Inverse Problems for Scalar Elliptic Equations and Systems

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Abstract

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In this thesis, we discuss inverse boundary value problems for scalar equations and for systems. First we introduce the famous Calderón problem and its recent developments. We focus on deriving the stability estimate of the conductivities from the partial Cauchy data. Second, we consider inverse boundary value problems for Stokes and Navier–Stokes equations and demonstrate the global uniqueness for viscosity in dimension two.

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GLOSSARY

Ω : a bounded open domain of $\mathbb{R}^n$, $n \geq 2$.

$\partial\Omega$: the boundary of Ω.

$H^S(\Omega)$: Sobolev space of order s over Ω.

S^{N-1} : the unit sphere in $\mathbb{R}^N$.

$\partial\Omega_{+,\varepsilon}$: For $\varepsilon > 0$, $\eta \in S^{n-1}$, $\partial\Omega_{+,\varepsilon} = \{x \in \partial\Omega : \ \eta \cdot \nu(x) > \varepsilon\}$ where $\nu(x)$ is the unit outer normal at $x \in \partial\Omega$.

$\partial\Omega_{-,\varepsilon}$: $\partial\Omega_{-,\varepsilon} = \partial\Omega \setminus \overline{\partial\Omega_{+,\varepsilon}}$.

$\lesssim$: there exists a positive constant for which the estimate holds whenever the right hand side of the estimate is multiplied by that constant.

CGO : complex geometrical optics.

DN MAP : Dirichlet-to-Neumann map.

IBVP : inverse boundary value problems.

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DEDICATION

to my parents
Tse-Hsien Lai and Chin-Yeh Ko

Chapter 1

INTRODUCTION

In 1980 A. P. Calderón published a short paper entitled "On an inverse boundary value problem" [9]. This pioneering contribution motivated many developments in inverse problems, in particular in the construction of "complex geometrical optics" (CGO) solutions of partial differential equations to solve inverse problems. The problem that Calderón considered was whether one can determine the electrical conductivity of a medium by making voltage and current measurements at the boundary of the medium. This inverse method is known as *Electrical Impedance Tomography* (EIT). EIT arises in geophysical prospection (see [57]) and also in medical imaging (See [23], [24] and [34]).

There are several directions in Calderón problem which are interesting in mathematics theory and applications such as uniqueness, stability, reconstruction, partial data, etc.. Uniqueness means that the mapping from the conductivity to DN map is injective. We call it stability if two DN maps are close and one can show that the corresponding conductivities of these two DN maps are also close in a suitable sense. Reconstruction is to find a procedure to reconstruct the conductivity. When we only consider boundary measures on a subset of $\partial\Omega$, it is the partial data problem.

The uniqueness issue for C^2 conductivities was first settled by Sylvester and Uhlmann [51]. In 2013, Haberman and Tataru [17] extended the uniqueness result to C^1 conductivities or small in the $W^{1,\infty}$ norm. In 2014, Haberman [16] improved their result to $W^{1,n}(\overline{\Omega}), n = 3, 4$. About two months later that Caro and Rogers [10] showed uniqueness holds in dimension $n \geq 3$ for Lipschitz conductivities.

As for the stability result with full data, it is well-known that the Calderón problem is severely ill-posed. A log-type stability estimate was derived by Alessandrini [1] and it has been shown in Mandache's paper [43] that this estimate is optimal. For the stability result with partial data, since less data was known on the boundary, one can expect that the stability estimate with partial data would be worse than that for the case of full data. Instead of getting log type stability estimate as in the full data case, we derive a log-log type stability estimate for less regular conductivities with partial data in Section 2.3. Moreover, we will discuss a new type of stability estimate which can demonstrate that stability will increase when we let the frequency grow. This phenomenon is called increasing stability behavior.

Another interesting inverse problem is to consider inverse boundary value problems (IBVP) for systems instead of a scalar equation. We will focus on IBVP for the Stokes and Navier–Stokes equations in Chapter 3. First we show the global uniqueness for viscosity for Stokes equations in dimension two. The key idea is to reduce IBVP for Stokes equations to IBVP for the first order system. Second, to study the Navier–Stokes equations, we apply the linearization method due to Isakov [29] and the result in Li-Wang [41] for three-dimensional Navier–Stokes equations to prove the uniqueness for viscosity for Navier–Stokes equations in dimension two.

This thesis is organized as follows. In Chapter 2, we will briefly introduce the Calderón Problem and its development over the past decades. In addition, we prove a new result showing that stability estimate with partial data holds when the conductivity has less regularity than previously known. We will also discuss a new aspect in inverse problems, increasing stability behavior, which has been numerically and rigorously demonstrated in different settings. In Chapter 3, we consider inverse boundary value problems for systems. More precisely, we are interested in inverse problems for Stokes equations and Navier–Stokes equations. We show the global identifiability for viscosity for both systems in dimension two.

Chapter 2

THE CALDERÓN PROBLEM

2.1 Introduction

The problem of determining the electrical conductivity of a medium by making voltage and current measurements at the boundary of the medium is known as Electrical Impedance Tomography (EIT). It has been recently proposed as a possible diagnostic tool in medical imaging. The mathematical formulation of this problem is due to Calderón's pioneering contribution [9] and is referred to as the *Calderón Problem* in the mathematical literature of inverse problems. Calderón's fundamental paper [9] on inverse problems has continued to have a crucial impact in the field and has motivated many developments.

We now describe the Calderón problem in more detail. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a bounded domain with smooth boundary. The electrical conductivity of Ω is represented by a bounded and positive function $\gamma \in C^2(\overline{\Omega})$. Given the voltage potential $f \in H^{1/2}(\partial\Omega)$ on the boundary, the equation for the potential $u \in H^1(\Omega)$ is given by

$$\operatorname{div}(\gamma \nabla u) = 0 \ \text{ in } \Omega; \qquad u|_{\partial\Omega} = f. \tag{2.1}$$

The Dirichlet-to-Neumann (DN) map, or voltage-to-current map, is defined by

$$\Lambda_\gamma f = \gamma \frac{\partial u}{\partial \nu}|_{\partial\Omega},$$

where ν is the unit outer normal to $\partial\Omega$. The Calderón problem is to determine the conductivity function γ from the knowledge of the boundary measurement Λ_γ. That is, if the measured current $\Lambda_\gamma f$ is known for all boundary voltages $f \in H^{1/2}(\partial\Omega)$, one would like to determine the conductivity γ. There are several aspects of this inverse problem which are interesting both for mathematical theory and practical applications such as uniqueness, stability, reconstruction and partial data. For Calderón problem, uniqueness means that the mapping from the conductivity to DN map is injective. If one can derive an estimate which demonstrates that when two DN maps are close, the corresponding conductivities of these two DN maps are also close in a suitable sense. We call it stability estimate. Reconstruction is to find a procedure to reconstruct the conductivity. When we only consider boundary measures on a subset of $\partial\Omega$, it is the partial data problem.

A crucial ingredient in the solution of Calderón's problem is the construction of complex geometrical optics (CGO) solutions. A breakthrough result was achieved by Sylvester and Uhlmann in [51] where they constructed CGO solutions of (2.1) for C^2 conductivities in dimensions $n \geq 2$. More precisely, CGO solutions of (2.1) have the form

$$u = \gamma^{-1/2} e^{x\cdot\rho}(1 + \psi(x)),$$

where $\psi \in L^2_\delta(\mathbb{R}^n)$, $-1 < \delta < 0$ and $\rho \in \mathbb{C}^n$ with $\rho \cdot \rho = 0$ and $|\rho|$ large. Moreover, $\psi \in H^s_\delta(\mathbb{R}^n)$ and for $0 \leq s \leq 2$, there exists $C = C(n, s, \delta) > 0$ such that

$$\|\psi\|_{H^s_\delta} \leq \frac{C}{|\rho|^{1-s}}.$$

Using the change of variable, $v = \gamma^{1/2}u$, one can also reduce the inverse problem with sufficiently regular conductivities to the inverse boundary value problem for the Schrödinger equation

$$(\Delta - q)v = 0 \ \text{ in } \Omega; \qquad v|_{\partial\Omega} = g,$$

where

$$q = \frac{\Delta(\gamma^{1/2})}{\gamma^{1/2}}.$$

The corresponding DN map is defined by

$$\Lambda_q(g) = \frac{\partial v}{\partial \nu}\Big|_{\partial\Omega}.$$

2.2 Uniqueness and Stability Results with Full Data

The determination of the conductivity from the DN map is known as a non-linear inverse problem. The first result was proven by Sylvester and Uhlmann in [51] where they applied CGO solutions to solve the uniqueness issue for C^2 conductivities in dimension $n \geq 3$. In addition, this result has led to further developments. A new result for uniqueness was given by Haberman and Tataru [17] for continuously differentiable conductivities. The main idea in [17] is to prove the remainder ψ of the CGO solutions has certain decay properties in Bourgain type spaces and the use of averaging technique. Recently, Haberman [16] improved their result to $W^{1,n}(\overline{\Omega}), n = 3, 4$. In two dimensions, the uniqueness for the Calderón problem was completely solved by Astala and Päivärinta for $L^\infty(\Omega)$ conductivities.

For the Schrödinger equation, Bukhgeim [7] showed that any $L^p(\Omega),\ p > 2$ potential in the two-dimensional case can be determined uniquely from the Cauchy data. A new type of CGO solutions was constructed in [7] by taking solutions of the form

$$u = e^{\tau z^2}(1 + O(\tau^{-1})),$$

where $\tau > 0$, $z = x_1 + ix_2$ so that the phase function has a non-degenerate critical point at zero. A inversion formula for potentials was also given in [7].

As for the stability result, it is well-known that the Calderón problem is severely ill-posed. A log-type stability estimate was derived by Alessandrini [1] and it has been shown in Mandache's paper [43] that this estimate is optimal. Since then, many people have worked on reducing the regularity of the conductivity. A recent result due to Caro, García and Reyes in [11] is for conductivities $\gamma \in C^{1,\varepsilon}(\overline{\Omega}), 0 < \varepsilon < 1$. They used Haberman and Tataru's ideas in [17] to derive the stability estimate in a Lipschitz domain.

2.2.1 *Increasing Stability Phenomena*

The study of increasing stability behavior is an important direction in inverse problems and has attracted also a lot of attention. The main reason is that the logarithmic stability estimate cannot be used to design a reliable reconstruction algorithms in practice since small errors in the data of the inverse problem result in large error in numerical reconstruction of physical properties of the medium. It has been observed numerically that the stability increases if one increases the frequency in some cases. Moreover, the papers ([25, 30, 31, 32, 33, 45, 48, 49]) rigorously demonstrated the increasing stability phenomena in different settings.

In the rest of this section, we briefly introduce two different methods of deriving estimates which can be viewed as an evidence of increasing stability behavior for Schrödinger and acoustic equation.

First, in [31], Isakov considered the Schrödinger equation

$$-\Delta u - k^2 u + cu = 0 \text{ in } \Omega$$

He proved that the stability of the $W^{1,\infty}(\Omega)$ potential coefficient c increases in the Schrödinger equation from boundary measurement when the frequency is growing.

Theorem 1 (Isakov [31]). *Assume* $\|c_j\|_{W^{1,\infty}} \leq M$ *for* $j = 1, 2$. *Let* $\varepsilon = \|\Lambda_{c_1} - \Lambda_{c_2}\|$, $E = -\log\varepsilon \geq 2$.

1. *There are constants* C_0, C_Ω *such that if*

$$k \leq \frac{E^2}{2} - \frac{E}{4},\ C_0^2 M < \Big(\frac{E^2}{2} - \frac{E}{4}\Big)^{1/2} + 2k^2 + 4,$$

then

$$\|c_1 - c_2\|_{L^2(\Omega)} \leq C_0 M^3 (E+k)^{-1/4} + \frac{C_0 M}{(E+k)^{1/2}} + C_\Omega E^2 (E^2 + M^2) \varepsilon^{1-2^{-1/2}}.$$

2. *There are constants* C_0, C_Ω *such that if*

$$E \leq k,\ C_0^2 M < k^2 + 2,$$

then

$$\|c_1 - c_2\|_{L^2(\Omega)} \leq C_0 M^3 (E+k)^{-1/4} + \frac{C_0(M^3 + M)}{(E+k)^{2/5}} + C_\Omega k(k + M^2)\varepsilon.$$

The idea is to use complex and real-valued geometrical optics solutions to deal with the low frequencies and high frequencies, respectively.

Second, Nagayasu, Uhlmann and Wang [45] considered the problem of stability for the refractive index in the space $H^s(\Omega), s > n/2+1$ in the acoustic equation, that is,

$$\Delta u + k^2 q(x) u = 0 \text{ in } \Omega.$$

The main result in [45] is the following theorem.

Theorem 2 (Nagayasu, Uhlmann and Wang [45])**.** *Let* $s > n/2+1$, $\|q_j\|_{H^s(\Omega)} \le M$ *and* $\operatorname{supp}(q_1 - q_2) \subset \Omega$*. Then there exists a constant* C_1*, depending on* n, s *and* Ω*, such that if* $k^2 \ge 1/(C_1 M)$ *and* $\varepsilon \le 1/e$*, then*

$$\|q_1 - q_2\|_{H^{-s}(\mathbb{R}^n)} \le Ck^2 e^{Ck^2}\varepsilon + C(k^2 + E)^{n-2s} \tag{2.2}$$

with $C > 0$ *depending on* n, s, Ω *and* M *and* $\operatorname{supp}(q_1 - q_2)$*.*

The stability estimate (2.2) was valid for all ranges of the frequency by using the complex geometrical optics (CGO) solutions constructed in [51]. Recently, Isakov, Nagayasu, Uhlmann and Wang [32] proved the increasing stability behavior in the Schrödinger equation by using similar computations in [45] and the CGO solutions in [18]. The novelty in [32] is the coefficient in front of ε grows only polynomially in k.

2.3 Uniqueness and Stability Results with Partial Data

The reason to study inverse problems with partial data is because one can measure currents and voltages only on part of the boundary in several applications in EIT. A general uniqueness result with partial data was first obtained by Bukhgeim and Uhlmann [8] for $C^2(\overline{\Omega})$ conductivities when the Neumann data were taken on part of $\partial\Omega$ which is slightly larger than the half of the boundary. Their result was improved in [36] where the Cauchy data can be taken on any part of the boundary. The regularity assumption on the conductivity was relaxed to $W^{3/2+r,2n}(\Omega)$ for some $r > 0$ by Knudsen in [37]. The latest result is proven by

Zhang in [56] where he showed the uniqueness result for $C^1(\overline{\Omega}) \cap H^{3/2}(\Omega)$ conductivities by using the idea in [17] and following the argument in [37].

For the stability result with partial data, since less data was known on the boundary, one can expect that the stability estimate with partial data would be worse than that for the case of full data. Instead of getting log type stability estimate in the full data case, Heck and Wang [21] obtained log-log type stability estimates. They improved their result to the log type stability in the paper [22] in 2007 by considering special domains. Using the idea in [17] to construct CGO solutions in Bourgain type spaces, allowed us to obtain a log-log type stability estimate in [38] for less regular conductivities. This result reduces the smoothness assumptions on the coefficients and on the boundary of the domain in Heck and Wang's paper [21].

2.3.1 Main Results

In the following subsections, we derive a log-log type stability estimate for less regular conductivities. To state the main result, we first introduce several notations. Picking a $\eta \in S^{n-1}$ and letting $\varepsilon > 0$, we define

$$\partial\Omega_{+,\varepsilon} = \{x \in \partial\Omega : \eta \cdot \nu(x) > \varepsilon\}, \quad \partial\Omega_{-,\varepsilon} = \partial\Omega \backslash \overline{\partial\Omega_{+,\varepsilon}}.$$

The local Dirichlet-to-Neumann map is given by

$$\tilde{\Lambda}_\gamma : f \mapsto \gamma \partial_\nu u|_{\partial\Omega_{-,\varepsilon}}.$$

So $\tilde{\Lambda}_\gamma$ is an operator from $H^{1/2}(\partial\Omega)$ to $\tilde{H}^{-1/2}(\partial\Omega_{-,\varepsilon})$, the restriction of $H^{-1/2}(\partial\Omega)$ onto $\partial\Omega_{-,\varepsilon}$. The operator norm of $\tilde{\Lambda}_\gamma$ is denoted by $\|\tilde{\Lambda}_\gamma\|_*$.

Theorem 3 (Lai [38]). *Let $\Omega \subset \mathbb{R}^n, n \geq 3$, be an open, bounded domain with C^2 boundary. Let $\gamma_j \in C^{1,\sigma}(\overline{\Omega}) \cap H^{\frac{3}{2}+\sigma}(\Omega)$ with $0 < \sigma < 1$ such that $\gamma_j > \gamma_0 > 0$ and*

$$\|\gamma_j\|_{C^{1,\sigma}(\overline{\Omega})} + \|\gamma_j\|_{H^{\frac{3}{2}+\sigma}(\Omega)} \leq M$$

for $j = 1, 2$ *and some constants* $\gamma_0,\ M > 0$. *Suppose that*

$$\gamma_1 = \gamma_2 \quad \textit{and} \quad \partial_\nu \gamma_1 = \partial_\nu \gamma_2 \quad \textit{on } \overline{\partial\Omega_{+,\varepsilon}}.$$

Then there exist constants $\theta, \tilde{\theta}, \tilde{\sigma} \in (0,1)$ *and constant* K *such that*

$$\|\gamma_1 - \gamma_2\|_{C^{0,\tilde{\sigma}}(\overline{\Omega})} \lesssim \left(\|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|_*^{\theta} + \|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|_*^{1-\theta} + \frac{1}{K} \log|\log \|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|_*^{\theta}|^{-\frac{1}{\tilde{\theta}}} \right)^{\frac{\tilde{\theta}(1-\tilde{\sigma})}{n}}. \tag{2.3}$$

Along our discussion we follow a recent improvement of the classical method introduced by Sylvester and Uhlmann in [51] and based on the construction of CGO solutions. This new improvement is due to Haberman and Tataru (see [17]) and it has allowed us to improve Heck and Wang's result in [21] relaxing the smoothness of the coefficients and the smoothness of the boundary of the domain. To deriving the estimate (2.3), we adapt Zhang's argument [56] to the case $\tilde{\Lambda}_{\gamma_1} \neq \tilde{\Lambda}_{\gamma_2}$. Then we will get an estimate of the Fourier transform of $q := (ik)\nabla v + \nabla(\log\sqrt{\gamma_1} + \log\sqrt{\gamma_2})\nabla v$ on some subset of $\mathbb{R}^n$ where $v = \log\sqrt{\gamma_1} - \log\sqrt{\gamma_2}$. Since q can be treated as a compactly supported function, its Fourier transform is real analytic. We use Vessella's stability estimate for analytic continuation [55] to our case here. This idea was first introduced in [21] to get the log-log type stability estimate with partial measurements.

2.3.2 Preliminary result

Let $n \geq 3$ and $\Omega \subset \mathbb{R}^n$ be an open bounded domain with C^2 boundary $\partial\Omega$ throughout the paper. Assume that $\gamma_j \in C^{1,\sigma}(\overline{\Omega}) \cap H^{\frac{3}{2}+\sigma}(\Omega)$ with $0 < \sigma < 1$ and $\gamma_j > \gamma_0 > 0$ for $j = 1, 2$. Let $\overline{\Omega} \subset B$. We can extend γ_j to be the function in $\mathbb{R}^n$ such that $\gamma_j \in C^{1,\sigma}(\mathbb{R}^n)$ with positive lower bound and $\gamma_j - 1 \in H^{\frac{3}{2}+\sigma}(\mathbb{R}^n)$ with $\text{supp}(\gamma_j - 1) \subset \overline{B}$.

Let $\Psi_t = t^n \Psi(tx)$ where $\Psi \in C_0^\infty(\mathbb{R}^n)$ supported on the unit ball and $\int \Psi = 1$. Denote that $\phi = \log\gamma$ and $A = \nabla \log\gamma$. Define $\phi_t = \Psi_t * \phi$ and $A_t = \Psi_t * A$. Then the following results are from [37] and [47].

Lemma 1. *Let* $\gamma \in C^{1,\sigma}(\mathbb{R}^n)$ *for* $0 \le \sigma \le 1$ *and* $\gamma - 1 \in H^{\frac{3}{2}+\sigma}(\mathbb{R}^n)$ *with compact support. Then*

$$\begin{aligned} &\|\nabla \cdot A_t\|_{L^\infty(\mathbb{R}^n)} \le Ct^{1-\sigma}, \\ &\|\phi_t - \phi\|_{L^\infty(\mathbb{R}^n)} \le Ct^{-1-\sigma}, \\ &\|A_t - A\|_{L^\infty(\mathbb{R}^n)} \le Ct^{-\sigma}, \end{aligned}$$

and

$$\begin{aligned} &\|\nabla \cdot A_t\|_{L^2(\mathbb{R}^n)} \le Ct^{\frac{1}{2}-\sigma}, \\ &\|\phi_t - \phi\|_{L^2(\mathbb{R}^n)} \le Ct^{-\frac{3}{2}-\sigma}, \\ &\|A_t - A\|_{L^2(\mathbb{R}^n)} \le Ct^{-\frac{1}{2}-\sigma}. \end{aligned}$$

The following lemma is taken from [56].

Lemma 2 (Zhang [56])**.** *Let* $\Omega \subset \mathbb{R}^n, n \ge 2$*, be a bounded domain with* C^2 *boundary and* $u \in H^1(\Omega)$*. Then there exists a constant* C *such that*

$$\int_{\partial\Omega} u^2 dS \le C \left\{ \left(\int_\Omega u^2 dx \right)^{1/2} \left(\int_\Omega |\nabla u|^2 dx \right)^{1/2} + \int_\Omega u^2 dx \right\}.$$

We will need the stable determination of the conductivity at points on the boundary of Ω. Since the stability estimate derived in [2] is local, the same estimates hold for the local Dirichlet-to-Neumann map. This result can be proved by the same arguments as in [2].

Theorem 4. *Let* $\gamma_j \in C^{1,\sigma}(\overline{\Omega})$ *satisfy* $\gamma_j > \gamma_0 > 0$ *for* $j = 1, 2$*. Then*

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\partial\Omega)} \lesssim \|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|_* \tag{2.4}$$

and

$$\sum_{|\alpha|=1} \|\partial^\alpha \gamma_1 - \partial^\alpha \gamma_2\|_{L^\infty(\partial\Omega)} \lesssim \|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|_*^\theta \tag{2.5}$$

for some $0 < \theta < 1$ *depending only on* σ*. Here the implicit constants depend on* $n, \Omega, \sigma, \gamma_0$ *and* $\|\gamma_j\|_{C^{1,\sigma}(\overline{\Omega})}$ *for* $j = 1, 2$*.*

We will use the following theorem to obtain the stability estimate on a large ball $B(0,R)$ by controlling an open subset of $B(0,R)$. This idea was introduced in [21].

Proposition 1 (Vessella [55])**.** *Let* $\tau_0, d_0 > 0$*. Let* $D \subset \mathbb{R}^n$ *be an open, bounded and connected set such that* $\{x \in D : d(x, \partial D) > \tau\}$ *is connected for any* $\tau \in [0, \tau_0]$*. Let* $E \subset D$ *be an open set such that* $d(E, \partial D) \geq d_0$*. If* f *is an analytic function with*

$$\|\partial^\alpha f\|_{L^\infty(D)} \leq \frac{M\alpha!}{\rho^{|\alpha|}}, \quad \textit{for all } \alpha \in \mathbb{N}^n$$

for some $M, \rho > 0$*, then*

$$|f(x)| \leq (2M)^{1-\tilde{\theta}(|E|/|D|)} (\|f\|_{L^\infty(E)})^{\tilde{\theta}(|E|/|D|)},$$

where $\tilde{\theta} \in (0,1)$ *depends on* $d_0, \operatorname{diam} D, \tau_0, n, \rho$ *and* $d(x, \partial D)$*.*

2.3.3 Complex geometrical optics solutions

In this section, we will review the construction of CGO solutions for the conductivity equation following the arguments presented in [56], but with the conductivity in $C^{1,\sigma}(\overline{\Omega}) \cap H^{\frac{3}{2}+\sigma}(\Omega), 0 < \sigma < 1$. Note that the regularity assumption $H^{\frac{3}{2}+\sigma}(\Omega)$ is used to control the $H^{1/2}$ norm of the conductivities on the boundary. The detailed discussion will be presented in Section 4.

First, we introduce the spaces $\dot{X}^b_\zeta$ and X^b_ζ which are defined by the norm

$$\|u\|_{\dot{X}^b_\zeta} = \||p_\zeta(\xi)|^b \hat{u}(\xi)\|_{L^2}$$

and

$$\|u\|_{X^b_\zeta} = \|(|\zeta| + |p_\zeta(\xi)|)^b \hat{u}(\xi)\|_{L^2},$$

respectively. Here $p_\zeta(\xi) = -|\xi|^2 + 2i\zeta \cdot \xi$ is the symbol of $\Delta + 2\zeta \cdot \nabla$.

Let Ω be an open bounded domain in $\mathbb{R}^n, n \geq 3$ with C^2 boundary. Let $\gamma \in C^{1,\sigma}(\overline{\Omega})$ and let u be the solution of $\nabla \cdot \gamma \nabla u = 0$ in Ω. Then u satisfies

$$(-\Delta - A \cdot \nabla)\, u = 0 \ \text{ in } \Omega, \tag{2.6}$$

where $A = \nabla \log \gamma \in C^{0,\sigma}(\overline{\Omega})$. Suppose that the CGO solutions of (2.6) are of the form

$$u = e^{-\frac{\phi_t}{2}} e^{x\cdot\zeta} \left(1 + w(x,\zeta)\right),$$

with $\phi_t = \Psi_t * \phi$ and $\zeta \in \mathbb{C}^n,\ \zeta\cdot\zeta = 0$. Here we denote $\phi = \log\gamma$. Then the function w satisfies the following equation

$$(-\Delta + (A_t - A)\cdot\nabla + q_t)\left(e^{x\cdot\zeta}(1+w)\right) = 0, \tag{2.7}$$

where $q_t = \frac{1}{2}\nabla\cdot A_t - \frac{1}{4}(A_t)^2 + \frac{1}{2}A\cdot A_t$. Equivalently, w is the solution of

$$(-\Delta_\zeta + (A_t - A)\cdot\nabla_\zeta + q_t)\, w = (A - A_t)\cdot\zeta - q_t, \tag{2.8}$$

where $-\Delta_\zeta = \Delta + 2\zeta\cdot\nabla$ and $\nabla_\zeta = \nabla + \zeta$.

We let $\eta \in S^{n-1}$. Fix $k \in \mathbb{R}^n$ satisfying $\eta\cdot k = 0$. Let $\eta_1 \in S^{n-1}$ such that $k\cdot\eta_1 = \eta\cdot\eta_1 = 0$. We choose $\zeta_1 = -s\eta - i\left(\frac{k}{2} - r\eta_1\right)$ and $\zeta_2 = s\eta - i\left(\frac{k}{2} + r\eta_1\right)$ such that $|k|^2/4 + r^2 = s^2$, $\zeta_i\cdot\zeta_i = 0$ and $\zeta_1 + \zeta_2 = -ik$.

The following lemma lists some inequalities between the norms in ordinary Sobolev spaces and the spaces X^b_ζ. The inequalities in this lemma are taken from Lemma 2.2 in [17] and Lemma 3.3 in [56].

Lemma 3. *Let Φ_B be a fixed Schwartz function and write $u_B = \Phi_B u$. Then the following estimates hold:*

$$\|u_B\|_{L^2(\mathbb{R}^n)} \lesssim s^{-1/2}\|u\|_{\dot X^{1/2}_\zeta}; \quad \|u_B\|_{H^{1/2}(\mathbb{R}^n)} \lesssim \|u\|_{\dot X^{1/2}_\zeta};$$
$$\|u_B\|_{H^1(\mathbb{R}^n)} \lesssim s^{1/2}\|u\|_{\dot X^{1/2}_\zeta}; \qquad \|u\|_{X^{-1/2}_\zeta} \lesssim s^{-1/2}\|u\|_{L^2(\mathbb{R}^n)}.$$

The following result is contained in Lemma 3.4 and 3.5 in [56].

Theorem 5 (Zhang [56])**.** *Let $\gamma_i \in C^1(\mathbb{R}^n)$ with $\gamma_i > \gamma_0 > 0$ and $\gamma_i = 1$ outside a ball. Then for any fixed $k \in \mathbb{R}^n$, there exists a sequence $\zeta_i^{(n)}$ with $|\zeta_i^{(n)}| = \sqrt{2}s_n$ such that*

$$\|w_i^{(n)}\|_{\dot X^{1/2}_{\zeta_i^{(n)}}} \lesssim \|\,(A_{is_n} - A_i)\cdot\zeta_i^{(n)} + q_{s_n}\|_{\dot X^{-1/2}_{\zeta_i^{(n)}}} \to 0 \quad \text{as } s_n \to \infty. \tag{2.9}$$

Moreover,

$$\|w_i^{(n)}\|_{L^2(\Omega)} \lesssim s_n^{-1/2}\|w_i^{(n)}\|_{\dot{X}^{1/2}_{\zeta_i^{(n)}}}; \quad \|w_i^{(n)}\|_{H^1(\Omega)} \lesssim s_n^{1/2}\|w_i^{(n)}\|_{\dot{X}^{1/2}_{\zeta_i^{(n)}}}$$

and

$$\|w_i^{(n)}\|_{H^{1/2}(\Omega)} \lesssim \|w_i^{(n)}\|_{\dot{X}^{1/2}_{\zeta_i^{(n)}}}; \quad \|w_i^{(n)}\|_{H^2(\Omega)} \lesssim s_n^{3/2}\|w_i^{(n)}\|_{\dot{X}^{1/2}_{\zeta_i^{(n)}}},$$

where $w_i^{(n)}$ *is a solution of (2.8) with* $t = s_n$ *and* $A_i = \nabla\phi_i = \nabla \log \gamma_i$ *for* $i = 1, 2$.

From Theorem 5, we take the CGO solutions

$$u_1^{(n)} = e^{-\frac{\phi_{1s_n}}{2}} e^{x\cdot\zeta_1^{(n)}} \left(1 + w_1^{(n)}\right)$$

and

$$u_2^{(n)} = e^{-\frac{\phi_{2s_n}}{2}} e^{x\cdot\zeta_2^{(n)}} \left(1 + w_2^{(n)}\right).$$

The CGO solutions can also be written as

$$u_i^{(n)} = e^{-\frac{\phi_{is_n}}{2}} e^{x\cdot\zeta_i^{(n)}} \left(1 + w_i^{(n)}\right) = \gamma_i^{-1/2} e^{x\cdot\zeta_i^{(n)}} \left(1 + \psi_i^{(n)}\right) \tag{2.10}$$

for $i = 1, 2$. Here $\psi_i^{(n)} = \sqrt{\gamma_i}\left(e^{-\frac{\phi_{is_n}}{2}} - \gamma_i^{-1/2}\right) + \sqrt{\gamma_i}e^{-\frac{\phi_{is_n}}{2}} w_i^{(n)}$. For simplicity, we will not write the superscripts (n) and the subscripts of s_n unless otherwise particularly specified.

Note that by lemma 1 and Theorem 5, we have

$$\|\psi_i\|_{L^2(\Omega)} \lesssim s^{-1-\sigma} + s^{-1/2}\|w_i\|_{\dot{X}^{1/2}_{\zeta_i}}. \tag{2.11}$$

Lemma 4 (Lai [38]). *For* $0 < \sigma < 1$, *if* λ *is sufficiently large we have*

$$\frac{1}{\lambda}\int_{S^{n-1}}\int_{\lambda}^{2\lambda} \|(A_s - A)\cdot\zeta + q_s\|^2_{\dot{X}^{-1/2}_{\zeta}} ds d\eta \lesssim \lambda^{-2\sigma} + \lambda^{-1}. \tag{2.12}$$

Proof. Let Φ be a cut-off function on the support of A_s and A. Then, by Lemma 2.2 in [17] and Lemma 3, we have

$$\|(A_s)^2\|^2_{\dot{X}^{-1/2}_{\zeta}} = \|\Phi(A_s)^2\|^2_{\dot{X}^{-1/2}_{\zeta}} \lesssim \|(A_s)^2\|^2_{X^{-1/2}_{\zeta}} \lesssim s^{-1},$$

$$\|A\cdot A_s\|^2_{\dot{X}^{-1/2}_{\zeta}} = \|\Phi(A\cdot A_s)\|^2_{\dot{X}^{-1/2}_{\zeta}} \lesssim \|A\cdot A_s\|^2_{X^{-1/2}_{\zeta}} \lesssim s^{-1}.$$

Observing that $|(\nabla\cdot A_s)\hat{(}\xi)| = |\xi\cdot\hat{A}_s| = |\xi\cdot\hat{\Psi}(\frac{\xi}{s})\hat{A}(\xi)| \leq \|\hat{\Psi}(\frac{\xi}{s})\|_{L^\infty(\mathbb{R}^n)}|\xi\cdot\hat{A}| \lesssim |(\nabla\cdot A)\hat{(}\xi)|$. Then $\|\nabla\cdot A_s\|^2_{\dot{X}_\zeta^{-1/2}} \lesssim \|\nabla\cdot A\|^2_{\dot{X}_\zeta^{-1/2}}$. Let $h=\sqrt{\lambda}$ and $\Psi_h = h^n\Psi(hx)$ as in Lemma 1, we have

$$\begin{aligned}\frac{1}{\lambda}\int_{S^{n-1}}\int_{\lambda}^{2\lambda}\|\nabla\cdot A_s\|^2_{\dot{X}_\zeta^{-1/2}}dsd\eta &\lesssim \frac{1}{\lambda}\int_{S^{n-1}}\int_{\lambda}^{2\lambda}\|\nabla\cdot A\|^2_{\dot{X}_\zeta^{-1/2}}dsd\eta\\ &\lesssim \frac{1}{\lambda}\int_{S^{n-1}}\int_{\lambda}^{2\lambda}\|\nabla\cdot(\Psi_h * A)\|^2_{\dot{X}_\zeta^{-1/2}}dsd\eta\\ &\quad+\frac{1}{\lambda}\int_{S^{n-1}}\int_{\lambda}^{2\lambda}\|\nabla\cdot(\Psi_h * A - A)\|^2_{\dot{X}_\zeta^{-1/2}}dsd\eta. \end{aligned}\tag{2.13}$$

Using Lemma 3.1 in [17] and Lemma 1, (2.13) follows that

$$\begin{aligned}\frac{1}{\lambda}\int_{S^{n-1}}\int_{\lambda}^{2\lambda}\|\nabla\cdot A_s\|^2_{\dot{X}_\zeta^{-1/2}}dsd\eta &\lesssim \frac{1}{\lambda}\|\nabla\cdot(\Psi_h * A)\|^2_{L^2(\mathbb{R}^n)} + \|\Psi_h * A - A\|^2_{L^2(\mathbb{R}^n)}\\ &\lesssim \frac{1}{\lambda}h^{1-2\sigma} + h^{-1-2\sigma} \lesssim \lambda^{-\frac{1}{2}-\sigma}.\end{aligned}\tag{2.14}$$

By the definition of q_s, we can deduce that

$$\begin{aligned}&\frac{1}{\lambda}\int_{S^{n-1}}\int_{\lambda}^{2\lambda}\|q_s\|^2_{\dot{X}_\zeta^{-1/2}}dsd\eta\\ &\lesssim \frac{1}{\lambda}\int_{S^{n-1}}\int_{\lambda}^{2\lambda}\|\nabla\cdot A_s\|^2_{\dot{X}_\zeta^{-1/2}} + \|(A_s)^2\|^2_{\dot{X}_\zeta^{-1/2}} + \|A\cdot A_s\|^2_{\dot{X}_\zeta^{-1/2}}dsd\eta\\ &\lesssim \lambda^{-\frac{1}{2}-\sigma} + \lambda^{-1}.\end{aligned}\tag{2.15}$$

Applying Lemma 2.2 in [17] and Lemma 3, we get

$$\|(A_s - A)\cdot\zeta\|^2_{\dot{X}_\zeta^{-1/2}} \lesssim s^2\|\Phi(A_s-A)\|^2_{\dot{X}_\zeta^{-1/2}} \lesssim s^2\|A_s - A\|^2_{X_\zeta^{-1/2}} \lesssim s\|A_s - A\|^2_{L^2(\mathbb{R}^n)}.$$

Thus we derive

$$\frac{1}{\lambda}\int_{S^{n-1}}\int_{\lambda}^{2\lambda}\|(A_s-A)\cdot\zeta\|^2_{\dot{X}_\zeta^{-1/2}}dsd\eta \lesssim \lambda^{-2\sigma}\tag{2.16}$$

from Lemma 1. The proof is completed. □

Note that $\|w\|^2_{\dot{X}_\zeta^{1/2}} \lesssim \|(A_s - A)\cdot\zeta + q_s\|^2_{\dot{X}_\zeta^{-1/2}}$. By lemma 4, we obtain the following estimate

$$\frac{1}{\lambda}\int_{S^{n-1}}\int_{\lambda}^{2\lambda}\|w\|^2_{\dot{X}_\zeta^{1/2}}dsd\eta \lesssim \lambda^{-2\sigma} + \lambda^{-1}.\tag{2.17}$$

The following Carleman estimate is deduced by Zhang by using the Carleman estimate in the paper [37].

Theorem 6 (Zhang [56]). *Let* $\eta \in S^{n-1}$ *and* $u \in H^2(\Omega)$. *Suppose that* $\gamma \in C^1(\Omega)$. *Then there exists a constant* $s_0 > 0$ *such that for* $s \geq s_0$, *we have*

$$\begin{aligned} & C\left(s^2\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}\right) - C_1 s^2 \int_{\partial\Omega} |u|^2 dS \\ & - C_2 \int_{\partial\Omega} \overline{u}\partial_\nu u dS + \int_{\partial\Omega} 4s\Re(\partial_\nu u \partial_\eta \overline{u}) - 2s(\nu\cdot\eta)|\nabla u|^2 + 2s^3(\nu\cdot\eta)|u|^2 dS \\ & \leq \|e^{-x\cdot s\eta}\left(-\Delta + (A_s - A)\cdot\nabla + q_s\right)\left(e^{x\cdot s\eta}u\right)\|^2_{L^2(\Omega)}. \end{aligned} \tag{2.18}$$

We also need the following result.

Proposition 2 (Knudsen [37]). *Suppose* $\gamma_j \in C^1(\overline{\Omega})$ *and* $u_j \in H^1(\Omega)$ *satisfy* $\nabla\cdot\gamma_j\nabla u_j = 0$ *in* Ω *for* $j = 1, 2$. *Suppose that* $\tilde{u}_1 \in H^1(\Omega)$ *satisfies* $\nabla\cdot\gamma_1\nabla\tilde{u}_1 = 0$ *with* $\tilde{u}_1 = u_2$ *on* $\partial\Omega$. *Then*

$$\begin{aligned} & \int_\Omega \left(\sqrt{\gamma_1}\nabla\sqrt{\gamma_2} - \sqrt{\gamma_2}\nabla\sqrt{\gamma_1}\right)\cdot\nabla\left(u_1u_2\right)dx \\ & = \int_{\partial\Omega} \gamma_1\partial_\nu\left(\tilde{u}_1 - u_2\right)u_1 dS + \int_{\partial\Omega}(\gamma_1 - \sqrt{\gamma_1\gamma_2})(u_1\partial_\nu u_2 - u_2\partial_\nu u_1)dS, \end{aligned} \tag{2.19}$$

where the integral is understood in the sense of the dual pairing between $H^{1/2}(\partial\Omega)$ *and* $H^{-1/2}(\partial\Omega)$.

Note that this proposition is slightly different from the Lemma 4.1 in [37] due to different assumptions on $\gamma|_{\partial\Omega}$. In [37], they have $\gamma_1 = \gamma_2$ on $\partial\Omega$, so the second term on the right hand side of (2.19) vanishes.

Using Theorem 4 and the trace theorem, we get

$$\begin{aligned} \left|\int_{\partial\Omega_{-,\varepsilon}} ((\gamma_1 - \gamma_2)\partial_\nu u_2)u_1 dS\right|^2 & \lesssim \|\gamma_1 - \gamma_2\|^2_{L^\infty(\partial\Omega)}\|\nabla u_2\|^2_{H^1(\Omega)}\|u_1\|^2_{H^1(\Omega)} \\ & \lesssim \|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|^2_*\|u_2\|^2_{H^2(\Omega)}\|u_1\|^2_{H^1(\Omega)}. \end{aligned} \tag{2.20}$$

Note that since $\gamma_2 \in C^{1,\sigma}$, the elliptic regularity theorem implies that $u_2 \in H^2(\Omega)$. By using the equality that

$$\gamma_1 \partial_\nu (\tilde{u}_1 - u_2) u_1 = (\gamma_1 \partial_\nu \tilde{u}_1 - \gamma_2 \partial_\nu u_2) u_1 + ((\gamma_1 - \gamma_2)\partial_\nu u_2) u_1$$

and (2.20), we have

$$\left| \int_{\partial\Omega_{-,\varepsilon}} \gamma_1 \partial_\nu (\tilde{u}_1 - u_2) u_1 dS \right|^2 \lesssim \|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|_*^2 \|u_2\|_{H^2(\Omega)}^2 \|u_1\|_{H^1(\Omega)}^2. \tag{2.21}$$

Proposition 2 and (3.42) imply that

$$\begin{aligned} &\left| \int_\Omega \left(\sqrt{\gamma_1}\nabla\sqrt{\gamma_2} - \sqrt{\gamma_2}\nabla\sqrt{\gamma_1}\right) \cdot \nabla\left(u_1 u_2\right) dx \right|^2 \\ &\lesssim \left| \int_{\partial\Omega_{+,\varepsilon}} \gamma_1 \partial_\nu \left(\tilde{u}_1 - u_2\right) u_1 dS \right|^2 + \|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|_*^2 \|u_2\|_{H^2(\Omega)}^2 \|u_1\|_{H^2(\Omega)}^2. \end{aligned} \tag{2.22}$$

In the remaining part of this section, we will estimate the first term on the right hand side of (2.22). Denote $u_0 = e^{\frac{\phi_{1s}}{2}}\left(\tilde{u}_1 - u_2\right)$ and $\delta u = \left(e^{\frac{\phi_{1s}}{2}} - e^{\frac{\phi_{2s}}{2}}\right) u_2$. Let $u = u_0 + \delta u$. Observing that

$$\begin{aligned} \left| \int_{\partial\Omega_{+,\varepsilon}} \gamma_1 \partial_\nu (\tilde{u}_1 - u_2) u_1 dS \right|^2 &\lesssim \|1 + w_1\|_{L^2(\partial\Omega_{+,\varepsilon})}^2 \int_{\partial\Omega_{+,\varepsilon}} e^{-2x\cdot s\eta} |\partial_\nu (\tilde{u}_1 - u_2)|^2 dS \\ &\lesssim \left(\int_{\partial\Omega_{+,\varepsilon}} e^{-2x\cdot s\eta} |\partial_\nu u|^2 dS + \int_{\partial\Omega_{+,\varepsilon}} e^{-2x\cdot s\eta} |\partial_\nu \delta u|^2 dS \right), \end{aligned} \tag{2.23}$$

here we use the face that if s is large, $\|w_1\|_{\dot{X}_\zeta^{1/2}}^2$ is small compared to 1 according to Theorem 5. Thus

$$\|1 + w_1\|_{L^2(\partial\Omega_{+,\varepsilon})}^2 \lesssim 1 + \|w_1\|_{\dot{X}_\zeta^{1/2}}^2 \lesssim 1$$

by applying Lemma 2.

Lemma 5 (Lai [38])**.** *Let $\Omega \subset \mathbb{R}^n, n \geq 3$, be an open and bounded domain with C^2 boundary. For $i = 1, 2$, let $\gamma_i \in C^{1,\sigma}(\overline{\Omega}) \cap H^{\frac{3}{2}+\sigma}(\Omega)$ be a real-valued function and $\gamma_i > \gamma_0 > 0$. Suppose* *that* $\gamma_1|_{\partial\Omega_{+,\varepsilon}} = \gamma_2|_{\partial\Omega_{+,\varepsilon}}$ *and* $\partial_\nu\gamma_1|_{\partial\Omega_{+,\varepsilon}} = \partial_\nu\gamma_2|_{\partial\Omega_{+,\varepsilon}}$. *If* s *is large, then*

$$\int_{\partial\Omega_{-,\varepsilon}} e^{-2x\cdot s\eta}|\nabla\delta u|^2 dS \lesssim \left(s^{-2\sigma} + \|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|_*^{2\theta} + s^2\|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|_*^2\right), \tag{2.24}$$

$$\int_{\partial\Omega_{-,\varepsilon}} e^{-2x\cdot s\eta}|\delta u|^2 dS \lesssim \left(s^{-2-2\sigma} + \|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|_*^2\right). \tag{2.25}$$

Moreover, we have

$$\int_{\partial\Omega_{+,\varepsilon}} e^{-2x\cdot s\eta}|\nabla\delta u|^2 dS \lesssim s^{-2\sigma}, \tag{2.26}$$

$$\int_{\partial\Omega_{+,\varepsilon}} e^{-2x\cdot s\eta}|\delta u|^2 dS \lesssim s^{-2-2\sigma} \tag{2.27}$$

when s *is sufficiently large.*

Proof. We will prove the estimate for $\int_{\partial\Omega_{-,\varepsilon}} e^{-2x\cdot s\eta}|\nabla\delta u|^2 dS$ first. We consider

$$\begin{aligned}\int_{\partial\Omega_{-,\varepsilon}} e^{-2x\cdot s\eta}|\nabla\delta u|^2 dS &\lesssim \int_{\partial\Omega_{-,\varepsilon}} e^{-2x\cdot s\eta}\left|\nabla\left(e^{\frac{\phi_{1s}}{2}} - e^{\frac{\phi_{2s}}{2}}\right)\right|^2 |u_2|^2 dS \\ &\quad + \int_{\partial\Omega_{-,\varepsilon}} e^{-2x\cdot s\eta}\left|e^{\frac{\phi_{1s}}{2}} - e^{\frac{\phi_{2s}}{2}}\right|^2 |\nabla u_2|^2 dS.\end{aligned} \tag{2.28}$$

Using Theorem 4 and Lemma 1, the first term of the right side of (2.28) can be written as

$$\begin{aligned}&\int_{\partial\Omega_{-,\varepsilon}} e^{-2x\cdot s\eta}\left|\nabla\left(e^{\frac{\phi_{1s}}{2}} - e^{\frac{\phi_{2s}}{2}}\right)\right|^2 |u_2|^2 dS \\ &\lesssim \int_{\partial\Omega_{-,\varepsilon}} e^{-2x\cdot s\eta}\left(\left|\nabla\left(e^{\frac{\phi_{1s}}{2}} - \sqrt{\gamma_1}\right)\right|^2 + |\nabla(\sqrt{\gamma_1} - \sqrt{\gamma_2})|^2 + \left|\nabla\left(e^{\frac{\phi_{2s}}{2}} - \sqrt{\gamma_2}\right)\right|^2\right)|u_2|^2 dS \\ &\lesssim \sum_{j=1}^{2}\left(\|A_{js} - A_j\|_{L^\infty(\overline{\Omega})}^2 + \|\nabla(\gamma_1 - \gamma_2)\|_{L^\infty(\partial\Omega)}^2 + \|\gamma_1 - \gamma_2\|_{L^\infty(\partial\Omega)}^2 + \|\phi_{js} - \phi_j\|_{L^\infty(\overline{\Omega})}^2\right) \\ &\quad\left(\|1 + w_2\|_{L^2(\partial\Omega)}^2\right) \\ &\lesssim \left(s^{-2\sigma} + s^{-2-2\sigma} + \|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|_*^{2\theta} + \|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|_*^2\right)\left(1 + \|w_2\|_{\dot{X}_{\zeta_2}^{1/2}}^2\right).\end{aligned}$$

We use similar arguments to estimate the second term of (2.28).

$$\begin{aligned}
&\int_{\partial\Omega_{-,\varepsilon}} e^{-2x\cdot s\eta}\left|e^{\frac{\phi_{1s}}{2}}-e^{\frac{\phi_{2s}}{2}}\right|^2|\nabla u_2|^2dS\\
&\lesssim \int_{\partial\Omega_{-,\varepsilon}}\left(\left|e^{\frac{\phi_{1s}}{2}}-\sqrt{\gamma_1}\right|^2+|\sqrt{\gamma_1}-\sqrt{\gamma_2}|^2+\left|e^{\frac{\phi_{2s}}{2}}-\sqrt{\gamma_2}\right|^2\right)|\nabla w_2|^2dS\\
&\quad+s^2\int_{\partial\Omega_{-,\varepsilon}}\left(\left|e^{\frac{\phi_{1s}}{2}}-\sqrt{\gamma_1}\right|^2+|\sqrt{\gamma_1}-\sqrt{\gamma_2}|^2+\left|e^{\frac{\phi_{2s}}{2}}-\sqrt{\gamma_2}\right|^2\right)\left(1+|w_2|^2\right)dS\\
&\lesssim \sum_{j=1}^{2}\left(\|\phi_{js}-\phi_j\|^2_{L^\infty(\overline{\Omega})}+\|\sqrt{\gamma_1}-\sqrt{\gamma_2}\|^2_{L^\infty(\partial\Omega)}\right)\left(\|\nabla w_2\|^2_{L^2(\partial\Omega)}+s^2\left(1+\|w_2\|^2_{L^2(\partial\Omega)}\right)\right)\\
&\lesssim \left(s^{-2\sigma}+s^2\|\tilde\Lambda_{\gamma_1}-\tilde\Lambda_{\gamma_2}\|_*^2\right)\left(1+\|w_2\|^2_{\dot X^{1/2}_{\zeta_2}}\right).
\end{aligned}$$

Thus we have

$$\int_{\partial\Omega_{-,\varepsilon}} e^{-2x\cdot s\eta}|\nabla\delta u|^2dS\lesssim\left(s^{-2\sigma}+\|\tilde\Lambda_{\gamma_1}-\tilde\Lambda_{\gamma_2}\|_*^{2\theta}+s^2\|\tilde\Lambda_{\gamma_1}-\tilde\Lambda_{\gamma_2}\|_*^2\right)\left(1+\|w_2\|^2_{\dot X^{1/2}_{\zeta_2}}\right).$$

Since $\gamma_1|_{\partial\Omega_{+,\varepsilon}}=\gamma_2|_{\partial\Omega_{+,\varepsilon}}$ and $\partial_\nu\gamma_1|_{\partial\Omega_{+,\varepsilon}}=\partial_\nu\gamma_2|_{\partial\Omega_{+,\varepsilon}}$, the estimate of $\int_{\partial\Omega_{+,\varepsilon}}e^{-2x\cdot s\eta}|\nabla\delta u|^2dS$ does not contain the $\|\tilde\Lambda_{\gamma_1}-\tilde\Lambda_{\gamma_2}\|_*$ terms. Thus,

$$\int_{\partial\Omega_{+,\varepsilon}} e^{-2x\cdot s\eta}|\nabla\delta u|^2dS\lesssim s^{-2\sigma}\left(1+\|w_2\|^2_{\dot X^{1/2}_{\zeta_2}}\right).$$

Similarly, we can deduce that

$$\int_{\partial\Omega_{-,\varepsilon}} e^{-2x\cdot s\eta}|\delta u|^2dS\lesssim\left(s^{-2-2\sigma}+\|\tilde\Lambda_{\gamma_1}-\tilde\Lambda_{\gamma_2}\|_*^2\right)\left(1+\|w_2\|^2_{\dot X^{1/2}_{\zeta_2}}\right)$$

and

$$\int_{\partial\Omega_{+,\varepsilon}} e^{-2x\cdot s\eta}|\delta u|^2dS\lesssim s^{-2-2\sigma}\left(1+\|w_2\|^2_{\dot X^{1/2}_{\zeta_2}}\right).$$

Since $\|w_2\|^2_{\dot X^{1/2}_{\zeta_2}}$ is small compared to 1 when s is large, we complete the proof. ☐

Lemma 6 (Lai [38])**.** *Under the same assumption as Lemma 5, we have*

$$\begin{aligned}
\int_{\partial\Omega_{+,\varepsilon}} e^{-2x\cdot s\eta}|\partial_\nu u|^2dS &\lesssim s^{-2\sigma}+s^{-1}+\|\tilde\Lambda_{\gamma_1}-\tilde\Lambda_{\gamma_2}\|_*^{2\theta}+s^2\|\tilde\Lambda_{\gamma_1}-\tilde\Lambda_{\gamma_2}\|_*^2\\
&\quad+e^{cs}(\|\tilde\Lambda_{\gamma_1}-\tilde\Lambda_{\gamma_2}\|_*+\|\tilde\Lambda_{\gamma_1}-\tilde\Lambda_{\gamma_2}\|_*^2)\|u_2\|^2_{H^2(\Omega)}
\end{aligned}\tag{2.29}$$

for some $0<\theta<1$ when s is sufficiently large.

Proof. Since $\gamma_1 > \gamma_0 > 0$, we have

$$|\partial_\nu(\tilde{u}_1 - u_2)|^2 \leq |\gamma_1\partial_\nu\tilde{u}_1 - \gamma_2\partial_\nu u_2|^2 + |(\gamma_1 - \gamma_2)\partial_\nu u_2|^{\cdot}$$

The interpolation theory implies that

$$\begin{aligned}\|\gamma_1\partial_\nu\tilde{u}_1 - \gamma_2\partial_\nu u_2\|^2_{L^2(\partial\Omega_{-,\varepsilon})} &\lesssim \|\gamma_1\partial_\nu\tilde{u}_1 - \gamma_2\partial_\nu u_2\|_{H^{1/2}(\partial\Omega_{-,\varepsilon})}\|(\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2})u_2\|_{H^{-1/2}(\partial\Omega_{-,\varepsilon})}\\ &\lesssim (\|\tilde{u}_1\|_{H^2(\Omega)} + \|u_2\|_{H^2(\Omega)})\|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|_*\|u_2\|_{H^1(\Omega)}.\end{aligned}$$

Thus we can deduce

$$\begin{aligned}\|\partial_\nu(\tilde{u}_1 - u_2)\|^2_{L^2(\partial\Omega_{-,\varepsilon})} &\lesssim (\|\tilde{u}_1\|_{H^2(\Omega)} + \|u_2\|_{H^2(\Omega)})\|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|_*\|u_2\|_{H^1(\Omega)}\\ &\quad + \|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|^2_*\|u_2\|^2_{H^2(\Omega)}\end{aligned}$$

from Theorem 4. By elliptic regularity theorem and $\tilde{u}_1|_{\partial\Omega} = u_2|_{\partial\Omega}$, $\|\tilde{u}_1\|_{H^2(\Omega)} \lesssim \|u_2\|_{H^2(\Omega)}$. Thus we have

$$\begin{aligned}\int_{\partial\Omega_{-,\varepsilon}} e^{-2x\cdot s\eta}|\partial_\nu u_0|^2 dS &\lesssim \int_{\partial\Omega_{-,\varepsilon}} e^{-2x\cdot s\eta}\left|\partial_\nu(\tilde{u}_1 - u_2)\right|^2 dS\\ &\lesssim e^{cs}(\|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|_* + \|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|^2_*)\|u_2\|^2_{H^2(\Omega)} \qquad (2.30)\end{aligned}$$

by using the fact that $u_0|_{\partial\Omega} = 0$. Let $v = e^{-x\cdot s\eta}u$. We substitute v into the Carleman estimate in Theorem 6, then we get that

$$\begin{aligned}\int_{\partial\Omega_{+,\varepsilon}} 4\Re(\partial_\nu v\partial_\eta\overline{v}) - 2(\nu\cdot\eta)|\nabla v|^2 dS &\lesssim s\int_{\partial\Omega}|v|^2 dS + \int_{\partial\Omega} s^2(\nu\cdot\eta)|v|^2 dS + \frac{1}{s}\int_{\partial\Omega}\overline{v}\partial_\nu v dS\\ &\quad + \frac{1}{s}\|e^{-x\cdot s\eta}(-\Delta + (A_{1s} - A_1)\cdot\nabla + q_{1s})(e^{x\cdot s\eta}v)\|^2_{L^2(\Omega)}\\ &\quad + \int_{\partial\Omega_{-,\varepsilon}} 4\Re(\partial_\nu v\partial_\eta\overline{v}) - 2(\nu\cdot\eta)|\nabla v|^2 dS\\ &=: I + II + III + IV + V.\end{aligned}$$

For I and II, since $u_0|_{\partial\Omega} = 0$ and Lemma 5, it follows that

$$s\int_{\partial\Omega}|v|^2 dS \lesssim s\int_{\partial\Omega} e^{-2x\cdot s\eta}|\delta u|^2 dS \lesssim \left(s^{-1-2\sigma} + s\|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|^2_*\right)$$

and

$$s^2\int_{\partial\Omega}(\nu\cdot\eta)|v|^2dS \lesssim s^2\int_{\partial\Omega}e^{-2x\cdot s\eta}|\delta u|^2dS \lesssim \left(s^{-2\sigma}+s^2\|\tilde{\Lambda}_{\gamma_1}-\tilde{\Lambda}_{\gamma_2}\|_*^2\right).$$

To estimate III, first we observe that

$$\begin{aligned}\frac{1}{s}\int_{\partial\Omega}e^{-2x\cdot s\eta}\overline{\delta u}\partial_\nu u dS &\lesssim \frac{1}{s}\int_{\partial\Omega}e^{-2x\cdot s\eta}|\delta u|^2dS+\frac{1}{s}\int_{\partial\Omega_{-,\varepsilon}}e^{-2x\cdot s\eta}|\partial_\nu\delta u|^2dS\\ &\quad+\frac{1}{s}\int_{\partial\Omega_{-,\varepsilon}}e^{-2x\cdot s\eta}|\partial_\nu u_0|^2dS+\frac{1}{s}\int_{\partial\Omega_{+,\varepsilon}}e^{-2x\cdot s\eta}|\partial_\nu u|^2dS\\ &\lesssim s^{-1}\left(s^{-2\sigma}+\|\tilde{\Lambda}_{\gamma_1}-\tilde{\Lambda}_{\gamma_2}\|_*^{2\theta}+s^2\|\tilde{\Lambda}_{\gamma_1}-\tilde{\Lambda}_{\gamma_2}\|_*^2\right)\\ &\quad+\frac{1}{s}\int_{\partial\Omega_{-,\varepsilon}}e^{-2x\cdot s\eta}|\partial_\nu u_0|^2dS+\frac{1}{s}\int_{\partial\Omega_{+,\varepsilon}}e^{-2x\cdot s\eta}|\partial_\nu u|^2dS.\end{aligned}$$

Since $u_0|_{\partial\Omega}=0$, we derive that

$$\begin{aligned}III&=\frac{1}{s}\int_{\partial\Omega}e^{-x\cdot s\eta}\overline{\delta u}\partial_\nu(e^{-x\cdot s\eta}u)dS\\ &=-(\nu\cdot\eta)\int_{\partial\Omega}e^{-2x\cdot s\eta}|\delta u|^2dS+\frac{1}{s}\int_{\partial\Omega}e^{-2x\cdot s\eta}\overline{\delta u}\partial_\nu udS\\ &\lesssim s^{-1}\left(s^{-2\sigma}+\|\tilde{\Lambda}_{\gamma_1}-\tilde{\Lambda}_{\gamma_2}\|_*^{2\theta}+s^2\|\tilde{\Lambda}_{\gamma_1}-\tilde{\Lambda}_{\gamma_2}\|_*^2\right)\\ &\quad+\frac{1}{s}\int_{\partial\Omega_{-,\varepsilon}}e^{-2x\cdot s\eta}|\partial_\nu u_0|^2dS+\frac{1}{s}\int_{\partial\Omega_{+,\varepsilon}}e^{-2x\cdot s\eta}|\partial_\nu u|^2dS.\end{aligned}$$

Next we estimate IV,

$$\begin{aligned}IV&\le\frac{1}{s}\int_\Omega e^{-2x\cdot s\eta}|(-\Delta+(A_{1s}-A_1)\cdot\nabla+q_{1s})e^{x\cdot\zeta_2}(1+w_2)|^2dx\\ &\le\frac{1}{s}\int_\Omega e^{-2x\cdot s\eta}|((A_{1s}-A_1)-(A_{2s}-A_2))\cdot\nabla(e^{x\cdot\zeta_2}(1+w_2))\\ &\quad+(q_{1s}-q_{2s})e^{x\cdot\zeta_2}(1+w_2)|^2dx.\end{aligned}$$

Then we deduce that

$$
\begin{aligned}
IV &\lesssim s\int_\Omega \sum_{j=1}^2 |A_{js}-A_j|^2dx + s\int_\Omega \sum_{j=1}^2 |A_{js}-A_j|^2|w_2|^2dx + \frac{1}{s}\int_\Omega \sum_{j=1}^2 |A_{js}-A_j|^2|\nabla w_2|^2dx \\
&\quad + \frac{1}{s}\int_\Omega |q_{2s}-q_{1s}|^2dx + \frac{1}{s}\int_\Omega |q_{2s}-q_{1s}|^2|w_2|^2dx \\
&\lesssim s\sum_{j=1}^2 \|A_{js}-A_j\|_{L^2}^2 + \sum_{j=1}^2 \|A_{js}-A_j\|_{L^\infty}^2\|w_2\|^2_{\dot{X}^{1/2}_{\zeta_2}} + \frac{1}{s}\|q_{2s}-q_{1s}\|^2_{L^2} \\
&\quad + \frac{1}{s^2}\|q_{2s}-q_{1s}\|^2_{L^\infty}\|w_2\|^2_{\dot{X}^{1/2}_{\zeta_2}} \\
&\lesssim s^{-2\sigma} + s^{-1} + s^{-2\sigma}\|w_2\|^2_{\dot{X}^{1/2}_{\zeta_2}}
\end{aligned}
$$

from Lemma 1.

Finally, for V, since $u_0|_{\partial\Omega}=0$ implies that $\nabla u_0 = \partial_\nu u_0$ on $\partial\Omega$, we have

$$
\begin{aligned}
&\left|\int_{\partial\Omega_{-,\varepsilon}} 4\Re(\partial_\nu v\partial_\eta \overline{v}) - 2(\nu\cdot\eta)|\nabla v|^2dS\right| \\
&\lesssim \int_{\partial\Omega_{-,\varepsilon}} |\nabla v|^2dS \\
&\lesssim s^2\int_{\partial\Omega_{-,\varepsilon}} e^{-2x\cdot s\eta}|\delta u|^2dS + \int_{\partial\Omega_{-,\varepsilon}} e^{-2x\cdot s\eta}|\nabla\delta u|^2dS + \int_{\partial\Omega_{-,\varepsilon}} e^{-2x\cdot s\eta}|\partial_\nu u_0|^2dS \\
&\lesssim \left(s^{-2\sigma} + \|\tilde{\Lambda}_{\gamma_1}-\tilde{\Lambda}_{\gamma_2}\|_*^{2\theta} + s^2\|\tilde{\Lambda}_{\gamma_1}-\tilde{\Lambda}_{\gamma_2}\|_*^2\right) + \int_{\partial\Omega_{-,\varepsilon}} e^{-2x\cdot s\eta}|\partial_\nu u_0|^2dS.
\end{aligned}
$$

Combining the estimates from I to V, we obtain

$$
\begin{aligned}
&\int_{\partial\Omega_{+,\varepsilon}} 4\Re(\partial_\nu v\partial_\eta \overline{v}) - 2(\nu\cdot\eta)|\nabla v|^2dS \\
&\lesssim s^{-2\sigma} + s^{-1} + \|\tilde{\Lambda}_{\gamma_1}-\tilde{\Lambda}_{\gamma_2}\|_*^{2\theta} + s^2\|\tilde{\Lambda}_{\gamma_1}-\tilde{\Lambda}_{\gamma_2}\|_*^2 \\
&\quad + \int_{\partial\Omega_{-,\varepsilon}} e^{-2x\cdot s\eta}|\partial_\nu u_0|^2dS + \frac{1}{s}\int_{\partial\Omega_{+,\varepsilon}} e^{-2x\cdot s\eta}|\partial_\nu u|^2dS \qquad (2.31)
\end{aligned}
$$

since by using Theorem 5, $\|w_2\|^2_{\dot{X}^{1/2}_{\zeta_2}}$ can be neglected if s is sufficiently large.

Moreover, for $(\nu\cdot\eta) > \varepsilon > 0$, we have

$$
\begin{aligned}
&\int_{\partial\Omega_{+,\varepsilon}} 4\Re(\partial_\nu v\partial_\eta \overline{v}) - 2(\nu\cdot\eta)|\nabla v|^2dS \\
&\geq \int_{\partial\Omega_{+,\varepsilon}} (\nu\cdot\eta)e^{-2x\cdot s\eta}|\partial_\nu u|^2dS - s^2\int_{\partial\Omega_{+,\varepsilon}} e^{-2x\cdot s\eta}|\delta u|^2dS - \int_{\partial\Omega_{+,\varepsilon}} e^{-2x\cdot s\eta}|\nabla\delta u|^2. \qquad (2.32)
\end{aligned}
$$

Combining (2.30), (2.31) and (2.32) and Lemma 5, the proof is completed. $\square$

From (2.23), Lemma 5 and Lemma 6, we can deduce

$$\left|\int_{\partial\Omega_{+,\varepsilon}} \gamma_1 \partial_\nu(\tilde{u}_1 - u_2) u_1 dS\right|^2 \lesssim s^{-2\sigma} + s^{-1} + \|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|_*^{2\theta} + s^2 \|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|_*^2$$
$$+ e^{cs}(\|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|_* + \|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|_*^2)\|u_2\|^2_{H^2(\Omega)}. \tag{2.33}$$

Note that $\|u_2\|^2_{H^2(\Omega)} \lesssim e^{cs}$ and $\|u_1\|^2_{H^2(\Omega)} \lesssim e^{cs}$. Therefore,

$$\left|\int_\Omega (\sqrt{\gamma_1}\nabla\sqrt{\gamma_2} - \sqrt{\gamma_2}\nabla\sqrt{\gamma_1}) \cdot \nabla (u_1 u_2)\, dx\right|^2$$
$$\lesssim s^{-2\sigma} + s^{-1} + \|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|_*^{2\theta} + e^{cs}(\|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|_* + \|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|_*^2). \tag{2.34}$$

from (2.22) and (2.33).

2.3.4 Stability estimate

We consider the function $v := \log\sqrt{\gamma_1} - \log\sqrt{\gamma_2} \in H^1(\Omega)$. This function v is a weak solution of

$$\Delta v + \nabla(\log\sqrt{\gamma_1} + \log\sqrt{\gamma_2})\nabla v = F \quad \text{in } \Omega \tag{2.35}$$
$$v|_{\partial\Omega} = (\log\sqrt{\gamma_1} - \log\sqrt{\gamma_2})|_{\partial\Omega},$$

with $F \in H^{-1}(\Omega)$.

Since v is also a weak solution of the elliptic equation $\nabla \cdot (\sqrt{\gamma_1}\sqrt{\gamma_2})\nabla v = (\sqrt{\gamma_1}\sqrt{\gamma_2}) \cdot F$ in Ω, we get the following estimate

$$\|v\|_{H^1(\Omega)} \lesssim \|F\|_{H^{-1}(\Omega)} + \|v\|_{H^{1/2}(\partial\Omega)}. \tag{2.36}$$

Using interpolation theory, Theorem 4 and $\gamma_j \in H^{\frac{3}{2}+\sigma}(\Omega)$, we get

$$\|v\|_{H^{1/2}(\partial\Omega)} \lesssim \|v\|^{1/2}_{L^2(\partial\Omega)}\|v\|^{1/2}_{H^1(\partial\Omega)} \lesssim \|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|_*^{1/2}. \tag{2.37}$$

Hence, we obtain

$$\|v\|_{H^1(\Omega)} \lesssim \|F\|_{H^{-1}(\Omega)} + \|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|_*^{1/2}. \tag{2.38}$$

The stability will now follow after treating $\|F\|_{H^{-1}(\Omega)}$. Following the argument in [19] and (2.35), let $g = \nabla(\log\sqrt{\gamma_1} + \log\sqrt{\gamma_2})$ and denote by $\tilde{f}$ the extension of $f \in L^2(\Omega)$ by zero to $\mathbb{R}^n$. Then for $\varphi \in H_0^1(\Omega)$ we have

$$\begin{aligned}
\langle F, \varphi\rangle &= \int_\Omega -\nabla v\nabla\overline{\varphi} + (g\nabla v)\overline{\varphi}dx \\
&= \int_{\mathbb{R}^n} -\widetilde{\nabla v}\nabla\overline{\tilde{\varphi}} + (g\widetilde{\nabla v})\overline{\tilde{\varphi}}dx \\
&= \int_{\mathbb{R}^n} \left((ik)\mathcal{F}\widetilde{\nabla v} + \mathcal{F}(g\widetilde{\nabla v})\right)\overline{\mathcal{F}\tilde{\varphi}}dk.
\end{aligned}$$

Hence

$$|\langle F, \varphi\rangle| \leq \left(\int_{\mathbb{R}^n} \left|(ik)\mathcal{F}\widetilde{\nabla v} + \mathcal{F}(g\widetilde{\nabla v})\right|^2 \left(1+|k|^2\right)^{-1} dk\right)^{\frac{1}{2}} \|\tilde{\varphi}\|_{H^1(\mathbb{R}^n)}.$$

Here $\mathcal{F}$ denotes the Fourier transform. Since $\gamma_i \in H^{\frac{3}{2}+\sigma}(\Omega)$, it follows that

$$\begin{aligned}
\|F\|^2_{H^{-1}(\Omega)} \leq &\int_{|k|\leq R} \left|(ik)\mathcal{F}\widetilde{\nabla v} + \mathcal{F}(g\widetilde{\nabla v})\right|^2 \left(1+|k|^2\right)^{-1} dk \\
&+ \int_{|k|>R} \left|(ik)\mathcal{F}\widetilde{\nabla v} + \mathcal{F}(g\widetilde{\nabla v})\right|^2 \left(1+|k|^2\right)^{-1} dk \\
\lesssim &\, R^n\|(ik)\mathcal{F}\widetilde{\nabla v} + \mathcal{F}(g\widetilde{\nabla v})\|^2_{L^\infty(B(0,R))} \\
&+ \frac{1}{R^2}\|g\widetilde{\nabla v}\|_{L^2(\mathbb{R}^n)} + \int_{|k|>R} \left(1+|k|^2\right)^{\frac{1}{2}} \left|\mathcal{F}\widetilde{\nabla v}\right|^2 \left(1+|k|^2\right)^{-\frac{1}{2}} dk \\
\lesssim &\, R^n\|(ik)\mathcal{F}\widetilde{\nabla v} + \mathcal{F}(g\widetilde{\nabla v})\|^2_{L^\infty(B(0,R))} \\
&+ \frac{1}{R^2}\|g\widetilde{\nabla v}\|_{L^2(\mathbb{R}^n)} + \frac{1}{R}\|\nabla v\|^2_{H^{\frac{1}{2}}(\Omega)}.
\end{aligned} \tag{2.39}$$

Now we need to estimate $\|(ik)\mathcal{F}\widetilde{\nabla v} + \mathcal{F}(g\widetilde{\nabla v})\|^2_{L^\infty(B(0,R))}$. Denote $q = (ik)\widetilde{\nabla v} + (g\widetilde{\nabla v})$.

Substituting $u_i = \sqrt{\gamma_i}^{-1} e^{x\cdot\zeta_i}(1+\psi_i), i = 1, 2$, into (2.34), we obtain that

$$\begin{aligned}|\mathcal{F}(q)(k)|^2 &= \left|\int_\Omega e^{-ik\cdot x}\left(ik\nabla\left(\log\sqrt{\gamma_1} - \log\sqrt{\gamma_2}\right) + \left(\nabla\log\sqrt{\gamma_1}\right)^2 - \left(\nabla\log\sqrt{\gamma_2}\right)^2\right) dx\right|^2 \\ &\leq \left|\int_\Omega \left(\sqrt{\gamma_1}\nabla\sqrt{\gamma_2} - \sqrt{\gamma_2}\nabla\sqrt{\gamma_1}\right)\cdot\nabla\left(\frac{1}{\sqrt{\gamma_1}\sqrt{\gamma_2}}e^{-ik\cdot x}\left(\psi_1+\psi_2+\psi_1\psi_2\right)\right) dx\right|^2 \\ &\quad + s^{-2\sigma} + s^{-1} + \|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|_*^{2\theta} + e^{cs}(\|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|_* + \|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|_*^2). \qquad (2.40)\end{aligned}$$

We use the product rule to estimate the first term on the right-hand side of (2.40),

$$\begin{aligned}&\left|\int_\Omega \left(\sqrt{\gamma_1}\nabla\sqrt{\gamma_2} - \sqrt{\gamma_2}\nabla\sqrt{\gamma_1}\right)\cdot\nabla\left(\frac{1}{\sqrt{\gamma_1}\sqrt{\gamma_2}}e^{-ik\cdot x}\left(\psi_1+\psi_2+\psi_1\psi_2\right)\right) dx\right|^2 \\ &\lesssim \left|\int_\Omega \left(\sqrt{\gamma_1}\nabla\sqrt{\gamma_2} - \sqrt{\gamma_2}\nabla\sqrt{\gamma_1}\right)\cdot\nabla\left(\frac{1}{\sqrt{\gamma_1}\sqrt{\gamma_2}}e^{-ik\cdot x}\right)\left(\psi_1+\psi_2+\psi_1\psi_2\right) dx\right|^2 \\ &\quad + \left|\int_\Omega \left(\sqrt{\gamma_1}\nabla\sqrt{\gamma_2} - \sqrt{\gamma_2}\nabla\sqrt{\gamma_1}\right)\cdot\left(\frac{1}{\sqrt{\gamma_1}\sqrt{\gamma_2}}e^{-ik\cdot x}\right)\left(\nabla\psi_1+\nabla\psi_2+\nabla\left(\psi_1\psi_2\right)\right) dx\right|^2 \\ &=: I + II.\end{aligned}$$

For I, using Theorem 5 and the definition of $\psi_i = \sqrt{\gamma_i}\left(e^{-\frac{\phi_{is}}{2}} - \sqrt{\gamma_i}^{-1}\right) + \sqrt{\gamma_i}e^{-\frac{\phi_{is}}{2}}w_i =: \psi^{i1} + \psi^{i2}$, we can deduce from (2.11) that

$$\begin{aligned}I &\lesssim \left(|k|^2+1\right)\left(\|\psi_1\|^2_{L^2(\Omega)} + \|\psi_2\|^2_{L^2(\Omega)} + \|\psi_1\|^2_{L^2(\Omega)}\|\psi_2\|^2_{L^2(\Omega)}\right) \\ &\lesssim \left(|k|^2+1\right)\left(s^{-2-2\sigma} + s^{-1}\left(\|w_1\|^2_{\dot{X}^{1/2}_{\zeta_1}} + \|w_2\|^2_{\dot{X}^{1/2}_{\zeta_2}}\right)\right) \\ &\lesssim |k|^2\left(s^{-2-2\sigma} + s^{-1}\left(\|w_1\|^2_{\dot{X}^{1/2}_{\zeta_1}} + \|w_2\|^2_{\dot{X}^{1/2}_{\zeta_2}}\right)\right).\end{aligned}$$

To estimate II, we divide it into two parts.

$$\begin{aligned}II &\lesssim \left|\int_\Omega \left(\sqrt{\gamma_1}\nabla\sqrt{\gamma_2} - \sqrt{\gamma_2}\nabla\sqrt{\gamma_1}\right)\cdot\left(\frac{1}{\sqrt{\gamma_1}\sqrt{\gamma_2}}e^{-ik\cdot x}\right)\left(\nabla\psi^{11}+\nabla\psi^{21}+\nabla(\psi_1\psi_2)\right) dx\right|^2 \\ &\quad + \left|\int_\Omega \left(\sqrt{\gamma_1}\nabla\sqrt{\gamma_2} - \sqrt{\gamma_2}\nabla\sqrt{\gamma_1}\right)\cdot\left(\frac{1}{\sqrt{\gamma_1}\sqrt{\gamma_2}}e^{-ik\cdot x}\right)\left(\nabla\psi^{12}+\nabla\psi^{22}\right) dx\right|^2 \\ &=: J_1 + J_2.\end{aligned}$$

For J_1, using Lemma 1,

$$\begin{aligned} J_1 &\lesssim s^{-2\sigma} + \|\psi_1\|_{L^2}\|\nabla\psi_2\|_{L^2} + \|\psi_2\|_{L^2}\|\nabla\psi_1\|_{L^2} \\ &\lesssim s^{-2\sigma} + \left(s^{-1-\sigma} + s^{-\frac{1}{2}}\|w_1\|_{\dot{X}^{1/2}_{\zeta_1}}\right)\left(s^{-\sigma} + s^{\frac{1}{2}}\|w_2\|_{\dot{X}^{1/2}_{\zeta_2}}\right) \\ &\quad + \left(s^{-1-\sigma} + s^{-\frac{1}{2}}\|w_2\|_{\dot{X}^{1/2}_{\zeta_2}}\right)\left(s^{-\sigma} + s^{\frac{1}{2}}\|w_1\|_{\dot{X}^{1/2}_{\zeta_1}}\right). \end{aligned}$$

To estimate J_2, first we have

$$J_2 \lesssim \|w_1\|^2_{L^2(\Omega)} + \|w_2\|^2_{L^2(\Omega)} + \left|\int_{\mathbb{R}^n} \Phi_B \nabla w_1 dx\right|^2 + \left|\int_{\mathbb{R}^n} \Phi_B \nabla w_2 dx\right|^2.$$

Note that since $\gamma_j \in H^{3/2}(\Omega)$, the function Φ_B has compact support and is in the space $H^{1/2}(\mathbb{R}^n)$. Then $\left|\int_{\mathbb{R}^n} \Phi_B \nabla w_1 dx\right|^2 \lesssim \|\Phi_B\|^2_{H^{1/2}(\mathbb{R}^n)}\|\Phi_B w_1\|^2_{H^{1/2}(\mathbb{R}^n)}$. We derive

$$J_2 \lesssim s^{-1}\left(\|w_1\|^2_{\dot{X}^{1/2}_{\zeta_1}} + \|w_2\|^2_{\dot{X}^{1/2}_{\zeta_2}}\right) + \left(\|w_1\|^2_{\dot{X}^{1/2}_{\zeta_1}} + \|w_2\|^2_{\dot{X}^{1/2}_{\zeta_2}}\right)$$

by applying $\|w\|_{L^2(\Omega)} \lesssim s^{-1/2}\|w\|_{\dot{X}^{1/2}_{\zeta_1}}$ from Lemma 3.

Based on the argument above, we have the estimate

$$\begin{aligned} |\mathcal{F}(q)(k)|^2 &\lesssim |k|^2\left(s^{-2-2\sigma} + s^{-1}\left(\|w_1\|^2_{\dot{X}^{1/2}_{\zeta_1}} + \|w_2\|^2_{\dot{X}^{1/2}_{\zeta_2}}\right)\right) + \left(\|w_1\|^2_{\dot{X}^{1/2}_{\zeta_1}} + \|w_2\|^2_{\dot{X}^{1/2}_{\zeta_2}}\right) \\ &\quad + s^{-\frac{1}{2}-\sigma}\|w_j\|_{\dot{X}^{1/2}_{\zeta_j}} + \|w_1\|_{\dot{X}^{1/2}_{\zeta_1}}\|w_2\|_{\dot{X}^{1/2}_{\zeta_2}} \\ &\quad + s^{-2\sigma} + s^{-1} + \|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|^{2\theta}_* + e^{cs}(\|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|_* + \|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|^2_*). \end{aligned} \tag{2.41}$$

Integrating on both sides of (2.41), we get

$$\begin{aligned} |\mathcal{F}(q)(k)|^2 &\lesssim |k|^2\left(\lambda^{-2-2\sigma} + \frac{1}{\lambda}\int_{S^{n-1}}\int_{\lambda}^{2\lambda} s^{-1}\left(\|w_1\|^2_{\dot{X}^{1/2}_{\zeta_1}} + \|w_2\|^2_{\dot{X}^{1/2}_{\zeta_2}}\right)\right) ds d\eta \\ &\quad + \frac{1}{\lambda}\int_{S^{n-1}}\int_{\lambda}^{2\lambda}\left(\|w_1\|^2_{\dot{X}^{1/2}_{\zeta_1}} + \|w_2\|^2_{\dot{X}^{1/2}_{\zeta_2}}\right) ds d\eta \\ &\quad + \frac{1}{\lambda}\int_{S^{n-1}}\int_{\lambda}^{2\lambda}\left(s^{-\frac{1}{2}-\sigma}\|w_j\|_{\dot{X}^{1/2}_{\zeta_j}} + \|w_1\|_{\dot{X}^{1/2}_{\zeta_1}}\|w_2\|_{\dot{X}^{1/2}_{\zeta_2}}\right) ds d\eta \\ &\quad + \lambda^{-2\sigma} + \lambda^{-1} + \|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|^{2\theta}_* + e^{c\lambda}(\|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|_* + \|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|^2_*). \end{aligned}$$

Applying estimate (2.17)

$$\frac{1}{\lambda}\int_{S^{n-1}}\int_{\lambda}^{2\lambda}\|w\|^2_{\dot{X}^{1/2}_{\zeta}}dsd\eta \lesssim \lambda^{-2\sigma}+\lambda^{-1},$$

we have

$$\begin{aligned}|\mathcal{F}(q)(k)|^2 \lesssim{}& |k|^2\left(\lambda^{-1-2\sigma}+\lambda^{-2}\right)+\lambda^{-2\sigma}+\lambda^{-1}\\ &+\|\tilde{\Lambda}_{\gamma_1}-\tilde{\Lambda}_{\gamma_2}\|_*^{2\theta}+e^{c\lambda}(\|\tilde{\Lambda}_{\gamma_1}-\tilde{\Lambda}_{\gamma_2}\|_*+\|\tilde{\Lambda}_{\gamma_1}-\tilde{\Lambda}_{\gamma_2}\|_*^2). \qquad (2.42)\end{aligned}$$

Let η vary in a small conic neighborhood $U_\eta \in S^{n-1}$, we get the estimate (2.42) uniformly for all $k \in E = \{k \in \mathbb{R}^n : k \text{ orthogonal to some } \tilde{\eta} \in U_\eta\}$.

Fixed $R > 0$ and $k \in \mathbb{R}^n$. Let $f(k) = \mathcal{F}(q)(Rk)$. Since q is compactly supported, $\mathcal{F}(q)$ is analytic by the Paley-Wiener theorem and

$$|D^\alpha f(k)| \le \|q\|_{L^1(\Omega)}\frac{R^{|\alpha|}}{(\mathrm{diam}(\Omega)^{-1})^{|\alpha|}} \le C\frac{R^{|\alpha|}}{\alpha!(\mathrm{diam}(\Omega)^{-1})^{|\alpha|}}\alpha! \le C\frac{e^{nR}}{(\mathrm{diam}(\Omega)^{-1})^{|\alpha|}}\alpha!$$

for any $\alpha \in \mathbb{N}^n$. Let $D = B(0,2)$ and $\tilde{E} = E \cap B(0,1)$ with $M = Ce^{nR}$ and $\rho = \mathrm{diam}(\Omega)^{-1}$. From Proposition 1, there exists $\tilde{\theta} \in (0,1)$ such that

$$|\mathcal{F}(q)(k)| = |f(k/R)| \le Ce^{nR(1-\tilde{\theta})}\|f\|^{\tilde{\theta}}_{L^\infty(\tilde{E})} \le Ce^{nR(1-\tilde{\theta})}\|\mathcal{F}(q)(k)\|^{\tilde{\theta}}_{L^\infty(E)} \qquad (2.43)$$

for all $k \in B(0,R)$.

Using (2.43), together with (2.42) and (2.39), we get

$$\begin{aligned}\|F\|^2_{H^{-1}(\Omega)} \lesssim{}& R^n e^{2nR(1-\tilde{\theta})}\Big(\lambda^{-2\sigma}+\lambda^{-1}+\|\tilde{\Lambda}_{\gamma_1}-\tilde{\Lambda}_{\gamma_2}\|_*^{2\theta}\\ &+e^{c\lambda}(\|\tilde{\Lambda}_{\gamma_1}-\tilde{\Lambda}_{\gamma_2}\|_*+\|\tilde{\Lambda}_{\gamma_1}-\tilde{\Lambda}_{\gamma_2}\|_*^2)\Big)^{\tilde{\theta}}+R^{-1}\end{aligned}$$

if $\lambda > R^2 > 1$. Thus,

$$\begin{aligned}\|F\|^{\frac{2}{\tilde{\theta}}}_{H^{-1}(\Omega)} \lesssim{}& R^{\frac{n}{\tilde{\theta}}}e^{2nR\frac{1-\tilde{\theta}}{\tilde{\theta}}}\lambda^{-2\beta}+R^{\frac{n}{\tilde{\theta}}}e^{2nR\frac{1-\tilde{\theta}}{\tilde{\theta}}}\|\tilde{\Lambda}_{\gamma_1}-\tilde{\Lambda}_{\gamma_2}\|_*^{2\theta}\\ &+R^{\frac{n}{\tilde{\theta}}}e^{2nR\frac{1-\tilde{\theta}}{\tilde{\theta}}}e^{c\lambda}(\|\tilde{\Lambda}_{\gamma_1}-\tilde{\Lambda}_{\gamma_2}\|_*+\|\tilde{\Lambda}_{\gamma_1}-\tilde{\Lambda}_{\gamma_2}\|_*^2)+R^{-\frac{1}{\tilde{\theta}}}. \qquad (2.44)\end{aligned}$$

Here we denote

$$\begin{cases}\beta=\sigma & \text{if } 0<\sigma\le\frac{1}{2},\\ \beta=\frac{1}{2} & \text{if } \frac{1}{2}<\sigma<1\ .\end{cases}$$

Choosing

$$\lambda = \left(R^{n+1} e^{2nR(1-\tilde{\theta})}\right)^{\frac{1}{2\beta\tilde{\theta}}}$$

such that

$$R^{\frac{n}{\tilde{\theta}}} e^{2nR\frac{1-\tilde{\theta}}{\tilde{\theta}}} \lambda^{-2\beta} = R^{-\frac{1}{\tilde{\theta}}},$$

the estimate (2.44) is bounded by

$$\begin{aligned} \|F\|^{\frac{2}{\tilde{\theta}}}_{H^{-1}(\Omega)} \lesssim{}& R^{\frac{n}{\tilde{\theta}}} e^{2nR\frac{1-\tilde{\theta}}{\tilde{\theta}}} \|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|^{2\theta}_{*} \\ &+ R^{\frac{n}{\tilde{\theta}}} e^{2nR\frac{1-\tilde{\theta}}{\tilde{\theta}}} e^{c\lambda} (\|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|_{*} + \|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|^{2}_{*}) + R^{-\frac{1}{\tilde{\theta}}}. \end{aligned} \tag{2.45}$$

Using the fact that

$$\begin{aligned} R^{\frac{n}{\tilde{\theta}}} e^{2nR\frac{1-\tilde{\theta}}{\tilde{\theta}} + c\lambda} &= R^{\frac{n}{\tilde{\theta}}} e^{2nR\frac{1-\tilde{\theta}}{\tilde{\theta}} + c\left(R^{n+1} e^{2nR(1-\tilde{\theta})}\right)^{\frac{1}{2\beta\tilde{\theta}}}} \\ &\leq \exp\left(e^{\left[\frac{n}{\tilde{\theta}} + 2n\frac{1-\tilde{\theta}}{\tilde{\theta}} + c + \frac{n+1}{2\beta\tilde{\theta}} + \frac{n(1-\tilde{\theta})}{\beta\tilde{\theta}}\right] R} \right) \quad \text{for all } R > 0. \end{aligned}$$

Setting $K = \frac{n}{\tilde{\theta}} + 2n\frac{1-\tilde{\theta}}{\tilde{\theta}} + c + \frac{n+1}{2\beta\tilde{\theta}} + \frac{n(1-\tilde{\theta})}{\beta\tilde{\theta}}$, (2.45) leads to

$$\|F\|^{\frac{2}{\tilde{\theta}}}_{H^{-1}(\Omega)} \lesssim e^{e^{KR}} \left(\|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|^{2\theta}_{*} + \|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|_{*} + \|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|^{2}_{*} \right) + R^{-\frac{1}{\tilde{\theta}}}. \tag{2.46}$$

The arguments above are valid if $\lambda \geq \lambda_0$. There exists a small δ such that if $\|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|_{*} < \delta$ and $R = \frac{1}{K} \log|\log \|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|^{\theta}_{*}|$, we have $\lambda \geq \lambda_0$. To be more precise, if

$$\lambda_0 \leq \lambda = \left(R^{n+1} e^{2nR(1-\tilde{\theta})}\right)^{\frac{1}{2\beta\tilde{\theta}}} \leq \left(e^{R(n+1)} e^{2nR(1-\tilde{\theta})}\right)^{\frac{1}{2\beta\tilde{\theta}}},$$

then

$$R \geq \frac{2\beta\tilde{\theta}}{3n + 1 - 2n\tilde{\theta}} \log \lambda_0 =: R_0.$$

We take $0 < \delta \leq \delta_0 < 1$ with $\delta_0^{\theta} \leq e^{-e^{K \exp R_0}}$. Thus

$$\|F\|_{H^{-1}(\Omega)} \lesssim \left(\|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|^{\theta}_{*} + \|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|^{1-\theta}_{*} + \frac{1}{K} \log|\log \|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|^{\theta}_{*}|^{-\frac{1}{\tilde{\theta}}} \right)^{\frac{\tilde{\theta}}{2}}. \tag{2.47}$$

For any $f \in L^\infty(\mathbb{R}^n)$ and $0 < \tilde{\sigma} < 1$, we deduce that

$$|f(x)|^{\frac{n}{1-\tilde{\sigma}}} \leq \|f\|_{L^\infty(\mathbb{R}^n)}^{\frac{n}{1-\tilde{\sigma}}-2} |f(x)|^2$$

for almost every $x \in \mathbb{R}^n$. Then we have

$$\|\gamma_1 - \gamma_2\|_{W^{1,\frac{n}{1-\tilde{\sigma}}}(\Omega)} \lesssim \|\gamma_1 - \gamma_2\|_{H^1(\Omega)}^{\frac{2(1-\tilde{\sigma})}{n}}. \tag{2.48}$$

From Theorem 5 in Ch. 5 in [13], we obtain that

$$\|\gamma_1 - \gamma_2\|_{C^{0,\tilde{\sigma}}(\overline{\Omega})} \lesssim \|\gamma_1 - \gamma_2\|_{W^{1,\frac{n}{1-\tilde{\sigma}}}(\Omega)}. \tag{2.49}$$

Applying (2.38), (2.47), (2.48) and (2.49), the estimate

$$\|\gamma_1 - \gamma_2\|_{C^{0,\tilde{\sigma}}(\overline{\Omega})} \lesssim \left(\|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|_*^{\theta} + \|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|_*^{1-\theta} + \frac{1}{K} \log|\log \|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|_*^{\theta}|^{-\frac{1}{\tilde{\theta}}} \right)^{\frac{\tilde{\theta}(1-\tilde{\sigma})}{n}}$$

holds.

Now if $\|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|_* \geq \delta > 0$, then we have

$$\|\gamma_1 - \gamma_2\|_{C^{0,\tilde{\sigma}}(\overline{\Omega})} \leq \frac{C}{\delta^{\frac{\theta\tilde{\theta}(1-\tilde{\sigma})}{n}}} \delta^{\frac{\theta\tilde{\theta}(1-\tilde{\sigma})}{n}} \lesssim \|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|_*^{\frac{\theta\tilde{\theta}(1-\tilde{\sigma})}{n}} \tag{2.50}$$

for some $C > 0$. The proof of Theorem 1.1 is completed.

Chapter 3

INVERSE BOUNDARY VALUE PROBLEMS FOR THE STOKES AND THE NAVIER–STOKES EQUATIONS

3.1 Introduction

Inverse boundary value problem for the Stokes and the Navier–Stokes equations is closely related to the conductivity inverse problem arising in the modeling of EIT. However, the mathematical approach that is used to recover the conductivity in EIT cannot be applied in solving the inverse problems for Stokes equations and Navier–Stokes equations. This is due to the fact that we have a system instead of a scalar equation.

The Stokes equations is formulated as follows. Let Ω be an open bounded domain in $\mathbb{R}^n$ with smooth boundary. Assume that Ω is filled with an incompressible fluid. Let $u = (u_1, \cdots, u_n)^T$ be the velocity vector field satisfying the Stokes equations

$$\begin{cases} \operatorname{div} \sigma(u,p) = 0 & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \end{cases} \tag{3.1}$$

where $\sigma(u,p) = 2\mu\varepsilon - pI_n$ is the stress tensor and $\varepsilon = ((\nabla u) + (\nabla u)^T)/2$, μ is the viscosity and p is the pressure. Here the notation I_n is the $n \times n$ identity matrix.

Physically, zero viscosity is observed only in superfluids the have the ability to self-propel and travel in a way that defies the forces of gravity and surface tension. Otherwise all fluids have positive viscosities. Thus, we can assume that $\mu > 0$ in $\overline{\Omega}$. The second equation of (3.1) is the incompressibility condition. Because of the conservation of mass, the incompressibility condition is equivalent to the material derivative of the density function ρ to be zero, that is,

$$\frac{D\rho}{Dt} =: \frac{\partial \rho}{\partial t} + u \cdot \nabla \rho = 0. \tag{3.2}$$

When ρ is constant, (3.2) is satisfied. The above equation also holds for nonconstant density functions. We can conclude that a nonconstant viscosity μ is possible since the viscosity function is a function of density. A fluid with nonconstant viscosity is called a non-Newtonian fluid which is relatively common, such as blood, shampoo and custard.

Let $g \in H^{3/2}(\partial\Omega)$ satisfy the compatibility condition

$$\int_{\partial\Omega} g \cdot \mathbf{n} dS = 0, \tag{3.3}$$

where $\mathbf{n}$ is the unit outer normal to $\partial\Omega$. This condition leads to the uniqueness of (3.1), that is, there exists a unique solution $(u, p) \in H^2(\Omega) \times H^1(\Omega)$ (p is unique up to a constant) of (3.1) and $u|_{\partial\Omega} = g$. We could define the Cauchy data for the Stokes equations (3.1) by

$$C_\mu = \left\{ (u, \sigma(u, p)\mathbf{n}) \,|_{\partial\Omega} : (u, p) \text{ satisfies } (3.1) \right\}.$$

The inverse problems we consider in this paper is to determine μ from the knowledge of the Cauchy data C_μ.

In higher dimensions, the global uniqueness of identifying the viscosity using the Cauchy data has been well-studied. For the Stokes equations, the uniqueness for the inverse boundary problem was established by Heck, Li and Wang [20] in dimension three. In [41], Li and Wang proved the unique determination of μ for the three-dimensional Navier–Stokes equations. In two dimensions, the first result was given by Imanuvilov and Yamamoto [27] in 2014, where they studied the same inverse problem with the DN map defined by

$$\Lambda_\mu(g) = \left(\frac{\partial u}{\partial \nu}, p \right) \Big|_{\partial\Omega}.$$

They showed that the knowledge of the DN map uniquely determines the viscosity μ of the Navier–Stokes equations. In a joint work with Gunther Uhlmann and Jenn-Nan Wang [39] in 2014, we discussed the Stokes and the Nevier–Stokes equations with Cauchy data in the two-dimensional case. Recently, Imanuvilov and Yamamoto proved the DN map and the Cauchy data are equivalent in [28].

We proved the global identifiability of the viscosity in an incompressible fluid by making boundary measurements $(u, \sigma(u, p)\mathbf{n}) \,|_{\partial\Omega}$.

The main result of this chapter is the following global uniqueness result. Note that the following theorem also holds for the Navier-Stokes equations.

Theorem 7 (L.-Uhlmann-Wang [39]). *Let Ω be a simply connected bounded domain in $\mathbb{R}^2$ with smooth boundary. Suppose that μ_1 and μ_2 are two viscosity functions for the Stokes equations. Assume that $\mu_j \in C^3(\overline{\Omega})$ and $\mu_j > 0$ with*

$$\partial^\alpha \mu_1|_{\partial\Omega} = \partial^\alpha \mu_2|_{\partial\Omega} \ \textit{ for all } |\alpha| \leq 1.$$

Let C_{μ_1} and C_{μ_2} be the Cauchy data associated with μ_1 and μ_2, respectively. If $C_{\mu_1} = C_{\mu_2}$, then $\mu_1 = \mu_2$ in Ω.

To study the Navier–Stokes equations, Heck, Li and Wang [20] applied the linearization technique due to Isakov [29]. The idea is to reduce the semilinear inverse boundary value problem to the corresponding linear one. When applying the linearization method to the Navier–Stokes equation, the difficulty is to show the existence of particular solutions to the Navier–Stokes equations with certain controlled asymptotic properties. The idea of linearization used in [41] is independent of the spatial dimension. It works for the two-dimensional case as well. Therefore, the global identifiability for the 2D Stokes equations implies that for the 2D Navier–Stokes equations. We will briefly describe the result in Section 3.4.

Like the 2D Calderón problem, the inverse boundary value problem for the 2D Stokes equations is formally determined. The global identifiability can not be proved by just using complex geometrical optics solutions in high frequencies as we did for the 3D Stokes equations in [20]. Our first strategy for proving Theorem 7 is to show that the inverse boundary value problem for the 2D Stokes equations and that for the thin plate-like are equivalent. The equivalence is known to hold for the 2D isotropic elastic equation and the thin plate equation. Recently, Kang, Milton and Wang [35] gave explicit formulas showing that the Cauchy data of the elasticity system determines the Cauchy data of the thin plate equations, and vice versa (see also [26]). Since the Stokes equations can be viewed as an elasticity system with incompressibility, we can prove a similar equivalence by using the similar arguments in [35]. Having established the equivalence of two inverse boundary value problems, we then transform the thin plate equations into a first order system. Albin, Guillarmou, Tzou and Uhlmann [4] showed that the Cauchy data of the first order system $D + V$ uniquely determine V if V is diagonal, where D is an operator with ∂ or $\bar{\partial}$ at its diagonal. When V is not diagonal, they reduced it to the diagonal case so that the similar result holds for the non-diagonal one. For the Stokes equation, the potential V contains the function μ up to the second order derivative, we apply their result and the assumption on the boundary of μ to deduce the global uniqueness.

3.2 Equivalence of boundary data for the plate and for the 2D Stokes equations

In this section we would like to connect the inverse boundary value problem for the thin plate equations to that for the Stokes equations. We define the 4-th order tensor $\mathcal{R}$ by

$$\mathcal{R}M = R_\perp^T M R_\perp$$

for any 2×2 matrix M, where

$$R_\perp = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Hereafter, for any function u, the notation $u_{,j}$ means the derivative of u with respect to x_j, $j = 1, 2$. Denote $\sigma = (\sigma_{ij})$.Componentwise, the first equation of (3.1) is equivalent to

$$\sigma_{11,1} + \sigma_{12,2} = 0, \quad \sigma_{21,1} + \sigma_{22,2} = 0.$$

It follows that there exist potentials ψ_1 and ψ_2 such that

$$\sigma_{11} = \psi_{1,2}, \quad \sigma_{12} = -\psi_{1,1}, \quad \sigma_{21} = \psi_{2,2}, \quad \sigma_{22} = -\psi_{2,1}. \tag{3.4}$$

Since $\sigma_{12} = \sigma_{21}$, we have

$$\psi_{1,1} + \psi_{2,2} = 0.$$

Thus there exists a potential ϕ such that

$$\psi_1 = \phi_{,2}, \quad \psi_2 = -\phi_{,1}. \tag{3.5}$$

The potential ϕ is called the Airy stress function. Substituting (3.5) into (3.4), we see that

$$\sigma = \mathcal{R}\nabla^2\phi = \begin{pmatrix} \phi_{,22} & -\phi_{,12} \\ -\phi_{,21} & \phi_{,11} \end{pmatrix}, \tag{3.6}$$

where $\nabla^2\phi$ denotes the Hessian of ϕ, i.e.,

$$\nabla^2\phi = \begin{pmatrix} \phi_{,11} & \phi_{,12} \\ \phi_{,21} & \phi_{,22} \end{pmatrix}.$$

In light of $\sigma = 2\mu\varepsilon - pI_2$ and (3.6), we get

$$\varepsilon = \frac{1}{2\mu}\left(\mathcal{R}\nabla^2\phi + pI_2\right). \tag{3.7}$$

The divergence-free condition $\operatorname{div} u = 0$ implies that

$$0 = u_{1,1} + u_{2,2} = \operatorname{tr}(\varepsilon) = \frac{1}{2\mu}\left(\Delta\phi + 2p\right),$$

thus

$$p = -\frac{\Delta\phi}{2}. \tag{3.8}$$

Note that the physical significance of the pressure p is that $-p$ is the mean of the two normal stresses at a point, that is,

$$p = -\frac{1}{2}(\sigma_{11} + \sigma_{22}).$$

From (3.7) and (3.8), it follows that

$$\begin{aligned} 0 = \operatorname{div}\operatorname{div}(\mathcal{R}\varepsilon) &= \operatorname{div}\operatorname{div}\left(\frac{1}{2\mu}\mathcal{R}\left(\mathcal{R}\nabla^2\phi + pI_2\right)\right) \\ &= \operatorname{div}\operatorname{div}\left(\frac{1}{2\mu}\left(\nabla^2\phi - \frac{\Delta\phi}{2}I_2\right)\right) \\ &= \operatorname{div}\operatorname{div}\left(\frac{1}{2\mu}\begin{pmatrix} \frac{1}{2}(\phi_{,11} - \phi_{,22}) & \phi_{,12} \\ \phi_{,12} & \frac{1}{2}(\phi_{,22} - \phi_{,11}) \end{pmatrix}\right). \end{aligned} \tag{3.9}$$

Conversely, if $\operatorname{div}\operatorname{div}(\mathcal{R}\varepsilon) = 0$, then

$$\varepsilon_{22,11} + \varepsilon_{11,22} - 2\varepsilon_{12,12} = 0,$$

where $\varepsilon = (\varepsilon_{ij})$. If Ω is simply connected, then there exists a function u such that $\varepsilon = \left((\nabla u) + (\nabla u)^T\right)/2$. (For the proof of the existence of such function u, we refer to [14], page 99-103). Based on (3.9), the function u also satisfies $\operatorname{div} u = 0$. Let $p = -\Delta\phi/2$, then (u, p) satisfies the Stokes equations (3.1). Thus we have proved that the two systems (3.1) and (3.9) are equivalent if Ω is simply connected.

Next we would like to discuss the equivalence of the Cauchy data. We define the operator $P_\mu(\phi)$ by

$$P_\mu(\phi) := \operatorname{div}\operatorname{div}\left(\frac{1}{2\mu}\begin{pmatrix} \frac{1}{2}(\phi_{,11} - \phi_{,22}) & \phi_{,12} \\ \phi_{,12} & \frac{1}{2}(\phi_{,22} - \phi_{,11}) \end{pmatrix}\right)$$

and denote $u_{,n} = \nabla u \cdot \mathbf{n}$ and $u_{,t} = \nabla u \cdot \mathbf{t}$, where $\mathbf{n} = (n_1, n_2)$ is the unit normal and $\mathbf{t} = (-n_2, n_1) = R_\perp^T \mathbf{n}$ is the unit tangent vector field along $\partial\Omega$ in the positive orientation. The Dirichlet data associated with (3.9) is described by the pair $\{\phi, \phi_{,n}\}$ and the Neumann data by the pair

$$\begin{aligned} M_n &:= \mathbf{n} \cdot \left(\frac{1}{2\mu}\left(\nabla^2\phi - \frac{\Delta\phi}{2} I_2\right)\right)\mathbf{n}, \\ (M_t)_{,t} &:= \operatorname{div}\left(\frac{1}{2\mu}\left(\nabla^2\phi - \frac{\Delta\phi}{2} I_2\right)\right) \cdot \mathbf{n} + \left(\mathbf{t} \cdot \frac{1}{2\mu}\left(\nabla^2\phi - \frac{\Delta\phi}{2} I_2\right)\mathbf{n}\right)_{,t}. \end{aligned}$$

We define the Cauchy data for (3.9) by

$$C_\mu^* = \left\{(\phi, \phi_{,n}, M_n, (M_t)_{,t})\,|_{\partial\Omega} : \phi \in H^4(\Omega), P_\mu(\phi) = 0\right\}.$$

We now adopt the arguments used in [35] to show that $\sigma\mathbf{n}|_{\partial\Omega}$ determines $\{\phi, \phi_{,n}\}$ on $\partial\Omega$ and $u|_{\partial\Omega}$ determines $\{M_n, (M_t)_{,t}\}$ on $\partial\Omega$, and vice versa. Therefore, the Cauchy data C_μ for the Stokes equations and the Cauchy data C_μ^* for (3.9) are equivalent. Assume for the moment that $u \in C^{2+\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$. It follows from (3.6) that

$$R_\perp^T \sigma\mathbf{n} = (\nabla^2\phi)\mathbf{t} = \begin{pmatrix} \nabla\phi_{,1} \cdot \mathbf{t} \\ \nabla\phi_{,2} \cdot \mathbf{t} \end{pmatrix}. \tag{3.10}$$

For $j = 1, 2$, we integrate $\nabla\phi_{,j} \cdot \mathbf{t}$ along $\partial\Omega$ from some point $x_0 \in \partial\Omega$, we recover $\nabla\phi$ (up to a constant) on $\partial\Omega$. Hence $\phi_{,n}$ and $\phi_{,t}$ are recovered. We integrate $\phi_{,t}$ along $\partial\Omega$, then ϕ on $\partial\Omega$ is known (also up to a constant). The appearance of integrating constants is evident from (3.9). In other words, the traction $\sigma\mathbf{n}$ uniquely determines the Dirichlet data ϕ and $\phi_{,n}$. On the other hand, if ϕ and $\phi_{,n}$ are given, then $\nabla\phi$ is known. Hence, the boundary traction $\sigma\mathbf{n}$ is recovered via (3.10).

To show that M_n and $(M_t)_{,t}$ can be recovered from u. Since $\varepsilon = ((\nabla u) + (\nabla u)^T)/2$, we get that

$$\mathcal{R}\varepsilon = R_\perp^T \varepsilon R_\perp = \begin{pmatrix} u_{2,2} & -\frac{1}{2}(u_{2,1} + u_{1,2}) \\ -\frac{1}{2}(u_{2,1} + u_{1,2}) & u_{1,1} \end{pmatrix}$$

and thus

$$\begin{aligned} \operatorname{div}(\mathcal{R}\varepsilon) &= \begin{pmatrix} u_{2,21} - \frac{1}{2}(u_{2,12} + u_{1,22}) \\ -\frac{1}{2}(u_{2,11} + u_{1,21}) + u_{1,12} \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} u_{2,12} - u_{1,22} \\ u_{1,12} - u_{2,11} \end{pmatrix} = \frac{1}{2} R_\perp^T \nabla \left(u_{1,2} - u_{2,1}\right). \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} \operatorname{div}(\mathcal{R}\varepsilon) \cdot \mathbf{n} &= \frac{1}{2}(R_\perp \mathbf{n}) \cdot \nabla \left(u_{1,2} - u_{2,1}\right) = -\frac{1}{2}\mathbf{t} \cdot \nabla \left(u_{1,2} - u_{2,1}\right) \\ &= \frac{1}{2}\left(u_{2,1} - u_{1,2}\right)_{,t}. \end{aligned}$$

Recall that $(M_t)_{,t} = \operatorname{div}(\mathcal{R}\varepsilon) \cdot \mathbf{n} + (\mathbf{t} \cdot (\mathcal{R}\varepsilon)\mathbf{n})_{,t}$ and therefore

$$(M_t)_{,t} = \frac{1}{2}\left(u_{2,1} - u_{1,2}\right)_{,t} + (\mathbf{t} \cdot (\mathcal{R}\varepsilon)\mathbf{n})_{,t}. \tag{3.11}$$

Integrating (3.11) along $\partial\Omega$ from some point $x_0 \in \partial\Omega$ and choosing an appropriate $(u_{2,1} - u_{1,2})(x_0)$, we obtain

$$M_t = \frac{1}{2}\left(u_{2,1} - u_{1,2}\right) + \mathbf{t} \cdot (\mathcal{R}\varepsilon)\mathbf{n}. \tag{3.12}$$

We observe that

$$\frac{1}{2}\left(u_{2,1} - u_{1,2}\right) = \left(R_\perp^T \begin{pmatrix} 0 & \frac{1}{2}(u_{1,2} - u_{2,1}) \\ \frac{1}{2}(u_{2,1} - u_{1,2}) & 0 \end{pmatrix} R_\perp \mathbf{n} \right) \cdot \mathbf{t}.$$

The second term on the right side of (3.12) can be written as

$$\mathbf{t} \cdot (\mathcal{R}\varepsilon)\mathbf{n} = \left(R_\perp^T \begin{pmatrix} u_{1,1} & \frac{1}{2}(u_{1,2} + u_{2,1}) \\ \frac{1}{2}(u_{1,2} + u_{2,1}) & u_{2,2} \end{pmatrix} R_\perp \mathbf{n} \right) \cdot \mathbf{t}.$$

Thus we have

$$\begin{aligned} M_t &= \left(R_\perp^T \begin{pmatrix} u_{1,1} & u_{1,2} \\ u_{2,1} & u_{2,2} \end{pmatrix} R_\perp \mathbf{n} \right) \cdot \mathbf{t} \\ &= -R_\perp^T (\nabla u)\mathbf{t} \cdot \mathbf{t} \\ &= -\mathbf{n} \cdot (\nabla u)\mathbf{t}. \end{aligned} \tag{3.13}$$

Moreover, using the definition of M_n, we get

$$M_n = \mathbf{n} \cdot \mathcal{R}\varepsilon\mathbf{n} = \mathbf{n} \cdot R_\perp^T \varepsilon R_\perp \mathbf{n} = \mathbf{t} \cdot (\nabla u)\mathbf{t}. \tag{3.14}$$

From (3.13) and (3.14), we deduce that

$$u_{,t} = -M_t \mathbf{n} + M_n \mathbf{t}, \tag{3.15}$$

which implies the Neumann data M_n and M_t can be recovered from $u_{,t}$. On the other hand, we use the formula (3.15) and integrate $-M_t\mathbf{n} + M_n\mathbf{t}$ along $\partial\Omega$. Thus, the velocity field u is determined.

By a density argument the above discussion holds for the slightly relaxed regularity assumption on the boundary data $g \in H^{3/2}(\partial\Omega)$. Hence, we can remove the assumption that $u \in C^{2+\alpha}(\overline{\Omega})$. We therefore conclude that knowing the Cauchy data of the Stokes equations is equivalent to knowing that of the thin plate-like equations (3.9).

3.3 Global uniqueness for the Stokes equations

From the previous section, we have concluded that to study the inverse boundary value for the Stokes equations (3.1), it suffices to consider the same question for the plate-like equation (3.9). Our strategy now is to deduce a first order system $DU + VU = 0$ from (3.9). The most nontrivial property that we will show is that C_μ^* determines the Cauchy data of the first order system $DU + VU = 0$. Having obtained this result, the global identifiability of μ for the Stokes equations is reduced to the uniqueness problem for this first order system. The global uniqueness of the inverse boundary value problem for such a first order system was recently studied by Albin, Guillarmou, Tzou and Uhlmann in [4]. Consequently, the proof of the uniqueness question for the Stokes equations follows from their result.

3.3.1 $(\partial_{\bar z}^2, \partial_z^2)$ system

As usual, we define $z = x + iy$,

$$\partial_z = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \partial_{\bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

The complex version of Gauss integral formulas are given by

$$\int_\Omega \partial_{\bar z} w(z)dxdy = \frac{1}{2i}\int_{\partial\Omega} w(z)dz, \quad \int_\Omega \partial_z w(z)dxdy = -\frac{1}{2i}\int_{\partial\Omega} w(z)d\bar z \tag{3.16}$$

for $w \in C^1(\Omega) \cap C(\overline{\Omega})$ lead to the Cauchy Pompeiu representations

$$w(z) = \frac{1}{2\pi i}\int_{\partial\Omega} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_\Omega \partial_{\bar\zeta} w(\zeta)\frac{d\xi d\eta}{\zeta - z}, \quad z \in \Omega, \tag{3.17}$$

$$w(z) = -\frac{1}{2\pi i}\int_{\partial\Omega} w(\zeta)\frac{d\bar\zeta}{\overline{\zeta - z}} - \frac{1}{\pi}\int_\Omega \partial_\zeta w(\zeta)\frac{d\xi d\eta}{\overline{\zeta - z}}, \quad z \in \Omega, \tag{3.18}$$

where $\zeta = \xi + i\eta$. Iterations of these formulas give the following higher order representations

$$w(z) = \frac{1}{2\pi i}\int_{\partial\Omega} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{2\pi i}\int_{\partial\Omega} \partial_{\bar\zeta} w(\zeta)\frac{\overline{\zeta - z}}{\zeta - z}d\zeta + \frac{1}{\pi}\int_\Omega \partial_{\bar\zeta}^2 w(\zeta)\frac{\overline{\zeta - z}}{\zeta - z}d\xi d\eta \tag{3.19}$$

and

$$w(z) = -\frac{1}{2\pi i}\int_{\partial\Omega} w(\zeta)\frac{d\bar\zeta}{\overline{\zeta - z}} + \frac{1}{2\pi i}\int_{\partial\Omega} \partial_\zeta w(\zeta)\frac{\zeta - z}{\overline{\zeta - z}}d\bar\zeta + \frac{1}{\pi}\int_\Omega \partial_\zeta^2 w(\zeta)\frac{\zeta - z}{\overline{\zeta - z}}d\xi d\eta. \tag{3.20}$$

for $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ (see [7, Page 272]). In the sequel, we need a technical lemma.

Lemma 7 (L.-Uhlmann-Wang [39])**.** *Let Ω be an open bounded domain in $\mathbb{C}$ and $f \in C^k(\overline{\Omega})$ for $k \geq 2$. Define*

$$u(z) = \frac{1}{\pi} \int_\Omega f(\zeta) \frac{\overline{z-\zeta}}{z-\zeta} d\xi d\eta$$

Then $u(z)$ is in $C^k(\Omega)$ and satisfies

$$\partial_{\bar{z}}^2 u(z) = f(z) \tag{3.21}$$

in Ω. Likewise, if we define

$$u(z) = \frac{1}{\pi} \int_\Omega f(\zeta) \frac{z-\zeta}{\overline{z-\zeta}} d\xi d\eta,$$

then $u(z)$ is in $C^k(\Omega)$ and satisfies

$$\partial_z^2 u(z) = f(z)$$

in Ω.

Proof. We adopt the proof of a similar result in [12, Theorem 2.1.2] to our case here. We only prove the first part of the lemma, the other part is treated similarly. We first consider $f \in C_0^k(\mathbb{C})$. Changing variable $\zeta' = z - \zeta$ in u and differentiation under the integral sign implies that $u \in C^k(\mathbb{C})$. To verify (3.21), we apply Gauss integral formula twice and (3.19) (note that f is compactly supported). We get

$$\partial_{\bar{z}}^2 u(z) = \frac{1}{\pi} \int_{\mathbb{C}} f(\zeta) \partial_{\bar{\zeta}}^2 \frac{\overline{z-\zeta}}{z-\zeta} d\xi d\eta = \frac{1}{\pi} \int_{\mathbb{C}} \partial_{\bar{\zeta}}^2 f(\zeta) \frac{\overline{z-\zeta}}{z-\zeta} d\xi d\eta = f(z).$$

For the general situation, let $z_0 \in \Omega$ and $\chi \in C_0^\infty(\mathbb{C})$, $0 \leq \chi \leq 1$, $\chi = 1$ in some neighborhood V of z_0 and $\operatorname{supp} \chi \subset \Omega$. Thus,

$$\begin{aligned} u(z) &= \frac{1}{\pi} \int_\Omega f(\zeta) \frac{\overline{z-\zeta}}{z-\zeta} d\xi d\eta \\ &= \frac{1}{\pi} \int_\Omega \chi f(\zeta) \frac{\overline{z-\zeta}}{z-\zeta} d\xi d\eta + \frac{1}{\pi} \int_\Omega (1-\chi(\zeta)) f(\zeta) \frac{\overline{z-\zeta}}{z-\zeta} d\xi d\eta \\ &=: u_1(z) + u_2(z). \end{aligned}$$

Since $\partial_{\bar z}^2 u_2 = 0$ in V, from the previous argument for $\Omega = \mathbb{C}$, we have

$$\partial_{\bar z}^2 u(z) = \partial_{\bar z}^2 u_1(z) + \partial_{\bar z}^2 u_2(z) = \chi(z) f(z) = f(z)$$

for $z \in V$.

$\square$

Lemma 8 (L.-Uhlmann-Wang [39]). *Let Ω be an open bounded domain with smooth boundary $\partial\Omega$. Suppose that $f, g \in C^2(\Omega) \cap C^1(\overline{\Omega})$. Suppose that the compatibility condition*

$$\partial_{\bar z}^2 f = \partial_z^2 g \quad \text{in } \Omega \tag{3.22}$$

is satisfied. Then there exists a function $w \in C^2(\Omega)$ satisfies

$$\begin{cases} \partial_z^2 w = f & \text{in } \Omega, \\ \partial_{\bar z}^2 w = g & \text{in } \Omega. \end{cases} \tag{3.23}$$

Proof. Let us make an ansatz

$$\begin{aligned} w(z) = {} & \frac{1}{\pi}\int_\Omega g(\zeta)\frac{\overline{z-\zeta}}{z-\zeta} d\xi d\eta + \frac{1}{\pi}\int_\Omega f(\zeta)\frac{z-\zeta}{\overline{z-\zeta}} d\xi d\eta \\ & - \frac{1}{\pi^2}\int_\Omega \left(\int_\Omega \partial_{\bar\lambda}^2 f(\lambda)\frac{\overline{\zeta-\lambda}}{\zeta-\lambda} ds dt\right)\frac{z-\zeta}{\overline{z-\zeta}} d\xi d\eta + \phi_1(z) + \phi_2(z), \end{aligned}$$

where

$$\begin{aligned} \phi_1(z) = {} & -\frac{1}{(2\pi i)^2}\int_{\partial\Omega}\left(\int_{\partial\Omega}\partial_{\bar\lambda} f(\lambda)\frac{\overline{\zeta-\lambda}}{\zeta-\lambda} d\lambda\right)(z-\zeta)\log(\overline{z-\zeta}) d\zeta \\ & - \frac{1}{(2\pi i)^2}\int_{\partial\Omega}\left(\int_{\partial\Omega} f(\lambda)\frac{1}{\zeta-\lambda} d\lambda\right)(z-\zeta)\log(\overline{z-\zeta}) d\zeta \\ & - \frac{1}{(2\pi i)^2}\int_{\partial\Omega}\left(\int_{\partial\Omega}\partial_{\bar\lambda} f(\lambda)\frac{1}{\zeta-\lambda} d\lambda\right)|z-\zeta|^2\log(\overline{z-\zeta}) d\zeta \end{aligned}$$

and

$$\phi_2(z) = -\frac{1}{2\pi i}\int_{\partial\Omega}\partial_\zeta g(\zeta)|z-\zeta|^2\log(z-\zeta) d\bar\zeta - \frac{1}{2\pi i}\int_{\partial\Omega} g(\zeta)(\overline{z-\zeta})\log(z-\zeta) d\bar\zeta.$$

Here we take the principal value for the log. Since $z-\zeta$ does not vanish for all $z \in \Omega$ and $\zeta \in \partial\Omega$, $h(z,\zeta) = \log(z-\zeta)$ is well-defined on $\Omega \times D$ where $D = \{\zeta \in \partial\Omega, 0 < \arg(z-\zeta) < 2\pi\}$.

Moreover, for fixed $\zeta \in \partial\Omega$, the function $h(z,\zeta)$ is holomorphic in Ω. We can interchange the differentiation and the integral sign see Chapter 8 in [40] and get

$$\partial_z^2\phi_1(z) = 0, \quad \partial_{\bar z}^2\phi_2(z) = 0 \text{ in } \Omega.$$

On the other hand, we can compute that

$$\begin{aligned}\partial_{\bar z}^2\phi_1(z) = &\frac{1}{(2\pi i)^2}\int_{\partial\Omega}\left(\int_{\partial\Omega}\partial_{\bar\lambda}f(\lambda)\frac{\overline{\zeta-\lambda}}{\zeta-\lambda}d\lambda\right)\partial_{\bar\zeta}\left(\frac{z-\zeta}{\overline{z-\zeta}}\right)d\zeta\\ &+\frac{1}{(2\pi i)^2}\int_{\partial\Omega}\left(\int_{\partial\Omega}f(\lambda)\frac{1}{\zeta-\lambda}d\lambda\right)\partial_{\bar\zeta}\left(\frac{z-\zeta}{\overline{z-\zeta}}\right)d\zeta\\ &-\frac{1}{(2\pi i)^2}\int_{\partial\Omega}\left(\int_{\partial\Omega}\partial_{\bar\lambda}f(\lambda)\frac{1}{\zeta-\lambda}d\lambda\right)\frac{z-\zeta}{\overline{z-\zeta}}d\zeta,\end{aligned}$$

and

$$\partial_z^2\phi_2(z) = \frac{1}{2\pi i}\int_{\partial\Omega}g(\zeta)\partial_\zeta\left(\frac{\overline{z-\zeta}}{z-\zeta}\right)d\bar\zeta - \frac{1}{2\pi i}\int_{\partial\Omega}\partial_\zeta g(\zeta)\frac{\overline{z-\zeta}}{z-\zeta}d\bar\zeta.$$

Using the compatibility condition (3.22), Lemma 7, and Gauss's formula (3.16) twice, we can see that

$$\begin{aligned}&\partial_z^2\left(\frac{1}{\pi^2}\int_\Omega\left(\int_\Omega\partial_{\bar\lambda}^2f(\lambda)\frac{\overline{\zeta-\lambda}}{\zeta-\lambda}dsdt\right)\frac{z-\zeta}{\overline{z-\zeta}}d\xi d\eta\right)\\ &=\partial_z^2\left(\frac{1}{\pi^2}\int_\Omega\left(\int_\Omega\partial_\lambda^2g(\lambda)\frac{\overline{\zeta-\lambda}}{\zeta-\lambda}dsdt\right)\frac{z-\zeta}{\overline{z-\zeta}}d\xi d\eta\right) = \frac{1}{\pi}\int_\Omega\partial_\zeta^2g(\zeta)\frac{\overline{z-\zeta}}{z-\zeta}d\xi d\eta\\ &=\frac{1}{\pi}\int_\Omega g(\zeta)\partial_\zeta^2\left(\frac{\overline{z-\zeta}}{z-\zeta}\right)d\xi d\eta + \frac{1}{2\pi i}\int_{\partial\Omega}g(\zeta)\partial_\zeta\left(\frac{\overline{z-\zeta}}{z-\zeta}\right)d\bar\zeta\\ &\quad-\frac{1}{2\pi i}\int_{\partial\Omega}\partial_\zeta g(\zeta)\frac{\overline{z-\zeta}}{z-\zeta}d\bar\zeta.\end{aligned}$$

By the above relation and the ansatz, we then deduce

$$\begin{aligned}\partial_z^2w(z) = &\frac{1}{\pi}\int_\Omega g(\zeta)\partial_z^2\left(\frac{\overline{z-\zeta}}{z-\zeta}\right)d\xi d\eta + f(z)\\ &-\partial_z^2\left(\frac{1}{\pi^2}\int_\Omega\left(\int_\Omega\partial_\lambda^2g(\lambda)\frac{\overline{\zeta-\lambda}}{\zeta-\lambda}dsdt\right)\frac{z-\zeta}{\overline{z-\zeta}}d\xi d\eta\right) + \partial_z^2\phi_1(z) + \partial_z^2\phi_2(z)\\ = &f(z).\end{aligned}$$

On the other hand, from (3.19), we have that

$$\int_\Omega \partial^2_{\overline{\lambda}} f(\lambda)\frac{\overline{\zeta-\lambda}}{\zeta-\lambda}dsdt = f(\zeta)\pi + \frac{1}{2i}\int_{\partial\Omega} f(\lambda)\left(\frac{1}{\zeta-\lambda}\right)d\lambda + \frac{1}{2i}\int_{\partial\Omega}\partial_{\overline{\lambda}} f(\lambda)\frac{\overline{\zeta-\lambda}}{\zeta-\lambda}d\lambda,$$

which implies that

$$\begin{aligned}
h(z) &:= \partial^2_{\overline{z}}\left(\frac{1}{\pi^2}\int_\Omega\left(\int_\Omega \partial^2_{\overline{\lambda}} f(\lambda)\frac{\overline{\zeta-\lambda}}{\zeta-\lambda}dsdt\right)\frac{z-\zeta}{\overline{z-\zeta}}d\xi d\eta\right)\\
&= \frac{1}{\pi}\int_\Omega f(\zeta)\partial^2_{\overline{z}}\left(\frac{z-\zeta}{\overline{z-\zeta}}\right)d\xi d\eta\\
&\quad + \frac{1}{2\pi^2 i}\int_\Omega\left(\int_{\partial\Omega} f(\lambda)\left(\frac{1}{\zeta-\lambda}\right)d\lambda + \int_{\partial\Omega}\partial_{\overline{\lambda}} f(\lambda)\frac{\overline{\zeta-\lambda}}{\zeta-\lambda}d\lambda\right)\partial^2_{\overline{z}}\left(\frac{z-\zeta}{\overline{z-\zeta}}\right)d\xi d\eta.
\end{aligned}$$

Applying (3.16) twice yields

$$\begin{aligned}
h(z) &= \frac{1}{\pi}\int_\Omega f(\zeta)\partial^2_{\overline{z}}\left(\frac{z-\zeta}{\overline{z-\zeta}}\right)d\xi d\eta\\
&\quad + \frac{1}{(2\pi i)^2}\int_{\partial\Omega}\left(\int_{\partial\Omega} f(\lambda)\left(\frac{1}{\zeta-\lambda}\right)d\lambda\right)\partial_{\overline{\zeta}}\left(\frac{z-\zeta}{\overline{z-\zeta}}\right)d\zeta\\
&\quad + \frac{1}{(2\pi i)^2}\int_{\partial\Omega}\left(\int_{\partial\Omega}\partial_{\overline{\lambda}} f(\lambda)\frac{\overline{\zeta-\lambda}}{\zeta-\lambda}d\lambda\right)\partial_{\overline{\zeta}}\left(\frac{z-\zeta}{\overline{z-\zeta}}\right)d\zeta\\
&\quad - \frac{1}{(2\pi i)^2}\int_{\partial\Omega}\left(\int_{\partial\Omega}\partial_{\overline{\lambda}} f(\lambda)\frac{1}{\zeta-\lambda}d\lambda\right)\frac{z-\zeta}{\overline{z-\zeta}}d\zeta.
\end{aligned}$$

In view of Lemma 7 and h, we conclude that

$$\begin{aligned}
\partial^2_{\overline{z}} w(z) &= g(z) + \int_\Omega f(\zeta)\partial^2_{\overline{z}}\left(\frac{z-\zeta}{\overline{z-\zeta}}\right)d\xi d\eta\\
&\quad - \partial^2_{\overline{z}}\left(\frac{1}{\pi^2}\int_\Omega\left(\int_\Omega \partial^2_{\overline{\eta}} f(\eta)\frac{\overline{\zeta-\eta}}{\zeta-\eta}dsdt\right)\frac{z-\zeta}{\overline{z-\zeta}}d\xi d\eta\right) + \partial^2_{\overline{z}}\phi_1(z) + \partial^2_{\overline{z}}\phi_2(z)\\
&= g(z).
\end{aligned}$$

□

Note that the above lemma also holds when $f, g \in H^2(\Omega)$ since we can approximate a H^2 function by a sequence in $C^\infty(\overline{\Omega})$ in the $H^2(\Omega)$ space.

3.3.2 $\partial_{\overline{z}}$ system

Let A and B be two 2×2 matrices. We define $A \cdot B = \operatorname{tr}(AB^T)$. We write equation (3.9) in nondivergence form

$$\begin{aligned} 0 &= \operatorname{div}\operatorname{div}\left(\frac{1}{2\mu}\left(\nabla^2\phi - \frac{\Delta\phi}{2}I_2\right)\right) \\ &= \frac{1}{4\mu}\Delta^2\phi + \frac{1}{2}\nabla\left(\frac{1}{\mu}\right)\cdot\nabla(\Delta\phi) + \frac{1}{2}\nabla^2\left(\frac{1}{\mu}\right)\cdot\left(\nabla^2\phi - \frac{\Delta\phi}{2}I_2\right). \end{aligned}$$

Since $\mu > 0$, the equation above is equivalent to

$$\Delta^2\phi + 2\mu\nabla\left(\frac{1}{\mu}\right)\cdot\nabla\left(\Delta\phi\right) + 2\mu\nabla^2\left(\frac{1}{\mu}\right)\cdot\left(\nabla^2\phi - \frac{\Delta\phi}{2}I_2\right) = 0, \tag{3.24}$$

which implies that

$$\partial_{\overline{z}}^2\partial_z^2\phi + \alpha\partial_z^2\partial_{\overline{z}}\phi + \beta\partial_z^2\phi + \overline{\alpha}\partial_z\partial_{\overline{z}}^2\phi + \overline{\beta}\partial_{\overline{z}}^2\phi = 0, \tag{3.25}$$

where

$$\alpha = \mu\partial_{\overline{z}}\left(\frac{1}{\mu}\right), \quad \beta = \frac{\mu}{2}\partial_{\overline{z}}^2\left(\frac{1}{\mu}\right). \tag{3.26}$$

With equation (3.25) in mind, we define a first order system $D + V$ acting on functions with values in $\mathbb{C}^4$ as follows

$$D + V = \begin{pmatrix} \partial_{\overline{z}} & 0 & 0 & 0 \\ 0 & \partial_{\overline{z}} & 0 & 0 \\ 0 & 0 & \partial_z & 0 \\ 0 & 0 & 0 & \partial_z \end{pmatrix} + \begin{pmatrix} \alpha & \beta & \overline{\alpha} & \overline{\beta} \\ -1 & 0 & 0 & 0 \\ \alpha & \beta & \overline{\alpha} & \overline{\beta} \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{3.27}$$

The corresponding Cauchy data of $D + V$ is

$$C_{D+V} = \left\{U|_{\partial\Omega} : U \in H^1(\Omega, \mathbb{C}^4), U \text{ is a solution of (D+V)U=0}\right\}.$$

The next key step is to show that the Cauchy data C_μ^* for (3.9) determine C_{D+V}. To do so, we begin the following lemma saying that C_μ^* determines all derivatives of the solution on the boundary up to third order under suitable assumption.

Lemma 9 (L.-Uhlmann-Wang [39]). *Assume that* $\partial^\kappa \mu_1|_{\partial\Omega} = \partial^\kappa \mu_2|_{\partial\Omega}$ *for all* $|\kappa| \le 1$. *If* $C^*_{\mu_1} = C^*_{\mu_2}$, *i.e.,*

$$\{\phi_1, \phi_{1,n}, M_{1,n}, (M_t)_{1,t}\} = \{\phi_2, \phi_{2,n}, M_{2,n}, (M_t)_{2,t}\},$$

where ϕ_j *is the solution to the equation* $P_{\mu_j}(\phi_j) = 0$, $j = 1, 2$, *then*

$$\partial^\kappa \phi_1 = \partial^\kappa \phi_2 \quad \text{on } \partial\Omega \text{ for } |\kappa| \le 3.$$

Proof. The equalities $\phi_1 = \phi_2$ and $\phi_{1,n} = \phi_{2,n}$ gives $\nabla\phi_1 = \nabla\phi_2$ on $\partial\Omega$, i.e.,

$$\phi_{1,1} = \phi_{2,1}, \quad \phi_{1,2} = \phi_{2,2} \quad \text{on } \partial\Omega$$

and thus

$$\nabla\phi_{1,k} \cdot \mathbf{t} = \nabla\phi_{2,k} \cdot \mathbf{t}, \qquad k = 1, 2 \quad \text{on } \partial\Omega. \tag{3.28}$$

Moreover, since $M_{1,n} = M_{2,n}$, by the definition of M_n and the hypothesis $\mu_1|_{\partial\Omega} = \mu_2|_{\partial\Omega}$, we obtain

$$(n_1^2 - n_2^2)(\phi_{1,11} - \phi_{2,11}) - (n_1^2 - n_2^2)(\phi_{1,22} - \phi_{2,22}) + 4n_1n_2(\phi_{1,12} - \phi_{2,12}) = 0. \tag{3.29}$$

From (3.28) and (3.29), we have

$$AU := \begin{pmatrix} -n_2 & n_1 & 0 \\ 0 & -n_2 & n_1 \\ n_1^2 - n_2^2 & 4n_1n_2 & n_2^2 - n_1^2 \end{pmatrix} \begin{pmatrix} \phi_{1,11} - \phi_{2,11} \\ \phi_{1,12} - \phi_{2,12} \\ \phi_{1,22} - \phi_{2,22} \end{pmatrix} = 0 \quad \text{on } \partial\Omega. \tag{3.30}$$

Since the matrix A is invertible, we get that $\phi_{1,ij} = \phi_{2,ij}$ on $\partial\Omega$ for $1 \le i, j \le 2$.

With $\phi_{1,ij} = \phi_{2,ij}$ on $\partial\Omega$, we can deduce

$$\nabla\phi_{1,ij} \cdot \mathbf{t} = \nabla\phi_{2,ij} \cdot \mathbf{t},$$

that is,

$$-n_2\phi_{1,1ij} + n_1\phi_{1,2ij} = -n_2\phi_{2,1ij} + n_1\phi_{2,2ij}. \tag{3.31}$$

Using the condition $(M_t)_{1,t} = (M_t)_{2,t}$ and $\phi_{1,ij} = \phi_{2,ij}$ on $\partial\Omega$ for $1 \le i, j \le 2$, it follows that

$$\operatorname{div}\left(\frac{1}{2\mu_1}(\nabla^2\phi_1 - \frac{\Delta\phi_1}{2}I_2)\right)\cdot\mathbf{n} = \operatorname{div}\left(\frac{1}{2\mu_2}(\nabla^2\phi_2 - \frac{\Delta\phi_2}{2}I_2)\right)\cdot\mathbf{n}. \tag{3.32}$$

Putting (3.31), (3.32) together and using the boundary assumption of μ, we obtain that

$$\begin{pmatrix} -n_2 & n_1 & 0 & 0 \\ 0 & -n_2 & n_1 & 0 \\ 0 & 0 & -n_2 & n_1 \\ n_1 & n_2 & n_1 & n_2 \end{pmatrix}\begin{pmatrix} \phi_{1,111} - \phi_{2,111} \\ \phi_{1,112} - \phi_{2,112} \\ \phi_{1,122} - \phi_{2,122} \\ \phi_{1,222} - \phi_{2,222} \end{pmatrix} = 0.$$

Since the matrix above is invertible, we deduce that $\phi_{1,ijk} = \phi_{2,ijk}$ for $1 \le i, j, k \le 2$.

□

We are now ready to prove the crucial step.

Lemma 10 (L.-Uhlmann-Wang [39])**.** *Assume that $\mu \in C^3(\overline{\Omega})$. Suppose that $\partial^\kappa\mu_1|_{\partial\Omega} = \partial^\kappa\mu_2|_{\partial\Omega}$, $\forall\ |\kappa| \le 1$. The Cauchy data C^*_μ of P_μ determines the Cauchy data C_{D+V} of $D+V$.*

Proof. Assume that $C^*_{\mu_1} = C^*_{\mu_2}$ with two parameters μ_1 and μ_2. Let $U_1 = (u_1, u_2, u_3, u_4)^T$ be a solution of $(D + V_1)U_1 = 0$, then

$$(D+V_1)U_1 = \left(\begin{pmatrix} \partial_{\overline{z}} & 0 & 0 & 0 \\ 0 & \partial_{\overline{z}} & 0 & 0 \\ 0 & 0 & \partial_z & 0 \\ 0 & 0 & 0 & \partial_z \end{pmatrix} + \begin{pmatrix} \alpha_1 & \beta_1 & \overline{\alpha}_1 & \overline{\beta}_1 \\ -1 & 0 & 0 & 0 \\ \alpha_1 & \beta_1 & \overline{\alpha}_1 & \overline{\beta}_1 \\ 0 & 0 & -1 & 0 \end{pmatrix}\right)\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = 0,$$

where α_j, β_j are defined in (3.26) with respect to μ_j, $j = 1, 2$, respectively. The 2nd and 4th equations of the system $(D + V_1)U_1 = 0$ gives

$$\partial_{\overline{z}}u_2 = u_1, \quad \partial_z u_4 = u_3. \tag{3.33}$$

Likewise, the 1st and 3rd equations of $(D + V_1))U_1 = 0$ implies

$$\partial_{\overline{z}}u_1 = \partial_z u_3 \tag{3.34}$$

It immediately follows from (3.33) and (3.34) that

$$\partial_{\bar{z}}^2 u_2 = \partial_z^2 u_4. \tag{3.35}$$

In view of (3.35) and Lemma 8, there exists a function Φ_1 satisfying

$$\partial_z^2 \Phi_1 = u_2, \quad \partial_{\bar{z}}^2 \Phi_1 = u_4. \tag{3.36}$$

Substituting (3.36) into (3.33) gives

$$u_1 = \partial_{\bar{z}} u_2 = \partial_z^2 \partial_{\bar{z}} \Phi_1, \quad u_3 = \partial_z u_4 = \partial_z \partial_{\bar{z}}^2 \Phi_1. \tag{3.37}$$

The 1st equation of $(D + V_1)U_1 = 0$, i.e.,

$$\partial_{\bar{z}} u_1 + \alpha u_1 + \beta u_2 + \overline{\alpha} u_3 + \overline{\beta} u_4 = 0$$

with $u_1, \cdots, u_4$ replaced by (3.36) and (3.37) above, is equivalent to

$$P_{\mu_1}(\Phi_1) = 0 \quad \text{in} \quad \Omega$$

(cf. (3.25)). Similarly, for V_2 and U_2 satisfying $(D + V_2)U_2 = 0$ in Ω associated with μ_2, we obtain a Φ_2 solving $P_{\mu_2}(\Phi_2) = 0$ in Ω where the components of U_2 and Φ_2 satisfy corresponding equations like (3.36), (3.37). The assumption $C^*_{\mu_1} = C^*_{\mu_2}$ implies

$$\{\Phi_1, \Phi_{1,n}, M_{1,n}, (M_t)_{1,t}\} = \{\Phi_2, \Phi_{2,n}, M_{2,n}, (M_t)_{2,t}\}$$

and Lemma 9 gives

$$\partial^\kappa (\Phi_1 - \Phi_2)|_{\partial\Omega} = 0 \quad \text{for } |\kappa| \le 3.$$

Since $U_2 = (\partial_z^2 \partial_{\bar{z}} \Phi_2, \partial_z^2 \Phi_2, \partial_z \partial_{\bar{z}}^2 \Phi_2, \partial_{\bar{z}}^2 \Phi_2)^T$, we obtain $(U_1 - U_2)|_{\partial\Omega} = 0$ and thus $C_{D+V_1} = C_{D+V_2}$.

□

3.3.3 Proof of the uniqueness result

We denote

$$A = \begin{pmatrix} \alpha & \beta \\ -1 & 0 \end{pmatrix} \text{ and } Q = \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix},$$

then the system $D + V$ can be represented as

$$D + V = \begin{pmatrix} \partial_{\overline{z}} I_2 & 0 \\ 0 & \partial_z I_2 \end{pmatrix} + \begin{pmatrix} A & \overline{Q} \\ Q & \overline{A} \end{pmatrix}.$$

In the following lemma, we show that μ is uniquely determined by the Cauchy data C_{D+V}.

Lemma 11 (L.-Uhlmann-Wang [39])**.** *Let* $(\alpha_j, \beta_j), j = 1, 2$ *be in* $C^1(\overline{\Omega})$*. Assume that* $\partial^\kappa \mu_1|_{\partial\Omega} = \partial^\kappa \mu_2|_{\partial\Omega}$ *for all* $|\kappa| \leq 1$*. If* $C_{D+V_1} = C_{D+V_2}$*, then* $\mu_1 = \mu_2$ *in* Ω*.*

Proof. Using that $C_{D+V_1} = C_{D+V_2}$, we apply Theorem 4.1 in [4] to obtain that there exist invertible matrices $F_j \in C^1(\Omega, \mathbb{C}^2 \oplus \mathbb{C}^2)$ such that $F_1 = F_2$ on $\partial\Omega$. Moreover,

$$\partial_{\overline{z}} F_j = F_j A_j \text{ and } Q_1 = \overline{F} Q_2 F^{-1}, \tag{3.38}$$

where $F := F_1^{-1} F_2$ is an invertible matrix.

Let us denote the two rows of the matrix F_j^{-1} by a_j and b_j, then the first relation of (3.38) implies $\partial_{\overline{z}} F_j^{-1} = -A_j F_j^{-1}$ and hence

$$F_j^{-1} = \begin{pmatrix} \partial_{\overline{z}} b_j \\ b_j \end{pmatrix} \tag{3.39}$$

with the help of the form of A_j. We now write

$$F^{-1} = \begin{pmatrix} h & v \\ m & r \end{pmatrix}.$$

Using the condition $Q_1 = \overline{F} Q_2 F^{-1}$, we have that

$$\overline{m}\alpha_1 = \overline{m}\beta_1 = 0$$

and

$$\overline{h}\alpha_1 = h\alpha_2 + m\beta_2, \quad \overline{h}\beta_1 = v\alpha_2 + r\beta_2. \tag{3.40}$$

Then $m = 0$ in Ω', where $\Omega' = \{x \in \Omega : \alpha_1(x) \neq 0 \text{ or } \beta_1(x) \neq 0\}$. Note that if x is in the complement of Ω', then $(\alpha_2(x), \beta_2(x))$ must be zero by (3.40) since F is invertible. Thus $\alpha_1 = \alpha_2 = 0$ in the complement of Ω'. If Ω' is empty, then $\alpha_1 = 0 = \alpha_2$ in Ω. By the boundary condition $\partial^\kappa \mu_1 = \partial^\kappa \mu_2$ for $|\kappa| \leq 1$, we conclude that $\mu_1 = \mu_2$. Actually, in this case, we obtain that $\mu_1 = \mu_2 = \text{constant}$.

Now we suppose that Ω' is a nonempty open set. Since $m = 0$ in Ω', F^{-1} can be rewritten as

$$F^{-1} = \begin{pmatrix} h & v \\ 0 & r \end{pmatrix}$$

in Ω'. Using $F = F_1^{-1}F_2$ and (3.39), we can deduce that

$$h\partial_{\overline{z}}b_1 + vb_1 = \partial_{\overline{z}}b_2, \quad rb_1 = b_2, \quad \text{in } \Omega',$$

which implies

$$F^{-1} = \begin{pmatrix} r & \partial_{\overline{z}}r \\ 0 & r \end{pmatrix} \quad \text{in } \Omega'. \tag{3.41}$$

In deriving (3.41), we used the fact that $\partial_{\overline{z}}b_1$ and b_1 are linearly independent due to the invertibility of F_1^{-1}. Note that since F is invertible, r never vanishes at any point in Ω'.

We observe that

$$\begin{aligned} \partial_{\overline{z}}F^{-1} = \partial_{\overline{z}}(F_2^{-1}F_1) &= \partial_{\overline{z}}F_2^{-1}F_1 + F_2^{-1}\partial_{\overline{z}}F_1 \\ &= -A_2F^{-1} + F^{-1}A_1, \end{aligned}$$

then it follows that

$$2\partial_{\overline{z}}r = (\alpha_1 - \alpha_2)r \tag{3.42}$$

and

$$\partial_{\overline{z}}^2 r = (\beta_1 - \beta_2)r - \alpha_2\partial_{\overline{z}}r. \tag{3.43}$$

From (3.42), we have

$$2\partial_{\bar{z}}^2 r = r\partial_{\bar{z}}(\alpha_1 - \alpha_2) + (\alpha_1 - \alpha_2)\partial_{\bar{z}} r. \tag{3.44}$$

Substituting ($\partial_{\bar{z}} r = (\alpha_1 - \alpha_2)r/2$) into (3.43) and (3.44) gives

$$2\partial_{\bar{z}}^2 r = (2\beta_1 - 2\beta_2 - \alpha_2(\alpha_1 - \alpha_2))\, r = \left(\partial_{\bar{z}}(\alpha_1 - \alpha_2) + (\alpha_1 - \alpha_2)^2/2\right) r,$$

which implies

$$2\beta_1 - 2\beta_2 - \alpha_2(\alpha_1 - \alpha_2) = \partial_{\bar{z}}(\alpha_1 - \alpha_2) + (\alpha_1 - \alpha_2)^2/2. \tag{3.45}$$

Note that r does not vanish in Ω'. By direct computation and the definition of α_j and β_j in (3.26), it follows that

$$2\beta_j = \partial_{\bar{z}}\alpha_j + \alpha_j^2. \tag{3.46}$$

Then we obtain

$$\alpha_1^2 = \alpha_2^2 \ \text{ in } \ \Omega' \tag{3.47}$$

by substituting (3.46) into (3.45). Combining (3.47) and the previously derived fact

$$\alpha_1 = \alpha_2 = 0 \ \text{ in } \ \Omega \setminus \Omega',$$

we have that

$$\alpha_1^2 = \alpha_2^2 \ \text{ in } \ \Omega,$$

which is equivalent to

$$(\nabla \log \mu_1)^2 = (\nabla \log \mu_2)^2 \ \text{ in } \ \Omega.$$

Since $\mu_1|_{\partial\Omega} = \mu_2|_{\partial\Omega}$ and by the continuity of μ_j and $\nabla\mu_j$, $j = 1, 2$, we obtain

$$\nabla \log \mu_1 = \nabla \log \mu_2 \ \text{ in } \ \Omega.$$

Using the boundary condition $\mu_1|_{\partial\Omega} = \mu_2|_{\partial\Omega}$ again, we finally conclude that $\mu_1 = \mu_2$ in Ω.

□

Proof of theorem 7. From Section 3.2 we have known that the Cauchy data for the Stokes equations and that for the equation $P_\mu(\phi) = 0$ are equivalent, that is, $C_{\mu_1} = C_{\mu_2}$ is equivalent to $C^*_{\mu_1} = C^*_{\mu_2}$. Therefore, Theorem 7 follows from Lemma 10 and Lemma 11.

3.4 Global uniqueness for the stationary Navier–Stokes equations

In this section we consider the unique determination of the viscosity in an incompressible fluid described by the stationary Navier–Stokes equations. In higher dimensions, this problem has been solved by Li and Wang in [41] using the linearization technique. Since their methods are independent of spatial dimensions, we could apply their ideas to show the uniqueness result of μ for the Navier–Stokes equations in the two dimensional case.

Let $u = (u_1, u_2)^T$ be the velocity vector field satisfying the stationary Navier–Stokes equations

$$\begin{cases} \operatorname{div} \sigma(u,p) - (u \cdot \nabla)u = 0 & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \end{cases} \tag{3.48}$$

and the corresponding Cauchy data is denoted by

$$\tilde{C}_\mu = \left\{ (u, \sigma(u,p)\mathbf{n})\,|_{\partial\Omega} : (u,p) \text{ satisfies } (3.48) \right\}.$$

Let $u|_{\partial\Omega} = \phi \in H^{3/2}(\partial\Omega)$ satisfy (3.3). We choose $\phi = \varepsilon\psi$ with $\psi \in H^{3/2}(\partial\Omega)$ and let $(u_\varepsilon, p_\varepsilon) = (\varepsilon v_\varepsilon, \varepsilon q_\varepsilon)$ satisfy (3.48). The problem (3.48) is reduced to

$$\begin{cases} \operatorname{div} \sigma(v_\varepsilon, q_\varepsilon) - \varepsilon(v_\varepsilon \cdot \nabla)v_\varepsilon = 0 & \text{in } \Omega, \\ \operatorname{div} v_\varepsilon = 0 & \text{in } \Omega, \\ v_\varepsilon = \psi & \text{on } \partial\Omega. \end{cases} \tag{3.49}$$

We are looking for a solution of (3.49) with the form $v_\varepsilon = v_0 + \varepsilon v$ and $q_\varepsilon = q_0 + \varepsilon q$, where (v_0, q_0) satisfies the Stokes equations

$$\begin{cases} \operatorname{div} \sigma(v_0, q_0) = 0 & \text{in } \Omega, \\ \operatorname{div} v_0 = 0 & \text{in } \Omega, \\ v_0 = \psi & \text{on } \partial\Omega, \end{cases} \tag{3.50}$$

and (v, q) satisfies

$$\begin{cases} -\operatorname{div} \sigma(v,q) + \varepsilon(v_0 \cdot \nabla)v + \varepsilon(v \cdot \nabla)v_0 + \varepsilon^2(v \cdot \nabla)v = f & \text{in } \Omega, \\ \operatorname{div} v = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.51}$$

with $f = -(v_0 \cdot \nabla)v_0$.

In [41], it is shown that for any $\psi \in H^{3/2}(\partial\Omega)$, let $(v_0, q_0) \in H^2(\Omega) \times H^1(\Omega)$ be the unique solution (q_0 is unique up to a constant) of the Stokes equations (3.50). There exists a solution $(u_\varepsilon, p_\varepsilon)$ of (3.48) of the form

$$u_\varepsilon = \varepsilon v_0 + \varepsilon^2 v, \quad p_\varepsilon = \varepsilon q_0 + \varepsilon^2 q$$

with the boundary data $u_\varepsilon|_{\partial\Omega} = \varepsilon\psi$ for all $|\varepsilon| \le \varepsilon_0$, where ε_0 depends on $\|\psi\|_{H^{3/2}(\partial\Omega)}$. Here (v, p) is a solution of (3.51) and satisfies the regularity result

$$\|v\|_{H^2(\Omega)} + \|q\|_{H^1(\Omega)/\mathbb{R}} \le C \sum_{j=2}^{16} \|\psi\|^j_{H^{3/2}(\partial\Omega)}$$

where C is independent of ε and $\|q\|_{H^1(\Omega)/\mathbb{R}} := \inf_{c\in\mathbb{R}} \|q + c\|_{H^1(\Omega)}$. Hence, we have

$$\begin{aligned} \|\varepsilon^{-1}u_\varepsilon - v_0\|_{H^2(\Omega)} &= \|\varepsilon v\|_{H^2(\Omega)} \to 0, \\ \|\varepsilon^{-1}p_\varepsilon - q_0\|_{H^1(\Omega)/\mathbb{R}} &= \|\varepsilon q\|_{H^1(\Omega)/\mathbb{R}} \to 0, \end{aligned}$$

as $\varepsilon \to 0$, which imply

$$\|\varepsilon^{-1}u_\varepsilon|_{\partial\Omega} - v_0|_{\partial\Omega}\|_{H^{3/2}(\partial\Omega)} \to 0, \tag{3.52}$$

and

$$\|\varepsilon^{-1}\sigma(u_\varepsilon, p_\varepsilon)\mathbf{n}|_{\partial\Omega} - \sigma(v_0, q_0)\mathbf{n}|_{\partial\Omega}\|_{H^{1/2}(\partial\Omega)} \to 0, \tag{3.53}$$

provided

$$\int_\Omega p_\varepsilon dx = \int_\Omega q_0 dx = 0.$$

From (3.52) and (3.53), we can deduce that the Cauchy data $\tilde{C}_\mu$ of the Navier–Stokes equations uniquely determines the Cauchy data C_μ of the Stokes equations. In other words, $\tilde{C}_{\mu_1} = \tilde{C}_{\mu_2}$ implies $C_{\mu_1} = C_{\mu_2}$. Therefore, the uniqueness of the viscosity for the Navier–Stokes equations follows from Theorem 7. We have the following theorem.

Theorem 8 (L.-Uhlmann-Wang [39]). *Let* Ω *be a simply connected bounded domain in* $\mathbb{R}^2$ *with smooth boundary. Suppose that* μ_1 *and* μ_2 *are two viscosity functions for the Navier–Stokes equations. Assume that* $\mu_j \in C^3(\overline{\Omega})$ *and* $\mu_j > 0$ *with*

$$\partial^\kappa \mu_1|_{\partial\Omega} = \partial^\kappa \mu_2|_{\partial\Omega} \ \textit{ for all } |\kappa| \leq 1.$$

Let $\tilde{C}_{\mu_1}$ *and* $\tilde{C}_{\mu_2}$ *be the Cauchy data associated with* μ_1 *and* μ_2*, respectively. If* $\tilde{C}_{\mu_1} = \tilde{C}_{\mu_2}$*, then* $\mu_1 = \mu_2$ *in* Ω.

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