Which of the following quantities is well-defined for the incident particle?

      i. The magnitude of the linear momentum, p. Explain.

      ii. The orbital angular momentum L_z. Explain.

iii. The total angular momentum squared, L^2. Explain.

iv. Write the incident wave function $\psi_{in}(r,\theta,\phi) = Ne^{ikz}$ for this particle in terms of a superposition of $|k,l,m\rangle$, where l and m are the quantum numbers associated with L^2 and L_z, respectively. Do the coefficients in your expression depend on k, l, or m? Explain.

g. Recall that the wave function for this particle after it has been scattered may be written $\psi_{sc}(r,\theta,\phi) = Nf(\theta)\frac{e^{ikr}}{r}$, for large values of the radial distance from the scattering center, r.

i. Write the scattered wave function for this particle in terms of a superposition of $|k,l,m\rangle$. Do the coefficients in your expression depend on k, l, or m? Explain.

Consider a measurement of the orbital angular momentum squared (corresponding to the quantum number l) of both the incident and the scattered particle.

ii. Is the probability that a given value of l is measured for the scattered particle *greater than*, *less than*, or *equal to* the probability that the same value of l is measured for the incident particle? Explain.

iii. Use the answer to the previous question to relate the coefficients in your expressions for the incident (question a. iii.) and scattered (question b) wave functions. Identify a similarity and a difference between these coefficients. Explain.

Appendix D
Exam Questions

"There are more sad light quantum mice than heavy mice. Correlation between how much the quantum mouse eats and how happy it is? What do quantum mice eat?"

—Sophomore-level quantum mechanics student, final exam

This appendix includes copies of all exam questions discussed throughout this dissertation. Every variant for which the results are discussed is included separately. The exams are reproduced exactly as they were given to students.

Each sub-section of this appendix (D.I, D.II, and D.III) includes exam questions described in the corresponding part of this dissertation: part I (for quantum measurements and time dependence), part II (for vector spaces), and part III (for angular momentum).

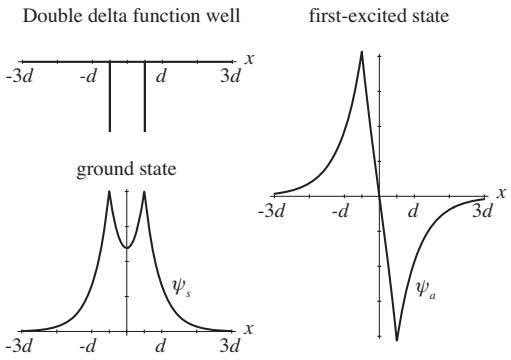
D.I – Exam Questions for Part I

The exam questions discussed throughout part I of this dissertation are included here. The questions are arranged by topic.

Name _____*Student ID* _____ *Score*_____
last first

Part III. [22 Pts.]

Consider a system with the potential shown at right consisting of two delta function wells separated by a distance, *d,* centered about the origin. The symmetric ground state wave function, ψ_s, and the anti-symmetric first excited state wave function, ψ_a, for this system are also shown at right. Assume that the energy of the anti-symmetric first excited state is four times the energy of the symmetric ground state, $E_a = 4E_s$.



The system is initially in a state given by.

$$\Psi(x, t = 0) = \frac{1}{\sqrt{2}}[\psi_s + i\psi_a]$$

1. [5 pts.] Is the initial probability of finding the particle to the left of the origin *greater than, less than,* or *equal to* the probability of finding the particle to the right of the origin? Explain.

2. [3 pts.] Is this state a stationary state? Explain how you know for full credit.

3. [4 pts.] Is there a time when the wave function is equal to: (i) the ground state wave function, ψ_s? (ii) the first excited state wave function, ψ_a? If so, give a time when this is true? Give your answers in terms of the variables given or describe a limiting case. Explain.

Name _____*Student ID* _____ *Score*_____
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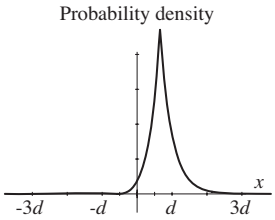
4. [3 pts.] Is there a time when the probability distribution is given by a flat line? If so, give a time when this is true? Give your answer in terms of the variables given or describe a limiting case. Explain.

5. [7 pts.] Is there a time when the probability distribution looks like the graph at right? If so, give a time when this is true? Give your answer in terms of the variables given or describe a limiting case. Explain. (*Hint:* Be mindful of the initial state given above.)



Name _____*Student ID* _____ *Score*_____
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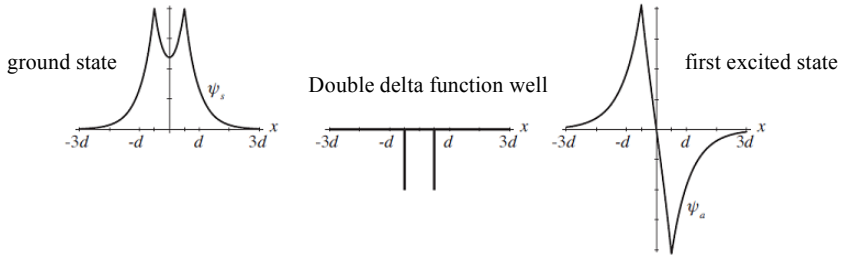
Part III. [30 Pts.]

Consider the potential shown below middle consisting of two delta function wells separated by a distance, *d,* centered about the origin. The symmetric ground state wave function, ψ_s, and the anti-symmetric first excited state wave function, ψ_a, for this potential are also shown. Assume that the energy of the first excited state is five times the energy of the ground state, $E_a = 5E_s$.



Consider a system initially prepared in a state given by: $\Psi(x,t=0) = [\psi_s + i\psi_a]/\sqrt{2}$

1. [10 pts.] Is the probability at $t = 0$ of finding the particle to the left of the origin *greater than, less than,* or *equal to* the probability of finding the particle to the right of the origin? Explain.

2. Consider again a system initially prepared in a state given by: $\Psi(x,t=0) = [\psi_s + i\psi_a]/\sqrt{2}$

   A. [4 pts.] Is this state a stationary state? Explain.

   B. [6 pts.] Is there a time when the wave function is equal to the ground state wave function, ψ_s? If so, give a time when this is true. Give your answers in terms of the variables given or describe a limiting case. Explain.

Name _____ *Student ID* _____ *Score*_____
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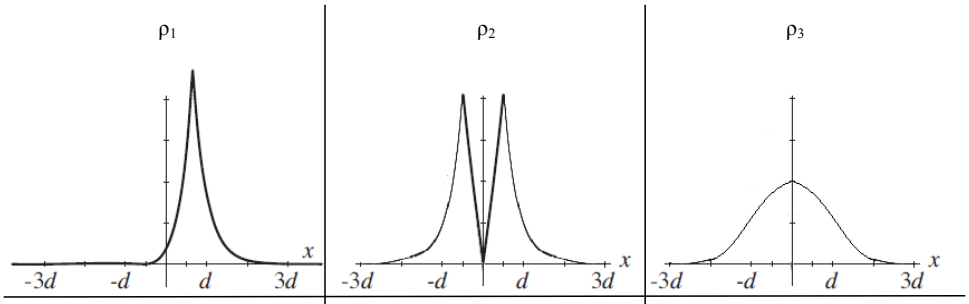
3. [10 pts.] Consider the three graphs below. Is there a time when the probability distribution for Ψ looks like each graph? If so, give a possible time when this is true or describe the wave function at this time (*i.e.* describe the relative phase between ψ_s and, ψ_a). If not, explain why the probability density can never have this form. Give your answers in terms of the variables given or describe a limiting case. In either case, explain your answers carefully. (*Hint:* Be mindful of the initial state from parts 1 and 2.)



Name _____
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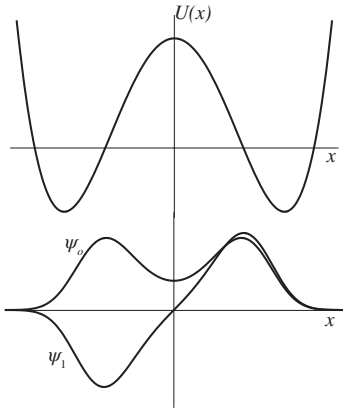
3. [30 pts.] Consider a system with the potential shown at right given by: $U(x) = ax^4 - bx^2 + c$. The symmetric ground state wave function, ψ_o, and the anti-symmetric first excited state wave function, ψ_1, for this system are also shown at right. Assume that the energy of the first excited state, E_1, is twice the energy of the ground state, E_o.



   a) The system is initially in a state given by: $\Psi(x,t=0) = [\psi_o + i\psi_1]/\sqrt{2}$

      1) (5 pts.) Write an expression for the probability density for this state at time $t = 0$.

      2) (5 pts.) Roughly sketch the probability density for this state at time $t = 0$.

   b) Consider again the state given by: $\Psi(x,t=0) = [\psi_o + i\psi_1]/\sqrt{2}$

      1) (5 pts.) Is this state a stationary state? Explain.

      2) (5 pts.) Is there a time when the wave function is exactly equal to the ground state wave function, ψ_o? If not, explain why not. If so, give a time when this is true, expressing your answers in terms of the variables given or describe a limiting case.

Name _____
last first

c) Consider again the state given by: $\Psi(x,t=0) = [\psi_o + i\psi_1]/\sqrt{2}$

1) (5 pts.) An energy measurement is made on this state at a later time t_1. What is the probability that the result of this measurement is E_1? If your answer depends on the time t_1, state so explicitly and describe the time dependence.

2) (5 pts.) Assume that the energy measurement above results in E_1. If another measurement is made at a much later time t_2, what is the probability that the result of this measurement is E_0? If your answer depends on time, state so explicitly and describe the time dependence.

Name _____*Student ID* _____ *Score*_____
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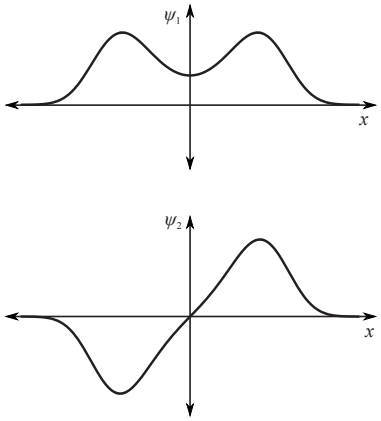
Part III. [30 Pts.]

Consider two particles, A and B, in identical quantum mechanical systems. The symmetric ground state wave function, ψ_1, and the anti-symmetric first excited state wave function, ψ_2, for this system are shown at right (both wave functions are entirely real). Assume that the energy of the first excited state, E_2, is **twice** the energy of the ground state, E_1.



The initial wave functions for the two particles are given below:

$$\Psi_A(x,t=0)=\frac{\psi_2-\psi_1}{\sqrt{2}} \quad \Psi_B(x,t=0)=\frac{\psi_2-i\psi_1}{\sqrt{2}}$$

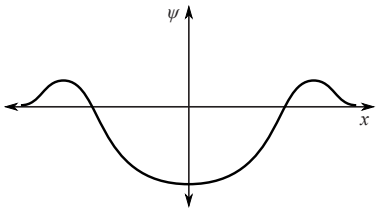
1. [7 pts.] Suppose the position of each particle were measured at $t = 0$. Is the probability that **particle A** is found to be on the right half of the well *greater than, less than,* or *equal to* the probability that **particle B** is found to be on the right half of the well? Explain.

2. [7 pts.] Is there a time when the wave function for **particle B** is exactly equal to the ground state wave function, ψ_1? If so, give a time when this is true. Give your answer in terms of the variables given or describe a limiting case. Explain. (Assume that no measurements were made during the previous question.)

Name _____*Student ID* _____ *Score*_____
last first

3. [8 pts.] Is there a time when the probability that **particle A** is measured to be on the right half of the well is equal to the probability that **particle A** is measured to be on the left half of the well? If so, give a time when this is true. Give your answer in terms of the variables given or describe a limiting case. Explain. (Assume that no measurements were made during the previous questions.)

A different particle in the same quantum mechanical system is known to have the wave function shown at right.



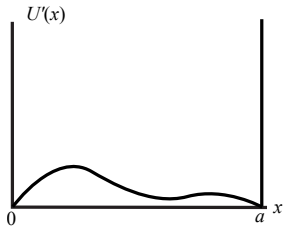
4. [8 pts.] Suppose the energy of this particle were measured. Is the probability that the particle is measured to have E_1 *greater than, less than,* or *equal to* the probability that it is measured to have E_2? If you do not have enough information to answer, state so explicitly. Explain.

Name _____*Student ID* _____ *Score*_____
last first

Question 3 [30 Pts.]

Consider a particle in an infinite square well of width a. Recall that the energy eigenvalues of this system are given by $E_n = n^2 E_1$, where E_1 is the ground-state energy. Suppose that this particle is initially in the **second excited-state ($n = 3$)**.

At $t = 0$, the small perturbation $U'(x)$ shown at right is turned on suddenly.



A. [7 pts.] Suppose you were to measure the energy of the particle at some time $t > 0$. Rank the probabilities, P_n, that the energy of the particle is determined to be E_n (the n^{th} **unperturbed** energy) for the first four values of n (*i.e.,* $n = 1, 2, 3, 4$). If any probabilities are equal to 1 or 0, state so explicitly. Explain your reasoning.

At $t = T$, the perturbation is turned off (instantaneously), such that the potential is the same as it was for $t < 0$. Assume that *no measurements* were made on the particle from $0 < t < T$.

B. [8 pts.] Suppose you were to measure the energy of the particle at some time $t > T$. Rank the probabilities, P_n, that the energy of the particle is determined to be E_n (the n^{th} **unperturbed** energy) for the first four values of n (*i.e.,* $n = 1, 2, 3, 4$). If any probabilities are equal to 1 or 0, state so explicitly. Explain your reasoning.

Name _____*Student ID* _____ *Score*_____
last first

Recall that this particle is initially in the **second excited-state (*n* = 3)** of the unperturbed potential. Assume that no other measurements have been made on this system.

C. [7 pts.] Is there a value of T such that the probability of measuring E_1, the **unperturbed** ground state energy, is equal to 0 after the perturbation is turned off? If so, determine at least one such time. Explain your reasoning.

Suppose that the time dependence of the perturbed potential (from $0 < t < T$) has the form $H'(x,t) = U'(x)\cos\left(7.1E_1 t/\hbar\right)$, where $U'(x)$ is the same potential as above. Recall that this particle is initially in the **second excited-state (*n* = 3)** of the unperturbed potential. Assume that no other measurements have been made on this system.

D. [8 pts.] Suppose you were to measure the energy of the particle at some time $t > T$. Rank the probabilities, P_n, that the energy of the particle is determined to be E_n (the n^{th} **unperturbed** energy) for the first four values of n (*i.e.,* n = 1, 2, 3, 4). If any probabilities are equal to 1 or 0, state so explicitly. Explain your reasoning.

Name _____*Student ID* _____ *Score*_____
last first

An electron is placed in a system with a constant magnetic field is turned on in the *z*-direction, such that the energy of the state $|+\rangle_z$ is $+\omega_p\hbar$ and the energy of the state $|-\rangle_z$ is $-\omega_p\hbar$.

1. Write down the Hamiltonian for this system in matrix form. Explain your reasoning.

At $t = 0$ the particle is measured to have spin-down in the *z*-direction.

2. Write an expression for the state of this particle as a function of time. If the state does not change in time, state so explicitly. Explain your reasoning.

3. Is this state a stationary state? If you do not have enough information to tell, state so explicitly. Explain your reasoning.

Name _____*Student ID* _____ *Score*_____
last first

An electron is initially in the state $\psi(t=0)=\frac{1}{\sqrt{5}}\begin{pmatrix}-i\\2\end{pmatrix}$. At $t = 0$, a constant magnetic field is turned on in the z-direction, such that the energy of the state $|+\rangle_z$ is $+\omega_p\hbar$ and the energy of the state $|-\rangle_z$ is $-\omega_p\hbar$.

4. At $t = 0$, is the probability of measuring spin up in the z-direction *greater than, less than,* or *equal to* the probability of measuring spin up in the x-direction? Explain your reasoning.

5. Is there a time $t_1 > 0$ when the probability of measuring **spin up** in the z-direction is *equal to zero?* If so, determine a time when this occurs and explain your reasoning. If not, explain why not.

6. Is there a time $t_1 > 0$ when the probability of measuring **spin up** in the x-direction is *equal to zero?* If so, determine a time when this occurs and explain your reasoning. If not, explain why not.

Name _____*Student ID* _____ *Score*_____
last first

An electron is placed in a system with a constant magnetic field is turned on in the *z*-direction, such that the energy of the state $|+\rangle_z$ is $+\omega_p\hbar$ and the energy of the state $|-\rangle_z$ is $-\omega_p\hbar$.

1. Write down the Hamiltonian for this system in matrix form. Explain your reasoning.

At $t = 0$ the particle is measured to have spin-down in the *z*-direction.

2. Write an expression for the state of this particle as a function of time. If the state does not change in time, state so explicitly. Explain your reasoning.

3. Is this state a stationary state? If you do not have enough information to tell, state so explicitly. Explain your reasoning.

Name _____*Student ID* _____ *Score*_____
last first

An electron is initially in the state $\psi(t=0)=\frac{1}{\sqrt{5}}\begin{pmatrix} i \\ -2 \end{pmatrix}$. There is a constant magnetic field in the *z*-direction, such that the energy of the state $|+\rangle_z$ is $+\omega_p\hbar$ and the energy of the state $|-\rangle_z$ is $-\omega_p\hbar$.

4. At $t = 0$, is the probability of measuring spin up in the *z*-direction *greater than, less than,* or *equal to* the probability of measuring spin up in the *x*-direction? Explain your reasoning.

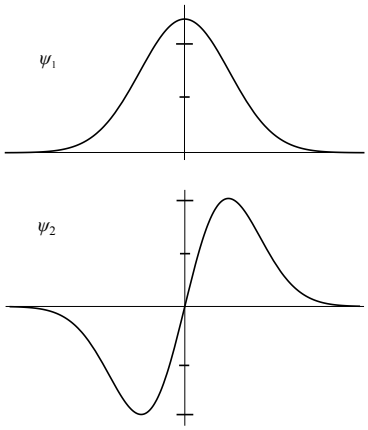
5. Is there a time $t_1 > 0$ when the probability of measuring **spin up** in the *z*-direction is *equal to zero?* If so, determine a time when this occurs and explain your reasoning. If not, explain why not.

6. Is there a time $t_1 > 0$ when the probability of measuring **spin up** in the *x*-direction is *equal to zero?* If so, determine a time when this occurs and explain your reasoning. If not, explain why not.

Name _____ *Student ID* _____ *Score*_____
last first

3. [30 Pts.]

Consider a system with a symmetric ground state wave function, ψ_1, and an anti-symmetric first excited state wave function, ψ_2, shown at right (both wave functions are entirely real). Assume that the energy of the first excited state, E_2, is **three times** the energy of the ground state, E_1.



The system is initially in a state given by: $\Psi(x,t=0)=\dfrac{\psi_1-\psi_2}{\sqrt{2}}$

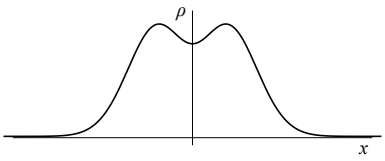
a) [5 pts.] Is this state a stationary state? Explain.

b) [5 pts.] Sketch the probability density for this state at time $t = 0$. Explain.

c) [5 pts.] Is there a time when the wave function is exactly equal to the ground state wave function, ψ_1? If so, give a time when this is true. Give your answers in terms of the variables given or describe a limiting case. Explain.

Name _____ *Student ID* _____ *Score*_____
last first

d) [5 pts] Could the system ever have the probability distribution shown at right? If so, give an approximate time in terms of E_1 and relevant constants when the system has the probability distribution shown and **explain**. If not, explain why not.



e) [5 pts] Suppose you were to measure the energy of the system at $t = 0$. What is the probability that the result of this measurement is E_1? Explain.

f) [5 pts] Suppose instead that you were to measure the energy of the system at time $t_1 = E_2/\hbar$ (no measurement was made at $t = 0$). Is the probability that the result of this measurement is E_1 *greater than, less than,* or *equal to* your answer to the previous question? Explain.

Name _____*Student ID* _____ *Score*_____
last first

3. (30 pts) Consider a particle in the quantum mechanical infinite square well of width *a.* At $t = 0$, the normalized wave function for the particle is given by $\Psi(x, t=0) = \sqrt{\frac{15}{a^5}}\left(x^2 - ax\right)$ for $0 < x < a$ and 0 elsewhere. **You may receive full credit on this problem without evaluating any integrals; however, writing down an integral without explaining where it comes from will not result in credit.**

   A. (5 pts) Suppose you were to measure the *energy* of this particle at $t = 0$. Determine the probability that the energy is equal to E_1, the energy of the ground state. Explain your reasoning.

   B. (5 pts) Does the probability that the energy is equal to E_1 depend on the time at which the measurement is made? Explain.

   C. (5 pts) Suppose you were to measure the *position* of this particle at $t = 0$ (no other measurements have been made). Determine the probability that the position of the particle is measured to be between $x = 0$ and $x = a/3$. Explain your reasoning.

Name _____*Student ID* _____ *Score*_____
last first

D. (5 pts) Does the probability that the position of this particle is measured to be between $x = 0$ and $x = a/3$ depend on the time at which the measurement is made? Explain.

Now suppose that the position of the particle is measured to be at $x = a/3$ at time t_1, where $t_1 > 0$.

E. (5 pts) Suppose you were to measure the position of this particle at t_2, where $t_2 > t_1$ (assume no measurements have been made between t_1 and t_2). Does the probability that the position of this particle is measured to be between $x = 0$ and $x = a/3$ depend on the time at which the measurement is made? Explain your reasoning.

F. (5 pts) Suppose instead you were to measure the energy of this particle at t_2 ,where $t_2 > t_1$ (assume no measurements have been made between t_1 and t_2). Does the probability that the energy of this particle is E_1 (the ground state energy) depend on the time at which the measurement is made? Explain your reasoning.

Name _____*Student ID* _____ *Score*_____
last first

Part III. [30 Pts.]

Consider an electron in the hydrogen atom potential. The particle's initial state is given by the expression $|\Psi(t=0)\rangle = \frac{|2,1,-1\rangle - |3,2,0\rangle}{\sqrt{2}}$, where $|n,l,m_l\rangle$ are the energy eigenstates of the system.

A. [10 pts.] For the particle defined above, would the probability of measuring a particular value for each of the following quantities depend on time: 1) *energy,* 2) *position,* or 3) *momentum?* Explain your reasoning for full credit. If you do not have enough information to tell, state so explicitly.

Name _____*Student ID* _____ *Score*_____
last first

Suppose that the energy of the particle initially defined above is measured and determined to be equal to the energy of the second excited state (E_3).

B. [10 pts.] After the measurement described above, would the probability of measuring a particular value for each of the following quantities depend on time: 1) *energy,* 2) *position,* or 3) *momentum?* Explain your reasoning for full credit. If you do not have enough information to tell, state so explicitly.

Suppose a different particle, also in the hydrogen atom potential, is initially in the state given by the expression $\left|\Phi(t=0)\right\rangle = \dfrac{\left|2,1,-1\right\rangle + i\left|3,2,1\right\rangle}{\sqrt{2}}$.

C. [10 pts.] For the state defined above, list the possible outcomes of the following two measurements: 1) *the z-component of orbital angular momentum (L_z)* and 2) *the x-component of orbital angular momentum (L_x)*. Explain your reasoning for full credit. If your answers depend on the time at which the measurement is made, or you do not have enough information to tell, state so explicitly.

D.II – Exam Questions for Part II

The exam questions discussed throughout part II of this dissertation are included here. The questions are arranged by topic.

Name: _____

4. [20 pts total] Given the states: $|\alpha\rangle = \frac{1}{\sqrt{2}}|+\rangle_z + \frac{1}{\sqrt{2}}|-\rangle_z$, $|\beta\rangle = \frac{1}{\sqrt{3}}|+\rangle_x - \sqrt{\frac{2}{3}}|-\rangle_x$, $|\gamma\rangle = \frac{1}{\sqrt{2}}|+\rangle_x - \frac{1}{\sqrt{2}}|-\rangle_x$, answer the following questions:

   a. [5 pts] What is $\langle\alpha|\beta\rangle$?

   b. [5 pts] If the spin was measured along the z-direction, rank the probability of obtaining $+\hbar/2$ for the three states. If any of the states have a probability equal to zero or one, state so explicitly.

THE WKB APPROXIMATION

I. The shape of the wave function

In this tutorial, you will only consider wave functions which are energy eigenstates. For a state with energy E_n in a system with potential $U(x)$, the wave function, $\psi_n(x)$, will satisfy the equation

$$\frac{\partial^2 \psi_n}{\partial x^2} = -\frac{2m}{\hbar^2}\left[E_n - U(x)\right]\psi_n(x)$$

A. Consider a region where the total energy, E_n, is *less than* the potential energy, $U(x)$. What is the general shape of the function, $\psi_n(x)$, in such a region? *(i.e.,* is the wave function linear, exponential, sinusoidal, quadratic, *etc.?)* Explain.

B. Now consider a region where the total energy is *equal to* the potential energy. What is the general shape of the function, $\psi_n(x)$, in such a region? Explain.

C. Now consider a region where the total energy is *greater than* the potential energy. What is the general shape of the function, $\psi_n(x)$, in such a region? Explain.

D. In those region(s) in which the wave function is sinusoidal, it may be expressed in the form, $\psi_n(x) = A \cos kx + B \sin kx$. The constant k is often called the *wave number*.

1. How is k related to the quantities in the equation at the top of the page? (*Hint:* Insert the form given for $\psi_n(x)$ into the equation at the top of the page, and solve for k.)

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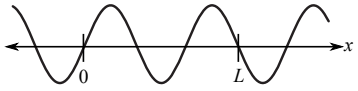
2. Find k in terms of L for the wave at right.
Sketch waves with wave numbers of $2k$ and $k/2$.



3. Give an interpretation of the wave number. (*Hint:* How is k related to *wavelength?*)

E. Consider the student discussion below.

Student 1: "The wave number is proportional to energy. The more oscillations per unit length, the more energy the wave has."

Student 2: "I think k is proportional to the difference between total energy and potential energy. So higher wave number means higher *kinetic energy,* not just energy."

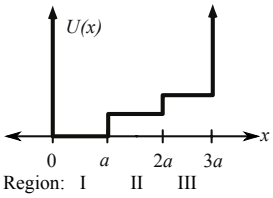
With which student do you agree, if either? Explain.

✓ Check your results with a tutorial instructor.

II. Sketching wave functions

A particle is in a *highly-excited* energy eigenstate of the potential well shown at right.

A. Sketch a line representing the total energy of this particle on the graph at right. Explain how you arrived at your answer.



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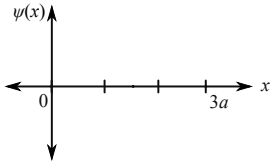
B. What general shape do you expect for the function $\psi_n(x)$ in each region of the well? (*i.e.*, Is the wave function linear, exponential, sinusoidal, *etc.)* Explain.

C. In which region is the *wave number* of the function $\psi_n(x)$ largest? In which region is it smallest? Explain.

D. In which region is the *amplitude* of the function $\psi_n(x)$ largest? In which region is it smallest? Explain.

E. Sketch a graph of this highly-excited eigenstate on the figure at right.



In the WKB approximation, the potential energy as a function of position may be represented by dividing space into small regions of uniform width within which the potential is nearly flat. The wave function for a highly-excited energy eigenstate may be represented as sinusoidal within each small region, $\psi(x) \approx \frac{A}{\sqrt{|p(x)|}} e^{\pm i \int p(x)dx/\hbar}$, with amplitude and wave number determined by the form of the (spatially-dependent) momentum, $p(x) = \sqrt{\frac{2m(E-V(x))}{\hbar^2}}$.

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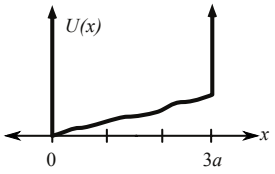
QM 4 *The WKB Approximation*

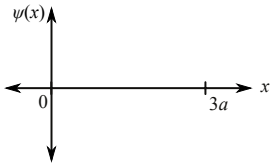
III. Approximate wave functions

Consider a particle in a *highly-excited* energy eigenstate of the different potential well shown at right.

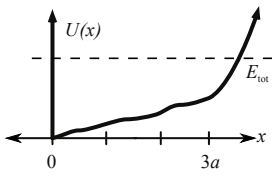


A. Explain why it is reasonable to use the WKB approximation for this system.

B. Sketch an approximate wave function for this particle on the figure at right. Explain your reasoning.



Suppose the infinite wall on the right is replaced by an increasing function, as shown in the graph at right. The dashed line indicates the total energy of the particle.



C. The intersection of the total energy with the potential energy curve is known as the *classical turning point*. Explain why this name is appropriate for this position.

D. What general shape do you expect for the wave function for values of x significantly greater than the classical turning point? Explain.

E. Using the WKB approximation, what happens to the amplitude of the wave function near the classical turning point?

Explain why the WKB approximation cannot be used to determine the wave function for regions of space near the classical turning points.

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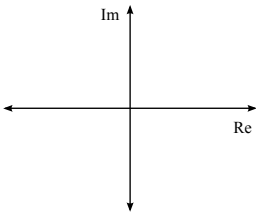
TWO-STATE TIME DEPENDENCE

QM 1

I. Complex numbers

Consider the complex number $z = 1 - i$.



A. Plot this number on the set of axes shown at right.

B. Draw a vector from the origin to the plotted number.

1. What does the length of this vector represent about the complex number *z*? Determine the value in this case.

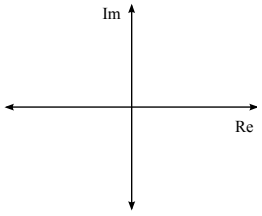
2. What does the angle between the real axis and this vector represent about the complex number *z*? Determine the value in this case. (*Note:* Is the angle positive or negative?)

3. Rewrite *z* as a complex exponential. Describe how you used the numbers you determined in the previous two questions.

Consider the complex function of time $f(t) = e^{-2it}$.

C. Find the smallest positive time, $T > 0$, such that $f(T) = f(0)$ (this time is known as the period). (*Hint:* Recall that $e^{\pm 2\pi i} = 1$)

D. Graph $f(t)$ on the axes at right, following the steps below.



1. Plot $f(t)$ for the points $t = 0$, $T/8$, $T/4$, $T/2$, $3T/4$, and T.
2. Connect these points with a dashed line. Be sure to indicate the direction of increasing *t*.
3. As *t* increases, does the length of the vector between the origin and the point described by $f(t)$ change? If so, describe how it changes.

4. As *t* increases, does the angle between the positive real axis and the vector from the origin to the point described by $f(t)$ change? If so, describe how it changes.

✓ Please wait for the class discussion.

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QM 2 *Two-state time dependence*

II. Time dependence for a quantum state

Consider an electron in the spin state $|\chi\rangle \equiv \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. (*Note:* when no basis is given, use the *z*-basis.)

A. Determine the inner product ${}_x\langle + | \chi \rangle$ in terms of α and β.

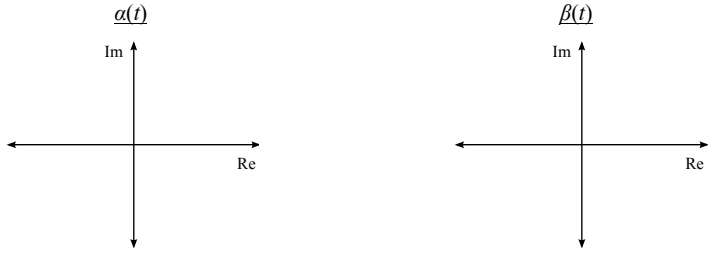
B. Is this inner product related to a probability? If so, which probability? If not, explain why not.

C. An electron's spin is measured just before $t = 0$ and determined to be $+\hbar/2$ in the ***x*-direction**.

1. Determine the state of this electron as a column vector at $t = 0$. Explain.

At $t = 0$, a constant magnetic field B_o is turned on in the ***z*-direction**. Recall that, at $t > 0$, the state of a particle in a magnetic field is given by $|\chi(t)\rangle \equiv \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix} = \begin{pmatrix} \alpha(0)e^{-i\omega_o t/2} \\ \beta(0)e^{+i\omega_o t/2} \end{pmatrix}$, where ω_o depends on the strength of both the magnetic field and the particle's magnetic moment.

2. For the electron described above, graph $\alpha(t)$ and $\beta(t)$ on the axes below, following the procedure from the previous section. Indicate the phase and the magnitude of $\alpha(t)$ and $\beta(t)$ at several instants in time *(e.g., $t = T/8$, $t = T/4$, etc.)*.



Describe how $\alpha(t)$ and $\beta(t)$ are changing in time. Discuss any similarities and differences between them.

Are $\alpha(t)$ and $\beta(t)$ changing in time at the same rate?

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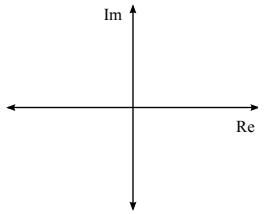
3. Write an expression for the inner product ${}_x\langle +|\chi(t)\rangle$ as a function of time.

4. On the graph at right, plot the inner product ${}_x\langle +|\chi(t)\rangle$ at several different instants *(e.g., $t = T/8$, $t = T/4$, etc.)* for this particle.



D. Suppose that you were to measure the ***x*-component** of the electron's spin at a later time, $t > 0$.

1. Does the probability that the result of this measurement is $+\hbar/2$ depend on time? Explain.

2. Use your graph of ${}_x\langle +|\chi\rangle$ to determine the smallest time $t > 0$ at which the probability of measuring $+\hbar/2$ in the x-direction is *equal to* the probability at $t = 0$. Explain.

How does this time compare to T, the period of $\alpha(t)$ and $\beta(t)$?

3. Write an expression for the probability of measuring $+\hbar/2$ in the x-direction in terms of $\alpha(t)$ and $\beta(t)$.

Suppose instead that you were interested in the probability that the ***z*-component** of spin is equal to $+\hbar/2$. Assume that no measurement was made in the previous question.

4. Write an expression for this probability in terms of $\alpha(t)$ and $\beta(t)$. Does this probability depend on time? Explain.

E. Use your answers from this section to fill out the first two columns of the handout.

✓ Please wait for the class discussion.

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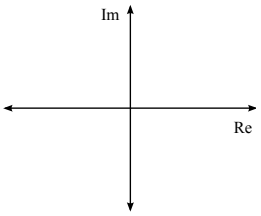
QM 4 *Two-state time dependence*

III. Time dependence for a stationary state

Now suppose that an electron's spin is measured and determined to be $+\hbar/2$ in the z-direction. As in the previous section, a constant magnetic field B_o is turned on in the z-direction at $t = 0$

A. Determine the state of this electron as a function of time in the form $\begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix}$. (*Hint:* What is $\alpha(t)$ at $t = 0$?)

B. Graph $\alpha(t)$ on the set of axes at right by following the procedure from the previous sections.



C. What is the probability that a measurement of the z-component of this electron's spin results in $+\hbar/2$? Justify your answer using your graph.

D. What is the probability that a measurement of the ***x*-component** of this electron's spin results in $+\hbar/2$? (Assume no measurements were made in the previous question.)

E. Which of the probabilities in this section, if any, depend on time? Explain. Use your answers to fill out the rest of the handout.

The term *stationary state* may be used to describe the state of the electron in this section.

F. Justify the use of this term for this state. For a stationary state, does the state depend on time?

G. Consider the student discussion below.

Student 1: "The probability of measuring the spin in the z-direction was independent of time for both of the states we considered in this tutorial, so they are both stationary states."

Student 2: "I disagree. The state that started as spin up in the z-direction is stationary because *all* probabilities are independent of time, even though the state does depend on time."

Student 3: "I think it depends on what we measure. Any state is stationary if we measure the spin in the z-direction, but not stationary if we were measure the spin in the x-direction."

With which student(s) do you agree, if any? For each student who is incorrect, identify the flaw(s) in that student's reasoning. Explain.

✓ Please wait for the class discussion.

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HANDOUT: TWO-STATE TIME DEPENDENCE

Fill out the chart below using answers you provided on the tutorial.

	For states starting in the $\left\|+\right\rangle_x$ state		For particles starting in the $\left\|+\right\rangle_z$ state	
	Does the probability depend on time?	Write an expression for the probability	Does the probability depend on time?	Write an expression for the probability
Measuring $+\hbar/2$ in the x-direction.				
Measuring $-\hbar/2$ in the x-direction.				
Measuring $+\hbar/2$ in the z-direction				
Measuring $-\hbar/2$ in the z-direction				

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Name _____*Student ID* _____ *Score*_____
last first

1. Consider a quantum mechanical mouse. You can observe the *weight* of the mouse (using the operator $\hat{W}$) and the *mood* of the mouse (using the operator $\hat{M}$). There are only two possibilities for the weight of the mouse and two possibilities for the mood of the mouse.

   The eigenequations for weight are: $\hat{W}|light\rangle = 1|light\rangle \quad \hat{W}|heavy\rangle = 10|heavy\rangle$.

   The eigenequations for mood are: $\hat{M}|happy\rangle = 1|happy\rangle \quad \hat{W}|sad\rangle = -1|sad\rangle$.

   Each set of eigenstates is orthonormal.

   A. Consider the general mouse state $|\phi\rangle = \frac{3}{5}|happy\rangle - \frac{4i}{5}|sad\rangle$. What is the expectation value of the *mood*? Show your work or explain your reasoning.

   B. Consider many identical copies of a quantum mouse, each in the *mood* eigenstate $|happy\rangle$. You measure the *weight* of each mouse and find 75% are $|light\rangle$and 25% are $|heavy\rangle$. Do the operators $\hat{W}$ and $\hat{M}$ commute? Explain your reasoning, and if you need more information to answer, state that explicitly.

   C. It is found that for a new collection of identical quantum mice, the state $|happy\rangle = \frac{\sqrt{3}}{2}|light\rangle - \frac{i}{2}|heavy\rangle$. What is the inner product between *light* and *sad*? (*i.e.,*what is $\langle light|sad\rangle$?) Show your work, and if you need more information to answer, state that explicitly.

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7. Consider a quantum mechanical mouse. You can observe the *weight* of the mouse (using the operator $\hat{W}$) and the *mood* of the mouse (using the operator $\hat{M}$). There are only two possibilities for the weight of the mouse and two possibilities for the mood of the mouse.

   The eigenequations for weight are: $\hat{W}|light\rangle = 1|light\rangle \qquad \hat{W}|heavy\rangle = 5|heavy\rangle$.

   The eigenequations for mood are: $\hat{M}|happy\rangle = 1|happy\rangle \qquad \hat{W}|sad\rangle = -1|sad\rangle$.

   Assume that the eigenstates of $\hat{M}$ are orthonormal, and that the eigenstates of $\hat{W}$ are also orthonormal.

   A. Consider a mouse described by the state $|\phi\rangle = \frac{3}{5}|happy\rangle - \frac{4i}{5}|sad\rangle$. What is the expectation value of the mood of the mouse? Show your work.

   Consider many identical copies of the mouse described by the state above. Suppose you were to measure the weight of each mouse and determine that every mouse had a weight of 1 (*i.e.,* "light").

   B. Determine the value of the inner product between a "light" mouse and a "happy" mouse. Explain how you arrived at your answer.

   Now consider a mouse that is known to have a mood of –1 (*i.e.,* "sad").

   C. If you were to measure the weight of this mouse, what is the probability that this mouse has weight 5 (*i.e.,* "heavy")? Explain how you arrived at your answer.

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5. [18 pts.] For this problem, you will consider a 4-dimensional Hilbert space. The basis vectors $|\chi_i\rangle$ are orthonormal. Consider the two state vectors defined below:

$$|A\rangle = |\chi_1\rangle + 2|\chi_3\rangle + i|\chi_4\rangle \qquad |B\rangle = 2i|\chi_1\rangle + |\chi_2\rangle + |\chi_4\rangle$$

A. Is the magnitude of state vector *A greater than, less than,* or *equal to* the magnitude of state vector *B?* If you do not have enough information to answer, state so explicitly. Explain your reasoning.

B. Evaluate the expression $\langle B|A\rangle$. Show your work.

A different vector is defined by the following expression, where γ is an unknown constant:

$$|C\rangle = i|\chi_2\rangle - i|\chi_3\rangle + \gamma|\chi_4\rangle$$

C. Determine the value of γ such that state vector *C* is orthogonal to state vector *A*. Explain how you arrived at your answer.

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5. [18 pts.] For this problem, you will consider a 4-dimensional Hilbert space. The basis vectors $|\chi_i\rangle$ are orthonormal. Consider the two state vectors defined below:

$$|A\rangle = |\chi_1\rangle + 2|\chi_3\rangle + i|\chi_4\rangle \qquad |B\rangle = 2i|\chi_1\rangle + |\chi_2\rangle + |\chi_4\rangle \qquad |C\rangle = |\chi_2\rangle + |\chi_3\rangle + (1+i)|\chi_4\rangle$$

A. Rank these three state vectors by magnitude from largest to smallest. Explain your reasoning.

B. Evaluate the expression $\langle C|A\rangle$. Show your work.

A different vector is defined by the following expression, where γ is an unknown constant:

$$|D\rangle = i|\chi_2\rangle - i|\chi_3\rangle + \gamma|\chi_4\rangle$$

C. Determine the value of γ such that state vector D is orthogonal to state vector A. Explain how you arrived at your answer.

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5. [18 pts.] For this problem, you will consider a 4-dimensional Hilbert space. The basis vectors $|\chi_i\rangle$ are orthonormal. Consider the three state vectors defined below:

$$|A\rangle = |\chi_1\rangle + 2|\chi_3\rangle + i|\chi_4\rangle \qquad |B\rangle = 2i|\chi_1\rangle + |\chi_2\rangle + |\chi_4\rangle \qquad |C\rangle = i|\chi_2\rangle - i|\chi_3\rangle - 2|\chi_4\rangle$$

A. Rank these three state vectors by magnitude from largest to smallest. Explain your reasoning.

B. Evaluate the expression $\langle A|B\rangle$. Show your work.

C. Determine which of the three state vectors $|A\rangle, |B\rangle, |C\rangle$, if any, are orthogonal to each other. Explain your reasoning.

D.III – Exam Questions for Part III

The exam questions discussed throughout part III of this dissertation are included here. The questions are arranged by topic.

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In the questions on this page, the state of a particle will be represented using the $|l,m_l;s,m_s\rangle$ basis.

5. [30 pts] Consider the state of a particle given by: $|\psi\rangle = \frac{3}{5}|3,2;1,0\rangle - \frac{4}{5}i|2,0;1,1\rangle$.

   A. [5 pts.] Suppose you measured the *z*-component of the *orbital* angular momentum (L_z). What are the possible results of this measurement for this system, and what is the probability of each? Explain.

   Let $\vec{J} = \vec{L} + \vec{S}$ be the total angular momentum of the particle in the same initial state.

   B. [5 pts.] Suppose you measured the *z*-component of the *total* angular momentum (J_z). What are the possible results of this measurement for this system, and what is the probability of each? Explain.

   C. [5 pts.] Suppose you measured the *total* angular momentum squared (J^2). What are the possible results of this measurement for this system? Explain.

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Recall that $\vec{J} = \vec{L} + \vec{S}$ represents the *total* angular momentum of a system. Suppose that a particle is in the state given by $\left| l, s; j, m_j \right\rangle = \left| 3,1;4,2 \right\rangle$. (Note: this is different notation from the previous page.)

D. [5 pts.] Suppose you measured the *z*-component of the *orbital* angular momentum (L_z). What are the possible results of this measurement for this system? Explain.

E. [5 pts.] Suppose instead that you measured the ***x*-component** of the total angular momentum (J_x) of the state given above (*i.e.,* no other measurements have been made on this particle). What are the possible results of this measurement? Explain.

F. [5 pts.] Suppose the measurement above (of J_x) results in $2\hbar$. Suppose that you then measure the ***z*-component** of the total angular momentum (J_z). What are the possible results of this measurement? Explain.

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3. Consider the state of a particle given by: $|\psi\rangle = |l, m_l; s, m_s\rangle = |1,-1; 3/2, 1/2\rangle$, which is represented using the $|l, m_l; s, m_s\rangle$ basis. Let total angular momentum be defined by $\vec{J} = \vec{L} + \vec{S}$.

   A. [10 pts.] Determine the possible results of a measurement of each of the following quantities on the particle in the state above, and describe how to calculate the probability of each result. Explain your reasoning for full credit.

      1. The *z*-component of the **total** angular momentum, J_z.

      2. The **total** angular momentum squared, J^2.

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Recall that $\vec{J} = \vec{L} + \vec{S}$ represents the *total* angular momentum of a system. Suppose that a different particle is in the state: $\left| l, s; j, m_j \right\rangle = \left| 3,1;4,1 \right\rangle$. (Note: this notation is different from the previous page.)

B. [10 pts.] Determine the possible results of a measurement of each of the following quantities on the particle in the state above. You do not need to find probabilities, but do not include values for which the probability is zero. Explain your reasoning for full credit.

1. The *z*-component of the **orbital** angular momentum, L_z.

2. The **orbital** angular momentum squared, L^2.

C. [10 pts.] Suppose that the ***x*-component** of the total angular momentum (J_x) is measured to be $+\hbar$. After this measurement, determine the possible results of a measurement of the ***z*-component** of the total angular momentum (J_z), and describe how to calculate the probability of each result. Explain your reasoning for full credit.