ON BROWNIAN EXCURSIONS IN LIPSCHITZ DOMAINS PART II. LOCAL ASYMPTOTIC DISTRIBUTIONS

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1. Introduction. In this paper, we continue the study initiated in Burdzy and Williams (1986) of the local properties of Brownian excursions in Lipschitz domains. The focus in part I was on local *path* properties of such excursions. In particular, a necessary and sufficient condition was given for Brownian excursions in a Lipschitz domain to share the local path properties with Brownian excursions in a half-space. This condition holds for $C^{1,\alpha}$-domains ($\alpha > 0$), but there is a C^1-domain for which it fails. Here we consider the *distributions* of a selection of local events for excursions. In particular, we focus on the asymptotics of these distributions as the region of locality shrinks to a point. We show that when a Lipschitz domain is locally approximated by a half-space, the asymptotics for excursions in the two domains are comparable.

Our choice of which local events to examine was influenced by the desire to give a representation for Brownian local time on the boundary of a Lipschitz domain. The definition of local time we use here may seem somewhat unorthodox, however, it is quite natural in the context of exit systems. Specifically, we define the local time to be the continuous additive functional of Brownian motion whose Revuz measure (relative to Lebesgue measure as invariant measure) is equal to the surface area measure on the boundary of the Lipschitz domain (Revuz (1970)). Using the results of parts I and II, for a class of Lipschitz domains that includes the $C^{1,\alpha}$-domains ($\alpha > 0$), we give several representations for Brownian local time in terms of limits of numbers of excursions of a certain "size".

Representation theorems of this type are well known for one-dimensional Brownian motion (see Fristedt and Taylor (1983) or Williams (1979)) and have been obtained for multi-dimensional *reflected* Brownian motion in a C^3-domain (see Hsu (1986), Theorem 6.1). Although we consider unconstrained Brownian motion rather than reflected Brownian motion, we require less regularity of the boundary than Hsu (1986), and it seems that a modification of our approach would yield a relaxation of his assumptions. Another closely related paper is that of Bass (1984) who discussed convergence of *continuous* additive functionals of Brownian motion. His results can be used to give some representation theorems for Brownian local time on a Lipschitz surface. In contrast, the representations we give are in terms of limits of *discontinuous* additive functionals that count the number of excursions of a certain size.

The key to our representations is a link we establish between the local behavior of excursions and the boundary behavior of the Green function. The potential of this connection has not been exhausted here, for further representation theorems could be obtained almost automatically once more about the Green function was known. In fact, some results of this nature have recently been obtained and will be discussed in a forthcoming paper by Bañuelos and Burdzy (1988).

The remainder of this paper is organized as follows. Section 2, introduces notation and reviews some known results. Section 3 presents explicit formulas for some distributions of excursions in a half-space. In Section 4 we show that when a Lipschitz domain is locally approximated by a half-space, the asymptotics of the distributions of a selection of local events for excursions in the two domains are comparable. These results are an improvement of those in Chapter 4 of Burdzy (1987). Representation theorems for local time are proved in Section 5. Section 6 contains some potential theoretic results which are needed in Section 4. This section relies heavily on the results of Fabes et al. (1986) and does not use probability, hence it is relegated to the end of the paper.

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2. Preliminaries. In this section we establish notation and review some known results that are fundamental to our study. The reader is advised to consult Doob (1984) concerning Brownian motion and potential theory. Fabes et al. (1986) is an excellent reference concerning the boundary behavior of parabolic functions. We use the version of excursion theory presented in Burdzy (1987) which is based on more general results of Maisonneuve (1975).

Since the paper of Fabes et al. (1986) will be referred to repeatedly in Section 6, we adopt its notation with slight modifications.

Let $\mathbb{R}^n$ denote n-dimensional Euclidean space for some $n \geq 2$. The points of $\mathbb{R}^n$ will be denoted as x, y, etc. In describing local properties at different points on the boundary of a domain in $\mathbb{R}^n$, it will be convenient to use orthonormal coordinate systems that vary from point to point. The notation $(x_1, \ldots, x_n)$ will be used to denote the coordinates of a point x in $\mathbb{R}^n$ in some orthonormal coordinate system, where the choice of coordinate system will always be made clear in the context. We will then write $\widetilde{x} = (x_1, \ldots, x_{n-1})$, in particular $\widetilde{0} = (0, \ldots, 0) \in \mathbb{R}^{n-1}$. The complement of a set D in $\mathbb{R}^n$ will be denoted $D^c = \mathbb{R}^n \setminus D$.

A set $D \subset \mathbb{R}^n$ will be called a Lipschitz domain if it is a non-empty domain and there exists $\lambda < \infty$ such that for each $x \in \partial D$ there is a non-empty neighborhood U of x, an orthonormal coordinate system $CS(x)$, and a Lipschitz function $\varphi_x : \mathbb{R}^{n-1} \to \mathbb{R}$ with constant λ, satisfying $D \cap U = \left\{ y \in U : y_n > \varphi_x(\widetilde{y}) \right\}$ in $CS(x)$. Then for $r > 0$, $x \in \partial D$ and $t \in \mathbb{R}$, let

$$\Psi_r(x,t) = \left\{(y,u) \in D \times \mathbb{R} : |x-y| < r, |t-u| < r^2\right\},$$
$$\Delta_r(x,t) = \overline{\Psi_r(x,t)} \cap (\partial D \times \mathbb{R}),$$

where $\overline{\Psi_r(x,t)}$ denotes the closure of $\Psi_r(x,t)$ in $\mathbb{R}^{n+1}$, and for $x = (x_1, \ldots, x_n)$ in $CS(x)$, let

$$\overline{A}_r(x,t) = \left(\widetilde{x}, x_n + r, t + 2r^2\right),$$
$$\underline{A}_r(x,t) = \left(\widetilde{x}, x_n + r, t - 2r^2\right).$$

For a domain $U \subset \mathbb{R}^n$ and open interval $I \subset \mathbb{R}$, a function $f : U \times I \to \mathbb{R}$ will be called parabolic if the first and second partial derivatives of f in U and first partial derivative of f in I are continuous on $U \times I$, and

$$\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} f(x,t) - \frac{\partial}{\partial t} f(x,t) = 0 \quad \text{for} \quad (x,\ t) \in U \times I.$$

For $(y,u) \in \overline{\Psi_r(x,t)}$, the caloric measure $\mu_{(y,u)}$ on $\partial\Psi_r(x,t)$ is the unique Borel probability measure that does not charge $\{(z,s) \in \partial\Psi_r(x,t) : s = t + r^2\}$ and satisfies

$$f(y,u) = \int_{\partial\Psi_r(x,t)} f(z,s)\mu_{(y,u)}(dz,ds)$$

for every parabolic function f in $\Psi_r(x,t)$ which is continuous in $\overline{\Psi_r(x,t)}$ (Fabes et al. (1986)).

The Green function of a Lipschitz domain $D \subset \mathbb{R}^n$ will be denoted $G_D(\,\cdot\,,\,\cdot\,)$.

Let Ω be the space of paths $\omega : [0,\infty) \to \mathbb{R}^n \cup \{\delta\}$ which are continuous on $[0,R)$ for some $R(\omega) \le \infty$ and such that $\omega(t) = \delta$ for $t \ge R$. Thus, R denotes the lifetime of a path, which may be infinite. Let X be the canonical process i.e., $X_t(\omega) \equiv \omega(t)$. Denote $\mathcal{F} = \sigma\{X_t, t \ge 0\}, \mathcal{F}_t = \sigma\{X_s, s \le t\}$. For a stopping time T let $\mathcal{F}_T$ denote the usual σ-field of pre-T-events and let θ_t, $t \ge 0$, be the shift operators on Ω. For a set $A \subset \mathbb{R}^n$ let

$$T_A = T(A) = \inf\{t > 0 : X_t \in A\}$$

and

$$T(A-) = \inf\{t > 0 : \lim_{s \uparrow t} X_s \in A\}.$$

Let P^x denote a measure on $(\Omega, \mathcal{F})$ which makes X the standard n-dimensional Brownian motion starting from x. Analogously, P_D^x will denote the distribution of Brownian motion in D, i.e., Brownian motion killed at $T(D^c)$.

An excursion law H^x in $D \subset \mathbb{R}^n$ is a σ-finite measure on $(\Omega, \mathcal{F})$ which has the following properties:

(i) $H^x(X_0 \neq x) = 0$,

(ii) H^x is strong Markov for the P_D^x-transition probabilities, i.e.,

$$H^x(a \cdot b(\theta_T)) = H^x\left(a \cdot P_D^{X(T)}(b)\right)$$

for all stopping times $T > 0$, nonnegative and $\mathcal{F}$-measurable b, and nonnegative and $\mathcal{F}_T$-measurable a.

If $D \subset \mathbb{R}^n$ is a Lipschitz domain and $x \in \partial D$ then there exists an excursion law H^x in D.

The following is a version of the exit system theorem. See Maisonneuve (1970) for more details on exit systems and see Revuz (1970) or Williams (1979) for the definition and properties of continuous additive functionals (CAF's).

Suppose that $D \subset \mathbb{R}^n$ is a Lipschitz domain and let μ denote the surface area measure on ∂D. Let L be the CAF of the Brownian motion X (with associated probability measures $\{P^x, x \in \mathbb{R}^n\}$), whose Revuz measure (relative to Lebesgue measure as invariant measure) is given by μ , i.e.,

$$\mu(A) = \lim_{t \downarrow 0} \frac{1}{t} E^v \left[\int_0^t 1_A(X_s) dL_s \right],$$

for all Borel sets $A \subset \mathbb{R}^n$, where υ denotes Lebesgue measure on $\mathbb{R}^n$. Fix some nonpolar compact set $B \subset D$. For μ-almost all points $x \in \partial D$, the unit inward normal vector N_x is well defined and $\lim_{\varepsilon \to 0} \varepsilon^{-1} P_D^{x+\varepsilon N_x}(T_B < \infty)$ exists. For such x let H^x be the excursion law in D with the property that $H^x(T_B < \infty)$ is equal to the above limit. For all other x, let $H^x = 0$. Then the pair (dL, H) is an exit system in D in the following sense.

For u such that $X_u \in \partial D$ let e_u be the excursion of X in D i.e.,

$$e_u(t) = \begin{cases} X(u+t) & \text{if } \inf\{s > u : X_s \in D^c\} > u + t, \\ \delta & \text{otherwise.} \end{cases}$$

For u such that $X_u \notin \partial D$, define $e_u \equiv \delta$. Then (Burdzy (1987), Theorem 7.2),

$$E^{\cdot}\left(\sum_{0<u<\infty} Z_u \cdot (f \circ e_u)\right) = E^{\cdot}\left(\int_0^\infty Z_s H^{X(s)}(f) dL_s\right) \tag{2.1}$$

for all universally measurable functions f on Ω which vanish on excursions $e_u \equiv \delta$ and nonnegative $\mathcal{F}_t$-predictable processes Z.

3. Some explicit formulas for excursions in a half-space.

Let $D_* = \{x \in \mathbb{R}^n : x_n > 0\}$. There exists a unique excursion law H_*^0 in D_* such that $H_*^0(T_B < \infty) = 1$ where $B = \{x \in D_* : x_n = 1\}$ (see Burdzy (1987) Theorem 3.1). Denote $S_u = \{x \in D_* : |x| = 1\}, S_\ell = \{x \in \partial D_* : |x| < 1\}, S = \overline{S_u} \cup \overline{S_\ell}$, and $T = \min(T_S, R)$. In the right members below, the symbol dx will denote the differential of Lebesgue measure in $\mathbb{R}^n$, $d\tilde{x}$ will denote the differential of $(n-1)$-dimensional Lebesgue measure (surface measure) on the hyperplane ∂D_*, and $d\sigma = d\sigma(x)$ will denote the differential of surface area measure on the semisphere S_u.

Theorem 3.1.

(i) $H_*^0(X_t \in dx) = 2^{-(n-2)/2}\pi^{-n/2}t^{-(n+2)/2}x_n e^{-|x|^2/(2t)}dx$ *for* $t > 0$, $x \in D_*$,

(ii) $H_*^0(|X_t| \in dr) = 2^{-(n-2)/2}\pi^{-1/2}(\Gamma((n+1)/2))^{-1}t^{-(n+2)/2}r^n e^{-r^2/(2t)}dr$ *for* $t > 0$, $r > 0$,

(iii) $H_*^0(R \in dt, X(R-) \in dx) = (2\pi)^{-n/2}t^{-(n+2)/2}e^{-|x|^2/(2t)}dt d\tilde{x}$ *for* $t > 0, x \in \partial D_*$,

(iv) $H_*^0(R \in dt) = \left(2\pi t^3\right)^{-1/2} dt$ *for* $t > 0$,

(v) $H_*^0(R > t) = 2^{1/2}(\pi t)^{-1/2}$ *for* $t > 0$,

(vi) $H_*^0(X(R-) \in dx) = \Gamma(n/2)\pi^{-n/2}|x|^{-n}d\tilde{x}$ *for* $x \in \partial D_*$,

(vii) $H_*^0(|X(R-)| > r) = 2\pi^{-1/2}[\Gamma(n/2)/\Gamma((n-1)/2)]r^{-1}$ *for* $r > 0$,

(viii) $H_*^0\left(\sup_{t\in(0,R)} |X_t| \in dr\right) = 2\pi^{-1/2}[\Gamma((n+2)/2)/\Gamma((n+1)/2)]r^{-2}dr$ *for* $r > 0$,

(ix) $H_*^0\left(\sup_{t\in(0,R)} |X_t| > r\right) = 2\pi^{-1/2}[\Gamma((n+2)/2)/\Gamma((n+1)/2)]r^{-1}$ *for* $r > 0$,

(x) $H_*^0(X(T-) \in dx) = \Gamma(n/2)\pi^{-n/2}\left(|x|^{-n} - 1\right) d\tilde{x}$ *for* $x \in S_\ell$,

(xi) $H_*^0(X(T-) \in dx) = 2\Gamma((n+2)/2)\pi^{-n/2}x_n d\sigma$ *for* $x \in S_u$,

(xii) $H_*^0(X(T-) \in S_u) = 2\pi^{-1/2}[\Gamma((n+2)/2)/\Gamma((n+1)/2)]$,

(xiii) *The random variables T and $X(T-)$ are conditionally independent under H_*^0 given $\{X(T-) \in S_u\}$. The H_*^0-distribution of T given $\{X(T-) \in S_u\}$ is the same as the distribution of the hitting time of the unit sphere $\left\{x \in \mathbb{R}^{n+2} : |x| = 1\right\}$ by the $(n+2)$-dimensional Brownian motion starting at* 0.

Proof. The proofs of parts (i), (iii), (iv), (vi), (viii), (x)-(xiii) were given in Burdzy (1987), Theorem 5.1. Parts (ii), (v), (vii) and (ix) are straightforward consequences of (i), (iv), (vi) and (viii). □

We would like to use this occasion to present some formulas for h-processes, related to excursion laws. Let $P^{y,x}_{D_*}$ denote the distribution of the h-process (i.e. conditioned Brownian motion) in D_* which starts at y and converges to x; $E^{y,x}_{D_*}$ will denote the corresponding expectation. See Doob (1984) for the definition of an h-process.

Theorem 3.2.

(i) $P^{0,x}_{D_*}(R \in dt) = 2^{-n/2}(\Gamma(n/2))^{-1}|x|^n t^{-(n+2)/2} e^{-|x|^2/(2t)} dt$ *for* $t > 0, x \in \partial D_*$,

(ii) $E^{0,x}_{D_*}(R) = \begin{cases} |x|^2/(n-2) & \text{if } n \geq 3, \\ \infty & \text{if } n = 2, \end{cases}$ *for* $x \in \partial D_*$,

(iii) $P^{0,x}_{D_*}\left(\sup_{t\in(0,R)} |X_t| \in dr\right) = n|x|^n r^{-(n+1)} dr$ *for* $r > |x|$, $x \in \partial D_*$,

(iv) $P^{0,x}_{D_*}\left(\sup_{t\in(0,R)} |X_t| > r\right) = (|x|/r)^n$ *for* $r \geq |x|, x \in \partial D_*$.

Proof. Let $T_\varepsilon = \min(\varepsilon, \inf\{t > 0 : |X_t| = \varepsilon\})$ and apply the strong Markov property at T_ε to obtain for $t > \varepsilon$,

$$P^{0,x}_{D_*}(R \in dt) = \int_0^t \int_{D_*} P^{y,x}_{D_*}(R \in dt - s) P^{0,x}_{D_*}\left(T_\varepsilon \in ds, X\left(T_\varepsilon\right) \in dy\right). \tag{3.1}$$

Suppose that the following limit exists

$$\lim_{\substack{y\to 0\\ y\in D_*\\ s\to 0\\ s>0}} P^{y,x}_{D_*}(R \in dt - s). \tag{3.2}$$

Then (3.1) shows that $P^{0,x}_{D_*}(R \in dt)$ is equal to the limit in (3.2).

Observe that

$$P^{y,x}_{D_*}(R \in dt) = \frac{P^y_{D_*}(R \in dt, X(R-) \in dx)}{P^y_{D_*}(X(R-) \in dx)}. \tag{3.3}$$

The hitting time of 0 by the 1-dimensional Brownian motion X_n starting from y_n has the density $y_n\left(2\pi t^3\right)^{-1/2} \exp\left(-y_n^2/(2t)\right)$ for $t > 0$. This is the $P^y_{D_*}$-density of R. Given $\{R = t\}$, the $P^y_{D_*}$-distribution of $X(R-)$ is normal with the density

$$(2\pi t)^{-(n-1)/2} \exp(-|\widetilde{x} - \widetilde{y}|^2/(2t))$$

for $x \in \partial D_*$. Multiply the last two formulas to obtain the numerator in (3.3). The denominator is obtained by integration of the numerator over t. It remains to take the limit, as indicated in (3.2) to obtain part (i) of the theorem. Part (ii) follows from (i) by integration.

Although parts (iii) and (iv) may be obtained in a similar, elementary but tedious way, let us point out that using the notation of Theorem 3.1,

$$P_{D_*}^{0,x}\left(\sup_{t\in(0,R)}|X_t|<1\right)=\frac{H_*^0(X(T-)\in dx)}{H_*^0(X(R-)\in dx)}$$

for $x\in\partial D_*, |x|<1$. Then Theorem 3.1 (vi) and (x) and scaling can be used to obtain (iii) and (iv). □

Remark 3.1. The above formulas should be compared with (8.1)-(8.3) of Hsu (1986), although the normalizing constants are not the same.

4. Convergence of excursion laws. Denote $B_1(r)=\{y\in\mathbb{R}^n : |y|\geq r\}, B_2(r,v)=\{y\in\mathbb{R}^n : y\cdot v\geq r\}$, where v is a vector in $\mathbb{R}^n$ satisfying $|v|=1$, and $y\cdot v$ stands for the scalar product. For a set $D\subset\mathbb{R}^n$ let $B_3(r,D)=\{y\in\mathbb{R}^n :\ \text{dist}\ (y,\partial D)\geq r\}$. Consider the following events:

$$\begin{aligned}
A_1(t,r)&=\{|X_t|>r\}\,,\\
A_2(t,r)&=\{R>t,|X(R-)|>r\},\\
A_3(t)&=\{R>t\},\\
A_4(r)&=\{|X(R-)|>r\},\\
A_5(r)&=\{T\left(B_1(r)\right)<\infty\}\,,\\
A_6(r,v)&=\{T\left(B_2(r,v)\right)<\infty\}\,,\\
A_7(r,D)&=\{T\left(B_3(r,D)\right)<\infty\}\,.
\end{aligned}$$

Suppose that $D\subset\mathbb{R}^n$ is a Greenian domain. Let $f_k(x,t)=P_D^x\left(A_k\right)$ for $1\leq k\leq 7$. Of course, f_k depends also on D and r.

For $\varepsilon>0$, let $B(x,\varepsilon)=\{y\in\mathbb{R}^n : |x-y|<\varepsilon\}$ and $T_{x,\varepsilon}=\min\left(\varepsilon^2,T(\partial B(x,\varepsilon))\right)$. Apply the strong Markov property at $T_{x,\varepsilon}$ to see that

$$f_k(x,t)=\int_0^t\int_D f_k(y,t-u)P_D^x\left(X\left(T_{x,\varepsilon}\right)\in dy,T_{x,\varepsilon}\in du\right)$$

for $1\leq k\leq 7$, provided $\varepsilon^2<t, B(x,\varepsilon)\subset D$ and $B(x,\varepsilon)\subset D\setminus B_{k-4}$ for $k=5,6,7$. This averaging property means that the functions f_k are parabolic in $D\times(0,\infty)$ for $k=1,2,3,4$ and in $(D\setminus B_{k-4})\times(0,\infty)$ for $k=5,6,7$ (see Doob (1984), p.276).

The next proposition contains a comparison result for certain distributions of Brownian motion in a Lipschitz domain that is locally approximable by a half-space.

Proposition 4.1. *Let $D_*=\{y\in\mathbb{R}^n : y\cdot v>0\}$ for some $v\in\mathbb{R}^n$ satisfying $|v|=1$, and let f_k^* and f_k correspond to domains D_* and D (the last one is described below). For positive λ,r,u,α and ε, there exist $\rho=\rho(n,\lambda,r,\varepsilon,u,\alpha)<\min(\sqrt{u},\alpha)$ and $\varepsilon_1=\varepsilon_1(\rho)$ with the following property.*

Suppose that φ is a Lipschitz function with constant λ and D is a domain such that

$$\{y\in D:|y|<1/\varepsilon_1\}=\{y\in D:|y|<1/\varepsilon_1,y_n>\varphi(\widetilde{y})\}.$$

Assume that

$$\{y\in\partial D:|y|<1/\varepsilon_1\}\subset\{y\in\mathbb{R}^n:|y|<1/\varepsilon_1,y\cdot v\in(-\varepsilon_1,\varepsilon_1)\}\,.$$

Then

$$\frac{f_k\left(\underline{A}_{\rho/2}(0,u)\right)}{f_k\left(\overline{A}_{\rho/2}(0,u)\right)} \geq 1/2, \tag{4.1}$$

$$\frac{f_k^*\left(\underline{A}_{\rho/2}(0,u)\right)}{f_k^*\left(\overline{A}_{\rho/2}(0,u)\right)} \geq 1/2 \tag{4.2}$$

and

$$\frac{f_k((\widetilde{0},\rho/32),u)}{f_k^*((\widetilde{0},\rho/32),u)} \in (1-\varepsilon, 1+\varepsilon) \tag{4.3}$$

for $1 \leq k \leq 7$.

Remark 4.1. The fraction $\rho/32$ appears here because it is used in later estimates.

Proof. First consider (4.2). It obviously holds for $k = 4, 5, 6$ and 7 since in these cases f_k^* does not depend on t.

Recall the following explicit formulas from Section 3 and the proof of Theorem 5.1 of Burdzy (1987). Here we use a coordinate system which makes D_* the half-space $\{y \in \mathbb{R}^n : y_n > 0\}$.

$$\begin{aligned}
&P^x_{D_*}\left(X_t \in dy\right) = (2\pi t)^{-n/2} \exp\left(-|x-y|^2/\,2t\right)\left(1-\exp\left(-2x_n y_n/\,t\right)\right)dy,\\
&P^x_{D_*}(R \in dt) = \left(2\pi t^3\right)^{-1/2} x_n \exp\left(-x_n^2/\,2t\right)dt,\\
&P^x_{D_*}(R \in dt, X(R-) \in dy)\\
&\qquad = \left(2\pi t^3\right)^{-1/2} x_n \exp\left(-x_n^2/\,2t\right)(2\pi t)^{-(n-1)/\,2} \exp(-|\widetilde{x}-\widetilde{y}|^2/\,2t)dt\,d\widetilde{y}.
\end{aligned}$$

Given these explicit formulas, it is elementary to check that for fixed r and u,

$$\frac{f_k^*\left(\underline{A}_q(0,u)\right)}{f_k^*\left(\overline{A}_q(0,u)\right)} \to 1$$

as $q \to 0$, for $k = 1, 2, 3$. Choose $\rho \in (0, \min(\sqrt{u}, \alpha))$ so that (4.2) holds even with 1/2 replaced by 3/4.

Now for the proof of (4.1) and (4.3), let

$$D^m = \{y \in \mathbb{R}^n : |y| < 1/\varepsilon_1, y \cdot v > \varepsilon_1\}$$

and

$$D^M = \{y \in \mathbb{R}^n : |y| > 1/\varepsilon_1 \text{ or } y \cdot v > -\varepsilon_1\}.$$

Observe that $D^m \subset D_* \subset D^M$ and $D^m \subset D \subset D^M$. Let f_k^m and f_k^M correspond to D^m and D^M. The continuity of probability implies that for a fixed $x \in D_*$ and $t > 0$,

$$f_k^m(x,t) - f_k^M(x,t) \to 0 \tag{4.4}$$

as $\varepsilon_1 \to 0$ for $1 \le k \le 7$. Since f_k is a monotone function of D for $k = 1, 3, 5, 6$, the formula (4.4) implies that for these values of k and fixed $x \in D_*$ and $t > 0$,

$$f_k^*(x,t) - f_k(x,t) \to 0 \text{ as } \varepsilon_1 \to 0. \tag{4.5}$$

It is easy to see that for any fixed x, the P^x probability of the union of the events

$$\{[T_{\partial D} > t \text{ and } |X(T_{\partial D})| > r] \text{ and } [T_{\partial D_*} \le t \text{ and } |X(T_{\partial D_*})| \le r]\}$$

and

$$\{[T_{\partial D_*} > t \text{ and } |X(T_{\partial D_*})| > r] \text{ and } [T_{\partial D} \le t \text{ or } |X(T_{\partial D})| \le r]\}$$

tends to zero as $\varepsilon_1 \to 0$. It follows that

$$f_k^*(x,t) - f_k(x,t) \to 0 \text{ as } \varepsilon_1 \to 0 \tag{4.6}$$

for $k = 2$ and, for similar reasons, for $k = 4$ and 7.

Let $(x,t) = ((\widetilde{0}, \rho/32), u)$ in (4.5) and (4.6) to see that (4.3) holds for suitably small ε_1.

When $\underline{A}_{\rho/2}(0,u)$ and $\overline{A}_{\rho/2}(0,u)$ are substituted for (x,t) in (4.5) and (4.6) then these formulas, together with (4.2) (recall this holds with 3/4 in place of 1/2), imply that (4.1) holds for small ε_1. □

Let D be a Lipschitz domain in $\mathbb{R}^n$. Fix some $z^0 \in D$ and let

$$B = \left\{y \in D : G_D\left(y, z^0\right) \ge 1\right\}.$$

Theorem 4.1. *For each $\varepsilon > 0$ there exists $\varepsilon_1 = \varepsilon_1(\varepsilon, n, \lambda)$ such that the following holds.*

Suppose that $a > 0$ satisfies

$$\{y \in D : |y| < a/\varepsilon_1\} = \{y \in D : |y| < a/\varepsilon_1, y_n > \varphi(\widetilde{y})\} \subset D \setminus B$$

where φ is a Lipschitz function with constant λ, $\varphi(0) = 0$, and for some v with $|v| = 1$,

$$\{y \in \partial D : |y| < a/\varepsilon_1\} \subset \{y \in \mathbb{R}^n : |y| < a/\varepsilon_1, y \cdot v \in (-a\varepsilon_1, a\varepsilon_1)\}.$$

Let H^0 be an excursion law in D with $H^0(T_B < \infty) \in (0, \infty)$. In the definitions of events $A_k, 1 \le k \le 7$, let $r = a, t = a^2$, and v and D be as above. Then

$$H^0(A_k) \cdot \frac{G_D\left(av, z^0\right)}{H^0(T_B < \infty)} \in (d_k(1-\varepsilon), d_k(1+\varepsilon))$$

for $1 \le k \le 7$. Here the d_k's are given by

$$d_1 = \int_1^\infty 2^{-(n-2)/2}\pi^{-1/2}(\Gamma((n+1)/2))^{-1} s^n e^{-s^2/2} ds,$$

$$d_2 = \int_1^\infty \int_1^\infty 2^{-(n-2)/2}\pi^{-1/2}(\Gamma((n-1)/2))^{-1} s^{-(n+2)/2} r^{n-2} e^{-r^2/(2s)} ds dr,$$

$$d_3 = (2/\pi)^{1/2},$$

$$d_4 = 2\pi^{-1/2}[\Gamma(n/2)/\Gamma((n-1)/2)],$$

$$d_5 = 2\pi^{-1/2}[\Gamma((n+2)/2)/\Gamma((n+1)/2)],$$

$$d_6 = d_7 = 1.$$

Proof. Denote $f_8^*(x,t) = x \cdot v$ for $x \in D_* \stackrel{\text{df}}{=} \{y \in \mathbb{R}^n : y \cdot v > 0\}, t > 0$, and $f_8(x,t) = aG_D\left(x, z^0\right) / G_D\left(av, z^0\right)$ for $x \in D, t > 0$. Let $\varepsilon_2, \varepsilon_3 > 0$ be small constants which will be specified later.

By the proof of Proposition 4.1 and scaling it is possible to choose $\rho < \min(1, 32\varepsilon_2/a)$ and $\varepsilon_1 > 0$ small enough so that for $1 \le k \le 7$,

$$\frac{f_k\left(\underline{A}_{a\rho/2}\left(0,a^2\right)\right)}{f_k\left(\overline{A}_{a\rho/2}\left(0,a^2\right)\right)} \ge 1/2, \tag{4.7}$$

$$\frac{f_k^*\left(\underline{A}_{a\rho/2}\left(0,a^2\right)\right)}{f_k^*\left(\overline{A}_{a\rho/2}\left(0,a^2\right)\right)} \ge 1/2 \tag{4.8}$$

and

$$\frac{f_k((\widetilde{0}, a\rho/32), a^2)}{f_k^*((\widetilde{0}, a\rho/32), a^2)} \in (1-\varepsilon_2, 1+\varepsilon_2)\,. \tag{4.9}$$

We have obviously

$$\frac{f_8^*(\underline{A}_{a\rho/2}(0,a^2))}{f_8^*(\overline{A}_{a\rho/2}(0,a^2))} = 1 \tag{4.10}$$

and

$$\frac{f_8(\underline{A}_{a\rho/2}(0,a^2))}{f_8(\overline{A}_{a\rho/2}(0,a^2))} = 1. \tag{4.11}$$

The Green function in a half-space behaves near the boundary like a linear function. It is easy to see that for fixed x and z^0, $G_D(x,z^0) \to G_{D_*}(x,z^0)$ as $\varepsilon_1 \to 0$ and, therefore, for ε_1 small enough

$$\frac{f_8^*((\widetilde{0}, a\rho/32), a^2)}{f_8((\widetilde{0}, a\rho/32), a^2)} = \frac{((\widetilde{0}, a\rho/32)\cdot v)G_D(av, z^0)}{G_D((\widetilde{0}, a\rho/32), z^0)a} \in (1-\varepsilon_2, 1+\varepsilon_2)$$

and this together with (4.9) implies that

$$\frac{f_k((\widetilde{0}, a\rho/32), a^2)}{f_k^*((\widetilde{0}, a\rho/32), a^2)}\,\frac{f_8^*((\widetilde{0}, a\rho/32), a^2)}{f_8((\widetilde{0}, a\rho/32), a^2)} \in ((1-\varepsilon_2)^2, (1+\varepsilon_2)^2). \tag{4.12}$$

Now Corollary 6.1 will be applied. Its assumptions are satisfied due to (4.7)-(4.12). Let ε_2 (and consequently ε_1) be so small that (6.21) holds with $c = \varepsilon_3$ for the functions f_k, f_k^*, f_8 and f_8^* i.e.,

$$\lim_{\substack{(x,t)\to(0,a^2)\\ x\in D\\ t>0}} \frac{f_k(x,t)}{f_8(x,t)} \cdot \lim_{\substack{(y,u)\to(0,a^2)\\ y\in D_*\\ u>0}} \frac{f_8^*(y,u)}{f_k^*(y,u)} \in (1-\varepsilon_3, 1+\varepsilon_3)\,.$$

Define d_k by declaring that the second limit in the above formula is equal to a/d_k.

Choose $0 < \varepsilon_4 < a$ so small that $\{x : |x| \le \varepsilon_4\} \cap B = \emptyset$ and

$$\frac{f_k(x,t)a}{f_8(x,t)d_k} \in (1 - 2\varepsilon_3, 1 + 2\varepsilon_3) \tag{4.13}$$

for $x \in D$, $|x| < \varepsilon_4$, $\left|t - a^2\right| < \varepsilon_4^2$. Denote

$$T = \min\left(\varepsilon_4^2, T\left(\{y \in \mathbb{R}^n : |y| \ge \varepsilon_4\}\right)\right).$$

Apply the strong Markov property at T, and use (4.13) together with the definition of f_8, to see that

$$\begin{aligned}
H^0(A_k) &= \int_0^{\varepsilon_4^2} \int_D f_k\left(y, a^2 - s\right) H^0(X(T) \in dy,\ T \in ds) \\
&\le \int_0^{\varepsilon_4^2} \int_D (d_k/a)(1 + 2\varepsilon_3) f_8\left(y, a^2 - s\right) H^0(X(T) \in dy,\ T \in ds) \\
&= (d_k/a)(1 + 2\varepsilon_3)\, a/G_D\left(av, z^0\right) \int_0^{\varepsilon_4^2} \int_D G_D\left(y, z^0\right) H^0(X(T) \in dy,\ T \in ds) \\
&= d_k(1 + 2\varepsilon_3)/G_D\left(av, z^0\right) \int_0^{\varepsilon_4^2} \int_D P^y(T_B < \infty) H^0(X(T) \in dy,\ T \in ds) \\
&= \left[d_k(1 + 2\varepsilon_3)/G_D\left(av, z^0\right)\right] H^0(T_B < \infty).
\end{aligned}$$

To obtain the second last equality we have used the fact that $G_D(y, z^0)$ and $P^y(T_B < \infty)$ are equal on $D \backslash B$, both being harmonic there with the same boundary values and vanishing at infinity if D is unbounded. Analogously,

$$H^0(A_k) \ge \left[d_k(1 - 2\varepsilon_3)/G_D\left(av, z^0\right)\right] H^0(T_B < \infty).$$

Set ε_3 to $\varepsilon/2$ to obtain the desired result. As for the d_k's, note that $d_k/a = H_*^0(A_k)$ and apply Theorem 3.1 to find their values. □

5. Local time representations. Let D be a Lipschitz domain in $\mathbb{R}^n$. Recall the definition of an excursion e_t of X in D and of the local time L of the Brownian motion X, $\{P^x, x \in \mathbb{R}^n\}$, from Section 2.

Theorem 5.1. *Suppose $\varepsilon_1 > 0$ and $h_k : \mathbb{R}^{n-1} \to \mathbb{R}$, $k = 1, 2$, are Lipschitz functions with constant $\lambda > 0$, satisfying $h_1 \le 0 \le h_2$ and*

$$\int_{\{x \in \mathbb{R}^{n-1} : |x| < 1\}} |h_k(x)|\,|x|^{-n} dx < \infty, \quad k = 1, 2. \tag{5.1}$$

Further suppose that for each $x \in \partial D$ there is a Lipschitz function $\varphi_x : \mathbb{R}^{n-1} \to \mathbb{R}$ with constant λ such that $\varphi_x(0) = 0$, and in a suitable orthonormal coordinate system $CS(x)$ where $x = 0$, we have

$$\begin{aligned}
\{y \in \partial D : |y| < \varepsilon_1\} &= \{y \in \mathbb{R}^n : |y| < \varepsilon_1, y_n = \varphi_x(\widetilde{y})\} \\
&\subset \{y \in \mathbb{R}^n : |y| < \varepsilon_1, h_1(\widetilde{y}) \le y_n \le h_2(\widetilde{y})\}.
\end{aligned}$$

Let $N_t^k(\varepsilon)$ be the number of excursions e_s of X in D such that $s \le t$ and $e_s \in A_k$. Here the A_k are the events defined in Section 4; in their definition we take $r = \varepsilon, t = \varepsilon^2$, and v to be the inward unit normal vector at $e_s(0)$, if it exists, otherwise $v = (1, 0, \dots, 0)$.

Then for each $t > 0, 1 \le k \le 7$,

$$\lim_{\varepsilon \to 0} \varepsilon \cdot N_t^k(\varepsilon)/d_k = L_t$$

where the convergence holds in P^x-probability (for each $x \in \mathbb{R}^n$). See Theorem 4.1 for formulas for the d_k.

Proof. First we prove an asymptotic comparison result (5.14) for the Green function in D. Fix some $x \in \partial D$ and use the coordinate system $CS(x)$. Let

$$\begin{aligned} D_k &= \{y \in \mathbb{R}^n : |y| < \varepsilon_1, y_n > h_k(\widetilde{y})\}, k = 1, 2, \\ z^0 &= (\widetilde{0}, \varepsilon_1/2), \\ V &= \{y \in D : y = (\widetilde{0}, b), b > 0\}. \end{aligned}$$

Theorem 4.2 of Burdzy and Williams (1986) implies in view of (5.1) that

$$\lim_{\substack{z \to 0 \\ z \in V}} G_{D_k}\left(z^0, z\right)/|z| = q_k \in (0, \infty), \qquad k = 1, 2. \tag{5.2}$$

Let $\varepsilon_2 > 0$; its value will be specified later. Find $\varepsilon_3 > 0$ so small that one has (use (5.2))

$$G_{D_1}\left(z^0, z\right) / \left(|z| q_1\right) \in (1 - \varepsilon_2, 1 + \varepsilon_2) \tag{5.3}$$

and

$$\frac{G_{D_1}\left(z^0, z\right)}{G_{D_2}\left(z^0, z\right)} \in \left(\frac{q_1}{q_2}\left(1 - \varepsilon_2\right), \frac{q_1}{q_2}\left(1 + \varepsilon_2\right)\right) \tag{5.4}$$

for $z \in V, |z| < \varepsilon_3 \varepsilon_1$.

It follows from the elliptic boundary Harnack principle (see the version presented in Theorem 2.2 of Burdzy (1987)) that there exists $\varepsilon_4 > 0$ such that if g_1 and g_2 are positive harmonic functions in

$$D_k^a \stackrel{\text{df}}{=} \{y \in \mathbb{R}^n : |y| < \varepsilon_1 a, y_n > h_k(\widetilde{y})\}, \qquad a < 1/2,$$

which vanish on $\partial D_k^a \setminus D_k$, and $y^m \in D_k^a, |y^m| < \varepsilon_1 a \varepsilon_4, m = 1, 2$, then

$$\frac{g_1\left(y^1\right)}{g_1\left(y^2\right)} \frac{g_2\left(y^2\right)}{g_2\left(y^1\right)} \in \left(1 - \varepsilon_2, 1 + \varepsilon_2\right). \tag{5.5}$$

The constant ε_4 depends only on n, λ and ε_2.

It is elementary to prove that (5.1) implies that

$$\lim_{y \to 0} h_k(y)/|y| = 0, \qquad k = 1, 2. \tag{5.6}$$

Denote $M_k^a = \partial D_k^a \cap D_k$. We can choose a sufficiently small that

$$\frac{P_{D_1^a}^{z^1}\left(T\left(M_1^a-\right)<\infty\right)}{P_{D_2^a}^{z^1}\left(T\left(M_2^a-\right)<\infty\right)} \in \left(1-\varepsilon_2, 1+\varepsilon_2\right) \tag{5.7}$$

where $z^1 \in V, \left|z^1\right| < \varepsilon_1 \min(\varepsilon_3, \varepsilon_4 a)/2$. Apply (5.5) to see that

$$\frac{P_{D_1^a}^{z^1}\left(T\left(M_1^a-\right)<\infty\right)}{P_{D_1^a}^{z}\left(T\left(M_1^a-\right)<\infty\right)} \frac{G_{D_1}\left(z^0, z\right)}{G_{D_1}\left(z^0, z^1\right)} \in \left(1-\varepsilon_2, 1+\varepsilon_2\right) \tag{5.8}$$

for $z \in D_1, |z| < \varepsilon_1 \varepsilon_4 a$ and

$$\frac{P_{D_2^a}^{z}\left(T\left(M_2^a-\right)<\infty\right)}{P_{D_2^a}^{z^1}\left(T\left(M_2^a-\right)<\infty\right)} \frac{G_{D_2}\left(z^0, z^1\right)}{G_{D_2}\left(z^0, z\right)} \in \left(1-\varepsilon_2, 1+\varepsilon_2\right)$$

for $z \in D_2, |z| < \varepsilon_1 \varepsilon_4 a$. The last two formulas imply that

$$\begin{aligned} &\frac{P_{D_2^a}^{z}\left(T\left(M_2^a-\right)<\infty\right)}{P_{D_1^a}^{z}\left(T\left(M_1^a-\right)<\infty\right)} \frac{P_{D_1^a}^{z^1}\left(T\left(M_1^a-\right)<\infty\right)}{P_{D_2^a}^{z^1}\left(T\left(M_2^a-\right)<\infty\right)} \times \\ &\qquad \times \frac{G_{D_1}\left(z^0, z\right)}{G_{D_2}\left(z^0, z\right)} \frac{G_{D_2}\left(z^0, z^1\right)}{G_{D_1}\left(z^0, z^1\right)} \in \left(\left(1-\varepsilon_2\right)^2, \left(1+\varepsilon_2\right)^2\right) \end{aligned} \tag{5.9}$$

for $z \in V, |z| < \varepsilon_1 \varepsilon_4 a$. Let $\varepsilon_5 = \min\left(\varepsilon_3, \varepsilon_4 a\right)$. Then (5.4), (5.7) and (5.9) imply

$$\frac{P_{D_2^a}^{z}\left(T\left(M_2^a-\right)<\infty\right)}{P_{D_1^a}^{z}\left(T\left(M_1^a-\right)<\infty\right)} \in \left(\frac{\left(1-\varepsilon_2\right)^3}{\left(1+\varepsilon_2\right)^2}, \frac{\left(1+\varepsilon_2\right)^3}{\left(1-\varepsilon_2\right)^2}\right) \tag{5.10}$$

for $z \in V, |z| < \varepsilon_1 \varepsilon_5$.

Let $D^a = \left\{y \in D : |y| < \varepsilon_1 a\right\}, M^a = \partial D^a \cap D$. Since $D_1^a \subset D^a \subset D_2^a$, one has

$$P_{D_2^a}^{z}\left(T\left(M_2^a-\right)<\infty\right) \leq P_{D^a}^{z}\left(T\left(M^a-\right)<\infty\right) \leq P_{D_1^a}^{z}\left(T\left(M_1^a-\right)<\infty\right).$$

It follows from this and (5.10) that

$$\frac{P_{D^a}^{z}\left(T\left(M^a-\right)<\infty\right)}{P_{D_1^a}^{z}\left(T\left(M_1^a-\right)<\infty\right)} \in \left(\frac{\left(1-\varepsilon_2\right)^3}{\left(1+\varepsilon_2\right)^2}, \frac{\left(1+\varepsilon_2\right)^3}{\left(1-\varepsilon_2\right)^2}\right) \tag{5.11}$$

for $z \in V, |z| < \varepsilon_1 \varepsilon_5$. Combine (5.3), (5.8) and (5.11) to see that

$$\frac{P_{D^a}^{z}\left(T\left(M^a-\right)<\infty\right)}{|z|} \frac{G_{D_1}\left(z^0, z^1\right)}{q_1 P_{D_1^a}^{z^1}\left(T\left(M_1^a-\right)<\infty\right)} \in \left(\frac{\left(1-\varepsilon_2\right)^4}{\left(1+\varepsilon_2\right)^3}, \frac{\left(1+\varepsilon_2\right)^4}{\left(1-\varepsilon_2\right)^3}\right) \tag{5.12}$$

for $z \in V, |z| < \varepsilon_1 \varepsilon_5$. By (5.5), for any $z^2 \notin D^a$,

$$\frac{G_D\left(z^2, z\right)}{G_D\left(z^2, z^1\right)} \frac{P_{D^a}^{z^1}\left(T\left(M^a-\right)<\infty\right)}{P_{D^a}^{z}\left(T\left(M^a-\right)<\infty\right)} \in \left(1-\varepsilon_2, 1+\varepsilon_2\right),$$

which combined with (5.12) yields

$$\frac{G_D\left(z^2,z\right)}{|z|}\frac{\left|z^1\right|}{G_D\left(z^2,z^1\right)}\in\left(\frac{\left(1-\varepsilon_2\right)^9}{\left(1+\varepsilon_2\right)^8},\frac{\left(1+\varepsilon_2\right)^9}{\left(1-\varepsilon_2\right)^8}\right) \tag{5.13}$$

for $z \in V,\ |z| < \varepsilon_1\varepsilon_5$.

Now, given any $\varepsilon_6 > 0$, by choosing ε_2 sufficiently small, it follows from the above that there exists $\varepsilon_7 > 0$ which depends on n, λ and ε_6 (but not x) such that for any $z^2 \in D,\ \left|z^2 - x\right| > \varepsilon_1/2, z^3 \in V, \left|z^3\right| \le \varepsilon_7$,

$$\frac{\left|z^3\right|}{G_D\left(z^3,z^2\right)}\lim_{\substack{z\to 0\\ z\in V}}\frac{G_D\left(z^2,z\right)}{|z|}\in\left(1-\varepsilon_6,1+\varepsilon_6\right). \tag{5.14}$$

The limit above exists according to Theorem 4.2 of Burdzy and Williams (1986) whose assumptions are satisfied due to (5.1).

We now use (5.14) to prove the local time representation result. Recall the definition of an exit system (dL, H) in D, from Section 2. The continuous additive functional L has the surface area measure on ∂D as its Revuz measure. Fix some $z^2 \in D$ with $\operatorname{dist}\left(z^2, \partial D\right) > \varepsilon_1/2$ so that the assumptions of (5.14) are satisfied for every $x \in \partial D$. Let

$$B = \left\{y \in D : G_D\left(z^2, y\right) \ge 1\right\}.$$

Then $G_D\left(z^2,y\right) = P_D^y\left(T_B < \infty\right)$ for $y \in D\backslash B$. It follows from the comments following (5.14) that for each $x \in \partial D$

$$\lim_{\substack{z\to 0\\ z\in V}} P_D^z\left(T_B<\infty\right)/|z|$$

exists (the formula is expressed in $CS(x)$). The excursion laws are normalized so that

$$H^x\left(T_B<\infty\right) = \lim_{\substack{z\to 0\\ z\in V}} P_D^z\left(T_B<\infty\right)/\left|z\right| \text{ in } CS(x).$$

In view of (5.6), Theorem 4.1 and (5.14) imply that for each $\varepsilon_8 > 0$ one may choose $\varepsilon_9 > 0$ so that for $1 \le k \le 7$

$$H^x\left(A_k\right)\in\left(d_k\left(1-\varepsilon_8\right)/\varepsilon, d_k\left(1+\varepsilon_8\right)/\varepsilon\right) \tag{5.15}$$

if $r = \varepsilon \le \varepsilon_9$ and $t = \varepsilon^2$ in the definition of A_k.

Denote $\sigma(s) = \inf\{t > 0 : L_t > s\}$. Theorem T4 from Chapter II of Brémaud (1981) and the exit system formula (2.1) imply that for $\varepsilon < \varepsilon_9$ the process $s \to N^k_{\sigma(s)}(\varepsilon)$ is Poisson with a random intensity which by (5.15) is bounded below by $d_k\left(1-\varepsilon_8\right)/\varepsilon$ and above by $d_k\left(1+\varepsilon_8\right)/\varepsilon$. When $\varepsilon \to 0$, one may let ε_8 go to 0 as well and for a fixed s, $\varepsilon \cdot N^k_{\sigma(s)}(\varepsilon)/d_k$ converges in probability to s; this may be easily deduced for example from formula (1.9) of Chapter II of Brémaud (1981). It is now elementary to see that $\varepsilon \cdot N^k_t(\varepsilon)/d_k$ converges in probability to L_t, for a fixed t. □

Remark 5.1. The above representation theorem works, for example, for $C^{1,\alpha}$ domains with $\alpha > 0$, i.e., for domains which have boundaries represented locally by functions whose first partial derivatives are α-Hölder continuous.

6. A parabolic boundary Harnack principle. The following result is a stronger version of Lemma 2.1 of Burdzy (1987).

In the sequel, inequalities involving zero divisors are to be interpreted as those obtained by multiplication by the divisors.

Lemma 6.1. *Suppose that $b, c, d \in (0,1)$, and f_1, f_2, g_x, g_y are real-valued, nonnegative measurable functions defined on a set $W = U \cup V$ where U and V are disjoint measurable sets. Let ν be an arbitrary positive measure on W. Assume that*

$$\frac{f_k(v)}{f_{3-k}(v)} \geq c \frac{f_k(w)}{f_{3-k}(w)} \quad \textit{for all } v, w \in W,\ k = 1, 2, \tag{6.1}$$

and

$$\frac{g_x(v)}{g_y(v)} \geq d\, \frac{g_x(w)}{g_y(w)} \quad \textit{for all } v, w \in V. \tag{6.2}$$

Let

$$h_k(z) = \int_W f_k g_z d\nu \overset{\mathrm{df}}{=} \int_W f_k(v) g_z(v) d\nu(v)$$

and

$$\widetilde{h}_k(z) = \int_V f_k g_z d\nu \overset{\mathrm{df}}{=} \int_V f_k(v) g_z(v) d\nu(v)$$

for $k = 1, 2$ and $z = x, y$. Suppose that

$$\infty > \widetilde{h}_k(z) \geq b h_k(z) \tag{6.3}$$

for $k = 1, 2$ and $z = x, y$. Then

$$\frac{h_2(x)}{h_1(x)} \geq \frac{h_2(y)}{h_1(y)} \left[c + b^2 d^2 (1 - c) \right].$$

Proof. Choose $v_0 \in V$ so that

$$g_x(v_0) \,/\, g_y(v_0) \overset{\mathrm{df}}{=} q \in (0, \infty).$$

It is easy to see that if such a v_0 does not exist then the lemma trivially holds. By (6.2), $g_x(v) \geq dq\ g_y(v)$ for all $v \in V$. It follows that $\widetilde{g}(v) \geq 0$ for all $v \in V$, where

$$\widetilde{g}(v) \overset{\mathrm{df}}{=} g_x(v) - dq\, g_y(v).$$

By (6.2), $g_y(v) \geq g_x(v) d/q$. Apply this inequality to see that

$$\widetilde{h}_1(y) = \int_V f_1 g_y d\nu \geq \int_V f_1 g_x (d/q) d\nu = \widetilde{h}_1(x) d/q. \tag{6.4}$$

By (6.1),

$$f_2(v) f_1(w) \geq c f_1(v) f_2(w) \quad \text{for all } v, w \in W. \tag{6.5}$$

Hence,

$$\left(\int_V f_2\widetilde{g}d\nu\right)\left(\int_V f_1 g_y d\nu\right) \geq c\left(\int_V f_1\widetilde{g}d\nu\right)\left(\int_V f_2 g_y d\nu\right),$$

or equivalently

$$\widetilde{h}_1(y)\int_V f_2\widetilde{g}d\nu \geq c\widetilde{h}_2(y)\int_V f_1\widetilde{g}d\nu. \tag{6.6}$$

In an analogous way, we obtain from (6.5) the following inequalities.

$$\widetilde{h}_1(y)\int_U f_2 g_x d\nu \geq c\widetilde{h}_2(y)\int_U f_1 g_x d\nu \tag{6.7}$$

and

$$h_2(x)\int_U f_1 g_y d\nu \geq c h_1(x)\int_U f_2 g_y d\nu. \tag{6.8}$$

By the definition of $\widetilde{g}$, (6.6) and (6.4),

$$\begin{aligned}
\widetilde{h}_2(x)\widetilde{h}_1(y) &= \left(\int_V f_2\widetilde{g}d\nu + dq\int_V f_2 g_y d\nu\right)\widetilde{h}_1(y)\\
&= \left(\int_V f_2\widetilde{g}d\nu + dq\widetilde{h}_2(y)\right)\widetilde{h}_1(y)\\
&\geq \left(c\int_V f_1\widetilde{g}d\nu + dq\widetilde{h}_1(y)\right)\widetilde{h}_2(y)\\
&= \left(c\left(\int_V f_1 g_x d\nu - dq\int_V f_1 g_y d\nu\right) + dq\widetilde{h}_1(y)\right)\widetilde{h}_2(y)\\
&= (c\widetilde{h}_1(x) + dq(1-c)\widetilde{h}_1(y))\widetilde{h}_2(y)\\
&\geq (c\widetilde{h}_1(x) + d^2(1-c)\widetilde{h}_1(x))\widetilde{h}_2(y)\\
&= (c + d^2(1-c))\widetilde{h}_1(x)\widetilde{h}_2(y).
\end{aligned} \tag{6.9}$$

By (6.7), (6.9) and (6.3),

$$\begin{aligned}
h_2(x)\widetilde{h}_1(y) &= \left(\int_U f_2 g_x d\nu + \int_V f_2 g_x d\nu\right)\widetilde{h}_1(y)\\
&= \left(\int_U f_2 g_x d\nu + \widetilde{h}_2(x)\right)\widetilde{h}_1(y)\\
&\geq \left(c\int_U f_1 g_x d\nu + (c + d^2(1-c))\widetilde{h}_1(x)\right)\widetilde{h}_2(y)\\
&= (ch_1(x) + d^2(1-c)\widetilde{h}_1(x))\widetilde{h}_2(y)\\
&\geq (ch_1(x) + d^2(1-c)bh_1(x))\widetilde{h}_2(y)\\
&= (c + bd^2(1-c))h_1(x)\widetilde{h}_2(y).
\end{aligned}$$

Then, by the last inequality, (6.8) and (6.3), we obtain

$$\begin{aligned} h_2(x)h_1(y) &= \left(\int_U f_1 g_y d\nu + \int_V f_1 g_y d\nu\right) h_2(x) \\ &= \left(\int_U f_1 g_y d\nu + \widetilde{h}_1(y)\right) h_2(x) \\ &\geq \left(c\int_U f_2 g_y d\nu + (c + bd^2(1-c))\widetilde{h}_2(y)\right) h_1(x) \\ &= (ch_2(y) + bd^2(1-c)\widetilde{h}_2(y))h_1(x) \\ &\geq (ch_2(y) + b^2d^2(1-c)h_2(y))h_1(x) \\ &= (c + b^2d^2(1-c))h_2(y)h_1(x). \quad \square \end{aligned}$$

Suppose that D is a Lipschitz domain and moreover $D = \{x \in \mathbb{R}^n : x_n > \varphi(\widetilde{x})\}$ where $|\varphi(\widetilde{x}) - \varphi(\widetilde{y})| \leq \lambda|\widetilde{x} - \widetilde{y}|$ and $\varphi(\widetilde{0}) = 0$.

Recall the definitions of $\Psi, \underline{A}, \overline{A}$ and Δ from Section 2.

Theorem 6.1. *There exists a function* $c = c(a, b, n, \lambda, \varepsilon)$, $a, b, \lambda, \varepsilon > 0, n \geq 2$, *with the following properties.*

(i) $c \in (0, 1)$, *c is decreasing in ε and increasing in a and b.*

(ii) For fixed a, b, n and λ,

$$\lim_{\varepsilon \downarrow 0} c(a, b, n, \lambda, \varepsilon) = 1.$$

(iii) Let $s > 0$ and $0 < r < \sqrt{s}$ and suppose that $f_1(x, t)$ and $f_2(x, t)$ are positive and parabolic in $\Psi_r(0, s)$ and they vanish continuously on $\Delta_r(0, s)$. Then, for $(x, t), (y, u) \in \Psi_\varepsilon(0, s)$, $\varepsilon < r/16, k = 1, 2$, we have

$$\frac{f_k(x,t)}{f_{3-k}(x,t)} \geq \frac{f_k(y,u)}{f_{3-k}(y,u)} \cdot c$$

where

$$c = c\left(\frac{f_1\left(\underline{A}_{r/2}(0,s)\right)}{f_1\left(\overline{A}_{r/2}(0,s)\right)}, \frac{f_2\left(\underline{A}_{r/2}(0,s)\right)}{f_2\left(\overline{A}_{r/2}(0,s)\right)}, n, \lambda, \varepsilon\right).$$

Proof. We will suppress $(0, s)$ in the notation i.e., $\Psi_\rho = \Psi_\rho(0, s), \underline{A}_\rho = \underline{A}_\rho(0, s)$ etc. We first establish some inequalities so that we can apply Lemma 6.1 and then we use induction to obtain the theorem.

By Theorem 1.6 (see also inequality (1.28)) of Fabes et al. (1986) we have for $k = 1, 2$, and (x, t), $(y, u) \in \Psi_{r/16}$,

$$\frac{f_k(x,t)}{f_{3-k}(x,t)} \geq \frac{f_k(y,u)}{f_{3-k}(y,u)} \frac{f_1\left(\underline{A}_{r/2}\right)}{f_1\left(\overline{A}_{r/2}\right)} \frac{f_2\left(\underline{A}_{r/2}\right)}{f_2\left(\overline{A}_{r/2}\right)} c_1^2 \tag{6.10}$$

where $c_1 = c_1(\lambda, n) > 0$. Now let

$$c_2 = \frac{f_1\left(\underline{A}_{r/2}\right)}{f_1\left(\overline{A}_{r/2}\right)} \frac{f_2\left(\underline{A}_{r/2}\right)}{f_2\left(\overline{A}_{r/2}\right)} c_1^2.$$

Fix some $\rho < r/16$ and assume that there is a constant $c_3 > 0$ such that

$$\frac{f_k(x,t)}{f_{3-k}(x,t)} \geq \frac{f_k(y,u)}{f_{3-k}(y,u)}\, c_3 \tag{6.11}$$

for $k = 1,2,\ (x,t),(y,u) \in \Psi_\rho$. Let $\mu_{(x,t)}$ denote the caloric measure on $\partial\Psi_\rho$ for $(x,t) \in \overline{\Psi_\rho}$ and $\underline{\Delta}_\rho \overset{\text{df}}{=} \left\{(x,t) \in \partial\Psi_\rho : t = s - \rho^2\right\}$. Recall that $\mu_{(x,t)}(\,\cdot\,)$ does not charge $\{(y,u) \in \partial\Psi_\rho : u = s + \rho^2\}$. For a Borel measurable set $B \subset \underline{\Delta}_\rho$, the function $(x,t) \to \mu_{(x,t)}(B)$ is parabolic in Ψ_ρ and vanishes continuously on

$$\left\{(x,t) \in \partial\Psi_\rho : t \in \left(s - \rho^2, s + \rho^2\right)\right\}.$$

It follows from Corollary 2.2 of Fabes et al. (1986) that

$$\mu_{\underline{A}_{\rho/8}}(B) \geq c_4 \mu_{\overline{A}_{\rho/8}}(B). \tag{6.12}$$

The constant c_4 depends only on n and λ although in the paper of Fabes et al. (1986) it depends on the diameter of Ψ_ρ as well. The last dependence may be removed by scaling.

Let B, C be Borel measurable sets in $\underline{\Delta}_\rho$. If $\mu_\cdot(B) > 0$ and $\mu_\cdot(C) > 0$ on Ψ_ρ, then by Theorem 1.6 of Fabes et al. (1986) and (6.12) we have

$$\begin{aligned}
\frac{\mu_{(x^1,t^1)}(B)}{\mu_{(x^1,t^1)}(C)} &\geq c_1 \frac{\mu_{\underline{A}_{\rho/8}}(B)}{\mu_{\overline{A}_{\rho/8}}(C)} \\
&\geq c_1 c_4^2 \frac{\mu_{\overline{A}_{\rho/8}}(B)}{\mu_{\underline{A}_{\rho/8}}(C)} \\
&\geq c_1^2 c_4^2 \frac{\mu_{(x^2,t^2)}(B)}{\mu_{(x^2,t^2)}(C)}
\end{aligned} \tag{6.13}$$

for $\left(x^1,t^1\right), \left(x^2,t^2\right) \in \Psi_{\rho/64}$. By the forward and backward Harnack principles (see Theorem 0.2 and Theorem 2.1 of Fabes et al. (1986)), if $\mu_{(x,t)}(B) = 0$ for some $(x,t) \in \Psi_\rho$, then $\mu_\cdot(B) \equiv 0$ in Ψ_ρ, and similarly for $\mu_\cdot(C)$. Thus it follows by our convention for zero divisors that (6.13) holds for all Borel measurable sets $B, C \subset \underline{\Delta}_\rho$. Fix some $\left(x^0,t^0\right) \in \partial\Psi_\rho$ such that $t^0 = s + \rho^2, \left|x^0\right| < \rho, x^0 \in D$. Then for each $(x,t) \in \Psi_\rho$, the caloric measure $\mu_{(x,t)}$ is absolutely continuous with respect to $\mu_{(x^0,t^0)}$ (Fabes et al. (1986), page 540). Let $g_{(x,t)}$ denote the Radon-Nikodym derivative $d\mu_{(x,t)}/\,d\mu_{(x^0,t^0)}$ on $\partial\Psi_\rho$. Then (6.13) implies that

$$\frac{g_{(x^1,t^1)}\left(y^1,u^1\right)}{g_{(x^1,t^1)}\left(y^2,u^2\right)} \geq c_1^2 c_4^2 \frac{g_{(x^2,t^2)}\left(y^1,u^1\right)}{g_{(x^2,t^2)}\left(y^2,u^2\right)} \tag{6.14}$$

for $\left(x^k,t^k\right) \in \Psi_{\rho/64}, \left(y^k,u^k\right) \in \underline{\Delta}_\rho, k = 1,2$. As above, we assume our convention about zero divisors here. Although strictly speaking (6.14) only holds for $\mu_{(x^0,t^0)}$-a.e. $(y^k,u^k) \in \underline{\Delta}_\rho$, by changing $g_{(x^1,t^1)}$ and $g_{(x^2,t^2)}$ on a set of $\mu_{(x^0,t^0)}$-measure zero (possibly depending on $(x^1,t^1),\ (x^2,t^2)$), we can make (6.14) hold for all $(y^k,u^k),\ k = 1,2$, as indicated.

Fix a point $y^0 \in D$ with $|y^0| > r$ and let $G_D(\cdot, \cdot)$ be the Green function in D. Then $G(x,t) \overset{\mathrm{df}}{=} G_D\left(y^0, x\right)$ is parabolic in Ψ_r and vanishes on Δ_r. Choose $c_5 = c_5(n,\lambda) > 0$ so small that the ball B_1 with center $(\widetilde{0}, \rho/2)$ and radius $2\rho c_5$ is contained in D. Let B_2 be the concentric ball with half the radius of B_1. By the elliptic Harnack principle, $G(x,t) \geq c_6 G(y,u)$ for $x, y \in B_2,\ t, u > 0$ and $c_6 = c_6(n)$.

Apply Theorem 1.6 of Fabes et al. (1986) to see that

$$\frac{f_k(x^1,t^1)}{G(x^1,t^1)} \geq c_1 \frac{f_k(\underline{A}_{r/2})}{G(\overline{A}_{r/2})},$$

$$\frac{G(x^2,t^2)}{f_k(x^2,t^2)} \geq c_1 \frac{G(\underline{A}_{r/2})}{f_k(\overline{A}_{r/2})},$$

and, therefore,

$$\begin{aligned} \frac{f_k(x^1,t^1)}{f_k(x^2,t^2)} &\geq c_1^2 \frac{f_k(\underline{A}_{r/2})}{f_k(\overline{A}_{r/2})} \frac{G(\underline{A}_{r/2})}{G(\overline{A}_{r/2})} \frac{G(x^1,t^1)}{G(x^2,t^2)} \\ &\geq c_1^2 c_6^2 \frac{f_k(\underline{A}_{r/2})}{f_k(\overline{A}_{r/2})} \end{aligned} \tag{6.15}$$

for $\left(x^1,t^1\right), \left(x^2,t^2\right) \in \Psi_{r/16}, x^1, x^2 \in B_2$.

For $k = 1, 2, (x,t) \in \Psi_\rho$, let

$$\widetilde{f}_k(x,t) = \int_{\underline{\Delta}_\rho} f_k(y,u) d\mu_{(x,t)}(y,u). \tag{6.16}$$

The function $\widetilde{f}_k$ is parabolic in Ψ_ρ and vanishes continuously on Δ_ρ, since f_k vanishes on $\Delta_\rho \subset \Delta_r$. Let $\underline{B}_2 = \left\{(x,t) \in \underline{\Delta}_\rho : x \in B_2\right\}$. It is easy to see that

$$\mu_{\underline{A}_{\rho/2}}\left(\underline{B}_2\right) \geq c_7 = c_7(n,\lambda) > 0. \tag{6.17}$$

By (6.15),

$$f_k(y,u) \geq f_k(\overline{A}_{\rho/2}) c_1^2 c_6^2 \frac{f_k(\underline{A}_{r/2})}{f_k(\overline{A}_{r/2})} \tag{6.18}$$

for $(y,u) \in \underline{B}_2, k = 1, 2$. Combine (6.16), (6.17) and (6.18) to see that

$$\widetilde{f}_k(\underline{A}_{\rho/2}) \geq c_7 f_k(\overline{A}_{\rho/2}) c_1^2 c_6^2 \frac{f_k(\underline{A}_{r/2})}{f_k(\overline{A}_{r/2})}.$$

Theorem 1.6 of Fabes et al. (1986) implies that

$$\begin{aligned} \frac{\widetilde{f}_k(x,t)}{f_k(x,t)} &\geq c_1 \frac{\widetilde{f}_k(\underline{A}_{\rho/2})}{f_k(\overline{A}_{\rho/2})} \\ &\geq c_1^3 c_6^2 c_7 \frac{f_k(\underline{A}_{r/2})}{f_k(\overline{A}_{r/2})} \end{aligned} \tag{6.19}$$

for $(x,t) \in \Psi_{\rho/16}, k = 1,2$.

Now Lemma 6.1 will be applied with $f_1(v), f_2(v), g_x(v), g_y(v), W$ and V replaced by $f_1(z,v), f_2(z,v), g_{(x,t)}(z,v), g_{(y,u)}(z,v), \partial\Psi_\rho$ and $\underline{\Delta}_\rho$. Note that

$$f_k(x,t) = \int_{\partial\Psi_\rho} f_k(z,v) d\mu_{(x,t)}(z,v) = \int_{\partial\Psi_\rho} f_k(z,v) g_{(x,t)}(z,v) d\mu_{(x_0,t_0)}(z,v),$$

for $k = 1,2, (x,t) \in \Psi_\rho$. Let

$$c_8 = c_1^3 c_6^2 c_7 \min\left(\frac{f_1(\underline{A}_{r/2})}{f_1(\overline{A}_{r/2})}, \frac{f_2(\underline{A}_{r/2})}{f_2(\overline{A}_{r/2})}\right).$$

Observe that (6.11) extends to $(x,t), (y,u) \in \partial\Psi_\rho$ by the continuity of $f_k, k = 1,2$, and our convention about zero divisors. With this, (6.14) and (6.19), the hypotheses of Lemma 6.1 are verified and so

$$\frac{f_k(x,t)}{f_{3-k}(x,t)} \geq \frac{f_k(y,u)}{f_{3-k}(y,u)} c_9$$

for $(x,t), (y,u) \in \Psi_{\rho/64}, k = 1,2$ and $c_9 = c_3 + (1 - c_3)\, c_1^4 c_4^4 c_8^2$.

It then follows by induction from the above that

$$\frac{f_k(x,t)}{f_{3-k}(x,t)} \geq \frac{f_k(y,u)}{f_{3-k}(y,u)} c_{10}$$

for $k = 1,2, (x,t), (y,u) \in \Psi_{r \cdot 2^{-6m}}$ and

$$\begin{aligned} &c_{10} = c_{10}(m), \\ &c_{10}(1) = c_2, \\ &c_{10}(m+1) = c_{10}(m) + (1 - c_{10}(m))\, c_1^4 c_4^4 c_8^2, \quad m \geq 1. \end{aligned}$$

Note that $c_1, c_4, c_8 \in (0,1)$, so $c_{10}(m)$ is increasing as $m \to \infty$ and, moreover, $c_{10}(m) \to 1$. It is easy to check that c_{10} depends only on $m, n, \lambda, f_1(\underline{A}_{r/2})/\, f_1(\overline{A}_{r/2})$ and
$f_2(\underline{A}_{r/2})/\, f_2(\overline{A}_{r/2})$ and it is an increasing function of $f_1(\underline{A}_{r/2})/\, f_1(\overline{A}_{r/2})$ and
$f_2(\underline{A}_{r/2})/\, f_2(\overline{A}_{r/2})$. □

Suppose that for $k = 1,2, \varphi^k$ is a Lipschitz function with constant $\lambda, \varphi^k : \mathbb{R}^{n-1} \to \mathbb{R}, \varphi^k(0) = 0, D^k = \{x \in \mathbb{R}^n : x_n > \varphi^k(\widetilde{x})\}$ and let $\Psi_r^k(x,t), \Delta_r^k(x,t)$ etc. be defined relative to D^k.

Corollary 6.1. *There exists $c = c(a_1, a_2, a_3, a_4, n, \lambda, \varepsilon)$ with the following properties.*

(i) $c \in (0,1)$, c is increasing in ε and decreasing in a_1, a_2, a_3 and a_4.

(ii) For fixed a_1, a_2, a_3, a_4, n and λ,

$$\lim_{\varepsilon \downarrow 0} c(a_1, a_2, a_3, a_4, n, \lambda, \varepsilon) = 0.$$

(iii) Suppose that for $k = 1, 2$, *functions* f_k *and* g_k *are positive and parabolic in* $\Psi_r^k(0, s)$ *and vanish continuously on* $\Delta_r^k(0, s)$, *where* $s > 0,\ 0 < r < \sqrt{s}$. *Assume that*

$$\frac{f_1\left(x^1, t^1\right)}{f_2\left(x^1, t^1\right)} \frac{g_2\left(x^1, t^1\right)}{g_1\left(x^1, t^1\right)} \in (1 - \varepsilon, 1 + \varepsilon) \tag{6.20}$$

for some $\left(x^1, t^1\right) \in \Psi_\varepsilon^1(0, s) \cap \Psi_\varepsilon^2(0, s), \varepsilon < r/16$. *Then*

$$\lim_{\substack{(x,t)\to(0,s)\\(x,t)\in\Psi_r^1(0,s)}} \frac{f_1(x, t)}{g_1(x, t)} \cdot \lim_{\substack{(y,u)\to(0,s)\\(y,u)\in\Psi_r^2(0,s)}} \frac{g_2(y, u)}{f_2(y, u)} \in (1 - c, 1 + c) \tag{6.21}$$

where

$$c = c\left(\frac{f_1\left(\underline{A}_{r/2}(0, s)\right)}{f_1\left(\overline{A}_{r/2}(0, s)\right)}, \frac{f_2\left(\underline{A}_{r/2}(0, s)\right)}{f_2\left(\overline{A}_{r/2}(0, s)\right)}, \frac{g_1\left(\underline{A}_{r/2}(0, s)\right)}{g_1\left(\overline{A}_{r/2}(0, s)\right)}, \frac{g_2\left(\underline{A}_{r/2}(0, s)\right)}{g_2\left(\overline{A}_{r/2}(0, s)\right)}, n, \lambda, \varepsilon\right).$$

In particular, the limits in (6.21) exist.

Proof. Let c_1 denote the constant c in Theorem 6.1 (iii) with f_1, g_1 in place of f_1, f_2 there. Then,

$$\frac{g_1\left(x^1, t^1\right)}{f_1\left(x^1, t^1\right)} \frac{f_1(x, t)}{g_1(x, t)} \in \left(c_1, c_1^{-1}\right) \tag{6.22}$$

for $(x, t) \in \Psi_\varepsilon^1(0, s)$. Similarly, let c_2 denote the constant c obtained in Theorem 6.1 (iii) with f_2, g_2 in place of f_1, g_1. Then

$$\frac{f_2\left(x^1, t^1\right)}{g_2\left(x^1, t^1\right)} \frac{g_2(y, u)}{f_2(y, u)} \in \left(c_2, c_2^{-1}\right), \tag{6.23}$$

for $(y, u) \in \Psi_\varepsilon^2(0, s)$. By multiplying (6.22) and (6.23) and using (6.20) we obtain

$$\frac{f_1(x, t)}{g_1(x, t)} \frac{g_2(y, u)}{f_2(y, u)} \in \left(c_1 c_2 (1 - \varepsilon), c_1^{-1} c_2^{-1} (1 + \varepsilon)\right) \tag{6.24}$$

for $(x, t) \in \Psi_\varepsilon^1(0, s)$ and $(y, u) \in \Psi_\varepsilon^2(0, s)$. The existence of the limits in (6.21) follows immediately from Theorem 6.1. The existence of a c such that (i)-(iii) hold then follows from (6.24) and the properties of c_1, c_2. □

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