Essays on International Demographic Economics

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A dissertation

submitted in partial fulfillment of the

requirements for the degree of

Doctor of Philosophy

University of Washington

2016

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Department of Economics

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Abstract

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This dissertation focuses on including realistic demographic structures in macroeconomic frameworks in order to circumvent existing modeling issues, explain current international trends, and forecast the implications of new social policies. The first chapter covers the closure of the small open economy model and a comparison of demographic structures. Closing the small open economy model has been a stumbling block in studying the dynamic implications of such models since the typical procedure of equating the after-tax return on traded bonds to the rate of time preference involves imposing constraining knife-edge conditions. This paper replaces the infinitely-lived representative agent framework with a plausible demographic structure. This yields a well-behaved macrodynamic equilibrium without imposing any knife-edge conditions.

The second chapter develops a two-country overlapping generations neoclassical growth model including a realistic demographic structure for the purpose of analyzing the impact of country-level asymmetries in demographic and structural characteristics on cross-country interdependence. I find that an increase in the relative life expectancy of a population will produce a positive per-capita net foreign asset position. Furthermore, I demonstrate how cross-country differences in the rate of time preference will augment

the decline of the American net foreign asset position generated by the demographic transition. Lastly, I present how an adjustment in the pension benefit of a pay-as-you-go social security structure will induce a change in the simulated net foreign asset position.

Lastly, in the third chapter, I develop a modified version of the Mierau and Turnovsky (2015) model to determine the impact of the reversal of China's state fertility policy commonly known as the "one-child policy". In order to estimate the impact of the policy reversal on the labor market, I forecast the survival function forward 20 years to determine the old-age dependency rate. I augment the analysis by including a simulation estimating the impact of the proposed 5-year extension of the retirement age. From this analysis I develop two key results. The first is that the government will be forced to reduce the national pension benefit by 2.85 percent if they continue with their established policies. The second is that the proposed 5 year retirement age extension will be sufficient to keep the pension system solvent during the demographic transition

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Chapter 1: Closing the Small Open Economy Model: A Demographic Approach

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Abstract

Closing the small open economy model has been a stumbling block in studying the dynamic evolution of such models. The typical procedure of equating the after-tax return on traded bonds to the rate of time preference involves imposing an arbitrary and constraining knife-edge condition. This paper replaces the infinitely-lived representative agent framework with a plausible demographic structure. This yields a well-behaved macrodynamic equilibrium without imposing any knife-edge conditions. The equilibrium dynamics generated by the Rectangular survival function, characteristic of the Samuelson-Diamond model, closely track those corresponding to an empirically-estimated survival function. However, the Blanchard survival function tracks the data poorly in terms of absolute levels, while the closeness of its relative dynamics (following a structural change) depends on the source of the structural change.

Keywords: Demographic structure, small open economy, macrodynamic equilibrium

JEL Classification: F32, F40, F41, J11

*Turnovsky's research was supported in part by the Van Voorhis endowment at the University of Washington. We wish to acknowledge the comments on this paper received from Hippolyte d'Albis, Neil Bruce, Santanu Chatterjee, Yu-chin Chen, David de la Croix, Oksana Leukhina, and Jochen Mierau. The paper has also benefited from comments received from seminar presentations at the National Taiwan University and the National University of Singapore.

1. Introduction

The canonical model of a small open economy assumes that it faces an exogenously fixed rate of return on traded bonds. Coupled with the conventional assumption of a constant rate of time preference, together with a constant tax rate on capital income, an interior equilibrium in a stationary economy can be sustained if and only if the implied constant after-tax return on bonds equals the given rate of time preference. This introduces a "knife-edge condition", which although invoked by much of the international macrodynamics literature based on utility-maximizing representative agents, raises several awkward issues.[1] First, there is no compelling reason why the required equality should hold. With the return on bonds determined in the rest of the world and the tax rate presumably set by the domestic government, there is absolutely nothing to ensure that the after-tax rate of return should equal the rate of time preference, which reflects the degree of impatience of domestic consumers. Second, requiring that this equality must always hold precludes the ability to analyze the "pure" effects of specific structural changes. For example, starting from an initial interior equilibrium, an increase in the foreign interest rate must be accompanied by either an appropriately set increase in the rate of time preference or a reduction in the tax rate in order for the knife-edge condition to be sustained and a new interior equilibrium to be reached. At best, one can determine some composite effect of the rise in the foreign interest rate, with the effect depending upon whether the necessary accommodation occurs via the rate of time preference or the tax rate. Third, the introduction of the knife-edge condition seriously limits the ability of the model to generate plausible transitional dynamics. In the absence of physical capital with associated adjustment costs, the only sustainable equilibrium is for the economy always to be in steady state, determined in part by its initial stock of traded bonds. The accumulation of traded bonds violates the transversality conditions, thus precluding any current account dynamics.

These are serious drawbacks and several approaches to circumventing these

[1] Early examples imposing the knife-edge condition include Obstfeld (1983), Brock (1988), and Sen and Turnovsky (1989). For textbook discussions of this issue see e.g. Blanchard and Fischer (1989) and Turnovsky (1997). The term "knife-edge" was originally introduced in a different context by Harrod (1939). It refers to the need to constrain some subset of independently set parameters in order for an equilibrium condition to be maintained. The small open economy model has several knife-edge characteristics, including some associated with generating endogenous growth; see Turnovsky (2002).

unappealing consequences have been adopted. One of the earliest was to modify the assumption of a fixed rate of time preference by assuming an Uzawa (1968) utility function; see e.g. Obstfeld (1981). This approach has been criticized in that to generate plausible transitional dynamics one needs to make the counter-intuitive assumption that wealthier agents are more impatient; see Blanchard and Fischer (1989). Another early approach was to assume that the purchase of traded bonds involves convex transactions costs, typically specified as being quadratic. This assumption, first introduced by Turnovsky (1985), later by Beningo (1989), and used recently by Cacciatore and Ghironi (2012), while convenient, is arbitrary. An alternative assumption, which dates back to Bardhan (1967), is to drop the assumption of a fixed given world interest rate and to assume that the borrowing or lending rate facing an individual country depends upon its net asset position in the world economy. While in many instances this modification is entirely plausible, particularly in the case of developing economies that are accumulating debt, the country can no longer be characterized as being a pure small economy, at least insofar as the international financial market is concerned.[2]

While the assumptions of a fixed world interest rate and constant rate of time preference are restrictive, the adoption of the infinitely-lived representative agent framework is also problematic. Indeed, Blanchard's (1985) celebrated overlapping generations (OLG) model does not require any knife-edge condition to hold. Instead, for an arbitrarily set rate of time preference, capital income tax, and foreign interest rate, his model yields a convergent dynamic equilibrium time path along which the small open economy gradually adjusts its holdings of traded assets. While Blanchard's model has the virtue of transparency and tractability, this comes at the price of being based on an unrealistic demographic survival function which has the property that mortality is independent of age and therefore cannot address lifecycle issues, as Blanchard himself acknowledged.

In this paper we build on the demographic structure pioneered by Blanchard and assume a general survival function, having the property that mortality increases with age,

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[2] For further discussions of alternative ways to close small open economy models, see Turnovsky (2002) and Schmitt-Grohé and Uribe (2003). Some of the more recent literature avoids the knife-edge condition by introducing financial frictions of various kinds; see e.g. Christiano, Trabandt, and Walentin (2011).

as intuition clearly suggests and empirical evidence strongly supports. We derive the macrodynamic equilibrium of a small open economy and show that its local dynamics can be approximated by a differential equation system which involves the evolution of: (i) the per capita accumulation of traded bonds, (ii) per capita consumption, (iii) human capital at birth, and (iv) the marginal propensity to consume at birth, the latter two capturing the role of the changing demographic structure as the economy develops over time. We show that the introduction of a general demographic structure eliminates the need for imposing any knife-edge condition.

Because of the relative complexity of the aggregate dynamic system, we conduct our analysis of specific structural changes numerically. To do so, we employ the survival function introduced by Boucekkine, de la Croix, and Licandro (2002) (BCL) as a benchmark. This function is highly tractable and tracks the empirical data on survival for most Western economies remarkably successfully. One of the reasons for its success is that it turns out to be a very good first order approximation to the Gompertz (1825) function, which demographers find to be the most accurate representation of mortality, but being a double exponential function its dynamics are computationally intractable in a general equilibrium framework.[3]

We compare the predictions of the BCL survival function to those of two alternative survival functions, parameterizing them so that all three yield the same life expectancy, (E), as that implied by the BCL model. The first is a "Rectangular" survival function in which agents live with certainty until age E is reached, at which time they all die. There is no uncertainty about life expectancy, so comparing with the BCL function gives some indication of the impact of uncertainty with respect to lifespan. This comparison is of some interest for at least two reasons. First, this "one hoss shay" representation of lifespan is characteristic of the early seminal work on OLG models by Samuelson (1958) and Diamond (1965). Second, there is a growing interest in the demographic literature on the "rectangularization of the survival function".[4] Indeed, for our benchmark parameterization, as well as across a wide range of parameter variations,

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[3] See Bruce and Turnovsky (2013b).

[4] This pertains to the trend toward a more rectangular shape of the survival function due to increased survival and concentration of deaths around the mean survival age; see e.g. Nuseelder and Mackenbach (1996) and Rossi, Rousson, and Paccaud (2012).

we find that the dynamic aggregate adjustments under the BCL and Rectangular survival functions track each other remarkably closely. This suggests that the latter, despite its simplicity, may in fact serve as a reasonable approximation to the empirical evidence on survival, insofar as its macroeconomic implications are concerned.

The other comparison is with the Blanchard survival function and here we find that the equilibria and the transitional dynamics it generates deviate significantly from either of the other two. This is a reflection of the "perpetual youth" assumption that it embodies, and its resulting convex rather than concave (with respect to the origin) survival function, which implies an unrealistic degree of uncertainty with respect to lifespan, leading to an unrealistic level of savings. Given the fact that the other two survival functions both track empirical data much more closely suggests that, despite its analytical convenience, it is desirable to move beyond the Blanchard specification of mortality.

There is a growing literature seeking to incorporate a more realistic demographic structure into macroeconomic models. Virtually all of these papers do so within the context of a closed economy model. Some studies introduce very general mortality structures to study issues pertaining to existence and uniqueness of steady-state equilibria; see e.g. Bommier and Lee (2003), d'Albis (2007), and Lau (2009). Others adopt empirically plausible mortality functions to analyze structural and demographic changes; see e.g. Boucekkine et al. (2002), Faruqee (2003), Heijdra and Romp (2008), Heijdra and Mierau (2012), and Bruce and Turnovsky (2013a, 2013b). In general, the global dynamics of an OLG model having a realistic demographic structure is represented by a high order transcendental equation and is intractable with a neoclassical production structure; see d'Albis and Augeron-Véron (2009). In response to this, Mierau and Turnovsky (2014b) propose a linear approximation, which enables them to represent the aggregate dynamics locally as indicated above.

Of the recent literature, the present paper is closest to Heijdra and Romp (2008), who also consider a small open economy. However, their focus is very different. The present concern is with the macrodynamic equilibrium and showing how the demographic structure avoids the knife-edge condition, characteristic of much of the small open economy literature. Heijdra and Romp restrict their analysis to the steady state and the distributional consequences of various fiscal policies, something that we address briefly as well. Finally, they assume that output grows exogenously with population, whereas this paper endogenizes output through the labor-leisure choice.

The remainder of the paper is structured as follows. Following this introduction, Section 2 sets out the analytical framework, with its demographic structure. Section 3 derives the macrodynamic equilibrium, while Section 4 discusses the steady state. Section 5 briefly revisits the knife-edge issue in the demographic economy. Sections 6 describes the calibration of the model, while numerical simulations, relating to the aggregate dynamics and the distribution across cohorts are reported in Sections 7 and 8. Section 9 concludes, while detailed derivations are relegated to the Appendix.

2. Analytical Framework

In describing the model, it is important to distinguish between calendar time and an agent's age. In general, the variable $X(v,t)$ refers to an agent born at time v, at calendar time *t*, when he has reached age, $(t-v)$. The partial derivative with respect to calendar time, t, represented by $\partial X/\partial t \equiv X_t(v,t)$ describes the change in X over time for a given cohort, as it ages. The partial derivative across cohorts, represented by $\partial X/\partial v \equiv X_v(v,t)$ describes the change across cohorts at a given point in time.

2.1 Individual Household Behavior

Consider an individual born at time *v*. The probability that this individual lives to become $(t-v)$ years old is described by the survival function, $S(t-v)$, where $S'(s) \equiv dS(s)/ds < 0$ decreases with age. The corresponding mortality rate, or instantaneous probability of death, is given by

$$\mu(t-v) = -\frac{S'(t-v)}{S(t-v)} > 0 \tag{1}$$

The probability that the individual dies before reaching age $(t-v)$ is determined by the cumulative mortality rate:

$$M(t-v) = \int_0^{t-v} \mu(\tau)d\tau \tag{2}$$

Combining these two relationships, the survival function and the mortality function are related by

$$S(t-v)=e^{-M(t-v)} \tag{3}$$

where we assume $S(0)=e^{-M(0)}=1,\ S(D)=e^{-M(D)}=0$, indicating that the probability of survival of a newborn is 1, while D defines the maximum age that an individual can attain.[5]

Specifying the mortality function as in (3), enables us to express the discounted expected lifetime utility of an individual born at time v, having an isoelastic function, by

$$\int_{v}^{v+D}\frac{1}{\gamma}\left(C(v,t)l(v,t)^{\theta}\right)^{\gamma}e^{-\rho(t-v)-M(t-v)}dt$$

(4a)

where $C(v,t)$ denotes consumption of a traded good, and $l(v,t)$ denotes leisure, at time t of an individual born at time v, with θ parameterizing their relative weight in utility. In addition, γ, is related to the intertemporal elasticity of substitution, σ, by $\sigma=1/(1-\gamma)$, ρ is the pure rate of time discount of a newborn, while $\rho+\mu(t-v)$ is the rate of time discount at age $(t-v)$.[6]

Each individual is endowed with a unit of time that he can allocate either to leisure or to supplying labor to firms at a wage rate, $w(t)$. These individuals also own the firms, and as an owner each one receives a share of the profit, $\Pi(t)$, which is distributed uniformly across all cohorts, with total factor income being taxed at the constant rate τ_y.[7] In addition to choosing leisure and consuming the traded good, the agent also accumulates internationally traded bonds, subject to his instantaneous budget

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[5] We shall treat D as being finite, though the extension to an infinite D (as in Blanchard, 1985) is straightforward.

[6] From (1)-(3) the mortality rate and therefore the discount rate increases with age if and only if $SS''<(S')^2$ which is certainly met if the survival function is concave.

[7] An alternative assumption is to assume that the share of profit received by the agent, $\Pi(v,t)$, is proportional to the labor he provides, namely $\Pi(v,t)=\left((1-l(v,t))/(1-l(t))\right)\Pi(t)$, where $l(t)$ denotes aggregate leisure. Since this assumption, while arguably more realistic, has no bearing on the main focus of this paper, namely the role of the demographic structure in closing the small open economy model, we prefer to adopt the simpler assumption.

constraint:

$$F_t(v,t) \equiv \frac{\partial F(v,t)}{\partial t} = (1-\tau_y)\Big[w(t)\big(1-l(v,t)\big)+\Pi(t)\Big]+[(1-\tau_r)r^* + \mu(\mathrm{t}-v)]F(v,t)-C(v,t)-T(t) \tag{4b}$$

where $F(v,t)$ denotes the traded bonds held at time *t* by an individual born at time *v*, r^* is the given world rate of return on traded bonds, the income of which is taxed at the constant rate τ_r and $T(t)$ are lump sum taxes paid or rebates received, assumed to be uniform across cohorts.

Individuals are born without assets, have no bequest motive, and are not permitted to have debt if they reach the maximum attainable age *D*. These conditions at the beginning and end of life imply $F(v,v)=0=F(v,v+D)$, and individuals fully annuitize their assets. The annuities are actuarially-fair life-insured financial assets that pay conditional on the survival of the individuals. Individuals receive a premium on these annuities equal to their instantaneous probability of death, $\mu(t-v)$, and in return when an agent dies his assets flow to the insurance company. Thus the overall rate of return received by an agent on his assets is $[(1-\tau_r)r^* + \mu(\mathrm{t}-v)]$.[8] Alternatively, an individual may borrow. In that case he pays a premium of $\mu(t-v)$ and if he dies his debts are cancelled.

Optimizing (4a) subject to (4b) with respect to $C(v,t)$, $l(v,t)$, and $F(v,t)$ yields:

$$\left(C(v,t)l(v,t)^{\theta}\right)^{\gamma-1} l(v,t)^{\theta} = \lambda(v,t) \tag{5a}$$

$$\left(C(v,t)l(v,t)^{\theta}\right)^{\gamma-1} \theta l(v,t)^{\theta-1}C(v,t) = \lambda(v,t)w(t)(1-\tau_y) \tag{5b}$$

[8] The assumption of an actuarially-fair annuities market originated with Yaari (1965) and is a central feature of the Blanchard (1985) model and the vast literature that it has spawned. It provides a convenient mechanism whereby the financial wealth of decedents is recycled to the survivors in the economy. In the absence of this assumption, when agents die they leave accumulated financial capital as unintended bequests. The nature of the equilibrium depends upon how these unintended bequests are reallocated across survivors; see e.g. Hansen and İmrohoroğlu (2008), Heijdra and Mierau (2012), and Bruce and Turnovsky (2013b) for examples of the consequences of annuities market imperfections.

$$\rho - \frac{\lambda_t(v,t)}{\lambda(v,t)} = (1-\tau_r)r^* \tag{5c}$$

Equation (5a) equates the marginal utility of consumption to the agent's shadow value of wealth, $\lambda(v,t)$, while (5b) equates the marginal utility of leisure to the utility value of the wage income foregone. Dividing (5b) by (5a) yields

$$\frac{C(v,t)}{l(v,t)} = \frac{1}{\theta} w(t)(1-\tau_y) \tag{6a}$$

which implies that agents of all ages choose the same time-varying ratio of consumption to leisure. Equation (5c) asserts that, for any agent, his rate of return on consumption must equal the after-tax rate of return on the traded bond. Solving this equation forward in time implies

$$\lambda(v,t) = \lambda(v,v)e^{(\rho - r^*(1-\tau_r))(t-v)} \tag{6b}$$

from which we see that the marginal utility of wealth of all agents grows indefinitely at the common constant rate $\rho - r^*(1-\tau_r)$, which is independent of their date of birth or calendar time.

In the case of the infinitely-lived representative agent economy this is the source of the knife-edge problem. The need to impose the equality $\rho = r^*(1-\tau_r)$ in that case arises in order to ensure that the long-run equilibrium remains bounded, as is required for a non-growing economy to be sustainable over time.[9] This implies that the marginal utility of wealth remains constant over time. Moreover, for the frequently adopted assumption that labor is supplied inelastically, the knife-edge condition is expressed in terms of the agent's consumption growth rate and implies complete consumption smoothing. In contrast, with heterogeneous cohorts who have finite mortality, the steady growth of marginal utility of each cohort is perfectly compatible with a bounded, sustainable, aggregate equilibrium.

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[9] In an endogenous growth context, the knife-edge condition need not apply and an equilibrium in which consumption grows indefinitely can be sustained if the country is sufficiently patient; see Turnovsky (1996).

Substituting for $l(v,t)$ from (6a) into (5a), differentiating with respect to *t*, and combining with (5c) yields the agent's growth rate of consumption

$$\frac{C_t(v,t)}{C(v,t)} = \frac{(1-\tau_r)r^* - \rho - \theta\gamma\left(\dot{w}(t)/w(t)\right)}{1-\gamma\left(1+\theta\right)} \equiv \psi(t) \tag{7}$$

As a result of the elastically supplied labor, each agent's growth rate of consumption varies with the growth rate of the wage rate, at a common rate across cohorts.[10] Solving (7), we can obtain the agent's consumption level at any arbitrary point in time, τ, relative to some earlier point in time, *t*:

$$C(v,\tau) = C(v,t)e^{\int_t^\tau \psi(s)ds} \tag{8}$$

To express the agent's consumption in terms of his financial resources we integrate the budget constraint (4b) forward from time *t* and impose the transversality condition $F(v, v+D) = 0$. This yields the intertemporal budget constraint

$$\begin{aligned} F(v,t) + \int_t^{v+D} \left\{(1-\tau_y)\left[w(\tau)\left(1-l(v,\tau)\right)+\Pi(\tau)\right] - T(\tau)\right\} e^{-(1-\tau_r)r^*(\tau-t)+M(t-v)-M(\tau-v)} d\tau \\ = \int_t^{v+D} C(v,\tau) e^{-(1-\tau_r)r^*(\tau-t)+M(t-v)-M(\tau-v)} d\tau \end{aligned} \tag{9}$$

which asserts that the present value of the agent's consumption discounted also for survival equals the discounted present value of after-tax resources. Using (6a), (9) can be expressed equivalently:

$$\begin{aligned} F(v,t) + \int_t^{v+D} \left\{(1-\tau_y)\left[w(\tau)+\Pi(\tau)\right] - T(\tau)\right\} e^{-(1-\tau_r)r^*(\tau-t)+M(t-v)-M(\tau-v)} d\tau \\ = (1+\theta)\int_t^{v+D} C(v,\tau) e^{-(1-\tau_r)r^*(\tau-t)+M(t-v)-M(\tau-v)} d\tau \end{aligned} \tag{9'}$$

Substituting (8) into (9) yields the following expression for the consumption of an agent

[10] If $\theta\gamma < 0$ so that leisure and consumption are Edgeworth substitutes (labor supply and consumption are complements), an increase in the growth rate of wages reduces leisure and increases the consumption growth rate, and correspondingly if $\theta\gamma > 0$.

born at time *v*, at calendar time *t*, $C(v,t)$

$$C(v,t) = \frac{F(v,t) + H(v,t)}{\Delta(v,t)} \tag{10a}$$

where

$$H(v,t) \equiv \int_t^{v+D} \left\{ (1-\tau_y)\left[w(\tau)\left(1-l(v,\tau)\right) + \Pi(\tau) \right] - T(\tau) \right\} e^{-(1-\tau_r)r^*(\tau-t) - M(\tau-v) + M(t-v)} d\tau \tag{10b}$$

denotes the agent's discounted future income (human wealth) at time *t*, and

$$\Delta(v,t) \equiv \int_t^{v+D} e^{\int_t^\tau \psi(s)ds - (1-\tau_r)r^*(\tau-t) - M(\tau-v) + M(t-v)} d\tau \tag{10c}$$

is the inverse of the marginal propensity to consume out of total wealth [financial wealth $F(v,t)$ plus human wealth $H(v,t)$]. Setting $t = v$, yields the corresponding quantities at birth, where with no bequests $F(v,v) = 0$.

2.2 Aggregate Demography

At each instant, a cohort of size $P(v,v) = \beta P(v)$ is born, where $P(v,v)$ is the size of the cohort, $P(v)$ is the size of the total population at time v, and β is the (crude) birth rate, as measured by the average number of births per population size, and taken as constant.[11] The number of individuals of cohort v alive at time t is $P(v,t) = \beta P(v) e^{-M(t-v)}$. Summing over all surviving cohort members, the population at time t is $P(t) = \beta \int_{t-D}^{t} P(v) e^{-M(t-v)} dv$. Alternatively, knowing $P(v)$, the population alive at time v, the population at time t is equal to $P(t) = P(v)e^{n(t-v)}$, where n is the population growth rate, assumed to be constant. Equating the two expressions for $P(t)$ leads to the following relationship that uniquely connects the mortality rate, the birth rate, and the population growth rate:

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[11] It is straightforward to modify the specification of the birth rate to allow the child-bearing population to be some subset of the overall population.

$$\frac{1}{\beta} = \int_{t-D}^{t} e^{-n(t-v)-M(t-v)} dv. \tag{11}$$

Under our assumptions (i) β and n are constants and (ii) mortality depends only on age and is independent of calendar time, we can eliminate t from (11), and rewrite it in the form:

$$\frac{1}{\beta} = \int_{0}^{D} e^{-n(s)-M(s)} ds \tag{11'}$$

where s indexes age; (11') is commonly referred to as the demographic steady state (Lotka, 1998, p.60). Dividing $P(v,t)$ by $P(t)$, the relative size of each cohort is:

$$p(t-v) \equiv \frac{P(v,t)}{P(t)} = \beta e^{-n(t-v)-M(t-v)} = \beta e^{-ns-M(s)}, \tag{12}$$

which, in the demographic steady state also depends only on age $s \equiv t-v$, but not on calendar time t. The dynamics of (12) are given by

$$\frac{p_t(t-v)}{p(t-v)} = -\left[n + \mu(t-v)\right] \tag{13}$$

so that the decline in the relative size of each cohort over time reflects both its mortality rate and the overall population growth rate.

To obtain aggregate per capita values, we sum across cohorts by employing the following generic aggregator function

$$x(t) \equiv \int_{t-D}^{t} p(t-v) X(v,t) dv = \beta \int_{t-D}^{t} e^{-n(t-v)-M(t-v)} X(v,t) dv \tag{14}$$

Taking the time derivative of (14), and using (13) and the fact that $p(0) = \beta, p(D) = 0$, the evolution of $x(t)$ is given by:

$$\dot{x}(t) \equiv \beta X(t,t) - nx(t) + \int_{t-D}^{t} p(t-v) X_t(v,t) dv - \int_{t-D}^{t} \mu(t-v) p(t-v) X(v,t) dv.$$

The rate of change of any aggregate variable, $x(t)$, depends upon the difference between the contribution of the newborn relative to the average, plus the rate of change across the survivors, less the amount released by the dying.

Using this notation, aggregate per capita consumption is:

$$c(t) \equiv \int_{t-D}^{t} p(t-v)C(v,t)dv = \beta \int_{t-D}^{t} e^{-n(t-v)-M(t-v)} C(v,t)dv. \quad (15)$$

Taking the time derivative of this expression and using (7) and (13) the dynamics of aggregate per capita consumption can be expressed in the form:

$$\dot{c}(t) \equiv \beta C(t,t) + \left[\psi(t) - \mu_c(t-v_1) - n\right] c(t) \quad (16)$$

where we define:[12]

$$\mu_c(t-v_1) \equiv \frac{\int_{t-D}^{t} \mu(t-v)\, p(t-v)C(v,t)dv}{\int_{t-D}^{t} p(t-v)C(v,t)dv} = \frac{1}{c(t)} \int_{t-D}^{t} \mu(t-v)\, p(t-v)C(v,t)dv \quad v_1 \in (t-D,t) \quad (17)$$

From (17) we see that $\mu_c(t-v_1)$ is the ratio of the consumption given up by the dying to aggregate consumption and can be interpreted as providing an estimate of average mortality over the period $(t-D,t)$, from the consumption profile across the cohorts.[13] For the general mortality function, $\mu_c(t-v_1)$ varies with time, although in the special case of the Blanchard (1985) survival function it is time-invariant; see (35c) below.

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[12] Equation (17) is a statement of the first mean value theorem of integration. We should note that the intermediate value $v_1 \in (t-D,t)$ will in general be a function of *t*, which case $\mu_c(t-v_1)$ should be written as $\mu_c(t-v_1(t))$. We refrain from representing this explicitly, so as not to clutter notation.

[13] The quantity $\left[\mu_c(t-v_1)+n\right]c(t) - \beta C(t,t)$ reflects the reduction in aggregate per capita consumption growth, below that of each cohort due to the arrival of newborn agents with no accumulated assets and the departure due to death of agents with assets. Mierau and Turnovsky (2014a) identify this as the "generational turnover term".

Applying (14) to the cohort holdings of traded bonds, the aggregate per capita holdings of traded bonds are $f(t) \equiv \int_{t-D}^{t} p(t-v)F(v,t)dv$. Taking the time derivative and using (4b) and (13) together with the fact that $F(t,t)=0$, yields

$$\dot{f}(t) = -\int_{t-D}^{t}[\mu(t-v)+n]p(t-v)F(v,t)dv$$
$$+\int_{t-D}^{t} p(t-v)\left\{(1-\tau_y)\left[w(t)\left(1-l(v,t)\right)+\Pi(t)\right]+\left[(1-\tau_r)r^* + \mu(t-v)\right]F(v,t)-C(v,t)-T(t)\right\}dv$$

(18)

We shall assume that the domestic government maintains a balanced budget by collecting taxes from the cohorts and rebating the revenues uniformly. In per capita terms this is expressed by

$$\int_{t-D}^{t} p(t-v)\left\{\tau_y\left[w(t)(1-l(v,t)+\Pi(t)\right]+\tau_r r^* F(v,t)+T(t)\right\}dv = 0 \qquad (19)$$

which performing the aggregation implies

$$\tau_y\left[w(t)\left(1-l(t)\right)+\Pi(t)\right]+\tau_r r^* f(t)+T(t)=0 \qquad (19')$$

where $l(t)$, defined in accordance with (14), specifies aggregate per capita leisure and $\Pi(t)$, per capita profit, is defined analogously.

Summing (6a) across the surviving cohorts, we immediately see that the following analogous relationship applies to the aggregates

$$\frac{c(t)}{l(t)} = \frac{1}{\theta} w(t)(1-\tau_y) \qquad (20)$$

2.3 Firms

Output is produced by a single representative firm using labor, $L(t)$, in accordance with a Cobb-Douglas production function. We assume that the labor market clears, in which case $L(t)=1-l(t)$, and we may write per capita output as

$$y\left[L(t)\right] = A\left(1-l(t)\right)^{\alpha} \qquad 0<\alpha<1 \qquad (21)$$

In the absence of physical capital, the firm's optimization problem is simple; it chooses labor to maximize profit $\Pi(t) \equiv y\left[L\right] - wL$, so that the equilibrium wage rate and profit are[14]

$$w(t) = \alpha A(1-l(t))^{\alpha-1} \tag{22a}$$

$$\Pi(t) = (1-\alpha)A(1-l(t))^{\alpha} \tag{22b}$$

Taking the time derivative of (22a) we obtain

$$\frac{\dot{w}(t)}{w(t)} = (1-\alpha)\frac{\dot{l}(t)}{1-l(t)} \tag{22a'}$$

so that increasing leisure (reducing labor supply) over time is associated with a growing wage rate.

Substituting for the equilibrium wage rate and profit from (22a) and (22b), and using the government budget constraint, (19'), the aggregate accumulation equation, (18) reduces to the conventional current account relationship

$$\dot{f}(t) = A(1-l(t))^{\alpha} - c(t) + (r^{*} - n) f(t) \tag{23}$$

where the first term on the right hand side of (23) measures the trade balance.

3. Aggregate Equilibrium

Equations (16) and (23) yield the dynamic equations for $\dot{c}$ and $\dot{f}$. In the familiar infinitely-lived representative agent economy this suffices to determine an equilibrium, in which the economy is always in steady state; see Turnovsky (1997). However, with a demographic structure the changing consumption patterns across the cohorts is an intrinsic component of the dynamics and needs to be taken into account.

[14] The firm's total profit is given by $\Pi(t)P(t) \equiv (y[L] - wL)P(t)$ and clearly maximizing total profit is equivalent to maximizing per capita profit.

3.1 Infinitely-Lived Representative Agent Economy

This is characterized by setting $\beta = n, \mu_c = 0, C(t,t) = c(t)$, in which case (16) reduces to $\dot{c}(t) = \psi(t)c(t)$. Taking the time derivative of (20), combining with (22a'), and recalling the definition of $\psi(t)$ in (7), it is straightforward to show that the dynamics of per capita consumption in the infinitely-lived representative agent model is

$$\frac{\dot{c}(t)}{c(t)} = \frac{1-\alpha l}{[1-\gamma(1+\theta)](1-l)+l(1-\alpha)(1-\gamma)}\left[(1-\tau_r)r^* - \rho\right] \qquad (24)$$

in which case the knife-edge condition required to ensure boundedness implies $\dot{c}(t) \equiv 0$. With $c(t)$ constant, the optimality condition (20) implies $l(t)$ is constant. Assuming dynamic efficiency, $r^* > n$, solving the accumulation equation (23), and applying the transversality condition then requires that $f(t)$ remain constant as well, so that the economy is always in steady state, ruling out any transitional dynamics; see Turnovsky (1997).[15]

3.2 Demographic Economy

With the introduction of a general demographic structure, we see from (16) that the consumption of newborns, $C(t,t)$, as well as individual consumption growth, $\psi(t)$, play a critical role in determining the dynamics of aggregate consumption, and need to be taken into account. Setting $v = t$ in (10a), and recalling $F(t,t) = 0$, this in turn involves the dynamics of the newborns' human capital and marginal propensity to consume

$$C(t,t) = \frac{H(t)}{\Delta(t)} \qquad (10a')$$

where for notational convenience (and using (6a)) we write

[15] As Turnovsky (1997) discusses at length, with a perfect international capital market and a fixed rate of time preference we need to introduce physical capital requiring adjustment costs in order to generate transitional dynamics.

$$H(t) \equiv H(t,t) = \int_t^{t+D} \Big[(1-\tau_y)\big[w(\tau)+\Pi(\tau)\big] - \theta C(t,\tau) - T(\tau)\Big] e^{-(1-\tau_r)r^*(\tau-t)-M(\tau-t)} d\tau \quad \text{(10b')}$$

$$\Delta(t) \equiv \Delta(t,t) = \int_t^{t+D} e^{\int_t^\tau \psi(s)ds-(1-\tau_r)r^*(\tau-t)-M(\tau-t)} d\tau \quad \text{(10c')}$$

Differentiating (10b') with respect to t yields the following relationship between $\dot{H}(t)$ and $\dot{\Delta}(t)$

$$(1+\theta)\dot{H}(t) - \frac{\theta H(t)}{\Delta(t)}\dot{\Delta}(t)$$

$$= -(1-\tau_y)[w(t)+\Pi(t)] + \frac{\theta H(t)}{\Delta(t)} + T(t) + \Big((1-\tau_r)r^* + \mu_H(\tau_1 - t) + \theta\psi(t)\Big)H(t)$$

(25)

where

$$\mu_H(\tau_1 - t) \equiv \frac{1}{H(t)}\int_t^{t+D} \mu(\tau-t)[(1-\tau_y)[w(\tau)+\Pi(t)] - \theta C(t,\tau) - T(\tau)] e^{-(1-\tau_r)r^*(\tau-t)-M(\tau-t)} d\tau$$

$$\tau_1 \in (t, t+D) \quad (26)$$

From (26) we see that $\mu_H(\tau_1 - t)$ is the ratio of the human wealth given up by the dying to aggregate human wealth and can be interpreted as providing an estimate of average mortality over the period $(t-D,t)$, from information on human wealth across the cohorts. Details of the derivation of (25) and (26) are provided in the Appendix.

Finally, differentiating (10c') with respect to t yields the dynamics of the marginal propensity to consume, namely

$$\dot{\Delta}(t) = -1 + \Big[(1-\tau_r)r^* + \mu_\Delta - \psi(t)\Big]\Delta(t) \quad (27)$$

where analogously we define:

$$\mu_\Delta(\tau_2 - t) \equiv \frac{1}{\Delta(t)}\int_t^{t+D} \mu(\tau-t) e^{\int_t^\tau \psi(s)ds-(1-\tau_r)r^*(\tau-t)-M(\tau-t)} d\tau \qquad \tau_2 \in (t, t+D) \quad (28)$$

Thus equations (16), (23), (25), and (27), which specify the evolution of per capita consumption, $c(t)$, per capita holdings of foreign assets, $f(t)$, together with human capital at birth, $H(t)$, and the corresponding marginal propensity to consume, $\Delta(t)$, describe the core equilibrium macroeconomic dynamics in the demographic small open economy. However, these relationships involve: (i) the equilibrium wage $w(t)$ and its growth rate, $\dot{w}(t)/w(t)$, (ii) equilibrium leisure, $l(t)$, (iii) equilibrium lump-sum taxes, $T(t)$, and (iv) the individuals' growth rate of consumption, $\psi(t)$. Thus the complete aggregate equilibrium needs to take account of (7), (19'), (20), (21), (22a), and (22b). Omitting details, the macrodynamic equilibrium can be expressed by the following autonomous fourth order system

$$\begin{pmatrix} 1+\frac{\theta\gamma}{1-\gamma(1+\theta)}\left[\frac{l(c)(1-\alpha)}{1-\alpha l(c)}\right] & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{\theta^2\gamma}{1-\gamma(1+\theta)}\left[\frac{l(c)(1-\alpha)}{1-\alpha l(c)}\right]\frac{H}{c} & 0 & 1+\theta & -\theta\frac{H}{\Delta} \\ -\frac{\theta\gamma}{1-\gamma(1+\theta)}\left[\frac{l(c)(1-\alpha)}{1-\alpha l(c)}\right]\frac{\Delta}{c} & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \dot{c}(t) \\ \dot{f}(t) \\ \dot{H}(t) \\ \dot{\Delta}(t) \end{pmatrix} \tag{29}$$

$$= \begin{pmatrix} \beta\frac{H(t)}{\Delta(t)}+\left(\frac{(1-\tau_r)r^*-\rho}{1-\gamma(1+\theta)}-n-\mu_c\right)c(t) \\ A(1-l(c))^\alpha+(r^*-n)f(t)-c(t) \\ -A(1-l(c))^\alpha-A\alpha(1-\tau_y)(1-l(c))^{\alpha-1}l+\theta\frac{H(t)}{\Delta(t)}-\tau_r r^* f(t)+\left((1-\tau_r)r^*+\mu_H+\frac{\theta[(1-\tau_r)r^*-\rho]}{1-\gamma(1+\theta)}\right)H(t) \\ -1+\left((1-\tau_r)r^*+\mu_\Delta-\frac{(1-\tau_r)r^*-\rho}{1-\gamma(1+\theta)}\right)\Delta(t) \end{pmatrix}$$

where $l(c)$ is obtained by solving the equation obtained by combining (20) and (21)

$$l=\frac{\theta c}{(1-\tau_y)A\alpha(1-l)^{\alpha-1}}$$

so that

$$l_c\equiv\frac{\partial l}{\partial c}=\frac{l}{c}\left[\frac{1-l(c)}{1-\alpha l(c)}\right]>0$$

It is important to stress that the dynamics described by (29) are functions of μ_c, μ_H, and μ_Δ which are defined by the integrals (17), (26), and (28), and are therefore in general functions of time. However, as Mierau and Turnovsky (2014b) show, given the assumption of the demographic steady state, they in fact vary only slightly over time.

Their contribution to the dynamics is of second order and for practical purposes these terms can be treated as constants.[16] Moreover, being estimates of mortality rates they are uniformly small, and can be approximated by their respective steady state levels, as derived from (33) below.[17] Accordingly, we approximate the aggregate per capita dynamics, (29), by assuming that μ_c, μ_H, and μ_Δ remain constant at these values.

Thus, linearizing (29) around the steady-state values $(\tilde{c}, \tilde{f}, \tilde{H}, \tilde{\Delta})$, the local dynamics are described by

$$\begin{pmatrix} 1+\frac{\theta\gamma}{1-\gamma(1+\theta)}\left[\frac{l(\tilde{c})(1-\alpha)}{1-\alpha l(\tilde{c})}\right] & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{\theta^2\gamma}{1-\gamma(1+\theta)}\left[\frac{l(\tilde{c})(1-\alpha)}{1-\alpha l(\tilde{c})}\right]\frac{\tilde{H}}{\tilde{c}} & 0 & 1+\theta & -\theta\frac{\tilde{H}}{\tilde{\Delta}} \\ -\frac{\theta\gamma}{1-\gamma(1+\theta)}\left[\frac{l(\tilde{c})(1-\alpha)}{1-\alpha l(\tilde{c})}\right]\frac{\tilde{\Delta}}{\tilde{c}} & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{c}(t) \\ \dot{f}(t) \\ \dot{H}(t) \\ \dot{\Delta}(t) \end{pmatrix} \tag{30}$$

$$= \begin{pmatrix} -\beta\frac{\tilde{H}}{\tilde{\Delta}\tilde{c}} & 0 & \frac{\beta}{\tilde{\Delta}} & -\beta\frac{\tilde{H}}{\tilde{\Delta}^2} \\ -1-\tilde{w}l_c & r^*-n & 0 & 0 \\ \left[\frac{\tau_y(1-\alpha l(\tilde{c}))-\tilde{l}(1-\alpha)}{1-l(\tilde{c})}\right]\tilde{w}l_c & -\tau_r r^* & \frac{\tilde{y}+(1-\tau_y)\tilde{w}l(\tilde{c})+\tau_r r^*\tilde{f}}{\tilde{H}} & -\theta\frac{\tilde{H}}{\tilde{\Delta}^2} \\ 0 & 0 & 0 & \frac{1}{\tilde{\Delta}} \end{pmatrix} \begin{pmatrix} c-\tilde{c} \\ f-\tilde{f} \\ H-\tilde{H} \\ \Delta-\tilde{\Delta} \end{pmatrix}$$

This is the system employed to analyze the local dynamics. We assume that foreign bonds are accumulated gradually, while per capita consumption, human capital at birth, and the marginal propensity to consume at birth can all adjust instantaneously. This system will have a unique bounded stable transitional path if and only if there is one negative and three positive eigenvalues, in which case the local stable manifold will be one-dimensional. In principle one can establish formal conditions that ensure this required configuration of eigenvalues, although in practice these conditions are

[16] The details of this procedure are spelled out in Mierau and Turnovsky (2014b, pp. 872-874).

[17] In the case of the Blanchard survival function, which assumes a constant mortality rate, μ_b across cohorts, $\mu_c = \mu_H = \mu_\Delta = \mu_b$ are in fact constant over time. As a result Δ is constant, in which case (29) reduces to a third order dynamic system; see Blanchard (1985).

uninformative.

We should also point out that while we take the demographic structure to be exogenous there is an extensive literature that endogenizes this as part of the economic decision. For example, Becker (1981) and Becker and Barro (1988) choose fertility, together with consumption, to maximize a dynastic utility function. Manuelli and Seshadri (2009) relate fertility and mortality differences across countries to differences in productivity and labor income tax rates. While the interdependence between the economic and demographic structures is important, it is tangential to the main issue being addressed in this paper.

4. Steady State

In steady state, the distributions of consumption, foreign bond accumulation, relative cohort size, survival, and mortality depend only upon age, $u \equiv t - v$, and not on calendar time. With no long-run per capita growth, per capita consumption, average leisure, per capita stock of traded bonds, and the wage rate all remain constant over time. Hence, recalling (7) each cohort's consumption grows at the constant rate $\tilde{\psi} \equiv \frac{r^*(1-\tau_r) - \rho}{1-\gamma(1+\theta)}$ with age, so that the consumption level of an individual of age u is equal to:

$$\tilde{C}(u) = \tilde{C}_0 e^{\tilde{\psi} u} \tag{31}$$

where steady-state consumption at birth, $\tilde{C}_0$, is given by

$$\tilde{C}_0 = \frac{\tilde{H}}{\tilde{\Delta}} \tag{32a}$$

and

$$\tilde{H} = \int_0^D [\tilde{y} + \tilde{w}\tilde{l}(1-\tau_y) + \tau_r r^* \tilde{f} - \theta \tilde{C}_0 e^{\tilde{\psi} u}] e^{-r^*(1-\tau_r)u - M(u)} du \tag{32b}$$

$$\tilde{\Delta} = \int_0^D e^{[\tilde{\psi} - (1-\tau_r) r^*]u - M(u)} du \tag{32c}$$

$$\tilde{c} = \beta \tilde{C}_0 \int_0^D e^{(\tilde{\psi} - n)u - M(u)} du \tag{32d}$$

$$\tilde{y} = A(1 - \tilde{l})^{\alpha} \tag{32e}$$

$$\tilde{w} = A\alpha(1 - \tilde{l})^{\alpha - 1} \tag{32f}$$

$$\tilde{w}(1 - \tau_y)\tilde{l} = \theta \tilde{c} \tag{32g}$$

$$\tilde{c} = \tilde{y} + (r^* - n)\tilde{f} \tag{32h}$$

Equation (32b) is obtained by substituting the government budget constraint, (19'), and (22b) into (10b'); (32c), (32e), (32f), and (32g) follow directly from (10c'), (23), (22a), and (20), respectively; (32d) follows from (15) together with (31). Given the steady-state growth rate across agents, $\tilde{\psi}$, together with the mortality function, $e^{-M(u)}$, equations (32a)-(32h) determine the steady-state values of $\tilde{C}_0, \tilde{H}, \tilde{\Delta}, \tilde{w}, \tilde{c}, \tilde{l}, \text{and } \tilde{f}$. The specific steady-state values corresponding to the three specific survival functions that we employ in our numerical simulations, are then obtained by substituting (35a)-(35c) in turn [see below] into (32b)-(32d) and evaluating. The resulting expressions for the three mortality functions we consider are reported in the Appendix.

Having determined these equilibrium quantities, the corresponding steady-state values of the mortality rates, $\mu_c, \mu_H, \text{and } \mu_\Delta$ are obtained from the steady-state relationships corresponding to the dynamic equations (29) and are respectively

$$\tilde{\mu}_c = \beta \frac{\tilde{H}}{\tilde{\Delta}\tilde{c}} + \tilde{\psi} - n \tag{33a}$$

$$\tilde{\mu}_H = \frac{\tilde{y} + (1 - \tau_y)\tilde{w}\tilde{l} + \tau_r r^* \tilde{f}}{\tilde{H}} - \frac{\theta}{\tilde{\Delta}} - \theta\tilde{\psi} - (1 - \tau_r)r^* \tag{33b}$$

$$\tilde{\mu}_\Delta = \frac{1}{\tilde{\Delta}} + \tilde{\psi} - (1 - \tau_r)r^* \tag{33c}$$

5. The Knife-Edge Revisited

As noted in Section 2.1, the source of the knife-edge problem associated with assuming an infinitely-lived representative agent in a small open economy is that under these conditions the marginal utility of wealth for all agents, and thus for the whole economy, evolve in accordance with

$$\frac{\dot{\lambda}(t)}{\lambda(t)} = \rho - (1-\tau_r)r^*$$

and that with ρ, τ_r, and r^* all being constant, $\lambda(t)$ will be bounded if and only if $r^*(1-\tau_r) = \rho$. This imposes severe constraints in analyzing structural changes to the economy. For example, an increase in the foreign interest r^* needs to be accompanied by either an increase in ρ or τ_r if boundedness is to be maintained following the change. This makes it impossible to address the "pure" effect of an increase in r^*, since the effect on the economy will depend upon whether the accompanying adjustment is achieved through ρ or τ_r. And likewise in considering the impact of changes in τ_r.

To consider whether similar constraints can apply in the demographic economy we apply the aggregator (14) to $\lambda(v,t)$ to yield the aggregate per capita marginal utility of wealth

$$\lambda(t) \equiv \int_{t-D}^{t} p(t-v)\lambda(v,t)dv = \beta\int_{t-D}^{t} e^{-n(t-v)-M(t-v)}\lambda(v,t)dv$$

Taking the time derivative of this expression and using (5c), and the derivative of (3) yields

$$\dot{\lambda}(t) = \beta\lambda(t,t) + \left(\rho - (1-\tau_r)r^* - n\right)\lambda(t) - \beta\int_{t-D}^{t} S'(t-v)e^{-n(t-v)}\lambda(v,t)dv$$

Integrating by parts and simplifying we obtain

$$\frac{\dot{\lambda}(t)}{\lambda(t)} = \left(\rho - (1-\tau_r)r^*\right) + \beta\int_{t-D}^{t} e^{-n(t-v)-M(t-v)}\left(\frac{\lambda_v(v,t)}{\lambda(t)}\right)dv \qquad (34)$$

In general it is clear that as long as λ_v/λ varies across cohorts, (34) does not imply a knife-edge condition for a bounded solution for $\lambda(t)$ to obtain, and that is the case for any arbitrary demographic structure. However, if λ_v/λ is constant across cohorts the last term on the right hand of (34) will be a constant in which case (34) will impose a constraint on ρ, τ_r, and r^* in order for (34) to yield a bounded solution. But a constant marginal growth rate λ_v/λ across cohorts implies a degenerate demographic structure, which essentially behaves like a representative agent.

6. Numerical Simulations

To obtain further insights we supplement the formal analysis with numerical simulations of the local dynamics and the steady-state demographic equilibrium.[18] As a benchmark we use the parametric survival function proposed by Boucekkine *et al.* (2002):

$$S\left(t-v\right) \equiv e^{-M(t-v)} = \frac{\mu_0 - e^{\mu_1(t-v)}}{\mu_0 - 1}, \qquad \text{(for } 0 \le t-v \le D\text{)}, \qquad \mu_0 > 1,\ \mu_1 > 0, \tag{35a}$$

where μ_0 and μ_1 are parameters governing "youth" and "old age" mortality, respectively. The maximum attainable age D, is determined by $S\left(t-v\right)=0$ and equals $\ln \mu_0 / \mu_1$. We estimate the two parameters, μ_0 and μ_1, by nonlinear least squares, using US cohort data for 2006.[19] The estimates reported in Table 1.1 highlight how we obtain a remarkably tight fit ($\bar{R}^2 = 0.996$), with highly significant parameter estimates.

For comparative purposes we also employ the Rectangular survival function

$$\begin{aligned} S^{RECT}(t-v) \equiv e^{-M(t-v)} &= 1, & 0 \le t-v < D \\ &= 0, & t-v \ge D \end{aligned} \tag{35b}$$

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[18] A simulation comparing the dynamics of the BCL and Blanchard Models to observed European consumption and output data during the recent financial crisis was previously analyzed. Because of the stylized nature of the theoretical framework the comparisons were inconclusive.

[19] Human Mortality Database. University of California, Berkeley (USA), and Max Planck Institute for Demographic Research, Rostock (Germany). Available at www.mortality.org or www.humanmortality.de.

characteristic of the seminal Samuelson (1958)-Diamond (1965) model, as well as the well known Blanchard (1985) survival function

$$S^{BLAN}(t-v) \equiv \mathrm{e}^{-M(t-v)} = e^{-\mu_b(t-v)} \tag{35c}$$

These three survival functions, together with the US data for 2006, are illustrated in Figure 1.1.[20] Since we do not consider childhood and education, we normalize the functions so that birth corresponds to age 18. As can be seen in the figure, the BCL function tracks the actual survival data for the United States closely from age 18 until around 90. Beyond that age, its concavity does not match the data particularly well. However, we do not view that as serious since only 0.7% of the US population exceed 90 and these individuals are generally retired and are relatively inactive in the economy.[21] Our estimated BCL function implies a maximum attainable age of 95.06 and life expectancy at age 18 of 78.38. These are a little low, reflecting the fact that, as Fig. 2 illustrates, the function fails to capture the outliers beyond age 90. We take the population growth rate to be 1.00% which, given the survival function, implies a birth rate of 2.24%.[22] This is a little high because the population growth rate also takes into account immigration.

To preserve comparability we parameterize the Rectangular and Blanchard functions so that they both imply the same life expectancy (78.38). In the case of the Rectangular function, death occurs with certainty at that same age, and comparison with the BCL function enables us to assess the impact of mild demographic uncertainty. Being extremely concave it tracks the data much better than does the Blanchard survival

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[20] An alternative approach that would avoid specifying a specific survival function would be to use observed mortality data. To follow this line we would need to conduct the analysis in discrete time, with relevant integrals being computed by substituting directly for the survival rates, from the mortality tables. While this has some appeal from a computational standpoint, we find that for our purposes introducing a specific surival function is more convenient. First, it is clearly more practical for addressing the issue of circumventing the knife-edge. Second, it is convenient for parameterizing demographic structural changes; see Boucekkine et al. (2002). Third, with the BCL function tracking the mortality data so closely, we can view it as providing a smoothed version of the raw data, thereby enabling us to enjoy the analytical advantages of continuous-time modelling. In any event, the choice raises issues regarding the tradeoffs involved in the choice between using discrete time versus continuous time in modelling macrodynamic systems.

[21] With this in mind, it might be more appropriate to refer to D as the maximum attainable economic age. We should also note that the ability of the BCL data to track mortality data closely is not restricted to the US. It does just as well for the Netherlands, for example; see Heijdra and Mierau (2012).

[22] UN Population Predictions 2010. Available at: www.un.org/esa/population

function, which being convex, understates the survival rate for much of the distribution, and thus provides a poor match.

Table 1.1 summarizes the key structural parameters for the baseline economy, most of which are quite standard. The productive elasticity of labor in the Cobb-Douglas production function is $\alpha = 0.65$, the world interest rate is taken to be 5%, while the two tax rates are set at 10%. With respect to preferences, we set the intertemporal elasticity of substitution to 0.4, well within the consensus range reported by Guvenen (2006), while the elasticity of leisure in utility is set at $\theta = 1.75$, consistent with the RBC literature. As noted, in general, the rate of time preference increases with age. Hence we take $\rho = 0.035$ to be the pure rate of time preference at birth, implying a discount rate of 0.0388 for the individual of average age for the BCL survival function.

One final aspect of the calibration concerns the implications for the lifetime consumption growth rate. An implication of the assumption of perfect annuities markets is that in steady-state equilibrium, consumption increases at a constant rate over agents' lifetime, contrary to the standard life-cycle theory of consumption. In contrast, evidence from the comprehensive National Transfer Accounts study is consistent with mild steady consumption growth over the life cycle; see the studies in Lee and Mason (2011). The implied equilibrium lifetime consumption growth rate of 0.2% per annum, is generally consistent with much of this evidence; see Tung (2011).

The implied equilibrium economic variables for the three survival functions are summarized in the bottom panel of Table 1.1. Focusing on the BCL survival function, it implies a ratio of foreign assets to GDP of around 1.73, of the same order of magnitude of those of Hong Kong, Luxembourg, Singapore, and Switzerland, all of which are prototype small open economies, for which the present analysis is most directly applicable.[23] In addition, it implies that the fraction of time allocated to leisure to be around 0.762, typical of the RBC literature, while the marginal propensity to consume at birth out of wealth is approximately 0.048, which is also well within the plausible range. We may also note the following dynamic characteristics. First, the linearized system does indeed have one stable eigenvalue $\lambda_1 = -0.019$ and in addition yields the values

[23] These data are obtained from the IMF elibrary database, http://www.elibrary-data.imf.org.

$\mu_c = 0.0131,\ \mu_H = 0.0041,\ \mu_\Delta = 0.0048$, suggesting that $d\mu_i(\tau_1 - t)/dt$ will indeed be very small.[24]

Comparing the benchmark equilibrium economic variables corresponding to the three survival functions two things stand out. First, the aggregate equilibrium values for the Rectangular and BCL survival functions are generally quite close to each other, while the equilibrium corresponding to the Blanchard survival function is very far removed. This reflects its fundamental difference due to its convex rather than concave shape. It also suggests that the Rectangular survival function may in fact serve as a workable approximation to describing aggregate behavior.[25] The second observation is that as we move from the Rectangular through the BCL to the Blanchard survival functions we are increasing the degree of uncertainty associated with survival. As a result agents save more, particularly as we move to the Blanchard case, where potentially an agent can live indefinitely. With a constant world interest rate but diminishing marginal productivity of labor, long-lived agents are able to accumulate their resources by investing abroad, enabling them to reduce their labor supply and domestic output, while increasing consumption.

7. Structural Changes

From the initial baseline equilibrium we analyze the transitional and long-run effects of the following structural changes: (i) a 25% increase in productivity A, (ii) an increase in the tax rate on ordinary income from 10% to 15%, (iii) an increase in the world interest rate from 5% to 5.5%, and (iv) an increase in the rate of time preference from 3.5% to 4%.[26] The long-run responses corresponding to the three demographic structures are reported in Table 1.2, while the dynamic adjustments of key variables, relative to their respective initial steady-state equilibria, are illustrated in Figs. 2.

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[24] The corresponding equilibrium values for the Rectangular distribution are $\mu_c = 0.0129,\ \mu_H = 0.0027$, and $\mu_\Delta = 0.0035$. In the case of the Blanchard distribution they all remain constant throughout the transition, equal to 0.0128.

[25] See also Bruce and Turnovsky (2013a) who conduct similar comparisons in the context of an endogenous growth model, although they consider the de Moivre function rather than the BCL function as the benchmark survival function.

[26] An increase in the tax rate on interest income is comparable to that of a decrease in r^* and can be easily inferred.

7.1 Increase in Productivity

In the long run, a 25% increase in productivity leads to a 25% increase in the holding of traded bonds, output, consumption, and human wealth. Leisure and the trade deficit expressed as a proportion of output, namely $(\tilde{c}-\tilde{y})/\tilde{y}$, remain unchanged. These long-run adjustments are identical for all three survival functions. While these responses also dominate in the short run, there is a weak offsetting factor due to the higher productivity creating an immediate incentive to increase labor supply, thereby reducing leisure and reducing the increase in consumption, while augmenting the increase in output; see Figs 2.A(d), 2.A(e), 2.A(c). Assuming that the economy starts out with a balance of payments equilibrium, this creates an immediate trade surplus, causing the economy to begin accumulating foreign bonds. Figs. 2.A(a), 2.(b). As the economy accumulates wealth its marginal utility declines, causing consumption and leisure to increase, slowing down the accumulation of traded bonds, and causing the economy to converge to its long-run equilibrium.

7.2 Increase in Tax on Income

Qualitatively, this structural change illustrated in Fig. 2.B, is approximately a mirror image of the productivity increase. In all cases, the long-run holdings of traded bonds, output, consumption, and human wealth all decline in the same proportions (approximately 2.8%), while leisure increases slightly (around 1%), and the relative trade deficit remains unchanged. But in contrast to the productivity increase the proportionality factors vary slightly between the three mortality rates. The long-run contraction is triggered by the fact that the higher tax rate reduces the after-tax real wage, causing an initial decrease in labor supply (increase in leisure) and income. With consumption and leisure being Edgeworth complements in utility, consumption declines by less than output, causing an immediate increase in the trade deficit so that the holdings of traded bonds begins to decline. This causes consumption to continue to decline, accompanied by declining leisure, leading to a partial reversal in output as the economy converges to its new steady state.

7.3 Increase in Foreign Interest Rate

In all cases an increase in the foreign interest rate raises the long-run stock of traded bonds, increases leisure and consumption, reduces human wealth and output. In the short run an increase in the foreign interest rate by reducing the present discounted value of future wage income impacts the economy by reducing human wealth at birth. This reduces consumption and leisure and hence output increases, albeit slightly in all cases. The increase in interest income creates a current account surplus and the economy begins to accumulate traded bonds. This leads to an increase in consumption, accompanied by an increase in leisure leading to a decline in output. This causes a trade deficit, which however is dominated by the interest earned on the foreign bond holdings, causing an overall current account surplus and accumulation of traded bonds over time. In contrast to the first two shocks, there are some sharp differences in the long-run responses for the Blanchard survival function and the other two. For example, under Blanchard output declines by 5.67% almost three times the 2.0% reduction under BCL.

7.4 Increase in Rate of Time Preference

This is essentially a mirror image of the foreign interest rate shock. In all cases an increase in the rate of time preference reduces the long-run stock of traded bonds, decreases leisure and consumption, and human wealth, but raises output. In the short run an increase in the rate of time preference impinges on the economy by raising the marginal propensity to consume. With human wealth remaining unaffected, this raises consumption and leisure thereby reducing output. The resulting trade deficit creates a current account surplus deficit and the economy begins to decumulate traded bonds. This leads to a decline in consumption, accompanied by a decline in leisure leading to an increase in output. This leads to a trade surplus, which however is dominated by the decline in interest earned on the foreign bond holdings, causing an overall current account deficit and deccumulation of traded bonds over time. Again, the long-run responses for the Blanchard survival function may deviate sharply. In this case, output increases by 5.87% as compared to 1.52% for BCL.

We may summarize the comparative responses to the four structural changes, under the alternative survival functions as follows. In all cases, the BCL and the Rectangular survival functions track each other remarkably closely, both in absolute terms as well as in terms relative to their respective equilibria. In contrast, the Blanchard survival function deviates sharply in terms of its levels from both. For example, it implied level of output is 5.8% below that of the BCL, while its level of foreign bonds is 177% higher. In terms of the relative dynamics, the comparability between the Blanchard and other two depends upon the source of the structural change and how closely they interact with the demographic structure. In the case of the productivity increase and the general income tax increase, which interact only indirectly with the survival function, the dynamics relative to their respective equilibria of all three demographic structures follow one another closely. In contrast, for the interest rate and rate of time preference shocks the dynamics of the Blanchard survival function deviates substantially. This is because in either case the shock interacts directly with the mortality function via the time discounting element. With the Blanchard function substantially understating the survival rate for much of the distribution across cohorts, this magnifies the size of the corresponding shock relative to the two other survival functions.

8. Demographic Structure and Natural Rates of Wealth and Income Inequality

An early article by Atkinson (1971) suggests that the changing savings behavior of agents over their life-cycle generates an inherent wealth inequality. While this cannot be accommodated in the typical representative agent model, the overlapping generations structure of the demographic model, enables us to trace out the development of assets over the individual life-cycle. This is seen in Figure 1.3, where the asset path is hump-shaped over the life-cycle. Individuals begin with zero assets, then build up assets for intertemporal consumption smoothing and, toward the end of their lives, deplete their assets so as to assure that assets are zero exactly at the maximum attainable life time, *D*.

The fact that individuals at different stages of their life-cycle possess different levels of wealth, knowing the size of the various cohorts enables us to calculate standard wealth inequality measures, such as the Gini coefficient. This measure indicates the degree of inequality inherent in an economy purely due its age composition and abstracts from any within-cohort inequality arising from differential endowments or skill levels. In this sense, it can be termed the "natural rate of wealth inequality".[27]

The first row of Table 1.3 reports the Gini coefficient of wealth inequality for the benchmark parameterization corresponding to the three survival functions. They increase from 0.278 for the BCL through 0.289 for the Rectangular to 0.428 for the Blanchard survival function. While the BCL and Rectangular functions are very close, the substantially more old population in the Blanchard economy means more very old wealthy people and therefore more wealth inequality. That these estimates are well below the US wealth Gini coefficient of 0.80 is to be expected, since it is reflecting only one factor influencing wealth inequality, namely the demographic composition.

In addition to generating aggregate dynamics, the demographic structure has consequences for the distributional dynamics. Rows 2-6 of the upper panel report the long-run changes in the wealth Gini coefficients in response to various structural changes. Thus while an increase in productivity raises wealth of all age groups, it does so in a neutral fashion, so that the Gini remains unchanged. Likewise raising the general income tax rate, τ_y, from 10% to 15% reduces proportionately the wealth of all groups again leaving the Gini unchanged. In contrast, raising the return to traded bonds by benefiting more the older agents with more wealth raises wealth inequality, while increasing the rate of time preference reduces the wealth Gini.

The lower panel of Table 1.3 reports the corresponding income Gini coefficients. The point here is that the impact of the demographic structure on income inequality is almost negligible. This reflects two factors in the model. First, all agents regardless of age receive the same wage rate; second profit income is distributed uniformly across cohorts. The only element of income that varies across cohorts is labor supply. Since this variation is also limited, the overall variation in income across age groups is very small, as Fig. 3 clearly illustrates.

[27] Mierau and Turnovsky (2014a) calculate a similar measure for an endogenous growth model of a closed economy.

9. Conclusion

Closing the small open economy model has been a stumbling block in studying the dynamic evolution of such models. The typical procedure of equating the after-tax return on traded bonds to the rate of time preference involves imposing a knife-edge condition, which severely constrains the ability to address the dynamic characteristics of small open economy models. Existing procedures to circumvent these difficulties involve endogenizing either the rate of time preference or the rate of return on traded bonds. While both approaches may resolve the problem, each is associated with its own set of issues. Endogenizing the rate of time preference typically leads to implausible equilibrium dynamics, while endogenizing the rate of return is essentially dispensing with an important aspect of the small open economy.

In this paper we have adopted an alternative approach, namely replacing the infinitely-lived representative agent framework with a plausible demographic structure. Introducing a general survival function enables us to derive a well-behaved plausible macrodynamic equilibrium without imposing any restrictive knife-edge relationship between rates of return and preferences. We have characterized the equilibrium and traced out the dynamics in response to several alternative classes of structural change. Two general conclusions stand out. First, for our plausible parameterization the simple Rectangular survival function, characteristic of the seminal Samuelson-Diamond model, tracks more general survival functions, estimated from real demographic data, surprisingly closely, both in levels and in its relative dynamics. Second, the Blanchard survival function tracks the data poorly in terms of absolute levels, while the closeness of its relative dynamics is dependent on the source of the structural change.

Finally, by focusing on a realistic demographic structure as an appealing way to close the small open economy model, this model has not been developed to match observed data trends for anything other than the age-dependent survival probabilities, and therefore is highly stylized in all other dimensions. In particular, we have simplified the production structure insofar as possible. While this is an obvious limitation, we nevertheless believe that the model suffices to make our point in a transparent way. As noted, the small open economy model may be closed using simpler ad-hoc techniques such as cost of adjustment functions and unrealistic overlapping-generations structures. However these approaches are arbitrary and lack any firm theoretical underpinning. With this paper we attempt to develop an approach that both simultaneously relaxes the knife-edge constraint, and increases the realism of the basic framework. While it is true that it may increase the complexity of the modeling environment, we find the benefits from removing the knife-edge restrictions in this realistic way to be quite compelling. At the same time, clearly an important extension of this framework is to introduce physical capital. Expanding the model in this direction will enable us to subject the significance of the demographic structure to empirical testing, something that the present model is too stylized to do effectively. It will also enable us to address a current issue of importance, specifically the impact of differential demographic structures on the international flow of capital; see Ghironi (2006), Backus et al. (2014). The approach developed in the present paper promises to provide a fruitful one for further investigation of this topical issue.

Chapter 2: Analysis of Demographic Trends on International Interdependence

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Abstract

This paper develops a two-country overlapping generations neoclassical growth model including a realistic demographic structure for the purpose of analyzing the impact of country-level asymmetries in demographic and structural characteristics on cross-country interdependence. I develop two modeling frameworks, with and without a pay-as-you-go social security system and a mandatory retirement age. I find that an increase in the relative life expectancy of a population will produce a positive per-capita net foreign asset position. This is generated by the fact that the country will be comprised of individuals who save relatively more in order to smooth consumption over their extended lifetimes. Furthermore, I demonstrate how cross-country differences in the rate of time preference will augment the decline of the American net foreign asset position generated by the demographic transition. Lastly, I present how an adjustment in the pension benefit of a pay-as-you-go social security structure will induce a change in the simulated net foreign asset position.

Keywords: Demographic transition, Net foreign assets, International capital flows
JEL Classification: D91, F21, F41, H55, J11

1. Introduction

Over the last two decades foreign holdings of US assets have significantly surpassed American claims on the rest of the world's, generating a current account deficit and a negative net foreign asset (NFA) position. From 1980 to 2007, the US real NFA position has dropped by a substantial 530 percent.[28] Conversely, some of the United States' largest trading partners, including Canada, China, Germany, and Japan, have all experienced significant increases in their NFA position, with Japan experiencing the largest increase of over 7,000 percent.

Over this same period, a significant demographic transition, centering on a decrease in mortality and a fall in fertility rates, has occurred in many regions across the world. For instance, from 1980 to 2010, the European Union experienced an increase in life expectancy by 6.8 years and a fall in the population growth rate by 46 percent.[29] For many countries this transition is a cause for concern. Countries, such as Japan and Germany, are experiencing a rapid aging of their population, straining social programs dependent upon a large taxable employment base.

This study demonstrates how demographic asymmetries across countries, including differences in age distributions and population growth rates, have a substantial impact on current account balances and NFA positions and can explain a significant portion of their observed trends. I augment the demographic analysis by incorporating observed structural changes, including the evolution of country-specific total factor productivity (TFP), capital share production parameters, and social security policies. Furthermore, I include a brief examination of the impact of an international asymmetry in the pure rate of time preference. Finally, the "natural rate of inequality", as defined by Mierau and Turnovsky (2014a), is measured in order to examine how the associated demographic and structural changes impact the distribution of wealth across the life cycle.

To analyze these changes, I develop a two-country neoclassical growth model including a realistic demographic structure. The two-country model framework is composed of the United States and a population-weighted average of trading partners

[28] NFA data retrieved from Lane and Milesi-Ferretti (2007).
[29] US Census: International Database.

including Canada, France, Germany, Japan, and the United Kingdom effectively making the structure a one-country one-region model. China is excluded from the list due to the unreliable nature of its demographic data.[30] Each economy is populated by an overlapping generations (OLG) of individuals that differ with respect to their age and their employment status. Two frameworks are utilized in order to decompose the separate demographic and structural influences on the international NFA position. The baseline model consists of agents born into the workforce and employed for their entire finite lifetime. I extend this analysis by including an additional framework that incorporates an exogenous retirement age and a pay-as-you-go social security policy. The social security system is funded through a wage income tax and pays a benefit to the retirees proportional to the per-capita wage level. For tractability, all agents are born as workers, do not reproduce, and do not receive bequests.

Through the use of an empirically estimated survival function the observed life tables specifying the probability of death per age are accurately modeled. The parameterized demographic survival functions are estimated through the use of nonlinear least squares to match country mortality data using the realistic yet tractable function developed by Boucekkine et al. (2002). This allows for the introduction of a credible age-varying probability of death that influences the saving behavior of the modeled economic agent. The international transition occurring during the period from 1980 to 2010 is the focus of the analysis. This time frame is long enough to generate significant demographic change. Because of the thorough integration of both the financial and goods markets of the chosen countries, cross-country interdependence is modeled through the use of perfect capital markets and a single traded consumption good.

Due to the complicated nature of the dynamic system, numerical simulations are used to analyze the impact of the 30-year demographic and structural transition. I first analyze the impact of the fall in mortality rates associated with an increase in life expectancy. Structural changes are then included in the analysis along with the mortality transition. I then analyze the impact of the inclusion of a retirement age and social security system. The retirement age and the gross replacement rate are set exogenously,

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[30] Gu and Cai (2009) state that the underreporting rate in some provinces reached 37.3 percent of newborns.

thereby allowing the tax rate to be endogenously determined through the interaction of the labor force participation rate and the per-capita wage level. Finally, for the baseline model, the Gini coefficient is estimated measuring the wealth inequality of the countries. The coefficient is calculated for each steady state, reflecting the equilibrium age distribution of assets before and after each shock.

Using these frameworks I obtain the following results. The per-capita NFA position is directly linked to the relative life expectancies of the population and the fertility rates of the countries. Regardless of the inclusion of a retirement period, the country that experiences a relatively higher life expectancy will save more and accumulate more wealth. Due to the increase in the relative saving rate, a positive NFA position is generated for the country. Furthermore, the country that experiences a relatively higher rate of time preference will produce a negative NFA position due to their inclination for current consumption. Over the 30-year period the average life expectancy of the region (Canada, France, Japan, Germany, and the UK) remained higher and increased by more than that of the US.

Including a social security system and matching the observed gross benefit replacement rates for the period drives the American NFA position to a positive value during the transition once again. However, the massive reduction in the region's replacement rate over the time interval forces the region's inhabitants to increase their saving, pushing the American NFA position to decline significantly.

The remainder of the paper is structured as follows. Section 2 discusses relevant literature. Section 3 explains recent demographic and employment trends experienced by the sampled countries. Section 4 and 5 lay out the two analytical frameworks. Sections 6 and 7 describe the numerical simulation and Section 8 concludes the paper.

2. Related Literature

While the inclusion of realistic demographic structures in international models is uncommon, this study is not the first to utilize them. Attanasio and Violante (2000) study the impact of a demographic transition on factor returns and cohort welfare levels as countries move from an autarkic state to a perfect open capital market. They find that the liberalization of the capital market will exacerbate the flows of capital associated with country-specific differentials in the rate of return due to asymmetric demographic structures across countries. While their analysis is close to mine, they calibrate the modeled regions to resemble the US and Europe as one region and Latin America as another. Additionally, I focus on the international setting post-liberalization. I model how recent developments in demographic and structural trends impact the flows of capital after the rates of return have been equalized.

Feroli (2003) develops an open economy model calibrated to match the G-7 nations. Using a similar model with different countries he is able to match certain trends in the NFA position for the US and Japan. The main difference in his modeling technique is his estimated demographic framework and the regional structure. The modeled demographics in his study are calibrated to match the observed and predicted population counts for 5-year intervals for the years 1950 to 2050 from the US Census Bureau's International Database. Agents in his model live with certainty for 12 five-year periods from age 20 to 80.

Ghironi (2006) develops a two-country overlapping generations model to study the role of NFA's in the transmission of productivity shocks. He highlights the mechanism by which the generational structure allows for the determinacy of the asset position in an incomplete markets setup. He exhibits the negative effect of certain assumptions associated with the removal of current account dynamics. His model is similar to mine, yet his focus is very different. His analysis focuses on the importance of using the overlapping generations framework in order to find a stationary steady state for the NFA position. Additionally, in order to increase tractability, he removes realistic demographic structures instead relying upon infinitely lived agents, more closely resembling the Blanchard (1985) model.

Ferrero (2010) uses a multi-country model to decompose the US trade balance into demographic and productivity factors. He simplifies the demographic structure by disentangling the survival probability from the age of the agent. However, he does include a stochastic retirement age but eschews from including any social security system. With this framework he is able to generate a strikingly realistic declining trade balance for the US. Additionally, he is able to mimic the decline in the international real interest rate produced by the increase in the world supply of savings.

Lastly, Backus et al. (2014) develop a multi-country open economy model looking at the impact of simulated demographic trends on the capital flows across countries. They calibrate the model to match the following countries: China, Germany, Japan, and the US. Through the modeling of an approximated demographic transition they are able to show that cross-country demographic differences have a significant impact on international capital flows. Unlike their analysis, I focus on differentiating the effects of the specific aspects of the demographic and structural trends in order to observe their individual impact on capital flows. I am also able to generate a more realistic fall in the NFA position and transition for the US, and simulate the impact of relative preference differences and the presence of a social security structure.

3. International Demographic Trends

This section briefly describes the significant population and employment trends experienced by the sampled countries in a greater level of detail. Through medical and lifestyle revolutions during the period from 1980 to 2010 a majority of countries experienced an increase in life expectancy. For the included countries the average life expectancy at birth in 1980 and 2010 was 74.2 and 80.7, respectively. This is an increase in life expectancy of 1.3 years per five years on average. The increase in life expectancy is a function of the decrease in the age-specific probability of death over the life cycle. This trend has been styled the "rectangularization of the survival function".[31] It refers to the survival function becoming a more box-like or rectangular shape as shown by Figure 1 for the US from 1980 to 2010.

The population growth rate utilized in this study is calculated by the World Bank as including all residents of the country regardless of legal status. The fertility rate is measured by the number of children that would be born to a specific woman living to the end of her childbearing years. Given that the fertility rates for the included countries remained largely stagnant and, with the exception of France and the UK, the annual population growth rates declined, this has led many countries to reach population growth rates significantly below their replacement rate, leading to the general aging of the population.

[31] See Rossi, I.A., V. Rousson, and F. Paccaud (2013).

The aging of the population has become a serious challenge for the funding of national pensions schemes for many countries. On average, as shown by Table 2.8, during the same time period the age dependency ratio, as measured by the percentage of people aged 64 and older to the working age population, has increased from 18.9 to 26.3 percent. Due to the fact that the effective retirement age has stayed fairly constant, with the exception of France where it has decreased by 4.0 years, the gross pension replacement rates have fallen substantially for all the sampled countries excluding Canada. With the exception of Canada, the average percent change in gross pension benefits amounted to a 24 percent decrease for the countries included in the trading region as opposed to a decrease by 12 percent for the US during the 30-year period.

Much of the reduction in retirement benefits is due to the trend of falling labor force participation rates. The labor force participation rate is defined as the fraction of the population that is employed in a country. As shown by Table 2.10, every country in my sample has experienced a decrease in the participation rate in the last twenty years, with the UK experiencing the largest decrease of 8.2 percent and the overall average decrease amounting to 6.4 percent. This change in the labor force will alter the tax base associated with the national pension scheme. A falling participation rate requires an increase in the social security tax in order to maintain constant benefit levels.

4. Baseline Analytical Framework

The description of the model will use the standard two-country model regional descriptors, “Home” and “Foreign”. In the case of the simulation, the US will represent “Home” and the trading region will be represented by “Foreign”. The foreign region’s specifications are identical unless otherwise specified and denoted with the use of the “*” notation.

Within each economy the cohort’s age at any random time, t, is given by $t-v$. Agents born at time, v, have a finite lifetime and die at age, D. The cohort variables are denoted by $X(v,t)$, where, v, denotes the cohort vintage and, t, denotes calendar time. The time derivative of a variable at the cohort level is specified as $\partial X(v,t)/\partial t = X_t(v,t)$.

4.1 Demographic Structure

The probability of survival of an individual that is born at time v for age $(t-v)$ is given by a general survival function $S(t-v)=e^{-M(t-v)}$. At the age of birth and death the following survival probabilities are given, $S(0)=e^{-M(0)}=1$ and $S(D)=e^{-M(D)}=0$. The probability of dying at each age, or the instantaneous probability of death, is given by $-S'(t-v)/S(t-v)=\mu(t-v)$.

As stated, the exogenous demographic structure that will be utilized in this paper was developed by Boucekkine et. al. (2002) (BCL) and is given by:

$$e^{-M(t-v)}=\frac{\mu_0-e^{\mu_1(t-v)}}{\mu_0-1} \tag{4.1}$$

The age-dependent instantaneous probability of death increases realistically over an individual's lifetime and is given by the following:

$$\mu(t-v)=\frac{\mu_1 e^{\mu_1(t-v)}}{\mu_0-e^{\mu_1(t-v)}} \tag{4.2}$$

Equation (4.1) is a highly tractable and accurate survival function specification. Unlike other tractable survival functions, for instance the perpetual youth specification utilized in Blanchard (1985), this function will produce an age-varying survival probability that realistically increases with age. The BCL demographic structure has two parameters that determine the life expectancy of the agent and the shape of their survival probability distribution. The μ_0 parameter regulates the death rates of young agents. The μ_1 parameter specifies the death rates for the elderly agents. For instance, if the μ_0 parameter increases then the life expectancy of the agents will increase due to a drop in death rates for the youth of the country, while a decrease in μ_1 will cause the death rates for the elderly to drop. Differences in these parameters allows for the modeling of cross-country disparities in overall health, as might be depicted by increased life expectancies. In the numerical simulations below, these parameters will be chosen such that the life expectancy and age-specific survival probabilities will match the data for the included countries.

Figure 2.2 exhibits the 2010 age survival data and the estimated parameterized BCL survival function for that year. The survival data has been normalized at age 18, corresponding to the birth age of the agents in the simulation. The estimated survival function produces an excellent fit for the majority of the individual's life cycle excluding the longest-living cohorts. Given that these individuals constitute a small percentage of the population, it is unlikely that the BCL approximation will significantly detract from the results of the simulation. As shown by Bruce and Turnovsky (2013b), the BCL function's accuracy is due to the fact that it is an approximation of the highly realistic yet intractable Gompertz function.

4.2 Production

Each country is populated by a representative competitive firm that employs workers and rents capital to produce a homogenous internationally traded good through the following constant returns to scale function:

$$Y(t) = AF\left(K(t), L(t)\right) \tag{4.3}$$

where $Y(t)$ is the aggregate output at time t, $K(t)$ is aggregate capital located in the Home country, and $L(t)$ is the aggregate supply of labor. The production function satisfies the standard conditions: $F_L > 0, F_K > 0, F_{LK} > 0,$ and $F_{LL}, F_{KK} < 0$. Per-worker output may be expressed by the following:

$$y(t) = \frac{Y(t)}{L(t)} = AF\left(\frac{K(t)}{L(t)}, 1\right) \equiv Af\left(k(t)\right) \tag{4.4}$$

Output per-worker for the home country is denoted by $Af(k)$, where $k(t)$ is the capital-labor ratio and A is the Hicks neutral total factor productivity (TFP) parameter that may differ across countries. The firm rents capital and hires labor such that the following marginal products are equalized with the price of the input:

$$Af'\left(k(t)\right) = r(t) \tag{4.5}$$

$$Af\left(k(t)\right) - Af'\left(k(t)\right)k(t) = w(t) \tag{4.6}$$

Where $r(t)$ is the endogenously determined interest rate and $w(t)$ is the wage rate paid to all workers regardless of age. For tractability I have removed the depreciation rate of capital stock.

$k(t)$ and $k^*(t)$ are the per-worker capital stock located in each country. These capital holdings are owned by both home and foreign agents such that:

$$k(t) = k_d(t) + k_f(t) \tag{4.7}$$

and

$$k^*(t) = k_d^*(t) + k_f^*(t) \tag{4.8}$$

The d or f subscript denotes the domestic or foreign agent, while the "*" notation, denotes where the capital is domiciled.

For the baseline model, it is assumed that agents enter and exit life as workers, causing the labor supply to equate to the population. This translates into the per-worker output equating to the per-capita output and a labor force participation rate equaling one. However, for the augmented model featuring a retirement period, a fraction of the population has exited from the labor force causing the labor supply to be a comprised of the population younger than the mandatory retirement age. Given that $P(t)$ is the size of the population and $L(t)$ is the labor force, the labor force participation rate is given by $l(t) = L(t)/P(t)$. The per-worker variables, $k_d(t)$ and $k_f(t)$, are therefore defined as the aggregate capital owned by the home and foreign agents per home worker.

4.3 The Household

The home country agent maximizes their expected lifetime utility:

$$E\big(U(v)\big) = \int_v^{v+D} \frac{C(v,t)^{1-1/\sigma} - 1}{1 - 1/\sigma} e^{-\rho(t-v) - M(t-v)} dt \tag{5.1}$$

An individual of cohort v maximizes expected utility generated from the consumption of a traded generic consumption good, $C(v,t)$ and has no bequest motive. σ is the intertemporal elasticity of substitution, ρ is the pure rate of time discount, while $\rho + \mu(t-v)$ is the overall rate of time discount at age $(t-v)$. The consumption choice for the agent has been simplified for two reasons. The first reason is to keep the model transparent and to maximize tractability. The second reason is associated with the terms of trade effect between countries. With the home and foreign countries producing an identical good, the terms of trade remain constant and equal to one. This removes any impact of price adjustments on the dynamics of the current account. This allows for the isolation of the "pure" effect of the demographic and structural transition on the dynamics of the model.

The agent maximizes their utility subject to the instantaneous budget constraint:

$$\frac{\partial K_d(v,t)}{\partial t} + \frac{\partial K_d^*(v,t)}{\partial t} = \left(r(t) + \mu(t-v)\right) K_d(v,t) + \left(r^*(t) + \mu(t-v)\right) K_d^*(v,t) + w(t) - C(v,t) \tag{5.2}$$

Each individual supplies an inelastic labor time quantity set to unity and earns wage, $w(t)$, that is set by the representative firm. The consumer accumulates two types of capital, domestic, $K_d(v,t)$, and foreign, $K_d^*(v,t)$.

I extend the use of a complete and competitive annuities market as set out in Yaari (1965) to the two-country case. The agent receives a domestic premium on the rate of return of their capital equal to their probability of death, $\mu(t-v)$. Each agent in their respective country earns a premium proportional to their accumulated capital level with the agreement that they will transfer all of their wealth to an insurance company at the time of their death. In a two-country framework, there exists a separate annuities market in each country. The agent of one country will hold capital domiciled in both regions, but the insurance company located in the agent's country will distribute the acquired assets upon death only in the country of the individual's origin.

Through maximization of (5.1) subject to (5.2), the following Euler condition will be produced:

$$\frac{C_t(v,t)}{C(v,t)} = \sigma\left(r(t) - \rho\right) \equiv \psi(t) \tag{5.3}$$

$\psi(t)$ denotes the growth rate of consumption. With the assumption of a perfect and competitive annuities market, agents will experience a common consumption growth rate, regardless of age, which adjusts with the endogenously determined international interest rate. If an imperfection is included in the annuities premium, as studied by Heijdra and Mierau (2012) and Bruce and Turnovsky (2013), the consumption growth rate will include the instantaneous probability of death, causing the consumption path to be "hump-shaped". If the instantaneous probability of death is accurate, the consumption path will be concave, reaching a maximum during the middle of an agent's life and declining in the later ages. While many might argue that a concave consumption profile is more realistic, a positive private consumption growth rate has been observed and studied by Tung (2011). My choice of a full annuities market is driven by the tractability gains associated with the ability to remove a bequest structure. The growth rate of consumption is also influenced by the pure rate of time preference, ρ. Adjustments in the rate of time preference will tilt the consumption path over the life cycle. A relatively higher home time preference causes the agent to place a larger value on current consumption. In the international setting, differential time preference rates will influence the NFA position for a country as a preference for higher current or future consumption will effect an agent's age-dependent saving profile.

Using equation (5.3) I derive the path of cohort consumption for the arbitrary time interval from t to τ:

$$C(v,\tau) = C(v,t)e^{\sigma\left(R(t,\tau)-\rho(\tau-t)\right)} \tag{5.4}$$

Where $R(t,\tau) = \int_t^\tau r(s)ds$.

I define nonhuman wealth of the home agent as their total age-dependent capital holdings consisting of the accumulated capital from both regions:

$$W(v,t) = K_d(v,t) + K_d^*(v,t) \tag{5.5}$$

and for the foreign agent:

$$W^*(v,t) = K_f(v,t) + K_f^*(v,t) \tag{5.6}$$

In order to enforce the solvency of the cohort, the transversality condition is upheld with the equality: $W(v,v+D) = 0$. Additionally, because there is no bequest motive in this economy, the agents receive no assets at the time of their birth or $W(v,v) = 0$.

Integrating the cohort's instantaneous budget constraint over their finite lifespan, imposing the capital international arbitrage condition, and the transversality condition, I am able to derive the cohort's intertemporal budget constraint:

$$W(v,t)+\int_{t}^{v+D} w(\tau)e^{-R(t,\tau)-M(\tau-v)+M(t-v)}d\tau=\int_{t}^{v+D} C(v,\tau)e^{-R(t,\tau)-M(\tau-v)+M(t-v)}d\tau \tag{5.7}$$

Substituting the consumption path (5.4) into the intertemporal budget constraint (5.7), and solving for consumption, I am able to show that cohort consumption at any time t, is a function of the present discounted value of lifetime wages also known as human wealth.

$$C(v,t)=\frac{W(v,t)+\int_{t}^{v+D} w(\tau)e^{-R(t,\tau)-M(\tau-v)+M(t-v)}d\tau}{\int_{t}^{v+D} e^{(\sigma-1)R(t,\tau)-\sigma\rho(\tau-t)-M(\tau-v)+M(t-v)}d\tau} \tag{5.8}$$

I define human wealth as:

$$H(v,t)\equiv\int_{t}^{v+D} w(\tau)e^{-R(t,\tau)-M(\tau-v)+M(t-v)}d\tau \tag{5.9}$$

The above is the discounted future labor income at time t for cohort of vintage v. Unlike models employing the representative agent framework, the discounting is a function of the uncertain life expectancy of the agent. Human wealth levels across countries are able to diverge due to differences in the mortality of the countries even with symmetric wages.

I define the inverse of the marginal propensity to consume (MPC) as the following:

$$\Delta(v,t)\equiv\int_{t}^{v+D} e^{(\sigma-1)R(t,\tau)-\sigma\rho(\tau-t)-M(\tau-v)+M(t-v)}d\tau \tag{5.10}$$

Unlike human wealth, the MPC increases as the life expectancy falls. Differences in the life expectancy across countries will cause asymmetric age-consumption profiles across countries solely through changes in the MPC.

4.4 Aggregation

I define the total population at time t as $P(t)$, the population's crude birth rate as φ, and the population growth rate as n. Abstracting from immigration, the number of living members of cohort v at any time t is given by:

$$P(v,t)=\varphi P(v)e^{-M(t-v)} \tag{6.1}$$

Where the relative weight of each cohort to the total population is defined as:

$$\frac{P(v,t)}{P(t)} = \varphi e^{-n(t-v)-M(t-v)} \equiv p(t-v) \tag{6.2}$$

And the dynamics of the weight is given by:

$$\frac{\partial p(t-v)/\partial t}{p(t-v)} = -\left[\varphi + \mu(t-v)\right] \tag{6.3}$$

The decrease in the relative size of the cohort is due to the overall increase in the population and the death of members of the cohort over time.

In order to obtain the aggregate per-capita variables, I employ the standard aggregator:

$$x(t) = \int_{t-D}^{t} p(t-v)X(v,t)dv \tag{6.4}$$

To determine the dynamics of the per-capita variable I take the time derivative of (6.4) which yields:

$$\dot{x}(t) = \varphi X(t,t) + \int_{t-D}^{t} p(t-v)X_t(v,t)dv + \int_{t-D}^{t} \frac{\partial p(t-v)}{\partial t} X(v,t)dv \tag{6.5}$$

and simplifies to:

$$\dot{x}(t) = \varphi X(t,t) + \int_{t-D}^{t} p(t-v)X_t(v,t)dv - \int_{t-D}^{t} \left[n + \mu(t-v)\right] X(v,t)dv \tag{6.6}$$

In (6.5) the following birth and death survival relations have been used: $p(0) = \varphi$ and $p(D) = 0$. Equation (6.6) exhibits the fact that the dynamics are dependent upon the effect of the addition of newborns and the growth of the economy, less the amount given up by the dying.

Using this method I derive the following per-capita home consumption level:

$$c(t) = \int_{t-D}^{t} p(t-v)C(v,t)dv \tag{6.7}$$

Taking the time derivative of (6.7), home consumption dynamics are given by:

$$\dot{c}(t) = \varphi C(t,t) + \left(\sigma\left[r(t) - \rho\right] - n - \mu_c(t-\upsilon_1)\right)c(t) \tag{6.8}$$

Equation (4.9) results from using the mean value theorem of integration on equation (6.7). μ_c is interpreted as the ratio of the consumption given up by the dying to per-capita consumption.

$$\mu_c(t-\upsilon_1) \equiv \frac{\int_{t-D}^{t} \mu(t-v)\,p(t-v)C(v,t)dv}{\int_{t-D}^{t} p(t-v)C(v,t)dv} \tag{6.9}$$

From equation (6.8) the dynamics of per-capita consumption includes the "generational turnover term", $\Phi(t)$, as defined in Mierau and Turnovsky (2014b):

$$\Phi(t) \equiv \int_{t-D}^{t} \mu(t-v)\,p(t-v)C(v,t)dv - \varphi C(t,t) + nc(t) \tag{6.10}$$

This term reduces the consumption level due to the arrival of penniless newborns into the economy. It is the difference between the consumption given up by the dying and the relative consumption of newborns to the growth of the per-capita consumption. Additionally, in the small open economy framework, this term would link aggregate consumption to human wealth path, breaking the indeterminacy of the steady state often plaguing models of this type. This is outlined in Ghironi (2006) for a two-country model and in the working paper Oxborrow and Turnovsky (2015) for the small open economy.

Applying the same method, the per-capita home nonhuman wealth dynamics are given by:

$$\dot{W}(t) = \big(r(t)-n\big)W(t) + w(t) - c(t) \tag{6.11}$$

If the available assets were limited to one capital type, (6.11) would reduce to the standard per-capita capital equation of motion of the neoclassical growth model.

Deriving the dynamics of human wealth and the inverse of the MPC at birth is slightly different. Starting out with human wealth, I set $v = t$ and define human wealth at birth as $H(t) \equiv H(t,t)$.

$$H(t) = \int_{t}^{t+D} w(\tau)e^{-R(t,\tau)-M(\tau-t)}d\tau \tag{6.12}$$

Taking the time derivative and applying the mean value theorem produces the dynamic equation for human wealth:

$$\dot{H}(t) = -w(t) + \left[r(t) + \mu_H(\tau_1 - t)\right]H(t) \tag{6.13}$$

Where:

$$\mu_H(\tau_1 - t) = \frac{\int_{t-D}^{t} \mu(t-v)p(t-v)H(v,t)dv}{\int_{t-D}^{t} p(t-v)H(v,t)dv} \tag{6.14}$$

(6.14) is interpreted similarly as (6.9) as the ratio of human wealth given up by the dying to the per-capita human wealth level.

Now looking at the inverse of the MPC out of human wealth, I set $v = t$ and define $\Delta(t) \equiv \Delta(t,t)$.

$$\Delta(t) = \int_{t}^{t+D} e^{(\sigma-1)R(t,\tau) - \sigma\rho(\tau-t) - M(\tau-t)} d\tau \tag{6.15}$$

Once again using the method outlined by (6.6), I derive the dynamics of the MPC inverse:

$$\dot{\Delta}(t) = -1 + \left[(1-\sigma)r(t) + \sigma\rho + \mu_\Delta(\tau_2 - t)\right]\Delta(t) \tag{6.16}$$

Where:

$$\mu_\Delta(\tau_2 - t) = \frac{\int_{t-D}^{t} \mu(t-v)p(t-v)\Delta(v,t)dv}{\int_{t-D}^{t} p(t-v)\Delta(v,t)dv} \tag{6.17}$$

Note that τ_1 and τ_2 are values of τ that will be determined by the mean value theorem.

4.5 Wealth and the Net Foreign Asset Position

I define the NFA position as the difference between the per-capita holdings of foreign capital by domestic agents and the holdings of domestic capital by foreign agents:

$$N(t) \equiv k_d^*(t) - k_f(t) \tag{7.1}$$

This expression can be related to per-capita wealth by the following:

$$N(t) = W(t) - k(t) \tag{7.2}$$

The NFA position of either economy can be positive, representing a lender of capital, or negative, representing a borrower of capital and in aggregate must sum to zero.

Taking the time derivative of (7.1) and (4.7):

$$\dot{N}(t) = \dot{k}_d^*(t) - \dot{k}_f(t) \tag{7.3}$$

$$\dot{k}(t) = \dot{k}_d(t) + \dot{k}_f(t) \tag{7.4}$$

Using equations (7.1)-(7.4) I can transform equation (6.11) into the per-capita current account equation:

$$\dot{N}(t) = Af\left(k(t)\right) - nk(t) + \left(r(t) - n\right)N(t) - c(t) - \dot{k}(t) \tag{7.5}$$

Equation (7.5) states that the rate of the accumulation of home NFA's depends on the sum of home production and the per-capita return on those assets less domestic per-capita absorption. Defining the first four terms as national per-capita savings and assuming a constant labor force participation rate, this expression simplifies to the difference of saving and investment.

4.6 The World Equilibrium

The world equilibrium is competitive and involves a fully integrated goods market with no costs of adjustment. Because of the complete integration, I sum equation (6.11) for Home and Foreign and the world per-capita market clearing condition becomes:

$$\dot{k}(t) + \dot{k}^*(t) = Af\left(k(t)\right) + A^* f^*\left(k^*(t)\right) - nW(t) - n^*W^*(t) - c(t) - c^*(t) \tag{8.1}$$

Either region may instantaneously alter their capital stock by entering the world market, however the per-capita wealth of each country remains sluggish. Equation (8.1) states that the sum of accumulated capital, wealth, and private consumption must equal the total production output. In order to derive an equilibrium the population growth rates of the countries are equalized. This allows me to assume that the US and the region have the same size population and that the per-capita equates to the aggregate. As observed in Table 2.11, the countries have maintained relatively constant populations aged 18 to 90, making this an innocuous assumption. Furthermore, if needed, the equilibrium may be adjusted with a population relative size factor in order to account for a cross-country size difference.

4.7 Steady State Equilibrium

In steady state, the variables are dependent upon age and not calendar time. The dynamics converge to their long run levels. With no additional capital accumulation the per-capita market clearing condition becomes:

$$Af\left(\tilde{k}\right)+A^{*}f^{*}\left(\tilde{k}^{*}\right)=\tilde{c}+\tilde{c}^{*}+n\tilde{W}+n^{*}\tilde{W}^{*} \qquad (9.1)$$

With a perfect capital market, capital is allocated until the per-worker marginal products of production are equalized internationally. This causes international interest rates to equalize implying the following firm optimality conditions:

$$\tilde{r}=Af'\left(\tilde{k}\right)=A^{*}f^{*\prime}(\tilde{k}^{*}) \qquad (9.2)$$

Differences in wages across countries will be generated through changes in the firm's productivity term and the capital share parameter.

$$\tilde{w}=(1-\alpha)Af\left(\tilde{k}\right) \qquad (9.3)$$

Human Wealth and inverse MPC distributions depend upon the age distribution of the population for each country:

$$\tilde{H}=\tilde{w}\int_{0}^{D}e^{-\tilde{r}u-M(u)}du \qquad (9.4)$$

The inverse MPC at steady state is given by:

$$\tilde{\Delta}=\int_{0}^{D}e^{(\sigma-1)\tilde{r}u-\sigma\rho u-M(u)}du \qquad (9.5)$$

In order to have a stable population distribution at steady state, I include a condition linking the survival function, death age, D, population growth rate, n, and the birth rate, φ, as specified by Lotka (1998):

$$\varphi\int_{0}^{D}e^{-nu-M(u)}du=1 \qquad (9.6)$$

By equating the survival function to zero and solving for the corresponding age I find the age of death. As stated above, an increase in the μ_0 parameter or a decrease in the μ_1, will cause the life expectancy for an agent to increase.

$$D = \ln\left(\mu_0\right) / \mu_1 \tag{9.7}$$

Steady state consumption is the per-capita aggregate of the consumption-age profile, where equilibrium consumption over age is increasing by $\sigma(\tilde{r} - \rho)$ from the birth consumption level of $\tilde{C}_0$:

$$\tilde{c} = \tilde{C}_0 \tilde{\varphi} \int_0^{\tilde{D}} e^{\left(\sigma\left(\tilde{r}-\rho\right)-n\right)u - M(u)} du, \quad \tilde{C}_0 = \frac{\tilde{H}}{\tilde{\Delta}} \tag{9.8}$$

The initial consumption level, $\tilde{C}_0$, is derived from the product of the present discounted value of human wealth and the MPC at birth. At birth the individual owns nothing except for the present discounted value of their future wage income stream. Per-capita consumption is constant in steady state, however the individual consumption varies depending on an agent's age. Given the positive consumption growth rate, consumption increases as an individual ages.

Steady state individual per-capita country wealth equations relating consumption, wealth, and output is given by:

$$(\tilde{r} - n)\tilde{W} + \tilde{w} = \tilde{c} \tag{9.9}$$

The above equation represents the steady state current account balance for the home country. Per-capita wealth is composed of the accumulated holdings of home and foreign capital holdings. The no arbitrage condition equates the domestic and foreign interest rate to the international interest rate, however the return to wealth is influenced by the dilution associated with a growing population.

4.8 Linearization

In order to characterize the local dynamics of the economy around the steady state the following equations for home and foreign are linearized for the baseline model (the linearization and the resulting matrices can be found in the appendix for the retirement model): (6.8), (6.11), (6.13), (6.16), and (8.1).

$$\dot{c}(t) = \varphi \frac{H(t)}{\Delta(t)} + \left(\sigma\left[r(t) - \rho\right] - n - \mu_c(t - \upsilon_1)\right)c(t) \tag{10.1}$$

$$\dot{c}^*(t) = \varphi^* \frac{H^*(t)}{\Delta^*(t)} + \left(\sigma^*\left[r(t) - \rho^*\right] - n - \mu_c^*(t - \upsilon_1^*)\right)c^*(t) \tag{10.2}$$

$$\dot{H}(t) = -w(t) + \left[r(t) + \mu_H(\tau_1 - t)\right]H(t) \tag{10.3}$$

$$\dot{H}^*(t) = -w^*(t) + \left[r(t) + \mu_H^*(\tau_1^* - t)\right]H^*(t) \tag{10.4}$$

$$\dot{\Delta}(t) = -1 + \left[(1-\sigma)r(t) + \sigma\rho + \mu_\Delta(\tau_2 - t)\right]\Delta(t) \tag{10.5}$$

$$\dot{\Delta}^*(t) = -1 + \left[(1-\sigma^*)r(t) + \sigma^*\rho^* + \mu_\Delta^*(\tau_2^* - t)\right]\Delta^*(t) \tag{10.6}$$

$$\dot{W}(t) = \left(r(t) - n\right)W(t) + w(t) - c(t) \tag{10.7}$$

$$\dot{W}^*(t) = \left(r(t) - n\right)W^*(t) + w^*(t) - c^*(t) \tag{10.8}$$

$$\dot{k}(t) + \dot{k}^*(t) = Af\left(k(t)\right) + A^* f^*\left(k^*(t)\right) - nW(t) - nW^*(t) - c(t) - c^*(t) \tag{10.9}$$

The dynamics associated with equations (10.1) to (10.9) are functions of the mortality variables μ_c, μ_H, and μ_Δ for both countries. These variables adjust slightly as the demographic age distributions change over the transition. The steady state relationships of these variables for the US are given by:

$$\tilde{\mu}_c = \frac{\tilde{\varphi}\tilde{H}}{\tilde{\Delta}\tilde{c}} + \sigma\left(\tilde{r} - \rho\right) - n \tag{10.10}$$

$$\tilde{\mu}_H = \frac{\tilde{w}}{\tilde{H}} - \tilde{r} \tag{10.11}$$

$$\tilde{\mu}_\Delta = \frac{1}{\tilde{\Delta}} - \left(1 - \sigma\right)\tilde{r} - \sigma\rho \tag{10.12}$$

I approximate these mortality variables as constants during the transition as their influence would be minimal.[32] The mortality values for the baseline model's initial equilibrium for the US are $\mu_c = 0.017$, $\mu_H = 0.0012$, and $\mu_\Delta = 0.002$. Their values for the final equilibrium are given by $\mu_c = 0.016$, $\mu_H = 0.0009$, and $\mu_\Delta = 0.0015$. Therefore, during the mortality decline of the baseline model, the average change in the American mortality variables is only 0.0006, justifying their approximation as constants during the transition.

Noting the equalization of the marginal products of capital, I transform the above system of dynamic equations into functions of per-capita domestic capital, $k(t)$. This allows for the simplification of the system of equations, as shown in the Appendix. The resulting linearized matrix is found below in vector form, a more detailed version is found in the appendix.

$$\begin{pmatrix} \dot{c}(t) \\ \dot{c}^*(t) \\ \dot{H}(t) \\ \dot{H}^*(t) \\ \dot{\Delta}(t) \\ \dot{\Delta}^*(t) \\ \dot{W}(t) \\ \dot{W}^*(t) \\ \dot{k}(t) \end{pmatrix} = \Omega \begin{pmatrix} c(t) - \tilde{c} \\ c^*(t) - \tilde{c}^* \\ H(t) - \tilde{H} \\ H^*(t) - \tilde{H}^* \\ \Delta(t) - \tilde{\Delta} \\ \Delta^*(t) - \tilde{\Delta}^* \\ W(t) - \tilde{W} \\ W^*(t) - \tilde{W}^* \\ k(t) - \tilde{k} \end{pmatrix} \qquad (10.13)$$

This system will have a unique bounded path if the linearized matrix of dimension 9x9, given by Ω, yields a positive determinant with two negative and seven positive eigenvalues. The stable eigenvalues are represented by λ_1 and λ_2. I assume that the per-capita nonhuman wealth for each country, $W(t)$ and $W^*(t)$, are both sluggish. Capital stock, consumption, human wealth at birth, and the MPC at birth all respond instantaneously to shocks.

[32] For a more detailed analysis of the impact of varying mortality variables, see Mierau and Turnovsky (2104b).

4.9 General solution

From the linearized matrix above, Ω, I derive the following solution:

$$W(t) = \tilde{W} + C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \tag{11.1}$$

$$W^*(t) = \tilde{W}^* + C_1 V_{1W^*} e^{\lambda_1 t} + C_2 V_{2W^*} e^{\lambda_2 t} \tag{11.2}$$

$$c(t) = \tilde{c} + C_1 V_{1c} e^{\lambda_1 t} + C_2 V_{2c} e^{\lambda_2 t} \tag{11.3}$$

$$c^*(t) = \tilde{c}^* + C_1 V_{1c^*} e^{\lambda_1 t} + C_2 V_{2c^*} e^{\lambda_2 t} \tag{11.4}$$

$$H(t) = \tilde{H} + C_1 V_{1H} e^{\lambda_1 t} + C_2 V_{2H} e^{\lambda_2 t} \tag{11.5}$$

$$H^*(t) = \tilde{H}^* + C_1 V_{1H^*} e^{\lambda_1 t} + C_2 V_{2H^*} e^{\lambda_2 t} \tag{11.6}$$

$$\Delta(t) = \tilde{\Delta} + C_1 V_{1\Delta} e^{\lambda_1 t} + C_2 V_{2\Delta} e^{\lambda_2 t} \tag{11.7}$$

$$\Delta^*(t) = \tilde{\Delta}^* + C_1 V_{1\Delta^*} e^{\lambda_1 t} + C_2 V_{2\Delta^*} e^{\lambda_2 t} \tag{11.8}$$

$$k(t) = \tilde{k} + C_1 V_{1k} e^{\lambda_1 t} + C_2 V_{2k} e^{\lambda_2 t} \tag{11.9}$$

Where the arbitrary constants, C_1 and C_2, are derived by solving the system:

$$\begin{pmatrix} 1 & 1 \\ V_{1W^*} & V_{2W^*} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} W_0 - \tilde{W} \\ W_0^* - \tilde{W}^* \end{pmatrix} \tag{11.10}$$

The variable, V_{ij}, is the eigenvector coefficient associated with the stable eigenvalue $i = 1$ or 2 for λ_1 or λ_2 respectively and the variable $j = W(t), W^*(t), c(t), c^*(t)$ etc. In solving for the arbitrary coefficients I assume that the system starts from an initial nonhuman wealth position, W_0 or W_0^*, for each country. Each country's initial wealth position may be different depending on the similarities of cross-country demographic characteristics.

4.10 Net Foreign Asset Position Dynamics

Starting from the dynamic equation (7.5) describing the home country's current account as a function of the accumulation of capital with the assumption of a constant labor force participation rate:

$$\dot{N}(t) = f(k) - nk + \left(r(t) - n\right)N(t) - c(t) - \dot{k}(t) \tag{11.11}$$

Substituting in the dynamic world equilibrium equation, I simplify the expression to a simple separable first order differential equation:

$$\begin{aligned}\dot{N}(t) &= (1-\chi)f(k) - \chi f^{*}(k^{*}) - nk + \left(r(t) - n\right)N - c(t) \\ &+ \chi n W(t) + \chi n W^{*}(t) + \chi c(t) + \chi c^{*}(t)\end{aligned} \tag{11.12}$$

Where χ is defined as:

$$\chi \equiv \left[\left(\frac{\alpha A}{\alpha^{*}A^{*}}\right)^{\frac{1}{\alpha^{*}-1}}\left(\frac{\alpha-1}{\alpha^{*}-1}\right)k^{\frac{\alpha-\alpha^{*}}{\alpha^{*}-1}} + 1\right]^{-1} \tag{11.13}$$

Linearizing this around the steady state:

$$\begin{aligned}\dot{N}(t) &= \left[(1-\chi)\, f'\left(\tilde{k}\right) + f''\left(\tilde{k}\right)\tilde{N} - n - \chi\frac{\alpha-1}{\alpha^{*}-1}\frac{\tilde{k}^{*}}{\tilde{k}}\tilde{r}\right]\left(k(t) - \tilde{k}\right) \\ &+ \chi n\left(W(t) - \tilde{W}\right) + \chi n\left(W^{*}(t) - \tilde{W}^{*}\right) \\ &+ \left(\chi - 1\right)\left(c(t) - \tilde{c}\right) + \chi\left(c^{*}(t) - \tilde{c}^{*}\right) \\ &+ \left(f'\left(\tilde{k}\right) - n\right)\left(N(t) - \tilde{N}\right)\end{aligned} \tag{11.14}$$

Defining:

$$\phi_k \equiv (1-\chi)\, f'\left(\tilde{k}\right) + f''\left(\tilde{k}\right)\tilde{N} - n - \chi\frac{\alpha-1}{\alpha^{*}-1}\frac{\tilde{k}^{*}}{\tilde{k}}\tilde{r} \tag{11.15}$$

$$\epsilon_1 \equiv \left(\phi_k V_{1k} + \chi n + \chi n V_{1W^{*}} + \left(\chi - 1\right)V_{1c} + \chi V_{1c^{*}}\right)C_1 \tag{11.16}$$

$$\epsilon_2 \equiv \left(\phi_k V_{2k} + \chi n + \chi n V_{2W^{*}} + \left(\chi - 1\right)V_{2c} + \chi V_{2c^{*}}\right)C_2 \tag{11.17}$$

Using the transversality condition I solve for the NFA position of the home country for any arbitrary time period t.

$$N(t)=\tilde{N}+\frac{\epsilon_1}{\lambda_1-(\tilde{r}-n)}e^{\lambda_1 t}+\frac{\epsilon_2}{\lambda_2-(\tilde{r}-n)}e^{\lambda_2 t} \tag{11.18}$$

The NFA position for either country can be either positive or negative representing a net lender or debtor, respectively. In the situation where the two countries are identical with respect to structural parameters and demographic characteristics the NFA position for each country will be zero. Only when there is an asymmetry in the saving behavior will the NFA position for the country become nonzero. International asset market clearance requires that the summation of the total NFA position be zero, or $N(t)+N^*(t)=0, \forall\, t$. With the NFA's path defined, I can derive the current account transition path, which is defined as the time derivative of the NFA position.

Taking the time derivative of (11.18) I derive the following current account time path for the home country for any arbitrary time t:

$$CA(t)\equiv\dot{N}(t)=\frac{\lambda_1\epsilon_1}{\lambda_1-(\tilde{r}-n)}e^{\lambda_1 t}+\frac{\lambda_2\epsilon_2}{\lambda_2-(\tilde{r}-n)}e^{\lambda_2 t} \tag{11.19}$$

4.11 Gradual versus Instantaneous Transition

Structural change does not occur rapidly. In order to compare the gradual versus instantaneous adjustment of the productivity parameter, the baseline transitional paths are augmented to include the ongoing structural change of the economy. This modeling technique forces the variable to follow a continuous transition, eliminating discrete jumps. Because of the interconnected nature of the two countries, a change in either of these rates for one country will have an impact on both countries.

Within the baseline dynamics, I have built in the dynamic process accounting for the gradual productivity change with the following functions:

$$A(t)=\tilde{A}+\left(A_0-A\right)e^{-at} \tag{12.1}$$

$$A^*(t)=\tilde{A}^*+\left(A_0^*-A^*\right)e^{-a^*t} \tag{12.2}$$

Where a is the rate of change for the TFP parameter, A. The impact of a gradual increase in productivity of the country versus an immediate change (corresponding to an infinite a value) on the dynamics of the countries will be substantial.

5. Augmented Model Including Retirement

This section details the required modifications to the model in order to include a social security system and retirement period. As stated, with a retirement period incorporated into the individual's life cycle, the population no longer equals the labor supply. I therefore have a choice to conduct the equilibrium analysis in per-worker or per-capita terms. A shown below, denominating the equilibrium in per-worker terms will impact the dynamics through adjustments in the labor force participation rate. An identical preference structure to the baseline model is used in this analysis, but the agent's budget constraint will now differ. In order to incorporate the national pension system, an additional governmental sector must be included in the framework. Lastly, the dynamics of human wealth will need to be approximated. This is due to the discontinuity in the human wealth equation caused by the retirement period.

5.1 The Household

While preferences remain identical to the baseline model, the instantaneous budget constraint takes on a slightly different form due to the presence of a retirement period and the social security benefit payment:

$$\begin{aligned}\frac{\partial K_d(v,t)}{\partial t}+\frac{\partial K_d^*(v,t)}{\partial t}&=\left(r(t)+\mu(t-v)\right)K_d(v,t)+\left(r^*(t)+\mu(t-v)\right)K_d^*(v,t)\\&\quad+(1-\tau_s)w(t)I(R)+B(t)\left(1-I(R)\right)-C(v,t)\end{aligned}\tag{13.1}$$

$I(R)$ is an indicator function equaling one during the employment period. Similar to the baseline setup, during the employment period, the agent earns the aggregate wage regardless of their age. The individual is required to pay an income tax, τ_s, in order to fund the national pension system. Given that labor is supplied inelastically, the tax rate's distortionary impact is minimized. The individual earns this after-tax wage until the date of their mandatory and exogenous retirement age, R. Following this period, the agent receives income from the social security benefit, $B(t)$. The agent receives this benefit for a period of expected length $D-R$.

Through the implementation of the transversality condition and the inclusion of (5.4) the individual consumption at an arbitrary time t may once again be written in terms of capital wealth, human wealth, and the MPC:

$$C(v,t) = \frac{W(v,t) + H(v,t)}{\Delta(v,t)} \tag{13.2}$$

The human wealth equation is now augmented to account for the retirement period and social security system.

$$H(v,t) \equiv \int_{t}^{v+R} (1-\tau_s) w(\tau) e^{-R(t,\tau)-M(\tau-v)+M(t-v)} d\tau + \int_{v+R}^{v+D} \mathrm{B}(\tau) e^{-R(t,\tau)-M(\tau-v)+M(t-v)} d\tau \tag{13.3}$$

Unlike the baseline human wealth definition, (13.3) includes the discounted after tax aggregate wage, $(1-\tau_s)w(t)$, and the pension benefit, $\mathrm{B}(t)$, that the individual earns during the associated period of employment or retirement.

5.2 The Government

A government sector is included to conduct the transfer of income from the employed to the retirees. The government is represented by a typical pay-as-you-go transfer scheme. To fund this system the government taxes labor income at rate τ_s for all working individuals. All individuals are taxed at the same rate while they are employed regardless of age. Taxes are used solely to fund the benefit that retired agents receive and therefore the government maintains a balanced budget each period given by the following relation:

$$\int_{t-R}^{t} p(t-v)\tau_s w(t) dv = \int_{t-D}^{t-R} p(t-v) B(t) dv \tag{13.4}$$

The benefit retirees receive, $B(t)$, is proportional to the aggregate wage rate such that, $B(t) = \beta w(t)$. In the two-country setting each country can differ with respect to their retirement age, tax rate, and retirement benefit. Given that the government is maintaining a balanced budget, if the simulated tax rate is matched to the observed value, the benefit will be endogenously determined.

5.3 Equilibrium Modifications

Due to the fact that the population diverges from the labor supply, differentiation is needed between the per-capita and per-worker aggregate. For any general cohort variable, $X(v,t)$, I can derive its per-worker variable, $\bar{x}(t)$:

$$\bar{x}(t)=\frac{1}{l}\int_{t-D}^{t}p(t-v)X(v,t)dv \tag{13.5}$$

Where l is the labor force participation rate and is defined as:

$$l=\int_{t-R}^{t}p(t-v)dv=\frac{\varphi\int_{0}^{R}e^{-nu-M(u)}du}{\varphi\int_{0}^{D}e^{-nu-M(u)}du} \tag{13.6}$$

This specifies the fraction of the employed population and exhibits the fact that, due to the demographic steady state, the labor force participation rate is time-invariant as long as the exogenous retirement age and demographic structure is constant throughout the transition. The labor force participation rate is dependent upon the retirement age, population growth rate, and the survival function. A longer period of retirement will decrease the participation rate while an increase in the population growth rate will increase the rate due to additional agents entering the economy as workers. A change in the mortality of the population could have an ambiguous effect on the participation rate. While in one case, a decrease in mortality can lengthen the period of employment due to improved health; it also may increase the length of retirement. In this framework, the retirement age is set exogenously in order to match observed country characteristics. Therefore the length of the working period will be fixed exogenously, causing a decrease in mortality to increase the retirement length and decrease the labor force participation rate.

Equilibrium human wealth in the retirement model is given by:

$$\tilde{H}=\int_{0}^{R}(1-\tau_s)\tilde{w}e^{-\tilde{r}(u)-M(u)}du+\int_{R}^{D}\beta\tilde{w}e^{-\tilde{r}(u)-M(u)}du \tag{13.7}$$

If the benefit proportion of the wage, β, is equal to the proportion of after-tax wages kept by the individual, $1-\tau_s$, then the augmented human wealth relation will simplify to an after tax version of the baseline human wealth definition.

5.4 Modifications to the Dynamics

The time derivative of (13.5) yields the per-worker dynamics of the model:

$$\dot{\bar{x}}(t)=\frac{\varphi}{l}X(t,t)+\frac{1}{l}\int_{t-D}^{t}p(t-v)X_t(v,t)dv+\frac{1}{l}\int_{t-D}^{t}\frac{\partial p(t-v)}{\partial t}X(v,t)dv-\frac{\dot{l}(R)}{l(R)}\bar{x}(t) \tag{13.8}$$

The final term accounts for the adjustment of the labor supply to a change in the retirement age or the demographic structure. This term is naturally absent in the baseline model due to the fact that the labor force participation rate is constant. The final term will be nonzero if the retirement age, R, or the demographic variables vary during the transition. Assuming an instantaneous adjustment for the demographic parameters and the exogenous retirement age, l will remain fixed at its final value and the final term above will drop out.

The time derivative of the human wealth equation (13.3) is given by:

$$\dot{H}(t) = (1-\tau_s)\Big[w(t+R)e^{-R(t,t+R)-M(R)} - w(t) + \int_t^{t+R} [r(t)+\mu(\tau-t)] w(\tau) e^{-R(t,\tau)-M(\tau-t)} d\tau \Big]$$
$$+\beta\Big[-w(t+R)e^{-R(t,t+R)-M(R)} + \int_{t+R}^{t+D} [r(t)+\mu(\tau-t)] w(\tau) e^{-R(t,\tau)-M(\tau-t)} d\tau \Big] \tag{13.9}$$

Due to the discontinuity associated with (13.9), we use the following approximated version of the dynamics of human wealth represented by:

$$\dot{H}(t) = -(1-\tau_s)w(t) + [r(t)+\mu_H(\tau-t)]H(t) - [\beta-(1-\tau_s)]w(t)e^{\left[\frac{\dot{w}(t)}{w(t)}-r(t)\right]R} \tag{13.10}$$

As stated in Mierau and Turnovsky (2015), the final term in (13.10) is an approximation of the wage income evolution occurring at the anticipated retirement age. A detailed derivation can be found in the Appendix.

6. Baseline Simulation

To determine the impact of international asymmetric demographic trends I include numerical simulations modeling the transitional dynamics around the steady state. In order to simulate the interaction between these six countries in a two-country framework, I develop a population-weighted average of the trading partners for multiple exogenous factors including: the survival function parameters, the total factor productivity, and the production capital share parameter. I parameterize the BCL survival function to match the included countries for the years 1980 and 2010.[33] The BCL survival function parameters, μ_0 and μ_1, have been estimated with nonlinear least squares using age-survival data from the Human Mortality Database.[34] I focus the analysis on the impact of cross-country differences in demographic characteristics including differences in birth rates and the decline in mortality. Additionally, I estimate the impact of the demographic and structural transition on the natural rate of wealth inequality present in an OLG economy through the calculation of the wealth Gini coefficient. Lastly, I analyze the impact of the presence of a retirement period and social security structure on the NFA position of the countries.

Table 2.2 summarizes the key baseline parameter values including the demographic parameter estimates. Given that my focus is the impact of demographic change, I set the parameter values such that I isolate its effect. To that end, the intertemporal elasticity of substitution is set to 0.5, consistent with the range given by Guvenen (2006). The pure rate of time preference is 0.035. Output is produced by a Cobb-Douglas function, $F(K,L) = AK^{\alpha}\bar{L}^{1-\alpha}$, for each country, where $\bar{L}$ denotes the inelastic labor supply, A is the productivity parameter that has been initially normalized for the US and set to 1.0, and the capital share parameter is initially given by $\alpha = 0.35$. The productivity rate variable, impacting the transitional speed for the gradual transition paths in the baseline model, is set to 5.0 percent.

The estimated BCL function tracks observed mortality data remarkably well for most Western nations as it does here. The function produces an accurate approximation of the survival data resulting in an adjusted R^2 value of 0.99. As a childhood period is not

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[33] Arguments can be made for the insertion of the observed mortality data directly. This would require the analysis to be performed in discrete time. While this is an option, given the close fit of the BCL function with the data, the parameterization of the survival function does not significantly harm the accuracy of the analysis. Additionally, with this continuous time framework, the analysis can be executed with the use of differential equation rather than difference equation techniques. Lastly, I am performing a two-country analysis that is similar to a comparative-static approach. The use of this modeling framework allows the analysis to be performed in a much more direct and simple manner.

[34] Human Mortality Database. University of California, Berkeley (USA), and Max Planck Institute for Demographic Research, Rostock (Germany). Available at www.mortality.org.

included in the model, the estimation is performed for the age interval from 18 to 90. In order to parameterize the survival function for the region, I estimate the mortality parameters for each trading partner. I then create an overall weighted average utilizing weights associated with the population of each country for the specified age interval. It is important to note once again that in order to create a stable per-capita equilibrium, population growth rates across countries are assumed to be equal. This allows for a key tractability assumption allowing the per-capita aggregate to equate to the overall aggregate. While this may seem constraining, from 1980 to 2010, the targeted population proportions of the countries have stayed relatively constant as can be seen from Table 2.11. Since this model lacks immigration, any increase in the proportion of the elderly due to a decrease in mortality will be reflected in the birth rate. The population growth rate for both countries is set to 1.0 percent.

6.1 Decrease in the Mortality Rate

I first focus on the impact of a decrease in mortality (an increase in the survival probability for each age) on the NFA position of the countries. Figure 2.3 exhibits the transitional dynamics associated with the asymmetric decrease in mortality between the US and the trading region. Over the 30-year period, the life expectancy for the US increases from 74.1 to 78.6 years and from 76.4 to 81.5 years for the region due to the decline in mortality. The lifetime extension generates a significant negative NFA position for the US. It is important to note that, while the overall population growth rate is constant at 1.0 percent, due to the change in the mortality parameters, the fertility rate drops for the US from 1.93 to 1.85 percent and from 1.89 to 1.81 percent for the trading region. This has a slightly positive effect on the per-capita variables.

The longer life interval generates an increase in wealth due to the extended lifespan during which capital accumulates. Given that each country experiences an increase in expected life, per-capita wealth for all countries increase over time. The immediate impact of the life expectancy increase is to decrease consumption in order to satisfy the goods market clearing constraint for the longer-lived population. As agents accumulate capital, output increases, and consumption will follow suit. The difference in saving across countries manifests itself in the NFA position. Due to their relatively shorter life span, the US enjoys slightly higher per-capita consumption and a lower savings level, which translates into a negative and decreasing American NFA position. The decline in mortality causes the NFA position to decrease by 13 percent during the 30-year interval. While this change is substantially lower than the amount observed by the US during that period, as shown in Table 2.6, the model generates a negative and declining position solely because of the demographic transition, something that has not been accomplished in other studies, see Backus et al. (2014). In the long run, measured from the initial to the final steady states, the NFA position for the US declines by 45 percent.

6.2 Structural Change

Figures 2.4-2.6 illustrate the impact of structural changes in addition to the mortality decline. The two structural changes being analyzed are an asymmetric increase of the total factor productivity, shown in Figure 2.4, and an increase in the capital share parameter of the production function, shown in Figure 2.5. For the trading region I use a population-weighted average for the TFP and production share parameters. The combined effect of both a TFP and share parameter change is found in Figure 2.6. Over the 30-year time period, the US experienced a 34 percent increase in productivity, while the trading region experienced an average productivity increase of 4 percent. The trading region’s increase in productivity is largely due to the substantial increase experienced by the UK of 31 percent and the modest 8 percent gain experienced by Japan. All other countries included in the region experienced a productivity decrease. These values can be seen in Table 2.8. Additionally, following the global trend, the labor share of production has fallen for both the US and the trading region by 8.1 percent and by 6.9 percent, respectively.

The permanent TFP gain immediately increases consumption for the US, however the most substantial change is the jump and slope reversal of the instantaneous adjustment NFA transitional path. Taken by itself, a permanent change in productivity will not generate a NFA position as it will not impact the saving behavior of a country. A rise in productivity increases the accumulation of capital in the affected country while simultaneously increasing consumption, wages, and wealth by the same percentage. As shown by Ghironi et al. (2005), a productivity change will only impact the NFA holdings if there previously exists a current non-zero asset position. For their study, they use heterogeneity in time preferences to induce a nonzero NFA position in order to analyze the effect of a productivity shock. For this study, an asymmetric productivity shock amplifies the nonzero NFA position generated by the asymmetric demographic transition.

Including the productivity changes with differential demographics causes the NFA path to remain negative similar to the path exhibited by the data. However, the TFP change generates an overshooting of the initial jump that causes the path to exhibit a positive trend, opposite of what has been observed for the US during that period. The substantial negative jump in the NFA position is due to the large increase in the investment into the US. With the added impact of the region's relatively high saving rate, associated with the longer life expectancy, the long run NFA position for the US stays negative and decreases by 90 percent.

The capital shares of production for the US and the region are similar in magnitude, 0.432 and 0.439 for the US and the trading region in 1980 respectively, and 0.467 and 0.47 in 2010.[35] During the 30-year time frame, the shares increased as production further emphasized capital inputs, with the trading region weighting capital slightly more during the entire period. The isolated impact of an increase in the home country's capital share is to draw in investment from overseas, decreasing the NFA position of the country. Ceteris paribus, the observed share positions cause the US to generate a positive NFA position. If the transition of the capital shares is included with the mortality change, in the first 30 years the NFA position for the US falls by 1,061 percent as calculated from the initial equilibrium. The American NFA position experiences a decrease by 1,063 percent in the long run.

Lastly, Figure 2.6 shows the impact of the combined structural effects. The capital share adjustment exacerbates the overshooting, leading to a transitional path that experiences a more negative position, but a steeper positive slope. In the long run, the NFA position for the US falls by 1,561 percent. The increase in productivity benefits the countries as they both experience an increase in consumption and an accumulation of wealth.

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[35] Capital shares are taken from Nieman (2013).

6.3 Asymmetric Time Preference

In the previous analysis the pure rate of time preference was held to be constant throughout the transition. However, relative differences between preference rates will generate a nonzero NFA position due to the international disparity in the valuation of current versus future consumption.[36] In their empirical cross-country study, Wang et al. (2011) find that rates of time preference vary substantially across countries. They test this by surveying economics students from 45 countries, questioning whether the student would prefer to receive a smaller payment immediately or a larger payment in the future. With the exception of France, which was not included in their sample, a higher proportion of students from all of the countries included in the trading region (Canada, France, Germany, Japan, and the UK) would prefer to wait than would students from the US. In order to see the effect of differential time preferences in the face of demographic change, I adjust the time preference rate for the US to 4.0 percent (an increase from 3.5 percent) while holding the region's rate constant at 3.5 percent. Ceteris paribus, I find that a relative increase in the rate of time preference generates a significant negative NFA position. If the American rate of time preference is held constant at 4 percent, the long run decrease in the NFA position amounts to a fall of 21 percent in response to the demographic transition. If the American rate of time preference is set to decrease from 3.5 to 4.0 percent, the long run NFA position falls by 694 percent. During the first 30 periods, the American NFA position falls by 177 percent. The substantial change in the American time preference towards current consumption augments the impact of the relatively higher saving behavior of the region, additionally decreasing the NFA position for the US.

6.5 Wealth Inequality across Countries

I now turn to estimating the inequality of the age-dependent distribution of assets in each country and the effect of the demographic transition and structural change on that allocation. Assuming international arbitrage with the rate of return on capital and using

[36] See Buiter (1981) and Ghironi et al. (2005) among others.

the definition of nonhuman wealth, the individual's budget constraint at the steady state is given by the following:

$$\dot{\tilde{W}}(u) = \left(\tilde{r} + \mu(u)\right)\tilde{W}(u) + \tilde{w} - \tilde{C}(u) \tag{13.11}$$

Integrating this and using (9.8), I solve for the steady state wealth distribution across the cohorts:

$$\tilde{W}(u) = e^{\tilde{r}u + M(u)}\left\{\tilde{w}\int_0^u e^{-\tilde{r}u - M(u)}du - \frac{\tilde{H}}{\tilde{\Delta}}\int_0^u e^{((\sigma-1)\tilde{r} - \sigma\rho)u - M(u)}du\right\} \tag{13.12}$$

From the steady state individual asset equation, (13.12), I am able to derive the cohort wealth Gini coefficient for each country; a measure of inequality attributed to individuals being at different points of their savings life cycle, Mierau and Turnovsky (2013). This measure relates how unequal the distribution of assets is across the generations. According to Davies et al. (2007), the concentration of wealth within countries is often high. A common interval of wealth Gini coefficients for a country will lie between 0.6-0.8. In comparison, the Gini coefficient associated with disposable income often falls within the 0.3 to 0.5 interval. Using the estimated demographic characteristics and the observed structural changes from 1980 to 2010, the simulated model produces the following "natural" wealth Gini coefficients for the US and the region as shown by Table 2.3.

Obviously the observed Gini coefficients, as shown in Table 2.7, are far different from the smaller estimated statistics for the modeled countries. This is to be expected due to the fact that this inequality estimate is solely measuring the wealth inequality associated with an age-dependent saving difference across individuals. I find that the mortality decline, irrespective of the observed productivity change, causes a slight decrease in inequality. The extended lifetime accompanying a decrease in mortality increases the proportion of wealthy individuals in the economy. Additionally, because I hold the population growth rate constant while decreasing mortality, there is a minor fall in the birth rate necessary to maintain the steady state demographic distribution. This slight fall in the birth rate enhances the downward pressure on the Gini coefficient. Conversely, a pure decrease in old age mortality will increase the Gini coefficient.

Moving to the included Gini statistics associated with the structural changes, I first consider the productivity shock. As stated, each country experiences a TFP increase during the time period, with the US experiencing a relatively larger gain. This increase in productivity triggers an accumulation of capital that generates a complementary effect on wages in both regions. This wage-driven income effect on inequality is negligible due to the fact that all individuals receive the same wage. However, the increase in the return of capital slightly benefits the older capital-laden individuals, which causes an increase in inequality. Overall, the fall in the birth rate, as stated above, dominates this effect and the Gini coefficient falls slightly.

The transition towards capital in the production process dramatically increases the inequality measure. The increase in the share of capital employed by the firm lowers the overall wage level negatively impacting the younger agent's income more so than an older individual with higher capital holdings. Given that the older, wealthier individuals are hurt less by the fall in wage income and benefit more from the firm's increasing capital demand, the Gini coefficient increases.

Finally, an increase the pure rate of time preference decreases inequality. If individuals weigh current consumption to a greater extent over time, their savings behavior declines. This decreases the asset holdings of the wealthy older individuals naturally flattening the asset distribution over the life cycle, causing inequality to fall.

7. Retirement Model Simulation

I augment the previous analysis by including a model incorporating a retirement period and social security system. This framework allows for the determination of the effect of an asymmetric aging of the workforce of the economies, as exhibited by the falling labor force participation rate. Additionally, with the inclusion of a pay-as-you-go social security structure, I am able to examine how the decline in retirement benefits and increase in income taxes will impact the saving behavior over the life cycle. The maximum retirement period for each economy is 25 years, assuming the agent reaches their maximum age of death, with an expected retirement period of 10 years. The modeled ages of retirement and pension replacement rates are exogenous and have been set to match the observed values. With the exclusion of Canada, over the 30-year time period all countries experienced a fall in their pension benefit level, with the US experiencing a decline of 12 percent and the trading region experiencing a decline of 29.7 percent.[37]

The dynamics of the macroeconomic equilibrium can be approximated by the system of per-capita equations given by (6.8), (6.11), (6.16), (7.5), and (13.10) for the home country and a similar system of equations for the foreign region. As shown above, if the dynamic system is specified as per-worker, the dynamic equations associated with consumption and wealth both include the labor supply adjustment term. With an instantaneous adjustment, the labor force participation rate is constant during transition and the term is removed.

7.1 Mortality Decrease – Retirement Model

Figure 2.7 displays the transitional dynamics associated with a decrease in mortality. In anticipation of the extended retirement period during which no wage income is earned, per-capita savings for both countries increase. The slightly longer life expectancy for the region causes its agents to save relatively more than Americans. Due to this, the per-capita wealth for the region increases by 1.14 percent, while the US only experiences an increase of 0.88 percent. In consequence of the increase in the saving rate, agents decrease their consumption expenditure and per-capita consumption for both countries falls. The relative frugality of the region causes the NFA position for the Americans to develop a negative and decreasing value over time. In the short run, from the initial equilibrium to year 30, the per-capita NFA position for the US decreases by 7.6 percent. In the long run, Americans experience an NFA position decrease of 66.7 percent.

7.2 Demographic Transition with Social Security Structure

Figure 2.8-A displays the per-capita dynamics associated with the mortality change and the introduction of a social security structure. Including the social security system entails a couple of important adjustments to the model. The most visible change is the establishment of a governmental sector collecting a wage income tax and transferring the proceeds to the retired. In this model, the tax rate is found endogenously through the

[37] Pension rates can be found in Table 2.8.

interaction of the per-capita wage rate, identical for all workers within a country, and the labor force participation rate, which determines the tax base. Over the time period the exogenously set replacement rates experience a substantial fall amounting to a decrease of 12 percent for the US and 23 percent for the region. The dramatic decrease in retirement benefits allow the American tax rate to decrease by 1.73 percent and the region’s tax rate to fall by 12.7 percent.

In this simulation, because the observed effective retirement ages for the included countries are relatively constant, I set the retirement age at 65. Taken in isolation, the country that enacts an increase in the retirement age will experience a long run increase in per-capita wealth and consumption. Due to the longer time frame during which an individual can accumulate savings, the NFA position for the country increases.

As shown by Figure 2.8-A, the inclusion of the social security system causes the US to experience a positive but declining NFA position for the following reasons. The relatively higher life expectancy increases the savings of the region above that of the Americans, placing positive pressure on the region’s NFA position. Additionally, the substantial decline in the region’s relative pension benefit places additional negative pressure on the American NFA position. The massive reduction in benefits for the trading region’s retirees causes the region’s agents to save and the American NFA position to decrease by a larger magnitude than the decrease associated solely with the mortality decline. Taken together, these effects cause the long run American NFA position to fall by 76 percent.

Figure 2.8-B exhibits the model including the social security structure and a gradual productivity change. The gradual productivity increase causes the countries to accumulate capital, positively impacting the per-capita consumption level. The NFA position exhibits a significantly different path depending on if the productivity change occurs instantaneously or over time. An immediate change in the productivity level causes an overshooting of the long run steady state, causing the path to exhibit a positive slope, opposite of what is observed. Forcing a gradual productivity transition produces a more realistic path. With the NFA position declining during the entire 30-year time interval.

8. Conclusion

This paper has developed a two-country neoclassical growth model including a realistic demographic structure for the main purpose of analyzing the impact of asymmetries of demographic characteristics on cross-country interdependence. Unlike many models in international economics, transitional dynamics are generated primarily by the evolution of demographic variables. In this analysis I have exhibited a few major points. The first is that a relative increase in the life expectancy of the population will exert a positive force on the NFA position of a country. This is generated by the fact that individuals who experience an extension of their lifetime will increase their savings. In the international setting, the higher relative saving rate of the country will naturally lead it to be a lender in the international capital market. This result holds whether or not a retirement period is included in the model. The retirement period will limit the interval during which agents are able to accumulate wealth, but an increase in the saving behavior of the workers will cause the overall result to hold.

Secondly, the pension benefit will impact the generated NFA position due to its impact on the savings of the individual. Over the 30-year time frame, the region experienced an average fall in their pension benefit. While this did generate a fall in the tax rate imposed on the workers, the substantial decline the in the benefit caused agents to increase their savings in order to compensate for the benefit loss. Due to the region's compensating saving behavior, including the social security program augmented the American NFA position's decline.

The third is that the rate of time preference has a significant impact on the NFA position through its impact on consumption behavior. A relatively high time preference rate generates a negative NFA position due to the population's partiality towards current consumption. With the empirical results from the study by Wang et al. (2011) suggesting that the US might experience a relatively higher time preference rate than the other countries included in this study, I model the impact of a higher American preference rate during the 30 year period. The transition to a higher preference rate for the US causes a significant decline in the long run American NFA position, reinforcing the effect of the region's high saving rate.

These findings relate to the natural wealth inequality for the baseline model as follows. A fall in mortality causes an increase in inequality due to the presence of longer living wealthy individuals; however, with a constant population growth rate, the birth rate will also decline in order to maintain the demographic steady state thereby decreasing inequality. Structural changes that benefit capital holders, such as an increase the share of capital in the production process, will increase inequality. Increasing the pure rate of time preference decreases inequality, due to the fact that a preference towards current consumption limits the saving behavior of an individual, flattening the life-cycle path of asset accumulation.

While the demographic framework of this model is complex, I realize that the overall model does simplify many structures to a sizeable extent. One important extension of this work is to compare the resulting equilibria and dynamics with a limited and asymmetric annuities structure present in each country. Not only would this potentially increase the accuracy of consumption across generations, as it would cause the consumption-age profile to be "hump shaped", but the implications for the consumption and saving behavior and therefore the NFA position would be significantly influenced. Additionally, the inclusion of a labor choice and age-dependent productivity would also strongly affect the nature of the cross-country interdependence. International differences in the age-productivity profile would reinforce demographic asymmetries across countries. The relative proportion of youth in the population would inherently change the productivity of the country's factors of production impacting international capital flows. With the inclusion of elastic labor, due to the complementarity of labor and capital in the production process, cross-country capital flows would influence the labor-leisure choice of the worker, directly impacting the aggregated savings of the economy. These complexities would increase the realism of the analytical framework and add additional international asymmetries to analyze.

Chapter 3: The Impact of the Reversal of China's One-child Policy

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Abstract

In this paper I have developed a modified version of the Mierau and Turnovsky (2015) model to determine the impact of the reversal of China's state fertility policy commonly known as the "one-child policy". In order to estimate the impact of the policy reversal on the labor market, I forecast the survival function forward 20 years to determine the old-age dependency rate. I augment the analysis by including a simulation estimating the impact of the proposed 5-year extension of the retirement age. Finally, I compare the implications of the model with and without an annuities market. From this analysis I develop three key results. The first is that the government will be forced to reduce the national pension benefit by 2.85 percent if they continue with their established policies. The second is that the proposed 5 year retirement age extension will be sufficient to keep the pension system solvent during the demographic transition. Lastly, I find that the lack of annuities causes the modeled individual to accumulate less capital over their life cycle. While the transitional paths of the models with and without an annuities market are generally analogous in their shape and convergence rate, the model without annuities generates an overall variable level that is relatively lower.

Keywords: One-child Policy, Social Security, Annuities

JEL Classification: H55, J11, E10

1. Introduction

In an astonishing announcement in November 2013, the Chinese government declared that it would relax its one child per family fertility restriction commonly known as the "one-child policy" (OCP). In this announcement, the government officially stated that if either member of a couple happened to be an only child they could now have two children. This policy was extended to all couples in October 2015. China's National Health and Family Planning Commission estimates that 90 million couples are eligible for this new two-child policy.[38] The reasons for this reversal are primarily of economic origin. The Chinese birthrate has been falling for the last 50 years from its peak in the mid 1960's. Currently, the World Health Organization approximates the average Chinese fertility rate at 1.7 children per woman, below the replacement rate of 2.1 births needed to maintain the current distribution of the population.[39] This has led to the overall aging of the population, stressing the national pension system through the degeneration of the tax base and forcing the government to consider new policies designed to return the social security system to solvency.

The solvency of a social security system is a concern for most advanced economies utilizing a pay-as-you-go transfer mechanism. With the added pressure of declining birth rates increasing the number of dependents per employee, governments across the world are scrambling for new politically feasible policies to help fund their pension systems. In this paper, I examine the impact of the reversal of the OCP and an extension of the retirement age using a closed economy neoclassical growth model with a realistic demographic framework. Through the use of numerical simulations, I derive estimates of feasible pension benefits associated with the proposed changes in China's social security policy and new fertility regulations.

While the introduction of fertility-constraining policies is not new to China, the OCP is the most recognized and restrictive in Chinese history, officially estimated to have prevented approximately 400 million births.[40] The policy dates back to 1980 when the government officially limited the allowed number of children to one for most urban families for the sake of economic development. The policy did not treat all families equally. Families that were located in rural areas or were members of certain ethnic minorities were eventually allowed by law to have more than one child. Overall, roughly two-thirds of Chinese couples are affected by the policy (Gu et al. (2007)). Families in violation of the law face punishment in the form of fines, loss of employment, and forced

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[38] http://en.nhfpc.gov.cn/2015-10/30/content_22313939.htm
[39] http://www.economist.com/node/18651512
[40] http://www.bbc.com/news/world-asia-china-34674444

abortions. Recently, Chinese authorities stated that 336 million abortions and 196 million sterilizations had been performed from the early 1970's to 2013 in order to enforce the policy.[41]

Determining the policy's impact on population growth is haphazard at best. Serious cases of misreporting birth statistics in order to achieve population targets are pervasive in some of the participating provinces. In a 1992 survey, two provinces underreported the number of newborns by 37.3 percent (Gu and Cai, 2009). While underreporting is a problem, the fertility rate of China has dropped below the replacement level in all of the subsequent studies leading demographers to assume that the 1992 reported statistics, while skewed, support the trend of an aging population. Officially the government has credited the policy as effectively decreasing the population and promoting the growth of the economy. Although the policy has been in effect for over 30 years, it was not meant to be permanent and the gradual lifting of restrictions has now begun.[42] The new official fertility rate target is 1.8 children per woman, aimed at limiting the official Chinese population to 1.5 billion.[43]

As a result of the low population replacement rate, the solvency of the national pension system is at risk as the population proportion of the retired increases. Since 1990 the proportion of the population aged 65 and older has steadily increased, reaching above 9 percent in 2014. Additionally, and more alarmingly, the old-age dependency ratio, as measured by the World Bank as the ratio of older dependents, people older than 64, to the working-age population, those aged 15-64, has increased to 12.47 percent in that same year. This percentage is relatively low as compared to other developed countries, however, the World Bank forecasts that by 2050 the population percentage of individuals aged 60 and older will reach 34 percent and by that time only 1.6 workers on average will support one retiree, down from the 4.9 workers per retiree currently.[44] Furthermore, the UN projects that the median age for the Chinese population will reach 43 years by 2030 overtaking the estimate for the US population of 40 years.[45] Due to these trends, in order to maintain the solvency of the pension system, the Chinese government plans to increase its official retirement age from 60 for males and 55 for females, gradually over an undisclosed period of time by a few

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[41] http://www.businessinsider.com/what-happened-when-people-violated-the-one-child-policy-2015-10
[42] Central Committee of the Chinese Communist Party. September 1980. *An Open Letter to Members of the Chinese Communist Party and Chinese Communist Youth League on Controlling Population Growth*.
[43] http://news.xinhuanet.com/english/china/2013-11/15/c_132891920.htm
[44] http://www.bloomberg.com/bw/articles/2013-10-31/chinese-rage-at-the-pension-system
[45] United Nations Department of Economic and Social Affairs/Population Division World Population Prospects: The 2015 Revision, Key Findings and Advance Tables

months each year.[46] Despite this overall ambiguity, some government officials have recently supported a proposal increasing the retirement age for men to age 65 by the year 2030, however the formal and explicit policy is set for announcement in 2017.[47]

In order to accurately model the effect of the reversal and social security policy change, a rich demographic structure must be included to estimate representative proportions of the employed and retired segments of the population. I use a modified version of the Mierau and Turnovsky (2015) framework that incorporates a realistic demographic structure through the inclusion of the survival function developed by Boucekkine et al (2002) (BCL). I estimate the BCL survival function's parameters with nonlinear least squares utilizing data from the World Health Organization's life table database, specifying the survival probability for each age. Unlike more tractable commonly used survival functions, the BCL function exhibits a realistic age-varying probability of death that accurately reflects the likelihood of dying at each age and will precisely model the expected lifetime of the simulated individuals. In order to estimate the impact of the policy reversal on the labor market and the national pension system, I include a social security system and an exogenous and mandatory period of retirement. The system taxes the employed at a common rate and transfers the benefit to the retired as a proportion of the aggregate wage rate. Due to the fact that the additional newly born children will take a number of years to enter the workforce, I forecast the life table data forward 20 years and re-estimate the BCL parameters to accurately represent the demographic structure when the fertility policy reversal will impact the labor market. I extend this analysis to include specifications with and without an annuities market. When included, the life annuities market is complete and perfectly competitive and offers a premium on acquired capital equal to the agent's age-dependent probability of death. In the specification without annuities, individuals save through the accumulation of capital and leave unintended bequests that are distributed throughout the economy by means of a lump sum transfer.[48] The removal of annuities generates a realistic "hump-shaped" life-cycle consumption profile as opposed to the monotonically increasing profile produced with the inclusion an actuarially fair annuities market.[49] The inclusion of a realistic life-

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[46] http://blogs.wsj.com/chinarealtime/2015/03/10/china-sets-timeline-for-first-change-to-retirement-age-since-1950s/

[47] http://blogs.wsj.com/chinarealtime/2013/08/21/chinas-retirement-age-sets-experts-at-odds/

[48] Lump sum transfer specification developed by Bruce and Turnovsky (2013).

[49] This is studied by: Bruce and Turnovsky (2013) using an endogenous growth model with a social security system, Bütler (2001) in a partial-equilibrium model, Hansen, G.D. and S. İmrohoroğlu, (2008) in a general equilibrium exogenous growth model, and Heijdra and Mierau (2012) in a endogenous growth model without social security.

cycle consumption path significantly impacts the aggregate economy due to the reduction of capital necessary to smooth the low level of consumption in the later ages.

Using this framework I derive the following results. Maintaining a constant pension tax rate set at the current observed value of 8 percent of wage earnings, I find that an average birth rate of 2 children per woman, as specified by the new state policy, will increase the labor force participation rate to such an extent that the national pension system will remain solvent. Additionally, with the influx of additional workers, the government will be able to increase the pension benefit by almost 18 percent while maintaining a balanced budget. Furthermore, the proposed extension of the retirement age to 65 will expand the tax base so significantly such that the benefit can be increased by 24 percent, even without the reversal of the fertility policy.

The remainder of this paper is structured as follows. Section 2 describes the analytical framework of the model. Sections 3 and 4 lay out the resulting equilibrium and dynamics. Section 5 describes the specification of the model including annuities. Section 6 covers the numerical simulations and Section 7 concludes the paper.

2. Analytical Framework

Due to the similarity with Mierau and Turnovsky (2015), I briefly describe the analytical model. Within the economy the cohort's age at any random time, t, is given by $t-v$. Agents born at time, v, have a finite lifetime and die at age, D. The cohort variables are denoted by $X(v,t)$, where, v, denotes the cohort vintage and, t, denotes calendar time. The time derivative of a variable at the cohort level is specified as $\partial X(v,t)/\partial t = X_t(v,t)$.

2.1 Demographic Structure

The probability of survival of an individual that is born at time v for age $(t-v)$ is given by a general survival function $S(t-v)=e^{-M(t-v)}$. At the age of birth and death the following survival probabilities are given, $S(0)=e^{-M(0)}=1$ and $S(D)=e^{-M(D)}=0$. The probability of dying at each age, or the instantaneous probability of death, is given by $-S'(t-v)/S(t-v)=\mu(t-v)$.

As stated, the exogenous demographic structure that will be utilized in this paper was developed by Boucekkine et. al. (2002) (BCL) and is given by:

$$e^{-M(t-v)} = \frac{\mu_0 - e^{\mu_1(t-v)}}{\mu_0 - 1} \tag{14.1}$$

The age-dependent instantaneous probability of death increases realistically over an individual's lifetime and is given by the following:

$$\mu(t-v) = \frac{\mu_1 e^{\mu_1 (t-v)}}{\mu_0 - e^{\mu_1 (t-v)}} \tag{14.2}$$

Equation (4.1) is a highly tractable and accurate survival function specification. The BCL demographic structure has two parameters that determine the life expectancy of the agent and the shape of their survival probability distribution. The μ_0 parameter regulates the death rates of young agents. The μ_1 parameter specifies the death rates for the elderly agents. In the numerical simulations below, these parameters will be chosen such that the life expectancy and age-specific survival probabilities will match the mortality trends of China.

2.2 Production

The economy is populated by a representative competitive firm that employs workers and rents capital to produce a consumption good through the following constant returns to scale function:

$$Y(t) = AF\left(K(t), L(t)\right) \tag{15.1}$$

where $Y(t)$ is the aggregate output at time t, $K(t)$ is aggregate capital located in the Home country, and $L(t)$ is the aggregate supply of labor. The production function satisfies the standard conditions: $F_L > 0, F_K > 0, F_{LK} > 0,$ and $F_{LL}, F_{KK} < 0$. Per-worker output may be expressed by the following:

$$y(t) = \frac{Y(t)}{L(t)} = AF\left(\frac{K(t)}{L(t)}, 1\right) \equiv Af\left(k(t)\right) \tag{15.2}$$

Output per worker is denoted by $Af(k)$, where $k(t)$ is the capital-labor ratio and A is the Hicks neutral total factor productivity (TFP) parameter. The firm rents capital and hires labor such that the following marginal products are equalized with the price of the input:

$$Af'\left(k(t)\right) = r(t) + \delta \tag{15.3}$$

$$Af\left(k(t)\right) - Af'\left(k(t)\right)k(t) = w(t) \tag{15.4}$$

Where $r(t)$ is the endogenously determined interest rate, δ is the rate of depreciation, and $w(t)$ is the wage rate paid to all workers regardless of age.

2.3 Household

Agents maximize their expected lifetime utility:

$$E\left(U(v)\right)=\int_{v}^{v+D}\ln\left(C(v,t)\right)e^{-\rho(t-v)-M(t-v)}dt \tag{15.5}$$

An individual of cohort v maximizes expected utility generated from the consumption of a generic consumption good $C(v,t)$ and has no bequest motive. ρ is the pure rate of time discount, while $\rho+\mu(t-v)$ is the overall rate of time discount at age $(t-v)$.

The individual's budget constraint is given by the following:

$$\frac{\partial K(v,t)}{\partial t}=r(t)K(v,t)+(1-\tau_s)w(t)I(R)+B(t)\left(1-I(R)\right)-C(v,t)+T(v,t) \tag{15.6}$$

$I(R)$ is an indicator function equaling one while the individual works. During the employment period, the agent earns the aggregate wage regardless of their age. Every worker is required to pay an income tax, τ_s, in order to fund the national pension system. The individual earns this after-tax wage until the date of their mandatory and exogenous retirement age, R. Following this period, the agent receives income from the social security benefit, $B(t)$. Additionally, the agent earns a lump sum tax $T(v,t)$ at each age. In order to account for the unintended bequests in the absence of an annuities market, I use the technique developed by Bruce and Turnovsky (2013) and define the lump sum transfer as the following:

$$T(v,t)\equiv\mu(t-v)K(v,t) \tag{15.7}$$

Performing the optimization leads to the following first order conditions:

$$\frac{1}{C(v,t)}=\lambda(v,t) \tag{15.8}$$

$$\rho-\frac{\partial\lambda(v,t)/\partial t}{\lambda(v,t)}+\mu(t-v)=r(t) \tag{15.9}$$

Equation (15.8) equates the marginal utility of consumption to the shadow value of wealth. Equation (15.9) equates the consumption rate of return to the return on capital. Taking the time derivative of equation (15.8) and combining it with equation (15.9) yields the growth rate of consumption:

$$\frac{C_t(v,t)}{C(v,t)}=r(t)-\rho-\mu(t-v) \tag{15.10}$$

The growth rate of consumption, as exhibited by equation (15.10), includes the instantaneous probability of death. As individuals age, the probability of death at each age realistically increases, which causes the consumption growth rate to eventually become negative. This leads the consumption path, as shown by equation (5.4), to become "hump shaped" as shown by Figure 3.5. The most substantial effect of the removal of annuities is the reduction in the level of accumulated capital. Agents lacking annuities consume less during the later years of their lifespan and therefore need to save less for the future.

Using equation (15.10) I derive the path of cohort consumption for the arbitrary time interval from t to τ:

$$C(v,\tau) = C(v,t)e^{R(t,\tau)-\rho(\tau-t)-M(\tau-t)} \qquad (15.11)$$

Where $R(t,\tau) = \int_t^\tau r(s)ds$.

Through the implementation of the transversality condition and the inclusion of equation (5.4) the individual consumption at an arbitrary time t may once again be written in terms of capital wealth, human wealth, and the MPC:

$$C(v,t) = \frac{K(v,t)+H(v,t)}{\Delta(v,t)} \qquad (15.12)$$

The human wealth equation is defined as the following:

$$H(v,t) \equiv \int_t^{v+R} (1-\tau_s)w(\tau)e^{-R(t,\tau)-M(\tau-v)+M(t-v)}d\tau + \int_{v+R}^{v+D} \text{B}(\tau)e^{-R(t,\tau)-M(\tau-v)+M(t-v)}d\tau \qquad (15.13)$$

Equation (13.3) includes the discounted after tax aggregate wage, $(1-\tau_s)w(t)$, and the pension benefit, $\text{B}(t)$, that the individual earns during the associated period of employment or retirement. I assume that the individual, at birth, holds no financial wealth.

I define the inverse of the marginal propensity to consume (MPC) as the following:

$$\Delta(v,t) \equiv \int_t^{v+D} e^{-\rho(\tau-t)-M(\tau-v)-M(\tau-t)+M(t-v)}d\tau \qquad (15.14)$$

Unlike human wealth, the MPC increases as the life expectancy falls.

2.4 Aggregation

I define the total population at time t as $P(t)$, the population's crude birth rate as φ, and the population growth rate as n. Abstracting from immigration, the number of living members of cohort v at any time t is given by:

$$P(v,t) = \varphi P(v) e^{-M(t-v)} \tag{15.15}$$

Where the relative weight of each cohort to the total population is defined as:

$$\frac{P(v,t)}{P(t)} = \varphi e^{-n(t-v)-M(t-v)} \equiv p(t-v) \tag{15.16}$$

And the dynamics of the weight is given by:

$$\frac{\partial p(t-v)/\partial t}{p(t-v)} = -\left[\varphi + \mu(t-v)\right] \tag{15.17}$$

The decrease in the relative size of the cohort is due to the overall increase in the population and the death of members of the cohort over time.

In order to obtain the aggregate per-capita variables, I employ the standard aggregator:

$$\overline{x}(t) = \int_{t-D}^{t} p(t-v) X(v,t) dv \tag{15.18}$$

Due to the fact that the population diverges from the labor supply, differentiation is needed between the per-capita and per-worker aggregate. For any general cohort variable, $X(v,t)$, I can derive its per-worker variable, $x(t)$:

$$x(t) = \frac{1}{l} \int_{t-D}^{t} p(t-v) X(v,t) dv \tag{15.19}$$

Where l is the labor force participation rate and is defined as:

$$l = \int_{t-R}^{t} p(t-v) dv = \frac{\varphi \int_{0}^{R} e^{-nu-M(u)} du}{\varphi \int_{0}^{D} e^{-nu-M(u)} du} \tag{15.20}$$

This specifies the fraction of the employed population and exhibits the fact that, due to the demographic steady state, the labor force participation rate is time-invariant as long as the exogenous retirement age and demographic structure is constant throughout the transition. The labor force participation rate is dependent upon the retirement age, population growth rate, and the survival function.

The time derivative of (13.5) yields the per-worker dynamics of the model:

$$\dot{x}(t)=\frac{\varphi}{l}X(t,t)+\frac{1}{l}\int_{t-D}^{t}p(t-v)X_{t}(v,t)dv+\frac{1}{l}\int_{t-D}^{t}\frac{\partial p(t-v)}{\partial t}X(v,t)dv-\frac{\dot{l}(R)}{l(R)}x(t) \quad (15.21)$$

The final term accounts for the adjustment of the labor supply to a change in the retirement age or the demographic structure. The term will be nonzero if the retirement age, R, or the demographic variables vary during the transition. Assuming an instantaneous adjustment for the demographic parameters and the exogenous retirement age, l will remain fixed at its final value and the final term above will drop out.

Using this method I derive the following per-worker home consumption level:

$$c(t)=\frac{1}{l}\int_{t-D}^{t}p(t-v)C(v,t)dv \quad (15.22)$$

Taking the time derivative of (6.7), home consumption dynamics are given by:

$$\dot{c}(t)=\frac{\varphi}{l}C(t,t)+\left(r(t)-\rho-n-2\mu_{c}(t-\upsilon_{1})\right)c(t)-\frac{\dot{l}}{l}c(t) \quad (15.23)$$

Equation (6.9) results from using the mean value theorem of integration on equation (6.7). μ_c is interpreted as the ratio of the consumption given up by the dying to per-capita consumption.[50]

$$\mu_{c}(t-\upsilon_{1})\equiv\frac{\int_{t-D}^{t}\mu(t-v)p(t-v)C(v,t)dv}{\int_{t-D}^{t}p(t-v)C(v,t)dv} \quad (15.24)$$

The approximation of the time derivative of the human wealth equation (13.3) is given by:

$$\dot{H}(t)=-(1-\tau_{s})w(t)+\left[r(t)+\mu_{H}(\tau-t)\right]H(t)-\left[\beta-(1-\tau_{s})\right]w(t)e^{\left[\frac{\dot{w}(t)}{w(t)}-r(t)\right]R} \quad (15.25)$$

As stated in Mierau and Turnovsky (2015), the final term in (13.10) is an approximation of the wage income evolution occurring at the anticipated retirement age.

[50] Mortality variables associated with human wealth and the MPC inverse are defined analogously as:

$$\mu_{H}(\tau_{1}-t)\equiv\frac{\int_{t-D}^{t}\mu(t-v)p(t-v)H(v,t)dv}{\int_{t-D}^{t}p(t-v)H(v,t)dv} \text{ and } \mu_{\Delta}(\tau_{2}-t)=\frac{\int_{t-D}^{t}\mu(t-v)p(t-v)\Delta(v,t)dv}{\int_{t-D}^{t}p(t-v)\Delta(v,t)dv}$$

Now looking at the inverse of the MPC out of human wealth, I set $v = t$ and define $\Delta(t) \equiv \Delta(t,t)$.

$$\Delta(t) = \int_{t}^{t+D} e^{-\rho(\tau-t)-2M(\tau-t)} d\tau \qquad (15.26)$$

Once again using the method, I derive the dynamics of the MPC inverse:

$$\dot{\Delta}(t) = -1 + \left[\rho + 2\mu_{\Delta}(\tau_2 - t)\right]\Delta(t) \qquad (15.27)$$

Equations (6.8) and (6.16) both include a multiple of the mortality variable as opposed to the case including annuities. This is due to the consumption growth rate being a function of the instantaneous probability of death.

2.5 Government

A government sector is included to conduct the transfer of income from the employed to the retirees. The government is represented by a typical pay-as-you-go transfer scheme. To fund this system the government taxes labor income at rate τ_s for all working individuals. All individuals are taxed at the same rate while they are employed regardless of age. Taxes are used solely to fund the benefit that retired agents receive and therefore the government maintains a balanced budget each period given by the following relation:

$$\int_{t-R}^{t} p(t-v)\tau_s w(t) dv = \int_{t-D}^{t-R} p(t-v) B(t) dv \qquad (15.28)$$

The benefit retirees receive, $B(t)$, is proportional to the aggregate wage rate such that, $B(t) = \beta w(t)$. Given that the government is maintaining a balanced budget, if the simulated tax rate is matched to the observed value, the benefit will be endogenously determined.

3. Equilibrium

In steady state, the variables are dependent upon age and not calendar time. The dynamics converge to their long run levels. With no additional capital accumulation the per-worker market-clearing condition becomes:

$$Af\left(\tilde{k}\right) = \tilde{c} + \left(n + \delta\right)\tilde{k} \qquad (16.1)$$

Firm optimality conditions include the marginal product of capital and labor equating to their respective prices.

$$\tilde{r}+\delta = Af'\left(\tilde{k}\right) \tag{16.2}$$

$$\tilde{w} = (1-\alpha)Af\left(\tilde{k}\right) \tag{16.3}$$

Equilibrium human wealth:

$$\tilde{H} = \int_0^R (1-\tau_s)\tilde{w}e^{-\tilde{r}(u)-M(u)}du + \int_R^D \beta\tilde{w}e^{-\tilde{r}(u)-M(u)}du \tag{16.4}$$

The inverse MPC at steady state is given by:

$$\tilde{\Delta} = \int_0^D e^{-\rho u-2M(u)}du \tag{16.5}$$

In order to have a stable population distribution at steady state, I include a condition linking the survival function, death age, D, population growth rate, n, and the birth rate, φ, as specified by Lotka (1998):

$$\varphi\int_0^D e^{-nu-M(u)}du = 1 \tag{16.6}$$

By equating the survival function to zero and solving for the corresponding age I find the age of death.

$$D = \ln\left(\mu_0\right)/\mu_1 \tag{16.7}$$

Steady state consumption is the per-worker aggregate of the consumption-age profile, where equilibrium consumption over age is exponentially adjusting by $(\tilde{r}-\rho-\mu(t-v))$ from the birth consumption level of $\tilde{C}_0$:

$$\tilde{c} = \tilde{C}_0 \frac{\tilde{\varphi}}{l}\int_0^{\tilde{D}} e^{(\tilde{r}-\rho-n)u-2M(u)}du, \quad \tilde{C}_0 = \frac{\tilde{H}}{\tilde{\Delta}} \tag{16.8}$$

The initial consumption level, $\tilde{C}_0$, is derived from the product of the present discounted value of human wealth and the MPC at birth. At birth the individual owns nothing except for the present discounted value of their future wage income stream.

The government budget constraint holds:

$$\tau_s = \beta\tilde{d} \tag{16.9}$$

Where the dependency rate is calculated as:

$$\tilde{d} = \frac{1-\tilde{l}}{\tilde{l}} \tag{16.10}$$

The government's ability to fund the pension benefit is dependent upon the tax rate, the size of the labor force, and the size of the retired population. Taken in isolation, an increase in the dependency rate, as produced by a decrease in mortality, will increase the expected size of the retired population. Lacking an increase in the tax rate, the increase in the dependency rate will case the per person benefit to fall in order to maintain the solvency of the pension system.

4. Dynamics

The complete per-worker dynamic system is given by the following equations:

$$\dot{k}(t) = f\left(k(t)\right) - c(t) - \left(\delta + n\right)k(t) - \frac{\dot{l}}{l}k(t) \tag{17.1}$$

$$\dot{c}(t) = \frac{\varphi}{l}\frac{H(t)}{\Delta(t)} + \left(r(t) - \rho - n - 2\mu_c(t - \upsilon_1)\right)c(t) - \frac{\dot{l}}{l}c(t) \tag{17.2}$$

$$\dot{H}(t) = -(1-\tau_s)w(t) + \left[r(t) + \mu_H(\tau - t)\right]H(t) - \left[\beta - (1-\tau_s)\right]w(t)e^{\left[\frac{\dot{w}(t)}{w(t)} - r(t)\right]R} \tag{17.3}$$

$$\dot{\Delta}(t) = -1 + \left[\rho + 2\mu_\Delta(\tau_2 - t)\right]\Delta(t) \tag{17.4}$$

The above system includes the mortality variables, μ_c, μ_H, and μ_Δ. During the transition these variables vary minutely from their initial equilibrium values.[51] As extensively discussed by Mierau and Turnovsky (2014b), approximating these terms as constants during the transition does not significantly impact the implications of the model for most specifications. Furthermore, I employ the standard assumption that the mortality function remains fixed at its new equilibrium level throughout the transition. Equation (17.1) and (17.2) both include the labor supply adjustment term $\dot{l}/l$. This term is removed in the case of an instantaneous adjustment of the retirement age and demographic structure as shown in the following section. A description of the gradual adjustment process can be found in the appendix.

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[51] For the case without annuities the initial mortality values are: $\mu_c = 0.006$, $\mu_H = 0.0004$, $\mu_\Delta = 0.002$.

4.1 Linearized System

Linearizing equations (17.1) to (17.4) around the steady state values, $\tilde{k}$, $\tilde{c}$, $\tilde{H}$, and $\tilde{\Delta}$, the local dynamics can be expressed in a general for by equation (17.5):

$$\dot{X}(t) = L\left(X(t) - \tilde{X}\right) \tag{17.5}$$

Which can be written explicitly as:

$$\begin{pmatrix} \dot{k}(t) \\ \dot{c}(t) \\ \dot{H}(t) \\ \dot{\Delta}(t) \end{pmatrix} = \begin{pmatrix} \alpha A\tilde{k}^{\alpha-1} - \delta - n & -1 & 0 & 0 \\ \tilde{c}(\alpha-1)\alpha A\tilde{k}^{\alpha-2} & -\frac{\varphi\tilde{H}}{\tilde{l}\tilde{\Delta}\tilde{c}} & \frac{\varphi}{\tilde{l}\tilde{\Delta}} & -\frac{\varphi\tilde{H}}{\tilde{l}\tilde{\Delta}^2} \\ L_{31} & L_{32} & L_{33} & 0 \\ 0 & 0 & 0 & \frac{1}{\tilde{\Delta}} \end{pmatrix} \begin{pmatrix} k(t) - \tilde{k} \\ c(t) - \tilde{c} \\ H(t) - \tilde{H} \\ \Delta(t) - \tilde{\Delta} \end{pmatrix} \tag{17.6}$$

$$L_{31} = (\alpha-1)\alpha A\tilde{k}^{\alpha-2}\left(\left((1-\tau_s)\tilde{k} + \tilde{H}\right) + \left(\beta - (1-\tau_s)\right)\left(\tilde{k} + \tilde{c}R\right)e^{-rR}\right) \tag{17.7}$$

$$L_{32} = -\tilde{k}(\alpha-1)\alpha A\tilde{k}^{\alpha-2}\left(\beta - (1-\tau_s)\right)e^{-rR} \tag{17.8}$$

$$L_{33} = \frac{\tilde{w}}{\tilde{H}}\left((1-\tau_s) + \left(\beta - (1-\tau_s)\right)e^{-rR}\right) \tag{17.9}$$

This system is characterized for the instantaneous transition of the mortality and pension system characteristics. Per-worker capital is constrained to evolve sluggishly with its initial value equaling the per-capita holdings of capital. Consumption, human wealth, and the inverse MPC are able to adjust discretely. This system will generate a bounded saddle path transition if there are one negative and three positive eigenvalues.

4.2 General Solution

The solution to the locally linearized system specified by equation (17.6) is given by the following:

$$k(t) = \tilde{k} + C_1 e^{\lambda_1 t} \tag{17.10}$$

$$c(t) = \tilde{c} + V_c C_1 e^{\lambda_1 t} \tag{17.11}$$

$$H(t) = \tilde{H} + V_H C_1 e^{\lambda_1 t} \tag{17.12}$$

$$\Delta(t) = \tilde{\Delta} + V_\Delta C_1 e^{\lambda_1 t} \tag{17.13}$$

Where:

$$C_1 = \frac{k(0)l_0}{l} - \tilde{k} \tag{17.14}$$

$$V_c = L_{11} - \lambda_1 \tag{17.15}$$

$$V_H = \frac{1}{\lambda_1 - L_{33}}\left(L_{31} + L_{32}V_c\right) \tag{17.16}$$

$$V_\Delta = \frac{L_{41}}{\lambda_1 - L_{44}} \tag{17.17}$$

The negative eigenvalue of the linearized system is given by λ_1. I assume the initial position of capital is given by the per capita holdings as specified by $k(0)l_0$ in equation (17.14). The normalized eigenvector coefficients are given by equations (17.15) to (17.17).

4.3 Gradual Transition

In order to compare the gradual versus instantaneous adjustment of the birth rate increase and the social security policy change, the transitional paths are augmented to include the gradual adjustment of the variables. This modeling technique forces the specified variable to follow a sluggish continuous transition, eliminating discrete jumps. Within the dynamics, I have built in the dynamic process accounting for the gradual adjustment of the mandatory retirement age, benefit payout, and birth rate with the following functions, where θ is the rate of change per year:

$$R(t) = \tilde{R} + \left(R_0 - \tilde{R}\right)e^{-\theta_R t} \tag{17.18}$$

$$\beta(t) = \tilde{\beta} + \left(\beta_0 - \tilde{\beta}\right)e^{-\theta_\beta t} \tag{17.19}$$

$$\varphi(t) = \tilde{\varphi} + \left(\varphi_0 - \tilde{\varphi}\right)e^{-\theta_\varphi t} \tag{17.20}$$

This sluggish adjustment process allows for a more realistic approximation of the gradual implementation of the retirement age extension currently being considered in China. Furthermore, it produces a realistic response time of a family planning for an increase in the size of their family, through the gradual transition of the birth rate.

5. Model Comparison: The Inclusion of Annuities

Including an annuities market augments the return on capital to include the age-dependent probability of death. The lump sum transfer $T(v,t)$ of the previous specification equates to zero. This causes the agent's budget constraint to become the following:

$$\frac{\partial K(v,t)}{\partial t} = \left(r(t)+\mu(t-v)\right)K(v,t)+(1-\tau_s)w(t)I(R)+B(t)\left(1-I(R)\right)-C(v,t) \qquad (18.1)$$

The consumption growth rate becomes age-invariant. All individuals share a common consumption growth rate regardless of age.

$$\frac{C_t(v,t)}{C(v,t)} = r(t)-\rho \qquad (18.2)$$

Using equation (18.2) I derive the path of cohort consumption for the arbitrary time interval from t to τ:

$$C(v,\tau) = C(v,t)e^{R(t,\tau)-\rho(\tau-t)} \qquad (18.3)$$

Equation (18.3) displays a consumption path that is no longer "hump shaped". As long as the interest rate remains larger than the pure rate of time preference, the consumption path will be monotonically increasing.

The human wealth equation (13.3) remains identical as in no annuities case, but the MPC inverse is augmented due to the adjusted consumption growth rate.

$$\Delta(v,t) \equiv \int_t^{v+D} e^{-\rho(\tau-t)-M(\tau-v)+M(t-v)}d\tau \qquad (18.4)$$

The survival function discounts the inverse MPC to a lesser extent due to the fact that the consumption path is less influenced by the probability of death at each age.

6. Numerical Simulations

In order to determine the impact of the one-child policy reversal I include a number of numerical simulations. I approximate the impact of the policy reversal by increasing the exogenous birth rate from the baseline value of 1.7 births per woman, which generates a population growth rate of 0.78, to 2.0 births per woman. The baseline survival function parameter specification is estimated through nonlinear least squares using the World Health Organization's 2012 life table data. The estimation produces a remarkably tight fit resulting in an R^2 of .99. The baseline parameterization produces a life expectancy of 79.51 years. I then linearly forecast the life table data forward 20 years and re-estimate the BCL survival function's parameters in order to approximate the age-dependent survival probabilities during the period when the children born resulting from the policy reversal are entering the labor market. This new specification increases the life expectancy to 82.4 years. I exogenously set the pension tax rate at 8 percent to match China's employee contribution rate for their national pension system.[52] This produces a baseline social security replacement rate of 0.1228. Furthermore, I augment the analysis by including a section detailing the effect of the proposed retirement age increase. Currently, the Chinese government is considering gradually increasing the retirement age for men from 60 to 65. The baseline retirement age of 60 generates an initial labor force participation rate of 0.61. The baseline equilibrium values associated with this specification can be found in Table 1 with and without annuities available.

6.1 Mortality Reduction

The evolution of the survival function causes the life expectancy to increase by 3.61 percent. As a result of the constant mandatory retirement age and the mortality decline primarily impacting the older individuals, the lifetime extension primarily impacts the expected retirement length. Even though the fertility rate is held constant, due to the decline in morality, the population growth rate increases slightly. This is due to the demographic equilibrium condition relating the population growth rate, birth rate, and the average mortality rate. This condition is given by $n = \varphi - \bar{\mu}$, where $\bar{\mu}$ is the average mortality of the population. A fall in the mortality rate, even if the birth rate is held constant, will increase the overall population growth rate. The increase in the proportion of retirees due to the decrease in mortality causes the dependency rate to increase from 0.65 to 0.67. The lifetime extension drives workers to accumulate more capital in order to smooth consumption over a longer lifetime. Given that the inclusion of the annuities market produces a positive consumption growth rate over the life cycle, workers accumulate more capital in comparison with the model specification lacking annuities. The higher proportion of retirees causes the per-capita output and consumption to fall in the long run. Due to the constant tax rate set at 8 percent and the increase in

[52] http://www.china-briefing.com/news/2012/02/21/mandatory-social-welfare-benefits-for-chinese-employees.html

the dependency rate, the pension benefit falls by 2.85 percent in order to maintain the solvency of the system. The fall in the pension benefit causes the average consumption per retiree to decrease. Due to the constant birth rate and retirement age, the gradual and instantaneous transitional paths lack any significant divergence.

6.2 Decrease in Mortality and Birth Rate Increase

In addition to the mortality transition, the fertility rate is now adjusted to reflect the higher level associated with the reversal of the OCP. The higher fertility rate increases the size of the labor force, decreasing the dependency rate by 15.23 percent. Due to the fact that agents are born as workers, the increase in the birth rate causes per-worker capital to fall in the long run. The higher employed proportion of the population drives per-capita capital and output to increased levels. Due to the larger workforce and the constant taxation rate, the pension benefit increases by 17.9 percent. While the average per-capita consumption of workers falls due to the increase of their population, the average per-capita consumption of the retired increases in response to the more generous pension benefit.

6.3 Decrease in Mortality and Retirement Age Extension

I now consider the increase in the mandatory retirement age along with the decrease in mortality while holding the birth rate constant. I find that the change in the retirement age dominates the effect associated with the decrease in mortality. The increased population proportion of the workforce once again decreases the per-worker holdings of capital by 6.82 and 4.77 percent with and without the inclusion of an annuities market, respectively. Furthermore, the labor supply increase causes the per-worker consumption to fall. However, the per-capita levels of capital and consumption both increase as the population proportion of workers increases. The longer lifetime employment period coupled with the decrease in mortality causes the labor force participation rate to increase from 60.6 percent to 65.6 percent. Due to the constant tax rate and the increased tax base, the pension benefit increases by a significant 24.27 percent.

6.4 Total Effect

Including the fall in the birth rate, the decrease in mortality, and the postponement of the retirement age produces an output similar to the combined effects of figures 3.2 and 3.3. The collective effect causes the labor force participation rate to increase from 60.6 to 69.9 percent. The larger labor force causes per-worker capital and output to fall, but per-capita capital, output, and consumption to increase. Due to the larger tax base, the constant tax rate, and the later retirement age, the pension benefit increases by 51.52 percent. This significantly increases retiree welfare, as exhibited by the average per-capita consumption of the retired, by over 20 percent for either annuities specification.

7. Conclusion

In this paper I have utilized a modified version of the Mierau and Turnovsky (2015) model to determine the impact of the reversal of China's state fertility policy commonly known as the "one-child policy". I approximate the effect of the fertility policy relaxation by increasing the birth rate, as defined by the number of children born per woman, from its current value of 1.7 to 2 children per woman. In order to estimate the impact of the policy reversal on the labor market, I forecast the survival function forward 20 years to determine the population proportions of the workforce and the retired and the subsequent the dependency rate. I augment the analysis by including a simulation estimating the impact of the proposed 5-year extension of the retirement age in order to maintain the solvency of the national pension system. Finally, I compare the impact of the inclusion of an annuities market. From this analysis I have developed three key results.

The first result is that the reversal of the one-child policy will substantially increase the labor force participation rate of the country. As determined by the forecast of the age-dependent survival profile, China is experiencing an overall aging of its population, as exhibited by the population proportion of the retired increasing over the next 30 years. In the face of the increase of the dependency ratio, the government will be forced to adjust the pension system in order for it to retain its future solvency. I estimate that with the forecasted survival trend and the given contribution rate, the pension benefit must decrease by 2.85 percent in order for the government to maintain a balanced budget. The increase in the birth rate attributed to the fertility policy reversal could remove the need for this fiscal tightening. Even in the face of the survival function transition, if the birth rate were to increase to 2 children per woman, the labor force increase would expand the tax base enough to warrant an increase in the benefit payout of almost 18 percent.

The second result is that the proposed 5 year retirement age extension will be sufficient to keep the pension system solvent during the demographic transition without the added help of an increase in the birth rate. The delayed retirement successfully increases the tax base to such an extent that the pension benefit increases by 24 percent. This significantly increases the average consumption of the retirees and their resulting welfare.

Lastly, I include specifications with and without an annuities market. The absence of annuities generates a more realistic consumption profile at the expense of tractability. By comparing these specifications, I am able to determine how the results may differ if the modeler were to include annuities solely for the tractability gains. I find that the lack of annuities causes the modeled individual to accumulate less capital over their life cycle. This is due to the fact that the consumption profile is concave and substantially decreases at later ages. While the transitional paths are generally analogous in their shape and convergence rate, the model without annuities experiences a level that is often significantly lower than the case with annuities.

While this modeling framework has incorporated a complex demographic structure it relies upon an exogenously set birth rate. One of the main questions that many in China currently ask is whether the relaxation of the fertility policy will have any effect on the birth rate at all. The families that are most heavily impacted by this new policy are those that are relatively well off and located in primarily urban environments. Due to the high cost of large families in these settings, while many might want to increase the number of children in their household, it may not be a feasible consideration. Furthermore, the length of the forecast may lead to inaccuracies in the estimated working-retired population ratio. Due to the extreme health risks associated with the high levels of carcinogens found in air pollution in many Chinese cities, future survival functions may look entirely different than those estimated in this paper. In any case, this analysis offers a concrete foundation for a first attempt at estimating the impact of the fertility policy reversal.

Figures: Chapter 1

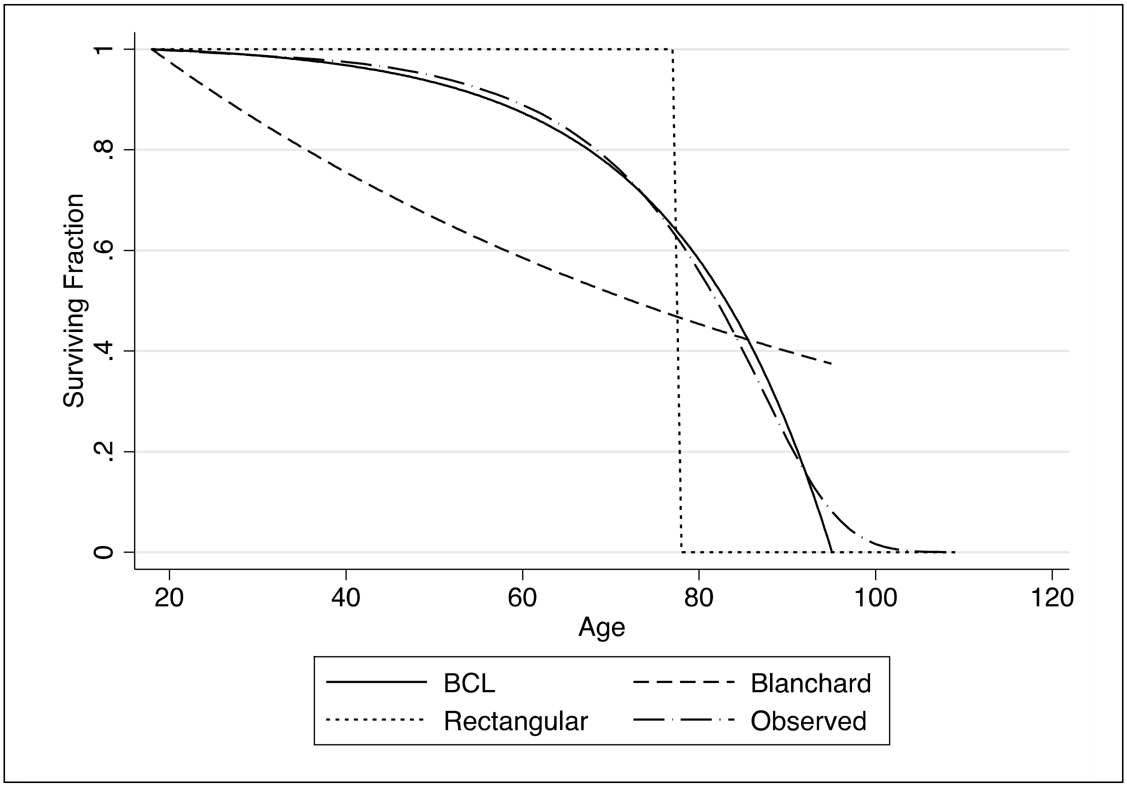

Figure 1.1: Survival functions per demography
Based on 2006 US data from www.mortality.org

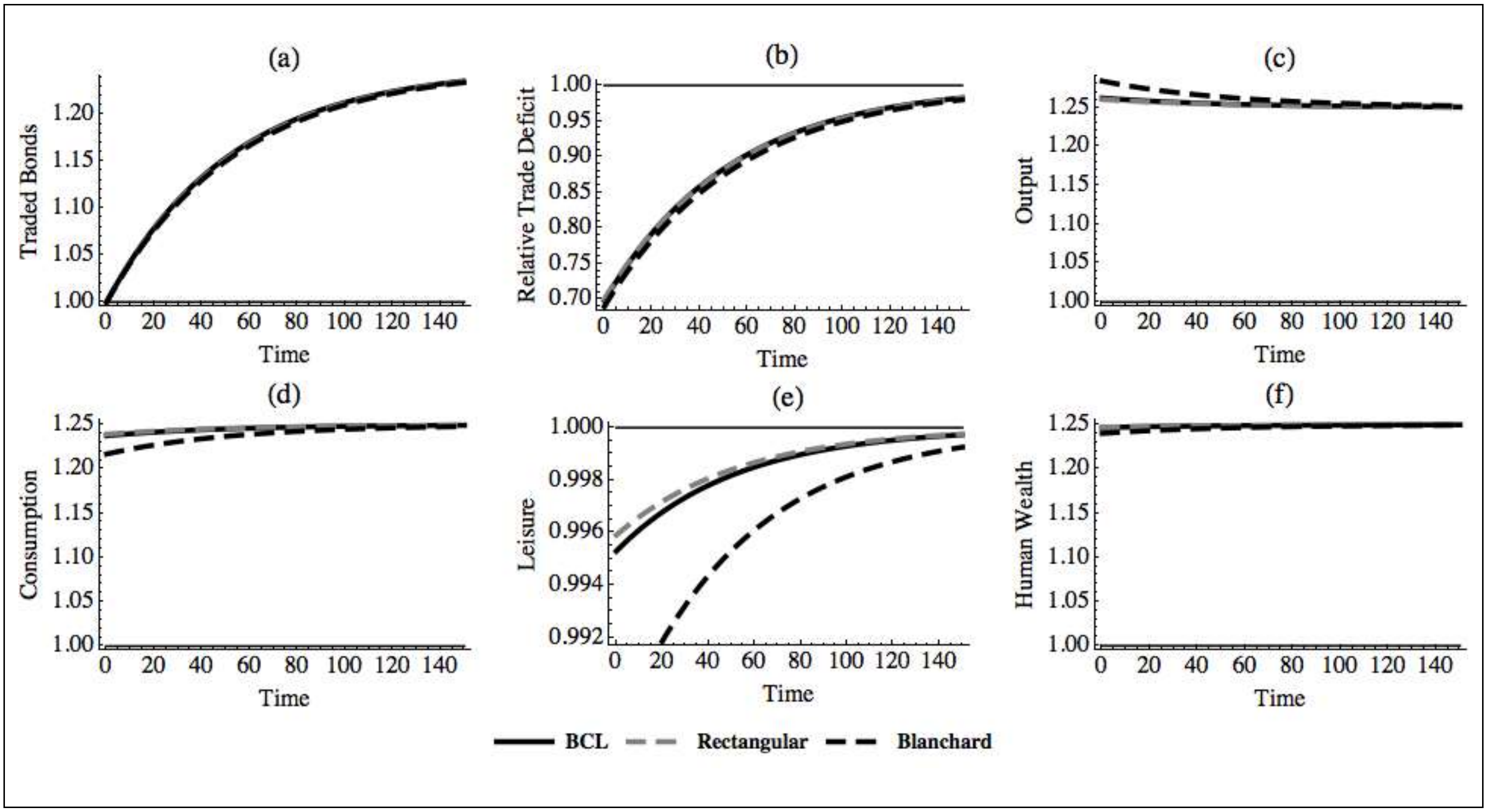

Figure 1.2.A: Per-capita Transition Paths – Increase in Productivity from A=1 to A=1.25

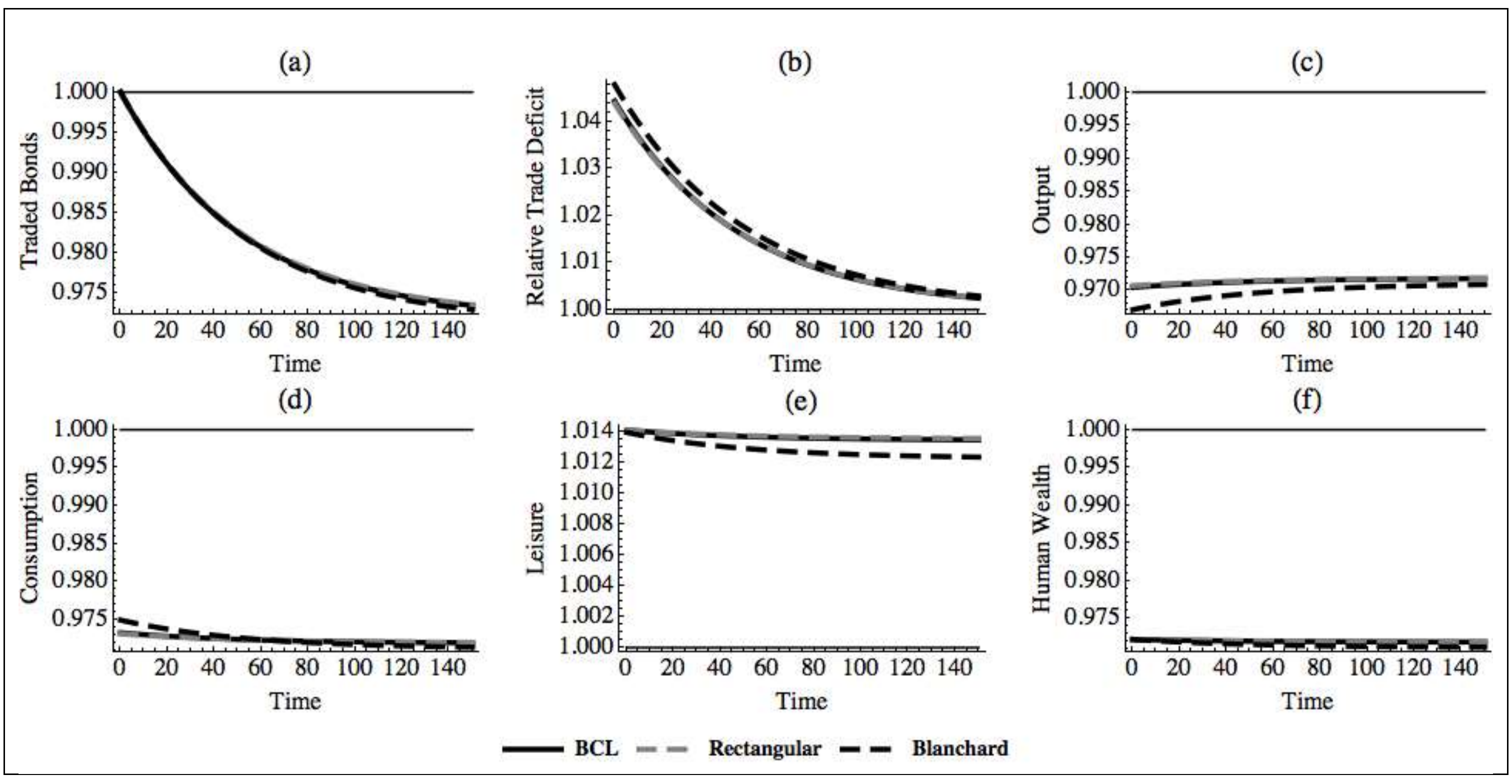

Figure 1.2.B: Per-capita Transition Paths – Increase in Income tax from τy=10% to τy=15%

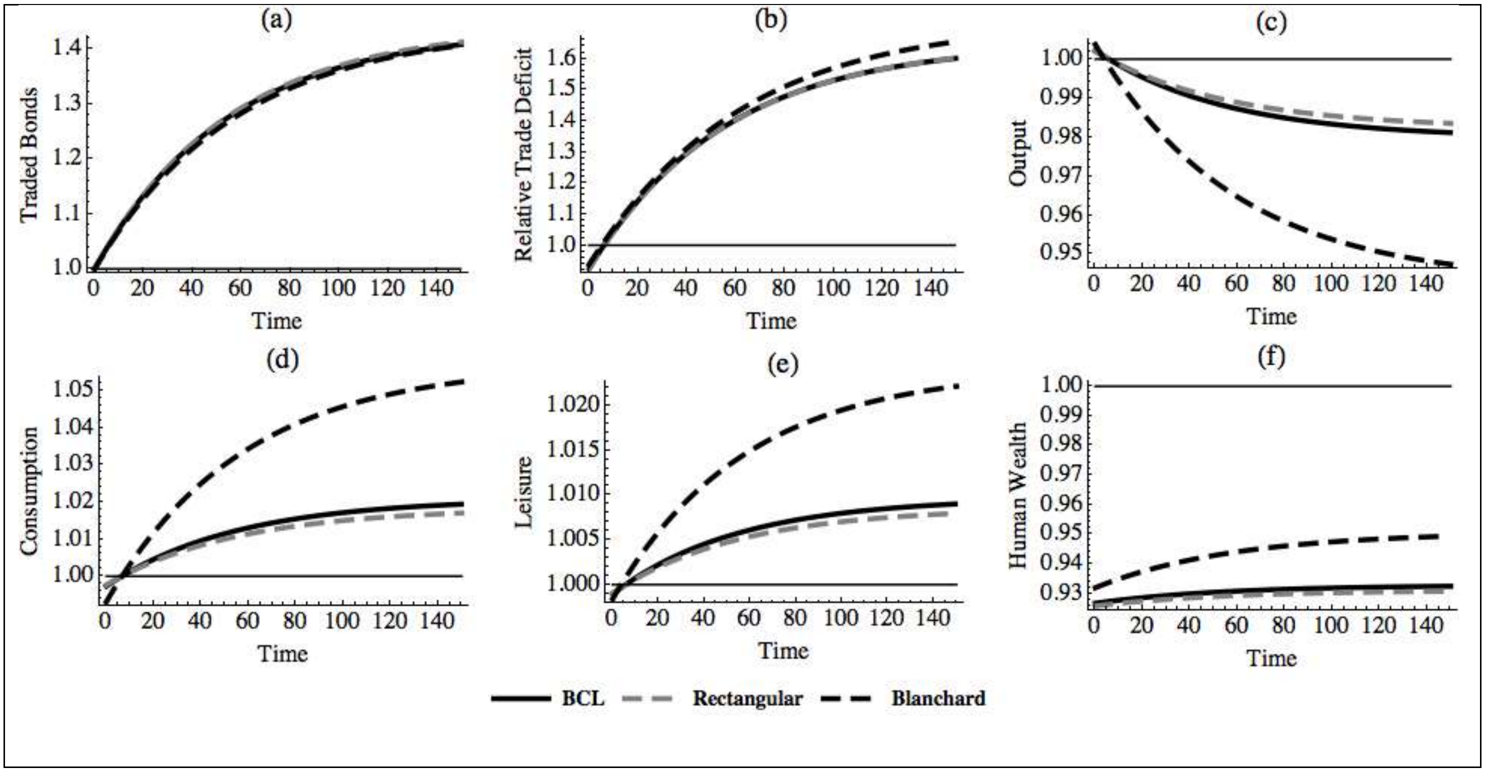

Figure 1.2.C: Per-capita Transition Paths – Increase in Foreign Interest Rate from r=5% to r=5.5%

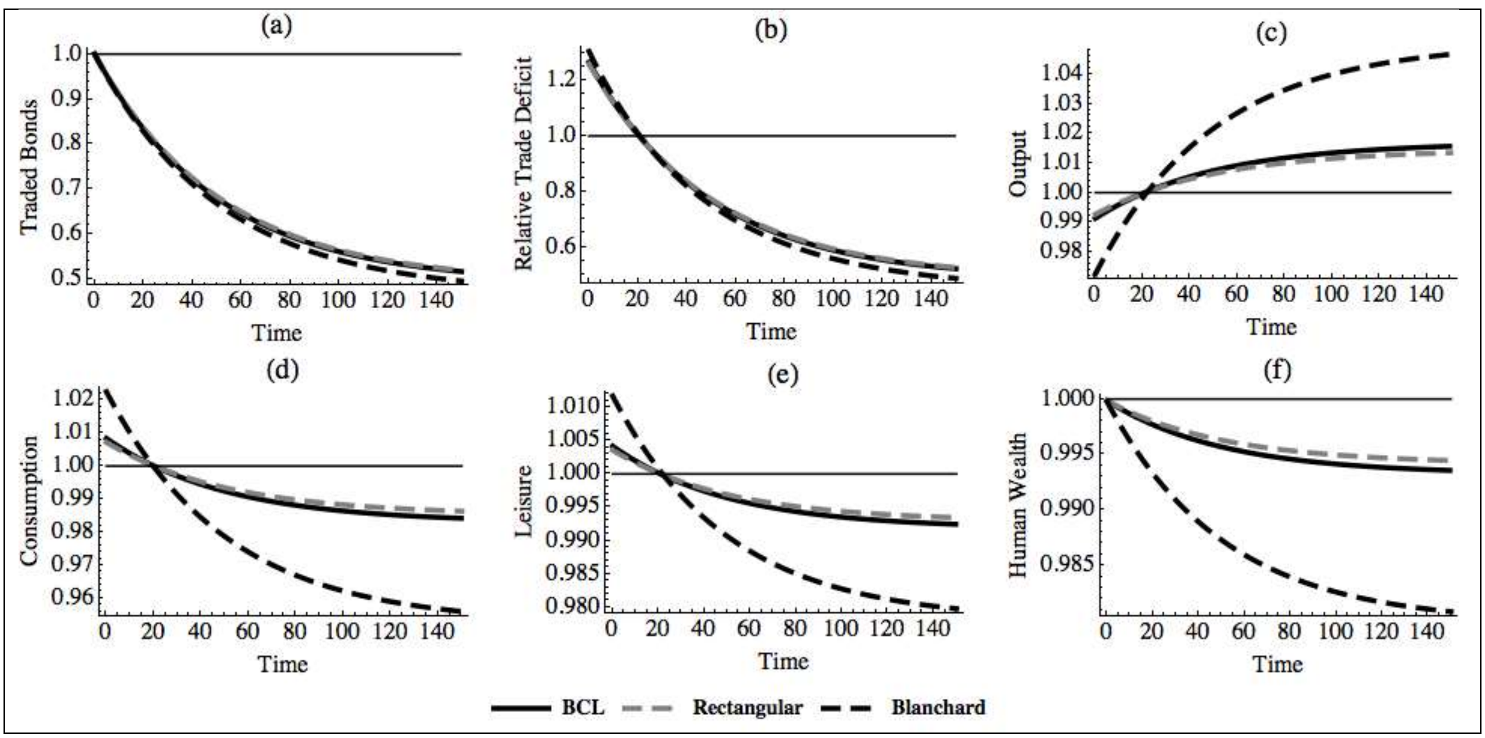

Figure 1.2.D: Per-capita Transition Paths – Increase in Rate of Time Preference from ρ=3.5% to ρ =4%

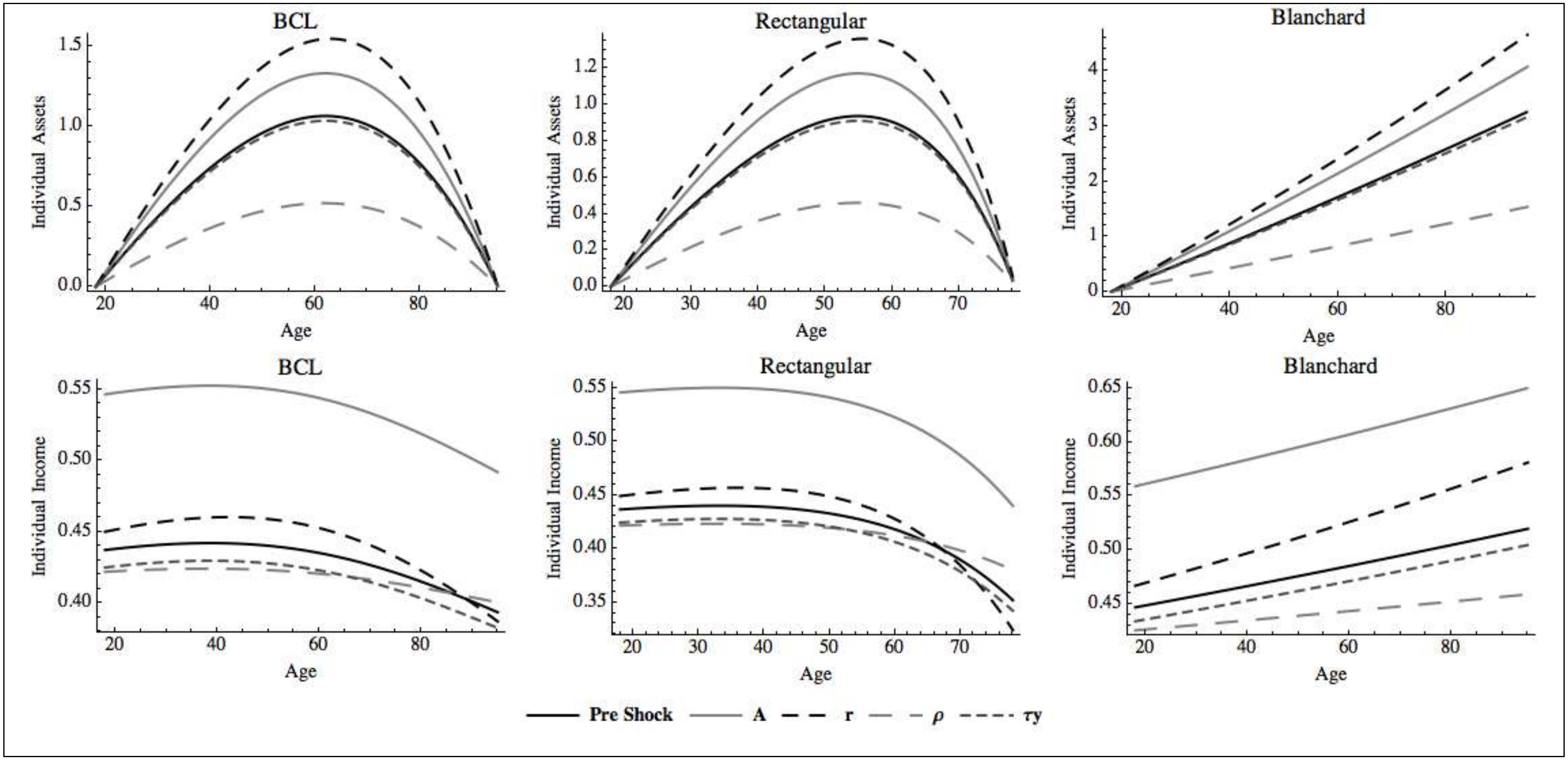

Figure 1.3: Distribution of assets and income over ages by shock

Where: A represents the porductivity shock, r the interest rate shock, ρ the time pref. shock, τ_y the income tax shock

Figures: Chapter 2

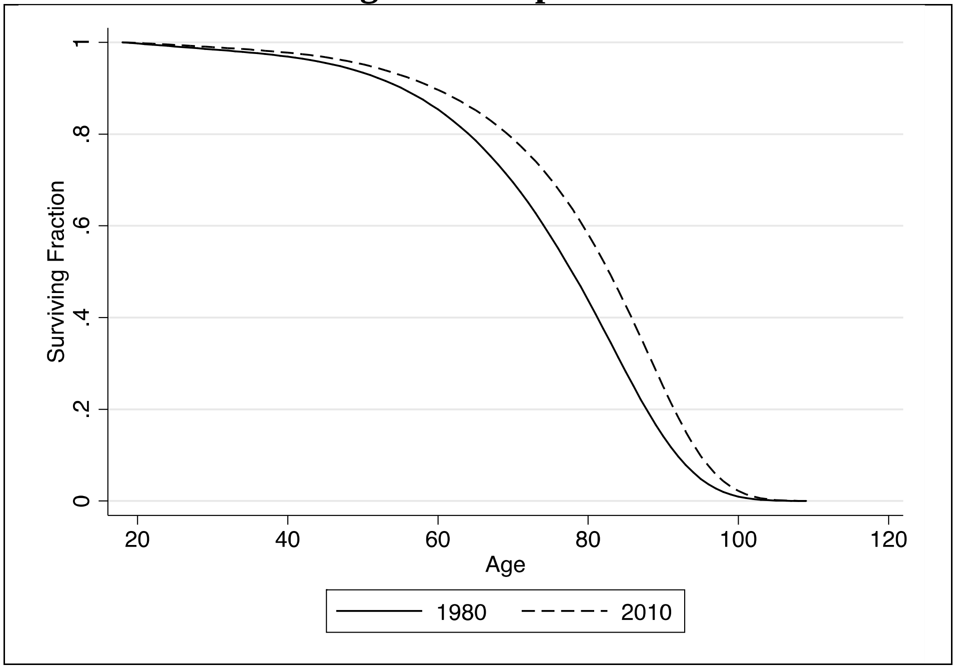

Figure 2.1: US observed survival data 1980-2010
Data retrieved from www.mortality.org

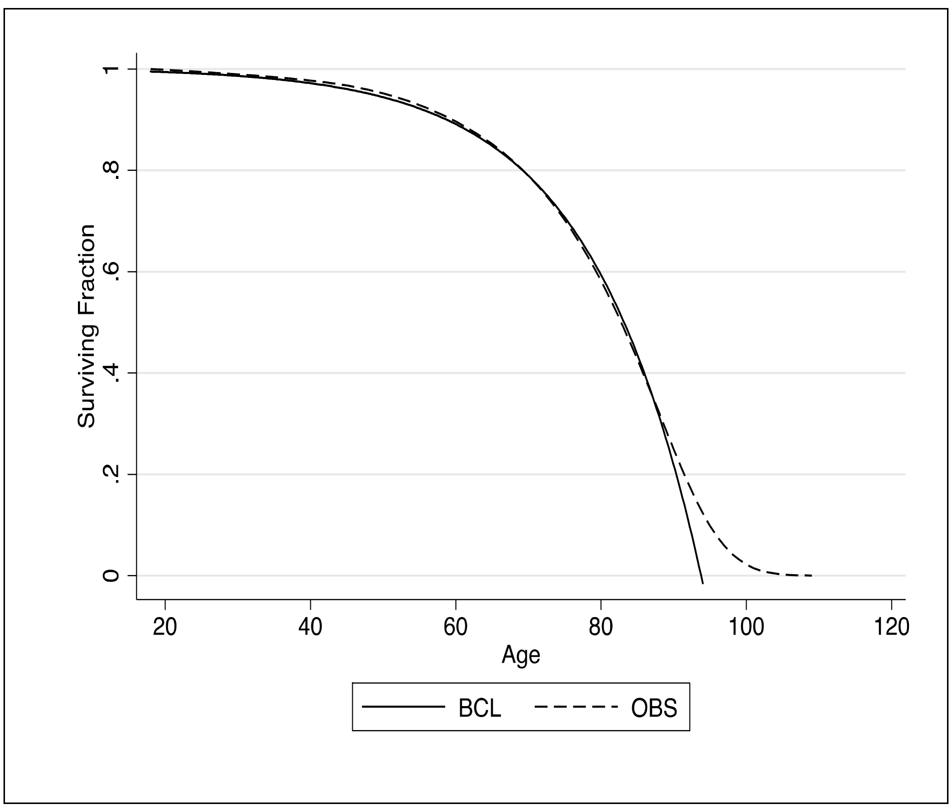

Figure 2.2: Estimated BCL function and 2010 US survival data
Data retrieved form: www.mortality.org

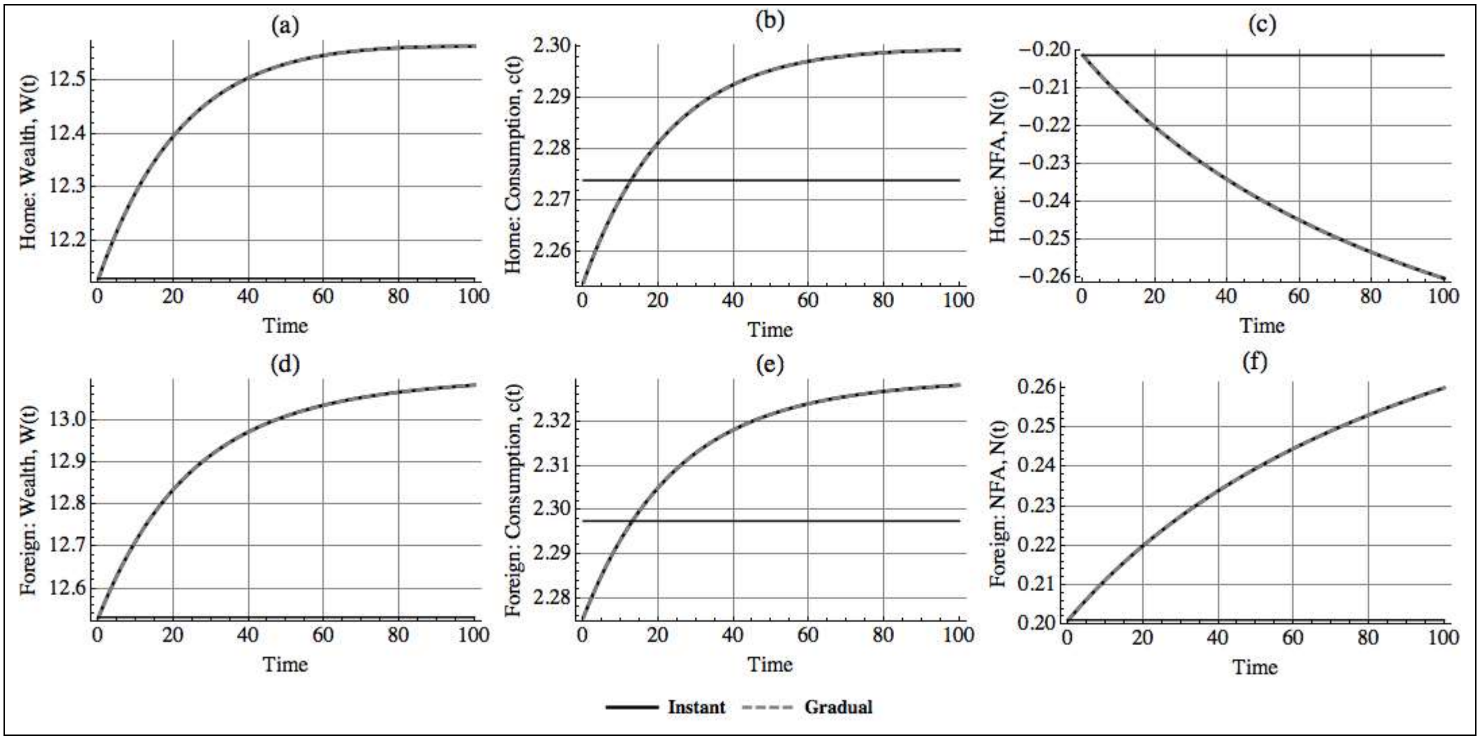

Figure 2.3: Baseline Model - US (Home) and multi-country (Foreign) mortality decrease

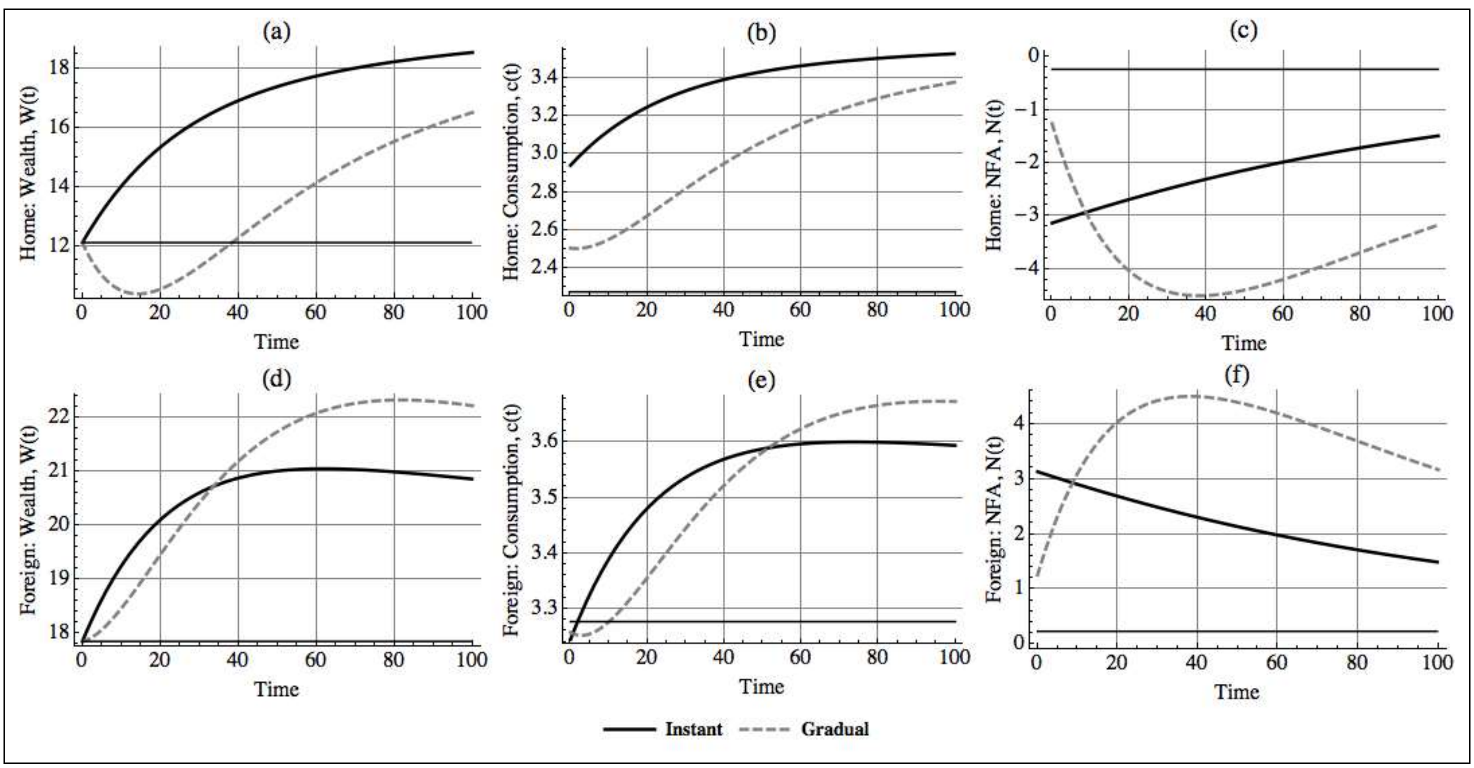

Figure 2.4: Baseline Model - US (Home) and multi-country (Foreign) productivity increase

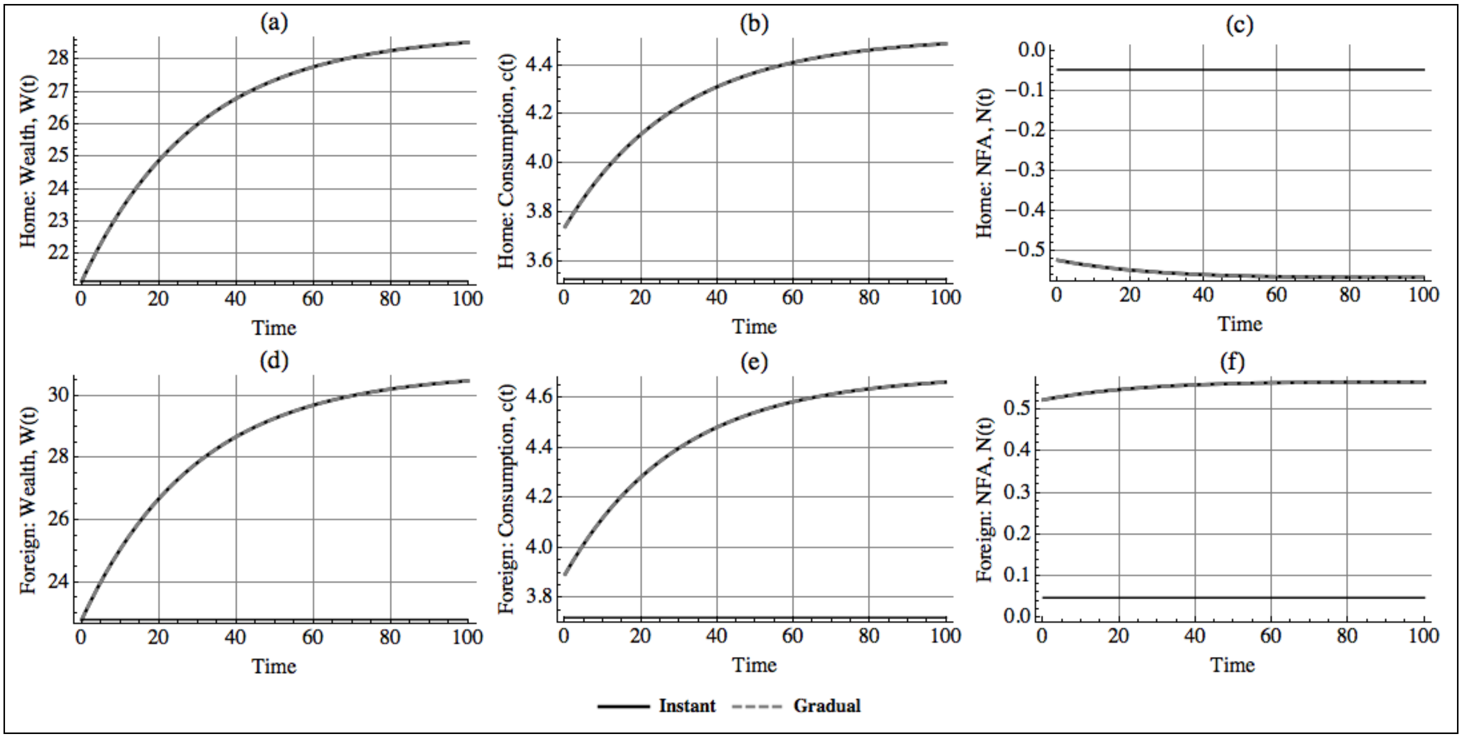

Figure 2.5: Baseline Model - US (Home) and multi-country (Foreign) capital share increase

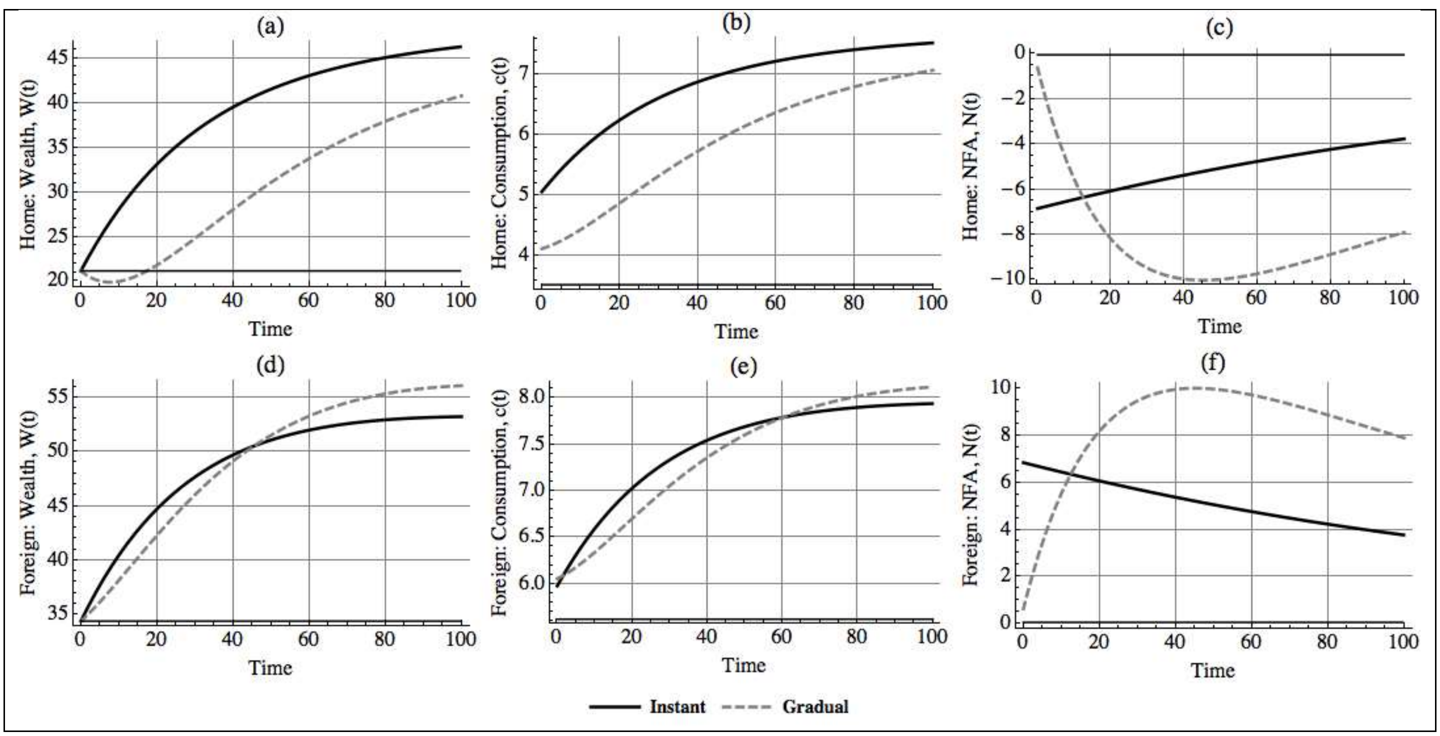

Figure 2.6: Baseline Model - US (Home) and multi-country (Foreign) productivity and capital share increase

*TFP data from PWT release 8.1.

**Capital share data available at: http://faculty.chicagobooth.edu/loukas.karabarbounis/research/index.html.

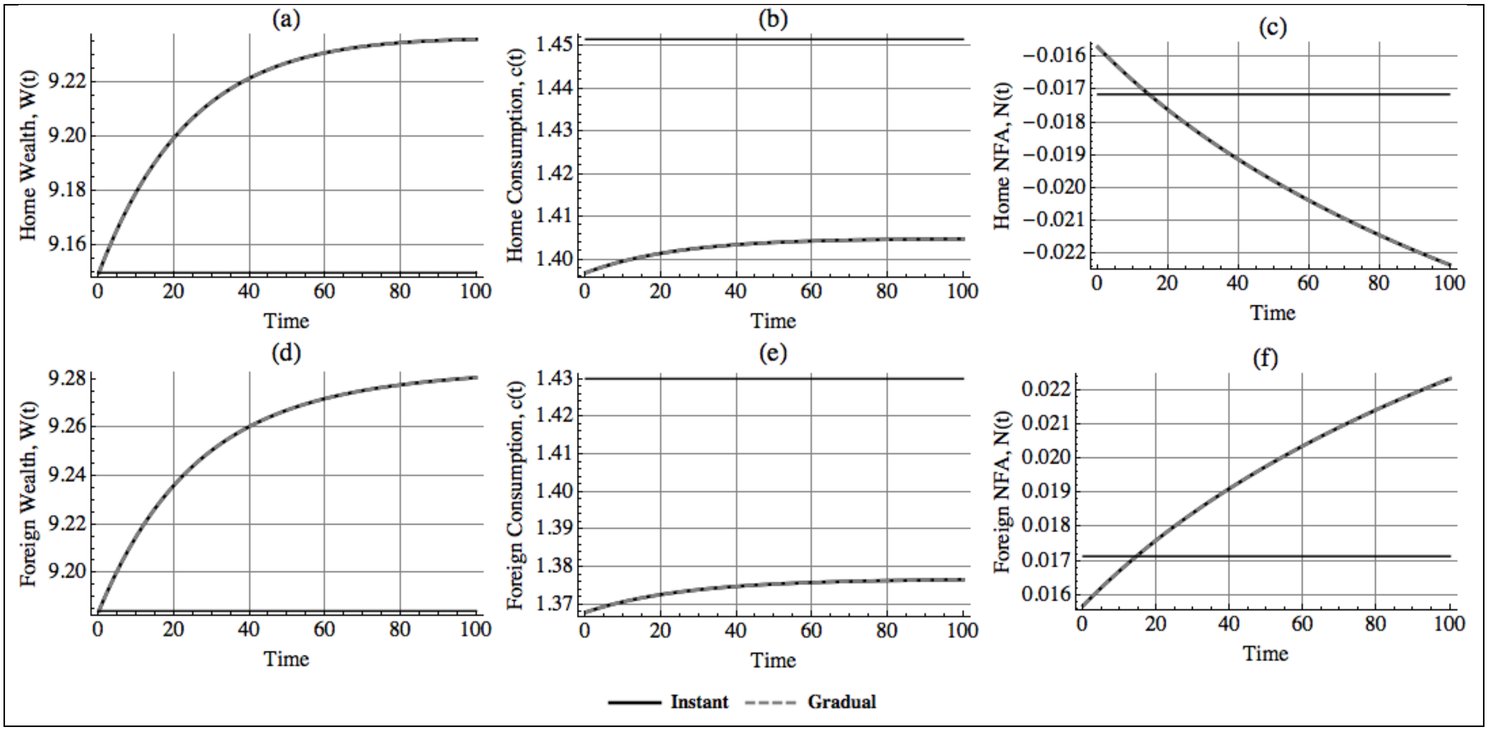

Figure 2.7: Retirement model - mortality decrease

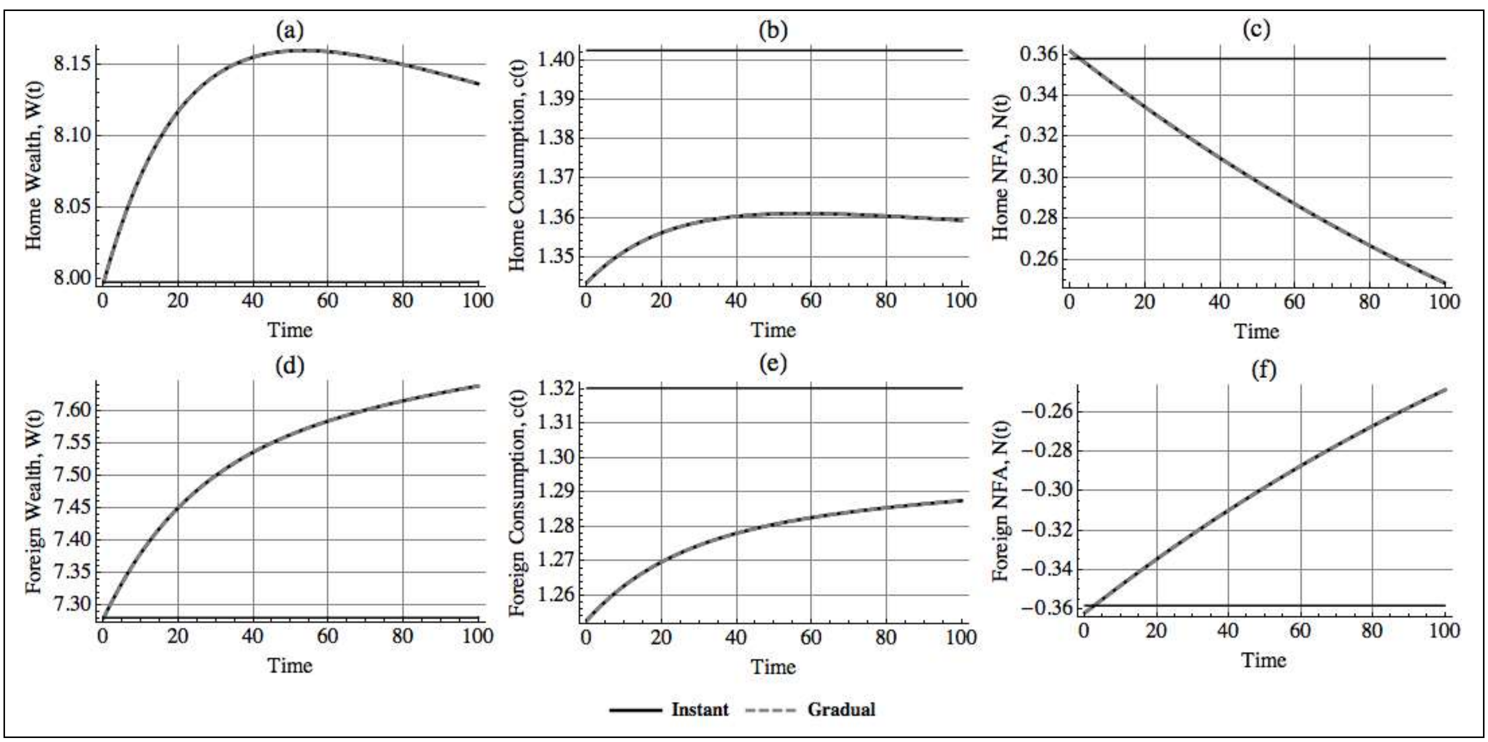

Figure 2.8-A: Retirement model - Demographic change with social security structure

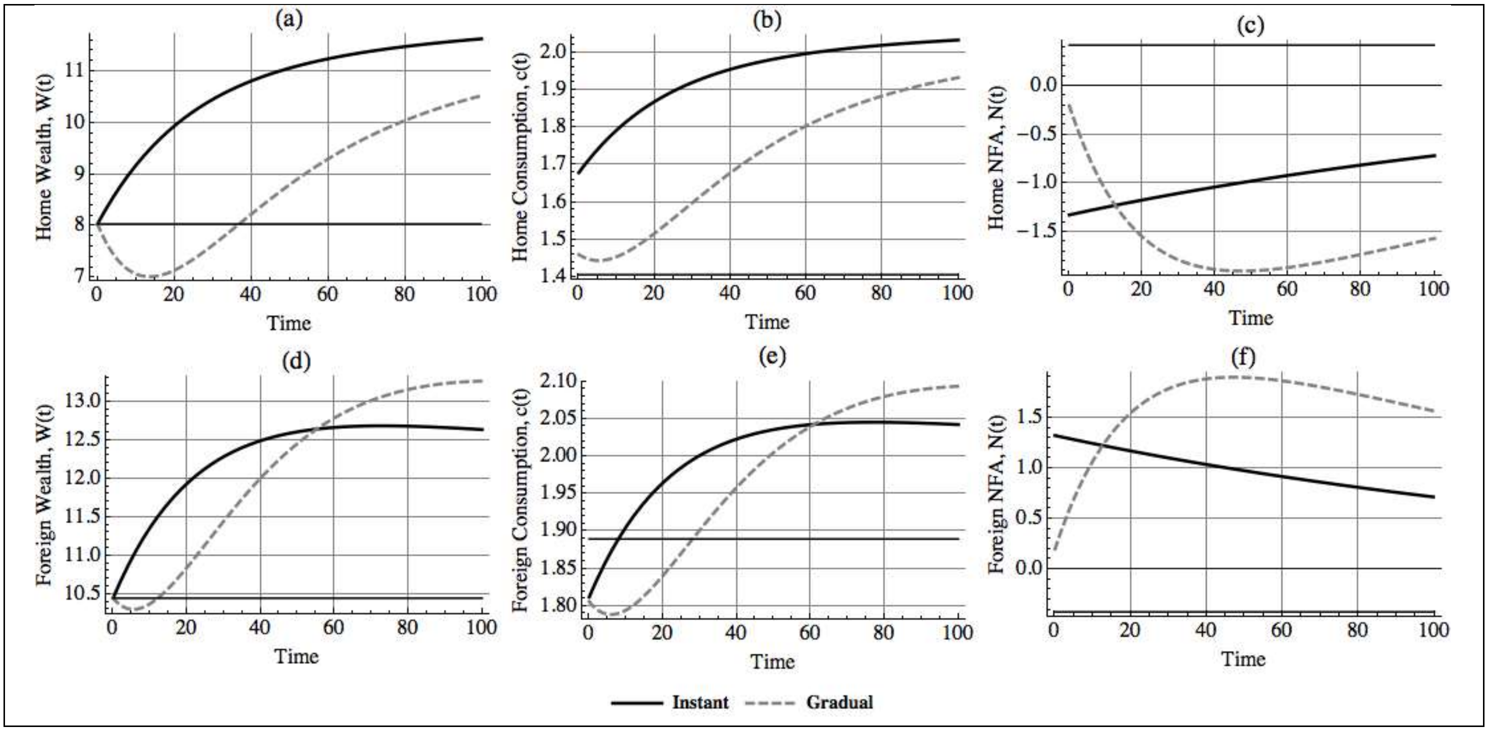

Figure 2.8-B: Retirement model - Demographic change with social security structure and TFP adjustment

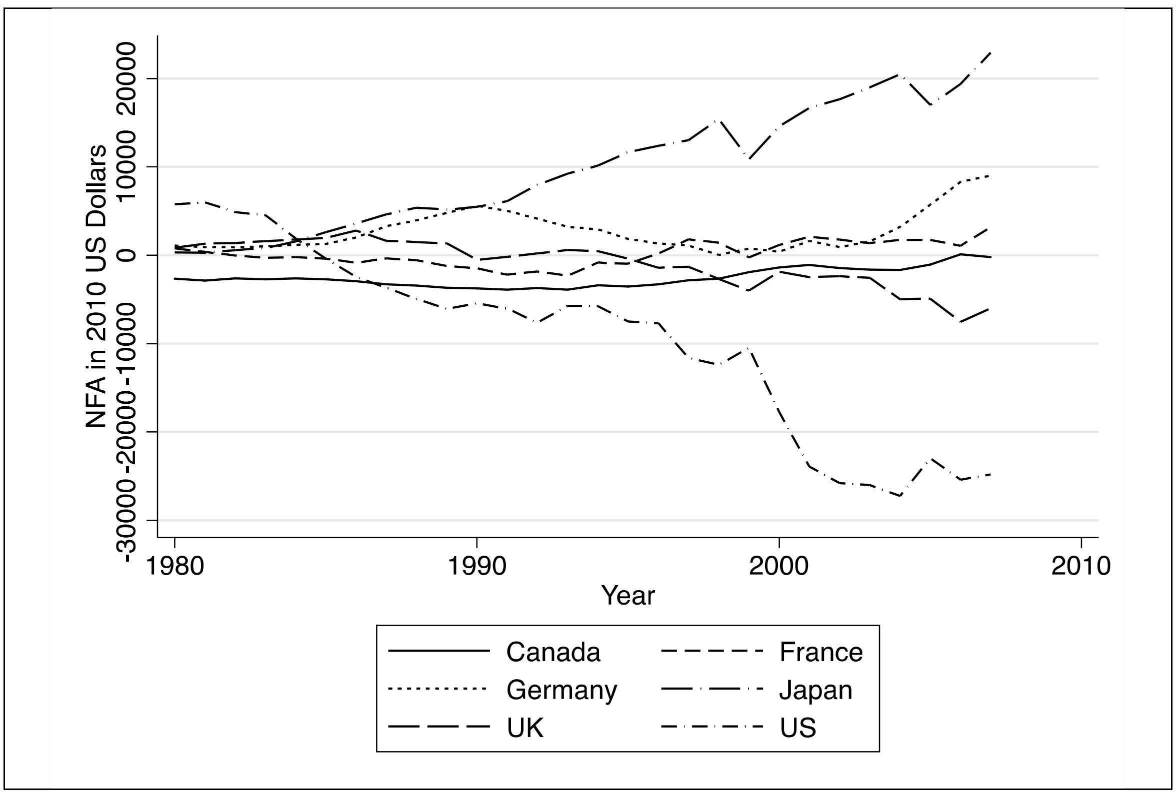

Figure 2.9: Net foreign asset position in 2010 US dollars (hundreds of millions)
Data from Lane, Milessi, and Ferreti (2007).

Figures: Chapter 3

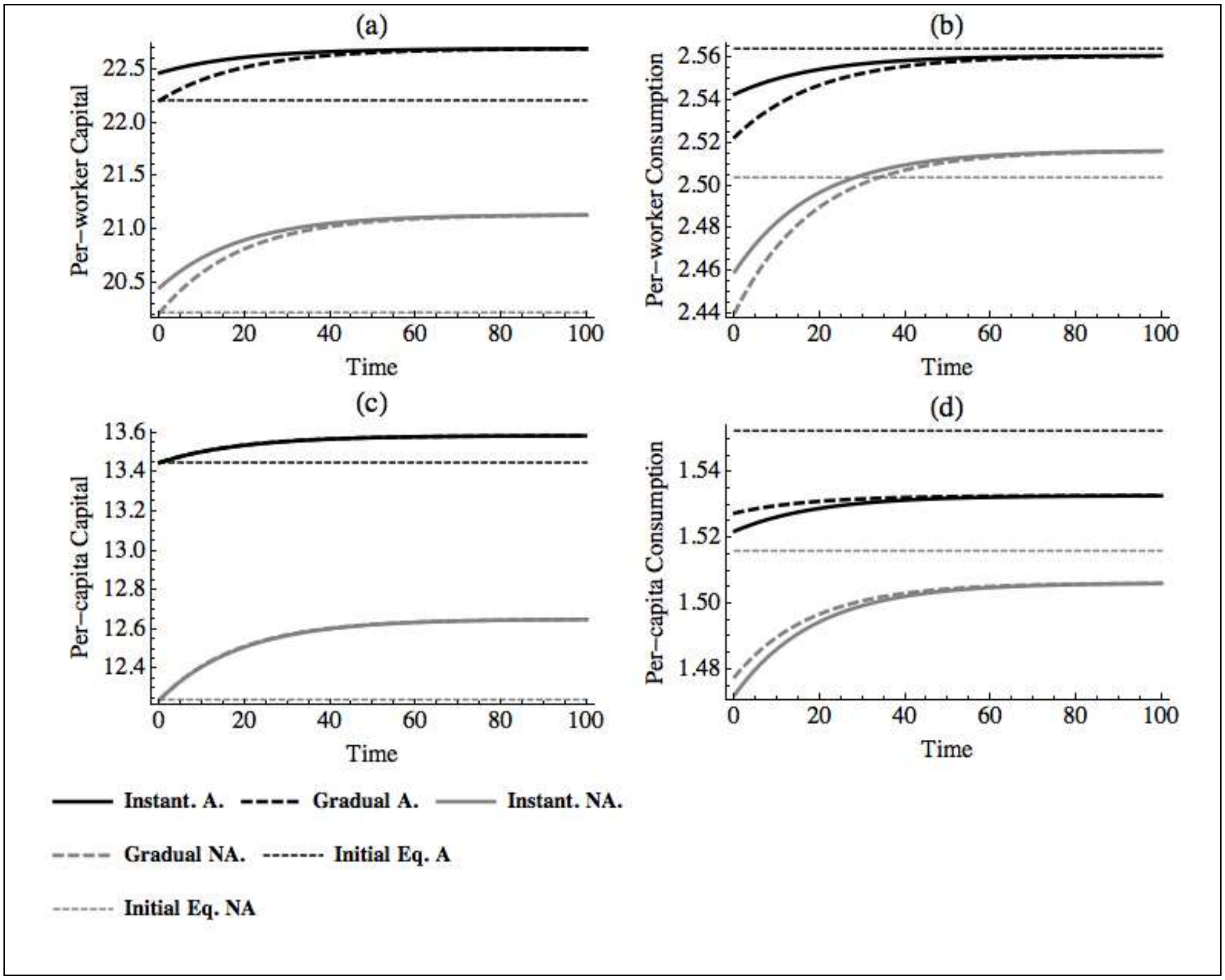

Figure 3.1: 20 Year Forecast – Mortality Decrease

$\tau_s = 0.08$, $\mu_0 = 1225.113 \rightarrow 7691.228$, $\mu_1 = 0.07693 \rightarrow 0.09649$, A: Annuities Market Included, NA: Annuities Market Absent

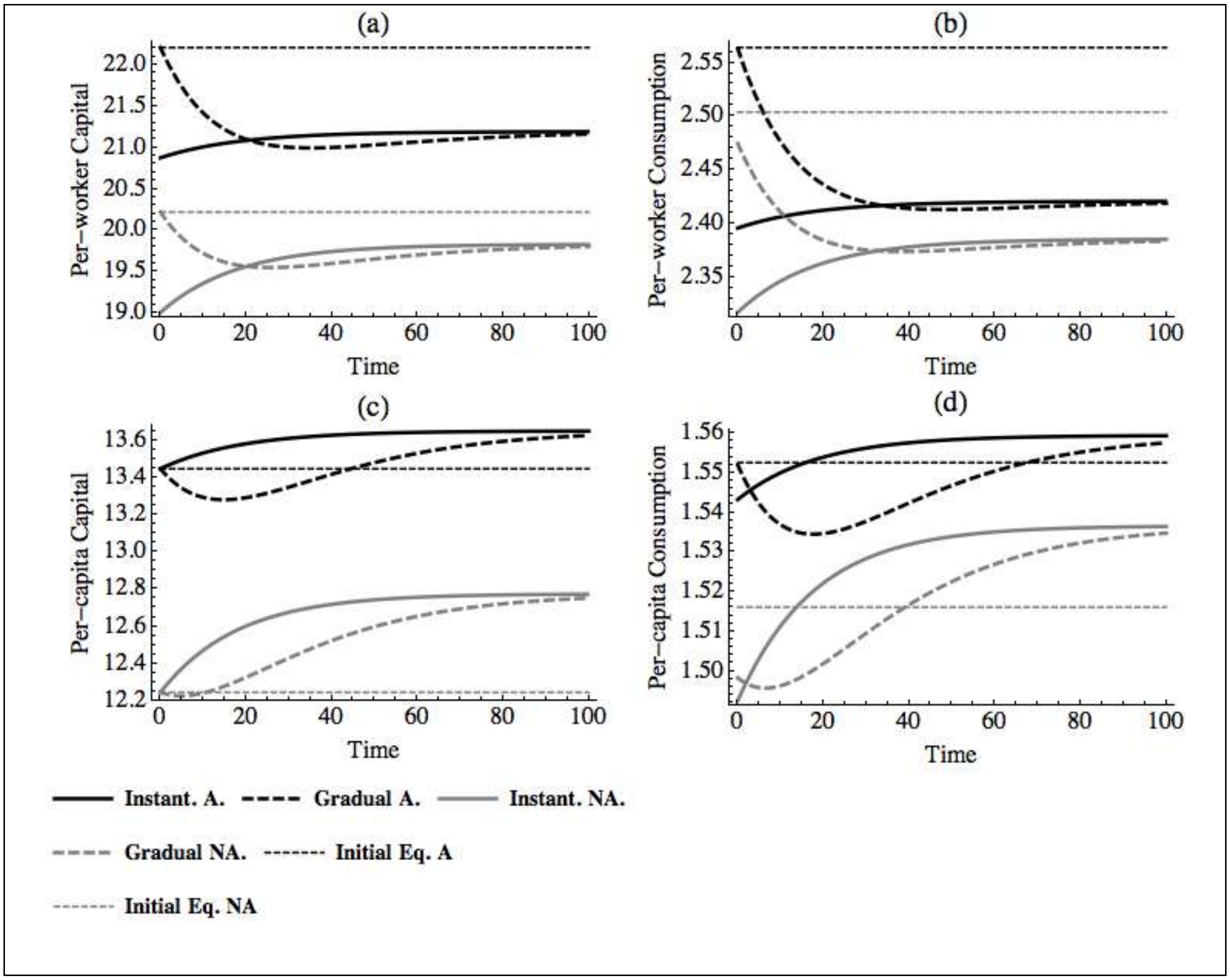

Figure 3.2: 20 Year Forecast – Mortality Decrease and Increase in Fertility from 1.7 to 2

$\tau_s = 0.08$, $\mu_0 = 1225.113 \to 7691.228$, $\mu_1 = 0.07693 \to 0.09649$, $\varphi = 1.7 \to 2$,

A: Annuities Market Included, NA: Annuities Market Absent

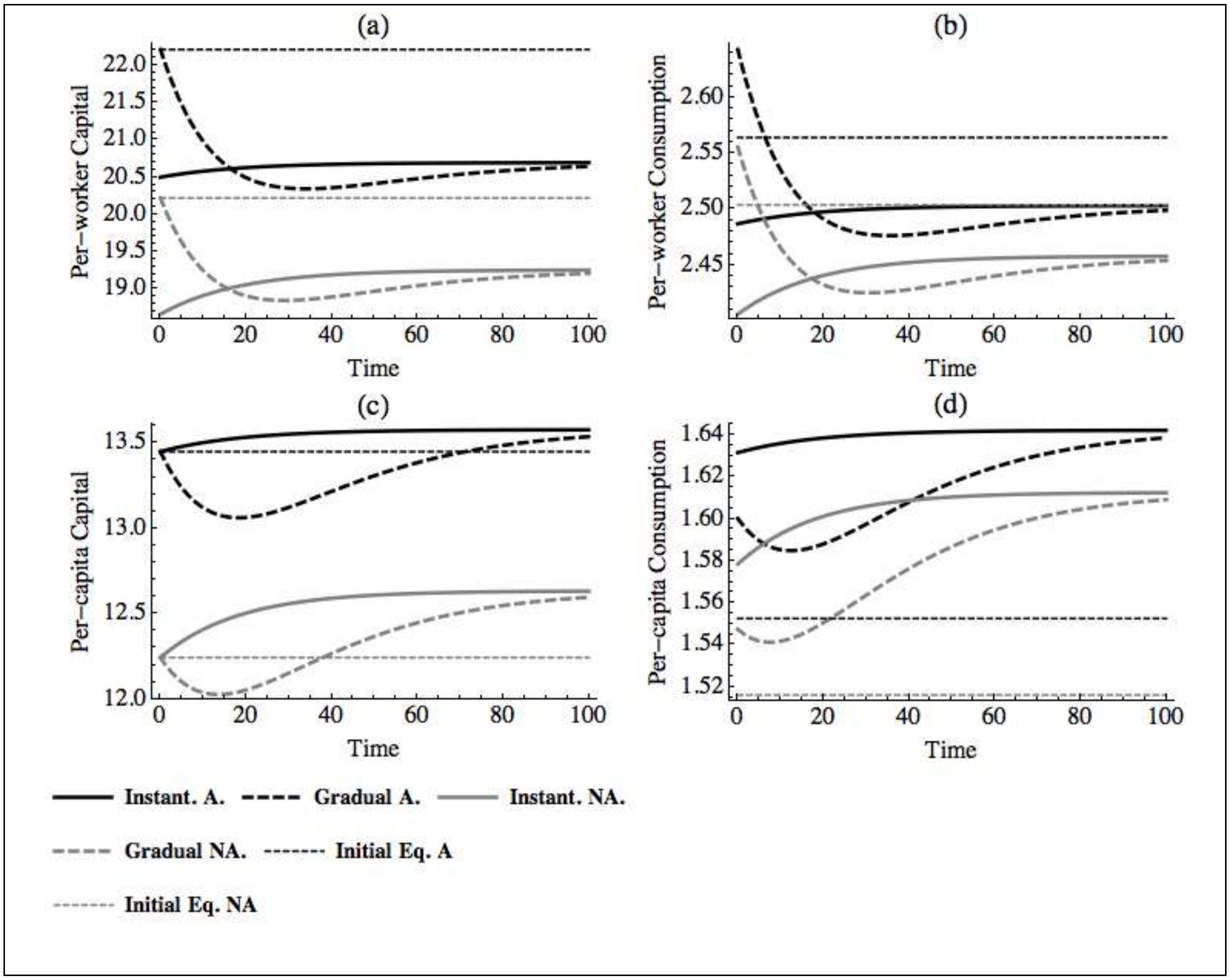

Figure 3.3: 20 Year Forecast – Mortality Decrease and Retirement Age Increase from 60 to 65

$\tau_s = 0.08$, $\mu_0 = 1225.113 \rightarrow 7691.228$, $\mu_1 = 0.07693 \rightarrow 0.09649$, $R = 60 \rightarrow 65$,

A: Annuities Market Included, NA: Annuities Market Absent

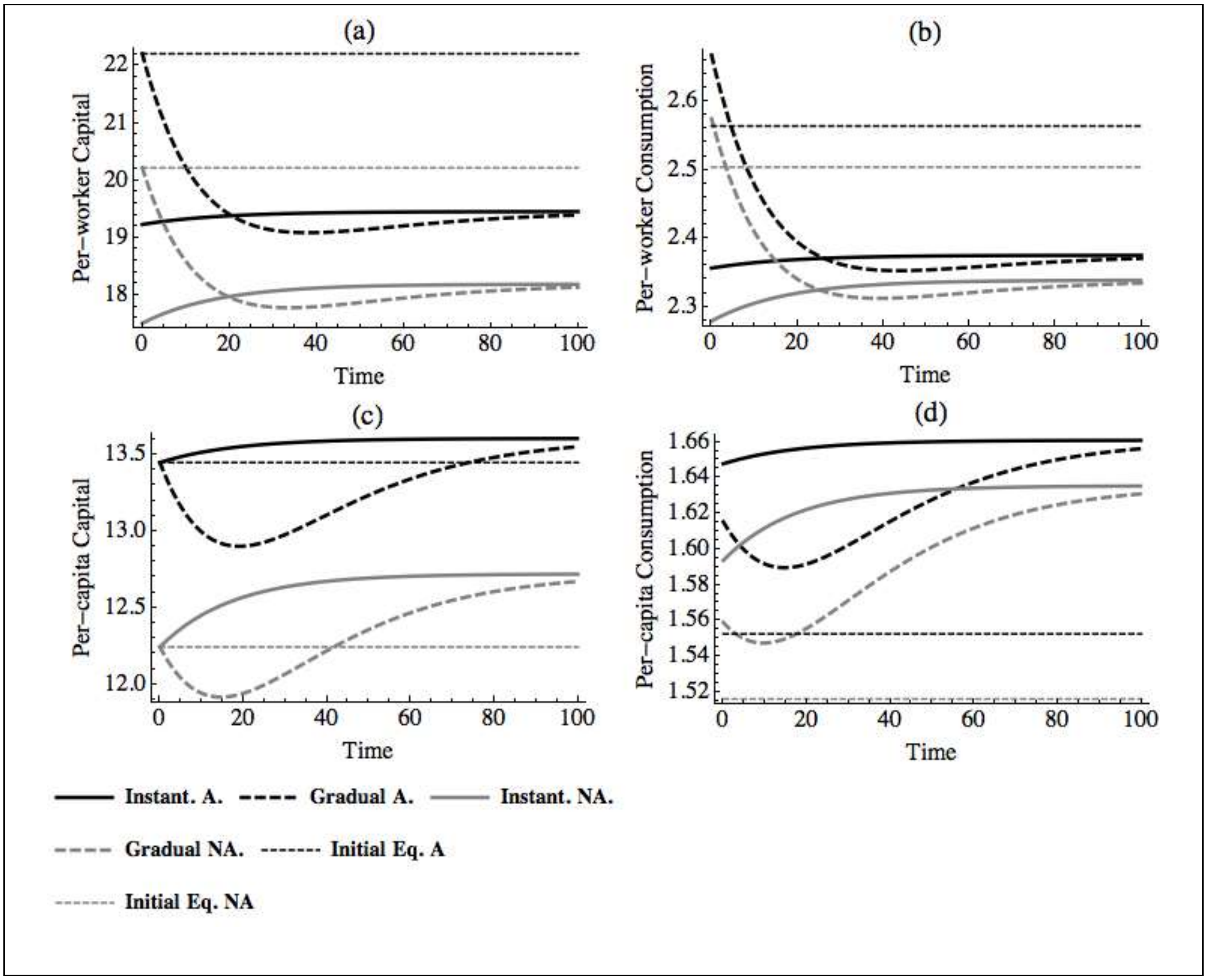

Figure 3.4: 20 Year Forecast - Mortality Decrease, Fertility Increase from 1.7 to 2, and Retirement Age Increase from 60 to 65

$\tau_s = 0.08$, $\mu_0 = 1225.113 \rightarrow 7691.228$, $\mu_1 = 0.07693 \rightarrow 0.09649$, $\varphi = 1.7 \rightarrow 2$, $R = 60 \rightarrow 65$,

A: Annuities Market Included, NA: Annuities Market Absent

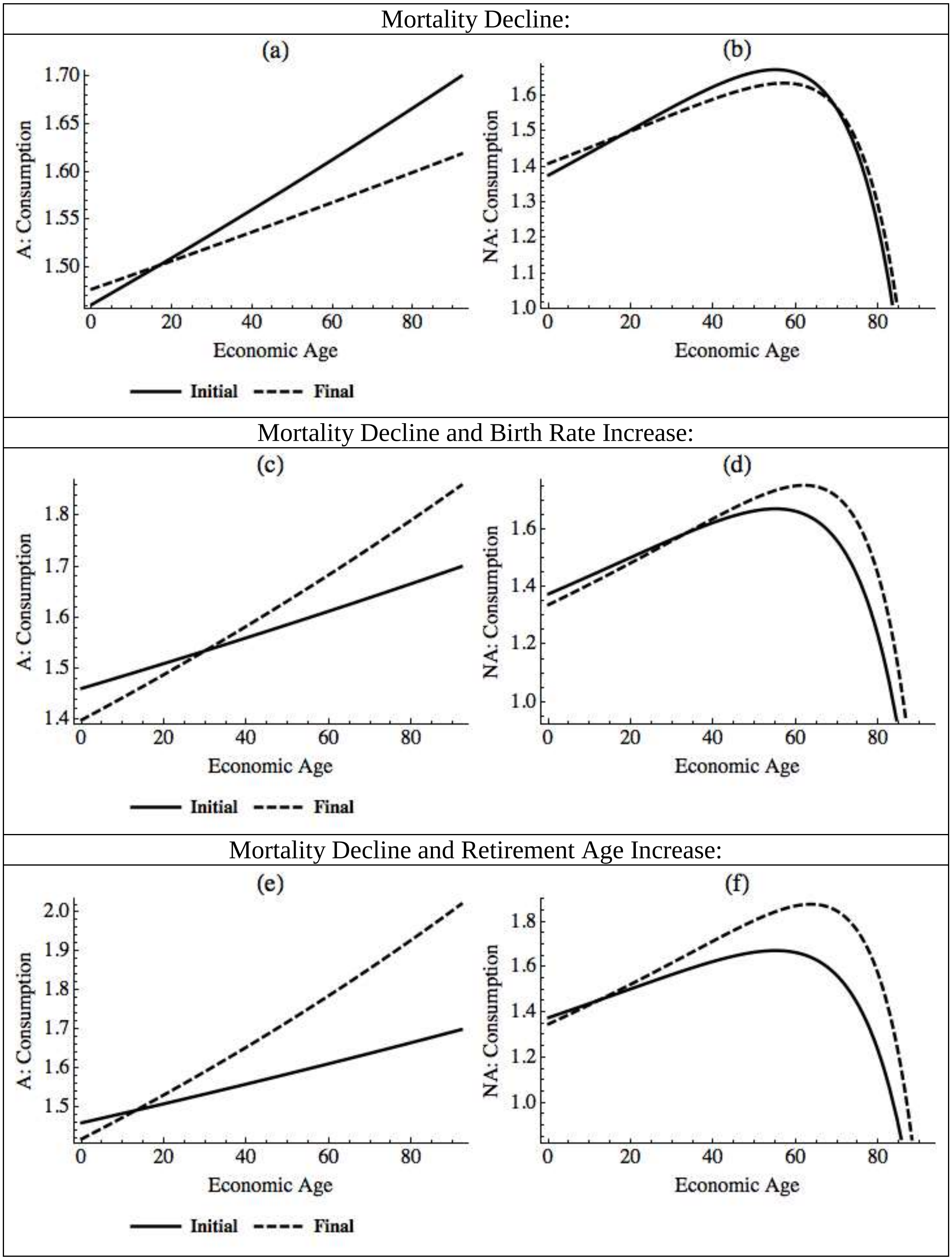

Figure 3.5-A: Consumption Profile with and without Annuities Per-worker Equilibrium

A: Annuities Market Included, NA: Annuities Market Absent.

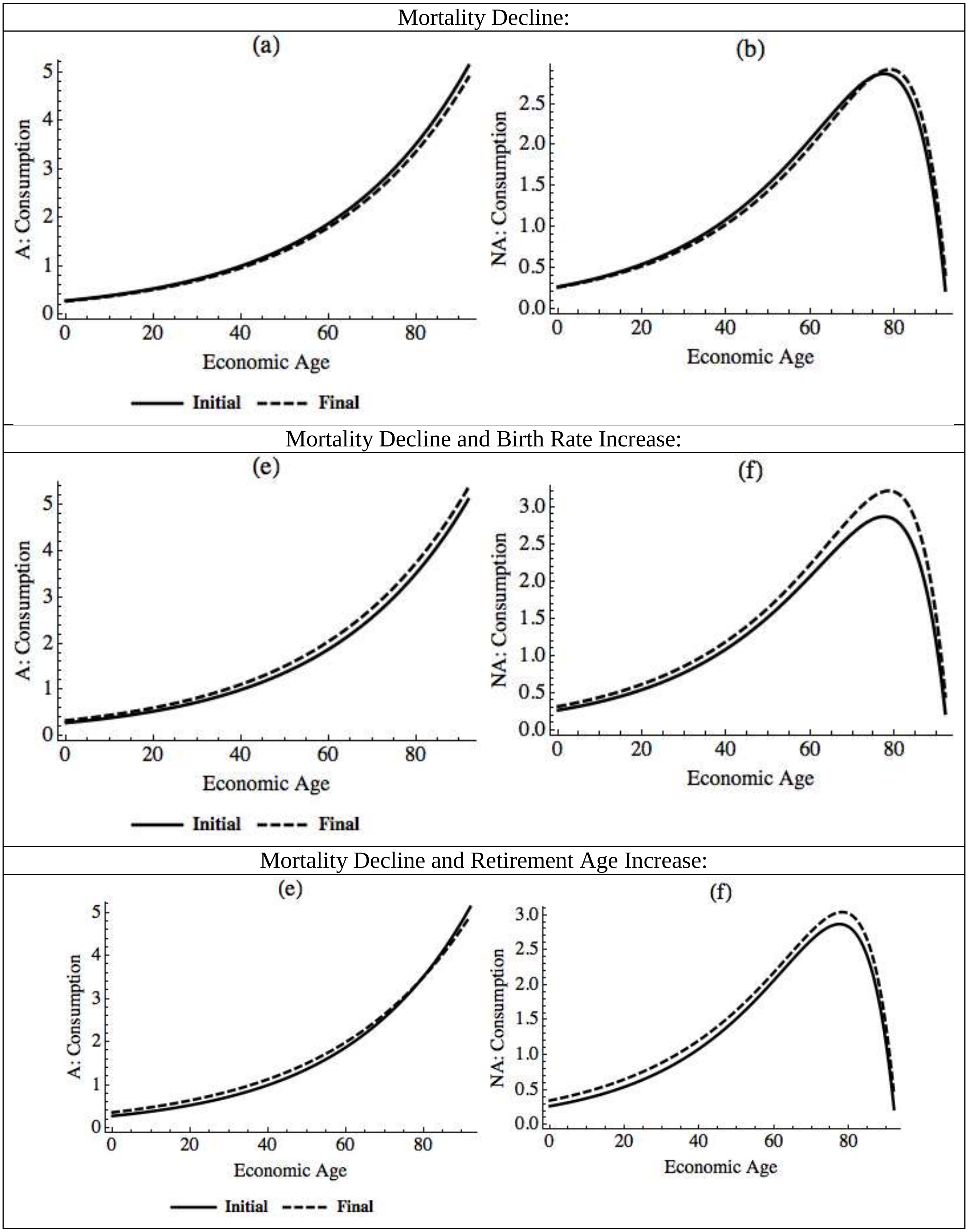

Figure 3.5-B: Consumption Profile with and without Annuities Per-capita Equilibrium

A: Annuities Market Included, NA: Annuities Market Absent.

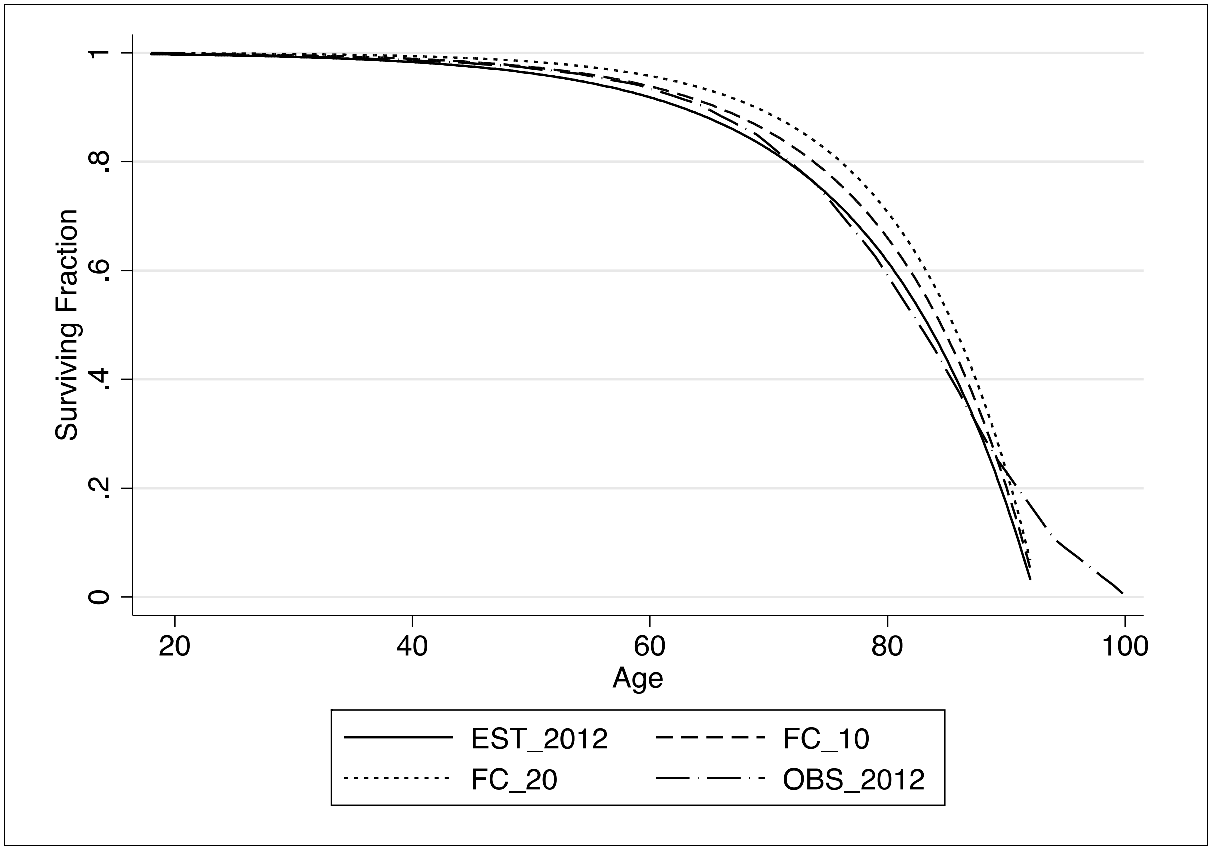

Figure 3.6: BCL Estimation – China Observed and Predicted
Data from World Health Organization Life Tables. Est_2012: Estimated Survival Function for 2012, FC: Forecasted Estimate (10 and 20 years forward)

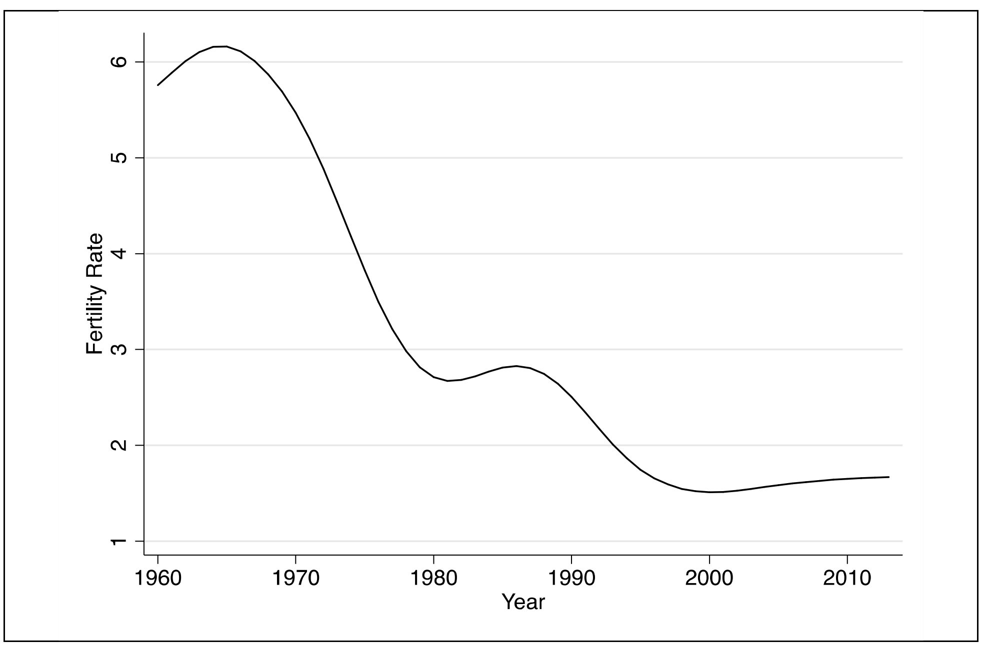

Figure 3.7: Chinese Fertility Rate
Data from the World Bank.

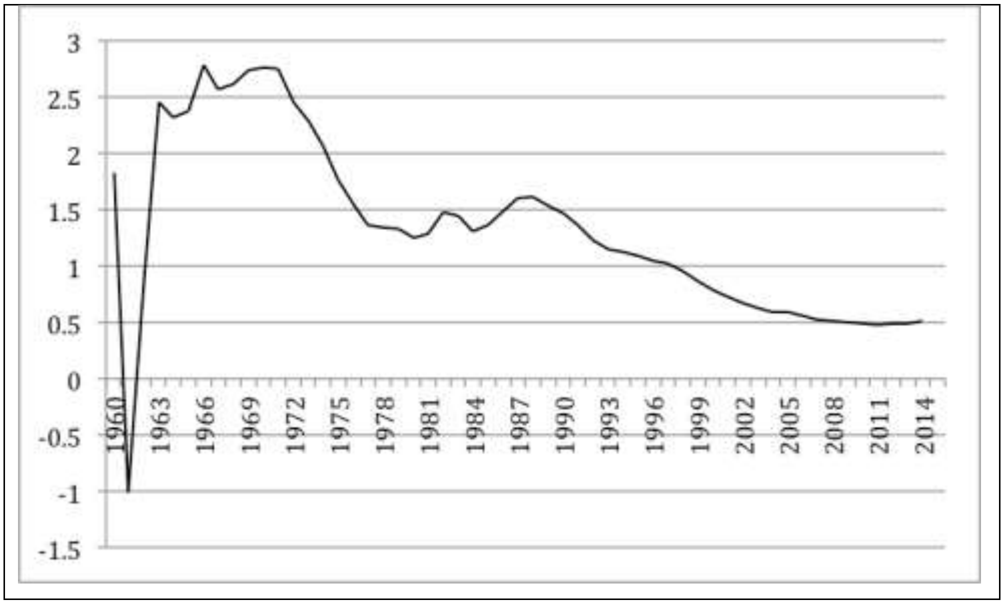

Figure 3.8: Chinese Annual Pop. Growth Rate
Data from the World Bank.

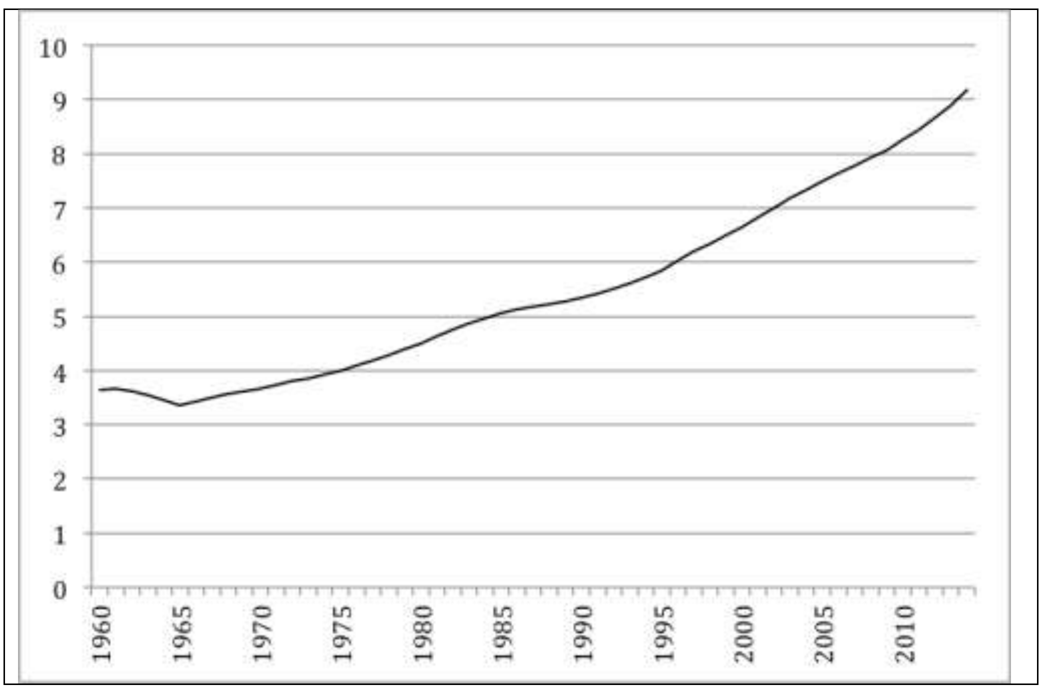

Figure 3.9: Chinese Population Aged 65 and Above (% of total)
Data from the World Bank.

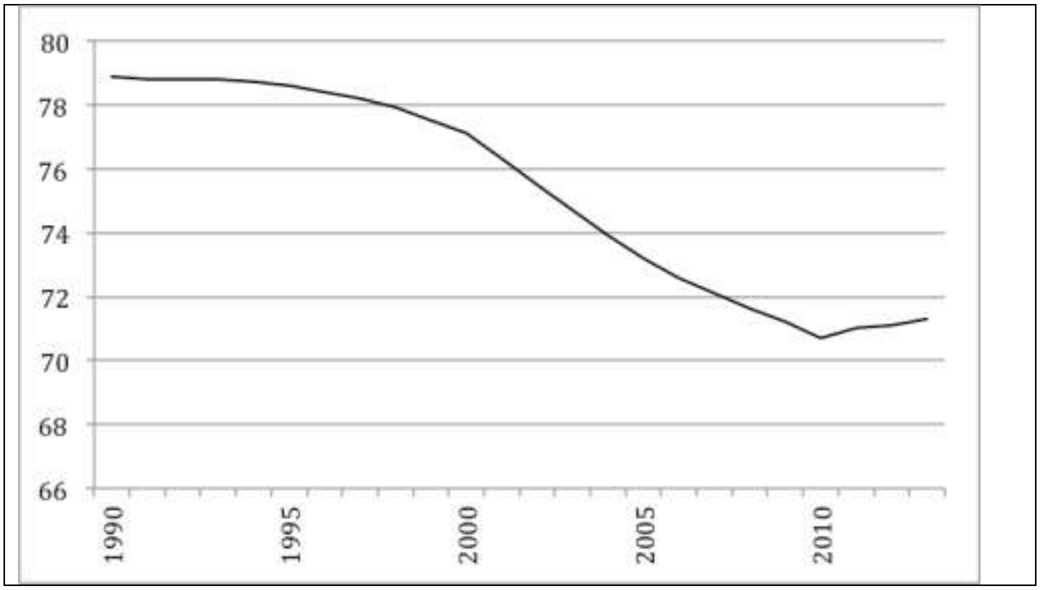

Figure 3.10: Chinese Labor Force Participation Rate
Data from the World Bank.

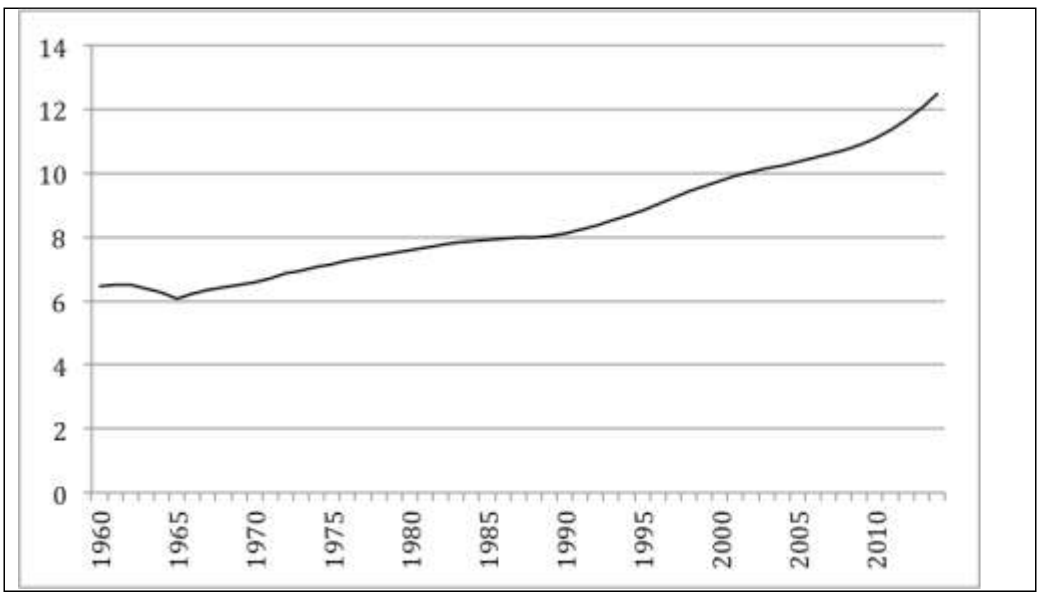

Figure 3.11: Chinese Age Dependency Ratio in %
Data from the World Bank.

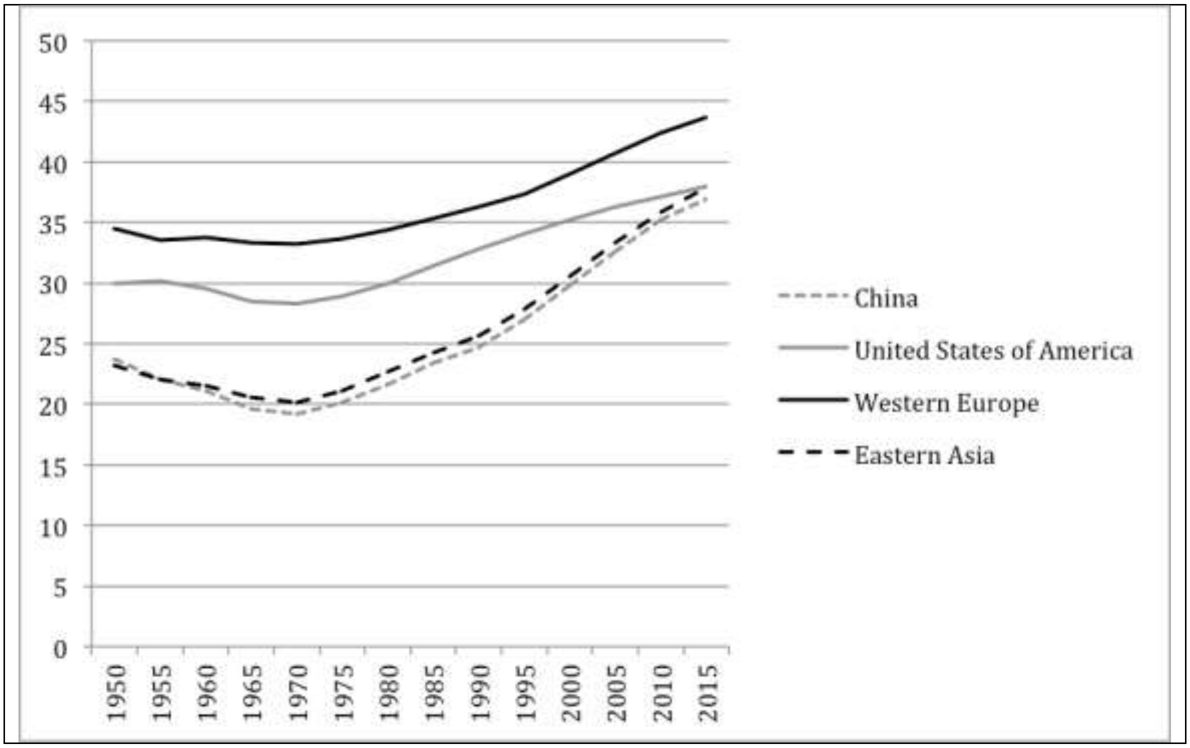

Figure 3.12: Median Age of Population (UN Estimate)
Data from the UN World Population Prospects: The 2015 Revision.

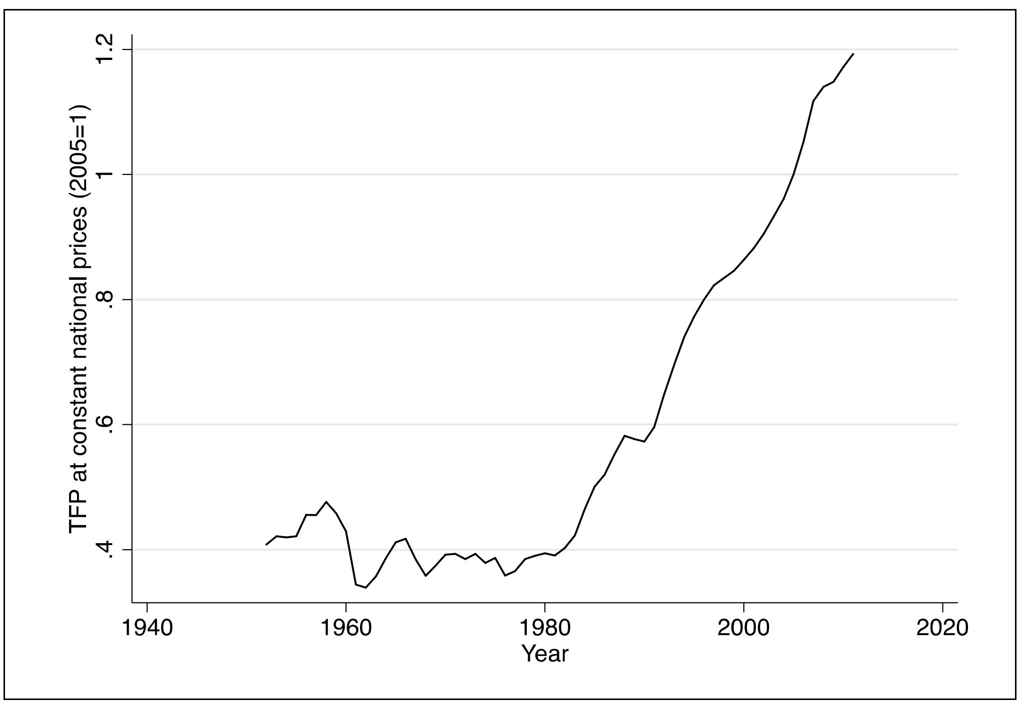

Figure 3.13: China Total Factor Productivity in Constant National Prices
Data from Penn World Table 8.0.

Tables: Chapter 1

Table 1.1:
Baseline Parameter and Equilibrium Values

Structural Parameters	**Value:**
Total factor productivity, A	1
Labor share of output, α	0.65
Intertemporal Elasticity of Substitution, $\sigma = 1/(1-\gamma)$	0.40
Elasticity of leisure in utility, θ	1.75
Pure rate of time preference, ρ	3.5%
World interest rate, r^*	5.0%
Population growth rate, n	1.0%
Distortionary Tax on Wage Income, τ_y	10.0%
Distortionary Tax on Interest Income, τ_r	10.0%

	Survival Function		
Demographic Parameters:	**Rectangular**	**BCL**	**Blanchard**
Life expectancy at age 18, L_{18}	78.38	78.38	78.38
Implied maximum age, D	78.38	95.06	∞
Time pref. of average individual $\rho + \mu(\bar{u})$	3.50%	3.88%	4.78%
Youth mortality, μ_0	0	78.3618 (6.0193)	N/A
Old age mortality, μ_1	0	0.0566 (0.0011)	0.01275
Birth Rate (Implied), β	2.21%	2.24%	2.28%
Lifetime consumption growth rate	0.20%	0.20%	0.20%
Numbers in parentheses are standard errors			

	Survival Function		
Implied Economic Variables:	**Rectangular**	**BCL**	**Blanchard**
Output, $\tilde{y}$	0.395	0.394	0.371
Traded Bonds, $\tilde{f}$	0.592	0.682	1.89
Relative trade deficit, $(\tilde{c}-\tilde{y})/\tilde{y}$	0.0599	0.0693	0.203
Leisure, $\tilde{l}$	0.760	0.762	0.783
Wage rate, $\tilde{w}$	1.071	1.074	1.110
Human Wealth, $\tilde{H}$	8.54	8.32	7.31
Consumption, $\tilde{c}$	0.419	0.421	0.446
Marginal Propensity to Consume, $1/\tilde{\Delta}$	0.0465	0.0478	0.0558
Stable Eigenvalue, Λ	-0.0188	-0.0191	-0.0184

Table 1.2: Long-run Effects of Permanent Structural Changes

A. Rectangular survival function

	Traded bonds	Output	Leisure	Consumption	Human wealth	Relative Trade Deficit
Initial Equilibrium Rectangular	**0.592**	**0.395**	**0.760**	**0.419**	**8.54**	**0.0599**
Increase in A from 1 to 1.25	0.740	0.494	0.760	0.524	10.67	0.0600
Increase in τ_y from 10% to 15%	0.575	0.384	0.771	0.407	8.30	0.0599
Increase in r from 5% to 5.5%	0.853	0.388	0.767	0.427	7.95	0.0990
Increase in ρ from 3.5% to 4%	0.292	0.401	0.755	0.410	8.49	0.3473

B. BCL Survival function

	Traded bonds	Output	Leisure	Consumption	Human wealth	Relative Trade Deficit
Initial Equilibrium BCL	**0.682**	**0.394**	**0.762**	**0.421**	**8.32**	**0.0693**
Increase in A from 1 to 1.25	0.852	0.492	0.762	0.526	10.40	0.0693
Increase in τ_y from 10% to 15%	0.662	0.383	0.772	0.409	8.09	0.0690
Increase in r from 5% to 5.5%	0.979	0.386	0.769	0.430	7.76	0.1142
Increase in ρ from 3.5% to 4%	0.335	0.400	0.756	0.414	8.27	0.0334

C. Blanchard Survival Function

	Traded bonds	Output	Leisure	Consumption	Human wealth	Relative Trade Deficit
Initial Equilibrium Blanchard	**1.89**	**0.371**	**0.783**	**0.446**	**7.31**	**0.203**
Increase in A from 1 to 1.25	2.36	0.464	0.783	0.560	9.14	0.203
Increase in τ_y from 10% to 15%	1.83	0.360	0.792	0.433	7.10	0.203
Increase in r from 5% to 5.5%	2.72	0.350	0.801	0.470	6.95	0.350
Increase in ρ from 3.5% to 4%	0.880	0.390	0.765	0.420	7.16	0.091

Table 1.3:
GINI Coefficients for Wealth and Income

	Wealth GINI Coefficients:		
	BCL	Rectangular	Blanchard
Initial Equilibrium	0.278	0.289	0.438
A=1 to A=1.25	0.278	0.289	0.438
r=5% to 5.5%	0.284	0.293	0.442
ρ=3.5% to ρ=4%	0.277	0.288	0.434
τ_r =10% to τ_r =15%	0.275	0.287	0.436
τ_y =10% to τ_y =15%	0.278	0.289	0.438
	Income GINI Coefficients:		
	BCL	Rectangular	Blanchard
Initial Equilibrium	0.010	0.025	0.023
A=1 to A=1.25	0.010	0.025	0.023
r=5% to 5.5%	0.013	0.035	0.034
ρ=3.5% to ρ=4%	0.005	0.013	0.012
τ_r =10% to τ_r =15%	0.008	0.019	0.017
τ_y =10% to τ_y =15%	0.01	0.025	0.023

Tables: Chapter 2

Table 2.1: Cross-country demographic characteristics

	Birth Life Expectancy		Pop. Growth Rate		Fertility Rate	
Country	1980	2010	1980	2010	1980	2010
Canada	75.1	80.9	1.3	1.1	1.7	1.6
France	74.1	81.7	0.4	0.5	1.9	2.0
Germany	72.7	80.0	0.2	-0.2	1.4	1.4
Japan	76.1	82.8	0.8	0.0	1.8	1.4
UK	73.7	80.4	0.1	0.8	1.9	1.9
US	73.7	78.5	1.0	0.8	1.8	1.9

Data retrieved from the World Bank World Development Indicators.

Table 2.2: Parameter and demographic values

Preference Parameters:	Value:	
Time preference rate, ρ	3.5 %	
Intertemporal Elasticity of Substitution, σ	.5	
Initial Specification (1980 Estimate):	USA	Region
Life expectancy at age 18, L_{18}	74.1	76.4
Implied maximum age, D	91.1	91.6
Youth mortality, μ_0	184.1073	378.0784771
Old age mortality, μ_1	0.0572393	0.064820507
Birth Rate (Implied), φ	1.93 %	1.6 %
Population growth rate, n	1.0 %	1.0 %
TFP, A	1	1.26
Capital share production parameter, α	0.432	0.439
Retirement age, R	66.4	65.9
Pension replacement rate, β	0.44	0.515
Final Specification (2010 Estimate):	USA	Region
Life expectancy at age 18, L_{18}	78.6	81.5
Implied maximum age, D	93.8	94.6
Youth mortality, μ_0	451.192	1323.2226
Old age mortality, μ_1	0.0651832	0.075948
Birth Rate (Implied), φ	1.73 %	1.37 %
Population growth rate, n	1.0 %	1.0 %
TFP, A	1.34	1.31
Capital share production parameter, α	0.467	0.469
Retirement age, R	65.5	64.9
Pension replacement rate, β	0.387	0.397
Speed Parameter:	USA	Region
TFP rate speed, a	5 %	5 %

Table 2.3: Gini coefficient adjustment

	US		Region	
	Initial	Final	Initial	Final
Mortality Decline	0.3563	0.354	0.3552	0.3544
Mortality & TFP	0.3562	0.3541	0.3551	0.3544
Mortality & Shares	0.3791	0.3853	0.3780	0.3856
Mortality, TFP, & Shares	0.3791	0.3853	0.3779	0.3857
Mortality & ρ	0.3563	0.3510	0.3552	0.3601

Table 2.4: Baseline Model - Equilibrium values for each shock

		Output	Output*	Total Capital	Total Capital*	Cons.	Cons.*	Wealth	Wealth*	NFA	NFA*	r
Mortality	Initial	2.409	2.409	12.331	12.331	2.274	2.298	12.13	12.532	-0.201	0.201	0.068
	Final	2.443	2.443	12.831	12.831	2.298	2.331	12.54	13.123	-0.291	0.291	0.067
Mort. & TFP	Initial	2.410	3.439	12.347	17.619	2.273	3.277	12.111	17.855	-0.236	0.236	0.068
	Final	3.832	3.701	20.125	19.436	3.605	3.532	19.676	19.885	-0.449	0.449	0.066
Mort. & α	Initial	3.742	3.945	21.214	22.745	3.527	3.721	21.167	22.793	-0.048	0.048	0.076
	Final	4.846	4.955	29.353	30.189	4.515	4.691	28.795	30.747	-0.558	0.558	0.077
Mort., TFP & α	Initial	3.742	5.958	21.219	34.355	3.526	5.619	21.162	34.413	-0.057	0.057	0.076
	Final	8.391	8.245	50.820	50.223	7.819	7.806	49.873	51.169	-0.947	0.947	0.077
Mort. & ρ	Initial	2.409	2.409	12.331	12.331	2.274	2.298	12.130	12.532	-0.201	0.201	0.068
	Final	2.405	2.405	12.267	12.267	2.188	2.376	10.671	13.864	-1.596	1.596	0.068

"*" Denotes foreign region.

Table 2.5: Retirement model - Equilibrium values for each shock

		Output	Output*	Total Capital	Total Capital*	Cons.	Cons.*	Wealth	Wealth*	NFA	NFA*	LFPR	LFPR*	r
Mortality	Initial	1.544	1.521	9.167	9.167	1.452	1.430	9.150	9.184	-0.0171	0.0171	0.711	0.700	0.083
	Final	1.499	1.468	9.260	9.260	1.404	1.377	9.231	9.289	-0.0285	0.0285	0.688	0.674	0.082
Mort. & SS	Initial	1.449	1.427	7.639	7.639	1.402	1.320	7.997	7.281	0.358	-0.358	0.711	0.700	0.093
	Final	1.418	1.388	7.892	7.892	1.346	1.302	7.978	7.806	0.086	-0.086	0.688	0.674	0.091
Mort., SS, & TFP	Initial	1.448	2.033	7.617	10.869	1.407	1.890	8.038	10.448	0.421	-0.421	0.711	0.700	0.094
	Final	2.224	2.103	12.381	11.958	2.111	1.973	12.514	11.825	0.133	-0.133	0.688	0.674	0.091

"*" Denotes foreign region.

Table 2.6: Comparison of NFA positions

Observed NFA position in hundreds of millions of 2010 US Dollars			
Country:	1980	2007	% Change
Canada	-2638.329	-226.7005	91.40742114
France	782.4547	3204.705	309.5706755
Germany	1081.599	9013.133	733.3155818
Japan	302.6127	22930.07	7477.365391
United Kingdom	850.539	-5999.803	-805.4118624
United States	5767.76	-24794.17	-529.8752029

Data from Lane, Milessi, and Ferreti (2007).

Table 2.7: 2000 wealth Gini coefficient by country

Country:	Wealth Gini Estimate:
Canada	0.688
France	0.73
Germany	0.667
Japan	0.547
United Kingdom	0.697
United States	0.801

Data retrieved from: Davies et al. (2007)

Table 2.8: Cross-country employment characteristics

	Age Dependency Ratio		Effective Retirement Age		Gross Replacement Rates %	
Country	1980	2010	1980	2010	1980	2010
Canada	13.8	20.4	64.9	63.4	34.0	44.0
France	21.8	26.4	63.5	59.4	66.0	49.0
Germany	23.9	31.3	60.0	62.0	49.0	43.2
Japan	13.4	36.0	70.7	70.1	54.0	35.7
United Kingdom	23.3	24.5	66.0	64.1	43.0	31.9
United States	17.2	19.4	66.4	65.5	44.0	38.7

Data retrieved from the World Bank World Development Indicators.
2010 replacement data from www.oecd.org.
1980 replacement data from Aldrich (1982).

Table 2.9: Productivity and labor share parameters

	TFP		Total labor share**	
	1980	2010	1980	2010
Canada	1.00	0.93	0.543	0.504
France	0.99	0.96	0.56	0.533
Germany	1.13	1.02	0.621	0.567
Japan	0.92	0.99	0.543	0.506
UK	0.74	0.97	0.536	0.543
US	0.76	1.01	0.568	0.533

*TFP values from the Penn World Table 8.1

**Share parameters from Nieman (2013)

Table 2.10: Labor force participation rates

Country:	1990	2010
Canada	76.0	71.5
France	65.2	62.0
Germany	69.6	66.3
Japan	77.2	71.6
UK	74.7	68.6
US	75.4	69.8

Data from the World Bank.

Table 2.11: Population Aged 18-90

Country	1980	2010	1980 Fraction	2010 Fraction
Canada	17265015	26719393	0.04	0.05
France	38913224	48600924	0.10	0.09
Germany	46842940	67919917	0.12	0.13
Japan	83225332	104909011	0.21	0.20
UK	41389953	48781077	0.11	0.09
US	161618776	232754319	0.42	0.44
Total	389255240	529684641	1	1

Data from the Human Mortality Database.

Tables: Chapter 3

Table 3.1: Steady State Results - With and Without Annuities

Annuities	β	l	k	c	y	$\bar{k}$	$\bar{c}$	$\bar{y}$	c_l	c_r
Baseline	**0.123**	**0.606**	**22.21**	**2.56**	**2.96**	**13.45**	**1.55**	**1.79**	**1.51**	**1.62**
M.D.	0.119	0.599	22.70	2.56	2.98	13.59	1.53	1.79	1.51	1.57
% Change	-2.85%		2.19%	-0.13%	0.76%	1.02%	-1.27%	-0.39%	-0.22%	-2.84%
M.D. and Birth	0.145	0.644	21.19	2.42	2.91	13.65	1.56	1.88	1.49	1.69
% Change	17.95%		-4.58%	-5.59%	-1.63%	1.52%	0.44%	4.66%	-1.62%	4.65%
M.D. and Ret. Age	0.153	0.656	20.70	2.50	2.89	13.58	1.64	1.89	1.55	1.82
% Change	24.27%		-6.82%	-2.37%	-2.44%	0.96%	5.78%	5.70%	2.43%	12.77%
M.D., Birth, and Ret. Age	0.186	0.699	19.46	2.38	2.83	13.61	1.66	1.98	1.53	1.97
% Change	51.52%		-12.40%	-7.37%	-4.53%	1.17%	6.98%	10.26%	1.26%	21.61%
No Annuities	β	l	k	c	y	$\bar{k}$	$\bar{c}$	$\bar{y}$	c_l	c_r
Baseline	**0.123**	**0.606**	**20.22**	**2.50**	**2.86**	**12.24**	**1.52**	**1.73**	**1.50**	**1.54**
M.D.	0.119	0.599	21.14	2.52	2.91	12.65	1.51	1.74	1.50	1.52
% Change	-2.85%		4.54%	0.50%	1.57%	3.34%	-0.65%	0.40%	-0.18%	-1.37%
M.D. and Birth	0.145	0.644	19.83	2.39	2.84	12.77	1.54	1.83	1.48	1.64
% Change	17.95%		-1.93%	-4.73%	-0.68%	4.34%	1.35%	5.67%	-1.47%	6.61%
M.D. and Ret. Age	0.153	0.656	19.26	2.46	2.82	12.63	1.61	1.85	1.55	1.74
% Change	24.27%		-4.77%	-1.83%	-1.69%	3.18%	6.36%	6.51%	2.92%	13.15%
M.D., Birth, and Ret. Age	0.186	0.699	18.19	2.34	2.76	12.72	1.64	1.93	1.53	1.88
% Change	51.52%		-10.04%	-6.60%	-3.64%	3.90%	7.87%	11.29%	1.82%	22.40%
M.D.-Mortality Decrease, Birth-Birth Rate Set to 2%, Ret. Age-Increase to 65										

Table 3.2: Parameter and demographic values

Unchanging Parameters:	Value:
Time preference rate, ρ	3.5 %
Capital share production parameter, α	.35
TFP, A	1
Social Security Tax Rate, τ_s	8.0 %
Depreciation Rate, δ	1.0 %
Initial Specification (2012 Estimate):	
Life expectancy at age 18, L_{18}	79.51
Implied maximum age, D	92.43
Youth mortality, μ_0	1225.113
Old age mortality, μ_1	0.0769306
Birth Rate, φ	1.7
Population growth rate (Implied), n	0.78 %
Retirement age, R	60
Final Specification (2032 Estimate):	
Life expectancy at age 18, L_{18}	82.38
Implied maximum age, D	92.73
Youth mortality, μ_0	7691.228
Old age mortality, μ_1	0.0964957
Birth Rate, φ	2
Population growth rate (Implied), n	0.85 %
Retirement age, R	65
Speed Parameters:	
Retirement age extension speed, θ_R	5 %
Birth rate speed, θ_φ	5 %
Benefit change speed, θ_β	5 %

Table 3.3: Chinese Demographic Characteristics (2013)

Annual Pop. Growth Rate	.5 %*
Life Expectancy at Birth	75 Years
Fertility Rate (per woman)	1.7 Births
LFPR	71 % of pop aged 15+

Data from World Bank. *2014 Estimate.

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Appendix: Chapter 1

A.1.1 Derivation of Equation (25)

We begin with equation (10b'), repeated here as

$$H(t)=\int_t^{t+D}\Big[(1-\tau_y)[w(\tau)+\Pi(\tau)]-\theta C(t,\tau)-T(\tau)\Big]e^{-(1-\tau_r)r^*(\tau-t)-M(\tau-t)}d\tau \qquad \text{(A.1)}$$

Differentiating with respect to t yields

$$\dot{H}(t)=-(1-\tau_y)[w(t)+\Pi(t)]+\theta C(t,t)+T(t)+(1-\tau_r)r^*H(t) \qquad \text{(A.2)}$$

$$+\int_t^{t+D}\mu(\tau-t)[(1-\tau_y)[w(\tau)+\Pi(\tau)]-\theta C(t,\tau)-T(\tau)]e^{-(1-\tau_r)r^*(\tau-t)-M(\tau-t)}d\tau$$

$$-\theta\int_t^{t+D}C_t(t,\tau)e^{-(1-\tau_r)r^*(\tau-t)-M(\tau-t)}d\tau$$

Recalling

(i) $\quad C(t,t)=\dfrac{H(t)}{\Delta(t)}$

(ii) $\quad C(t,\tau)=C(t,t)e^{\int_t^\tau \psi(s)ds}=\left(\dfrac{H(t)}{\Delta(t)}\right)e^{\int_t^\tau \psi(s)ds}$

we have that

$$C_t(t,\tau)=\left[\frac{d}{dt}\left(\frac{H(t)}{\Delta(t)}\right)-\left(\frac{H(t)}{\Delta(t)}\right)\psi(t)\right]e^{\int_t^\tau \psi(s)ds}$$

so that utilizing the definition of $\Delta(t)$

$$\int_t^{t+D}C_t(t,\tau)e^{-(1-\tau_r)r^*(\tau-t)-M(\tau-t)}d\tau=\left[\frac{d}{dt}\left(\frac{H(t)}{\Delta(t)}\right)-\left(\frac{H(t)}{\Delta(t)}\right)\psi(t)\right]\int_t^{t+D}e^{\int_t^\tau \psi(s)ds-(1-\tau_r)r^*(\tau-t)-M(\tau-t)}d\tau$$

$$=\left[\frac{d}{dt}\left(\frac{H(t)}{\Delta(t)}\right)-\left(\frac{H(t)}{\Delta(t)}\right)\psi(t)\right]\Delta(t)$$

and thus

$$\int_t^{t+D}C_t(t,\tau)e^{-(1-\tau_r)r^*(\tau-t)-M(\tau-t)}d\tau=\dot{H}(t)-\frac{H(t)}{\Delta(t)}\dot{\Delta}(t)-\psi(t)H(t) \qquad \text{(A.3)}$$

Also, we define

$$\mu_H(\tau_1-t)\equiv\frac{\int_t^{t+D}\mu(\tau-t)[(1-\tau_y)[w(\tau)+\Pi(\tau)]-\theta C(t,\tau)-T(\tau)]e^{-(1-\tau_r)r^*(\tau-t)-M(\tau-t)}d\tau}{\int_t^{t+D}[(1-\tau_y)[w(\tau)+\Pi(\tau)]-\theta C(t,\tau)-T(\tau)]e^{-(1-\tau_r)r^*(\tau-t)-M(\tau-t)}d\tau}$$

$$=\frac{1}{H(t)}\int_t^{t+D}\mu(\tau-t)[(1-\tau_y)w(\tau)-\theta C(t,\tau)-T(\tau)]e^{-(1-\tau_r)r^*(\tau-t)-M(\tau-t)}d\tau \qquad \text{(A.4)}$$

as the average mortality rate as derived from the rate of human capital; c.f. (26) of the text. Then substituting (A.3) and (A.4) into (A.2) yields

$$(1+\theta)\dot{H}(t)-\frac{\theta H(t)}{\Delta(t)}\dot{\Delta}(t)=-(1-\tau_y)[w(\tau)+\Pi(\tau)]+\frac{\theta H(t)}{\Delta(t)}+T(t)+\left((1-\tau_r)r^*+\mu_H+\theta\psi(t)\right)H(t) \quad \text{(A.5)}$$

which is (25) of the text.

A.1.2 Mortality-specific functions

Here we substitute the three mortality function into (33b)-(33d) to yield the specific expressions for $\tilde{H},\ \tilde{\Delta},\ \tilde{c}$:

Human wealth at birth:

BCL:

$$\begin{aligned}\tilde{H}=&\frac{\tilde{y}+\tilde{w}\tilde{l}(1-\tau_y)+\tau_r r^*\tilde{f}}{\mu_0-1}\left\{\frac{\mu_0}{(1-\tau_r)r^*}\left(1-e^{-(1-\tau_r)r^*D}\right)-\frac{1}{(1-\tau_r)r^*-\mu_1}\left(1-e^{\left[\mu_1-(1-\tau_r)r^*\right]D}\right)\right\}\\&-\theta\frac{\tilde{H}}{\tilde{\Delta}}\frac{1}{\mu_0-1}\left\{\frac{\mu_0}{(1-\tau_r)r^*-\tilde{\psi}}\left(1-e^{\left[\tilde{\psi}-(1-\tau_r)r^*\right]D}\right)-\frac{1}{(1-\tau_r)r^*-\mu_1-\tilde{\psi}}\left(1-e^{\left[\mu_1+\tilde{\psi}-(1-\tau_r)r^*\right]D}\right)\right\}\end{aligned} \quad \text{(A.6a)}$$

Rectangular:

$$\tilde{H}=\frac{\tilde{y}+\tilde{w}\tilde{l}(1-\tau_y)+\tau_r r^*\tilde{f}}{(1-\tau_r)r^*}\left(1-e^{-(1-\tau_r)r^*D}\right)-\theta\frac{\tilde{H}}{\tilde{\Delta}}\left(\frac{1}{(1-\tau_r)r^*-\tilde{\psi}}\right)\left(1-e^{\left[\tilde{\psi}-(1-\tau_r)r^*\right]D}\right) \quad \text{(A.6b)}$$

Blanchard:

$$\tilde{H}=\frac{\tilde{y}+\tilde{w}\tilde{l}(1-\tau_y)+\tau_r r^*\tilde{f}}{(1-\tau_r)r^*+\mu_b}-\theta\frac{\tilde{H}}{\tilde{\Delta}}\frac{1}{(1-\tau_r)r^*+\mu_b-\tilde{\psi}} \quad \text{(A.6c)}$$

Inverse of the MPC:

BCL:

$$\tilde{\Delta}=\frac{1}{\mu_0-1}\left\{\frac{\mu_0}{(1-\tau_r)r^*-\tilde{\psi}}\left(1-e^{[\tilde{\psi}-(1-\tau_r)r^*]\tilde{D}}\right)-\frac{1}{(1-\tau_r)r^*-\tilde{\psi}-\mu_1}\left(1-e^{[\tilde{\psi}-(1-\tau_r)r^*+\mu_1]\tilde{D}}\right)\right\} \quad \text{(A.7a)}$$

Rectangular: $$\tilde{\Delta} = \left(\frac{1}{(1-\tau_r)r^* - \tilde{\psi}} \right) \left(1 - e^{[\tilde{\psi} - (1-\tau_r)r^*]\tilde{D}} \right) \qquad \text{(A.7b)}$$

Blanchard: $$\tilde{\Delta} = \left(\frac{1}{(1-\tau_r)r^* + \mu_b - \tilde{\psi}} \right) \qquad \text{(A.7c)}$$

Per-capita consumption:

BCL: $$\tilde{c} = \tilde{\beta} \frac{\tilde{H}}{\tilde{\Delta}} \left(\frac{1}{\mu_0 - 1} \right) \left(\frac{\mu_0}{n - \tilde{\psi}} \left[1 - e^{(\tilde{\psi} - n)\tilde{D}} \right] - \frac{1}{n - \tilde{\psi} - \mu_1} \left[1 - e^{(\tilde{\psi} - n + \mu_1)\tilde{D}} \right] \right) \qquad \text{(A.8a)}$$

Rectangular: $$\tilde{c} = \tilde{\beta} \frac{\tilde{H}}{\tilde{\Delta}} \left(\frac{1}{n - \tilde{\psi}} \right) \left(1 - e^{(\tilde{\psi} - n)\tilde{D}} \right) \qquad \text{(A.8b)}$$

Blanchard: $$\tilde{c} = \frac{\tilde{\beta}\tilde{H}}{\tilde{\Delta}} \left(\frac{1}{n + \mu_b - \tilde{\psi}} \right)$$

Appendix: Chapter 2

A.2.1 Baseline Model - Dynamic equation system as a function of domestic capital:

Noting from the no arbitrage condition that $r(t) = r^*(t)$ and using the fact that

$$f'(k) = f^{*}{}'(k^*) = r(t) \tag{A.9}$$

which using the explicit production function gives the capital relation:

$$k^* = \left(\frac{\alpha A}{\alpha^* A^*} k^{\alpha - 1} \right)^{\frac{1}{\alpha^* - 1}} \tag{A.10}$$

and

$$f''(k)\dot{k} = f^{*}{}''(k^*)\dot{k}^* \tag{A.11a}$$

which gives us:

$$\dot{k}^* = \left(\frac{\alpha A}{\alpha^* A^*} \right)^{\frac{1}{\alpha^* - 1}} \left(\frac{\alpha - 1}{\alpha^* - 1} \right) k^{\frac{\alpha - \alpha^*}{\alpha^* - 1}} \dot{k} \tag{A.11b}$$

$$\dot{c}(t) = \varphi \frac{H(t)}{\Delta(t)} + \left(\sigma \left[\alpha A k(t)^{\alpha - 1} - \rho \right] - n - \mu_c (t - \upsilon_1) \right) c(t) \tag{A.12a}$$

$$\dot{c}^*(t) = \varphi^* \frac{H^*(t)}{\Delta^*(t)} + \left(\sigma^* \left[\alpha A k(t)^{\alpha - 1} - \rho^* \right] - n^* - \mu_c^* (t - \upsilon_1^*) \right) c^*(t) \tag{A.12b}$$

$$\dot{H}(t) = -(1 - \alpha) A k(t)^{\alpha} + \left[\alpha A k(t)^{\alpha - 1} + \mu_H (\tau_1 - t) \right] H(t) \tag{A.12c}$$

$$\dot{H}^*(t) = -(1 - \alpha^*) \left(\frac{\alpha A}{\alpha^* A^{*\frac{1}{\alpha^*}}} \right)^{\frac{\alpha^*}{\alpha^* - 1}} k^{\frac{\alpha^*(\alpha - 1)}{\alpha^* - 1}} + \left[\alpha A k(t)^{\alpha - 1} + \mu_H^* (\tau_1^* - t) \right] H^*(t) \tag{A.12d}$$

$$\dot{\Delta}(t) = -1 + \left[(1 - \sigma) \alpha A k(t)^{\alpha - 1} + \sigma \rho + \mu_\Delta (\tau_2 - t) \right] \Delta(t) \tag{A.12e}$$

$$\dot{\Delta}^*(t) = -1 + \left[(1 - \sigma^*) \alpha A k(t)^{\alpha - 1} + \sigma^* \rho^* + \mu_\Delta^* (\tau_2^* - t) \right] \Delta^*(t) \tag{A.12f}$$

$$\dot{W}(t) = \left(\alpha A k(t)^{\alpha - 1} - n \right) W(t) + (1 - \alpha) A k(t)^{\alpha} - c(t) \tag{A.12g}$$

$$\dot{W}^*(t) = \left(\alpha A k(t)^{\alpha-1} - n^*\right) W^*(t) + (1-\alpha^*)\left(\frac{\alpha A}{\alpha^* A^{*\frac{1}{\alpha^*}}}\right)^{\frac{\alpha^*}{\alpha^*-1}} k^{\frac{\alpha^*(\alpha-1)}{\alpha^*-1}} - c^*(t) \qquad \text{(A.12h)}$$

$$\dot{k}(t) = \chi\left(A k(t)^{\alpha} + A^*\left(\frac{\alpha A}{\alpha^* A^*} k^{\alpha-1}\right)^{\frac{\alpha^*}{\alpha^*-1}} - nW(t) - n^* W^*(t) - c(t) - c^*(t) \right) \qquad \text{(A.12i)}$$

Linearized Matrix of Baseline Model:

Where Ω is:

$$\begin{pmatrix}
-\frac{\varphi \tilde{H}}{\tilde{\Delta}\tilde{c}} & 0 & \frac{\varphi}{\tilde{\Delta}} & 0 & -\frac{\varphi \tilde{H}}{\tilde{\Delta}^2} & 0 & 0 & 0 & \sigma(\alpha-1)\tilde{c}\frac{\tilde{r}}{\tilde{k}} \\
0 & -\frac{\varphi^* \tilde{H}^*}{\tilde{\Delta}^* \tilde{c}^*} & 0 & \frac{\varphi^*}{\tilde{\Delta}^*} & 0 & -\frac{\varphi^* \tilde{H}^*}{\tilde{\Delta}^{*2}} & 0 & 0 & \sigma^*(\alpha-1)\tilde{c}^*\frac{\tilde{r}}{\tilde{k}} \\
0 & 0 & \frac{\tilde{w}}{\tilde{H}} & 0 & 0 & 0 & 0 & 0 & (\alpha-1)\frac{\tilde{r}\tilde{H}}{\tilde{k}} - \alpha\frac{\tilde{w}}{\tilde{k}} \\
0 & 0 & 0 & \frac{\tilde{w}}{\tilde{H}^*}\left(\frac{A}{A^*}\right)^{\frac{1}{\alpha-1}} & 0 & 0 & 0 & 0 & (\alpha-1)\frac{\tilde{r}\tilde{H}^*}{\tilde{k}} - \frac{\alpha^*(\alpha-1)}{\alpha^*-1}\frac{\tilde{w}^*}{\tilde{k}} \\
0 & 0 & 0 & 0 & \frac{1}{\tilde{\Delta}} & 0 & 0 & 0 & (1-\sigma)(\alpha-1)\frac{\tilde{r}}{\tilde{k}}\tilde{\Delta} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{\tilde{\Delta}^*} & 0 & 0 & (1-\sigma^*)(\alpha-1)\frac{\tilde{r}}{\tilde{k}}\tilde{\Delta}^* \\
-1 & 0 & 0 & 0 & 0 & 0 & \tilde{r}-n & 0 & (\alpha-1)\frac{\tilde{r}}{\tilde{k}}\tilde{W} + \alpha\frac{\tilde{w}}{\tilde{k}} \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & \tilde{r}-n^* & (\alpha-1)\frac{\tilde{r}}{\tilde{k}}\tilde{W}^* + \frac{\alpha^*(\alpha-1)}{\alpha^*-1}\frac{\tilde{w}^*}{\tilde{k}} \\
-\chi & -\chi & 0 & 0 & 0 & 0 & -n\chi & -n^*\chi & \chi\left(\tilde{r} + \frac{\alpha^*(\alpha-1)}{\alpha^*-1}\frac{f^*(\tilde{k})}{\tilde{k}}\right)
\end{pmatrix}$$

(A.13)

Where:

$$\chi = \left[\left(\frac{\alpha A}{\alpha^* A^*}\right)^{\frac{1}{\alpha^*-1}}\left(\frac{\alpha-1}{\alpha^*-1}\right) k^{\frac{\alpha-\alpha^*}{\alpha^*-1}} + 1\right]^{-1}$$

A.2.2 Retirement Model – Derivation of Equation (13.10)

$$H(v,t) \equiv \int_t^{v+R} (1-\tau_s) w(\tau) e^{-R(t,\tau)-M(\tau-v)+M(t-v)} d\tau + \int_{v+R}^{v+D} B(\tau) e^{-R(t,\tau)-M(\tau-v)+M(t-v)} d\tau \qquad \text{(A.14)}$$

$$H(t) = H(t,t) = \int_t^{t+R} (1-\tau_s) w(\tau) e^{-R(t,\tau)-M(\tau-t)} d\tau + \int_{t+R}^{t+D} B(\tau) e^{-R(t,\tau)-M(\tau-t)} d\tau$$

$$\dot{H}(t) = (1-\tau_s) w(t+R) e^{-R(t,t+R)-M(R)} - (1-\tau_s) w(t) + \int_t^{t+R} \left[r(t) + \mu_{H1} \right] (1-\tau_s) w(\tau) e^{-R(t,\tau)-M(\tau-t)} d\tau$$
$$-\beta w(t+R) e^{-R(t,t+R)-M(R)} + \int_{t+R}^{t+D} \left[r(t) + \mu_{H2} \right] \beta w(\tau) e^{-R(t,\tau)-M(\tau-t)} d\tau$$

$$\dot{H}(t) = \left[(1-\tau_s) - \beta \right] w(t+R) e^{-R(t,t+R)-M(R)} - (1-\tau_s) w(t) + \left[r(t) + \mu_{H1} \right] H(t)$$
$$-\left[(1-\tau_s) - \beta \right] \left[\mu_{H2} - \mu_{H1} \right] \int_{t+R}^{t+D} w(\tau) e^{-R(t,\tau)-M(\tau-t)} d\tau$$

Given that the difference of the mortality is approximately zero, the last term drops out.

Now employing the log-linear procedure of Mierau and Turnovsky (2015):

$$Z(t+R) \equiv w(t+R) e^{-R(t,t+R)-M(R)}$$

$$\ln Z(t+R) = \ln w(t+R) - \int_t^{t+R} r(s) ds - \int_0^R \mu(s) ds \qquad \text{(A.15)}$$

$$\ln Z(t+R) \approx \ln Z(t) + \left. \frac{d \ln Z(t+R)}{dR} \right|_{R=0} R$$

This is the extrapolation from age 0 to age Rand then discounting it back to age 0.

Applying this method to **Error! Reference source not found.** generates:

$$w(t+R) e^{-R(t,t+R)-M(R)} \approx w(t) e^{\left[\frac{\dot{w}(t)}{w(t)} - r(t) \right] R} \qquad \text{(A.16)}$$

A.2.3 Retirement Model – Per-capita Dynamic Equations and linearization:

$$\dot{c}(t) = \varphi \frac{H(t)}{\Delta(t)} + \left(\sigma\left[r(t) - \rho\right] - n - \mu_c\right)c(t) \tag{A.17a}$$

$$\dot{c}^*(t) = \varphi^* \frac{H^*(t)}{\Delta^*(t)} + \left(\sigma^*\left[r(t) - \rho^*\right] - n - \mu_c^*\right)c^*(t) \tag{A.17b}$$

$$\dot{H}(t) = -(1-\tau_s)w(t) + \left[r(t) + \mu_H(\tau - t)\right]H(t) - \left[\beta - (1-\tau_s)\right]w(t)e^{\left[\frac{\dot{w}(t)}{w(t)} - r(t)\right]R} \tag{A.17c}$$

$$\dot{H}^*(t) = -(1-\tau_s^*)w^*(t) + \left[r(t) + \mu_H^*(\tau - t)\right]H^*(t) - \left[\beta^* - (1-\tau_s^*)\right]w(t)e^{\left[\frac{\dot{w}^*(t)}{w^*(t)} - r(t)\right]R^*} \tag{A.17d}$$

$$\dot{\Delta}(t) = -1 + \left[(1-\sigma)r(t) + \sigma\rho + \mu_\Delta(\tau - t)\right]\Delta(t) \tag{A.17e}$$

$$\dot{\Delta}^*(t) = -1 + \left[(1-\sigma^*)r(t) + \sigma^*\rho^* + \mu_\Delta^*(\tau - t)\right]\Delta^*(t) \tag{A.17f}$$

$$\dot{W}(t) = \left(r(t) - n\right)W(t) + w(t) - c(t) \tag{A.17g}$$

$$\dot{W}^*(t) = \left(r(t) - n\right)W^*(t) + w^*(t) - c^*(t) \tag{A.17h}$$

$$\dot{k}(t) + \dot{k}^*(t) = f\left(k(t)\right) + f^*\left(k^*(t)\right) - nW(t) - nW^*(t) - c(t) - c^*(t) \tag{A.17i}$$

Linearization:

$$\begin{pmatrix} \dot{c}(t) \\ \dot{c}^*(t) \\ \dot{H}(t) \\ \dot{H}^*(t) \\ \dot{\Delta}(t) \\ \dot{\Delta}^*(t) \\ \dot{W}(t) \\ \dot{W}^*(t) \\ \dot{k}(t) \end{pmatrix} = \Omega \begin{pmatrix} c(t) - \tilde{c} \\ c^*(t) - \tilde{c}^* \\ H(t) - \tilde{H} \\ H^*(t) - \tilde{H}^* \\ \Delta(t) - \tilde{\Delta} \\ \Delta^*(t) - \tilde{\Delta}^* \\ W(t) - \tilde{W} \\ W^*(t) - \tilde{W}^* \\ k(t) - \tilde{k} \end{pmatrix} \tag{A.18a}$$

Where Ω is:

$$\begin{pmatrix}
-\frac{\varphi\tilde{H}}{\tilde{\Delta}\tilde{c}} & 0 & \frac{\varphi}{\tilde{\Delta}} & 0 & -\frac{\varphi\tilde{H}}{\tilde{\Delta}^2} & 0 & 0 & 0 & \sigma(\alpha-1)\tilde{c}\frac{\tilde{r}}{\tilde{k}} \\
0 & -\frac{\varphi^*\tilde{H}^*}{\tilde{\Delta}^*\tilde{c}^*} & 0 & \frac{\varphi^*}{\tilde{\Delta}^*} & 0 & -\frac{\varphi^*\tilde{H}^*}{\tilde{\Delta}^{*2}} & 0 & 0 & \sigma^*(\alpha-1)\tilde{c}^*\frac{\tilde{r}}{\tilde{k}} \\
L_{31} & L_{32} & L_{33} & 0 & 0 & 0 & L_{37} & L_{38} & L_{39} \\
L_{41} & L_{42} & 0 & L_{44} & 0 & 0 & L_{47} & L_{48} & L_{49} \\
0 & 0 & 0 & 0 & \frac{1}{\tilde{\Delta}} & 0 & 0 & 0 & (1-\sigma)(\alpha-1)\frac{\tilde{r}}{\tilde{k}}\tilde{\Delta} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{\tilde{\Delta}^*} & 0 & 0 & (1-\sigma^*)(\alpha-1)\frac{\tilde{r}}{\tilde{k}}\tilde{\Delta}^* \\
-1 & 0 & 0 & 0 & 0 & 0 & \tilde{r}-n & 0 & (\alpha-1)\frac{\tilde{r}}{\tilde{k}}\tilde{W}+\alpha\frac{\tilde{w}}{\tilde{k}} \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & \tilde{r}-n & (\alpha-1)\frac{\tilde{r}}{\tilde{k}}\tilde{W}^*+\alpha\left(\frac{A}{A^*}\right)^{\frac{1}{\alpha-1}}\frac{\tilde{w}}{\tilde{k}} \\
-\chi & -\chi & 0 & 0 & 0 & 0 & -n\chi & -n\chi & \tilde{r}
\end{pmatrix} \quad \text{(A.18b)}$$

$$L_{31} = \frac{1}{\tilde{k}}\left[\beta-\left(1-\tau_s\right)\right]\tilde{w}\alpha\chi R e^{-\tilde{r}R} \quad \text{(A.19a)}$$

$$L_{32} = L_{31} \quad \text{(A.19b)}$$

$$L_{33} = \frac{\tilde{w}}{\tilde{H}}\left(\left(1-\tau_s\right)\left(1-e^{-\tilde{r}R}\right)+\beta e^{-\tilde{r}R}\right) \quad \text{(A.19c)}$$

$$L_{37} = \frac{1}{\tilde{k}}\left[\beta-\left(1-\tau_s\right)\right]\tilde{w}\alpha\chi Rne^{-\tilde{r}R} \quad \text{(A.19d)}$$

$$L_{38} = L_{37} \quad \text{(A.19e)}$$

$$L_{39} = \frac{1}{\tilde{k}}\left(\left(\alpha-1\right)\tilde{r}\tilde{H}-\left(1-\tau_s\right)\alpha\tilde{w}-\left[\beta-\left(1-\tau_s\right)\right]\tilde{w}e^{-\tilde{r}R}\left(\alpha+\tilde{r}R\right)\right) \quad \text{(A.19f)}$$

$$L_{41} = \left(\frac{A}{A^*}\right)^{\frac{1}{\alpha-1}}\frac{1}{\tilde{k}}\left[\beta^*-\left(1-\tau_s^*\right)\right]\tilde{w}\alpha\chi R^* e^{-\tilde{r}R^*} \quad \text{(A.19g)}$$

$$L_{42} = L_{41} \quad \text{(A.19h)}$$

$$L_{44} = \left(\frac{A}{A^*}\right)^{\frac{1}{\alpha-1}} \frac{\tilde{w}}{\tilde{H}^*}\left(\left(1-\tau_s^*\right)\left(1-e^{-\tilde{r}R^*}\right)+\beta^* e^{-\tilde{r}R^*}\right) \tag{A.19i}$$

$$L_{47} = \left(\frac{A}{A^*}\right)^{\frac{1}{\alpha-1}} \frac{1}{\tilde{k}}\left[\beta^* - \left(1-\tau_s^*\right)\right]\tilde{w}\alpha\chi R^* n e^{-\tilde{r}R^*} \tag{A.19j}$$

$$L_{48} = L_{47} \tag{A.19k}$$

$$L_{49} = \frac{1}{\tilde{k}}\left(\left(\alpha-1\right)\tilde{r}\tilde{H}^* - \left(\frac{A}{A^*}\right)^{\frac{1}{\alpha-1}}\left(1-\tau_s^*\right)\alpha\tilde{w} - \left[\beta^* - \left(1-\tau_s^*\right)\right]\left(\frac{A}{A^*}\right)^{\frac{1}{\alpha-1}} \tilde{w}e^{-\tilde{r}R^*}\left(\alpha+\tilde{r}R^*\right)\right) \tag{A.19l}$$

Appendix: Chapter 3

A.3 Gradual Adjustment System

Linearized system:

$$\begin{pmatrix} \dot{k}(t) \\ \dot{c}(t) \\ \dot{H}(t) \\ \dot{\Delta}(t) \\ \dot{R}(t) \\ \dot{\beta}(t) \\ \dot{\varphi}(t) \end{pmatrix} = \begin{pmatrix} L_{11} & L_{12} & 0 & 0 & L_{15} & 0 & 0 \\ L_{21} & L_{22} & L_{23} & L_{24} & L_{25} & 0 & L_{27} \\ L_{31} & L_{32} & L_{33} & 0 & L_{35} & L_{36} & 0 \\ 0 & 0 & 0 & L_{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\theta \end{pmatrix} \begin{pmatrix} k(t)-\tilde{k} \\ c(t)-\tilde{c} \\ H(t)-\tilde{H} \\ \Delta(t)-\tilde{\Delta} \\ R(t)-\tilde{R} \\ \beta(t)-\tilde{\beta} \\ \varphi(t)-\tilde{\varphi} \end{pmatrix} \quad \text{(A.20a)}$$

Where:

$$L_{15} = \frac{\tilde{k}}{l}\theta\varphi e^{-nR-M(R)} \quad \text{(A.20b)}$$

$$L_{25} = \frac{\varphi}{l}\left[c\theta - \frac{\varphi}{l}\frac{\tilde{H}}{\tilde{\Delta}}\right]e^{-nR-M(R)} \quad \text{(A.20c)}$$

$$L_{35} = er\left[\beta - \left(1-\beta d\right)\right]e^{-rR} - w\left(1-e^{-rR}\right)\frac{\beta}{l^2}\varphi e^{-nR-M(R)} \quad \text{(A.20d)}$$

$$L_{27} = w\left(d - \left(1-d\right)e^{-rR}\right) \quad \text{(A.20e)}$$

$$L_{36} = \frac{\tilde{H}}{l\tilde{\Delta}} \quad \text{(A.20f)}$$

Gradual General Solution

$$k(t) = \tilde{k} + C_2 e^{\lambda_1 t} + V_{kR}\left(R_0 - \tilde{R}\right)e^{-\theta_R t} + V_{k\beta}\left(\beta_0 - \tilde{\beta}\right)e^{-\theta_\beta t} + V_{k\varphi}\left(\varphi_0 - \tilde{\varphi}\right)e^{-\theta_\varphi t} \quad \text{(A.21a)}$$

$$c(t) = \tilde{c} + V_c C_2 e^{\lambda_1 t} + V_{cR}\left(R_0 - \tilde{R}\right)e^{-\theta_R t} + V_{c\beta}\left(\beta_0 - \tilde{\beta}\right)e^{-\theta_\beta t} + V_{c\varphi}\left(\varphi_0 - \tilde{\varphi}\right)e^{-\theta_\varphi t} \quad \text{(A.21b)}$$

$$H(t) = \tilde{H} + V_H C_2 e^{\lambda_1 t} + V_{HR}\left(R_0 - \tilde{R}\right)e^{-\theta_R t} + V_{H\beta}\left(\beta_0 - \tilde{\beta}\right)e^{-\theta_\beta t} + V_{H\varphi}\left(\varphi_0 - \tilde{\varphi}\right)e^{-\theta_\varphi t} \quad \text{(A.21c)}$$

$$\Delta(t) = \tilde{\Delta} + V_\Delta C_2 e^{\lambda_1 t} + V_{\Delta R}\left(R_0 - \tilde{R}\right)e^{-\theta_R t} + V_{\Delta\beta}\left(\beta_0 - \tilde{\beta}\right)e^{-\theta_\beta t} + V_{\Delta\varphi}\left(\varphi_0 - \tilde{\varphi}\right)e^{-\theta_\varphi t} \quad \text{(A.21d)}$$

Where:

$$C_2 = k_0 - \tilde{k} - V_{kR}\left(R_0 - \tilde{R}\right) - V_{k\beta}\left(\beta_0 - \tilde{\beta}\right) - V_{k\varphi}\left(\varphi_0 - \tilde{\varphi}\right) \quad \text{(A.22)}$$

The eigenvectors V_{ij}, where i is the variable and j is the gradual mover are solved like so:

$$\begin{pmatrix} L_{11}+\theta_R & L_{12} & 0 & 0 \\ L_{21} & L_{22}+\theta_R & L_{23} & L_{24} \\ L_{31} & L_{32} & L_{33}+\theta_R & 0 \\ 0 & 0 & 0 & L_{44}+\theta_R \end{pmatrix} \begin{pmatrix} V_{kR} \\ V_{cR} \\ V_{HR} \\ V_{\Delta R} \end{pmatrix} = \begin{pmatrix} -L_{15} \\ -L_{25} \\ -L_{35} \\ 0 \end{pmatrix} \quad \text{(A.23a)}$$

$$\begin{pmatrix} L_{11}+\theta_\beta & L_{12} & 0 & 0 \\ L_{21} & L_{22}+\theta_\beta & L_{23} & L_{24} \\ L_{31} & L_{32} & L_{33}+\theta_\beta & 0 \\ 0 & 0 & 0 & L_{44}+\theta_\beta \end{pmatrix} \begin{pmatrix} V_{k\beta} \\ V_{c\beta} \\ V_{H\beta} \\ V_{\Delta\beta} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -L_{36} \\ 0 \end{pmatrix} \quad \text{(A.23b)}$$

$$\begin{pmatrix} L_{11}+\theta_\varphi & L_{12} & 0 & 0 \\ L_{21} & L_{22}+\theta_\varphi & L_{23} & L_{24} \\ L_{31} & L_{32} & L_{33}+\theta_\varphi & 0 \\ 0 & 0 & 0 & L_{44}+\theta_\varphi \end{pmatrix} \begin{pmatrix} V_{k\varphi} \\ V_{c\varphi} \\ V_{H\varphi} \\ V_{\Delta\varphi} \end{pmatrix} = \begin{pmatrix} 0 \\ -L_{27} \\ 0 \\ 0 \end{pmatrix} \quad \text{(A.23c)}$$