Three Applications of Conformal Geometry Techniques to Planar Geometry Problems

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Abstract

Three Applications of Conformal Geometry Techniques
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We talk about how to use conformal geometry techniques to solve problems in three different areas. For the first application, we introduce a Collet-Eckmann type condition for the unicritical laminations on the unit circle. We prove that this condition implies the lamination admits a Hölder continuous conformal welding which produces a Julia set for some unicritical polynomial. For the second application, We consider the solution of $-\Delta u = 1$ on convex domains $\Omega \subset \mathbb{R}^2$ subject to Dirichlet boundary conditions $u = 0$ on $\partial\Omega$. We consider the two shape optimization problems $\|\nabla u\|_{L^\infty}/|\Omega|^{1/2}$ and $\|\nabla u\|_{L^\infty}/\mathcal{H}^1(\partial\Omega)$. We prove that (1) either the extremal domain does not have a $C^{2+\varepsilon}$ boundary or (2) there exists an infinite set of points on $\partial\Omega$ where the curvature vanishes. Finally, we present a result about electrostatic skeleton. Eremenko conjectured that every convex polygon admits a unique electrostatic skeleton. We will prove the conjecture for quadrilaterals with a line of symmetry using arguments from conformal geometry. We will also discuss a natural condition that implies the existence of electrostatic skeletons.

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Most of my thesis research was carried out on the unceded traditional lands of the Coast Salish peoples. I recognize the privilege of pursuing mathematical research in this place and express my gratitude to the land and to the Indigenous communities who have stewarded it since time immemorial.

Chapter 1

CONFORMAL WELDING OF UNICRITICAL LAMINATIONS

1.1 Introduction

1.1.1 Motivation & Background

We investigate the geometry of Julia sets and their corresponding polynomials. We focus on the cases where polynomials are unicritical ($p(z) = z^d + c$) and Julia sets are topologically "trees" (locally connected unique geodesic spaces, or *dendrites*). In [28], the authors introduced what is now called the *Collett-Eckmann (CE) condition* for rational maps. In [39], J. Graczyk and S. Smirnov proved that CE condition is sufficient for Fatou components of the polynomials to be Hölder domains. The converse of this statement is proven in [87]. In particular, this implies that the Julia sets for such polynomials are (quasi-)conformally removable [47]. S. Smirnov [98] has also shown that with the exception of a set of zero harmonic measure, all the points on the boundary of the Mandelbrot set correspond to CE polynomials. This implies that CE is a generic condition for unicritical polynomials and implies nice geometric properties for their Julia sets. The CE condition has been extensively studied for rational maps. One might refer to [1, 2, 3, 12, 19, 32, 36, 40, 69, 85, 86, 91] and references therein.

There is a topological invariant analogue of CE called the *topological Collett-Eckmann (TCE) condition*, constructed in [70] and [90]. For unicritical polynomials, CE and TCE are equivalent (see for example, [89]). However, TCE is generally a stronger condition for rational maps. It has been also broadly studied in the context of dynamical systems acted by hyperbolic rational functions [25, 46]. In [88], the authors show that every rational map satisfying TCE possesses a unique invariant measure carrying certain statistical properties of the map. In [29], it is shown that TCE rational maps satisfy some large-deviation principles.

Using the Riemann mapping, one can translate the geometry information of a Julia set into a flat equivalence relation (or a *lamination*) on the unit circle $\mathbb{T}$. This was first studied by B. Thurston in the mid 1980's, whose manuscript was unpublished until included in [97] (see also [13]). Compared to the intricate geometric features of the Julia set, the corresponding lamination can also be described easily in terms of symbolic dynamics, and is parametrized by points on $\mathbb{T}$ and degree of the unicritical polynomial (see the example in Figure 1.1). A. Douady and J. Hubbard [30] first discovered that polynomial Julia sets can be encored symbolically using what are now called the *Hubbard trees*. Since then, a lot of progress has been made studying polynomial Julia sets using symbolic dynamics. One might refer to [14, 15, 16, 17, 18, 20, 21, 27, 48, 51, 53, 67, 80, 109], and references therein.

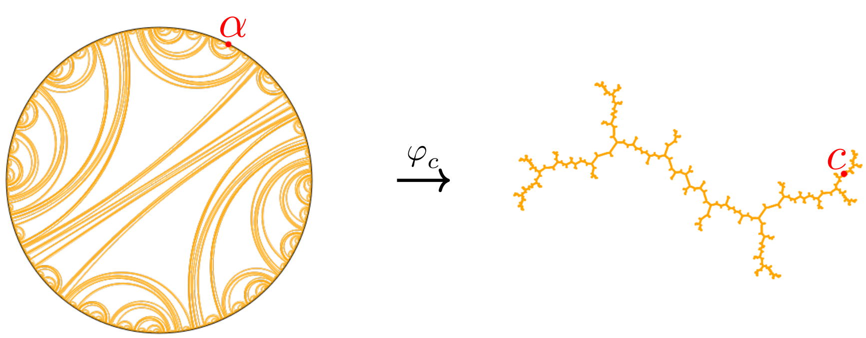

Figure 1.1: Quadratic lamination $\approx_\alpha^{(2)}$ with $\alpha \approx e^{0.35\pi i}$ and its corresponding quadratic Julia set J_c with $c \approx -0.326 + 0.102i$. Riemann map φ_c maps α to c and "glues" $\approx_\alpha^{(2)}$.

1.1.2 Statement of Results

Given a degree $d \geq 2$, we denote the aforementioned one-parameter family of laminations by $\{\approx_\alpha^{(d)}\}_{\alpha\in\mathbb{T}}$ (or just $\{\approx_\alpha\}_{\alpha\in\mathbb{T}}$ when the degree is clear in the context). The main goal of this paper is to find a "generic" condition analogous to CE on $\{\approx_\alpha^{(d)}\}_{\alpha\in\mathbb{T}}$. We want such a condition to guarantee the conformal welding reconstruction and ensure that the reconstructed Julia sets are *Hölder trees* (dendrites whose complements are Hölder domains). We draw inspiration from [59] and define the *combinatorial Collett-Eckmann (CCE) condition* on $\{\approx_\alpha^{(d)}\}_{\alpha\in\mathbb{T}}$, which is sufficient for reconstructing Hölder tree Julia sets.

Theorem 1.1.1. *If $\alpha \in \mathbb{T}$ satisfies the combinatorial Collet-Eckmann (CCE) Condition for degree 2, then the lamination $\approx_\alpha^{(2)}$ admits a conformal embedding, which reconstructs the corresponding Hölder tree Julia set, along with its unicritical polynomial $p_c(z) = z^2 + c$.*

The author does not see any obstruction against generalizing this theorem to higher degrees. We will only prove the case of degree 2 as the notations are much easier to work with. We shall also show that for the degree 2 case, CCE condition is satisfied by generic points on the unit circle.

Theorem 1.1.2. *Almost all $\alpha \in \mathbb{T}$ satisfies the CCE condition for degree 2. Thus $\approx_\alpha$ admits a Hölder continuous conformal welding, which reconstructs the corresponding Hölder tree Julia set, along with its unicritical polynomial $p_c(z) = z^2 + c$.*

In Section 1.2, we will go through the preliminaries needed to prove the results and also give the overview of the proofs. We will introduce the definition of the CCE condition and prove Theorem 1.1.1 in Section 1.3. In Section 1.4, we will discuss the *Strongly Recurrent (SR)* condition devised by S. Smirnov in [98], which is associated with the *semi-Combinatorial Collet-Eckmann (semi-CCE)* condition which we will define. In order to combine CCE and semi-CCE to prove Theorem 1.1.2, we will introduce the notion of *Generalized Cylinder Sets* in Section 1.5. With them, we will prove Theorem 1.1.2 in Section 1.6. One might find the flowchart in Figure 1.2 helpful in understanding the layout of the paper.

1.2 Preliminaries

We denote the Riemann sphere as $\hat{\mathbb{C}}$ and the unit circle as $\mathbb{T}$. For a Borel set $U \subseteq \mathbb{T}$, we denote its $\mathcal{H}^1$-measure as $|U|$. Given an alphabet $\mathcal{A}$, we denote the set of all finite words of this alphabet $\mathcal{A}^*$ and the set of all infinite words $\mathcal{A}^\infty$. If w is a word in $\mathcal{A}^* \cup \mathcal{A}^\infty$, we

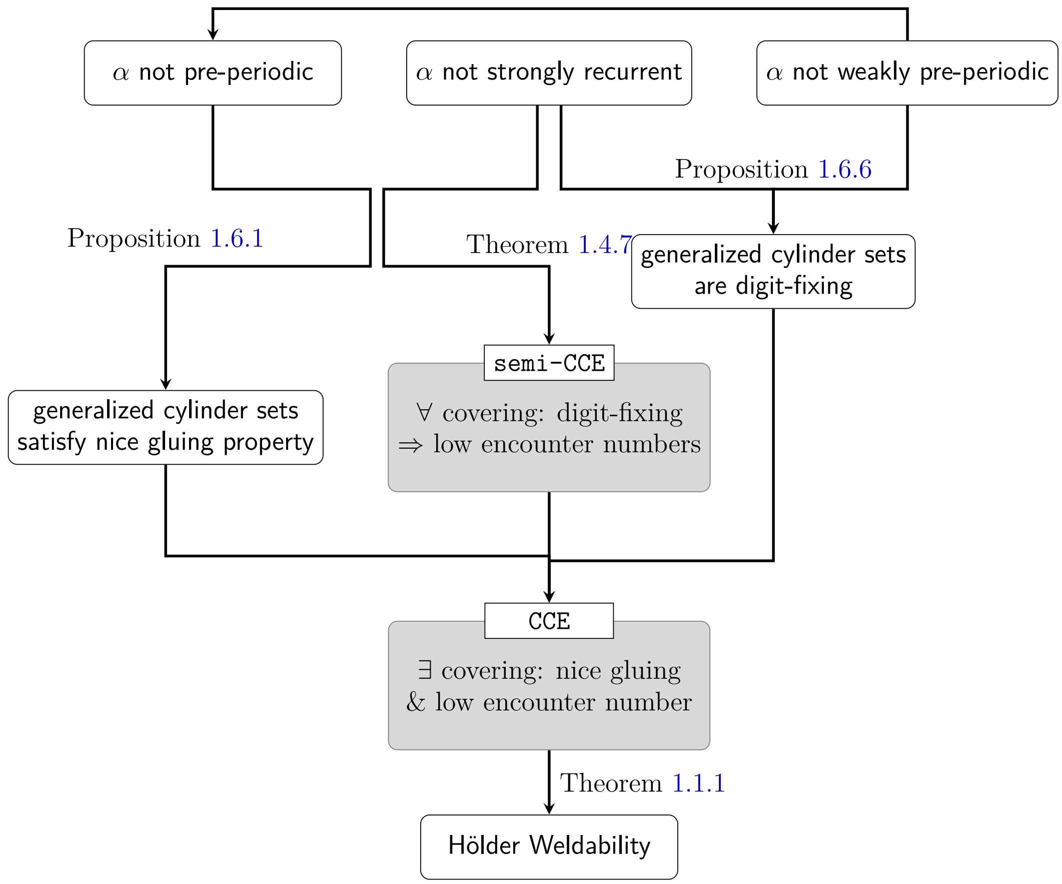

Figure 1.2: Outline for the proof of Theorem 1.1.2

denote its n-th letter by $w[n] \in \mathcal{A}$ and with some abuse of notation, we denote its length by $|w| \in \mathbb{N} \cup \{\infty\}$. Given a word w and an interval $I \subseteq (0, \infty)$, we denote the subword with letters on the indices $I \cap \mathbb{N}$ as $w|_I$. Given $a, b \in \mathbb{T}$, we denote the open proper arc between the two by $a \frown b$. When $a = -b$, we let $a \frown b$ be the arc going from a to b counterclockwise.

1.2.1 Laminations on $\mathbb{T}$ and Julia sets

Given a unicritial polynomial $p_c(z) = z^d + c$ and its corresponding Julia set J_c, if J_c is a dendrite, by the Riemann mapping theorem, we can find a unique Riemann map $\varphi_c : \hat{\mathbb{C}} \backslash \overline{\mathbb{D}} \to \hat{\mathbb{C}} \backslash J_c$ such that $\varphi_c(\infty) = \infty$ and $\varphi_c(z) = z + O(1/z)$ near ∞ (we call it the *hydrodynamically* *normalized* Riemann map). Note that $\varphi_c^{-1} \circ p_c \circ \varphi_c$ is a d-to-1 map of $\hat{\mathbb{C}} \backslash \overline{\mathbb{D}}$ that is $z^d + o(z^d)$ near ∞. It follows that

$$\varphi_c^{-1} \circ p_c \circ \varphi_c(z) = z^d.$$

Since J_c is locally connected, by the Carathéodory-Torhorst theorem [108], the map φ_c has a unique continuous extension which is surjective from $\hat{\mathbb{C}} \backslash \mathbb{D}$ onto $\hat{\mathbb{C}}$. The extended φ_c defines a equivalence relation $\equiv_c$ on $\mathbb{T}$ by

$$x \equiv_c y \quad \Leftrightarrow \quad \varphi_c(x) = \varphi_c(y).$$

One can check that $\equiv_c$ is *forward-invariant* in the sense that $x^d \equiv_c y^d$ whenever $x \equiv_c y$. Now given an angle $\alpha \in \mathbb{T}$, we set $\star_{d+1} = \star_1, \dots, \star_d$ be the d roots of $z^d = \alpha$ ordered counterclockwise from 1. Note that $z \mapsto z^d$ maps each $\star_i \frown \star_{i+1}$ bijectively to $\mathbb{T} \backslash \{\alpha\}$. We let $\widetilde{L_i} : \mathbb{T} \backslash \{\alpha\} \to \star_i \frown \star_{i+1}$ denote the inverse of such a map. We can define a new forward-invariant lamination $\sim_\alpha$ with the following:

1. We set $\{\star_i\}_{i=1}^d \subseteq [\star_i]_{\sim_\alpha}$.

2. For $x \sim_\alpha y$ and $\alpha \notin \{x, y\}$, we set $\widetilde{L_i}(x) \sim_\alpha \widetilde{L_i}(y)$ for all $i = 1, \dots, d$.

3. If $x_n \to x$, $y_n \to y$ and $x_n \sim_\alpha y_n$ via (1) or (2), we set $x \sim_\alpha y$.

We say an equivalence relation on $\mathbb{T}$ is a *lamination* if it does not have crossings. The following lemma tells us that $\sim_\alpha$ is in fact a lamination.

Lemma 1.2.1 ([107]). *Suppose $x_1 \sim_\alpha y_1$ and $x_2 \sim_\alpha y_2$. If the line segment connecting x_1 and y_1 intersects that connecting x_2 and y_2, then $x_1 \sim_\alpha y_1 \sim_\alpha x_2 \sim_\alpha y_2$.*

Additionally, $\sim_\alpha$ is a d-invariant unicritical lamination as in [107]. Since $z \mapsto z^d$ conjugates with p_c via φ_c, by checking the external rays, we have the following:

Lemma 1.2.2 ([59]). *If J_c is a dendrite, for any $\alpha \in \varphi_c^{-1}(\{c\})$, we have $\sim_\alpha \subseteq \equiv_c$.*

We call $\varphi_c^{-1}(c)$ the *special angles* of polynomial p_c.

1.2.2 Symbolic Dynamics and Cylinder Sets

The lamination $\sim_\alpha$ is not easy to work with in our setting with how it is defined using limits. In this section, we will introduce an equivalent definition. Given $\alpha \in \mathbb{T}$, let $\{\star_i\} \subseteq \mathbb{T}$ and $g_i : \mathbb{T}\setminus\{\alpha\} \to \star_i \frown \star_{i+1}$ be the points and functions defined above.

Definition 1.2.3 (Itinerary Maps, [58])**.** An *itinerary map* I^α is a map from $\mathbb{T}$ to the set of infinite words with alphabet $\{L_1, \dots, L_d, \star\}$ such that for any $x \in \mathbb{T}$,

$$I^\alpha(x)[n] := \begin{cases} \star, & \text{if } x^{d(n-1)} \in \{\star_i\}, \\ L_i, & \text{if } x^{d(n-1)} \in \star_i \frown \star_{i+1}, \end{cases} \qquad \text{for all } n > 0.$$

In other words, $I^\alpha(x)$ tracks where x jumps when iteratively applying $z \mapsto z^d$. One can check that

1. $I^\alpha(x)|_{[t+1,\infty)} = I^\alpha(x^{td})$.

2. If $x \neq \alpha$, then $I^\alpha(\widetilde{L_i}(x)) = L_i I^\alpha(x)$.

3. If x is periodic under $z \mapsto z^d$, then $I^\alpha(x)$ is periodic. However, the period of x can be a multiple of the period of $I^\alpha(x)$.

Therefore, we can define an equivalence relation $\approx_\alpha$ as follows

$$x \approx_\alpha y \quad \Leftrightarrow \quad \text{for all } n \in \mathbb{N}, \quad I^\alpha(x)[n] = I^\alpha(y)[n] \quad \text{or} \quad \star \in \{I^\alpha(x)[n], I^\alpha(y)[n]\}.$$

One immediate consequence is that $\widetilde{L_i}(x) \approx_\alpha \widetilde{L_i}(y)$ if $x \approx_\alpha y$ and $\alpha \notin \{x, y\}$. That is, $\widetilde{L_i}$ preserves $\approx_\alpha$. This equivalence relation is the one in Theorems 1.1.1 and 1.1.2 and it satisfies the following:

Lemma 1.2.4 ([7, 107])**.**

1. *If J_c is a dendrite, then for any $\alpha \in \varphi_c^{-1}(\{c\})$, we have $\equiv_c \subseteq \approx_\alpha$.*

2. *We have that $\sim_\alpha \subseteq \approx_\alpha$. In the case when α is aperiodic under $z \mapsto z^d$, each equivalence class of $\approx_\alpha$ is finite and $\sim_\alpha = \approx_\alpha$.*

In particular, if J_c is a dendrite and c is aperiodic under p_c, then $\sim_\alpha$, $\approx_\alpha$ and $\equiv_c$ are the same lamination.

Given a finite word $w \in \{L_1, \ldots, L_d\}^n$, we can then associate a map $\widetilde{w}$ such that

$$\widetilde{w} = \widetilde{w[1]} \circ \widetilde{w[2]} \circ \cdots \circ \widetilde{w[n]}.$$

Since each $\widetilde{L_i}$ is well-defined on $\mathbb{T}$ except at α, $\widetilde{w}$ is well-definded everywhere on $\mathbb{T}$ except possibly at $\{\alpha^{di}\}_{i=0}^{|w|}$.

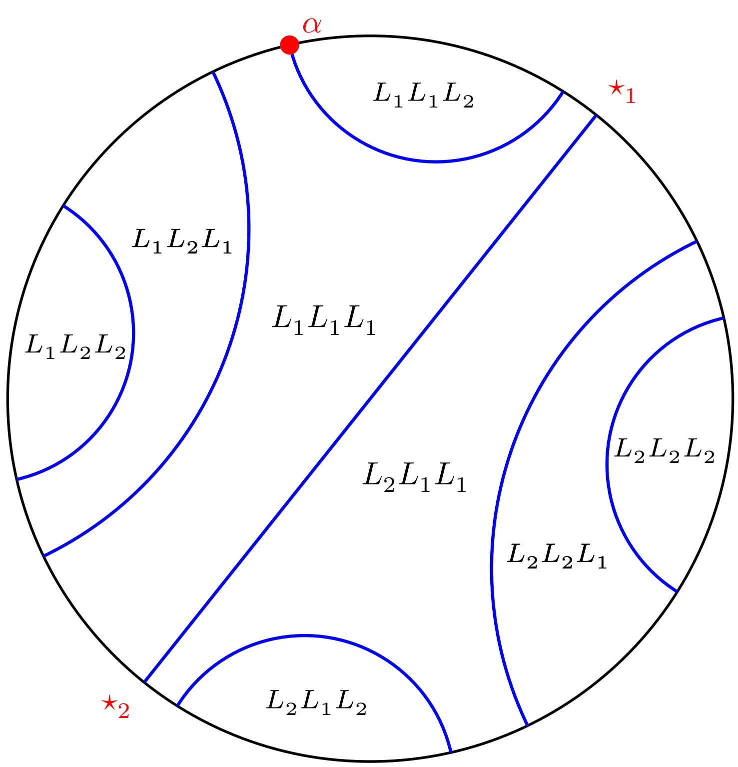

Figure 1.3: The cylinder sets for $\alpha = e^{4\pi i/7}$ associated with words of length 3 with $d = 2$.

Definition 1.2.5 (Cylinder Sets, [58])**.** Given a finite word $w \in \{L_1, \dots, L_d\}^*$, the *cylinder set* $C(w)$ associated with w is the closure of the image of $\widetilde{w}$ (see Figure 1.3). Alternatively, if we define $C(L_i) = \overline{\star_i \frown \star_{i+1}}$, then

$$C(w) = \left\{x \in \mathbb{T} : x^{d(i-1)} \in C(w[i]) \quad \text{for all } i = 1, \dots, n\right\}.$$

In consequence, we have that $x \in C(w)$ if $I^\alpha(x)[i] \in \{w[i], \star\}$ for all $i = 1, \dots, |w|$.

1.2.3 Gluing Links

In light of Lemma 1.2.4, we would only consider aperiodic α such that $\sim_\alpha$ (defined dynamically) and $\approx_\alpha$ (defined with itineraries) are the same. It is clear that there are only countable periodic angles so this assumption is harmless. It is easy to see that the cylinder sets defined above are saturated closed set with respect to the quotient map defined by $\approx_\alpha$. More generally, they are *gluing links*, whose definition is as follows:

Definition 1.2.6 (Gluing links)**.** A gluing link D (with respect to $\approx_\alpha$) is a finite union of disjoint closed arcs $\sqcup_{i=1}^n \overline{a_1 \frown b_1}$ with $a_1, b_1, \dots, a_n, b_n$ ordered counterclockwise such that $b_i \approx_\alpha a_{i+1}$ with $a_{n+1} = a_1$.

In other words, a gluing link is a finite set of arcs whose end points are "linked" by $\approx_\alpha$. With invariant properties of $\approx_\alpha$, one can easily verify the following properties of gluing links. To take pre-images more easily, we set $h(z) = z^d$ ("h" for "hopping").

Proposition 1.2.7 (see Figure 1.4)**.** *Given a gluing link $D \subseteq T$ with n arcs, we have*

1. *The image $h(D)$ is either the full circle or a gluing link.*

2. *If $\alpha \notin D$, then*
$$h^{-1}(D) = \bigsqcup_{i=1}^{d} \widetilde{L_i}(D),$$
where each $\widetilde{L_i}(D)$ is a gluing link with n arcs and $|\widetilde{L_i}(D)| = |D|/d$.

3. If $\alpha \in D$, then

$$h^{-1}(D) = \overline{\bigsqcup_{i=1}^{d} \widetilde{L_i}(D \setminus \{\alpha\})},$$

is a gluing link with dn arcs and $|h^{-1}(D)| = |D|$.

As a consequence of Lemma 1.2.7, given a gluing link D and integer $i \geq 0$, $h^{-i}(D)$ is a disjoint union of gluing links (Here we define h^0 as Id). We call each of them an *α-component* of $h^{-i}(D)$. If $x \in h^{-i}(D)$, we denote the α-component containing x by $\operatorname{Comp}_x^\alpha h^{-i}(D)$. To justify the notation, we observe that under the quotient topology of $\mathbb{T}$ given by $\approx_\alpha$,

$$\operatorname{Comp}_x^\alpha h^{-i}(D) = \operatorname{Comp}_x^{\mathbb{T}/\approx_\alpha} h^{-i}(D).$$

That is, α-components are just topological connected components of $h^{-i}(D)$.

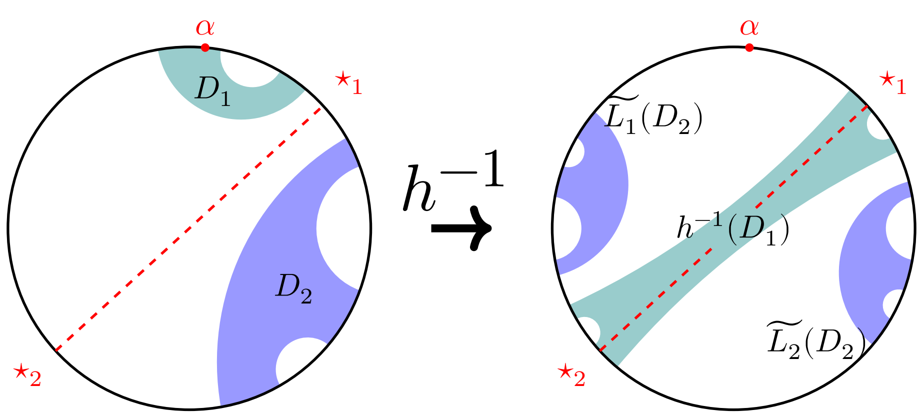

Figure 1.4: Picture for Proposition 1.2.7 where $d = 2$. The gluing link D_1 contains α and thus its preimage has one component, whereas the gluing link D_2's preimage has two separate gluing links as it does not contain α.

1.2.4 Pulling back Gluing Links

In this section, we will introduce a specific way to construct families of gluing links using Proposition 1.2.7. The proofs of the main theorems require us to extract the microscopic information of $\approx_\alpha$ at each point. In order to do that, we want to find gluing links of different scales, which we can do using the following construction: Suppose first that we have a covering of $\mathbb{T}$ by gluing links. We would like to denote this covering by $\{D(x)\}_{x\in\mathbb{T}}$, where for each point $x \in \mathbb{T}$, we assign a unique gluing link $D(x) \ni x$.

Lemma 1.2.8. *Given $x \in \mathbb{T}$, for all $n > 0$, we have $x \in h^{-n}(D(h^n(x)))$. As a consequence, we can set*

$$D^{(n)}(x) := \operatorname{Comp}_x^\alpha h^{-n}(D(h^n(x))),$$

and for each n, $\left\{D^{(n)}(x)\right\}_{x\in\mathbb{T}}$ forms a new covering of $\mathbb{T}$ by gluing links.

This statement is rather trivial as $h^n(x) \in D(h^n(x))$ by definition. Nevertheless, we would like to list it as a lemma for its usefulness. Note that by Proposition 1.2.7, $\left\{D^{(n)}(x)\right\}$ are generally much smaller than $\{D(x)\}$. Nevertheless, we do not necessarily have $D^{(n)}(x) \subseteq D(x)$. We can think of $D^{(n)}(x)$ as arising from the sequence

$$\begin{aligned} D(h^n(x)) := D^{(0)}(h^n(x)) \to D^{(1)}(h^{n-1}(x)) \to \cdots \\ \cdots \to D^{(n-1)}(h^1(x)) \to D^{(n)}(x), \end{aligned} \tag{1.1}$$

where at each step, we take the preimage of $D^{(i)}(h^{n-i}(x))$ and pick the α-component that contains $h^{n-i-1}(x)$. A consequence of Proposition 1.2.7 is as follows:

Corollary 1.2.9. *Consider the sequence of gluing links in (1.1). For each $i \in [1, n] \cap \mathbb{N}$, we have*

$$h(D^{(n-i+1)}(h^{i-1}(x))) = D^{(n-i)}(h^i(x)).$$

Moreover, if $D^{(n-i)}(h^i(x))$ does not contain α, then $D^{(n-i+1)}(h^{i-1}(x))$ is a rescaled image of $D^{(n-i)}(h^i(x))$ under some $\widetilde{L_j}$. If $D^{(n-i)}(h^i(x))$ does contain α, then

$$D^{(n-i+1)}(h^{i-1}(x)) = h^{-1}(D^{(n-i)}(h^i(x))).$$

For our purpose, we would like to think that there is a property P for $\approx_\alpha$ preserved by maps $\widetilde{L_j}$. This is quite natural considering that we have $\widetilde{L_i}(x) \approx_\alpha \widetilde{L_i}(y)$ whenever $x \approx_\alpha y$. We suppose

1. if a gluing link D with $|D| \asymp d^{-n}$ is "nice" for P and $\alpha \notin D$, then $\widetilde{L_j}(D)$ is a gluing link with size $\asymp d^{-n-1}$ still nice for P.

2. If a gluing link D with $|D| \asymp d^{-n}$ is nice for P but $\alpha \in D$, then $h^{-1}(D)$ is a gluing link with size $\asymp d^{-n-1}$ that is "slightly worse" for P.

Inspired by this, given $\{D(x)\}$, we set the *encounter number* to be

$$N(x,n) := \#\left\{i \in [1,n] \cap \mathbb{N} \mid \alpha \in D^{(n-i)}(h^i(x))\right\}.$$

Intuitively, if all of $\{D(x)\}$ are nice for the property P with size $\asymp 1$, then $D^{(n)}(x)$ is "acceptable" for P with size about d^{-n} as long as $N(x,n)$ is small.

1.2.5 Structure of proofs

In Section 1.3, we will describe the "nice gluing" property for a gluing link introduced by P. Lin and S. Rohde in [59]. This "nice gluing" property is in fact preserved by $\widetilde{L_j}$ and thus we can use the strategy mentioned above. The CCE condition can be roughly described as

1. there exists a covering $\{D(x)\}$ of $\mathbb{T}$ by gluing links where each $D(x)$ is of comparable size and satisfies the "nice gluing property" at the macro level.

2. For each $x \in \mathbb{T}$, the encounter number $N(x,n)$ is small for many $n \in \mathbb{N}$.

A consequence of the second condition is that for each x, we will have many $D^{(n)}(x)$ of different scales that are acceptable in regards to the nice gluing property. This will imply that $\approx_\alpha$ admits a conformal welding by Theorem 1.1 of [59]. We will provide the details and give the full proof of Theorem 1.1.1 in Secion 1.3.

To prove Theorem 1.1.2, we want to show that for a generic α, we can find a covering $\{D(x)\}$ that satisfies CCE. This is rather challenging and we will dedicate Sections 1.4,1.5 and 1.6 to finding such a covering. In Section 1.4, we will show that for a generic α, $\approx_\alpha$ satisfies the *semi-CCE* condition, which can be roughly described as

> For any covering $\{D(x)\}$ of $\mathbb{T}$ by gluing links (with respect to $\approx_\alpha$), if each $D(x)$ satisfies the *digit-fixing* condition, then for each $x \in \mathbb{T}$, $N(x, n)$ is small for many $n \in \mathbb{N}$.

This digit-fixing condition (defined in Section 1.4.2) is described using itinerary map $I^\alpha(x)$ so it is relatively trackable. With CCE and semi-CCE conditions, we are aimed to find a good candidate for $\{D(x)\}$ that has nice gluing property and satisfies the digit-fixing condition. In Section 1.5, we will discuss a special type of gluing links called *generalized cylinder sets*. Like cylinder sets, the generalized cylinder sets are defined using itinerary map $I^\alpha(x)$. In consequence, we can explicitly check for which α, the generalized cylinder sets satisfy the digit-fixing condition. To check when the generalized cylinder sets satisfy the nice gluing property, we will follow a similar strategy outlined in [59] and use the symbolic information provided by itineraries to infer geometric nicety inside each generalized cylinder sets. In Section 1.6, we will show that for a generic α, we can find a covering of $\mathbb{T}$ by the generalized cylinder sets that satisfy the two properties, which implies that $\approx_\alpha$ admits a Hölder-continuous conformal welding.

1.3 Combinatorial Collet-Eckmann (CCE) Condition

We will give the definition of the combinatorial Collet-Eckmann condition and prove Theorem 1.1.1 in this section.

1.3.1 Nice gluing circuits

Definition 1.3.1 (Nice gluing circuits [59])**.** Given a lamination $\approx$ and a triple (N, C, r), we define an (N, C, r)-*nice gluing circuit* to be a set of disjoint closed arcs $A_1, A_1', \dots, A_n, A_n'$ ordered counterclockwise in $\mathbb{T}$ with $A_{n+1} = A_1, A_{n+1}' = A_1'$ and $n \leq N$ such that

1. For each j, we have

$$\operatorname{diam}(A_i \cup A_i') \asymp_C r, \quad \operatorname{diam} A_i \asymp_C \operatorname{diam}(A_i \cup A_i') \asymp_C \operatorname{diam} A_i'.$$

2. For each j, there exist (Borel) probability measures μ_j and μ'_j supported on A_j and A'_j respectively such that

$$\max(\mathcal{E}(\hat{\mu}_j), \mathcal{E}(\hat{\mu}'_j)) \leq C,$$

where $\hat{\mu}_j$ the push-forward of μ_j under the map $x \mapsto x/\operatorname{diam}(A_j)$ (the same for $\hat{\mu}'_j$) and $\mathcal{E}(\mu)$ is the *logarithmic energy* of μ defined as

$$\mathcal{E}(\mu) = \int_{\operatorname{supp}(\mu)^2} -\log|x-y| d\mu(x) d\mu(y).$$

We call $\mathcal{E}(\hat{\mu})$ the *rescaled logarithmic energy* for μ.

3. For each j, there exists a μ_j-full measure set $B_j \subseteq A_j$ and a μ'_{j+1}-full measure set $B'_{j+1} \subseteq A'_{j+1}$ along with a $(\mu_j|_{B_j}, \mu'_{j+1}|_{B'_{j+1}})$-measure preserving bijection $\phi_j : B_j \to B'_{j+1}$ such that

$$y \approx \phi_j(y), \quad \text{for all } y \in B_j.$$

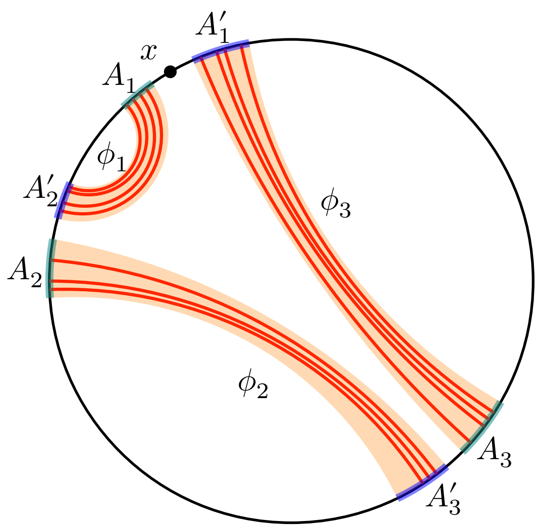

Figure 1.5: A nice gluing circuit around x

We say a nice gluing circuit is *around* $x \in \mathbb{T}$ if x is between some A_i and A'_i for some i (see Figure 1.5). Intuitively, one can think of a nice gluing circuit as a series circuit in the context of electrical networks with the pairs (A_j, A'_{j+1}) acting as resistors in the circuit, and A_j and A'_j connected by zero-resistance paths. The parameter N bounds the number of resistors in the circuit, whereas the parameter C controls the resistance for each of them (r describes the physical scale of the circuit). To prove Theorem 1.1.1, we will use the following criterion for the existence of Hölder continuous conformal welding.

Theorem 1.3.2 ([58, Theorem 1.1]). *Given a lamination, if there are N, C, r and $P > 1$ such that for every $x \in \mathbb{T}$, there exists an increasing sequence of integers $\{n_j\}$ with $n_j \le Pj$ such that there exists an $(N, C, 2^{-n_j}r)$-nice gluing circuit around x for each i, then the lamination admits a Hölder continuous conformal welding.*

The removability of Hölder trees then implies that the resultant dendrite of the conformal welding for $\approx_\alpha$ has to be a Julia set (One can refer Chapter 5 of [58] for details). Note that the base 2 in the statement can be change to any real number $d > 2$ as we can replace P with $\lceil \log d/\log 2 \rceil P$.

1.3.2 Definition of CCE Condition

The TCE condition is a topological condition on polynomials implying that the corresponding Julia set is a Hölder tree.

Definition 1.3.3 ([90, 98]). Given $c \in \mathbb{C}$ and consider the polynomial $p_c = z^d + c$ with corresponding Julia set J_c, we say p_c satisfies the *topological Collet-Eckmann (TCE) condition* if there exist (r, P, M) such that for any $x \in J_c$, there exists an increasing sequence of integers $\{n_j\}$ with $n_j \le Pj$ and

$$\#\left\{0 \le i < n_j : c \in \mathrm{Comp}_{p_c^{n_j - i}(x)}\, p_c^{-i}(B_{p_c^{n_j}(x)}(r))\right\} \le M.$$

The definition of CCE condition is inspired by the TCE condition, where gluing links play a similar role to euclidean balls intersecting the Julia set J_c and the map h playing the role of the polynomial p_c.

Definition 1.3.4. Given $\alpha \in \mathbb{T}$ and consider the lamination $\approx_\alpha$, we say α satisfies the *combinatorial Collet-Eckmann (CCE) condition* if there exists a family of gluing links $\{D(y)\}_{y\in\mathbb{T}}$ with $y \in D(y)$ that satisfies the following two conditions:

1. there exist (N, C, r) such that for any $x \in \mathbb{T}$, there exists an (N, C, r)-nice gluing circuit around x contained in $D(x)$;

2. there exist (P, M) such that for any $x \in \mathbb{T}$, there exists an increasing sequence of integers $\{n_j\}$ with $n_j \leq Pj$ and

$$N(x, n_j) = \#\left\{0 \leq i < n_j : \alpha \in \mathrm{Comp}^{\alpha}_{h^{n_j-i}(x)}\, h^{-i}(D(h^{n_j}(x)))\right\} \leq M.$$

We will call the first condition of $\{D(x)\}$ the *nice gluing* condition.

1.3.3 Proof of Theorem 1.1.1

Following the strategy in Section 1.2.4, we need to show that the nice gluing condition is preserved under $\widetilde{L_i}$ and will only be slightly worse when $\alpha \in D^{(n-i)}(h^i(x))$ and we need to take the preimage. Again we will only consider the case where $d = 2$. Therefore we have $h(z) = z^2$. The argument should work for $d \geq 3$ with some slight modification.

Lemma 1.3.5. *Suppose that a collection $\{D(x)\}_{x\in\mathbb{T}}$ of gluing links satisfies Condition 1 of Definition 1.3.4 with parameters (N, C, r). Given any integer M, there exist N_1, C_1 depending only on (N, C, M, d) such that for any $x \in \mathbb{T}$, if $N(x, n) \leq M$, then we can find an $(N_1, C_1, 2^{-n}r)$-nice gluing circuit in $D^{(n)}(x)$ around x.*

Proof. We set $L := L_1$ and $R := L_2$. Starting with a nice gluing circuit around $h^n(x)$ in $D(h^n(x))$, we will iterate the following algorithm to get a gluing circuit around x.

For $i \in [n]$, suppose we have an (N, C, r)-nice gluing circuit in $D^{(i)}(h^{n-i}(x))$ around $h^{n-i}(x)$. We have the following three scenarios:

1. When α is outside $D^{(i)}(h^{n-i}(x))$ or is between some A_j and A'_{j+1}:

In this case, suppose $h^{n-i-1}(x) = \widetilde{L}(h^{n-i}(x))$. Note that since both rescaled logarithmic energy and $\approx_\alpha$ are preserved under $\widetilde{L}$, $\widetilde{L}$ maps the (N, C, r)-nice gluing circuit around $h^{n-i}(x)$ to an $(N, C, r/2)$-nice gluing circuit around $h^{n-i-1}(x)$ in $D^{(i+1)}(h^{n-i-1}x) = \widetilde{L}(D^{(i)}(h^{n-i}x))$. The case for $\widetilde{R}$ is similar. Thus we get an $(N, C, r/2)$-nice gluing circuit around $h^{n-i-1}(x)$ in $D^{(i+1)}(h^{n-i-1}(x))$.

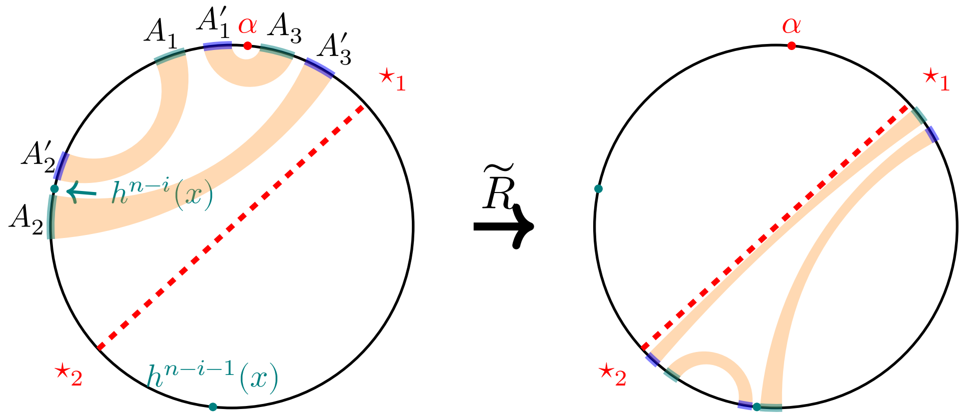

Figure 1.6: Case 1 where $h^{n-i-1}(x) = \widetilde{R}(h^{n-i}(x))$

2. When α is in $D^{(i)}(x)$ and between some A_j and A'_j:

   Without loss of generality, we assume $j = 1$ and denote this arc between A_1 and A'_1 by Γ_α. Then $h^{-1}(\Gamma_\alpha)$ consists of two arcs J_1, J_2 of length $|\Gamma_\alpha|/2$ contained in $D^{(i+1)}(h^{n-i-1}(x))$ covering $\star_1, \star_2$ respectively. Moreover, the gluing link $D^{(i+1)}(h^{n-i-1}(x))$ also contains all $\widetilde{L}(A_j)$, $\widetilde{L}(A'_j)$, $\widetilde{R}(A_j)$ and $\widetilde{R}(A'_j)$. Note that $\widetilde{L}(B_j)$ and $\widetilde{L}(B'_{j+1})$ are glued by $\widetilde{L} \circ \phi_i \circ h$ and same for $\widetilde{R}(B_j)$ and $\widetilde{R}(B'_{j+1})$. We have the corresponding gluing measures which are push-forwards of the original measures by $\widetilde{L}$ and $\widetilde{R}$. Going counterclockwise from $\widetilde{L}(A'_1)$, $\widetilde{L}(A_1)$ to $\widetilde{R}(A'_1)$, $\widetilde{R}(A_1)$, we denote the arcs as $C'_1, C_2, C'_2, \dots,$ C'_{2n}, C_1 and the push-forward measures by $\nu'_1, \nu_2, \dots, \nu'_{2n}, \nu_1$ respectively. Since $\widetilde{L}$ and $\widetilde{R}$ are linear, the push forward measures have the same rescaled logarithmic energy as before and we have a measure preserving map gluing each C_j with C'_{j+1}. Moreover, apart from (C_1, C'_1) and (C_n, C'_n) which are connected by J_1 and J_2 respectively, all other $C_j \sqcup C'_j$ are rescaled images of some $A_j \sqcup A'_j$ by $\widetilde{L}$ or $\widetilde{R}$. Thus we have $\operatorname{diam}(C_j \cup C'_j) \in [r/(2C), rC/2]$ for all j. This implies that $\{C_j\} \cup \{C'_j\}$ form a $(2N, C, r/2)$-nice gluing circuit around $h^{n-i-1}(x)$ in $D^{(i+1)}(x)$.

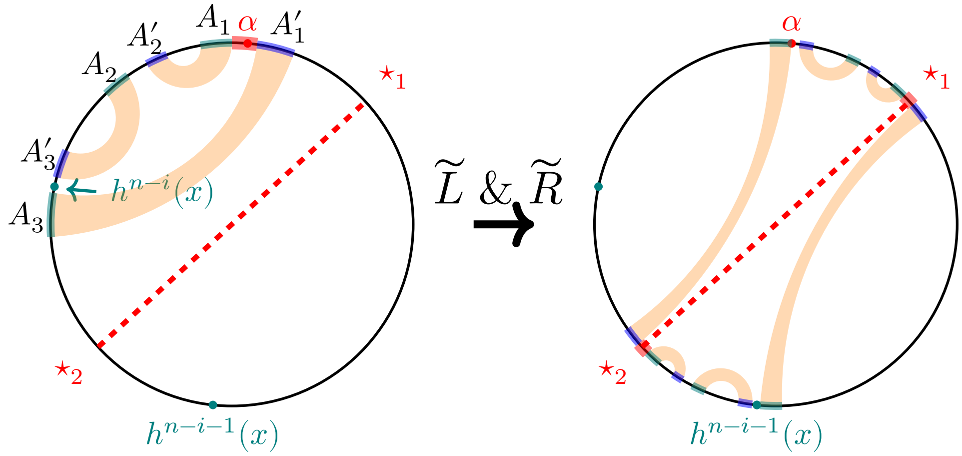

Figure 1.7: Case 2 where $h^{n-i-1}(x) = \widetilde{R}(h^{n-i}(x))$, arcs Γ_α, J_1 and J_2 are highlighted in red.

3. When α is in some A_j or A'_j:

   We might assume the former case as the argument works for the latter with little modification. Thus we can also assume that α is in A_1. We cut A_1 at α into two sets C_1, D_1 such that α is between C_1 and A'_2 and also between D_1, A'_1. Since ϕ_1 respects the lamination, it has to be monotone. We extend it arbitrarily to be a monotone function from A_1 to A'_2 and take $\alpha' = \phi_1(\alpha)$. We can then cut A'_2 at α' into two sets C'_2 and D'_2 such that $\phi_1(C_1) \subseteq C'_2$ and $\phi_1(D_1) \subseteq D'_2$ (see Figure 1.8). We let G_1 and G_2 be the two components of $h^{-1}(A_1)$ containing $\star_1$ and $\star_2$ respectively. Then going counterclockwise, the preimage of $(\cup A_i) \cup (\cup A'_i)$ under h consists of $2n$ disjoint closed arcs (see Figure 1.9)

$$G_1, \widetilde{L}(A'_2), \widetilde{L}(A_2), \dots, \widetilde{L}(A'_1), G_2, \widetilde{R}(A'_2), \widetilde{R}(A_2), \dots, \widetilde{R}(A'_1).$$

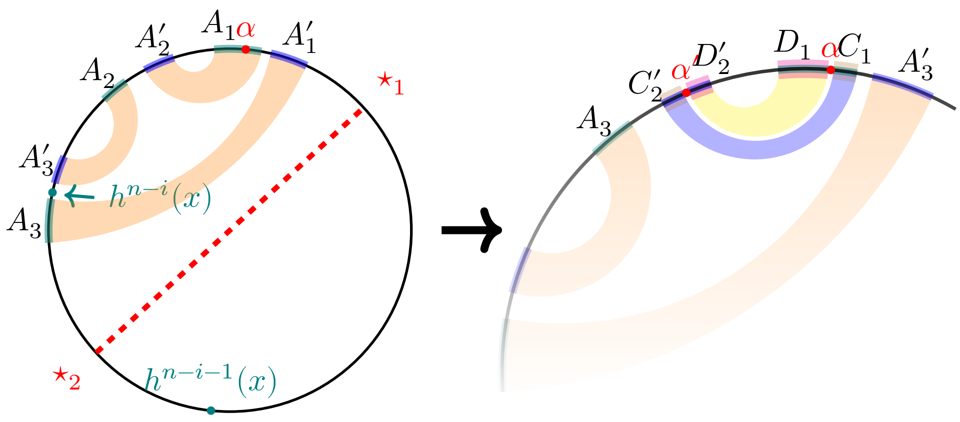

Figure 1.8: For Case 3, we cut A_1 into two sets along α and cut A_2' in the compatible way.

We also have that

$$\operatorname{diam} G_1 = \operatorname{diam} G_2 = \frac{1}{2} \operatorname{diam} A_1,$$
$$\operatorname{diam}(G_1 \cup \widetilde{R}(A_1')) = \operatorname{diam}(G_2 \cup \widetilde{L}(A_1')) = \frac{1}{2} \operatorname{diam}(A_1 \cup A_1').$$

Without loss of generality, we assume that $h^{n-i-1}(x) \in C(L)$. Then we have the following two scenarios:

(a) $\mu_1(C_1) \geq 1/2$ and thus $\mu_2'(C_2') \geq 1/2$:
We set $m_1 = \mu_1|_{C_1}/\mu_1(C_1)$ and $m_2' = \mu_2|_{C_2'}/\mu_2'(C_2')$. Then m_1 as a measure on A_1 (and m_2' on A_2') has rescaled logarithmic energy bounded by $4C$. We let ν_1 and ν_2 be the push-forwards of m_1 and m_2' under $\widetilde{L}$. Additionally, $\widetilde{L} \circ \phi_1 \circ h : \widetilde{L}(C_1) \to \widetilde{L}(C_2')$ will be (ν_1, ν_2) measure-preserving. Then

$$(\widetilde{L}(A_1'), G_2), (\widetilde{L}(A_2'), \widetilde{L}(A_2)), \dots, (\widetilde{L}(A_n'), \widetilde{L}(A_n))$$

form a $(N, 4C, r/2)$-nice gluing circuit around $h^{n-i-1}(x)$.

(b) $\mu_1(D_1) > 1/2$ and thus $\mu_2'(D_2') > 1/2$:
We set $m_1 = \mu_1|_{D_1}/\mu_1(D_1)$ and $m_2' = \mu_2|_{D_2'}/\mu_2'(D_2')$. Then m_1 as a measure

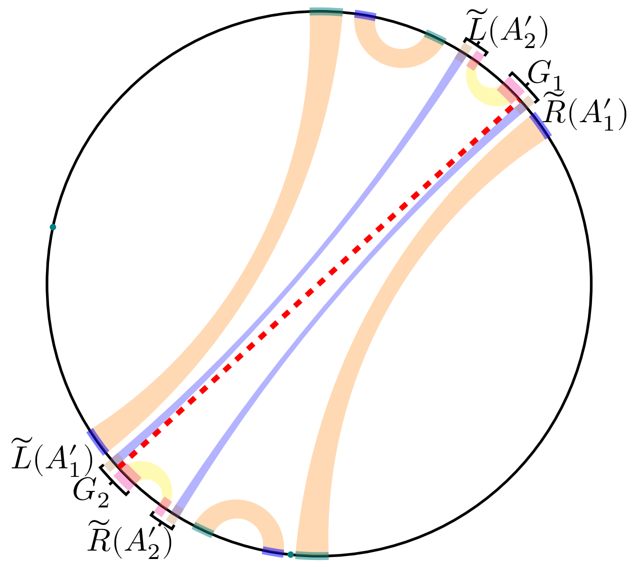

Figure 1.9: The preimages of $A_i, A_i', C_1, D_1, C_2', D_2'$. If $\mu_1(C_1)$ is large (blue tunnels are more "conductive"), we choose the gluing circuit in one semi-circle similar to Case 1. If $\mu_1(D_1)$ is large (yellow tunnels are more "conductive"), we glue two gluing circuits like Case 2.

on A_1 (and m_2' on A_2') has rescaled logarithmic energy bounded by $4C$. We let ν_1^L and ν_2^L be the pushforward of m_1 and m_2' under $\widetilde{L}$. Then ν_1^L is supported on G_1 and ν_2^L on $\widetilde{L}(A_2')$. We similarly define ν_1^R and ν_2^R supported on G_2 and $\widetilde{R}(A_2')$ respectively. Note that all of the four have rescaled logarithmic energy bounded by $4C$ and $\widetilde{L} \circ \phi_1 \circ h : \widetilde{L}(D_1) \to \widetilde{L}(D_2')$ is (ν_1^L, ν_2^L) measure-preserving and $\widetilde{R} \circ \phi_1 \circ h : \widetilde{R}(D_1) \to \widetilde{R}(D_2')$ is (ν_1^R, ν_2^R) measure-preserving. It follows that

$$(\widetilde{R}(A_1'), G_2), (\widetilde{L}(A_2'), \widetilde{L}(A_2)), \dots, (\widetilde{L}(A_n'), \widetilde{L}(A_n)),$$
$$(\widetilde{L}(A_1'), G_2), (\widetilde{R}(A_2'), \widetilde{R}(A_2)), \dots, (\widetilde{R}(A_n'), \widetilde{R}(A_n)),$$

form a $(2N, 4C, r/2)$-nice gluing circuit around $h^{n-i-1}(x)$.

In summary, each iteration, we will divide r by 2. If α is outside $D^{(i)}(x)$, we keep N, C

unchanged. Otherwise, at worst we need to increase N to $2N$ and increase C to $4C$. Since $N(x,n) < M$, α is in at most M of $D^{(i)}(h^{n-i}(x))$'s for $k = 1, \dots, n$. Thus we will end up with a $(2^M N, 4^M C, 2^{-n} r)$-nice gluing circuit around x. $\square$

Proof of Theorem 1.1.1. Let $\{D(x)\}$ be a covering of $\mathbb{T}$ by gluing links satisfies the two conditions as in Definition 1.3.4. That is, there exist (N, C, r) such that each $D(x)$ contains an (N, C, r)-nice gluing circuit around x. Moreover, there exist (P, M) such that for each $x \in \mathbb{T}$, we have a sequence $\{n_j\}$ with $n_j \leq Pj$ such that $N(x, n_j) \leq M$. By Lemma 1.3.5, there exist (N_1, C_1) such that there exists $(N_1, C_1, 2^{-n_j})$-nice gluing circuit around x contained in $D^{(n_j)}(x)$ for each n_j. The result then follows from Theorem 1.3.2. $\square$

1.4 Strong Recurrence and Digit-Fixing Set

In this section, we will discuss how to guarantee low encounter number $N(x,n)$ for a covering $\{D(x)\}_{x\in\mathbb{T}}$. We will define the *digit-fixing* condition inspired. To define this condition, we will first introduce the *strong recurrence* condition.

1.4.1 Strong Recurrence

Definition 1.4.1 (Duplicating intervals and digits)**.** Let α be fixed. Given any finite or infinite word g, we define the following.

1. Fix a set $\mathcal{R} \subseteq \mathbb{N}$. We say an interval $[a,b]$ of digits of g is *$\mathcal{R}$-duplicating* if $g|_{[a,b]}$ coincides with $I^{\alpha}(\alpha)|_{[1,b-a+1]}$ except possibly in the digits of the former contained in $\mathcal{R} \cap [a,b]$. We also call a $\emptyset$-duplicating interval a *(proper) duplicating interval.*

2. Given an integer D and a set $\mathcal{R} \subseteq \mathbb{N}$. We say the i-th digit of g is *$(D,\mathcal{R})$-duplicating* if i is covered by some $\mathcal{R}$-duplicating interval I such that I either covers the last letter of g or has length exceeding D.

Here we introduce the notion of strong recurrence by S. Smirnov in [98]. We say a set $\mathcal{R} \subseteq \mathbb{N}$ is *D-rare* if $\#(\mathcal{R} \cap [i, i+D]) \leq 3$ for any $i \in \mathbb{N}$.

Definition 1.4.2 (Strong recurrence [98])**.** Given an angle $\alpha \in \mathbb{T}$ and its itinerary $I^\alpha(\alpha)$, we say α is *strongly recurrent (SR)* if for every $D > 0$ and $\tau < 1$, there exist integer $n > D$ and a D-rare set $\mathcal{R}_n \subseteq [n]$ such that

$$\#\left\{0 < i \leq n : i\text{-th digit of } I^\alpha(\alpha)|_{[1,n]} \text{ is } (D, \mathcal{R}_n)\text{-duplicating}\right\} > \tau n.$$

Intuitively, if α is strongly recurrent, then the word $I^\alpha(\alpha)$ frequently repeats its starting pattern. As a result, $I^\alpha(\alpha)$ should "contain less information" than a generic infinite word in $\{L_1, \dots, L_d\}^\infty$. We have the following result on the rarity of strongly recurrent angles:

Proposition 1.4.3 ([98])**.** *The set of SR angles has Hausdorff dimension zero.*

Note that in Smirnov's original paper, a D-rare set is defined as a set $\mathcal{R} \subseteq \mathbb{N}$ with the property that $\#(\mathcal{R} \cap [i, i + D]) \leq 2$ for any i. For our modified definition, one can check that the proof for this Proposition still works by essentially changing some constants.

1.4.2 Digit-Fixing Property

Given a covering $\{D(x)\}_{x\in\mathbb{T}}$ of $\mathbb{T}$ by gluing links and a integer $n > 1$, we can construct a new covering by (usually smaller) gluing links by setting

$$D^{(n)}(x) = \operatorname{Comp}_x^\alpha h^{-n}(D(h^n(x))).$$

Definition 1.4.4 (digit-fixing cover)**.** We say the covering $\{D(x)\}_{x\in\mathbb{T}}$ of gluing links is *L-Digit-Fixing*, if the following condition holds:

> For each n, if both x_1, x_2 are points in $D^{(n)}(y)$ for some $y \in \mathbb{T}$, then the words $I^\alpha(x_1)|_{[1,n+1]}$ and $I^\alpha(x_2)|_{[1,n+1]}$ differ only on an increasing sequence of digits $\{n_i\}$ (depending on the choice of x_1 and x_2) such that $n_{i+1} - n_i > L$.

Intuitively speaking, if the covering $\{D(x)\}_{x\in\mathbb{T}}$ is digit-fixing, then each $D^{(n)}(x)$ only contains points with almost the same itinerary up to $(n + 1)$-th digit. This definition is inspired by the following observation.

Lemma 1.4.5 ([98, Lemma 2.1])**.** *Let $J \subseteq \mathbb{C}$ be a dendrite Julia set for polynomial $p_c(z) := z^d + c$. Then for any integer L, there exists $r > 0$ such that for any r-radius ball B_r that intersects J, if points $x, y \in \mathbb{T}$ are such that $\varphi_c(x)$ and $\varphi_c(y)$ belong to the same component of $p_c^{-n}(B_r)$, then $I^\alpha(x)|_{[1,n+1]}$ and $I^\alpha(y)|_{[1,n+1]}$ differ only on an increasing sequence of digits $\{n_i\}$ such that $n_{i+1} - n_i > L$.*

In a way, the lemma implies that small Euclidean balls intersecting the Julia set are digit-fixing. Again, in our setting, the gluing links play the role of Euclidean balls and the map h plays the role of the polynomial p_c.

1.4.3 Criteria for Strong Recurrence and Digit-Fixing

Essentially only using Lemma 1.4.5, S. Smirnov proves that the special angle of a non-TCE polynomial satisfies SR condition. Therefore, we can extract from the result a more general criterion for SR. To do so, we first introduce the definition of semi-CCE condition:

Definition 1.4.6. Given $\alpha \in \mathbb{T}$ and $h : z \to z^d$, we says α satisfies *semi-CCE condition* if there exist (L, P, M) such that the following is satisfied:

> If $\{D(y)\}_{y \in \mathbb{T}}$ is a L-digit-fixing covering of gluing links, then for any $x \in \mathbb{T}$ there exists an increasing sequence of integers $\{n_j\}$ with $n_j \leq Pj$ and $N(x, n_j) \leq M$.

In other words, semi-CCE condition guarantees that any digit-fixing cover of gluing links will give low encounter numbers (recall we need this for Condition 2 of Definition 1.3.4). The relation between semi-CCE and SR is as follows.

Theorem 1.4.7 ([98])**.** *If α is not pre-periodic and does not satisfy semi-CCE condition, it is strongly recurrent.*

Here we repeat the proof in Smirnov's paper for the general setting.

Proof. We first observe that since α is aperiodic, the collection of equivalence classes $\left\{[x]_{\approx_\alpha}\right\}_{x \in \mathbb{T}}$ are gluing links covering $\mathbb{T}$ that are digit-fixing for any L. Hence, there always exist some

L-digit-fixing cover $\{D(x)\}_{x\in\mathbb{T}}$. Given arbitrary $D>0$ and $\tau<1$, we shall show that if α does not satisfy semi-CCE, then there exist integer $n>D$ and a D-rare set $\mathcal{R}\subseteq[n]$ such that

$$\#\left\{0<i\leq n : i\text{-th digit of } I^{\alpha}(\alpha)|_{[1,n]} \text{ is } (D,\mathcal{R})\text{-duplicating}\right\} > \tau n.$$

We let $L=D+1$ and set M,P such that $M>2D+2$ and $(1-\tau)(M/2)(1-1/P)>1$. Since α does not satisfy semi-CCE, there exists L-digit-fixing cover $\{D(y)\}_{y\in T}$ and a point $x\in\mathbb{T}$ and $N>0$ such that at least $(1-1/P)N$ integers $n\in[N]$ satisfy $N(x,n)>M$. Now we define the following function :

$$\begin{aligned} l :&[N-1]\cup\{0\}\to[N]\cup\{-\infty\} \\ &i\mapsto \max\left\{i\leq j\leq N : \alpha\in D^{(j-i)}(h^{i}(x))\right\}, \end{aligned}$$

where we set $\max\emptyset=-\infty$. In other words, $l(i)$ is the largest integer j not exceeding N such that the gluing link of $h^{i}(x)$ we get from pulling back $D(h^{j}(x))$ contains α. Since $\{D(x)\}$ is L-digiting fixing, it follows that $I^{\alpha}(h^{i}(x))_{[1,l(i)-i+1]}(=I^{\alpha}(x)|_{[i,l(i)]})$ and $I^{\alpha}(\alpha)|_{[1,l(i)-i+1]}$ match except possibly at a set of digits $S_i\subseteq[i,l(i)]$, where digits in S_i are at least D digits apart. We set $\mathcal{J}=\{[i,l(i)]\}_{i=0}^{N-1}$.

Now we observe that if $N(x,n)>M$ for some integer n, then α is contained in at least $(M+1)$ of $D^{(i)}(h^{n-i}(x))'s$. When $\alpha\in D^{(i)}(h^{n-i}(x))$, we know that $l(n-i)\geq n$ and thus $n\in[n-i,l(n-i)]$. Hence, it follows that $\mathcal{J}$ covers the integer n at least $M+1$ times. Therefore, since at least $(1-1/P)N$ integers in $[N]$ satisfy $N(x,n)>M$, $\mathcal{J}$ covers at least $(1-1/P)N$ integers at least $(M+1)$ times.

Note that all the intervals in $\mathcal{J}$ start at different points. Thus we can order the intervals by their left ends. By removing intervals in $\mathcal{J}$ with length not greater than D, then we can a new collection $\mathcal{J}'$ that covers those integers at least $(1-1/P)N$ integers in $[N]$ at least $[M+1-(D+1)]>M/2$ times. Thus we have

$$\sum_{I\in\mathcal{J}'}|I| > \frac{M}{2}\cdot\left(1-\frac{1}{P}\right)N \geq \frac{N}{1-\tau}. \tag{1.2}$$

Given an interval $[i, l(i)] \in \mathcal{J}'$, we set

$$\mathcal{J}_i := \{[j, l(j)] \in \mathcal{J}' : i < j \leq l(i)\}$$

to be the collection of intervals in $\mathcal{J}'$ intersecting $[i, l(i)]$ from the right. The standard Besicovitch covering argument implies that there exists $\mathcal{J}'_i \subseteq \mathcal{J}_i$ such that $\mathcal{J}'_i$ covers the same integers in $\mathcal{J}_i$ but each integer is covered at most twice. Note that by the definition of S_k above, we have

$$I^\alpha(x)[m] = I^\alpha(\alpha)[m - k + 1] \quad \text{for all} \quad m \in [k, l(k)] \backslash S_k.$$

It follows that for any $[j, l(j)]$ in $\mathcal{J}'_i$, $I^\alpha(\alpha)|_{[i,l(i)]\cap[j,l(j)]-(i-1)}$ matches well with $I^\alpha(x)|_{[i,l(i)]\cap[j,l(j)]}$ (except at S_i), which in turn matches well with $I^\alpha(\alpha)|_{[j,l(j)]-(j-1)}$ (except at S_j). Therefore, the interval $[i, l(i)] \cap [j, l(j)] - (i - 1)$ is $(S_i \cap S_j - (i - 1))$-duplicating for $I^\alpha(\alpha)$. Since $\mathcal{J}'_i$ covers each integer at most twice, the set

$$\mathcal{R} := S_i \cup \left(\bigcup_{[j,l(j)] \in \mathcal{J}'_i} S_j \right)$$

is rare and so is $\mathcal{R} - (i - 1)$. Therefore, for each $[j, l(j)] \in \mathcal{J}'_i$, $[i, l(i)] \cap [j, l(j)] - (i - 1)$ is a $(\mathcal{R} - (i - 1))$-duplicating interval of $I^\alpha(\alpha)|_{[i,l(i)]-(i-1)}$, either of length greater than D or containing the $(l(i) - i + 1)$-th digit. If $\mathcal{J}_i$ (thus $\mathcal{J}'_i$) covers more than $\tau(l(i) - i)$ integers in $[i, l(i)]$, then more than $\tau(l(i) - i)$ digits of $I^\alpha(\alpha)|_{[1,l(i)-i+1]}$ are $(D, \mathcal{R} - (i - 1))$-duplicating and we are done (recall that $l(i) - i > D$).

To recapitulate, if there exists $[i, l(i)] \in \mathcal{J}'$ such that $\mathcal{J}_i$ defined above covers more than $\tau(l(i) - i)$ integers, then we are done. If such an interval does not exist, then for all $[i, l(i)]$'s, at most $\tau(l(i) - i)$ integers in the interval can be covered by some $[j, l(j)]$ with $j > i$. Given an integer k in $[N]$, we denote the number of intervals in $\mathcal{J}'$ covering k by $m(k)$. Then among all the intervals covering k, only for the rightmost one, k is not covered by any interval right to it. This implies that

$$\sum_{k=1}^{N}(m(k)-1) = \sum_{I\in\mathcal{J}'} \#\{i \in I : i \text{ is covered by some interval right to } I\}$$
$$\leq \sum_{I\in\mathcal{J}'} \tau|I| = \tau \sum_{I\in\mathcal{J}'} |I|.$$

Using the fact that $\sum_{k=1}^{N} m(k) = \sum_{I\in\mathcal{J}'} |I|$, we conclude $\sum_{I\in\mathcal{J}'} |I| \leq N/(1-\tau)$, which contradicts (1.2). Thus the result follows. □

Corollary 1.4.8. *Semi-CCE condition is satisfied by all points on $\mathbb{T}$ except for a set of Hausdorff dimension zero.*

Here we also provide a criterion to check whether a covering $D(x)_{x\in\mathbb{T}}$ is digit-fixing. This criterion is relatively easy to check for the covering we will use in Section 1.6. It is inspired by the proof of Lemma 2.1 in [98].

Proposition 1.4.9. *Let $\{D(x)\}_{x\in\mathbb{T}}$ be a cover of $\mathbb{T}$ by gluing links. If for all $x \in \mathbb{T}$ and integer n with $\alpha \in D^{(n)}(x)$, we have*

$$D^{(n)}(x) \cap \left\{h^i(\alpha)\right\}_{i=1}^{L} = \emptyset,$$

then $\{D(x)\}$ is L-digit-fixing.

Proof. Given such a covering $\{D(x)\}$, suppose x_1 and x_2 are both in $D^{(n)}(y)$. If the j-th digits of $I^\alpha(x_1)$ and $I^\alpha(x_2)$ differ with $j \leq n+1$, then either one of $h^{j-1}(x_1)$ and $h^{j-1}(x_2)$ is in $\{\star_1, \star_2\}$ or the two are separated by $\{\star_1, \star_2\}$. Therefore, $h^{j-1}(D^{(n)}(y)) = D^{(n-j+1)}(h^{j-1}(y))$, which contains $\left\{h^{j-1}(x_1), h^{j-1}(x_2)\right\}$, has to intersect $\{\star_1, \star_2\}$ as well. It follows that $\alpha \in D^{(n-j)}(h^j(y))$. By our assumption, this would mean

$$D^{(n-j)}(h^j(y)) \cap \left\{h^i(\alpha)\right\}_{i=1}^{L} = \emptyset.$$

Now if $I^\alpha(x_1)$ and $I^\alpha(x_2)$ also differ at the digit $k \in [j-L, j)$, by the same argument, we have $\alpha \in D^{(n-k)}(h^k(y))$ and thus

$$h^{j-k}(\alpha) \in h^{j-k}(D^{(n-k)}(h^k(y))) = D^{(n-j)}(h^j(y)),$$

which contradicts our assumption. Thus the digits at which the two words differ have to be at least L digit apart. □

1.5 Generalized Cylinder Sets and Legal Words

To recapitulate, we have shown that CCE condition implies the existence of Hölder continuous welding and non-SR implies semi-CCE. If we can find some gluing link covering $\{D(x)\}$ that satisfies the nice gluing property and is digit-fixing, then we can link two results. In the case when $d = 2$, the candidate we have is the *generalized cylinder sets*. In this section, we will present their definition and give a nice cover of $\mathbb{D}$ by such sets.

1.5.1 Generalized Cylinder Sets

From now on, we assume $d = 2$ and again set $L = L_1$ and $R = L_2$. The definitions in this section can be generalized to $d \geq 3$ but for now we will restrict our attention for simplicity.

Lemma 1.5.1. *Given a finite word $u \in \{L, R\}^*$, since α is aperiodic, $\tilde{u}(\star_1)$ and $\tilde{u}(\star_2)$ are well-defined. We set l_u to be the hyperbolic geodesic in $\overline{\mathbb{D}}$ connecting $\tilde{u}(\{\star_1, \star_2\})$. Then for $u_1 \neq u_2$, the two geodesics l_{u_1} and l_{u_2} are disjoint.*

Proof. Suppose we have $u_1 \neq u_2$ such that l_{u_1} and l_{u_2} intersect. By the non-crossing property in Lemma 1.2.1, we have

$$\widetilde{u_1}(\star_1) \approx_\alpha \widetilde{u_2}(\star_1).$$

However, we have

$$I^\alpha(\widetilde{u_1}(\star_1)) = u_1 \star I^\alpha(\alpha), \quad I^\alpha(\widetilde{u_2}(\star_1)) = u_2 \star I^\alpha(\alpha).$$

This would imply that $u_1 = u_2$, causing contradiction. □

Definition 1.5.2 (Generalized Cylinder Sets)**.** Given l_u defined above, we set $L_n = \{l_u : |u| = n\}$. The geodesics in L_n cut $\overline{\mathbb{D}}$ into several components. We denote this collection of components by $\mathcal{D}_n$ and set $\mathcal{GCS}_n = \{D \cap \mathbb{T} : D \in \mathcal{D}_n\}$. Since $\tilde{u}(\star_1) \approx_\alpha \tilde{u}(\star_2)$, one can easily check that

$\mathcal{GCS}_n$ is a collection of gluing links. We call each member in this collection a *generalized cylinder set of degree* n.

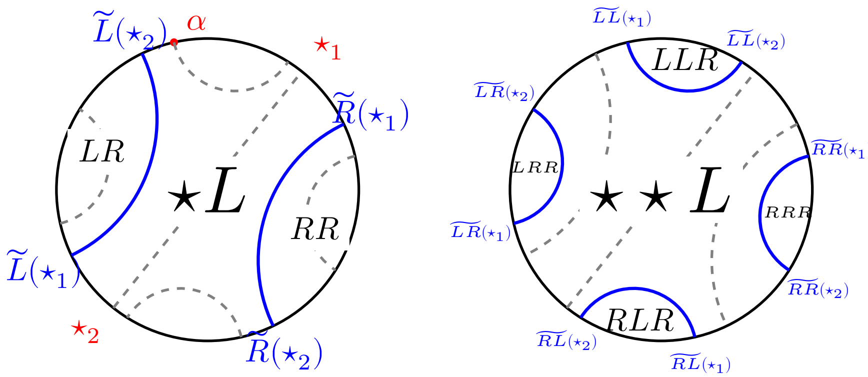

Figure 1.10: The generalized cylinder sets for $\alpha = 4\pi/7$ associated with words of length not greater than 3, where $\alpha \in C(LLR) \subseteq C(\star L) \subseteq C(L)$. The dashed lines indicates the boundary of the cylinder sets in Figure 1.3. Note that $C(\star\star L)$ is a union of cylinder sets $C(LLL)$, $C(LRL)$, $C(RRL)$ and $C(RLL)$.

Lemma 1.5.3. *Given* $x \in \mathbb{T}$ *such that* $h^{n+1}(x) \neq \alpha$*, there exists a unique generalized cylinder set in* $\mathcal{GCS}_n$ *containing* x*, which we call* $GCS_n(x)$*.*

Proof. It suffices to check that x is not $\tilde{u}(\star_i)$ for some $|u| = n$, which is guaranteed by our assumption as $h^{n+1}(\tilde{u}(\star_i)) = \alpha$ for any $|u| = n$. □

It is also easy to check that generalized cylinder sets behave well in the forward iteration. That is,

$$h(GCS_n(x)) = GCS_{n-1}(h(x)).$$

In fact, so do they in the backward iteration as indicated by the following.

Lemma 1.5.4. *For any* $x \in \mathbb{T}$ *such that* $h^{n+2}(x) \neq \alpha$*, we have*

$$GCS_n^{(1)}(x) = \mathrm{Comp}_x^{\alpha}\, h^{-1}(GCS_n(h(x))) = GCS_{n+1}(x).$$

In consequence, for any $i > 0$, if $h^{n+i+1}(x) \neq \alpha$, we have

$$GCS_n^{(i)}(x) = GCS_{n+i}(x).$$

Proof. Note that both $GCS_n^{(1)}(x)$ and $GCS_{n+1}(x)$ are gluing links containing x whose end points are in $\{\tilde{u}(\star_i) |\ |u| = n+1\}$. Such a gluing link is unique. $\square$

Note that given a integer n, not all the points x have corresponding $GCS_n(x)$. Therefore, $\{GCS_n(x)\}$ does not form a covering of $\mathbb{T}$.

1.5.2 Legal Words and Free Digits

The following proposition tells us that like cylinder sets, the information of a generalized cylinder set can be encoded by words.

Proposition 1.5.5 (Representation of generalized cylinder sets)**.** *For each $C \in \mathcal{GCS}_n$, there exists a unique word $u \in \{L, R, \star\}^{n+1}$ (see Figure 1.10) such that*

$$\begin{aligned} C &= \left\{x \in \mathbb{T} : \textit{if } u[i] \in \{L, R\}, \textit{ then } h^{i-1}(x) \textit{ lies in the semi-circle } C(u[i])\right\} \\ &= \left\{x \in \mathbb{T} : \textit{if } u[i] \in \{L, R\}, \textit{ then } I^\alpha(x)[i] = u[i]\right\}. \end{aligned}$$

In this case, we write $C = C(u)$.

Remark 1.5.6. Given any $u \in \{L, R, \star\}^*$, we can always define

$$C(u) := \left\{x \in \mathbb{T} : \text{if } u[i] \in \{L, R\}, \text{then } h^{i-1}(x) \text{ lies in the semi-circle } C(u_i)\right\}.$$

When u does not contain any $\star$, $C(u)$ will be the cylinder set for u. Thus our notation is therefore consistent.

Proof of Proposition 1.5.5. We prove this by induction. For $n = 0$, $\mathcal{GCS}_0$ are just $\{C(L), C(R)\}$. Now if take any $C \in \mathcal{GCS}_n$ and suppose $h(C) \in \mathcal{GCS}_{n-1}$ is represented by $C(u)$ for some $u \in \{L, R, \star\}^n$. We have the following two cases

1. If $\alpha \notin C(u)$, then C is either $\widetilde{L}(C(u))$ or $\widetilde{R}(C(u))$.Therefore, we can represent C by Lu or Ru respectively.

2. If $\alpha \in C(u)$, then C is the union of the closures of $\widetilde{L}(C(u))$ and $\widetilde{R}(C(u))$. We represent it by $\star u$.

$\square$

Corollary 1.5.7. *Let $C(u)$ be a generalized cylinder set with $u \in \{L, R, \star\}^n$. Then for any $k \in [n]$, $C(u|_{[k,n]})$ is also a well-defined generalized cylinder set. Moreover, $u[k] = \star$ if and only if $\alpha \in C(u|_{[k+1,n]})$.*

Definition 1.5.8 (Free digits & legal words)**.** If $u \in \{L, R, \star\}^*$ can represent some generalized cylinder set, we call u a *legal word* and call all the digits of u at which we have $\star$ the *free digits* of u.

Lemma 1.5.9. *Given $g \in \{L, R\}^n$, there exists a unique $u \in \{L, R, \star\}^n$ obtained by replacing some digits of g by $\star$ such that u is legal and $C(u)$ contains $g(\{\star_1, \star_2\})$. We call u the legal word associated with g and write $u = \mathrm{Legal}(g)$. In particular,*

$$GCS_n(x) = C(\mathrm{Legal}(I^\alpha(x)|_{[1,n+1]})).$$

Proof. Note that by the non-crossing property of $\approx_\alpha$, $\tilde{g}(\star_1)$ and $\tilde{g}(\star_2)$ belongs to the same generalized cylinder set $C \in \mathcal{GSC}_{n-1}$. The legal word $\mathrm{Legal}(g)$ associated with g is then just the word representing C from Proposition 1.5.5.

We can construct $\mathrm{Legal}(g)$ manually from g (See Figure 1.11). Corollary 1.5.7 implies that $\mathrm{Legal}(g)[k] = \star$ if and only if $I^\alpha(\alpha)|_{[1,|g|-k]}$ matches the legal word $\mathrm{Legal}(g|_{[k+1,|g|]})$ except at the free digits. We can start by checking from the last digit of g and looking backwards. If the last digit matches the first digit of $I^\alpha(\alpha)$, we replace the second-to-last digit of g with $\star$ to get $\mathrm{Legal}(g|_{[|g|-1,|g|]})$. For the k-th step, we check whether the last k digits of modified g ($\mathrm{Legal}(g|_{[|g|-k,|g|]})$) matches the first k of $I^\alpha(\alpha)$ except at the free digits.

If that is the case, we replace the $(n-k)$-th digit of g with $\star$. It follows from Lemma 1.5.7 that we will in fact end up with a legal word of g. $\square$

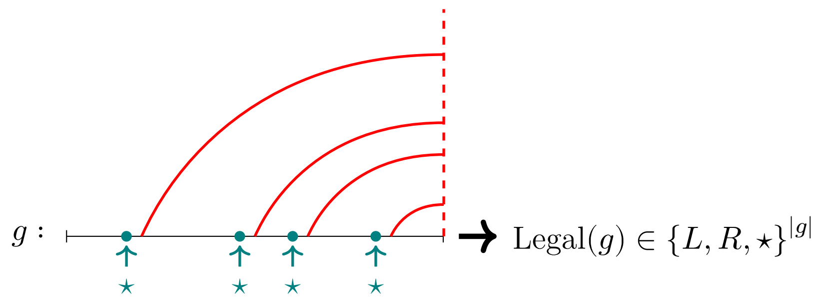

Figure 1.11: Constructing a legal word. Red arcs indicate where $\text{Legal}(g|_{[k,n]})$ matches with $I^\alpha(\alpha)|_{[1,n-k+1]}$ up to non-free digits.

Given $g \in \{L, R\}^n$, if we concatenate g with $h \in \{L, R\}^k$, it might happen that $\text{Legal}(gh)|_{[1,n]} \neq \text{Legal}(g)$. This would be the case only when $\alpha \in C(\text{Legal}(h))$ and thus $\text{Legal}(gh)[n] = \star$, even though $\text{Legal}(g)[n] = g[n]$ will never be $\star$.

Lemma 1.5.10. *Given $g \in \{L, R\}^n$ and $h \in \{L, R\}^k$, if $\text{Legal}(gh)[n] \neq \star$ or equivalently $\alpha \notin C(\text{Legal}(h))$, then $\text{Legal}(gh)|_{[1,n]} = \text{Legal}(g)$.*

1.5.3 Cover of $\mathbb{T}$ by Generalized Cylinder Sets

From now on, we only consider α that are not pre-periodic. That is, none of $h^n(\alpha)$ is periodic. Again we are only excluding a countable set of points. In this case, we have

1. The itinerary of the angle $I(\alpha) := I^\alpha(\alpha)$ does not have letter $\star$. Thus it belongs to the set of infinite words $\{L, R\}^\infty$;

2. For any $x \in \mathbb{T}$, there is at most one $\star$ in $I^\alpha(x)$;

Lemma 1.5.11 ([58, Proposition 2.10])**.** *We denote the infinite word where we replace this $\star$ in $I^\alpha(x)$ with L by $I^L(x)$ and define $I^R(x)$ similarly (If $I^\alpha(x)$ does not contain $\star$, then*

$I^L(x), I^R(x)$ and $I^\alpha(x)$ are just the same). If α is not preperiodic, we have

$$[x]_{\approx_\alpha} = \bigcap_{n=1}^{\infty} C(I^L(x)|_{[1,n]}) = \bigcap_{n=1}^{\infty} C(I^R(x)|_{[1,n]}).$$

Lemma 1.5.12 ([7, Theorem 7])**.** *Suppose α is not pre-periodic. Then $[\alpha]_{\approx_\alpha}$ contains at most 2 points and thus $[\star_i]_{\approx_\alpha}$ has either 2 or 4 points. Moreover, in the case when $|[\alpha]_{\approx_\alpha}| = 2$ if we denote the other point of $[\alpha]_{\approx_\alpha}$ by β, then the forward orbit $\{h^i(\alpha)\}_{i=1}^{\infty}$ does not intersect the arcs $\alpha \frown \beta$, $\widetilde{L}(\alpha \frown \beta)$ or $\widetilde{R}(\alpha \frown \beta)$.*

In the case when $[\alpha]_{\approx_\alpha} = \{\alpha, \beta\}$, we let $\star_1 \frown b_1$ and $\star_2 \frown b_2$ denote the two arcs of $\mathbb{T}\backslash[\star_1]_{\approx_\alpha}$ that are not $\widetilde{L}(\alpha \frown \beta)$ or $\widetilde{R}(\alpha \frown \beta)$. Let C_* be the gluing link containing $[\star_i]_{\approx_\alpha}$ made of $\overline{\widetilde{L}(\alpha \frown \beta)}$ and $\overline{\widetilde{R}(\alpha \frown \beta)}$ (see Figure 1.12). In the case, where $[\alpha]_{\approx_\alpha} = \{\alpha\}$, we define $C_* = \{\star_1, \star_2\}$. Intuitively, we want to think of C_*, as well as its images

$$\left\{\tilde{u}(C_*) | u \in \{L, R\}^*\right\},$$

as "bad" sets that we want to stay away from.

Lemma 1.5.13. *Given an integer $K > 0$, there exists $\delta > 0$ such that the following holds: for each $x \in T$, there exists at least one generalized cylinder set C in $\mathcal{GCS}_K \cup \mathcal{GCS}_{K+1}$ such that $x \in C$ and for any $\tilde{g}(\{\star_1, \star_2\}) \subseteq \partial C$, we have*

$$dist(x, \tilde{g}(C_*)) > \delta.$$

We denote this gluing link C by $C^K(x)$.

Proof. Consider first the case when $[\alpha]_{\approx_\alpha} = \{\alpha, \beta\}$. We observe that since $\alpha \frown \beta$ does not contain any $h^i(\alpha)$, any $\tilde{g}$ is well-defined on $\alpha \frown \beta$ and contracts this arc. Thus, if $|\star_1 \frown b_2| = |\star_2 \frown b_1| = \ell$. Then $\tilde{g}(C_*)$ is a gluing link made of two arcs $\overline{\tilde{g}(\star_1 \frown b_2)}$ and $\overline{\tilde{g}(\star_2 \frown b_1)}$ with length $\ell/2^{|g|}$.

By the non-crossing property of $\approx_\alpha$, for any two distinct $g_1, g_2 \in \{L, R\}^*$, either $\tilde{g}_1(C_*)$ and $\tilde{g}_2(C_*)$ are disjoint or one contains the other. When $-1 \leq |g_1| - |g_2| \leq 1$, the containment

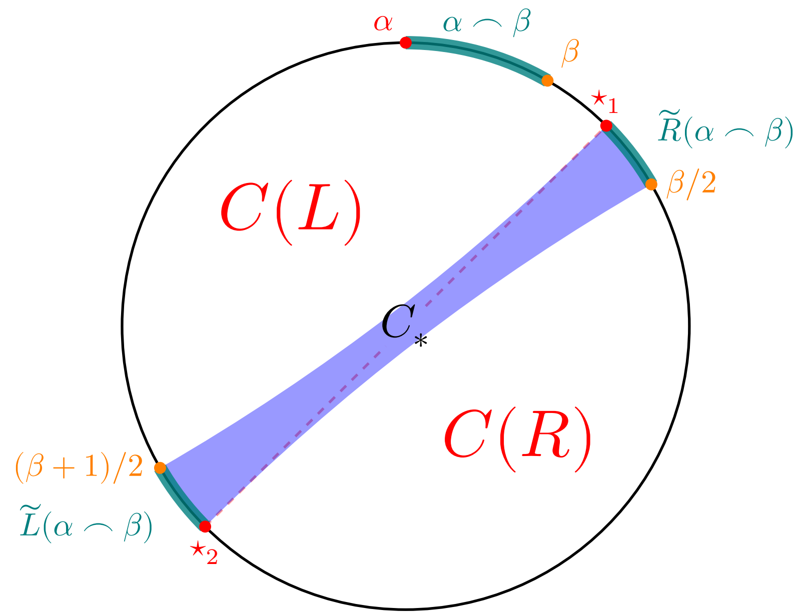

Figure 1.12: When $|[\star_1]_{\approx_\alpha}| = 4$, we have the above gluing link C_*. In the case when $\beta \in (0, \alpha)$, we have that $\star_1 \frown b_1 \subseteq C(L)$ and $\star_2 \frown b_2 \subseteq C(R)$.

is impossible and thus $\tilde{g}_1(C_*)$ and $\tilde{g}_2(C_*)$ have to be disjoint. Therefore, given an integer $K > 1$, there exists $\delta > 0$ such that

$$\mathrm{dist}(\tilde{g}_1(C_*), \tilde{g}_2(C_*)) > 2\delta, \quad \text{for any distinct } g_1, g_2 \text{ in } \{L, R\}^{K-1} \cup \{L, R\}^K.$$

Note that $\mathcal{GCS}_K \cup \mathcal{GCS}_{K+1}$ is a finite collection of gluing links whose boundaries are $\tilde{g}\{\star_1, \star_2\}$'s with $|g| \in \{K-1, K\}$, the result follows. In the case where $[\alpha]_{\approx_\alpha} = \{\alpha\}$, we can get an analogous result by replacing C_* above with $\{\star_1, \star_2\}$. $\square$

In summary, given an integer K, $\{C^K(x)\}$ is a finite closed covering of $\mathbb{T}$ by generalized cylinder sets where each x is uniformly far away from the bad sets $\tilde{g}(C_*)$ at the boundary of $C^K(x)$. This is the covering of gluing links that we will use to prove Theorem 1.1.2.

1.6 Proof of Theorem 1.1.2

1.6.1 When $\{C^K(x)\}$ Satisfies Nice Gluing Property

In this section, we will show that when $d = 2$, if α is not strongly recurrent, then each $C^K(x)$ contains a nice gluing circuit around x.

Proposition 1.6.1. *If α is not strongly recurrent as in Definition 1.4.3, then for any integer K, there exist (N, C, r) depending only on K such that for any $x \in \mathbb{T}$, there is an (N, C, r)-nice gluing circuit around x in $C^K(x)$.*

We will need the following lemma inspired by Lemma 4.2 in [58].

Lemma 1.6.2. *Suppose that α is not strongly recurrent. Then we have*

1. *If $|[\star_1]_{\approx_\alpha}| = 2$, then we can find two sequence of periodic points $\{x_n\}$ and $\{y_n\}$ in $C(L)$ such that*

$$x_n \approx_\alpha y_n, \quad x_n \to \star_1, \quad y_n \to \star_2.$$

2. *If $|[\star_1]_{\approx_\alpha}| = 4$, then we can find two sequence of periodic points $\{x_n\}$ and $\{y_n\}$ in $C(L)\backslash C_*$ such that*

$$x_n \approx_\alpha y_n, \quad x_n \to \star_1, \quad y_n \to b_1.$$

Moreover, for each n and m, there exist two measures μ and μ' of finite logarithmic energy such that

$$\operatorname{supp} \mu \subseteq x_n \frown x_m, \quad \operatorname{supp} \mu' \subseteq y_n \frown y_m,$$

and there exists $\phi : \operatorname{supp} \mu \to \operatorname{supp} \mu'$ such that $\mu' = \phi \# \mu$ and $x \approx_\alpha \phi(x)$.

Note that by symmetry, we can replace $C(L)$ and b_1 in the statements with $C(R)$ and b_2.We first recall some useful properties of (proper) duplicating intervals.

Lemma 1.6.3. *Given a finite word $g \in \{L, R\}^*$, we have*

1. *For a cylinder set $C(g)$, $\partial C(g)$ is the union of all $\tilde{u}\{\star_1, \star_2\}$'s where u is an initial word $g|_{[1,i]}$ such that $[i+2, |g|]$ is a duplicating interval. We recall that this means*

$$g|_{[i+2,|g|]} = I(\alpha)|_{[1,|g|-i-1]}.$$

In this case, we call the hyperbolic geodesic l_u a boundary leaf of $C(g)$. In particular, $l_{g|_{[1,|g|-1]}}$ is a boundary leaf since $[|g|+1, |g|] = \emptyset$ is duplicating.

2. *Let $\tilde{g}$ be the inverse map associated with g. Then $\tilde{g}$ is well-defined everywhere except exactly at $h^i(\alpha)$ where $[|g|-i+1, |g|]$ is a duplicating interval. In particular, $\tilde{g}$ is always ill-defined at $h^0(\alpha) = \alpha$.*

Proof. For part (1), we note that $\tilde{u}\{\star_1, \star_2\} \subseteq \partial C(g)$ exactly when

$$\alpha \in h^{|u|+1}(C(g)) = C(g|_{[|u|+1,|g|]}).$$

For part (2), we note that $\tilde{g}$ is ill-defined at $h^n(\alpha)$ if and only if there exists some i between 2 and $|g|+1$ such that

$$\alpha = \widetilde{g|_{[i,|g|]}}(h^n(\alpha)).$$

This only happens when $g|_{[i,|g|]}$ matches exactly with the itinerary of α up to the first $n-1$ steps (here the first step is the original position of α). $\square$

Proof of Lemma 1.6.2. We suppose $|[\star_i]| = 4$. The cases for $|[\star_i]| = 2$ can be proven using a similar argument and we will omit the details here.

Since α is not strongly recurrent, there exists integer D such that for any D-rare set $\mathcal{R}$, there exist integer $P > 2$ and an increasing sequence of integers $\{n_i\}$ with $n_i \leq Pi$ such that n_i-th digit of $I(\alpha)$ is not $(D, \mathcal{R})$-duplicating.

Let $m_i := n_{3Di}$. Then $\{m_i\}$ is also increasing and each m_i is at least $3D$ digits apart. Since we have $n_i \leq Pi$, it follows that $m_i \leq 3PDi$. Thus there exists an increasing sequence of indices $\{i_k\}$ such that

$$3D \leq m_{i_k} - m_{i_k - 1} \leq 3PD$$

By our assumption, for any $i \in \mathbb{N}$, there does not exist any $\mathscr{R}$-duplicating interval with length exceeding D that covers the m_i-th digit of $I(\alpha)$. In particular, there cannot be any proper duplicating interval with length exceeding D that covers the m_i-th digit. For any $k \in \mathbb{N}$ and the cylinder set $C(LI^\alpha|_{[1,m_{i_k}]})$, it follows that each boundary leaf l_{Lu} (aside from $\{\star_1, \star_2\}$) satisfies that

$$m_{i_k} - D \leq |u| \leq m_{i_k} - 1.$$

Moreover, since $m_{i_k - 1} > m_{i_k} - 3PD$, we have $[|u| - 3PD, |u|]$ contains $m_{i_k - 1}$. Therefore, there cannot be a duplicating interval of I^α covering $[|u| - 3PD, |u|]$ (which has length greater than D). It follows that if $\widetilde{Lu}$ is ill-defined at some $h^K(\alpha)$ with $K > 0$, we have

$$|u| - 3PD \leq |u| - K \leq |u| - 1.$$

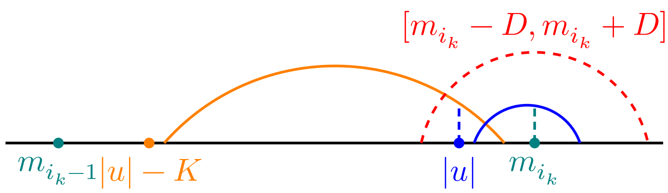

Figure 1.13: Duplicating intervals around m_{i_k}. Note that the duplicating interval associated with u covers m_{i_k} (in blue) and thus cannot have length exceeding D; and the duplicating interval associated with $|u| - K$ ends in $[m_{i_k} - D, m_{i_k}]$ (in orange) and cannot cover $m_{i_k - 1}$.

Therefore $\widetilde{Lu}$ is well-defined outside $\{h^i(\alpha)\}_{i=0}^{3PD}$. Let Γ_i denote the components of $C(L) \setminus \{h^i(\alpha)\}_{i=0}^{3PD}$ that contain $\star_i$ respectively. We shrink each Γ_i slightly such that $\Gamma_1 \cup \Gamma_2$ are positive distance away from $\{h^i(\alpha)\}_{i=0}^{3PD}$ while still contains b_1 and let J be the arc $C(L) \setminus \overline{\Gamma_1 \cup \Gamma_2}$.

Note that $C(LI^\alpha|_{[1,m_{i_k}]})$ shrinks to $\{\star_1, \star_2, b_1\}$ monotonely as $k \to \infty$. Since we have $\{h^i(\alpha)\}_{i=0}^{3PD} \subseteq \overline{J} \subsetneq \star_1 \frown b_1$, we can find k sufficiently large such that

$$C(LI^\alpha|_{[1,m_{i_k}]}) \cap \overline{J} = \emptyset.$$

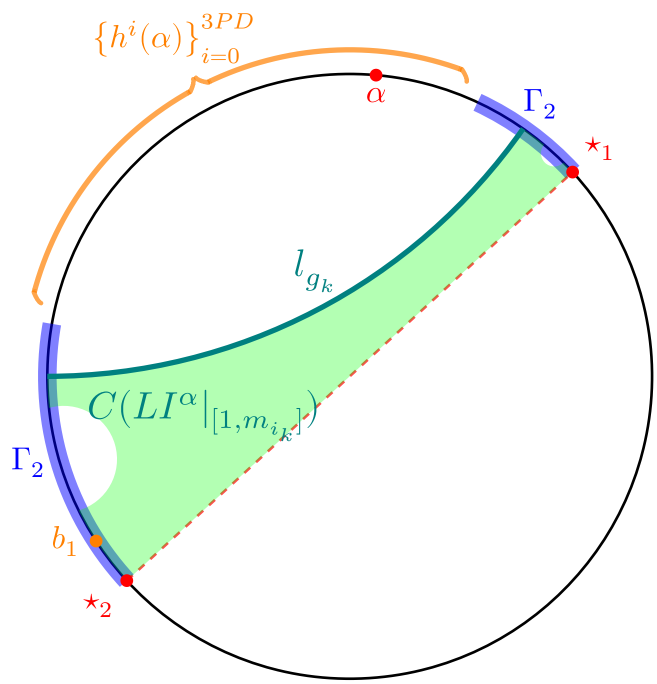

Figure 1.14: I_i, $C(LI^\alpha|_{[1,m_{i_k}]})$ and l_{g_k} in the proof of Lemma 1.6.2 when $\star_1 \frown b_1 \subseteq C(L)$

For such a cylinder set, there exists a boundary leaf l_{Lu} that connects Γ_i. We denote $g_k = Lu$ and we have shown that $\widetilde{g_k}$ is well-define outside $\{h^i(\alpha)\}_{i=0}^{3PD}$. In particular, $\widetilde{g_k}$ is a continuous on $\overline{\Gamma_1 \cup \Gamma_2}$ and maps $\{\star_1, \star_2\}$ to $\widetilde{g_k}(\{\star_1, \star_2\})$. It follows that $\widetilde{g_k}^2$ is a contraction on each $\overline{\Gamma_i}$ and thus each $\overline{\Gamma_i}$ contains a periodic point with itinerary $\overline{g_k}$. Having the same itinerary, the two periodic points are thus glued by $\approx_\alpha$.

Note that we have $|g_k| \in [m_{i_k} - D + 1, m_{i_k}]$. Thus $|g_k| \to \infty$ as we increase k. According to the proof of Lemma 4.2 of [59], since α is not pre-periodic, for any k, there exists a $j > k$ such that $\widetilde{g_k}^2$ and $\widetilde{g_j}^2$ fix different points and thus correspond to different gluing pairs of periodic points.

By the proof of Proposition 4.3 in [58], between any two gluing pairs of periodic points we constructed above (say associated with g_k and g_j respectively), we can find a pair of generalized Cantor set between the two pairs by the iterative function system $\left\{\widetilde{g_k}^2, \widetilde{g_j}^2\right\}$ such that the pair is glued up by a linear transformation. The result then follows. $\square$

Proof of Proposition 1.6.1. Given any $C^K(x) \in \{C^K(y)\}_{y\in\mathbb{T}}$, let $\delta > 0$ be the constant in Lemma 1.5.13 which indicates the distance between x and the "bad sets" in $C^K(x)$. We will show that there exists an (N, C, r)-nice gluing circuit $\{A_i\}\cup\{A'_i\}$ in $C^K(x)$ with the property that for all $A \in \{A_i\} \cup \{A'_i\}$, there exists a boundary leaf l_g such that $\mathrm{dist}(A, \tilde{g}(C_*)) \le \delta$. This would imply that $\{A_i\} \cup \{A'_i\}$ is a nice gluing circuit around x.

Suppose that the boundary of our $C^K(x)$ is the union of $\{\tilde{g}_n\{\star_1, \star_2\}\}_{n=1}^N$ ordered counterclockwise. Given any boundary leaf l_{g_n} of $C^K(x)$, since α is aperiodic and $\widetilde{g_n}$ is only ill-defined at a subset of $\{h^i(\alpha)\}_{i=0}^{|g_n|}$, $\widetilde{g_n}$ is always defined and continuous at some neighborhood of C_*. Then Lemma 1.6.2 implies that we can find two sequences of periodic points $\{x_i\}$ and $\{y_i\}$ such that $\widetilde{g_n}(x_i) \approx_\alpha \widetilde{g_n}(y_i)$, and the geodesics connecting $\widetilde{g_n}(x_i)$ and $\widetilde{g_n}(y_i)$ approaches $\widetilde{g_n}(C_*)$. In particular, we can find two pairs $\{\widetilde{g_n}(x_N), \widetilde{g_n}(y_N)\}$ and $\{\widetilde{g_n}(x_M), \widetilde{g_n}(y_M)\}$ in this sequence that are less than δ away from $\widetilde{g_n}(C_*)$. Also by Lemma 1.6.2, we can find two measures μ and μ' of finite logarithmic energy supported in $x_N \frown x_M$ and $y_N \frown y_M$ respectively that are glued together via $\approx_\alpha$. We set $A_n = \widetilde{g_n}(x_N \frown x_M)$, $A'_n = \widetilde{g_n}(y_N \frown y_M)$, $B_n = \mathrm{supp}\,\widetilde{g_n}\#\mu \subseteq A_n$ and $B'_n = \mathrm{supp}\,\widetilde{g_n}\#\mu' \subseteq A'_n$.

Note that this collection $\{A_n\} \cup \{A'_n\}$ for each $C^K(x)$ is finite and there are also only finitely many different gluing links in $\{C^K(x)\}$. Thus we can find some constants (N, C, r) such that for each $C^K(x)$, $\{A_n\}\cup\{A'_n\}$ is a (N, C, r)-nice gluing circuit. By our construction, for each $C^K(x)$, the gluing circuit $\{A_n\}\cup\{A'_n\}$ is less than δ away from the bad sets $\{\tilde{g}(C_*)\}$, thus $\{A_n\} \cup \{A'_n\}$ should be around x. $\square$

1.6.2 When $\{C^K(x)\}$ is Digit-Fixing

For $\{C^K(x)\}$ to be digit-fixing, we need to introduce a weak pre-periodicity condition.

Definition 1.6.4 (Weak pre-periodicity)**.** We say $\alpha \in \mathbb{T}$ is *weakly pre-periodic* with period k with there exists integer m such that $I(\alpha)[m + kn]$ is the same letter for all integer $n \in \mathbb{N}$.

Lemma 1.6.5. *Weakly pre-periodic points on $\mathbb{T}$ has measure zero.*

Proof. Given $m, k \in \mathbb{N}$ and $X \in \{L, R\}$, we define

$$S_{m,k,X} := \left\{u \in \{L,R\}^{\infty} : u[m+nk] = X \quad \text{for all} \quad n\right\}.$$

Under the metric of the space of sequences, we have that

$$\dim_{\mathcal{H}}(S_{m,k,X}) \leq \frac{k-1}{k}.$$

According to Proposition 1 of [98] (see also [24, Section 16]), pulling back by the map $\varphi : \alpha \mapsto I^{\alpha}$ does not increase Hausdorff dimension. The result follows from the fact that

$$\{\alpha \text{ is weakly pre-periodic}\} = \bigcup_{(m,k,X)\in\mathbb{N}^2\times\{L,R\}} \varphi^{-1}(S_{m,k,X}).$$

$\square$

Proposition 1.6.6. *If α is not strongly recurrent nor is it weakly pre-periodic, then for any integer L, there exists $L_{\alpha} > 0$ such that $\left\{C^{L_{\alpha}}(x)\right\}$ is L-digit-fixing.*

Note that if $\alpha \in (C^K)^{(n)}(x)$, then the gluing link $(C^K)^{(n)}(x)$ is either $GCS_{n+K}(\alpha)$ or $GCS_{n+K+1}(\alpha)$ by the property of generalized cylinder sets. Thus by the criterion in Proposition 1.4.9, the proposition above holds due to the following.

Proposition 1.6.7. *Suppose that α satisfies the same condition as in Theorem 1.1.2. Then given any integer $L > 0$, there exists integer $L_{\alpha} > 0$ such that*

$$GCS_n(\alpha) \cap \left\{h^i(\alpha)\right\}_{i=1}^{L} = \emptyset \quad \textit{for all} \quad n \geq L_{\alpha}.$$

Note that if we replace the generalized cylinder sets $GCS_n(\alpha)$ (which is represented by $\mathrm{Legal}(I(\alpha)|_{[1,n+1]})$) with proper cylinder sets $C(I(\alpha)|_{[1,n+1]})$, then it is easy to see that the result is true so long as α is not pre-periodic. I suspect that Proposition 1.6.7 holds even without the non-weakly pre-periodic condition. In that case, all points on $\mathbb{T}$, except for a set of Hausdorff dimension zero, would satisfy CCE. It might be worth looking into.

Conjecture 1.6.8. *Suppose α is not strongly recurrent nor is it pre-periodic. Then the statement in Proposition 1.6.7 still holds.*

For now to prove the proposition, we need the following lemma.

Lemma 1.6.9. *Suppose that α is not weakly pre-periodic. Then for any pre-periodic $x \in \mathbb{T}$ with period sufficiently large (depending on α), there exists N_x such that*

$$\alpha \notin GCS_n(x), \quad \textit{for all} \quad n \geq N_x.$$

The proof of this lemma is done by arguing that the legal word $\mathrm{Legal}(I^\alpha(x)|_{[1,n+1]})$ associated with the $GCS_n(x)$ should be approximately pre-periodic. Therefore, if α is in infinitely many $GCS_n(x)$, it should also be approximately pre-periodic. However, without further information, it is hard to control the positions of the free digits for $\mathrm{Legal}(I^\alpha(x)|_{[1,n+1]})$, even for pre-periodic x. Therefore, instead of dealing with the free digits directly, the proof here uses an algorithm to carefully truncate $\mathrm{Legal}(I^\alpha(x)|_{[1,n+1]})$ and we will use some facts about functional graphs to circumvent such difficulty.

Proof of Lemma 1.6.9. Let k be large enough such that $I(\alpha)$ does not start with k same letters. It follows that for any generalized cylinder set $C(u)$, u cannot have k consecutive free digits. We claim that the lemma holds for all pre-periodic x with period greater than k. Suppose such an N_x does not exist and we have an increasing sequence of indices $\{n_l\}$ such that

$$\alpha \in GCS_{n_l-1}(x) = C(\mathrm{Legal}(I^\alpha(x)|_{[1,n_l]})).$$

Since x is pre-periodic, we can write $I^\alpha(x)$ as $u'\bar{u}$ for some $u, u' \in \{L, R\}^*$ and $|u| > k$. We let $v^i \in \{L, R, \star\}^{2|u|-i}$ be the legal word associated with $(uu)|_{[1,2|u|-i]}$. We define a map

$$\begin{aligned} \phi :&[k] \cup \{0\} \to [k] \cup \{0\} \\ &i \mapsto \text{the number of consecutive } \star\text{'s right before } v^i[|u|+1]. \end{aligned}$$

The map ϕ defines a directed graph G on $[k] \cup \{0\}$ where each vertex has out-degree 1. Note that by definition, the $(|u| - \phi(i))$-th digit (or $(|u| - i + \phi(i) + 1)$-th-to-last digit) of v^i is not free. Now we can associate each n_l with a finite walk $\{V_k\}_{k=0}^{p_l-1}$ in G as follows. We shall write

$$I^\alpha(x)|_{[1,n_l]} = u'u^{p_l}r,$$

for some word r with length less than $|u|$ and some integer p_l. We set $w_0 = I^\alpha(x)|_{[1,n_l]}$ and V_0 be the number of consecutive $\star$'s proceeding the last r digits of Legal(w_0). We will start our walk at V_0, remove the last $V_0 + |r|$ digits of $I^\alpha(x)|_{[1,n_l]}$ (denoted by w_1). Since $(V_0 + |r| + 1)$-th-to-last digit of Legal(w_0) is not free, by Lemma 1.5.10 we have

$$C(\text{Legal}(w_0)) \subseteq C(\text{Legal}(w_1)).$$

Now for each step, the walker goes from V_{j-1} to $V_j := \phi(V_{j-1})$. We truncate the last $(|u| - V_{j-2}) + V_{j-1}$ digits of w_{j-1} to get w_j. Note that w_{j-1} ends with $u|_{[1,|u|-V_{j-2}]}$ and thus Legal(w_{j-1}) ends with $v^{V_{j-2}}$. Since $(|u| - V_{j-1})$ is not a free digit of $v^{V_{j-2}}$, the $(|u| - V_{j-2} + V_{j-1} + 1)$-th-to-last digit of Legal(w_{j-1}) cannot be free. Therefore using Lemma 1.5.10 again, we conclude that

$$C(\text{Legal}(w_{j-1})) \subseteq C(\text{Legal}(w_j)).$$

After $p_l - 1$ steps, we terminate the walk at the vertex V_{p_l-1} and make the final truncation, which results in

$$w_{p_l} = u'(u|_{[1,|u|-V_{p_l-1}]}).$$

Note that by our assumption, $\alpha \in C(\text{Legal}(w_0)) \subseteq C(\text{Legal}(w_k))$. Therefore,

$$\text{Legal}(I(\alpha)|_{[1,|w_k|]}) = \text{Legal}(w_k).$$

By our construction,

$$|w_k| = |u'| + (p_l - k)|u| - V_{k-1}.$$

This implies that given our walk $\{V_k\}$, we have

$$I(\alpha)[|u'| - V_{k-1} + (p_l - k)|u|] = I^\alpha(x)[|u'| - V_{k-1} + (p_l - k)|u|].$$

Now among all the walks, there are infinitely many of them that are trapped in the same cycle $\sigma \subset G$ and end at the same vertex V. For those walks, we always have $V_{p_l-1-k|\sigma|} = V$ and thus

$$|w_{p_l-k|\sigma|}| = |u'| - V_{p_l-1-k|\sigma|} + (p_l - (p_l - 1 - k|\sigma| + 1))|u| = |u'| - V + k|\sigma||u|$$

It follows that for any $k \geq 1$, the $(|u'|-|V|+k|\sigma||u|)$-th digits of $I^\alpha(x)$ and $I(\alpha)$ coincide. Since they all have to be the same letter by the pre-periodicity of x, this contradicts the fact that α is not weakly pre-periodic. $\square$

Proof of Proposition 1.6.7. Given any $i \in [L]$, we can choose an integer n_i large enough such that $h^i(\alpha)$ is not in the cylinder set $C(I^\alpha|_{[1,n_i]})$. Since $\alpha \in C(I^\alpha|_{[1,n_i]})$, we can select a boundary pair $\tilde{u}(\{\star_1, \star_2\})$ of the cylinder set $C(I^\alpha|_{[1,n_i]})$ that l_u separates $h^i(\alpha)$ and α. Let k be the integer in Lemma 1.6.2. Note that there exist only finitely many periodic points in $\mathbb{T}$ with period not greater than k.

We first suppose that $|[\star_i]_{\approx_\alpha}| = 2$. Note that $\tilde{u}$ will be continuous near $\{\star_1, \star_2\}$. By Lemma 1.6.2, we can find a gluing pair $\{x_1, x_2\}$ of periodic points with itinerary $\overline{g}$ arbitrarily close to $\{\star_1, \star_2\}$ with period larger than k. For such a pair, $\tilde{u}(\{x_1, x_2\})$ will be a pair of pre-periodic points with itinerary $u\overline{g}$ that is close to $\tilde{u}(\{\star_1, \star_2\})$ and thus the geodesic connecting $\tilde{u}(\{x_1, x_2\})$ also separates $h^i(\alpha)$ and α.

If $|[\star_i]_{\approx_\alpha}| = 4$ instead. The function $\tilde{u}$ will be continuous in a neighborhood of C_* on $\mathbb{T}$. Since the lamination $\approx_\alpha$ satisfies the non-crossing property, $\tilde{u}(C_*)$ will separate $h^i(\alpha)$ and α. It follows that one of $\tilde{u}(\{\star_i, b_i\})$ will also separate the two. For this i, by Lemma 1.6.2, we can find a gluing pair $\{x_1, x_2\}$ of periodic points with itinerary $\overline{g}$ arbitrarily closed to $\{\star_i, b_i\}$ larger than k. Hence, $\tilde{u}(\{x_1, x_2\})$ will be a pair of pre-periodic points with itinerary $u\overline{g}$ that is close to $\tilde{u}(\{\star_i, b_i\})$ and thus the geodesic connecting $\tilde{u}(\{x_1, x_2\})$ separates $h^i(\alpha)$ and α.

By Lemma 1.6.9, there exists L_i such that

$$\alpha \notin GCS_n(\tilde{u}(x_1)) = GCS_n(\tilde{u}(x_2)) \quad \text{for all} \quad n \geq L_i.$$

Thus there is a generalized cylinder set of any degree greater than L_i that is between $h^i(\alpha)$ and α. It follows that

$$h^i(\alpha) \notin GCS_n(\alpha) \quad \text{for all} \quad n \geq N_i.$$

The result follows by choosing $L_\alpha = \max_{i \in [L]} L_i$. $\square$

1.6.3 Proving Theorem 1.1.2

The proof of Theorem 1.1.2 is outlined in Figure 1.2.

Proof of Theorem 1.1.2. Since α is not strongly recurrent, by Theorem 1.4.7, there exist (L, P, M) such that α satisfies semi-CCE condition for any L-digit-fixing covering $\{D(x)\}_{x\in\mathbb{T}}$ with respect to parameters (L, P, M). Since we additionally assume that α is not weakly pre-periodic, by Proposition 1.6.6, there exists L_α such that $\{C^{L_\alpha}(x)\}$ is a L-digit-fixing covering of gluing links. By Proposition 1.6.1, there exist (N, C, r) such that each $C^{L_\alpha}(x)$ contains an (N, C, r)-nice chain around x. Thus α satisfies CCE condition with the covering $\{C^{L_\alpha}(x)\}_{x\in\mathbb{T}}$ and parameters (N, C, r, P, M). The result then follows. $\square$

Chapter 2

GRADIENT MAXIMIZING DOMAIN OF TORSION FUNCTION

2.1 Introduction and Results

2.1.1 Introduction

We study a variational problem involving the torsion function. For any bounded, convex domain $\Omega \subset \mathbb{R}^2$, the *torsion function* u on Ω is the unique solution to the Dirichlet problem:

$$\begin{aligned} -\Delta u &= 1 \quad \text{inside } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

This is a classical object in Elasticity Theory and was already studied by Saint Venant [8] in 1856. From the very beginning, there has been substantial interest in trying to understand where $|\nabla u|$ is the largest; these points are the one's subjected to the most stress. Saint Venant called them 'the dangerous points' ('*les points dangereux*' [8]). These were also of interest to Lord Kelvin and Tait (see their 1867 *Treatise on Natural Philosophy* [106]). Additional early work was undertaken by Boussinesq [22], Filon [35] and Griffith-Taylor [41]. Pólya [81] showed in 1930 that the point maximizing $|\nabla u|$ is always on the boundary. The partial differential equation $-\Delta u = 1$, being particularly simple and nonetheless exhibiting very nontrivial behavior, has been studied for a very long time in a wide variety of different settings. Much is known about the partial differential equation $-\Delta u = 1$, we refer to [4, 5, 6, 9, 10, 23, 31, 43, 49, 50, 54, 56, 57, 61, 62, 65, 66, 72, 73, 74, 75, 76, 77, 78, 79, 92, 99, 100, 101, 103, 104] and references therein.

2.1.2 The Problem

We are concerned with a very simple question: how large can the gradient be, what are the best bounds for $\|\nabla u\|_{L^\infty(\Omega)}$? It is relatively easy to see that the gradient need not be bounded in general domains; however, the expression is always finite in convex domains. There is a scaling symmetry, if we rescale the domain $\Omega \to \lambda\Omega$ then the gradient rescales by a factor of $\sqrt{\lambda}$. Accounting for this symmetry leads to two natural shape optimization problems

$$\sup_{\Omega \text{ convex, bounded}} \frac{\|\nabla u\|_{L^\infty(\Omega)}}{|\Omega|^{1/2}} \qquad \text{and} \qquad \sup_{\Omega \text{ convex, bounded}} \frac{\|\nabla u\|_{L^\infty(\Omega)}}{\mathcal{H}^1(\partial\Omega)}.$$

The first problem has been studied for a very long time. A fundamental insight by Sperb [99] is that the expression $|\nabla u|^2 + 2u$ attains its maximum at the boundary. Since u vanishes on the boundary, we deduce the inequality

$$\|\nabla u\|_{L^\infty(\Omega)}^2 = \left\|\frac{\partial u}{\partial \mathbf{n}}\right\|_{L^\infty(\partial\Omega)}^2 \leq 2\|u\|_{L^\infty(\Omega)}.$$

It is a celebrated result of Pólya [82] that $\|u\|_{L^\infty(\Omega)}$ is maximized when the domain is a ball and a direct computations gives $\|u\|_{L^\infty(\Omega)} \leq |\Omega|/(4\pi)$. Altogether,

$$\frac{\|\nabla u\|_{L^\infty(\Omega)}}{|\Omega|^{1/2}} \leq \frac{1}{\sqrt{2\pi}} \approx 0.398\ldots$$

It is known that this upper bound is not tight. A simple computation using ellipses (for which u is merely a quadratic polynomial) gives an example for which the functional is at least 0.321. Two independent numerical approaches [44] both suggest the optimal value to be roughly 0.358. The second functional, which was suggested by Krzysztof Burdzy, can be bounded in terms of the first: using the isoperimetric inequality and the previous bound, one has

$$\frac{\|\nabla u\|_{L^\infty(\Omega)}}{\mathcal{H}^1(\partial\Omega)} \leq \frac{\|\nabla u\|_{L^\infty(\Omega)}}{|\Omega|^{1/2}} \frac{|\Omega|^{1/2}}{\mathcal{H}^1(\partial\Omega)} \leq \frac{1}{\sqrt{2\pi}} \frac{1}{2\sqrt{\pi}} = \frac{1}{\sqrt{8}\pi}.$$

Explicit examples, see Figure 2.1, shows that both inequalities are nearly tight, the first being as little as 10% away from the optimal value, the second at most 13%.

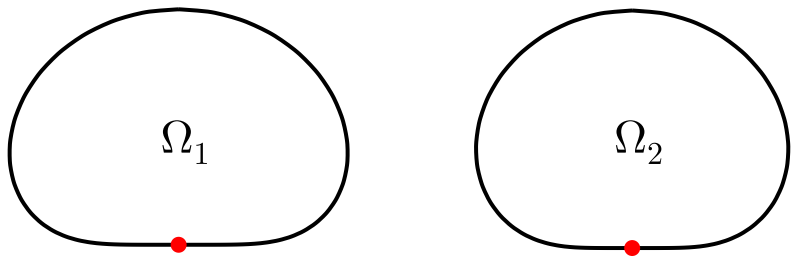

Figure 2.1: Left: an example of a convex domain for which $\|\nabla u\|_{L^\infty}/|\Omega|^{1/2} \sim 0.8926/\sqrt{2\pi}$. Right: am example of a domain where one has $\|\nabla u\|_{L^\infty}/\mathcal{H}^1(\Omega) \sim 0.8723/(\sqrt{8}\pi)$. Both are from a family of domains described by Sweers (see [44] and Section 6). The red dots are the point with largest $|\nabla u(z)|$

We are motivated by the extremal domains for these two functionals. The shape shown in Figure 2.1 for the first functional was deduced in two different ways, once via explicit numerics and once via conformal geometry [44], and the results appear to be consistent and very close to what is shown in Figure 1. Since we are maximizing an $L^\infty-$norm, the asymmetry is perhaps not too surprising. The extremal domains appear to be smooth (though hypothetical non-smooth extremizers would be difficult to find numerically) and they appear to be relatively flat close to the point where $|\nabla u|$ attains its maximum. Virtually nothing seems to be known about these domains. Standard results could be used to show that the inradius cannot be too small or that the curvature cannot be uniformly very large, however, the existing techniques do not seem to easily allow for the study of these domains.

2.1.3 Statement of Results

We prove two structure statements about the possible shape of these extremizers. The first result states that the maximizer of $\|\nabla u\|_{L^\infty}/|\Omega|^{1/2}$ cannot have 'sharp corners' (meaning corners with an angle of more than 90 degrees).

Theorem 2.1.1 (No Sharp Corners)**.** *If Ω is a convex maximizer of $\|\nabla u\|_{L^\infty}/|\Omega|^{1/2}$ and if the boundary is $\partial\Omega$ is $C^{1+\alpha}$ around any maximizing point for some $\alpha > 0$, then $\partial\Omega$ cannot* *admit two adjacent $C^{1+\beta}$-smooth subarcs that meet at an angle sharper than $\pi/2$ for any $\beta \in (0,1)$.*

This rules out extremal examples like the square, however, one would naturally be inclined to believe that this limitation is an artifact of the method; in fact, the result follows from showing that the Riemann map $f : \mathbb{D} \to \Omega$ has to satisfy $f' \in L^2(\partial\mathbb{D})$ which rules out corners sharper than 90 degrees. The second statement is our main result. It shows that the extremizer of either of these two problems is guaranteed to be fairly unusual (and thus fairly interesting).

Theorem 2.1.2 (Main Result). *Let Ω be a convex maximizer of either of the two functionals. Then*

1. *either the boundary is not $C^{2+\varepsilon}$ for some $\varepsilon > 0$ or*

2. *or the points of zero curvature on $\partial\Omega$ accumulate around the point in which the gradient is maximal.*

Both cases would be *very* interesting. Given the nature of the problem, one is naturally inclined to believe that the extremal domain is going to be smooth. There are natural questions of the precise type of regularity (real-analytic, Gevrey, C^∞, ...) but there is nothing in the problem statement that would suggest the possibility of $\partial\Omega$ being not, say, C^3. The second alternative would mean that $\partial\Omega$ is 'flat' around the point in which $|\nabla u|$ is maximal. This is natural and to be expected, however, the natural expectation is surely that the curvature κ vanishes, say, quadratically, or to some higher order. Having a sequence of points with zero curvature in every neighborhood is very atypical.

2.1.4 The techniques

Our techniques rely entirely on classic arguments from complex analysis. This is partially inspired by the hope that one might one day be able to characterize the extremal domains.

Given their unusual shape, the only hope of possibly achieving this might be by describing it in terms of the Riemann map. Instead of optimizing over convex domains Ω, we optimize over conformal maps $f : \mathbb{D} \to \Omega$. For example, if $f(\mathbb{D}) = \Omega$ has a $C^{1+\alpha}$-boundary and if $|\nabla u|$ is maximal in $f(1) \in \partial\Omega$, then

$$\frac{\|\nabla u\|_{L^\infty(\Omega)}}{|\Omega|^{1/2}} = \frac{\int_{\mathbb{D}} |f'(w)|^2 P(w) |dw|^2}{2\pi |f'(1)| \cdot \left(\int_{\mathbb{D}} |f'(w)|^2 |dw|^2 \right)^{1/2}}, \qquad (\diamond).$$

where $P : \mathbb{D} \to \mathbb{C}$ is an explicit function. We prove a precise equivalence between the PDE problem and its analogue in conformal geometry. One persistent obstacle is that convexity of Ω is non-trivially encoded in the behavior of the conformal map $f : \mathbb{D} \to \Omega$ (Study's Theorem etc). The main idea is then to use an Euler-Lagrange approach within a suitable function space and with a very precisely constructed deformation. It would be nice if the extremal domain for either of the two problems had a nice representation in terms of the conformal map or, equivalently, if the functional $(\diamond)$ had a nice extremizers; once more, the global restriction of $f(\mathbb{D})$ being convex poses a significant challenge.

2.2 Preliminaries

2.2.1 Conformal Geometry on Convex Domains

In this paper, we will follow [11, 33, 37] and use the term 'conformal map' to refer to analytic univalent maps. We will use the complex variable w for area integrals and ζ for line integrals. That is, we will write $|dw|^2$ for the Lebesgue measure on $\mathbb{R}^2 = \mathbb{C}$ and $|d\zeta|$ for the Hausdorff measure $\mathcal{H}^1(\Gamma)$ on a rectifiable curve (or loop) $\Gamma \subseteq \mathbb{C}$.

Given a Jordan domain $\Omega \subseteq \mathbb{C}$, as it is simply connected, there exists a conformal map $f : \mathbb{D} \to \Omega$. We refer to this map as a Riemann map of Ω. By Carathéodory's theorem [26], f extends uniquely to a homeomorphism between $\overline{\mathbb{D}}$ and $\overline{\Omega}$. Convexity of a domain in $\mathbb{R}^2$ is a purely geometric condition: there are a number of classical results explaining how this notion is reflected in the Riemann map and we summarize three such results.

Lemma 2.2.1 (Characterization of Convexity, see [33, Duren, Section 2.5])**.** *Let $f : \mathbb{D} \to \mathbb{C}$ be a locally univalent analytic function (that is, $f'(z) \neq 0$). Then f is a conformal map onto a convex domain if and only if*

$$\operatorname{Re}\left(1 + z\frac{f''(z)}{f'(z)}\right) > 0 \quad \textit{for all} \quad z \in \mathbb{D}.$$

There is another characterization of convex conformal maps by concentric disks.

Lemma 2.2.2 (Study's Theorem, [102, Section 13])**.** *Let $f : \mathbb{D} \to \Omega$ be a locally univalent function. Then f is a conformal map onto convex domain if and only if for any $r \in (0, 1)$, $f(r\mathbb{D})$ is a convex domain.*

Proof. Lemma 2.2.1 allows for a very short proof: let $f_r(z) := f(rz)$. If f is a conformal map of a convex domain, then we have $f_r'(z) = rf'(rz) \neq 0$ and

$$\operatorname{Re}\left(1 + z\frac{f_r''(z)}{f_r'(z)}\right) = \operatorname{Re}\left(1 + z\frac{r^2 f''(rz)}{rf'(rz)}\right) = \operatorname{Re}\left(1 + rz\frac{f''(rz)}{f'(rz)}\right) > 0.$$

□

The third characterization of convex conformal maps is stated as follows.

Lemma 2.2.3 (Radial Increase, [71, Proposition 2.13])**.** *Let $f : \mathbb{D} \to \Omega$ be a conformal map onto a simply connected domain Ω. Then Ω is convex if and only if*

$$r|f'(re^{i\theta})| \quad \textit{is an increasing function of } r \in [0, 1) \textit{ for all } \theta \in [-\pi, \pi].$$

Proof. For the convenience of the reader, we sketch a quick proof using only Lemma 2.2.1. Given $\theta \in [-\pi, \pi]$, we let $h_\theta(r) := r^2|f'(re^{i\theta})|^2$. Then we have

$$\begin{aligned}
h_\theta'(r) &= 2r|f'(re^{i\theta})|^2 + r^2\frac{d}{dr}f'(re^{i\theta})\overline{f'(e^{i\theta})} \\
&= 2r|f'(re^{i\theta})|^2 + r^2\left(e^{i\theta}f''(re^{i\theta})\right)\overline{f'(e^{i\theta})} + r^2 f'(e^{i\theta})\overline{(e^{i\theta}f''(re^{i\theta}))} \\
&= 2r|f'(re^{i\theta})|^2 + 2r^2\operatorname{Re}\left(e^{i\theta}f''(re^{i\theta})\overline{f'(re^{i\theta})}\right) \\
&= 2r|f'(re^{i\theta})|^2\operatorname{Re}\left(1 + \frac{re^{i\theta}f''(re^{i\theta})}{f'(re^{i\theta})}\right).
\end{aligned}$$

The result then follows from Lemma 2.2.1. □

2.2.2 Two Types of Perturbations

Given a Riemann map f of a bounded convex domain Ω, in this paper, we will introduce two ways to "perturb" the map f to generate new convex domains. First, we upgrade Lemma 2.2.2.

Corollary 2.2.4. *Let f be a Riemann map of bounded convex domain. Then for any disk $B \subset \mathbb{D}$, $f(B)$ is also a convex domain.*

Proof. We divide the proof into two cases: the case $\overline{B} \subseteq \mathbb{D}$ and the case where ∂B touches $\partial\mathbb{D}$. We start with the case $\overline{B} \subseteq \mathbb{D}$. Here, we will show that we can map B to some $r\mathbb{D}$ using a Möbius transformation of the unit disk. Consider the Möbius transformation $\varphi : \mathbb{D} \to \mathbb{H}$ given by

$$\phi(z) = i\frac{1+z}{1-z}.$$

This map (see Figure 2.2) satisfies that

$$\varphi(r\mathbb{D}) = B_{\frac{2r}{1-r^2}}\left(i\frac{1+r^2}{1-r^2}\right).$$

Setting $\lambda = (1+r^2)/(1-r^2)$, we have $\varphi(r\mathbb{D}) = B_{\sqrt{\lambda^2-1}}(\lambda i)$, where $r \in (0,1)$ corresponds to $\lambda \in (1,\infty)$ bijectively (we write $r = r(\lambda)$). Suppose φ maps B to a disk $B_c(a+bi)$ in $\mathbb{H}$.

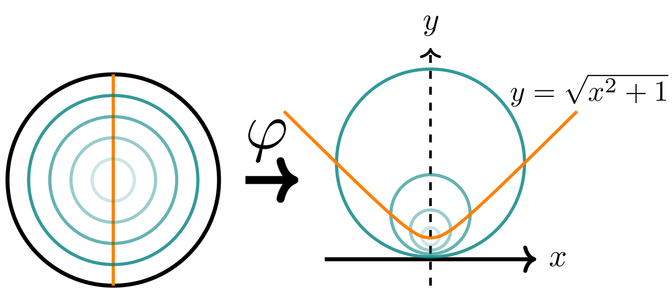

Figure 2.2: Image of concentric circles under φ

Since $\overline{B} \subseteq \mathbb{D}$, we have that $c < b$. Note that

$$\varphi_0(B_c(a+bi)) = B_{\frac{c}{b^2-c^2}}\left(\frac{bi}{b^2-c^2}\right) = B_{\sqrt{\lambda'^2-1}}(\lambda' i) = \varphi(r'\mathbb{D}),$$

where $\lambda' = b/(b^2 - c^2)$, $r' = r(\lambda')$ and

$$\varphi_0(z) = \frac{z-a}{b^2 - c^2}.$$

is an automorphism of $\mathbb{H}$. Then $\varphi_0(B_r(a+bi))$ is the image of some $r'\mathbb{D}$ under φ. Hence, $B_c(a+bi)$ is the image of some $r'\mathbb{D}$ under $\varphi_0^{-1} \circ \varphi : \mathbb{D} \to \mathbb{H}$. This shows that B is an image of some $r\mathbb{D}$ under the Möbius transform $\varphi_B = \varphi^{-1} \circ \varphi_0 \circ \varphi \in \mathrm{Aut}(\mathbb{D})$. Note that $f \circ \varphi_B^{-1}$ is a conformal map with convex image. Thus $f \circ \varphi_B^{-1}(r\mathbb{D}) = f(B)$ is an analytic convex domain by Lemma 2.2.2. It remains to deal with the case where ∂B touches $\partial\mathbb{D}$: without the loss of generality, we suppose that $\partial B \cap \partial\mathbb{D} = \{1\}$. Let $t > 0$ be such that $\overline{B} - t \subseteq \mathbb{D}$. Then we have that $f(B)$ is a topological limit of $f(B-t)$ as $t \to 0$. The result follows from the first case. $\square$

By restricting f to a smaller disk $B \subseteq \mathbb{D}$, we can get a convex sub-domain of the original convex domain and its corresponding Riemann map. We refer to this method for producing new convex maps as *domain restriction* (see Figure 2.3).

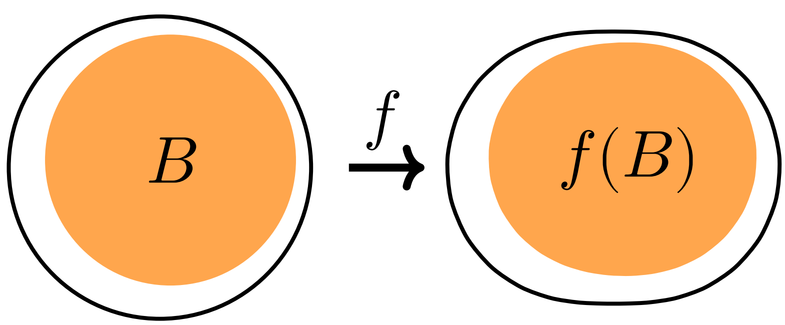

Figure 2.3: Domain restriction.

Secondly, we consider adding a noise to the map f in the direction of a new function $g : \mathbb{D} \to \mathbb{C}$. We set

$$v_f(z) := \mathrm{Re}\left(1 + z\frac{f''(z)}{f'(z)}\right).$$

Note that $v_f(z)$ is a harmonic function. We can set

$$k_f(\theta) := \liminf_{\mathbb{D} \ni z \to e^{i\theta}} v_f(z).$$

We call $k_f : [-\pi, \pi] \to [-\infty, \infty]$ the *curvature signature* of f. We will discuss its relation to the curvature in Section 2.4.1. Note that

$$v_f(0) = \mathrm{Re}\left(1 + 0 \cdot \frac{f''(0)}{f'(0)}\right) = 1.$$

It follows that if k_f is a continuous extention of v_f to $\partial\mathbb{D}$, then we have

$$\int_{-\pi}^{\pi} k_f(\theta) d\theta = 2\pi v_f(0) = 2\pi.$$

Corollary 2.2.5 (Characterization by the curvature signature)**.** *Let $f : \mathbb{D} \to \mathbb{C}$ be a locally univalent analytic function. Then f is a conformal map onto a convex domain if and only if $k_f(\theta) \geq 0$.*

Proof. It is clear that $k_f \geq 0$ if $v_f > 0$. Suppose that k_f is nonnegative. Then v_f is lower-bounded. Thus by the maximum (minimum) principle, we have

$$v_f(z) \geq \inf_{\theta \in [-\pi, \pi]} k_f(\theta) \geq 0.$$

Since v_f is a non-negative harmonic function that is strictly positive at 1, it is strictly positive on $\mathbb{D}$. $\square$

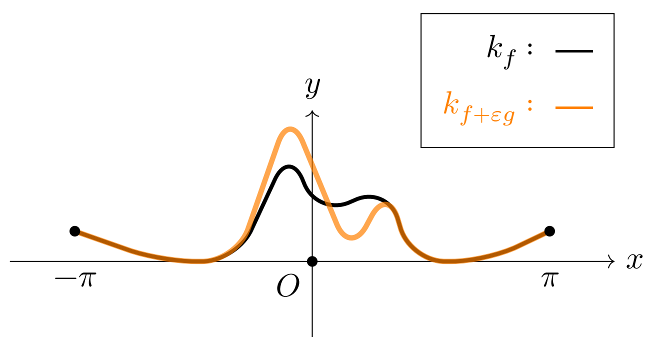

Figure 2.4: Changing the curvature signature.

Corollary 2.2.5 gives us the second way to perturb a conformal map f: When adding a noise function g to the original conformal map f, $f + \varepsilon g$ is a conformal map onto a convex domain so long as it is locally univalent and the curvature signature $k_{f+\varepsilon g}$ remains non-negative (see Figure 2.4).

2.2.3 Setup for Torsion Functions

Given a Jordan domain Ω and its torsion function u, we set

$$\mathbb{L}_1(\Omega) := \frac{\|\nabla u\|_{L^\infty(\Omega)}}{|\Omega|^{1/2}} \qquad \text{and} \qquad \mathbb{L}_2(\Omega) := \frac{\|\nabla u\|_{L^\infty(\Omega)}}{\mathcal{H}^1(\partial\Omega)}. \tag{2.1}$$

Recall that we are interested in maximizing $\mathbb{L}_i$ over all convex bounded domains. Note that we have

$$\|\nabla u\|^2_{L^\infty(\Omega)} = \left\|\frac{\partial u}{\partial \mathbf{n}}\right\|^2_{L^\infty(\partial\Omega)}.$$

We call the points on $\partial\Omega$ where $\|\partial u/\partial \mathbf{n}\|_{L^\infty(\partial\Omega)}$ are attained the *extremal points*. We will rewrite the quantities $\mathbb{L}_i(\Omega)$ in terms of a Riemann map f of Ω. Let us first recall Kellogg's theorem, which states the connection between the differentiability of f on $\overline{\mathbb{D}}$ and the smoothness of $\partial\Omega$.

Lemma 2.2.6 (Kellogg's Theorem, [52])**.** *Suppose $f : \mathbb{D} \to \Omega$ is a conformal map onto a Jordan domain Ω. Let $k \geq 1$ and $0 < \alpha < 1$. Then the following are equivalent:*

(a) $\partial\Omega$ is a $C^{k+\alpha}$-smooth curve.

(b) $\arg f' \in C^{k-1+\alpha}(\partial\mathbb{D})$.

(c) $f \in C^{k+\alpha}(\overline{\mathbb{D}})$ and $f' \neq 0$ on $\partial\mathbb{D}$.

If $k \geq 1$ and $\alpha = 0$, then (a) and (b) are still equivalent and (c) implies both. However, (a) or (b) does not imply (c).

With Kellogg's theorem, we can find a representation of the maximal normal derivative purely in terms of the Riemann map for convex $C^{1+\alpha}$ domains.

Lemma 2.2.7. *Let Ω be a bounded convex $C^{1+\alpha}$ domain for some $\alpha > 0$. Then for any Riemann map $f \in C^{1+\alpha}(\overline{\mathbb{D}})$ of Ω that sends 1 to an extremal point on $\partial\Omega$,*

$$\left\|\frac{\partial u}{\partial \mathbf{n}}\right\|_{L^\infty(\partial\Omega)} = \frac{1}{2\pi|f'(1)|}\int_{\mathbb{D}} |f'(w)|^2 \frac{1-|w|^2}{|1-w|^2}|dw|^2.$$

Proof. Suppose that Ω is $C^{1+\alpha}$ with torsion function u. Let $f : \mathbb{D} \to \Omega$ be the Riemann map of Ω. By Kellogg's theorem, we have $f \in C^{1+\alpha}(\overline{\mathbb{D}})$ and thus f' is α-Hölder continuos on $\overline{\mathbb{D}}$. Given the function $v = u \circ f$, we have

$$\begin{aligned}\frac{\partial^2 v}{\partial x^2}(x,y) = {}& \frac{\partial^2 u}{\partial x^2}(f(x,y)) \cdot \left(\frac{\partial \operatorname{Re} f}{\partial x}(x,y)\right)^2 + \frac{\partial^2 u}{\partial y^2}(f(x,y)) \cdot \left(\frac{\partial \operatorname{Im} f}{\partial x}(x,y)\right)^2 \\ &+ 2\frac{\partial^2 u}{\partial x \partial y}(f(x,y)) \cdot \frac{\partial \operatorname{Re} f}{\partial x}(x,y) \cdot \frac{\partial \operatorname{Im} f}{\partial x}(x,y) \\ &+ \frac{\partial u}{\partial x}(f(x,y)) \cdot \frac{\partial^2 \operatorname{Re} f}{\partial x^2}(x,y) + \frac{\partial u}{\partial y}(f(x,y)) \cdot \frac{\partial^2 \operatorname{Im} f}{\partial x^2}(x,y)\end{aligned}$$

as well as

$$\begin{aligned}\frac{\partial^2 v}{\partial y^2}(x,y) = {}& \frac{\partial^2 u}{\partial x^2}(f(x,y)) \cdot \left(\frac{\partial \operatorname{Re} f}{\partial y}(x,y)\right)^2 + \frac{\partial^2 u}{\partial y^2}(f(x,y)) \cdot \left(\frac{\partial \operatorname{Im} f}{\partial y}(x,y)\right)^2 \\ &+ 2\frac{\partial^2 u}{\partial x \partial y}(f(x,y)) \cdot \frac{\partial \operatorname{Re} f}{\partial y}(x,y) \cdot \frac{\partial \operatorname{Im} f}{\partial y}(x,y) \\ &+ \frac{\partial u}{\partial x}(f(x,y)) \cdot \frac{\partial^2 \operatorname{Re} f}{\partial y^2}(x,y) + \frac{\partial u}{\partial y}(f(x,y)) \cdot \frac{\partial^2 \operatorname{Re} f}{\partial y^2}(x,y).\end{aligned}$$

By the Cauchy-Riemann equations, we have

$$\frac{\partial \operatorname{Re} f}{\partial x} = \frac{\partial \operatorname{Im} f}{\partial y} \qquad \text{and} \qquad \frac{\partial \operatorname{Re} f}{\partial y} = -\frac{\partial \operatorname{Im} f}{\partial x}.$$

By plugging in the equations, it follows that v solves the Poisson equation.

$$\begin{aligned} -\Delta v &= |f'|^2 \quad \text{inside } \mathbb{D}, \\ v &= 0 \quad \text{on } \partial\mathbb{D}. \end{aligned} \tag{2.2}$$

The unique solution to (2.2) is given in terms of the Green's function $G(z,w)$ on $\mathbb{D}$:

$$v(z) = \int_{\mathbb{D}} G(z,w)|f'(w)|^2|dw|^2,$$

where the Green's function on $\mathbb{D}$ has the expression (see, for example, [37])

$$G(z,w) = \frac{1}{2\pi} \log \frac{|z-w|}{|1-\bar{w}z|}.$$

Since $|f'|^2$ is Hölder-continuous on the closed unit disk $\overline{\mathbb{D}}$, [38, Lemma 4.1] tells us that the normal derivative $\partial v/\partial \mathbf{n}$ exists in the strong sense on $\partial\mathbb{D}$ and is continuous. It has the representation

$$\frac{\partial v}{\partial \mathbf{n}}(e^{i\theta}) = -\int_{\mathbb{D}} P(e^{i\theta}, w)|f'(w)|^2|dw|^2,$$

where

$$P(\zeta, w) := \frac{\partial G(\zeta, w)}{\partial \mathbf{n}_\zeta} = \frac{1}{2\pi}\frac{1-|w|^2}{|\zeta - w|^2}$$

is the Poisson kernel of $\mathbb{D}$. Note that by composition, we have

$$\nabla v(z) = f'(z) \cdot \nabla u(f(z)), \tag{2.3}$$

where the gradients are treated as complex numbers. Since both u and v vanish on the $C^{1+\alpha}$ boundary, the regular level set theorem [55] tells us that

$$|\nabla v(e^{i\theta})| = -\frac{\partial v}{\partial \mathbf{n}}(e^{i\theta}), \quad |\nabla u(f(e^{i\theta}))| = -\frac{\partial u}{\partial \mathbf{n}}(f(e^{i\theta})).$$

It follows that

$$\frac{\partial v}{\partial \mathbf{n}}(e^{i\theta}) = |f'(e^{i\theta})|\frac{\partial u}{\partial \mathbf{n}}(f(e^{i\theta})).$$

This shows that $-\partial u/\partial \mathbf{n}$ is continuous (in fact it has to be as $\Delta u \in C^\infty(\overline{\Omega})$ and $u \in C^\infty(\partial\Omega)$). Thus the extremal points exist. Using the conformal map $f : \mathbb{D} \to \Omega$ mapping 1 to one of the extremal points, we have

$$\left\|\frac{\partial u}{\partial \mathbf{n}}\right\|_{L^\infty(\partial\Omega)} = -\frac{1}{|f'(1)|}\frac{\partial v}{\partial \mathbf{n}}(1).$$

The result then follows. □

With some abuse of the notation, we also refer to

$$P(w) = \frac{1-|w|^2}{|1-w|^2} = \frac{1}{1-\bar{w}} - \frac{1}{1-1/w} = \frac{1}{1-w} + \frac{1}{1-\bar{w}} - 1$$

as the Poisson kernel. Note that for a $C^{1+\alpha}(\overline{\mathbb{D}})$ Riemann map $f : \mathbb{D} \to \Omega$ of a Jordan domain Ω with $\alpha \geq 0$, we have that

$$|\Omega| = \int_{\mathbb{D}} |f'(w)|^2|dw|^2, \quad \mathscr{H}^1(\partial\Omega) = \int_{\partial\mathbb{D}} |f'(\zeta)||d\zeta|.$$

It follows that we can express $\mathbb{L}_1(\Omega)$ and $\mathbb{L}_2(\Omega)$ in terms of a Riemann map of Ω.

Proposition 2.2.8. *Let Ω be a bounded convex $C^{1+\alpha}$ domain for some $\alpha \in (0,1)$. For any Riemann map f of Ω that sends 1 to an extremal point on $\partial\Omega$, we have*

$$L_1(f) := \frac{\int_{\mathbb{D}} |f'(w)|^2 P(w) |dw|^2}{2\pi |f'(1)| \left(\int_{\mathbb{D}} |f'(w)|^2 |dw|^2\right)^{1/2}} = \mathbb{L}_1(\Omega)$$

and

$$L_2(f) := \frac{\int_{\mathbb{D}} |f'(w)|^2 P(w) |dw|^2}{2\pi |f'(1)| \int_{\partial\mathbb{D}} |f'(\zeta)||d\zeta|} = \mathbb{L}_2(\Omega).$$

2.2.4 L_i is well-defined.

It will be important for us to be able to use Proposition 2.2.8 also in a slightly rougher setting. Indeed, we do not necessarily need the conformal map f to be $C^{1+\alpha}(\overline{\mathbb{D}})$, the functional tells us what is needed: we require that

1. $|f'(z)|^2$ is integrable on $\mathbb{D}$.

2. $|f'(z)|^2 P(z)$ is integrable on $\mathbb{D}$.

3. $f'(z)$ is defined at 1.

4. $f'(z)$ is defined almost everywhere on $\partial\mathbb{D}$ and $|f'(z)|$ is integrable on $\partial\mathbb{D}$.

If f is a conformal map of a bounded domain, Condition (1) is always true and we always have $\int_{\mathbb{D}} |f'(w)|^2 |dw|^2 = |f(\mathbb{D})|$. We will now show that in our setting condition (4) is also automatically satisfied. It then suffices to check Conditions (2) and (3). To check Conditions (3) and (4), we need the concept of *nontangential limits*. One can refer to [37] for a more detailed introduction for nontangential limits. For $e^{i\theta} \in \partial\mathbb{D}$ and $\alpha > 1$, we define the *cone*

$$C_\alpha(\theta) = \left\{ z \in \mathbb{D} : |z - e^{i\theta}| < \alpha(1 - |z|) \right\}$$

Note that $C_\alpha(\theta)$ is an increasing family of subsets in α. The cone $C_\alpha(\theta)$ is asymptotic to a sector with vertex $e^{i\theta}$ and angle $2\sec^{-1}(\alpha)$ that is symmetric about the radius $[0, e^{i\theta}]$ (See Figure 2.5).

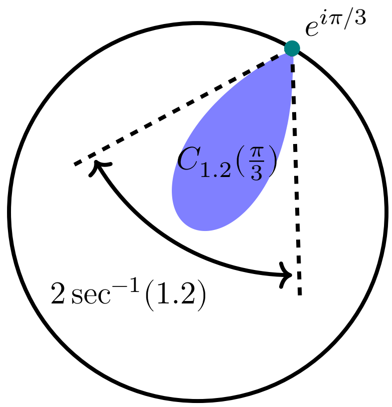

Figure 2.5: Cone $C_{1.2}(\pi/3)$ highlighted in blue

Definition 2.2.9 (Nontangential limit)**.** We say a function $u : \mathbb{D} \to \mathbb{C}$ has a *nontangential limit* A at $e^{i\theta} \in \partial\mathbb{D}$ if for all $\alpha > 1$, we have

$$\lim_{C_\alpha(\theta) \ni z \to e^{i\theta}} u(z) = A.$$

A celebrated result about nontangential limits of conformal maps is the following.

Lemma 2.2.10 ([37])**.** *Let Ω be a Jordan domain and f a conformal map from $\mathbb{D}$ onto Ω. Then the curve $\partial\Omega$ is rectifiable if and only if f' is in the Hardy space $\mathbf{H}^1$. If $f' \in \mathbf{H}^1$, then f' admits has a nontangential limit $f'(e^{i\theta})$ almost everywhere on $\partial\mathbb{D}$ and*

$$\int_{\partial\mathbb{D}} |f'(\zeta)||d\zeta| = \mathscr{H}^1(\partial\Omega).$$

Lemma 2.2.10 implies that if f is a conformal map of a bounded convex domain (recall that closed convex curves are rectifiable), then Condition 4 is always satisfied and the integral $\int_{\partial\mathbb{D}} |f'(\zeta)||d\zeta|$ is always the length of $f(\partial\mathbb{D})$.

Definition 2.2.11 (Families of convex conformal maps)**.** Let $\mathbf{F}$ be the set of all conformal maps of bounded convex domains. In particular, since the area is finite and given by $\int_{\mathbb{D}} |f'(z)|^2 dz$, we automatically know that all functions of interest satisfy $f' \in L^2(\mathbb{D})$. That is, by Lemma 2.2.1,

$$\mathbf{F} := \left\{ f \text{ analytic on } \mathbb{D} \mid f' \neq 0, v_f > 0 \text{ on } \mathbb{D}, \quad f' \in L^2(\mathbb{D}) \right\},$$

Let $\mathbf{F}_0$ is a subfamily of $\mathbf{F}$ such that for each $f \in \mathbf{F}_0$, $|f'(z)|^2 P(z)$ is integrable on $\mathbb{D}$ and $f'(z)$ has a nontangential limit at 1. That is,

$$\mathbf{F_0} := \left\{ f \in \mathbf{F} \mid f' \text{ has a nontangential limit at 1 and } |f'|^2 P \in L^1(\mathbb{D}) \right\}.$$

The functional L_1 and L_2 are well-defined functionals on $\mathbf{F}_0$. We will show in Section 2.5 that

$$\sup_{f \in \mathbf{F}_0} L_1(f) = \sup_{\Omega \text{ convex, bounded}} \mathbb{L}_1(\Omega),$$
$$\sup_{f \in \mathbf{F}_0} L_2(f) = \sup_{\Omega \text{ convex, bounded}} \mathbb{L}_2(\Omega).$$

For the ease of computation, we abbreviate

$$\operatorname{Area}(f) := \int_{\mathbb{D}} |f'(w)|^2 |dw|^2 \qquad \text{and} \qquad \operatorname{Len}(f) := \int_{\partial\mathbb{D}} |f'(\zeta)||d\zeta|$$

whose names are a consequence of the fact that for sufficiently nice Riemann maps, Area(f) is the area of the image and Len(f) is the length of the boundary of the image. Motivated by the preceding considerations we also introduce the functional

$$\operatorname{Poi}(f) := \int_{\mathbb{D}} |f'(w)|^2 P(w) |dw|^2.$$

With these abbreviations in place, we can write the two functionals concisely as

$$L_1(f) = \frac{\operatorname{Poi}(f)}{2\pi |f'(1)| (\operatorname{Area}(f))^{1/2}} \qquad \text{and} \qquad L_2(f) = \frac{\operatorname{Poi}(f)}{2\pi |f'(1)| \operatorname{Len}(f)}.$$

We quickly note variation identities for Area$(\cdot)$ and Poi$(\cdot)$.

Lemma 2.2.12. *Given $f \in \mathbf{F_0}$ and g holomorphic on $\mathbb{D}$ and C^1 on $\overline{\mathbb{D}}$, we have*

$$\begin{aligned}
\frac{d}{dt}\operatorname{Area}(f+tg)\Big|_{t=0} &= 2\int_{\mathbb{D}} \operatorname{Re}(\overline{f'(w)}g'(w))|dw|^2, \\
\frac{d}{dt}\operatorname{Poi}(f+tg)\Big|_{t=0} &= 2\int_{\mathbb{D}} \operatorname{Re}(\overline{f'(w)}g'(w))P(w)|dw|^2, \\
\frac{d}{dt}|f'(1)+tg'(1)|^2\Big|_{t=0} &= 2\operatorname{Re}(\overline{f'(1)}g'(1)).
\end{aligned}$$

Proof. Note that the functionals $\operatorname{Area}(\cdot)$ and $\operatorname{Poi}(\cdot)$ as well as $f \mapsto |f'(1)|^2$ can be thought of as "L^2 energies" of f' with respective to measures $|dw|^2$, $P(w)|dw|^2$ and δ_1 respectively. Thus, it is somewhat straightforward to compute their first variations. For instance, we have that

$$\begin{aligned}
\frac{d}{dt}\operatorname{Area}(f+tg)\Big|_{t=0} &= \frac{d}{dt}\int_{\mathbb{D}} |f'(w)+tg'(w)|^2|dw|^2\Big|_{t=0} \\
&= \frac{d}{dt}\int_{\mathbb{D}} \left(2t\operatorname{Re}(\overline{f'(w)}g'(w)) + t^2|g'(w)|^2\right)|dw|^2\Big|_{t=0} \\
&= 2\int_{\mathbb{D}} \operatorname{Re}(\overline{f'(w)}g'(w))|dw|^2.
\end{aligned}$$

The cases for $\operatorname{Poi}(\cdot)$ and $f \mapsto |f'(1)|^2$ follow similarly. □

The functional $\operatorname{Len}(\cdot)$ is not an L^2 energy for f', we defer the computation of its first variation to Section 2.4.1.

2.2.5 Structure of the proofs

We have converted the optimization problems of $\mathbb{L}_i$ over convex domains to ones of L_i over conformal maps. The purpose of this short section is to explain the different steps that underlie our proof of Theorem 2.1.1 and Theorem 2.1.2.

Proof of Theorem 2.1.1. In Section 2.5, we will present Lemma 2.5.7 which shows that if $f(\partial\mathbb{D})$ has a smooth corner at $f(e^{i\theta_0})$ with angle $\pi\alpha$, then we have

$$|f'(z)| \asymp |z-e^{i\theta_0}|^{\alpha-1}.$$

Inspired by this fact, we aim to show in Section 2.3 that the optimizer f of L_1 satisfies that $f' \in L^2_{\text{loc}}(\partial\mathbb{D}\setminus\{1\})$ (Proposition 2.3.4) and thus $f(\partial\mathbb{D})$ cannot have smooth acute corners away from $f(1)$. To show this,

(1.1) we will first construct an increasing family of disks $\{D_a\}_{a\in(0,1]}$ in $\mathbb{D}$ (see Figure 2.6). These disks correspond to the level sets of the Poisson kernel P.

(1.2) Using domain restriction (Lemma 2.2.4), we will then show that for a conformal map $f:\mathbb{D}\to\Omega$ onto a convex domain, $\left\{f|_{D_a}\right\}$ corresponds to an increasing family of convex subsets in Ω.

(1.3) We then compute the change of L_1 over $\left\{f|_{D_a}\right\}$ for the maximizing conformal map f of L_1 (Proposition 2.3.4). Because of the relation between $\{D_a\}$ and P, the change of $\mathrm{Poi}(f)$ can be computed explicitly along with that of $\mathrm{Area}(f)$.

(1.4) Since f is the maximizer of L_1, the value for L_1 should increase as $\{D_a\}$ increase. This would imply the local square integrability of f' (Proposition 2.3.4).

Proof of Theorem 2.1.2. We will show that for the optimizer f of either L_1 or L_2, $f(\partial\mathbb{D})$ has many "almost zero-curvature" points around $f(1)$ (Proposition 2.4.8) in Section 2.4. To show this,

(2.1) we will define *conformal curvature*, which generalizes the notion of curvature to non-$C^{2+\alpha}$ domains (Proposition 2.4.2).

(2.2) We will develop a way to apply an Euler-Lagrange argument to the optimizer f with a noise function g. We will show how to ensure the nonnegativity of the curvature signature when perturbing f. In particular, we will show that by setting the noise function to be $g=\int_{[0,z]} f'h$, we can relate the change of the curvature signature to the outward pointing normal vector of the domain $h(\mathbb{D})$ (Proposition 2.4.4).

(2.3) We will then construct a specific class $\{h_b\}$ (Lemma 2.4.6). We will prove that if the curvature signature of $f(\mathbb{D})$ were positive near $f(0)$, then we could use $\{h_b\}$ to perturb f.

(2.4) We will then prove that if we could use $\{h_b\}$ to perturb f, then the Euler-Lagrange argument applied to $L_i(f)$ would imply that f is not the maximizer. Thus we can deduce that the curvature signature k_f will have zero points accumulated at 0 (Proposition 2.4.9).

(2.5) This will imply that $f(\partial\mathbb{D})$ has many zero conformal curvature points around $f(1)$ (Lemma 2.4.3).

Translating the conformal functional L_i *to* $\mathbb{L}_i$. So far, we have only worked with the optimal conformal map for L_i instead of the optimal domain for $\mathbb{L}_i$. In Section 2.5,

(3.1) we will show that L_i and $\mathbb{L}_i$ attain the same supremum (Theorem 2.5.3).

(3.2) Using that, we will then show that if f maximizes L_i and is $C^{1+\alpha}$ near 1, then $f(\mathbb{D})$ is in fact the optimal domain for $\mathbb{L}_i$ with $f(1)$ being an extremal point (Section 2.5.3).

(3.3) Finally, we will show that if f is $C^{2+\alpha}$ near 1, then the points with zero conformal curvature on $f(\partial\mathbb{D})$ near 1 are exactly the ones with zero Euclidean curvature, proving Theorem 2.1.2 (Section 2.5.4).

2.3 Domain Restriction

2.3.1 Analytic Domains

For a domain Ω, we denote the set of analytic functions on Ω by $\mathrm{Hol}(\Omega)$ and let $\mathrm{Hol}(\overline{\Omega})$ denote the set of functions in $\mathrm{Hol}(\Omega)$ that can be extended analytically to an open set containing $\overline{\Omega}$. If one (therefore all) Riemann maps of Ω are in $\mathrm{Hol}(\overline{\mathbb{D}})$, then $\partial\Omega$ is a real analytic loop (thus Ω is a real analytic domain). Recall that the family of all Riemann maps of bounded convex domain can be characterized as

$$\mathbf{F} := \left\{ f \in \mathrm{Hol}(\mathbb{D}) |\ f'(z) \neq 0 \text{ and } v_f(z) > 0 \text{ on } \mathbb{D}, f' \in L^2(\mathbb{D}) \right\}. \tag{2.4}$$

We also recall that for L_i to be well-defined, we would like to consider the following subfamily of $\mathbf{F}$:

$$\mathbf{F_0} = \{ f \in \mathbf{F} : \mathrm{Poi}(f) < \infty,\ f \text{ has nontangential limit at } 1 \}.$$

Note that Lemma 2.2.3 implies $f'(1) \neq 0$ whenever it is defined. Then $\mathbf{F_0}$ is the family of all functions in $\mathbf{F}$ where L_1 and L_2 are well-defined in $(0, \infty)$. Using domain restriction with Lemma 2.4.2, we can approximate $L_i(f)$ using functions in $\mathrm{Hol}(\overline{\mathbb{D}})$:

Lemma 2.3.1. *For any $f \in \mathbf{F_0}$, there exists a sequence of functions $\{f_n\} \subseteq \mathbf{F}_0 \cap \mathrm{Hol}(\overline{\mathbb{D}})$ such that*

1. *The sequence $\{f_n\}$ converges to f normally (uniformly on compact sets).*

2. *The sequence $\{f_n(\mathbb{D})\}$ is an increasing family of domains such that*
$$\bigcup_{n=1}^{\infty} f_n(\mathbb{D}) = f(\mathbb{D}).$$

3. *The perimeters of $\{f_n(\mathbb{D})\}$ also increase to that of $f(\mathbb{D})$.*

4. *Let u_n be the torsion function for domain $\Omega_n := f_n(\mathbb{D})$. Then*
$$\mathbb{L}_1(\Omega_n) \geq \frac{\left|\frac{\partial u_n}{\partial \mathbf{n}}(f_n(1))\right|}{|\Omega_n|^{1/2}} = L_1(f_n) \to L_1(f),$$
$$\mathbb{L}_2(\Omega_n) \geq \frac{\left|\frac{\partial u_n}{\partial \mathbf{n}}(f_n(1))\right|}{\mathcal{H}^1(\partial\Omega_n)} = L_2(f_n) \to L_2(f).$$

   We note that if $f_n(1)$ is an extremal point of Ω_n, then the previous inequalities are equations.

Proof. Given $r \in (0,1]$, we define $f_r(z) = f(rz)$. Then Lemma 2.2.2 implies $f_r \in \mathbf{F} \cap \mathrm{Hol}(\overline{\mathbb{D}})$ for all r and Lemma 2.2.3 implies that for all $z \in \mathbb{D}$, $|f_r'(z)|^2 = r^2|f'(rz)|^2$ increases to $|f'(z)|^2$ as $r \to 1$. Note that $\mathrm{Poi}(f)$, $\mathrm{Area}(f)$ and $\mathrm{Len}(f)$ are integrals of $|f'|^2$. Thus by the monotone convergence theorem, we have $\mathrm{Poi}(f_r) \uparrow \mathrm{Poi}(f)$ as well as $\mathrm{Area}(f_r) \uparrow \mathrm{Area}(f)$ and $\mathrm{Len}(f_r) \uparrow \mathrm{Len}(f)$, as r tends to 1. Since for $r \in (0,1)$, $f'(r)$ tends to the nontangential limit $f'(1)$, we also have that $|f_r'(1)| \uparrow |f'(1)|$. Therefore, we have $L_1(f_r) \to L_1(f)$ and $L_2(f_r) \to L_2(f)$. However, for $r < 1$, f_r is analytic on $\overline{\mathbb{D}}$ (in particular it is $C^\infty(\overline{\mathbb{D}})$). The proof of Lemma 2.2.7 implies that

$$\left|\frac{\partial u_r}{\partial \mathbf{n}}(f_r(1))\right| = \frac{\int_{\mathbb{D}} |f_r'(w)|^2 P(w)|dw|^2}{2\pi |f_r'(1)|} = \frac{\mathrm{Poi}(f_r)}{2\pi|f_r'(1)|},$$

where u_r is the torsion function for $f(r\mathbb{D})$. Thus the result follows by taking the sequence with $r_n = 1 - 1/n$. $\square$

We know from Proposition 2.2.8 and Part (4) of Lemma 2.3.1 that

Corollary 2.3.2. *We have*

$$\sup_{f \in \mathbf{F}_0} L_1(f) = \sup_{\Omega \text{ convex, bounded, analytic}} \mathbb{L}_1(\Omega) \leq \frac{1}{\sqrt{2\pi}},$$
$$\sup_{f \in \mathbf{F}_0} L_2(f) = \sup_{\Omega \text{ convex, bounded, analytic}} \mathbb{L}_2(\Omega) \leq \frac{1}{\sqrt{8\pi}}.$$

2.3.2 Family of Shrinking Disks

Now let us consider the following family of disks inside $\mathbb{D}$:

$$D_a = B_a(1-a), \quad a \in (0,1].$$

As we will see soon, these are the level sets of Poisson kernel with $\partial D_a = P^{-1}(\frac{1-a}{a})$. This is the property of this disk family that we want to leverage. For any $f \in \mathbf{F}_0$, $\{D_a\}$ give us a family of conformal maps:

$$\begin{aligned} f_a :& \mathbb{D} \to f(D_a) \\ & z \mapsto f(az + (1-a)). \end{aligned} \tag{2.5}$$

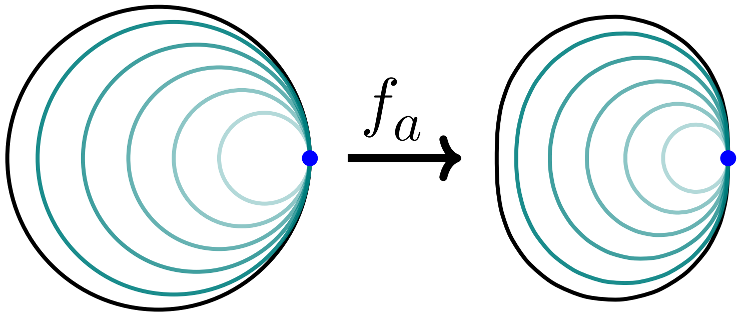

Figure 2.6: Increasing family of disks $\{D_a\}$ and corresponding family of convex domains $\{f(D_a)\}$

For each $a \in (0,1)$, f_a is a conformal map of bounded convex domain by Lemma 2.2.4. It follows that for each $f \in \mathbf{F}_0$, $\{f_a\}$ are a family of maps in $\mathbf{F}$ with $f_1 = f$. Additionally, each $f_a'(z) = af'(az + (1-a))$ has a nontangential limit $af'(1)$ at 1. We want to analyze how $L_1(f_a)$ changes along a. It is clear that $|f_a'(1)| = a|f'(1)|$ and thus $\frac{d}{da}|f_a'(1)| = |f'(1)|$. For $\mathrm{Area}(\cdot)$ and $\mathrm{Poi}(\cdot)$, we have

$$\mathrm{Area}(f_a) = \int_{\mathbb{D}} |f_a'(w)|^2 |dw|^2 = \int_{\mathbb{D}} a^2 |f'(aw + (1-a))|^2 |dw|^2 = \int_{D_a} |f'(w)|^2 |dw|^2$$

and

$$\begin{aligned}\mathrm{Poi}(f_a) &= \int_{\mathbb{D}} |f_a'(w)|^2 P(w) |dw|^2 = \int_{\mathbb{D}} a^2 |f'(aw + (1-a))|^2 P(w) |dw|^2 \\ &= \int_{D_a} |f'(w)|^2 P\left(\frac{w-(1-a)}{a}\right) |dw|^2.\end{aligned}$$

The following lemma gives the formula for the derivatives of $\mathrm{Area}(f_a)$ and $\mathrm{Poi}(f_a)$.

Lemma 2.3.3. *For $a \in (0,1)$, we have that*

$$\frac{d}{da}\mathrm{Area}(f_a) = \int_{\partial D_a} |f'(\zeta)|^2 \,\mathrm{Re}\left(1 - \frac{\zeta}{|\zeta|}\right) |d\zeta|$$

and

$$\liminf_{h \downarrow 0} \frac{\mathrm{Area}(f_1) - \mathrm{Area}(f_{1-h})}{h} \geq \int_{\partial \mathbb{D}} |f'(\zeta)|^2 \,\mathrm{Re}(1-\zeta) |d\zeta|.$$

For $a \in (0,1]$, we also have that

$$\frac{d}{da}\operatorname{Poi}(f_a) = \int_{D_a} |f'(w)|^2 \left(P(w)+1\right)|dw|^2 > 0,$$

where we use the left derivative for $a = 1$. In particular, $\operatorname{Poi}(f_a)$ *is increasing in a. Therefore, if $f \in \mathbf{F_0}$, the the family $\{f_a\}_{a\in(0,1]}$ is also in $\mathbf{F_0}$.*

Proof. Let us first look at $\operatorname{Area}(f_a)$. Using the parametrization $w = re^{i\theta} + (1-r)$ where $r \in (0,1)$ and $\theta \in (-\pi,\pi)$, we have

$$\operatorname{Area}(f_a) = \int_{D_a} |f'(w)|^2 |dw|^2 = \int_0^a \int_{-\pi}^{\pi} |f'(re^{i\theta} + (1-r))|^2 (1-\cos\theta) r d\theta dr.$$

For $a \in (0,1)$, we have $|f'|^2 \in C^\infty(\overline{D_a})$. It follows that

$$\begin{aligned}\frac{d}{da}\operatorname{Area}(f_a) &= \int_{-\pi}^{\pi} |f'(ae^{i\theta} + (1-a))|^2 (1-\cos\theta) a d\theta \\ &= \int_{\partial D_a} |f'(w)|^2 \operatorname{Re}\left(1 - \frac{\zeta}{|\zeta|}\right)|d\zeta|.\end{aligned}$$

Note that $\operatorname{Area}(f_a)$ is increasing, therefore if we set

$$X = \liminf_{h\downarrow 0} \frac{\operatorname{Area}(f_1) - \operatorname{Area}(f_{1-h})}{h},$$

we have

$$\begin{aligned}X &= \liminf_{h\downarrow 0} \frac{\int_{1-h}^{1}\int_{-\pi}^{\pi} |f'(re^{i\theta} + (1-r))|^2 (1-\cos\theta) r d\theta dr}{h} \\ &= \liminf_{h\downarrow 0} \int_{-\pi}^{\pi} \frac{\int_{1-h}^{1} |f'(re^{i\theta} + (1-r))|^2 r dr}{h} (1-\cos\theta) d\theta,\end{aligned}$$

where the limit can be infinite. Since f' has nontangential limits almost everywhere on $\partial\mathbb{D}$, it follows that

$$\lim_{h\downarrow 0} \frac{\int_{1-h}^{1} |f'(re^{i\theta} + (1-r))|^2 r dr}{h} = |f'(e^{i\theta})|^2,$$

for almost all $\theta \in (-\pi,\pi)$. Thus it follows from Fatou's lemma that

$$X \geq \int_{-\pi}^{\pi} |f'(e^{i\theta})|^2 (1-\cos\theta) d\theta = \int_{\partial\mathbb{D}} |f'(\zeta)|^2 \operatorname{Re}(1-\zeta)|d\zeta|.$$

Now for $\mathrm{Poi}(f_a)$, we note that for $0 < h \leq 1-a$,

$$\begin{aligned}
\mathrm{Poi}(f_{a+h}) - \mathrm{Poi}(f_a) = &\int_{D_{a+h}} |f'(w)|^2 P\left(\frac{w-(1-a-h)}{a+h}\right) |dw|^2 \\
&- \int_{D_a} |f'(w)|^2 P\left(\frac{w-(1-a)}{a}\right) |dw|^2 \\
&= \int_{D_{a+h}\setminus D_a} |f'(w)|^2 P\left(\frac{w-(1-a-h)}{a+h}\right) |dw|^2 \\
&+ \int_{D_a} |f'(w)|^2 \left(P\left(\frac{w-(1-a-h)}{a+h}\right) - P\left(\frac{w-(1-a)}{a}\right)\right) |dw|^2.
\end{aligned}$$

A short computation shows that

$$P\left(\frac{w-(1-a-h)}{a+h}\right) - P\left(\frac{w-(1-a)}{a}\right) = h(P(w)+1).$$

On the other hand, by change of variables, the integral

$$Y = \int_{D_{a+h}\setminus D_a} |f'(w)|^2 P\left(\frac{w-(1-a-h)}{a+h}\right) |dw|^2$$

can be written as

$$Y = (a+h)^2 \int_{\mathbb{D}\setminus D_{\frac{a}{a+h}}} |f'((a+h)w + (1-a-h))|^2 P(w) |dw|^2.$$

Note that ∂D_a are level sets of $P(z)$ and $P \equiv (1-a)/a$ on ∂D_a and, as a consequence, $P(w) \leq (1-a)/a$ for $w \in \mathbb{D}\setminus D_a$. Therefore,

$$\begin{aligned}
Y &\leq (a+h)^2 \int_{\mathbb{D}\setminus D_{\frac{a}{a+h}}} |f'((a+h)w + (1-a-h))|^2 \frac{1-\frac{a}{a+h}}{\frac{a}{a+h}} |dw|^2 \\
&= \frac{(a+h)^2 h}{a} \int_{\mathbb{D}\setminus D_{\frac{a}{a+h}}} |f'((a+h)w + (1-a-h))|^2 |dw|^2 \\
&= \frac{h}{a} \int_{D_{a+h}\setminus D_a} |f'(w)|^2 |dw|^2 = \frac{h}{a} \int_{\mathbb{D}} |f'(w)|^2 \mathbf{1}_{D_{a+h}\setminus D_a}(w) |dw|^2,
\end{aligned}$$

which is $o(h)$ by the dominated convergence theorem. Therefore, it follows that $\frac{d}{da}\mathrm{Poi}(f_a) = \int_{D_a} |f'(w)|^2 (P(w)+1) |dw|^2$ for all $a \in (0,1]$. □

2.3.3 Local Square Integrability on $\partial\mathbb{D}$

In this section, we will show using domain restriction that extremizers cannot have a very large derivative on $\partial\mathbb{D}$.

Proposition 2.3.4. *Suppose f is a conformal map in $\mathbf{F}_0$ that maximizes L_1. Then we have that*

$$2(\operatorname{Area}(f))^2 \geq \operatorname{Poi}(f) \int_{\partial\mathbb{D}} |f'(w)|^2 \operatorname{Re}(1-w)|dw|.$$

In particular, we have $f' \in L^2_{\text{loc}}(\partial\mathbb{D}\setminus\{1\})$.

Proof. Suppose f maximizes L_1 in $\mathbf{F}_a$. Then for any $a \in (0,1)$ we have $L_1(f_a) \leq L_1(f)$. It follows that

$$\limsup_{a\uparrow 1} \frac{\log L_1(f_1) - \log L_1(f_a)}{1-a} \geq 0.$$

By Lemma 2.3.3, we have

$$\begin{aligned}\limsup_{a\uparrow 1} \frac{\log L_1(f_1) - \log L_1(f_a)}{1-a} &= \frac{\operatorname{Area}(f)}{\operatorname{Poi}(f)} - \frac{\liminf_{h\downarrow 0} \frac{\operatorname{Area}(f_1)-\operatorname{Area}(f_{1-h})}{h}}{2\operatorname{Area}(f)} \\ &\leq \frac{\operatorname{Area}(f)}{\operatorname{Poi}(f)} - \frac{\int_{\partial\mathbb{D}} |f'(w)|^2 \operatorname{Re}(1-w)|dw|}{2\operatorname{Area}(f)}\end{aligned}$$

Hence,

$$\frac{\operatorname{Area}(f)}{\operatorname{Poi}(f)} - \frac{\int_{\partial\mathbb{D}} |f'(w)|^2 \operatorname{Re}(1-w)|dw|}{2\operatorname{Area}(f)} \geq 0.$$

The result then follows. $\square$

2.4 Curvature Signature

2.4.1 Conformal Curvature

The purpose of this section is to recall some facts about the curvature of the domain for the convenience of the reader and to introduce a new geometric notion that we call *conformal curvature*. We are not aware of this notion existing in the literature, however, it is sufficiently natural that it may well appear somewhere. We give a self-contained exposition of the relevant properties. Let Ω be a Jordan domain and $f : \mathbb{D} \to \Omega$ be one of its Riemann maps. Recall that we set

$$v_f(z) = \mathrm{Re}\left(1 + z\frac{f''(z)}{f'(z)}\right) \qquad \text{and} \qquad k_f(\theta) = \liminf_{\mathbb{D}\ni z\to e^{i\theta}} v_f(z).$$

The function $v_f : \mathbb{D} \to \mathbb{R}$ is a harmonic function for locally univalent f and curvature signature k_f is lower semicontinuous. If $f \in C^2(\overline{\mathbb{D}})$, then $t \mapsto f(e^{it})$ is a proper parametrization of the C^2 curve $\partial\Omega$. Note that for a properly parametrized curve $\gamma(t) = (x(t), y(t))$ the (signed) Euclidean curvature is given by

$$\kappa(t) = \frac{x'(t)y''(t) - y'(t)x''(t)}{|x'(t)^2 + y'(t)^2|^{3/2}}.$$

In our case, we have $x(t) = \mathrm{Re}\, f(e^{it})$ and $y(t) = \mathrm{Im}\, f(e^{it})$. By plugging them in, we have that the Euclidean curvature of $\partial\Omega$ at $f(e^{i\theta})$ is given by

$$\frac{1}{|f'(e^{i\theta})|}\,\mathrm{Re}\left(1 + e^{i\theta}\frac{f''(e^{i\theta})}{f'(e^{i\theta})}\right) = \frac{v_f(e^{i\theta})}{|f'(e^{i\theta})|}. \tag{2.6}$$

For a $C^{2+\alpha}$ domain, we have $f \in C^{2+\alpha}(\partial\mathbb{D})$ for any Riemann map f and thus the curvature on the boundary can be calculated using (2.6). For non-$C^{2+\alpha}$ domain, we cannot guarantee that f is $C^2(\partial\mathbb{D})$. Nevertheless, as we will see, it is natural to set the curvature of $f(\partial\mathbb{D})$ at point $f(e^{i\theta})$ to be

$$K_{f(\mathbb{D})}(f(e^{i\theta})) = \liminf_{\mathbb{D}\ni z\to e^{i\theta}} \frac{v_f(z)}{|f'(z)|}. \tag{2.7}$$

We call $K_\Omega(p)$ the *conformal curvature* of point $p \in \partial\Omega$. It follows immediately from the definition that K_Ω is a lower semicontinuous function on $\partial\Omega$. One relation between the conformal curvature K_Ω and curvature signature k_f is as follows:

Lemma 2.4.1. *Let f be a conformal map onto a Jordan domain Ω. Then there exists a constant $C_f \in (0, \infty)$ such that for all $\theta \in [-\pi, \pi]$,*

$$K_\Omega(f(e^{i\theta})) \le C_f k_f(\theta). \tag{2.8}$$

Proof. We set

$$C_f = \frac{2}{\inf_{\theta\in[-\pi,\pi]}\{|f'(e^{i\theta}/2)|\}} \in (0, \infty).$$

If f is a Riemann map of a convex domain, then Lemma 2.2.3 implies that

$$|f'(re^{i\theta})| \geq \frac{|f'(e^{i\theta}/2)|}{2r} \geq \frac{C_f^{-1}}{r} \quad \text{for all} \quad r \in (\frac{1}{2}, 1).$$

It follows that

$$K_\Omega(f(e^{i\theta})) = \liminf_{\mathbb{D} \ni z \to e^{i\theta}} \frac{v_f(z)}{|f'(z)|} \leq C_f \liminf_{\mathbb{D} \ni z \to e^{i\theta}} \frac{v_f(z)}{|z|} = C_f k_f(\theta).$$

$\square$

2.4.2 Conformal curvature is well-defined.

In this section, we will prove that conformal curvature is well-defined, it is an *intrinsic* geometric object that is independent of the choice of the Riemann map f.

Proposition 2.4.2. *Given any bounded convex domain Ω, the conformal curvature $K_\Omega : \partial\Omega \to [0, \infty]$ is well-defined and independent of the choice of Riemann map $f : \mathbb{D} \to \Omega$. When Ω is a $C^{2+\alpha}$ domain, K_Ω coincides with the Euclidean curvature.*

To prove Proposition 2.4.2, it suffices to show invariance under automorphisms.

Lemma 2.4.3. *Given an automorphism φ of $\mathbb{D}$ and $g = f \circ \varphi$, we have*

$$\liminf_{\mathbb{D} \ni z \to \varphi^{-1}(e^{i\theta})} \frac{v_f(z)}{|f'(z)|} = \liminf_{\mathbb{D} \ni z \to e^{i\theta}} \frac{v_g(z)}{|g'(z)|}$$

for all θ.

Proof. Note that $g'(z) = f'(\varphi(z))\varphi'(z)$ and

$$g''(z) = f''(\varphi(z))(\varphi'(z))^2 + f'(\varphi(z))\varphi''(z).$$

Therefore, we have

$$v_g(z) = \mathrm{Re}\left(z\varphi'(z)\frac{f''(\varphi(z))}{f'(\varphi(z))}\right) + \mathrm{Re}\left(1 + z\frac{\varphi''(z)}{\varphi'(z)}\right).$$

Since φ is a Riemann map of $\mathbb{D}$ which has constant curvature 1 on the boundary, we have

$$\lim_{\mathbb{D}\ni z\to e^{i\theta}} \frac{1}{|\varphi'(z)|}\operatorname{Re}\left(1+z\frac{\varphi''(z)}{\varphi'(z)}\right) = \frac{1}{|\varphi'(e^{i\theta})|}\operatorname{Re}\left(1+e^{i\theta}\frac{\varphi''(e^{i\theta})}{\varphi'(e^{i\theta})}\right) = 1.$$

It follows that

$$\begin{aligned}\liminf_{\mathbb{D}\ni z\to e^{i\theta}} \frac{v_g(z)}{|g'(z)|} &= \liminf_{\mathbb{D}\ni z\to e^{i\theta}} \frac{\operatorname{Re}\left(z\varphi'(z)\frac{f''(\varphi(z))}{f'(\varphi(z))}\right)+|\varphi'(z)|}{|f'(\varphi(z))\varphi'(z)|}\\ &= \liminf_{\mathbb{D}\ni z\to e^{i\theta}} \frac{1+\operatorname{Re}\left(\frac{z\varphi'(z)}{|\varphi'(z)|}\frac{f''(\varphi(z))}{f'(\varphi(z))}\right)}{|f'(\varphi(z))|}.\end{aligned}$$

However, we have

$$\lim_{\mathbb{D}\ni z\to e^{i\theta}} \frac{z\varphi'(z)}{\varphi(z)|\varphi'(z)|} = \frac{e^{i\theta}\varphi'(e^{i\theta})}{\varphi(e^{i\theta})|\varphi'(e^{i\theta})|} = 1.$$

It follows that

$$\begin{aligned}\liminf_{\mathbb{D}\ni z\to e^{i\theta}} \frac{v_g(z)}{|g'(z)|} &= \liminf_{\mathbb{D}\ni z\to e^{i\theta}} \frac{1+\operatorname{Re}\left(\varphi(z)\frac{f''(\varphi(z))}{f'(\varphi(z))}\right)}{|f'(\varphi(z))|}\\ &= \liminf_{\varphi^{-1}(\mathbb{D})\ni w\to \varphi^{-1}(e^{i\theta})} \frac{1+\operatorname{Re}\left(w\frac{f''(w)}{f'(w)}\right)}{|f'(w)|}\\ &= \liminf_{\mathbb{D}\ni w\to \varphi^{-1}(e^{i\theta})} \frac{1+\operatorname{Re}\left(w\frac{f''(w)}{f'(w)}\right)}{|f'(w)|}.\end{aligned}$$

□

2.4.3 A Euler-Lagrange argument

In this section, we will show that following:

Proposition 2.4.4. *Given $f \in \mathbf{F_0}$ and $h \in \operatorname{Hol}(\mathbb{D})\cap C^1(\overline{\mathbb{D}})$, suppose for all $t>0$ sufficiently small, we have*

$$k_f(\theta) \geq -t\operatorname{Re}(e^{i\theta}h'(e^{i\theta})) + 2t^2\|h\|_{L^\infty(\mathbb{D})}\|h'\|_{L^\infty(\mathbb{D})} \quad \textit{almost everywhere.}$$

Then the following holds: If f maximizes L_1, then

$$2\frac{\int_{\mathbb{D}}|f'(w)|^2\operatorname{Re}(h(w))P(w)|dw|^2}{\operatorname{Poi}(f)} - \frac{\int_{\mathbb{D}}|f'(w)|^2\operatorname{Re}(h(w))|dw|^2}{\operatorname{Area}(f)} \leq \operatorname{Re}(h(1)). \tag{2.9}$$

If f maximizes L_2, then

$$2\frac{\int_{\mathbb{D}}|f'(w)|^2\operatorname{Re}(h(w))P(w)|dw|^2}{\operatorname{Poi}(f)}-\frac{\int_{\partial\mathbb{D}}|f'(w)|\operatorname{Re}(h(w))|dw|}{\operatorname{Len}(f)}\le\operatorname{Re}(h(1)). \tag{2.10}$$

Remark 2.4.5. Suppose $f\in\mathbf{F}_0$ is the maximizer of either L_i. Using Euler-Lagrange argument for $\log L_i$, we know that for any holomorphic $g:\mathbb{D}\to\mathbb{C}$ such that

$$f+tg\in\mathbf{F_0}\quad\text{for all } t>0 \text{ sufficiently small,} \tag{2.11}$$

we need to have

$$\frac{d}{dt}\log L_i(f+tg)\bigg|_{t=0}\le 0.$$

Proof of Proposition 2.4.4. Let us first look at when Condition (2.11) will be true. Suppose g is a primitive of $f'h$ for some $h\in\operatorname{Hol}(\mathbb{D})\cap C^1(\overline{\mathbb{D}})$ (we write it as $\int f'h$). For $f\in\mathbf{F}$, $|f'|$ is lower-bouned away from 0 in $\mathbb{D}$. Since $(f+t\int f'h)'=f'(1+th)$, it follows that

$$\left|\left(f+t\int f'h\right)'(z)\right|>|f'(z)|\left(1-t\|h\|_{L^\infty(\mathbb{D})}\right)>0,$$

for all $t>0$ sufficiently small. Moreover, for any $f\in\mathbf{F_0}$, we have

$$\operatorname{Poi}\left(f+t\int f'h\right)\le\left(1+t\|h\|_{L^\infty(\mathbb{D})}\right)^2\operatorname{Poi}(f)<\infty.$$

Thus to check $f+t\int f'h\in\mathbf{F_0}$, it suffices to check that

$$\begin{aligned}0<v_{f+t\int f'h}(z)&=\operatorname{Re}\left(1+z\frac{f''(z)(1+th(z))+tf'(z)h'(z)}{f'(z)(1+th(z))}\right)\\&=\operatorname{Re}\left(1+z\frac{f''(z)}{f'(z)}\right)+t\operatorname{Re}\left(z\frac{h'(z)}{1+th(z)}\right)\\&=v_f(z)+t\operatorname{Re}\left(zh'(z)\right)-t^2\operatorname{Re}\left(\frac{zh(z)h'(z)}{1+th(z)}\right).\end{aligned}$$

Since h' is also bounded on $\mathbb{D}$, we have

$$\left|\operatorname{Re}\left(\frac{zh(z)h'(z)}{1+th(z)}\right)\right|\le\frac{\|h\|_{L^\infty(\mathbb{D})}\|h'\|_{L^\infty(\mathbb{D})}}{\left|1-t\|h\|_{L^\infty(\mathbb{D})}\right|}.$$

Therefore, Condition 2.11 holds if

$$v_f(z)>-t\operatorname{Re}(zh'(z))+2t^2\|h\|_{L^\infty(\mathbb{D})}\|h'\|_{L^\infty(\mathbb{D})}\quad\text{for all } t>0 \text{ sufficiently small.}$$

Since both sides are harmonic functions and the right-hand side is continuous up to $\partial\mathbb{D}$, the maximum principle implies that Condition 2.11 is true if for $t > 0$ sufficiently small, $k_f(\theta)$ dominate $-t\,\mathrm{Re}\left(e^{i\theta}h'(e^{i\theta})\right) + t^2\|h\|_{L^\infty(\mathbb{D})}\|h'\|_{L^\infty(\mathbb{D})}$ almost everywhere. Note that for $g' = f'h$, by Lemma 2.2.12, we have that

$$\begin{aligned}\frac{d}{dt}\log L_1(f+tg)\Big|_{t=0} &= 2\frac{\int_{\mathbb{D}}|f'(w)|^2\,\mathrm{Re}(h(w))P(w)|dw|^2}{\mathrm{Poi}(f)} \\ &\quad - \frac{\int_{\mathbb{D}}|f'(w)|^2\,\mathrm{Re}(h(w))|dw|^2}{\mathrm{Area}(f)} - \mathrm{Re}(h(1)).\end{aligned}$$

As for L_2, we need to check for the functional $\mathrm{Len}(\cdot)$. We have that

$$\begin{aligned}\mathrm{Len}(f+tg) &= \int_{\partial\mathbb{D}}|f'(\zeta)+tg'(\zeta)||d\zeta| = \int_{\partial\mathbb{D}}|f'(\zeta)||1+th(\zeta)||d\zeta| \\ &= \int_{\partial\mathbb{D}}|f'(\zeta)|\left(1+t\,\mathrm{Re}(h(\zeta))+o(t)\left\|h\right\|_{L^\infty(\partial\mathbb{D})}\right)|d\zeta|.\end{aligned}$$

Thus we have

$$\frac{d}{dt}\mathrm{Len}(f+tg)\Big|_{t=0} = \int_{\partial\mathbb{D}}|f'(\zeta)|\,\mathrm{Re}(h(\zeta))|d\zeta|.$$

Therefore,

$$\begin{aligned}\frac{d}{dt}\log L_2(f+tg)\Big|_{t=0} &= 2\frac{\int_{\mathbb{D}}|f'(w)|^2\,\mathrm{Re}(h(w))P(w)|dw|^2}{\mathrm{Poi}(f)} \\ &\quad - \frac{\int_{\partial\mathbb{D}}|f'(\zeta)|\,\mathrm{Re}(h(\zeta))|d\zeta|}{\mathrm{Len}(f)} - \mathrm{Re}(h(1)).\end{aligned}$$

The result then follows. □

2.4.4 A nice family of test functions

We now construct a family of test functions $\{h_b\}$ that will be later used in the Euler-Lagrange argument.

Lemma 2.4.6. *There exists a family of $C^\infty(\overline{\mathbb{D}})$ conformal maps $\{h_b\}_{b\in(0,1)}$ that satisfies the following properties.*

1. $|h_b(z)| \leq 10$.

2. $\mathrm{Re}(h_b(1)) < -0.1$.

3. $\mathrm{Re}(h_b(z)) \to 0$ *pointwise on* $\mathbb{D}$ *as* b *tends to* 1.

4. *for each* b, *there exists a closed interval* $I(b) \subseteq (0, \pi)$ *such that*

$$\left\{\theta \in [-\pi, \pi] \middle| \mathrm{Re}\left(e^{i\theta} h_b'(e^{i\theta})\right) < 0\right\} \subseteq I(b)$$

and

$$\inf_{\theta \in [-\pi,\pi] \setminus I(b)} \mathrm{Re}\left(e^{i\theta} h_b'(e^{i\theta})\right) > 0.$$

Before proving this, we make the following observation:

Remark 2.4.7. If h is a conformal map in $C^1(\overline{\mathbb{D}})$, then it is angle-preserving on $\partial\mathbb{D}$ and $h(\partial\mathbb{D})$ is a C^1 curve. Let $\mathbf{n}_\Omega : \partial\Omega \to \partial\mathbb{D}$ be the normal vector of Ω going outward. Since h is angle-preserving, $\mathbf{n}_{h(\mathbb{D})}(\zeta)$ is in the direction of velocity vector of $t \mapsto h(te^{i\theta})$ where $\theta = \arg h^{-1}(\zeta)$. Then we have

$$\mathrm{Re}\left(e^{i\theta} h'(e^{i\theta})\right) = |h'(e^{i\theta})| \,\mathrm{Re}\left(\mathbf{n}_{h(\mathbb{D})}(h(e^{i\theta}))\right) = |h'(e^{i\theta})| \mathbf{n}_{h(\mathbb{D})}(h(e^{i\theta})) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

This implies that the negativity of $\mathrm{Re}\left(e^{i\theta} h'(e^{i\theta})\right)$ for the definition of $I_{-}(b)$ depends only on outward pointing normal vectors of $h_b(\mathbb{D})$ around $h(e^{i\theta})$ (note that $h' \neq 0$ by Kellogg's theorem).

Proof of Lemma 2.4.6. We set $h : \mathbb{D} \to B_1(-1)$ be the biholomorphism such that $h(-1) = 0$, $h(1) = e^{i\pi/4} - 1$ and $\mathrm{Arg}(h(0) + 1) = \pi/8$. Note that there are two such functions but both are in $C^\infty(\overline{\mathbb{D}})$. For $b \in [0, 1)$, we define

$$h_b(z) = h\left(\frac{z - b}{1 - bz}\right)$$

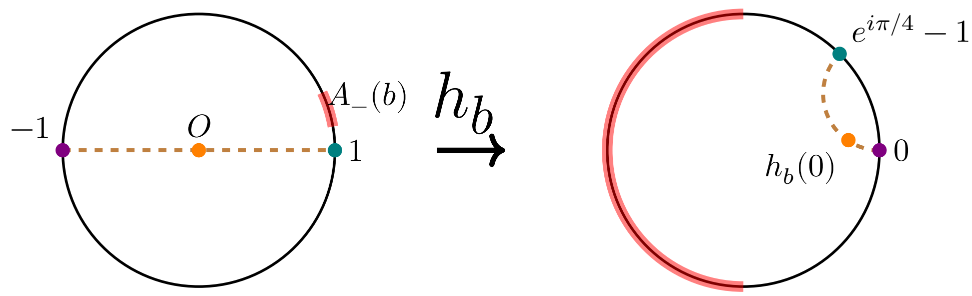

Figure 2.7: Conformal map h_b with circular arcs $A\_(b)$. Note that as $b \to 1$, $h_b(0) \in h((-1,1))$ approaches 0. Thus the harmonic measure of the left semi-circle tends to 0 and consequently the arc $A\_(b)$ shrink to $\{1\}$.

Then each h_b is still a $C^\infty(\overline{\mathbb{D}})$ map from $\mathbb{D}$ onto $B_1(-1)$ and maps -1 and 1 to 0 and $e^{i\pi/4}-1$ respectively. We also have $h_b([-1,1]) = h([-1,1])$ (see Figure 2.7). Moreover, we have

$$\lim_{b\to 1} h_b(z) = h(-1) = 0,$$

for all $z \in \overline{\mathbb{D}} \setminus \{1\}$. Now we define

$$A\_(b) = h_b^{-1}(\{e^{i\theta}-1 | \theta \in (\pi/2, 3\pi/2)\}).$$

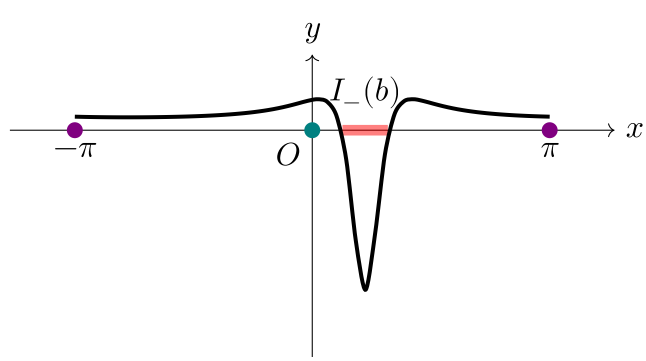

Figure 2.8: Graph of $\mathrm{Re}\,(e^{i\theta}h_b'(e^{i\theta}))$. Note that since $A\_(b)$ shrink to 1 as $b \to 1$, the interval $I\_(b)$ will shrink to 0.

Then it follows that $\mathcal{H}^1(A\_(b))/(2\pi)$ is the harmonic measure of the left semi-circle of $\partial B_1(-1)$ with respect to $h_b(0)$. We denote the interval on $[-\pi,\pi]$ corresponding to $A\_(b)$ as $I\_(b)$ (see Figure 2.8). Then we have $|I\_(b)| = \mathcal{H}^1(A\_(b))$ and according to the discussion in Remark 2.4.7,

$$\operatorname{Re}\left(e^{i\theta}h_b'(e^{i\theta})\right)\begin{cases} < 0, & \text{when} \quad \theta \in I\_(b), \\ = 0, & \text{when} \quad e^{i\theta} \in h_b^{-1}(\{-\pi/2,\pi/2\}), \\ > 0, & \text{otherwise.} \end{cases}$$

In fact, h_b is a translation of Möbius transformation so one can compute this function explicitly. Using the fact that $h_b(0) \to 0$ (and thus the harmonic measure on $\partial B_1(-1)$ with respect to $h_b(0)$ converges to δ_{-i}), we can conclude that $I\_(b) \subseteq (0,\pi)$ and as b tends to 1, we have $\sup I\_(b) \to 0$. One can take a closed interval $I(b) \subseteq (0,\pi)$ slightly larger than $I\_(b)$ such that $\operatorname{Re}\left(e^{i\theta}h_b'(e^{i\theta})\right)$ is lower-bounded away from 0 outside $I(b)$. The result then follows.

□

2.4.5 A statement for flatness

Using these properties of h_b, we can restrict the behavior of the conformal curvature k_f around 0 for a maximizer $f \in \mathbf{F}_0$ to prove

Proposition 2.4.8. *Suppose f is a conformal map in $\mathbf{F}_0$ that maximizes either L_1 or L_2. Then the boundary of the domain $f(\mathbb{D})$ has infinitely many points with zero conformal curvature accumulated around $f(1)$.*

This will help us show Theorem 2.1.2. Note that by Lemma 2.4.1, we can restate the proposition in terms of $k_f(\theta)$.

Proposition 2.4.9. *Suppose $f \in \mathbf{F}_0$ maximizes either L_i. Then we can find an increasing negative sequence $\{n_j\}$ and decreasing positive sequence $\{p_j\}$, both approaching 0, such that*

$$k_f(n_j) = k_f(p_j) = 0.$$

As $k_f(\theta)$ is lower semicontinuous, we also have $k_f(0) = 0$.

Proof. We will only show the existence of the decreasing positive sequence. The existence of the negative sequence can be proven similarly.

Suppose for the contradiction that $k_f(\theta) > 0$ on some interval $(0, a)$ with $a > 0$. By lower semi-continuity, $k_f(\theta)$ is lower-bounded away from 0 on all compact subsets of $(0, a)$. We take the family of test functions $\{h_b : \mathbb{D} \to B_{-1}(1)\}_{b\in(0,1)}$ in Lemma 2.4.6. The lemma implies that there exists b_0 such that for all $b \in (0, b_0)$, $I(b)$ is a closed interval in $(0, a)$. It follows that $\inf_{\theta\in I(b)} k_f(\theta) > 0$. Since $\theta \mapsto \operatorname{Re}\left(e^{i\theta}h_b'(e^{i\theta})\right)$ is smooth, for any t such that

$$0 < t < \min\left(\sqrt{\frac{\inf_{\theta\in I(b)} k_f(\theta)}{4\|h\|_{L^\infty(\mathbb{D})}\|h'\|_{L^\infty(\mathbb{D})}}}, \frac{\inf_{\theta\in I(b)} k_f(\theta)}{2\sup_{\theta\in I(b)} |\operatorname{Re}\left(e^{i\theta}h_b'(e^{i\theta})\right)|}\right),$$

we have

$$k_f(\theta) > -t\operatorname{Re}\left(e^{i\theta}h_b'(e^{i\theta})\right) + 2t^2\|h\|_{L^\infty(\mathbb{D})}\|h'\|_{L^\infty(\mathbb{D})},$$

for all θ inside $I(b)$. On the other hand, for any

$$0 < t < \frac{\inf_{\theta\in[-\pi,\pi]\setminus I(b)} \operatorname{Re}\left(e^{i\theta}h_b'(e^{i\theta}\right)}{4\|h\|_{L^\infty(\mathbb{D})}\|h'\|_{L^\infty(\mathbb{D})}},$$

we have for all $\theta \in [-\pi, \pi]\setminus I(b)$

$$k_f(\theta) \geq 0 > -t\operatorname{Re}\left(e^{i\theta}h_b'(e^{i\theta})\right) + 2t^2\|h\|_{L^\infty(\mathbb{D})}\|h'\|_{L^\infty(\mathbb{D})}.$$

Therefore, if f maximizes L_1, by inequality (2.9) in Proposition 2.4.4, we have

$$2\frac{\int_{\mathbb{D}} |f'|^2 \operatorname{Re}(h_b)P}{\operatorname{Poi}(f)} - \frac{\int_{\mathbb{D}} |f'|^2 \operatorname{Re}(h_b)}{\operatorname{Area}(f)} \leq \operatorname{Re}(h_b(1)) < -0.1,$$

for all $b \in (b_0, 1)$. However, $\{\operatorname{Re} h_b\}$ is a uniformly bounded sequence of functions converging to 0 on $\overline{\mathbb{D}}\setminus\{1\}$. Thus by dominated convergence theorem, the left-hand hand of the inequality tends to 0 as b tends to 1, causing a contradiction.

Similarly, if f maximizes L_2, by inequality (2.10), we have

$$2\frac{\int_{\mathbb{D}} |f'|^2 \operatorname{Re}(h_b)P}{\operatorname{Poi}(f)} - \frac{\int_{\partial\mathbb{D}} |f'| \operatorname{Re}(h_b)}{\operatorname{Len}(f)} \leq \operatorname{Re}(h_b(1)) < -0.1,$$

for all $b \in (0, b_0)$. Again by dominated convergence theorem, the left-hand hand of the inequality tends to 0 as b tends to 1, causing a contradiction. ∎

2.5 Going Back to Torsion Function

2.5.1 Normal Derivative

Suppose $f : \mathbb{D} \to \Omega$ is a Riemann map of a bounded convex domain Ω. Since $\partial\Omega$ is rectifiable, we know that $f \in \mathbf{H}^1$. In fact, we can conclude something stronger:

Lemma 2.5.1 (F. and M. Riesz theorem, [94])**.** *Let Ω be a Jordan domain with rectifiable boundary and let $f : \mathbb{D} \to \Omega$ be a Riemman map of Ω. Then for any $\varphi \in L^1(\partial\Omega)$, we have*

$$\int_{\partial\Omega} \varphi = \int_{\partial\mathbb{D}} \varphi(f(\zeta))|f'(\zeta)||d\zeta|.$$

Using F. and M. Riesz theorem, we will show the following:

Proposition 2.5.2. *Let Ω be a bounded convex domain and let $f : \mathbb{D} \to \Omega$ be a Riemann map of Ω. Then for almost every $\theta \in [-\pi, \pi)$*

$$\frac{\partial u}{\partial \mathbf{n}}(f(\zeta)) = -\frac{\int_{\mathbb{D}} P(\zeta, w)|f'(w)|^2|dw|^2}{|f'(\zeta)|} > -\infty.$$

Proof. We set

$$u_N(f(\zeta)) = -\frac{\int_{\mathbb{D}} P(\zeta, w)|f'(w)|^2|dw|^2}{|f'(\zeta)|}.$$

We want to show that for all $\varphi \in C^\infty(\mathbb{R}^2)$, u_N satisfies the Gauss-Green formula:

$$\int_{\partial\Omega} \varphi u_N = \int_{\Omega} (\varphi\Delta u + u\Delta\varphi) = \int_{\Omega} (-\varphi + u\Delta\varphi). \tag{2.12}$$

Then it follows that $u_N = \partial u/\partial\mathbf{n}$ almost everywhere on $\partial\Omega$ by the uniqueness of weak derivatives. Note that Lemma 2.5.1 allows us to rewrite (2.12) as

$$\int_{\partial\mathbb{D}} \varphi(f(\zeta))u_N(f(\zeta))|f'(\zeta)||d\zeta| = \int_{\mathbb{D}} -\varphi(f(w)) + u(f(w))\Delta\varphi(f(w))|f'(w)|^2|dw|^2.$$

For $r \in (0, 1)$, we set $f_r(z) = f(rz)$ as in the proof of Lemma 2.3.1 and let v_r be the solution to

$$-\Delta v_r = |f_r'|^2 \quad \text{inside} \quad \mathbb{D},$$

$$v_r = 0 \quad \text{on} \quad \partial\mathbb{D}.$$

Note that a unique solution exists in $C^\infty(\overline{\mathbb{D}})$ as $|f_r'|^2 \in C^\infty(\overline{\mathbb{D}})$. This implies that

$$v_r(z) = \int_{\mathbb{D}} G(z,w)|f_r'(w)|^2|dw|^2,$$
$$\frac{\partial v_r}{\partial \mathbf{n}}(e^{i\theta}) = -\int_{\mathbb{D}} P(e^{i\theta},w)|f_r'(w)|^2|dw|^2.$$

Recall that for each $z \in \mathbb{D}$, $|f_r'(z)|$ increases to $|f'(z)|$. Since both $G(z,w)$ and $P(\zeta,w)$ are positive, the monotone convergence theorem implies that

$$v_r(z) \uparrow v(z), \quad \frac{\partial v_r}{\partial \mathbf{n}}(e^{i\theta}) \downarrow v_N(e^{i\theta}),$$

where

$$v_N(e^{i\theta}) = -\int_{\mathbb{D}} P(e^{i\theta},w)|f'(w)|^2|dw|^2 = |f'(e^{i\theta})|u_N(f(e^{i\theta})),$$

which a priori could be $-\infty$. Note that $v_r \in C^\infty(\overline{\mathbb{D}})$ and $\frac{\partial v_r}{\partial \mathbf{n}} \in C^\infty(\partial\mathbb{D})$. By the Gauss-Green formula, for any $\psi \in C^\infty(\overline{\mathbb{D}})$, we have

$$\int_{\partial\mathbb{D}} \psi(\zeta)\frac{\partial v_r}{\partial \mathbf{n}}(\zeta)|d\zeta| = \int_{\mathbb{D}} \left(-\psi(w)|f_r'(w)|^2 + v_r(w)\Delta\psi(w)\right)|dw|^2. \tag{2.13}$$

The right-hand side of (2.13) is dominated by the integral

$$\max(\|\psi\|_{L^\infty(\mathbb{D})}, \|\Delta\psi\|_{L^\infty(\mathbb{D})}) \int_{\mathbb{D}} \left(|f'(w)|^2 + v(w)\right)|dw|^2.$$

Therefore, it converges as $r \to 1$ to

$$\int_{\mathbb{D}} \left(-\psi(w)|f'(w)|^2 + v(w)\Delta\psi(w)\right)|dw|^2.$$

In particular, for any positive $\psi \in C^\infty(\overline{\mathbb{D}})$, it follows that

$$\begin{aligned}\int_{\partial\mathbb{D}} \psi(\zeta)v_N(\zeta)|d\zeta| &= \lim_{r\to 1}\int_{\partial\mathbb{D}} \psi(\zeta)\frac{\partial v_r}{\partial \mathbf{n}}(\zeta)|d\zeta| \\ &= \int_{\mathbb{D}} \left(-\psi(w)|f'(w)|^2 + v(w)\Delta\psi(w)\right)|dw|^2 > -\infty.\end{aligned}$$

It follows that $v_N \in L^1(\partial\mathbb{D})$ and thus $|u_N|$ is finite almost everywhere on $\partial\Omega$. Since every $\psi \in C^\infty(\overline{\mathbb{D}})$ is lower-bounded, it follows that

$$\int_{\partial\mathbb{D}} \psi(\zeta)v_N(\zeta)|d\zeta| = \int_{\mathbb{D}} \left(-\psi(w)|f'(w)|^2 + v(w)\Delta\psi(w)\right)|dw|^2, \tag{2.14}$$

holds for all $\psi \in C^\infty(\overline{\mathbb{D}})$. Now for $\varphi \in C^\infty(\mathbb{R}^2)$, we know that $\varphi \circ f_r \in C^\infty(\overline{\mathbb{D}})$ for all $r \in (0,1)$. Plugging it to (2.14), we have

$$\int_{\partial\mathbb{D}} \varphi(f(r\zeta))v_N(\zeta)|d\zeta| = \int_{\mathbb{D}} -\varphi(f(rw))|f'(w)|^2 + v(w)r^2|f'(rw)|^2\Delta\varphi(f(rw))|dw|^2.$$

Note that $(\varphi \circ f_r)v_N$ is dominated by $\|\varphi\|_{L^\infty(\overline{\mathbb{D}})} |v_N| \in L^1(\partial\mathbb{D})$ while the integrant on the right-hand side is dominated by

$$\left(\|\varphi\|_{L^\infty(\overline{\mathbb{D}})} + \|\Delta\varphi\|_{L^\infty(\overline{\mathbb{D}})} \|u\|_{L^\infty(\Omega)}\right)|f'|^2 \in L^1(\mathbb{D}).$$

Applying the dominated convergence theorem to both sides, we have

$$\int_{\partial\mathbb{D}} \varphi(f(\zeta))v_N(\zeta)|d\zeta| = \int_{\mathbb{D}} -\varphi(f(w))|f'(w)|^2 + v(w)|f'(w)|^2\Delta\varphi(f(w))|dw|^2,$$

which is

$$\int_{\partial\mathbb{D}} \varphi(f(\zeta))\frac{v_N(\zeta)}{|f'(\zeta)|}|f'(\zeta)||d\zeta| = \int_{\mathbb{D}} \left(-\varphi(f(w)) + v(w)\Delta\varphi(f(w))\right)|f'(w)|^2|dw|^2.$$

By change of variables, we have

$$\int_{\partial\Omega} \varphi u_N = \int_{\Omega} \left(-\varphi + u\Delta\varphi\right).$$

The result then follows. □

We recall that by Corollary 2.3.2, we have

$$\begin{aligned}\sup_{f\in\mathbf{F_0}} L_1(f) &= \sup_{\Omega \text{ convex, bounded, analytic}} \mathbb{L}_1(\Omega)\\ \sup_{f\in\mathbf{F_0}} L_2(f) &= \sup_{\Omega \text{ convex, bounded, analytic}} \mathbb{L}_2(\Omega).\end{aligned}$$

Proposition 2.5.2 gives us the equivalence between L_i and $\mathbb{L}_i$.

Theorem 2.5.3. *We have*

$$\begin{aligned}\sup_{f\in\mathbf{F_0}} L_1(f) &= \sup_{\Omega \text{ convex, bounded}} \mathbb{L}_1(\Omega)\\ \sup_{f\in\mathbf{F_0}} L_2(f) &= \sup_{\Omega \text{ convex, bounded}} \mathbb{L}_2(\Omega).\end{aligned}$$

Proof. It is suffice to show that

$$\sup_{f\in\mathbf{F_0}} L_i(f) \geq \sup_{\Omega \text{ convex, bounded}} \mathbb{L}_i(\Omega).$$

Suppose that there exists a bounded convex domain Ω such that

$$\mathbb{L}_1(\Omega) \geq \sup_{f\in\mathbf{F_0}} L_1(f) + \varepsilon_0,$$

for some $\varepsilon_0 > 0$. It follows from Proposition 2.5.2 that given a Riemann map $f \to \Omega$, there exists $\zeta_0 \in \partial\mathbb{D}$ such that $f'(\zeta_0)$ exists and

$$\infty > -\frac{\partial u}{\partial \mathbf{n}}(f(\zeta_0)) = \frac{\int_{\mathbb{D}} P(\zeta_0, w)|f'(w)|^2 |dw|^2}{|f'(\zeta_0)|} > \left(\sup_{f\in\mathbf{F_0}} L_1(f) + \varepsilon_0/2\right) |\Omega|^{1/2}.$$

We set $f_0(z) = f(\zeta_0 z)$. Then we have

$$\begin{aligned}\mathrm{Poi}(f_0) &= \int_{\mathbb{D}} P(w)|f_0'(w)|^2|dw|^2 = \int_{\mathbb{D}} P(w/\zeta_0)|f'(w)|^2|dw|^2 \\ &= \int_{\mathbb{D}} 2\pi P(\zeta_0, w)|f'(w)|^2|dw|^2 = -2\pi|f'(\zeta_0)|\frac{\partial u}{\partial \mathbf{n}}(f(\zeta_0)) < \infty.\end{aligned}$$

It follows that $f_0 \in \mathbf{F}_0$. Besides,

$$L_1(f_0) = \frac{\mathrm{Poi}(f_0)}{2\pi|f'(\zeta_0)||\Omega|^{1/2}} = -\frac{\frac{\partial u}{\partial \mathbf{n}}(f(\zeta_0))}{|\Omega|^{1/2}} > \sup_{f\in\mathbf{F_0}} L_1(f) + \varepsilon_0/2.$$

This causes a contradiction. The proof for L_2 and $\mathbb{L}_2$ follows similarly. □

We will now show that there is a natural relationship between the maximizers of L_i and $\mathbb{L}_i$ if they exist. Maximizers of L_i can be used to generate a sequence of convex domains for which $\mathbb{L}_i$ has the same limit and, conversely, maximizing convex domains Ω give rise to Riemann maps for which L_i converges to the same limit.

Theorem 2.5.4. *For $i \in \{1, 2\}$,*

1. *Suppose $f \in \mathbf{F}_0$ maximizes L_i. Then if we set $\Omega_r = f(r\mathbb{D})$, then $\{\Omega_r\}$ forms a increasing sequence of analytic convex domains such that*

$$\mathbb{L}_i(\Omega_r) \to \sup_{\Omega \text{ convex, bounded}} \mathbb{L}_1(\Omega) \quad as \quad r \to 1.$$

2. *Suppose Ω is a convex bounded domain that maximizes $\mathbb{L}_i$. Then there exists a sequence of Riemann maps $f_n : \mathbb{D} \to \Omega$ in $\mathbf{F}_0$ such that*

$$L_i(f_n) \to \sup_{f \in \mathbf{F_0}} L_1(f) \quad as \quad n \to \infty.$$

Proof. The first part follows immediately from Part (5) of Lemma 2.3.1. For the second part, we pick any point $z_0 \in \Omega$. For any $p \in \partial\Omega$, we set f_p be the Riemann map of Ω such that

$$f_p(0) = z_0, \quad f_p(1) = p.$$

Then we know from Proposition 2.5.2 that for almost all $p \in \partial\Omega$, we have $f_p \in \mathbf{F}_0$ and

$$\frac{\partial u}{\partial \mathbf{n}}(p) = -\frac{\int_{\mathbb{D}} P(1,w)|f_p'(w)|^2|dw|^2}{|f_p'(1)|} = -\frac{\mathrm{Poi}(f_p)}{2\pi|f_p'(1)|}.$$

Then we can define a sequence of points $\{p_n\} \subset \partial\Omega$ such that the above equality holds and $-\partial u/\partial\mathbf{n}(p_n)$ converges to $\|\partial u/\partial\mathbf{n}\|_{L^\infty(\partial\Omega)}$. □

2.5.2 Local Smoothness Results

We need a local version of Kellogg's theorem.

Lemma 2.5.5 ([37, Theorem 4.1])**.** *Suppose $f : \mathbb{D} \to \Omega$ is a conformal map onto a Jordan domain Ω and $\gamma' = f(\gamma)$ an open subarc of $\partial\Omega$. Let $k \geq 1$ and $0 < \alpha < 1$. If γ' is $C^{k+\alpha}$, then f is $C^{k+\alpha}$ on compact subsets of $\mathbb{D} \cup \gamma$ and $f' \neq 0$ on γ.*

We also need the following result for the Poisson equation:

Lemma 2.5.6 ([60, Theorem 5.1] & [38, Lemma 4.2])**.** *Given $h \in L^1(\mathbb{D})$, the Poisson equation*

$$-\Delta u = h \quad inside \quad \mathbb{D},$$

$$u = 0 \quad on \quad \partial\mathbb{D}.$$

admits a unique weak solution with $u \in W_0^{1,p}(\mathbb{D})$ for all $1 \le p < 2$. Let $\zeta \in \partial\mathbb{D}$ be any point such that h is Hölder-continuous on a neighborhood U of ζ. Then ∇u is continuous at ζ with

$$|\nabla u(\zeta)| = \int_{\mathbb{D}} P(\zeta, w) h(w) |dw|^2.$$

To prove Theorem 2.1.1, we need to extract information from $C^{1+\alpha}$-corners of the domain. For that we have the following:

Lemma 2.5.7 ([84, Theorem 3.9])**.** *If $\partial\Omega$ has two Dini-smooth subarcs (that is, curves parametrized by functions with Dini-continuous derivative), which meet at $\zeta \in \partial\Omega$ with an angle of $\pi\alpha$ for $0 < \alpha \le 1$, then the function $|f'(z)|/|z-\zeta|^{\alpha-1}$ is continuous and bounded away from 0 around ζ.*

Note that Hölder-smooth open arcs are Dini-smooth as well.

2.5.3 Proof of Theorem 2.1.1

Suppose u be the torsion function of Ω. Let $f : \mathbb{D} \to \Omega$ be the Riemann map of Ω such that $f(1)$ is an extremal point contained in some $C^{1+\alpha}$-smooth subarc γ of $\partial\Omega$. We also set $v = u \circ f$, which satisfies (2.2). By Lemma 2.5.5, f' is α-Hölder-continuous near 1. Then Lemma 2.5.6 implies that

$$\begin{aligned} L_1(f) &= \frac{\int_{\mathbb{D}} |f'(w)|^2 P(w) |dw|}{2\pi |f'(1)|} |\Omega|^{-1/2} \\ &= \frac{\int_{\mathbb{D}} P(1,w) |f'(w)|^2 |dw|}{|f'(1)|} |\Omega|^{-1/2} = \frac{|\nabla(u \circ f)(1)|}{|f'(1)|} |\Omega|^{-1/2}. \end{aligned}$$

Note that Lemma 2.5.6 also tells us that $\nabla(u \circ f)(z) = f'(z) \cdot \nabla u(f(z))$ is continuous at 1. Since f' is continuous at 1 and $f'(1) \neq 0$. It follows that $\nabla u(f(z))$ is continuous at 1 and

$$\nabla u(f(1)) = f'(1)^{-1} \cdot \nabla(u \circ f)(1).$$

Therefore,

$$L_1(f) = \frac{|\nabla(u \circ f)(1)|}{|f'(1)||\Omega|^{1/2}} = \frac{|\nabla u(f(1))|}{|\Omega|^{1/2}} = \frac{\left|\frac{\partial u}{\partial \mathbf{n}}(f(1))\right|}{|\Omega|^{1/2}} = \frac{\left\|\frac{\partial u}{\partial \mathbf{n}}\right\|_{L^\infty(\partial\Omega)}}{|\Omega|^{1/2}} = \mathbb{L}_1(\Omega).$$

It follows from Theorem 2.5.3 that f maximizes L_1. By Proposition 2.3.4, we have $f' \in L^2_{\text{loc}}(\partial\mathbb{D}\setminus\{1\})$. Note that $|z-\zeta|^{\alpha-1} \notin L^2_{\text{loc}}(\partial\mathbb{D}\setminus\{1\})$ for any $\zeta \in \partial\mathbb{D}\setminus\{1\}$ and $\alpha < 1/2$. Therefore, once we have the L^2-integrability of f' away from 1, the theorem will follow from Lemma 2.5.7.

2.5.4 Proof of Theorem 2.1.2

Given the Riemann map $f : \mathbb{D} \to \Omega$ of Ω, we might assume that $f(1)$ is an extremal point contained in a $C^{2+\alpha}$ subarc. Then f is $C^{2+\alpha}$ near 1. Using the same argument as above, we have that $f \in \mathbf{F_0}$ and

$$L_i(f) = \mathbb{L}_i(\Omega) = m_i.$$

Thus f maximizes L_i. In this case, $t \to f(e^{it})$ is a C^2 parametrization of $\partial\Omega$ near $f(1)$. Therefore, the conformal curvature we defined in (2.7) is the actual Euclidean curvature of $\partial\Omega$ near $f(1)$. Thus the result then follows from Proposition 2.4.8.

2.6 Numerics

In this section, we will briefly describe the numerical construction of the two "nearly optimal" domains in Figure 2.1. The construction is due to Guido Sweers [44]. Consider the domain

$$E_q = \left\{ (x,y) \in \mathbb{R}^2 | x^2 + \frac{y^2}{q^2} < 1 \right\}$$

and the map $h_q : E_q \to \mathbb{C}$ given by

$$h_q = z + \frac{3q^2+1}{4q^4+5q^2+1} z^2 + \frac{1}{3(4q^2+1)} z^3.$$

The domain $h_q(E_q)$ is convex for $q \in (1,2)$ and $h_q(-1)$ is the unique extremal point. Let u_q be the torsion function of $h_q(E_q)$ and $v_q = u_q \circ h_q$. Then we have

$$-\Delta v_q = |h_q'|^2 \quad \text{inside } E_q,$$
$$v_q = 0 \quad \text{on } \partial E_q.$$

It follows that v_q is a polynomial of degree 6 with a factor $(x^2 + (y/q)^2 - 1)$. We can solve v_q explicitly. Then

$$\frac{\partial u_q}{\partial \mathbf{n}}(h_q(-1)) = \frac{1}{|h_q'(-1)|}\frac{\partial v_q}{\partial \mathbf{n}}(-1).$$

The area and perimeter of $h_q(E_q)$ are given by

$$|h_q(E_q)| = \int_{E_q} |h_q'(w)|^2 |dw|^2 \qquad \text{and} \qquad \mathcal{H}^1(h_q(\partial E_q)) = \int_{\partial E_q} |h_q'(\zeta)||d\zeta|.$$

Thus we can evaluate $L_1(h_q(E_q))$ and $L_2(h_q(E_q))$ numerically. One finds that $L_1(h_q(E_q))$ is maximized for $q \approx 1.3866$ and $L_2(h_q(E_q))$ is maximized for $q \approx 1.2278$. This gives us the two domains in Figure 2.1.

Chapter 3

ELECTROSTATIC SKELETON AND CONDITION OF STRICT DESCENT

3.1 Introduction and Results

3.1.1 Introduction

Given a finite, compactly supported positive Borel measure μ, the *logarithmic potential* of μ is defined as $U^\mu : \mathbb{C} \to (-\infty, \infty]$ where

$$U^\mu(z) = \int \log \frac{1}{|z-w|} \, d\mu(w).$$

An electrostatic skeleton is a positive measure supported on a tree inside the polygon that recovers the Green's function with its potential and, in particular, it produces the polygon as a level set of its potential. A formal definition is as follows.

> **Definition.** A positive measure μ is the *electrostatic skeleton* for a precompact domain Ω if μ generates the same logarithmic potential as the equilibrium measure μ_E outside Ω and $\operatorname{supp}\mu$ is inside $\overline{\Omega}$ and contains no simple loops.

Note that the condition $U^\mu|_{\Omega^C} = U^{\mu_E}|_{\Omega^C}$ is equivalent to requiring that $-U^\mu - \log \operatorname{Cap}(\Omega)$ is the Green's function of Ω on $\mathbb{C}\backslash\Omega$. The question of understanding which sets have an electrostatic skeleton was first raised by E.B. Saff in 2003 [64]. Saff asked whether every convex polygon admits an electrostatic skeleton. Since then, Lundberg and Ramachandran [63] proved the existence and uniqueness of electrostatic skeleton for triangles (and also simplices in higher dimensional generalization) using the subharmonic extension of the Green's function (see also [34]). The existence result for regular polygons was proven by Lundberg and Totik [64], who also mentioned its relation to the concept of *lemniscate growth.* In particular,

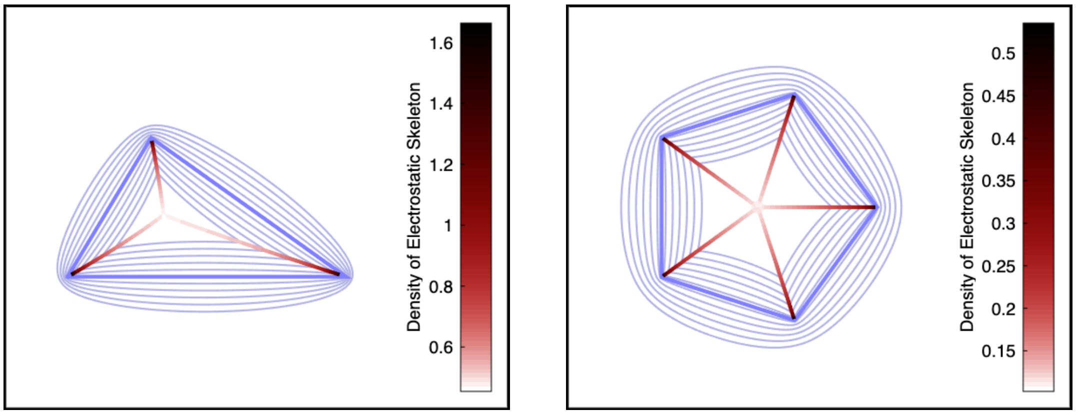

Figure 3.1: Electrostatic skeletons for a triangle and the regular pentagon, with level sets of the logarithmic potentials in blue.

they show that if the support of the skeleton does not intersect the boundary of the domain, then domain can be shrunk in a way the preserve the Green's function. In a sense, the electrostatic skeleton outlines the maximum domain of conformal invariance. In an example of this phenomenon is for the ellipse which can be shrunk to the linear segment between foci. The example skeletons for a triangle and a regular pentagon are plotted in Figure 3.1.

The task of finding the electrostatic skeleton is essentially the one of finding fictitious charges on a tree-like subset inside the domain which can replace the equilibrium charges. This relates to the classic technique in the study of electrostatics called the *method of image charges* [45]. In the context of potential theory, the measure for the electrostatic skeleton would generate the equilibrium measure as the *balayage* on the boundary of the domain. A similar question with potentials generated by Lebesgue measure was also considered by Gustafsson in [42]. Electrostatic skeletons have also been shown by Díaz, Saff and Stylianopoulos [68] to relate to Bergman orthogonal polynomials.

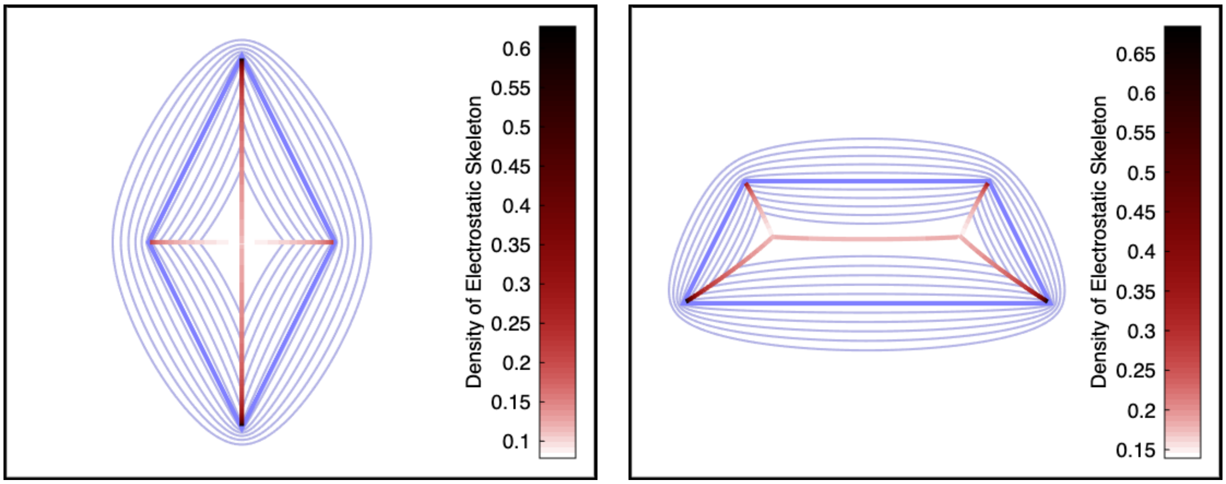

Figure 3.2: Electrostatic skeletons for a kite shape and an isosceles trapezoid, with level sets of the logarithmic potentials in blue

3.1.2 Geometric Results

We prove the existence of electrostatic skeletons for symmetric convex quadrilaterals: we will first show that the convex kite shapes (convex quadrilaterals symmetric with respect to one of their diagonals) admit electrostatic skeletons (see Fig. 3.2).

Theorem 3.1.1. *Each convex kite shape admits an electrostatic skeleton.*

Our second result deals with a second family of quadrilaterals: we consider the isosceles trapezoid where the line of symmetry does not pass through any vertex and show that all these shapes admit electrostatic skeletons.

Theorem 3.1.2. *Each isosceles trapezoid admits an electrostatic skeleton.*

For proofs of both Theorems 3.1.1 and 3.1.2, we leverage heavily with the line of symmetry. Thus it is hard to generalize the methods for a generic convex polygon.

3.1.3 Structural Results

We now discuss the main result: all convex polygons that satisfy what we call the "strict descent" condition admit electrostatic skeletons. First note that the Green's function g of a polygon Ω is a harmonic function defined on the outside of the polygon. The defining property of g is that it is a non-negative function with logarithmic growth outside the polygon and vanishes on the edges of the polygon. Given an edge $[a, b]$ of the polygon, we can reflect the Green function using Schwarz reflection with respect to each side by setting

$$\tilde{g}(z) = -g\left(a + \frac{b-a}{\overline{b-a}}(\overline{z-a})\right).$$

Given a convex polygon Ω, we denote its vertices as $z_1, \ldots, z_n \in \mathbb{C}$ and edges which are finite line segments $L_1, \ldots, L_n$ (see Figure 3.3) such that

$$L_i \cap L_{i+1} = \{z_i\}.$$

We shall denote the Schwarz reflection of g with respect to L_i by g_i. We denote the line containing the line segment L_i by ℓ_i. Note that given each pair of g_i and g_j, we can extend both functions to the component of $\mathbb{C}\backslash(\ell_i \cup \ell_j)$ that contains Ω. We denote the closure of this component (in $\mathbb{C}$) by $W_{i,j}$. Note that both g_i and g_j can be further extended continuously to $\partial W_{i,j}$ (on ℓ_i for instance, we have $g_i(z) = -g(z)$). We set

$$S_{i,j} := \left\{z \in W_{i,j} : g_i(z) = g_j(z)\right\}.$$

Definition 3.1.3 (Condition of Strict Descent)**.** We say a convex polygon Ω satisfies *the strict descent condition* if given any two distinct Schwarz reflections g_i and g_j, for any $z \in S_{i,j}$ such that $\nabla g_i(z)$ is parallel to $\nabla g_j(z)$, we have $\nabla g_i(z) \cdot \nabla g_j(z) < 0$.

The motivation for the strict descent condition is that we want to construct the electrostatic skeleton on where different Schwarz reflections coincide. Loosely speaking, this condition implies that we can build the skeleton by looking at level sets of the Schwarz reflection with increasingly smaller value.

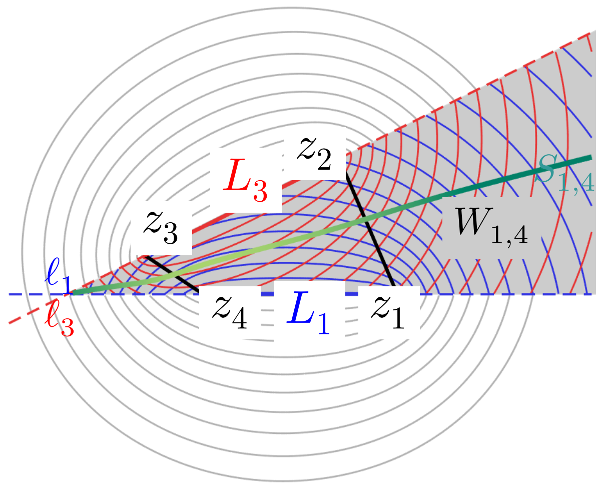

Figure 3.3: Level sets of g_1 and g_3 are highlighted in red and blue respectively. The wedge $W_{1,4}$ is gray area between ℓ_1 and ℓ_3 that contains the polygon.

Theorem 3.1.4 (Main Result)**.** *Each polygon that satisfies the strict descent condition admits an electrostatic skeleton. Moreover, the support of the skeleton is piecewise analytic and consists of at most $2n-3$ analytic curves and the measure is equivalent to $\mathcal{H}^1$ on the support.*

Theorem 3.1.4 implies that the existence of electrostatic skeleton can be reduced to a somewhat explicit analytic property of the Green's function of the polygon. The bound $2n-3 = n+(n-3)$ comes from a correspondence between the shape of the skeleton and a partition of n-gon. Thus the number of arcs that are not rooted at the vertices is bounded by the number of non-crossing matching (see Figure 3.4).

We argue that the strict descent condition is quite natural and should hold for generic polygons. The author has not found any convex polygon that does not satisfy the strict descent condition. This suggests the following strenghtening of the existing conjecture (see above).

Conjecture 3.1.5. *All convex polygons satisfy the strict descent condition.*

Whether or not a convex polygon satisfies the strict descent condition is a property of its Riemann map: as such, it can be explicitly computed for any arbitrary example. While, in principle, being completely explicit (using only the exterior Schwarz-Christoffel mapping and Schwarz reflection), we have been unable to deal with the problem at a general level.

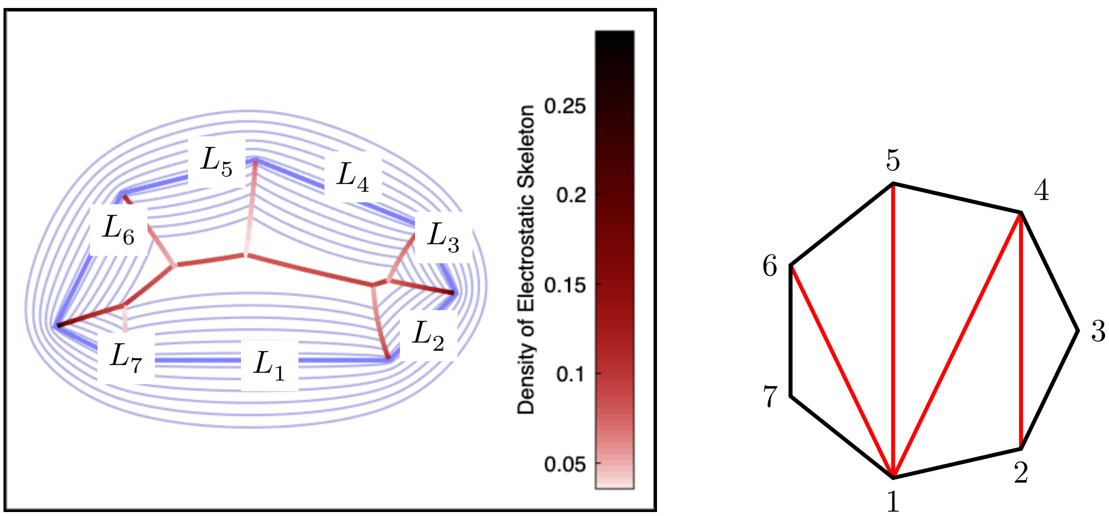

Figure 3.4: Electrostatic skeleton for a heptagon and its corresponding partition of (regular) heptagon.

3.2 *Preliminary*

3.2.1 Logarithmic Potential and Green's Function

We review some basic properties of logarithmic potentials that will be used for the proofs. We refer to Saff's survey [95] and the book of Saff and Totik [96] for further details.

Lemma 3.2.1 (Riesz decomposition, see [95, Theorem 2.4])**.** *Given a finite, compactly supported Borel measure μ,*

(a) U^μ is superharmonic. In particular, it is lower-semicontinuous,

(b) U^μ is finite and harmonic on $\mathbb{C}\setminus \operatorname{supp}\mu$.

Conversely, given a superharmonic function $f : \mathbb{C} \to (-\infty, \infty]$ *which is harmonic outside a compact set* K*, there exists a unique positive Borel measure*

$$\mu = -(2\pi)^{-1}\Delta f,$$

which supported on K *such that* $U^\mu - f$ *is constant.*

Here the Laplacian is defined in the distributional sense as in [93, Section 3.7]. Given a compact set $D \subseteq \mathbb{C}$ with positive capacity, the Green's function $g : \mathbb{C}\backslash \operatorname{Int} D \to \mathbb{R}$ is the unique harmonic function that vanishes on ∂D and satisfies $g(z) = \log|z| + O(1)$. When $\hat{\mathbb{C}}\backslash D$ is simply-connected, we have $g(z) = \log|f(z)|$, where $f : \hat{\mathbb{C}}\backslash D \to \hat{\mathbb{C}}\backslash \mathbb{D}$ is the unique Riemann mapping function satisfying $f(\infty) = \infty$ and $f'(\infty) > 0$.

Lemma 3.2.2 (Equipotential Lines for Convex Domains [83, Theorem 2.9])**.** *Let* D *be a compact convex set and* $f : \hat{\mathbb{C}}\backslash D \to \hat{\mathbb{C}}\backslash \overline{\mathbb{D}}$ *be the biholomorphic mapping such that* $f(\infty) = \infty$ *and* $f'(\infty) > 0$*. Then for any* $r > 1$*, the set* $\{z : |f(z)| = r\}$*, which is the equipotential line of the Green's function, has strictly convex interior.*

The task of finding an electrostatic skeleton is related to finding a subharmonic extension of g inside the original domain Ω, as indicated by the following Lemma.

Lemma 3.2.3. *Given a Jordan domain* Ω *with its Green's function* $g : \mathbb{C}\backslash\Omega \to \mathbb{R}$*, suppose* $\tilde{g}$ *is a subharmonic function over* $\mathbb{C}$ *such that* $\tilde{g}|_{\mathbb{C}\backslash\Omega} = g$*. Then we have* $g(z) = -U^\mu(z) - \log \operatorname{Cap}(\Omega)$ *for* $z \in \mathbb{C}\backslash\Omega$*, where* $\mu = (2\pi)^{-1}\Delta\tilde{g}$*.*

Proof. Let g_0 be the extension of g such that $g_0(z) = 0$ for $z \in \Omega$. Then we know that the equilibrium measure $\mu_E = (2\pi)^{-1}\Delta g_0$ satisfies that $g = -U^{\mu_E} - \log\operatorname{Cap}(\Omega)$ outside Ω. So it suffices to prove that $U^{\mu_E} = U^\mu$ for any subharmonic extension $\tilde{g}$ of g and $\mu = (2\pi)^{-1}\Delta\tilde{g}$. Given any C^2 Jordan domain $\mathcal{O}$ containing $\overline{\Omega}$ and any z outside $\mathcal{O}$, the function $w \mapsto -\log|z - w|$ is harmonic inside $\mathcal{O}$. By the Gauss-Green formula, we have

$$U^\mu(z) = \int \log\frac{1}{|z-w|}d\mu(w) = -\int_{\partial\mathcal{O}} \log\frac{1}{|z-\zeta|}\frac{\partial\tilde{g}}{\partial\mathbf{n}}(\zeta)|d\zeta|.$$

However, $\tilde{g}$ equals g on $\partial\mathcal{O}$. Thus the value of U^μ outside $\mathcal{O}$ is the same as U^{μ_E}. Since $\mathcal{O}$ is arbitrary, the result then follows. $\square$

3.2.2 Geometry of the zero sets

Recall the notations established in Section 3.1.3.

Lemma 3.2.4. *$S_{i,j}$ consists of a single analytic curve. If $\ell_i \cap \ell_j = \{z_{i,j}\}$, then $S_{i,j}$ intersects $\partial W_{i,j}$ only at $z_{i,j}$. If ℓ_i and ℓ_j are parallel (intersect at ∞), then $S_{i,j}$ is disjoint from $\partial W_{i,j}$ and it intersects every line perpendicular to ℓ_i.*

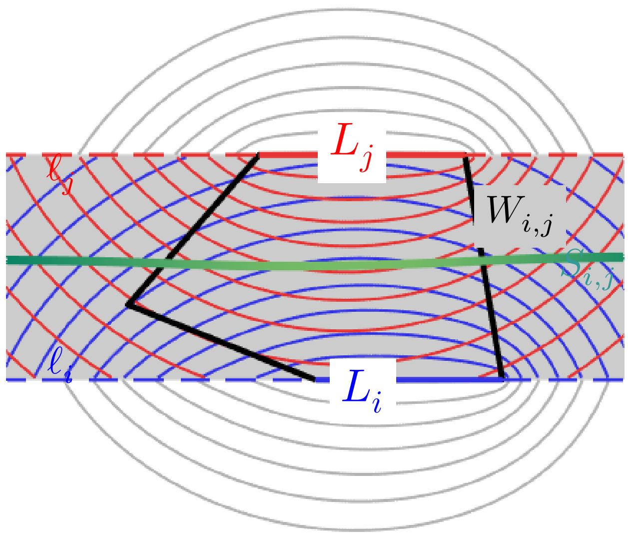

Figure 3.5: The case where ℓ_i and ℓ_j are parallel

Proof. Note that in the case when ℓ_i and ℓ_j intersects at $z_{i,j}$, we have

$$-g_i(z_{i,j}) = -g_j(z_{i,j}) = g(z_{i,j}).$$

Thus we have $z_{i,j} \in S_{i,j}$. Given any $z \in (\ell_i \cap W_{i,j}) \setminus \{z_{i,j}\}$, we set z' to be the reflection of z with respect to ℓ_j (see Figure 3.6). We then have that

$$-g_i(z) = g(z), \quad -g_j(z) = g(z').$$

Since the support of the equilibrium μ_E (which is $\partial\Omega$) lies on z's side of $\mathbb{C}\backslash\ell_j$, every point on $\partial\Omega$ is closer to z than z'. Let μ_E be the equilibrium measure of Ω. We thus have

$$\begin{aligned} g(z) &= -\log \mathrm{Cap}(\Omega) - \int \log \frac{1}{|z-w|} d\mu_E(w) \\ &< -\log \mathrm{Cap}(\Omega) - \int \log \frac{1}{|z'-w|} d\mu_E(w) \\ &= g(z'). \end{aligned}$$

This implies that $z \notin S_{i,j}$. Hence, $S_{i,j}$ does not intersect $(\ell_i \cap W_{i,j})$ except possibly at $z_{i,j}$. Similarly, it can be shown that $S_{i,j}$ does not intersect $(\ell_j \cap W_{i,j})$, except possibly at $z_{i,j}$.

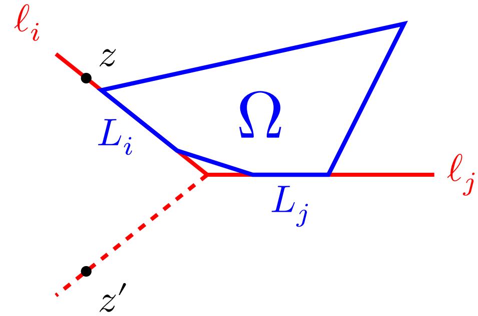

Figure 3.6: Idea behind the proof of Lemma 3.2.4.

Now we prove the rest of the assertions. Take any component C of $W_{i,j}\backslash S_{i,j}$, if $C \cap \partial W_{i,j} = \emptyset$, then we have $\partial C \subseteq S_{i,j}$. Since $S_{i,j}$ is the zero set of a harmonic function, we have that C cannot be bounded. Without loss of generality, we assume that $g_i - g_j$ is positive in C. Then the function

$$u(z) := \begin{cases} g_i(z) - g_j(z), & z \in C, \\ 0, & z \in W_{i,j}\backslash C. \end{cases}$$

Since u is harmonic on $W_{i,j}\backslash\partial C$ and satisfies the mean value inequality on ∂C for sufficiently small $r > 0$:

$$u(z) = 0 \leq \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta}) dr.$$

We have that u is a subharmonic function vanishes on the sector boundary $\partial W_{i,j}$ and satisfies the logarithmic growth condition

$$u(z) = O(\log|z|).$$

By the Phragmén-Lindelöf theorem for subharmonic functions (see [105]), we have that $u \equiv 0$ which would imply $g_i \equiv g_j$. But this contradicts the fact that $S_{i,j} \neq W_{i,j}$. Therefore, all the components of $W_{i,j} \backslash S_{i,j}$ intersect either ℓ_i or ℓ_j and are unbounded. Since $S_{i,j}$ is disjoint from ℓ_i and ℓ_j except for the two lines' intersection, we have that there is only one component of $W_{i,j} \backslash S_{i,j}$ that intersects ℓ_i and one intersecting ℓ_j. We denote the components of $W_{i,j} \backslash S_{i,j}$ that contains L_i by C_i and define C_j similarly. It follows that C_i and C_j are the only two components and we have that $C_i \neq C_j$. The result then follows. □

3.3 Electrostatic Skeleton for Quadrilaterals

3.3.1 Proof of Theorem 3.1.1

Theorem 3.1.1 can be considered as an immediate consequence of Lemma 3.2.4, as the following proof illustrates. Note that a kite shape is a quadrilateral symmetric with respect to one of its diagonals.

Proof. Given a convex kite shape Ω, without loss of generality, we assume the line of symmetry ℓ passes z_1 and z_3. By symmetry, we then have that both $S_{1,2}$ and $S_{3,4}$ are contained in ℓ. Lemma 3.2.4 implies that there exists a unique (closed) segment of $S_{2,3}$ that connects z_2 to ℓ. We shall denote this segment by K_1 and denote the point when K_1 and ℓ intersect by c. By symmetry, there exists a unique segment K_1' of $S_{1,4}$ which connects z_4 with ℓ and also reaches ℓ at c. The segment K_1' is the reflection of K_1 with respect to ℓ.

We set the segment of ℓ inside the kite shape by L. Then the three curves K_1, K_1' and L divide Ω into four components at c. We shall denote the union of these four arcs by K. Then one can extend g continuously to $\mathbb{C} \backslash K$ using g_i's on the four components. That is, the extension coincides g_i on the component touching L_i, which we denote by Ω_i, which we

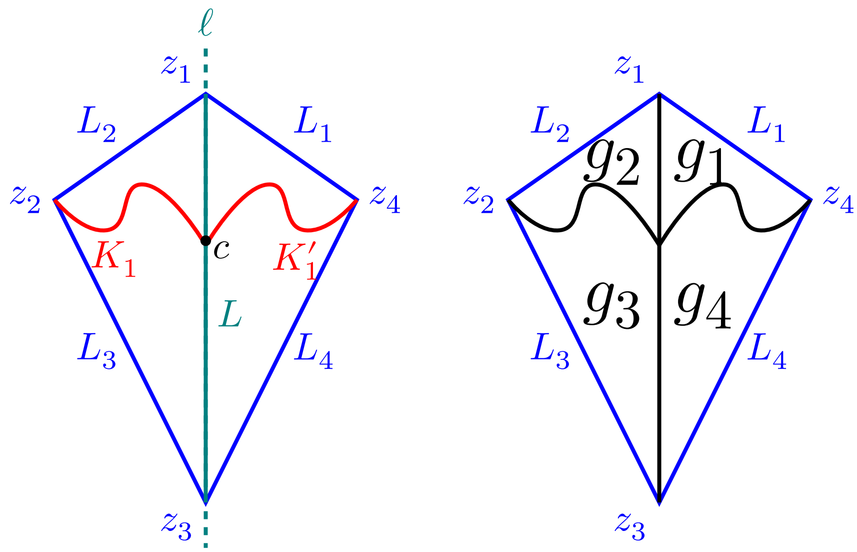

Figure 3.7: Sketch of the proof for Theorem 3.1.1.
Left: skeleton constructed from $S_{i,j}$'s on the left.
Right: subharmonic extension of g that generates the skeleton.

denote by $\tilde{g}$. Note that on $\Omega_i \cup \Omega_{i+1}$, $\tilde{g}$ coincides $\max(g_i, g_{i+1})$ as Ω_i and Ω_{i+1} are contained in different nodal regions of $g_i - g_{i+1}$, which is $W_{i,i+1} \backslash S_{i,i+1}$. It follows that $\tilde{g}$ satisfies the mean value inequality on K for all sufficiently small radii. This shows that g after extension is globally subharmonic and continuous which is harmonic on $\mathbb{C} \backslash K$. Thus we have $\Delta\tilde{g}$ is a positive Borel measure compactly supported inside K. In particular, $\operatorname{supp} \Delta\tilde{g}$ does not contain simple loops. The result hence follows from Lemma 3.2.3 $\square$

As one can see, the main idea of the proof is to construct the extension of g using the mirror reflections g_i's. So long as the switching between g_i and g_j happens only on $S_{i,j}$ and respects the nodal regions of $g_i - g_j$, we will end up with a subharmonic extension. The main challenge here is that in order for the subharmonic extension to admit the electrostatic skeleton, the "switching arcs" need to form a tree. This is easy in the case of kite shapes due to symmetry but much harder in general.

3.3.2 Isosceles Trapezoids

The simplicity of the proof to Theorem 3.1.1 stems from the fact that two arcs in the electrostatic skeleton are on ℓ, while the other two arcs are symmetric. Therefore, we effectively only need to look at the behavior of one arc to construct the skeleton. This is a special property for kite shapes that unfortunately cannot be generalized. To prove the existence of the skeleton for isosceles trapezoids, we need the following proposition:

Lemma 3.3.1. *For any isosceles trapezoid Ω, we parametrize each analytic curve $S_{i,i+1}$ by a function γ_i with non-vanishing derivative oriented from z_i to ∞. We denote the line of symmetric for Ω by ℓ, if t is a critical point of $g_i \circ \gamma_i$, then z_i and $\gamma_i(t)$ are separated by ℓ.*

In particular, before hitting ℓ, g_i is monotone along γ_i. And it makes sense to define the first time when γ_i and γ_{i+1} meet.

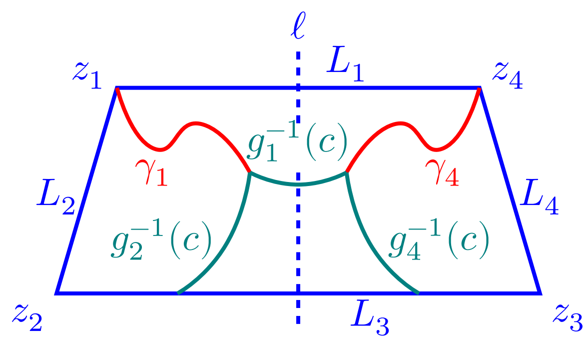

Figure 3.8: Sketch of proof of Lemma 3.1. If t_0 is a critical point of the real function $g_1 \circ \gamma_1 = g_2 \circ \gamma_1$ and $\gamma_1(t)$ is on the left of ℓ, then the level sets $g_1^{-1}(c)$, $g_2^{-1}(c)$ and $g_4^{-1}(c)$ will intersect tangentially for $c = g_1(\gamma_1(t_0))$. This is geometrically impossible.

Proof. Without loss of generality, we assume that z_1 and z_4 are symmetric vertices for Ω. Suppose for the contradiction that there exists a critical point t_0 for $g_1 \circ \gamma_1$ such that z_1 and $\gamma_1(t_0)$ lie on the same side of ℓ. We set $c = g_1(\gamma_1(t_0))$. Since t_0 is a critical point of

$g_1 \circ \gamma_1 = g_2 \circ \gamma_1$, we have

$$\nabla g_1(\gamma_1(t_0)) \cdot \gamma_1'(t_0) = \nabla g_2(\gamma_1(t_0)) \cdot \gamma_1'(t_0) = 0.$$

In particular, ∇g_1 and ∇g_2 are parallel at $\gamma_1(t_0)$. It follows that the convex curves $g_1^{-1}(c)$ and $g_2^{-1}(c)$ intersect tangentially at $\gamma_1(t_0)$. However, by symmetry, $g_1^{-1}(c)$ and $g_4^{-1}(c)$ also intersect tangentially at some point symmetric to $\gamma_1(t_0)$. This would allow us to construct the C^1 convex curve symmetric with respect to ℓ by combining the segments of three level sets, which starts and ends outside Ω. This is impossible for that Ω is convex.

□

Now we are ready to prove Theorem 3.1.2. Essentially, Lemma 3.3.1 guarantees that it makes sense to ask where $S_{1,2}$ and $S_{2,3}$ (parametrized by γ_1 and γ_2) meet for the first time before hitting ℓ. This helps us separate cases and build the skeleton accordingly.

Proof of Theorem 3.1.2. Given the isosceles trapezoid Ω, we again denote the line of symmetry by ℓ and assume the setup in Lemma 3.3.1. We denote the component of $\Omega\backslash\ell$ that touches L_1 by Ω_1. We also denote the component of $S_{1,2} \cap \Omega_1$ (paramtrized by γ_1) that contains z_1 by S_1 and that of $S_{2,3} \cap \Omega_1$ that contains z_2 by S_2. Then by Lemma 3.3.1, $g_1 \circ \gamma_1$ (with respect to $g_2 \circ \gamma_2$) monontonely decreases when $\gamma_1(t) \in S_1$ (with respect to $\gamma_2(t) \in S_2$).

We will consider first the case when S_1 and S_2 are disjoint. We will denote their intersections with ℓ by w_1 and w_2 respectively. Then by symmetry, γ_1 and γ_4 meet at w_1 and γ_2 and γ_3 meet at w_2. Additionally, w_1 and w_2 are in turn connected by a segment of $S_{2,4}$, which we denote by S'. Let S_1' and S_2' be the reflection of S_1 and S_2 with respect to ℓ which contain part of $S_{1,4}$ and $S_{3,4}$ respectively. Then we can easily verify that

$$K := S_1 \cup S_2 \cup S' \cup S_1' \cup S_2'$$

forms a tree which allows g extends continuously to a subharmonic function globally.

Now we consider the case when S_1 intersects S_2. Note that if S_1 and S_2 only intersect on ℓ, it can fall under the previous case such that w_1 and w_2 are the same point. Thus we

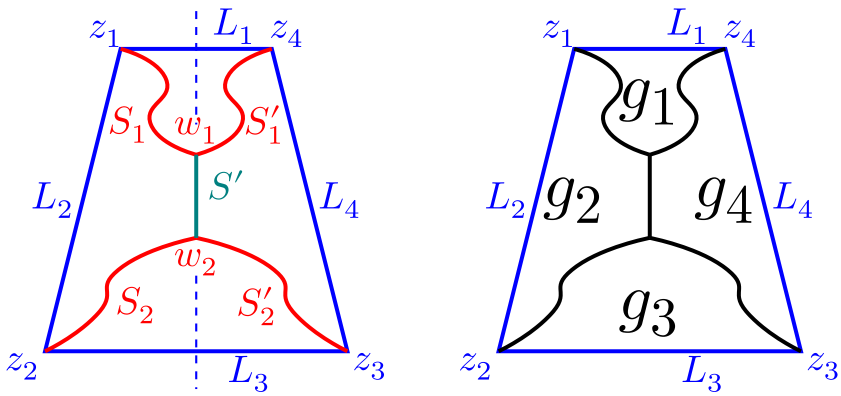

Figure 3.9: First case. Left: Skeleton. Right: Subharmonic extension.

assume otherwise. Since γ_1 and γ_2 are monotone on S_1 and S_2 respectively, we can order the points in $S_1 \cap S_2$ in terms of the value of $g_1 \circ \gamma_1$. We denote the point with the largest value as w. That is, w is where two curves "meet for the first time". By definition, we have

$$g_1(w) = g_2(w) = g_3(w)$$

and we denote this value by a. This implies that $w \in S_{1,3}$. Note that since S_1 and S_2 are distinct analytic curves, $S_1 \cap S_2$ is a discrete set near w. Thus such a w is unique. Let K_3 be the unique component of $S_{1,3}$ that connects w to ℓ. Note that this component is unique because due to symmetry, $S_{1,3}$ cannot intersect ℓ twice.

We set K_1 and K_2 to be the images of γ_1 and γ_2 up to w respectively. Then for any $z \in K_1 \cap K_3$, we also have $g_1(z) = g_2(z) = g_3(z)$, but the only point on K_1 that would satisfy this equality is w. Thus we have

$$K_i \cap K_j = \{w\}, \quad i \neq j \in \{1, 2, 3\}.$$

We now set K_1', K_2' and K_3' to be the reflections of K_1, K_2 and K_3 against ℓ respectively. Then the compact set

$$K := K_1 \cup K_2 \cup K_3 \cup K_1' \cup K_2' \cup K_3'$$

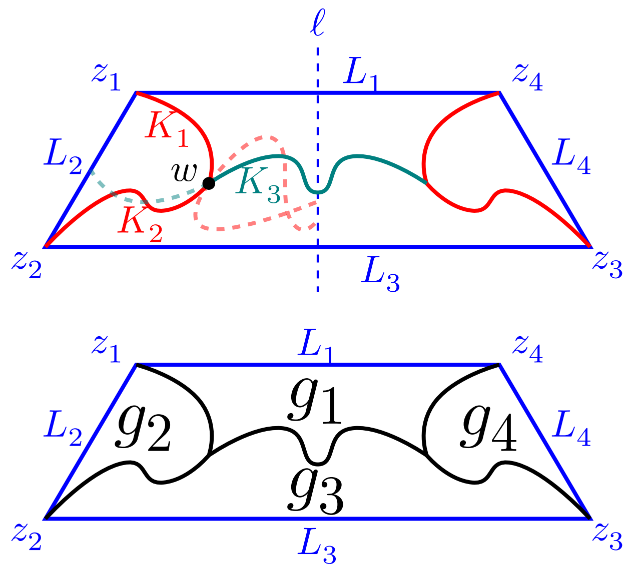

Figure 3.10: Second case. Top: Skeleton. Bottom: Subharmonic extension

forms a tree which allows g to extend to a globally continuous subharmonic function. The result then follows. $\square$

3.4 Strict descent condition

3.4.1 Motivating example: generic quadrilateral

It is still an open problem if every convex quadrilateral admits an electrostatic skeleton. Nevertheless, if we set $g_{\max} := \max\{g_1, g_2, g_3, g_4\}$, numerical simulations suggest that the measure $(2\pi)^{-1}\Delta g_{\max}$ has a tree-like support, making such a measure the skeleton. In this section, we will explore the structure of this measure, which will motivate the algorithm used to construct the electrostatic skeleton under the condition of strict descent.

The important observation here is that the support of $\Delta g_{\max}$ consists of corners of the level sets $g_{\max}^{-1}(-t)$, where $t > 0$ is small enough that the level set is an analytic quadrilateral, with each side being a segment of $g_i^{-1}(t)$. As we increase t, one of the following three things will happen to $g_{\max}^{-1}(-t)$:

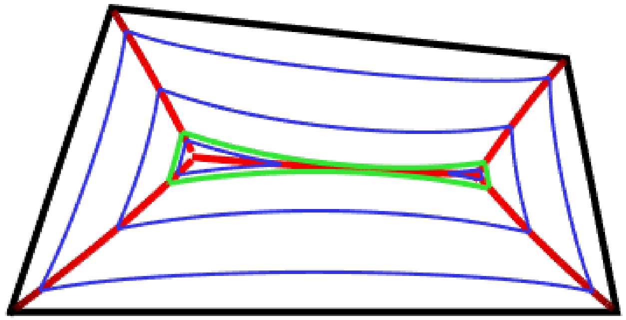

Figure 3.11: Shrinking level sets in a thin quadrilateral, phase transition highlighted in green

1. it vanishes to a single point (convex kite shapes),

2. it splits into two analytic triangles (quadrilaterals that are thin, see Figure 3.11),

3. it converts into one single analytic triangle (quadrilaterals with a significantly shorter side, see Figure 3.12).

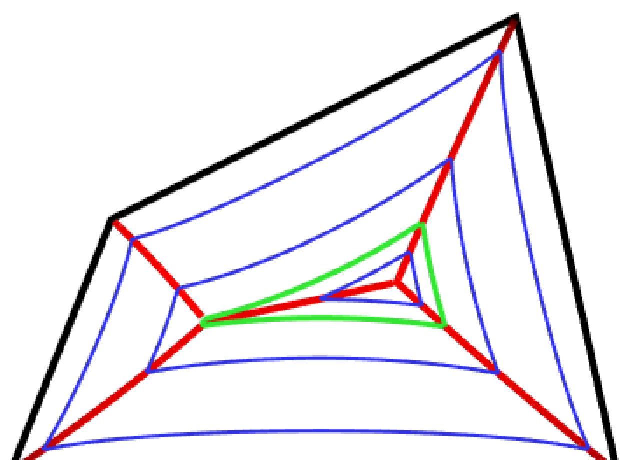

Figure 3.12: Shrinking level sets in a quadrilateral with a significantly shorter side, phase transition highlighted in green

It is worth paying attention to the critical time when the phase transitions of the latter two cases happen. For the second case, the phase transition happens when two non-adjacent analytic arcs meet. For the third case, it happens when two adjacent vertices meet. In either case, the vertices of the analytic triangles are still in the support in $\Delta g_{\max}$

We note that the analytic quadrilaterals and triangles that arise as the level sets of $g_{\max}$ share some geometric properties:

1. they are both piecewise analytic Jordan domains enclosed by finitely many segments of level sets $g_i^{-1}(-t)$'s for a fixed $t \geq 0$,

2. on the segment of level set $g_i^{-1}(-t)$, the gradient ∇g_i points toward the outside of the domain (the edges concave inwards),

3. at each vertex, the internal angles of the domain is strictly less than π.

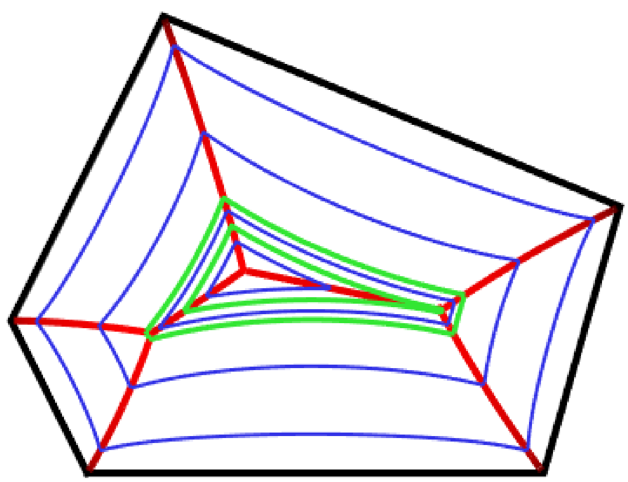

Figure 3.13: Multiple phase transitions occur when shrinking the level sets inside the pentagon

We will call the analytic polygons that satisfy these three properties the *regular loops* of the polygon. The vertices of the regular loops will be the building blocks for electrostatic skeletons. The main idea of the skeleton building algorithm is to find a subharmonic extension $\tilde{g}$ of g inside the convex polygon that satisfies the following:

> the level set $\tilde{g}^{-1}(-t)$ is a regular loop which shrinks as we increase t. The level set $\tilde{g}^{-1}(-t)$ will go through phase transitions and decompose into smaller regular loops (see Figure 3.13). After finitely many phase transitions, all the regular loops vanish to single points.

We define a *critical loop* as the curve that arises during the phase transition of a regular loop. The more rigorous definitions of regular and critical loops will be introduced in Section 3.5 after we introduce the proper notations.

The condition of strict descent is posed to assure that during the shrinking, the internal angles of the loops remain strictly less than π. This, in practice, always happens for the level sets of $\max\{g_1, \dots, g_n\}$. However, showing such a property for $\max\{g_1, \dots, g_n\}$ (or any subharmonic extensions of g) is difficult and might require delicate estimates of Schwarz–Christoffel maps. Thus in this paper, we will focus on showing the condition of strict descent implies the existence of electrostatic skeleton, instead of proving the condition.

3.4.2 Parametrization of the zero sets

As we can see in the proof of Theorem 3.1.2, it is convenient to construct the electrostatic skeleton when we can control when two zero sets intersect for the first time. In this section, we will look further into the geometry of the zero sets with the strict descent condition. Again we assume the notations from Section 3.1.3.

Lemma 3.4.1. *The Schwarz reflection g_i has a unique maximum on $S_{i,j}$ for all $j \neq i$.*

Proof. If L_i and L_j are adjacent edges (w.l.o.g. we assume $j = i + 1$), it is clear that $g_i < 0$ in the interior of $W_{i,j}$ and $g_i(z_i) = 0$. So the result follows trivially. Now suppose L_i and L_j are not adjacent. We note that the function

$$t \mapsto d(\{z : g_i(z) \geq -t\}, \{z : g_j(z) \geq -t\})$$

is continuous. Let $t_0 > 0$ be the first time this decreasing function is zero. Then $\{z : g_i(z) \geq -t\}$ and $\left\{z : g_j(z) \geq -t\right\}$ intersect when $t = t_0$ and are disjoint when $t < t_0$. It follows that $g_i^{-1}(-t_0)$ and $g_j^{-1}(-t_0)$ are tangent. By Lemma 3.2.2, $g_i^{-1}(-t_0)$ and $g_j^{-1}(-t_0)$ are convex curves and thus can only be tangent at a unique point. That tangent point is thus the global maximum of both g_i and g_j on $S_{i,j}$. □

The reflection functions g_i and g_j share the same global maximum on $S_{i,j}$. We shall denote this maximum by $m_{i,j}$. When $j = i + 1$, the unique maximum is $z_i = m_{i,i+1}$. When L_i and L_j are disjoint, $m_{i,j}$ is in the interior of $W_{i,j}$. Lemma 3.2.4 and Lemma 3.4.1 together imply the following.

Proposition 3.4.2 (Strict descent condition reformatted). *Given a convex polygon Ω that satisfies the strict descent condition, the following conditions hold:*

1. *for any pair of adjacent edges L_i and L_{i+1}, we can parametrize $S_{i,i+1}$ as a curve $\gamma_{i,i+1}$ such that*

$$\gamma_{i,i+1}(0) = z_i, \quad g_i(\gamma_{i,i+1}(t)) = g_{i+1}(\gamma_{i,i+1}(t)) = -t,$$

2. *for any pair of non-adjacent edges L_i and L_j, let $m_{i,j}$ be the unique global maximum shared by g_i and g_j on $S_{i,j}$ and set $t_{i,j} = -g_i(m_{i,j})$. Then we can parametrize $S_{i,j}$ as two curves $\gamma_{i,j}$ and $\eta_{i,j}$ meeting at $m_{i,j}$ such that*

$$g_i(\gamma_{i,j}(t)) = g_j(\gamma_{i,j}(t)) = g_i(\eta_{i,j}(t)) = g_j(\eta_{i,j}(t)) = -t,$$
$$\gamma_{i,j}(t_{i,j}) = \gamma_{i,j}(t_{i,j}) = \eta_{i,j}(t_{i,j}) = \eta_{i,j}(t_{i,j}) = m_{i,j}.$$

Proof. We first note that for $z \in S_{i,j}$, if $\nabla g_i(z)$ and $\nabla g_j(z)$ are pointed at the exact opposite direction, then have that $g_i^{-1}(g_i(z))$ and $g_j^{-1}(g_i(z))$ must be tangential at z. This implies that $z = m_{i,j}$. Therefore the condition of strict descent as in Definition 3.1.3 implies that on $S_{i,j} \backslash m_{i,j}$, ∇g_i and ∇g_j cannot be parallel.

Note that away from the global maximum $m_{i,j}$ of g_i (and g_j) on $S_{i,j}$, we can parametrize $S_{i,j}$ by the smooth curve β. Then we have

$$\frac{\partial}{\partial t} g_i(\beta(t)) = \nabla g_i(\beta(t)) \cdot \beta'(t) = \nabla g_j(\beta(t)) \cdot \beta'(t).$$

Since $S_{i,j}$ is smooth, we can assume that $\beta'(t)$ is nowhere vanishing. Since $\beta(t)$ is away from $m_{i,j}$, $\nabla g_i(z)$ and $\nabla g_j(z)$ are not parallel. Thus we have

$$\nabla g_i(\beta(t)) \cdot \beta'(t) = \nabla g_j(\beta(t)) \cdot \beta'(t) \neq 0.$$

This implies that g_i and g_j are strictly monotone on $S_{i,j}$ away from $m_{i,j}$. Thus the result follows. □

For the notational convenience, we always assume $\gamma_{i,j}$ parametrizes one of the unbounded component of $S_{i,j} \backslash \{m_{i,j}\}$. In the case where both components of $S_{i,j} \backslash \{m_{i,j}\}$ are unbounded (as is the case where L_i and L_j are parallel), we assign $\gamma_{i,j}$ and $\eta_{i,j}$ arbitrarily. In consequence, the domain of $\gamma_{i,j}$ is always $[t_{i,j}, \infty)$ while if $\eta_{i,j}$ exists, its domain is a half-open subinterval of $[t_{i,j}, \infty)$ containing $t_{i,j}$.

The strict descent condition thus significantly restricts the behavior of $S_{i,j}$'s and how they intersect one another. Under such a condition, given two adjacent edges L_i and L_{i+1} and any $t > 0$, $\gamma_{i,i+1}(t)$ is the unique intersection between $g_i^{-1}(-t)$ and $g_j^{-1}(-t)$. Meanwhile, the condition of strict descent also implies that for any two non-adjacent edges L_i and L_j, the images of $\eta_{i,j}$ and $\gamma_{i,j}$ are separated by

$$g_i^{-1}([-t_{i,j}, 0]) \cup g_i^{-1}([-t_{i,j}, 0])$$

since the two sets above intersect uniquely at $m_{i,j}$ (see Figure 3.14).

Corollary 3.4.3 (Geometry of the level sets)**.** *Under the condition of strict descent, consider the level set $g_i^{-1}(-t)$ inside the wedge $W_{i,j}$,*

1. *When $0 \leq t < t_{i,j}$, $g_i^{-1}(-t)$ is completely contained in one of the components of $W_{i,j} \backslash S_{i,j}$.*

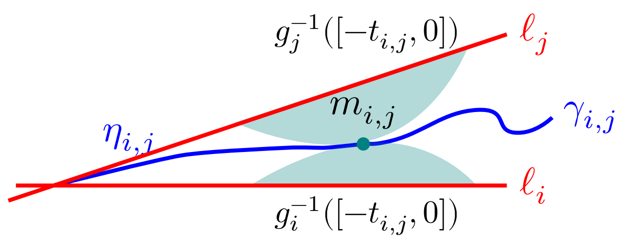

Figure 3.14: Sketch of $J^{\gamma}_{i,j}(t)$ and $U^{\gamma}_{i,j}(t)$

2. *When $t = t_{i,j}$, $g_i^{-1}(-t)$ is completely contained in the closure of one of the components of $W_{i,j} \backslash S_{i,j}$, and intersects $S_{i,j}$ only at $m_{i,j}$.*

3. *When $t > t_{i,j}$ and $\eta_{i,j}(t)$ is well-defined, $g_i^{-1}(-t)$ consists of one segment connecting ℓ_i to $\gamma_{i,j}(t)$, one segment connecting ℓ_i to $\eta_{i,j}(t)$ and one connecting $\gamma_{i,j}(t)$ and $\eta_{i,j}(t)$. The first two segments are separated from the last by $S_{i,j}$.*

4. *When $t > t_{i,j}$ and t is outside the domain of $\eta_{i,j}$, $g_i^{-1}(-t)$ consists of one segment connecting ℓ_i to $\gamma_{i,j}(t)$ and the other connecting $\gamma_{i,j}(t)$ to ℓ_j. The two are separated by $S_{i,j}$.*

In particular, if we assume the strict descent condition, then for any $t > t_{i,j}$, $g_i^{-1}(-t)$ will only intersect the image of $\gamma_{i,j}$ exactly once at $\gamma_{i,j}(t)$ (and that of $\eta_{i,j}$ at $\eta_{i,j}(t)$ if it exists). Let $\beta \in \{\gamma, \eta\}$. If t is in the domain of $\beta_{i,j}$, it follows that one of the components of

$$W_{i,j} \backslash \left(g_i^{-1}([-t,0]) \cup g_j^{-1}([-t,0]) \right)$$

contains the entirety of the curve $\beta_{i,j}$ after time t. We denote that component as $U^{\beta}_{i,j}(t)$ and set

$$J^{\beta}_{i,j}(t) = \partial U^{\beta}_{i,j}(t) \backslash (\ell_i \cup \ell_j).$$

Then $J^{\beta}_{i,j}(t)$ is a curve consisting of a segment of $g_i^{-1}(-t)$ connecting ℓ_i to $\beta_{i,j}(t)$ and that of $g_j^{-1}(-t)$ connecting ℓ_j to $\beta_{i,j}(t)$. It also cuts $W_{i,j}$ into two components (see Figure 3.15).

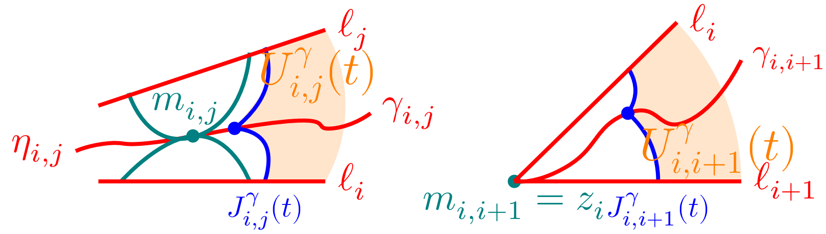

Figure 3.15: Sketch of $J^{\gamma}_{i,j}(t)$ and $U^{\gamma}_{i,j}(t)$

Note that $J^{\beta}_{i,j}$ is a continuous function on the space of compact sets on $\mathbb{C}$ under the Hausdorff topology. Moreover, $J^{\beta}_{i,j}(t_{i,j}) := \lim_{t\downarrow t_{i,j}} J^{\beta}_{i,j}(t)$ is either $L_i \cup L_j$ (when L_i and L_j are adjacent), or consisting of parts of $g_i^{-1}(-t_{i,j})$ and $g_j^{-1}(-t_{i,j})$ forming a cusp. The next lemma gives us some geometric intuition about the curve $J^{\beta}_{i,j}$ and domain $U^{\beta}_{i,j}$.

Lemma 3.4.4. *Under the condition of strict descent as in Proposition 3.4.2, let $t \geq t_{i,j}$ be such that $\beta_{i,j}(t)$ is a well-defined point in $W_{i,j}$. Then the angle $J^{\beta}_{i,j}(t)$ makes toward $U^{\beta}_{i,j}(t)$ is strictly less than π.*

Proof. It is easy to check that in the case when $t = t_{i,j}$, the angle $J^{\beta}_{i,j}(t)$ makes is either 0 (when $1 < |i-j| < n-1$) or an inner angle of the polygon (when $|i-j| \in \{1, n-1\}$). Thus we assume $t > t_{i,j}$. Note that since $W_{i,j}\backslash S_{i,j}$ has two components, the one touching ℓ_i, which we denote by Ω_i, satisfies

$$\Omega_i = \left\{z \in W_{i,j} : g_i(z) > g_j(z)\right\}.$$

Note that the curve $\beta_{i,j}$ is smooth. We can let $T_{i,j}(t)$ be the normalized velocity vector of the curve at $\beta_{i,j}(t)$ and $N_{i,j}(t)$ the normal vector at $\beta_{i,j}(t)$ pointing toward Ω_i. Then the density of $\Delta \max(g_i, g_j)$ at $\beta_{i,j}(t)$ is given by

$$-\nabla g_i(\beta_{i,j}(t)) \cdot N_{i,j}(t) + (-\nabla g_j(\beta_{i,j}(t)) \cdot (-N_{i,j}(t)) = \nabla(g_i - g_j)(\beta_{i,j}(t)) \cdot N_{i,j}(t).$$

This quantity should be non-negative as $\max(g_i, g_j)$ is subharmonic at $\beta_{i,j}(t)$. Meanwhile, the strict descent condition implies ∇g_i and ∇g_j are not parallel at $\beta_{i,j}(t)$. Thus

$$\nabla g_i(\beta_{i,j}(t)) \cdot T_{i,j}(t) = \nabla g_j(\beta_{i,j}(t)) \cdot T_{i,j}(t) < 0.$$

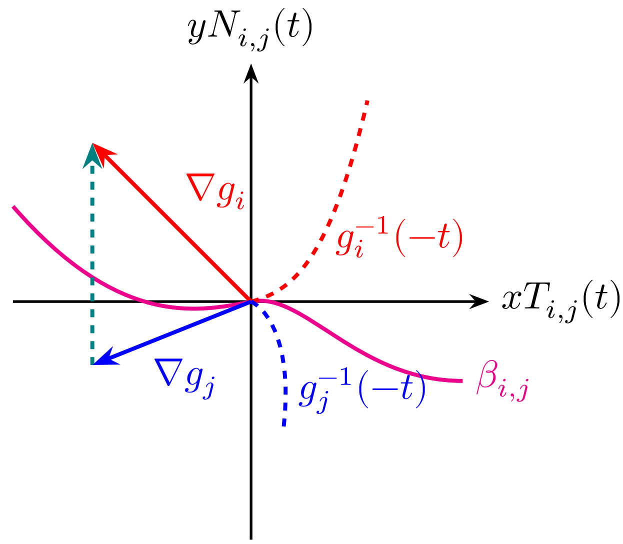

Figure 3.16: Coordinate plane by $T_{i,j}(t)$ and $N_{i,j}(t)$. Note that ∇g_i and ∇g_j do not necessarily lie in different quadrants

It follows that in the coordinate plane $(T_{i,j}(t), N_{i,j}(t))$, $\nabla g_i(\beta_{i,j}(t))$ has the same x-coordinate as $\nabla g_j(\beta_{i,j}(t))$ but strictly larger y-coordinate (see Figure 3.16). The statement hence follows. $\square$

3.5 Skeleton building algorithm

In this section, we will prove Theorem 3.1.4 to show that the strict descent condition implies the existence of electrostatic skeleton. To prove the theorem, we first need to properly define the concepts of regular and critical loops.

3.5.1 Regular loops

Note that if $t > 0$ is in the domains of both $\gamma_{i,j}$ and $\gamma_{j,k}$, then $\gamma_{i,j}(t)$ and $\gamma_{j,k}(t)$ are two points on $g_j^{-1}(-t)$. We shall denote the closed segment of $g_j^{-1}(-t)$ connecting the two points by the tuple $(t, \gamma_{i,j}, g_j, \gamma_{j,k})$ (see Figure 3.17).

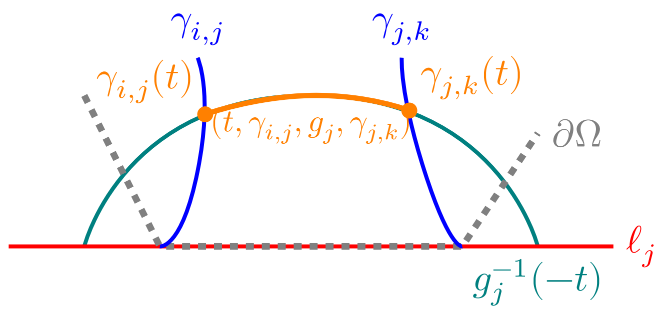

Figure 3.17: Sketch of $(t, \beta_{i,j}, g_j, \beta_{j,k})$, in the case when $i+1=j=k-1$

Similarly, we can define $(t, \gamma_{i,j}, g_j, \eta_{j,k})$ and $(t, \eta_{i,j}, g_j, \gamma_{i,k})$. We set

$$\begin{aligned}(t, \beta_{i_0,i_1}, g_{i_1}, \beta_{i_1,i_2}, g_{i_2}, \cdots, \beta_{i_{n-1},i_n}, g_{i_n}, \beta_{i_n,i_{n+1}}) \\ = \bigcup_{k=1}^{n} (t, \beta_{i_{k-1},i_k}, g_{i_k}, \beta_{i_k,i_{k+1}}),\end{aligned} \tag{3.1}$$

where $\beta \in \{\gamma, \eta\}$ and we assume t is in the domain of each $\beta_{i_k,i_{k+1}}$. When $\beta_{i_0,i_1} = \beta_{i_n,i_{n+1}}$, this is a loop that consists of segments of level sets of g_i's. To simplify the notation, we also write the left-hand side of (3.1) as $(t, \boldsymbol{\beta}, \boldsymbol{g})$ where

$$\begin{aligned}\boldsymbol{\beta} &= (\beta_{i_0,i_1}, \beta_{i_1,i_2}, \cdots, \beta_{i_n,i_{n+1}}), \\ \boldsymbol{g} &= (g_{i_1}, g_{i_2}, \cdots, g_{i_n}).\end{aligned}$$

Definition 3.5.1 (Regular loops)**.** Given a tuple $(t, \boldsymbol{\beta}, \boldsymbol{g})$ such that $\beta_{i,j}(t)$ is well-defined for each $\beta_{i,j} \in \boldsymbol{\beta}$ and the first and last elements of $\boldsymbol{\beta}$ coincide (hence it represents a loop), we say $(t, \boldsymbol{\beta}, \boldsymbol{g})$ represents a *regular loop* if all of the following conditions are satisfied:

1. The loop represented by the tuple is a Jordan curve and $\boldsymbol{g}$ orders the arcs counter-clockwise, and

2. Each arc $(t, \beta_{i,j}, g_j, \beta_{j,k})$ has positive length ($\beta_{i,j}(t) \neq \beta_{j,k}(t)$), and

3. On each arc $(t, \beta_{i,j}, g_j, \beta_{j,k})$, ∇g_j points outside the loop, and

4. For each $\beta_{i,j} \in \boldsymbol{\beta}$, the internal angle of the loop at $\beta_{i,j}(t)$ is in $[0, \pi)$.

It is easy to see that the arc $(t, \beta_{i,j}, g_j, \beta_{j,k})$ is a continuous function over t in the space of compact sets on $\mathbb{C}$ under the Hausdorff topology. As a consequence, if $(t, \boldsymbol{\beta}, \boldsymbol{g})$ is well-defined for $[a, b]$ and $(a, \boldsymbol{\beta}, \boldsymbol{g})$ represents a loop, $t \mapsto (t, \boldsymbol{\beta}, \boldsymbol{g})$ is a continuous function on $[a, b]$ of loops as well. Lemma 3.4.4 implies an equivalent definition of a regular loop:

Corollary 3.5.2. *Given a tuple $(t, \boldsymbol{\beta}, \boldsymbol{g})$ that represents a loop and satisfies Conditions (1), (2) and (3) in Definition 3.5.1, $(t, \boldsymbol{\beta}, \boldsymbol{g})$ satisfies Condition (4) if and only if for each pair of adjacent arcs $(t, \beta_{i,j}, g_j, \beta_{j,k})$ and $(t, \beta_{j,k}, g_k, \beta_{k,l})$ intersecting at $\beta_{j,k}(t)$, we have*

$$(t, \beta_{i,j}, g_j, \beta_{j,k}) \cup (t, \beta_{j,k}, g_k, \beta_{k,l}) \subseteq J^{\beta}_{j,k}(t). \tag{3.2}$$

Proof. Without loss of generality, we assume that the loop goes from $\beta_{i,j}(t)$ to $\beta_{j,k}(t)$ then to $\beta_{k,l}(t)$ in a counterclockwise fashion. Thus by Conditions (1) and (3), both ∇g_j on $(t, \beta_{i,j}, g_j, \beta_{j,k})$ and ∇g_k on $(t, \beta_{j,k}, g_k, \beta_{k,l})$ point to the right of the curve. Note that Corollary 3.4.3 implies that under the strict descent condition, for any $t > t_{i,j} = -\sup_{z \in S_{i,j}} g_i(z)$, the level set $g_i^{-1}(-t)$ and $g_j^{-1}(-t)$ intersect at most twice, both in a non-tangential manner. It thus follows that (3.2) holds if and only if the internal angle at $\beta_{j,k}(t)$ is strictly smaller than π (see Figure 3.18).

□

Lemma 3.5.3. *Under the strict descent condition, given a regular loop $(t_0, \boldsymbol{\beta}, \boldsymbol{g})$, there exists $\varepsilon > 0$ such that for all $t \in (t_0, t_0 + \varepsilon)$, the curve $(t, \boldsymbol{\beta}, \boldsymbol{g})$ is a regular loop inside $(t_0, \boldsymbol{\beta}, \boldsymbol{g})$.*

Proof. Suppose $(t_0, \boldsymbol{\beta}, \boldsymbol{g})$ is a regular loop. Then Condition (4) would imply in particular that $J^{\beta}_{i,j}(t_0)$ is a well-defined curve with positive length for all $\beta_{i,j} \in \boldsymbol{\beta}$. Therefore, $\beta_{i,j}$ should be well defined for all $(t_0, t_0 + \varepsilon)$ and all $\beta_{i,j} \in \boldsymbol{\beta}$ given $\varepsilon > 0$ sufficiently small. Suppose $\varepsilon > 0$ is small enough that for all $\beta_{i,j} \in \boldsymbol{\beta}$, $\beta_{i,j}(t)$ is well-defined for $t \in (t_0, t_0 + \varepsilon)$. On $[t_0, t_0 + \varepsilon]$, the loop $(t, \boldsymbol{\beta}, \boldsymbol{g})$ evolves continuously. Thus we can choose a smaller ε such that on $(t_0, t_0 + \varepsilon)$, each arc of $(t, \boldsymbol{\beta}, \boldsymbol{g})$ has positive length and non-adjacent arcs remain positive

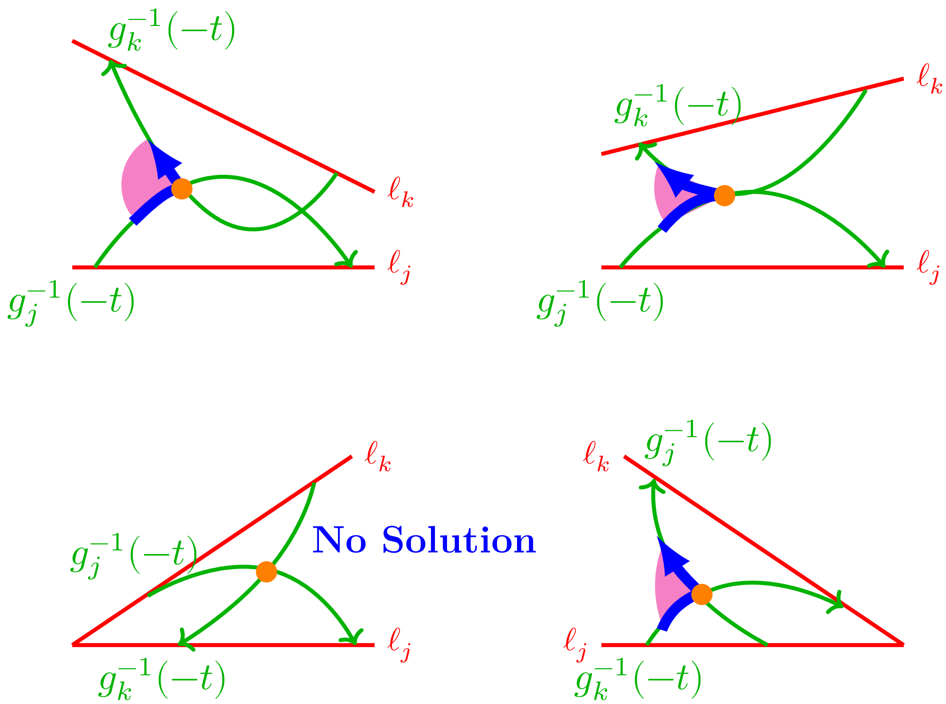

Figure 3.18: The unique solution for a curve going from $g_j^{-1}(-t)$ to $g_k^{-1}(-t)$ at $\beta_{j,k}(t)$ (highlighted in orange), assuming ∇g_j and ∇g_k point to the right of the curve and that the left turn angle is less than π, is $J^{\beta}_{j,k}(t)$.

distance away from each other. In consequence, $(t, \boldsymbol{\beta}, \boldsymbol{g})$ is still a Jordan curve with the same counter-clockwise orientation. Note that on each $(t, \beta_{i,j}, g_j, \beta_{j,k})$, ∇g_j never vanishes and always points to one side of the curve by convexity. Thus Condition (3) should also hold for all $t \in (t_0, t_0 + \varepsilon)$. In summary, given $\varepsilon > 0$ sufficiently small, Conditions (1), (2) and (3) hold for all $(t, \boldsymbol{\beta}, \boldsymbol{g})$ with $t \in (t_0, t_0 + \varepsilon)$.

Consider a pair of adjacent arcs $(t_0, \beta_{i,j}, g_j, \beta_{j,k})$ and $(t_0, \beta_{j,k}, g_k, \beta_{k,l})$. Note that the strict descent condition implies that $\beta_{i,j}((t_0, t_0 + \varepsilon])$ does not intersect $g_j^{-1}(-t_0)$. The fact that $(t, \beta_{i,j}, g_j, \beta_{j,k})$ is non-degenerate for $t \in (t_0, t_0 + \varepsilon)$ implies that $\beta_{i,j}([t_0, t_0 + \varepsilon])$ does not intersect the entire image of $\beta_{j,k}$. Thus it follows that $\beta_{i,j}([t_0, t_0 + \varepsilon])$ is entirely contained in the component of $U^{\beta}_{j,k}(t_0) \backslash S_{j,k}$ that touches ℓ_j. Likewise, $\beta_{k,l}([t_0, t_0 + \varepsilon])$ is entirely contained in the component of $U^{\beta}_{j,k}(t_0)\backslash S_{j,k}$ that touches ℓ_k. Therefore Corollary 3.4.3 would imply for each $t \in [t_0, t_0+\varepsilon]$, both $(t, \beta_{i,j}, g_j, \beta_{j,k})$ and $(t, \beta_{j,k}, g_k, \beta_{k,l})$ are in $J^{\beta}_{j,k}(t)$. Thus Condition (4) holds for all $t \in [t_0, t_0+\varepsilon]$ by Corollary 3.5.2.

Now we claim that $(t, \boldsymbol{\beta}, \boldsymbol{g})$ is disjoint from $(t_0, \boldsymbol{\beta}, \boldsymbol{g})$ for all $t \in (t_0, t_0+\varepsilon)$ with sufficiently small $\varepsilon > 0$. It suffices to show that for each $(t_0, \beta_{i,j}, g_j, \beta_{j,k}) \subseteq (t_0, \boldsymbol{\beta}, \boldsymbol{g})$, $(t, \beta_{i,j}, g_j, \beta_{j,k})$ and $(t_0, \boldsymbol{\beta}, \boldsymbol{g})$ are disjoint for all $t \in (t_0, t_0+\varepsilon)$ with sufficiently small $\varepsilon > 0$. Note that if $(t_0, \beta_{i,j}, g_j, \beta_{j,k})$ and $(t_0, \beta_{i',j'}, g_{j'}, \beta_{j',k'})$ are disjoint, then $(t, \beta_{i,j}, g_j, \beta_{j,k})$ and $(t_0, \beta_{i',j'}, g_{j'}, \beta_{j',k'})$ are also disjoint given $t - t_0$ sufficiently small. Thus we only need to consider the two arcs that share a vertex with $(t_0, \beta_{i,j}, g_j, \beta_{j,k})$. Suppose $(t_0, \beta_{j,k}, g_k, \beta_{k,l})$ is the one that shares vertex $\beta_{j,k}(t)$ with it. Thus by Condition (4), $(t, \beta_{i,j}, g_j, \beta_{j,k})$ and $(t_0, \beta_{j,k}, g_k, \beta_{k,l})$ lie in the different components of $W_{j,k}\backslash S_{j,k}$, implying that they are disjoint. Therefore, given $\varepsilon > 0$ sufficiently small, for all $t \in (t_0, t_0+\varepsilon)$, $(t, \beta_{i,j}, g_j, \beta_{j,k})$ is disjoint from all the arcs in $(t_0, \boldsymbol{\beta}, \boldsymbol{g})$, proving the claim. Since the angle $J^{\beta}_{i,j}(t_0)$ makes toward $\beta_{i,j}([t_0, t_0+\varepsilon])$ is the internal angle of $(t_0, \boldsymbol{\beta}, \boldsymbol{g})$, it follows that $\beta_{i,j}(t)$ is inside $(t_0, \boldsymbol{\beta}, \boldsymbol{g})$ for $t \in (t_0, t_0+\varepsilon)$. Thus $(t, \boldsymbol{\beta}, \boldsymbol{g})$ is inside $(t_0, \boldsymbol{\beta}, \boldsymbol{g})$, proving the lemma. □

3.5.2 Critical loops

Given a regular loop $(t_0, \boldsymbol{\beta}, \boldsymbol{g})$, it is clear that $\beta_{i,j}(t)$ is well-defined on some half-closed interval $(t_0, t_0+\varepsilon]$ for all $\beta_{i,j} \in \boldsymbol{\beta}$. Thus $(t, \boldsymbol{\beta}, \boldsymbol{g})$ is a well-defined (not necessarily simple) loop on such a interval. We set

$$t_c = \inf\left\{t \in (t_0, \infty) | (t, \boldsymbol{\beta}, \boldsymbol{g}) \text{ is not well-defined or it is not a regular loop}\right\}.$$

By Lemma 3.5.3, t_c is strictly greater than t_0. The finiteness of t_c and well-defined-ness of $(t_c, \boldsymbol{\beta}, \boldsymbol{g})$ are guaranteed by the following:

Lemma 3.5.4. *Assuming the condition of strict descent and that $(t_0, \boldsymbol{\beta}, \boldsymbol{g})$ is a regular loop, we have $t_c < \infty$ and $(t_c, \boldsymbol{\beta}, \boldsymbol{g})$ is well-defined. Additionally, $t \mapsto (t, \boldsymbol{\beta}, \boldsymbol{g})$ is a continuously shrinking family of Jordan curves on $[t_0, t_c)$ and either of the following conditions needs to hold for the loop $(t_c, \boldsymbol{\beta}, \boldsymbol{g})$:*

(a) Some of $(t_c, \beta_{i,j}, g_j, \beta_{j,k})$*'s are degenerate to points, or*

(b) $(t_c, \boldsymbol{\beta}, \boldsymbol{g})$ *is not a Jordan curve.*

Proof. Note that for all $t \in [t_0, t_c)$, any non adjacent arcs of $(t, \boldsymbol{\beta}, \boldsymbol{g})$ are disjoint and all arcs have strictly positive length. The Jordan curve theorem implies that such a loop $(t, \boldsymbol{\beta}, \boldsymbol{g})$ is in fact a Jordan curve. According to Lemma 3.5.3, the function

$$t \mapsto d((t_0, \boldsymbol{\beta}, \boldsymbol{g}), (t, \boldsymbol{\beta}, \boldsymbol{g}))$$

is a locally increasing function as $(t, \boldsymbol{\beta}, \boldsymbol{g})$ shrinks locally over t. It follows from continuity that this function increases globally and therefore $(t_1, \boldsymbol{\beta}, \boldsymbol{g})$ is always inside $(t_2, \boldsymbol{\beta}, \boldsymbol{g})$ for $t_1 > t_2$ in $[t_0, t_c)$. Since inside $(t_0, \boldsymbol{\beta}, \boldsymbol{g})$, each g_i is lower-bounded, we cannot have $\beta_{i,j}(t)$ inside $(t_0, \boldsymbol{\beta}, \boldsymbol{g})$ for arbitrarily large t. This implies that $t_c < \infty$.

Now suppose that Condition (2) holds for $(t_c, \boldsymbol{\beta}, \boldsymbol{g})$. Again on each $(t, \beta_{i,j}, g_j, \beta_{j,k})$, ∇g_j never vanishes and always points to one side of the curve. Therefore, Condition (3) should also hold for all $t \in [t_0, t_c]$ by the continuity of ∇g_i's. Moreover, using the same argument as in Lemma 3.5.3, we have that Condition (2) holds for all $t \in [t_0, t_c]$ implies that Condition (4) also holds for all $t \in [t_0, t_c]$. Therefore, if $(t_c, \boldsymbol{\beta}, \boldsymbol{g})$ is a Jordan curve, $\boldsymbol{g}$ would induce a counterclockwise parametrization and it would thus imply that it is a regular loop. This would contradiction the definition of t_c.

In conclusion, Conditions (3) and (4) will not fall before non-adjacent arcs of $(t, \boldsymbol{\beta}, \boldsymbol{g})$ touch or some of the arcs become degenerate. □

Given a regular loop $(t_0, \boldsymbol{\beta}, \boldsymbol{g})$, we call the loop $(t_c, \boldsymbol{\beta}, \boldsymbol{g})$ a *critical loop* originated from $(t_0, \boldsymbol{\beta}, \boldsymbol{g})$. The important fact the enables us to explicitly build the electrostatic skeleton is as follows:

Proposition 3.5.5. *Under the strict descent condition, given a regular loop* $(t_0, \boldsymbol{\beta}, \boldsymbol{g})$*, the critical loop* $(t_c, \boldsymbol{\beta}, \boldsymbol{g})$ *is either a point or a union of regular loops intersecting each other on their vertices.*

To prove this proposition, we will need the following lemma to deal with the degenerate arcs in $(t_c, \boldsymbol{\beta}, \boldsymbol{g})$.

Lemma 3.5.6. *Under the strict descent condition, if we delete the degenerate arcs in $(t_c, \boldsymbol{\beta}, \boldsymbol{g})$ to form a new tuple $(t_c, \boldsymbol{\beta}_c, \boldsymbol{g}_c)$ (which represents the same loop), then $(t_c, \boldsymbol{\beta}_c, \boldsymbol{g}_c)$ satisfies Condition (4) in Definition 3.5.1.*

Proof. Let $\beta_0, \beta_1, \dots, \beta_K \in \boldsymbol{\beta}$ be such that $\beta_0(t), \beta_1(t), \dots, \beta_K(t)$ are vertices of $(t_c, \boldsymbol{\beta}, \boldsymbol{g})$ ordered counter-clockwise and

$$\beta_0(t_c) \neq \beta_1(t_c) = \beta_2(t_c) = \cdots = \beta_{K-1}(t_c) \neq \beta_K(t_c).$$

Then there exists $\delta > 0$ such that the lengths of the arc between $\beta_0(t)$ and $\beta_1(t)$ and the arc between $\beta_{K-1}(t)$ and $\beta_K(t)$ are lower-bounded by 2δ for $t \in [t_0, t_c]$. We let $\alpha_0(t) = \alpha_0(t, \delta)$ be the point on the arc between $\beta_0(t)$ and $\beta_1(t)$ that is δ away from $\beta_1(t)$ in arc length and set $\alpha_K(t) = \alpha_K(t, \delta)$ be the point between $\beta_{K-1}(t)$ and $\beta_K(t)$ that is δ away from $\beta_{K-1}(t)$ in arc length (see Figure 3.19). Then by convexity, we have

$$|\alpha_0(t) - \beta_1(t)| < \delta, \quad \text{and} \quad |\beta_{K-1}(t) - \alpha_K(t)| < \delta \quad \text{for all } t \in [t_0, t_c].$$

Now due to the orientation of the regular loops, for any $t \in [t_0, t_c)$, the curve segment $\mathcal{C}_t$ of $(t, \boldsymbol{\beta}, \boldsymbol{g})$ that goes from $\alpha_0(t)$ to $\alpha_K(t)$ lies on the left of the piecewise linear path $\mathcal{L}_t = \mathcal{L}_t(\delta)$ joining $\alpha_0(t), \beta_1(t), \dots, \beta_{K-1}(t)$ and $\alpha_K(t)$. It follows that since $\mathcal{C}_t$ is a simple curve, $\mathcal{L}_t$ cannot turn left in the angle of π or enclose a domain in a counter-clockwise manner. However, we know that as $\mathcal{L}_t$ evolves continuously over t,

1. the first and last linear segments of $\mathcal{L}_t$ have lower-bounded length, and

2. all other segments in between vanish.

Now we note that the inside of $(t_0, \boldsymbol{\beta}, \boldsymbol{g})$ is precompact, and that $\beta_{i,j}$'s are continuous and ∇g_i's are smooth. Therefore, by uniform continuity, there exist $\varepsilon_0, \delta_0 > 0$ such that for all $t \in [t_0, t_c]$,

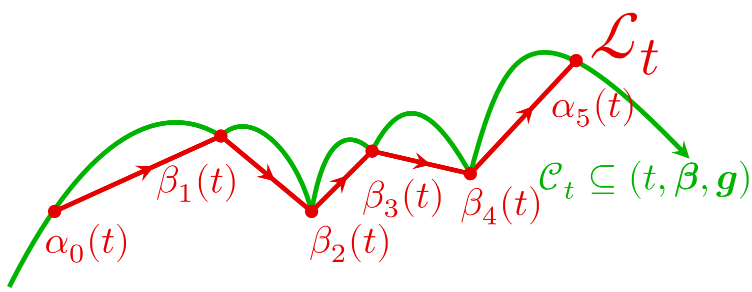

Figure 3.19: $\mathcal{L}_t$ is highlighted in red and $(t, \boldsymbol{\beta}, \boldsymbol{g})$ is highlighted in green

1. The angle any $J^{\beta}_{i,j}(t)$ makes toward $U^{\beta}_{i,j}(t)$ is upper bounded by $\pi - 2\varepsilon_0$;

2. For any $J^{\beta}_{i,j}(t)$ and $z_1, z_2 \in J^{\beta}_{i,j}(t)$ separated by $\beta_{i,j}(t)$, $|z_1 - z_2| \leq 2\delta_0$ implies that the angle of $[z_1, \beta_{i,j}(t)] \cup [\beta_{i,j}(t), z_2]$ is ε_0 away from the angle of $J^{\beta}_{i,j}(t)$. In particular, the angle of $[z_1, \beta_{i,j}(t)] \cup [\beta_{i,j}(t), z_2]$ would be upper bounded by $\pi - \varepsilon_0$.

Now we suppose t is sufficiently close to t_c such that $|\beta_k(t) - \beta_{k+1}(t)| < \delta_0$ for all $k = 1, \dots, K-2$. This implies that the sum of left turns for $\mathcal{L}_t$ from $\beta_2(t)$ to $\beta_{K-2}(t)$ is at least $(K-3)\varepsilon_0 \geq 0$. In particular, this means that the sum of left turns at $\beta_1(t)$ and $\beta_K(t)$ is upper bounded by $\pi - (K-3)\varepsilon_0$, otherwise $\mathcal{L}_t$ will enclose the domain counter-clockwise. Therefore, we can conclude that at $\beta_1(t_c) = \dots = \beta_{K-1}(t_c)$, $\mathcal{L}_{t_c}$ cannot make the left turn for more than $\pi - (K-3)\varepsilon_0$.

On the other hand, if we set δ in the definition of α_0 and α_K to be less than δ_0, then the left turns of $\mathcal{L}_t$ at $\beta_1(t)$ and $\beta_{K-1}(t)$ are also lower-bounded by $\varepsilon_0 > 0$ given t sufficiently close to t_c. As a result, the left turn of $\mathcal{L}_{t_c}$ at $\beta_1(t_c) = \dots = \beta_{K-1}(t_c)$ is at least $(K-1)\varepsilon_0$.

In conclusion, the angle $\mathcal{L}_{t_c}$ makes at $\beta_1(t_c) = \dots = \beta_{K-1}(t_c)$ (equal to π minus the left turn angle) is between $(K-3)\varepsilon_0$ and $\pi - (K-1)\varepsilon_0$. Thus the internal angle of $(t_c, \boldsymbol{\beta}, \boldsymbol{g})$, which is upper-bounded by the angle of $\mathcal{L}_{t_c}$, is less than $\pi - (K-1)\varepsilon_0$.

Thus, we rewrite $(t_c, \boldsymbol{\beta}, \boldsymbol{g})$ as $(t_c, \boldsymbol{\beta}_c, \boldsymbol{g}_c)$ by removing degenerate arcs. The new tuple $(t_c, \boldsymbol{\beta}_c, \boldsymbol{g}_c)$ would satisfy Condition (4). $\square$

Proof of Proposition 3.5.5. Assume that $(t_c, \boldsymbol{\beta}, \boldsymbol{g})$ is not a point. Note that since $\{(t, \boldsymbol{\beta}, \boldsymbol{g})\}_{t\in[t_0,t_c)}$ is a family of continuously shrinking Jordan curves, Carathéodory kernel theorem implies that $(t_c, \boldsymbol{\beta}, \boldsymbol{g})$ has a simply connected outside on $\hat{\mathbb{C}}$ and that its inside, which combined with the loop, is also simply connected. In particular, since ∇g_i's are continuous inside the polygon, ∇g_j should still point toward the outside of $(t_c, \boldsymbol{\beta}, \boldsymbol{g})$ on any non-degenerate $(t_c, \beta_{i,j}, g_j, \beta_{j,k})$ and $\boldsymbol{g}$ is still ordered in a counter-clockwise fashion. As a consequence, on each non-degenerate arc $(t_c, \beta_{i,j}, g_j, \beta_{j,k})$, going from $\beta_{i,j}(t_c)$ to $\beta_{j,k}(t_c)$, ∇g_j points to the right of the curve. Therefore, by Lemma 3.5.6, we can re-write $(t_c, \boldsymbol{\beta}, \boldsymbol{g})$ as a new loop $(t_c, \boldsymbol{\beta}_c, \boldsymbol{g}_c)$ in such a way that none of its arcs are degenerate and Conditions (4) holds. Then if the new loop is a Jordan curve, it has to be regular per Definition 3.5.1.

Now suppose that $(t_c, \boldsymbol{\beta}_c, \boldsymbol{g}_c)$ is not a Jordan curve. Carathéodory kernel theorem then implies that there exists a conformal map $f : \mathbb{D} \to \hat{\mathbb{C}}$ that is continuous on $\overline{\mathbb{D}}$ such that

$$f(\partial\mathbb{D}) = (t_c, \boldsymbol{\beta}_c, \boldsymbol{g}_c), \quad f(\mathbb{D}) = \bigcup_{t\in[t_0,t_c)} \text{exterior of } (t, \boldsymbol{\beta}, \boldsymbol{g}).$$

Then $\theta \mapsto f(e^{-i\theta})$ is a counter-clockwise parametrization of $(t_c, \boldsymbol{\beta}_c, \boldsymbol{g}_c)$. Since each arc of $(t_c, \boldsymbol{\beta}_c, \boldsymbol{g}_c)$ only intersects each other finitely many times, the equivalence relation induced by f on $\partial\mathbb{D}$ has only finitely many non-singleton equivalence classes $\{[z_1], [z_2], \dots, [z_n]\}$. This equivalance relation is non-crossing. This implies that $(t_c, \boldsymbol{\beta}_c, \boldsymbol{g}_c)$ along with its inside is simply connected and consists of finitely many Jordan domains intersecting each other only at $f([z_i])$. Note that given $t \in [t_0, t_c)$, the external angle of the loop $(t, \boldsymbol{\beta}, \boldsymbol{g})$ going counter-clockwise is π almost everywhere and always not less than π according to Lemma 3.4.4. By the continuity of $\beta_{i,j}$'s and ∇g_i's, this also holds when $t = t_c$. Therefore, at each $f([z_i]) \in (t_c, \boldsymbol{\beta}_c, \boldsymbol{g}_c)$ where the curve has multiple angles facing outside of $(t_c, \boldsymbol{\beta}_c, \boldsymbol{g}_c)$, the externals angle can only be π and thus $|[z_i]| = 2$. Therefore, if two Jordan curves in $(t_c, \boldsymbol{\beta}_c, \boldsymbol{g}_c)$ intersect at a, the internal angles of both at a should be 0 and the counter-clockwise curve parametrizing either of them will turn left in an angle of π.

In conclusion, for each Jordan curve in $(t_c, \boldsymbol{\beta}_c, \boldsymbol{g}_c)$,

1. when parametrized counter-clockwise, the curve turns left for less than π at $\beta_{i,j}(t_c)$

for $\beta_{i,j} \in \boldsymbol{\beta}_c$ and turns right everywhere else, and

2. at the point where the curve intersects another Jordan curve, it turns left in the angle of π.

Since each Jordan curve consists of segments of level sets $g_i^{-1}(-t_c)$'s, this implies that they are regular loops and that the points where they intersect are vertices. $\square$

From the proof of Proposition 3.5.5, we can also infer the following important property of the decomposition of a critical loop:

Corollary 3.5.7 (Non-crossing matching)**.** *Let $(t_c, \boldsymbol{\beta}_1, \boldsymbol{g}_1), \dots, (t_c, \boldsymbol{\beta}_l, \boldsymbol{g}_l)$ be the set of regular loops that make up the critical loop $(t_c, \boldsymbol{\beta}, \boldsymbol{g})$. Suppose each element of $\boldsymbol{g} \in \{g_1, \dots, g_n\}^m$ is distinct and we identify each element in $\boldsymbol{g}$ with a vertex in a convex m-gon P in a counter-clockwise manner. Then each tuple $\boldsymbol{g}_i$ represents a sub-polygon P_i inside P. Moreover, $P_1, \dots, P_l$ have disjoint interior (see Figure 3.20).*

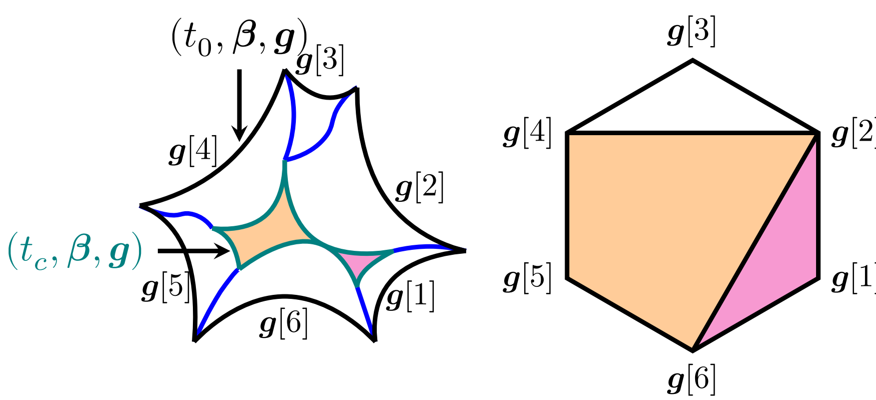

Figure 3.20: Identifying the phase transition with a partition of polygon. Note that $\boldsymbol{g}[i]$ stands for the i-component of $\boldsymbol{g}$ (not to be confused with g_i). Each regular loop in $(t_c, \boldsymbol{\beta}, \boldsymbol{g})$ corresponds to a sub-polygon in the partition (color matched).

Proof. Note that we have a conformal map $f : \mathbb{D} \to \hat{\mathbb{C}}$ continuous on $\overline{\mathbb{D}}$ such that

$$f(\partial\mathbb{D}) = (t_c, \boldsymbol{\beta}, \boldsymbol{g}), \quad f(\mathbb{D}) = \text{exterior of } (t_c, \boldsymbol{\beta}, \boldsymbol{g}),$$
$$|f^{-1}(z)| \in \{1, 2\}, \quad \text{for all } z \in (t_c, \boldsymbol{\beta}, \boldsymbol{g}).$$

Additionally, the equivalence relation on $\partial\mathbb{D}$ induced by f is non-crossing. Since each element in $\boldsymbol{g}$ is unique, we can denote their corresponding arcs as $C_1, \dots, C_m$ in order. Let $F : [0, 2\pi]$ be a reparametrization of $\theta \mapsto f(e^{-i\theta})$ such that

$$F(\left[\frac{2\pi(k-1)}{m}, \frac{2\pi k}{m}\right]) = C_k.$$

Note that since some of C_k's might be degenerate, this reparametrization is not reversible. That is, there exists a increasing function r such that

$$F(\theta) = f(e^{-ir(\theta)}),$$

but r might not be bijective. Nevertheless, since the orientation is preserved, the equivalence relation on $[0, 2\pi)$ induced by F is still non-crossing. Note that this means for $a_1, a_2, b_1, b_2 \in [0, 2\pi)$ if

$$a_1 \sim a_2, \quad b_1 \sim b_2, \quad a_1 < a_2, \quad b_1 < b_2, \quad a_1 \leq b_1$$

then $a_1 \leq b_1 < b_2 \leq a_2$. Now let $v : [0, 2\pi) \to [0, 2\pi)$ be a piecewise linear continuous function such that if $\theta \in [2\pi(k-1)/m, 2\pi k/m]$ is equivalent to some $\theta' \neq \theta$, we have

$$v(\theta) = 2\pi(k-1)/m.$$

Then non-crossing implies that v is increasing. Let $\approx$ be the equivalence relation on $[0, 2\pi)$ induced by $F \circ v$. Then $\approx$ is a non-crossing equivalence relation on $\{2\pi k/m\}_{k=0}^{m-1}$ such that $2\pi k_1/m \approx 2\pi k_2/m$ if and only if C_{k_1+1} and C_{k_2+1} are two non-degenerate arcs that intersect in their interiors. Now for $k_1 < k_2$ we also set $2\pi k_1/m \approx 2\pi k_2/m$, if C_k is degenerate for all $k = k_1 + 2, k_1 + 2, \dots, k_2$. Since none of the degenerate arcs can intersect other arcs in the interior, the equivalence relation is still non-crossing.

In summary, we have identified each C_k with $2\pi(k-1)/m$. Note that since C_k's follows the order of $\boldsymbol{g}$, we label the arcs in the counter-clockwise manner as k increases. For $k_1 < k_2$, C_{k_1} is related to C_{k_2} under $\approx$ if they are both non-degenerate arcs and either

1. C_{k_1} and C_{k_2} intersect in the interiors of both arcs, or

2. C_k is degenerate for all $k = k_1 + 1, \dots, k_2 - 1$.

On the other hand, as a non-crossing matching on $\left\{e^{2\pi k/m}\right\}_{k=0}^{m-1}$, $\approx$ partitions the regular m-gon $\mathrm{Conv}\{z|z^m = 1\}$ into sub-polygons. Then each regular loop in $(t_c, \boldsymbol{\beta}, \boldsymbol{g})$ can be identified with a unique sub-polygon under the partition. The result then follows. $\square$

Since a triangle does not admit a non-crossing matching, it follows that

Corollary 3.5.8. *For a regular loop $(t_0, \boldsymbol{\beta}, \boldsymbol{g})$ with $|\boldsymbol{g}| = 3$, $(t_c, \boldsymbol{\beta}, \boldsymbol{g})$ is a single point.*

3.5.3 Building the skeleton

Definition 3.5.9 (Loop measure)**.** Given a loop $(t, \boldsymbol{\beta}, \boldsymbol{g})$ inside a convex polygon Ω, the loop measure μ of $(t, \boldsymbol{\beta}, \boldsymbol{g})$ is the unique measure such that

$$\mu(U) = \sum_{(t,\beta_{i,j},g_j,\beta_{j,k})\subseteq(t,\boldsymbol{\beta},\boldsymbol{g})} \frac{1}{2\pi} \int_{(t,\beta_{i,j},g_j,\beta_{j,k})\cap U} |\nabla g_j(\zeta)||d\zeta|.$$

Note that since we are integrating along the level sets, we have

$$|\nabla g_j(\zeta)| = \left|\frac{\partial g_j}{\partial \mathbf{n}}(\zeta)\right|, \quad \zeta \in (t, \beta_{i,j}, g_j, \beta_{j,k}).$$

The equilibrium measure μ_E of the polygon is the loop measure of the regular loop

$$\partial\Omega = (0, (\gamma_{1,2}, \gamma_{2,3}, \dots, \gamma_{n-1,n}, \gamma_{1,n}), (g_2, g_3, \dots, g_n, g_1)).$$

Lemma 3.5.10. *Given a regular loop $(t_0, \boldsymbol{\beta}, \boldsymbol{g})$ with loop measure μ_0 and its corresponding critical loop $(t_c, \boldsymbol{\beta}, \boldsymbol{g})$ with loop measure μ_c, if we set λ to be a measure supported on the annulus between $(t_0, \boldsymbol{\beta}, \boldsymbol{g})$ and $(t_c, \boldsymbol{\beta}, \boldsymbol{g})$ such that*

$$\lambda(U) = \sum_{\beta_{i,j}\in\boldsymbol{\beta}} \int_{\beta_{i,j}([t_0,t_c])\cap U} |\nabla g_i(\zeta) - \nabla g_j(\zeta)||d\zeta|,$$

then U^{μ_0} and $U^{\mu_c+\lambda}$ coincide on the exterior of $(t_0, \boldsymbol{\beta}, \boldsymbol{g})$. That is, μ_0 is the balayage of $\mu_c + \lambda$ on $(t_0, \boldsymbol{\beta}, \boldsymbol{g})$.

Proof. Given an arc $(t_0, \beta_{i,j}, g_j, \beta_{j,k}) \subseteq (t_0, \boldsymbol{\beta}, \boldsymbol{g})$, we set $SQ(t_0, \beta_{i,j}, g_j, \beta_{j,k})$ (see Figure 3.21) to be the Jordan domain enclosed by

$$(t_0, \beta_{i,j}, g_j, \beta_{j,k}) \cup \beta_{i,j}([t_0, t_c]) \cup (t_c, \beta_{i,j}, g_j, \beta_{j,k}) \cup \beta_{j,k}([t_0, t_c]).$$

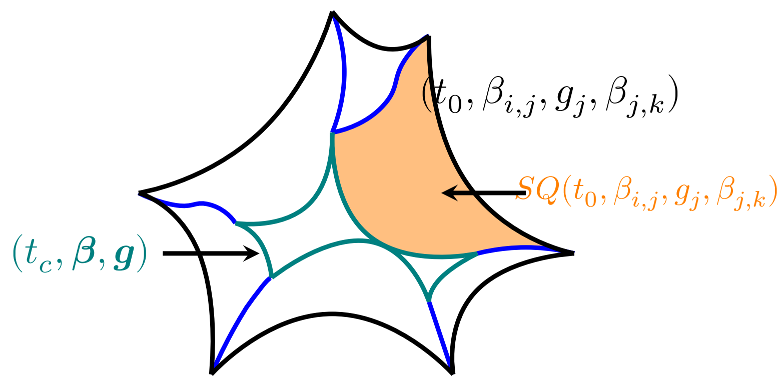

Figure 3.21: Given an arc arc $(t_0, \beta_{i,j}, g_j, \beta_{j,k})$ from a regular loop, the Jordan domain $SQ(t_0, \beta_{i,j}, g_j, \beta_{j,k})$ is the analytic square (or triangle) highlighted in orange

Then the function

$$u(z) = \begin{cases} -t_0, & z \text{ is outside } (t_0, \beta_{i,j}, g_j, \beta_{j,k}), \\ g_j(z), & z \in SQ(t_0, \boldsymbol{\beta}, \boldsymbol{g}), \\ -t_c, & z \text{ is inside } (t_c, \beta_{i,j}, g_j, \beta_{j,k}), \end{cases}$$

is continuous on $\mathbb{C}$ and harmonic away from

$$\bigcup_{(t_0, \beta_{i,j}, g_j, \beta_{j,k}) \subseteq (t_0, \boldsymbol{\beta}, \boldsymbol{g})} \partial SQ(t_0, \beta_{i,j}, g_j, \beta_{j,k}).$$

By the Green-Gauss formula, for any $f \in C^\infty(\mathbb{C})$, we have

$$\int_{\mathbb{C}} f \Delta u = \sum_{(t_0, \beta_{i,j}, g_j, \beta_{j,k}) \subseteq (t_0, \boldsymbol{\beta}, \boldsymbol{g})} \int_{\partial SQ(t_0, \beta_{i,j}, g_j, \beta_{j,k})} f(\zeta) \frac{\partial g_j}{\partial \mathbf{n}_{sq_j}}(\zeta) |d\zeta|, \tag{3.3}$$

where $\boldsymbol{n}_{sq_j}$ is the normal vector on $\partial SQ(t_0, \beta_{i,j}, g_j, \beta_{j,k})$ pointing outward. Now on $(t_0, \beta_{i,j}, g_j, \beta_{j,k})$, ∇g_j is the normal vector pointing toward outside of $SQ(t_0, \beta_{i,j}, g_j, \beta_{j,k})$ and on $(t_c, \beta_{i,j}, g_j, \beta_{j,k})$, ∇g_j points inward, thus each summand in (3.3) is equal to

$$\int_{(t_0,\beta_{i,j},g_j,\beta_{j,k})} f(\zeta)|\nabla g_j(\zeta)||d\zeta| - \int_{(t_c,\beta_{i,j},g_j,\beta_{j,k})} f(\zeta)|\nabla g_j(\zeta)||d\zeta|$$
$$+ \int_{\beta_{i,j}([t_0,t_c])} f(\zeta)\frac{\partial g_j}{\partial \mathbf{n}_{sq_j}}(\zeta)|d\zeta| + \int_{\beta_{j,k}([t_0,t_c])} f(\zeta)\frac{\partial g_j}{\partial \mathbf{n}_{sq_j}}(\zeta)|d\zeta|.$$

Note that we have

$$\beta_{i,j}'(t) \cdot (\nabla g_i(\beta_{i,j}(t)) - \nabla g_j(\beta_{i,j}(t))) = 0.$$

Thus the fact that the angle of $J_{i,j}^{\beta}(t)$ makes in the descending direction of $\beta_{i,j}$ is strictlt less than π implies that on $\beta_{i,j}([t_0, t_c])$,

$$\frac{\partial g_i}{\partial \mathbf{n}_{sq_i}}(\zeta) + \frac{\partial g_j}{\partial \mathbf{n}_{sq_j}}(\zeta) = -|\nabla g_i(\zeta) - \nabla g_j(\zeta)| \leq 0.$$

In summary, we have

$$\int_{\mathbb{C}} f\Delta u = \sum_{(t_0,\beta_{i,j},g_j,\beta_{j,k})\subseteq(t_0,\boldsymbol{\beta},\boldsymbol{g})} \int_{(t_0,\beta_{i,j},g_j,\beta_{j,k})} f(\zeta)|\nabla g_j(\zeta)||d\zeta|$$
$$- \sum_{(t_c,\beta_{i,j},g_j,\beta_{j,k})\subseteq(t_c,\boldsymbol{\beta},\boldsymbol{g})} \int_{(t_c,\beta_{i,j},g_j,\beta_{j,k})} f(\zeta)|\nabla g_j(\zeta)||d\zeta|$$
$$- \sum_{\beta_{i,j}\in\boldsymbol{\beta}} \int_{\beta_{i,j}([t_0,t_c])} f(\zeta)|\nabla g_i(\zeta) - \nabla g_j(\zeta)||d\zeta|.$$

It follows that we have

$$\Delta u = \mu_0 - \mu_c - \lambda.$$

Since u is constant outside $(t_0, \boldsymbol{\beta}, \boldsymbol{g})$, so is $U^{\Delta u}$. Thus $U^{\Delta u}$ vanishes outside $(t_0, \boldsymbol{\beta}, \boldsymbol{g})$ and the result then follows. □

Given a loop $(t, \boldsymbol{\beta}, \boldsymbol{g})$ either regular or critical, we denote the closure of the inside of the loop by $P(t, \boldsymbol{\beta}, \boldsymbol{g})$.

Lemma 3.5.11. *Given a regular loop* $(t_0, \boldsymbol{\beta}, \boldsymbol{g})$ *and its corresponding critical loop* $(t_c, \boldsymbol{\beta}, \boldsymbol{g})$*, there exists a retraction*

$$r : P(t_0, \boldsymbol{\beta}, \boldsymbol{g}) \to P(t_c, \boldsymbol{\beta}, \boldsymbol{g}) \cup \bigcup_{\beta_{i,j} \in \boldsymbol{\beta}} \beta_{i,j}([t_0, t_c]).$$

Proof. This follows from the fact that for each $(t_0, \beta_{i,j}, g_j, \beta_{j,k}) \subseteq (t_0, \boldsymbol{\beta}, \boldsymbol{g})$, we can construct a retraction

$$r_j : \overline{SQ(t_0, \beta_{i,j}, g_j, \beta_{j,k})} \to \beta_{i,j}([t_0, t_c]) \cup (t_c, \beta_{i,j}, g_j, \beta_{j,k}) \cup \beta_{j,k}([t_0, t_c]).$$

We can glue r_j's with the identity map on $P(t_c, \boldsymbol{\beta}, \boldsymbol{g})$ using the pasting lemma. $\square$

Corollary 3.5.12. *Suppose a measure* μ *and a compact set* K *inside the polygon satisfy that*

1. U^{μ} *and* U^{μ_E} *coincide outside of the polygon,*

2. $\operatorname{supp} \mu \subseteq K$,

3. K *is simply connected,*

4. *there exists a regular loop* $(t_0, \boldsymbol{\beta}, \boldsymbol{g})$ *inside* K *such that* $\mu|_{P(t_0, \boldsymbol{\beta}, \boldsymbol{g})}$ *is the loop measure of* $(t_0, \boldsymbol{\beta}, \boldsymbol{g})$,

5. *for such a regular loop, we have*

$$\overline{K \backslash P(t_0, \boldsymbol{\beta}, \boldsymbol{g})} \cap P(t_0, \boldsymbol{\beta}, \boldsymbol{g}) \subseteq \left\{\beta_{i,j}(t_0) : \beta_{i,j} \in \boldsymbol{\beta}\right\},$$

Then we can find a new measure μ' *and a compact subset* $K' \subseteq K$ *strict smaller in area such that*

1. μ' *and* K' *coincide with* μ *and* K *respectively on the exterior of* $(t_0, \boldsymbol{\beta}, \boldsymbol{g})$,

2. μ' *and* K' *satisfy the first three conditions above,*

3. the critical loop $(t_c, \boldsymbol{\beta}, \boldsymbol{g})$ *is inside* K' *and* $\mu'|_{P(t_c,\boldsymbol{\beta},\boldsymbol{g})}$ *is the loop measure of* $(t_c, \boldsymbol{\beta}, \boldsymbol{g})$*,*

4. we have

$$\overline{K' \backslash P(t_c, \boldsymbol{\beta}, \boldsymbol{g})} \cap P(t_c, \boldsymbol{\beta}, \boldsymbol{g}) \subseteq \left\{ \beta_{i,j}(t_c) : \beta_{i,j} \in \boldsymbol{\beta} \right\},$$

5. μ' *is equivalent to* $\mathcal{H}^1$ *on*

$$K' \cap (P(t_0, \boldsymbol{\beta}, \boldsymbol{g}) \backslash P(t_c, \boldsymbol{\beta}, \boldsymbol{g})),$$

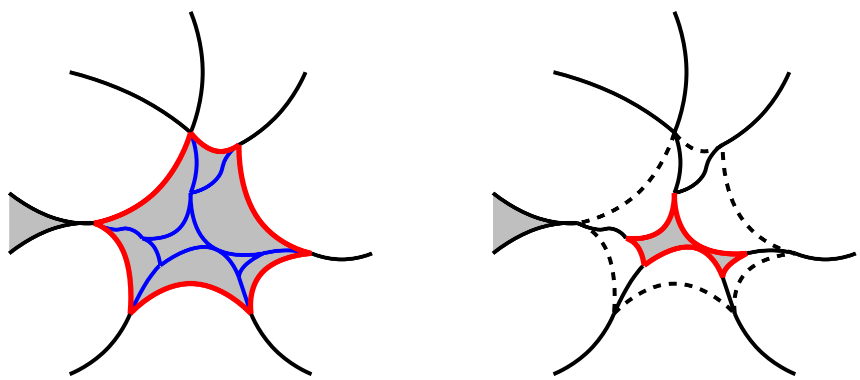

Figure 3.22: Replace (K, μ) with (K', μ') where the loop measures on $(t_0, \boldsymbol{\beta}, \boldsymbol{g})$ and $(t_c, \boldsymbol{\beta}, \boldsymbol{g})$ are highlighted in red

Proof. The idea here is illustrated by Figure 3.22. We replace μ with

$$\mu' = \mu|_{P(t_0,\boldsymbol{\beta},\boldsymbol{g})^C} + \mu_c + \lambda, \text{where } \mu_c \text{ and } \lambda \text{ are measures in Lemma 3.5.10},$$

and K with

$$K' = (K \backslash P(t_0, \boldsymbol{\beta}, \boldsymbol{g})) \cup P(t_c, \boldsymbol{\beta}, \boldsymbol{g}) \cup \bigcup_{\beta_{i,j} \in \boldsymbol{\beta}} \beta_{i,j}([t_0, t_c]).$$

Then μ' generates the same logarithmic potential as μ outside $(t_0, \boldsymbol{\beta}, \boldsymbol{g})$ due to Lemma 3.5.10. According to Lemma 3.5.11, we can construct a retraction from K to K', implying K' is simply connected. Moreover, the strict descent condition implies that the density of λ, which is $|\nabla g_i - \nabla g_j|$, is strictly positive on $\beta_{i,j}([t_0, t_c])$. Thus λ is equivalent to $\mathcal{H}^1$ on $\bigcup_{\beta_{i,j} \in \boldsymbol{\beta}} \beta_{i,j}([t_0, t_c])$. $\square$

3.5.4 Proof of Theorem 3.1.4

Given a convex polygon Ω, we note that μ_E is the loop measure of

$$\partial\Omega = (0, (\gamma_{1,2}, \gamma_{2,3}, \cdots, \gamma_{n-1,n}, \gamma_{1,n}), (g_2, g_3, \cdots, g_n, g_1)).$$

Thus the pair $(\mu_E, \overline{\Omega})$ satisfies the five conditions in Corollary 3.5.12, implying that we can replace the pair with a new one (μ', K') which contains the loop measure of the critical loop

$$(t_c, (\gamma_{1,2}, \gamma_{2,3}, \cdots, \gamma_{n-1,n}, \gamma_{1,n}), (g_2, g_3, \cdots, g_n, g_1)). \tag{3.4}$$

Note that if this critical loop is not a point, then it is a union of regular loops intersecting each other only at the vertices per Proposition 3.5.5. Moreover, the restriction of the loop measure on each regular loop is, according to Definition 3.5.9, the loop measure of the regular loop. Thus if the critical loop in (3.4) is not a point, (μ', K') again satisfies the five conditions in Corollary 3.5.12. It follows that we can repeatedly apply Corollary 3.5.12 to get a new pair (μ, K) where K is a strict subset of the previous one. Note that after each replacement, K is always a union of finitely many $P(t_0, \boldsymbol{\beta}, \boldsymbol{g})$'s and $\beta_{i,j}([t_0, t_c])$'s.

According to Corollary 3.5.7, each time we replace a regular loop $(t_0, \boldsymbol{\beta}, \boldsymbol{g})$ with its corresponding critical loop $(t_c, \boldsymbol{\beta}, \boldsymbol{g})$, each regular loop inside $(t_c, \boldsymbol{\beta}, \boldsymbol{g})$ (if any) corresponds to a sub-polygon of the polygon represented by $\boldsymbol{g}$. Since we cannot partition sub-polygons further than triangles, by Corollary 3.5.8 the process terminates after finitely many steps where all critical loops we arrive at are degenerate. After the process terminates, K will consist of solely analytic curves of the form $\beta_{i,j}([t_0, t_c])$. And the density of μ is strictly positive on each curve.

Now we bound the number of analytic curves on the resultant K. Note that when we rewrite a critical loop $(t_c, \boldsymbol{\beta}, \boldsymbol{g})$ as a union of regular loops $\{(t_c, \boldsymbol{\beta}_k, \boldsymbol{g}_k)\}_{k=1}^{K}$, $\beta_{i,j}$ is in $\boldsymbol{\beta}_k$ for some k but not in original $\boldsymbol{\beta}$ if and only if either

1. arcs corresponding to g_i and g_j in $(t_c, \boldsymbol{\beta}, \boldsymbol{g})$ intersect in their interiors, or

2. arcs corresponding to g_i and g_j in $(t_c, \boldsymbol{\beta}, \boldsymbol{g})$ intersect at their end points (arcs in between become degenerate).

Each analytic curve in K is either the image of one of $\gamma_{i,i+1}$'s or that of a new $\beta_{i,j}$ added in either case above. Moreover, in the first case, we are adding both a segment of $\gamma_{i,j}$ and of $\eta_{i,j}$ to K but their images are connected at $m_{i,j}$ and thus make up a singular analytic curve. Therefore, when shrinking $(t_0, \boldsymbol{\beta}, \boldsymbol{g})$ to $(t_c, \boldsymbol{\beta}, \boldsymbol{g})$, the number of analytic curves we add into K is upper bounded by the number of matching pairs in $\approx$ per Corollary 3.5.7. As a result, the total number of arcs added is bounded by the number of non-crossing matching pairs of a convex n-gon, which is $n-3$. Therefore, K contains at most $2n-3$ analytic curves.

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