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Algorithmic Design-for-Manufacturing and Programmability of Metamaterials

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Abstract

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Modern digital design tools offer vast creative freedom, yet a persistent gap remains between conceptual designs and their physical realization, particularly for advanced materials and designs. This challenge is acute for metamaterials, whose novel properties are derived from complex, fine-scale architecture that is often difficult to design for and manufacture. This dissertation argues that design tools should directly translate a user’s high-level intent into physical form by managing complexity on two fronts: abstracting complex material behavior into simple, programmable controls, and embedding the physics of the fabrication process directly into the design environment. This approach treats physical realization not as a downstream constraint but as an integral design parameter, enabling more expressive, accessible, and efficient workflows.

To substantiate this claim, this thesis presents three interlocking contributions that integrate computational design with advanced manufacturing:

First, to address the design complexity of metamaterials, I introduce a framework that abstracts complex mechanical behaviors into simple, programmable primitives. This work uses compliant straight-line mechanisms (SLMs) as reconfigurable building blocks that explicitly encode zero-energy deformation modes. By coordinating these SLMs with planar symmetries, I demonstrate the ability to deterministically program and smoothly interpolate between all 2D extremal classes (nullmode, unimode, bimode, and trimode), enabling in-situ,

reversible tuning of emergent properties like Poisson’s ratio and chirality without costly re-computation.

Second, I present Fabrication-Directed Entanglement (FDE), which combines the previous work on simplifying metamaterial design with topology optimization and viscous thread printing (VTP) to computationally design and fabricate monolithic foam metamaterials with spatially patterned entanglement. By translating intended directions of compliance and rigidity into optimized density fields and controlled VTP coiling patterns, FDE produces single-filament structures exhibiting targeted anisotropy, extreme Poisson’s ratios, and significant chirality-driven normal-shear coupling—behaviors previously inaccessible in uniform or multi-material entangled foams.

Third, to demonstrate the generality of this fabrication-aware approach beyond mechanics, I extend the VTP-based design methodology to the optical domain. I present a computational pipeline for creating foam-based lithophanes where light transmission is controlled by spatially varying the foam’s porosity. This work leverages a calibrated physical simulation of the VTP process and a photorealistic rendering pipeline to automate the translation of digital images into manufacturable, uniform-thickness structures with programmed optical properties.

Collectively, these projects demonstrate that abstracting low-level fabrication and material physics into process-aware computational tools expands the space of what can be designed and built. By algorithmically integrating the means of production into the act of creation, this work provides a pathway toward more powerful and intuitive design systems for complex physical objects.

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DEDICATION

To my parents, my wife, and my daughter. Thank you for your steadfast belief in me.

Chapter 1

INTRODUCTION

For as long as humans have been creating, a perennial struggle has existed between envisioning a solution and realizing it in physical form. Designs are conceived in the mind with infinite possibilities, but must confront the realities of physics, materials, tools, and processes—constraints that often force compromise or, worse, total abandonment before the idea ever manifests physically. While modern digital tools have made the visualization and simulation of ideas far more sophisticated, they have not fully bridged this gap; many designs still fail to translate neatly into the physical world without extensive reworking or specialist knowledge.

Before the advent of computer-aided design (CAD), engineers and draftspeople were required to simultaneously carry fabrication knowledge *in mente*, organically integrating the manufacturing process as heuristics in their minds. They sketched by hand, built scale mock-ups, and relied on shop-floor intuition to bake manufacturability into every drawing. Even as the technology improved to encompass high-fidelity modeling software, the boundary between “design” and “fabrication” remained fluid: designers had to understand cutting-tool limitations, jig setups, and assembly sequences to ensure their concepts would look, feel, fit, and function as intended once built.

Today, however, advances in simulation, materials, and manufacturing have increasingly decoupled design from fabrication as each of them become more technically demanding and require their own subject matter expertise (SME). High-fidelity computational geometry and physical modeling unlock vast digital design spaces—yet there is not guarantee the designs can be realized. When real-world constraints aren’t incorporated by the designer from the outset, powerful tools can mislead, forcing multiple costly redesigns once manufacturability is considered. At the same time, novel materials—from advanced composites to metamaterials—

offer unprecedented functionality but introduce complex, non-homogeneous behaviors that are difficult to design for and furthermore complicate design-for-manufacture because they don't fit neatly into traditional manufacturing workflows.

To address the increasingly complicated product lifecycle pipeline, this thesis makes the following claim:

Unifying algorithmic design-for-manufacturing with programmable abstractions of complex physical phenomena translates high-level design intent directly into physical reality, enabling next-generation metamaterial design across multiple physical domains.

To substantiate this claim, this thesis introduces three interlocking research directions:

1. *Chapter 3 Reprogrammable Extremal Materials via Straight-Line Mechanisms:* Here I illustrate how a simple design process can enable sophisticated metamaterial behavior by introducing a general framework that uses compliant straight-line mechanisms (SLMs) embedded in planar symmetries to explicitly program and reprogram zero-energy deformation modes. By rotating these SLMs in situ, centimeter-scale 2D lattices transition smoothly among all 2D theoretical extremal classes—nullmodes, unimodes, bimodes, and trimodes—while continuously varying properties such as Poisson's ratio from negative to positive.
2. *Chapter 4 Fabrication-Directed Entanglement for Tunable Foams:* I leverage the extremal material theory with combine viscous thread printing (VTP) and topology optimization (TO) to computationally explore the emergent material property space of single filament entangled materials. A single filament spatially varied between “dense” and “sparse” entanglements which produces monolithic foam metamaterials exhibiting targeted anisotropy, extreme Poisson's ratios (0.06–0.56), and significant chirality-driven normal-shear coupling ($\eta_{212} \approx 0.72$)—behaviors inaccessible to uniform or multi-material foams.
3. *Chapter 5 Computational Design of 3D Printed Foam Lithophane Structures:* Here I

develop (alongside a co-author) a computational framework for leveraging the controlled porosity of VTP to produce 2D lithophane structures. We utilize physical simulation of the VTP process and photo-realistic rendering to rapidly explore the design space of VTP parameters (V^* , H^* , Δs) which produce a range of perceived grayscale values (G). We then perform optimization to develop an inverse mapping which will process grayscale images, predict their realistic brightness and contrast, and realize them via 3D printing all within a single user-friendly tool.

Collectively, these contributions demonstrate that algorithmically embedding fabrication and material behavior into the design process enables more expressive, accessible, and efficient workflows. By treating physical realization—not just as a downstream constraint, but as a design-level concern—this work expands the space of what can be designed and built, while reducing the expertise required to do so. Though grounded in mechanical metamaterials, the approach generalizes: simplifying complex design tasks by abstracting low-level material and fabrication considerations into process-aware design primitives.

Chapter 2

BACKGROUND

2.1 *Metamaterials*

Metamaterials are artificially structured materials whose macroscopic properties arise primarily from their engineered microarchitecture rather than the intrinsic properties of their constituent materials [44, 17, 108, 32, 110, 60]. By designing the size, shape, and arrangement of unit elements at length scales below the relevant characteristic length of the physics targeted, metamaterials can exhibit responses—such as negative refractive index, negative Poisson’s ratio, or extreme stiffness contrasts—not observed in conventional materials.

Mechanical Metamaterials Mechanical metamaterials apply this architectural paradigm to tailor quasi-static and dynamic mechanical behavior. Through careful unit-cell design and topology control, one can achieve features such as auxetic behavior (negative Poisson’s ratio) [17, 45], tunable stiffness [127, 122, 123], bandgap engineering for vibration isolation [60, 86, 29], and handed-shearing auxetic behaviors [71]. These systems leverage mechanisms including bistable elements, re-entrant geometries, and hierarchical lattices to realize responses beyond the capabilities of homogeneous solids.

Extremal Mechanical Metamaterials Within the broader landscape of mechanical metamaterials lies a subclass termed *extremal materials* whose effective responses coincide with the theoretical bounds of linear elasticity, as dictated by the eigenvalues and eigenvectors of the homogenized elasticity tensor C [78]. These bounds emerge from discrete, low-energy “zero” modes — deformation patterns that cost vanishing elastic energy — whose activation, coordination, or reconfiguration can, in principle, approximate any admissible linear-elastic behavior [78]. Extremal materials are conventionally classified by the number of compliant directions (near-zero eigenvalues) where a *nullmode* is perfectly rigid, a *unimode* has one compliant direction, a *bimode* has two compliant directions and so on up to *hexamodes* in 3D.

Because, in 2D, the elasticity matrix is 3×3 the highest mode is a trimode. Furthermore, because null- and trimodes represent the rigid and fully compliant limits, most practical attention centers on uni- and bimode topologies.

Early demonstrations relied on bespoke unit-cell geometries derived from analytical insight or targeted optimization. Pentamode lattices, for example, achieve extreme bulk-to-shear modulus ratios by enforcing binary phase distributions with high stiffness contrast [59]. Conversely, isotropic unimode designs minimize Poisson’s ratio under symmetry constraints [7]. While these strategies produced seminal proofs of concept — including mechanical cloaking [25, 84, 121], acoustic cloaking [66, 30], and out-of-plane shear-wave polarizers [117] — each new target response demanded fresh, case-specific effort, and fabrication realities such as minimum-feature-size limits often curtailed performance [59, 58].

Inverse homogenization (IH), a density-based topology-optimization (TO) technique, has since emerged as a powerful computational framework for discovering extremal microstructures and discussed in more detail in Section 2.3. IH can directly match a prescribed tensor C or optimize scalar proxies such as Poisson’s ratio or bulk-to-shear ratios, while optionally enforcing symmetry classes (isotropic, orthotropic, chiral, etc.) [105, 7]. Nevertheless, density-based methods suffer from gray-scale artifacts, checkerboard instabilities, and tenuous connections, necessitating filters, penalization, and constraints to recover binary, manufacturable lattices [7, 46, 11]. These challenges motivate a broader trend toward algorithmically integrating fabrication constraints into the design loop from the outset, thereby producing structures that are not only extremal in theory but also robust in practice.

Recent work has pushed beyond static architectures to realize switchable lattices that toggle between predefined zero-mode sets, hinting at reprogrammable or adaptive functionality [50]. Yet the field still lacks a unified framework capable of specifying arbitrary zero-mode combinations while simultaneously guaranteeing manufacturability across varied symmetry classes [21]. Addressing this gap remains critical for translating extremal mechanical metamaterials from bespoke curiosities into broadly deployable engineering materials.

2.2 Viscous Thread Printing (VTP)

Rheology and Material Prerequisites Viscous Thread Printing (VTP) exploits the viscous-thread instability (VTI)—the coiling seen when honey drizzles onto toast—where a falling filament spontaneously loops instead of depositing linearly. Extensive work has characterized VTI via experiments, theory, and simulation [125, 97, 98, 24, 23, 81, 75, 12], yet almost all studies treat one-dimensional coils in low-surface-tension fluids that interpenetrate. Such assumptions exclude thermally driven polymer melts extruded during 3D printing, where successive coils sit on, rather than merge with, previous layers. VTP overcomes this limitation by extruding semi-viscous or molten polymer filaments whose paths are continuously perturbed by already-deposited strands, generating fully three-dimensional, open-cell architectures from any standard material that avoids self-penetration [69, 125, 37, 70, 39]. Unlike other printed foams that depend on bespoke chemistries or post-processing [85, 18, 114, 82], VTP produces entangled structures rapidly through a single continuous strand whose stochastic morphology emerges from self-interaction. Because the network’s properties arise from filament entanglement rather than polymer formulation, VTP provides a printer-controlled platform for studying and exploiting entanglement at the mesoscale.

Physically entangled networks underpin functions across scales, from protein folding to engineering foams [101, 35, 73, 34]. While microscale entanglement can be programmed chemically [109, 33], macroscopic analogues—metallic-wire foams and non-woven fiber mats—lack spatial precision, limiting mechanical tunability [128, 112, 65]. By enabling voxel-scale placement of an entangled filament, VTP bridges this gap, allowing mechanical performance to be tuned through path design rather than material chemistry alone.

Printer Hardware and Process Parameters The coiling behavior is parameterized via two dimensionless groups [125]: the height and speed ratios

$$H^* = \frac{H}{D_T} \quad \text{and} \quad V^* = \frac{F}{U} \quad (2.1)$$

where H is the nozzle height above the substrate, D_T is the extruded filament diameter (including die swell effects), F is the nozzle translation speed, and U is the speed of the filament leaving the nozzle. By fixing D_T and U and varying H and F (and thus H^* and

V^*), VTP can produce coils ranging from tight, high-density loops to loose, open structures, controlling local stiffness with minimal parameters. A detailed derivation of translating (V^*, H^*) into G-Code is given in Appendix C.

Examples of VTP in Published Work Early VTP studies established the relationship between homogeneous (V^*, H^*) and bulk stiffness [70, 37]. Subsequent work introduced spatial gradients in (V^*, H^*) to locally tailor stiffness and developed models to predict effective modulus and layer height as functions of (V^*, H^*) [38, 39]. These advances expose a broad, compatible design space for multi-phase VTP foams—achieving stiffness contrasts on the order of $10^2:1$ —yet leave open the question of how best to pattern this space to realize prescribed mechanical responses. Manual, forward-design approaches (e.g., soft-robotics actuators with circular, trapezoidal, or triangular stiffness regions [79]) have shown promise, but their heuristic geometries offer no guarantee of optimality. Moreover, these strategies underutilize VTP’s inherent ability to continuously vary coil parameters, instead relying on a limited set of canonical shapes with discrete boundaries rather than leveraging the full spectrum of spatial variability that VTP can encode.

2.3 Inverse Homogenization (IH)

Inverse Homogenization casts microstructure design as a PDE-constrained optimization on a periodic unit cell to match prescribed effective properties captured by an effective elasticity matrix of the metamaterial. The typical formulation minimizes an objective

$$\begin{aligned}
 \min_{\rho(\mathbf{x})} \quad & f(C(\rho)) \\
 \text{s.t.} \quad & \nabla \cdot \sigma = \mathbf{0} \\
 & \sigma = C(\rho) \varepsilon \\
 & \rho(\mathbf{x}) \in [0, 1] \\
 & g_i(\rho) \leq 0
 \end{aligned}$$

where $\mathbf{x}$ is the spatial coordinate, ρ is the material density field, $C(\rho)$ the homogenized matrix, f is some scalar function on $C(\rho)$, e.g., deviations from a target tensor C^{target} or scalar

properties such as Poisson’s ratio ν , and g_i represent some number of additional constraints $0 \leq i < N$ such as an explicit isotropy constraint or minimum bulk modulus [105, 6, 10]. Key challenges include avoiding gray regions and checkerboard patterns, typically mitigated via density penalization, filtering, and projection schemes [7, 115]. High phase contrast designs approach theoretical bounds but exacerbate manufacturability issues—thin walls and disconnected features—necessitating further regularization [46, 11].

Extremal Materials via IH To create extremal materials, researchers often optimize scalar proxies—such as Poisson’s ratio or the bulk-to-shear modulus ratio—while imposing additional constraints to guide the microstructure toward the boundaries of feasible extremal behavior [7, 28]. However, since extremal materials arise directly from the eigenstructure of the elasticity tensor C , it is conceptually more natural to design by targeting its eigenvalues and eigenvectors rather than derived scalar summaries. Despite this, few studies leverage eigenanalysis of C as the primary optimization driver. One notable example [26] formulates objectives based on eigenvalues, but still relies on extensive constraint enforcement to achieve valid extremal designs. In contrast, by directly optimizing over a suitably transformed tensor (e.g., the compliance tensor $S = C^{-1}$), one can encode material symmetries and stiffness bounds intrinsically within the objective, reducing dependence on ad hoc constraints. Although the full tensor optimization involves higher dimensionality—six independent components in 2D and twenty-one in 3D—this approach often converges more rapidly and yields robust, manufacturable microstructures without the gray-area and connectivity issues common to proxy-based methods and remains relatively unexplored compared to methods using intrinsic material properties.

2.4 *Image-driven Surface Design*

Lithophane design is driven by the goal of displaying some provided imagery and, specifically, achieves this goal via manipulating occlusion to change light transmission and reflection.

Prior work that directly manipulates the surface geometry to change occlusion includes shadow art and self-shadows [80, 14, 91], height fields [107], surface microfacets [118], and irregular surface pit placement [4, 3]. There are also techniques that achieve image display

through an additional layer of optical effect, including caustics effects [102, 90], lenticular lens surfaces [126], reflective surface design [100], scratch-based reflection art [104], and computational highlight holography [96].

View-dependent approaches extend this concept by creating surfaces that display different images depending on viewing angle or lighting conditions. Examples include 3D printing to create reflectors [100] and height-varying structures [94], 3D printing with magnetic [92] or goniochromatic [1] materials, and illusion knitting that reveals different patterns by occluding certain stitches from different viewpoints [129].

Reliefs represent another branch of this research, where the goal is also to encode images through controlled surface variations but displaying the image does not require being lit from the back. Examples include high relief synthesis from 3D scenes [8] and 3D woven relief [120].

2.5 *Simulation of Discrete Elastic Rods (DERs).*

Given the prevalence and importance of rod-like structures throughout science and engineering, extensive work addresses modeling and simulating elastic cables using techniques such as finite differences [42, 63, 62] and finite elements [74, 76, 87, 124, 43, 67], where a key consideration is whether and how to model bending and torsion deformation and dynamics. Typical geometric discretizations of cables include mass-spring networks [103, 99, 52], Cosserat curves [88, 113], piecewise helices [15] or clothoids [16], and Kirchhoff rods [13, 12]. This work employs the Kirchhoff rod model because this approach was thoroughly validated by its authors, and has been applied to a variety of problems in both physics [24] and computational fabrication [93, 89, 95, 56, 57, 9].

Chapter 3

DESIGN AND REPROGRAMMABILITY OF ZERO MODES IN 2D MATERIALS FROM A SINGLE ELEMENT

Note on Publication

Portions of this chapter are based on work accepted for publication in *Advanced Science*.

3.1 Introduction

Here I show an expressive and reprogrammable method to directly engineer zero modes for two-dimensional (2D) extremal materials using compliant straight-line mechanisms (SLMs) and planar symmetry. I base the design on the SLM because it is inherently extremal. The SLM constrains motion to a single straight-line trajectory [61, 48, 49, 47, 51], and thus operates as an explicitly definable zero mode unit cell. I also design the SLMs to be innately reprogrammable through a rotationally symmetric construction, allowing them to pivot in place and modify the zero modes of the metamaterial without altering the global structure. To realize the SLMs as a comprehensive metamaterial, the global construction and coordination of the zero modes is performed using planar symmetry which integrates multiple SLMs onto 2D lattices to achieve the desired extremal behaviors.

This approach achieves centimeter-scale extremal materials of various material symmetries — isotropic, orthotropic, and chiral. It also includes the ability to reprogram these materials in situ to smoothly and reversibly interpolate between different extremal modes (e.g., unimode to bimode) and emergent properties (e.g., negative to positive Poisson’s ratio). Unlike other methods, my framework enables rapid, arbitrary, continuous, and spatially independent adjustment of engineered zero modes. Thus, I am able to uniquely realize Milton and Cherkaev’s concept of 2D extremal material lamination [78] and facilitate dynamic adjustments of material properties on demand without costly computational methods or lengthy reprogramming procedures. I design, simulate, fabricate, and validate the SLMs and

symmetry as a programmable and tunable framework to define the zero modes of extremal materials, demonstrating how SLM metamaterials can access all possible 2D mechanical extremal modes. By examining both the engineering constants (Young’s moduli (E_1, E_2), shear modulus (G_{12}), Poisson’s ratios (ν_{12}, ν_{21}), the normal-shear coupling ratio (η_{121}, η_{122}), and the zero modes of the extremal materials’ homogenized elastic matrix C , I discover that the orientation of the SLM and the employed symmetry pattern comprehensively determine the emergent properties and extremal behavior.

The core contributions of this work are: (1) the use of SLMs as continuously reprogrammable zero-mode metamaterial cells, (2) the use of symmetry to coordinate zero modes and realize extremal types of varied material symmetries (isotropic, orthotropic, chiral), and (3) a demonstration of smoothly reprogramming extremal mode and emergent properties of my extremal materials. Leveraging these insights and contributions, I extend and realize the design space of reprogrammable 2D extremal metamaterials and demonstrate their temporal and spatial variability.

3.2 Composition and Analysis of SLM Unit Cells

3.2.1 SLMs are Reprogrammable Unimode Unit Cells

SLMs are the foundational element of my extremal materials. Among the various SLM designs [61], I devised a compliant double-sided Robert’s mechanism, inspired by prior work on generating a compliant single degree of freedom (DOF) motion [51] (see **Figure 3.1a** and b). They are compliant exclusively along their straight-line DOF and rigid against orthogonal deformations, with an approximate 30-fold decrease in stiffness between rigid and compliant loading shown in Figure 3.1c. SLMs are therefore inherently extremal, specifically unimodal, possessing one intrinsic zero mode. As shown by [78] unimodes are sufficient to form the basis for all other extremal materials, thus my SLM-based framework is well-suited for extremal material construction.

A key innovation of my approach is the SLM’s rotationally symmetric design, which allows in situ reorientation of the compliant direction (see Figure 3.1d); this tunes the zero modes of my extremal materials. The idealized SLM compliance can be expressed as the deformation

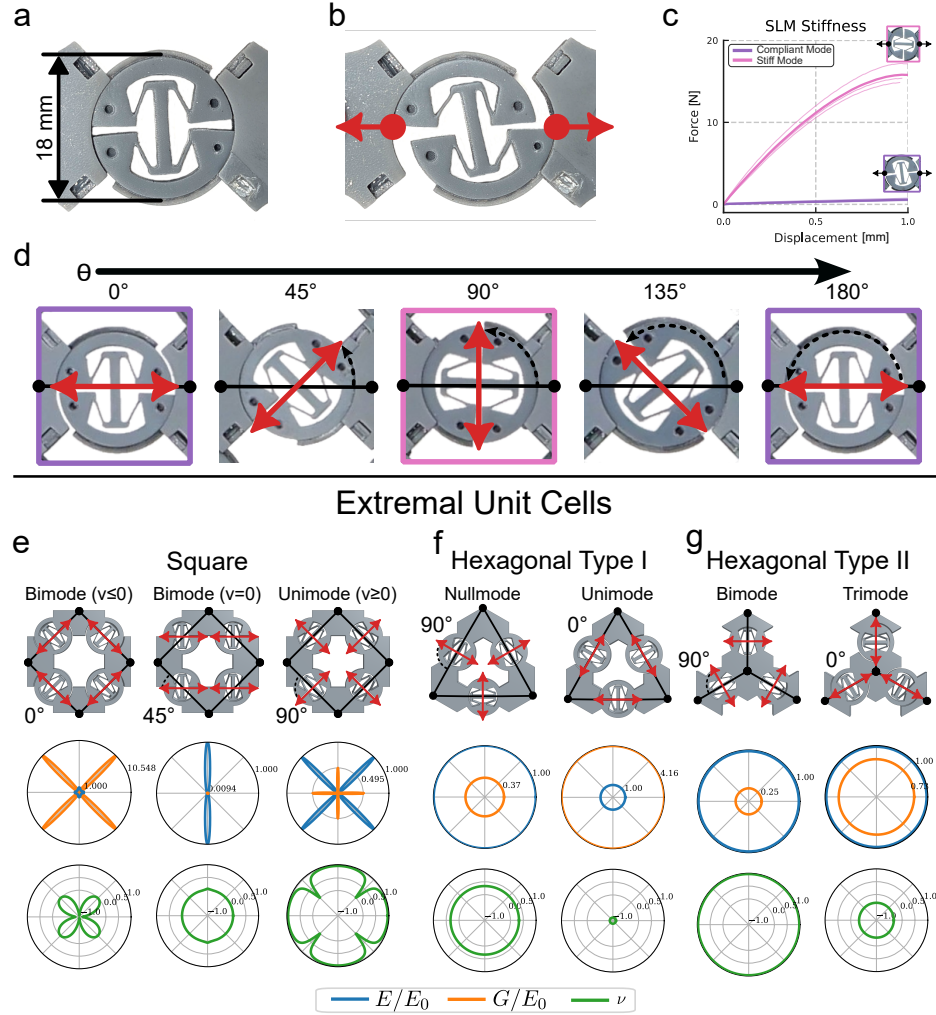


Figure 3.1: The SLM is shown at rest (a) and under horizontal loading (b) showing linear displacement along the straight-line trajectory when loaded horizontally. (c) A single fabricated SLM shows an approximate 30-fold reduction in stiffness from rigid loading (pink) to compliant loading (purple). Measured data is shown as lighter lines with averages ($n=5$) shown as bold lines. (d) The SLM is reprogrammed by pivoting in place and is parameterized by θ , the angle of the zero mode relative to the lattice connection direction. The SLM can be embedded onto 2D lattices using square symmetry (e) or two different types of hexagonal symmetry (f and g). Each of these configurations (e-g) is shown as CAD unit cell overlaid with its abstract lattice representation and FEA homogenized data shown in the polar plots below. Raw values for moduli are provided in Table A.1 and Table A.2

gradient F of the SLM cell base and the subsequent linear strain $\varepsilon = \frac{1}{2}(F + F^T) - I$

$$F = \begin{bmatrix} 1 + \alpha \cos \theta & 0 \\ \alpha \sin \theta & 1 \end{bmatrix} \quad (3.1)$$

and

$$\varepsilon = \frac{\alpha}{2} \begin{bmatrix} 2 \cos \theta & \sin \theta \\ \sin \theta & 0 \end{bmatrix} \quad (3.2)$$

where α is the non-dimensional length scale of the SLM deformation and $\theta \in [0^\circ, 180^\circ)$ is the angle of the SLM DOF relative to the connective direction, shown in Figure 3.1d. The linear strain presented here is equivalent to the unimode strain expressed in Milton's and Cherkaev's foundational work [78].

Notably, the SLM and its deformation gradient exhibit rotational symmetry (C2 point group), whereas the strain shows mirror symmetry (D2 point group) for θ values of 0 and 90 degrees. Therefore, even if the symmetry pattern is solely rotationally symmetric, such as 632, the material properties can display characteristics requiring mirror symmetry, like isotropy.

Thus, my SLM fulfills the original criterion for extremal material construction from unimode and can simultaneously adapt its extremal material response — from shear to normal deformation and back — through simple rotational adjustments. These unique characteristics enable in situ tuning of the mechanical behavior without requiring global structural change or lengthy reprogramming processes.

3.2.2 Composition of SLM Zero Modes

Naïvely tiling unimodal SLMs can only result in unimodal materials with compliant strains described by Equation 3.2. Thus, introducing more DOF-wise independent SLMs is necessary to achieve a broader range of behaviors; however, each adjoined SLM introduces constraints, (i.e., rigidity) along with its DOF as expressed by the zero value in Equation 3.2. Crucially, the DOFs and constraints must be coordinated to ensure that they do not interfere and restrict deformation, which requires simultaneously satisfying each individual SLM's deformation gradient.

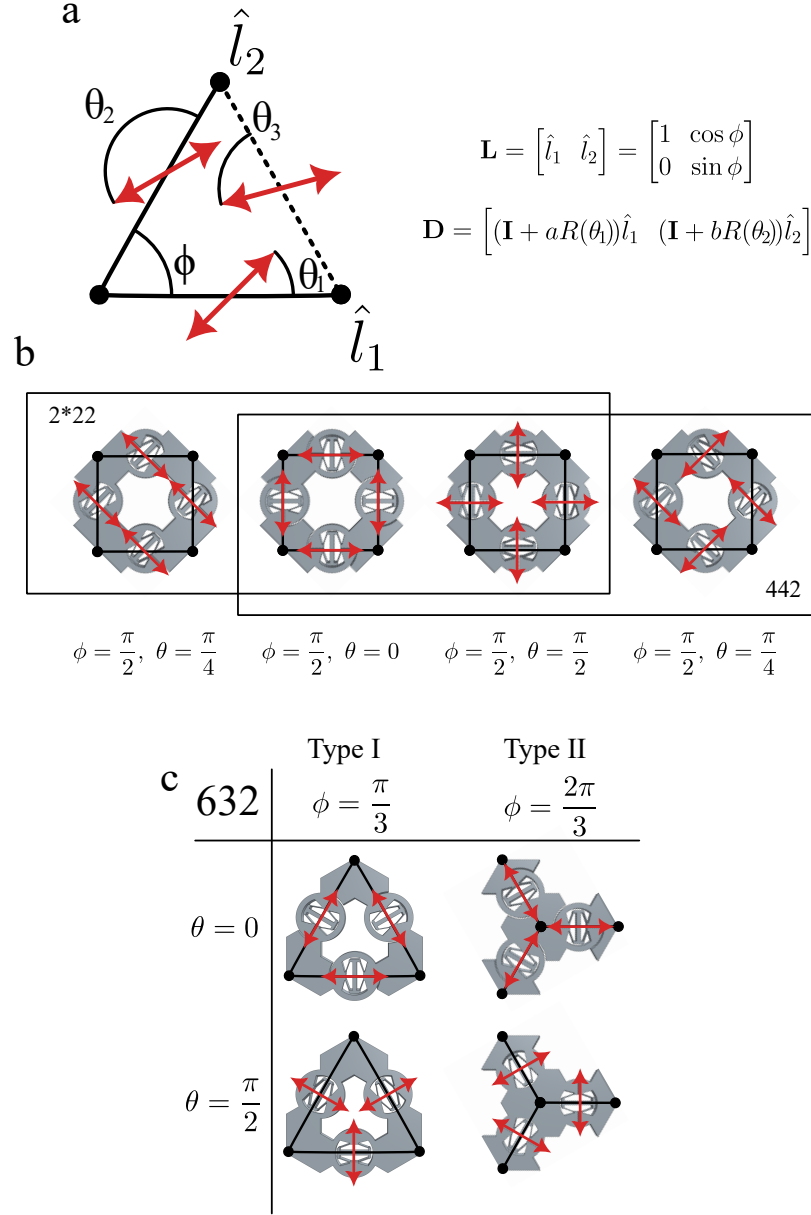


Figure 3.2: The generalized form of constructing SLM lattices. a) The two lattice vectors form matrix $\mathbf{L}$ and the deformation gradient, parameterized by a and b is found as rotational offsets from the lattice vectors. Under symmetry the SLM rotation angle θ is assumed to be the same for every SLM. b) The two square symmetries use $\phi = \frac{\pi}{2}$ for their lattice construction, but vary in how the symmetry is applied to realize the other SLM orientations. They are identical for $\theta = 0, \frac{\pi}{2}$ where they converged into $*442$ symmetry. c) 632 symmetry uses two different lattice vector sets, $\phi = \frac{\pi}{3}, \frac{2\pi}{3}$, for Type I and Type II cells respectively. Unlike the square symmetry case there is an additional SLM to account for in the deformation gradient.

A general method of constructing compliant extremal materials and analytically determining their zero modes is developed and illustrated in Figure 3.2 and algebraically shown in the supplementary file `extremal_algebra.m` (Appendix D). I assume periodic tiling using the lattice vectors $\mathbf{L} = [\hat{l}_1 \quad \hat{l}_1]$. I am free to choose the orientation of the coordinate system and make $\hat{l}_1 = [1, 0]^T$ for simplicity. All other vectors (lattice and deformation) can be derived as various rotations of this vector. I find the Cartesian deformation gradient $\mathbf{F}$ for each unit cell and then can analyze unique deformation modes as well as the linear strain zero modes.

The generalized lattice vectors are defined with $\hat{l}_1$ oriented along the x-axis. The second lattice vector $\hat{l}_2$ is generated by rotating $\hat{l}_1$ by ϕ radians

$$\hat{l}_2 = \mathbf{R}(\phi)\hat{l}_1 = \begin{bmatrix} \cos \phi \\ \sin \phi \end{bmatrix}$$

where $\mathbf{R}(\phi)$ is the standard 2D rotation matrix

$$\mathbf{R}(\phi) = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix}$$

(Figure 3.2a). Square lattices used in this work have $\phi = \frac{\pi}{2}$ and hexagonal lattices have $\phi = \frac{n\pi}{3}$ where $n = 1, 2$ (Figure 3.2c). For 632 symmetry, a third vector $\hat{l}_3$ must be considered to account for additional DOFs and constraints, which can be incorporated as terms into the first two.

The deformation vector of an SLM is a rotation of the lattice vector it is defined on

$$d_i = \mathbf{I} + \xi \mathbf{R}(\theta_i) \cdot \hat{l}_i$$

where ξ represents the free parameters associated with that SLM. The deformation gradient of an entire unit cell is then

$$\mathbf{D} = \begin{bmatrix} d_1 & d_2 \end{bmatrix} = \begin{bmatrix} (\mathbf{I} + a\mathbf{R}(\theta_1) \cdot \hat{l}_1) & (\mathbf{I} + b\mathbf{R}(\theta_2) \cdot \hat{l}_2) \end{bmatrix}. \quad (3.3)$$

The SLM deformation vectors are defined by the individual θ_i for $i = 1, 2, 3$. For $\theta_i = 0$ the SLM DOF is aligned with the lattice vector it is attached to. This allows me to again use a rotation of the lattice vector to define the deformation vectors d_1 and d_2 .

Table 3.1: Symmetry enforces different constraints on the lattice shown here for the four types of symmetry explored in this work. θ_1 is the driven design parameter that defines all other SLM orientations. Square symmetry was restricted to tiling only two SLMs, making θ_3 undefined for 2*22 and 442 symmetry.

	2*22	442	632 - Type I	632 - Type II
ϕ	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{3}$	$\frac{2\pi}{3}$
θ_2	$-\theta_1$	θ_1	θ_1	θ_1
θ_3	-	-	θ_1	θ_1

The deformation vectors have the free parameters a and b for square symmetry as well as c in 632 symmetry. These parameters are assumed to lie within the interval $[0, 1]$ and represent the length scale of displacement of the DOF. The unit cell deformation gradient must simultaneously satisfy all individual SLM DOFS and constraints.

The deformation gradient $\mathbf{D}$ expresses global deformations in terms lattice vector lines, which may not necessarily be the global Cartesian space. In general, global deformation gradient can be found from the lattice deformation gradient

$$\mathbf{F} = \mathbf{L}\mathbf{F}'\mathbf{L}^{-1}$$

where $\mathbf{F}'$ is the deformation gradient of the lattice in lattice coordinates and the lattice coordinates $\mathbf{L}$ work as transformation matrices between the two. $\mathbf{D}$ already expresses global deformation in lattice coordinates $\mathbf{D} = \mathbf{L}\mathbf{F}'$, so to find the Cartesian deformation gradient I use

$$\mathbf{F} = \mathbf{D}\mathbf{L}^{-1}. \quad (3.4)$$

This approach allows me to articulate deformations in a universal coordinate system, thereby generalizing the model to accommodate various lattice symmetries and configurations.

3.2.3 Applying Symmetry

Applying symmetry introduces constraints to the generalized lattice and deformation construction seen in Table 3.1. Pure rotational symmetry uses the lattice construction with

the constraint $\theta_i = \theta_j, \forall i, j$. Mirror symmetry has the constraint of $\theta_2 = -\theta_1$, but for mirror symmetry there is no way to define θ_3 and be mirror symmetric with both θ_1 and θ_2 . This means that the 632 wallpaper group can not, in general, produce mirror symmetric deformation gradients across θ , which is observed on the extremal gamut. The trajectory of an isotropic extremal material from one state to another passes through the anisotropic regime, implying that isotropy is not allowed while maintaining symmetry.

I use 2D planar symmetries, called the *wallpaper groups*, to introduce multiple SLMs and coordinate their zero modes. Wallpaper groups ensure complete tiling of the plane for metamaterial construction [31]. They also effectively arrange each individual SLM zero mode to avoid the combinatorial complexity seen in other efforts to design zero mode materials [21]. This setup creates a two-dimensional material where the number and characteristics of zero modes are determined by the SLM orientation and applied symmetry. Due to its uniform structure, my reprogrammable SLM fits seamlessly into regular square and hexagonal lattices. Significantly, in contrast to other reprogrammable zero mode materials [50], the overall global lattice structure is preserved regardless of the microstructure configuration (i.e., SLM orientation), demonstrated in Figure 3.1e-g. This allows the lattice to act as a scaffold, where the nodes serve as connection points for the SLMs positioned along the edges. The loading of the structure is transferred through the lattice connections, but deformations are allowed only in line with the SLM orientations. This results in a range of emergent behaviors based on a single lattice, which are coordinated by the orientation of zero modes relative to the lattice lines.

3.2.4 Analyzing Zero Modes

I can construct an idealized deformation gradient for the unit cells by linearly combining multiple SLM deformation gradients (Equation 3.1) and resolving any constraints. This is shown in Figure 3.3 and derived in `extremal_algebra.m` (Appendix D, where the deformation gradient is derived from the generalized lattice formulation in each case. There are two independent SLMs in square symmetry (red and blue) and three independent SLMs for hexagonal symmetry (red, blue, and green). These are color coded to highlight how each

SLM is allowed to independently deform and illuminate the relationship with the deformation gradient.

For example, Figure 3.3a is the 2*22 $\theta = 0$ bimodal material. This material allows horizontal deformation along the horizontal lattice lines and vertical deformation along vertical lattice lines. This is shown as the independent red and blue arrows, but also conveyed in the deformation gradient with the independent parameters a and b each allowed to modify the deformation gradient. The Type I hexagonal materials have a third SLM which operates as a constraint on the unit cell by ensuring $|\hat{l}_1| = |\hat{l}_2|$. Thus, the deformation gradient actually loses free parameters, seen in Figure 3.3e and f having only the free parameter $a = b$. Meaning, that if the SLM that corresponds to a deforms then the SLM corresponding to b must also deform.

Analyzing the deformation gradient directly can provide insight into the material that is lost in linear elasticity. For example, the isotropic nullmode material (Figure 3.3e) has deformation gradient

$$\begin{bmatrix} 1 & -a \\ a & 1 \end{bmatrix}$$

indicating rotation that is possible in the unit cell. The eigenvectors (Q) and eigenvalues (Λ) of this material's deformation gradient are

$$Q = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 1 - ai & 0 \\ 0 & 1 + ai \end{bmatrix}$$

confirming that rotational deformation of the material is possible. However, this information is lost in the translation to Cauchy elasticity which does not account for rotations. This is seen by the compliant strain equaling the zero vector in Figure 3.3, seemingly indicating that no deformation is allowed. The use of polar DOFs is of substantial interest [41, 84, 121] and an examination using micropolar or Cosserat elasticity theories would be required to explore this concept further.

The compliant linear strain provides key insight into a material's behavior by revealing the zero modes of a linear elastic material. I find the zero modes by converting the deformation

gradient to strain which, in the limit of infinite compliance, satisfies

$$\sigma = 0 = C\varepsilon \quad (3.5)$$

implying that the compliant strains form the nullspace of C . In reality, the elasticity matrix C is always symmetric positive definite, meaning that Equation 3.5 can only be satisfied when $\varepsilon = [0, 0, 0]^T$. However, if the compliance is weak enough it will still approximately satisfy Equation 3.5. Equivalently, the compliant strains are the eigenvectors corresponding to zero-value eigenvalues, meaning that the number of independent zero-energy strain vectors (i.e., the nullity of C) are the extremal mode of the material. This is seen in Figure 3.3, where nullmode materials have no eigenstrains (outside the trivial zero vector), unimodal materials have one eigenstrain, bimodal materials have two, and the trimodal materials have three.

3.2.5 An Analytical Model for the SLM Compliance Matrix S

Derivation The analytical compliance matrix can be developed using the same linear strain defined in Equation 3.2. I express the Voigt strain $\varepsilon = [\varepsilon_{xx}, \varepsilon_{yy}, \gamma_{xy}]^T$ in terms of the strain-DOF operator matrix $B(\theta)$ and the DOFs $q = [a, b]^T$, where a and b are the DOFs for the individual SLMs shown in Equation 3.3, as

$$\varepsilon(\theta) = B(\theta)q.$$

The matrix $B(\theta)$ dictates the allowable deformation modes of the lattice irrespective of the actual SLM deformation length scales. Each SLM stiffness is modeled as an axial spring k_i , collected in $K = \text{diag}(k_a, k_b)$. To determine the macroscale compliance, I minimize the internal strain energy

$$U(q) = \frac{1}{2}q^T K q$$

subject to the kinematic constraint

$$B(\theta)q = \bar{\varepsilon},$$

where $\bar{\varepsilon}$ is the imposed macro strain.

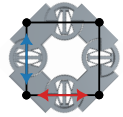
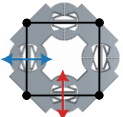
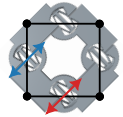
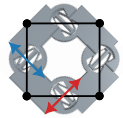
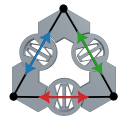
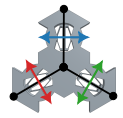
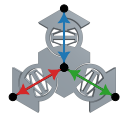
	 a b c	Deformation Gradient (F)	Compliant Strains (ε)	Eigenstrains
a		$\begin{bmatrix} 1+a & 0 \\ 0 & 1+b \end{bmatrix}$	$\begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$
b		$\begin{bmatrix} 1 & a \\ b & 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ a-b \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$
c		$\begin{bmatrix} a + \frac{a}{\sqrt{2}} & \frac{b}{\sqrt{2}} \\ \frac{a}{\sqrt{2}} & 1 + \frac{b}{\sqrt{2}} \end{bmatrix}$	$\begin{bmatrix} \frac{\sqrt{2}a}{2} \\ \frac{\sqrt{2}b}{2} \\ \frac{\sqrt{2}a}{2} + \frac{\sqrt{2}b}{2} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$
d		$\begin{bmatrix} a + \frac{a}{\sqrt{2}} & \frac{-b}{\sqrt{2}} \\ \frac{a}{\sqrt{2}} & 1 + \frac{b}{\sqrt{2}} \end{bmatrix}$	$\begin{bmatrix} \frac{\sqrt{2}a}{2} \\ \frac{\sqrt{2}b}{2} \\ \frac{\sqrt{2}a}{2} - \frac{\sqrt{2}b}{2} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$
e		$\begin{bmatrix} 1 & -a \\ a & 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
f		$\begin{bmatrix} 1+a & 0 \\ 0 & 1+a \end{bmatrix}$	$\begin{bmatrix} a \\ a \\ 0 \end{bmatrix}$	$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$
g		$\begin{bmatrix} 1 + \frac{\sqrt{3}c}{6} - \frac{\sqrt{3}a}{6} & -\frac{a}{6} - \frac{2b}{3} - \frac{c}{6} \\ \frac{a}{2} + \frac{c}{2} & 1 + \frac{\sqrt{3}a}{6} - \frac{\sqrt{3}c}{6} \end{bmatrix}$	$\begin{bmatrix} -\frac{\sqrt{3}(a-c)}{6} \\ \frac{\sqrt{3}(a-c)}{6} \\ \frac{a}{3} - \frac{2b}{3} + \frac{c}{3} \end{bmatrix}$	$\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$
h		$\begin{bmatrix} 1 + \frac{a}{2} + \frac{c}{2} & \frac{\sqrt{3}(a-c)}{6} \\ \frac{\sqrt{3}(a-c)}{6} & 1 + \frac{a}{6} + \frac{2b}{3} + \frac{c}{6} \end{bmatrix}$	$\begin{bmatrix} \frac{a}{6} + \frac{c}{2} + \frac{2b}{3} + \frac{c}{6} \\ \frac{\sqrt{3}(a-c)}{3} \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Figure 3.3: The three symmetry groups examined in this study are presented at specific angles in Table 3.1. Deformation vectors are indicated by colored arrows, with a legend above explaining the correspondence of each vector to the variables a , b , or c . Deformation gradients are calculated by fulfilling all independent SLMs within the cell concurrently. Compliant strains are determined as linear strains and are displayed in vector form. These strains can also be expressed as linearly independent eigenstrains, which are the unique strains compliant with a material constructed with the specified DOF.

I form the Lagrangian

$$\mathcal{L}(q, \sigma) = \frac{1}{2}q^\top K q + \sigma^\top (\bar{\varepsilon} - B(\theta)q),$$

where the Lagrange multiplier σ is power-conjugate to $\bar{\varepsilon}$ and is therefore identifiable as the macro stress. Stationarity with respect to q gives

$$\frac{\partial \mathcal{L}}{\partial q} = Kq - B(\theta)^\top \sigma = 0 \quad \Rightarrow \quad q = K^{-1}B(\theta)^\top \sigma.$$

Substituting back into the constraint yields

$$\bar{\varepsilon} = B(\theta)K^{-1}B(\theta)^\top \sigma.$$

Thus the effective compliance matrix is

$$S(\theta) = B(\theta)K^{-1}B(\theta)^\top. \quad (3.6)$$

If all the SLMs have equal stiffness $k_i = k$, Equation 3.6 reduces to $S(\theta) = \frac{1}{k}B(\theta)B(\theta)^\top$.

Note, the analytical model assumes $\hat{l}_1$ to be along the global $\hat{x}$ direction. Thus a coordinate system rotation may be necessary depending on the orientation of the lattice relative to the global coordinate system.

Fitting Analytical Models to Data Analytical models were fit to both the measured data and the FEA square-symmetric (2*22 and 442) simulated data for direct comparison. This provides a robust and comprehensive method of directly comparing the material characteristics and behavior across θ rather than on a point-by-point basis. First, the analytical models were derived for the 2*22 and 442 materials in the lattice orientation used for testing and simulation. The $B(\theta)$ matrices for each respectively are

$$B_{2*22}(\theta) = \begin{bmatrix} \cos(\theta) & 0 \\ 0 & \cos(\theta) \\ \sin(\theta) & \sin(\theta) \end{bmatrix} \quad (3.7)$$

and

$$B_{442}(\theta) = \begin{bmatrix} \cos(\theta) & 0 \\ 0 & \cos(\theta) \\ \sin(\theta) & -\sin(\theta) \end{bmatrix} \quad (3.8)$$

which have the orthogonal lattice lines oriented along the global $\hat{x}$ and $\hat{y}$ directions.

Rotating the coordinate space by $\pi/4$ (to align with the test and simulation coordinate system) and assuming $k_i = k = 1$, the characteristic behaviors from Equation 3.6 are then

$$S(\theta)_{2*22} = \begin{bmatrix} \frac{1}{2}(1 + \sin(2\theta)) & \frac{1}{2}\cos(2\theta) & 0 \\ \frac{1}{2}\cos(2\theta) & \frac{1}{2}(1 - \sin(2\theta)) & 0 \\ 0 & 0 & \cos(2\theta) + 1 \end{bmatrix} \quad (3.9)$$

and

$$S(\theta)_{442} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2}\cos(2\theta) & -\frac{1}{2}\sin(2\theta) \\ \frac{1}{2}\cos(2\theta) & \frac{1}{2} & \frac{1}{2}\sin(2\theta) \\ -\frac{1}{2}\sin(2\theta) & \frac{1}{2}\sin(2\theta) & \cos(2\theta) + 1 \end{bmatrix} \quad (3.10)$$

. I generalize the models to optimize over a set of linear coefficients C

$$S(\theta)_{2*22}^{opt} = \begin{bmatrix} c_1 \sin(2\theta) + c_2 & c_3 \cos(2\theta) + c_4 & 0 \\ c_3 \cos(2\theta) + c_4 & c_2 - c_1 \sin(2\theta) & 0 \\ 0 & 0 & c_5 \cos(2\theta) + c_6 \end{bmatrix} \quad (3.11)$$

and

$$S(\theta)_{442}^{opt} = \begin{bmatrix} c_1 & c_2 \cos(2\theta) + c_3 & c_4 \sin(2\theta) \\ c_2 \cos(2\theta) + c_3 & c_1 & -c_4 \sin(2\theta) \\ c_4 \sin(2\theta) & -c_4 \sin(2\theta) & c_5 \cos(2\theta) + c_6 \end{bmatrix} \quad (3.12)$$

optimizing over all $c_i \in C$ for each material set independently. The optimization function then becomes

$$\begin{aligned} \min_C \quad & \sum_{\theta_i \in \Theta} \frac{(S(\theta_i; C)^{opt} - \hat{S}(\theta_i))^2}{(\hat{S}(\theta_i))^2} \\ \text{s.t.} \quad & \det(S(\phi)) > 0, \quad \phi = \{0^\circ, 1^\circ, 2^\circ, \dots, 180^\circ\} \end{aligned} \quad (3.13)$$

where $\hat{S}(\theta_i)$ represents either the measured data or the FEA modeled data for each independent model fit respectively, Θ represents the set of angles that resolve the data (0° to 180° in 22.5° increments for measured data and 5° increments for FEA data) and the constraint on the determinant is applied to a more fine angular increment ϕ every 1° to the model to ensure the it is physically realistic at all points. The final optimized coefficients for the modeled data are provided in Table 3.2.

Table 3.2: Analytical model fit coefficients for each measured and FEA data set. Reference Equation 3.9 and Equation 3.10 for placement of coefficients in S matrices.

Data Set	c_1	c_2	c_3	c_4	c_5	c_6
2*22 Measured	2.606	3.018	2.336	0.072	3.720	4.908
2*22 FEA	3.092	3.150	3.061	-0.003	6.197	6.321
442 Measured	2.868	2.591	-0.120	-2.346	2.753	3.727
442 FEA	2.904	2.826	0.064	-2.855	5.636	5.755

While not shown here, an analytical model for the hexagonal tiled lattices can be similarly derived.

Figure 3.6 shows the FEA simulated and FEA fit model. Notably, the FEA model predicts a higher stiffness than the measured results indicate; which is also borne out in the larger coefficients of Table 3.2. However, the general trends are all qualitatively well-matched (e.g., 2*22 peaks in moduli or asymptotes in Poisson’s ratio; 442 constant Young’s moduli sinusoidal normal-shear coupling). This then validates the models derived in Equation 3.9 and Equation 3.10 as characteristic of the materials’ behavior across the full range of θ and indicates suitability for use against the measured data later.

3.3 Methods

Simulation Computational homogenization was performed using ANSYS Material Designer. Representative volume elements (RVEs) were created in PTC OnShape CAD software and simulated with periodic boundary conditions in 3D to extract the full 6×6 3D elasticity matrix. This matrix was then reduced to the 3×3 elasticity matrix for 2D plane stress materials. In this work, all analysis of simulated data was done on the reduced 2D matrix.

RVEs for 2*22 and 442 materials were generated to simulate data for $\theta \in [0^\circ, 180^\circ)$ in 5° steps, using symmetry to minimize the required number of simulations. 632 models were simulated with only $\theta = 0^\circ$ and 90° due to ANSYS Material Designer’s limitations of (1) rectilinear periodic boundary conditions and (2) the entire simulated body needing to be one continuous object.



Figure 3.4: The 4×18 array under tensile loading (a) and shear loading (b). Red markers are used for optical tracking to extract local strain information in the array.

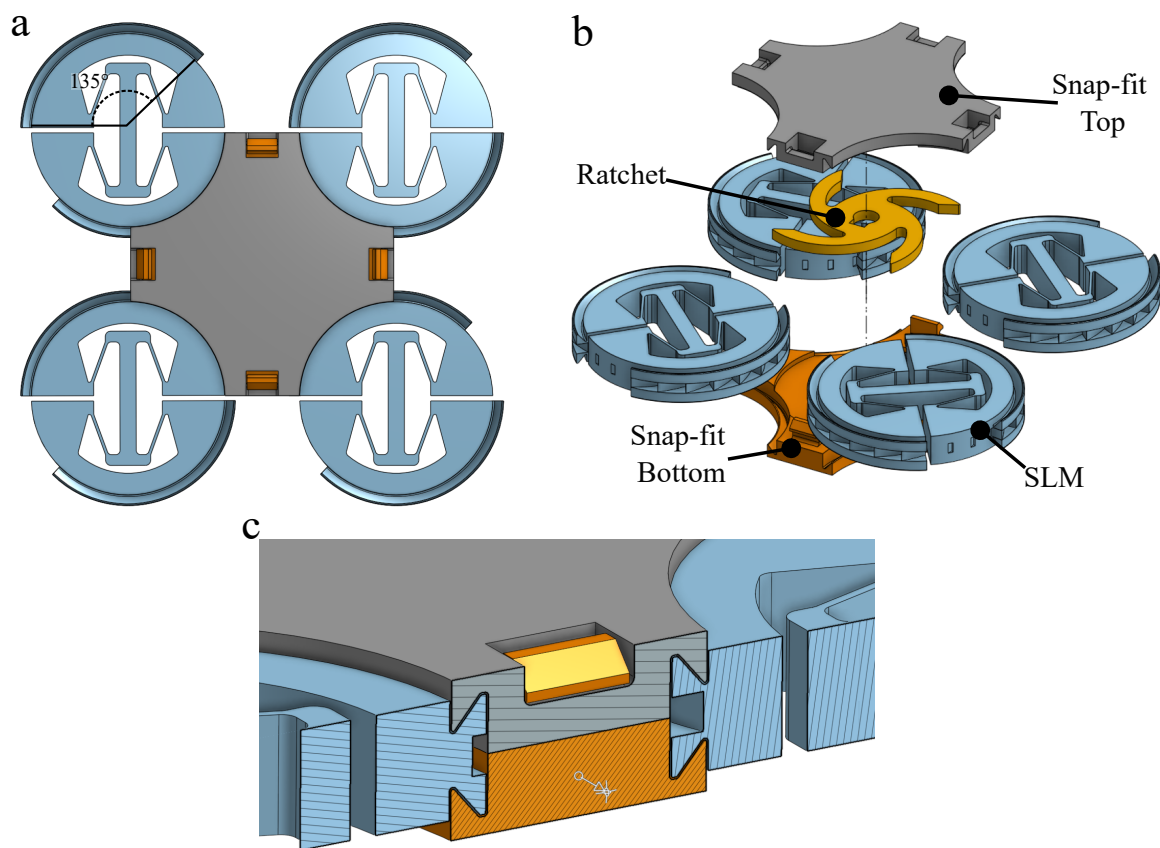


Figure 3.5: CAD of the SLM reprogrammable array unit cell. SLMs are arranged on a square lattice (a) and attached by running a dovetail through a snap-fit enclosure (b) so that the SLM is captured at least on one side. There is a hidden, internal ratchet spring system to help align and secure the SLMs during testing. (c) shows a close-up section view of the dovetail, illustrating how the SLM is allowed to pass through the enclosure, while maintaining contact.

Fabrication All SLM samples were fabricated using a Carbon 3D M1 printer with UMA-90 material using standard print settings. The SLMs were printed in plane with the print bed and with a Carbon 3D film release coating applied to the print bed. Post-processing followed the manufacturer’s recommendations.

Array Construction A 4×18 square lattice array was fabricated to validate and characterize the 2*22 and 442 materials (Figure 3.4). The design incorporates snap-fit connectors and a rotational dovetail, facilitating reconfigurable and easy assembly (Figure 3.5). This modular construction allows for the easy replacement of parts because no parts are rigidly connected, instead relying on friction and other forces to maintain contact. A ratcheting scheme aligns and secures the SLMs during testing (Figure 3.5b). The close fit of the dovetail seen in the section view Figure 3.5c enables the SLM to pivot while maintaining its connection during loading for a range of θ .

The dovetail’s length determines the array’s available configuration space. Dovetails in the samples extend over 135° (Figure 3.5a), allowing testing within $\theta \in [0, 135^\circ]$. However, this precludes reprogrammability within $\theta \in (135^\circ, 180^\circ)$, where the dovetails contact the snap-fit base, immobilizing the SLM and eliminating all compliance. This restriction serves as a tunable parameter, offering potential areas for future investigation.

Test, Measurement, and Characterization The array was tested on a universal testing machine (UTM) (Instron 5965) in a quasi-static manner. For each rotation angle five tests were performed; I report the median value, with error bars spanning the interquartile range (25th-75th percentiles). Two symmetry patterns (2*22 and 442) were tested, in two different loading conditions (tensile and shear), with approximately five tests per θ value for $\theta \in 0, 22.5^\circ, 45^\circ, 67.6^\circ, 90^\circ, 112.5^\circ, 135^\circ$. Measured data could not be captured for $\theta = 157.5^\circ$ due to the dovetail mechanisms construction and data at $\theta = 180^\circ$ is identical to 0° by construction, thus this data is not included in the measured data.

The force vs. displacement curves and videos of the array deforming array were recorded for testing at various θ configurations. The global force was used for modulus characterization (E_1, E_2, G_{12}), while the middle third of the optical trackers were used to capture deformation

behavior and characterize strain coupling ratios ($\nu_{12}, \nu_{21}, \eta_{121}, \eta_{122}$). The middle third of the trackers were used to avoid clamped boundary effects at the top and bottom end of the array.

3.4 Results & Discussion

3.4.1 SLM Orientation and Symmetry Provide Different Emergent Properties

We can observe a variety of emergent and extremal properties by constructing materials using wallpaper groups 2*22, 632, and 442 expressed here in the orbifold notation [31] (cmm, p6, p4 in crystallographic notation) and shown in Figure 3.1e-g. In all cases, altering the orientation of the SLM within a single lattice type maintains the global structure, but leads to the development of distinct extremal materials.

Figure 3.1e shows the topological layout and FEA simulated data for three variations of the 2*22 symmetry for $\theta \in \{0^\circ, 45^\circ, 90^\circ\}$, which create three distinct orthotropic extremal materials: an auxetic bimode, a zero Poisson's ratio bimode, and a positive Poisson's ratio unimode. In each case, the behavior of the material is described by Equation 3.2 which prescribes the directions of compliance relative to the lattice lines.

The angles of Figure 3.1e show a wide range of simulated FEA properties. For $\theta = 0^\circ$ the zero modes of a material will lie diagonally along the lattice lines, without any shear compliance. With SLMs along each diagonal of the lattice, this material is a bimode, compliant with extension along each of the diagonals independently. This is demonstrated in the moduli plot along the diagonal, where a low Young's modulus and a high shear modulus confirm the expected allowable deformations. Additionally, there is a zero Poisson's ratio along those diagonals with the DOFs, but the diverging nature of the DOFs causes a negative Poisson's ratio (auxetic) along the principal horizontal and vertical directions. The 2*22 $\theta = 45^\circ$ structure is also a bimodal material, allowing independent horizontal and shear deformations due to the alignment of zero modes in the horizontal direction. This results in a low Young's modulus horizontally, a high Young's modulus vertically, a low shear modulus in the plane, and a zero Poisson's ratio in the plane. Finally, the 2*22 structure with $\theta = 90^\circ$ exhibits unimodal properties. Complementary to the $\theta = 0^\circ$ case, here only the off diagonal

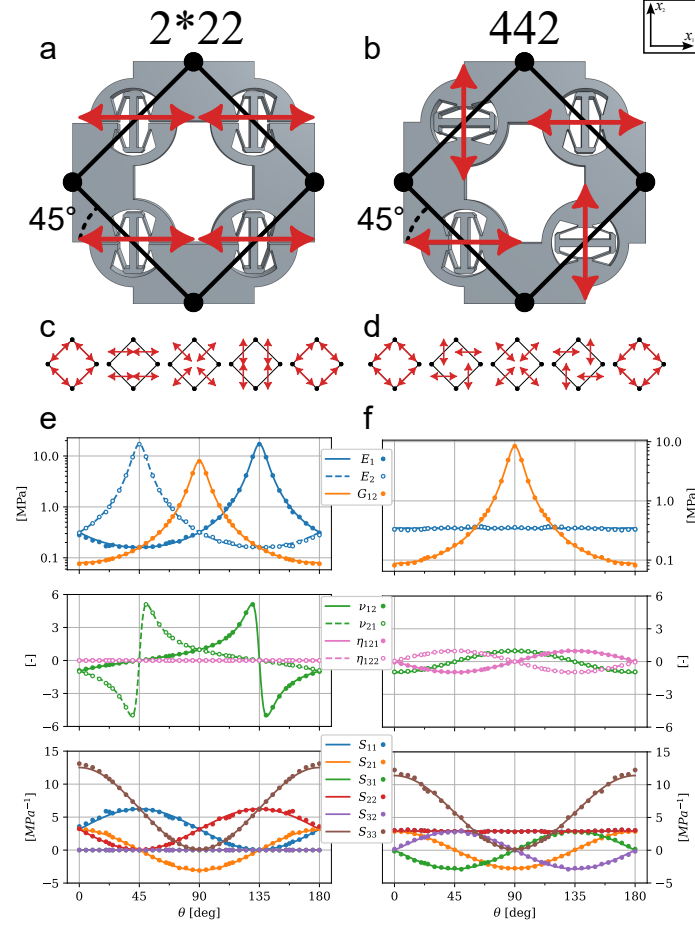


Figure 3.6: FEA simulated and FEA fit model data. CAD models used for FEA for 2*22 (a) and 442 (b) square symmetry materials shown at a $\theta = 45^\circ$ configuration. The illustrations below each image (c and d) show the DOF alignment for each symmetry as a function of θ and correspond to the major x-axis grid lines in the plots below. The data shown in (e) and (f) show experimental data (dots) and corresponding analytical fits (solid/dashed lines). From top to bottom: the directional Young's and shear moduli (E_1 , E_2 and G_{12}); the Poisson's and normal-shear coupling ratios (ν_{12} , ν_{21} and η_{121} , η_{122}); and finally the individual S matrix components which were used to fit the model directly. Interquartile ranges are not shown due to each data point having only one simulation run.

elements of Equation 3.2 are non-zero, meaning that the material does not extend and can only shear relative to the lattice lines. This is seen with the low shear modulus along the diagonals. Furthermore, in contrast to $\theta = 0^\circ$, the DOFs are now aligned to produce a positive Poisson’s ratio in the horizontal and vertical directions. Taken together, these three orientations prove that a single lattice can be dialed from stretch, to stretch-shear, and then shear-only behavior simply by rotating the SLMs, demonstrating full, on-demand control over its deformation modes.

My analytical and FEA models indicate that hexagonal 632 symmetry can realize all four isotropic extremal types — null-, uni-, bi-, and trimode — via two tiling variants (Figure 3.1f and g). Strikingly, the hexagonal lattices only need two orthogonal orientations ($\theta = 0^\circ$ and 90°) to span this full isotropic extremal set. In this case the Type I tiling produces both nullmode and unimode materials (90° and 0° respectively), while Type II tiling produces bimode and trimode materials (90° and 0° respectively).

The Type I construction uses an equilateral triangle cell layout, which adds a geometric constraint that all sides must be equal upon deformation. The combination of this constraint and zero modes oriented perpendicular to the lattice lines ($\theta = 90^\circ$) results in a material that resists extension or shear deformation, making it a nullmode material. While nullmode materials are often considered trivial, my SLM-based design produces a non-trivial internal rotational motion due to the coordinated DOFs of its zero modes. Notably, this rotational DOF is fully described by the SLM deformation gradient (Equation 3.1), but is not captured in the linear elastic theory (Equation 3.2).

The Type I material with SLMs oriented along the lattice lines ($\theta = 0^\circ$) numerically demonstrates an isotropic unimode. The zero modes align with the lattice lines meaning that only expansion and contraction along the lattice dimensions are allowed. This produces a material that compresses and expands easily without shear, i.e., has a low bulk modulus and high shear modulus, a key feature of isotropic unimodes. Consequently, this also results in an FEA simulated negative Poisson’s ratio of $\nu \approx -0.88$ which approaches the theoretical limit of -1 for isotropic 2D elastic materials.

Type II tiling covers the isotropic bimode-trimode materials. Similar to the nullmode material, the bimode material has DOFs perpendicular to the lattice ($\theta = 90^\circ$); however,

Type II tiling does not have the additional equilateral triangle constraint from Type I. Because of this, the bimode material is able to shear along the lattice lines, producing the opposite effect of the Type I unimode, that is a material with a high bulk modulus and low shear modulus. This bimode material is the 2D analog of the 3D pentamode material [78, 59], operating as a “metafluid”, resisting bulk compression, but compliant to shear or “flow.” Therefore as expected, this material approaches the theoretical limit of +1 for isotropic 2D elastic materials with a simulated value of $\nu = 0.97$.

Finally, the trimode material, like the nullmode, is normally considered trivial as it is completely compliant; however, my SLM trimode material has DOFs along the lattice lines ($\theta = 0^\circ$). As with the previous materials with DOFs along the lattice, this trimode material has a negative Poisson’s ratio, a feature that has not been previously seen in other compliant extremal materials. The sum total of the isotropic materials is that I have shown a method of constructing all four types of isotropic extremal materials using SLMs.

3.4.2 The Property Space is Interpolative

In Figure 3.1 we can see discrete states of the lattice; however, unlike previous work [59, 50, 117], these extremal materials are continuously reprogrammable. By rotating the SLMs I produce smooth variations in their properties. This produces an interpolative property space parameterized by the continuous variable $\theta \in [0, 180^\circ)$. This variability arises from both the lattice symmetry and the SLM orientation, each independently influencing the material’s properties. Thus, by using a single lattice I can enable or disable properties, like chirality or auxetics, through reprogramming of the SLM orientations.

The two materials shown in Figure 3.7 demonstrate how the same square lattice can produce distinct interpolative material property characteristics by enforcing different symmetry types (mirror vs. rotational) on the SLMs. Figure 3.7a and b show the two square symmetry patterns, $2*22$ and 442 , in a $\theta = 45^\circ$ configuration. Shown directly below (Figure 3.7c and d) is an illustration for how the DOFs differ due to mirror vs. rotational symmetry as θ varies is shown directly, which correspond to the x-axis gridlines of the below plots. Finally, I show the measured data (dots) and fit model data (solid/dashed lines) in Figure 3.7e and f. The

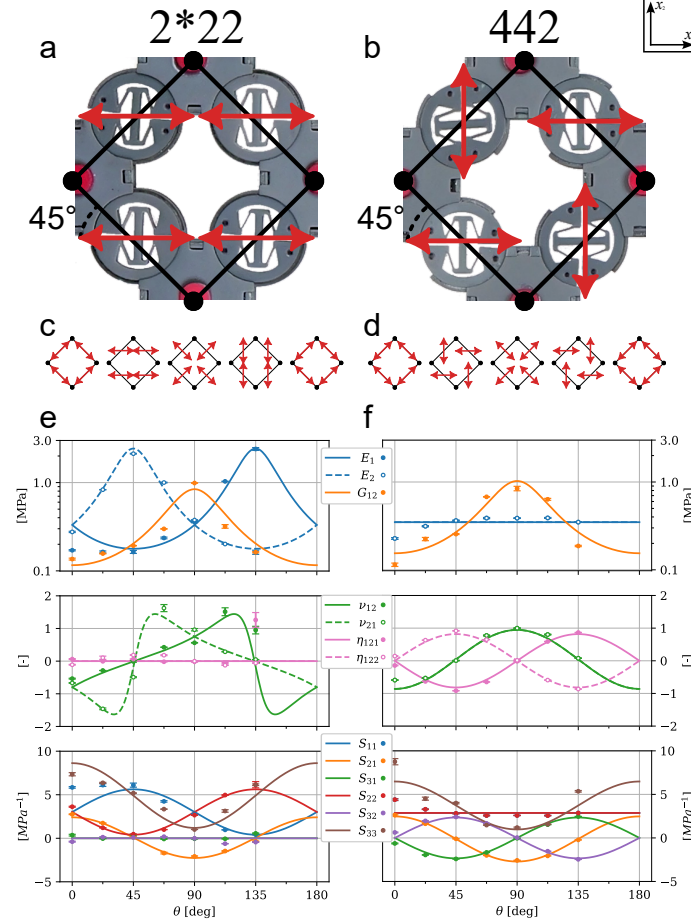


Figure 3.7: Fabricated $2*22$ (a) and 442 (b) square symmetry materials shown in close-up at a $\theta = 45^\circ$ configuration. The illustrations below each image (c and d) show the DOF alignment for each symmetry as a function of θ and correspond to the major x-axis grid lines in the plots below. The data shown in (e) and (f) show experimental data (dots) and corresponding analytical fits (solid/dashed lines). From top to bottom: the directional Young's and shear moduli (E_1, E_2 and G_{12}); the Poisson's and normal-shear coupling ratios (ν_{12}, ν_{21} and η_{121}, η_{122}); and finally the individual S matrix components which were used to fit the model directly. Markers show the median of five repeats, $n=5$; vertical bars denote interquartile range. Data was not gathered for 157.5° due to limitations of the dovetail construction. Similarly, data was not gathered for 180° due to it having an identical construction to 0° and so was considered redundant.

measured data was captured every 22.5° while the analytical model is fit to these points but interpolated along the full span of θ . Each θ configuration was tested with five physical replicates with median values reported and vertical bars representing the interquartile range (25th-75th percentiles). Data is not shown for 157.5° or 180° . Data was not captured for 157.5° due to limitations of the dovetail mechanism, while data for 180° was not captured due to being identical by construction to the 0° configuration and so was considered redundant in nature. Regardless, the physical performance is expected to follow the relationships driven by the symmetry of the material, predicted by the analytical model, and demonstrated numerically with the FEA data (Figure 3.6).

For the 2*22 material in Figure 3.7e, the Young's modulus varies from low to high depending on the alignment of the SLMs, low when all the DOFs are aligned with a principal direction and high in the orthogonal direction, while the shear modulus peaks simply once for $\theta = 90^\circ$. Notably, where the Young's moduli peak (45° (E_2) and 135° (E_1)) the corresponding Poisson's ratio has an asymptotic behavior, due to the SLM rigid directions all being aligned in the respective principal direction, x_1 or x_2 . While the shear modulus demonstrates a single peak at 90° , the normal-shear coupling is minimal by the mirror symmetric nature of the 2*22 pattern, with the exception at 135° , which is an artifact of testing at the asymptotic point highlighted earlier for Poisson's ratio. Significantly, the moduli can be varied by an order of magnitude and along with that we can see the Poisson's ratio vary from negative (auxetic) to positive.

In contrast to the 2*22 material, the 442 Young's modulus remains relatively flat (Figure 3.7f). This is expected because all four DOFs of the unit cell never fully align and, by rotating together, lead to a constant level of stiffness as a function of θ . The shear modulus, however, follows the same trend as the 2*22 material and also varies about one order of magnitude. Here the Poisson's ratio does not exhibit the asymptotic behavior, instead producing a cosine wave, while the normal-shear couplings produce sine waves of opposite signs ($\eta_{121} = -\eta_{122}$). This normal-shear coupling is expected due to the rotational symmetry of the material which, lacking mirror symmetry, allows for non-zero S_{13} and S_{23} values, a feature absent with the 2*22 material. Importantly, we can see the chirality of the material changes as a function of θ with varying η_{121} from -1 to $+1$. This allows me to turn

on and off the coupling as well as change the handedness of the material.

The bottom plot of Figure 3.7e and f show the measured data and corresponding analytical model of the S matrix. The specific coefficients of the analytical model are then optimized to minimize the relative error across all S values equally and are provided in Table 3.2. Overall, there is good agreement between the analytical model and measured data with several key idealizations were made in generating the models. First for the 2*22 material, the values of S_{31} and S_{32} were assumed to be zero, due to the natural symmetry of the material. This is demonstrated in the analytically derived model and FEA data. While the data does not reflect zero values, it is still relatively small when compared to other values of S , most likely arising from noise in the motion capture system. For the 442 material, the values of S_{11} and S_{22} were assumed to be constant, again supported by the analytical model and corroborated by the FEA simulated data. These simplifying assumptions enable the model to capture the dominant trends in material behavior across all θ , while smoothing out local noise and inconsistencies in the measured data—resulting in a compact, robust representation that reflects both the symmetry-driven physics of the system and the empirical observations.

3.4.3 The Extremal Materials Span the Eigenvalue Gamut

To analyze the transitions of SLM lattices, I define the “extremal material gamut” (Figure 3.8) which provides a stiffness- and coordinate-invariant space to compare materials. Here I use normalized eigenvalues of C sorted smallest to largest, constraining the values within the upper-left triangle of the unit square. Specifically, if the eigenvalues of the 3×3 C are $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3$ then the normalized eigenvalues are $\hat{\lambda}_i = \lambda_i/\lambda_3$ for $i = 1, 2, 3$. Because $\hat{\lambda}_3 = 1$ by definition, I plot the relationships in a 2D space $\hat{\lambda}_2$ vs. $\hat{\lambda}_1$. In this space, the diagonal represents $\lambda_2 = \lambda_1$ while the top line represents $\lambda_2 = \lambda_3$. The remaining left-most boundary forms the extremal boundary, where there is always one small relative eigenvalue ($\lambda_1 \ll \lambda_3$) and defines the ideal unimode and bimode materials at points $(0, 1)$ and $(0, 0)$ respectively. Importantly, because I normalize each material configuration at θ to *its own* largest eigenvalue we can compare unimode and bimode materials directly in terms of their extremal characteristics independent of their inherent stiffness. The tradeoff for this

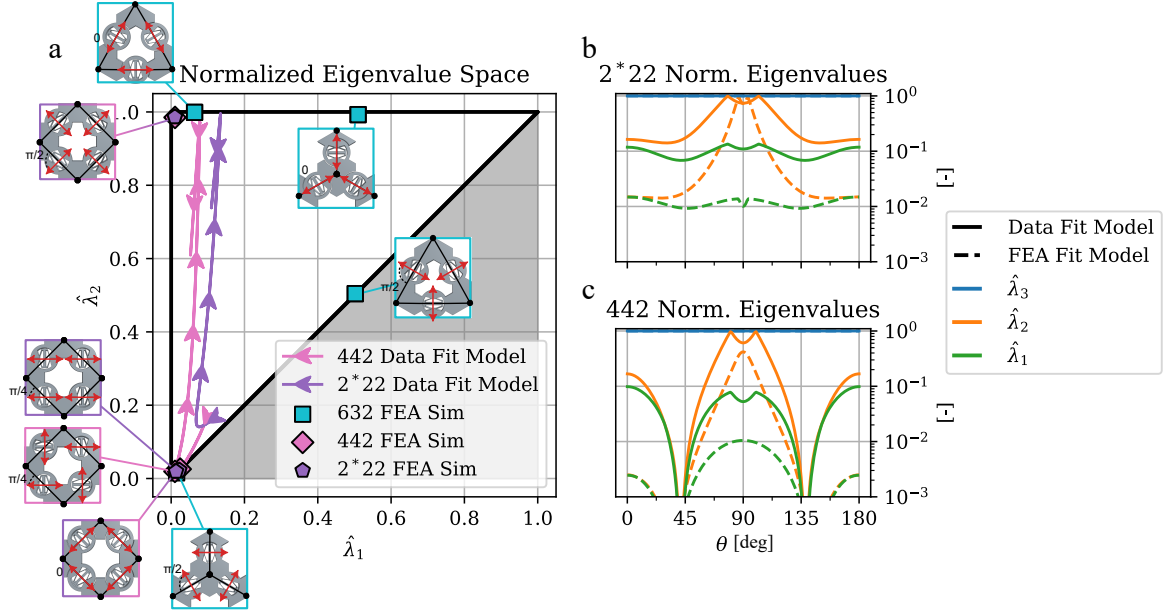


Figure 3.8: (a) Normalized FEA simulated eigenvalues of the 2*22, 442 and 632 materials plotted as $\hat{\lambda}_2$ vs. $\hat{\lambda}_1$. Additionally, the transition from bimode to unimode of the fitted data models for the 2*22 (purple) and 442 (pink) are presented for $\theta = [0^\circ, 90^\circ]$. The arrowheads indicate the direction of the trajectory as θ transitions from 0° to 90° . (b and c) The normalized eigenvalues from the experimental data and FEA data analytical fits for the 2*22 and 442 materials respectively. The line style indicates the measured data and FEA models (solid and dashed) and the line color indicates the normalized eigenvalue being plotted for a given model ($\hat{\lambda}_3$ is blue, $\hat{\lambda}_2$ is orange, $\hat{\lambda}_1$ is green). Absolute and relative eigenvalues for all FEA points are tabulated in Table A.1.

normalization is that it obscures materials with three small, but relatively close, eigenvalues, such as the 632 trimode discussed below.

As shown in Figure 3.7, just as the material properties smoothly vary over θ , so do the eigenvalues. This is seen by the purple and pink trajectories for the measured-data model in Figure 3.8a. The significance of the trajectories is two-fold: 1) both material sets (2*22 and 442) maintain relative extremality and 2) the extremal space is continuously variable. Here, the trajectories span $\theta = 0^\circ$ to $\theta = 90^\circ$ and follow the flow of the arrows. Figure 3.8b and c show the normalized eigenvalues of these trajectories as a function of θ . In each case the analytical 2*22 and 442 materials consistently have one small eigenvalue ($\hat{\lambda}_1 \lesssim 10^{-1}$) while $\hat{\lambda}_2$ varies from low to high, nearly equal with $\hat{\lambda}_3$ as $\theta \rightarrow 90^\circ$. This means the trajectories in the gamut (Figure 3.8a) show the square-symmetry materials transitioning from bimode to unimode by tracking along the far-left extremal boundary. The back-tracking seen toward the end of these trajectories, while remaining extremal, indicates a reordering of the eigenvalues relative to the eigenvectors, which is an artifact of the eigenvalue normalization process.

Beyond the measured-data model, I fit the FEA simulated data in the same manner for direct comparison. These are represented as the scatter data points on Figure 3.8a and the dashed lines in Figure 3.8b and c. Here we can see all the examples from Figure 3.1e-g, now shown in an extremal landscape. The 2*22 and 442 $\theta = 0^\circ$ and $\theta = 45^\circ$ materials are all bimodal, existing at the bottom left corner, noting that the 2*22 and 442 $\theta = 0^\circ$ materials are identical in DOFs and so overlap here. The 2*22 and 442 $\theta = 90^\circ$ material (again identical in DOFs) is shown at the top-left corner clearly exhibiting unimodal characteristics. Significantly, this demonstrates that the square-symmetric materials are capable of maintaining extremality while transitioning from bimode to unimode.

The 632 materials from Figure 3.1f and g are all shown. The null mode material (Type I tiling, $\theta = 0^\circ$) lies plainly in the middle of the space with no small eigenvalues approximately at (0.5, 0.5). For the extremal materials, the isotropic unimode (Type I tiling, $\theta = 0^\circ$) lies at the top left corner, while the bimode (Type II tiling, $\theta = 90^\circ$) lies at the bottom left corner, matching their desired extremal behavior. Notably, the trimode material (Type II tiling, $\theta = 0^\circ$) does not lie on an extremal boundary. While counterintuitive, this is an artifact of the normalization process. This gamut obscures materials where all the eigenvalues are

small relative to the base material, as in the 632 trimode case, which lies near $(0.5, 1)$ despite have three simulated small eigenvalues, as shown Table A.2. Regardless, the fact that all the unimode and bimode materials lie in their respective domains, and that the trajectories of the square-symmetric materials track along the extremal boundary, confirms the desired outcome of 1) programming all types of extremal behavior and 2) the ability to smoothly interpolate between extremal points.

3.5 *Spatially Varying Orientations Make Programmable Composites*

To demonstrate the utility of the reprogrammable SLM framework I implemented a 2D column with spatially tailored material properties. By globally coordinating the changes in the angle θ of the SLMs, I control the properties of the entire lattice; however, the angle θ also provides me with a field to spatially vary the local properties of the lattice. I take advantage of this tunability by reprogramming local stiffness in a finite array to provide different modes of actuation, as shown in Figure 3.9.

To load the array, I affixed two cables at the top, routed them downward through the body, and connected them to the array throughout the length to prevent buckling (Figure 3.9a).

When all SLMs are aligned with the loading direction (Figure 3.9b), the material compresses in the vertical direction. This lattice has the properties of the 2*22 material previously described in Figure 3.7a for $\theta = 135^\circ$; namely low E_2 and $\nu_{21} \approx 0$).

In contrast, reprogramming one column of SLMs perpendicularly partially emulates the 2*22 material for $\theta = 45^\circ$ along that vertical band (i.e., high E_2). The outcome of the striping process is a material that exhibits low stiffness on one side and high stiffness on the other in relation to the loading direction. Notably, because both of these configurations have low and equal shear modulus (seen in Figure 3.7e) the material is allowed to bend easily without significant shear stresses. This results in lateral bends as only one side of the array compresses, and by selecting the right or left most column for stiffening, I can control the direction in which the array bends (Figure 3.9c and d).

Leveraging the spatial reprogrammability even further, the design allows for more sophisticated behaviors, as illustrated by the S-bend curve pattern in Figure 3.9e. I take the lattice in Figure 3.9a, split it vertically into two halves, and program the upper half like

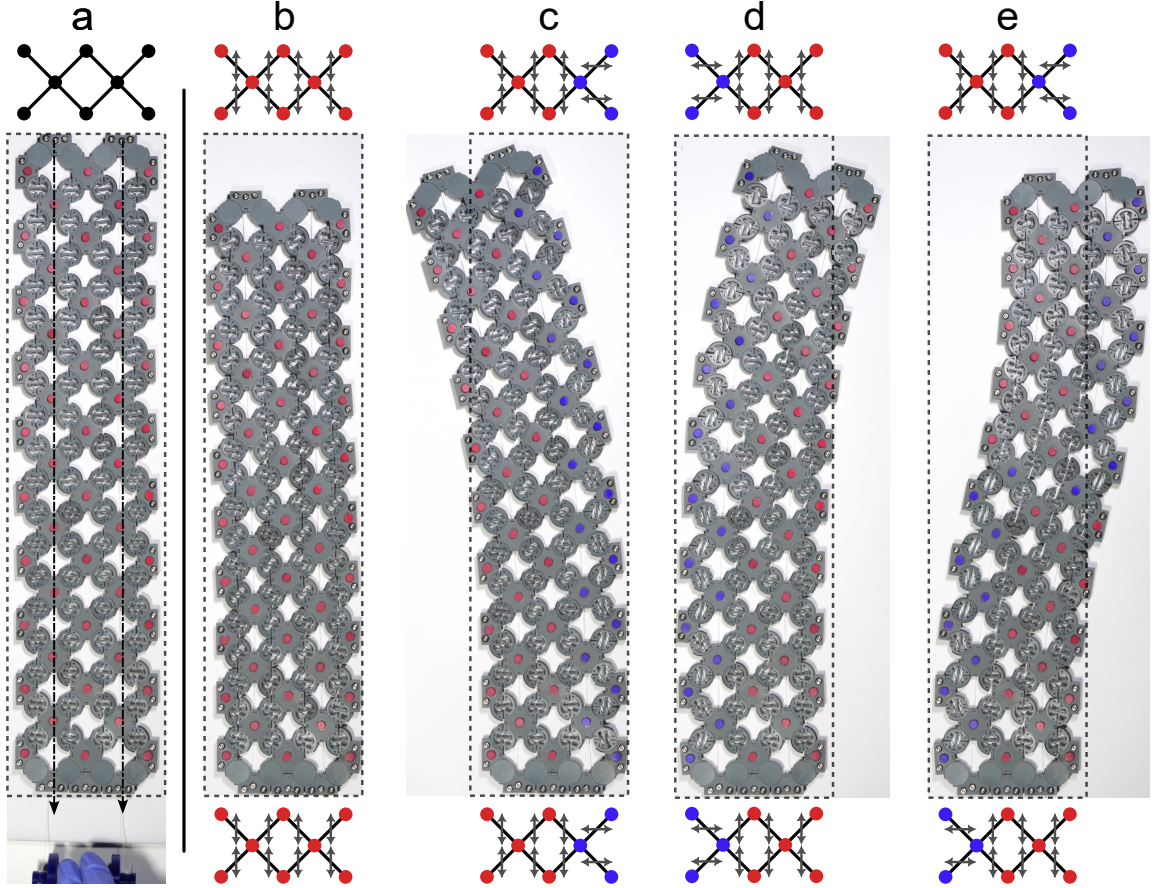


Figure 3.9: A single, cable-driven array can be spatially reprogrammed to provide unique actuation modes. (a) depicts an illustration of the lattice structure in the undeformed state while (b)-(e) show deformed states. Markers are physically red; in each panel I recolor subsets to match the DOF groupings shown in the lattice illustrations. SLM orientations are depicted by gray arrows. (b) has all the SLMs oriented vertically and compresses with $\nu \approx 0$ under tension. (c) and (d) have one column of horizontally oriented SLMs, right and left respectively causing the lateral bends. (e) uses different SLM orientations on the top and bottom halves of the array to create the composite S-bend.

Figure 3.9c and the bottom like Figure 3.9d. This leads to an S-shaped curve when compressed with complementary bending. The adaptability shows that my SLM-based materials enable the development of spatially programmable mechanical metamaterials within a single lattice and furthermore is not restricted to simple configurations. Significantly, a myriad of behaviors including auxetics and chirality are technically achievable as long as the DOFs and constraints are resolved without frustration.

3.6 Conclusion

In this study, I introduced a novel, reprogrammable approach to engineering 2D extremal materials using straight-line mechanisms and planar symmetry. SLMs provide a geometric basis for explicitly programming the zero modes of extremal materials, enabling experimental construction for square lattices and numerical validation for hexagonal lattices across nullmode through trimode responses, dictated by SLM orientation (θ), lattice type (square vs. hexagonal), and orientation symmetry (rotational vs. mirror). By controlling the SLM orientation, I spatially vary elastic properties across a given lattice, allowing selective tuning of Poisson's ratio, bulk-to-shear stiffness ratio, and chirality within a unified framework. I demonstrated the breadth of this approach by simulating and experimentally validating the square-symmetric unimode and bimode emergent material properties and extremal behaviors a stiffness- and coordinate-invariant extremal material gamut. I also demonstrate numerically how two types of hexagonal tilings can produce all four types of 2D isotropic extremal materials. This work not only realizes distinct extremal states but also demonstrates smooth, continuous transitions between them, representing the first such achievement in extremal materials.

While my results demonstrate the ability of a single reconfigurable unit to access all four extremal deformation classes, practical limitations remain. My simulations were primarily restricted to rectangular tilings due to ANSYS limitations with non-rectilinear periodic boundary conditions, leading me to fabricate only square-symmetric specimens. Future work will experimentally confirm the predicted continuous behavior of hexagonal specimens. Although experiments were conducted within the linear elastic regime (under 2% global strain), local large strains, geometric nonlinearities, buckling, or friction were not explicitly

modeled and may influence behavior at larger deformations. This reliance on standard linear elasticity also meant I could not fully validate intriguing degrees of freedom, such as the internal rotational motion observed rotationally symmetric materials. These uncaptured behaviors suggest a promising direction for future theoretical and experimental work exploring connections to micropolar/Cosserat elasticity and their potential for phenomena like mechanical cloaking [84, 121]. Finally, long-term durability and fatigue resistance of compliant SLM joints were not characterized. Future efforts will address material fatigue, creep, and statistical variation across printed samples through higher-throughput testing and extended mechanical cycling to validate performance in real-world scenarios.

This work significantly expands the design space for mechanical metamaterials, enabling geometrically intuitive, on-demand, dynamic adjustments of mechanical properties without structural alterations or costly computational methods. The continuous reprogrammability and arbitrary construction of zero modes unlocked by my SLM-based framework offer unprecedented control over material behavior. This paves the way for practical applications demanding adaptive mechanical responses, from next-generation soft robotics and underwater acoustics to advanced wearable technologies, fundamentally shifting how we design and utilize materials with tailored mechanical properties.

Chapter 4

FABRICATION-DIRECTED ENTANGLEMENT FOR DESIGNING CHIRAL AND ANISOTROPIC METAMATERIAL FOAMS

Note on Publication

Portions of this chapter are based on work currently under review for *Advanced Engineering Materials* and available as a preprint on *arXiv* (arXiv:2505.03064).

4.1 Introduction

Here I introduce fabrication-directed entanglement (FDE), a new approach leveraging computational design (Topology Optimization, TO [10]) and additive manufacturing (Viscous Thread Printing, VTP [70, 125]) to create spatially patterned entanglement within macroscopic structures, targeting prescribed elastic behaviors. FDE uniquely enables the programming of complex mechanical responses like targeted anisotropy and chirality *within* a continuous, single-material entangled foam structure—a capability previously unrealized in this material class. While the underlying VTP process involves certain trade-offs, notably resolution limits (~ 1 mm) and moderate achievable stiffness contrast ($\sim 30:1$) compared to some other additive multi-material manufacturing techniques, this integrated approach allows TO to strategically guide VTP coiling density, placing regions of dense or sparse entanglement to achieve the targeted macroscopic response.

In contrast to sequence-based chemical encoding or statistically homogeneous fabrication methods, FDE enables direct control over the local entangled network structure during VTP deposition, offering spatial modulation of density and connectivity from a single filament. This programmability is key: it allows tailoring the macroscopic mechanical response in ways inaccessible through simple material mixing or uniform VTP structures, demonstrating the value of programmed heterogeneity even within the inherent constraints of the VTP system. Furthermore, the resulting monolithic foam structure provides a direct physical analogue to TO’s continuum assumption, potentially mitigating connectivity challenges

sometimes faced in discrete solid-void realizations. The practical outcome is therefore a broadened design space for monolithic entangled foams, directly evidenced later in this work by achieving targeted directional stiffness and significant programmed normal-shear coupling (Section 4.4.1, 4.4.2)—complex behaviors arising from the FDE-patterned network structure.

In this work, I systematically explore how higher-order, spatially controlled entanglement patterns affect the mechanical response of 2D foam materials. I show that FDE-based structures surpass simple mixtures of homogeneous materials in terms of accessible property space, establishing FDE as a method for programming mechanical properties through controlled deposition. Specifically, I:

1. Introduce and implement the FDE method, using computational guidance (via TO) to inform VTP deposition.
2. Experimentally validate the FDE workflow by fabricating TO designs and demonstrating targeted mechanical behaviors like anisotropy and chirality.
3. Show, through simulation and experimental examples, an expanded property space compared to uniform VTP foams, including programmable directional stiffness and normal-shear coupling.

4.2 Fabrication-Directed Entanglement Overview

The FDE methodology, illustrated in Figure 4.1, integrates VTP and IH to impose higher-order patterns onto the inherently stochastic VTP coiling process. By spatially varying VTP parameters according to an IH-derived design, I create quasi-two-phase entangled structures with programmed local density and tailored macroscopic properties.

Viscous Thread Printing (VTP) Enables Control of Local Entanglement As highlighted previously, I can control characteristic entanglement (dense vs. sparse) via Equation 2.1, controlled by nozzle height H and speed F when filament diameter D_T and extrusion speed U are fixed, thereby tuning local stiffness independently of the global toolpath. I selected two compatible (V^*, H^*) pairs (Section 4.3.1) yielding dense and sparse

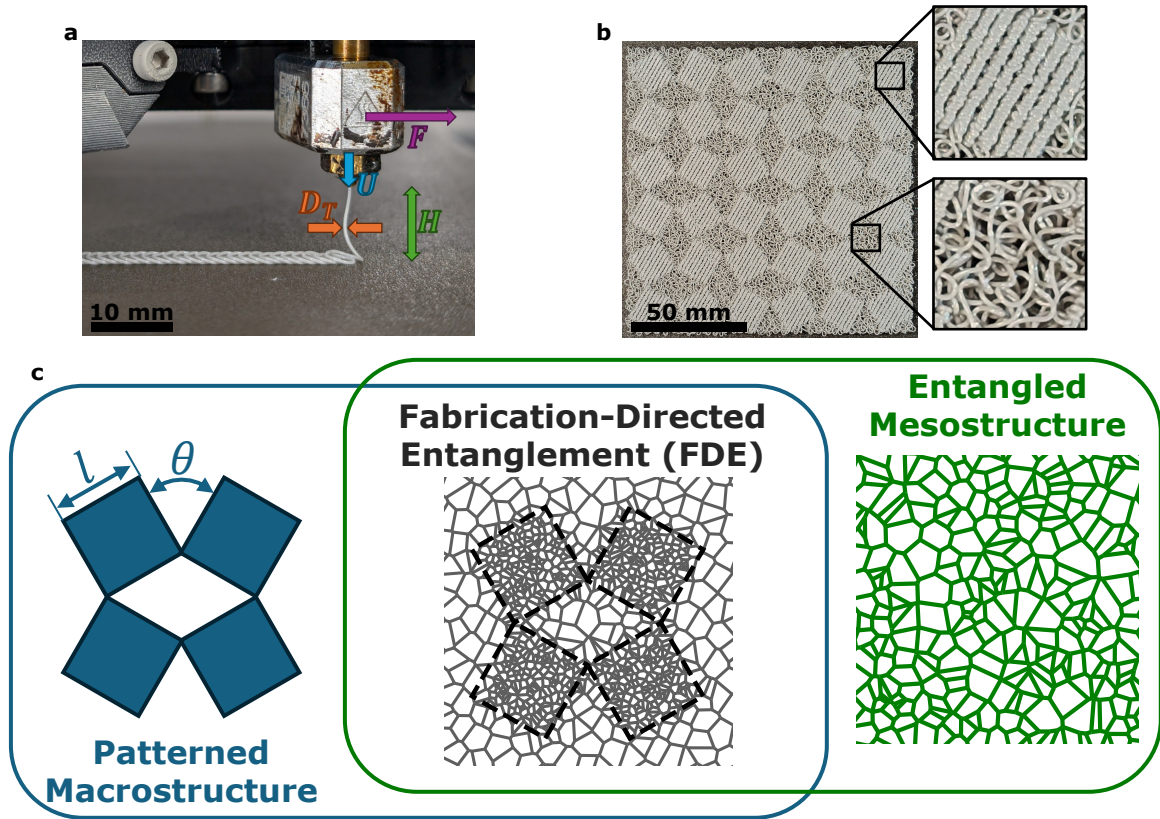


Figure 4.1: a) The VTP process depositing a line of stacked coils with key parameters highlighted: F (translational speed), U (extrusion speed), H (print height), and D_T (thread diameter). b) A “rotating squares” metamaterial produced with FDE by changing the coiling parameters. Zoomed insets show the different densities and characteristics of dense and sparse coiling. c) The FDE process takes an explicitly defined macrostructure and combines it with an implicit stochastic microstructure (schematically represented here by a Voronoi diagram), to produce variable-density entangled structures.

“phases” with an approximate measured stiffness ratio of $\sim 30:1$. Additionally, the selection of the print parameters resulted in a minimum feature width ~ 1.2 mm.

By spatially varying these parameters during the build, dense or sparse regions can be patterned on demand; however, the inherent coiling dynamics of VTP involve a trade-off. While gaining complex filament entanglement as an emergent property of the deposition dynamics, VTP cannot resolve features (e.g., hinges) smaller than the characteristic filament coil size. Furthermore, the limited stiffness ratio of the dense and sparse coils ultimately restricts the possible range of achievable material properties. Despite these limitations, with a single, homogeneous thread, this process yields a quasi-two-phase structure that expands the possible range of mechanistic behaviors of monolithic entangled foam metamaterials—particularly anisotropy and chirality.

Inverse Homogenization Enables Higher-Order Patterning of Local Entanglement

To strategically pattern the dense and sparse entangled regions created by VTP, I employ an IH framework following conventional density-based TO implementations in mechanical metamaterials [11] with two key differences: 1) I interpret TO low and high density regions as sparse and dense entanglements and 2) I optimize for a desired eigenstructure (i.e., principal directions and stiffnesses) instead of specific material properties.

First, the optimized density field ρ directly informs the VTP process, where $\rho = 0$ corresponds to the sparse, compliant (V^*, H^*) pair, and $\rho = 1$ corresponds to the dense, rigid (V^*, H^*) pair (Figure 4.1b). Whereas conventional TO often uses a solid-void material set for the density field, with stiffness contrasts such as $10^9:1$, the FDE approach operates with a quasi-two-phase entanglement system with stiffness $\sim 30:1$. While this stiffness ratio is substantially lower than conventional TO work, restricting the available property space, it is nevertheless comparable to those found in traditional composites [55], indicating that an engineered response is achievable. Furthermore, since FDE translates the density map into variations along a single, continuous filament path via VTP, the resulting structure is inherently continuous and well-connected—directly realizing the domain continuity in the TO FEM formulation—and is thus free from the disconnected features or ill-defined interfaces that can plague conventional TO.

Second, to target complex material behaviors I develop a generalized framework based on principal deformation modes (eigenstructure) (V, Λ) , where $T^* = V\Lambda V^\top$. This approach, inspired by extremal materials theory [78], allows designers to directly specify desired principal directions (V) and their relative stiffnesses (Λ) rather than deriving them from bulk properties. This direct specification greatly facilitates the exploration of diverse anisotropic and coupled behaviors; for instance, the chiral designs herein emerged unexpectedly when targeting specific normal-shear coupled strains using this method.

Here $V \in \mathbb{R}^{3 \times 3}$ where $V^\top V = I$ contains the desired principal directions (a design input) while the matrix $\Lambda = \text{diag}(a, 1, 1)$ for $a \ll 1$ defines the target eigenvalues. Throughout this work, the eigenvector strains of V are expressed using as $v_i = [\varepsilon_{11}, \varepsilon_{22}, \gamma_{12}]^\top$ and the specific library of V matrices used in this work is provided in Table 4.2. Thus by design, T^* possesses one relatively small eigenvalue corresponding to the first column vector of V , which I term the “leading direction,” $\mathbf{v}_{lead}$, with remaining columns corresponding to larger eigenvalues and span the orthonormal “supporting directions.” Importantly, the eigenstructure also implicitly encodes material symmetry, biasing the optimizer for certain topological symmetries, but not explicitly enforcing them.

The core optimization then seeks to match the eigenstructure of the homogenized material properties M ($M = C$ for unimodes and $M = S = C^{-1}$ for bimodes) to this target T^* . The homogenized stiffness matrix C is used directly for unimodes, as targeting the single compliant direction of v_{lead} fits the definition of a unimode material and simultaneously drives rigidity in the supporting directions. For bimodes requiring two compliant directions, I instead use the homogenized compliance matrix S . This sets v_{lead} to be a direction of rigidity, not compliance, and therefore drives the two supporting directions to now be directions of compliance. Specifically, I solve:

$$\begin{aligned} \min_{\rho} \quad & \|\hat{M} - T^*\|_F^2 \\ \text{s.t.} \quad & K(\rho)u = f \\ & g_n(\rho) \leq 0 \end{aligned}$$

Here, $K(\rho)u = f$ is the static equilibrium constraint, $g_n(\rho)$ is any number of additional desired constraints (e.g., volume fraction, explicit symmetry), and $\hat{M} = M/\|M\|_2$ is the normalized

homogenized matrix, which allows the optimizer to focus on aligning the eigenstructure irrespective of the overall material distribution and stiffness.

FDE combines VTP and IH to solve the joint problems of local entanglement and its spatial distribution. This results in a system that achieves directed mechanical performance by encoding characteristic entanglement as an optimized density field using targets defined by extremal material theory. While the moderate stiffness contrast achievable with VTP precludes reaching true extremal bounds, FDE employs topology optimization guided by the principles from extremal theory (e.g., targeting specific eigenstructures) to effectively utilize the available contrast for designing foams with programmed anisotropy and chirality.

4.3 Methods

4.3.1 Materials & Fabrication

I fabricated the samples using NinjaTek NinjaFlex® thermoplastic polyurethane (TPU) (density 1.19 g/cc, tensile modulus 12 MPa, 85 Shore A) as the base material stored in and printed from a filament dryer. The extrusion temperature was 240°C, bed temperature 70°C, on a Prusa Mk4 fused filament fabrication (FFF) printer with a 0.4 mm nozzle. The samples were printed with the plane of the pattern being the XY -plane of the printer; however, with the toolpath traversing 45° to the sample principal axes. I used the Gaussian process regression model from [39] to select two (V^*, H^*) pairs for low-density (LD, (0.4, 14.81)) and high-density (HD, (0.15, 6.93)) regions with line spacing (dL) 1.2 mm and layer height (dz) 1.27 mm.

Two types of samples were fabricated: dense and sparse homogeneous cubes (~ 40 mm side length) using the specified LD/HD parameters, and 6×6 metamaterial unit cell arrays yielding samples size $\sim 150 \times 150 \times 28$ mm³. The cubic samples took approximately 15 hours to print and the metamaterial samples took approximately 40 hours to print. Preparing the metamaterial designs involved projecting the density fields from IH into binary layouts using a filter and threshold approach with TO projection parameters $\beta = 64$ (projection strength) and $\chi = 0.5$ (dilation factor) based on previous work [115, 7]. These resulting binary bitmaps were then converted into CAD files for slicing and fabrication. Both types of

samples were then processed with a custom slicer to generate the VTP specific G-Code [83].

4.3.2 Physical Characterization of FDE Materials

Modulus Measurement via Compression Both homogeneous and metamaterial samples were tested under uniaxial compression using an Instron 6844 universal testing machine, capturing force-displacement data and simultaneous time-lapse images (1 frame/sec) for subsequent DIC analysis (see Section 4.3.2). Effective stress and engineering strain were calculated using initial sample dimensions. Homogeneous cubic samples exhibited linear stress-strain behavior up to 10% strain and were tested accordingly to determine their Young’s modulus ($E_{HD} \approx 3.4$ MPa, $E_{LD} \approx 0.11$ MPa) via linear fits over this range. Metamaterial samples were also tested up to 10% strain, but buckling became apparent above approximately 5% strain. To avoid confounding effects from this instability, the effective properties for metamaterials (summarized in Table 4.3) were extracted solely from the initial 0-2.5% strain region. Stress-strain curves confirm this region remained appropriately linear for material property estimation.

Poisson’s and Normal-Shear Coupling Ratios via Digital Image Correlation

Digital Image Correlation (DIC) was used to measure displacement and strain fields during compression testing using the freely available MATLAB package Ncorr [19]. Images were taken approximately every one second over a compression range of 0-10%. The homogeneous cube samples were characterized in the full 0-10% linear strain range, similar to modulus measurements. The metamaterial samples were characterized in the linear 0-2.5% strain range due to buckling observed at higher strains.

A region of interest (ROI) was selected covering the inner portion of the sample face (e.g., central 4×4 unit cell region for the 6×6 metamaterial arrays; similar ratio for homogeneous cubes) to minimize boundary effects (see Figure 4.2). For DIC analysis, the coordinate system was defined with the positive y -direction aligned with compressive loading and positive x -direction toward the right (see Figure 4.5c-d). To determine directional material properties such as E_1 vs. E_2 the material axes (x_1, x_2) were mapped based on sample orientation.

First, the full discretized Lagrangian strain fields $\varepsilon_x(x_i, y_i, t)$, $\varepsilon_y(x_i, y_i, t)$, and $\varepsilon_{xy}(x_i, y_i, t)$

Table 4.1: DIC Parameters

Parameter	Value
Radius	20
Strain Radius	5
Subset Spacing	4
Norm Cutoff	1e-6
Iteration Cutoff	100
DIC Subset Truncation	Disabled
Strain Subset Truncation	Disabled

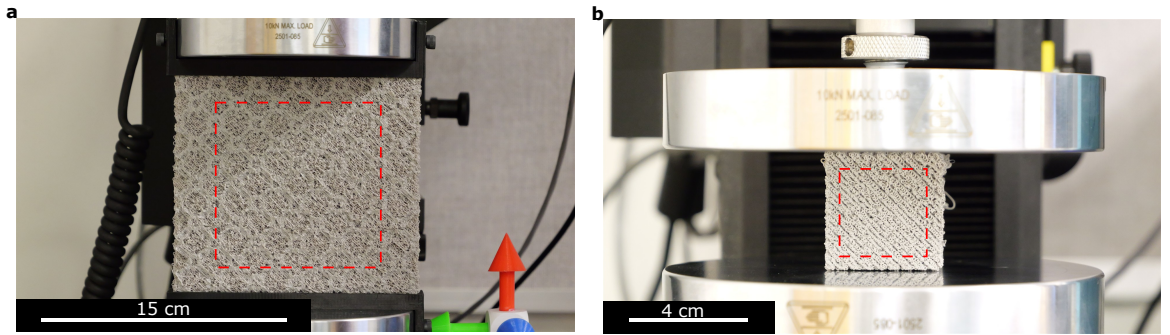


Figure 4.2: The approximate ROI placement used for the metamaterial (a) and homogeneous (b) samples.

were determined for the ROI using the parameters in Table 4.1. At each time step t , homogenized strain values were calculated by averaging the respective strain fields over the Region of Interest (ROI) for all N points in the ROI:

$$\begin{aligned}\bar{\varepsilon}_x(t) &= \frac{1}{N} \sum_{i=1}^N \varepsilon_x(x_i, y_i, t) \\ \bar{\varepsilon}_y(t) &= \frac{1}{N} \sum_{i=1}^N \varepsilon_y(x_i, y_i, t) \\ \bar{\varepsilon}_{xy}(t) &= \frac{1}{N} \sum_{i=1}^N \varepsilon_{xy}(x_i, y_i, t).\end{aligned}$$

The effective Poisson's ratio $\bar{\nu}_{yx}$ was determined by performing a linear fit on the ROI-averaged strain data, plotting $\bar{\varepsilon}_x(t)$ against $\bar{\varepsilon}_y(t)$ for the range where $\bar{\varepsilon}_y(t)$ is between 0% and 2.5%. The effective Poisson's ratio is the negative slope of this fit: $\bar{\nu}_{yx} = -d\bar{\varepsilon}_x/d\bar{\varepsilon}_y$. An example for the dense homogeneous samples is shown in Figure 4.3.

The normal-shear coupling coefficient $\bar{\eta}_{yxy}$ was calculated similarly for non-chiral samples. A linear fit was performed on the $\bar{\varepsilon}_{xy}(t)$ vs. $\bar{\varepsilon}_y(t)$ data over the same 0-2.5% strain range for $\bar{\varepsilon}_y(t)$. The slope gives the coefficient: $\bar{\eta}_{yxy} = d\bar{\varepsilon}_{xy}/d\bar{\varepsilon}_y$. Non-zero values occurred for non-chiral samples (Figure 4.4); however, this was due to the 45° print angle of the samples rather than any significant shear effects. Note this definition gives half of the engineering strain $d\bar{\gamma}_{xy}/d\bar{\varepsilon}_y$ definition. See Figure 4.3 for illustration of linear fitting.

For the mirrored chiral sample (Figure 4.4d), a modified approach was needed because the mirroring of the array results in opposite signed shear strains, thus meaning the naïvely averaged shear strain in the ROI would zero out. While the ROI is approximately centered on the array, the ROI midline may not exactly align with the sample midline. Thus, I located the sample midline in ROI coordinates y_{mid} by averaging the rows of shear strain and finding the index of the minimum value closest to the center of the ROI: $\min |\text{mean}(\varepsilon_{xy}(x, y_{row}, t))|$. Then, to account for the mirror symmetry, a modified shear strain field $\varepsilon_{xy,mod}$ was created by multiplying the shear strain field values below the midline by -1:

$$\varepsilon_{xy,mod}(x, y, t) = \begin{cases} \varepsilon_{xy}(x, y, t) & \text{if } y \geq y_{mid} \\ -\varepsilon_{xy}(x, y, t) & \text{if } y < y_{mid}. \end{cases}$$

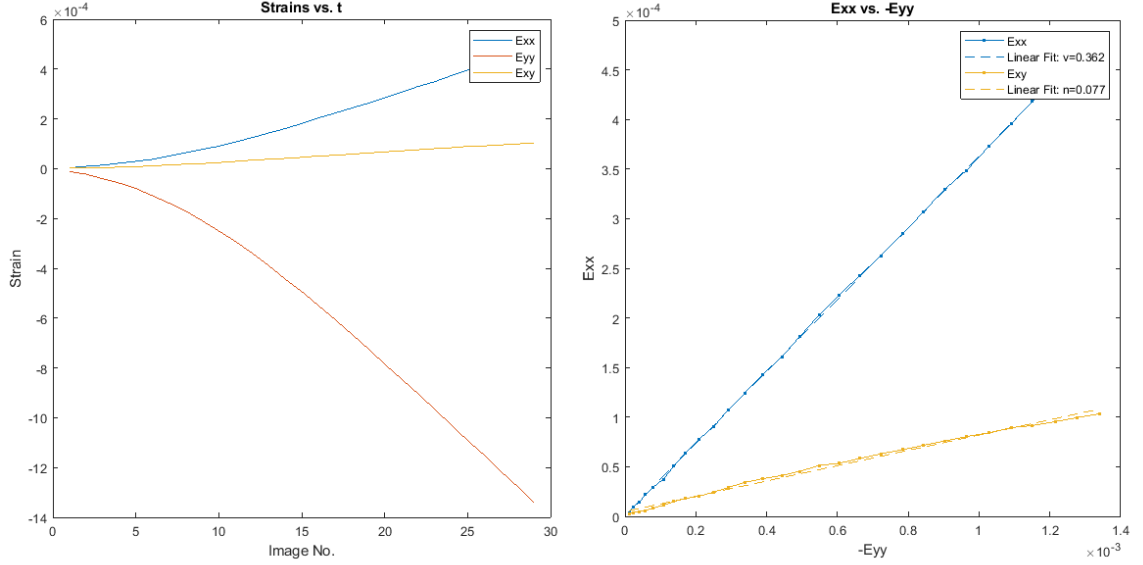


Figure 4.3: Example for curve fitting of time-dependent strain values.

The ROI-averaged modified shear strain, $\bar{\varepsilon}_{xy,mod}(t)$, was calculated by averaging the $\varepsilon_{xy,mod}(x, y, t)$ field over the ROI. Finally, a linear fit was performed on the $\bar{\varepsilon}_{xy,mod}(t)$ vs. $\bar{\varepsilon}_y(t)$ data (over the 0-2.5% strain range for $\bar{\varepsilon}_y(t)$). The slope of this fit yielded the normal-shear coupling coefficient $\bar{\eta}_{yxy}$ for this sample.

4.3.3 Computational Methods

Finite element simulations using periodic boundary conditions were implemented in FEniCS [5] on a 2D cross-diagonal triangular mesh (unit cell domain side length = 1). Periodic homogenization was applied to extract the effective elasticity matrix C [6, 105]. Optimization was performed using NLOpt [54] using the Method of Moving Asymptotes (MMA) [111].

The IH approach employs standard density-based TO informed by extremal material theory (see Section 4.2). The target eigenstructure (V, Λ) is user-defined: V provides the orthonormal basis for the desired principal directions (Table 4.2), and $\Lambda = \text{diag}([a, 1, 1])$ sets the relative eigenvalue ratio with $a \ll 1$.

Based on the measured stiffness contrast between dense and sparse homogeneous VTP foams, the corresponding normalized stiffness values used for material interpolation within

Table 4.2: The list of orthonormal V matrices used in this work to construct the optimization target $T^* = V\Lambda V^T$.

Lead Dir.	Bulk	Normal-Shear	Pure-Shear	Simple-Shear	Vertical
V	$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

TO were $E_{max} = E(\rho = 1) = 1 = E_{HD}/E_{HD}$ and $E_{min} = E(\rho = 0) = 1/30 \approx E_{LD}/E_{HD}$. I used the minimum stiffness and an approximate lower bound to set the target eigenvalue ratio $a = 1/30$. An effective Poisson's ratio $\nu = 0.4$ was assumed for both phases, representing an intermediate value between the measured $\nu_{LD} \approx 0.45$ and $\nu_{HD} \approx 0.35$. While standard TO constraints (e.g., volume fraction) can be included ($g_n(\rho)$), they were found unnecessary to achieve distinct, well-connected designs with this FDE framework.

The target eigenstructures carry implicit material symmetries. For example, a 2D plane stress isotropic material will have the eigenvectors V_{bulk} (see Table 4.2) corresponding with the eigenvalues $\Lambda_{iso} = \text{diag}(2K, 2G, 2G)$ where K is the plane stress bulk modulus [78]. This means that $T^* = V_{bulk}\Lambda V_{bulk}^T$ with the eigenvalue matrix of $\Lambda = \text{diag}(a, b, b)$ will loosely constrain the topology to produce isotropy. Likewise additional V carry their own symmetry.

4.4 Results & Discussion

I empirically demonstrate the FDE method by fabricating and measuring the properties of four optimized samples, shown in Figure 4.4. These designs represent different numbers of compliant or rigid modes and leading directions $\mathbf{v}_{lead}$. All samples were fabricated and tested under uniaxial compression (see Methods). Given their relatively thin geometry (aspect ratio approx. 5.4:1 based on the $150 \times 150 \times 28$ mm³ dimensions), the samples predictably exhibited global buckling under load at approximately 5% strain. Consequently, the effective elastic properties (Young's moduli E_1, E_2 ; Poisson's ratios $\nu_{ji} = -\epsilon_{ii}/\epsilon_{jj}$; and normal-shear coupling coefficients $\eta_{kij} = \gamma_{ij}/\epsilon_{kk}$) were extracted using data solely from the initial 0 – 2.5% strain range, prior to the onset of this instability. This strain range provided a clear linear

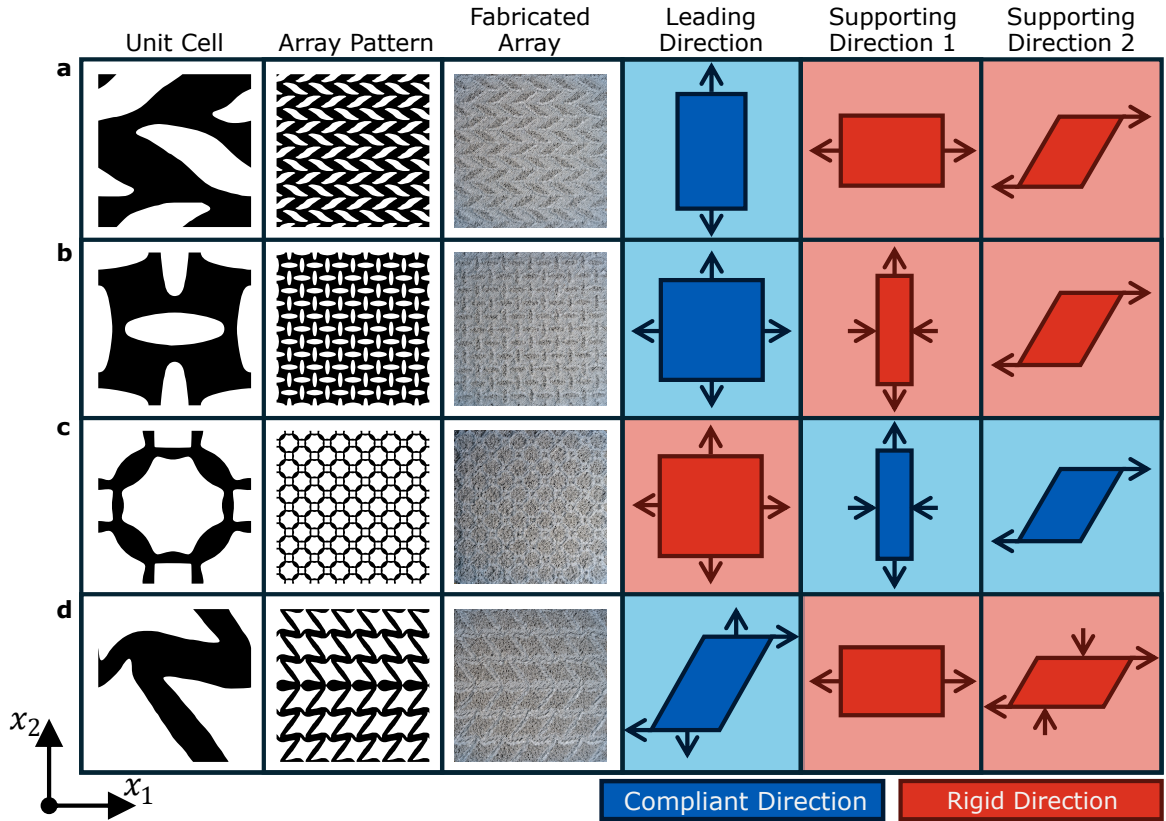


Figure 4.4: Topologies generated by optimization (pre-calibration, shown to illustrate the target fine features before accounting for fabrication limits discussed in Section 4.4.4) for materials targeting different modes of compliance (blue) and rigidity (red). a) Unimodal, compliant to vertical strain, rigid otherwise. b) Unimodal, compliant to bulk strain, rigid to shear. c) Bimodal, rigid to bulk strain, complaint to shear. d) Unimodal, compliant to positive vertical-shear coupling. This topology was mirrored vertically during fabrication to manage testing boundary conditions.

Table 4.3: Measured versus simulated effective properties for the four design examples depicted in Figure 4.4. Values are reported as Measured(Simulated). E values are normalized to the dense phase modulus (~ 3.4 MPa). Simulated values are for calibrated topologies (see Figure 4.7).

Family	E_1	E_2	ν_{12}	ν_{21}	η_{112}	η_{212}
Unimode, Vertical	0.54(0.48)	0.29(0.29)	0.24(0.38)	0.15(0.22)	0.08(−0.01)	0.00(0.00)
Unimode, Bulk	0.61(0.58)	0.47(0.46)	0.06(0.18)	0.06(0.14)	0.03(0.00)	0.03(0.00)
Bimode, Bulk	0.23(0.24)	0.22(0.24)	0.56(0.57)	0.52(0.57)	0.02(0.00)	0.02(0.00)
Unimode, Normal-Shear	0.37(0.41)	0.21(0.19)	0.30(0.31)	0.14(0.14)	0.03(0.14)	0.72(0.67)

elastic regime across all samples, suitable for determining effective properties via linear regression fits to the measured data. It should be noted, however, while this approach allows for characterizing the material’s intrinsic linear response, the early buckling onset suggests that the operational strain range of these specific large-aspect-ratio structures is likely limited in unconfined compression for applications requiring larger strains; achieving stability at higher strains could require geometric modifications or the application of external constraints.

Table 4.3 validates the FDE process by comparing the measured effective properties against simulation results that were calibrated post-fabrication by adjusting the TO projection dilation factor χ to account for fabrication minimum feature size (see Section 4.4.4 for full details). This calibration accounts for the scale-dependent effect of the VTP resolution limits (~ 1.2 mm coil diameter), which cause effective thickening of fine features in the fabricated samples compared to the idealized topology. Overall, the table shows reasonable agreement between the measured and simulated properties across the different designs.

4.4.1 Targeted Anisotropy and Bulk/Shear Response

The first example (Figure 4.4a) was optimized to be unimodal: compliant in the vertical direction while remaining rigid under horizontal and shear strains. Structurally, it features thick, diagonally oriented members connected by thin horizontal hinges. Testing yielded

an elastic modulus ratio $E_1/E_2 \approx 1.87$, $\nu_{12} \approx 0.24$, $\nu_{21} \approx 0.15$, and negligible normal-shear coupling (Table 4.3), demonstrating clear orthogonal anisotropy, consistent with the design goal. The ability to achieve such targeted anisotropy, isolating deformation primarily to one direction, highlights FDE’s capacity to move beyond the typically isotropic behavior of statistically homogeneous foams and is valuable in scenarios requiring precision motion control or minimal lateral bulging (e.g., mounts, seals).

The second test piece (Figure 4.4b) targeted unimodal compliance under bulk strain while remaining rigid to shear, yielding a topology resembling “rotating squares” [45]. Experimental testing confirmed the low Poisson’s ratio goal ($\nu_{12} \approx \nu_{21} \approx 0.06$), although it also revealed moderate elastic anisotropy ($E_1/E_2 \approx 1.29$). Nevertheless, this resulting anisotropy underscores that the T^* -guided optimization allows flexibility; while biasing towards certain symmetries, it doesn’t strictly enforce them, permitting anisotropic solutions even when more symmetric ones are possible or could be enforced via constraints. Importantly, achieving the low Poisson’s ratio by optimizing the eigenstructure, rather than ν directly, validates the effectiveness of targeting fundamental deformation modes.

Complementing the previous example, the third test piece (Figure 4.4c) targeted the inverse behavior: rigidity to bulk strain and compliance to pure and simple shears (bimodal). The resulting linked-octagonal unit cell is reminiscent of known extremal structures explored to maximize isotropic Poisson’s ratio [105]. Experimental testing revealed stiffnesses consistent with square symmetry ($E_1/E_2 \approx 1.01$) and high Poisson’s ratios ($\nu_{12} \approx 0.56$, $\nu_{21} \approx 0.52$). This contrasts sharply with the result for the complementary unimodal target (Figure 4.4b), further illustrating the optimizer’s flexibility. The high Poisson’s ratio achieved is consistent with the design goal of maximizing the bulk-to-shear moduli ratio (K/G), yielding values increased from the base material’s ($\nu_{base} = 0.4$).

These examples (Figure 4.4a-c) demonstrate that optimizing for specific eigenstructures allows FDE to realize fundamentally complementary mechanical responses (e.g., low ν vs. high ν). Targeting the eigenstructure effectively yielded designs that minimized or maximized Poisson’s ratios, achieving goals often pursued through direct design around scalar properties [7, 59, 77] but via a more fundamental, direction-driven approach.

4.4.2 Programmed Normal-Shear Coupling and Chirality

The fourth example (Figure 4.4d) demonstrates FDE’s ability to create chiral foams with programmed normal-shear coupled strains. The design target was unimodal compliance under simultaneous, similarly-signed vertical and shear strains. This directional coupling naturally leads to a chiral topology resembling a four-bar parallelogram linkage used in other metamaterials [71]. Experimental testing yielded $E_1/E_2 \approx 1.76$, $\nu_{12} \approx 0.30$, $\nu_{21} \approx 0.14$ (Table 4.3). Crucially, due to the handed nature of the design, non-zero normal-shear coupling terms emerge: $\eta_{212} \approx 0.72$, reflecting the design target for vertical-shear coupling, while the low value of $\eta_{112} \approx 0.03$ indicates minimal shear coupling when strained horizontally, consistent with the design’s aim to resist deformations other than the primary vertical-shear mode.

The targeted vertical-shear coupling is clearly visualized via digital image correlation (DIC) during compression testing, shown in Figure 4.5. The foam, compressed vertically (x_2), exhibits a clear shear deformation coupled with the compression, visible in the x- and y-displacement fields (Figure 4.5c-d) and the Euler-Almansi shear strain field (Figure 4.5e). Note, the tested sample was mirrored horizontally to mitigate boundary effects, thus the shear strains seen in Figure 4.5e have opposite signs from top to bottom. The measured relatively low Poisson’s ratio ($\nu_{21} \approx 0.14$) confirms that the observed horizontal displacement is dominated by the programmed normal-shear coupling, not the Poisson effect.

To my knowledge, this represents the first demonstration of programmed chirality—arising directly from encoding handed topologies via FDE—within monolithic foam materials built from a single continuous filament. Distinct from periodic lattice-based chiral metamaterials [2] or foams exhibiting merely statistical chirality, achieving these significant normal-shear coupling terms ($\eta_{212} \approx 0.72$, Table 4.3) purely through patterned mesostructure highlights a unique capability of FDE. This ability to program chirality expands the functional possibilities for entangled materials, moving beyond previously realized isotropic or simple anisotropic responses and suggesting potential uses in transducers or systems requiring engineered extension under shear/torsional load or vice versa.

Collectively, these fabricated examples validate that FDE, by manipulating local en-

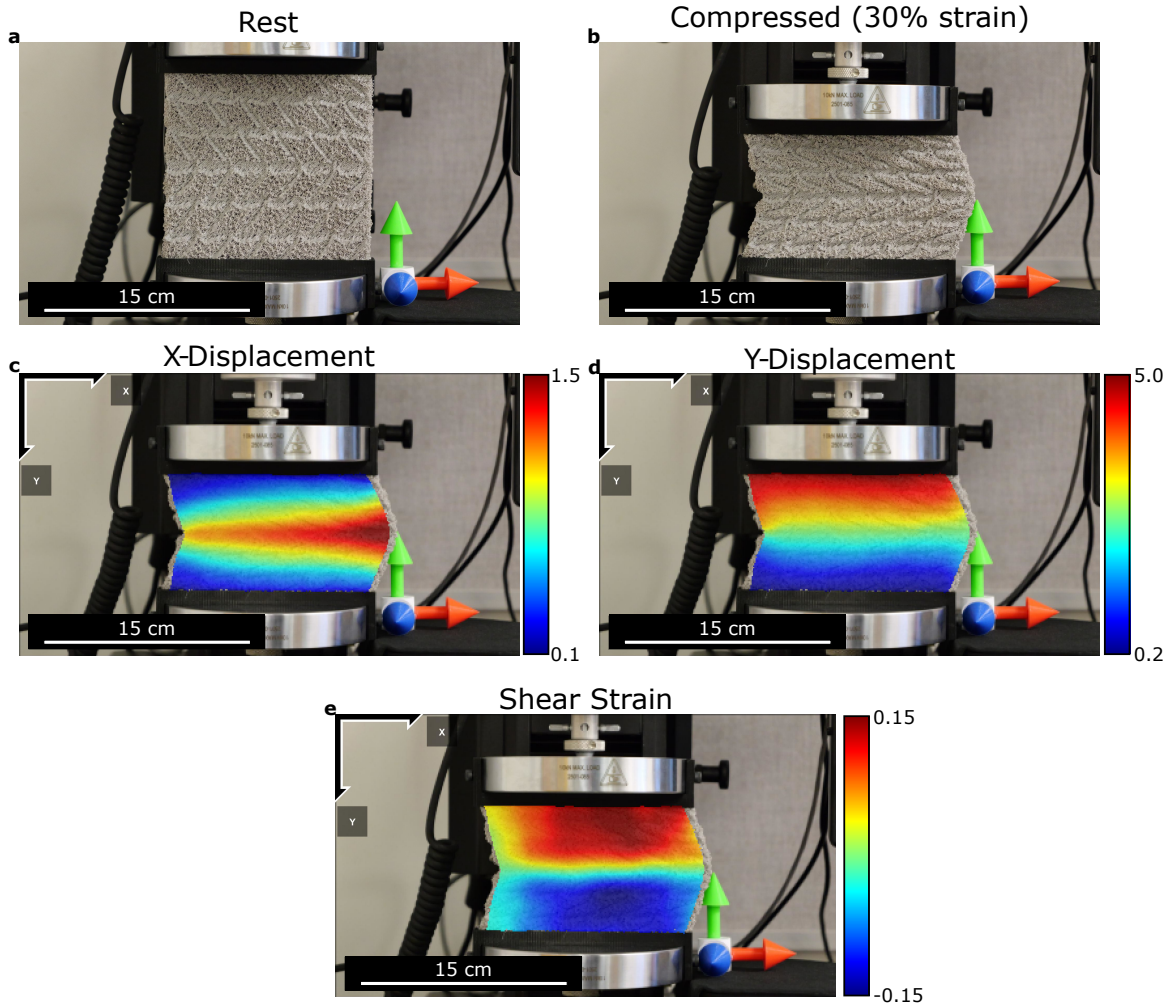


Figure 4.5: DIC analysis of a compression test of the handed shearing material (Figure 4.4c). The foam is mirrored about the horizontal axis to prevent shear stresses at the test boundaries. a-b) The material at rest and 30% compressive strain (45 mm of compression). c) X-Displacement of the foam showing the center of the material displacing to the right. The banded nature indicates how the material shifts uniformly on top and bottom. d) Y-Displacement of the foam showing uniform vertical compression. e) The tensorial Eulerian-Almansi shear strain ($\epsilon_{xy} = \gamma_{xy}/2$) showing symmetric shear strain mirrored about the middle. The RGB coordinate system in the bottom-right indicates the print orientation, whereas the XY arrows in the top-left indicate the measurement coordinate system.

tanglement based on targeted eigenstructures, allows the creation of monolithic foams exhibiting a range of tailored mechanical behaviors—including anisotropy and programmed chirality—that expand upon the capabilities of conventional entangled materials.

4.4.3 Comparison with Simulation

Comparing the measured initial effective properties with the fabrication-calibrated simulations (Table 4.3) reveals general agreement, validating the FDE approach’s ability to achieve targeted mechanical behaviors. When accounting for fabrication minimum feature size, the simulations accurately capture the primary stiffness responses (Young’s moduli E_1 , E_2) and crucially predict the presence or absence of significant normal-shear coupling (η_{212}) consistent with the chiral or symmetric design intent. Accounting for the fabrication minimum feature size required calibrating simulation parameters; additional details on the calibration methodology are given in Section 4.4.4.

Quantitative agreement is weaker for other secondary properties, notably the Poisson’s ratios (ν_{12} , ν_{21}) in certain cases (e.g., the Unimode Bulk design). Discrepancies also appear in lower normal-shear coupling terms. For instance, the simulated η_{112} (0.14) differs notably from the measured value (0.03), potentially influence by implementing the horizontal mirroring applied during fabrication to manage test boundary conditions. Furthermore, small non-zero shear coupling terms (e.g., $\eta_{112} \approx 0.01 - 0.04$) were measured for the nominally symmetric designs where simulations predict zero. I hypothesize these minor terms arise from toolpath orientation effects relative to the testing axes, which are perhaps incompletely canceled by the layer-by-layer stacking strategy. Overall, remaining discrepancies likely reflect the combined influence of inherent VTP variability, simplifications in the linear elastic material model, and geometry inaccuracies tied to the post-fabrication calibration step.

Therefore, this assessment suggests that when interpreting the simulated property space in Figure 4.7, the results for Young’s moduli (E_1 , E_2) and major normal-shear coupling effects (e.g., significant η_{212}) provide a good indication of achievable quantitative trends via FDE. Conversely, predictions involving Poisson’s ratios (ν_{ij}) and minor coupling terms should be viewed more qualitatively, reflecting the greater simulation-experimental discrepancies

observed for these properties.

4.4.4 Post-Fabrication Calibration

Comparisons between initial finite element simulations and experimental mechanical tests revealed discrepancies in the predicted effective properties of the fabricated metamaterials. These differences are primarily attributed to the resolution limitations inherent in the VTP process, which results in the effective thickening of the thinnest features in the physical samples compared to the idealized designs generated by TO.

To reconcile these results and improve the predictive capability of the simulations, an experimental calibration was performed. This involved a parametric optimization against key factors influencing the simulation output: the TO base material properties (E_{max} , E_{min} , ν) and TO projection parameters (β , χ). The optimization was set up to minimize a weighted sum of the squared normalized errors between simulated and measured homogenized properties. The properties included in the calibration were E_1 , E_2 , ν_{12} , and ν_{21} . Weights were applied to the squared normalized error of each property to balance their influence in the objective function: a weight of 1 was used for the errors in the Young's moduli (E_1 , E_2), and a weight of 0.1 was used for the errors in the Poisson's ratios (ν_{12} , ν_{21}). The objective function is defined as:

$$\min_{P \in \{E_{max}, E_{min}, \nu, \beta, \chi\}} \sum_{\text{designs } p \in \{E_1, E_2, \nu_{12}, \nu_{21}\}} \sum w_p \left(\frac{p_{\text{sim}}(P) - p_{\text{measured}}}{p_{\text{measured}}} \right)^2$$

where $w_p = 1$ for $p \in \{E_1, E_2\}$ and $w_p = 0.1$ for $p \in \{\nu_{12}, \nu_{21}\}$.

Normal-shear coupling is a material property characteristic of chiral structures. Given that the majority of the designs included in this calibration optimization were optimized for symmetric responses (e.g., uniaxial, bulk, or pure shear compliance/rigidity, as shown in Figure 4.4a-c), their ideal, intended behavior exhibits zero normal-shear coupling due to symmetry. Consequently, including η terms, which are only significant for explicitly chiral designs (Figure 4.4d), in a global calibration objective pooling data from all designs was deemed inappropriate. The calibration was thus focused on optimizing parameters to improve the agreement for properties (E_1 , E_2 , ν_{12} , ν_{21}) relevant to all design symmetries considered. Despite their exclusion from the optimization objective, reasonable agreement

was still observed between the measured and calibrated simulated η values across all designs, as shown in Table 4.3.

The optimization process identified the projection dilation factor (χ) as the dominant parameter significantly influencing the agreement between simulated and measured properties, with other parameters having only a marginal effect (referred to as η in [10, 115]). This finding aligns with the physical expectation that adjusting χ directly compensates for the fabrication-induced feature thickening by modifying the thickness of features within the simulation model. An error-minimizing optimal value of $\chi = 0.15$ was determined, a significant adjustment from the initial value of $\chi = 0.5$ used in preliminary simulations.

The effect of this calibration on the resulting topologies is illustrated in Table 4.4, which compares the original outputs of the TO process ($\chi = 0.5$) with the post-calibrated topologies ($\chi = 0.15$). The visual differences highlight how the adjustment of the dilation factor modifies the feature thickness in the simulation, aiming to better represent the fabricated structures.

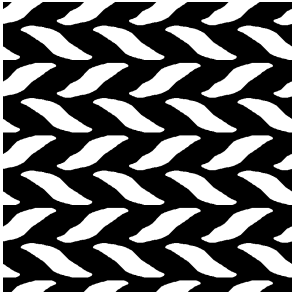
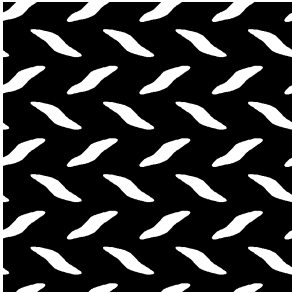
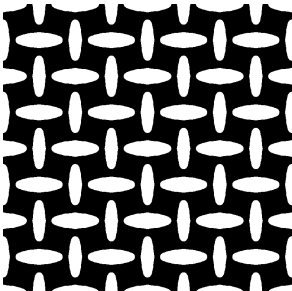
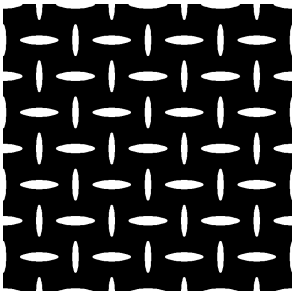
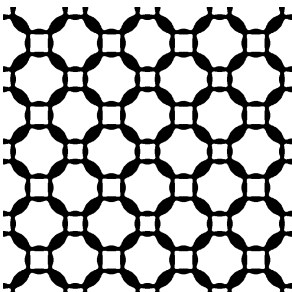
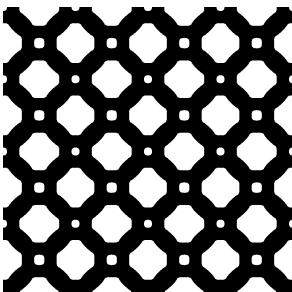
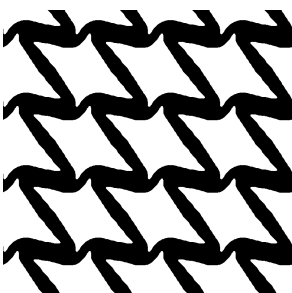
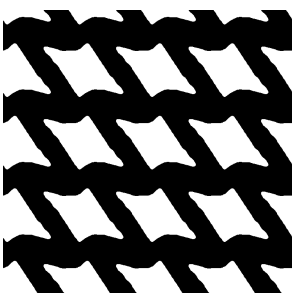
Figure 4.6 quantitatively demonstrates the impact of the calibration by comparing the measured effective properties against both the original ($\chi = 0.5$) and calibrated ($\chi = 0.15$) simulation results for E_1 , E_2 , ν_{12} , and ν_{21} . The figure shows a clear improvement in the agreement between simulated and measured values for these properties after applying the calibrated χ .

Consequently, all simulated properties reported (Table 4.3, Figure 4.7) were generated using the calibrated projection dilation factor of $\chi = 0.15$, while other TO parameters (E_{max} , E_{min} , ν , β) were kept at their initially assumed values.

4.4.5 Expanded Material Property Space

Figure 4.7 presents the expanded material property space predicted by the post-fabrication calibrated simulations, which incorporate corrections for fabrication resolution limits as detailed in Section 4.4.4. A key insight is that the FDE-patterned foams occupy regions unattainable by this naive blending, highlighting the expanded design freedom. The results (n=247 simulations) are summarized in Figure 4.7, plotting various effective property combinations. Each symbol type corresponds to a prescribed mode (circle is unimode, square

Table 4.4: Original outputs of the IH process (first column) and the post-calibrated topologies (second column) for the examples from Figure 4.4.

Original Topology ($\chi = 0.5$)	Dilated Topology ($\chi = 0.15$)
	
	
	
	

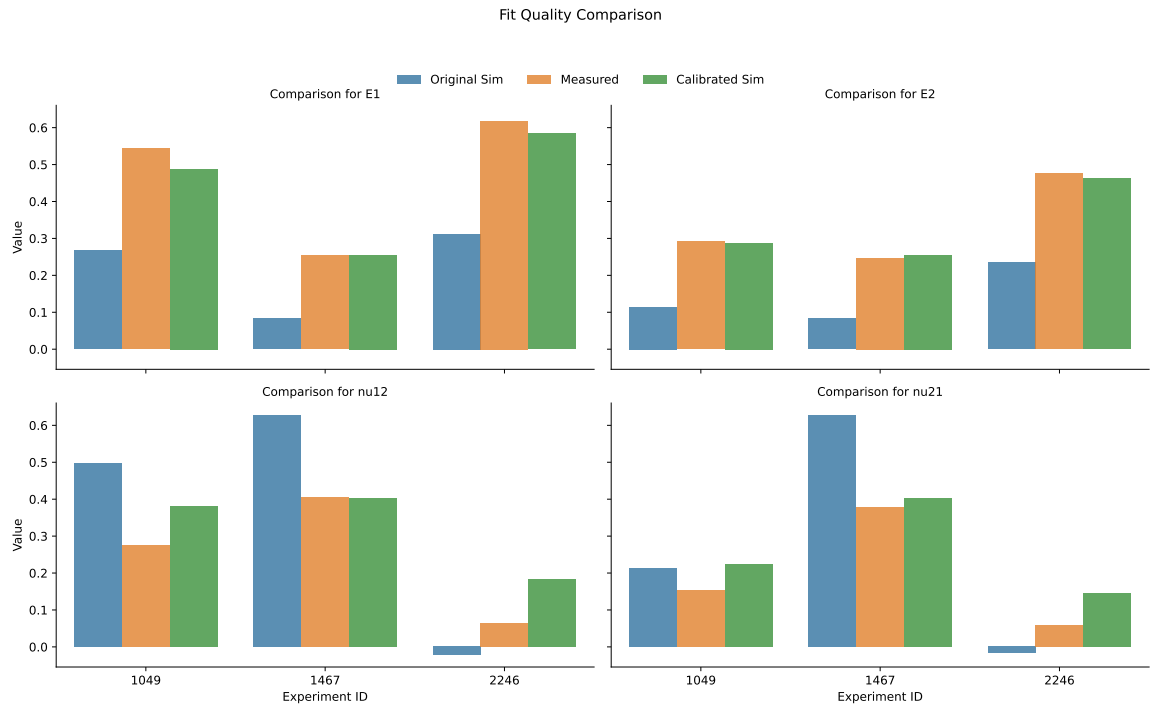


Figure 4.6: Comparison of simulated, measured, and calibrated material parameters for E_1 , E_2 , ν_{12} , and ν_{21} . The orange and green bars are the values shown in Table 4.3.

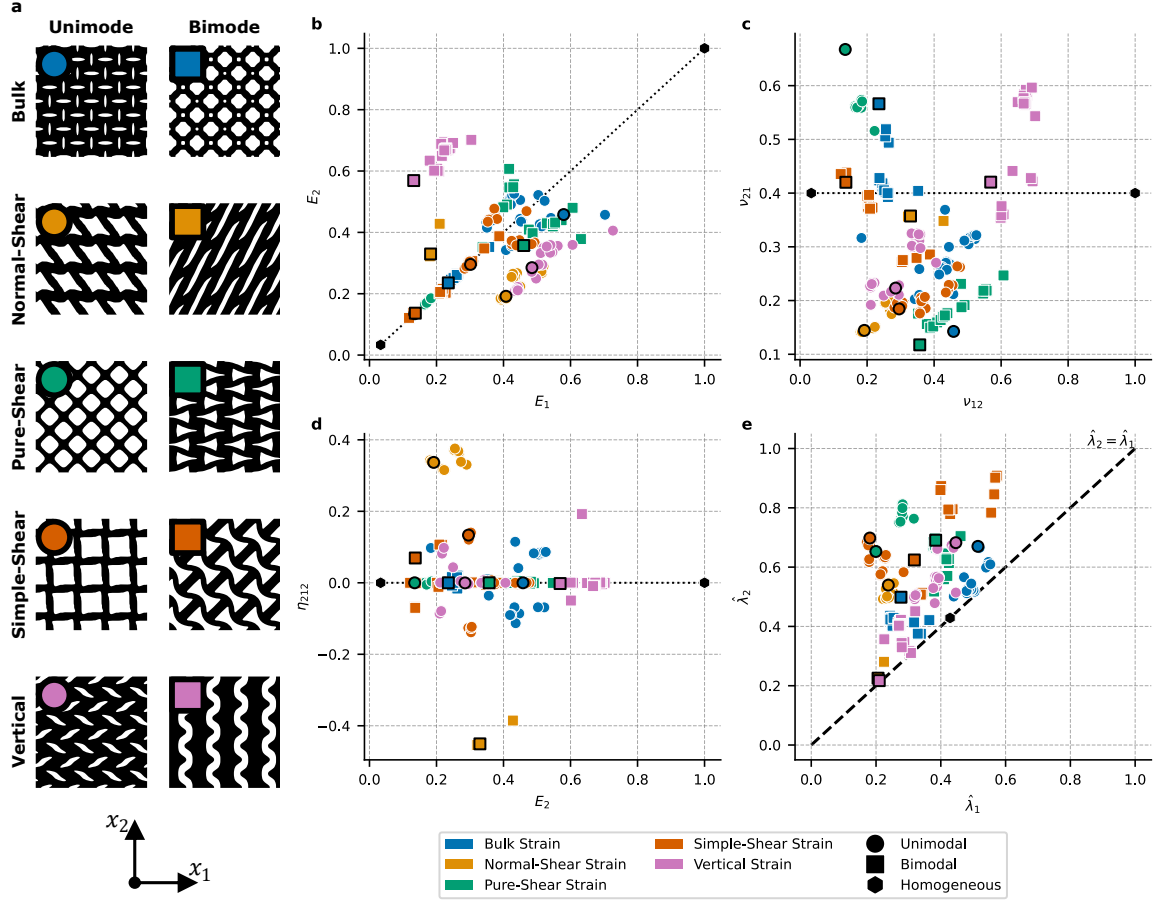


Figure 4.7: Simulated material parameter mappings from topology optimization runs (n=247), highlighting the expanded material property space achievable via FDE compared to homogeneous foams. Hexagonal black points indicate homogeneous foam samples, connected by a line representing linear interpolation. a) Marker legend identifying the four topology-calibrated designs from Fig. 2 denoted with a black marker border on the plots. The remaining topologies serve as illustrative examples of the other design families. b) Comparison of Young's modulus in principal directions (E_2 vs E_1), demonstrating one form of anisotropy. c) The vertical stiffness and resulting Poisson's ratio (ν_{21} vs. E_2), showing the expanded range of values compared to the approximated homogeneous baseline (constant $\nu = 0.4$). d) The vertical stiffness and resulting normal-shear coupling term (η_{212} vs. E_2), illustrating the creation of materials with positive and negative normal-shear coupling. e) Normalized eigenvalues of the elasticity matrix C , providing a coordinate-independent representation of the materials' principal stiffness magnitudes and overall anisotropy.

is bimode), while colors indicate the targeted strain direction for compliance or rigidity. The four fabricated designs (Figure 4.4) and additional designs are marked with black borders (Figure 4.7a). The black hexagons representing the homogeneous sparse and dense foams, and the dashed line connecting them represents a simple linear interpolation (mixing rule). A limitation of analyzing traditional material properties in plots Figure 4.7b-d is their implicit dependence on a coordinate system, potentially obfuscating material similarities. I address this in Figure 4.7e by analyzing normalized eigenvalues—a coordinate- and stiffness-invariant metric—to investigate those relationships.

Figure 4.7b compares principal moduli E_1 and E_2 to illustrate directional stiffness with points along the diagonal ($E_1 = E_2$) representing isotropic/square-symmetric responses. While many FDE designs cluster near this diagonal, several families exhibit distinct anisotropy consistent with design targets. For instance, vertically compliant unimodes (pink circles) lie below the diagonal ($E_1 > E_2$), while their complementary bimodes (pink squares) lie above it ($E_1 < E_2$). Normal-shear designs (yellow markers) also show this reflection across the diagonal based on their mode. Furthermore, several design families (e.g., pure-shear, simple-shear, bulk strain) contain solutions both on or near the diagonal ($E_1 \approx E_2$) and away from it ($E_1 \neq E_2$), corresponding to the optimizer finding either symmetric or asymmetric design rotated by 90° . The existence of both symmetric and asymmetric outcomes for the same functional target demonstrates that FDE can access diverse topologies without being restricted to a single symmetry type. Figure 4.7b thus indicates that despite the constraints of FDE (e.g., moderate stiffness range), the optimization framework can target distinct anisotropy, yielding predicted directional stiffness different from homogeneous base materials, while also revealing potential symmetries in the optimized topologies.

Figure 4.7c plots Poisson’s ratio ν_{21} against vertical modulus E_2 . The FDE designs show a wide distribution, expanding beyond the base material value ($\nu = 0.4$). This demonstrates that FDE optimization not only broadens the range of available Poisson’s ratios but also allows achieving these varied ratios across a considerable range of vertical stiffness (E_2 spanning approximately 0.15 to 0.70). For instance, the families targeting pure-shear strain (green markers) appear particularly effective at reaching the highest and lowest Poisson’s ratios (i.e., high and low K/G respectively) in this coordinate system, while the bulk strain

family (blue markers), despite also influencing K/G , does not reach the same extremes. This difference can likely be attributed to the implicit encoding of symmetry in the formulation of T^* where $V_{pure-shear}$ encodes square symmetry while V_{bulk} encodes the more restrictive isotropic symmetry. It follows that the choice of target eigenstructure, particularly the coordinate system and symmetry defined by V , directly influences the achievable range of properties like Poisson’s ratio within the FDE framework.

Figure 4.7d plots the normal-shear coupling coefficient η_{212} against E_2 , quantifying chirality. As intended, designs explicitly targeting normal-shear coupling (yellow markers) exhibit distinct positive (unimode) or negative (bimode) η_{212} values, clearly separated from zero and generally larger in magnitude than the incidental coupling seen in other families that do not target the coupling. This apparent coupling can arise from two distinct mechanisms: either the optimizer finds a truly topologically chiral solution lacking mirror symmetry, or a mirror-symmetric structure is measured in a coordinate system rotated away from its principal axes of symmetry. Both mechanisms are observed here. For example, the simple-shear targets (orange markers) produced both topologically chiral solutions (e.g., Figure 4.7a unimode example) and mirror-symmetric structures (e.g., the rotated “inverted honeycomb” topology in Figure 4.7a bimode example) that exhibit coupling in this coordinate frame. Similarly, the unimode bulk strain family (blue circles), aiming to minimize K/G (equivalent to minimizing Poisson’s ratio), often converged to topologically chiral solutions, consistent with established links between chirality and auxetic behavior [2]. Overall, the ability to achieve both targeted positive and negative coupling, alongside the emergence of incidental chirality (whether topological or coordinate-system-based), confirms FDE allows tuning of the handed response and provides multiple mechanisms for generating structures exhibiting normal-shear coupling.

Figure 4.7e provides a scale- and coordinate-invariant perspective using normalized eigenvalues, illuminating the fundamental range of intrinsic material behaviors that can be accessed by FDE. If $\lambda_1 \leq \lambda_2 \leq \lambda_3$ are the eigenvalues of a design’s elasticity matrix C , I project the materials behavior onto the pair $(\hat{\lambda}_1, \hat{\lambda}_2)$, where $\hat{\lambda}_i = \lambda_i/\lambda_3$ ($i = 1, 2, 3$). In this space, both homogeneous base materials collapse to a single point (0.43, 0.43), as they share the same Poisson’s ratio. The FDE designs, however, scatter across a wide area,

visually demonstrating the expansion of the intrinsic property space enabled by patterned entanglement.

This plot reveals underlying similarities obscured in other views; for instance, families spanning wide property ranges in other plots (e.g., unimode bulk-strain, blue circles) cluster more tightly, suggesting similar intrinsic stiffness ratios of C are achieved and that the spread in other plots is due to differences in volume fraction or eigenvector misalignment between V and T^* . Furthermore, overlaps between families targeting related deformations suggest topological similarity; for example, the simple- and pure-shear unimode designs (orange and green circles) cluster here, reflecting their linked mechanics and similar topologies (differing mainly by a 45° rotation, see Figure 4.7a). The tightness of clustering also varies (e.g., tight normal-shear clusters vs. broader bulk-strain clusters), hinting at the relative ease or difficulty of achieving certain targets and the potential for multiple distinct and dominating topological solutions. Thus, the substantial spread of FDE results compared to the single homogeneous point, along with the varied clustering and overlaps, underscores FDE’s ability to reshape stiffness distributions and access diverse intrinsic material behaviors, offering multiple design pathways to achieve specific mechanical objectives, even with moderate material contrast.

Overall, the simulation results (Figure 4.7) indicate that FDE-based foams occupy a far richer design space than uniform blends. The ability to tune anisotropy, Poisson’s ratio, and particularly normal-shear coupling underscores that, when properly controlled, entangled materials can have emergent properties not seen previously. While the $\sim 30:1$ stiffness contrast via VTP precludes finding truly extremal material properties at the bounds of elastic theory, the use of TO targeting specific eigenstructures (inspired by extremal materials theory) is designed to find optimal material arrangements. These optimized structures, achieving diverse and complex behaviors (Figure 4.7), serve as computational probes exploring the performance limits imposed by the finite material contrast.

4.5 Conclusion

This work introduces and validates Fabrication-Directed Entanglement (FDE), a methodology integrating topology optimization (TO) with viscous thread printing (VTP), to unlock

spatially programmed mechanical properties in monolithic foam materials. By computationally guiding VTP deposition to create quasi-two-phase (dense/sparse) regions, I successfully implemented and experimentally validated an FDE workflow. Fabricated prototypes realized targeted anisotropic responses, tuned Poisson’s ratios, and, critically, demonstrated chiral foam structures exhibiting significant programmed normal-shear coupling. Simulation results further illustrated that FDE broadens the accessible property space for VTP-based materials compared to uniform structures. By targeting specific eigenstructures inspired by extremal materials theory, the optimization framework finds tailored mesostructures that serve to explore the boundaries of this achievable space, making effective use of the VTP system’s $\sim 30:1$ contrast.

FDE provides a new capability for designing and fabricating hierarchical foam mechanical metamaterials from a single homogeneous filament. The key significance lies in demonstrating that this approach can unlock complex mechanical behaviors previously inaccessible in such monolithic foam structures, notably the patterned chirality and resultant normal-shear coupling. Achieving these handed responses purely through patterned mesostructure showcases the power of combining computational design with controlled VTP deposition. This integrated method provides a route to functional materials leveraging both designed mechanics and the inherent advantages of a continuous, monolithic build process, such as global network connectivity.

While FDE enables complex programmed mechanics in monolithic foams, key trade-offs arise from the underlying VTP process. The achievable stiffness contrast ($\sim 30:1$), while sufficient to demonstrate behaviors like chirality, restricts the property range compared to high-contrast TO methods and limits access to theoretically extremal materials. Similarly, although the continuous filament path ensures inherent network connectivity—a potential advantage over discrete fabrication approaches—the process resolution (~ 1.2 mm coil diameter) fundamentally limits fine feature realization and poses challenges for miniaturization. Additionally, the current reliance on post-fabrication calibration (Section 4.4.4), necessitated by resolution-limited feature fidelity (i.e., effective feature thickening) at the demonstrated scale, currently limits the ability to reliably predict the quantitative properties of new FDE designs entirely a priori, necessitating further validation or potentially iterative calibration.

Finally, stability considerations, highlighted by the early buckling ($\sim 5\%$ strain) observed in the tested high-aspect ratio geometries, suggest that practical applications in unconfined compression may require operation at low strains or designs optimized for greater stability under load.

Looking ahead, future efforts should focus on both enhancing the current FDE methodology and expanding its scope. Immediate priorities include improving fabrication fidelity and developing more predictive simulation models that better capture VTP fabrication limitations a priori, thereby reducing reliance on post-hoc calibration and enhancing design accuracy. Alongside these refinements, exploring methods to increase the achievable stiffness contrast could push material properties closer to theoretical limits. Furthermore, a crucial next step involves extending the FDE framework to three dimensions to unlock its full potential for complex component design. Beyond refining the core method, the demonstrated control over Cauchy elasticity opens avenues for incorporating dynamics (e.g., targeting elastic bandgaps) or extending the chiral work to incorporate Cosserat elasticity. Finally, fully characterizing these hierarchical materials necessitates investigating potential entanglement-related properties, such as damping and toughness, in conjunction with the programmed elastic response explored here.

Chapter 5

COMPUTATIONAL DESIGN OF 3D PRINTED FOAM LITHOPHANE
STRUCTURES**Note on Publication**

Portions of this chapter are based on work submitted for publication to the *ACM Symposium on Computational Fabrication* 2025.

Note on Authorship

This chapter presents work conducted in collaboration with my co-author, Dr. Yuxuan Mei. In recognition of her significant contributions, I use the first-person plural (“we”) throughout this chapter.

5.1 Introduction

A lithophane [119] is a translucent panel that reveals an image when backlit, using variations in light transmission to encode grayscale intensity. Traditionally made from porcelain, lithophanes were popular in the 19th and 20th centuries as decorative lampshades and window panels. Today, lithophanes are commonly fabricated using 3D printing, where the effect is achieved by varying the thickness of a translucent material [36, 64, 106, 22]. This method makes lithophanes accessible as educational and artistic projects but comes with limitations: it restricts material choice, introduces structural non-uniformity, and limits its use in applications that depend on a constant thickness.

In this work, we ask: can we create 3D-printed lithophanes by varying porosity instead of thickness? This approach would offer several advantages. First, it enables greater light transmission, even when using materials that are not highly translucent. Second, it allows us to fabricate lithophanes with uniform thickness, which improves structural stability and keeps the image hidden when not backlit – making it valuable for privacy-preserving displays. Uniform thickness also supports a much broader range of applications: it simplifies

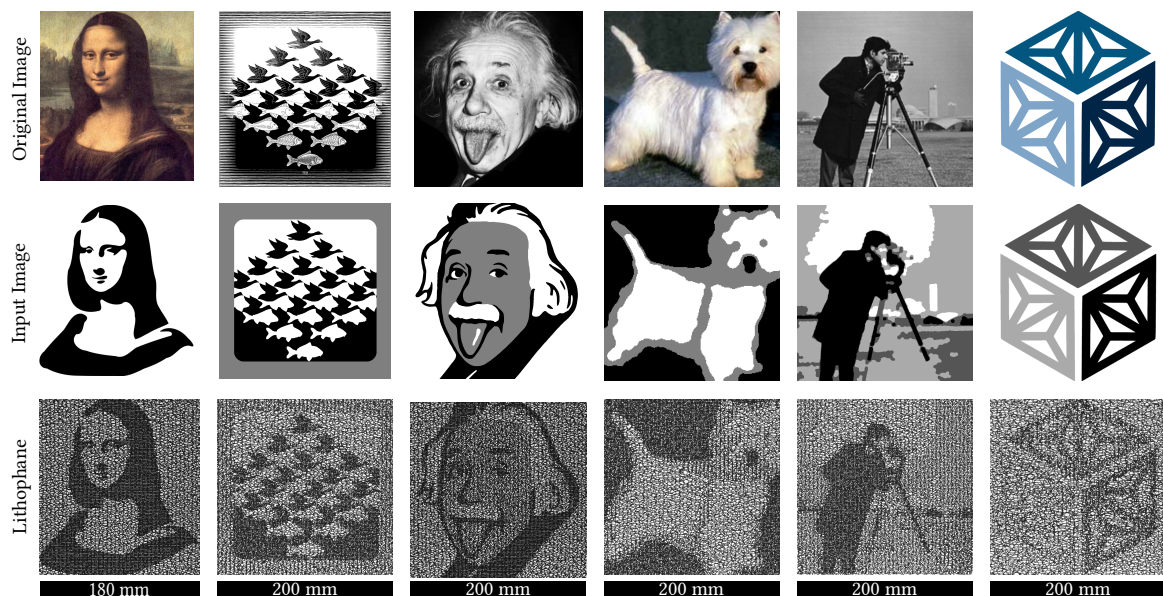


Figure 5.1: Lithophane designs fabricated with viscous thread printing. The three rows show, from top to bottom, the original photograph/artwork, the quantized input used for lithophane generation, and the finished lithophanes illuminated from the rear. Left to right: **Mona Lisa**, quantized to 2 levels; **M.C. Escher’s *Sky and Water I***, quantized to 3 gray levels; **Albert Einstein**, quantized to 3 gray levels; **West Highland Terrier**, quantized to 3 gray levels; **“Cameraman”**, quantized to 4 gray levels; **the Symposium on Computational Fabrication (SCF) Logo**, quantized to 4 gray levels.

integration into other components, enabling use in flat panel designs such as lampshades, signage, or architectural elements, and easier storage and transportation through folding or stacking the lithophanes when printed with soft materials like TPU.

However, introducing porosity through standard 3D printing typically comes at a cost to material efficiency and print speed, as it often requires printing holes or internal voids. These features usually demand extensive support structures, which increase both fabrication time and material waste. To avoid these drawbacks, we leverage viscous thread printing (VTP)—a process that exploits viscous thread instability (VTI), in which a stream of molten filament buckles and coils as it falls onto the print surface. By extruding the material above the surface and allowing it to accumulate in self-intersecting patterns, VTP forms lightweight, open-cell foam structures. The morphology of these coils can be precisely controlled by tuning parameters such as nozzle height and print speed, enabling spatially varying porosity—and thus light transmission—without changing the panel’s overall geometry or requiring post-processing.

The fundamental challenge in using VTP for lithophanes is controlling the process to produce a desired grayscale image. The relationship between print settings and the resulting brightness is nonlinear and difficult to predict, as the light transmission depends on the complex interaction of coiling behavior, material accumulation, and internal porosity. To address this, we develop a method to simulate the VTP process and a rendering pipeline (Section 5.3), modeling how different printing parameters affect the structure and light transmission of the resulting foam. Our simulation method builds upon Discrete Viscous Threads, with a freezing mechanism that dramatically reduces computational cost and keeps the cost per timestep constant even when many coils overlap (Section 5.3.2). We use the simulation to build a look-up table (LUT) that maps grayscale values to corresponding print parameters for each layer of the lithophane (Section 5.3.3). For values not present in the LUT, we interpolate between neighboring entries. This LUT enables a direct and automatic mapping from image pixels to printable toolpaths, forming the basis of our image-to-lithophane pipeline.

We demonstrate our system’s capabilities by producing a wide range of lithophanes from both continuous-tone grayscale images and quantized images with up to four clearly

distinguishable tones. Even at small sizes—for example, 20 cm $\times$ 20 cm panels—our lithophanes reveal detailed images, highlighting the high spatial resolution enabled by VTP.

5.2 System Overview

In this work, we present an automatic pipeline that takes an image as input and produces a gcode file that can be used to print a two-layer foam lithophane corresponding to the input image.

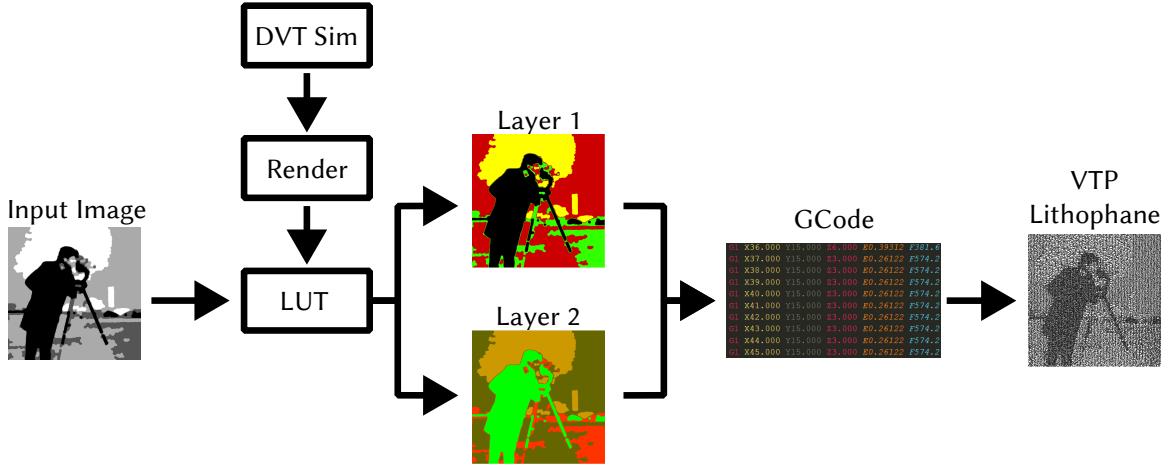


Figure 5.2: Our image-to-lithophane pipeline. A look-up table (LUT), precomputed using a simulation and rendering methodology described in Section 5.3.2, is used to map each pixel of the input image to (V^*, H^*) parameters that should be used for each lithophane layer (encoded here with V^* as the red channel and H^* as the green channel of an RGB image). These images are then sampled along toolpaths to produce corresponding G-Code. A VTP printer can then produce the desired lithophane.

The steps of our pipeline are illustrated in Figure 5.2. The user provides a grayscale input image. For each pixel of the image, we query a precomputed sparse look-up table (see Section 5.3.3) that maps from grayscale values to (V^*, H^*) parameters that should be used when printing each of the two lithophane layers. For grayscale values not in the table, we linearly interpolate the (V^*, H^*) values of the two nearest entries. For ease of

visualization, we encode (V^*, H^*) as the red and green channels of an RGB image (the layer's *parameter image*) in Figure 5.2. We then generate gcode for each of the two layers, using an empirically-determined optimal toolpath spacing Δs for each layer (see Section 5.3.3). We sample the parameter image (V^*, H^*) values to determine the print parameters that should be used for fabricating the lithophane designs (see Section 5.3.1).

The User Interface We provide a user interface (Figure 5.3) that gives guidance and feedback when generating the lithophane G-Code from a desired input image.

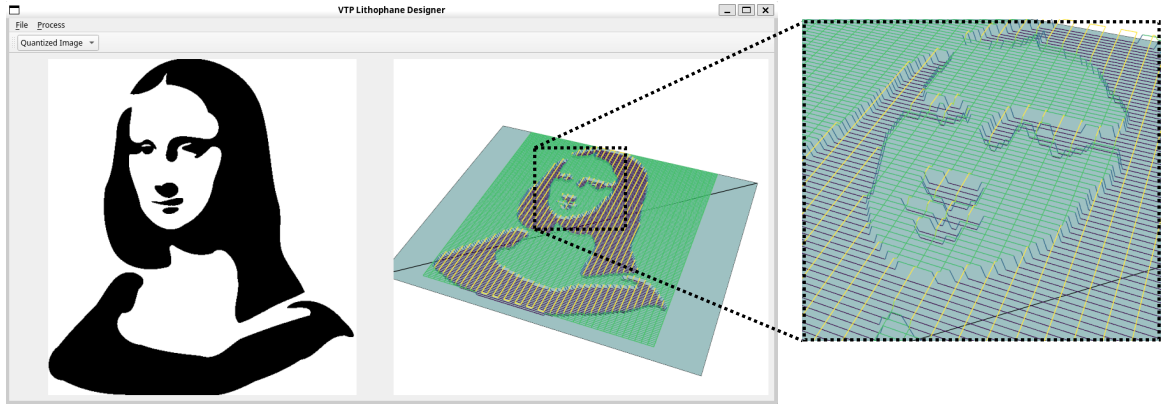


Figure 5.3: The GUI for the lithophane tool. Shown on the left is the input image with toolbar selection to view the input, quantized, and individual layer images. On the right is a rendering of the G-Code toolpath colored by z height (a proxy for H^*) for visual confirmation of size and correct image sampling. An expanded inset is shown of the toolpath for further clarity.

The UI is laid out as a two-window view. On the left is an image interface, where the user can import a target image and apply some basic preprocessing such as conversion to grayscale, median or Gaussian blurs, and color quantization. The image interface can also be used to view the RGB layer images representing the (V^*, H^*) parameters computed by our LUT. On the right, a 3D viewport visualizes the print bed and the toolpath that was produced using the input image. We color the toolpath according to nozzle height so that the user can ensure at a glance that the G-Code will fit within the bounds of the printer and

that the lithophane captures the qualitative appearance of the input image. A multi-tab settings interface accessed via the **Process** drop down menu (not shown) allows the user to specify printer and print-process parameters (e.g. allowed (V^*, H^*) upper and lower bounds, nozzle/bed temperatures, extrusion multipliers, etc.). Table B.1 lists the parameters for the printer that we used for all of the paper’s fabricated examples.

5.3 Methods

The design space of our two-layer foam lithophanes consists of the following parameters:

- the line spacing $\Delta s_1, \Delta s_2$ between toolpaths on each layer. Since we consider only a simple back-and-forth rasterization of the image rectangle, the line spacing parameter is constant for each entire layer;
- the VTP control parameters $V_1^*, H_1^*, V_2^*, H_2^*$ at each point on each layer.

Together, $(V^*, H^*, \Delta s)$ determine the overlap between neighboring coils and the porosity of the resulting structure. By co-tuning these parameters, we obtain fine control over the density and overlap of coiling patterns, and thus over the local light transmission of the final printed lithophane.

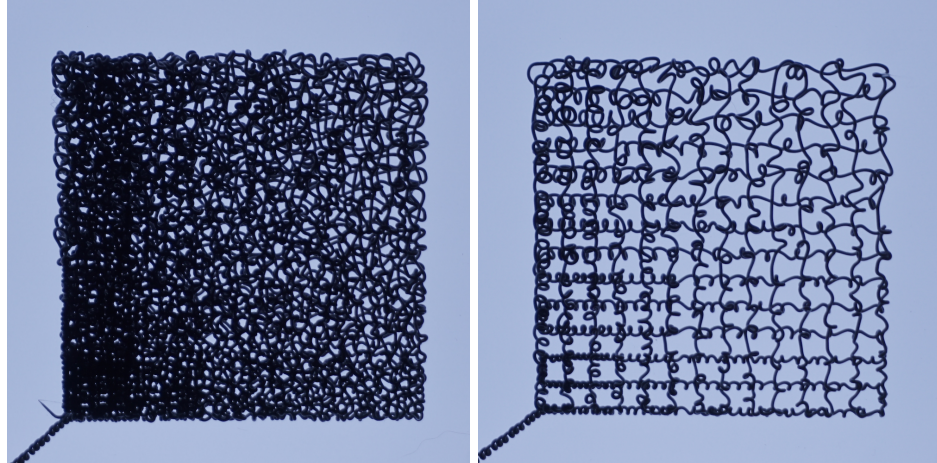
This design space is too large to search by trial and error. We therefore developed a physical simulation and rendering pipeline that predicts the light transmission through a two-layer foam lithophane given $(V^*, H^*, \Delta s)$ and tuned it on a small set of fabricated examples. We then used our calibrated physical simulation to search for the pair of $\Delta s_1, \Delta s_2$ values that allow for the largest possible variation in light transmission for different (V^*, H^*) values. Finally, we fixed $\Delta s_1, \Delta s_2$ to their discovered optimal values, and used the simulation to populate a lookup table (LUT in Figure 5.2) from grayscale pixel values to VTP control parameters (V^*, H^*) that should be used in that location on each of the two lithophane layers to achieve the desired amount of light transmission.

The rest of this section describes each of these steps in detail. First, we describe the toolpath generation process that takes VTP parameters $(V^*, H^*, \Delta s)$ as input and outputs a G-Code file (Section 5.3.1). Then we describe our algorithms for simulating viscous thread

printing from the G-Code file and rendering the simulation results, and how we bridged the sim-to-real gap by tuning the simulation parameters and render settings using a set of calibration prints and their photos taken with a fixed photography setup (Section 5.3.2). We end with a description of how we searched for the Δs combination for two layers that provides the best range of transmission, and then populated the LUT (Section 5.3.3).

5.3.1 Toolpath Generation

This subsection describes how we map from $(V^*, H^*, \Delta s)$ values to gcode. Figure 5.4 shows two single-layer printed VTP samples exhibiting a range of coiling patterns produced by stepping through different (V^*, H^*) combinations for two Δs values.



(a) Printed coiling patterns with $\Delta s = 1.5\text{mm}$. (b) Printed coiling patterns with $\Delta s = 4.5\text{mm}$.

Figure 5.4: Two printed samples ($60\text{ cm} \times 60\text{ cm}$) showing the coiling patterns with (V^*, H^*) varying within ranges of $V^* \in [0.25, 0.5]$, $H^* \in [5.0, 10.0]$. We sample five values each for V^* and H^* . Here, V^* increases at equal increments from left to right, and H^* increases at equal increments from bottom to top.

Each layer consists of evenly spaced raster lines. Because local transmission is directly tied to coiling morphology, we must ensure consistent coil size and density throughout the print.

Abrupt transitions from starts, stops, or directional changes can result in irregular coiling diameters that degrade optical fidelity. To suppress these artifacts, we construct a single continuous toolpath that travels through every raster line of every layer while continuously extruding, which helps maintain steady-state extrusion dynamics that is important for VTP. In addition, we rotate raster orientation 90° between layers to promote optical uniformity and prevent directional bias in coil geometry.

Once the the global raster scan path is generated, it is then subsampled at regular increments. At each point the corresponding V^* and H^* values are extracted from the parameter image to generate the appropriate G-Code commands (see Appendix C).

Algorithm Overview Each layer $k \in \mathbb{N}$ of the print (in this work $N = 2$) consists of a series of raster lines with consistent travel and step directions. Let W and H denote the width and height of the printing domain, and let Δx and Δy denote the fixed raster spacing in the x and y directions, respectively. We focus here on the planar rasterization logic and omit vertical positioning, as the z -coordinate is determined separately from the VTP parameter H^* .

The image is subdivided into local evaluation regions (the coarseness of which is a user parameter), and each region is mapped to image coordinates to extract a local grayscale value. This value is used to query the LUT (the precomputed lookup table) that maps grayscale to discrete sets of candidate $(V^*, H^*, \Delta s)$ combinations. The raster spacing Δs —the center-to-center spacing between adjacent deposition lines—is fixed within each layer but may vary across layers to modulate transmission dynamic range. When Δs approaches the coil diameter, coils merge and occlude more light. If spacing exceeds the coil radius R_c , gaps appear between lines, increasing transmission.

Layer Construction and Raster Line Positioning The first layer begins at the origin $\mathbf{r}_0^{(1)} = (0, 0)$ and proceeds with raster lines oriented in the $\hat{\mathbf{x}}$ direction and stepped in the $\hat{\mathbf{y}}$ direction. The number of raster lines required to fill a given layer in the step direction is

given by

$$n^{(k)} = \begin{cases} \left\lfloor \frac{H}{\Delta y} \right\rfloor + 1 & \text{if } k \bmod 2 = 1 \\ \left\lfloor \frac{W}{\Delta x} \right\rfloor + 1 & \text{if } k \bmod 2 = 0 \end{cases}.$$

Letting $j \in \{0, \dots, n^{(k)} - 1\}$ index the raster lines within layer k , the travel direction alternates as:

$$\mathbf{t}_j^{(k)} = (-1)^j \cdot \mathbf{t}_0^{(k)}.$$

The axis of travel alternates between layers:

$$\text{Axis}(\mathbf{t}_0^{(k)}) = \begin{cases} \hat{\mathbf{x}} & \text{if } k \bmod 2 = 1 \\ \hat{\mathbf{y}} & \text{if } k \bmod 2 = 0 \end{cases}.$$

For the first layer, we initialize $\mathbf{t}_0^{(1)} = \hat{\mathbf{x}}$ and $\mathbf{s}^{(1)} = \hat{\mathbf{y}}$. The sign of the step direction is chosen so that the raster lines advance into the image domain.

For each subsequent layer $k > 1$, the raster begins from the endpoint of the previous layer:

$$\mathbf{r}_0^{(k)} = \mathbf{r}_{\text{end}}^{(k-1)}.$$

The axis of travel follows the alternating pattern, but the sign of the initial travel direction $\mathbf{t}_0^{(k)}$ depends on the coordinate of the previous endpoint along the new raster axis. For example, if $\text{Axis}(\mathbf{t}_0^{(k)}) = \hat{\mathbf{y}}$ and the y -coordinate of $\mathbf{r}_{\text{end}}^{(k-1)}$ is H , then we set $\mathbf{t}_0^{(k)} = -\hat{\mathbf{y}}$ to raster downward.

The step direction $\mathbf{s}^{(k)}$ is orthogonal to the travel direction and its sign is set to move inward from the starting corner:

$$\mathbf{s}^{(k)} = \begin{cases} \pm \hat{\mathbf{y}} & \text{if } \text{Axis}(\mathbf{t}^{(k)}) = \hat{\mathbf{x}} \\ \pm \hat{\mathbf{x}} & \text{if } \text{Axis}(\mathbf{t}^{(k)}) = \hat{\mathbf{y}} \end{cases},$$

where the sign is determined by the remaining coordinate of $\mathbf{r}_0^{(k)}$. For example, if the raster begins at $x = 0$, then $\mathbf{s}^{(k)} = \hat{\mathbf{x}}$; if it begins at $x = W$, then $\mathbf{s}^{(k)} = -\hat{\mathbf{x}}$.

Raster lines proceed from the basepoint as:

$$\mathbf{r}_j^{(k)} = \mathbf{r}_0^{(k)} + j \cdot \Delta s^{(k)} \cdot \mathbf{s}^{(k)},$$

where

$$\Delta s^{(k)} = \begin{cases} \Delta y^{(k)} & \text{if Axis}(\mathbf{s}^{(k)}) = \hat{\mathbf{y}} \\ \Delta x^{(k)} & \text{if Axis}(\mathbf{s}^{(k)}) = \hat{\mathbf{x}} \end{cases}.$$

In the simplest case, the raster spacing is uniform across all layers, with $\Delta x^{(k)} = \Delta y^{(k)} = \delta$ for all k , where δ is a user-specified constant.

Each raster line j travels a distance $L_r^{(k)}$ in the direction $\mathbf{t}_j^{(k)}$, where the raster length is:

$$L_r^{(k)} = \begin{cases} W & \text{if Axis}(\mathbf{t}_j^{(k)}) = \hat{\mathbf{x}} \\ H & \text{if Axis}(\mathbf{t}_j^{(k)}) = \hat{\mathbf{y}} \end{cases}.$$

Parameter Sampling and G-Code Generation In this work, we fix both D_T and U in Equation 2.1 by prescribing the input filament feed rate, which leaves V^* and H^* controllable solely through F and H —two parameters that correspond directly to machine-level G-Code commands on the 3D printer. A full derivation of these expressions and their relationship to G-Code parameters is provided in Appendix C.

The raster pattern is subsampled at even increments Δr . At each of these refined points, the image’s red and green values are sampled for the current layer to determine the V^* and H^* at that point. The V^* and H^* pair are then used to determine relevant G-Code commands to realize the desired VTP structure.

Once the target (V^*, H^*) and Δs fields are assigned, the corresponding motion commands are generated by inverting the nondimensional parameters using the derivations in this appendix for $\mathbf{x}$ and $\mathbf{y}$ G-Code values and Appendix C for $\mathbf{z}$, $\mathbf{e}$, and $\mathbf{f}$ G-Code values.

The resulting G-Code file contains no retractions or travel moves to preserve steady-state coiling dynamics. Instead, a priming line is emitted from the print bed origin $(0, 0)$ to the image start location to initiate extrusion cleanly. The image is assumed to be centered on the bed, and all raster motion is continuous within and across layers.

Toolpath Continuity and Alternating Raster Orientation The endpoint of layer k is given by the termination point of the final raster line $j = n^{(k)} - 1$, and serves as the starting point for layer $k + 1$. This recursive construction defines a single continuous thread

that traverses every raster line of every layer, alternating raster orientation between layers while preserving path continuity and ensuring inward coverage of the domain.

This construction ensures that the full toolpath is continuous in space, traverses every raster line of every layer in a single uninterrupted thread, and rotates raster orientation between layers to homogenize optical transmission properties in the final print. The alternating pattern distributes coiling effects isotropically in the plane, promoting uniform light scattering and preserving the intended tonal values of the source image.

5.3.2 VTP Simulation, Rendering, and Calibration

Exploring the design space of $(V^*, H^*, \Delta s)$ parameters by printing physical samples is slow and expensive; a natural idea for accelerating the search is to perform it in silico. However, a useful simulator must overcome several challenges:

- **performance:** previous work has demonstrated that the discrete elastic rod (DER) formulation with time-parallel transport [13, 12, 53] can successfully model the viscous thread instability that underlies VTP. However, previous applications have involved only a few, moderate-length viscous threads. Scaling up DER simulation to an entire layer of overlapping coils becomes excruciatingly expensive, due to the large number of rod segments involved in the simulation and large number of self-contacts. Our initial experiments with using DER to simulate the VTP process took hours to complete—longer than a physical print!
- **sim-to-real gap:** the DER viscous thread model can simulate only a simplification of the mechanics and rheology of the VTI. Calibration of the simulation parameters against printed samples is required for the simulation to provide useful predictions.

In this section, we describe how we simulate the VTP process, render the simulation results, and calibrate the simulator and renderer parameters in order to overcome these challenges.

Simulation Details

A single filament is discretized into vertices x_i and associated material frames (d_{i1}, d_{i2}, d_{i3}) , with twist angles θ_i . Nozzle trajectories are ingested directly from G-Code, assuming perfect motion and extrusion. We discretize each filament into segments of rest length $\ell_k = 0.2$ mm and integrate dynamics using a leapfrog scheme with time step $\Delta t = 1 \times 10^{-4}$ s, which is sufficient to resolve the highest-frequency modes under strong damping.

Viscous Damping To capture viscous dissipation without resorting to a full fluid model, we incorporate linear damping forces at both translational and rotational degrees of freedom. Specifically, we apply vertex-based drag forces

$$\mathbf{F}_k = -\eta_1 \ell_k \dot{\mathbf{x}}^k$$

and torsional damping on edges via

$$\mathbf{M}_{dk} = -\eta_2 A^k \dot{\theta}^k,$$

where $\dot{\mathbf{x}}$ and $\dot{\theta}^k$ are the vertex and angular velocities, ℓ^k denotes the average rest edge length of the two edges adjacent to vertex k , A^k denotes the edge cross-sectional area, and η_1 and η_2 are the damping coefficients. This viscous formulation integrates cleanly into the explicit scheme and was sufficient to reproduce the rich range of coiling behaviors observed experimentally. The elastic and damping coefficients used in simulation are summarized in Table B.2.

Freezing Mechanism and Computational Efficiency As printing progresses, the number of rod segments and coil-coil and coil-floor contacts grow, as does the cost of simulating each time step. To dramatically reduce computational cost, we exploit the following insight: during VTP printing, the viscous thread rapidly cools and hardens after extrusion. Coils away from the print nozzle are essentially rigid.

This insight motivates the following rule: upon self-contact, contacting vertices and edges are marked *frozen*, removing them from subsequent force and kinematic updates. All edges “older” than the contacting edge are also frozen since they are bounded by frozen edges

and cannot move appreciably. As the thread accumulates contacts, the number of active degrees of freedom stabilizes, leading to a constant-time dynamics update per new extruded vertex. This freezing mechanism, combined with AABB tree collision detection, allows us to perform simulations orders of magnitude faster than without the rule, and 2x-5x faster than real-world printing.

Using G-Code as Simulation Input We simulate full print jobs by ingesting raw G-Code toolpaths directly. An extruding G-Code command takes the form `G1 Xx Yy Zz Ff Ee`, where `G1` and the capital letters are command keywords interpreted by the printer, and the lowercase letters denote numeric values: the 3D position x, y, z (in mm), the feed rate f (in mm/min), and the extrusion amount e (in mm of filament). The simulator, like the physical printer, uses the feed rate f to determine the nozzle speed, while the velocity direction is inferred from the difference between the current nozzle position and the target position specified in the command. In this way, the extrusion trajectory is fully specified by the G-Code file. We ensure the printer is properly calibrated, such that each position is reached accurately and each commanded millimeter of filament is physically extruded. This results in a “digital twin” of the printing process, though not accounting for higher-order effects such as mechanical backlash or position drift. Acceleration, jerk, and other motion planner effects are likewise ignored; we assume constant velocity along each segment with instantaneous direction changes. This simplification is justified by the fact that, in VTP, the nozzle moves at nearly constant speed during deposition, and coiling behavior is dominated by the steady-state feed rate.

Numerical Integration and Stability To ensure numerical stability in this regime, we use a typical rest edge length of $\ell^k = 0.2$ mm, balancing geometric accuracy and computational cost. The leapfrog time step is fixed at $\Delta t = 1 \times 10^{-4}$ s, chosen to stably resolve the highest-frequency modes present under strong damping. This resolution is sufficient to capture the coiling dynamics at relevant print speeds while maintaining convergence of steady-state coil morphologies across a wide range of (V^*, H^*) values.

Collision Detection and Contact Handling Our implementation uses continuous-time collision detection (CTCD) [40] combined with an axis-aligned bounding box (AABB) tree to efficiently identify candidate intersections [27]. This approach enables robust self-contact resolution even in tightly coiled configurations. The AABB tree maintains collision queries at $\mathcal{O}(n \log n)$ complexity but the freezing mechanism ensures that force and motion computations are effectively amortized over a fixed number of active elements, enabling efficient simulations.

Rendering and Calibration

Since our goal is to build a simulation-rendering pipeline that can predict the light transmission of a two-layer VTP print, we calibrate our simulation to maximize the grayscale similarity between the rendered images of the simulation results and photos of the corresponding physical prints.



We perform this calibration by printing VTP samples with parameter ranges $V^* \in [0.25, 0.5]$ and $H^* \in [5.0, 10.0]$, for three specific Δs values (1.5, 3.0, and 4.5 mm); Figure 5.4 shows two sheets of printed samples. The samples were photographed from the top using a fixed camera setup against a surface back light (see inset figure).

We replicated the physical photography setup in Blender [20, 72]. We thicken the thread centerline produced by the physical simulation into a cylindrical tube, and manually identified a suitable material and color for this tube, for the emissive strength and color of the back light, and for the camera parameters by referencing our photographs.

Notice that although the rendering and photograph are qualitatively similar, there is still a significant sim-to-real gap between the coiling pattern predicted by the simulation and the real printed pattern. We calibrate the simulator by tuning material and damping parameters to reproduce observed coiling morphologies (coil diameter, frequency, spatial density) in the simulation. Because we focus on characterizing the transmission response instead of accurately matching the coiling geometries, the metric we used to compare our

simulation-render results with actual photos is “Dark Pixel Percentage”: we convert the photos and renders to grayscale, and compute histograms of the grayscale values (0-255). For each histogram, we normalize it and threshold it with a dark pixel threshold (160) and a light pixel threshold (80) to get quantities representing the number of dark pixels N_{dark} and the number of light pixels N_{light} . So the Dark Pixel Percentage (DPP for short) is defined as

$$DPP = \frac{N_{dark}}{N_{dark} + N_{light}}.$$

We treat DPP as a proxy for the transmission response of the printed lithophanes: higher values mean darker grayscale colors in the resulting lithophane image.

Since one set of simulation parameters could not account for diverse coiling behaviors, we apply a (V^*, H^*) -dependent correction (see Appendix B). As shown in Figure 5.5b, the average DPP absolute percentage error after calibration is around 6.22% and the error increases as we increase Δs . This is due to a limitation in our simulation model where the simulated coils cannot replicate the curvature of the viscous threads in actual printing (as seen in Figure 5.5d), thereby increasing the amount of filament that is visible, which leads to higher DPP values. As Δs becomes larger, the printed structures become more porous, causing this discrepancy to be more obvious and therefore the DPP errors to increase.

5.3.3 Print Parameter Discovery

We now describe how we build the look-up table (LUT) with the toolpath generator and the calibrated simulation-rendering pipeline. Our goal is to search within the parameter space of $(V^*, H^*, \Delta s)$ for N layers of coiling to find a set of $(V^*, H^*, \Delta s)$ that correspond to transmission values that are most different from each other to be used as colors in the print parameter space.

Because $(V^*, H^*, \Delta s)$ are ranged continuous parameters ($V^* \in [0.25, 0.5]$, $H^* \in [5.0, 10.0]$, $\Delta s \in [1.5, 4.5]$), even for only two layers ($N = 2$), the number of combinations can be large depending on how we sample the ranges. The large number of combinations of $(V^*, H^*, \Delta s)$ could also map to very similar transmission values, and thus the mapping is not a bijective one. Thus, we determined a computational discovery of the parameter space was necessary as opposed to an empirical approach.

Although the search space seems vast, we can leverage domain knowledge about the problem. First, Δs is fixed per layer because it is not possible to vary the spacing within a single layer and maintain constant number of raster scans within our toolpathing scheme. Second, given that the Δs creates spacing between coils as it increases, it can be seen as a parameter that is more directly related to transmission differences. Therefore, we model this parameter search as a bilevel optimization problem. The outer loop searches over candidate raster spacings $\Delta s^{(k)}$, while the inner loop selects the (V^*, H^*) values to maximize differences of their corresponding transmission.

Δs pair sweep We begin with finding the pair of Δs values for two layers that provide the best range of transmission variation. Our strategy is to sample 10 values from the Δs range $[1.5, 4.5]$ and for each Δs , simulate 9 single-layer prints with 9 (V^*, H^*) combinations by sampling 3 values each from V^* range and H^* range. Then we organize the simulated geometries in a 9x9 grid in two configurations, and overlay the two grids, before rendering the composed two-layer grids (see Figure 5.6). This gives us an approximation of the range of transmission a given Δs pair provides. By analyzing the variance of the grayscale values of the rendered images, we have found that the combination of (1.5 mm, 4.5 mm) gives the maximum variance in average grayscale.

Although compositing single-layer simulation results and then rendering is fundamentally different from rendering two-layer simulations, we chose this approach because (1) we have verified that compositing two layers and then rendering does not cause a significant difference in grayscale or DPP (see Figure 5.7); (2) this reduces the number of simulations we need to run significantly. Furthermore, simulating a single-layer is also less costly than two-layer (27 single-layer simulations and 55 renderings for 55 Δs pairs, compared to 243 two-layer simulations and 243 renderings).

(V^*, H^*) sweep After finding the optimal Δs pair for two layers, we fix them and vary (V^*, H^*) for two layers. In this case, we sample 6 values each for V^* and H^* and run 72 single-layer simulations. Then we compose the 36 layer one simulations with the 36 layer two simulations and render 1,296 images, before finally analyzing the rendered images to find the

(V^*, H^*) combinations that correspond to most different transmissions (as represented by DPP). The resulting (V^*, H^*) pairs become the LUT used in our design workflow. Figure 5.8 shows the five most different (V^*, H^*) combinations based on DPP.

Calibrating DPP for the LUT To validate that the computationally discovered VTP parameters can actually act as the “print-parameter-space colors”, we printed a 200 mm × 40 mm array of five swatches, each using one of the five most distinct (V^*, H^*) combinations shown in Figure 5.8. We expect to see that the printed swatches have DPP values that the print parameters are supposed to correspond to, as listed in Figure 5.8.

Figure 5.9 shows the result of this analysis. In each of the subfigures, from left to right, it shows the resulting lithophane and the photo’s DPP value, the lithophane DPP map which is computed by sliding the DPP metric over the lithophane image with window size 33 (approximately 5 mm in physical space)—a radius large enough to average over several coils while preserving edge fidelity, and the histogram of the DPP Map.

In all cases, the histogram shows roughly a Gaussian distribution, and the printed structure is consistently lighter than expected, with mean differences between 1% to 13%. This difference is mostly consistent with the errors shown in the DPP error heatmap (Figure 5.5b) for the two-layer calibration samples of three different Δs , so we treat these discrepancies as a result of our simulation model’s limitation in not capturing some inherent properties about the VTP process. As such, we use these mean differences to correct for the higher DPP values predicted from the parameter discovery step (Table 5.1).

5.3.4 *Lithophane Fabrication Details*

Physical printing was performed on a Prusa Mk4 fused filament fabrication (FFF) printer equipped with a 0.4 mm nozzle using Polymatter Matte Grey PLA filament stored in a heated filament dryer to maintain low humidity and reduce rheological variability during extrusion. All prints were performed at a nozzle temperature of 230°C and a bed temperature of 50°C, with temperatures held constant throughout the print to ensure consistent coiling dynamics and stable extrusion viscosity. The extrusion multiplier was fixed at 1.06 for our printer, and the cooling fan was disabled throughout to support consistent coiling behavior. A summary

Table 5.1: We printed additional swatches according to the LUT we built using parameter discovery and analyzed the resulting discrepancies between the DPP values of a local sliding window of size 33 to find the correction term for the five predicted DPP values in Figure 5.8.

Swatch	Grayscale	Predicted DPP	Mean of	Difference
	Value	from Render	Measured DPP Map	
	255	0.663	0.576	0.087
	191	0.728	0.609	0.119
	128	0.793	0.666	0.127
	64	0.858	0.759	0.099
	0	0.923	0.909	0.014

of print parameters are provided in Table 5.2.

Nozzle translation speed and print height were controlled explicitly in the custom G-Code generated by our tool to modulate the (V^*, H^*) values locally across the part. No slicer-generated retraction, z-hop, or non-extruding travel commands were used; all toolpaths were generated programmatically to ensure precise correspondence between simulated and printed deposition. The custom G-Code is executed directly by the printer firmware without modification. This setup assumes accurate velocity control, negligible mechanical resonance, and precise execution of G-Code commands under steady-state conditions. The final lithophane is subsequently removed from the build platform and requires no additional post-processing prior to use.

Exact header and footer G-Code files used for our print production can be found in the code repository.

5.4 Results & Discussion

In this section, we evaluate our pipeline with fabricated examples and analysis of their similarity with input images (Section 5.4.1). In addition, we also compare to standard lithophanes achieved via varying local thicknesses (Section 5.4.2) and conduct additional experiments to evaluate our parameter discovery process (Section 5.4.3).

Parameter	Value
Nozzle Diameter	0.4 mm
Filament Diameter	1.75 mm
Nozzle Temperature	230 °C
Bed Temperature	50 °C
Extrusion Multiplier	1.06

Table 5.2: Summary of relevant printing parameters for the lithophanes in this work.

5.4.1 Fabricated Examples

Figure 5.10 presents a quantitative analysis of the samples from Figure 5.1, using the same analysis technique, but with the mean-adjusted DPP values established in Section 5.3.3. The analysis confirms that after correcting for the prediction error (Table 5.1), the fabricated lithophanes are an excellent structural match to the intended designs.

Visually, the DPP difference maps show a noisy but unbiased pattern, and the corresponding histograms are centered around a mean near zero. This indicates that the simple constant correction factors successfully account for the sim-to-real gap across a variety of images and quantization levels. While there are some discernable features these are mostly restricted to edges. We attribute these to registration and alignment mismatch between the photograph and input image, and not indicative of any latent error in the model.

Furthermore, to quantify the perceptual similarity between the expected and measured DPP maps, we use the structural similarity index measure (SSIM) [116]. Unlike absolute-error metrics such as mean-squared error (MSE) or peak signal-to-noise ratio (PSNR), SSIM is designed to better approximate human perception by comparing the structural information, luminance, and contrast between two images.

The SSIM is a scalar value between -1 and +1, where +1 indicates perfect similarity, 0 indicates no similarity, and -1 indicates perfect anticorrelation. For each of our samples we compare the the smoothed mean-adjusted expected DPP and the smoothed lithophane DPP. Each of our samples have values close approximately +0.7, indicating good similarity. We

Table 5.3: The structural similarity index measures (SSIMs) for DPP maps shown in Figure 5.10. A value of +1 indicates perfect correlation, 0 indicates no similarity, and -1 indicates perfect anti-correlation.

Example	DPP SSIM
Mona Lisa	0.6916
Sky and Water I	0.6951
Albert Einstein	0.6743
West Highland Terrier	0.7487
Cameraman	0.7213
SCF Logo	0.6157

report these values in Table 5.3.

5.4.2 Comparing with Standard Lithophanes

To understand the difference between VTP lithophanes and standard lithophanes, we used a web-based lithophane tool [36] to generate the geometry. The STL was sliced in PrusaSlicer using default PLA settings with a 0.4 mm nozzle. This lithophane was limited to three layers of 0.2 mm each, giving it a total thickness of 0.6 mm. Thicker prints appeared opaque under the available backlight, effectively collapsing the grayscale into a binary silhouette.

For each specimen we captured a RAW image with the lithophane in place and a second RAW image with the lithophane removed. Camera settings (ISO 100, $f/5.6$, shutter speed 1/30) were held constant. The RAW files were linearised with **rawpy**. The sensor’s per-channel black level (512 AUD) was subtracted on a pixel-wise basis, yielding a calibrated radiance field $L(x, y)$. Two 200 mm $\times$ 200 mm regions were extracted from every frame. The brightest portion of the lithophane (labeled "Light", L_{light}) and the darkest portion (labeled "Dark", L_{dark}). A third reference image was used to measure just the backlight (L_{white}). The percentage transmission for each patch was computed as

$$T = \frac{L_{\text{patch}}}{L_{\text{white}}}.$$

The VTP lithophane transmits 44.8% of the incident luminance in its brightest regions and 12.9% in its darkest regions (Table 5.4). Comparatively, the standard lithophane is much darker, admitting only 17.5% and 0.9% in the corresponding regions. This data confirms the qualitative impression in Figure 5.11: the camera JPG of the standard lithophane overstates its apparent contrast, while the RAW file reveals that the part blocks most of the light. Because the VTP lithophane encodes tone via controlled porosity rather than material thickness, it achieves higher overall luminance for the same backlight. Consequently, a conventional lithophane would require a significantly brighter light source to reach visual parity with the VTP sample.

Table 5.4: The measured transmittance of the two lithophanes from Figure 5.11.

Lithophane	Dark ROI Transmittance	Light ROI Transmittance
Ours	12.9%	44.8%
Standard	0.9%	17.5%

5.4.3 Additional Sim-to-Real Validation

In Section 5.3.3, we discussed how we tried building a mapping from $(V^*, H^*, \Delta s)$ to transmission response using the simulation-rendering pipeline. This mapping may not be bijective because there could be several $(V^*, H^*, \Delta s)$ for two layers mapping to roughly the same grayscale value. To validate that picking one of these sets of parameters would suffice for our purpose of constructing the look-up table (LUT) for mapping from grayscale to print parameters, we did a sampling test from the (V^*, H^*) sweep experiments. We picked five sets of parameters that map to the darkest values, five mapped to the middle values, and five mapped to the lightest values, and printed a sample using these parameters to inspect. As shown in Figure 5.12, each row represents the lightest, darkest, and the middle grayscale values respectively, and visual differences within each row are negligible.

5.5 *Limitations and Future Work*

This work proposes a novel workflow to design lithophanes fabricated with viscous thread printing, and there exist several interesting directions and limitations to overcome in future work.

First, the stochastic nature of our process inherently limits the lithophanes we produce. It limits our ability to produce clear and sharp grayscale values compared to traditional lithophanes. This randomness restricts our capacity to reproduce crisp lines and rapid transitions, naturally blurring the image. To achieve clear contrast levels, larger separations in grayscale values are required. This is a fundamental limitation of the VTP process that can only be overcome with higher-resolution printing. Using smaller strands could enable finer grayscale segmentation. We are also currently constrained in the size of our lithophanes. On the smaller end, spatial resolution is limited by the strand size and the inherent randomness of the process. Employing smaller strands could improve image resolution, though this would increase print time and may necessitate augmented extrusion techniques for finer nozzles. On the larger end, we are restricted by the build volume of the 3D printer. Although large-format 3D printers exist, developing a method for compositing VTP sections would be a valuable direction, enabling quilt-like construction of larger lithophanes.

Additionally, our current simulation method functions reliably only within a limited range of the VTP coiling regime, restricting the diversity of coiling behaviors and thus the light transmission characteristics of the printed lithophanes. By developing more advanced simulation models that incorporate higher-order viscosity effects, varying thread diameters, and thermal dynamics, we could expand the usable design space. These more sophisticated models would allow for greater clarity in the final lithophane images, though potentially at the cost of significantly increased computational effort.

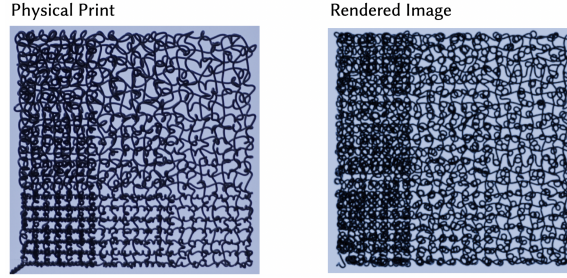
Furthermore, our simulations and fabrications are currently limited to flat surfaces. However, many practical applications, such as lampshades, require lithophanes to conform to curved surfaces. Future work could explore how deformation of flat lithophanes affects the displayed imagery and implement corrections during the toolpath generation stage to account for these effects.

Another limitation is in our calibration, we picked the metric “Dark Pixel Percentage” for characterizing the transmission response. However, it does not necessarily correspond to human perception of transmission, nor does it capture the level of fidelity of the produced lithophane imagery relative to the input image. In Figure 5.13, we did an experiment to compare the Δs pair that produced the maximum DPP (left), and the Δs pair that produced the maximum range in DPP (right, the pair found through the Δs pair sweep). This shows an interesting trade-off that humans might perceive differently. Thus, beyond building a mapping between photos and renders through a quantitative metric, it would also be interesting to explore building a perception mapping from print parameters to perceived transmission or available transmission range. The same parameter discovery methodology could be employed with enough data collected for this perception mapping and we could build a corresponding LUT.

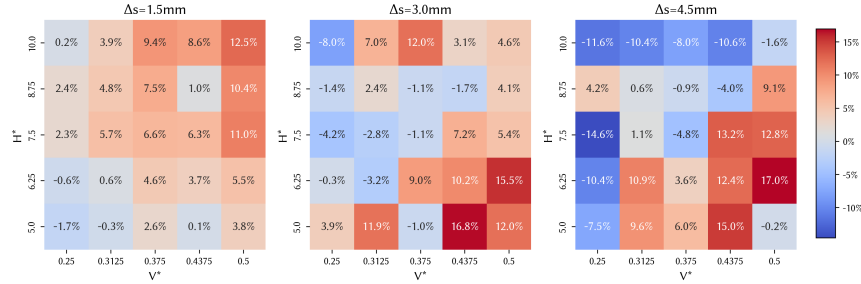
Lastly, there could be interesting applications when we embed the lithophanes into larger bulks. The stochastic nature of VTP foams brings challenges in developing a mechanism that can decode the lithophane image but also provides opportunities to embed information in a different way into bulk VTP structures. By expanding the choices of light sources to consider invisible light, such as infrared light, we may be able to produce machine-readable tags in bulk foams to provide additional meta-data and security.

5.6 Conclusion

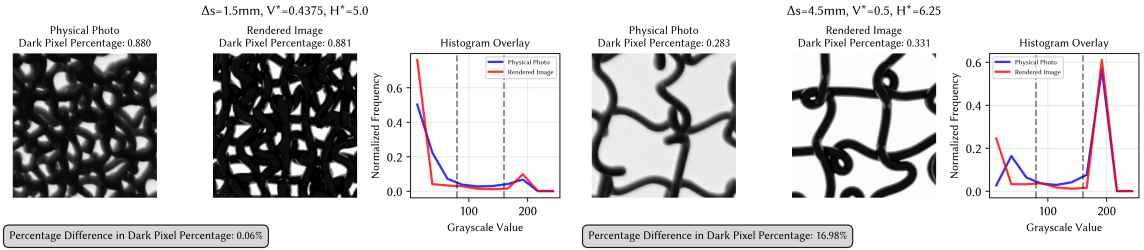
This work enables the fabrication of a new type of 3D-printed lithophanes that leverage the porous foam structure created by viscous thread printing to allow light to shine through, instead of the standard lithophanes that display an image by varying local thicknesses. Our design tool maps from input grayscale images to print parameters for VTP, where the mapping is found by parameter search with a simulation-rendering pipeline for VTP lithophanes that is calibrated against physical prints. We validate our system by fabricating a variety of examples and analyzed the fabricated lithophanes.



(a) Left: photo of a printed sample (60 cm $\times$ 60 cm) showing the coiling patterns with (V^*, H^*) uniformly sampled within ranges of $V^* \in [0.25, 0.5]$, $H^* \in [5.0, 10.0]$ (three values in each parameter). Right: rendered image of the simulation result using the G-Code that printed the left sample.



(b) Heatmap of percentage errors of DPP predicted by our simulation-rendering pipeline relative to the physical photos across the three print samples that we calibrated the simulator against. The average absolute percentage error across the $(V^*, H^*, \Delta s)$ combinations is 6.22%.



(c) The minimum DPP absolute percentage error from the heatmap. (d) The maximum DPP absolute percentage error from the heatmap.

Figure 5.5: We calibrated the rendering settings first (a) and then the simulator parameters (b-d). (a) After calibrating rendering settings, the render has similar visual quality but the coiling morphologies are still quite different. (b) The heatmap of the overall percentage errors of DPP across three calibration prints of different Δs . (c-d) Two comparisons between physical photos and rendered images (after converting to grayscale), representing the min and max percentage errors in DPP heatmap. The vertical dashed lines represent the light and dark pixel threshold values (80 and 160, respectively).

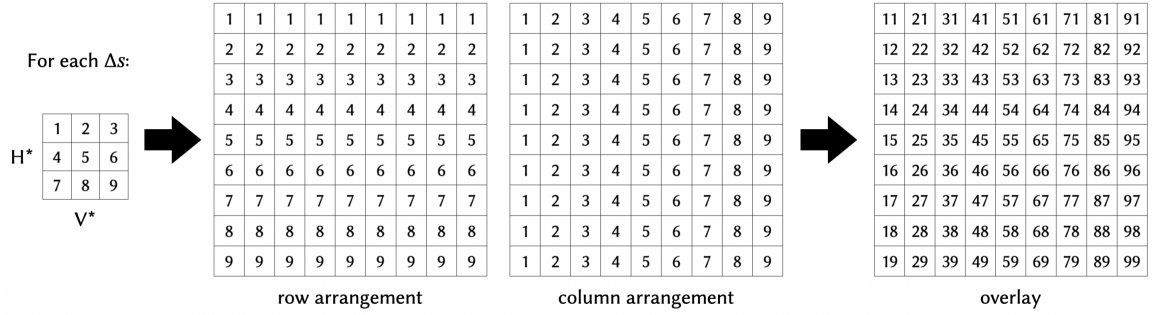


Figure 5.6: The arrangement of simulation results to produce all combinations of (V^*, H^*) for two layers.

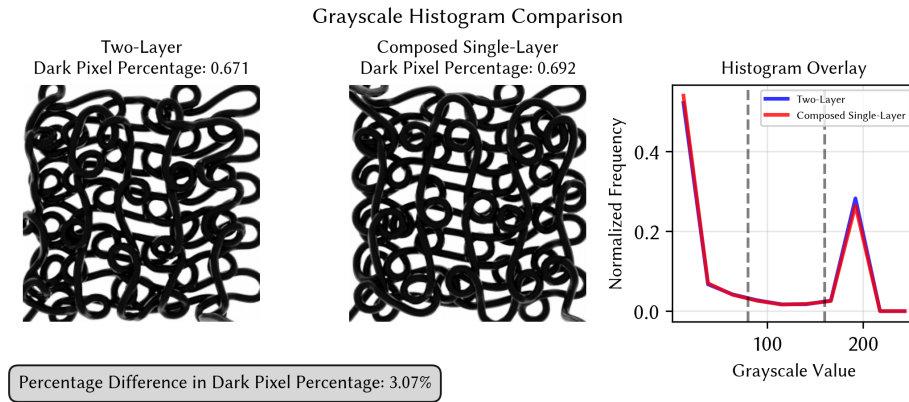


Figure 5.7: We compare the grayscale histograms of two rendered images. The left is the render of a two-layer simulation result, and the right is for a composite of two single-layer simulation results. They are highly similar in grayscale histograms and also DPP values.

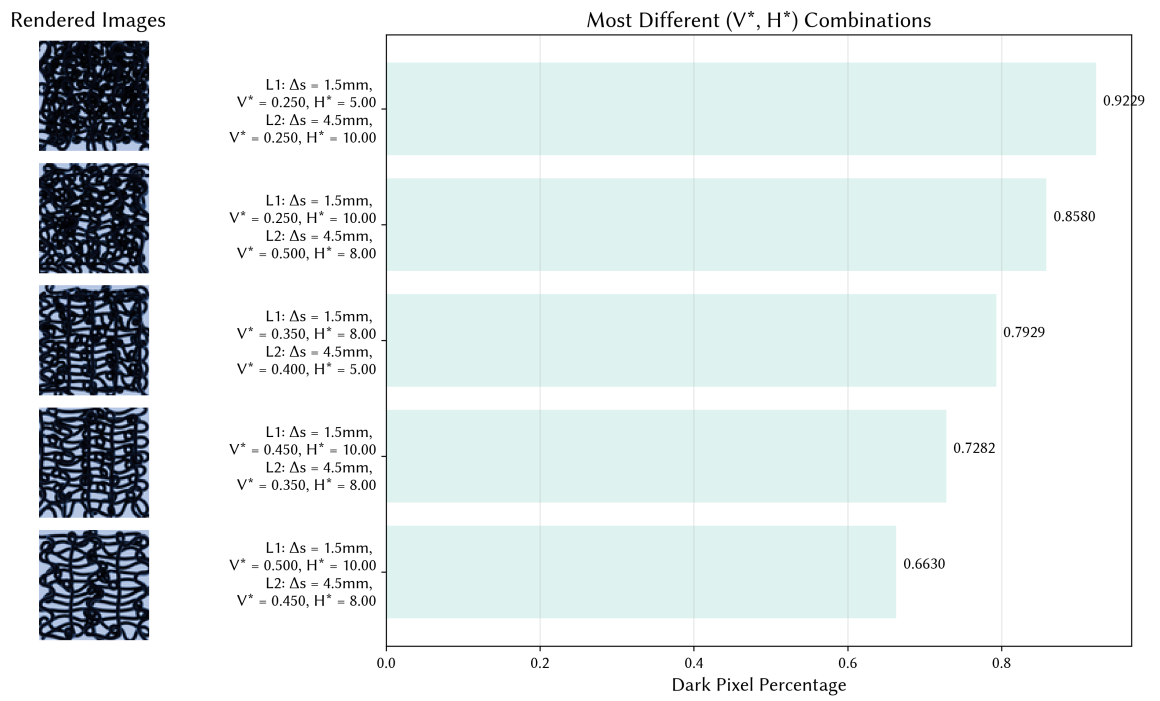


Figure 5.8: Five most different (V^* , H^*) combinations and their corresponding rendered images and predicted DPP values.

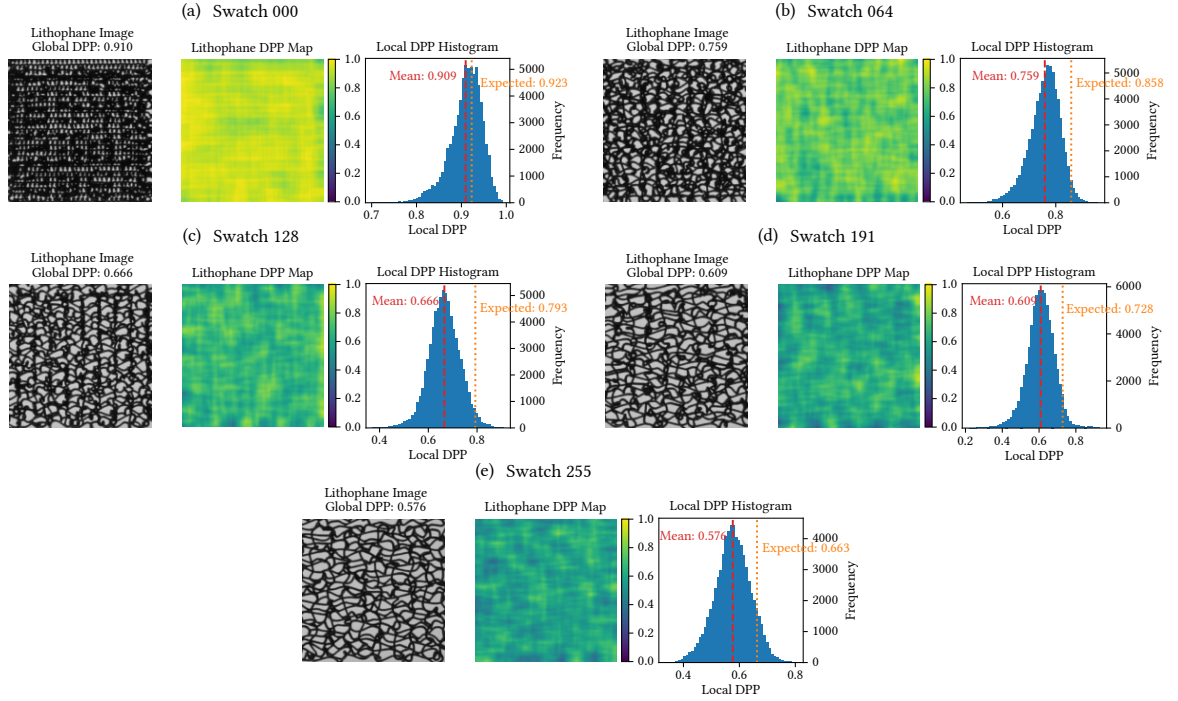


Figure 5.9: We compare the predicted and actual DPP for swatches printed with parameters that are intended to map to five grayscale values between white (0) and black (255). The mean of each histogram is consistently lower than the predicted DPP value, which shows the simulation-rendering pipeline consistently predicted darker values than reality. The variance in the DPP Map reveals the variation inherent to the stochastic VTP process. Thus, we use the discrepancies between the predicted DPP and the actual mean of the DPP Map to correct for the inherent variation in the process before evaluating the fabricated results against the input images.

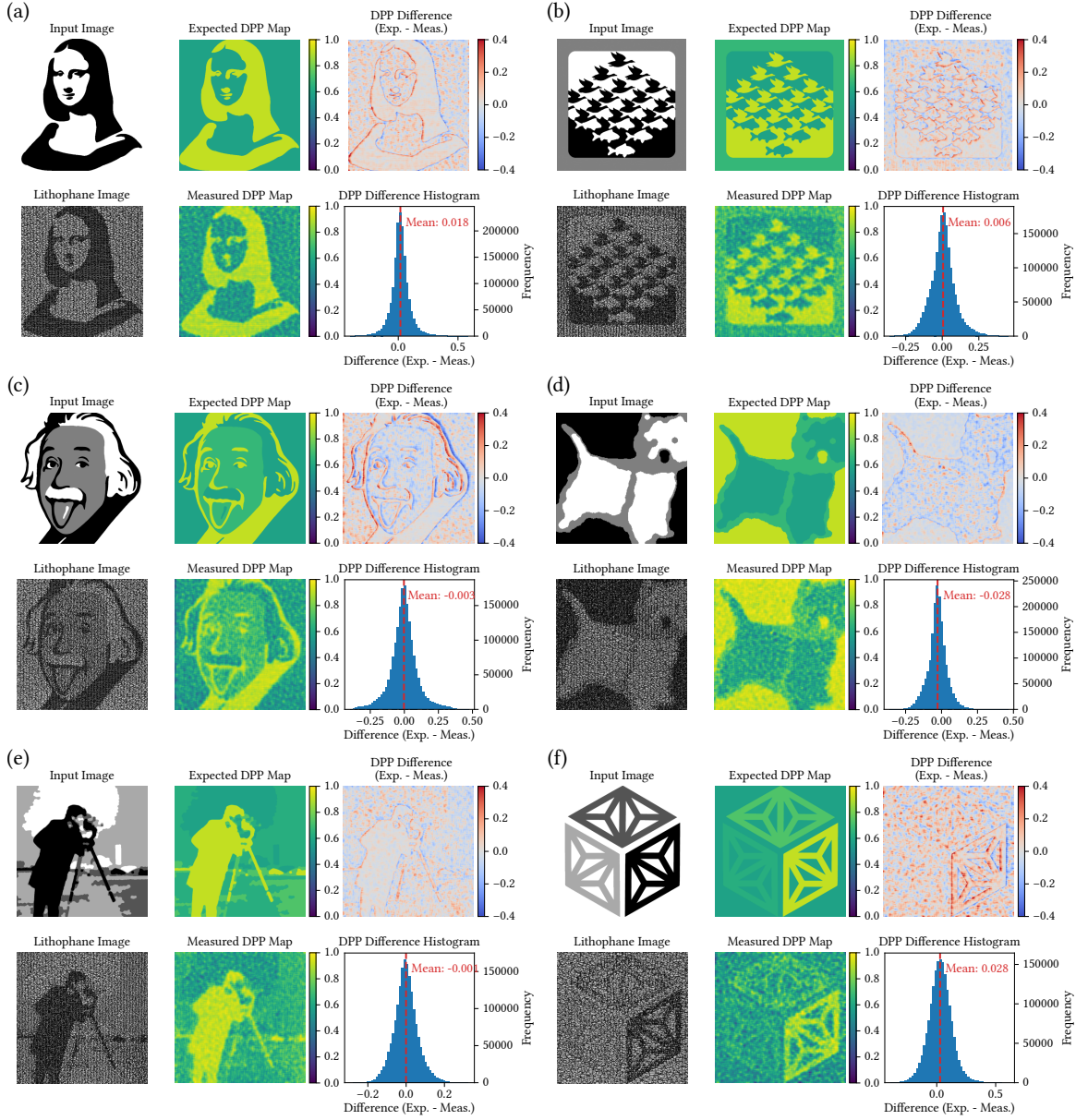


Figure 5.10: Quantitative analysis of the printed samples' dark percentages for each of the samples shown in Figure 5.1: (a) Mona Lisa, (b) M.C. Escher's *Sky and Water I*, (c) Albert Einstein, (d) West Highland Terrier, (e) the cameraman, and (f) the SCF logo. Each subfigure contains three columns. Column 1: The input image and the lithophane photograph. Column 2: The mean-adjusted expected DPP and the lithophane measured DPP. Column 3: The DPP difference ($DPP(x, y)_{\text{diff}} = DPP(x, y)_{\text{exp}} - DPP(x, y)_{\text{meas}}$) and a histogram of the same data. A more positive difference value corresponds to the lithophane being lighter than expected, while a more negative value corresponds to a lithophane that is darker than expected.

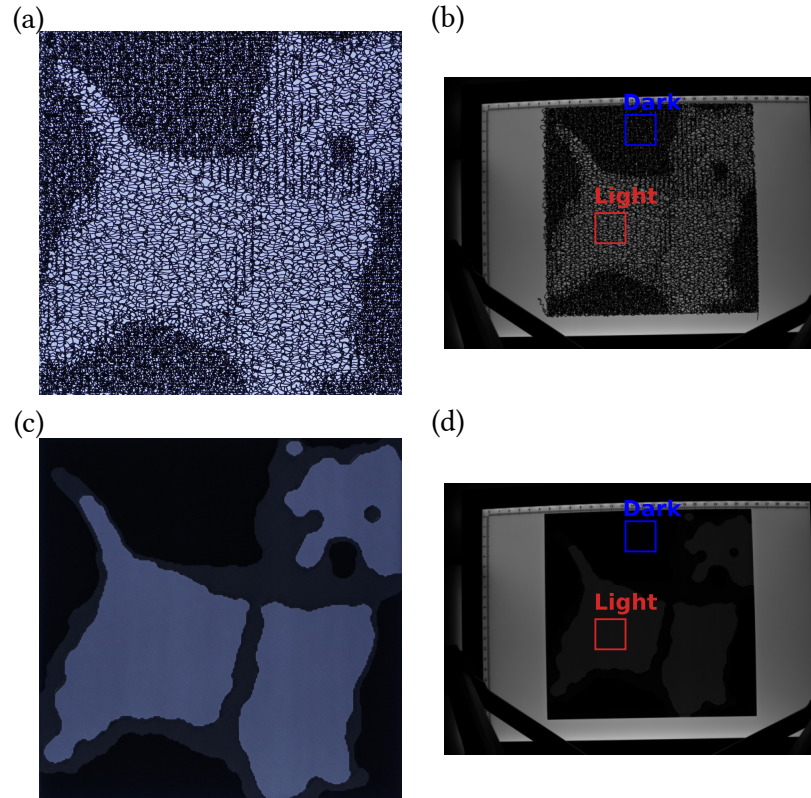
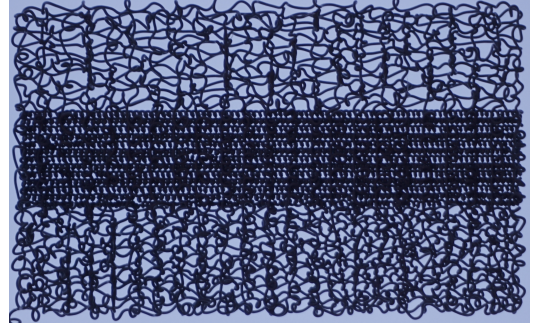
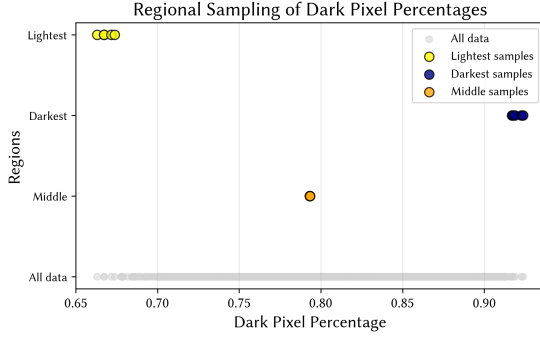


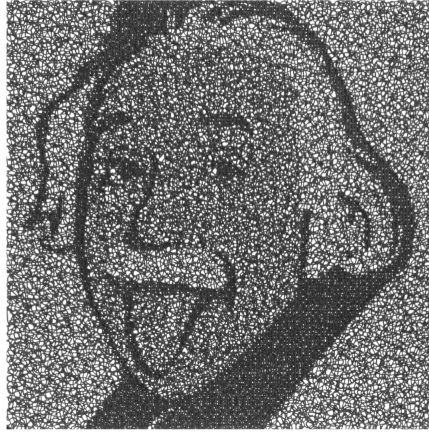
Figure 5.11: A comparison of VTP and standard lithophane transmittance when printed with the same PLA material. (a) A VTP lithophane terrier (JPG image) and (b) the RAW file used to calculate the transmittance. (c) A standard lithophane terrier and its RAW file (d). For each lithophane we capture two regions of interest (ROIs)—the darkest and lightest regions—and normalize it to the unobstructed light source to calculate the transmittance shown in Table 5.4.



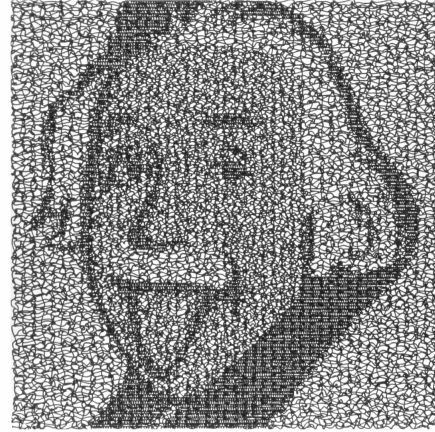
(a) We picked five samples each for the darkest, middle, and lightest values based on DPP values.

(b) A 3x5 sheet sample printed with the sampled parameter sets from the (V^*, H^*) sweep.

Figure 5.12: We further validated that selecting any of the parameter sets that map to roughly the desired grayscale value is sufficient given that this mapping is not bijective.



Δs pair (1.5 mm, 1.5 mm)



Δs pair (1.5 mm, 4.5 mm)

Figure 5.13: Two lithophanes of the quantized Albert Einstein image from Figure 5.1. The G-Code file for the left is generated using a new LUT built through the same (V^*, H^*) sweep process for Δs pair (1.5 mm, 1.5 mm).

Chapter 6

DISCUSSION AND CONCLUSION

The perennial challenge in engineering is translating an envisioned design into a physical object. While modern digital tools have granted designers unprecedented freedom in the virtual realm, a significant gap persists between what can be conceived and what can be reliably and efficiently manufactured. This gap is especially pronounced in the field of metamaterials, where extraordinary properties emerge from complex architectures that defy conventional design and fabrication paradigms. The central argument of this dissertation is that for design tools to successfully bridge the gap between user intent and physical reality, they must manage complexity on two fronts: by abstracting complex material behavior into simple, programmable controls, and by embedding the physics of the fabrication process directly into the design environment.

This concluding chapter synthesizes the contributions of the preceding research chapters to substantiate this claim. First, it articulates how the three distinct projects—the design of reprogrammable extremal materials, the development of fabrication-directed entanglement for foams, and the creation of porosity-based lithophanes—collectively form a coherent argument for this new mode of design. Second, it discusses the limitations of this work and outlines promising avenues for future research. Finally, it offers a concise summary of the dissertation’s overall contribution to the fields of computational design and advanced manufacturing.

6.1 *Synthesis of Contributions*

The core thesis of this work is that by treating physical realization not as a downstream constraint but as a design-level primitive, we can create more powerful and intuitive computational tools. This approach was investigated through three projects that build upon one another, first by developing a powerful abstraction for expressing design intent, and then by

integrating that abstraction into fabrication-aware workflows.

Chapter 3, on the design of reprogrammable extremal materials, first tackled the problem of design complexity by creating a new abstraction for mechanical behavior. Extremal materials offer a vast range of theoretical properties, but designing microstructures to achieve them is a non-trivial computational task. This work established a design vocabulary by abstracting the complex physics of zero-energy deformation modes into an intuitive primitive: the compliant straight-line mechanism (SLM). The SLM acts as a programmable building block whose orientation directly controls the material’s compliant direction. This abstraction makes an otherwise intractable design space accessible, allowing a user to express high-level intent (e.g., “I want this to extend horizontally and vertically, but not in shear”) through simple, vector-based mechanics, without needing to solve a complex inverse problem.

Chapter 4, on fabrication-directed entanglement (FDE), then demonstrated how this new design vocabulary could be integrated into a fabrication-aware workflow that treats the manufacturing process itself as a generative partner. This project addressed fabrication complexity by embracing the emergent properties of a stochastic process—viscous thread printing (VTP). Crucially, the user’s intent was specified using the exact abstraction developed in Chapter 3: the target eigenstructure, T^* , provided the formal specification for the desired directions of compliance and rigidity. The FDE method then embeds a model of the VTP process into a topology optimization framework to realize this intent. Rather than fighting the inherent variability of the process, the optimizer guides the statistical properties of the filament entanglement, designing a pattern of local coiling behavior that produces a monolithic foam with the desired anisotropy or chirality. This approach has a key advantage: the resulting continuous structure better reflects the continuum assumption of the optimizer, avoiding the fragile connections that can plague discrete solid-void designs. This demonstrates a powerful two-step process—first abstracting the goal, then using a process-aware tool to achieve it—where fabrication is an active and essential component of the design itself.

Chapter 5, on the computational design of foam lithophanes, served to demonstrate the generality of this entire philosophy by applying it to a new physical domain: optics. Here again, the key was to embrace the stochastic nature of VTP rather than fight it. The complex, random coiling of the filament is precisely what allows the structure to act as a

light diffuser; at a macro scale, these random patterns homogenize into an area of perceived luminance. This project controlled that effect by building a predictive model—a calibrated simulation and rendering pipeline—that abstracted the complex physics of light transmission through a stochastic foam into a simple, user-facing control: a desired grayscale value. This abstraction formed the basis of an automated, fabrication-aware workflow that translates a user’s intended a digital image into the machine instructions required to produce a physical lithophane with corresponding variations in porosity.

Taken together, these three contributions form a cohesive intellectual arc. Chapter 3 develops a language for expressing mechanical design intent and Chapter 4 uses that language as a formal input to a tool that understands the nuances of a specific fabrication process. Chapter 5 demonstrates that this entire methodology—abstracting complex physics into a simple user control and embedding a model of the manufacturing process—is a general and powerful pattern for the design of complex physical objects. The result is a unified vision for computational design where the user specifies high-level intent, and the tool handles the low-level physics of both the material and the machine to realize it.

6.2 Limitations

While this dissertation provides a strong proof-of-concept for this design paradigm, the work presented has several limitations that open clear avenues for future research.

The work on reprogrammable SLMs, while powerful, was primarily demonstrated in two dimensions. A significant and necessary extension of this research is the translation of these concepts into three-dimensional systems. This could involve direct 3D tiling of the existing SLMs or, more fundamentally, the development and integration of 3D analogs such as flat-plane mechanisms (FPMs) [68]. Such an extension would also open the door to exploring the much richer design space afforded by 3D spatial symmetry groups, including the targeted breaking of symmetries to achieve even more complex behaviors. Additionally, the current analysis relies on linear elasticity, which by its nature overlooks the rich non-linear dynamics and uncaptured degrees of freedom—such as the internal rotational modes observed in the nullmode material—that are present in these systems. A promising future direction would be to analyze these structures using more advanced mechanical models, such as Cosserat or

micropolar elasticity, which are equipped to handle such rotational effects and could unlock new insights into phenomena like mechanical cloaking.

The FDE and lithophane projects are both constrained by the capabilities of Viscous Thread Printing. The moderate stiffness contrast ($\sim 30:1$) and feature resolution (~ 1.2 mm) of the VTP process limit the achievable performance and fidelity of the final parts. Future work should explore integrating other manufacturing processes, particularly those with higher resolution or multi-material capabilities, into this design framework. Moreover, both of the current workflows rely on post-fabrication calibration steps to bridge the sim-to-real gap. A major goal for future research is the development of more predictive, a priori simulation models that can account for fabrication effects without this empirical tuning, leading to a more streamlined and reliable design process.

6.3 Future Directions

More broadly, this work points toward several exciting long-term research directions. The foundational idea of creating process-aware design tools built on user-friendly abstractions is highly generalizable. Future work could push this paradigm in three key directions:

1. expanding the scope of physical phenomena and manufacturing processes,
2. raising the level of design abstraction to be more intuitive and goal-oriented,
3. and leveraging machine learning to automate the discovery of novel structures.

Expansion to New Physics and Processes As shown, the core pattern—identifying governing physical systems, building a process-aware computational model, and creating a high-level abstraction—can be applied to numerous domains. A natural next step is to expand this methodology to new physics and manufacturing processes. In fluidics, one could design porous media with anisotropic permeability by controlling pore structure, targeting Darcy’s law instead of elastic stiffness. In thermal management, this approach could yield heat sinks with programmed conductive pathways, guiding heat away from critical components by optimizing over a desired thermal conductivity tensor. In electromagnetism, it could

enable the design of custom antennas or absorbers by structuring materials based on a target permittivity tensor. Each new domain would require coupling a relevant manufacturing process, such as multi-material resin-jetting or laser powder bed fusion, with a corresponding physical simulation to create a new generation of fabrication-aware design tools.

Natural Language Design Abstractions Another critical research avenue lies in developing higher-level design abstractions that move beyond the current low-level inputs. While the methods outline here are powerful they can still require domain expertise or interpretation. The ultimate goal is to allow users to specify functional intent directly. Future systems could leverage natural language interfaces, allowing a designer to simply state, “design a compliant gripper that applies uniform pressure to an object” or “create a privacy screen that reveals an image only from a specific angle.” Such a system would be responsible for translating this high-level functional goal into the appropriate low-level physical targets and then invoking the fabrication-aware optimization pipeline. This would represent a true paradigm shift, moving the designer’s focus entirely from specifying how a material should behave to what it should do.

Autonomous Design and Discovery Finally, by coupling these process-aware design tools with machine learning, we can pursue autonomous design and discovery. Instead of a human designer exploring the possibilities, a generative algorithm (such as a GAN or a reinforcement learning agent) could be tasked with creating and testing novel structures. In this model, the design tool becomes the “environment” where the agent acts by proposing a topology, and the process-aware simulation provides the “reward” by evaluating its performance against a high-level objective. This closed-loop system could not only optimize structures for known targets but could also be tasked with open-ended discovery, for example, “find the structure with the highest possible normal-shear coupling for a given volume fraction.” This moves beyond mere design assistance and transforms the computational tool into an engine more tuned for scientific discovery, capable of uncovering optimal and entirely unexpected material architectures.

6.4 *Final Summary*

This dissertation set out to address the fundamental disconnect between the freedom of digital design and the constraints of the physical world. Through three interconnected research thrusts, it has demonstrated that a powerful solution lies in creating computational tools that intelligently manage complexity. By abstracting the intricate behavior of materials and structures into intuitive, programmable components and by embedding the physics of fabrication directly into the design process, we can empower designers to translate their high-level creative intent into functional, physical reality more directly and effectively. This work provides not only a set of novel methods for creating advanced materials but also a conceptual framework for the next generation of design tools—tools that treat design and fabrication not as separate disciplines, but as a single, unified act of creation.

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Appendix A

APPENDIX FOR DESIGN AND REPROGRAMMABILITY OF ZERO MODES IN 2D MATERIALS FROM A SINGLE ELEMENT

Table of Material Values Material values for the materials seen in Figure 1 from the main document are shown in Table A.1 and Table A.2.

Table A.1: Selection of 2*22 Simulated Material Properties

θ	mode	E_1 [MPa]	E_2 [MPa]	G [MPa]	ν_{12}	η	λ_1 [MPa]	λ_2 [MPa]	λ_3 [MPa]	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$
0°	m=2	0.280	0.310	0.076	-0.909	0.001	0.150	0.153	6.740	0.022	0.023	1.000
45°	m=2	0.162	17.099	0.160	0.001	-0.001	0.162	0.320	17.100	0.009	0.019	1.000
90°	m=1	0.317	0.317	7.880	0.980	0.000	0.160	15.760	16.040	0.010	0.983	1.000

Table A.2: 632 Simulated Material Properties

θ	mode	E [MPa]	G [MPa]	K [MPa]	K/G	ν	λ_1 [MPa]	λ_2 [MPa]	λ_3 [MPa]	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$
90°	m=0	16.000	6.040	12.000	1.987	0.330	12.030	12.080	24.070	0.500	0.502	1.000
0°	m=1	0.526	2.190	0.140	0.064	-0.880	0.280	4.378	4.382	0.064	0.999	1.000
90°	m=2	0.394	0.102	6.680	65.490	0.970	0.200	0.204	13.360	0.015	0.015	1.000
0°	m=3	0.131	0.097	0.050	0.513	-0.320	0.099	0.193	0.195	0.510	0.993	1.000

Appendix B

APPENDIX FOR COMPUTATIONAL DESIGN OF 3D PRINTED FOAM LITHOPHANES

VTP Parameters All VTP specific parameters used in this work are listed in Table B.1. However, these do not encompass the general printer settings needed such as bed temperature, nozzle temperature, etc. Additionally, the die swell constant ($\alpha = 1.5$) and input feed rate ($\dot{e} = 50$ mm/s) are closely tied to using PLA as the base material. These numbers are generally material specific and may need to be tuned for other materials.

Table B.1: The VTP-specific parameters used in this work.

Parameter	Value
V^*	$0.25 \sim 0.5$
H^*	$5.0 \sim 10.0$
Raster spacing (Δs)	$1.5 \sim 4.5$
Effective layer thickness (Δz)	1.3 mm
Die swell constant (α)	1.5
Input feedrate ($\dot{e}$)	$50 \sim 150$ mm/s

Simulation Parameters The simulated material property values were calibrated to match real-world coiling dynamics, namely coil radius and frequency. One set of parameters was not able to capture the full range of (V^*, H^*) tested, thus multiple sets of material parameters were used to capture the full span of coiling morphologies. These values are summarized in Table B.2 and the detailed values are displayed in Table B.3.

Table B.2: Summary of simulation parameters used in DER-based VTP dynamics.

Coefficient	Value	General Notes
Lame's First Parameter (λ)	10 $\sim$ 300 Pa	Increases with increasing H^*
Lame's Second Parameter (μ)	20 Pa	Except when $H^* \in [8.75, 10]$, $V^* \in [0.25, 0.3125]$, $\Delta s \in [3.75, 4.5]$, $\mu = 50$
Vertex Dynamic Viscosity (η_1)	0.001 $\sim$ 0.1 Pa $\cdot$ s	Increases with decreasing H^* and varying V^*
Rotational Viscosity (η_2)	5e-5 Pa $\cdot$ s $\cdot$ m ³	Not used for adjustment

Table B.3: Detailed simulation parameters tuned for different Δs ranges.

Raster Spacing	Lame's First Parameter (μ)						Vertex Dynamic Viscosity (η_1)					
	H^*/V^*	0.25	0.3125	0.375	0.4375	0.5	H^*/V^*	0.25	0.3125	0.375	0.4375	0.5
$\Delta s \in [1.5, 2.25]$	10.00	300	300	300	300	300	10.00	0.002	0.002	0.002	0.002	0.002
	8.75	300	300	300	300	300	8.75	0.002	0.002	0.002	0.002	0.001
	7.50	250	250	250	250	250	7.50	0.005	0.005	0.005	0.005	0.002
	6.25	20	20	20	20	20	6.25	0.050	0.050	0.050	0.050	0.050
	5.00	20	20	20	10	10	5.00	0.100	0.100	0.100	0.050	0.050
$\Delta s \in [2.25, 3.75]$	H^*/V^*	0.25	0.3125	0.375	0.4375	0.5	H^*/V^*	0.25	0.3125	0.375	0.4375	0.5
	10.00	300	300	300	300	250	10.00	0.002	0.002	0.002	0.002	0.005
	8.75	300	300	300	300	300	8.75	0.002	0.002	0.002	0.002	0.002
	7.50	20	20	20	10	20	7.50	0.005	0.050	0.005	0.005	0.005
	6.25	20	20	10	20	20	6.25	0.050	0.050	0.001	0.002	0.005
	5.00	20	20	20	20	20	5.00	0.100	0.005	0.005	0.005	0.050
$\Delta s \in [3.75, 4.5]$	H^*/V^*	0.25	0.3125	0.375	0.4375	0.5	H^*/V^*	0.25	0.3125	0.375	0.4375	0.5
	10.00	250	300	300	300	300	10.00	0.005	0.005	0.005	0.005	0.002
	8.75	250	250	20	300	300	8.75	0.005	0.005	0.005	0.002	0.002
	7.50	20	20	20	250	250	7.50	0.005	0.005	0.005	0.005	0.005
	6.25	20	20	20	300	300	6.25	0.050	0.005	0.005	0.002	0.002
	5.00	20	20	10	10	20	5.00	0.100	0.100	0.050	0.050	0.100

Appendix C

HOW TO GENERATE FDM G-Code FROM VTP PARAMETERS (V^* , H^*)

Problem Statement To generate controlled VTP we must find a way to determine specific values of V^* and H^* that can be used with an FDM printer. This involves converting V^* and H^* into G-Code commands, which are the instructions digested by the FDM printer.

A G-Code command instructs the printer to move from its current position (x_0, y_0, z_0) to a target location (x, y, z) at a certain speed f , while extruding e millimeters of material (here we are assuming the positions are in absolute coordinates while the extrusion values are specified in relative increments). In G-Code this looks like `G1 Xx Yy Zz Ff Ee` where `G1` and capital letters are character literals used by the printer to parse the command and lowercase letters are values to be provided to the computer. For example, `G1 X10.0 Y20.0 Z30.0 F1000 E40.0` is the command that moves the printer extruder to the target position $(x, y, z) = (10.0 \text{ mm}, 20.0 \text{ mm}, 30.0 \text{ mm})$ at a speed of $f = 1000 \text{ mm/min}$ and push $e = 40 \text{ mm}$ of filament into the extruder (millimeters and minutes are the default units of many 3D printers). If an x , y , z , or f value is omitted from a command it is assumed to be held constant from the last command. If an e value is omitted from a command then it is assumed to be 0.

For our goal of converting (V^*, H^*) to G-Code, we have three unknown variables: e^i , f^i , and z^i where the superscript i denotes a value for the i -th command in a G-Code file. The (x, y) position is entirely dependent on the geometry and thus we do not consider derive those from (V^*, H^*) . To keep things simple, we'll drop the i superscript unless it is necessary to clear ambiguity. All quantities in the rest of the derivation should be understood to apply to a single G-Code command.

Derivation The final goal is to express e^i , f^i , and z in terms of independent variables and process constants. To do this we'll make some underlying assumptions

1. we are printing onto a (nominally) flat layer,
2. we have instantaneous conservation of mass through the nozzle (i.e., steady-state printing)
3. and the density of the input filament and the extruded thread remain nominally constant.

As given in [125], the equations for V^* and H^* are:

$$V^* := f/C$$

$$H^* := H/(\alpha D_N)$$

which leads to

$$f = CV^* \tag{C.1}$$

$$H = \alpha D_N H^*; \tag{C.2}$$

however, this results in two equations (Equation C.1, Equation C.2) for three unknowns (e, f, z), thus we must introduce a third equation.

By assuming conservation of mass $m_{in} = m_{out}$ we can derive an equation for e . Assuming cylindrical geometries for the input filament and the output thread, the mass fed into the nozzle during a G-Code command is $\rho_F V_{in} = e \rho_F A_F$ and the mass extruded from the nozzle during a print move is $\rho_T V_{out} = (\Delta l) \rho_T A_T$, where ρ_F and ρ_T are the filament and extruded thread intrinsic densities, Δl is the length of the extruded thread and A_F and $A_T = \alpha D_N$ are the filament and thread areas respectively. We now have have

$$f = CV^* \tag{C.3}$$

$$e = \Delta l \frac{\rho_T A_T}{\rho_F A_F} \tag{C.4}$$

$$H = A_T H^*. \tag{C.5}$$

First, we focus on deriving from V^* . C is defined as the linear speed of the material leaving the nozzle, meaning $C = \Delta l / \Delta t$ giving us

$$f = \frac{\Delta l}{\Delta t} V^* \quad (\text{C.6})$$

$$e = \Delta l \frac{\rho_T A_T}{\rho_F A_F} \quad (\text{C.7})$$

which now is an overdefined system (two equations, one unknown), thus we must introduce an additional equation. It is simplest to simply hold the volumetric flow rate constant in the system, which (assuming constant filament cross-section across the spool) gives $\dot{e} := e / \Delta t$ where now $\dot{e}$ will be a user specified process constant.

Assuming the $\rho_F \approx \rho_T$ and that during a print move at steady state the molten flow of material is constant $\frac{\Delta V_{in}}{\Delta t} = \frac{\Delta V_{out}}{\Delta t}$, we can divide Equation C.7 by Δt resulting in the equation:

$$\frac{e}{\Delta t} = \frac{\Delta l}{\Delta t} \frac{A_T}{A_F} = C \frac{A_T}{A_F} = \dot{e} \quad (\text{C.8})$$

where Δt is the time it takes to complete the printing move. Thus

$$C = \dot{e} \frac{A_F}{A_T} \quad (\text{C.9})$$

and plugging it into Equation C.1 we get

$$f = V^* \dot{e} \frac{A_F}{A_T}. \quad (\text{C.10})$$

Next, we'll solve for e . From the definition of $\dot{e}$

$$e = \dot{e} \Delta t.$$

The time it takes to complete an extruding move Δt depends on the length of the move L and the translational speed f ; $\Delta t = L / f$. Using this and Equation C.10 we arrive at

$$e = \frac{L}{V^*} \frac{A_T}{A_F}. \quad (\text{C.11})$$

Finally, assuming we print on a flat plane trivially solves for the z coordinate as equivalent to the nozzle height H so

$$z = H = A_T H^*. \quad (\text{C.12})$$

e and f , however, are defined on the geometry of the extruded thread, which we

Summary In conclusion, we derived equation to convert V^* and H^* into G-Code commands for an FDM printer. The key steps in the derivation involved making several assumptions, such as equating the z coordinate to the nozzle height, conserving mass and volume during the extrusion process, and maintaining a constant material flow rate. The resulting equations for the G-Code variables z , f , and e in terms of V^* and H^* are:

$$\begin{aligned} f^i &= V^* \dot{e} \frac{A_F}{A_T} \\ e^i &= \frac{L^i}{V^*} \frac{A_T}{A_F} \\ z^i &= \alpha D_N H^* \end{aligned}$$

These equations enable us to translate the desired extrusion properties into G-Code commands that can be executed by the printer. While the assumptions made may not be universally applicable, they provide a useful foundation for determining G-Code commands based on input variables V^* and H^* .

Appendix D

EXTREMAL_ALGEBRA.M

```

1 clear; close all; clc;
2
3 %% Setup
4 % a, b, & c are the __positive__ SLM deformations for any of the 3
5 % respective SLMs in a unit cell
6 syms a b c
7 % phi is the angle between the first lattice vector and the second
8 % theta is the angle between the lattice vector and the SLM direction
9 syms theta phi positive
10 assume(theta >= 0 & theta <= pi/2)
11 % Linear, Hexagonal, or Square tiling
12 assume(phi == 0 | phi == pi/3 | phi == pi/2)
13 assume(0 <= a & a <= 1)
14 assume(0 <= b & b <= 1)
15 assume(0 <= c & c <= 1)
16
17 % Generalized symbolic rotation matrix since everything is a rotation of
18 % everything else
19 R = @(t) [cos(t), -sin(t); sin(t), cos(t)];
20
21 %% Standard Lattice definition
22 % Lattice vectors We define l1_hat to be along the x-axis. The l2_hat
23 % lattice vector is then described by a rotation of l1_hat about the origin
24 % by phi radians. l3_hat (for cells with constraining symmetry like 632) is
25 % defined as the difference of the two
26 l1_hat = sym([1,0]');
27 l2_hat = R(phi)*l1_hat;
28 l3_hat = l2_hat - l1_hat;
29 L = [l1_hat, l2_hat];
30

```

```

31 % SLM deformed vectors [d1, d2, d3]. SLM DOF is a rotation off of the
32 % lattice vector. d{n} is the deformed version of the lattice vector, which
33 % is the lattice vector plus the deformation vector from the SLM
34 d1 = (eye(2) + a * R(theta)) * l1_hat; % deformation of l1
35 d2 = (eye(2) + b * R(theta)) * l2_hat; % deformation of l2
36 d3 = (eye(2) + c * R(theta)) * l3_hat; % deformation of l3
37 d2star = (eye(2) + c * R(-theta)) * l2_hat; % deformation of l2 for mirror
    symmetry
38
39 % D is the deformation gradient in the lattice coordinate system. For
40 % square lattices (e.g., 442) this is the global coordinate system
41 % already. For hex/tri lattices (e.g., 632) this is not the global
42 % coordinates and will need to be transformed via L to get cartesian
43 % deformation
44 D = simplify([d1, d2]);
45
46 % F0 is the generalized deformation gradient in the global coordinate
47 % system. We use F0 so we can substitute different values of theta and phi
48 % into it to generate a unique F for each sub case. The D/L (mrdivide(D,L))
49 % solves the matrix equation F*L = D where F is the deformation gradient in
50 % Cartesian coordinates, L is the description of the lattice vectors in
51 % Cartesian coordinates, and D is the deformation gradient of the lattice
52 % vectors in lattice coordinates. It is intuitive to describe the
53 % deformation in the (partial) lattice coordinate system (D) so we need to
54 % transform this into global coordinates. If the lattice is already in the
55 % Cartesian coordinate system (i.e., phi = pi/2) then L is the identity
56 % matrix and F0 = D. D is the partial lattice coordinates because
57 % describing the true deformation in the full lattice coordinates gets
58 % complicated, because the lattices are not necessarily orthogonal, but
59 % if we were to use D' as the deformation of the lattice in the lattice
60 % coordinates then we get F = L * D' * inv(L)
61 F0 = simplify(D/L)
62
63 %% Generalized Lattice
64 disp('=====GENERALIZED LATTICE=====')
65

```

```

66 % We have a lattice with independent SLMs connected with some angle phi
67 % between the two lattice vectors. This can result in a square or if phi is
68 % an acute angle it can result in a hex/triagonal grid.
69 %{
70
71
72 %}
73
74 F = F0
75 eps = cauchy_strain(F)
76 epsVec = [eps(1), eps(4), sqrt(2)*eps(2)]'
77 disp(['Conclusion: The easy mode of the generalized lattice ' ...
78       'is dependent on both phi and theta'])
79 disp('=====')
80
81 %% Generalized Square Lattice
82 disp('=====GENERALIZED SQUARE LATTICE=====')
83
84 % We have a lattice with square lattice spacing (phi = pi/2), but the SLM
85 % angle is allowed to rotate. Note, this is a spacing with rotational
86 % symmetry only, not mirror symmetry, so 442 symmetry, not 2*22 symmetry.
87 % To get 2*22 symmetry we would plug in -theta for the d2 vector
88 %{
89
90     0-----0
91     |           |
92 12 |           |
93     |phi        |
94     0-----0
95         11
96 phi = pi/2
97 %}
98
99 F = simplify(subs(F0,phi,pi/2))
100 eps = cauchy_strain(F)
101 epsVec = [eps(1), eps(4), sqrt(2)*eps(2)]'

```



```

102 disp('Conclusion: There is a smooth transition from bimode to unimode')
103 disp('=====')
104 %% Generalized 3-SLM Constrained Cell
105 disp('=====3 SLM CONSTRAINED CELL=====')
106 % We have 3 SLMs connected in a triangle with the SLM deformed vector d{n}
107 % defined as a rotated vector from the l{n} lattice vector. The
108 % significance is that this constrains the motion of the l1 and l2 vectors
109 % because the definition of d3 = d2 - d1 which must always be true. We
110 % don't have this in the other definitions because we assume that the cell
111 % is rotationally symmetric all the way around
112 %{
113     0
114     / \
115  12 /   \ 13
116     /     \
117  0-----0
118     11
119 %}
120
121 phi0 = pi/3;
122 % Solve for constraint of 3rd SLM on the other two to simplify the
123 % equations. This should give us two maps, g[1]:b->a and g[2]:c->a
124 g = simplify(subs(d3==d2-d1,phi,phi0));
125 S = solve(g,[b, c],'ReturnConditions',true)
126
127 % Substitute a in for b and c
128 F = simplify(subs(F0,[phi,b,c],[phi0,S.b,S.c]));
129
130 % Cauchy strain
131 eps = cauchy_strain(F);
132 epsVec = [eps(1),eps(4),eps(2)*sqrt(2)]'
133 disp("Conclusion: Easy mode is not dependent on lattice structure")
134 disp('=====')
135
136 %% Working from a lattice to S
137 q = [a; b];

```

```

138
139 F = simplify(subs(F0,phi,pi/2));
140 eps = cauchy_strain(F);
141 epsVoigt = [eps(1,1); eps(2,2); sqrt(2)*eps(1,2)];
142 B = simplify(jacobian(epsVoigt, q))
143 K = diag([1, 1])
144 S0 = B * K * B'
145 S = simplify(mandelT(pi/4) * S0 * mandelT(pi/4).')
146
147
148 %%
149 %{
150 Test case, a material with easy strains in the vertical and horizontal
151 directions. Also zero Poisson's ratio in these directions
152
153      0-----0
154      ^ |           |
155      | |           |
156      v |phi        |
157      0-----0
158      <----->
159
160 phi = pi/2
161 theta = 0
162 %}
163 clc
164 % 2*22 UMA 0 degrees, but rotated 45deg from material designer orientation
165 C = [ 335499.8067696      -499.99902046   -60103.98569627;
166      -499.99902046   335499.8067696      -60103.98569627;
167      -60103.98569627   -60103.98569627   7994999.91441055];
168 disp("Simulated 2*22 0 deg with lattice aligned with XY")
169 C = C./max(C(:))
170 disp("We clean this up to make it easier for us to solve")
171 C = [0.042 0 0;
172      0 0.042 0;
173      0 0 1]

```

```

174 [v,w] = eig(C)
175
176 % Assume first lattice vector is along the x-axis and parameterized by a
177 e1 = a*simplify(subs(e,phi,0));
178 % Second lattice vector is allowed to be phi radians rotated from l1 and is
179 % parameterized by b
180 e2 = b*e;
181
182 disp("We look at the linear combinations of the easy mode strains")
183 disp("e1 + e2 == v(:,n), where n is the vector for a small eigen value n")
184 disp("If we look at the linear combination for the first eigen vector")
185 g1 = simplify(e1+e2==v(:,1),"Steps",50);
186 sol1 = solve(g1,[a,b,theta,phi]);
187 disp("We get 4 solutions")
188 S = [a, b, theta, phi;
189      sol1(:).a, sol1(:).b, sol1(:).theta, sol1(:).phi]
190 disp("If we look at the solutions for the second eigen vector")
191 g2 = simplify(e1+e2==v(:,2),"Steps",50);
192 sol2 = solve(g2,[a,b,theta,phi]);
193 disp("We get 1 solution")
194 S = [a, b, theta, phi;
195      sol2(:).a, sol2(:).b, sol2(:).theta, sol2(:).phi]
196 disp("The lattice spacing must agree between the two solutions")
197 disp("So we get this")
198 S = [a, b, theta, phi;
199      1, 0, 0, pi/2;
200      0, 1, 0, pi/2]
201
202 disp("??? How do we solve for the entire thing at the same time ???")
203 % G = simplify(g1|g2,"Steps",50);
204 % sol3 = solve(G,[a,b,theta,phi]);
205 % S = [sol3(:).a, sol3(:).b, sol3(:).theta, sol3(:).phi]
206
207 %%
208 %{
209 Test case, a material with easy strains in the vertical and horizontal

```

```

210 directions. Also zero Poisson's ratio in these directions
211
212   0-----0
213   |         |
214 <-->|         |
215   |phi      |
216   0-----0
217       ^
218       |
219       v
220
221 phi = pi/2
222 theta = pi/2
223 %}
224 clc
225 % 2*22 UMA 90 degrees, but rotated 45deg from material designer orientation
226 C = [8402499.92910539    82499.99936396   -3535.53429604;
227      82499.99936396 8402499.92910539   -3535.53429604;
228      -3535.53429604   -3535.53429604   344999.99999574];
229 disp("Simulated 2*22 90 deg with lattice aligned with XY")
230 C = C./max(C(:))
231 disp("We clean this up to make it easier for us to analyze")
232 C = [1 0 0;
233      0 1 0;
234      0 0 .04]
235 [v,w] = eig(C)
236
237 % Assume first lattice vector is along the x-axis and parameterized by a
238 e1 = a*simplify(subs(e,phi,0));
239 % Second lattice vector is allowed to be phi radians rotated from l1 and is
240 % parameterized by b
241 e2 = b*e;
242
243 disp("We look at the linear combinations of the easy mode strains")
244 disp("e1 + e2 == v(:,n), where n is the vector for a small eigen value n")
245 disp("If we look at the linear combination for the first (and only) eigen

```

```

        vector")
246 g1 = simplify(e1+e2==v(:,1),"Steps",50);
247 sol1 = solve(g1,[a,b,theta,phi]);
248 disp("We get 1 solution")
249 S = [a, b, theta, phi;
250       sol1(:).a, sol1(:).b, sol1(:).theta, sol1(:).phi]
251 disp("This isn't right! I should get phi = pi/2 and theta = pi/2 and
        a==b==1")
252 disp("At least I'm pretty sure about this")
253
254
255 g2 = simplify(e1+e2==v(:,2),"Steps",50)
256 sol2 = solve(g2,[a,b,theta,phi])
257
258 return
259
260 %%
261 % What about for an isotropic matrix?
262 fprintf("\n\n")
263 disp('An isotropic unimode with nu=-1')
264 C = generate_isotropic_matrix(-.9999)
265 [v,w] = eig(C)
266 % Here we can't just solve for the single SLM strain because in this
267 % material the three SLMs are all constrained. This should give me a
268 % material with 3 SLMs, phi = pi/3 and theta = 0
269 % solve(subs(e==v(:,1),phi,0),theta,'Real',true)
270 syms theta1 theta2
271 e1 = subs(e,[phi, theta],[0, theta1])
272 e2 = subs(e,theta,theta2)
273 % solve(e1+e2==v(:,1),[phi,theta1,theta2])
274 solve(e1+e2==v(:,1))
275
276
277 %% 2*22 0 degree
278 disp('=====2*22 BI MODE=====')
279 % theta = sym(0); % SLM aligned with lattice vector

```

```

280 % phi = sym(pi/2); % Traditional X/Y Cartesian lattice
281 Dtemp=simplify(subs(D)) % Substitute theta and phi into D from above
282 F = simplify(subs(Dtemp / L)) % Simplify and transform Dtemp into global
    coordintes
283 eps = 1/2*(F + F.') - eye(2) % Cauchy strain
284 detEps = simplify(det(eps))
285 eps = [eps(1), eps(4), 2*eps(2)]'
286 disp('=====')
287
288 %% 2*22 45 degree
289 disp('=====2*22 45 Deg BI MODE=====')
290 theta = sym(pi/4);
291 phi = sym(pi/2);
292 Dtemp=simplify(subs(D));
293 % transforms the lattice coordinate system deformation Dtemp into the global
294 % coordinate displacements F
295 F = simplify(subs(Dtemp / L))
296 eps = 1/2*(F + F.') - eye(2)
297 detEps = simplify(det(eps))
298 eps = [eps(1), eps(4), 2*eps(2)]'
299 disp('=====')
300
301 %% 2*22 90 degree
302 disp('=====2*22 UNI MODE=====')
303 theta = sym(pi/2);
304 phi = sym(pi/2);
305 Dtemp=simplify(subs(D));
306 F = simplify(subs(Dtemp / L))
307 [v,w] = eig(F);
308 simplify(v)
309 simplify(w)
310 eps = 1/2*(F + F.') - eye(2)
311 detEps = simplify(det(eps))
312
313 eps = [eps(1), eps(4), 2*eps(2)]'
314 disp('=====')

```

```

315
316 %% Experimental Constrained Square Pattern
317 % TODO: Figure out which wallpaper group this is
318 disp('=====EXPERIMENT=====')
319 theta = sym(0);
320 phi = sym(pi/2);
321 Dtemp=simplify(subs(D))
322
323 % We constrain so that the length of the side with the 3rd SLM is defined
324 % by the other two SLMs
325 g = simplify(subs(d3==d2-d1))
326 S = solve(g)
327
328 Dtemp = simplify(subs(D,S))
329 F = simplify(subs(Dtemp / L))
330 [v,w] = eig(F)
331 eps = 1/2 * (F + F.') - eye(2)
332 detEps = simplify(det(eps))
333 eps = [eps(1),eps(4),eps(2)*2]'
334 return
335 %% 632-6 way 90 degrees
336 % NULL Mode
337 disp('=====NULL MODE=====')
338 theta = sym(pi/2);
339 phi = sym(pi/3);
340
341 g = simplify(subs(d3==d2-d1));
342 S = solve(g);
343
344 Dtemp = simplify(subs(D,S))
345 F = simplify(subs(Dtemp / L))
346 [v,w] = eig(F);
347 v=simplify(v)
348 w
349 eps = 1/2 * (F + F.') - eye(2)
350 detEps = simplify(det(eps))

```

```

351 eps = [eps(1),eps(4),eps(2)*2] '
352 % No easy strains!
353 disp('=====')
354
355 %% 632-6 way 0 degrees
356 % Uni Mode
357 disp('=====UNI MODE=====')
358 assume(a, 'positive')
359 % theta = sym(0);
360 % phi = sym(pi/3);
361
362 g = simplify(subs(d3==d2-d1));
363 S = solve(g);
364
365 Dtemp = simplify(subs(D,S));
366 F = simplify(subs(Dtemp / L))
367 [v,w] = eig(F);
368 v=simplify(v)
369 w
370 eps = 1/2 * (F + F.') - eye(2)
371 detEps = simplify(det(eps))
372 eps = [eps(1),eps(4),eps(2)*2] '
373 disp('=====')
374
375 %% 632-3 way Lattice setup
376 syms theta phi
377 eta = sqrt(sym(3))/3;
378 % eta = 1;
379 na = eta * [cos(phi/2),sin(phi/2)].';
380 nb = eta * [0, 1].';
381 nc = eta * [cos(phi/2),-sin(phi/2)].';
382 da = a * R(theta) * na;
383 db = b * R(theta) * nb;
384 dc = c * R(theta) * nc;
385 l1 = na + nc;
386 l2 = na + nb;

```



```

387 d1 = da + dc;
388 d2 = da + db;
389 D = simplify([l1 + d1, l2 + d2]);
390 %% 632-3 way 90 degrees
391 % Bi Mode
392 disp('=====BI MODE=====')
393
394 theta = sym(pi/2);
395 phi = sym(pi/3);
396
397 Dtemp = simplify(subs(D));
398 F = simplify(subs(Dtemp / L))
399 [v,w] = eig(F);
400 v=simplify(v)
401 w
402 eps = simplify(1/2 * ( F + F.') - eye(2))
403 detEps = simplify(det(eps))
404 eps = [eps(1),eps(4),eps(2)*2] '
405 disp('=====')
406
407 %% 632-3 way 0 degrees
408 % Tri Mode
409 disp('=====TRI MODE=====')
410
411 theta = sym(0);
412 phi = sym(pi/3);
413
414 Dtemp = simplify(subs(D));
415 F = simplify(subs(Dtemp/L))
416 [v,w] = eig(F);
417 v=simplify(v)
418 w
419 eps = simplify(1/2*(F+F.') - eye(2))
420 detEps=simplify(det(eps))
421 eps = [eps(1),eps(4),eps(2)*2] '
422 disp('=====')

```

```

423
424 %%
425 function eps = cauchy_strain(F)
426     assert(all(size(F)==2), "F is not 2x2 matrix")
427     eps = simplify(1/2 * (F + F.') - eye(2));
428 end
429
430 function hfig = plot_displacement(L, D, varargin)
431     L = subs(L);
432     plot([0, L(1)], [0, L(2)], "Marker", "o")
433     hold on
434     plot([0, L(3)], [0, L(4)], "Marker", "o")
435
436     set(gca, "XLim", [-.2, 1.2])
437     set(gca, "YLim", [-.2, 1.2])
438     axis square
439 end
440
441 function T = mandelT(theta)
442 c = cos(theta); s = sin(theta);
443 T = [ c^2, s^2, sqrt(2)*s*c ;
444       s^2, c^2, -sqrt(2)*s*c ;
445       -sqrt(2)*s*c, sqrt(2)*s*c, cos(2*theta) ];
446 end

```