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An Electrodynamic Inverse Problem  
in Chiral Media

by

Stephen R. McDowall

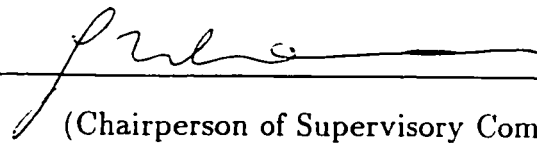
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Abstract

An Electrodynamic Inverse Problem  
in Chiral Media

by Stephen R. McDowall

Chairperson of Supervisory Committee

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We consider the inverse problem of determining the electromagnetic material parameters of a body from information obtainable only at the boundary of the body; such information comes in the form of a boundary map which we assume to be known. In particular we consider the question in the case of a chiral body. In such a body, the relationship between the electromagnetic fields depends not only on the conductivity, electric permittivity and magnetic permeability of the body, but further on the chirality.

We consider two problems. The first is determination of the parameters and their normal derivatives at the boundary of the body. We show that in both the chiral and non-chiral cases, such information is obtainable for all the parameters. We also show how a layer stripping algorithm may be derived to estimate the unknown parameters near the boundary in both situations. The approach is to calculate an explicit asymptotic expansion for the symbol of the boundary map which is shown to be a pseudo-differential operator; this expansion is shown in each case to determine the unknown parameters at the boundary.

The second problem is that of interior determination. We show that knowledge of the boundary map determines the electromagnetic parameters in the interior under the assumption that we know the parameters to infinite order at the boundary. We rewrite Maxwell's equations as a first order perturbation of the Laplacian and construct exponentially growing solutions, and obtain the result in the spirit of complex geometrical optics.

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## Chapter 1

### INTRODUCTION

The inverse problem considered herein is an *inverse boundary value problem*. Such a problem may be described as a problem of “non-destructive determination” which refers to determining information about the interior of a body without the need to destroy it in the process. The information one has to work with comes in the form of boundary measurements. An example is Electrical Impedance Tomography: Suppose you have a conducting body and you wish to determine how the conductivity varies throughout the body. By applying voltage potentials to the boundary and measuring the induced current flux at the boundary, can one reconstruct, or uniquely identify, the function which describes the conductivity? In [17], Sylvester and Uhlmann proved that the conductivity of a body can be uniquely identified from such information, obtained only from the boundary. It was assumed that the conductivity has certain smoothness and that it depends only on the position in the body, not on the direction through that position. That is to say, the conductivity is assumed to be *isotropic* and is thus a scalar valued function. All results of this work are for isotropic parameters. The questions of identifiability of anisotropic parameters are still largely open.

If instead of applying a constant (time-independent) potential the body is immersed into a time-dependent electric field, a time-dependent magnetic field is induced. The relationship between these fields is described by Maxwell’s equations, which depend on the electromagnetic parameters of the body (for example, the conductivity). In [16] Somersalo et al. presented a boundary map for time-harmonic

fields at a fixed frequency which maps the tangential component of the electric field on the boundary to the tangential component of the induced magnetic field on the boundary, and raised the question of whether the parameters describing the electromagnetic properties of the body could be determined from knowledge of this boundary map. They showed that these parameters could be recovered approximately provided they differed only slightly from known constants. In [11] this assumption was dropped and it was shown that the parameters are recoverable provided they are known in a small neighborhood of the boundary of the body.

In all these treatments, the constituent equations, which describe the dependence of the electric displacement and the magnetic induction on the electric and magnetic fields, do not take into account the *chirality* of the body. Instead, they depend only on the conductivity, electric permittivity and magnetic permeability of the body. Chirality is an asymmetry in the molecular structure; a molecule is chiral if it cannot be superimposed onto its mirror image, and a chiral material is one which has chiral molecules in its molecular structure. A basic example of a chiral molecule is a carbon based molecule, which is tetrahedral in structure; if there are four different atoms or molecules attached to a central carbon atom, interchanging any two results in a molecule which differs from the original in its handedness. Presence of chirality results in the rotation of electromagnetic fields. For example, if plane-polarized light is passed through a chiral liquid, the plane of polarization is rotated and the degree of rotation depends on the chirality of the liquid. Chirality is experimentally observable, particularly in the microwave range, and such experimental observations are used in physical chemistry to characterize molecular structures. For a detailed treatment of chirality and time-harmonic electromagnetic fields see [5].

In this work we treat the case of a chiral body, and so the constituent equations depend on a fourth parameter  $\beta$  which describes this chirality.

To have a well posed inverse problem for an electromagnetic body we must show that there is a well defined boundary map. For a non-chiral body, a boundary map

was introduced in [16]; in chapter 2 we prove that the analogous boundary map is well defined for a chiral body. The map is again that which maps the tangential component of the electric field on the boundary to the tangential component of the induced magnetic field on the boundary.

In order to prove results about the identifiability of parameters in the interior of a body, it is typically necessary to have some information about the parameters at the boundary. In the case of the conductivity problem, it was necessary to know the conductivity and its first normal derivative at the boundary in order to prove the interior result. For an electromagnetic body, the result of [11] assumed knowledge of the parameters of the body in a neighborhood of the boundary. In chapter 3 we show that knowledge of the boundary map for an electromagnetic body determines the electromagnetic parameters and their first normal derivatives at the boundary. We do this for both non-chiral and chiral bodies.

The main result is that of chapter 4 in which we show that knowledge of the boundary map for a chiral body uniquely determines all four electromagnetic parameters of the body, in particular the chirality. As mentioned earlier, all parameters are assumed to be isotropic, and we assume them to be smooth throughout the body. The precise statement of the result is given in chapter 4, and an outline of the proof is in the first section of that chapter.

## Chapter 2

### THE BOUNDARY ADMITTANCE MAP

Let  $\Omega$  be a bounded connected subset of  $\mathbb{R}^3$  with connected complement and with smooth boundary  $\partial\Omega$ . We restrict our interest to *time-harmonic* electromagnetic fields on  $\Omega$ , at fixed frequency  $\omega$ , i.e. if  $E$  and  $H$  are the electric and magnetic fields respectively then

$$E = e^{i\omega t} E(x), \quad H = e^{i\omega t} H(x).$$

For such time-harmonic fields, Maxwell's equations are

$$\nabla \wedge E = i\omega B, \quad \nabla \wedge H = -i\omega D. \quad (2.1)$$

Using the Born-Fedorov formulation for a chiral body, (see [5]) the magnetic induction  $B$  and the electric displacement  $D$  are related to  $E$  and  $H$  through the constituent equations

$$B = \tilde{\mu}(H + \tilde{\beta}\nabla \wedge H), \quad D = \tilde{\epsilon}(E + \tilde{\beta}\nabla \wedge E).$$

Here  $\tilde{\epsilon} = \sigma + (i/\omega)\gamma$  where  $\sigma$  is the electric permittivity and  $\gamma$  is the conductivity, and  $\tilde{\mu}$  is the magnetic permeability of the body. The chirality of the body is described by  $\tilde{\beta}$ . The parameters  $\sigma$ ,  $\gamma$ ,  $\tilde{\mu}$  and  $\tilde{\beta}$  are real-valued and we assume here that  $\tilde{\epsilon}$ ,  $\tilde{\mu}$  and  $\tilde{\beta}$  are smooth and are constant outside a compact set. We assume

$$\sigma \geq \sigma_0 > 0, \quad \gamma \geq 0, \quad \tilde{\mu} \geq \tilde{\mu}_0 > 0 \quad (2.2)$$

for constants  $\sigma_0$  and  $\tilde{\mu}_0$ . We shall be using an equivalent formulation but with

$$\epsilon = \frac{\tilde{\epsilon}}{1 - \omega^2 \tilde{\epsilon} \tilde{\mu} \tilde{\beta}^2}, \quad \mu = \frac{\tilde{\mu}}{1 - \omega^2 \tilde{\epsilon} \tilde{\mu} \tilde{\beta}^2}, \quad \beta = \frac{-i\omega \tilde{\epsilon} \tilde{\mu} \tilde{\beta}}{1 - \omega^2 \tilde{\epsilon} \tilde{\mu} \tilde{\beta}^2}.$$

We are assuming that  $1 - \omega^2 \bar{\varepsilon} \bar{\mu} \bar{\beta}^2 \neq 0$ ; this means that we assume that the electric and magnetic fields never become parallel. Given the bounds (2.2), there is  $\omega_0 > 0$  such that this assumption is satisfied for  $\omega \in (-\omega_0, \omega_0)$ ; if  $\omega \in (0, \omega_0)$ , then  $\varepsilon$  and  $\mu$  are bounded away from zero as for  $\bar{\varepsilon}$  and  $\bar{\mu}$ . With this change of parameters, we have the constituent equations

$$B = \mu H - \beta E, \quad D = \varepsilon E + \beta H. \quad (2.3)$$

We are assuming further that there are no magnetic poles or electric sinks or sources in  $\Omega$ ; that is to say we assume the induction and displacement to be divergence free:

$$\nabla \cdot B = \nabla \cdot (\mu H - \beta E) = 0, \quad \nabla \cdot H = \nabla \cdot (\varepsilon E + \beta H) = 0. \quad (2.4)$$

We remark that  $1 - \omega^2 \bar{\varepsilon} \bar{\mu} \bar{\beta}^2 \neq 0$  is equivalent to  $\varepsilon \mu + \beta^2 \neq 0$ .

If  $F$  is a function space, we denote by  $F^k$  the space of  $k$ -vectors whose components are in  $F$ , and by  $F^{k \times k}$  the space of  $k \times k$  matrices whose components are in  $F$ . We shall need the following function spaces:  $H^s(\Omega)^k$  consists of  $k$ -dimensional vector fields whose components are in the usual  $L^2$ -based Sobolev space  $H^s$ . Let  $\text{Div}$  denote the surface divergence on the boundary of  $\Omega$ , and  $\nu(x)$  be the outward unit normal vector at  $x \in \partial\Omega$ , and define the following space of tangential fields:

$$TH_{\text{Div}}^{\frac{1}{2}}(\partial\Omega) = \left\{ F \in H^{\frac{1}{2}}(\partial\Omega)^3 \mid \nu \cdot F = 0, \text{ and } \text{Div} F \in H^{\frac{1}{2}}(\partial\Omega) \right\}.$$

**THEOREM 2.1.** *Let  $F \in TH_{\text{Div}}^{\frac{1}{2}}(\partial\Omega)$ . There is a discrete set  $D$  containing no limit points in  $(0, \omega_0)$  such that for all  $\omega \in (0, \omega_0) \setminus D$  there exist unique  $(E, H) \in \mathcal{D}(\Omega)^3 \times \mathcal{D}(\Omega)^3$  solving the following boundary value problem:*

$$\left. \begin{aligned} \nabla \wedge E &= i\omega(\mu H - \beta E) \\ \nabla \wedge H &= -i\omega(\varepsilon E + \beta H) \\ \nu \wedge E|_{\partial\Omega} &= F. \end{aligned} \right\} \quad (2.5)$$

We may thus define the boundary map which, following [15], we term the *boundary admittance map*  $\Pi : TH_{\text{Div}}^{\frac{1}{2}}(\partial\Omega) \rightarrow TH_{\text{Div}}^{\frac{1}{2}}(\partial\Omega)$ . Given  $F \in TH_{\text{Div}}^{\frac{1}{2}}(\partial\Omega)$  let  $(E, H)$  solve (2.5) and define

$$\Pi F = \Pi(\nu \wedge E|_{\partial\Omega}) = \nu \wedge H|_{\partial\Omega}.$$

**Proof of Theorem 2.1:** We define the function spaces

$$\begin{aligned} H(\nabla\wedge) &= \{E \in L^2(\Omega)^3 \mid \nabla\wedge E \in L^2(\Omega)^3\} \\ \mathring{H}(\nabla\wedge) &= \left\{ E \in H(\nabla\wedge) \mid \int_{\Omega} \nabla\wedge E \cdot F = \int_{\Omega} E \cdot \nabla\wedge F \text{ for all } F \in H(\nabla\wedge) \right\} \end{aligned}$$

We shall use the equivalent Born-Fedorov formulation

$$\begin{aligned} \nabla\wedge E &= i\omega\tilde{\mu}H + i\omega\tilde{\mu}\tilde{\beta}\nabla\wedge H \\ \nabla\wedge H &= -i\omega\tilde{\varepsilon}E - i\omega\tilde{\varepsilon}\tilde{\beta}\nabla\wedge E \end{aligned}$$

which, following the presentation of [16], we may write as

$$(L - \omega - B) \begin{pmatrix} E \\ H \end{pmatrix} = 0$$

where

$$\begin{aligned} L &= \begin{bmatrix} -i\nabla\wedge & 0 \\ 0 & -i\nabla\wedge \end{bmatrix} : \mathcal{D}(L) \rightarrow L^2(\Omega)^3 \times L^2(\Omega)^3, \quad \text{and} \\ B &= \frac{\omega}{\omega^2\tilde{\varepsilon}\tilde{\mu}\tilde{\beta}^2 - 1} \begin{bmatrix} 1 & -\tilde{\mu} \\ \tilde{\varepsilon} & 1 \end{bmatrix}. \end{aligned}$$

The domain of  $L$  is  $\mathcal{D}(L) = \mathring{H}(\nabla\wedge) \times H(\nabla\wedge)$ ; on  $\mathcal{D}(L)$ ,  $L$  is self-adjoint (see [6]). In order to solve Maxwell's equations with

$$\nu \wedge E|_{\partial\Omega} = F \in TH_{\text{Div}}^{\frac{1}{2}}(\partial\Omega)$$

we write  $\tilde{E} = E - RF$  where  $R$  is the right inverse of the tangential trace mapping

$$\text{tr} : H^1(\Omega)^3 \rightarrow TH_{\text{Div}}^{\frac{1}{2}}(\partial\Omega), \quad \text{tr} : E \mapsto \nu \wedge E|_{\partial\Omega}.$$

Then the boundary value problem may be written

$$(L - \omega - B) \begin{pmatrix} \tilde{E} \\ H \end{pmatrix} = \begin{pmatrix} J \\ K \end{pmatrix}$$

where

$$J = i\nabla \wedge RF + \frac{\omega^3 \tilde{\epsilon} \tilde{\mu} \tilde{\beta}^2}{\omega^2 \tilde{\epsilon} \tilde{\mu} \tilde{\beta}^2 - 1} RF, \quad K = \frac{\omega \tilde{\epsilon}}{\omega^2 \tilde{\epsilon} \tilde{\mu} \tilde{\beta}^2 - 1} RF$$

LEMMA 2.2. *The range of  $L$  is closed, the mapping  $L^{-1} : \mathcal{R}(L) \rightarrow \mathcal{R}(L) \cap \mathcal{D}(L)$  exists, and  $L^{-1}$  is continuous and compact.*

This is proven in [6]. From this we have

$$L^2(\Omega)^3 \times L^2(\Omega)^3 = \text{Ker}(L) \oplus \mathcal{R}(L)$$

and there is a discrete set  $\mathcal{S} \subset \mathbb{R}$  containing no limit points such that  $(L - \omega)^{-1}$  exists and is compact for all  $\omega \in \mathbb{C} \setminus \mathcal{S}$ . The compactness follows from

$$(L - \omega)^{-1} = L^{-1} + \omega L^{-1}(L - \omega)^{-1}$$

and the compactness of  $L^{-1}$ . For  $\omega \notin \mathcal{S}$ , we want to solve

$$(I - (L - \omega)^{-1}B) \begin{pmatrix} \tilde{E} \\ H \end{pmatrix} = (L - \omega)^{-1} \begin{pmatrix} J \\ K \end{pmatrix}.$$

Recall that  $\omega^2 \tilde{\epsilon} \tilde{\mu} \tilde{\beta}^2 - 1 \neq 0$  for  $\omega \in \mathcal{A} = \mathbb{C} \setminus \{\omega \mid \omega \in \mathbb{R}, |\omega| \geq \omega_0\}$ , so on  $\mathcal{A} \setminus \mathcal{S}$ ,  $B$  is analytic and  $(L - \omega)^{-1}$  exists: at  $\omega = 0$ ,  $B = 0$ , so by the analytic Fredholm theorem (for example [13]),  $(I - (L - \omega)^{-1}B)^{-1}$  exists for all  $\omega \in \mathcal{A} \setminus (\mathcal{S} \cup \mathcal{S}')$  for some discrete set  $\mathcal{S}'$  containing no limit points in  $\mathcal{A} \setminus \mathcal{S}$ . The theorem follows with  $D = \mathcal{S} \cup \mathcal{S}'$ .  $\square$

## Chapter 3

# BOUNDARY DETERMINATION

### 3.1 Introduction

In this chapter we consider the inverse problem of determining the electromagnetic material parameters at the *boundary* of the body from knowledge of the boundary admittance map defined in the previous chapter. The approach is as in [3] where knowledge of the Dirichlet-to-Neumann map was shown to determine the conductivity and all its derivatives at the boundary of a conducting body subjected to a time-independent electric potential. The result in [3] was obtained for anisotropic conductivities; here we shall restrict ourselves to isotropic parameters. The results of this chapter are in [7].

The chapter is divided into two main sections. In section 3.2 we treat the situation where the body  $\Omega \subset \mathbb{R}^3$  has defined on it three material parameters, its conductivity, electric permittivity, and magnetic permeability. In section 3.3 we consider the situation of a chiral body.

We now define the problems more precisely. Take  $\Omega$  to be a non-chiral body, a smoothly bounded subset of  $\mathbb{R}^3$ . We shall assume that the conductivity  $\gamma$ , the electric permittivity  $\sigma$ , and the magnetic permeability  $\mu$  are smooth functions satisfying the following conditions:

$$\gamma \geq 0, \quad \sigma \geq \sigma_0 > 0, \quad \mu \geq \mu_0 > 0$$

throughout  $\Omega$ . Suppose that a time-harmonic electric field with fixed frequency  $\omega$  is applied to  $\Omega$  and that the tangential component of the induced magnetic field at the

boundary of  $\Omega$  is recorded. The electric field  $E$  and magnetic field  $H$  are related by Maxwell's equations, which in Euclidean coordinates take the form

$$\nabla \wedge H = (\gamma - i\omega\sigma)E, \quad \nabla \wedge E = i\omega\mu H$$

where  $\wedge$  denotes the  $\mathbb{R}^3$  vector product. The boundary condition is  $\nu \wedge E|_{\partial\Omega} = F$ , the tangential component of the applied field at the boundary (see section 3.2.2). The tangential component of the magnetic field is then  $\nu \wedge H$ . The boundary admittance map, defined as in chapter 2, is

$$\Lambda : F = \nu \wedge E|_{\partial\Omega} \mapsto \nu \wedge H|_{\partial\Omega}.$$

The inverse problem being considered for the non-chiral body is then the following: Suppose that we know the map  $\Lambda$ . From this information can we determine the unknown parameters  $\gamma$ ,  $\sigma$  and  $\mu$  at the boundary of  $\Omega$ ? This question has been answered in the affirmative in [15]. There Somersalo calculates the principal symbol of  $\Lambda$ , and that of the *impedance map* which, when restricted to the Div-spaces, is  $\Lambda^{-1}$ . Here we use an alternate method to calculate not only the principal symbol, but terms of lower orders of homogeneity in an asymptotic expansion for the full symbol of  $\Lambda$ . This approach yields the determination of not only the parameters, but further their normal derivatives at the boundary of  $\Omega$ . Furthermore, the technique can be applied to the case of chiral media, and in section 3.3 we prove the analogous boundary determination for a chiral body. In the case of a non-chiral body, *interior* determination of parameters was shown to be possible from the boundary map, assuming prior knowledge of the parameters near the boundary (see [11]). To remove this assumption in the unique determination of parameters throughout  $\Omega$ , it suffices to determine the parameters and their first normal derivatives at the boundary; the result of section 3.2 shows that such information is obtainable from  $\Lambda$ .

*Remark.* A Riccati-type differential equation for  $\Lambda$  can be derived by way of equations (3.7) and (3.15) (see sections 3.2.1 and 3.2.2), and one may derive a “layer

stripping” algorithm to estimate the parameters near the boundary, as was done in [15]. In section 3.3 we point out the analogous derivation of a Riccati equation for the boundary map in the case of a chiral body.

Section 3.2 is a treatment of the case of a non-chiral body and the details of the technique are presented. In section 3.2.1 we rewrite Maxwell’s equations in local coordinates as a second order system and derive a factorization by pseudodifferential operators. Section 3.2.2 expresses the admittance map in terms of these operators, and in section 3.2.3 we prove determination of the material parameters by way of calculating the symbol of  $\Lambda$  as a pseudodifferential operator.

Section 3.3 follows the format of 3.2, but in the case of a chiral body. The precise formulation of the problem is left to that section.

## 3.2 Non-Chiral Media

### 3.2.1 A Factorization of Maxwell’s Equations

Let  $\Omega$  be a bounded subset of  $\mathbb{R}^3$  with smooth boundary. Our treatment is local to a point  $p$  in the boundary of  $\Omega$ , and for the moment assume that  $p = 0 \in \partial\Omega$ ,  $\Omega \subset \{x_3 > 0\}$  locally, and that  $\partial\Omega$  is locally characterized by  $x_3 = 0$ . We consider a general boundary point at the end of section 3.2. Let  $(E, H) \in \mathcal{D}'(\Omega)^3 \times \mathcal{D}'(\Omega)^3$  be time harmonic electromagnetic fields at frequency  $\omega$ . In these Euclidean coordinates, such time-harmonic fields are related via the following form of Maxwell’s equations:

$$\nabla \wedge H = (\gamma - i\omega\sigma)E = -i\omega\varepsilon E \quad (3.1)$$

$$\nabla \wedge E = i\omega\mu H \quad (3.2)$$

Here we use  $\wedge$  to denote the  $\mathbb{R}^3$  vector product, and define  $i\omega\varepsilon = i\omega\sigma - \gamma$ . We also assume that

$$\nabla \cdot (\varepsilon E) = 0, \quad \nabla \cdot (\mu H) = 0. \quad (3.3)$$

which physically means that there are no electric charge sources or magnetic poles. Substituting (3.2) into (3.1) we obtain

$$-\Delta E + \nabla(\nabla \cdot E) - (\nabla \log \mu) \wedge \nabla \wedge E - \omega^2 \mu \varepsilon E = 0.$$

But  $\nabla \cdot (\varepsilon E) = 0$ , or  $\nabla \cdot E = -E \cdot \nabla \log \varepsilon$ , and so

$$-\Delta E - \nabla(E \cdot \nabla \log \varepsilon) - (\nabla \log \mu) \wedge \nabla \wedge E - \omega^2 \mu \varepsilon E = 0. \quad (3.4)$$

We introduce some notation: write  $\nabla \log \varepsilon = (d\varepsilon_1, d\varepsilon_2, d\varepsilon_3)$ ,  $\nabla \log \mu = (d\mu_1, d\mu_2, d\mu_3)$  and let  $\partial_j$  denote  $\partial x_j$  and  $D_x$ , denote  $-i\partial_j$ ,  $j = 1, 2, 3$ . Let  $I$  denote the  $3 \times 3$  identity matrix. Then if  $x' = (x_1, x_2)$  we may write (3.4) as

$$\mathcal{M}(x, D)(E) = (D_{x_3}^2 I - \Delta' I - M(x, D_{x'}) - iP(x)D_{x_3} - R(x)) E = 0 \quad (3.5)$$

with  $M$  a  $3 \times 3$  system of differential operators of order one in  $x'$  and depending smoothly on  $x_3$ , and  $P(x)$  and  $R(x)$  zero order matrix multipliers. Here,  $\Delta'$  is the two dimensional Laplacian in  $x_1$  and  $x_2$ .

**PROPOSITION 3.1.** *There is a pseudodifferential operator  $B(x, D_{x'})$  of order one in  $x'$  and depending smoothly on  $x_3$  such that*

$$\mathcal{M}(x, D_{x'}) = (D_{x_3} I - iP(x) - iB(x, D_{x'}))(D_{x_3} I + iB(x, D_{x'})) \quad (3.6)$$

*modulo a smoothing operator.*

For definitions and properties of pseudodifferential operators see [20].

**Proof:** We prove the existence of  $B$  by explicitly deriving its asymptotic expansion, which is made use of in the sequel. From (3.5) and (3.6), we have

$$\Delta' + i[D_{x_3}, B] + M + PB + B^2 + R = 0. \quad (3.7)$$

Let  $B(x, D_{x'}) = (B_{jk})$  have symbol  $b(x, \xi') = (b_{jk})$ . Here  $\xi_j$  is the dual variable to  $D_{x_j}$ . Recall that the symbol of  $B_{jl}B_{lk}$  is  $\sum_{\alpha} \frac{1}{\alpha!} (\partial_{\xi'}^{\alpha} b_{jl})(D_{x'}^{\alpha} b_{lk})$ , and so the symbol of  $B^2$  is the matrix

$$\left( \sum_l \sum_{\alpha} \frac{1}{\alpha!} (\partial_{\xi'}^{\alpha} b_{jl})(D_{x'}^{\alpha} b_{lk}) \right)$$

where  $j$  and  $k$  index the components of the matrix. Similarly one finds that the symbol of  $[D_{x_n}, B] = ([D_{x_n}, B_{jk}])$  is

$$\left( \frac{1}{i} \frac{\partial b_{jk}}{\partial x_n} \right).$$

The symbol of  $M$  is

$$m(x, \xi') = (m_{jk}) = \begin{pmatrix} i(d\varepsilon_1 \xi_1 - d\mu_2 \xi_2) & i(d\varepsilon_2 + d\mu_2) \xi_1 & i(d\varepsilon_3 + d\mu_3) \xi_1 \\ i(d\varepsilon_1 + d\mu_1) \xi_2 & i(d\varepsilon_2 \xi_2 - d\mu_1 \xi_1) & i(d\varepsilon_3 + d\mu_3) \xi_2 \\ 0 & 0 & -i(d\mu_1 \xi_1 + d\mu_2 \xi_2) \end{pmatrix},$$

and in terms of symbols. (3.7) becomes

$$-|\xi'|^2 \delta_{jk} + \partial_3 b_{jk} + m_{jk} + \sum_l P_{jl} b_{lk} + \sum_l \sum_\alpha \frac{1}{\alpha!} (\partial_{\xi'}^\alpha b_{jl}) (D_x^\alpha b_{lk}) + R_{jk} = 0. \quad (3.8)$$

We write

$$b_{jk} \sim \sum_{q \leq 1} b_{jk}^{(q)}(x, \xi')$$

with  $b_{jk}^{(q)}$  homogeneous of degree  $q$  in  $\xi'$ , and will determine the  $b_{jk}^{(q)}$  inductively in  $q$ , thus proving the proposition. To this end, the terms in (3.8) with homogeneity of order two give

$$-|\xi'|^2 \delta_{jk} + \sum_l b_{jl}^{(1)} b_{lk}^{(1)} = -|\xi'|^2 I + (b_{jk}^{(1)})^2 = 0$$

and we choose the solution  $(b_{jk}^{(1)}) = -|\xi'| I$ . Next the terms homogeneous of order one, together with  $b_{jk}^{(1)} = -|\xi'| \delta_{jk}$  give

$$\begin{aligned} 0 &= \partial_3 b_{jk}^{(1)} + m_{jk} + \sum_l P_{jl} b_{lk}^{(1)} + \sum_l (b_{jl}^{(1)} b_{lk}^{(0)} + b_{jl}^{(0)} b_{lk}^{(1)}) + \sum_{|\alpha|=1} \partial_{\xi'}^\alpha b_{jl}^{(1)} D_x^\alpha b_{lk}^{(1)} \\ &= m_{jk} - P_{jk} |\xi'| - 2|\xi'| b_{jk}^{(0)}. \end{aligned}$$

Since we are solving only modulo smoothing, we put

$$b_{jk}^{(0)} = \frac{m_{jk} - P_{jk} |\xi'|}{2|\xi'|}. \quad (3.9)$$

Continuing inductively, if  $q < 0$  (the case  $q = 0$  is similar), the terms homogeneous of order  $q$  in (3.8) are

$$0 = \partial_3 b_{jk}^{(q)} + \sum_l P_{jl} b_{lk}^{(q)} + \sum_l \sum_{0 \leq |\alpha| \leq -q+2} \frac{1}{\alpha!} \sum_{q+|\alpha|-1 \leq s \leq 1} \partial_{\xi'}^\alpha b_{jl}^{(s)} D_x^\alpha b_{lk}^{(q+|\alpha|-s)}.$$

The only terms involving  $b_{jk}^{(q-1)}$  are when  $|\alpha| = 0$  and  $s = 1$  or  $s = q - 1$  in which case

$$\sum_l (b_{jl}^{(1)} b_{lk}^{(q-1)} + b_{jl}^{(q-1)} b_{lk}^{(1)}) = -2|\xi'| b_{jk}^{(q-1)}$$

and thus we set  $b_{jk}^{(q-1)} =$

$$\frac{1}{2|\xi'|} \left( \partial_3 b_{jk}^{(q)} + \sum_l P_{jl} b_{lk}^{(q)} - \sum_l \sum_{0 \leq |\alpha| \leq -q+2} \frac{1}{\alpha!} \sum_{\substack{q+|\alpha|-1 \leq s \leq 1 \\ s \neq 1, q-1 \text{ when } |\alpha|=0}} \partial_{\xi'}^\alpha b_{jl}^{(s)} D_x^\alpha b_{lk}^{(q+|\alpha|-s)} \right).$$

□

### 3.2.2 The Boundary Admittance Map, $\Lambda$

It was shown in [16] that for all but a discrete set of  $\omega > 0$ , with no accumulation points, the following Dirichlet problem has a unique solution: for  $F \in TH_{\text{Div}}^{\frac{1}{2}}(\partial\Omega)$ , let  $(E, H) \in \mathcal{D}'(\Omega)^3 \times \mathcal{D}'(\Omega)^3$  be the solution to

$$\begin{aligned} \nabla \wedge H &= -i\omega \varepsilon E \\ \nabla \wedge E &= i\omega \mu H \quad \nu \wedge E|_{\partial\Omega} = F \end{aligned} \quad (3.10)$$

where  $\nu$  is the outward unit normal to the boundary of  $\Omega$ .

**PROPOSITION 3.2.** *If  $E$  solves (3.10), then*

$$\frac{\partial}{\partial x_3} E \Big|_{\partial\Omega} = BE|_{\partial\Omega} \quad (3.11)$$

*modulo a smoothing operator.*

**Proof:** Certainly,  $\mathcal{M}(x, D)E = 0$ . Let  $(x', x_3)$  be local coordinates, for  $x_3 \in [0, T]$ . Since the principal symbol of  $\mathcal{M}(x, D)$  is  $-|\xi|^2 I$ , the plane  $\{x_3 = 0\}$  is non-characteristic, and so  $\mathcal{M}$  is partially hypoelliptic with respect to this boundary (see [2] p107). Thus since  $E$  solves (3.10),  $E$  is smooth in the normal direction; that is,  $E \in C^\infty([0, T]; \mathcal{D}(\mathbb{R}^2))^3$  locally. By Proposition (3.1), (3.10) is locally equivalent to

$$(D_{x_3} I + iB)E = U, \quad \nu \wedge E|_{x_3=0} = F, \quad (3.12)$$

$$(D_{x_3} I - iP - iB)U = W \in C^\infty([0, T] \times \mathbb{R}^2)^3 \quad (3.13)$$

with  $E$  and  $U$  in  $C^\infty([0, T]; \mathcal{D}(\mathbb{R}^2))^3$  (again,  $(D_{x_3} I + iB)$  is hypoelliptic with respect to the boundary). We may view (3.13) as a backwards generalized heat equation; indeed with  $t = T - x_n$ , we have

$$\frac{\partial U}{\partial t} - (B + P)U = -iW. \quad (3.14)$$

By interior regularity for  $\mathcal{M}(x, D)$ ,  $E$  is smooth in the interior of  $\Omega$ , and hence so is  $U$ ; in particular,  $U|_{x_3=T}$  is smooth. Now the principal symbol of  $B$  is  $b^1 = -|\xi'| I$  and so the solution operator for (3.14) is smoothing for  $t > 0$  (see [19] p134), and

$$(D_{x_3} I + iB)E = U \in C^\infty([0, T] \times \mathbb{R}^2)^3$$

locally. In particular,

$$D_{x_3} E|_{x_3=0} = -i B E|_{x_3=0} + U|_{x_3=0}$$

which completes the proof since  $U|_{x_3=0}$  is smooth.  $\square$

Recall that if  $(E, H)$  solves (3.10) then the admittance map  $\Lambda = \Lambda(\gamma, \sigma, \mu)$  is the map  $\Lambda : \nu \wedge E|_{\partial\Omega} \mapsto \nu \wedge H|_{\partial\Omega}$  where  $\nu = (0, 0, -1)$  is the outward unit normal to the boundary  $\partial\Omega$ . From (3.2),

$$\begin{pmatrix} E_2 \\ -E_1 \\ 0 \end{pmatrix} \Big|_{x_3=0} \xrightarrow{\Lambda} \frac{1}{i\omega\mu} \begin{pmatrix} \partial_3 E_1 - \partial_1 E_3 \\ \partial_3 E_2 - \partial_2 E_3 \\ 0 \end{pmatrix} \Big|_{x_3=0}.$$

By Proposition 3.2.

$$\partial_3 E_j = B_{j1} E_1 + B_{j2} E_2 + B_{j3} E_3, \quad j = 1, 2, 3,$$

and from (3.3)

$$\partial_3 \varepsilon E_3 + \varepsilon \partial_3 E_3 = -(\varepsilon \partial_1 + \partial_1 \varepsilon) E_1 - (\varepsilon \partial_2 + \partial_2 \varepsilon) E_2;$$

combining these, we have

$$(\partial_3 \varepsilon + \varepsilon B_{33}) E_3 = -(\varepsilon B_{31} + \varepsilon \partial_1 + \partial_1 \varepsilon) E_1 - (\varepsilon B_{32} + \varepsilon \partial_2 + \partial_2 \varepsilon) E_2.$$

Let  $J(x, D_{x'}) = (\partial_3 \varepsilon + \varepsilon B_{33})$ , and  $K(x, D_{x'})$  be a pseudodifferential operator of order -1 in  $x'$  such that the composition  $KJ$  is the identity modulo smoothing. Then  $\Lambda$  is given by the  $2 \times 2$  system with components for  $j = 1, 2$

$$\begin{aligned} \Lambda_{j1} &= \frac{1}{i\omega\mu} ((\partial_j K - B_{j3} K)(\varepsilon B_{32} + \varepsilon \partial_2 + \partial_2 \varepsilon) + B_{j2}) \\ \Lambda_{j2} &= \frac{1}{i\omega\mu} ((-\partial_j K + B_{j3} K)(\varepsilon B_{31} + \varepsilon \partial_1 + \partial_1 \varepsilon) - B_{j1}). \end{aligned} \quad (3.15)$$

We write the symbol of  $J$  as  $j(x, \xi') \sim \sum_{l \leq 1} j_l(x, \xi')$  with  $j_l$  homogeneous of degree  $l$  in  $\xi'$  and given by

$$j_1 = -\varepsilon |\xi'|, \quad j_0 = \partial_3 \varepsilon + \varepsilon b_{33}^{(0)}, \quad j_l = \varepsilon b_{33}^{(l)}, \quad l < 0.$$

To compute the symbol  $k$  of  $K$  modulo  $S^{-\infty}$ , we write  $k \sim \sum_{r \leq -1} k_r(x, \xi')$  and consider the terms of decreasing homogeneity in the identity

$$1 = \sum_{|\alpha| \geq 0} \frac{1}{\alpha!} (\partial_{\xi'}^\alpha k(x, \xi')) (D_{x'}^\alpha j(x, \xi')) = \text{the symbol of } KJ.$$

### 3.2.3 The Symbol of $\Lambda$ and Boundary Determination of $\mu, \sigma, \gamma$

In this section we show that knowledge of the boundary admittance map  $\Lambda$  is sufficient to determine the unknown parameters  $\sigma, \gamma$  and  $\mu$  on the boundary of  $\Omega$ . Let

$$\lambda(x', \xi') = (\lambda_{jk}(x', \xi')) \sim \left( \sum_{q \leq 1} \lambda_{jk}^{(q)} \right)$$

be the symbol of  $\Lambda$ . Since we are assuming complete knowledge of  $\Lambda$ , we know the full symbol  $\lambda(x', \xi')$  for all  $(x', \xi') \in \mathbb{R}^2 \times \mathbb{R}^2$ .

First, calculating the terms of homogeneity one in (3.15), one finds that the principal symbol of  $\Lambda$  is

$$\lambda^{(1)}(x', \xi') = \frac{-1}{i\omega\mu(x', 0)|\xi'|} \begin{pmatrix} -\xi_1\xi_2 & -\xi_2^2 \\ \xi_1^2 & \xi_1\xi_2 \end{pmatrix}.$$

Thus the principal symbol of  $\Lambda$  determines  $\mu$  on the boundary. Note that  $\lambda^{(1)}$  then also determines the tangential derivatives of  $\mu$  on the boundary. We remark that this is the negative of the principal symbol calculated in [15], the reason for this being that in [15]  $\xi_j$  is taken to be the dual variable to  $i\partial_j = -D_{x_j}$ .

Next, we calculate the terms of homogeneity zero in (3.15). For example,

$$i\omega\mu\lambda_{11}^{(0)} = i\xi_1\varepsilon k_{-1}b_{32}^{(0)} - \xi_1\xi_2\varepsilon k_{-2} - \xi_1\xi_2 \sum_{|\alpha|=1} \partial_{\xi'}^{\alpha} k_{-1} D_{x', \varepsilon}^{\alpha} + i\xi_1\varepsilon k_{-1} d\varepsilon_2 + b_{12}^{(0)} - i\xi_2 b_{13}^{(0)} \varepsilon k_{-1}.$$

The symbols  $b_{32}^{(0)}$ ,  $b_{12}^{(0)}$  and  $b_{13}^{(0)}$  are given by (3.9), and (3.2.2) yields

$$k_{-1} = \frac{-1}{\varepsilon|\xi'|}, \quad k_{-2} = \frac{i}{2\varepsilon|\xi'|^3} (i d\varepsilon_1 |\xi'| + \xi_1 d\mu_1 + \xi_2 d\mu_2 + 2\xi_1 d\varepsilon_1 + 2\xi_2 d\varepsilon_2)$$

which gives

$$i\omega\mu\lambda_{11}^{(0)}(x', \xi') = \frac{i\xi_1}{2|\xi'|^3} (2|\xi'|^2 d\mu_2 - \xi_1\xi_2 d\mu_1 - \xi_2^2 d\mu_2 + i\xi_2 |\xi'| d\mu_3).$$

Since we have determined  $\mu$ , and hence  $d\mu_1$  and  $d\mu_2$ , on the boundary,  $\lambda_{11}^{(0)}$  determines  $d\mu_3$  there; this is the only new information to be gained from all four of the components of the matrix  $\lambda^{(0)}$ .

*Remark.* A similar analysis of the impedance map  $\Lambda^{-1}$  shows that the two highest terms of homogeneity in the expansion of the symbol for  $\Lambda^{-1}$  determine  $\varepsilon$  and its normal derivative at the boundary.

To determine higher order normal derivatives of the parameters at the boundary, it is necessary to continue and calculate the components of  $\lambda^{(-1)}(x', \xi')$ . The calculation is involved, but is straightforward in manner. To extract the desired information we choose to set  $\xi'$  equal to  $(1,0)$  and then  $(0,1)$ . In what follows,  $f_j(x')$  are *known* functions of  $x'$  (each is defined by functions already determined). We find:

$$2i\omega\mu\lambda_{11}^{(-1)}(x', (1,0)) - f_1 = d\mu_1 d\varepsilon_2 - \partial_1 d\varepsilon_2 \quad (3.16)$$

$$2i\omega\mu\lambda_{11}^{(-1)}(x', (0,1)) - f_2 = -\partial_1 d\varepsilon_2 \quad (3.17)$$

$$\begin{aligned} 2i\omega\mu\lambda_{12}^{(-1)}(x', (1,0)) - f_3 &= -d\mu_1 d\varepsilon_1 - (1+i)(2d\varepsilon_1^2 - d\varepsilon_2^2 - \partial_2 d\varepsilon_2) \\ &\quad - (1-2i)\partial_1 d\varepsilon_1 - 2\omega^2 \mu \varepsilon \end{aligned} \quad (3.18)$$

$$4i\omega\mu\lambda_{12}^{(-1)}(x', (0,1)) - f_4 = -\partial_3 d\mu_3 + 2\partial_1 d\varepsilon_1 - 2\omega^2 \mu \varepsilon \quad (3.19)$$

$$4i\omega\mu\lambda_{21}^{(-1)}(x', (1,0)) - f_5 = \partial_3 d\mu_3 - 2\partial_2 d\varepsilon_2 + 2\omega^2 \mu \varepsilon$$

$$\begin{aligned} 2i\omega\mu\lambda_{21}^{(-1)}(x', (0,1)) - f_6 &= d\mu_2 d\varepsilon_2 + (1+i)(2d\varepsilon_2^2 - d\varepsilon_1^2 - \partial_1 d\varepsilon_1) \\ &\quad + (1+2i)\partial_2 d\varepsilon_2 + 2\omega^2 \mu \varepsilon \end{aligned}$$

$$2i\omega\mu\lambda_{22}^{(-1)}(x', (1,0)) - f_7 = \partial_2 d\varepsilon_1 \quad (3.20)$$

$$2i\omega\mu\lambda_{22}^{(-1)}(x', (0,1)) - f_8 = -d\mu_2 d\varepsilon_1 + \partial_2 d\varepsilon_1 \quad (3.21)$$

It is easy to see that equations (3.16) to (3.21) determine the unknown parameters on the boundary: (3.16) and (3.17) determine  $d\varepsilon_2$ , and (3.20) and (3.21) determine  $d\varepsilon_1$ ; furthermore, the tangential derivatives of these functions are known. With this additional knowledge, (3.18) gives  $\varepsilon$ , and hence  $\sigma$  and  $\gamma$ , and then (3.19) gives  $\partial_3 d\mu_3$ .

Let us summarize these findings: recalling the definitions of  $d\varepsilon_j$  and  $d\mu_j$ , the highest three terms of homogeneity in the asymptotic expansion of  $\lambda$  determine  $\sigma$ ,  $\gamma$ ,  $\mu$  and  $\partial_\nu \mu$  on  $\partial\Omega$ , where  $\partial_\nu$  denotes the normal derivative. Since these functions are known on the boundary, we remark that their tangential derivatives are also known there. We remark further that  $\lambda^{(-1)}$  does not determine the normal derivatives of  $\sigma$  and  $\gamma$ ; it seems reasonable to expect  $\lambda^{(-2)}$  to determine these, however the calculations become considerably more cumbersome. In fact we can expect  $\lambda$  to

determine the derivatives of all the parameters, of all orders, at the boundary.

We now consider the situation where  $\partial\Omega$  is not flat near  $p$ . In local coordinates near  $p$ , if  $E = \sum_j E_j \frac{\partial}{\partial x_j}$ , denote by  $E^b$  the one-form  $E^b = \sum_j E_j dx^j$  obtained via duality. Then Maxwell's equations take the form

$$(*dE^b)^\# = i\omega\mu H, \quad (*dH^b)^\# = -i\omega\varepsilon E$$

where  $*$  is the Hodge-star operator for the metric induced from the Euclidean metric in  $\mathbb{R}^3$ . and  $\#$  reinterprets the one-form as a vector field via duality. We choose local coordinates near  $p$  to be boundary normal coordinates (see [3] p1101) with the coordinates for  $\partial\Omega$  being Riemann normal coordinates. Then  $\partial\Omega$  is locally characterized by  $x_3 = 0$  and the induced metric is Euclidean at  $p$  and has all tangential first derivatives vanishing at  $p$ . With this choice the calculations reduce to those of the flat case for the two highest order terms in the expansion of  $\Lambda$ . Thus by considering  $\Lambda$  and  $\Lambda^{-1}$ , the parameters and their first normal derivatives are obtainable at  $p$ .

### 3.3 Chiral Media

Suppose now that  $\Omega$  is a body with chirality described by a smooth function  $\beta$ . In this section we wish to apply the techniques of the previous section to determine  $\beta$  together with  $\mu$ ,  $\sigma$  and  $\gamma$  on the boundary of  $\Omega$ . The same arguments of section 3.2 regarding a general point on the boundary of  $\Omega$  versus the case of a flat boundary apply here and so we take  $\partial\Omega$  to be  $\{x_3 = 0\}$  near 0. Recall from chapter 2 that for a chiral body Maxwell's equations have the form

$$\nabla \wedge E = i\omega(\mu H - \beta E) \tag{3.22}$$

$$\nabla \wedge H = -i\omega(\varepsilon E + \beta H) \tag{3.23}$$

together with the divergence free conditions

$$\nabla \cdot (\mu H - \beta E) = 0, \quad \nabla \cdot (\varepsilon E + \beta H) = 0. \tag{3.24}$$

Here the parameters  $\mu$ ,  $\varepsilon$  and  $\beta$  come from making a change of variables in the Born-Fedorov formulation of Maxwell's equations, and determination of these parameters is equivalent to determination of the conductivity, electric permittivity, magnetic permeability and chirality. See chapter 2.

The system employed here is not exactly the same as that of section 3.2. It is no longer possible to use the divergence free conditions to decouple the system and find the analogue of equation (3.5) in terms of  $E$  alone. Instead we consider the electric and magnetic fields together. Taking the curl of (3.22) and (3.23), we may write

$$-\Delta \begin{pmatrix} E \\ H \end{pmatrix} + \nabla \begin{pmatrix} \nabla \cdot E \\ \nabla \cdot H \end{pmatrix} + i\omega \begin{pmatrix} -\nabla\mu \wedge H \\ \nabla\varepsilon \wedge E \end{pmatrix} = Z \begin{pmatrix} E \\ H \end{pmatrix} \quad (3.25)$$

where  $Z$  is a zero order matrix multiplier. The conditions (3.24) imply

$$\begin{pmatrix} \nabla \cdot E \\ \nabla \cdot H \end{pmatrix} = \frac{1}{\varepsilon\mu + \beta^2} \begin{pmatrix} (-\beta\nabla\beta - \mu\nabla\varepsilon) \cdot E + (\beta\nabla\mu - \mu\nabla\beta) \cdot H \\ (\varepsilon\nabla\beta - \beta\nabla\varepsilon) \cdot E + (-\beta\nabla\beta - \varepsilon\nabla\mu) \cdot H \end{pmatrix}.$$

Further, from (3.22) and (3.23),

$$i\omega \begin{pmatrix} -\nabla\mu \wedge H \\ \nabla\varepsilon \wedge E \end{pmatrix} = \frac{1}{\varepsilon\mu + \beta^2} \begin{pmatrix} -\varepsilon\nabla\mu \wedge & \beta\nabla\mu \wedge \\ -\beta\nabla\varepsilon \wedge & -\mu\nabla\varepsilon \wedge \end{pmatrix} \begin{pmatrix} \nabla \wedge E \\ \nabla \wedge H \end{pmatrix}$$

Combining these with (3.25), we obtain an equation of the form

$$\mathcal{N}(x, D) \begin{pmatrix} E \\ H \end{pmatrix} = (D_{x_3}^2 - \Delta' - N(x, D_{x'}) - iQ(x)D_{x_3} - S(x)) \begin{pmatrix} E \\ H \end{pmatrix} = 0 \quad (3.26)$$

where  $N(x, D)$  is a  $6 \times 6$  system of differential operators of order one in  $x'$  and depending smoothly on  $x_3$ , and  $Q(x)$  and  $S(x)$  are zero order matrix multipliers. As in Proposition 3.1, there is a pseudodifferential operator  $C(x, D_{x'})$  of order one in  $x'$  and depending smoothly on  $x_3$  factoring  $\mathcal{N}$  as

$$\mathcal{N}(x, D) = (D_{x_3} - iQ(x) - iC(x, D_{x'}))(D_{x_3} + iC(x, D_{x'})) \quad (3.27)$$

modulo smoothing.

### 3.3.1 The Admittance Map $\Pi$ for Chiral $\Omega$

Assuming that  $\omega$  is not a Dirichlet eigenvalue for the boundary value problem for  $(\Omega; \varepsilon, \mu, \beta)$ , from theorem (2.1) we have: if  $F \in TH_{\text{Div}}^{\frac{1}{2}}(\partial\Omega)$ , there is a unique  $(E, H) \in \mathcal{D}'(\Omega)^3 \times \mathcal{D}'(\Omega)^3$  solving

$$\begin{aligned} \nabla \wedge E &= i\omega(\mu H - \beta E) \\ \nabla \wedge H &= -i\omega(\varepsilon E + \beta H) \quad \nu \wedge E|_{\partial\Omega} = F \end{aligned}$$

where  $\nu$  is the outward unit normal to the boundary of  $\Omega$ . With the analogous proof of Proposition 3.2 we have for such solutions  $(E, H)$ ,

$$\frac{\partial}{\partial x_3} \begin{pmatrix} E \\ H \end{pmatrix} \Big|_{\partial\Omega} = C \begin{pmatrix} E \\ H \end{pmatrix} \Big|_{\partial\Omega} \quad (3.28)$$

modulo smoothing. From (3.24) and (3.28), on  $\partial\Omega$ ,

$$\left[ \begin{pmatrix} -\partial_3\beta & \partial_3\mu \\ \partial_3\varepsilon & \partial_3\beta \end{pmatrix} + \begin{pmatrix} -\beta & \mu \\ \varepsilon & \beta \end{pmatrix} \begin{pmatrix} C_{33} & C_{36} \\ C_{63} & C_{66} \end{pmatrix} \right] \begin{pmatrix} E_3 \\ H_3 \end{pmatrix} = \begin{pmatrix} (\partial_1\beta + \beta\partial_1 + \beta C_{31} - \mu C_{61})E_1 + (\partial_2\beta + \beta\partial_2 + \beta C_{32} - \mu C_{62})E_2 \\ -(\partial_1\mu + \mu\partial_1 - \beta C_{34} + \mu C_{64})H_1 - (\partial_2\mu + \mu\partial_2 - \beta C_{35} + \mu C_{65})H_2 \\ -(\partial_1\varepsilon + \varepsilon\partial_1 + \varepsilon C_{31} + \beta C_{61})E_1 - (\partial_2\varepsilon + \varepsilon\partial_2 + \varepsilon C_{32} + \beta C_{62})E_2 \\ -(\partial_1\beta + \beta\partial_1 + \varepsilon C_{34} + \beta C_{64})H_1 - (\partial_2\beta + \beta\partial_2 + \varepsilon C_{35} + \beta C_{65})H_2 \end{pmatrix}$$

or

$$\begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \begin{pmatrix} E_3 \\ H_3 \end{pmatrix} = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}, \quad \text{say.}$$

Let  $G$  be a pseudodifferential operator of order  $-1$  such that  $GF = Id$  modulo smoothing. We shall use  $\Pi = \Pi(\varepsilon, \mu, \beta)$  to denote the admittance map for a chiral

body: using (3.22) and (3.23) this is the map

$$\begin{pmatrix} E_2 \\ -E_1 \\ H_2 \\ -H_1 \end{pmatrix} \Big|_{x_3=0} \xrightarrow{\Pi} \begin{pmatrix} \frac{1}{i\omega\mu}(\partial_3 E_1 - \partial_1 E_3) \\ \frac{1}{i\omega\mu}(\partial_3 E_2 - \partial_2 E_3) \\ \frac{-1}{i\omega\varepsilon}(\partial_3 H_1 - \partial_1 H_3) \\ \frac{1}{i\omega\varepsilon}(\partial_3 H_2 - \partial_2 H_3) \end{pmatrix} \Big|_{x_3=0} + \begin{pmatrix} \frac{\beta}{\mu} E_2 \\ -\frac{\beta}{\mu} E_1 \\ \frac{\beta}{\varepsilon} H_2 \\ -\frac{\beta}{\varepsilon} H_1 \end{pmatrix} \Big|_{x_3=0}.$$

The components of the  $4 \times 4$  system  $\Pi$  can be computed in terms of  $C$  and  $G$ , for example.

$$\begin{aligned} \Pi_{11} &= \frac{1}{i\omega\mu} \left\{ C_{12} + (C_{13} + C_{16} - \partial_1) [G_{11}(\partial_2\beta + \beta\partial_2 + \beta C_{32} - \mu C_{62}) \right. \\ &\quad \left. - G_{12}(\partial_2\varepsilon + \varepsilon\partial_2 + \varepsilon C_{32} + \beta C_{62})] \right\} + \frac{\beta}{\mu} \\ \Pi_{12} &= \frac{-1}{i\omega\mu} \left\{ C_{11} + (C_{13} + C_{16} - \partial_1) [G_{11}(\partial_1\beta + \beta\partial_1 + \beta C_{32} - \mu C_{62}) \right. \\ &\quad \left. - G_{12}(\partial_1\varepsilon + \varepsilon\partial_1 + \varepsilon C_{32} + \beta C_{62})] \right\} \\ \Pi_{31} &= \frac{-1}{i\omega\varepsilon} \left\{ C_{42} + (C_{43} + C_{46} - \partial_1) [G_{21}(\partial_2\beta + \beta\partial_2 + \beta C_{32} - \mu C_{62}) \right. \\ &\quad \left. - G_{22}(\partial_2\varepsilon + \varepsilon\partial_2 + \varepsilon C_{32} + \beta C_{62})] \right\} \\ \Pi_{32} &= \frac{1}{i\omega\varepsilon} \left\{ C_{41} + (C_{43} + C_{46} - \partial_1) [G_{21}(\partial_1\beta + \beta\partial_1 + \beta C_{32} - \mu C_{62}) \right. \\ &\quad \left. - G_{22}(\partial_1\varepsilon + \varepsilon\partial_1 + \varepsilon C_{32} + \beta C_{62})] \right\} \end{aligned} \quad (3.29)$$

### 3.3.2 Boundary Determination of $\varepsilon$ , $\mu$ and $\beta$

Writing the symbol of  $\Pi$  as  $\pi(x', \xi') = \pi_{jk}(x', \xi') \sim \sum_{q \leq 1} \pi_{jk}^{(q)}$ , the principal symbol of  $\Pi$  is computed to be

$$\pi^{(1)}(x', \xi') = \frac{1}{i\omega|\xi'|} \begin{bmatrix} \frac{1}{\mu} \begin{pmatrix} \xi_1 \xi_2 & \xi_2^2 \\ -\xi_1^2 & -\xi_1 \xi_2 \end{pmatrix} & 0 \\ 0 & \frac{1}{\varepsilon} \begin{pmatrix} -\xi_1 \xi_2 & -\xi_2^2 \\ \xi_1^2 & \xi_1 \xi_2 \end{pmatrix} \end{bmatrix}$$

Clearly this determines  $\mu$  and  $\varepsilon$  (and hence  $\sigma$  and  $\gamma$ ) at the boundary. The computations for  $\pi^{(0)}$  are analogous to those for  $\lambda$ . We outline the arguments leading to the determination of the unknown parameters. In what follows,  $g_j(x)$  are known functions at each stage, and  $\xi'$  is chosen as appropriate.

$$\begin{aligned}\pi_{13}^{(0)}(x', (1, 0)) - g_1(x') &= \frac{\beta}{\varepsilon\mu + \beta^2} \\ \pi_{14}^{(0)}(x', (0, 1)) - g_2(x') &= \left( \frac{\partial_2\mu + i\partial_3\mu}{2\omega\mu} \right) \frac{\beta}{\varepsilon\mu + \beta^2}\end{aligned}\tag{3.30}$$

and so these combine to determine  $\partial_3\mu$ . Next,

$$\pi_{12}^{(0)}(x', (0, 1)) - g_3(x') = \left( \frac{\partial_2\mu + i\partial_3\mu}{2\omega} \right) \frac{1}{\varepsilon\mu + \beta^2}$$

determines  $\varepsilon\mu + \beta^2$  which, with (3.30), gives  $\beta$ . Finally

$$\pi_{11}^{(0)}\left(x', \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right) - g_4(x') = \frac{-i\partial_3\beta}{2\omega(\varepsilon\mu + \beta^2)}$$

determines  $\partial_3\beta$ .

To summarize, the two highest order terms in the expansion of  $\pi$  determine  $\beta$ ,  $\mu$ ,  $\sigma$  and  $\gamma$ , and the normal derivatives of each on  $\partial\Omega$ .

### 3.3.3 Remark: Layer Stripping for Chiral Bodies

For a non-chiral body, a *layer stripping* algorithm was proposed in [15] to estimate the parameters near the boundary. This algorithm essentially proceeds as follows: from the boundary map  $\Lambda$ , the parameters are determined at the boundary. One then *strips* away the known “surface layer” to expose a new boundary, and by way of a Riccati-type equation for  $\Lambda$ , an approximate boundary map for the new boundary is obtained. From this the parameters are estimated a little below the surface and the procedure can be repeated. The essential component of this algorithm is the existence of a Riccati-type equation with which to advance the boundary map in from the original surface. From (3.26) and (3.27), together with the fact that  $D_{x_3}f(x', 0) =$

0. we have the following Riccati-type equation for  $C$ :

$$\partial_3 C = -C^2 - QC - \Delta' - N - S .$$

Now we may calculate the composition of normal differentiation with  $\Pi$  by way of (3.29), in terms of operators which are determined from the symbol of  $\Pi$ . This yields a Riccati-type equation for  $\Pi$ , and layer stripping may be applied in the case of a chiral body.

## Chapter 4

# INTERIOR DETERMINATION FOR CHIRAL BODIES

### 4.1 *Introduction*

In [17], Sylvester and Uhlmann proved that the conductivity of a body can be uniquely identified from information obtained only from the boundary. If a time dependence is introduced to the electromagnetic fields, the equations governing these fields change from a single second order elliptic partial differential equation to the full Maxwell's equations. In [16] Somersalo et al. presented a boundary map for time-harmonic fields at a fixed frequency and raised the question of whether the parameters describing the electromagnetic properties of the body could be determined from knowledge of this boundary map. They showed that these parameters could be recovered approximately provided they differed only slightly from known constants. In [11] this assumption was dropped and it was shown that the parameters are recoverable provided they are known in a small neighborhood of the boundary of the body.

In all these treatments, chirality is neglected. In this work we treat the case of a chiral body, and so the constituent equations depend on a fourth parameter  $\beta$  which describes this chirality. In [12] Ola and Somersalo simplified the proof of interior identifiability in [11] by constructing a second order system of differential equations, which has as its principal part the Laplacian, in such a way that solutions to this system yield solutions to Maxwell's equations. They were able to construct a system with no first order part, that is a Schrödinger equation, and then use the results of

[17] to construct exponentially growing solutions. Here we follow this idea and show that in the chiral case we are able to construct a system with the Laplacian as its principal part which again yields solutions to Maxwell's equations, but which has a first order term. Nakamura and Uhlmann [9] have developed a technique to handle such first order perturbations of the Laplacian, and it is this technique we employ here to construct exponentially growing solutions. The ability to construct these solutions enables us to use *complex geometrical optics* to prove identifiability of all four of the material parameters, in particular the chirality, throughout a chiral body. The results of this chapter appear in [8].

In section 4.2 we state the problem precisely, present the main theorem (theorem 4.1) and briefly outline the proof which comprises the later sections. Section 4.3 sets up the second order system; in section 4.4 we construct the exponentially growing solutions. The proof of our result is brought together in section 4.5. Section 4.6 contains a more technical proof of a result used in the construction of the exponentially growing solutions.

## 4.2 Statement of the Result

Let  $\Omega$  be a bounded connected subset of  $\mathbb{R}^3$  with connected complement and with smooth boundary  $\partial\Omega$ . We restrict our interest to *time-harmonic* electromagnetic fields on  $\Omega$ , at fixed frequency  $\omega$ , i.e. if  $E$  and  $H$  are the electric and magnetic fields respectively then

$$E = e^{i\omega t} E(x), \quad H = e^{i\omega t} H(x).$$

For such time-harmonic fields, from chapter 2, Maxwell's equations are

$$\nabla \wedge E = i\omega(\mu H - \beta E), \quad \nabla \wedge H = -i\omega(\varepsilon E + \beta H) \quad (4.1)$$

together with the divergence free conditions

$$\nabla \cdot (\mu H - \beta E) = 0, \quad \nabla \cdot (\varepsilon E + \beta H) = 0. \quad (4.2)$$

Recall that we are assuming that  $\varepsilon\mu + \beta^2 \neq 0$ , which amounts to assuming the electric and magnetic fields never become parallel.

The problem considered herein can now be stated.

**THEOREM 4.1.** *Let  $(\Omega; \varepsilon_1, \mu_1, \beta_1)$  and  $(\Omega; \varepsilon_2, \mu_2, \beta_2)$  be two electromagnetic bodies with the same boundary  $\partial\Omega$ . Suppose that  $\Pi_1 = \Pi_2$ ; that is, if  $F \in TH_{Div}^{\frac{1}{2}}(\partial\Omega)$  and  $(E_j, H_j)$  solve (2.5) with parameters  $(\varepsilon_j, \mu_j, \beta_j)$  for  $j = 1, 2$ , then*

$$\Pi_1 F = \nu \wedge H_1|_{\partial\Omega} = \nu \wedge H_2|_{\partial\Omega} = \Pi_2 F.$$

*Assuming that  $\varepsilon_1 = \varepsilon_2$ ,  $\mu_1 = \mu_2$  and  $\beta_1 = \beta_2$  on  $\partial\Omega$ , and the same is true for all normal derivatives at  $\partial\Omega$ , then in  $\Omega$ ,*

$$\varepsilon_1 = \varepsilon_2, \quad \mu_1 = \mu_2, \quad \beta_1 = \beta_2.$$

*Remarks:* (1) From chapter 3 ([7]),  $\Pi$  determines the material parameters and their first normal derivatives at the boundary. It is expected that the technique of [7] would show that  $\Pi$  also determines all the higher order derivatives at the boundary, however the computations become unmanageable.

(2) The assumption that the parameters agree to all order at the boundary is necessary only in the construction of the intertwining operators (see section 4.4). These operators belong to the Shubin class which require smooth symbols. In [18] Tolmasky showed that such intertwining operators may be constructed for equations with non-smooth parameters. This technique should remove the necessity of the assumption at the boundary, and further should lower the regularity assumptions on the parameters throughout  $\Omega$ .

(3) In the case that  $\beta_1 = \beta_2 = 0$ , the result of [11] follows without any assumption on  $\varepsilon$  or  $\mu$  at  $\partial\Omega$ . The reason for this is that exponentially growing solutions are constructed without the need for intertwining operators, and so the parameters may be extended outside  $\Omega$  in a non-smooth way.

Under the assumption of the theorem we may extend the parameters smoothly to all of  $\mathbb{R}^3$ , constant outside  $\Omega$ , and so that  $\varepsilon_1 = \varepsilon_2$ ,  $\mu_1 = \mu_2$  and  $\beta_1 = \beta_2$  outside  $\Omega$ . Fundamental to the proof of theorem 4.1 is the following identity.

PROPOSITION 4.2. *Let  $(E_j, H_j)$  solve (4.1) for parameters  $(\varepsilon_j, \mu_j, \beta_j)$ ,  $j = 1, 2$ . If  $\Pi_1 = \Pi_2$ , then*

$$\int_{\Omega} ((\beta_1 - \beta_2)(H_1 \cdot E_2 + H_2 \cdot E_1) + (\varepsilon_1 - \varepsilon_2)E_1 \cdot E_2 + (\mu_2 - \mu_1)H_1 \cdot H_2) dx = 0 \quad (4.3)$$

*Proof.* Integrating by parts, and using the definition of  $\Pi$ .

$$\begin{aligned} \int_{\Omega} i\omega(\varepsilon_1 E_1 + \beta_1 H_1) \cdot E_2 dx &= - \int_{\Omega} \nabla \wedge H_1 \cdot E_2 dx \\ &= - \int_{\partial\Omega} \nu \wedge H_1 \cdot E_2 dS(x) - \int_{\Omega} H_1 \cdot \nabla \wedge E_2 dx \\ &= - \int_{\partial\Omega} \Pi_1 E_1 \cdot E_2 dS(x) - \int_{\Omega} H_1 \cdot i\omega(\mu_2 H_2 - \beta_2 E_2) dx \end{aligned}$$

and similarly

$$\int_{\Omega} i\omega(\varepsilon_2 E_2 + \beta_2 H_2) \cdot E_1 dx = - \int_{\partial\Omega} \Pi_2 E_2 \cdot E_1 dS(x) - \int_{\Omega} H_2 \cdot i\omega(\mu_1 H_1 - \beta_1 E_1) dx.$$

Thus

$$\begin{aligned} &i\omega \int_{\Omega} ((\beta_1 - \beta_2)(H_1 \cdot E_2 + H_2 \cdot E_1) + (\varepsilon_1 - \varepsilon_2)E_1 \cdot E_2 + (\mu_2 - \mu_1)H_1 \cdot H_2) dx \\ &= \int_{\partial\Omega} (\Pi_2 E_2 \cdot E_1 - \Pi_1 E_1 \cdot E_2) dS(x). \end{aligned}$$

The proposition follows if we show that

$$\int_{\partial\Omega} \Pi_2 E_2 \cdot E_1 dS(x) = \int_{\partial\Omega} E_2 \cdot \Pi_2 E_1 dS(x).$$

Let  $(E_0, H_0)$  be the solution to (2.5) with parameters  $(\varepsilon_2, \mu_2, \beta_2)$  and with

$F = \nu \wedge E_1|_{\partial\Omega}$ . Then

$$\begin{aligned}
\int_{\partial\Omega} (\Pi_2 E_2 \cdot E_1 - E_2 \cdot \Pi_2 E_1) dS(x) &= \int_{\partial\Omega} (\nu \wedge H_2 \cdot E_1 - E_2 \cdot \nu \wedge H_0) dS(x) \\
&= \int_{\partial\Omega} (-H_2 \cdot \nu \wedge E_0 - E_2 \cdot \nu \wedge H_0) dS(x) \\
&= \int_{\Omega} (\nabla \wedge H_2 \cdot E_0 - H_2 \cdot \nabla \wedge E_0 - E_2 \cdot \nabla \wedge H_0 + \nabla \wedge E_2 \cdot H_0) dx \\
&= \int_{\Omega} (-i\omega(\varepsilon_2 E_2 + \beta_2 H_2) \cdot E_0 - H_2 \cdot i\omega(\mu_2 H_0 - \beta_2 E_0) \\
&\quad + E_2 \cdot i\omega(\varepsilon_2 E_0 + \beta_2 H_0) + i\omega(\mu_2 H_2 - \beta_2 E_2) \cdot H_0) dx \\
&= 0.
\end{aligned}$$

□

The remainder of the chapter is devoted to constructing sufficiently many suitable solutions to Maxwell's equations to conclude from (4.3) the claim of theorem 4.1. We present now an outline of the proof.

The aim is to use *complex geometrical optics* in the manner of [17] and many subsequent papers; that is we wish to construct exponentially growing solutions depending on a complex parameter  $\rho$  and to examine the asymptotics as the size of  $\rho$  gets large. Rather than construct solutions to (4.1) directly, we follow the idea of Ola and Somersalo in [12] and introduce a new  $8 \times 8$  system

$$(P(\nabla) + V)(P(\nabla) + V')Y = (\Delta + N + Q)Y = 0$$

where  $P(\nabla)$  and  $N$  are first order differential operators, and  $V$ ,  $V'$  and  $Q$  are matrix multipliers. We shall do this in such a way that if  $Y$  is a solution to this system, and

$$X = (P(\nabla) + V')Y$$

is such that the first and last components of  $X$  are zero, then the vector fields  $((X_2, X_3, X_4)', (X_5, X_6, X_7)')$  will solve Maxwell's equations.

We then construct exponentially growing solutions to  $(\Delta + N + Q)Y_\rho = 0$  of the form

$$Y_\rho = e^{x \cdot \rho} (y_{0,\rho} + \psi_\rho)$$

with  $\rho \in \mathbb{C}^3$  satisfying  $\rho \cdot \rho = 0$ , with  $y_{0,\rho}$  an 8-vector which is constant in  $x$  and chosen to depend on  $\rho$  in a convenient way, and  $\psi_\rho$  constructed so that  $\psi_\rho \rightarrow 0$  in some sense as  $|\rho| \rightarrow \infty$ . In [12] where chirality was not taken into account ( $\beta = 0$ ) the system above included no first order term  $N$ , and so the authors were able to use the methods of [17] to construct exponentially growing solutions to a Schrödinger equation. When  $\beta \neq 0$ , such a reduction does not seem possible, and so here we must construct solutions to a first order perturbation of the Laplacian. The techniques employed are those of [9] where Nakamura and Uhlmann constructed solutions to a system of a similar form arising from elasticity.

The final ingredient is to set  $X_\rho = (P(\nabla) + V')e^{x \cdot \rho} (y_{0,\rho} + \psi_\rho)$  and show that we can choose  $y_{0,\rho}$  in such a way that  $X_\rho$  yields solutions to Maxwell's equations, and to use these solutions in (4.3) to prove the claim of theorem 4.1.

### 4.3 A Reformulation of Maxwell's Equations

In this section we introduce a new system of differential equations, the solutions of which, under certain restrictions yield solutions to Maxwell's equations. We first introduce the following  $8 \times 8$  operator

$$P(\nabla) = \begin{bmatrix} 0 & \nabla \cdot & 0 & 0 \\ \nabla & 0 & \nabla \wedge & 0 \\ 0 & -\nabla \wedge & 0 & \nabla \\ 0 & 0 & \nabla \cdot & 0 \end{bmatrix}$$

The domain of  $P(\nabla)$  is  $\mathcal{D}'(\mathbb{R}^3) \times \mathcal{D}'(\mathbb{R}^3)^3 \times \mathcal{D}'(\mathbb{R}^3)^3 \times \mathcal{D}'(\mathbb{R}^3)$ . We point out that

$$P(\nabla)P(\nabla) = \Delta.$$

Our aim is to find  $8 \times 8$  matrices  $V$  and  $V'$  and write

$$(P(\nabla) + V)(P(\nabla) + V') = \Delta + N + Q$$

with  $N$  a first order differential operator, and  $Q$  a zero order matrix multiplier. Then if  $Y$  solves

$$(\Delta + N + Q)Y = 0 \tag{4.4}$$

and we put

$$X = (P(\nabla) + V')Y$$

we would like (4.4) to imply that in some sense  $X$  solves Maxwell's equations. The advantage of this reformulation is that we are in the position of seeking solutions to (4.4) for which a method is known.

We introduce some notation: for  $X \in \mathcal{D}'(\mathbb{R}^3) \times \mathcal{D}'(\mathbb{R}^3)^3 \times \mathcal{D}'(\mathbb{R}^3)^3 \times \mathcal{D}'(\mathbb{R}^3)$  we shall write

$$X = (a, A, B, b)'$$

In order to have  $X$  a solution to Maxwell's equations, we will find  $Y$  is such a way that  $a = b = 0$ ; for the moment assume that this is the case. We must choose  $V$  so that (4.4) implies (4.1) and (4.2); in particular, the central 6 rows of (4.4) must imply (4.1) and the first and last rows must imply (4.2). Let

$$V_m = \begin{bmatrix} V_{22} & V_{23} \\ V_{32} & V_{33} \end{bmatrix}, \quad \text{and} \quad L = i\omega \begin{bmatrix} \varepsilon I_3 & \beta I_3 \\ -\beta I_3 & \mu I_3 \end{bmatrix}$$

where  $V_{jk}$  are the  $3 \times 3$  blocks in the center of  $V$  and  $I_3$  is the  $3 \times 3$  identity matrix. If in fact  $(A, B)$  are taken to be  $(E, H)$  (that is we don't rescale the fields in any way), then (4.4) is equivalent to

$$\begin{pmatrix} \nabla \wedge H \\ -\nabla \wedge E \end{pmatrix} = -V_m \begin{pmatrix} E \\ H \end{pmatrix}$$

and so taking  $V_m = L$  we obtain (4.1). Now set

$$M = \begin{bmatrix} -3I_3 & \mu I_3 \\ \varepsilon I_3 & 3I_3 \end{bmatrix}, \quad \nabla M \cdot = \begin{bmatrix} -\nabla\beta \cdot & \nabla\mu \cdot \\ \nabla\varepsilon \cdot & \nabla\beta \cdot \end{bmatrix}, \quad \text{and} \quad V_0 = \begin{bmatrix} \vec{v}_{12} & \vec{v}_{13} \\ \vec{v}_{42} & \vec{v}_{43} \end{bmatrix}$$

where  $\vec{v}_{12}$  is the 3-vector  $(v_{12}, v_{13}, v_{14})$ ,  $\vec{v}_{13} = (v_{15}, v_{16}, v_{17})$ ,  $\vec{v}_{42} = (v_{82}, v_{83}, v_{84})$ , and  $v_{43} = (v_{85}, v_{86}, v_{87})$  in  $V$ . Notice that under the condition that  $\varepsilon\mu + \beta^2 \neq 0$ ,  $M$  is invertible. Conditions (4.2) are equivalent to

$$M \begin{pmatrix} \nabla \cdot E \\ \nabla \cdot H \end{pmatrix} + \nabla M \cdot \begin{pmatrix} E \\ H \end{pmatrix} = 0, \quad \text{or} \quad \begin{pmatrix} \nabla \cdot E \\ \nabla \cdot H \end{pmatrix} = -M^{-1} \nabla M \cdot \begin{pmatrix} E \\ H \end{pmatrix},$$

and  $(P + V)X = 0$  implies

$$\begin{pmatrix} \nabla \cdot E \\ \nabla \cdot H \end{pmatrix} + V_0 \begin{pmatrix} E \\ H \end{pmatrix} = 0$$

so putting

$$V_0 = M^{-1} \nabla M \cdot = \frac{1}{\varepsilon\mu + \beta^2} \begin{bmatrix} (\mu\nabla\varepsilon + \beta\nabla\beta) \cdot & (\mu\nabla\beta - \beta\nabla\mu) \cdot \\ (\beta\nabla\varepsilon - \varepsilon\nabla\beta) \cdot & (\varepsilon\nabla\mu + \beta\nabla\beta) \cdot \end{bmatrix}$$

we have (4.2). At this point, assuming the first and last components  $a$  and  $b$  of  $X$  are zero, we have determined the central 6 columns of  $V$  so that (4.4) implies that the fields  $(A, B)$  satisfy (4.1). We now remove the assumption that  $a = b = 0$  and choose the rest of  $V$  and all of  $V'$  in such a way that

$$(P(\nabla) + V)(P(\nabla) + V') = \Delta + N + Q$$

has as simple a first order term  $N$  as possible. This term is determined by  $P(\nabla)V' + VP(\nabla)$ ; analyzing this row by row and making choices to eliminate first order terms, we find that we may choose

$$V = \begin{bmatrix} i\omega\mu & \vec{v}_{12} & \vec{v}_{13} & i\omega\beta \\ 0 & i\omega\varepsilon & i\omega\beta & 0 \\ 0 & -i\omega\beta & i\omega\mu & 0 \\ -i\omega\beta & \vec{v}_{42} & \vec{v}_{43} & i\omega\varepsilon \end{bmatrix}, \quad V' = \begin{bmatrix} -i\omega\varepsilon & \vec{v}_{23} & -\vec{v}_{22} & -i\omega\beta \\ 0 & -i\omega\mu & -i\omega\beta & 0 \\ 0 & i\omega\beta & -i\omega\varepsilon & 0 \\ i\omega\beta & \vec{v}_{33} & -\vec{v}_{32} & -i\omega\mu \end{bmatrix}$$

and obtain the first order term

$$N = \begin{bmatrix} \vec{v}_{12} \cdot \nabla & -\vec{v}_{13} \cdot \nabla \wedge & \vec{v}_{12} \cdot \nabla \wedge & \vec{v}_{13} \cdot \nabla \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \vec{v}_{42} \cdot \nabla & -\vec{v}_{43} \cdot \nabla \wedge & \vec{v}_{42} \cdot \nabla \wedge & \vec{v}_{43} \cdot \nabla \end{bmatrix}.$$

We remark that  $N$  has compact support since its components consist of derivatives of the parameters, which are constant outside of a compact set. The zero order term  $Q$  can be calculated easily, but as it will not be needed here we shall not present it explicitly.

*Remark:* A natural question to ask is, by rescaling the fields  $(E, H)$  can a system be found that has *no* first order term, in which case we would have a Schrödinger equation? Such a system was achieved in [12] for a non-chiral body by rescaling the fields. For a chiral body, however, the answer to this appears to be no; suppose that we write  $(A, B) = R(E, H)$  for some invertible matrix  $R$  of the form

$$R = \begin{bmatrix} r_{11}I_3 & r_{12}I_3 \\ r_{21}I_3 & r_{22}I_3 \end{bmatrix},$$

and we set

$$\tilde{V}_m = \begin{bmatrix} -V_{32} & -V_{33} \\ V_{22} & V_{23} \end{bmatrix}, \quad \text{and} \quad \tilde{L} = i\omega \begin{bmatrix} -\beta I_3 & \mu I_3 \\ -\varepsilon I_3 & -\beta I_3 \end{bmatrix}$$

then we find that to satisfy (4.1) we must set

$$\begin{aligned} \tilde{V}_m &= -\nabla R \wedge R^{-1} - R \tilde{L} R^{-1} \\ V_0 &= -\nabla R \cdot R^{-1} + R M^{-1} \nabla M \cdot R^{-1} \end{aligned}$$

(the notation should be interpreted in the way that makes sense), and this results in a first order term whose non-zero components are given by the components of

$$-2\nabla R \wedge R^{-1} + R M^{-1} \nabla M \cdot R^{-1};$$

we conjecture that there is no choice of matrix  $R$  which makes this zero. In [12] the rescaling matrix is

$$R = \begin{bmatrix} \varepsilon^{\frac{1}{2}} & 0 \\ 0 & \mu^{\frac{1}{2}} \end{bmatrix}$$

and an easy calculation shows that the first order term vanishes when  $\beta = 0$ . An interesting observation is that no matter what choice of  $R$  is made, the system obtained by following this construction always leads to solutions to Maxwell's equations. The proof of this is more involved than what is presented here, but the same program carries through.

#### 4.4 Construction of Solutions - Intertwining Operators

Recall that we wish to construct solutions to  $(\Delta + N + Q)Y = 0$  with  $Y$  of the form  $Y = e^{x \cdot \rho}(y_{0,\rho} + \psi_\rho)$ . For  $\rho \in \mathbb{C}^3$  with  $\rho \cdot \rho = 0$  define the operators

$$\Delta_\rho = e^{-x \cdot \rho} \Delta (e^{x \cdot \rho} \cdot), \quad \text{and} \quad N_\rho^+ = e^{-x \cdot \rho} (N + Q) (e^{x \cdot \rho} \cdot)$$

and so we wish to solve

$$(\Delta_\rho + N_\rho^+)(y_{0,\rho} + \psi_\rho) = 0. \quad (4.5)$$

We specify  $\psi_\rho$  later by prescribing its asymptotic behavior. Generally speaking, our approach is to construct pseudodifferential operators  $A_\rho$ ,  $B_\rho$  and  $C_\rho$  of order zero and depending on the parameter  $\rho$  so that

$$(\Delta_\rho + N_\rho^+)A_\rho(y_{0,\rho} + \psi_\rho) = B_\rho(\Delta_\rho + C_\rho)(y_{0,\rho} + \psi_\rho).$$

For sufficiently large  $\rho$ ,  $A_\rho$  is invertible, and we shall always take our operators to be properly supported, so that there is no problem defining compositions. This reduction to a zero order perturbation of the Laplacian enables us to use the extensive knowledge in the literature of constructing exponentially growing solutions.

Such solutions have been used extensively in identifiability results, initiated by the conductivity result of [17].

We introduce the class these “intertwining operators” belong to. Let  $Z = \{\rho \in \mathbb{C}^3 \mid |\rho| \geq 1, \rho \cdot \rho = 0\}$ , and denote by  $L^0(\mathbb{R}^3, Z)$  the *Shubin class* of order zero (see [14], section 9). We refer the reader to [9] for a discussion of the Shubin class of operators, and repeat some important properties here. Most importantly we define the symbol class of  $L^0(\mathbb{R}^3, Z)$ .

**DEFINITION 4.3.** Let  $\rho \in Z$ ; then  $a_\rho(x, \xi) \in S^0(\mathbb{R}^3, Z)$  if and only if

1.  $a_\rho \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$  for each fixed  $\rho \in Z$ , and
2. for any multi-indices  $\alpha, \delta$  and compact set  $K \subset \mathbb{R}^3$ , there exists a constant  $C_{\alpha, \delta, K} > 0$  such that

$$\sup_{x \in K} |\partial_\xi^\alpha \partial_x^\delta a_\rho(x, \xi)| \leq C_{\alpha, \delta, K} (1 + |\xi| + |\rho|)^{-|\alpha|}$$

for any  $\xi \in \mathbb{R}^3, \rho \in Z$ .

We say that  $a_\rho$  is the full symbol of  $A_\rho$  in the same way as for usual pseudodifferential operators. We say  $A_\rho \in L^0(\mathbb{R}^3, Z)$  is properly supported if there exists a closed set  $H \subset \mathbb{R}^3 \times \mathbb{R}^3$  such that the support of the Schwartz kernel of  $A_\rho$  is contained in  $H$  for all  $\rho \in Z$  and the projections of  $H$  onto each factor  $\mathbb{R}^3$  are proper. We note that if  $A_\rho \in L^0(\mathbb{R}^3, Z)$  is properly supported, then we may expand the symbol  $\tilde{\sigma}(A_\rho)(x, \xi)$  of  $A_\rho$  asymptotically as

$$\tilde{\sigma}(A_\rho)(x, \xi) \sim \sum_{\alpha} \frac{1}{\alpha!} \partial_\xi^\alpha D_y^\alpha a_\rho(x, y, \xi)|_{y=x}$$

**PROPOSITION 4.4.** Let  $\varphi \in C_c^\infty(\mathbb{R}^3)$  be such that  $\varphi$  is identically one on  $\Omega$ . Then there exist operators  $A_\rho, B_\rho$  and  $C_\rho$  in  $L^0(\mathbb{R}^3, Z)^{8 \times 8}$  such that

$$(\Delta_\rho + N_\rho^+)A_\rho = B_\rho(\Delta_\rho + \varphi C_\rho \varphi) \tag{4.6}$$

We leave the proof of this to a later section. Let

$$L^2_\delta = \{f \in L^2_{loc}(\mathbb{R}^3) : \|f\|_\delta^2 = \int (1 + |x|^2)^\delta |f(x)|^2 dx < \infty\},$$

and for  $s \in \mathbb{R}$  let  $H^s_\delta$  be the associated weighted Sobolev space. Assuming (4.6) we have the following proposition:

**PROPOSITION 4.5.** *Let  $-1 < \delta < 0$  and  $y_{0,\rho}$  be an 8-vector constant in  $x$  and bounded in  $\rho$ . Then for sufficiently large  $|\rho|$  there exists  $\psi_\rho \in H^2_\delta(\mathbb{R}^3)^8$ , and constant  $C$  depending only on  $\delta$ ,  $\varphi$  and  $C_\rho$  such that*

$$(\Delta_\rho + \varphi C_\rho \varphi)(y_{0,\rho} + \psi_\rho) = 0$$

and

$$\|\psi_\rho\|_{H^2_\delta} \leq \frac{C}{|\rho|}. \quad (4.7)$$

*Proof.* We have  $\Delta_\rho \psi_\rho = -\varphi C_\rho \varphi y_{0,\rho} - \varphi C_\rho \varphi \psi_\rho$ ; from [14]  $C_\rho : H^2(\mathbb{R}^3)^8 \rightarrow H^2(\mathbb{R}^3)^8$  continuously with operator norm independent of  $\rho$ , and since  $\varphi$  is compactly supported,  $\varphi C_\rho \varphi y_{0,\rho} \in H^2_{\delta+1}(\mathbb{R}^3)^8$ . Let  $r_0 > 0$ ; by [17], if  $|\rho| > r_0 > 0$ , we may solve

$$\Delta_\rho \psi_\rho^{(0)} = -\varphi C_\rho \varphi y_{0,\rho}$$

for  $\psi_\rho^{(0)} \in H^2_\delta(\mathbb{R}^3)^8$  and from the estimates for  $\Delta_\rho^{-1}$  in [17],

$$\|\psi_\rho^{(0)}\|_{H^2_\delta} \leq \frac{C(r_0, \delta)}{|\rho|} \|\varphi C_\rho \varphi y_{0,\rho}\|_{H^2_{\delta+1}}.$$

In general, for any  $j$ ,  $\varphi C_\rho \varphi \psi_\rho^{(j-1)} \in H^2_{\delta+1}(\mathbb{R}^3)^8$  and so for  $|\rho| > r_0$  we solve

$$\Delta_\rho \psi_\rho^{(j)} = -\varphi C_\rho \varphi \psi_\rho^{(j-1)}$$

with

$$\|\psi_\rho^{(j)}\|_{H^2_\delta} \leq \frac{C(r_0, \delta)}{|\rho|} \left( \frac{C'(\varphi, C_\rho)}{|\rho|} \right)^j \|\varphi C_\rho \varphi y_{0,\rho}\|_{H^2_{\delta+1}};$$

choosing  $|\rho|$  large enough and putting  $\psi_\rho = \sum_{j=0}^{\infty} \psi_\rho^{(j)}$ , we have  $\psi_\rho \in H^2_\delta(\mathbb{R}^3)^8$  for sufficiently large  $|\rho|$ , and

$$\|\psi_\rho\|_{H^2_\delta} \leq \frac{C}{|\rho|}.$$

Furthermore,  $(\Delta_\rho + \varphi C_\rho \varphi)(y_{0,\rho} + v_\rho) = 0$ . □

Thus we have a means to construct solutions to (4.4). We have

$$(\Delta_\rho + N_\rho^+) A_\rho(y_{0,\rho} + \psi_\rho) = 0,$$

and introducing a cut-off to gain compact support, we put

$$Y_\rho = e^{x^\rho} \varphi A_\rho(y_{0,\rho} + \psi_\rho); \quad (4.8)$$

then in  $\Omega$ ,  $(\Delta + N + Q)Y_\rho = 0$ . In order to construct solutions  $Y_\rho$  so that  $X_\rho = (P(\nabla) + V')Y_\rho$  are solutions to Maxwell's equations, we must ensure that the first and last components of  $X_\rho$ , namely  $(a, b)$  are zero. We introduce the notation  $P(\rho)$  to be the  $8 \times 8$  matrix where  $\rho$  replaces  $\nabla$  in  $P(\nabla)$ .

**PROPOSITION 4.6.** *If  $y_{0,\rho}$  is chosen so that the first and last components of  $P(\rho)A_\rho y_{0,\rho}$  are zero, then the first and last components  $(a, b)'$  of  $X_\rho = (P(\nabla) + V')Y_\rho$  are zero.*

*Proof.* We first show that  $(a, b)'$  is harmonic. Since  $(P(\nabla) + V)X_\rho = 0$ ,

$$\Delta \begin{pmatrix} a \\ b \end{pmatrix} + i\omega \nabla \cdot \begin{pmatrix} \varepsilon E + \beta H \\ -\beta E + \mu H \end{pmatrix} = 0.$$

But

$$\begin{aligned} \nabla \cdot \begin{pmatrix} \varepsilon E + \beta H \\ -\beta E + \mu H \end{pmatrix} &= \begin{bmatrix} \nabla \varepsilon \cdot & \nabla \beta \cdot \\ -\nabla \beta \cdot & \nabla \mu \cdot \end{bmatrix} \begin{pmatrix} E \\ H \end{pmatrix} + \begin{bmatrix} \varepsilon & \beta \\ -\beta & \mu \end{bmatrix} \begin{pmatrix} \nabla \cdot E \\ \nabla \cdot H \end{pmatrix} \\ &= \left\{ \begin{bmatrix} \nabla \varepsilon \cdot & \nabla \beta \cdot \\ -\nabla \beta \cdot & \nabla \mu \cdot \end{bmatrix} - \begin{bmatrix} \varepsilon & \beta \\ -\beta & \mu \end{bmatrix} M^{-1} \nabla M \cdot \right\} \begin{pmatrix} E \\ H \end{pmatrix} \\ &= 0, \end{aligned}$$

which a straight-forward calculation shows. Thus  $\Delta(a, b)' = 0$ . Next we use the fact that  $X_\rho = (P(\nabla) + V')Y_\rho$ :

$$\begin{aligned} X_\rho &= (P(\nabla) + V')e^{x \cdot \rho} \varphi A_\rho(y_{0,\rho} + \psi_\rho) \\ &= e^{x \cdot \rho} \{ P(\rho) \varphi A_\rho y_{0,\rho} + P(\rho) \varphi A_\rho \psi_\rho + P(\nabla) \varphi A_\rho (y_{0,\rho} + \psi_\rho) + \\ &\quad V' \varphi A_\rho (y_{0,\rho} + \psi_\rho) \} \\ &= e^{x \cdot \rho} \{ P(\rho) \varphi A_\rho y_{0,\rho} + X_s \}, \quad \text{say.} \end{aligned}$$

If we write

$$\begin{pmatrix} a \\ b \end{pmatrix} = e^{x \cdot \rho} \left\{ \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} + \begin{pmatrix} a_s \\ b_s \end{pmatrix} \right\}$$

then

$$\Delta_\rho \begin{pmatrix} a_s \\ b_s \end{pmatrix} = - \begin{pmatrix} a_0 \\ b_0 \end{pmatrix}.$$

Now  $X_s \in L^2_{loc}(\mathbb{R}^3)^8$  and has compact support, so in particular  $(a_s, b_s)' \in L^2_\delta(\mathbb{R}^3)^2$ . From [17],  $\Delta_\rho$  is invertible on  $L^2_\delta$  and since  $(a_0, b_0)' = (0, 0)'$ , the proposition follows.  $\square$

We will show later, in the proof of proposition 4.4, that the symbol  $a_\rho(x, \xi)$  of  $A_\rho$  is of the form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{17} & a_{18} \\ 0 & \ddots & & & 0 \\ \vdots & & I_6 & & \vdots \\ 0 & & & \ddots & 0 \\ a_{81} & a_{82} & \cdots & a_{87} & a_{88} \end{bmatrix}$$

where  $I_6$  is the  $6 \times 6$  identity matrix, and that  $a_\rho(x, \xi)$  is homogeneous of degree zero

in  $\xi$  and  $\rho$ . Thus

$$P(\rho)\varphi A_\rho y_{0,\rho} = \varphi \begin{bmatrix} \rho \cdot (y_2, y_3, y_4) \\ (\bar{a}_1 \cdot y_{0,\rho})\rho + \rho \wedge (y_5, y_6, y_7) \\ (\bar{a}_8 \cdot y_{0,\rho})\rho - \rho \wedge (y_2, y_3, y_4) \\ \rho \cdot (y_5, y_6, y_7) \end{bmatrix} \quad (4.9)$$

where  $y_j$  are the components of  $y_{0,\rho}$  and  $\bar{a}_1$  and  $\bar{a}_8$  are the first and last rows of  $a_\rho$ . To satisfy the conditions of proposition 4.6 we must therefore choose  $y_{0,\rho}$  so that  $\rho \cdot (y_2, y_3, y_4) = \rho \cdot (y_5, y_6, y_7) = 0$ .

#### 4.5 Proof of Theorem 4.1

We first investigate the asymptotics in  $\rho$  of  $A_\rho$ .

PROPOSITION 4.7. *If  $f \in L^2(\Omega)^8$  then for all  $x \in \Omega$ ,*

$$A_\rho f(x) = a_\rho(x, 0)f(x) + R_\rho f(x)$$

*modulo smoothing, and*

$$\|R_\rho f\|_{L^2(\Omega)} \leq \frac{C}{1 + |\rho|} \|f\|_{L^2(\Omega)}$$

*for a constant  $C > 0$  independent of  $\rho$  and  $f$ . Recall that  $a_\rho(x, \xi)$  is the symbol of  $A_\rho$ .*

*Proof.* Let  $\chi \in C_c^\infty(\mathbb{R}^3)$  be such that  $\chi(x) = 1$  on  $\{|x| \leq 1\}$ ,  $\chi(x) = 0$  on  $\{|x| \geq 2\}$ , and let  $\sigma(y) \in C_c^\infty(\mathbb{R}^3)$  be such that

$$\sigma(y) = 1 \text{ on } \{y \mid \exists x \in \Omega \text{ with } \chi(x - y) \neq 0\}.$$

For  $x \in \Omega$ ,

$$\begin{aligned} A_\rho f(x) &= \int e^{i(x-y)\cdot\xi} \chi(x-y) a_\rho(x, \xi) f(y) dy d\xi \\ &\quad + \int e^{i(x-y)\cdot\xi} (1-\chi)(x-y) a_\rho(x, \xi) f(y) dy d\xi \\ &= \int e^{i(x-y)\cdot\xi} \chi(x-y) a_\rho(x, \xi) (\sigma f)(y) dy d\xi + g_1(x) \end{aligned}$$

where  $g_1 \in C^\infty(\mathbb{R}^3)^8$  since the second integral is smoothing. Here we have used  $\sigma = 1$  where  $\chi(x - y) \neq 0$ . Expanding  $\chi$  in a Taylor series about  $y = x$  we have, modulo smoothing,

$$\begin{aligned} A_\rho f(x) &= \int e^{i(x-y)\cdot\xi} a_\rho(x, \xi) (\sigma f)(y) dy d\xi \\ &= \int e^{ix\cdot\xi} a_\rho(x, \xi) (\widehat{\sigma f})(\xi) d\xi \end{aligned}$$

where  $\widehat{g}$  denotes the Fourier transform of  $g$ . We now expand  $a_\rho(x, \xi)$  in a Taylor series about  $\xi = 0$  to obtain

$$\begin{aligned} A_\rho f(x) &= \int e^{ix\cdot\xi} a_\rho(x, 0) (\widehat{\sigma f})(\xi) d\xi \\ &\quad + \int e^{ix\cdot\xi} \sum_{j=1}^3 \xi_j \int_0^1 (\partial_{\xi_j} a_\rho)(x, t\xi) dt (\widehat{\sigma f})(\xi) d\xi \\ &= a_\rho(x, 0) f(x) + R_\rho^{(1)}(\sigma f)(x) \end{aligned}$$

modulo smoothing. Let  $\tilde{\sigma}(R_\rho^{(1)})$  denote the symbol of  $R_\rho^{(1)}$ ; on the one hand, since  $a_\rho \in S^0(\mathbb{R}^3, Z)^{8 \times 8}$

$$|\tilde{\sigma}(R_\rho^{(1)})| = \left| \sum_{j=1}^3 \xi_j \int_0^1 (\partial_{\xi_j} a_\rho)(x, t\xi) dt \right| \leq \frac{C|\xi|}{1+|\rho|}$$

for some constant  $C > 0$ , but on the other hand, since  $\tilde{\sigma}(R_\rho^{(1)}) = a_\rho(x, \xi) - a_\rho(x, 0)$ , it is homogeneous of degree 0 in  $\xi$  and  $\rho$ , and so

$$|\tilde{\sigma}(R_\rho^{(1)})| \leq \frac{C}{1+|\rho|}.$$

Therefore,

$$\|R_\rho^{(1)}(\sigma f)\|_{L^2(\Omega)} \leq \frac{C}{1+|\rho|} \|\sigma f\|_{L^2(\mathbb{R}^3)} \leq \frac{C}{1+|\rho|} \|f\|_{L^2(\Omega)}.$$

We have denoted  $R_\rho(f) = R_\rho^{(1)}(\sigma f)$ . □

COROLLARY 4.8. *If  $f \in H^2(\Omega)^8$  then there exists a constant  $C > 0$  independent of  $f$  and  $\rho$  such that*

$$\begin{aligned} P(\nabla)A_\rho f(x) &= (P(\nabla)a_\rho(x, 0))f(x) + a_\rho(x, 0)(P(\nabla)f)(x) \\ &\quad + R_\rho(P(\nabla)f)(x) + R'_\rho f(x) \end{aligned}$$

*modulo smoothing, and with*

$$\|R_\rho\|_{L^2(\Omega), L^2(\Omega)} + \|R'_\rho\|_{L^2(\Omega), L^2(\Omega)} \leq \frac{C}{1 + |\rho|}$$

*Here  $\|\cdot\|_{L^2(\Omega), L^2(\Omega)}$  denotes the operator norm.*

COROLLARY 4.9. *If  $X_\rho = (P(\nabla) + V')Y_\rho$  and  $Y_\rho$  is as in (4.8), then in  $\Omega$ ,*

$$X_\rho = e^{x \cdot \rho} (P(\rho)a_\rho(x, 0)y_{0,\rho} + W_\rho)$$

*and there is a constant  $C > 0$  such that*

$$\|W_\rho\|_{L^2(\Omega)} \leq C.$$

*Proof.* We have

$$X_\rho = e^{x \cdot \rho} (P(\rho)A_\rho y_{0,\rho} + X_{\rho,s})$$

where

$$X_{\rho,s} = P(\rho)A_\rho \psi_\rho + P(\nabla)(A_\rho y_{0,\rho} + A_\rho \psi_\rho) + V'A_\rho(y_{0,\rho} + \psi_\rho).$$

Applying proposition 4.7 and corollary 4.8 together with the estimate (4.7) from proposition 4.5, the corollary follows.  $\square$

Let  $F_1$  and  $F_2$  be the projections so that  $F_1 X = E = (X_2, X_3, X_4)$  and  $F_2 X = H = (X_5, X_6, X_7)$ . In constructing solutions to use in the identity (4.3), we must make choices of  $\rho_j$  and  $y_{0,\rho}$ , in such a way that, for example, the inner products  $H_1 \cdot E_2$  and  $H_2 \cdot E_1$  grow in  $|\rho|$  at a different rate from the products  $E_1 \cdot E_2$  and

$H_1 \cdot H_2$ . This will enable us to isolate the  $(\beta_1 - \beta_2)$  term. From corollary 4.9 and expression (4.9) the fields have the form

$$\begin{aligned} E &= e^{x \cdot \rho} F_1 P(\rho) a_\rho(x, 0) y_{0, \rho} + O(s^0) \\ &= e^{x \cdot \rho} [(\bar{a}_1 \cdot y_{0, \rho}) \rho + \rho \wedge (y_5, y_6, y_7)] + O(s^0) \end{aligned} \quad (4.10)$$

$$\begin{aligned} H &= e^{x \cdot \rho} F_2 P(\rho) a_\rho(x, 0) y_{0, \rho} + O(s^0) \\ &= e^{x \cdot \rho} [(\bar{a}_8 \cdot y_{0, \rho}) \rho - \rho \wedge (y_2, y_3, y_4)] + O(s^0) \end{aligned} \quad (4.11)$$

and so our choices for  $\rho$  and  $y_{0, \rho}$  are to control the growth in  $|\rho|$  in the inner product of such terms.

We now make these choices for  $\rho_j$  and  $y_{0, \rho_j}$ . Fix  $k \in \mathbb{R}^3$  and for  $s \in \mathbb{R}$ ,  $s > 0$ , let  $\eta, \xi \in \mathbb{R}^3$  be such that

$$\begin{aligned} \langle \eta, k \rangle &= \langle \eta, \xi \rangle = \langle k, \xi \rangle = 0 \\ |\eta|^2 &= \frac{|k|^2}{4} + s^2 \\ |\xi|^2 &= 1. \end{aligned}$$

Notice that  $\eta$  is of order one in  $s$  and  $\xi$  and  $k$  are bounded in  $s$ : as evident in the following expressions for  $\rho_j$ ,  $|\rho|$  grows with the parameter  $s$ . Set

$$\begin{aligned} \rho_1 &= \eta + i \left( \frac{k}{2} + s\xi \right) \\ \rho_2 &= -\eta + i \left( \frac{k}{2} - s\xi \right) \end{aligned}$$

so that

$$\rho_1 + \rho_2 = ik \quad (4.12)$$

$$\rho_j \cdot \rho_j = 0, \quad j = 1, 2.$$

To specify  $y_{0, \rho_j}$ , we define  $y_1, y_2 \in \mathbb{C}^3$  as

$$\begin{aligned} y_1 &= \frac{1}{s}(1+i)\xi - \frac{2}{|k|^2}(1+i)k \\ y_2 &= \frac{1}{s}(1-i)\xi + \frac{2}{|k|^2}(1-i)k \end{aligned}$$

and set

$$y_{0,\rho_1} = (0, 0, 0, 0, \dots, y_1 \dots, 0)$$

$$y_{0,\rho_2} = (0, \dots, y_2 \dots, 0, 0, 0, 0).$$

Observe that  $y_{0,\rho_j}$  is bounded in  $s$ ,  $j = 1, 2$ , and with this choice

$$\rho_j \cdot y_j = 0, \quad j = 1, 2.$$

This ensures that the conditions of proposition 4.6 are satisfied, which in turn ensures that the divergence free conditions of Maxwell's equations are satisfied by the constructed fields. As evident in the expressions (4.10) and (4.11), to compute  $E_1$  and  $H_2$ , we must compute  $\rho_1 \wedge y_1$  and  $\rho_2 \wedge y_2$ :

$$\begin{aligned} \rho_1 \wedge y_1 &= \left( \frac{|k|}{2s|\eta|} + \frac{2s}{|k||\eta|} \right) \eta + \frac{2|\eta|}{|k|} \xi + \frac{|\eta|}{s|k|} k \\ &\quad + i \left\{ \left( -\frac{|k|}{2s|\eta|} + \frac{2s}{|k||\eta|} \right) \eta + \frac{2|\eta|}{|k|} \xi + \frac{|\eta|}{s|k|} k \right\} \\ &= \frac{2s}{|k||\eta|} (1+i)\eta + \frac{2|\eta|}{|k|} (1+i)\xi + O(s^0) \end{aligned}$$

where we have used the fact that  $\eta$  is of order 1 in  $s$  and  $k$  and  $\xi$  are bounded in  $s$ .

Similarly,

$$\rho_2 \wedge y_2 = -\frac{2s}{|k||\eta|} (1+i)\eta + \frac{2|\eta|}{|k|} (1-i)\xi + O(s^0).$$

For  $j = 1, 2$ , given the chosen  $\rho_j$  let  $X_{\rho_j}$  be solutions constructed as in section 4.4 with parameters  $(\varepsilon_j, \mu_j, \beta_j)$ . Using (4.10), (4.11) and (4.12), we have

$$\begin{aligned} E_1 \cdot H_2 &= e^{x \cdot (\rho_1 + \rho_2)} F_1 P(\rho_1) a_{\rho_1}(x, 0) y_{0,\rho_1} \cdot F_2 P(\rho_2) a_{\rho_2}(x, 0) y_{0,\rho_2} + O(s^0) \\ &= e^{ix \cdot k} [(\bar{a}_{\rho_1,1} \cdot y_{0,\rho_1}) \rho_1 + \rho_1 \wedge y_1] \cdot [(\bar{a}_{\rho_2,2} \cdot y_{0,\rho_2}) \rho_2 - \rho_2 \wedge y_2] + O(s^0). \end{aligned}$$

In this expression we wish to calculate the highest order terms in  $s$ . To this end,

$$\begin{aligned} \rho_1 \cdot \rho_2 &= \frac{1}{2} |k|^2 = O(s^0), \\ \rho_1 \cdot (\rho_2 \wedge y_2) &= (\rho_1 \wedge \rho_2) \cdot y_2 \\ &= \left( -\frac{s|k|}{|\eta|} \eta - i|\eta||k|\xi \right) \cdot y_2 = O(s) \end{aligned}$$

and

$$\begin{aligned}
& (\rho_1 \wedge y_1) \cdot (\rho_2 \wedge y_2) \\
&= \left( \frac{2s}{|k||\eta|} (1+i)\eta + \frac{2|\eta|}{|k|} (1+i)\xi \right) \cdot \left( -\frac{2s}{|k||\eta|} (1+i)\eta + \frac{2|\eta|}{|k|} (1-i)\xi \right) + O(s) \\
&= \frac{8|\eta|^2}{|k|^2} (1-i) + O(s) \\
&= \frac{8s^2}{|k|^2} (1-i) + O(s).
\end{aligned}$$

Thus

$$E_1 \cdot H_2 = e^{ix \cdot k} \frac{8s^2}{|k|^2} (i-1) + O(s).$$

It is the second-order growth in  $s$  of this term which isolates it from the remaining inner products in (4.3): by the choices of  $y_{0,\rho}$ , and the above calculations, we have

$$\begin{aligned}
E_2 \cdot H_1 &= e^{ix \cdot k} F_1 P(\rho_2) a_{\rho_2}(x, 0) y_{0,\rho_2} \cdot F_2 P(\rho_1) a_{\rho_1}(x, 0) y_{0,\rho_1} + O(s^0) \\
&= e^{ix \cdot k} (\bar{a}_{\rho_2,1} \cdot y_{0,\rho_2}) \rho_2 \cdot (\bar{a}_{\rho_1,8} \cdot y_{0,\rho_1}) \rho_1 + O(s^0) \\
&= O(s^0),
\end{aligned}$$

and similarly

$$E_1 \cdot E_2 = O(s)$$

$$H_1 \cdot H_2 = O(s).$$

We may thus use these solutions and the  $s$ -asymptotics in the identity (4.3). We have

$$\begin{aligned}
0 &= \lim_{s \rightarrow \infty} \frac{1}{s^2} \int_{\Omega} e^{ix \cdot k} \left( (\beta_1 - \beta_2) \frac{8s^2}{|k|^2} (i-1) + O(s) \right) dx \\
&= \int e^{ix \cdot k} (\beta_1 - \beta_2) \frac{8}{|k|^2} (i-1) dx
\end{aligned}$$

where we have used the fact that  $\beta_1$  and  $\beta_2$  have been extended to be equal outside  $\Omega$ . This holds for all non-zero vectors  $k$  in  $\mathbb{R}^3$ , and so by inversion of the Fourier transform we have  $\beta_1 = \beta_2$  throughout  $\Omega$ .

With the knowledge that  $\beta_1 = \beta_2$ , identity (4.3) becomes

$$\int_{\Omega} ((\varepsilon_1 - \varepsilon_2)E_1 \cdot E_2 + (\mu_2 - \mu_1)H_1 \cdot H_2)dx = 0.$$

We must now make similar choices of  $\rho_j$  and  $y_{0,\rho_j}$  to isolate  $(\varepsilon_1 - \varepsilon_2)E_1 \cdot E_2$ . We choose  $\rho_j$  as before, and

$$\begin{aligned} y_{0,\rho_1} &= (0, 0, 0, 0, \dots, y_1 \dots, 0) \\ y_{0,\rho_2} &= (0, 0, 0, 0, \dots, y_2 \dots, 0). \end{aligned}$$

With these choices, computing as before we find

$$\begin{aligned} E_1 \cdot E_2 &= e^{ix \cdot k} \frac{8s^2}{|k|^2} (i - 1) + O(s), \\ H_1 \cdot H_2 &= O(s^0). \end{aligned}$$

We thus obtain  $\varepsilon_1 = \varepsilon_2$  in the same manner, and finally that  $\mu_1 = \mu_2$ .

#### 4.6 Proof of proposition 4.4

Let  $S_\rho = \text{Char}(\Delta_\rho) = \{\xi \in \mathbb{R}^3 \mid -|\xi|^2 + 2i\rho \cdot \xi = 0\}$ . In a neighborhood of  $S_\rho$ , we will construct  $A_\rho = B_\rho$ , and so in such a neighborhood, (4.6) is equivalent to

$$[\Delta_\rho, A_\rho] + N_\rho^+ A_\rho - A_\rho \varphi C_\rho \varphi = 0. \quad (4.13)$$

We define  $A_\rho$  by defining its symbol  $a_\rho(x, \xi) \in S^0(\mathbb{R}^3 \times \mathbb{R}^3 \times Z)^{8 \times 8}$ , an  $8 \times 8$  matrix.

Write

$$\rho = \eta + ik, \quad \text{with} \quad \eta, k \in \mathbb{R}^3.$$

Computing terms of homogeneity of order 1 in  $\xi$  and  $\rho$  in (4.13) we have

$$(L_1 + iL_2)a_\rho^{(0)} + \frac{1}{2|\rho|}n_\rho a_\rho^{(0)} = 0 \quad (4.14)$$

where  $a_\rho^{(0)}$  is the principal symbol of  $A_\rho$ ,

$$L_1 = \frac{1}{|\rho|} \sum_{j=1}^3 \eta_j \frac{\partial}{\partial x_j}, \quad L_2 = \frac{1}{|\rho|} \sum_{j=1}^3 (k_j + \xi_j) \frac{\partial}{\partial x_j}.$$

and  $n_\rho$  is the principal symbol of  $N_\rho^+$ . Observe that so long as  $L_1$  and  $L_2$  are linearly independent, there is a change of variables mapping  $L_1 + iL_2$  to  $\bar{\partial}$  where

$$\bar{\partial} = \frac{1}{2} \left( \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \right);$$

in some of the proofs that follow we shall assume that  $L_1 + iL_2 = \bar{\partial}$  to simplify the exposition. It is easy to see that  $L_1$  and  $L_2$  are linearly independent on and hence near  $S_\rho$ , which for fixed  $\rho = \eta + ik$  is the circle orthogonal to  $\eta$  of radius  $|k|$  and center  $-k$ .

We now describe a partition of unity of  $\mathbb{R}_\xi^3$ -space which depends smoothly on  $\rho$ , and which divides the space into a tubular neighborhood of  $S_\rho$  and the complement. Let

$$\begin{aligned} \mathcal{U}_{1,\rho} &= \left\{ \xi \in \mathbb{R}^3 \mid |\xi - S_\rho| < \frac{2}{3\sqrt{2}}|\rho| \right\}, & \mathcal{U}_{1,\rho}^0 &= \left\{ \xi \in \mathbb{R}^3 \mid |\xi - S_\rho| < \frac{1}{3\sqrt{2}}|\rho| \right\} \\ \mathcal{U}_{2,\rho} &= \left\{ \xi \in \mathbb{R}^3 \mid |\xi - S_\rho| > \frac{1}{3\sqrt{2}}|\rho| \right\}, & \mathcal{U}_{2,\rho}^0 &= \left\{ \xi \in \mathbb{R}^3 \mid |\xi - S_\rho| > \frac{2}{3\sqrt{2}}|\rho| \right\} \end{aligned}$$

For  $|\rho| = 1$  let  $\{\tilde{\varphi}_{1,\rho}, \tilde{\varphi}_{2,\rho}\}$  be a partition of unity subordinate to the open cover  $\{\mathcal{U}_{1,\rho}, \mathcal{U}_{2,\rho}\}$  of  $\mathbb{R}_\xi^3$ , depending smoothly on  $\rho$ , and such that

$$\tilde{\varphi}_{1,\rho} = 1 \text{ on } \mathcal{U}_{1,\rho}^0, \quad \text{and} \quad \tilde{\varphi}_{2,\rho} = 0 \text{ on } \mathcal{U}_{1,\rho}^0.$$

Then  $\{\xi \mid \tilde{\varphi}_{1,\rho}(\xi) = 1 \text{ and } \tilde{\varphi}_{2,\rho}(\xi) = 0\}$  is a tubular neighborhood of  $S_\rho$  of radius  $|\rho|/3\sqrt{2}$ . On this neighborhood,  $L_1$  and  $L_2$  are linearly independent. Now extend  $\tilde{\varphi}_{j,\rho}$  to all of  $\mathbb{R}_\xi^3 \times Z$  to be homogeneous of degree zero in  $\xi$  and  $\rho$  for  $|\rho| > 1$  say, and arbitrarily for  $|\rho| < 1$ ; that is define

$$\tilde{\varphi}_{j,\rho}(\xi) = \tilde{\varphi}_{j,\frac{\rho}{|\rho|}}\left(\frac{\xi}{|\rho|}\right),$$

so

$$\tilde{\varphi}_{j,\lambda\rho}(\lambda\xi) = \tilde{\varphi}_{j,\frac{\lambda\rho}{\lambda|\rho|}}\left(\frac{\lambda\xi}{\lambda|\rho|}\right) = \tilde{\varphi}_{j,\frac{\rho}{|\rho|}}\left(\frac{\xi}{|\rho|}\right) = \tilde{\varphi}_{j,\rho}(\xi).$$

**PROPOSITION 4.10.** *Let  $-1 < \delta < 0$ . There is a unique  $a_\rho^{(0)}(x, \xi) \in S^0(\mathbb{R}^3, Z)$  solving (4.14) with  $a_\rho^{(0)} - I \in L_\delta^2(\mathbb{R}_x^3)$ ; furthermore,  $a_\rho^{(0)}$  is invertible for large  $\rho$ .*

*Proof.* We shall only need the solution on the support of  $\tilde{\varphi}_{1,\rho}$  where  $L_1$  and  $L_2$  are linearly independent, and so we shall prove the result for  $\bar{\partial}$ :

$$\bar{\partial}a_\rho^{(0)} + \frac{1}{2|\rho|}n_\rho a_\rho^{(0)} = 0 \quad (4.15)$$

Write  $a_\rho^{(0)} = d_\rho + I$ , and  $\bar{d}_\rho = \bar{\partial}d_\rho$ ; thus we must solve

$$\left(I + \frac{1}{2|\rho|}n_\rho \bar{\partial}^{-1}\right) \bar{d}_\rho = -\frac{1}{2|\rho|}n_\rho. \quad (4.16)$$

We shall need the following lemmas.

**LEMMA 4.11.** *If  $-1 < \delta < 0$  then*

$$\frac{1}{2|\rho|}n_\rho \bar{\partial}^{-1} : L_{\delta+1}^2(\mathbb{R}_x^3) \rightarrow L_{\delta+1}^2(\mathbb{R}_x^3)$$

*is compact.*

*Proof.* From [10] Theorem 2.1 (with  $n = 3$ ,  $p = p' = 2$ ,  $\rho = \delta$ ,  $m = 1$ ,  $r = 0$ ), for  $v \in C_0^\infty(\mathbb{R}^3)$

$$\|v\|_{H_\delta^1} \leq C \|\bar{\partial}v\|_{L_{\delta+1}^2}$$

and since  $H_\delta^1$  is the completion of  $C_0^\infty(\mathbb{R}^3)$  in this norm, the same estimate holds for all  $v \in H_\delta^1(\mathbb{R}^3)$  such that  $\bar{\partial}v \in L_{\delta+1}^2$ . Thus

$$\bar{\partial}^{-1} : L_{\delta+1}^2 \rightarrow H_\delta^1$$

continuously. Since  $n_\rho$  is compactly supported (in  $x$ ), we have

$$L_{\delta+1}^2 \xrightarrow[\text{cts}]{\bar{\partial}^{-1}} H_\delta^1 \xrightarrow[\text{cts}]{n_\rho} H^1(\text{supp}(n_\rho)) \xrightarrow[\text{compact}]{\text{incl}} L^2(\text{supp}(n_\rho)) \xrightarrow[\text{cts}]{\text{incl}} L_{\delta+1}^2$$

□

LEMMA 4.12. *The equation*

$$\left( I + \frac{1}{2|\rho|} n_\rho \bar{\partial}^{-1} \right) \bar{d}_\rho = 0$$

has only the trivial solution in  $L^2_{\delta+1}(\mathbb{R}^3_x)$ .

*Proof.* With  $d_\rho = \bar{\partial}^{-1} \bar{d}_\rho$ , we show that  $d_\rho = 0$  is the unique solution in  $L^2_\delta$  to

$$\bar{\partial} d_\rho + \frac{1}{2|\rho|} n_\rho d_\rho = 0.$$

Since  $\text{supp}(n_\rho) \subset \{z \mid |z| \leq R\}$  for some  $R$ ,  $d_\rho$  is analytic for  $|z| > R$ . From

$$d_\rho(z) = \frac{-1}{2\pi i} \int_{|z| \leq R} \frac{1}{z-w} \frac{n_\rho(w) d_\rho(w)}{2|\rho|} dw \wedge d\bar{w}$$

it follows easily that  $d_\rho(z)$  decays to all orders at infinity; since  $d_\rho$  is also analytic in a neighborhood of infinity, it follows that  $d_\rho(z) = 0$  in a neighborhood of infinity. Now by (Cor. 5.3.8, [21]) unique continuation implies  $d_\rho(z)$  is identically zero.  $\square$

From the above lemmas and the Fredholm alternative, there is a unique  $\bar{d}_\rho \in L^2_{\delta+1}$  solving (4.16), or if we write  $a_\rho^{(0)} = I + \bar{\partial}^{-1} \bar{d}_\rho$ ,  $a_\rho^{(0)} - I \in L^2_\delta(\mathbb{R}^3_x)$  and  $a_\rho^{(0)}$  solves (4.15).

To prove that  $a_\rho^{(0)}$  is invertible we exploit the structure of  $n_\rho$  (see section 4.3)

$$n_\rho = \begin{bmatrix} \bar{v}_{12} \cdot (\rho + i\xi) & -\bar{v}_{13} \wedge (\rho + i\xi) & \bar{v}_{12} \wedge (\rho + i\xi) & \bar{v}_{13} \cdot (\rho + i\xi) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \bar{v}_{42} \cdot (\rho + i\xi) & -\bar{v}_{43} \wedge (\rho + i\xi) & \bar{v}_{42} \wedge (\rho + i\xi) & \bar{v}_{43} \cdot (\rho + i\xi) \end{bmatrix}.$$

This implies that  $a_\rho^{(0)}$  is of the form

$$\begin{bmatrix} a_{\rho,11}^{(0)} & a_{\rho,12}^{(0)} & \dots & a_{\rho,18}^{(0)} \\ 0 & \ddots & & 0 \\ \vdots & & I_6 & \vdots \\ 0 & & & \ddots & 0 \\ a_{\rho,81}^{(0)} & a_{\rho,82}^{(0)} & \dots & a_{\rho,88}^{(0)} \end{bmatrix}$$

where  $I_6$  is the  $6 \times 6$  identity matrix. It follows that

$$\det(a_\rho^{(0)}) = \det \begin{bmatrix} a_{\rho,11}^{(0)} & a_{\rho,18}^{(0)} \\ a_{\rho,81}^{(0)} & a_{\rho,88}^{(0)} \end{bmatrix} = \det(\tilde{a}_\rho^{(0)}), \text{ say,}$$

and  $\tilde{a}_\rho^{(0)}$  satisfies

$$\bar{\partial} \tilde{a}_\rho^{(0)} + \frac{1}{2|\rho|} \begin{bmatrix} n_{\rho,11} & n_{\rho,18} \\ n_{\rho,81} & n_{\rho,88} \end{bmatrix} \tilde{a}_\rho^{(0)} = 0$$

and so  $\det(\tilde{a}_\rho^{(0)})$  satisfies

$$\bar{\partial} \det(\tilde{a}_\rho^{(0)}) + \text{tr} \begin{bmatrix} n_{\rho,11} & n_{\rho,18} \\ n_{\rho,81} & n_{\rho,88} \end{bmatrix} \det(\tilde{a}_\rho^{(0)}) = 0. \quad (4.17)$$

Furthermore,  $a_\rho^{(0)} - I \in L_\delta^2(\mathbb{R}_x^3)$  implies  $|\det a_\rho^{(0)} - 1| \rightarrow 0$  as  $|x| \rightarrow \infty$ . From this and the compact support of  $n_\rho$ , (4.17) has unique solution with  $\det(\tilde{a}_\rho^{(0)}) - 1 \in L_\delta^2(\mathbb{R})$  given by  $\det(\tilde{a}_\rho^{(0)}) = e^{-\tau}$  where  $\bar{\partial}\tau = (1/2|\rho|)(n_{\rho,11} + n_{\rho,88})$ . Thus  $\det a_\rho^{(0)} = \det \tilde{a}_\rho^{(0)} \neq 0$  and  $a_\rho^{(0)}$  is invertible. The smoothness of  $a_\rho^{(0)}$  follows from differentiating equation (4.15) and from the fact that the change of coordinates transforming  $L_1 + iL_2$  to  $\bar{\partial}$  is smooth.

We define  $a_\rho^{(j)}$  for  $j < 0$  iteratively to be homogeneous of order  $j$  in  $\xi$  and  $\rho$  by considering terms of homogeneity  $j + 1$  in (4.13) and write  $a_\rho$  as an asymptotic sum of the  $a_\rho^{(j)}$ . This completes the proof of proposition 4.10.  $\square$

Recall that we have been restricting ourselves to a neighborhood of  $S_\rho$  where we may consider  $L_1 + iL_2$  to be  $\bar{\partial}$ ; now define  $a_\rho$  on all of  $\mathbb{R}_x^3 \times \mathbb{R}_\xi^3 \times Z_\rho$  by taking  $\check{\varphi}_{1,\rho} a_\rho + \check{\varphi}_{2,\rho} I$  - abusing notation, we shall call this  $a_\rho$ . Since  $\check{\varphi}_{j,\rho}$  are homogeneous of degree 0 in  $\xi$  and  $\rho$ ,  $a_\rho \in S^0(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3 \times Z_\rho)$ .

To achieve (4.6) we now define  $C_\rho \in L^0(\mathbb{R}^3, Z)$  by

$$A_\rho \varphi C_\rho \varphi = [\Delta_\rho, A_\rho] + N_\rho^+ A_\rho \quad (4.18)$$

for large  $|\rho|$  so that  $A_\rho$  is invertible. Next we define  $B_\rho \in L^0(\mathbb{R}^3, Z)$  by

$$B_\rho = \tilde{\varphi}_{1,\rho} A_\rho + \tilde{\varphi}_{2,\rho} (\Delta_\rho + N_\rho^+) A_\rho (\Delta_\rho + \varphi C_\rho \varphi)^{-1} \quad (4.19)$$

observing that  $\Delta_\rho + \varphi C_\rho \varphi$  is invertible on  $\text{supp} \tilde{\varphi}_{2,\rho}$  which is disjoint from  $S_\rho$ . To summarize, where  $\tilde{\varphi}_{1,\rho} = 1$ ,  $A_\rho = B_\rho$  and we have (4.6) via (4.13); where  $\tilde{\varphi}_{2,\rho} = 1$ , (4.19) gives (4.6), and in between,

$$\begin{aligned} B_\rho (\Delta_\rho + \varphi C_\rho \varphi) &= \tilde{\varphi}_{1,\rho} A_\rho (\Delta_\rho + \varphi C_\rho \varphi) + \tilde{\varphi}_{2,\rho} (\Delta_\rho + N_\rho^+) A_\rho \\ &= (\tilde{\varphi}_{1,\rho} + \tilde{\varphi}_{2,\rho}) (\Delta_\rho + N_\rho^+) A_\rho \end{aligned}$$

by (4.18). This completes the proof of proposition 4.4.  $\square$

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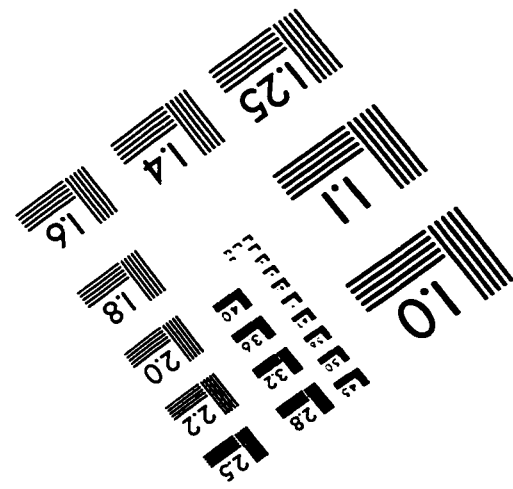
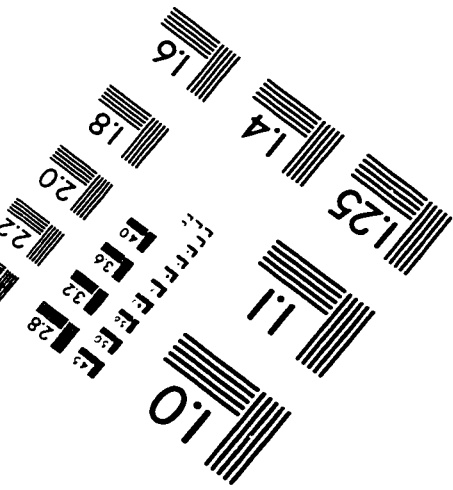
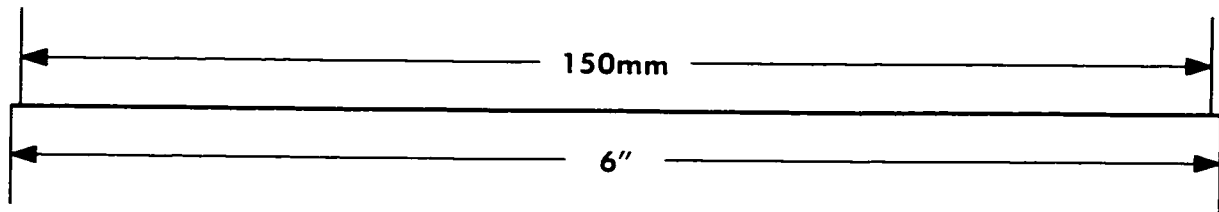
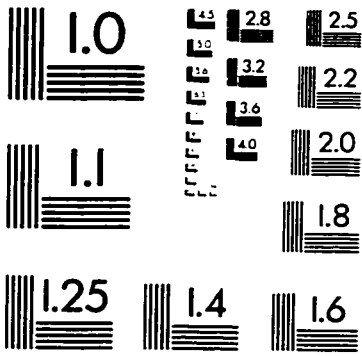
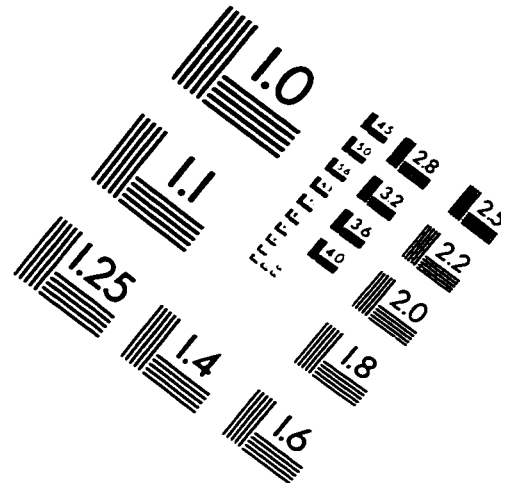
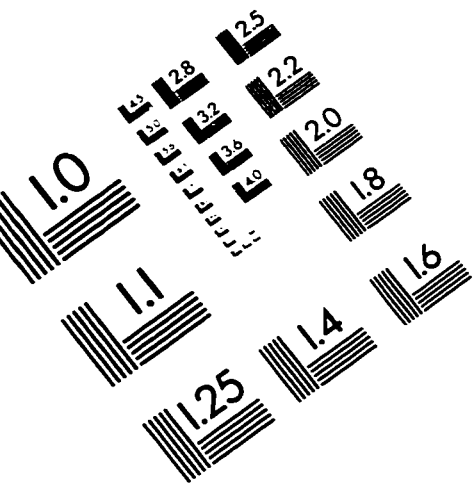
## Vita

Stephen McDowall was born on the 28th of December, 1969 in Wellington, New Zealand. He earned the degree of Bachelor of Science with first class honors in mathematics in May of 1992 at the University of Canterbury, Christchurch, New Zealand. In May of 1993 he earned a Master of Science with Distinction in mathematics at the same institution. In August of 1998 he recieved the Doctor of Philosophy degree in mathematics from the University of Washington, Seattle, Washington.

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1. *An Electromagnetic Inverse Problem in Chiral Media*, Trans. AMS., to appear.
2. *Boundary Determination of Material Parameters from Electromagnetic Boundary Information*. Inverse Problems **13** (1997) 153-163.
3. *Constructing Infinite Dimensional Polytopes using Gale Transforms*, SEAMS Bull. Math., to appear.

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