Cornered Asymptotically Hyperbolic Metrics

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Abstract

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This thesis considers asymptotically hyperbolic manifolds that have a finite boundary in addition to the usual infinite boundary – *cornered asymptotically hyperbolic manifolds*. A theorem of Cartan-Hadamard type near infinity for the normal exponential map on the finite boundary is proved, and this is used to construct a corner normal form, analogous to the usual asymptotically hyperbolic normal form, and suitable for studying questions near the corner. Formal expansion at the corner of a boundary value problem for the scalar Laplacian is then studied in the special case that the finite boundary makes constant angle $\frac{\pi}{2}$ with the infinite boundary, and a formal existence and uniqueness result is proved. The thesis then takes up the study of cornered asymptotically hyperbolic Einstein metrics, with a constant mean curvature umbilic boundary condition imposed at the finite boundary. First, recent work of Nozaki, Takayanagi, and Ugajin is generalized and extended showing that such metrics cannot have smooth compactifications for generic corners embedded in the infinite boundary. Then unique formal existence at the corner, up to order equal to the boundary dimension, of Einstein metrics in normal form and polyhomogeneous in polar coordinates is demonstrated for arbitrary smooth conformal infinity. Finally it is shown that, in the special case that the finite boundary is taken to be totally geodesic, there is an obstruction to existence beyond this dimension, which defines a conformal hypersurface invariant.

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Soli Deo gloria

DEDICATION

To Ivor and Kay McKeown, with love and gratitude.

Il est de l'or, et beaucoup de perles; mais les lèvres de science sont précieuses bagues.

Chapter 1

INTRODUCTION

Over the past thirty years, the analysis and geometry of asymptotically hyperbolic (AH) manifolds have been highly active areas of research. Two topics that have received much attention are the behavior and properties of AH Einstein metrics, and the scalar Laplacian. The field of AH Einstein metrics in its modern form was opened by [FG85], which established formal existence of AH Einstein metrics given a conformal infinity, with a view to studying conformal invariants. This was substantially expanded in [FG12], and the technique has generated myriad papers on conformal invariants. Global existence of AH Einstein metrics given a specified conformal infinity also quickly became a focus. Notable papers on the subject include [GL91], [Biq00], [Lee06], and a set of papers on four-manifolds by Michael Anderson described, for example, in [And08a]. Attention was also given to the boundary regularity of AH Einstein metrics. A result for $n = 4$ was given in [And03], and a global regularity result yielding an asymptotic expansion at the boundary in all dimensions was proved in [CDLS05]. Local regularity, again in the form of a specific asymptotic expansion at the boundary, was proved for odd boundary dimension in [Hel08], and for all dimensions in [BH14]. Concurrently, the properties of scalar elliptic operators on conformally compact spaces have been studied since the seminal paper [MM87], which described the resolvent of the Laplacian. An enormous number of papers have followed, such as [GZ03], which analyzed the Dirichlet problem at infinity.

We begin a study of two of these problems in the case when the infinite boundary itself has a boundary, so that there is a corner at infinity. Specifically we analyze formal Einstein metrics and formal solutions to Laplace's equation. There are many similarities between these problems and both the analogous problems on AH manifolds, on the one hand, and previous analysis of cornered spaces, on the other. The combination of both features, however, has received scant attention.

An AH manifold X^{n+1} always has a boundary at infinity, M^n. We define a cornered space as a manifold X^{n+1} with two boundary components M and Q that meet in a codimension-two corner $S = Q \cap M \neq \emptyset$, and a cornered AH (CAH) space as a cornered space equipped on the interior with a metric g_+ such that g_+ is smooth and nondegenerate at $Q \setminus S$ but asymptotically hyperbolic at M. Like a usual AH space, a CAH space has a conformal infinity $[h]$ on M. It will often be convenient to let $[h]$ be a conformal class of AH metrics on (M, S), though at times we will consider smooth classes; there is a canonical one-to-one correspondence between smooth and AH conformal classes, as discussed on page 19, so the distinction is largely one of convenience. In both of the problems we study, we will need to prescribe boundary conditions on Q in addition to the usual asymptotic boundary conditions on M. Throughout the paper, we will take $n \geq 2$.

The paradigm example of a CAH space is a portion of hyperbolic space bounded by a hyperplane. Let $\mathbb{H}^{n+1} = \left\{x^0 > 0\right\}$ be the upper half-space model of hyperbolic space, with g_+ the hyperbolic metric

$$g_+ = \frac{(dx^0)^2 + \cdots + (dx^n)^2}{(x^0)^2}. \tag{1.1}$$

Let $\alpha \in \mathbb{R}$, and $X = \{(x^0, \ldots, x^n) \in \overline{\mathbb{H}^{n+1}} : x^n \geq \alpha x^0\}$, with $Q = \left\{x^n = \alpha x^0\right\}$ and $M = \left\{x^0 = 0 \text{ and } x^n \geq 0\right\}$. The conformal infinity $[h]$ is that of the Euclidean metric on M. We will return to this example below.

The question of regularity of a cornered AH Einstein metric has received some attention in at least two prior papers. In [BH14], the authors proved local boundary regularity of an AH Einstein metric. The authors carried out analysis of an AH Einstein metric on a half-ball, where the flat face is the infinite boundary and the curved face a finite one. Their space therefore did feature a corner that had to be dealt with, but because their interest was primarily in regularity at points away from the corner, they introduced a family of function spaces with a two-parameter family of weights designed to avoid having to analyze detailed behavior at the corner.

The paper [NTU12] reflects an interest in this setting from physicists who wish to study the AdS/CFT correspondence when the conformal field theory is on a space with boundary; see the references given there. In the first part of that paper, the authors considered a conformally compact manifold X whose infinite boundary, $(M^n, [h])$, was a piece of $\mathbb{R}^2$ or $\mathbb{R}^3$ with smooth boundary S and endowed with the conformal class of the flat metric. They then added a finite boundary Q of X, intersecting M precisely at S, imposing the boundary condition that Q was umbilic with constant mean curvature (CMC umbilic) with respect to the hyperbolic metric g on the upper half-space. They proceeded to study constraints on S by performing a first-order expansion of the umbilic condition on Q, and concluded that, if $n = 3$ (or, they posited, larger), S must be a sphere or a plane. As we review in Section 6.1, this result actually follows for all $n \geq 2$ from the classical theorem characterizing the umbilic surfaces in hyperbolic space. This theorem, in particular, implies that Q is part of a sphere or a plane, and it follows that $S = Q \cap \mathbb{R}^n$ must itself be a sphere or a plane.

The authors of [NTU12] pointed out that this condition on S may be regarded as a consequence of the smoothness of the compactified metric $\bar{g} = \varphi^2 g_+$ up to the corner, where φ is any defining function for M in X. We consider such consequences in a general context in Section 6.1. Let $(M, [h])$ be a smooth compact manifold with boundary S, equipped with a conformal class. Suppose that (X, g_+) is a cornered AH Einstein space having $(M, [h])$ as its infinite boundary, with finite boundary Q satisfying the umbilic boundary condition $K_Q = \lambda g_+|_{TQ}$ with a constant coefficient λ, where K_Q is the second fundamental form of Q; and such that the compactified metric $\bar{g}$ is smooth up to the corner S. We ask what conditions on g_+ and Q may be derived at the corner from these conditions. We find that the condition that $\bar{g}$ be smooth imposes constraints on the geometry of S itself as a submanifold of M, just as we have seen in $\mathbb{R}^n$. For example, S must always be umbilic in M, which in the case that M is Euclidean space of dimension $n \geq 3$ implies the result mentioned in [NTU12] that S is a sphere or hyperplane. We also perform a higher-order analysis, which yields by the same method the same result in the case $n = 2$, and adds new conditions in higher dimensions. See Corollary 6.3 and Proposition 6.5 and the surrounding discussion.

Having found that the assumption that $\bar{g}$ is smooth puts onerous conditions on the geometry of g_+ and S, we next remove the condition of smoothness up to the corner and consider in general the regularity and generic formal radial asymptotic expansion at the corner of a CMC-umbilically cornered AH Einstein metric. This was considered to first order in the distance to S in the second half of [NTU12], where the authors allow the interior metric to be a specified type of infinitesimal perturbation of the four-dimensional hyperbolic metric. They found that infinitesimal perturbations can arise that have an expansion involving first powers in the distance. We prove a formal existence and uniqueness result for the Einstein equation, up to order n in a radial coordinate, for arbitrary conformal infinity. We follow [NTU12] in imposing a CMC-umbilic boundary condition at Q; this is one of the most natural boundary conditions, and requires perhaps the least data – only a single constant λ, as opposed to (say) a tensor field on Q if we wished to use a Dirichlet condition. As we will show in Section 6.1, this boundary condition ensures that Q and M make constant angle $\theta_0 \in (0, \pi)$ with respect to a compactified metric $\bar{g}$, and that $\lambda = -\cos(\theta_0)$; thus, we must have $\lambda \in (-1, 1)$.

In order to study expansions involving radial (and also angular) coordinates, we follow the usual expedient of blowing up X along S to obtain smooth polar coordinates, obtaining a blown up space $(\widetilde{X}, \widetilde{M}, \widetilde{Q}, \widetilde{S})$, with a blow-down map $b : \widetilde{X} \to X$. This has the properties that $b|_{\widetilde{X}\setminus\widetilde{S}} : \widetilde{X} \setminus \widetilde{S} \to X \setminus S$ is a diffeomorphism, as are $b|_{\widetilde{M}} : \widetilde{M} \to M$ and $b|_{\widetilde{Q}} : \widetilde{Q} \to Q$, while $b|_{\widetilde{S}} : \widetilde{S} \to S$ is a fibration with fibers diffeomorphic to the closed unit interval. We will denote such a diffeomorphism equivalence by $\widetilde{X} \setminus \widetilde{S} \approx X \setminus S$ (for example). We note that the blown-up face $\widetilde{S}$ has an edge structure in the sense of [Maz91], which meets the AH face $\widetilde{M}$, and we define 0-edge bundles as the natural bundles associated to this structure.

As seen earlier, in studying the Einstein problem we will need to use polar coordinates and allow metrics that are smooth on the blowup but not on the base. Thus, we study a somewhat wider class of metrics than those of the form b^*g_+ for g_+ a smooth cornered AH metric on X. (We call a CAH metric smooth or C^k if its compactifications $\varphi^2 g_+$ by smooth defining functions are smooth or C^k.) In Definition 3.2 we define admissible metrics, which differ from such a pullback by a perturbation that is smooth on $\widetilde{X}$ and vanishes in a 0-edge sense at $\widetilde{M}$ and $\widetilde{S}$. Thus, such a metric may be written $g = b^*g_+ + \mathcal{L}$ for appropriate $\mathcal{L}$. Given any admissible metric g, there is a well-defined angle function Θ on $\widetilde{S}$, which serves as a fiber coordinate. Any admissible metric induces a C^0 CAH metric g_X on X satisfying $g = b^*g_X$.

As is well known, the Einstein equations are invariant under the diffeomorphism group, and thus in general coordinates they are badly underdetermined. It is therefore necessary to gauge break the equations, and there are various ways to do this. Fefferman and Graham ([FG85]) put the metric into a normal form ([GL91]) that, subject to the boundary conditions they considered, breaks the gauge invariance. The same normal form has been of considerable importance in the subsequent study of AH metrics and their invariants, as well; see, e.g., [Gra00]. As a first order of business, then, we wish to construct a corner normal form for CAH metrics that will be similarly useful, and which we can use to gauge-break the Einstein equations.

It is natural to our space to consider polar-like coordinates. As motivation, then, we consider the hyperbolic metric (1.1) on the cornered space X, discussed above. In this space, the geodesics normal to Q are precisely the intersections with X of the circles $(x^0)^2 + (x^n)^2 = a^2$ (where $a \in \mathbb{R}^{>0}$), $x^1 = x^2 = \cdots = x^{n-1} = const$. The corner normal form in the hyperbolic case is obtained by introducing polar coordinates (θ, ρ), in which Q, M, and S are all given by constant coordinates:

$$x^0 = \rho \sin\theta, \quad x^n = \rho\cos\theta.$$

In these coordinates, the metric takes the form

$$g_+ = \csc^2(\theta)\left[d\theta^2 + \frac{d\rho^2 + (dx^1)^2 + \cdots + (dx^{n-1})^2}{\rho^2}\right]. \tag{1.2}$$

We will construct the corner normal form on general CAH spaces by studying geodesics leaving Q normally. The normal exponential map exp of $Q \setminus S \approx \widetilde{Q} \setminus \widetilde{S}$ is defined on the inward-pointing normal ray bundle $N_+(\widetilde{Q} \setminus \widetilde{S})$. With ν the inward-pointing unit normal field on $\widetilde{Q} \setminus \widetilde{S}$, this bundle has a natural decomposition $N_+(\widetilde{Q}\setminus\widetilde{S}) \approx [0,\infty)_t \times (\widetilde{Q}\setminus\widetilde{S})$ given by the prescription $(t, q) \mapsto t\nu_q$. We compactify $N_+(\widetilde{Q} \setminus \widetilde{S})$ by adding faces corresponding to $t = \infty$ and to $[0,\infty] \times (\widetilde{Q} \cap \widetilde{S})$, and we denote the compactification by $\widehat{N_+(\widetilde{Q} \setminus \widetilde{S})}$, a manifold with corners of codimension two.

Our first main result is as follows.

Theorem 1.1. *Let $(\widetilde{X}, \widetilde{M}, \widetilde{Q}, \widetilde{S})$ be the blowup of a cornered space, and g an admissible metric on $\widetilde{X}$. There is a neighborhood V of $\widetilde{Q} \cap \widetilde{S}$ in $\widetilde{Q}$ and a neighborhood $\widetilde{U}$ of $\widetilde{S}$ in $\widetilde{X}$ such that* exp *extends to a diffeomorphism* $\exp : \widehat{N_+(V \setminus \widetilde{S})} \to \widetilde{U}$.

One of the consequences of this theorem is that near S there is a distinguished representative of the conformal infinity $[h]$ on $M \setminus S$, which itself is conformally compact on (M, S). To see this, simply note that e^{-t} is a defining function for $\widetilde{M}$ via the diffeomorphism in the theorem, so that $e^{-2t} g|_{T\widetilde{M}}$ is a well-defined element of $(b|_{\widetilde{M}})^*[h]$ on $\widetilde{M} \approx M$ depending only on the geometry of $(\widetilde{X}, g)$. We call this the induced metric on M.

Theorem 1.1 allows us to prove two normal form theorems. The first applies generally, while the second requires that Q and M make a constant angle with respect to the compactified g_X, but gives a normal form with better properties.

Theorem 1.2. *Let $(\widetilde{X}, \widetilde{M}, \widetilde{Q}, \widetilde{S})$ be the blowup of a cornered space, and g an admissible metric on $\widetilde{X}$. For sufficiently small neighborhoods W of $\widetilde{M} \cap \widetilde{S}$ in $\widetilde{M}$, there exist a neighborhood $\widetilde{U}$ of $\widetilde{S}$ in $\widetilde{X}$ and a unique diffeomorphism $\zeta : [0,1]_u \times W \to \widetilde{U}$ such that $\zeta|_{\{0\}\times W} = \mathrm{id}_W$ and so that*

$$\zeta^* g = \frac{du^2 + h_u}{u^2}, \tag{1.3}$$

with h_u ($0 \le u \le 1$) a smooth one-parameter family of smooth conformally compact metrics on $(W, \widetilde{M} \cap \widetilde{S})$, and such that $\widetilde{M} = \zeta(\{u = 0\})$ and $\widetilde{Q} = \zeta(\{u = 1\})$.

We also prove a version in which we fix $\widetilde{Q}$ instead of $\widetilde{M}$; see Theorem 4.15.

Notice that (1.3) is in normal form in the usual asymptotically hyperbolic sense relative to $\widetilde{M}$, while under the subsitution $t = -\log u$, it is in the usual geodesic normal form relative to $\widetilde{Q}$. In particular, t is the distance to $\widetilde{Q}$.

The metrics h_u for fixed u are generally not asymptotically hyperbolic. The asymptotic curvature depends on both u and the angle between Q and M at the point of S approached. However, when the two boundary components M and Q make constant angle θ_0 with respect to g_X – a setting we call a constant-angle CAH space – we can say more. By making the change of variable $u = \frac{\csc\theta - \cot\theta}{\csc\theta_0 - \cot\theta_0}$, we obtain a normal form with AH slice metrics. We prove several versions of this result. Of particular interest is the following, which we will use to gauge-break the Einstein equations and state our results.

Theorem 1.3. *Let $(\widetilde{X}, \widetilde{M}, \widetilde{Q}, \widetilde{S})$ be the blowup of the cornered space (X, M, Q), and $g = b^*g_+ + \mathcal{L}$ an admissible metric on $\widetilde{X}$. Suppose that there is some $\theta_0 \in (0, \pi)$ such that, for any defining function φ for M, the boundary components M and Q make constant angle θ_0 with respect to the compactified metric $\varphi^2 g_X$.*

For sufficiently small neighborhoods W of $\widetilde{M} \cap \widetilde{S}$ in $\widetilde{M}$, there is a neighborhood $\widetilde{U}$ of $\widetilde{S}$ in $\widetilde{X}$ and a unique diffeomorphism $\zeta : [0, \theta_0]_\theta \times W \to \widetilde{U}$ such that $\zeta|_{\{0\}\times W} = \mathrm{id}_W$ and

$$\zeta^* g = \frac{d\theta^2 + h_\theta}{\sin^2 \theta}, \tag{1.4}$$

where h_θ $(0 \le \theta \le \theta_0)$ is a smooth one-parameter family of smooth asymptotically hyperbolic metrics on $(W, \widetilde{M} \cap \widetilde{S})$, and such that $\widetilde{M} = \zeta(\{\theta = 0\})$ and $\widetilde{Q} = \zeta(\{\theta = \theta_0\})$. Moreover, $\theta|_{[0,\theta_0]\times(\widetilde{M}\cap\widetilde{S})} = \zeta^*\Theta$, and $\partial_\theta \bar{h}_\theta|_{\rho=0} = 0$, where $\bar{h}_\theta = \rho^2 h_\theta$ and ρ is any defining function for $\widetilde{M} \cap \widetilde{S}$ in W.

Note that in this statement, ρ is defined on $W \subset \widetilde{M}$, which in turn defines it on $[0, \theta_0] \times W$ by extending it to be constant in θ. Notice also that in the case of hyperbolic space, (1.4) reduces to (1.2).

A normal form for edge spaces was constructed in [GK12]. However, the normal form derived there corresponds to a flow transverse to the edge boundary, whereas the flow generated by ∂_u in (1.3) is tangent to the edge face $\widetilde{S}$. Another difference is that the normal form constructed here takes a special form at two different faces, $\widetilde{M}$ and $\widetilde{Q}$, as opposed to one.

Having introduced CAH metrics and constructed a normal form for them, we study the Laplace operator on such spaces in Chapter 5, and in particular study formal existence near the corner of harmonic functions satisfying an inhomogeneous Dirichlet condition at $\widetilde{M}$ and a homogeneous Neumann condition at $\widetilde{Q}$. This is both interesting in its own right and a model problem for the analysis of the more complicated Einstein problem to follow.

The analysis proceeds in several steps. The key idea, as in general for such constructions, is to define and then study the *indicial operator* of the Laplacian, which is an operator on $C^\infty(\widetilde{S})$ defined by $I_\nu(u) = \rho^{-\nu}\Delta_g(\rho^\nu \tilde{u})|_{\rho=0}$, where ρ is a particular defining function for $\widetilde{S}$ in $\widetilde{X}$ and $\tilde{u}$ is an extension of u to $\widetilde{X}$. However, we here meet a significant difference from the usual AH case:

whereas the indicial operator is there an algebraic operator, due to the edge structure of g at $\widetilde{S}$, it here restricts to a second-order ordinary differential operator on each fiber of $\widetilde{S}$, with a regular singularity at $\theta = 0$:

$$I_\nu(u) = \sin^2(\theta)\partial_\theta^2 u + (1-n)\sin(\theta)\cos(\theta)\partial_\theta u + \nu(\nu+1-n)\sin^2(\theta)u \tag{1.5}$$

For any ν, the indicial roots of I_ν at $\theta = 0$ are 0 and n. The key content of section 5.1, then, is an analysis of this operator with the Dirichlet boundary condition at $\theta = 0$ and Neumann at $\theta = \theta_0$, for $0 < \theta_0 < \pi$. We study the mapping properties of the Green's operator, and also study the indicial roots of the Laplacian, or values of ν for which the indicial operator fails to be injective with the given boundary conditions. The latter is equivalent to studying the singular Sturm-Liouville eigenvalue problem for the operator $L = -\partial_\theta^2 + (n-1)\cot(\theta)\partial_\theta$. We estimate the lowest eigenvalue, and can characterize the eigenvalues in general as the roots of an equation involving hypergeometric functions. In the case that $\theta_0 = \frac{\pi}{2}$, we can calculate explicitly that they are $\lambda_k = \nu_k(\nu_k + 1 - n)$ for $\nu_k = n + 2k$ ($k \geq 0$). For this reason, we restrict our full analysis to the case $\theta_0 = \frac{\pi}{2}$: although similar ideas would apply in the general case, it would be difficult to be as specific as we can be when we know the eigenvalues explicitly. For that case, in Section 5.2 we formally construct a harmonic function order by order in ρ, solving at each order j an equation of the form $I_j u = f_j$. When $j = n + 2k$ is an indicial root, we show that we can proceed by including powers of $\log(\rho)$ in the solution, although uniqueness is lost. A formal solution could be uniquely parametrized by $u|_{\widetilde{M}}$ and by $\{\upsilon_k\}_{k=0}^\infty$, where $\upsilon_k \in C^\infty(S)$ parametrizes the formal freedom at order $n + 2k$.

We define the space $\mathcal{P}(\widetilde{X})$ to be functions on $\widetilde{X}$ that have an asymptotic expansion in ρ, θ, $\theta^n \log(\theta)$ to the first power, and $\rho^{n+2k}\log(\rho)^k$ ($k \geq 0$), where ρ is a defining function for $\widetilde{M} \cap \widetilde{S}$ in $\widetilde{M}$; see (3.15) for a precise definition. Our result is as follows.

Theorem 1.4. *Let $(\widetilde{X}^{n+1}, \widetilde{M}, \widetilde{Q}, \widetilde{S})$ be the blowup of a cornered space (X, M, Q), with $g = b^* g_X$ an admissible metric such that M and Q make constant angle $\frac{\pi}{2}$ with respect to g_X. Let $\psi \in C^\infty(\widetilde{M})$. There exists $u \in \mathcal{P}(\widetilde{X})$ such that $\Delta_g u = O(\rho^\infty)$, such that $u \equiv \psi$ along $\widetilde{M}$, and* *such that* $\partial_\nu u|_{\widetilde{Q}} \equiv 0$, *where* ∂_ν *is the normal derivative at* $\widetilde{Q}$. *If* u_1 *and* u_2 *are two such functions, then* $u_1 - u_2 = O(\rho^n)$, *and* $\rho^{-n}(u_1 - u_2)|_{\widetilde{S}} = \upsilon_0 \sin^n \Theta$, *where* $\upsilon_0 \in C^\infty(\widetilde{S})$ *is constant on fibers.*

Having constructed a normal form in Chapter 4 and studied the Laplace operator in Chapter 5, we turn finally to constructing Einstein metrics in Section 6.2. As mentioned earlier, the choice of boundary condition $K_{\widetilde{Q}} = \lambda g|_{\widetilde{Q}}$ ensures that our space will be constant-angle with angle $\theta_0 = \cos^{-1}(-\lambda)$. It is this that allows us to break the gauge using (1.4). In this gauge, the CMC umbilic condition at $\widetilde{Q}$ becomes the Neumann condition $\partial_\theta h_\theta|_{\theta=\theta_0} = 0$ on h; see Lemma 6.6. We define $\mathcal{M}(\theta_0, M)$ to be the set of families h_θ $(0 \leq \theta \leq \theta_0)$ of smooth AH metrics on M such that $\bar{h}_\theta = \rho^2 h_\theta$ is smooth in θ (in case n is odd or $n = 2$) or such that it is smooth in θ and $\theta^n \log(\theta)$ (if $n \geq 4$ is even). See page 27 for full details. In the following statement, $T = O_g(f)$ for T a tensor field means $|T|_g = O(f)$.

Theorem 1.5. *Let* M^n *be a manifold with boundary, let* $\lambda \in (-1, 1)$, *and let* $[h]$ *be a conformal class on* M. *Set* $\theta_0 = \cos^{-1}(-\lambda)$. *Then there exists a one-parameter family* $h_\theta \in \mathcal{M}(\theta_0, M)$ *of smooth AH metrics on* M, *such that if* g *is the normal-form metric*

$$g = \csc^2(\theta)\left[d\theta^2 + h_\theta\right]$$

on $\widetilde{X} = [0, \theta_0]_\theta \times M$, *then*

(a) $h_0 \in [h]$;

(b) $\partial_\theta \bar{h}_\theta|_{\rho=0} = 0$;

(c) *the second fundamental form of* $\widetilde{Q} \setminus \widetilde{S} = \{\theta_0\} \times (M \setminus S)$ *satisifes* $K_{\widetilde{Q}} = \lambda g|_{T\widetilde{Q}}$; *and*

(d) *the formal Einstein condition*

$$\mathrm{Ric}(g) + ng = O_g(\rho^n)$$

is satisfied.

Moreover, if h_θ, h'_θ are two such families, then $\bar{h}_\theta - \bar{h}'_\theta = O(\rho^n)$.

This theorem is concerned with smooth formal series in ρ. As we will show, the lowest non-negative indicial root γ_0 of the Einstein operator satisfies $n-1 < \gamma_0 < n$ if $\theta_0 > \frac{\pi}{2}$, $\gamma_0 = n$ if $\theta_0 = \frac{\pi}{2}$, and $n < \gamma_0 < \infty$ if $\theta_0 < \frac{\pi}{2}$. Thus, if $\frac{\pi}{2} < \theta_0 < \pi$, then we would expect additional solutions with leading asymptotics ρ^γ, where $n-1 < \gamma < n$. If $\theta_0 \leq \frac{\pi}{2}$, then uniqueness would hold mod $O(\rho^n)$ even allowing non-integral powers of ρ. If $\theta_0 = \frac{\pi}{2}$, then a term of the form $\rho^n \log(\rho)$ would generically appear; see Theorem 1.6 below. The form of the solution to higher order depends on indicial roots, which depend on θ_0. Uniqueness fails at order n in every case, however.

Notice that the given data is only $(M, [h])$ and λ, and that we get an Einstein metric in normal form that is unique up to order n. In particular, the induced metric $h_0 \in [h]$ is an AH representative of the conformal class that is invariantly defined to order n given only $[h]$ and λ.

Uniqueness in Theorem 1.5 should hold without (b), but we do not yet have a proof of this; it will entail uniqueness analysis of a system of nonlinear ODEs.

The proof of Theorem 1.5 is conceptually similar to that of the simpler Theorem 1.4, but significant complications arise in the Einstein setting: for example the operator we now study, the Einstein operator $E(g) = \mathrm{Ric}(g) + ng$, is nonlinear, acts on tensors instead of scalars, and has a gauge invariance. We define an indicial operator, as in the scalar case, by $I^\gamma(\varphi) = \rho^{-\gamma}(E(g + \rho^\gamma \varphi) - E(g))|_{\rho=0}$, where φ is a section of an appropriate bundle; and as in [GL91], we decompose it into its irreducible parts, in this case seven of them. Once again, and unlike in that paper, the indicial operator is a second-order system of regular singular ordinary differential operators as opposed to algebraic operators. As in the AH case, the part of the indicial operator acting on the trace-free part of the metric perturbations tangent to S is identical with the indicial operator of the scalar Laplacian. We construct the Einstein metric term-by-term in ρ. At each order, this gives us a system of second-order regular singular ordinary differential equations to solve, which is overdetermined because of the gauge-broken form (1.4). An additional complication in the analysis comes from the fact that, since the Einstein metric is unique to order n, then as observed after Theorem 1.1 and unlike in the case of the usual AH normal form, the induced metric in the conformal infinity is uniquely determined and cannot be chosen arbitrarily in the conformal class. These two problems are solved in tandem. For our boundary data, we take $h \in [h]$ to be arbitrary (but AH), and then impose the boundary condition $h_0 = \chi h$, where $\chi \in C^\infty(M)$ is some scalar function to be determined order by order in ρ along with g. Thus the induced metric is determined simultaneously with the metric g. At each order, we use four of the seven irreducible parts of the indicial operator to solve uniquely for the perturbation of the metric at that order. We then use the Bianchi identity to show that the remaining three equations are also satisfied. However, this turns out to be true only for a unique choice of the perturbation of χ, and thus we get uniqueness both for g and χh (within $[h]$) up to order n. The behavior of the system changes at order n. The trace-free part of the indicial operator has a set of eigenvalues going to infinity; in the special case that $\lambda = 0$ ($\theta_0 = \frac{\pi}{2}$), the first of these is at order n, as mentioned above, and this obstruction allows us to identify a new conformal hypersurface invariant.

Theorem 1.6. *Let M^n ($n \geq 2$) be a manifold with boundary S, and τ a smooth metric on M. Let $[h]$ be the AH conformal class corresponding to $[\tau]$. There is a generically nontrivial symmetric, trace-free 2-tensor field $\mathcal{K}(\tau)$ on S, defined by (6.38), whose nonvanishing obstructs the formal existence of a smooth normal-form metric $g = \csc^2(\theta)[d\theta^2 + h_\theta]$ on $[0, \frac{\pi}{2}] \times M$ satisfying (a) - (c) from Theorem 1.5, and also satisfying* $\mathrm{Ric}(g) + ng = O_g(\rho^{n+1})$.

Moreover, if $\hat{\tau} = \Omega^2\tau$ for $\Omega \in C^\infty(M)$, then $\mathcal{K}(\hat{\tau}) = (\Omega|_S)^{2-n}\mathcal{K}(\tau)$.

The thesis is organized as follows.

In Chapter 2, we review background results about AH metrics and conformal geometry.

In Chapter 3, we define CAH spaces and their blowups, as well as construct a class of special polar decompositions that will be ubiquitous in Chapter 4. We also develop the 0-edge structure point of view, and then use this to define and discuss admissible metrics. We then introduce function spaces that we will use in our later analysis. Finally, we define the notation that will be used throughout the rest of the thesis.

In Chapter 4, we study the basic properties of admissible metrics, analyze their geodesics, and prove Theorems 1.1 - 1.3 and other versions of normal form results. In Section 4.1, inspired by the convexity arguments of [BO69], we use a natural asymptotic solution to $\nabla_g^2 w = wg$ to derive the central properties of the g-geodesics leaving $\widetilde{Q}$ normally. Our result shows that they approximately generalize the behavior of the analogous geodesics in hyperbolic space, namely that they do not return to $\widetilde{Q}$ or $\widetilde{S}$ and that they approach $\widetilde{M}$ normally. In Section 4.2, we study the geodesic flow equations to extend the exponential map to the compactified normal bundle and show that the extended map is smooth and a local diffeomorphism. The extensive debt this chapter owes to [Maz86] is especially clear here, where we regularize the flow equations using the method developed there. The final substantial step, in Section 4.3, is to show that the normal exponential map is actually injective on a suitably restricted neighborhood of $\widetilde{S}$. Many of the previous (and elegant) Cartan-Hadamard-type proofs adapt with difficulty, if at all, to the noncomplete and local setting studied here. The homotopy-lifting approach of [Her63], however, adapts well to this setting, and it enables us to show injectivity. In Section 4.4, Theorems 1.1 through 1.3 and other normal form results follow quickly.

In Chapter 5, we study the scalar Laplacian on constant-angle CAH spaces. After calculating the scalar Laplace operator, we then compute its indicial operator (1.5), and prove theorems about its eigenvalues and the mapping properties of its Green's operator. We do this for general θ_0 and arbitrary integral powers of $\log(\theta)$, since although these features are unnecessary for our analysis of the linear scalar problem with $\theta_0 = \frac{\pi}{2}$, they will be used in the nonlinear Einstein setting. We then prove Theorem 1.4.

In Chapter 6, we turn to Einstein metrics. In Section 6.1, we study smooth Einstein metrics and deduce the compatibility conditions along S for their existence that were discussed earlier. In Section 6.2, we study formal existence for aribtrary S, enlarging the class of metrics from those smooth on X to those polyhomogeneous on $\widetilde{X} = [0, \theta_0] \times M$ in the form (1.4); and prove Theorems 1.5 - 1.6.

Finally, in Chapter 7, we discuss future research directions and questions raised by this work.

Chapter 2

BACKGROUND

We begin by reviewing some basic relevant results. Recall that, if $S \subset M$ is an embedded submanifold of a smooth manifold M, then $\varphi \in C^\infty(M)$ is a *defining function* for S if $S = \varphi^{-1}(\{0\})$ and $d\varphi|_S$ is nonvanishing as a section of T^*M.

Definition 2.1. *A* conformally compact manifold *is a compact manifold with boundary X^{n+1} with nonempty boundary M^n, such that the interior $\mathring{X}$ of X is equipped with a Riemannian metric g such that, for any smooth defining function φ for M, $\varphi^2 g$ extends to a Riemannian metric on X. We may denote a conformally compact manifold by (X, M, g).*

Remark. The definition can trivially be extended to the pseudo-Riemannian case, but we will exclusively deal with the Riemannian case unless otherwise stated. More substantively, the definition says nothing about the smoothness of the metric g, which *a priori* might be only a positive definite continuous section of $S^2(T\mathring{X})$. Throughout this thesis, smooth will mean C^∞. For the remainder of the chapter, we will refer to C^k (smooth) conformally compact manifolds to mean that the metric $\varphi^2 g$ extends to a C^k (smooth) metric on X, for any φ. If left unspecified, conformally compact manifold will henceforth imply a smooth metric.

Given a conformally compact manifold (X, M, g) and a defining function φ for M, the metric $h = \varphi^2 g|_{TM}$ is a Riemannian metric on M. The metric $h = h_\varphi$ depends on the choice of defining function φ, but if ψ is another such defining function, then $h_\psi = (\psi\varphi^{-1})^2 h_\varphi$. In particular, $h \mapsto h_\psi$ is a conformal transformation, so it follows that the conformal class $[h]$ is an invariant of (X, M, g). The class $[h]$ is called the *conformal infinity* of g.

The motivating example of a conformally compact manifold is the Poincaré ball model of hyperbolic space, which is the ball B^{n+1} with metric

$$ds^2 = \frac{4\sum_{i=1}^{n+1}(dx^i)^2}{(1-|x|^2)^2}.$$

The boundary is S^n, and the induced conformal class $[h]$ is the class of the round (canonical) metric on S^n.

It is useful to tighten the analogy to hyperbolic space further. Let φ be any defining function for the boundary M, and let $\bar{g}$ be the metric $\bar{g} = \varphi^2 g$ on X. A tedious but straightforward computation carried out in [Maz88] shows that

$$R_{ijkl} = -|d\varphi|^2_{\bar{g}}(g_{il}g_{jk} - g_{ik}g_{jl}) + O(\varphi^{-3}),$$

where R is the Riemann curvature tensor of g. This motivates the following.

Definition 2.2. *A C^2 conformally compact metric is called* asymptotically hyperbolic *(AH) if, for some defining function φ for M, and with $\bar{g} = \varphi^2 g$, the condition $|d\varphi|^2_{\bar{g}} = 1$ holds on M.*

Remark. It is easy to show that this condition holds for all defining functions if it holds for any.

By changing coordinates, an asymptotically hyperbolic metric can be put into a particularly nice form relative to any representative of the conformal class $[h]$ on M. The following normal form result is proved in [GL91], and is Proposition 4.3 in [FG12]; it has played a central role in the development and applications of the theory of AH spaces.

Theorem 2.3. *Let (X, M, g) be an asymptotically hyperbolic manifold with conformal infinity $[h]$. Let $h \in [h]$. Then there is a neighborhood U of M in X, and a unique diffeomorphism $\psi : M \times [0, \varepsilon) \to U$, such that if r is the coordinate on the second factor, then $r^2\psi^* g|_{TM} = \psi^* h$. In these coordinates, $\psi^* g$ takes the form*

$$\psi^* g = r^{-2}(dr^2 + g_r), \tag{2.1}$$

where g_r is a smooth one-parameter family of smooth metrics on M and $g_0 = h$. Thus, on U, $|dr|^2_{r^2 g} = 1$.

A metric in form (2.1) on $M \times [0, \varepsilon)$ will be said to be in *normal form* relative to h, and r will be called a *geodesic defining function* for M.

Definition 2.4. *An* asymptotically hyperbolic Einstein manifold *is an asymptotically hyperbolic manifold that satisfies the equation*

$$Ric(g) = -ng.$$

Remark. If (X, M, g) is Einstein and g is in normal form, then $\partial_r g_r|_{r=0} = 0$ as a first-order consequence of the Einstein equation.

The publication of [FG85] initiated the extremely fruitful study of relationships between the conformal geometry of $(M, [h])$ and the Riemannian geometry of $(\mathring{X}, g)$, where (X, M, g) is an asymptotically hyperbolic Einstein manifold. The project has led to insights into each geometry in terms of the other. The subject has become a topic of intense study in physics as well, where it provides the geometric setting for the so-called AdS/CFT correspondence whose study was initiated in [Mal98, Wit98].

The idea of [FG85], which was fully developed in [FG12], is that given a smooth compact conformal manifold $(M^n, [h])$, one can realize M as the boundary of an $(n+1)$-manifold X, and formally develop an AH Einstein metric g on X at M, in powers of a defining function φ and with coefficients determined by the geometry of $[h]$. To the extent that this procedure works and is well-defined, the Riemannian invariants of (X, g) can then be used to construct conformal invariants of $(M, [h])$. Since the construction is formal and completely local, the requirement that M be compact may in fact be removed.

An AH metric g in normal form on $M \times [0, \varepsilon)_r$ is called *even* if there exists a smooth extension of $\bar{g} = r^2 g$ to $M \times (-\varepsilon, \varepsilon)$ that is symmetric under the map $r \mapsto -r$.

The extent of success of the formal construction turns out to depend on the parity of the dimension. The primary relevant result of [FG85, FG12] is the following ([FG12], Theorem 4.8). Notice that regularity depends on the parity of the dimension.

Theorem 2.5. *Let $(M, [h])$ be a smooth conformal manifold of dimension n, and h a representative of its conformal class. Let t be a smooth symmetric 2-tensor field on M, such that $h^{ij}t_{ij} = 0$. Let $\widetilde{M} = M \times [0, \infty)$, and let r be the coordinate on the second factor.*

- *If $n = 2$ and if t satisfies $t_{ij,}{}^{j} = -\frac{1}{2}R,_{i}$, where R is the scalar curvature of h, then there exists an even formal AH Einstein metric g in normal form relative to h, and such that $\operatorname{tf}(\partial_r^2 g_r)_{r=0} = t$. These conditions uniquely determine g_r to infinite order at $r = 0$.*

- *If $n \geq 3$ is odd and if t satisfies $t_{ij,}{}^{j} = 0$, then there exists a formal AH Einstein metric metric in normal form relative to h, such that $\operatorname{tf}(\partial_r^n g_r|_{r=0}) = t$. These conditions uniquely determine g_r to infinite order at $r = 0$, and the solution satisfies $\operatorname{tr}(\partial_r^n g_r|_{r=0}) = 0$.*

- *If $n \geq 4$ is even, then there is a natural Riemannian invariant 1-form b_i of h so that if t satisfies $t_{ij,}{}^{j} = b_i$, then there exists a metric $g = r^{-2}(dr^2 + g_r)$ on some neighborhood M_+ of M in $\widetilde{M}$, such that $g_0 = h$ and $Ric(g) = -ng$ to infinite order along M, and such that g_r has an expansion of the form*

$$g_r \sim \sum_{N=0}^{\infty} g_r^{(N)} (r^n \log r)^N, \tag{2.2}$$

  where each $g_r^{(N)}$ is a smooth family of symmetric 2-tensors on M that is even in r, and $\operatorname{tf}_h \partial_r^n g_r^{(0)}|_{r=0} = t$. These conditions uniquely determine the $g_r^{(N)}$ to infinite order at $r = 0$. There is a symmetric two-tensor field $\mathcal{O}_{ij}$ on M, conformally invariant of weight $2 - n$, such that the solution g_r is smooth if and only if $\mathcal{O}_{ij}$ vanishes on M.

Remark. The conformally invariant two-tensor field $\mathcal{O}$ in the even-dimensional case of this theorem is called the obstruction tensor. It is discussed in Chapter 3 of [FG12]. In dimension $n = 4$ it is the classical Bach tensor.

This construction of a formal AH Einstein metric has been tremendously successful in the generation and classification of conformal invariants, beginning with $\mathcal{O}$ itself in even dimensions.

Notable examples include [BEG94], which classified scalar conformal invariants, and [GJMS92], which constructed important new conformally invariant differential operators.

In addition to the formal context that has led to such advances in conformal invariant theory, asymptotically hyperbolic metrics have been much studied. Global existence, for example, has been analyzed in [GL91],[Lee06], and [Biq00], among several others; it will not concern us here. Boundary regularity has also been studied, both locally and globally.

An Einstein metric is always smooth on the interior of a space in appropriate coordinates. The question of interest in this context, then, is how regular a AH Einstein metric must be at the boundary, given some *a priori* regularity. An early result was obtained by Anderson in [And03] for dimension four ($n = 3$). In [CDLS05], a global boundary regularity result was achieved that fully realized the suggestion of Theorem 2.5:

Theorem 2.6. *Let $(M^n, [h])$ be a smooth compact conformal manifold, let $U = M \times [0, 1]$ be a collar neighborhood of M, and let g be a C^2 AH Einstein metric on U. Let (x, ρ) be coordinates on U, so that ρ is a defining function for M. Suppose that $h = \rho^2 g|_{TM}$ is smooth. Then for any $0 < \lambda < 1$, there exists $R > 0$ and a $C^{1,\lambda}$ collar diffeomorphism $\Phi : M \times [0, R] \hookrightarrow U$ such that $\Phi|_M$ is the identity; such that $\Phi^* g$ can be written as*

$$\Phi^* g = r^{-2}(dr^2 + g_r);$$

such that g_r is a one-parameter family of metrics on M; such that $g_0 = h$; and such that

- *if n is odd or $n = 2$, then g_r extends smoothly to $M \times [0, R]$, so $\Phi^* g$ is asymptotically hyperbolic and smooth; and*

- *if $n \geq 4$ is even, then g_r can be written in the form*

  $$g_r = \psi(r, r^n \log r),$$

  where $\psi = \psi(r, z)$ is a two-parameter family of Riemannian metrics on M that is smooth in all its arguments as a function on $M \times [0, R] \times [R^n \log R, 0]$. Furthermore, $\Phi^* g$ is smooth if and only if $\partial_z \psi(0, 0)$ vanishes identically on M.

The question of global regularity being settled, it was natural next to investigate local regularity. A result for n odd came in [Hel08], where regularity was shown for metrics on a neighborhood near a point on the boundary of X^{n+1} with sufficient *a priori* smoothness. A local regularity result in all dimensions was obtained in [BH14]. Put coordinates $(y = x^0, x^1, \ldots, x^n)$ on $\mathbb{R}^{n+1}$. The main result is as follows:

Theorem 2.7. *Let X^{n+1} be the closed unit half ball $\{x \in \mathbb{R}^{n+1} : |x| \leq 1$ and $y \geq 0\}$. Let $D = \{x \in X : y = 0\}$ be the disk portion of its boundary. Suppose that the interior is equipped with a AH Einstein metric g with D the boundary at infinity, and such that the conformal infinity $[h]$ is smooth. Let $h \in [h]$, and suppose further that, if $g_0 = \frac{dy^2+h}{y^2}$, we have*

$$g - g_0 \in y^{\varepsilon} C^{1,\alpha}(g_0)$$

for some $\alpha \in (0, 1)$ and $\varepsilon > 0$, and where $C^{1,\alpha}$ is the appropriate Hölder space for g_0. Then there exist $\delta < 1$ and some half ball $Y_\delta = \{x \in \mathbb{R}^{n+1} : |x| \leq \delta$ and $y \geq 0\} \subset X$ such that

- *if n is odd or $n = 2$, then $g|_{Y_\delta}$ is smooth in appropriate coordinates; and*

- *if $n \geq 4$ is even, then in appropriate coordinates, $g|_{Y_\delta}$ has a polyhomogeneous expansion at the boundary.*

Establishing this result required a degree of control of the regularity of g near the corner $\{x \in D : |x| = 1\}$, but the authors used a two-parameter family of weighted spaces that let them avoid analyzing the corner regularity in detail.

Asymptotically hyperbolic manifolds are a simple example of a space with an *edge structure*, a concept introduced in [Maz91]. This perspective is often useful in carrying out analysis on AH spaces. An edge space is a manifold X with boundary M, such that M is the total space of a fibration $b : M \to Y$ with fiber F. Edge vector fields are those smooth vector fields V on X which lie tangent to the fibers of b along M. There is a natural bundle, ${}^{e}TX$, called the edge tangent bundle, of which edge vector fields are precisely the smooth sections.

In the case of AH manifolds, we let F and b be trivial, so that each fiber is simply a point. Then edge fields are those vector fields that vanish at M. If r is a defining function for M, and if $(\partial_{x^1}, \cdots, \partial_{x^{n+1}})$ is a frame for TX near some point of M, then $(r\partial_{x^1}, \cdots, r\partial_{x^{n+1}})$ is a frame for the edge bundle, which in this case we denote ${}^0TX$. The edge bundle has a dual bundle, ${}^0T^*X$, whose fibers are spanned by $\left(\frac{dx^1}{r}, \cdots, \frac{dx^{n+1}}{r}\right)$. Thus, for example, an AH metric can be viewed as a smooth section of $S^2({}^0T^*X)$.

Finally, it will be helpful to review the relationship between two types of conformal classes on a manifold with boundary. Let M be a manifold with boundary S. The first type is the usual conformal class, $[\tau]$, where τ is a smooth metric on M. Here, $[\tau]$ is the family of metrics τ' such that $\tau' = \Omega^2\tau$ for some nonvanishing $\Omega \in C^\infty(M)$. The second type is an AH conformal class, $[h]$, where h is an AH metric on M. Here, $[h]$ is precisely the set of metrics $\Psi^2 h$, where $\Psi \in C^\infty(M)$ is nonvanishing and $\Psi|_S \equiv 1$.

Observe that there is a one-to-one correspondence between ordinary conformal classes $[\tau]$ and AH conformal classes $[h]$. Given a conformal class $[\tau]$, let $\tau \in [\tau]$ and let $\varphi \in C^\infty(M)$ be any defining function for S in M such that $|d\varphi|_\tau = 1$ along S. Set $h = \varphi^{-2}\tau$. Then the conformal class $[h]$ is independent of the choices of φ and of τ. To see this, suppose ψ is some other defining function satisfying $|d\psi|_\tau = 1$ along S. Then $\psi^{-2}\tau = \psi^{-2}\varphi^2(\varphi^{-2}\tau)$. But $\Psi = \psi^{-2}\varphi^2$ extends smoothly to all of M, and $\Psi|_S \equiv 1$ by the choice of φ, ψ. Thus $[h]$ does not depend on φ. Similarly, suppose $\tau' = \Omega^2\tau$, and let φ' be a defining function for S such that $|d\varphi'|_{\tau'} = 1$ along S. Then it is easy to check that if $\varphi = \Omega^{-1}\varphi'$, then $|d\varphi|_\tau = 1$ along S; and $\varphi^{-2}\tau = (\varphi')^{-2}\tau'$. Thus the map taking $[\tau]$ to $[h]$ is well-defined. It is an easy exercise to reverse these steps and show that it is a bijection.

One may also consider conformal classes of conformally compact metrics. The same argument as above, without the requirement that $|d\varphi|_\tau = 1$ at S, shows that there is likewise a correspondence between conformally compact and smooth conformal classes.

Chapter 3
CORNERED SPACES AND BLOWUPS

3.1 Context

We take up a natural generalization of conformally compact manifolds by considering manifolds that have *finite* boundaries as well as the boundary at infinity; a simple example would be half of the Poincaré ball. In such spaces, the finite and infinite boundaries meet in a corner, which is at infinity.

We first give an intrinsic definition of this situation.

Definition 3.1. *A* cornered space *is a smooth manifold with codimension-two corners, X^{n+1}, such that*

(i) There are submanifolds with boundary $M^n \subset \partial X$ and $Q^n \subset \partial X$ of the boundary ∂X, such that $\emptyset \neq S = M \cap Q$ is the mutual boundary, and is the entire codimension-two corner of X, and such that $\partial X = M \cup Q$; and

(ii) the corner $S \subset M$ is a smooth, compact hypersurface in M.

We denote a cornered space by (X, M, Q), and we set $\mathring{X} = X \setminus (Q \cup M)$.

Given a cornered space (X, M, Q), a smooth (resp. C^k) cornered conformally compact metric *on X is a smooth Riemannian metric g_+ on $X \setminus M$ such that, for any smooth defining function φ for M, the metric $\varphi^2 g_+$ extends to a smooth (resp. C^k) metric on X. We call such a metric a* cornered asymptotically hyperbolic (CAH) metric *if for some (hence any) such defining function φ, the condition $|d\varphi|_{\varphi^2 g_+} = 1$ holds along M.*

A smooth (resp. C^k) cornered asymptotically hyperbolic (CAH) space *is a cornered space (X, M, Q) together with a smooth (resp. C^k) CAH metric g_+. We denote such a space by (X, M, Q, g_+). The definition for cornered conformally compact space is analogous.*

For a cornered conformally compact space (X, M, Q, g_+), the *conformal infinity* $[h]$ is the conformal class $[\varphi^2 g_+|_{TM}]$ on M, where φ is a defining function for M. Notice that a consequence of the fact that X is a manifold with corners is that the boundary components M and Q intersect transversely.

For each $x \in S$, we define $\theta_0(x)$ to be the angle between M and Q at X with respect to $\varphi^2 g_+$, where φ is any smooth defining function for M. Plainly $\theta_0 \in C^\infty(S)$.

It will be important to our analysis to be able to view X as a submanifold of a larger AH manifold without corner. By doubling across Q ([Mel96], Chapter 1) and using partitions of unity, we may construct a global AH manifold $(\breve{X}, \breve{g}_+)$ with boundary $\breve{M}$, such that $\mathring{X}$ is an open submanifold of $\breve{X}$ with $\partial\mathring{X} = M \cup Q$, where $M \subset \breve{M}$ and $Q \subset X$ is a hypersurface in $\breve{X}$, and such that $\breve{g}_+|_{X\setminus M} = g_+$. The extension $\breve{g}_+$ is not canonical, of course.

As we are planning to study polar-like coordinates at the codimension-two hypersurface S, and since such coordinates must be singular there, we employ the usual measure of blowing up X along S ([Mel08]). Let X be a cornered space, with M, Q, and S as in the definition. For $s \in S$, define $N_s S = T_s X / T_s S$, which is a vector space of dimension two. Let NS be the vector bundle $NS = \sqcup_{s \in S} N_s S$. Let $N_+ S \subset NS$ be the inward-pointing normal vectors (including those tangent to $\partial X = Q \cup M$). Thus $N_+ S$ is a bundle with fiber a closed cone in $\mathbb{R}^2$ and base S. Finally, let $\widetilde{S} = (N_+ S \setminus \{0\})/\mathbb{R}^+$, which is the total space of a fibration over S with fiber the closed interval $[0, 1]$. Set $\widetilde{X} = (X \setminus S) \sqcup \widetilde{S}$, and define the *blow-down map* $b : \widetilde{X} \to X$ by $b(x) = x$ $(x \in X \setminus S)$ and $b(\tilde{s}) = \pi(\tilde{s})$ $(\tilde{s} \in \widetilde{S})$, where π is the natural projection. Then as shown in [Mel08], $\widetilde{X}$ has a unique smooth structure as a manifold with corners of codimension two such that b is smooth, $b|_{\widetilde{X}\setminus\widetilde{S}} : \widetilde{X} \setminus \widetilde{S} \to X \setminus S$ is a diffeomorphism onto its image, and $db_{\tilde{s}}$ has rank n for $\tilde{s} \in \widetilde{S}$. Moreover, polar coordinates on X centered along S lift to smooth coordinates. We set $\widetilde{M} = \overline{b^{-1}(M \setminus S)}$ and $\widetilde{Q} = \overline{b^{-1}(Q \setminus S)}$. Then $b|_{\widetilde{M}} : \widetilde{M} \to M$ and $b|_{\widetilde{Q}} : \widetilde{Q} \to Q$ are diffeomorphisms.

Recall that an edge structure on a manifold with boundary is a fibration of the boundary, and the associated edge vector fields are the vector fields that are tangent to the fibers at the boundary ([Maz91]). An important special case is a 0-structure ([MM87]), for which the boundary fibers are points and the edge vector fields are those that vanish at the boundary. On our blowup space $\widetilde{X}$, the blown-up face $\widetilde{S}$ is the total space of the fibration $b|_{\widetilde{S}} : \widetilde{S} \to S$ with interval fibers, while we can view $b|_{\widetilde{M}} : \widetilde{M} \to M$ as a fibration whose fibers are points. We will refer to the structure defined by these two fibrations as a 0-edge structure, and the associated 0-edge vector fields are the smooth vector fields on $\widetilde{X}$ which are tangent to the fibers at $\widetilde{S}$, and which vanish at $\widetilde{M}$.

The 0-edge vector fields may be easily expressed in appropriate local coordinates. Let θ be a defining function for $\widetilde{M}$ whose restriction to each fiber of $\widetilde{S}$ is a fiber coordinate taking values in $[0, \pi)$; let ρ be any defining function for $\widetilde{S}$; and locally let x^s, $1 \leq s \leq n-1$, be the lifts to $\widetilde{X}$ of functions on X that restrict to local coordinates on S. Then the vector fields

$$\sin\theta \frac{\partial}{\partial\theta}, \quad \rho\sin\theta\frac{\partial}{\partial x^s}, \quad \rho\sin\theta\frac{\partial}{\partial\rho}$$

span the 0-edge vector fields over $C^\infty(\widetilde{X})$. As in the usual edge case, there is a well-defined vector bundle ${}^{0e}T\widetilde{X}$ whose smooth sections are the 0-edge vector fields. The smooth sections of the dual bundle ${}^{0e}T^*\widetilde{X}$ are locally spanned by

$$\frac{d\theta}{\sin\theta}, \quad \frac{dx^s}{\rho\sin\theta}, \quad \frac{d\rho}{\rho\sin\theta}. \tag{3.1}$$

By a 0-edge metric we will mean a smooth positive definite section g of $S^2({}^{0e}T^*\widetilde{X})$. This is equivalent to the condition that locally g may be written as

$$g = \begin{pmatrix} \frac{d\theta}{\sin\theta}, & \frac{dx^s}{\rho\sin\theta}, & \frac{d\rho}{\rho\sin\theta} \end{pmatrix} G \begin{pmatrix} \frac{d\theta}{\sin\theta} \\ \frac{dx^s}{\rho\sin\theta} \\ \frac{d\rho}{\rho\sin\theta} \end{pmatrix},$$

where G is a smooth, positive-definite matrix-valued function on $\widetilde{X}$. This allows us to define the class of metrics that we will study.

Definition 3.2. *An admissible metric on $\widetilde{X}$ is a 0-edge metric g on $\widetilde{X}$ which can be written in the form*

$$g = b^*g_+ + \mathcal{L},$$

where g_+ is a smooth cornered asymptotically hyperbolic metric on X and $\mathcal{L}$ is a smooth section of $S^2({}^{0e}T^*\widetilde{X})$ that vanishes on $\widetilde{S}$ and $\widetilde{M}$.

The latter condition is the same as saying that $\mathcal{L} = (\rho \sin\theta)\ell$, for some smooth section ℓ of $S^2({}^{0e}T^*\widetilde{X})$. We will see below that if g_+ is a smooth CAH metric on X, then b^*g_+ is a 0-edge metric.

Since $b|_{\widetilde{X}\setminus(\widetilde{M}\cup\widetilde{S})} : \widetilde{X} \setminus (\widetilde{M} \cup \widetilde{S}) \to X \setminus M$ is a diffeomorphism, an admissible g uniquely determines a smooth metric g_X on $X \setminus M$ satisfying $b^*g_X = g$ on $\widetilde{X} \setminus (\widetilde{M} \cup \widetilde{S})$. Since $\mathcal{L}$ vanishes on $\widetilde{S}$ and $\widetilde{M}$, it is not hard to see that g_X is a C^0 CAH metric on X. Thus we will call a metric g_X on $X \setminus M$ an admissible metric on X if b^*g_X extends to an admissible metric on $\widetilde{X}$.

Observe that an admissible metric g_X on X determines a well-defined angle function Θ on the blown-up face $\widetilde{S}$, which serves as a smooth fiber coordinate. Let $\tilde{s} \in \widetilde{S}$, with $s = b(\tilde{s}) \in S$. Then, under one interpretation, $\tilde{s}$ naturally represents a hyperplane $P_{\tilde{s}}$ in T_sX containing T_sS. The angle $\Theta(\tilde{s})$ between P_s and T_sM is well-defined. It can be computed as follows: let φ be any defining function for M, and $\bar{g}_X = \varphi^2 g_X$. Let $\bar{\nu}_M \in T_sM$ be normal to T_sS, inward pointing in M, and unit $\bar{g}_X$-length (this is uniquely defined and continuous, by the continuity of admissible metrics just observed). Similarly, let $\bar{\nu}_{P_{\tilde{s}}}$ be inward-pointing in $P_{\tilde{s}}$, normal to T_sS, and unit length. Then $\Theta(\tilde{s}) = \cos^{-1}(\bar{g}_X(\bar{\nu}_M, \bar{\nu}_{P_{\tilde{s}}}))$. We could also have defined Θ using g_+, and in particular, it is clear that $\Theta \in C^\infty(\widetilde{S})$. It is easy to show that this is defined independently of φ. Thus, Θ is well-defined.

Let g_+ be a smooth CAH metric on X. We construct a product identification on X that we will use extensively throughout, and we then use it to show that b^*g_+ is a 0-edge metric. Choose an extension $(\breve{X}, \breve{g}_+)$. To each representative $\breve{h}$ on $\breve{M}$ we can associate a neighborhood $\breve{U}$ of $\breve{M}$ in $\breve{X}$ and a unique diffeomorphism $\chi : [0,\varepsilon)_r \times \breve{M} \to \breve{U}$ such that $\chi|_{\breve{M}} = \mathrm{id}$ and $\chi^*\breve{g}_+ = r^{-2}(dr^2 + \breve{h}_r)$, where $\breve{h}_0 = \breve{h}$. Now let y be a geodesic defining function for S in $\breve{M}$ with respect to the metric $\breve{h}$ – that is, a solution near S on $\breve{M}$ to the equation $|dy|^2_{\breve{h}} = 1$ with $y|_S \equiv 0$. We choose $y > 0$ on M. Then there is a diffeomorphism ψ from $S \times (-\delta,\delta)_y$ to a neighborhood W of S in $\breve{M}$ such that $\psi^*\breve{h} = dy^2 + k_y$, where k_y is a smooth one-parameter family of metrics on S. Thus, we have shown that there is a neighborhood $\breve{U}$ of S in $\breve{X}$ and a unique diffeomorphism

$\varphi : [0,\varepsilon)_r \times S \times (-\delta,\delta)_y \to \breve{U}$, for which $\varphi|_{\{0\}\times S\times\{0\}} = \mathrm{id}_S$ and

$$\varphi^* \breve{g}_+ = \frac{dr^2 + \breve{h}_r}{r^2}, \tag{3.2}$$

where $\breve{h}_r$ is a one-parameter family of metrics on $S \times (-\delta,\delta)_y$ with

$$\breve{h}_0 = dy^2 + k_y. \tag{3.3}$$

We call this the product identification for $\breve{g}_+$ determined by $\breve{h}$, and we let $\pi_S : \breve{U} \to S$ be the projection onto S determined by it.

In cases where Q makes an obtuse angle with M, the values inside X of the functions r and y just constructed will depend on $\breve{g}_+$ outside X. We will use the product identification to analyze the behavior of geodesics in X, which of course is independent of the extension chosen.

We obtain smooth coordinates on the blowup $\widetilde{X}$ near $\widetilde{S}$ by introducing polar coordinates on X. Using the coordinates defined above, these are given by

$$r = \rho \sin\theta, \quad y = \rho\cos\theta, \text{ with } \rho \geq 0, \quad 0 \leq \theta < \pi. \tag{3.4}$$

Then locally, a product identification on the blowup is given by $p \mapsto (\theta(p), \pi_S(p), \rho(p))$. Observe that $\widetilde{S}$ is given precisely by $\rho = 0$ and $\widetilde{M}$ is given by $\theta = 0$. For any admissible metric g on $\widetilde{X}$ such that $g = b^* g_+ + \mathcal{L}$, this identification on the blowup will be called a *polar g-identification*, or depending on context, *polar g-coordinates*. Notice that by (3.2) and (3.3), $\theta|_{\widetilde{S}} = \Theta$.

Now by (3.4), we have

$$dr = \rho\cos\theta d\theta + \sin\theta d\rho$$
$$dy = -\rho\sin\theta d\theta + \cos\theta d\rho.$$

Let $\{x^s\}_{s=1}^{n-1}$ be local coordinates on S. Extend these into $\widetilde{X}$ near $\widetilde{S}$ using the product identification. Note by (3.3) that in (3.2), $\breve{h}_r = dy^2 + k_y + O(r)$. It is then straightforward to compute the metric in our new coordinates:

$$b^* g_+ = \left(\frac{d\theta}{\sin\theta}, \quad \frac{dx^s}{\rho\sin\theta}, \quad \frac{d\rho}{\rho\sin\theta} \right) G \begin{pmatrix} \frac{d\theta}{\sin\theta} \\ \frac{dx^s}{\rho\sin\theta} \\ \frac{d\rho}{\rho\sin\theta} \end{pmatrix}, \tag{3.5}$$

where

$$G = \begin{pmatrix} 1 + O(\rho \sin^3 \theta) & O(\rho \sin^2 \theta) & O(\rho \sin^2 \theta) \\ O(\rho \sin^2 \theta) & k_{\rho \cos \theta} + O(\rho \sin \theta) & O(\rho \sin \theta) \\ O(\rho \sin^2 \theta) & O(\rho \sin \theta) & 1 + O(\rho \sin \theta) \end{pmatrix} \tag{3.6}$$

Thus, $b^* g_+$ is a 0-edge metric. Notice that $k_{\rho \cos \theta} = k_\rho + O(\rho \sin^2 \theta)$. This yields the following.

Proposition 3.3. *In a polar identification, an admissible metric g on $\widetilde{X}$ takes the form*

$$g = \frac{1}{\sin^2(\theta)} \left[d\theta^2 + \frac{d\rho^2 + k_\rho}{\rho^2} \right] + (\rho \sin \theta)\ell, \tag{3.7}$$

where k_ρ is a one-parameter family of metrics on S and $\ell \in C^\infty(S^2({}^{0e}T^*\widetilde{X}))$.

We note that the statement that g can be written in the form (3.7) is equivalent to the statement that it can be written as

$$g = \frac{1}{\sin^2(\theta)} \left[d\theta^2 + \frac{d\rho^2 + k_{\theta,\rho}}{\rho^2} \right] + (\rho \sin \theta)\ell,$$

where ℓ is as before and where $k_{\theta,\rho}$ is a two-parameter family of metrics on S such that $k_{\theta,0}$ is independent of θ.

Notice that for the hyperbolic metric, (1.2) exhibits the form (3.7) with $k = |dx|^2$ and $\ell = 0$.

It will be useful to have equation (3.7) expressed in block form. On $\widetilde{X}$ in the coordinates (θ, x^s, ρ), the metric takes the form

$$g_{ij} = \csc^2(\theta) \begin{pmatrix} 1 + O(\rho \sin \theta) & O(\sin \theta) & O(\sin \theta) \\ O(\sin \theta) & \rho^{-2} k_\rho + O(\rho^{-1} \sin \theta) & O(\rho^{-1} \sin \theta) \\ O(\sin \theta) & O(\rho^{-1} \sin \theta) & \rho^{-2} + O(\rho^{-1} \sin \theta) \end{pmatrix}. \tag{3.8}$$

This may also be written

$$g_{ij} = \csc^2(\theta) A(\rho) \begin{pmatrix} 1 + O(\rho \sin \theta) & O(\rho \sin \theta) & O(\rho \sin \theta) \\ O(\rho \sin \theta) & k_\rho + O(\rho \sin \theta) & O(\rho \sin \theta) \\ O(\rho \sin \theta) & O(\rho \sin \theta) & 1 + O(\rho \sin \theta) \end{pmatrix} A(\rho),$$

where

$$A(\rho) = \begin{pmatrix} 1 & & \\ & \rho^{-1} & \\ & & \rho^{-1} \end{pmatrix}.$$

This allows us easily to use Cramer's rule to find that

$$g^{ij} = \sin^2(\theta) \begin{pmatrix} 1 + O(\rho \sin\theta) & O(\rho^2 \sin\theta) & O(\rho^2 \sin\theta) \\ O(\rho^2 \sin\theta) & \rho^2 k_\rho^{-1} + O(\rho^3 \sin\theta) & O(\rho^3 \sin\theta) \\ O(\rho^2 \sin\theta) & O(\rho^3 \sin\theta) & \rho^2 + O(\rho^3 \sin\theta) \end{pmatrix}. \tag{3.9}$$

3.2 Function Spaces

In this section we define function spaces that will be of use throughout Chapters 5 and 6.

First we define several spaces of functions on the interval $[0, \theta_0]$, where $\theta_0 \in (0, \pi)$ is fixed. For $n \geq 2$ and $k \geq 1$, define

$$\mathcal{A}_{n,k}(\theta_0) = C^\infty([0, \theta_0]) \oplus \bigoplus_{i=1}^{k} (\theta^n \log\theta)^i C^\infty([0, \theta_0]), \tag{3.10}$$

and

$$\mathcal{A}^0_{n,k}(\theta_0) = \left\{u \in \mathcal{A}_{n,k} : u(0) = 0 = u'(\theta_0)\right\}. \tag{3.11}$$

Next, define

$$\mathcal{B}_{n,1}(\theta_0) = \theta C^\infty([0, \theta_0]) \oplus \theta^{n+1} \log(\theta) C^\infty([0, \theta_0]), \tag{3.12}$$

and for $k \geq 2$, define

$$\mathcal{B}_{n,k}(\theta_0) = \mathcal{B}_{n,1}(\theta_0) \oplus \bigoplus_{i=2}^{k} (\theta^n \log\theta)^i C^\infty([0, \theta_0]). \tag{3.13}$$

In general, θ_0 will be fixed and clear from the context, and we will refer to these spaces simply as $\mathcal{A}_{n,k}$ and $\mathcal{B}_{n,k}$.

Now let Y be any manifold with or without boundary, and $\mathcal{V}$ a vector bundle over Y. We define spaces of one-parameter families of smooth sections $u_\theta : Y \to \mathcal{V}$ $(0 \leq \theta \leq \theta_0)$ as follows.

Let $C^\infty(Y, \mathcal{V})$ be the space of smooth sections of $\mathcal{V}$, i.e., smooth maps $\sigma : Y \to \mathcal{V}$ such that $\pi_{\mathcal{V}} \circ \sigma = \mathrm{id}_Y$. Set

$$\mathcal{A}_{n,k}(\theta_0, Y, \mathcal{V}) = C^\infty([0,\theta_0], C^\infty(Y, \mathcal{V})) \oplus \bigoplus_{i=1}^{k} (\theta^n \log\theta)^i C^\infty([0,\theta_0], C^\infty(Y, \mathcal{V})).$$

Thus, in particular, $\mathcal{A}_{n,k}(\theta_0)$ is canonically isomorphic to $\mathcal{A}_{n,k}(\theta_0, \{0\}, \mathbb{R})$. We similarly define $\mathcal{A}^0_{n,k}(\theta_0, Y, \mathcal{V})$ to be the space of functions $u_\theta \in \mathcal{A}_{n,k}(\theta_0, Y, \mathcal{V})$ satisfying $u_\theta|_{\theta=0} = 0$ and $\partial_\theta u_\theta|_{\theta=\theta_0} = 0$, and we set

$$\mathcal{B}_{n,1}(\theta_0, Y, \mathcal{V}) = \theta C^\infty([0,\theta_0], C^\infty(Y, \mathcal{V})) \oplus \theta^{n+1}\log(\theta) C^\infty([0,\theta_0], C^\infty(Y, \mathcal{V}))$$

and for $k \geq 2$,

$$\mathcal{B}_{n,k}(\theta_0, Y, \mathcal{V}) = \mathcal{B}_{n,1}(\theta_0, Y, \mathcal{V}) \oplus \bigoplus_{i=2}^{k} (\theta^n \log\theta)^i C^\infty([0,\theta_0], C^\infty(Y, \mathcal{V})).$$

In each of these function spaces, if $\mathcal{V}$ is omitted, then it is taken to be the trivial vector bundle $\mathbb{R} \times Y$.

Now if $n \geq 4$ is even, then we define $\mathcal{H}_{n,k}(\theta_0, Y, \mathcal{V}) = \mathcal{A}_{n,k}(\theta_0, Y, \mathcal{V})$; and we let $\mathcal{H}_n(\theta_0, Y, \mathcal{V})$ be the space of maps u in $C^\infty((0,\theta_0], C^\infty(Y, \mathcal{V})) \cap C^{n-1}([0,\theta_0], C^\infty(Y, \mathcal{V}))$ that have an infinite asymptotic expansion $u \sim \sum_{i=0}^\infty u_i$ at $\theta = 0$, where $u_i \in (\theta^n \log\theta)^i C^\infty([0,\theta_0], C^\infty(Y, \mathcal{V}))$. If $n = 2$ or n is odd, then for consistency of notation, we define $\mathcal{H}_{n,k}(\theta_0, Y, \mathcal{V})$ and $\mathcal{H}_n(\theta_0, Y, \mathcal{V})$ to be simply $C^\infty([0,\theta_0], C^\infty(Y, \mathcal{V}))$.

If M^n is a manifold with boundary, we define $\mathcal{M}(\theta_0, M) \subset \mathcal{H}_n(\theta_0, M, S^2({}^0T^*M))$ to be the sections h_θ in $\mathcal{H}_n(\theta_0, M, S^2({}^0T^*M))$ such that, for θ fixed, h_θ is a smooth AH metric on M.

We next define two families of function spaces on $[0,\theta_0]_\theta \times [0,\varepsilon)_\rho$. For $q \geq 0$ an integer, we let $\mathcal{E}_{n,q}$ be the space of functions η that can be written

$$\eta(\theta,\rho) = \rho^{n+q} \sum_{i=0}^{\lfloor \frac{q}{2}+1 \rfloor} \log(\rho)^i b_i(\theta),$$

where each $b_i \in \mathcal{A}^0_{n,1}$. Next, we let $\mathcal{F}_{n,q}$ be the space of functions η on $[0,\theta_0] \times [0,\varepsilon)$ that can be written

$$\eta(\theta,\rho) = \rho^{n+q} \sum_{i=0}^{\lfloor \frac{q+1}{2} \rfloor} \log(\rho)^i c_i(\theta),$$

where each $c_i \in \mathcal{B}_{n,1}$.

Finally, let $(\widetilde{X}, \widetilde{M}, \widetilde{Q}, \widetilde{S})$ be the blowup of a cornered space X^{n+1}. Let (θ, x, ρ), where $x \in S$, be a parametrization of $\widetilde{X}$ near $\widetilde{S}$ for which θ and ρ are defining functions for $\widetilde{M}$ and $\widetilde{S}$, respectively. An example would be the polar decompositions considered in the previous section. Then we define

$$\mathcal{R}(\widetilde{X}) = C^\infty(\widetilde{X}) + \theta^n \log(\theta) C^\infty(\widetilde{X}). \tag{3.14}$$

Notice that the definition is independent of the choice of θ. Then, define $\mathcal{P}(\widetilde{X})$ to be the space of functions $u \in C^\infty(\mathring{\widetilde{X}}) \cap C^{n-1}(\widetilde{X})$ that have an asymptotic expansion

$$u(\theta, x, \rho) \sim a_0(\theta, x, \rho) + \sum_{j=1}^{\infty} \rho^{n+2(j-1)} \log(\rho)^j a_j(\theta, x, \rho), \tag{3.15}$$

where each $a_j \in \mathcal{R}(\widetilde{X})$.

3.3 Notation

Throughout this thesis, unless otherwise noted, $(\widetilde{X}, \widetilde{M}, \widetilde{Q}, \widetilde{S})$ will be the blowup of a cornered space (X, M, Q), with blowdown map $b : \widetilde{X} \to X$. We let $g = b^* g_X = b^* g_+ + (\rho \sin\theta)\ell$ be an admissible metric on $\widetilde{X}$. Except where noted otherwise, θ and ρ will denote the polar coordinates in a polar g-coordinate system. Similarly, $r = \rho \sin\theta$ will denote the geodesic defining function on $X \subset \check{X}$ in terms of which they were defined. The projection onto the S factor will be denoted π_S.

X will be of dimension $n+1$, where unless otherwise specified, $n \geq 2$ always.

We use index notation in polar coordinates. When doing so, 0 will refer to the first factor θ, and n will refer to the last factor ρ. The indices $1 \leq s, t \leq n-1$ will refer to local coordinates on the second factor, S, while the indices $0 \leq i, j \leq n$ will run over all $n+1$ coordinates. Finally, $1 \leq \mu, \nu \leq n$ will run over S and the last factor.

The metric g will be used to raise and lower indices, except that g^{ij} is the inverse metric. We write $\bar{g} = \rho^2 \sin^2(\theta) g = r^2 g$ for the metric compactified with respect to polar g-coordinates. Note that $\bar{g}$ is degenerate along $\widetilde{S}$, as $\left|\frac{\partial}{\partial\theta}\right|_{\bar{g}} = 0$ there.

If $a > 0$, we define

$$\widetilde{Q}_a = \left\{ q \in \widetilde{Q} : 0 < \rho(q) < a \right\},$$

and

$$\underline{\widetilde{Q}_a} = \left\{ q \in \widetilde{Q} : 0 \leq \rho(q) < a \right\}.$$

For general open $V \subseteq \widetilde{Q}$, we define $\underline{V} = V \cup L$, where L is the interior in $\widetilde{Q} \cap \widetilde{S}$ of the set of limit points of V in $\widetilde{S}$. We also define

$$\widetilde{X}_a = \left\{ x \in \widetilde{X} : 0 < \rho(x) < a \right\}.$$

We let ν be the inward-pointing unit normal vector field on $\widetilde{Q} \backslash \widetilde{S}$ with respect to g. If $q \in \widetilde{Q} \backslash \widetilde{S}$, we let γ_q denote the g-geodesic that begins at q and has initial tangent vector $\gamma_q'(0) = \nu_q$.

If A is a covariant k-tensor, we write $A = O_g(f)$ to indicate that $|A|_g = O(f)$; or equivalently, that if $Y_1, \ldots, Y_k$ are g-unit vector fields, then $A(Y_1, \ldots, Y_k) = O(f)$ with constant independent of the Y_i. Similarly, if Y is a vector field, we write $Y = O_g(f)$ to indicate $|Y|_g = O(f)$. Note that this condition is independent of the particular admissible metric g.

Chapter 4

NORMAL FORM AND GEODESICS

In this chapter, we study geodesics off the finite boundary $\widetilde{Q}$ of a blown up space, and use our results to prove Theorems 1.1 - 1.3 and variations. We take $(\widetilde{X}, \widetilde{M}, \widetilde{Q}, \widetilde{S})$ to be the blowup of a fixed cornered space (X, M, Q), with $g = b^*g_+ + (\rho \sin\theta)\ell$ a fixed admissible metric.

4.1 Behavior of Normal Geodesics

As a starting point to our study of the normal exponential map over the finite boundary $\widetilde{Q}$ of the blowup, in this section we study the basic behavior of g-geodesics that leave $\widetilde{Q}$ normally, where g is an admissible metric on $(\widetilde{X}, \widetilde{M}, \widetilde{Q}, \widetilde{S})$. Throughout, we will work in a polar g-identification as constructed in the Section 3.1, and we will let $\widetilde{U}$ be a neighborhood of $\widetilde{S}$ in $\widetilde{X}$ on which such coordinates exist.

Our first task is to study the normal field to $\widetilde{Q} \setminus \widetilde{S}$ near $\widetilde{S}$.

Lemma 4.1. *Sufficiently near $\widetilde{S}$, the normal field ν on $\widetilde{Q} \setminus \widetilde{S}$ satisfies*

$$\nu = -\sin\theta \frac{\partial}{\partial\theta} + O_g(\rho). \tag{4.1}$$

In particular, ν extends smoothly to $\widetilde{Q} \cap \widetilde{S}$.

Proof. Using the implicit function theorem and the fact that $\frac{\partial}{\partial\theta}$ is transverse to $\widetilde{Q} \cap \widetilde{S}$, we may write $\widetilde{Q}$ near $\widetilde{S}$ smoothly as $\theta = \psi(x, \rho)$. Define $f(\theta, x, \rho) = \psi(x, \rho) - \theta$. Then using (3.9), we find in local coordinates that

$$\operatorname{grad} f = g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j} = (-\sin^2(\theta) + O(\rho)) \frac{\partial}{\partial\theta} + O(\rho^2),$$

while

$$|\operatorname{grad} f|_g = \sin\theta + O(\rho),$$

where we keep in mind that $\sin\theta$ is bounded away from 0 on $\widetilde{Q}$. As $\nu = \frac{\operatorname{grad} f}{|\operatorname{grad} f|_g}$, the result follows. ■

It has long been the case that convex functions are important in studying spaces of negative curvature; see [BO69]. Most of our interior analysis of the g-geodesics leaving Q will follow from the fact that the cotangent function on a cornered AH space has a Hessian of a very special form related to convexity. This Hessian equation actually has a history of its own in the negatively curved setting and beyond; see for example [CC96, HPW15].

Lemma 4.2. *Define $w \in C^\infty(\widetilde{U} \setminus \widetilde{M})$ by $w = \cot(\theta)$. Then*

$$\nabla_g^2 w = wg + O_g(\rho). \tag{4.2}$$

Remark. This result is motivated by the fact that, in the case of the hyperbolic upper half-space, the equation holds exactly.

Proof. We will need the Christoffel symbols Γ_{ij}^0. Computing from (3.8) and (3.9) using the equation $\Gamma_{ij}^k = \frac{1}{2}g^{kl}(\partial_i g_{lj} + \partial_j g_{li} - \partial_l g_{ij})$, we find that

$$\Gamma_{ij}^0 = \begin{pmatrix} -\cot(\theta) + O(\rho) & O(1) & O(1) \\ O(1) & \rho^{-2}\cot(\theta)k_{st} + O(\rho^{-1}) & O(\rho^{-1}) \\ O(1) & O(\rho^{-1}) & \rho^{-2}\cot(\theta) + O(\rho^{-1}) \end{pmatrix}.$$

Now $dw = -\csc^2(\theta)d\theta$, and we can use these Christoffel computations to find that

$$\begin{aligned} \nabla^2 w = \nabla(dw) &= \cot(\theta)g + \frac{(d\theta, dx^s, d\rho)}{\sin^2\theta} \begin{pmatrix} O(\rho) & O(1) & O(1) \\ O(1) & O(\rho^{-1}) & O(\rho^{-1}) \\ O(1) & O(\rho^{-1}) & O(\rho^{-1}) \end{pmatrix} \begin{pmatrix} d\theta \\ dx^t \\ d\rho \end{pmatrix} \\ &= \cot(\theta)g + \left(\frac{d\theta}{\sin\theta}, \frac{dx^s}{\rho\sin\theta}, \frac{d\rho}{\rho\sin\theta}\right) \begin{pmatrix} O(\rho) & O(\rho) & O(\rho) \\ O(\rho) & O(\rho) & O(\rho) \\ O(\rho) & O(\rho) & O(\rho) \end{pmatrix} \begin{pmatrix} \frac{d\theta}{\sin\theta} \\ \frac{dx^t}{\rho\sin\theta} \\ \frac{d\rho}{\rho\sin\theta} \end{pmatrix}. \end{aligned}$$

This proves the claim. ■

We next need two technical results.

Proposition 4.3. *Let $D = J \times \mathbb{R}^2_{(u,v)} \subseteq \mathbb{R}^3$, where J is an interval containing $[a,b)$. Let $f : D \to \mathbb{R}$ be continuous and locally Lipschitz, and weakly increasing in u. Define an ordinary differential operator L by $Lu(x) = u''(x) - f(x, u(x), u'(x))$. Suppose $\theta, \psi : [a,b) \to \mathbb{R}$ are C^2 functions such that the graphs of $t \mapsto (\theta(t), \theta'(t))$ and $t \mapsto (\psi(t), \psi'(t))$ lie in D. If*

- $\theta(a) \leq \psi(a)$*; and*
- $\theta'(a) \leq \psi'(a)$*; and*
- $L\theta \leq L\psi$ *on* $[a,b)$*,*

then $\theta(t) \leq \psi(t)$ and $\theta'(t) \leq \psi'(t)$ for all $t \in [a,b)$.

This is essentially Theorem 11.XVI in [Wal98].

Lemma 4.4. *(a) Let $0 < \delta < 1$, $a > 0$, and $b \in \mathbb{R}$, and set $f(t) = ae^t + be^{-t} + \delta$. Then there exists a continuous function $C = C(a,b,\delta) > 0$ such that, if $w(t) \geq f(t)$ for all $t \geq 0$, then $1 + w(t)^2 \geq C^{-2}e^{2t}$ for all $t \geq 0$.*

(b) Let $0 < \delta < 1$, $0 < a_1 < a_2$, and $b_1, b_2 \in \mathbb{R}$. Suppose that $a_1e^t + b_1e^{-t} + \delta \leq w(t) \leq a_2e^t + b_2e^{-t} - \delta$. Then there exists $D = D(a_1, a_2, b_1, b_2, \delta)$, continuous in its arguments, such that $1 + w(t)^2 \leq D^2e^{2t}$ for all $t \geq 0$.

Proof. (a) There exists $T > 0$ such that, for all $t \geq T$, we have $f(t) > \frac{1}{2}ae^t$. Let $\frac{2}{a} < C$ be such that $C^{-2}e^{2T} \leq 1$. The result follows immediately.

The proof of (b) is similar. ∎

We also recall the following result from [Maz86].

Proposition 4.5 (Propositions 1.8 and 1.9 in [Maz86])**.** *Let (X, M, g) be an asymptotically hyperbolic manifold, and φ a defining function for M. There exists $\varphi_0 > 0$ such that, if $p \in X$ with $0 < \varphi(p) < \varphi_0$, and if $\gamma : [0,\infty) \to \mathring{X}$ is a geodesic ray with $\gamma(0) = p$ and $(\varphi \circ \gamma)'(0) < 0$, then γ asymptotically approaches a well-defined point of M, normally with respect to $\varphi^2 g$.*

In the introduction, we discussed a subset (X, M, Q) of hyperbolic space $\mathbb{H}^{n+1}$ as an example of a CAH manifold, with Q a Euclidean plane. Recall that geodesics leaving Q normally are semicircles that approach the infinite boundary $\mathbb{R}^n$ orthogonally. The following theorem, which is the main result of this section, shows that this behavior is approximated by the geodesics in a general cornered AH space.

First some notation. We let d_S be the distance function on S with respect to k_0, where k_0 is as in (3.8), and $\pi_S : \widetilde{U} \to S$ be projection onto the S factor in the polar g-identification.

Proposition 4.6. *Let $(\widetilde{X}, \widetilde{M}, \widetilde{Q}, \widetilde{S})$ be the blowup of a cornered space (X, M, Q), and g an admissible metric. There exists $a > 0$ such that for each $q \in \widetilde{Q}_a$,*

- *γ_q exists for all $t \geq 0$ and $\gamma_q(t) \in \mathring{\widetilde{X}}$ for all $t > 0$;*
- *the limit $\lim_{t\to\infty} \gamma_q(t)$ exists and lies in $\widetilde{M} \setminus \widetilde{S}$, and γ_q approaches $\widetilde{M}$ $\bar{g}$-normally; and*
- *for all $t \geq 0$, $(\theta \circ \gamma_q)'(t) < 0$.*

Moreover, there exist $\varepsilon > 0$ and $A_1, A_2, C > 0$ such that, for all $q \in \widetilde{Q}_a$ and all $t \in [0, \infty)$,

(a) $\varepsilon\rho(q) < \rho(\gamma_q(t)) < C\rho(q)$;

(b) $d_S(\pi_S(q), \pi_S(\gamma_q(t))) < C\rho(q)$; *and*

(c) $A_1 e^{-t} < \sin(\theta(\gamma(t))) < A_2 e^{-t}$.

Note that we will improve this result in Proposition 4.7; in particular, it will imply that in (b) we could write $\rho(q)^2$ instead of $\rho(q)$, and that in (a) we can take $\frac{\varepsilon}{C} \to 1$ as $a \to 0$.

Proof. By Lemma 4.1, $\nu = -\sin(\theta)\frac{\partial}{\partial\theta} + O_g(\rho)$. Set $\nu_q^\theta = d\theta(\nu_q)$. Thus, for $q \in \widetilde{Q} \setminus \widetilde{S}$ sufficiently near $\widetilde{S}$, we have $-\csc^2(\theta(q))\nu_q^\theta = \csc(\theta(q)) + O(\rho(q))$, uniformly in q. Now for $\theta \in (0, \pi)$, we always have $\csc\theta > |\cot\theta|$. Let $0 < \delta < 1$ be such that there exists $\rho_0 > 0$ so that $\widetilde{X}_{\rho_0} \subset \widetilde{U}$ and such that

$$\alpha := \inf_{q\in\widetilde{Q}_{\rho_0}} (-\csc^2(\theta(q))\nu_q^\theta - |\cot(\theta(q))| - \delta) > 0.$$

Such a δ exists because θ is bounded away from 0 and π on $\widetilde{Q}$. Now let

$$\beta := \sup_{q\in\widetilde{Q}_{\rho_0}} \left(\max\left\{-\csc^2(\theta(q))\nu_q^\theta + |\cot(\theta(q))| + \delta, |\csc^2(\theta(q))\nu_q^\theta + \cot(\theta(q)) + \delta|\right\}\right) > 0,$$

which is finite because the cosecant and cotangent functions are bounded on $\widetilde{Q}$ near $\widetilde{S}$. Next, let $B > 0$ be large enough that, for all $Y \in T(\widetilde{X}_{\rho_0})$ with $|Y|_g = 1$, we have $|d\rho(Y)| < B\rho\sin\theta$; such B exists by (3.8).

For $q \in \widetilde{Q}_{\rho_0}$, define $\theta_q(t) = \theta(\gamma_q(t))$, and $\rho_q(t) = \rho(\gamma_q(t))$. Next, define $w_q(t) = \cot(\theta_q(t))$; thus w_q is defined on the same domain as the geodesic γ_q. Noting that $w_q(0) = \cot(\theta(q))$ and $\dot{w}_q(0) = -\csc^2(\theta)\nu_q^\theta$, let C be as defined in Lemma 4.4 and set

$$A := \sup_{q\in\widetilde{Q}_{\rho_0}} C\left(\frac{1}{2}(w_q(0) + \dot{w}_q(0) - \delta), \frac{1}{2}(w_q(0) - \dot{w}_q(0) - \delta), \delta\right) > 0,$$

which is finite. Also let D be as in Lemma 4.4 and set

$$\begin{aligned} E := \sup_{q\in\widetilde{Q}_{\rho_0}} D\Bigg(&\frac{1}{2}(w_q(0) + \dot{w}_q(0) - \delta), \frac{1}{2}(w_q(0) + \dot{w}_q(0) + \delta), \frac{1}{2}(w_q(0) - \dot{w}_q(0) - \delta), \\ &\frac{1}{2}(w_q(0) - \dot{w}_q(0) + \delta), \delta\Bigg) > 0, \end{aligned}$$

which is likewise finite. By shrinking it if necessary, we may assume that ρ_0 is small enough that, if $q \in \widetilde{Q}_{\rho_0}$, then

$$\left|\nu_q + \sin(\theta)\frac{\partial}{\partial\theta}\right|_g < \frac{\alpha E^{-1}}{8}. \tag{4.3}$$

We may similarly suppose, by (3.8), that on $\widetilde{Q}_{\rho_0}$,

$$\sin(\theta)|g_{00} - \csc^2(\theta)| < \min\left\{\frac{\alpha E^{-1}}{8A\beta}, 1\right\}, \tag{4.4}$$

and that if $Y \in T\widetilde{X}|_{\widetilde{Q}_{\rho_0}}$ with $d\theta(Y) = 0$, then

$$\left|\left\langle \sin\theta\frac{\partial}{\partial\theta}, Y\right\rangle\right| \leq \frac{\alpha E^{-1}}{8(1 + \sqrt{2}A\beta)}|Y|_g. \tag{4.5}$$

Now because γ_q is a geodesic, $\frac{d^2}{dt^2}w(\gamma_q(t)) = (\nabla_g^2 w)(\dot{\gamma}_q, \dot{\gamma}_q)$. It follows by Lemma 4.2 that

$$\ddot{w}_q = w_q + O(\rho_q(t)).$$

By shrinking ρ_0 if necessary, we assume that the $O(\rho)$ term in this equation is bounded by δ for $\rho \leq \rho_0$. Now let $0 < a < \frac{1}{2e^{AB}}\rho_0$. We henceforth assume $q \in \widetilde{Q}_a$.

Now let $f_\pm$ be the solutions to $\ddot{f}_\pm = f_\pm \pm \delta$, with $f_\pm(0) = w_q(0)$ and $\dot{f}_\pm(0) = \dot{w}_q(0)$. Then

$$f_\pm(t) = \frac{1}{2}\left(w_q(0) + \dot{w}_q(0) \pm \delta\right)e^t + \frac{1}{2}\left(w_q(0) - \dot{w}_q(0) \pm \delta\right)e^{-t} \mp \delta.$$

The leading coefficient is always positive, by our choice of δ. Moreover, we have

$$\ddot{f}_- - f_- = -\delta \leq \ddot{w}_q - w_q \leq \delta = \ddot{f}_+ - f_+,$$

so by Proposition 4.3, we have

$$f_-(t) \leq w_q(t) \leq f_+(t) \tag{4.6}$$

for all $t \geq 0$ such that $\rho_q(t) < \rho_0$ up to t, and so long as the geodesic continues to exist. Also by the same proposition,

$$\dot{f}_-(t) \leq \dot{w}_q(t) \leq \dot{f}_+(t), \tag{4.7}$$

subject to the same constraints. Since both bounding functions are positive, we conclude that $\dot{w}_q(t) > 0$ for all q, and for all $t \geq 0$ such that $\rho_q(t) < \rho_0$. This implies that $\dot{\theta} < 0$ for all such $t \geq 0$. In addition, the coefficients appearing in $f_\pm$ are uniformly bounded in q. It follows that we have shown that $\theta_q(t)$ goes to zero and $\cot(\theta_q(t))$ goes to infinity exponentially, *so long as* $\rho_q(t)$ remains bounded by ρ_0 and γ_q exists.

Now by Lemma 4.4 and the definition of f_-, we have

$$\csc^2(\theta_q(t)) = 1 + w_q(t)^2 \geq A^{-2}e^{2t}.$$

(This and the following continue, for now, to depend on the assumption that ρ remains bounded by ρ_0.) Hence, also,

$$\sin(\theta_q(t)) \leq Ae^{-t}. \tag{4.8}$$

Also by (4.6), by definition of E, and by Lemma 4.4, we have

$$\sin(\theta_q(t)) \geq E^{-1}e^{-t}. \tag{4.9}$$

It now follows from (4.8) that $\left|\frac{\dot{\rho}_q}{\rho_q}\right| < ABe^{-t}$, by definition of B. Hence, at least as long as $\rho \leq \rho_0$, we find by integrating that

$$e^{-AB}\rho_q(0) < \rho_q(t) < e^{AB}\rho_q(0). \tag{4.10}$$

But then, since $\rho_q(0) \leq a \leq \frac{\rho_0}{2e^{AB}}$, $\rho_q(t)$ must remain bounded by $\frac{\rho_0}{2}$; so a brief contradiction argument shows that, indeed, (4.6) – (4.10) hold for all time $t \geq 0$ such that γ_q exists. We have also shown that γ_q remains bounded away from $\widetilde{S}$, i.e., ρ is bounded away from 0. This shows that γ_q never reaches $\widetilde{S}$, and also, with (4.10), yields (a). Also, by (4.8) and (4.9), we have (c).

We next analyze the motion in the tangential directions along S. It follows from the definition and smoothness of k_ρ in (3.8) that there exists $K > 0$ such that, for unit-length $Y \in T\widetilde{X}_{\rho_0}$, we have $|d\pi_S(Y)|_{k_0} < K\rho\sin(\theta)$. It then follows, using (4.8) and (4.10), that

$$\int_0^\infty |d\pi_S(\dot{\gamma}(t))|_{k_0} dt < AKe^{AB}\rho(q),$$

which yields (b).

Since $\cot(\theta)$ eventually becomes positive with $\dot{\theta}$ negative, we conclude that $\dot{r}$ is ultimately negative, and so the fact that γ_q approaches a defined point of $\widetilde{M}$ normally, if it exists for all time, follows by the analysis of geodesics in the standard AH case, Proposition 4.5. Thus, we have established all desired behavior, except that the geodesic γ_q might leave $\mathring{\widetilde{X}}$ and return to $\widetilde{Q}$, ceasing to exist. Since the geodesic is unit speed, it either exists for all time or returns to $\widetilde{Q}$ in finite time, and so we have only to show that the latter does not happen.

Suppose by way of contradiction that γ_q does return to $\widetilde{Q}$, say at q' and at time $t_1 > 0$. Then $\rho(q') < \rho_0$, and we have $\langle \dot{\gamma}_q(t_1), \nu_{q'} \rangle \leq 0$. Now by (4.7), the definitions of f_- and α, and (4.9), we deduce that

$$\frac{\alpha E^{-1}}{2} \leq \frac{|\dot{\theta}_q(t)|}{\sin(\theta_q(t))} \tag{4.11}$$

for all times $t \geq 0$, and in particular t_1. Similarly, by (4.7), (4.8), the definition of f_+, and the definition of β, we get

$$\frac{|\dot{\theta}_q(t)|}{\sin(\theta_q(t))} \leq A\beta. \tag{4.12}$$

By (4.12) and (4.4), and because $|\sin\theta| \leq 1$, we find that at t_1,

$$\begin{aligned}\left\langle \dot{\theta}\frac{\partial}{\partial\theta}, \dot{\theta}\frac{\partial}{\partial\theta}\right\rangle &= \frac{\dot{\theta}^2}{\sin^2\theta} + (g_{00} - \csc^2(\theta))\dot{\theta}^2 \\ &\leq A^2\beta^2 + A^2\beta^2 = 2A^2\beta^2.\end{aligned}$$

Thus,

$$\left|\dot{\theta}\frac{\partial}{\partial\theta}\right|_g \leq \sqrt{2}A\beta. \tag{4.13}$$

By (4.3)-(4.5), (4.11)-(4.13), Cauchy-Schwartz, and the triangle inequality, we find that

$$\begin{aligned}\langle\dot{\gamma}_q(t_1), \nu_{q'}\rangle &= \left\langle \dot{\gamma}_q(t_1), -\sin(\theta)\frac{\partial}{\partial\theta}\right\rangle + \left\langle \dot{\gamma}_q(t_1), \nu_{q'} + \sin(\theta)\frac{\partial}{\partial\theta}\right\rangle \\ &= \frac{|\dot{\theta}_q(t_1)|}{\sin(\theta)} - (g_{00} - \csc^2(\theta))\sin(\theta)\dot{\theta}_q(t_1) + \left\langle \dot{\gamma}_q(t_1) - \dot{\theta}_q(t_1)\frac{\partial}{\partial\theta}, -\sin(\theta)\frac{\partial}{\partial\theta}\right\rangle \\ &\quad + \left\langle \dot{\gamma}_q(t_1), \nu_{q'} + \sin(\theta)\frac{\partial}{\partial\theta}\right\rangle \\ &\geq \frac{\alpha E^{-1}}{2} - \frac{\alpha E^{-1}}{8A\beta}(A\beta) - \frac{\alpha E^{-1}}{8(1+\sqrt{2}A\beta)}\left(1 + \left|\dot{\theta}\frac{\partial}{\partial\theta}\right|_g\right) \\ &\quad - |\dot{\gamma}_q|_g \cdot \left|\nu_q + \sin(\theta)\frac{\partial}{\partial\theta}\right|_g \\ &\geq \frac{\alpha E^{-1}}{8} > 0,\end{aligned}$$

which is a contradiction. Hence, as desired, γ_q does not return to $\widetilde{Q}$. ■

4.2 The Exponential Map

We continue our analysis of the geodesics leaving $\widetilde{Q}$ normally now by turning our attention to the mapping properties of the exponential map on the normal bundle to $\widetilde{Q}$. We ultimately must prove that this map is a diffeomorphism on a suitable space. For now, we content ourselves with more local properties.

Let $N_+(\widetilde{Q}\setminus\widetilde{S})$ be the inward-pointing half-closed normal *ray* bundle to $\widetilde{Q}\setminus\widetilde{S}$, so that $N_+(\widetilde{Q}\setminus\widetilde{S}) \approx [0,\infty)_t \times (\widetilde{Q}\setminus\widetilde{S})$ by the identification $t\nu_q \mapsto (t,q)$; and similarly for the normal bundle over subsets of $\widetilde{Q} \setminus \widetilde{S}$. We denote the normal exponential map by exp. We have shown in Proposition 4.6 that there is some $a > 0$ such that exp is defined on the entirety of $N_+\widetilde{Q}_a$, and takes its values in $\widetilde{X} \setminus (\widetilde{M} \cup \widetilde{S})$. Trivially, $\{0\} \times (\widetilde{Q} \setminus \widetilde{S})$ is mapped by exp to $\widetilde{Q} \setminus \widetilde{S}$ as the identity, and Proposition 4.6 also shows that $\exp|^{-1}_{N_+\widetilde{Q}_a}(\widetilde{Q}_a) = \{0\} \times \widetilde{Q}_a$ as well. In order to show that the exponential map induces a diffeomorphism with a neighborhood of $\widetilde{S}$, we will have to analyze it as $q \to \widetilde{S}$ and as $t \to \infty$. We thus introduce a partial compactification of the normal bundle that includes faces corresponding to $t = \infty$ and to $[0, \infty] \times (\widetilde{Q} \cap \widetilde{S})$, and we will show that the exponential map is defined and a local diffeomorphism on the entire space.

Let V be a neighborhood of $\widetilde{Q} \cap \widetilde{S}$ in $\widetilde{Q}$. Then as observed previously, $N_+(V \setminus \widetilde{S})$ has a natural identification, induced by ν, with $[0, \infty) \times (V \setminus \widetilde{S})$. Letting t be the coordinate on the first factor, we set $\tau = 1 - e^{-t}$, and hence obtain an identification with $[0, 1) \times (V \setminus \widetilde{S})$. We thus define the compactification $\widehat{N_+(V \setminus \widetilde{S})} = [0, 1] \times V$, and we regard $N_+(V \setminus \widetilde{S}) \subset \widehat{N_+(V \setminus \widetilde{S})}$ as a subspace via the identification just described. Note that we have added two new faces in this compactification: one corresponding to $t = \infty$, and one corresponding to $[0, 1] \times (\widetilde{Q} \cap \widetilde{S})$. We will consistently let τ be the coordinate on the first factor of $\widehat{N_+(V \setminus \widetilde{S})}$. The space $\widehat{N_+(V \setminus \widetilde{S})}$ has a natural smooth structure as a manifold with corners, and $T\widehat{N_+(V \setminus S)} \cong T[0, 1] \oplus TV$ canonically. We note that $\widehat{N_+(V \setminus \widetilde{S})}$ is not quite a compactification, since the interior boundary of V in $\widetilde{Q}$ is still not included.

With the compactification of the normal bundle in hand, we are ready to extend the exponential map to reach the boundary. In proving the following, we follow the approach in [Maz86]. For the statement, notice that $\theta \mapsto v(\theta) := \csc\theta - \cot\theta$ is a diffeomorphism of $(0, \pi)$ with $(0, \infty)$.

Proposition 4.7. *There exists $\rho_0 > 0$ such that the exponential map* $\exp : N_+\widetilde{Q}_{\rho_0} \to \mathring{\widetilde{X}}$ *extends smoothly to a map* $\exp : \widehat{N_+\widetilde{Q}_{\rho_0}} \to \widetilde{X}$*, and the extended map is a local diffeomorphism of manifolds with corners. For $q \in \widetilde{Q} \cap \widetilde{S}$,* exp *maps $[0, 1] \times \{q\}$ to the b-fiber of $\widetilde{S}$ containing q. For such q,* exp *satisfies*

$$v(\Theta(\exp(\tau, q))) = v(\Theta(q))(1 - \tau); \tag{4.14}$$

that is, in the v coordinate, exp *is a linear function of τ.*

Moreover, for $1 \le \mu \le n$ *and* $q \in \widetilde{Q}$ *and for any* τ*, the equation*

$$x^\mu(\exp(\tau, q)) = x^\mu(q) + O(\rho(q)^2) \tag{4.15}$$

holds uniformly in $\tau \in [0,1]$*.*

Finally, there exists $c > 0$ *such that, if* $Y \in T_q\widetilde{Q}_{\rho_0} \subset T[0,\infty) \oplus T\widetilde{Q}_{\rho_0} \cong TN_+\widetilde{Q}_{\rho_0}$ *and* $t\nu_q \in N_+\widetilde{Q}_{\rho_0}$*, then*

$$|d\exp_{t\nu_q}(Y)|_{\bar{g}} \ge c|Y|_{\bar{g}}. \tag{4.16}$$

Proof. We begin by recalling the equations for the geodesic flow on the cotangent bundle. Let $\{x^s\}$ be local coordinates for S, so that (θ, x^s, ρ) are coordinates on some neighborhood $\widetilde{U} \subseteq \widetilde{X}$ of a fiber F in $\widetilde{S}$. Let ρ_1 be small enough that Proposition 4.6 holds on $\widetilde{Q}_{\rho_1}$, and let $V \subseteq \underline{\widetilde{Q}_{\rho_1}}$ be a sufficiently small neighborhood of the point $\widetilde{Q} \cap F$ that normal geodesics off points in $V \setminus \widetilde{S}$ remain in $\widetilde{U}$.

The geodesic flow off points of $V \setminus \widetilde{S}$ then satisfies

$$\begin{aligned} \dot{x}^i &= g^{ij}\xi_j \\ \dot{\xi}_i &= -\frac{1}{2}\frac{\partial g^{kl}}{\partial x^i}\xi_k\xi_l. \end{aligned}$$

We also have

$$g^{ij}\xi_i\xi_j = 1. \tag{4.17}$$

We use this fact to rewrite the geodesic equations in terms of $\bar{g} = \rho^2\sin^2(\theta)g$ or, rather, $\bar{g}^{-1} = \rho^{-2}\csc^2(\theta)g^{-1}$, obtaining

$$\begin{aligned} \dot{x}^i &= \rho^2\sin^2(\theta)\bar{g}^{ij}\xi_j \\ \dot{\xi}_i &= -\frac{1}{2}\frac{\partial}{\partial x^i}\left[\rho^2\sin^2(\theta)\bar{g}^{kl}\right]\xi_k\xi_l \\ &= -\frac{\rho_i}{\rho} - \cot(\theta)\theta_i - \frac{1}{2}\rho^2\sin^2(\theta)\frac{\partial\bar{g}^{kl}}{\partial x^i}\xi_k\xi_l. \end{aligned} \tag{4.18}$$

This system is obviously degenerate at both $\theta = 0$ and $\rho = 0$. We thus introduce rescaled variables, setting

$$\begin{aligned} \bar{\xi}_\mu &= \rho\xi_\mu \quad (1 \le \mu \le n) \\ \bar{\xi}_0 &= \sin(\theta)\xi_0. \end{aligned} \tag{4.19}$$

Hence,

$$\dot{\bar{\xi}}_\mu = \dot{\rho}\xi_\mu + \rho\dot{\xi}_\mu = \frac{\dot{\rho}}{\rho}\bar{\xi}_\mu + \rho\dot{\xi}_\mu$$
$$\dot{\bar{\xi}}_0 = \cos(\theta)\dot{\theta}\xi_0 + \sin(\theta)\dot{\xi}_0 = \cot(\theta)\dot{\theta}\bar{\xi}_0 + \sin(\theta)\dot{\xi}_0. \tag{4.20}$$

Now

$$\dot{\rho} = \rho^2\sin^2(\theta)\bar{g}^{nj}\xi_j = \rho\sin^2(\theta)\bar{g}^{n\mu}\bar{\xi}_\mu + \rho^2\sin(\theta)\bar{g}^{n0}\bar{\xi}_0 \text{ and}$$
$$\dot{\theta} = \rho^2\sin^2(\theta)\bar{g}^{0j}\xi_j = \rho\sin^2(\theta)\bar{g}^{0\mu}\bar{\xi}_\mu + \rho^2\sin(\theta)\bar{g}^{00}\bar{\xi}_0. \tag{4.21}$$

Thus, rewriting our equations of motion (4.18) and (4.20) in terms of our new variables, we get

$$\begin{aligned}
\dot{x}^i &= \rho\sin^2(\theta)\bar{g}^{i\mu}\bar{\xi}_\mu + \rho^2\sin(\theta)\bar{g}^{i0}\bar{\xi}_0\\
\dot{\bar{\xi}}_\mu &= (\sin^2(\theta)\bar{g}^{n\nu}\bar{\xi}_\nu + \rho\sin(\theta)\bar{g}^{n0}\bar{\xi}_0)\bar{\xi}_\mu - \rho_\mu - \frac{1}{2}\rho\sin^2(\theta)\frac{\partial\bar{g}^{\sigma\lambda}}{\partial x^\mu}\bar{\xi}_\sigma\bar{\xi}_\lambda\\
&\quad - \rho^2\sin(\theta)\frac{\partial\bar{g}^{0\sigma}}{\partial x^\mu}\bar{\xi}_0\bar{\xi}_\sigma - \frac{1}{2}\rho^3\frac{\partial\bar{g}^{00}}{\partial x^\mu}\bar{\xi}_0^2\\
\dot{\bar{\xi}}_0 &= (\rho\sin(\theta)\cos(\theta)\bar{g}^{0\mu}\bar{\xi}_\mu + \rho^2\cos(\theta)\bar{g}^{00}\bar{\xi}_0)\bar{\xi}_0 - \cos(\theta) - \frac{1}{2}\sin^3(\theta)\frac{\partial\bar{g}^{\sigma\lambda}}{\partial\theta}\bar{\xi}_\sigma\bar{\xi}_\lambda\\
&\quad - \rho\sin^2(\theta)\frac{\partial\bar{g}^{0\sigma}}{\partial\theta}\bar{\xi}_0\bar{\xi}_\sigma - \frac{1}{2}\rho^2\sin(\theta)\frac{\partial\bar{g}^{00}}{\partial\theta}\bar{\xi}_0^2.
\end{aligned} \tag{4.22}$$

Now $\bar{g}^{00} = O(\rho^{-2})$; otherwise $\bar{g}^{-1}$ is smooth on $\widetilde{X}$ by (3.9). It follows that equations (4.22) have smooth coefficients all the way up to $\rho = 0$.

We already know that, for $\rho(q) \neq 0$, solutions exist for all time t. We turn to study the case when $\rho(q) = 0$, that is, when the geodesic starts from some point $q \in V \cap \widetilde{S}$. Using (3.9), and recalling that $\bar{g}_{st}$ is independent of θ at $\rho = 0$, note that when $\rho = 0$, the equations (4.22) are given by

$$\begin{aligned}
\dot{\theta} &= \sin(\theta)\bar{\xi}_0\\
\dot{x}^\mu &= 0\\
\dot{\bar{\xi}}_0 &= \cos(\theta)\bar{\xi}_0^2 - \cos(\theta)\\
\dot{\bar{\xi}}_\mu &= \sin^2(\theta)\bar{\xi}_n\bar{\xi}_\mu - \rho_\mu + \rho_\mu\bar{\xi}_0^2.
\end{aligned} \tag{4.23}$$

Our initial conditions for q, by (4.1), (4.17), and the above, are

$$\begin{aligned}
\theta(0) &= \theta(q)\\
x^s(0) &= x^s(q)\\
\rho(0) &= 0 \qquad (4.24)\\
\bar{\xi}_0(0) &= -1\\
\bar{\xi}_\mu(0) &= 0.
\end{aligned}$$

Let $\psi(v)$ be the inverse of the function $\theta \mapsto \csc\theta - \cot\theta$; thus ψ is defined on $(0,\infty)$ and is monotonic increasing from 0 to π. Then observe that the solution to (4.22) with the given initial conditions is given by

$$\begin{aligned}
\theta(t) &= \psi((\csc\theta(q) - \cot\theta(q))e^{-t})\\
x^s(t) &\equiv x^s(0)\\
\rho(t) &\equiv 0 \qquad (4.25)\\
\bar{\xi}_0(t) &\equiv -1\\
\bar{\xi}_\mu(t) &\equiv 0.
\end{aligned}$$

This exists for all $t \geq 0$, and it satisfies the properties that $\dot{\theta} < 0$ for all time and that $\lim_{t\to\infty}\theta(t) = 0$. Using smooth dependence of solutions on initial conditions, we conclude that solutions to the geodesic equations may be smoothly extended to $\rho = 0$ for all time $t \geq 0$. Thus, exp is smooth on $[0,\infty) \times V$; and by compactness of $\widetilde{S}$, on $[0,\infty)_t \times \underline{\widetilde{Q}_{\rho_0}}$ for some ρ_0.

We now turn our attention to $\theta = 0$, which corresponds to $t = \infty$. We compactify the normal bundle, as above, by setting $\tau = 1 - e^{-t}$, and we wish to show that the exponential map is smooth to $\tau = 1$. It will be important throughout to understand the asymptotic behavior of $\bar{\xi}_i$. Now by (4.17), we have

$$\begin{aligned}
1 &= g^{-1}(\xi,\xi)\\
&= \begin{pmatrix} \sin(\theta)\xi_0, & \rho\sin(\theta)\xi_\mu \end{pmatrix} B \begin{pmatrix} \sin(\theta)\xi_0 \\ \rho\sin(\theta)\xi_\mu \end{pmatrix},
\end{aligned}$$

where B is a smooth, uniformly positive definite matrix on $\widetilde{U}$. Thus, for some $c > 0$, we get

$$c(\sin^2(\theta)\xi_0^2 + \delta^{\mu\nu}\rho^2\sin^2(\theta)\xi_\mu\xi_\nu) \leq 1,$$

from which it follows that $\xi_0 = O(\csc\theta)$ and $\xi_\mu = O(\rho^{-1}\csc\theta)$. Hence, $\bar{\xi}_0 = O(1)$ and $\bar{\xi}_\mu = O(\csc\theta) = O(e^t)$, both uniformly in the starting point q. Putting this into (4.22), we find that $\dot{\bar{\xi}}_\mu = O(1)$, from which it follows that we can improve our estimate to $\bar{\xi}_\mu = O(t) = O(|\log\sin\theta|)$. Finally, we put this back into (4.17) and substitute (4.19) to conclude that $\bar{\xi}_0^2 \to 1$ as $t \to \infty$ or $\theta \to 0$, indeed, that $\bar{\xi}_0^2 = 1 + O(e^{-t}) = 1 + O(\sin(\theta))$. Due to the sign of $\dot{\theta}$, we may likewise conclude that

$$\bar{\xi}_0 = -1 + O(e^{-t}) = -1 + O(\sin(\theta)).$$

We here use that $\rho^2\bar{g}^{00} = 1 + O(\rho\sin(\theta))$.

Because $\dot{\theta} < 0$ for all t, we may reparametrize our geodesic equations by θ. This amounts, by the chain rule, to dividing by $\dot{\theta}$, and by (4.21), we have $\dot{\theta} = \rho\sin^2(\theta)\bar{g}^{0\mu}\bar{\xi}_\mu + \rho^2\sin(\theta)\bar{g}^{00}\bar{\xi}_0$. (We recall that $\bar{g}^{00} = O(\rho^{-2})$, and regard $\rho^2\bar{g}^{00}$ as a single smooth function up to $\rho = 0$, which however does not vanish.) Changing variables on the first equation in (4.22) we then get

$$\frac{dx^\mu}{d\theta} = \frac{\rho\sin^2(\theta)\bar{g}^{\mu\nu}\bar{\xi}_\nu + \rho^2\sin(\theta)\bar{g}^{\mu 0}\bar{\xi}_0}{\rho\sin^2(\theta)\bar{g}^{0\nu}\bar{\xi}_\nu + \rho^2\sin(\theta)\bar{g}^{00}\bar{\xi}_0} = \frac{\rho\sin(\theta)\bar{g}^{\mu\nu}\bar{\xi}_\nu + \rho^2\bar{g}^{\mu 0}\bar{\xi}_0}{\rho\sin(\theta)\bar{g}^{0\nu}\bar{\xi}_\nu + \rho^2\bar{g}^{00}\bar{\xi}_0}.$$

Now, the denominator is just $\frac{\dot{\theta}}{\sin(\theta)}$, which we know by Proposition 4.6 is nonzero as a function of t when t is finite, and thus as a function of θ when $\theta > 0$. On the other hand, when $\theta \to 0$, the denominator goes to -1 by our above computations of $\bar{\xi}_i$ asymptotics. Thus, the denominator is bounded away from zero, and the equation is smooth in a neighborhood of our solutions.

We next study $\frac{\partial\bar{\xi}_\mu}{\partial\theta}$. All but two of the terms in $\dot{\bar{\xi}}_\mu$ have a factor of $\sin(\theta)$ and thus yield to the same analysis we just performed. Focusing on the remaining terms, we have

$$\frac{\partial\bar{\xi}_\mu}{\partial\theta} = (smooth) - \frac{\rho_\mu + \frac{1}{2}\rho^3\frac{\partial\bar{g}^{00}}{\partial x^\mu}\bar{\xi}_0^2}{\rho\sin^2(\theta)\bar{g}^{0\mu}\bar{\xi}_\mu + \rho^2\sin(\theta)\bar{g}^{00}\bar{\xi}_0}.$$

Now if $\mu = s \neq n$, the numerator vanishes to order $\sin(\theta)$, so again the equations are smooth in a neighborhood of our solutions. If $\mu = n$, then the numerator is $1 - \bar{\xi}_0^2 + O(\sin(\theta))$. But

$\bar{\xi}_0^2 = 1 + O(\sin(\theta))$, so the numerator is $O(\sin(\theta))$, and this equation is smooth in a neighborhood of our solutions. Finally we study $\frac{\partial \bar{\xi}_0}{\partial \theta}$. Once again, only two terms of $\dot{\bar{\xi}}_0$ lack a factor of $\sin(\theta)$, their sum being $\cos(\theta)(\rho^2 \bar{g}^{00} \bar{\xi}_0^2 - 1)$. For the same reasons as before, this is in fact $O(\sin(\theta))$. Thus, the entire $(x^\mu, \bar{\xi}_i)$ system is smooth up to $\theta = 0$; and so the solutions are smooth as functions of θ, and depend smoothly on the initial point $q \in \underline{\widetilde{Q}_{\rho_0}}$. All of this analysis is uniform up to $\rho = 0$.

It remains to show that θ is smooth in τ up to $\tau = 1$, and depends smoothly on q; of course, we already know this for $\tau < 1$. We have just used $2n + 1$ of the equations in (4.22); the remaining equation, for $\dot{\theta}$, can now be written as a scalar ODE for θ in terms of t, since x^i and $\bar{\xi}_i$ depend smoothly on θ. Explicitly, the equation is

$$\dot{\theta} = \rho \sin^2(\theta) \bar{g}^{0\mu} \bar{\xi}_\mu + \rho^2 \sin(\theta) \bar{g}^{00} \bar{\xi}_0.$$

The right-hand side is $O(\theta)$, so write the equation as

$$\dot{\theta} = -\theta a(\theta), \tag{4.26}$$

Here a is smooth in θ all the way to $\theta = 0$. It is clear from our earlier analysis of $\bar{\xi}_0$ that $a(0) = 1$ for all q, and also that a is nonvanishing for θ along our curves. We reparametrize θ by $\tau = 1 - e^{-t}$, and (4.26) becomes

$$(\tau - 1)\frac{d\theta}{d\tau} = \theta a(\theta).$$

This is a separable equation; if we write $a(\theta)^{-1} = 1 + \theta b(\theta)$, then the equation has solution

$$\theta e^{\int_0^\theta b(\zeta) d\zeta} = c(1 - \tau),$$

which holds for $0 \le \tau \le 1$, and where b is smooth in both θ and q. Now by the implicit function theorem, this uniquely defines θ as a function of τ and q near $\tau = 1$, smoothly depending on both variables. Thus, as desired, θ – and, hence, the entire $(x^i, \bar{\xi}_i)$ system – exists and depends smoothly on both τ and q for $\tau \in [0, 1]$ and for q up to $\widetilde{S}$.

We have still to show that exp is a local diffeomorphism on $\widehat{N_+ \widetilde{Q}_{\rho_0}}$. Now it is elementary that, given a smooth map between manifolds with corner which takes the corner to the corner, the boundary interior to the boundary interior, and the interior to the interior, it is a local diffeomorphism if and only if its differential is nowhere singular. It suffices, then, to show that $d\exp$ is everywhere a bijection, or that $\det d\exp \neq 0$. Given this last formulation, it suffices to show this on $[0,1] \times (\widetilde{Q} \cap \widetilde{S}) \subset \widehat{N_+\widetilde{Q}_{\rho_0}}$, and it will then follow for $(\tau, q) \in [0,1] \times \underline{\widetilde{Q}_{\rho_0}}$ by shrinking ρ_0. Now it is clear from (4.23) that $\exp|_{[0,1]\times(\widetilde{Q}\cap\widetilde{S})} : [0,1] \times (\widetilde{Q} \cap \widetilde{S}) \to \widetilde{S}$ is a diffeomorphism. Moreover, by Proposition 4.6(a), $(d\exp)|_{\widehat{TN_+(\widetilde{Q}\setminus\widetilde{S})}|_{[0,1]\times(\widetilde{Q}\cap\widetilde{S})}}$ takes nonzero transverse vectors to nonzero transverse vectors. Thus, $d\exp$ is an isomorphism, and exp is a local diffeomorphism on all of $\widehat{N_+\widetilde{Q}_{\rho_0}}$ (for ρ_0 small).

Next we wish to demonstrate (4.15) using (4.22) and the equation of variation. We consider perturbations about the solution (4.25) starting from $q \in \widetilde{Q} \cap \widetilde{S}$, as q varies. Write (4.22) as $(x, \bar{\xi})^{\cdot} = F(x, \bar{\xi})$; let F^μ be the component of F corresponding to x^μ. Then the equation of variation tells us that

$$\frac{\partial}{\partial t}\frac{\partial x^\mu}{\partial \rho(q)} = \frac{\partial F^\mu}{\partial x^i}\frac{\partial x^i}{\partial \rho} + \frac{\partial F^\mu}{\partial \bar{\xi}_i}\frac{\partial \bar{\xi}_i}{\partial \rho}; \tag{4.27}$$

and because, at $t = 0$, x^μ are simply the coordinates of q, we have initial condition $\left.\frac{\partial x^\mu}{\partial \rho}\right|_{t=0} = \delta^{\mu n}$. There is additionally an initial condition for $\left.\frac{\partial \bar{\xi}_i}{\partial \rho}\right|_{t=0}$, smooth in q, but we do not need to write it explicitly.

We claim that the right-hand side of (4.27) is 0 along our solution. By (4.25), we have in this case $\rho = 0$ and $\bar{\xi}_\mu = 0$, and $\bar{\xi}_0 = -1$. First consider the first term of (4.27), involving $\frac{\partial F^\mu}{\partial x^i}$. By (4.22), $F^\mu = \rho \sin^2(\theta)\bar{g}^{\mu\nu}\bar{\xi}_\nu + \rho^2 \sin(\theta)\bar{g}^{\mu 0}\bar{\xi}_0$. Because $\bar{\xi}_\nu = 0$ along our solution, the derivative of the first term of F^μ vanishes easily, and the derivative of the second term vanishes because $\rho = 0$. Very similar considerations show that the second term of (4.27) vanish because $\rho = 0$ along the solution. Hence, the entire right-hand side of (4.27) vanishes identically along our solution. Thus, $\frac{\partial x^\mu}{\partial \rho(q)} = \delta^{\mu n} + O(\rho)$, which establishes (4.15).

We now turn to the final statement. Let $\widetilde{U}$, V, and $\widetilde{Q}_{\rho_0}$ be as above. Notice by (3.8) that $\bar{g}$ extends to $\widetilde{S}$ as a smooth symmetric positive semidefinite tensor field, and that along $\widetilde{S}$, we have $\ker \bar{g} = \operatorname{span}\left\{\frac{\partial}{\partial \theta}\right\}$. It follows that $\bar{g}|_{TV}$ is a metric. Now for any $(\tau, q) \in \widehat{N_+(V \setminus \widetilde{S})}$, we have $T_{(\tau,q)}\widehat{N_+(V \setminus \widetilde{S})} \cong \mathbb{R}\frac{\partial}{\partial \tau} \oplus T_q V$ canonically. For $0 \leq \tau \leq 1$, define $\exp_\tau : \overline{V} \to \widetilde{X}$ by

$\exp_\tau(q) = \exp(\tau, q)$. The function $f : [0,1] \times T\overline{V} \to \widetilde{X}$ given by $f(\tau, Y) = |d\exp_\tau(Y)|_{\bar{g}}$ is a smooth map. Now exp is a local diffeomorphism such that $0 \neq d\exp_{(\tau,q)}\left(\frac{\partial}{\partial \tau}\right) \in \operatorname{span} \frac{\partial}{\partial \theta} = \ker \bar{g}$ for $q \in V \cap \widetilde{S}$. We conclude that f is nonvanishing. Thus, it attains a positive minimum on the compact set $[0,1] \times S^1_{\bar{g}} T\overline{V}$. This yields the claim. ■

The above proof relied in a fundamental way on the behavior of the extension exp to the boundary $\widetilde{S}$ in order to show that exp is a local diffeomorphism. It is possible to give a proof on the interior that exp is a local diffeomorphism using Jacobi fields in a more general setting. The following result is unlikely to surprise practitioners in the area, but we did not find a published proof. Because of its potential applications in other settings, it may be worthwhile to record explicitly in the literature, so we state and prove the result, and then use it in Proposition 4.11 to give an alternate proof of the local diffeormorphism property in Proposition 4.7.

Proposition 4.8. *Let $\beta > 0$ and $0 < \kappa < \sqrt{\beta}$, and let (Z, g) be a Riemannian manifold with hypersurface Q having unit normal field ν. Suppose that $|g^{-1}K| \leq \kappa$ on Q, where K is the second fundamental form of Q and $|g^{-1}K|$ is the maximal absolute value of an eigenvalue of the shape operator. Moreover, suppose $W \subseteq N_+Q$ is an open subset of the one-sided normal bundle to Q having the property that whenever $Y \in W$, $tY \in W$ for $0 \leq t \leq 1$. Finally suppose that all sectional curvatures of g are bounded above by $-\beta$ on* $\exp(W)$. *Then* exp *is a local diffeomorphism on W, and if $\xi : (-\varepsilon, \varepsilon) \to Q$ is a smooth curve with $\nu_{\xi(s)} \in W$ and if $\Gamma : [0, a) \times (-\varepsilon, \varepsilon) \to Z$ is given by $\Gamma_t(s) := \Gamma(t, s) = \exp(t\nu_{\xi(s)})$, then for all $t \geq 0$ and $s \in (-\varepsilon, \varepsilon)$, $|\Gamma_t'(s)|_g \geq c|\xi'(s)|_g$, where $c = \sqrt{\frac{1}{2}\left(1 - \frac{\kappa^2}{\beta}\right)}$.*

Proof. Let $\pi : N_+Q \to Q$ be the basepoint map. For convenience, we assume that $\pi(W) = Q$ (or we could just restrict Q). For each $p \in Q$, we let γ_p be the geodesic in Z for which $\gamma(0) = p$ and $\gamma'(0) = \nu_p$.

Let $(t_0, p) \in [0, \infty) \times Q \approx N_+Q$ be fixed. Plainly $d\exp_{(t_0,p)}\left(\frac{d}{dt}\right) = \gamma_p'(t_0) \neq 0$. For $Y \in T_pQ$ with $|Y|_g = 1$ for convenience, let $\xi : (-\varepsilon, \varepsilon) \to Q$ be a smooth curve such that $\xi(0) = p$ and $\xi'(0) = Y$. Let $a > 0$ be sufficiently small that $a\nu_{\xi(s)} \in W$ for each $s \in (-\varepsilon, \varepsilon)$ (shrinking ε if necessary). For $(t, s) \in [0, a) \times (-\varepsilon, \varepsilon)$, define $\Gamma(t, s) = \gamma_{\xi(s)}(t)$. Then $d\exp_{(t_0,p)}(Y) =$

$\partial_s\Gamma(t_0, 0)$. (We are using the identification $T_{(t_0,p)}N_+Q \cong \mathbb{R} \oplus T_pQ$.) This is simply the Jacobi field along γ_p defined by the smooth variation Γ evaluated at t_0. Thus, since Y is arbitrary, it suffices to show that the Jacobi field $J(t) = \partial_s\Gamma(t, 0)$ is nonvanishing and is nowhere parallel to $\gamma_p'(t)$. At $t = 0$, we have $J(0) = \xi'(0) = Y \perp \nu_p$. Moreover, by the symmetry lemma we have

$$D_t J(t) = D_t\partial_s\Gamma(t, s)|_{s=0} = D_s\partial_t\Gamma(t, s)|_{s=0},$$

where D_t denotes covariant differentiation along the curve $t \mapsto \Gamma(t, s)$, and similarly for D_s. At $t = 0$, this gives

$$D_t J(0) = D_s\gamma_{\xi(s)}'(0)|_{s=0} = D_s\nu \perp \nu = \gamma_p'(0),$$

since ν is a unit vector field. Thus, at $t = 0$, both J and $D_t J$ are normal to γ_p', which implies that J is a normal Jacobi field. In particular, if nonvanishing, it is nowhere parallel to $\gamma_p'(t)$.

Set $f(t) = \langle J(t), J(t)\rangle_g$. Then

$$f'(t) = \frac{d}{dt}\langle J, J\rangle_g = 2\langle D_t J, J\rangle_g. \tag{4.28}$$

It follows by Cauchy-Schwartz that

$$|f'(t)| \leq 2|f(t)|^{\frac{1}{2}}|D_t J|_g. \tag{4.29}$$

As we have seen, $D_t J|_{t=0} = D_s\nu$, which is simply the shape operator applied to $Y = \xi'(0) = J(0)$. It follows that, due to our restriction of ρ_0, $\langle D_t J, J\rangle_g|_{t=0} \geq -\kappa$, so

$$f'(0) = \frac{d}{dt}\langle J, J\rangle|_{t=0} \geq -2\kappa.$$

Now by (4.28), (4.29), and the Jacobi equation, whenever $f(t) \neq 0$ we have

$$\begin{aligned} f''(t) = \frac{d^2}{dt^2}\langle J, J\rangle_g &= 2\langle D_t^2 J, J\rangle + 2\langle D_t J, D_t J\rangle \\ &\geq -2R(J(t), \gamma_p'(t), \gamma_p'(t), J(t)) + \frac{1}{2}\frac{f'(t)^2}{f(t)} \\ &> 2\beta f(t) + \frac{1}{2}\frac{f'(t)^2}{f(t)} \end{aligned} \tag{4.30}$$

since the sectional curvature is less than $-\beta$.

We briefly pause to define weighted hyperbolic trigonometric functions. For $\eta \in \mathbb{R}$, define $\cosh_\eta(t) = \frac{1}{2}(e^t + \eta e^{-t})$, and $\sinh_\eta(t) = \frac{1}{2}(e^t - \eta e^{-t})$. It is easy to show that $\cosh'_\eta(t) = \sinh_\eta(t)$ and $\sinh'_\eta(t) = \cosh_\eta(t)$, and also that $\sinh^2_\eta(t) = \cosh^2_\eta(t) - \eta$.

Now consider the second-order differential equation given by

$$h''(t) = 2\beta h(t) + \frac{1}{2}\frac{h'(t)^2}{h(t)},$$

with $h(0) = 1$ and $h'(0) = -2\kappa$. Let $A = \frac{1}{2}\left(1 - \frac{\kappa^2}{\beta}\right) > 0$, $B = \frac{1}{2}\left(1 - \frac{\kappa}{\sqrt{\beta}}\right)^2 > 0$, and $\eta = \frac{(\sqrt{\beta}+\kappa)^2}{(\sqrt{\beta}-\kappa)^2} > 0$. Then a solution to our initial-value problem is $h(t) = A + B\cosh_\eta(2\sqrt{\beta}t)$, as may be easily checked. We wish to apply Proposition 4.3 to show that $f(t) \geq h(t)$, and that thus f is bounded below by $A = \frac{1}{2}\left(1 - \frac{\kappa^2}{\beta}\right)$. Define $a : \mathbb{R}^2 \setminus (\{0\} \times \mathbb{R}) \to \mathbb{R}$ by

$$a(u, v) = 2\beta u + \frac{1}{2}\frac{v^2}{u}.$$

Now define $b : \mathbb{R}^2 \to \mathbb{R}$ by

$$b(u, v) = \begin{cases} 2\beta u + \frac{1}{2}\frac{v^2}{u} & |v| < 2\sqrt{\beta}|u| \\ 4\beta u & |v| \geq 2\sqrt{\beta}|u| \end{cases}.$$

Note that b is Lipschitz, and that whenever $u > 0$, we have $a \geq b$.

Now h satisfies $h''(t) - a(h(t), h'(t)) = 0$; and because $|h'(t)| < 2\sqrt{\beta}|h(t)|$, it also satisfies $h''(t) - b(h(t), h'(t)) = 0$. Similarly, because $f(0) \geq 0$ and because, whenever $f \neq 0$, f satisfies $f''(t) \geq a(f(t), f'(t))$, it follows that, at least up until f vanishes for the first time, we have $f''(t) \geq b(f(t), f'(t))$. But b is Lipschitz and is nondecreasing in u. Thus, by Proposition 4.3, $f \geq h$ on any interval $I = [0, t_0]$ such that f is nonvanishing on I. But since f is continuous, and h is bounded away from 0 by $A > 0$, we conclude that f must be everywhere greater than A. This yields the claim, taking $c = \sqrt{A} = \sqrt{\frac{1}{2}\left(1 - \frac{\kappa^2}{\beta}\right)}$. ∎

To apply this to our situation, we require two lemmas that will also be independently useful in applications.

Let $\widetilde{U}$ be the neighborhood on which a polar identification (θ, π_S, ρ) exists.

Lemma 4.9. *Let g be an admissible metric on $(\widetilde{X}, \widetilde{M}, \widetilde{Q}, \widetilde{S})$. If $\bar{g} = \rho^2 \sin^2(\theta) g$ is the compactified metric on the interior of $\widetilde{U}$, then the second fundamental form $\overline{K}$ of $(\widetilde{Q} \setminus \widetilde{S}) \cap \widetilde{U}$ with respect to $\bar{g}$ extends smoothly to $\widetilde{Q} \cap \widetilde{U}$.*

Proof. Notice once again that by (3.8), $\bar{g}$ extends smoothly to $\widetilde{U}$ as a smooth tensor field (although not as a metric). Let $\bar{\nu}$ be the inward unit normal vector field on $\widetilde{Q} \setminus \widetilde{S}$ with respect to $\bar{g}$. By Lemma 4.1, $\bar{\nu} = -\frac{1}{\rho}\frac{\partial}{\partial\theta} + O_{\bar{g}}(\rho)$. We next wish to consider $\overline{\Gamma}_{i0j} = \frac{1}{2}(\partial_i \bar{g}_{0j} + \partial_\theta \bar{g}_{ij} - \partial_j \bar{g}_{i0})$. Plainly this is smooth on $\widetilde{U}$. But moreover, by (3.8), we see that it is $O(\rho)$. (Remember that k_ρ is independent of θ). All other Christoffel symbols are likewise smooth. Now using Weingarten's equation, we have in coordinates

$$\begin{aligned}\overline{K}_{ij} &= -\bar{g}_{kj} \overline{\nabla}_i \bar{\nu}^k \\ &= -\bar{g}_{kj} \partial_i \bar{\nu}^k - \bar{\nu}^l \overline{\Gamma}_{ilj} \\ &= \bar{g}_{0j} \partial_i (\rho^{-1}) + (smooth).\end{aligned}$$

Since $\bar{g}_{0j} = O(\rho^2)$ for any $0 \le j \le n$, we conclude that $\overline{K}$ extends smoothly to $\widetilde{Q} \cap \widetilde{S}$. ■

Lemma 4.10. *Let g be an admissible metric on $(\widetilde{X}, \widetilde{M}, \widetilde{Q}, \widetilde{S})$ and R its curvature tensor. Then $R_{ijkl} + (g_{ik}g_{jl} - g_{il}g_{jk}) = O_g(\rho \sin\theta)$.*

Proof. We begin by showing that $T_{ijkl} := R_{ijkl} + (g_{ik}g_{jl} - g_{il}g_{jk}) = O_g(\rho)$, using a modification of the proof of Proposition 1.10 of [Maz86].

Let $r = \rho \sin\theta$, so that $\bar{g} = r^2 g$. Now the standard formula for conformal change of the Riemann tensor shows that

$$R_{ijkl} = r^{-2} \overline{R}_{ijkl} + r^{-3}\left(r_{jk}\bar{g}_{il} + r_{il}\bar{g}_{jk} - r_{ik}\bar{g}_{jl} - r_{jl}\bar{g}_{ik}\right) - |\nabla r|^2_{\bar{g}}\left(g_{il}g_{jk} - g_{ik}g_{jl}\right),$$

where r_{jk} represents the Hessian of r taken with respect to $\bar{g}$. Thus, the expression we are interested in takes the form

$$\begin{aligned}R_{ijkl} + (g_{ik}g_{jl} - g_{il}g_{jk}) = r^{-2}\overline{R}_{ijkl} &+ r^{-3}(r_{jk}\bar{g}_{il} + r_{il}\bar{g}_{jk} - r_{ik}\bar{g}_{jl} - r_{jl}\bar{g}_{ik}) \\ &+ (g_{il}g_{jk} - g_{ik}g_{jl})(1 - |\nabla r|^2_{\bar{g}}). \quad (4.31)\end{aligned}$$

Now using the fact that $r = \rho \sin\theta$ and (3.9), it follows immediately that $|\nabla r|_g^2 = 1 + O(\rho)$. Thus, the last term is $O_g(\rho)$.

We next turn to computing $\overline{R}_{ijkl}$. It will be convenient to use the formula

$$\overline{R}_{ijkl} = \frac{1}{2}\left(\partial^2_{jl}\bar{g}_{ik} + \partial^2_{ik}\bar{g}_{jl} - \partial^2_{il}\bar{g}_{jk} - \partial^2_{jk}\bar{g}_{il}\right) + \bar{g}^{pq}\left(\overline{\Gamma}_{jlp}\overline{\Gamma}_{ikq} - \overline{\Gamma}_{jkp}\overline{\Gamma}_{ilq}\right),$$

where $\overline{\Gamma}_{ijk} = \frac{1}{2}(\partial_i \bar{g}_{jk} + \partial_j \bar{g}_{ij} - \partial_k \bar{g}_{ij})$. We thus compute these Christoffel symbols. Using our polar g coordinates and (3.8), we find that

$$\begin{array}{lll}
\overline{\Gamma}_{000} = O(\rho^3) & \overline{\Gamma}_{00u} = O(\rho^3) & \overline{\Gamma}_{00n} = -\rho + O(\rho^2) \\
\overline{\Gamma}_{0s0} = O(\rho^2) & \overline{\Gamma}_{0su} = O(\rho) & \overline{\Gamma}_{0sn} = O(\rho) \\
\overline{\Gamma}_{0n0} = \rho + O(\rho^2) & \overline{\Gamma}_{0nu} = O(\rho) & \overline{\Gamma}_{0nn} = O(\rho) \\
\overline{\Gamma}_{st0} = O(\rho) & \overline{\Gamma}_{stu} = O(1) & \overline{\Gamma}_{stn} = O(1) \\
\overline{\Gamma}_{sn0} = O(\rho) & \overline{\Gamma}_{snu} = O(1) & \overline{\Gamma}_{snn} = O(\rho) \\
\overline{\Gamma}_{nn0} = O(\rho) & \overline{\Gamma}_{nnu} = O(1) & \overline{\Gamma}_{nnn} = O(1).
\end{array}$$

Now using these computations, (3.8), and (3.9), it follows straightforwardly that

$$\begin{aligned}
\overline{R}_{0\mu\nu 0} &= O(\rho) \\
\overline{R}_{i\mu\nu\sigma} &= O(1),
\end{aligned}$$

where $1 \leq \mu \leq n$ and $0 \leq i \leq n$. It follows, since $\sin(\theta)\frac{\partial}{\partial\theta}$ and $\rho\sin(\theta)\frac{\partial}{\partial x^\mu}$ are a basis of approximately g-unit vector fields, that $r^{-2}\overline{R} = O_g(\rho)$.

Finally, we compute that

$$r_{ij} = 2\rho_{(i}\theta_{j)}\cos(\theta) - \sin(\theta)\bar{g}^{nk}\overline{\Gamma}_{ijk} - \rho\cos(\theta)\bar{g}^{0k}\overline{\Gamma}_{ijk},$$

from which it follows that $r_{\mu\nu} = O(1)$, that $r_{0\mu} = O(\rho)$, and that $r_{00} = O(\rho^2)$. Thus, the second term of (4.31) is also $O_g(\rho)$, which yields the claim that $T = O_g(\rho)$.

Now, near neighborhoods in $\widetilde{M}$ away from $\widetilde{M} \cap \widetilde{S}$, it follows from the usual curvature result for asymptotically hyperbolic spaces, given in Proposition 1.10 of [Maz86], that $T = O_g(\sin(\theta))$. Thus, the result will follow if we can show that T is smooth as a section of the bundle $\otimes^4({}^{0e}T^*\widetilde{X})$.

But this follows from what we have already done, and in particular from (4.31). First, $r^{-2}\overline{R}_{ijkl}$ is smooth as a section of the bundle, since no more than two of the indices can be 0, and a smooth frame for ${}^{0e}T^*\widetilde{X}$ is given by (3.1). But the second term similarly is smooth, for we have just seen that $r_{jk} = O(\rho)$ whenever either index is 0. Thus, T is a smooth section and is $O_g(\rho)$ and $O_g(\sin(\theta))$. The result follows. ■

We can now give the alternate proof of the local diffeomorphism property.

Proposition 4.11. *Let $(\widetilde{X}, \widetilde{M}, \widetilde{Q}, \widetilde{S})$ be the blowup of a cornered space (X, M, Q), and let g be an admissible metric on $\widetilde{X}$. There exists $\rho_0 > 0$ such the the map* $\exp : N_+\widetilde{Q}_{\rho_0} \to \widetilde{X}$ *is a local diffeomorphism on the normal bundle $N_+\widetilde{Q}_{\rho_0}$. Moreover, there exists some $c > 0$ such that, if $\xi : (-\varepsilon, \varepsilon) \to \widetilde{Q}_{\rho_0}$ is a smooth curve and $\Gamma : [0, \infty) \times (-\varepsilon, \varepsilon) \to \widetilde{X}$ is given by $\Gamma_t(s) := \Gamma(t, s) = \exp(t\nu_{\xi(s)})$, then for all $t \geq 0$ and $s \in (-\varepsilon, \varepsilon)$, we have $|\Gamma_t'(s)|_g \geq c|\xi'(s)|_g$.*

Proof. It suffices to prove the second claim. Let $0 < \kappa < 1$ be such that $|\cos\theta| < \kappa$ on $\widetilde{Q} \cap \widetilde{S}$, which exists by compactness. Also let $\kappa < \beta < 1$.

By Lemma 4.10, there is some ρ_β such that the sectional curvatures of g are strictly less than $-\beta$ for all $x \in \mathring{\widetilde{X}}_{\rho_\beta}$. By Proposition 4.6, we can choose $\rho_0 > 0$ such that, for $q \in \widetilde{Q}_{\rho_0}$, γ_q remains in $\widetilde{X}_{\rho_\beta}$.

We begin by studying the eigenvalues of the second fundamental form of $\widetilde{Q}$, which we denote by $K(Y, Z) = \langle \nabla_Y Z, \nu \rangle_g$ (and correspondingly for $\overline{K}$ with respect to $\bar{g}$), and to do this we first compute the compactified second fundamental form $\overline{K}$. For $r \neq 0$, the unit $\bar{g}$-normal vector field to $\widetilde{Q}_{\rho_0}$ is given by $\bar{\nu} = r^{-1}\nu$. (Recall that $r = \rho\sin\theta$.) A straightforward computation shows that for any vector fields X, Y tangent to $\widetilde{Q}$, we have

$$\overline{\nabla}_X Y = \nabla_X Y + r^{-1}\left[dr(X)Y + dr(Y)X - \langle X, Y \rangle_{\bar{g}} \operatorname{grad}_{\bar{g}} r\right].$$

For $q \in \widetilde{Q}_{\rho_0}$ and $X, Y \in T_q\widetilde{Q}$, it follows (taking extensions where necessary) that

$$\begin{aligned}\overline{K}(X, Y) &= -\langle \overline{\nabla}_X(r^{-1}\nu), Y \rangle_{\bar{g}} \\ &= -r^{-1}\langle \nabla_X \nu + dr(X)\bar{\nu} + dr(\bar{\nu})X - \langle X, \bar{\nu} \rangle \operatorname{grad}_{\bar{g}} r - dr(X)\bar{\nu}, Y \rangle_{\bar{g}} \\ &= -r^{-1}(r^2 K(X, Y) - \bar{g}(X, Y) dr(\bar{\nu})).\end{aligned}$$

Now let $Y, Z \in TQ$ be g-unit vectors over the same point, and let $\overline{Y} = r^{-1}Y$ and $\overline{Z} = r^{-1}Z$ be the parallel $\bar{g}$-unit vectors. It follows that

$$K(Y, Z) = g(Y, Z)dr(\bar{\nu}) + r\overline{K}(\overline{Y}, \overline{Z}).$$

Now $dr = \sin\theta d\rho + \rho\cos\theta d\theta$, and by Lemma 4.1, $\bar{\nu} = (-\frac{1}{\rho} + O(1))\frac{\partial}{\partial\theta} + O(\rho)$. Thus, $|dr(\bar{\nu})| \to |\cos(\theta)| < \kappa$ as $\rho \to 0$. Since $\overline{K}$ is smooth on all of $\underline{\widetilde{Q}_{\rho_0}}$ by Lemma 4.9, it follows that for ρ small enough, the eigenvalues of the shape operator $g^{-1}K$ are bounded in absolute value by κ: $|\lambda| < \kappa$. We restrict ρ_0 if necessary to ensure this condition.

The result now follows straightforwardly by applying Proposition 4.8 with $Z = X$, with $Q = \widetilde{Q}_{\rho_0}$, and with $W = N_+\widetilde{Q}_{\rho_0}$. ■

4.3 Injectivity

In the preceding sections, we have shown that there is a neighborhood $\underline{\widetilde{Q}_{\rho_0}}$ of $\widetilde{S}$ in $\widetilde{X}$ such that exp : $\widehat{N_+\widetilde{Q}_{\rho_0}} \to \widetilde{X}$ is a local diffeomorphism. The remaining step to show that exp is a diffeomorphism onto its image is to prove injectivity.

We will first work on the interior or non-compactified normal bundle $N_+\widetilde{Q}$ near a fixed point of $\widetilde{Q} \cap \widetilde{S}$. We will then make the result global along $\widetilde{Q} \cap \widetilde{S}$ using a compactness argument.

We prove injectivity on the interior using a homotopy lifting argument whose structure is that of Theorem 2 in [Her63]. We first prove a lifting result. We let $\pi : N_+(\widetilde{Q} \setminus \widetilde{S}) \to (\widetilde{Q} \setminus \widetilde{S})$ be the basepoint map.

Proposition 4.12. *Let c, ρ_0 be as in Proposition 4.7. Let $W \subset \widetilde{Q}_{\frac{\rho_0}{2}}$ be open, $q \in W$, and let $x = \exp t_x\nu_q$ for some $t_x > 0$. Let $\alpha : [0, l] \to \mathring{\widetilde{X}}$ be a smooth curve such that $\alpha(0) = x$ and such that*

$$\alpha([0, l]) \cap \exp(\pi^{-1}(\partial W)) = \emptyset. \tag{4.32}$$

Then there is a unique smooth curve $\sigma : [0, l] \to N_+W$ such that $\sigma(0) = t_x\nu_q$ and $\exp\sigma(s) = \alpha(s)$. Let $\xi = \pi \circ \sigma : [0, l] \to W$. Then $L_g(\xi) \leq c^{-1}L_g(\alpha)$.

Moreover, if $\alpha : [0, 1] \times [0, l] \to \mathring{\widetilde{X}}$ is a homotopy of smooth curves such that, for each τ, $\alpha(\tau, 0) = x$ and the curve $s \mapsto \alpha(\tau, s)$ satisfies (4.32), then there is a unique lift of α to a homotopy *of curves based at $t_x\nu_q$. That is, there is a unique smooth map $\sigma : [0,1]\times[0,l] \to N_+W$ such that $\sigma(\tau,0) = t_x\nu_q$ for each τ and such that $\exp\circ\sigma = \alpha$.*

Remark. We are especially interested in the special case where $W = \widetilde{Q}_{\frac{\rho_0}{2}}$ itself.

Proof. Let x, α be as in the statement. By Proposition 4.7, $\exp : N_+\widetilde{Q}_{\frac{\rho_0}{2}} \to X$ is a local diffeomorphism. Hence, at least some opening interval of α may be lifted uniquely to a smooth curve σ beginning at $t_x\nu_q$ – that is, there is some $a > 0$ and a unique smooth $\sigma : [0,a] \to N_+\widetilde{Q}_{\frac{\rho_0}{2}}$ such that $\sigma(0) = t_x\nu_q$ and $\exp\circ\sigma = \alpha|_{[0,a]}$. Suppose we cannot lift the entire curve, and let b be the supremum of $a > 0$ such that we can uniquely lift $\alpha|_{[0,a]}$ in the preceding sense. Then there is a unique lift σ of $\alpha|_{[0,b)}$. By continuity and (4.32), σ takes values in $N_+(\overline{W}\setminus\widetilde{S})$.

As in the statement, define $\xi : [0,b) \to W$ by $\xi = \pi\circ\sigma$. By the canonical identification $N_+W \approx [0,\infty)\times W$, we can write $\sigma(s) = (t(s),\xi(s))$. Now for each $s < b$, $\exp\sigma(s) = \alpha(s)$; it follows that, for $s < b$, $(d\exp)(\sigma'(s)) = \alpha'(s)$; and thus that $|(d\exp)(\sigma'(s))|_g^2 = |\alpha'(s)|_g^2$. Under the identification $T_{(t,q)}N_+W \approx \mathbb{R}\frac{\partial}{\partial t}\oplus T_qW$, we can write $\sigma'(s) = \dot{t}(s)\frac{\partial}{\partial t} + \xi'(s)$. Let $A = \sup_{0\le s\le b}|\alpha'(s)|_g^2$. Then since $(d\exp)\left(\frac{\partial}{\partial t}\right) \perp (d\exp)(\xi'(s))$, we have

$$\begin{aligned} A \ge |\alpha'(s)|_g^2 &= \left|\dot{t}(s)(d\exp)\left(\frac{\partial}{\partial t}\right) + (d\exp)_{t\nu_\xi}(\xi'(s))\right|_g^2 \\ &= \dot{t}(s)^2 + |(d\exp)_{t\nu_\xi}(\xi'(s))|_g^2 \\ &\ge \dot{t}(s)^2 + c^2|\xi'(s)|_g^2 \text{ (by Proposition 4.11).} \end{aligned} \tag{4.33}$$

Thus, both $|\dot{t}(s)|$ and $|\xi'(s)|_g$ are bounded. It follows that $\lim_{s\to b}\xi(s)$ exists in $\overline{W}$, so ξ may be extended to exist continuously on $[0,b]$ (although *a priori* $\xi(b)$ may not lie in W). Because $\xi : [0,b] \to \overline{W}$ has finite length by (4.33), $\xi(b) \notin \widetilde{S}$.

Now also by (4.33), $\lim_{s\to b} t(s)$ exists; so $\lim_{s\to b}\sigma(s)$ exists, and σ may be continuously extended to $[0,b]$, possibly taking values in $N_+\overline{W} \supset N_+W$. However, by continuity we have $\exp\sigma(b) = \alpha(b)$. Because (4.32) holds, $\xi(s) \notin \partial W$ for any $0 \le s \le b$. Therefore, $\xi([0,b]) \subset W$, and hence $\sigma([0,b]) \subset N_+W$. Now by Proposition 4.11 and because $W \subseteq \widetilde{Q}_{\frac{\rho_0}{2}}$, exp is a local diffeomorphism on some ball about $\sigma(b)$, so it follows that σ can be smoothly and uniquely extended at least some distance beyond b. This is a contradiction, so σ can be extended smoothly and uniquely to all of $[0, l]$.

We now turn to homotopy lifting. Suppose that x is as above, and that $\alpha : [0,1] \times [0,l] \to \overset{\circ}{\widetilde{X}}$ is a smooth map such that $\alpha(\tau, 0) = x$ for all τ and such that, for fixed τ, the curve $s \mapsto \alpha(\tau, s) =: \alpha_\tau(s)$ meets condition (4.32). We wish to show that there is a lift $\sigma : [0,1] \times [0,l] \to N_+W$ such that $\exp \sigma = \alpha$. In the following, we will also use the notation $\alpha^s(\tau) = \alpha(\tau, s)$.

Let $\sigma_0 : [0,l] \to N_+W$ be a lift, as above, of α_0 beginning at $t_x v_q$. For each $s \in [0,l]$, let $\tau \mapsto \sigma(\tau, s)$ be the lift, starting at $\sigma_0(s)$, of the map $\tau \mapsto \alpha(\tau, s)$. Then $\sigma : [0,1] \times [0,l] \to N_+W$ and $\alpha(\tau, s) = \exp \circ \sigma(\tau, s)$. It is plain that σ is smooth in τ. We wish to show that it is smooth in s.

Let $s_0 \in (0, l]$. We will construct a small strip in $[0,1] \times [0,l]$, containing $[0,1] \times \{s_0\}$, such that σ is smooth on the strip. (The case $s_0 = 0$ is easy because exp is a local diffeomorphism and $\alpha_\tau(0) = x$ for all τ). Let $U \subset N_+W$ be a coordinate ball, containing $\sigma(0, s_0)$, such that $\exp|_U$ is a diffeomorphism. Then $\alpha^{-1}(\exp(U)) \subseteq [0,1] \times [0,l]$ is an open neighborhood of $(0, s_0)$. By continuity of σ_0, there is some $\varepsilon > 0$ such that $\sigma_0([s_0 - \varepsilon, s_0 + \varepsilon]) \subset U$. Now if $s \in [s_0 - \varepsilon, s_0 + \varepsilon]$ and $a > 0$ is small enough that $[0, a] \times \{s\} \subset \alpha^{-1}(\exp(U))$, then for $0 \le \tau \le a$, the map $\tau \mapsto (\exp|_U)^{-1}(\alpha(\tau, s))$ is a smooth lift of $\tau \mapsto \alpha(\tau, s)$ beginning at $\sigma_0(s)$. It follows by uniqueness of lifting that it is equal to $\tau \mapsto \sigma(\tau, s)$. Thus, on some neighborhood of $\{0\} \times [s_0 - \varepsilon, s_0 + \varepsilon]$, we have $\sigma = (\exp|_U)^{-1} \circ \alpha$, so in particular, σ is smooth on a neighborhood of $(0, s_0)$.

Set

$$b = \sup \{d \ge 0 : \sigma \text{ is smooth on a neighborhood of } [0, d] \times \{s_0\} .\} .$$

The preceding discussion shows that $b > 0$. Clearly $b \le 1$, We claim $b = 1$. Suppose not, by way of contradiction. Once more, let U be an open set containing $\sigma(b, s_0)$ such that $\exp|_U$ is a diffeomorphism. Then again, $\alpha^{-1}(\exp(U)) \subseteq [0,1] \times [0,l]$ is an open neighborhood of (b, s_0). Moreover, because σ^{s_0} is smooth in τ, we have $\sigma(a, s_0) \in U$ for all a sufficiently near b. Let $a_1 < b$ be sufficiently near. Then σ is smooth on some neighborhood of $[0, a_1] \times \{s_0\}$ by definition of b. Hence by continuity, we conclude that σ maps some neighborhood of (a_1, s_0) into U. We can choose some $\varepsilon > 0$ such that $\sigma(\{a_1\} \times [s_0 - \varepsilon, s_0 + \varepsilon]) \subset U$ and so that σ is smooth on a neighborhood of $[0, a_1] \times [s_0 - \varepsilon, s_0 + \varepsilon]$. Thus, for $s \in [s_0 - \varepsilon, s_0 + \varepsilon]$, $\sigma(a_1, s) = (\exp|_U)^{-1} \circ \alpha(a_1, s)$. Now, by shrinking ε if need be, we can choose $a_2 > b$ such that $[a_1, a_2] \times [s_0 - \varepsilon, s_0 + \varepsilon] \subseteq \alpha^{-1}(\exp(U))$. Fix $s \in [s_0 - \varepsilon, s_0 + \varepsilon]$. The map $\tau \mapsto (\exp|_U)^{-1}(\alpha(\tau, s))$ (where $a_1 \leq \tau \leq a_2$) is a smooth lift of the map $\tau \mapsto \alpha(\tau, s)$ beginning at $\sigma(a_1, s)$. Then by uniqueness of path lifts, we have $\sigma(\tau, s) = (\exp|_U)^{-1}(\alpha(\tau, s))$ on this rectangle. Thus, σ is smooth on $[0, a_2] \times [s_0 - \varepsilon, s_0 + \varepsilon]$, which is a contradiction since $a_2 > b$. Thus $b = 1$. We conclude that σ is smooth on all of $[0, 1] \times [0, l]$. ■

Remark. It follows from the proof that, if $W = \widetilde{Q}_{\frac{\rho_0}{2}}$ and $R = \left\{q \in \widetilde{Q} : \rho(q) = \frac{\rho_0}{2}\right\}$, then the hypothesis (4.32) could be replaced by $d_{\widetilde{Q}}(q, R) > c^{-1} L_g(\alpha)$.

This result in hand, we may prove interior injectivity.

Proposition 4.13. *There exists $a > 0$ such that* $\exp : N_+\widetilde{Q}_a \to \widetilde{X}$ *is injective.*

Proof. Let ρ_0 be small enough that Propositions 4.7 and 4.12 hold on $\widetilde{Q}_{\rho_0}$, then define $R \subset \widetilde{Q}$ by $R = \left\{q \in \widetilde{Q} : \rho(q) = \frac{\rho_0}{2}\right\}$.

We first prove a result with a topological hypothesis. By Proposition 4.6, there exists $\rho_1 < \frac{\rho_0}{2}$ such that if $q \in \widetilde{Q}_{\rho_1}$, then γ_q will not intersect $\exp(\pi^{-1}(R))$: for if $q' \in R$, then by Proposition 4.6(a), $\frac{\varepsilon\rho_0}{2} < \rho(\gamma_{q'}(t))$ for all t, whereas γ_q can be made to remain arbitrarily close to $\widetilde{S}$ by choosing ρ_1 sufficiently small.

We will show that if $V \subseteq \widetilde{Q}_{\rho_1}$ is connected, and $\widetilde{A} \subset \widetilde{X} \setminus (\widetilde{M} \cup \widetilde{S})$ is a simply connected open set such that $\widetilde{A} \cap \exp(\pi^{-1}(R)) = \emptyset$ and such that $\exp(N_+V) \subseteq \widetilde{A}$, then exp is injective on N_+V. Here, once again, $\pi : N_+(\widetilde{Q} \setminus \widetilde{S}) \to \widetilde{Q} \setminus \widetilde{S}$ is the basepoint map.

Suppose, by way of contradiction, that exp is not injective on N_+V. Then there exists some $x \in \widetilde{A}$, $u \neq v \in N_+V$ such that $\exp(u) = x = \exp(v)$. Let $\sigma : [0, 1] \to N_+V$ be a smooth path in N_+V from u to v, and let $\alpha = \exp \circ \sigma : [0, 1] \to \widetilde{A}$. Then α is a smooth loop segment at x; and since $\widetilde{A}$ is simply connected, there is a smooth homotopy $\tilde{\alpha} : [0, 1] \times [0, 1] \to \widetilde{A}$ such that $\tilde{\alpha}(0, s) = \alpha(s)$ and $\tilde{\alpha}(1, s) = x$. Now by construction, $\tilde{\alpha}(\tau, s)$ avoids $\exp(\pi^{-1}(R))$ for all τ, s; so by Proposition 4.12 with $W = \widetilde{Q}_{\frac{\rho_0}{2}}$, there exists a lifted homotopy $\tilde{\sigma} : [0, 1] \times [0, 1] \to N_+\widetilde{Q}_{\frac{\rho_0}{2}}$, based at u, such that $\exp\circ\tilde{\sigma} = \tilde{\alpha}$. Thus, $\exp\circ\tilde{\sigma}(\tau, 1) = x$ for all τ. Moreover, $\tilde{\sigma}(0, 1) = v$ and $\tilde{\sigma}(1, 1) = u$ (since the lift of a constant path is constant). Thus, defining $\zeta : [0, 1] \to N_+\widetilde{Q}_{\frac{\rho_0}{2}}$ by $\zeta(\tau) = \tilde{\sigma}(\tau, 1)$, the curve ζ must be a smooth path from v to u. On the other hand, $\exp\circ\zeta(\tau) = x$ for all τ. But as exp is a local diffeomorphism, $\exp^{-1}(\{x\})$ is discrete. Thus, ζ is a non-constant smooth map from a connected space to a discrete space, which is a contradiction. Hence, exp is injective on N_+V, which establishes our claim.

To allow a general topology, and in particular a full neighborhood of $\widetilde{S}$, first note that if $B \subseteq S$ is simply connected, then so is $b^{-1}(B)$. This is because $\widetilde{S} \to S$ is a trivial fibration, since S is the intersection of two globally defined hypersurfaces. For such B, and for $\kappa > 0$, we define

$$\widetilde{A}(B, \kappa) = \left\{(\theta, p, \rho) \in \widetilde{X} : p \in B \text{ and } 0 < \rho < \kappa\right\},$$

where we are using our polar identification. Notice that by taking κ small enough, we may always assure that $\exp(\pi^{-1}(R)) \cap \widetilde{A}(B, \kappa) = \emptyset$. Also, $\widetilde{A}(B, \kappa)$ will be simply connected for κ small. Now let $\delta > 0$ be less than the injectivity radius of S with respect to $k_0 = \bar{g}|_{TS}$. For $p \in S$, let $B_\delta(p)$ denote the δ-ball about p with respect to k_0. Thus, for each $p \in S$, $B_\delta(p) \subseteq S$ is simply connected. Let $\kappa_0 > 0$ be small enough for the above conditions to hold for $B_\delta(p)$ at every $p \in S$. Set $\widetilde{A}_p = \widetilde{A}(B_\delta(p), \kappa_0)$.

By Proposition 4.6, there are $\varepsilon > 0, \kappa_1 > 0$ such that for each $p \in S$, $\exp(N_+(\widetilde{A}(B_\varepsilon(p), \kappa_1) \cap \widetilde{Q})) \subseteq \widetilde{A}_p$. Set $V_p = \widetilde{A}(B_{\frac{\varepsilon}{4}}(p), \kappa_1) \cap \widetilde{Q}$. Then $\left\{\underline{V_p}\right\}_{p\in S}$ covers $\widetilde{S} \cap \widetilde{Q} \approx S$, and by compactness we may take a finite subcover $\left\{\underline{V_{p_i}}\right\}_{i=1}^{N}$. We will denote $V_i = V_{p_i}$. Now since there are finitely many V_i, we may apply Proposition 4.6(b) to conclude that by shrinking κ_1 if necessary, we may ensure that $\exp(N_+V_i) \cap \exp(N_+V_j) = \emptyset$ whenever $\overline{V_i} \cap \overline{V_j} = \emptyset$.

Set $V = \cup_i V_i$. We claim that exp is injective on N_+V. For suppose that there exist $u_1 = t_1 v_{q_1}, u_2 = t_2 v_{q_2} \in N_+V$ such that $\exp(u_1) = \exp(u_2)$. By what has just been said, we must have $q_1 \in V_i$ and $q_2 \in V_j$ where $\overline{V_i} \cap \overline{V_j} \neq \emptyset$. Let $p_i = \pi_S(q_i)$. Then by definition of V_i, V_j, we have $q_1, q_2 \in V_\varepsilon := \widetilde{A}(B_\varepsilon(p_1), \kappa_1) \cap \widetilde{Q}$; and since, by choice of ε, we have $\exp(V_\varepsilon) \subseteq \widetilde{A}_{p_1}$, and by the simply connected case, we may conclude that $q_1 = q_2$. Thus taking a small enough that $\widetilde{Q}_a \subseteq V$ yields the theorem. ∎

This result may be extended to the compactified bundle, $\widehat{N_+\widetilde{Q}_a}$.

Proposition 4.14. *There exists $a > 0$ such that* $\exp : \widehat{N_+\widetilde{Q}_a} \to \widetilde{X}$ *is injective.*

Proof. Let a be as in Proposition 4.13. As before, we label points in $\widehat{N_+\widetilde{Q}_a}$ by $(\tau, q) \in [0,1] \times \underline{\widetilde{Q}_a} \approx \widehat{N_+\widetilde{Q}_a}$. Throughout this proof, we regard exp as a function of $\tau \in [0,1]$ and $q \in \widetilde{Q}_a$. We already know that exp is injective when restricted to the set $\{0 \le \tau < 1, \rho(q) > 0\}$. We may quickly extend this to $[0,1] \times S$: the exponential map takes $[0,1] \times S$ injectively to $\widetilde{S}$ by the explicit solution (4.14), and by Proposition 4.6 no other points are mapped to $\widetilde{S}$. Moreover, since $\{1\} \times \widetilde{Q}_a$ is mapped to $\widetilde{M} \setminus \widetilde{S}$, its image is disjoint from the image of $([0,1) \times \widetilde{Q}_a) \cup ([0,1] \times (\widetilde{S} \cap \widetilde{Q}))$. Therefore, we need only show that for $q_1 \neq q_2 \in \widetilde{Q}_a$, $\exp(1, q_1) \neq \exp(1, q_2)$.

Suppose, by way of contradiction, that $\exp(1, q_1) = \exp(1, q_2)$, with $q_1, q_2 \in \widetilde{Q}_a$. Let $\widetilde{Q}_a \supseteq B_1 \ni q_1$ be open such that $q_2 \notin B_1$. Then $[0,1] \times B_1$ is open in $\widehat{N_+\widetilde{Q}_a}$, and so since exp is a local diffeomorphism, $\exp([0,1] \times B_1)$ is open in $\widetilde{X}$. Let $\hat{\gamma}_{q_2} : [0,1] \to \widetilde{X}$ be the rescaled geodesic given by $\hat{\gamma}_{q_2}(\tau) = \exp(\tau, q_2)$, which in particular is continuous. Thus, $\hat{\gamma}_{q_2}^{-1}(\exp([0,1] \times B_1))$ is nonempty and is open in $[0,1]$, and so there is some $\tau_2 < 1$ such that $\hat{\gamma}_{q_2}(\tau_2) \in \exp([0,1] \times B_1)$. Since $\tau_2 < 1$, we must have $\hat{\gamma}_{q_2}(\tau) \in \mathring{\widetilde{X}}$, and so there is some $q_3 \in B_1$, $\tau_3 \in [0,1)$ such that $\exp(\tau_3, q_3) = \exp(\tau_2, q_2)$. This contradicts interior injectivity and thus Proposition 4.13. ∎

4.4 Normal Form Theorems

Proof of Theorem 1.1. By Proposition 4.7, there exists $\rho_0 > 0$ such that the exponential map $\exp : N_+\widetilde{Q}_{\rho_0} \to \widetilde{X}$ extends to a local diffeomorphism $\exp : \widehat{N_+\widetilde{Q}_{\rho_0}} \to \widetilde{X}$. By Proposition 4.14, it is injective for ρ_0 small enough. Taking $V = \underline{\widetilde{Q}_{\rho_0}}$ and $\widetilde{U} = \exp(V)$, this yields the claim. ∎

We next state versions of Theorems 1.2 and 1.3 in which $\widetilde{Q}$, and not $\widetilde{M}$, is fixed. We prove these first, and derive Theorems 1.2 and 1.3 as consequences.

Theorem 4.15. *Let $(\widetilde{X}, \widetilde{M}, \widetilde{Q}, \widetilde{S})$ be the blowup of a cornered space, and g an admissible metric on $\widetilde{X}$. For sufficiently small neighborhoods V of $\widetilde{Q} \cap \widetilde{S}$ in $\widetilde{Q}$, there exist a neighborhood $\widetilde{U}$ of $\widetilde{S}$*

in $\widetilde{X}$ *and a unique diffeomorphism* $\psi : [0,1]_u \times V \to \widetilde{U}$ *such that* $\psi|_{\{1\}\times V} = \mathrm{id}_V$ *and*

$$\psi^* g = \frac{du^2 + h_u}{u^2}, \tag{4.34}$$

with h_u $(0 \le u \le 1)$ *a smooth one-parameter family of smooth conformally compact metrics on* $(V, \widetilde{Q} \cap \widetilde{S})$*, and such that* $\widetilde{M} = \psi(\{u = 0\})$ *and* $\widetilde{Q} = \psi(\{u = 1\})$*.*

Theorem 4.16. *Let* $(\widetilde{X}, \widetilde{M}, \widetilde{Q}, \widetilde{S})$ *be the blowup of the cornered space* (X, M, Q)*, and* $g = b^* g_+ + \mathcal{L}$ *an admissible metric on* $\widetilde{X}$*. Suppose that there is some* $\theta_0 \in (0, \pi)$ *such that, for any defining function* φ *for* M*, the boundary components* M *and* Q *make constant angle* θ_0 *with respect to the compactified metric* $\varphi^2 g_+$*.*

For sufficiently small neighborhoods V *of* $\widetilde{Q} \cap \widetilde{S}$ *in* $\widetilde{Q}$*, there is a neighborhood* $\widetilde{U}$ *of* $\widetilde{S}$ *in* $\widetilde{X}$ *and a unique diffeomorphism* $\psi : [0, \theta_0]_\theta \times V \to \widetilde{U}$ *such that* $\psi|_{\{\theta_0\}\times V} = \mathrm{id}_V$ *and*

$$\psi^* g = \frac{d\theta^2 + h_\theta}{\sin^2 \theta},$$

where h_θ $(0 \le \theta \le \theta_0)$ *is a smooth one-parameter family of smooth asymptotically hyperbolic metrics on* $(V, \widetilde{Q} \cap \widetilde{M})$*, and such that* $\widetilde{M} = \psi(\{\theta = 0\})$ *and* $\widetilde{Q} = \psi(\{\theta = \theta_0\})$*. Moreover,* $\theta|_{[0,\theta_0]\times(\widetilde{Q}\cap\widetilde{S})} = \psi^*\Theta$*. Also* $\partial_\theta \bar{h}_\theta|_{\rho=0} = 0$*, where* $\bar{h}_\theta = \rho^2 h_\theta$ *and* ρ *is any defining function for* $\widetilde{Q} \cap \widetilde{S}$ *in* V*.*

Proof of Theorems 4.15 and 4.16. By Theorem 1.1, we may take $\rho_0 > 0$ such that $\exp : \widehat{N_+ \widetilde{Q}_{\rho_0}} \to \widetilde{X}$ is a diffeomorphism onto its image. Let $V \subset \underline{\widetilde{Q}_{\rho_0}}$ be a neighborhood of $\widetilde{Q} \cap \widetilde{S}$, and set $\widetilde{U} = \exp(\widehat{N_+(V \setminus \widetilde{S})})$. Now under the canonical decomposition $N_+(V \setminus \widetilde{S}) \approx \mathbb{R}_t \times (V \setminus \widetilde{S})$, we have

$$(\exp|_{N_+(V\setminus\widetilde{S})})^* g = dt^2 + g_t, \tag{4.35}$$

where g_t is a one-parameter family of metrics on $V \setminus \widetilde{S}$. Now let $u = 1 - \tau = e^{-t}$, so that $t = -\log u$. Then in these coordinates,

$$(\exp|_{N_+(V\setminus\widetilde{S})})^* g = \frac{du^2 + u^2 g_{-\log u}}{u^2}. \tag{4.36}$$

Set $h_u = u^2 g_{-\log u}$, so that this takes the form (1.3). Now u obviously extends to a global coordinate on $\widehat{N_+(V \setminus \widetilde{S})}$, and we have already observed that exp is a diffeomorphism. Taking

$\psi(u,q) = \exp(1-u,q)$ and $\widetilde{U} = \psi(\widehat{N_+(V \setminus \widetilde{S})})$, we plainly have $\psi(\{u=0\}) = \widetilde{U} \cap \widetilde{M}$, $\psi(\{u=1\}) = \widetilde{U} \cap \widetilde{Q}$, and $\psi|_{\{1\}\times V} = \mathrm{id}$. Uniqueness of ψ follows from uniqueness of the form (4.35), because all the intermediate steps are reversible. To prove Theorem 4.15, it thus remains only to show that h_u extends smoothly down to $u=0$ and that each h_u is a conformally compact metric on V.

Since ψ is a diffeomorphism, we do our calculation on $\widetilde{U}$ and regard u as a function on $\widetilde{U}$. For $0 \le c \le 1$ set $V_c = \left\{x \in \widetilde{U} : u(x) = c\right\} \approx V$. Let $\{x^s\}$ be a local coordinate system on S, and (θ, x^s, ρ) a polar identification. Then locally, $u = u(\theta, x^s, \rho)$, so taking the exterior derivative of both sides of the equation $u = c$, we find that along V_c,

$$0 = du = \frac{\partial u}{\partial \theta} d\theta + \frac{\partial u}{\partial x^s} dx^s + \frac{\partial u}{\partial \rho} d\rho.$$

Now $\frac{\partial u}{\partial \theta} \neq 0$, and so for some smooth functions a, b_s, we have

$$d\theta = a d\rho + b_s dx^s \tag{4.37}$$

on V_c. Notice that $\frac{u}{\sin\theta}$ is smooth and nonvanishing on $\widetilde{U}$. Then by (3.7),

$$u^2 g = \frac{u^2}{\sin^2(\theta)}\left(d\theta^2 + \frac{d\rho^2 + k_\rho}{\rho^2}\right) + (\rho u^2 \sin(\theta)\ell) = \frac{u^2}{\sin^2(\theta)} \frac{\rho^2 d\theta^2 + d\rho^2 + k_\rho}{\rho^2} + (\rho u^2 \sin(\theta)\ell).$$

It is then clear by (4.37) that the restriction of $u^2 g$ to any V_c gives a smooth conformally compact metric on V depending smoothly on c all the way up to $c = 0$. Thus Theorem 4.15 is proved.

Now suppose that Q makes a constant angle θ_0 with M, so that $\widetilde{Q} \cap \widetilde{S}$ is given by $\{\theta = \theta_0\}$. Let $\alpha = (\csc\theta_0 - \cot\theta_0)^{-1}$, and define a coordinate ϕ on $\widehat{N_+(V \setminus \widetilde{S})}$ by $u = \alpha(\csc(\phi) - \cot(\phi))$. Notice that $\frac{du}{u} = \frac{d\phi}{\sin\phi}$. Thus, the metric (1.3) transforms to

$$\chi^* g = \frac{d\phi^2 + l_\phi}{\sin^2(\phi)}, \tag{4.38}$$

where we have defined $\chi : [0, \theta_0] \times V \to \widetilde{U}$ by $\chi(\phi, q) = \psi(u(\phi), q)$ and $l_\phi = \frac{\sin^2\phi}{u^2} h_{u(\phi)}$. We view ϕ as a function on $\widetilde{U}$ via the diffeomorphism χ. Now by (4.14), we see that $\theta = \phi$ on $[0,1] \times (\widetilde{Q} \cap \widetilde{S})$; or put differently, that $\theta = \phi + O(\rho)$. Because the level sets of ϕ and u are the same, we get (4.37) again, and so by an identical calculation to the preceding, we find that

$$\sin^2(\phi)g = \frac{\sin^2(\phi)}{\sin^2(\theta)}\frac{\rho^2 d\theta^2 + d\rho^2 + k_\rho}{\rho^2} + (\rho\sin^2(\phi)\sin(\theta)\ell),$$

which is asymptotically hyperbolic as desired on each V_c because $\frac{\sin\phi}{\sin\theta} = 1 + O(\rho)$.

We now prove the final claim. For this purpose, we define a change of coordinates (ϕ, y^μ) by setting $y^\mu = x^\mu$ on $\widetilde{Q}$ and extending y^μ to be constant along orbits of the exponential map. These are coordinates by Theorem 1.1. We wish to show that $(y^n)^2 l_\phi|_{\rho=0}$ is constant in ϕ. Now notice that it follows by (4.15) that for $1 \le \mu \le n$,

$$\frac{\partial}{\partial y^\mu} = \frac{\partial}{\partial x^\mu} + \frac{\partial\theta}{\partial y^\mu}\frac{\partial}{\partial\theta} + O(\rho).$$

Since $\frac{\partial}{\partial\theta} \in \ker\bar{g}$ at $\rho = 0$, we have, for $p \in \widetilde{U}$, Thus,

$$\bar{g}\left(\frac{\partial}{\partial y^\mu}, \frac{\partial}{\partial y^\nu}\right)\bigg|_p = \bar{g}\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right)\bigg|_p + O(\rho(p)).$$

The right-hand side is constant in ϕ for $\rho = 0$ by (3.8). But

$$(y^n)^2 l_\phi\left(\frac{\partial}{\partial y^\mu}, \frac{\partial}{\partial y^\nu}\right) = \bar{g}\left(\frac{\partial}{\partial y^\mu}, \frac{\partial}{\partial y^\nu}\right)$$

at $\rho = 0$, and as we have seen, the right-hand side is constant in ϕ there. This yields the claim.

Renaming ϕ by θ, χ by ψ, and l_ϕ by h_θ yields the result. ■

Proof of Theorems 1.2 and 1.3. We prove Theorem 1.2. Let $V \subset \widetilde{Q}$, $\widetilde{U} \subset \widetilde{X}$, and $\psi : [0,1] \times V \to \widetilde{U}$ be as in Theorem 4.15. Set $W = \psi(\{0\} \times V)$, a neighborhood in $\widetilde{M}$ of $\widetilde{M} \cap \widetilde{S}$. Let $\phi : V \to W$ be the diffeomorphism given by $\phi(q) = \psi(0, q)$. Define $\zeta(u, m) = \psi(u, \phi^{-1}(m))$.

Uniqueness follows from uniqueness in Theorem 4.15, the construction can be reversed to recover ψ from ζ. ■

For the statement of the next result, we let $[k] = \{h|_{TS} : h \in [h]\}$; so $[k]$ is a conformal class of metrics on S.

Corollary 4.17. *Let $(\widetilde{X}, \widetilde{M}, \widetilde{Q}, \widetilde{S})$, (X, M, Q), and g be as in Theorem 1.3, with again a constant angle θ_0 between Q and M. For any $k \in [k]$ and for sufficiently small $\varepsilon > 0$, there is a neighborhood $\widetilde{U}$ of $\widetilde{S}$ in $\widetilde{X}$ and a unique diffeomorphism $\chi : [0, \theta_0]_\theta \times S \times [0, \varepsilon)_\rho \to \widetilde{U}$ such that $b \circ \chi|_{\{0\}\times S\times\{0\}} = \mathrm{id}_S$ and*

$$\chi^* g = \frac{d\theta^2 + h_\theta}{\sin^2 \theta}, \tag{4.39}$$

where h_θ is a smooth one-parameter family of smooth AH metrics on $S \times [0, \varepsilon)$ with

$$h_0 = \frac{d\rho^2 + k_\rho}{\rho^2},$$

where k_ρ is a smooth one-parameter family of smooth metrics on S with $k_0 = k$, and where $\widetilde{M} = \chi(\{\theta = 0\})$, $\widetilde{Q} = \chi(\{\theta = \theta_0\})$, and $\widetilde{S} = \chi(\{\rho = 0\})$. Moreover, $\partial_\theta(\bar{h}_\theta|_{\rho=0}) = 0$, where $\bar{h} = \rho^2 h$.

Notice this generalizes the form of the hyperbolic metric in (1.2).

Proof. Let $W, \widetilde{U}, h_\theta$, and ζ be as in Theorem 1.3. Now h_0 is an asymptotically hyperbolic metric. So by the existence result for the standard geodesic normal form ([GL91]), there is a unique diffeomorphism $\varphi : S \times [0, \varepsilon)_\rho \to W$ such that $\varphi^* h_0 = \rho^{-2}(d\rho^2 + k_\rho)$, where k_ρ is a smooth one-parameter family of metrics on S such that $k_0 = k$; and such that $\varphi|_{\{0\}\times S} = id_S$. The desired diffeomorphism is then given by $\chi = \zeta \circ (\mathrm{id}_{[0,\theta_0]} \times \varphi)$. The claimed properties follow immediately. (Note that the ρ in Corollary 4.17 is not the same as that appearing in polar g-coordinates). ■

Chapter 5

THE LAPLACIAN AND ODE ANALYSIS

In this chapter, we study the scalar Laplacian on cornered asymptotically hyperbolic spaces and prove Theorem 1.4. We study the formal asymptotics of harmonic functions on a CAH space with given boundary conditions. Along the way, we prove results about some ODEs that will also be important in the next chapter.

Throughout, let (X, M, Q) be a cornered space and $(\widetilde{X}, \widetilde{M}, \widetilde{Q}, \widetilde{S})$ its blowup, with g an admissible metric on $\widetilde{X}$. We will let θ, x, ρ be coordinates in which g takes the form (3.8), with k_ρ as in that equation. We also introduce the following notation, motivated by [GL91]. We will write E^k to denote any polynomial of degree less than or equal to k in $\sin(\theta)\frac{\partial}{\partial\theta}$ and $\rho\sin(\theta)\frac{\partial}{\partial x^\mu}$ $(1 \leq \mu \leq n)$, with coefficients in $C^\infty(\widetilde{X})$.

We first compute the Laplace operator of g.

Lemma 5.1. *Let $(\widetilde{X}^{n+1}, \widetilde{M}, \widetilde{Q}, \widetilde{S})$ be the blowup of a cornered space, and g an admissible metric on $\widetilde{X}$ expressed in the form (3.8). Then for $u \in C^\infty(\widetilde{X} \setminus (\widetilde{M} \cup \widetilde{S}))$,*

$$\Delta_g u = \sin^2(\theta)\partial_\theta^2 u + (1-n)\sin(\theta)\cos(\theta)\partial_\theta u + \rho^2\sin^2(\theta)\partial_\rho^2 u + (2-n)\rho\sin^2(\theta)\partial_\rho u + \rho^2\sin^2(\theta)\Delta_{k_\rho} u + \rho\sin(\theta)E^2(u).$$

Proof. Using the formula $\Delta_g u = g^{-1/2}\partial_i(g^{ij}g^{1/2}\partial_j u)$, this follows easily from (3.8) and (3.9). ∎

We wish to carry out an asymptotic analysis of solutions to a boundary-value problem for the equation $\Delta_g u = 0$. To carry out our analysis, we will expand the solution order-by-order in ρ. Consequently, we work in terms of the *indicial operator* of Δ_g. For $\nu \in \mathbb{R}$, this is the operator $I_\nu : C^\infty(\widetilde{S}) \to C^\infty(\widetilde{S})$ defined by extending $u \in C^\infty(\widetilde{S})$ to $\tilde{u} \in C^\infty(\widetilde{X})$, and then setting $I_\nu(u) = \rho^{-\nu}\Delta_g(\rho^\nu\tilde{u})|_{\rho=0}$. The following is immediate from Lemma 5.1.

Lemma 5.2. *Let $\nu \in \mathbb{R}$. Then*

$$I_\nu(u) = \sin^2(\theta)\partial_\theta^2 u + (1-n)\sin(\theta)\cos(\theta)\partial_\theta u + \nu(\nu+1-n)\sin^2(\theta)u. \tag{5.1}$$

We now specialize to the constant-angle case, supposing that M and Q make constant angle θ_0 with respect to g_X. We will assume the metric is in the form given by Corollary 4.17; however, we will suppress the diffeomorphism χ and will simply regard (θ, x, ρ) as a parametrization of $\widetilde{X} = [0, \theta_0] \times S \times [0, \varepsilon)$ near $\widetilde{S}$. Thus, we will take g to be given by

$$g = \frac{1}{\sin^2(\theta)}\left[d\theta^2 + h_\theta\right], \tag{5.2}$$

where h_θ is a smooth family of smooth AH metrics on the hypersurface $\widetilde{M} = \{0\} \times S \times [0, \varepsilon)$ and where $h_0 = \frac{d\rho^2 + k_\rho}{\rho^2}$. We impose the inhomogeneous boundary condition $u|_{\widetilde{M}} = \psi$ at $\widetilde{M}$ (where $\psi \in C^\infty(\widetilde{M})$) and the homogeneous Neumann condition $\partial_\nu u = 0$ at $\widetilde{Q}$. Notice that for the metric (5.2), this last is equivalent to $\partial_\theta u|_{\theta=\theta_0} = 0$. These boundary conditions are motivated partially by their relevance to the Einstein problem in the next chapter.

Notice that, with the constant-angle hypothesis, the indicial operator (5.1) restricts to each fiber as an operator independent of the fiber, if each fiber is identified with the interval $[0, \theta_0]$. Since we will construct solutions order-by-order in ρ using the indicial operator, our problem turns on an analysis of the equation $I_\nu u = f$, with homogeneous Neumann boundary condition at $\theta = \theta_0$ and Dirichlet boundary condition at $\theta = 0$. We turn to an analysis of this equation.

5.1 ODE Analysis

We will refer to an *indicial root* as a value of the spectral parameter ν for which I_ν is not injective on the space of smooth functions u on $[0, \theta_0]$ satisfying $u(0) = 0 = u'(\theta_0)$. This is equivalent to saying it is a value of ν for which $\lambda_\nu := \nu(\nu+1-n)$ is an eigenvalue of the operator $L = -\partial_\theta^2 + (n-1)\cot(\theta)\partial_\theta = -\sin^{n-1}(\theta)\partial_\theta(\sin^{1-n}(\theta)\partial_\theta)$. This factoring of the eigenvalues of L is motivated by Lemma 5.2, and we call ν the spectral parameter. The operator L has a limit point singularity at $\theta = 0$, and we impose the homogeneous Neumann condition at $\theta = \theta_0$. It is a fact from the theory of singular Sturm-Liouville problems that the L^2 spectrum of L, spec L, is discrete when the boundary conditions are suitably interpreted; moreover the eigenfunctions, which are smooth, form an orthonormal basis for $L^2([0,\theta_0])$ with the measure $\sin^{1-n}(\theta)d\theta$. The discreteness of the spectrum will also follow from our results in this section. Since for all ν, the indicial roots of I_ν at $\theta = 0$ are 0 and n, all eigenfunctions of L satisfying the given boundary conditions are of the form $\sin^n(\theta)v(\theta)$ for some smooth $v(\theta)$.

We begin with the following elementary result.

Lemma 5.3. *Let $u, v \in C^2([0,\theta_0])$ be such that one is $O(\theta)$ and the other is $O(\theta^n)$, and such that $u'(\theta_0) = 0 = v'(\theta_0)$. Then*

(a) $\int_0^{\theta_0} (Lu)v\sin^{1-n}(\theta)d\theta = \int_0^{\theta_0} (\partial_\theta u)(\partial_\theta v)\sin^{1-n}(\theta)d\theta$;

(b) $\int_0^{\theta_0} (Lu)v\sin^{1-n}(\theta)d\theta = \int_0^{\theta_0} u(Lv)\sin^{1-n}(\theta)d\theta$.

Proof. Part (a) follows using integration by parts from the realization that we can write $Lu(\theta)$ as $Lu(\theta) = -\sin^{n-1}(\theta)\frac{d}{d\theta}\left[\sin^{1-n}(\theta)u'(\theta)\right]$. Part (b) follows from part (a) by symmetry. ■

We next study the eigenvalues of L. First, we state a singular Sturm comparison theorem from [Nai12].

Proposition 5.4 (Theorem 3 from [Nai12])**.** *Suppose that $u \in C^2((a,b))$ satisfies the equation*

$$(p(t)u')' + q(t)u = 0,$$

on the interval (a,b), where $p \in C^1((a,b))$ and $q \in C^0((a,b))$, and $p(t) \geq 0$. Suppose further that $\int_a \frac{1}{p(t)u(t)^2}dt = \infty$, and that $\int^b \frac{1}{p(t)u(t)^2}dt = \infty$. Suppose further that $u(t)$ has exactly $n-1$ zeros on the interval (a,b), where $n \in \mathbb{N}$.

Now let $P \in C^1((a,b))$ and $Q \in C^0((a,b))$ be such that $p(t) \geq P(t) \geq 0$ and $Q(t) \geq q(t)$ on (a,b), and that $Q(t) \not\equiv q(t)$. Suppose that $v \in C^2((a,b))$ satisfies the equation

$$(P(t)v')' + Q(t)v = 0.$$

Then $v(t)$ has at least n zeroes in (a,b).

Proposition 5.5. *Let* $n \geq 2$ *and* $\theta_0 \in (0, \pi)$. *Then the smallest eigenvalue of L for the boundary conditions* $u(0) = 0$ *and* $u'(\theta_0) = 0$

- *lies in* $(0, n)$ *if* $\frac{\pi}{2} < \theta_0 < \pi$*;*
- *is* n *if* $\theta_0 = \frac{\pi}{2}$*; and*
- *lies in* (n, ∞) *if* $0 < \theta_0 < \frac{\pi}{2}$.

Remark. In terms of the spectral parameter ν, this says that λ_ν is not an eigenvalue for $0 \leq \nu \leq n-1$, and that the first eigenvalue occurs for ν in $(n-1, n)$, at n, or in (n, ∞), respectively.

Proof. It is immediate from Lemma 5.3 that the lowest eigenvalue λ_0 satisfies $\lambda_0 \geq 0$. From the same lemma, it follows that the only solutions u to $Lu = 0$ are the constant functions. These, however, do not satisfy $u(0) = 0$ (or in short, they are not L^2 with respect to $\sin^{1-n}(\theta)$).

Suppose $\theta_0 \in (0, \frac{\pi}{2}]$ and that $\lambda \in (0, n)$. Suppose that $u \in C^\infty([0, \theta_0])$ satisfies our boundary conditions, and that that $Lu = \lambda u$. By considering once again the indicial roots of this equation, we can write $u(\theta) = \sin^n(\theta) v(\theta)$, where $v(\theta)$ is smooth. The equation then transforms to

$$v''(\theta) + (n+1)\cot(\theta) v'(\theta) + (\lambda - n) v(\theta) = 0, \tag{5.3}$$

and since $u'(\theta_0) = 0$, we conclude that

$$v'(\theta_0) = -n \cot(\theta_0) v(\theta_0). \tag{5.4}$$

Then we have

$$\sin^{n+1}(\theta) v''(\theta) + (n+1)\sin^n(\theta)\cos(\theta) v'(\theta) = (n - \lambda)\sin^{n+1}(\theta) v(\theta). \tag{5.5}$$

Thus,

$$\frac{d}{d\theta}\left[\sin^{n+1}(\theta) v'(\theta)\right] = (n-\lambda)\sin^{n+1}(\theta) v(\theta), \text{ so}$$
$$\int_0^{\theta_0} v(\theta) \frac{d}{d\theta}\left[\sin^{n+1}(\theta) v'(\theta)\right] d\theta = (n-\lambda)\int_0^{\theta_0} \sin^{n+1}(\theta) v(\theta)^2 d\theta.$$

Integrating by parts, we get

$$\sin^{n+1}(\theta)v(\theta)v'(\theta)\Big|_0^{\theta_0} - \int_0^{\theta_0} \sin^{n+1}(\theta)v'(\theta)^2 d\theta = (n-\lambda)\int_0^{\theta_0} \sin^{n+1}(\theta)v(\theta)^2 d\theta.$$

Applying our boundary condition gives

$$-n\sin^n(\theta_0)\cos(\theta_0)v(\theta_0)^2 - \int_0^{\theta_0} \sin^{n+1}(\theta_0)v'(\theta_0)^2 d\theta = (n-\lambda)\int_0^{\theta_0} \sin^{n+1}(\theta)v(\theta)^2 d\theta \geq 0$$

Since $\theta_0 \leq \frac{\pi}{2}$, it is plain that this equality can hold only if $v(\theta) \equiv 0$.

Now suppose that $\lambda = n$, and that (5.3) holds for some v satisfying the boundary condition (5.4). It follows immediately that $\sin(\theta)v''(\theta) + (n+1)\cos(\theta)v'(\theta) = 0$, from which we conclude that $v'(\theta) = b\sin^{-(n+1)}(\theta)$ for some b. Thus, for some a,

$$v(\theta) = a + b\int_{\theta_0}^{\theta} \sin^{-(n+1)}(\phi)d\phi.$$

Since v is smooth, we conclude that $b = 0$. Then $v \equiv a$ can be nonvanishing and satisfy (5.4) if and only if $\theta_0 = \frac{\pi}{2}$. In that case, we see that $u(\theta) = \sin^n(\theta)$ is a solution to $Lu = \lambda u$.

Before handling the last case, let $v(\theta) = u'(\theta) = \sin^{n-1}(\theta)\cos(\theta)$. Then by differentiating both sides of the equation $Lu = nu$, we find that v satisfies the equation

$$-v''(\theta) + (n-1)\cot(\theta)v'(\theta) + (1-n)\csc^2(\theta)v(\theta) = nv(\theta),$$

with boundary conditions $v(0) = 0 = v\left(\frac{\pi}{2}\right)$. Multiplying through by $-\sin^{1-n}(\theta)$, we can rewrite this equation as

$$(\sin^{1-n}(\theta)v'(\theta))' + [(n-1)\csc^{n+1}(\theta) + n\csc^{n-1}(\theta)]v(\theta) = 0.$$

Obviously, $v(\theta)$ has no zeros on $\left(0, \frac{\pi}{2}\right)$. Notice also that $\int_0 \frac{\sin^{n-1}(\theta)}{v(\theta)^2}d\theta = \infty = \int^{\frac{\pi}{2}} \frac{\sin^{n-1}(\theta)}{v(\theta)^2}d\theta$. Now let $\lambda > n$, and suppose that $w(\theta)$ satisfies

$$(\sin^{1-n}(\theta)w'(\theta))' + [(n-1)\csc^{n+1}(\theta) + \lambda\csc^{n-1}(\theta)]w(\theta) = 0$$

on $\left(0, \frac{\pi}{2}\right)$. Then w has at least one zero on $\left(0, \frac{\pi}{2}\right)$ by Theorem 5.4.

Now suppose that $\theta_0 \in \left(\frac{\pi}{2}, \pi\right)$. Let $\lambda_0 > 0$ be the lowest eigenvalue of L, with eigenfunction u_0 satisfying $u_0(0) = 0 = u_0'(\theta_0)$. Set $v_0(\theta) = u_0'(\theta)$. We have already shown that $\lambda_0 \neq n$. Now differentiating both sides of $Lu = \lambda_0 u$, we find that v satisfies the equation

$$-v''(\theta) + (n-1)\cot(\theta)v'(\theta) + (1-n)\csc^2(\theta)v(\theta) = \lambda_0 v(\theta) \tag{5.6}$$

with homogeneous Dirichlet conditions at both endpoints. Moreover, λ_0 must be the lowest eigenvalue of this boundary value problem as well, or we could produce a lower eigenvalue to $Lu = \lambda u$ by integration. Now as is well known, the lowest eigenfunction of a positive operator with homogeneous Dirichlet boundary values is nonvanishing away from the endpoints. This can be shown, for example, by adapting the proof of Proposition 5.2.4 of [Tay11] to the simpler ODE case, using the maximum principle given in Theorem 26.XVIII of [Wal98]. So we may conclude that $v(\theta)$ has no zeros on $(0, \theta_0)$, and in particular on $\left(0, \frac{\pi}{2}\right)$. Thus, it follows that $\lambda_0 < n$. ■

We can in fact characterize all of the eigenvalues of L.

Proposition 5.6. *The eigenvalues of L are the values λ_ν, where $\nu \geq 0$ runs over the non-negative solutions to the equation*

$$\frac{d}{d\theta}\left[\sin^n(\theta) F^{\frac{n}{2}+1}_{n-\nu,\nu+1}\left(\sin^2\left(\frac{\theta}{2}\right)\right)\right]\Bigg|_{\theta=\theta_0} = 0. \tag{5.7}$$

Here $F^c_{ab}(x)$ is the hypergeomtric function.

Note that by the identity $\frac{d}{dx}F^c_{ab}(x) = \frac{ab}{c}F^{c+1}_{a+1,b+1}(x)$, the equation (5.7) is equivalent to

$$n\cot(\theta_0) F^{\frac{n}{2}+1}_{n-\nu,\nu+1}\left(\sin^2\left(\frac{\theta_0}{2}\right)\right) = \frac{(\nu-n)(\nu+1)}{n+2} F^{\frac{n}{2}+2}_{n-\nu+1,\nu+2}\left(\sin^2\left(\frac{\theta_0}{2}\right)\right).$$

Proof. We look for solutions $u(\theta)$ to the equation $I_\nu u(\theta) = 0$, satisfying $u(0) = 0 = u'(\theta_0)$. As shown before, such a solution will necessarily be $O(\theta^n)$. We thus write $u(\theta) = \sin^n(\theta)v(\theta)$; it was shown earlier that v then satisfies equation (5.3). We introduce the substitution $x = \sin^2\left(\frac{\theta}{2}\right)$, and set $v(\theta) = l(x(\theta))$. Then setting $a = n - \nu$, $b = \nu + 1$, and $c = \frac{n}{2} + 1$, equation (5.3) transforms to

$$x(1-x)l''(x) + (c - (1+a+b)x)l'(x) - abl(x) = 0.$$

This is the hypergeometric equation, and the solution that is smooth at $x = 0$ is (up to scaling) the hypergeometric function $F_{ab}^{c}(x)$. Thus, we find $u(\theta) = \sin^n(\theta) F_{n-\nu,\nu+1}^{\frac{n}{2}+1}\left(\sin^2\left(\frac{\theta}{2}\right)\right)$. The claim then follows by differentiating u and requiring that $u'(\theta_0) = 0$. ■

In the case $\theta_0 = \frac{\pi}{2}$ we can say much more.

Proposition 5.7. *Let L be as in Proposition 5.5, and $\theta_0 = \frac{\pi}{2}$. Then the spectrum is given in terms of the spectral parameter by* $\operatorname{spec}(L) = \left\{\lambda_\nu : \nu = n + 2k, \text{ where } k \in \mathbb{Z}^{\geq 0}\right\}$, *and the eigenfunctions are given by $w_k(\theta) = \sin^n(\theta) C_{2k}^{\frac{1+n}{2}}(\cos(\theta))$, where C_j^α are the Gegenbauer polynomials.*

Proof. In the equation $Lu = \lambda_\nu u$, we assume a solution of the form $u(\theta) = \sin^n(\theta) v(\cos(\theta))$. This gives the equation

$$\sin^2(\theta) v''(\cos(\theta)) - (n+2)\cos(\theta) v'(\theta) + (\nu - n)(1+\nu) v(\cos(\theta)) = 0. \tag{5.8}$$

We make the substitution $x = \cos(\theta)$, and get the equation

$$(1 - x^2) v''(x) - (n+2) x v'(x) + (\nu - n)(1+\nu) v(x) = 0.$$

This is the Gegenbauer equation $(1-x^2)v''(x) - (1+2\alpha) x v'(x) + k(k+2\alpha) v(x) = 0$ with $\alpha = \frac{1+n}{2}$ and $k = \nu - n$. Recall that a solution to this equation for each integer $k \geq 0$ is given by the kth-degree Gegenbauer polynomial $C_k^\alpha(x)$; that the polynomials $\left\{C_k^\alpha(x)\right\}_{k=0}^{\infty}$ are an orthonormal basis for $L^2\left([-1,1], (1-x^2)^{\alpha/2-1}\right)$; and that each polynomial $C_k^\alpha(x)$ has the same parity as k, with the even polynomials being nonvanishing at 0. It follows that if we let $u_k(\theta) = \sin^n(\theta) C_k^\alpha(\cos(\theta))$, then u_k is a solution of the equation $Lu = \lambda_{n+k} u$, and that the solutions for k even satisfy the condition that $u_k'\left(\frac{\pi}{2}\right) = 0$. By the orthonormal basis property of the Gegenbauer polynomials, it follows that every solution to the equation is one of the u_k; and we conclude that precisely the solutions for even k satisfy our boundary conditions. ■

We now turn to the mapping properties of I_ν. Recall that the function spaces mentioned in the following proposition are defined in Section 3.2.

Proposition 5.8. *Fix $n \geq 2 \in \mathbb{N}$, and $\theta_0 \in (0, \pi)$. For $\nu \in \mathbb{R}^{\geq 0}$, and $k \geq 1$, I_ν given by (5.1) maps $\mathcal{A}^0_{n,k}$ to $\mathcal{B}_{n,k}$. If $\lambda_\nu \notin \operatorname{spec}(L)$, then $I_\nu : \mathcal{A}^0_{n,k} \to \mathcal{B}_{n,k}$ is bijective. Otherwise, $\dim\ker I_\nu = 1$ and the kernel is spanned by a smooth function w_ν; the image in that case is the orthogonal complement in $\mathcal{B}_{n,k}$ of w_ν with respect to the measure $\sin^{-(n+1)}(\theta)d\theta$.*

When $\nu \notin \operatorname{spec}(L)$, the inverse of I_ν is given by the Green's operator

$$Gf(\theta) = p(\theta)\int_0^\theta \frac{q(\phi)f(\phi)}{\sin^{n+1}(\phi)}d\phi + q(\theta)\int_\theta^{\theta_0} \frac{p(\phi)f(\phi)}{\sin^{n+1}(\phi)}d\phi, \tag{5.9}$$

where $q(\theta) = O(\theta^n)$ is smooth, and

- *if n is odd or $0 \leq \nu < n$ is integral, then $p(\theta)$ is smooth and even in θ; while*
- *if n is even and $\nu \notin \{0, 1, \cdots, n-1\}$, then $p(\theta) \in C^\infty([0, \theta_0]) + \theta^n \log(\theta) C^\infty([0, \theta_0])$, and is even up to order n.*

If n is odd and $f \in \mathcal{B}_{n,k}$ is even in θ through order $n+1$, then Gf is smooth.

Proof. For convenience, we set $I = [0, \theta_0]$.

It is straightforward that I_ν maps $\mathcal{A}^0_{n,k}$ into $\mathcal{B}_{n,k}$. The anomaly in the definition of $\mathcal{B}_{n,1}$ is because n is an indicial root of I_ν, so that $I_\nu(\theta^n u) = O(\theta^{n+1})$ for any $u \in C^\infty(I)$.

We first wish to identify independent global solutions to the equation $I_\nu u = 0$. As before, we set $x = \sin^2\left(\frac{\theta}{2}\right)$, and let $l(x)$ be defined by $u(\theta) = l(x(\theta))$. Then transforming the equation $I_\nu u = 0$ yields the equation

$$x(1-x)l''(x) + \left(1 - \frac{n}{2} - (2-n)x\right)l'(x) + \nu(\nu + 1 - n)l(x) = 0. \tag{5.10}$$

Let $a = -\nu$, let $b = \nu + 1 - n$, and let $c = 1 - \frac{n}{2}$. Then (5.10) becomes

$$x(1-x)l''(x) + (c - (1 + a + b)x)l'(x) - abl(x) = 0,$$

which is again the hypergeometric differential equation. The indicial roots of the equation are $s = 0$ and $s = \frac{n}{2}$. It follows that we can find two independent solutions u_0 and u_n having the following properties: $u_n \in C^\infty(I)$ and $u_n = O(\theta^n)$, while $u_0(0) \neq 0$. If n is odd, then $u_0 \in C^\infty(I)$, while if n is even, then it follows by standard hypergeometric theory (for example, section 15.10 of [NIS]) that $u_0 \in C^\infty(I)$ if $\nu = 0, 1, 2, \cdots, n-1$, and that $u_0 \in C^\infty(I) + \theta^n \log(\theta) C^\infty(I)$ otherwise, with nonzero logarithmic coefficient.

We already know that I_ν is injective if and only if $\lambda_\nu \notin \operatorname{spec} L$. We now show that I_ν is surjective whenever it is injective. Let $q(\theta)$ be the solution to $I_\nu q = 0$ with $\lim_{\theta\to 0^+} q(0)/\theta^n = 1$. Let $p(\theta)$ be the solution to $I_\nu p = 0$ with $p'(\theta_0) = 0$ and $p(0) = 1$ (this can be taken to be nonzero by the assumption of injectivity). It follows that p is some linear combination of u_0 and u_n with nontrivial coefficient for u_0. Now by Abel's identity, the Wronskian of p and q is $c \sin^{n-1}(\theta)$ for some $c \neq 0$. It thus follows from standard formulas that the Green's function for I_ν is given by

$$\Phi(\theta, \phi) = \begin{cases} \frac{q(\theta)p(\phi)}{c\sin^{n+1}(\phi)} & \theta < \phi \\ \frac{q(\phi)p(\theta)}{c\sin^{n+1}(\phi)} & \theta \geq \phi \end{cases},$$

and it is elementary to show that the Green's operator

$$Gf(\theta) = p(\theta)\int_0^\theta \frac{q(\phi)f(\phi)}{c\sin^{n+1}(\phi)}d\phi + q(\theta)\int_\theta^{\theta_0} \frac{p(\phi)f(\phi)}{c\sin^{n+1}(\phi)}d\phi \tag{5.11}$$

is a right inverse to I_ν when $f \in \mathcal{B}_{n,k}$, and a left inverse to I_ν when $f = I_\nu u$ with $u \in \mathcal{A}^0_{n,k}$. (For the formula in the statement, we may absorb c into q.) We wish to show that the equation $I_\nu u = f$ can be solved whenever $f \in \mathcal{B}_{n,k}$, and that in particular, G maps $\mathcal{B}_{n,k}$ to $\mathcal{A}^0_{n,k}$. It is easy to check that $(Gf)'(\theta_0) = 0$ and that, for $f \in \mathcal{B}_{n,k}$, $Gf(0) = 0$. Thus, if $Gf \in \mathcal{A}_{n,k}$, then $Gf \in \mathcal{A}^0_{n,k}$. It remains to show that $Gf(\theta) \in \mathcal{A}_{n,k}$.

We do this in two steps. We first show that G maps $\mathcal{B}_{n,k}$ into $\mathcal{A}_{n,k} + \theta^{(k+1)n}\log(\theta)^{k+1}C^\infty(I)$ by asymptotically expanding the integrands in (5.11) and considering the kinds of terms that can arise.

Suppose $f \in \theta C^\infty(I)$. Then since $q(\phi) = O(\phi^n)$, the integral in the first term of (5.11) is smooth. The factor $p(\theta)$ may contain a term with a factor of $\theta^n\log(\theta)$ if n is even. Thus, the first term is in $\mathcal{A}_{n,k}$.

Because, in general, $\theta^j \int_0^\theta \theta^{-k} d\theta$ is smooth for $0 \leq k \leq j$ unless $k = 1$, the integral in the second term is smooth unless either $p(\phi)$ has a $\phi^n \log\phi$ term in it (which could happen if n

is even), or there is a nonvanishing term of order ϕ^{-1} in the expansion of the integrand. In the first case, when $p(\phi)$ has a term in its expansion of the form $\phi^n \log(\phi)$, the term yields a log at order $\theta^{n+2}\log(\theta)$ and at higher powers in θ. When the integrand contains a term of order ϕ^{-1}, the second term yields a term of the form $\theta^n \log(\theta)$. Thus, in this case, $Gf \in \mathcal{A}_{n,k}$.

Now suppose that $f \in \theta^{n+1}\log(\theta)C^\infty(I)$. Then the first integral yields a log at order $\theta^{n+1}\log(\theta)$, and a possible $\log(\theta)^2$ at order $\theta^{2n+1}\log(\theta)^2$, if $p(\theta)$ has a term of the form $\theta^n \log(\theta)$.

When the integrand of the second term is expanded in an asymptotic series, we get several nonsmooth terms. First, at lowest order we get a term of the form $\theta^{n+1}\log(\theta)$, with logs at higher orders as well. Now, it is an elementary result that

$$\int_0^\theta \theta^j \log(\theta)^2 d\theta = \theta^{j+1}\left(2(1+j)^{-3} - 2(1+j)^{-2}\log(\theta) + (1+j)^{-1}\log(\theta)^2\right).$$

Thus, if $p(\phi)$ has a term of the form $\phi^n \log(\phi)$, then we get a term of the form $\theta^{2n+1}\log(\theta)^2$ as well. If $k = 1$, we have shown, as desired, that $Gf \in \mathcal{A}_{n,k} + \theta^{(k+1)n+1}\log(\theta)^{k+1}C^\infty(I)$. If $k > 1$, then in both the cases so far considered, we have seen that $Gf \in \mathcal{A}_{n,k}$.

Now suppose that $k > 1$ and that $f(\theta) \in (\theta^n \log(\theta))^j C^\infty(I)$ for some $2 \le j \le k$. Taking account of the possible $\theta^n \log(\theta)$ term in $p(\theta)$, an integral formula analogous to the above shows that the first term yields a function in $\oplus_{i=0}^{j+1}(\theta^n \log(\theta))^i C^\infty(I)$. This clearly lies in the desired space. It is easy to see that the second term lies in the same space.

Thus, we have seen that $Gf \in \mathcal{A}_{n,k} + \theta^{(k+1)n}\log(\theta)^{k+1}C^\infty([0,\theta_0])$. Our second step is now to show that, in fact, the last term does not arise. We have seen that $I_\nu Gf = f \in \mathcal{B}_{n,k}$. But for no $k \ge 1$ is $(k+1)n$ an incidial root of the operator I_ν at $\theta = 0$. Thus, if $0 \not\equiv u \in C^\infty(I)$, then $I_\nu(\theta^{(k+1)n}\log(\theta)^{k+1}u)$ yields a term in $\theta^{(k+1)n}\log(\theta)^{k+1}C^\infty(I)$, which is not canceled by any other term, and this preclues $I_\nu u$ from lying in $\mathcal{B}_{n,k}$. Thus, we must in fact have that $G : \mathcal{B}_{n,k} \to \mathcal{A}^0_{n,k}$.

Now I_ν is injective, and it has a right inverse. Thus, it is a bijection.

It remains to show that if I_ν is *not* injective, then it is also not surjective. Suppose that $\ker I_\nu \ne \emptyset$. Renaming q, let $0 \ne q \in \ker I_\nu \subseteq \mathcal{A}^0_{n,k}$. Then Lemma 5.3(b) makes it clear that any $f \in \operatorname{Im} I_\nu$ must be orthogonal to q with respect to measure $\sin^{-(1+n)}(\theta)d\theta$. Since $v = \sin(\theta)q \in \mathcal{B}_{n,k}$ is clearly not orthogonal to q, we conclude that I_ν is not surjective.

If q is orthogonal to f with respect to the measure $\sin^{-(n+1)}(\theta)$, then let p be any solution of the equation $I_\nu p = 0$ with $p(0) = 1$. Then it is easy to check that with p and q reinterpreted according to these definitions, equation (5.11) still gives a solution, of course not unique, to the equation $I_\nu u = f$, with $u(0) = 0 = u'(\theta_0)$. The hypothesis that q is orthogonal to f is necessary in showing that $u'(\theta_0) = 0$.

We finally turn to the last claim. If n is odd and $f \in \mathcal{B}_{n,k}$ is even through order $n+1$, then no term of the form θ^{-1} can appear in the integrand in (5.9), and no logarithms appear in p or q. This yields the claim. ■

This result will be used, for general k, in our formal solution of the Einstein equations in the next chapter.

5.2 The Laplacian

We now continue to study the Laplace boundary value problem at hand. Recall that we are letting $\widetilde{X} = [0, \theta_0] \times S \times [0, \varepsilon)$ be the pullback of the blowup of a cornered space by a diffeormophism as in Corollary 4.17, and are assuming that g is the pullback of a constant-angle admissible metric on the blowup, in the form (5.2). The goal of this section is to prove Theorem 5.1. This problem requires only the $k = 1$ case of Proposition 5.8 since it deals with a linear equation. For this section, we specialize to the case $\theta_0 = \frac{\pi}{2}$, for which we have a full, explicit solution to the eigenvalue problem. The techniques we use would be of relevance in studying other cases as well, although the behavior would depend crucially on the spectrum, which might display a variety of behaviors in general.

Now it is easy to show that for $v \in C^\infty(\widetilde{S})$,

$$\begin{aligned}\Delta_g \left(\rho^\nu \log(\rho)^k v\right) =&\rho^\nu \Big[\log(\rho)^k I_\nu v + k(1+\nu-k)\log(\rho)^{k-1}\sin^2(\theta)v \\ &+k(k-1)\log(\rho)^{k-2}\sin^2(\theta)v\Big] + o(\rho^\nu).\end{aligned}$$

Motivated by this, we define an operator J_{n+q} on $\mathcal{E}_{n,q}$ (see page 27) by setting

$$\begin{aligned}J_{n+q}(\rho^{n+q}\log(\rho)^k b) =&\rho^{n+q}(\log(\rho)^k I_{n+q}(b) + k(1+n+q-k)\log(\rho)^{k-1}\sin^2(\theta)b \\ &+ k(k-1)\sin^2(\theta)\log(\rho)^{k-2}b),\end{aligned}$$

where $b \in \mathcal{A}^0_{n,1}$, and extending linearly. For q even, we define

$$\mathcal{E}^0_{n,q} = \left\{ \eta = \rho^{n+q} \sum_{i=0}^{\frac{q}{2}+1} \log(\rho)^i b_i(\theta) : b_{\frac{q}{2}+1} \in \ker I_{n+q} \right\}.$$

For consistency of notation, we define $\mathcal{E}^0_{n,q} = \mathcal{E}_{n,q}$ when q is odd.

Proposition 5.9. *For $q \geq 1$ odd, $J_{n+q} : \mathcal{E}_{n,q} \to \mathcal{F}_{n,q}$ is an isomorphism. For $q \geq 0$ even, $J_{n+q} : \mathcal{E}^0_{n,q} \to \mathcal{F}_{n,q}$ is surjective, with a one-dimensional kernel spanned by $\rho^{n+q} w_{\frac{q}{2}}$, where $w_{\frac{q}{2}}$ is as in Proposition 5.7.*

Proof. That J_{n+q} has the given codomains follows immediately from the definitions of $\mathcal{E}_{n,q}$, $\mathcal{E}^0_{n,q}$, $\mathcal{F}_{n,q}$, and J.

For the isomorphism claim, we assume $q = 2j + 1$ is odd. We set $J = J_{n+q}$. By Propositions 5.7 and 5.8, $I_{n+2j+1} : \mathcal{A}^0_{n,1} \to \mathcal{B}_{n,1}$ is a bijection. So suppose we wish to solve $J\eta = f$, where $\eta = \rho^{n+2j+1} \sum_{i=0}^{j+1} \log(\rho)^i b_i(\theta)$ and $f = \rho^{n+2j+1} \sum_{i=0}^{j+1} \log(\rho)^i c_i(\theta)$. We will do this term by term, starting with the highest power of log and working down. We can uniquely solve $I_{n+2j+1} b_{j+1} = c_{j+1}$. Set $\eta^{(j+1)} = \rho^{n+2j+1} \log(\rho)^{j+1} b_{j+1}(\theta)$, and note that $f - J\eta^{(j+1)} = \rho^{n+2j+1} \sum_{i=0}^{j} \log(\rho)^i c_i^{(j+1)}(\theta)$, where $c_i^{(j+1)}$ may differ from c_i. Now suppose, by way of induction, that we have constructed $\eta^{(l)} \in \mathcal{E}_{n,2j+1}$ such that $f - J\eta^{(l)} = \rho^{n+2j+1} \sum_{i=0}^{l-1} \log(\rho)^i c_i^{(l)}$. Then we can uniquely solve the equation $I_{n+2j+1} b_{l-1} = c_{l-1}^{(l)}$, and setting $\eta^{(l-1)} = \eta^{(l)} + \rho^{n+2j+1} \log(\rho)^{l-1} b_{l-1}$, we easily see that $f - J\eta^{(l-1)} = \rho^{n+2j+1} \sum_{i=0}^{l-2} \log(\rho)^i c_i^{(l-1)}$. Thus, by induction, we can solve $J\eta = f$. At each step, the solution is unique, and so we see that J is an isomorphism.

We now address the even case, $q = 2j$. In this case, I_{n+2j} has a one-dimensional kernel spanned by $w_{\frac{q}{2}}$, that is, w_j. We wish to solve the equation $J\eta = f$, where we take $\eta = \rho^{n+2j} \sum_{i=0}^{j+1} \log(\rho)^i b_i(\theta)$ and $f = \rho^{n+2j} \sum_{i=0}^{j} \log(\rho)^i c_i(\theta)$. We cannot necessarily solve the equation $I_{n+2j} b_j = c_j$, since c_j may not be orthogonal to w_j, generically (see Proposition 5.8). However, we may solve this using our freedom in the $\log(\rho)^{j+1}$ term. First, notice that $\sin^2(\theta) w_j$ is not orthogonal to w_j. Now let $b_{j+1} = \frac{1}{(j+1)(n+j)} \frac{\langle c_j, w_j \rangle}{\langle \sin^2(\theta) w_j, w_j \rangle} w_j$, where $\langle , \rangle$ here refers to the L^2

norm on $[0,\theta_0]$ with measure $\sin^{-(n+1)}(\theta)d\theta$. (This is finite since $w_j = O(\theta^n)$ and $c_j = O(\theta)$.) Set $\eta^{(j+1)} = \rho^{n+2j}\log(\rho)^{j+1}b_j$. Then $f - J\eta^{(j+1)} = \rho^{n+2j}\sum_{i=0}^{j}\log(\rho)^i c_i^{(j+1)}$, where $c_j^{(j+1)}$ is orthogonal to w_j. We can now proceed by induction as in the odd case; except that at each step l, we uniquely add a multiple of w_j to b_l to ensure that $c_{l-1}^{(l)}$ is orthogonal to w_j. By induction, we may thus solve the equation. However, a solution is unique only up to addition of a multiple of w_j. ∎

The following lemma follows easily from Lemma 5.1, and particularly from the fact that 0 and n are indicial roots of the Laplacian at $\theta = 0$.

Lemma 5.10. *Suppose that $u \in \mathcal{P}(\widetilde{X})$. Then $\Delta_g u \in \mathcal{P}(\widetilde{X})$ as well. Moreover, suppose that for any fixed $x \in S$, $u(\theta, x, \rho) \in \mathcal{E}^0_{n,q}$ for some q independent of x. Then for each fixed x, there is some $f \in \mathcal{F}_{n,q}$ such that $\Delta_g u(\theta, x, \rho) = f(\theta, \rho) + o(\rho^{n+q})$.*

Proof of Theorem 1.4. We work in the decomposition given by Corollary 4.17. In particular, $\widetilde{U}$, θ, and ρ are as in that corollary. We write $\psi = \psi(x, \rho)$. Throughout the proof, primes will refer to a derivative with respect to θ.

Define $u_0 \in C^\infty(\widetilde{U})$ by $u_0(\theta, x, \rho) = \psi(x, \rho)$. Then by Lemma 5.1, $\Delta_g u_0 = O(\rho)$.

Now suppose that $0 \leq k < n-1$ and that u_k has been smoothly and uniquely defined in $\mathcal{R}(\widetilde{X})$ so that

(a) $\Delta_g u_k = O(\rho^{k+1})$;

(b) $u_k|_{\theta=0} = \psi$; and

(c) $u_k'(\theta_0) = 0$.

We wish to find $\varphi_{k+1} \in \mathcal{A}^0_{n,1}(\frac{\pi}{2}, S)$ so that $u_{k+1} = u_k + \rho^{k+1}\varphi_{k+1}$ satisfies each of these conditions with k replaced by $k+1$. Define $f_{k+1} \in \mathcal{B}_{n,1}\left(\frac{\pi}{2}, S\right)$ by $f_{k+1} = \rho^{-(k+1)}\Delta_g u_k|_{\rho=0}$. By Lemma 5.1, $f_{k+1} = O(\theta)$, and indeed, since $\Delta_g(\theta^n) = O(\theta^{n+1})$, $f_{k+1} \in \mathcal{B}_{n,1}(\frac{\pi}{2}, S)$ as desired. Fix $x \in S$. By Proposition 5.8, we can uniquely solve $I_{k+1}\varphi_{k+1}(\theta, x) = -f_{k+1}(\theta, x)$

in $\mathcal{A}^0_{n,1}$, with $\varphi(0,x) = 0$ and $\varphi'_{k+1}(\theta_0, x) = 0$. Plainly φ_{k+1} depends smoothly on x. Thus, $\varphi_{k+1} \in \mathcal{A}^0_{n,k}\left(\frac{\pi}{2}, S\right)$ is determined as desired. By induction and Proposition 5.8, we thus can construct a function $u_{n-1} \in \mathcal{R}(\widetilde{X})$, unique through order $n-1$, such that $u_{n-1}|_{\widetilde{M}} = \psi$, such that $u'_{n-1}(\theta_0) \equiv 0$, and such that $\Delta u_{n-1} = O(\rho^n)$.

At order n, I_n is not surjective, so our procedure generically fails unless, for each x, $f_n(\theta, x)$ is orthogonal to $\sin^n(\theta)$ with respect to the measure $\sin^{-(n+1)}(\theta)d\theta$, i.e., unless it is orthogonal to $\csc(\theta)$. To proceed, we must introduce logarithmic terms, and for this we will use Proposition 5.9. Let $f_n = \rho^{-n}\Delta_g u_{n-1}|_{\rho=0} \in \mathcal{B}_{n,1}$ as above, and fix x. Then notice that $\rho^n f_n(\cdot, x) \in \mathcal{F}_{n,0}$. Thus, by Proposition 5.9 with $q = 0$, there exists a solution $\Phi_n \in \mathcal{E}_{n,0}$ to $J_n\Phi_n(\theta) = \rho^n f_n(\theta, x)$, which however is determined only up to a term that is a multiple of $\rho^n \sin^n(\theta)$. Set $\varphi_n = \rho^{-n}\Phi_n$, and $u_n = u_{(n-1)} + \rho^n\varphi_n$. Then since this procedure is smooth in x, $u_n(\theta, x, \rho)$ satisfies $\Delta_g u = o(\rho^n)$, and the boundary conditions $u_n|_{\widetilde{M}} = \psi$ and $u'_n(\frac{\pi}{2}) = 0$ are satisfied. Notice that $u_n \in \mathcal{P}(\widetilde{X})$.

We proceed by induction. Suppose that $u_{n+2j} \in \mathcal{P}(\widetilde{X})$ has been successfully defined satisfying $\Delta_g u_{n+2j} = o(\rho^{n+2j})$, with both boundary conditions as desired, and containing $(j+1)$st powers of $\log(\rho)$. Then since Δ_g is linear, $\Delta_g u_{n+2j}$ will likewise contain at most $(j+1)$st powers of $\log(\rho)$. Fix $x \in S$. Let $F_{n+2j+1} \in \mathcal{F}_{n,2j+1}$ be such that $\Delta_g u_{n+2j}(\theta, x, \rho) = F_{n+2j+1}(\theta, \rho) + o(\rho^{n+2j+1})$. Such an F exists by Lemma 5.10. We wish to solve the equation $J_{n+2j+1}\Phi_{n+2j+1} = F_{n+2j+1}$; by Proposition 5.9, we may do so, and plainly the solution varies smoothly in x. Then set $u_{n+2j+1} = u_{n+2j} + \Phi_{n+2j+1}$. Clearly, $u_{n+2j+1} \in \mathcal{P}(\widetilde{X})$ satisfies $\Delta_g u_{n+2j+1} = o(\rho^{n+2j+1})$ and our boundary conditions.

Next we wish to find $u_{n+2(j+1)}$, satisfying our boundary conditions, such that $\Delta_g u_{n+2(j+1)} = o(\rho^{n+2(j+1)})$. But this is exactly the same as the odd case, except that by Proposition 5.9, the solution will be unique only up to a term of the form $\rho^{n+2(j+1)}w_{j+1}$, where w_{j+1} is as in Proposition 5.7. Hence, by induction, we get an infinite sequence $\{u_k\}_{k=0}^{\infty}$ such that $\Delta_g u_k = o(\rho^k)$, $u_k|_{\widetilde{M}} = \psi$, and $\partial_\theta u_k|_{\theta=\frac{\pi}{2}} = 0$, and such that each member of the sequence $\{u_k\}$ has at most $\left\lfloor \frac{k+1-n}{2} \right\rfloor$ powers of $\log(\rho)$.

Thus, by Borel's Lemma, as stated in [Erd56], there exists a function $u \in \mathcal{P}(\widetilde{X})$ such that $\Delta_g u = O(\rho^\infty)$, such that $\partial_\theta u|_{\theta=\frac{\pi}{2}} = 0$, and such that $u|_{\widetilde{M}} = \psi$. ■

Notice that, in order to uniquely determine a solution, we would need to specify not only ψ, but also a scalar function $\eta_k \in C^\infty(S)$ at order $n + 2k$ for all $k \geq 0$.

Chapter 6

EINSTEIN METRICS

In this chapter, we consider formal existence of CAH Einstein metrics on a cornered space (X, M, Q). This has famously been studied in the AH setting in [FG12], where the boundary data is a conformal class on the conformal infinity M. Here, we must additionally prescribe a boundary condition at Q. One of the most natural is that considered in [NTU12], which requires that Q be totally umbilic and of constant mean curvature, so that its second fundamental form K_Q satisfies $K_Q = \lambda g|_{TQ}$ away from the corner $S = Q \cap M$. We follow that paper in imposing this boundary condition. It requires less data than a Dirichlet condition would – only the constant λ is required, whereas that would require the prescription of an entire tensor field. Moreover, as we will see, it interacts particularly nicely with the geometry of CAH spaces.

In section 6.1, we prove basic results about Einstein spaces satisfying our boundary conditions, such as the fact that $\lambda \in (-1, 1)$ necessarily. We also study compatibility conditions that are imposed on the corner S if the Einstein metric g is to have smooth compactification. These prove to be severe with the given boundary conditions.

In section 6.2, we then pursue the question of formal existence of Einstein metrics in normal form, given an arbitrary manifold M with boundary S and a conformal class. Our setting is analogous to that of Theorem 2.5 in the AH case. As the results of [NTU12] and of section 6.1 show, we must allow metrics without smooth compactification if we are to hope to solve this problem. We construct instead metrics on the blowup in the normal form (1.4), and show that Einstein metrics are unique to order n in that form.

6.1 Smooth Compatibility Conditions

We first consider whether and when we can hope for a smooth CAH Einstein metric with CMC-umbilic boundary condition on a cornered space, given arbitrary initial data. Suppose a manifold M^n with boundary S is given, and is equipped with a Riemannian conformal class $[h]$, which throughout this section will be taken to be a smooth conformal class. In this section, we will investigate, if $(M, [h])$ is to be the conformal infinity of a *smooth* CAH Einstein space (X, M, Q, g_+) satisfying $K_Q = \lambda g|_{TQ}$, what we can deduce from M and S about the developments of g and Q; and what constraints are put on S. Because we will frequently want to use indices on g_+, and no blowups occur, for convenience we break in this section with the notation used in the rest of this thesis and write $g = g_+$. Elsewhere, g will remain an admissible metric on $\widetilde{X}$. As mentioned in the introduction, this situation has been analyzed in [NTU12] with the *a priori* assumption that g is the hyperbolic metric. We remove this assumption to investigate the general case. In this section, we will work directly on (X, M, Q) and not the blowup; thus, we will not use the machinery or notation of Chapter 4.

First, we state a classical result.

Theorem 6.1. *Let $n \geq 2$, and let $M^n \subset \mathbb{H}^{n+1}$ be an umbilic hypersurface. Then the umbilic coefficient $\lambda_p = \lambda$ is the same at all points $p \in M$. Moreover, as a subset of $\mathbb{R}^{n+1}$, M is either part of a sphere or part of a hyperplane. In particular, M falls into one of the following classes:*

(i) M is part of a geodesic sphere. In this case, M is part of a Euclidean sphere entirely contained in $\mathbb{H}^{n+1}$. In this case, $|\lambda| > 1$;

(ii) M is a horosphere: either it is part of a Euclidean sphere contained entirely in $\mathbb{H}^{n+1}$ except for one point, which lies on the boundary $\mathbb{R}^n$; or it is part of a Euclidean plane contained in $\mathbb{H}^{n+1}$ and parallel to $\mathbb{R}^n$. In either case, $|\lambda| = 1$;

(iii) M is totally geodesic. In this case, M is either part of a Euclidean hemisphere that meets the boundary $\mathbb{R}^n$ normally, or part of a Euclidean hyperplane that meets the boundary normally. In either event, $\lambda = 0$; or

(iv) *M is part of a Euclidean sphere or Euclidean hyperplane that intersects the boundary $\mathbb{R}^n$ non-normally. In the case of a sphere, M lies in the upper part of a sphere of radius r, such that its center (y, x) satisfies $0 < |y| < r$. Either case is called an equidistant hypersurface, because it lies at fixed distance from the totally geodesic hypersurface S such that the intersection of S with $\mathbb{R}^n$ is the same as that of the plane or sphere in which M lies. In this situation, $0 < |\lambda| < 1$.*

See [Spi99, chap. IV.7] for a discussion and proof.

We next prove a lemma in a more general context.

Lemma 6.2. *Let (X, g) be a Riemannian manifold with a smooth metric g, and with embedded hypersurfaces Q and M that intersect transversely in an embedded submanifold S. Denote by K the scalar second fundamental form with respect to a fixed unit normal vector. By K_S, we will mean the second fundamental forms of S considered as a submanifold of M, while K_M and K_Q will mean the second fundamental forms of M and Q as submanifolds of X. Then*

$$K_S = \left. \frac{K_Q - \langle \nu_Q, \nu_M \rangle K_M}{\langle \nu_Q, \nu_S \rangle} \right|_{TS}, \tag{6.1}$$

where ν_S is unit the g-normal to S in M, and ν_Q and ν_M are the unit g-normal vectors to Q and M in X.

Proof. Denote by II the vector second fundamental form, with the same conventions as for K in the statement: so, for example, II_S is the vector second fundamental form of S as a submanifold of $(M, g|_{TM})$. We compute each second fundamental form. We will use P to denote an orthogonal projection operator, while ∇ will denote the Levi-Civita connection on X. First, for $p \in S$ and $X, Y \in T_pS$, extended smoothly to a neighborhood,

$$\begin{aligned} II_S(X, Y) &= P_{TS^\perp} P_{TM} \nabla_X Y \\ &= \langle \nabla_X Y, \nu_S \rangle \nu_S. \end{aligned}$$

Next,

$$\begin{aligned} II_M(X,Y) &= P_{TM^\perp}\nabla_X Y \\ &= \langle \nabla_X Y, \nu_M\rangle \nu_M. \end{aligned}$$

Now write $\nu_Q = \nu_Q^S \nu_S + \nu_Q^M \nu_M$ (such a decomposition must be possible at S since $TS \subset TQ$). Then we have

$$\begin{aligned} II_Q(X,Y) &= \langle \nabla_X Y, \nu_Q\rangle \nu_Q \\ &= (\nu_Q^S \langle \nabla_X Y, \nu_S\rangle + \nu_Q^M \langle \nabla_X Y, \nu_M\rangle)(\nu_Q^S \nu_S + \nu_Q^M \nu_M) \\ &= (\nu_Q^S)^2 \langle \nabla_X Y, \nu_S\rangle \nu_S + \nu_Q^S \nu_Q^M (\langle \nabla_X Y, \nu_S\rangle \nu_M + \langle \nabla_X Y, \nu_M\rangle \nu_S) \\ &\quad + (\nu_Q^M)^2 \langle \nabla_X Y, \nu_M\rangle \nu_M. \end{aligned}$$

Hence,

$$\begin{aligned} \langle II_Q(X,Y), \nu_S\rangle &= (\nu_Q^S)^2 \langle \nabla_X Y, \nu_S\rangle + \nu_Q^S \nu_Q^M \langle \nabla_X Y, \nu_M\rangle \\ &= \langle \nu_Q, \nu_S\rangle^2 \langle II_S(X,Y), \nu_S\rangle + \langle \nu_Q, \nu_S\rangle \langle \nu_Q, \nu_M\rangle \langle K_M(X,Y), \nu_M\rangle, \end{aligned}$$

where the last line follows from our prior computations. Now since M and Q are transverse, $\langle \nu_Q, \nu_S\rangle \neq 0$. Moreover, $\langle II_Q(X,Y), \nu_S\rangle = K_Q(X,Y)\langle \nu_Q, \nu_S\rangle$. Thus, we find

$$K_Q(X,Y) = \langle \nu_Q, \nu_S\rangle K_S(X,Y) + \langle \nu_Q, \nu_M\rangle K_M(X,Y),$$

which yields the claim. ■

We now continue in the CAH Einstein context, where we get an immediate corollary from the preceding result.

Corollary 6.3. *Let $(M,[h])$ be a compact manifold with boundary S, equipped with a conformal class $[h]$. Suppose (X,M,Q,g,λ) is a CMC-umbilically cornered AH Einstein space, with smooth conformal infinity $[h]$. Further, suppose that, for smooth defining functions, the compactified metric $\bar{g}$ is smooth. Then S is umbilic in M with respect to any metric $h \in [h]$.*

Proof. It certainly suffices to consider only h, as umbilicity is a conformally invariant condition. For the same reason, it follows that Q is $\bar{g}$-umbilic. Because g is a CAH Einstein metric, M is $\bar{g}$-totally geodesic. Thus, since $h = \bar{g}|_{TM}$ (where r is an appropriate defining function), the claim follows directly from (6.1). ■

This corollary gives a substantial obstruction to the existence of a smooth CMC-umbilically cornered CAH Einstein space realizing $(M, [h])$ as its conformal infinity. For $n > 2$, for example, it provides a proof relying only on the boundary geometry that, if $\widetilde{X}$ is hyperbolic space $\mathbb{H}^{n+1}$ and M is a subset of $\mathbb{R}^n$, then the boundary S of M must be a sphere or a hyperplane, as these are the only umbilic surfaces in Euclidean space. (This proof does not work if $n = 2$, since every hypersurface of a 2-space is umbilic). We may even further characterize the geometry near the boundary as we further expand the umbilic condition.

First we define some helpful coordinates. Fix a metric $h \in [h]$. On a neighborhood V near a point of S in M, we may always choose geodesic normal coordinates $x^1, \ldots, x^n$ such that $S = \{x^n = 0\}$ and such that $\frac{\partial}{\partial x^n} \perp_h TS$, with $\left|\frac{\partial}{\partial x^n}\right| = 1$ and $\frac{\partial}{\partial x^n}(\langle \frac{\partial}{\partial x^n}, \frac{\partial}{\partial x^n} \rangle) = 0$ at S. Let r be a geodesic defining function for M, so that $h = r^2 g|_{TM}$, such that $|dr|^2_{\bar{g}} = 1$, and such that $x^0 = r, x^1, \ldots, x^n$ are coordinates for some $X \supseteq U \simeq [0, \varepsilon) \times V$, where $\frac{\partial}{\partial r} \perp_{\bar{g}} TM$.

When working with these coordinates, we will use the Roman indices $0 \leq i, j, k \leq n$ to label coordinates on X; the Greek indices $0 \leq \alpha, \beta, \gamma \leq n-1$ to label coordinates on Q; and the Roman indices $1 \leq s, t, u \leq n-1$ to label coordinates on S.

Before continuing to explore the consequences of smoothness, we state a useful result that is true more generally.

Proposition 6.4. *Let (X, M, Q) be a cornered space, and let g be a smooth CAH Einstein metric on X satisfying $K_Q = \lambda g|_{TQ}$, where K_Q is the second fundamental form of Q with respect to the inward-pointing normal vector. Let $\bar{\nu}_M$ be the X-inward $\bar{g}$-unit normal to M, and let $\bar{\nu}_S$ be the M-inward h-unit normal to S in M. Then at every point $p \in S$, $\cos(\theta_0(p)) = -\lambda$. In particular, $\lambda \in (-1, 1)$.*

Notice that the proof given here depends only on the continuity of K_Q up to S; thus, by Lemma

4.9, this proposition remains true if g is only admissible.

Proof. Let ν be the inward-pointing unit normal field on Q, and K the second fundamental form of Q, both with respect to g; and let $\bar{\nu} = \bar{\nu}_Q$ be the inward-pointing unit normal field on Q with respect to $\bar{g}$. Now the umbilic condition is equivalent by Weingarten to

$$\langle \nabla_X \nu, Y \rangle_g = -\lambda \langle X, Y \rangle_g$$

for all $X, Y \in C^\infty(TQ)$. For $r \neq 0$, the unit normal to Q with respect to $\overline{g}$ is given by $\overline{\nu} = r^{-1}\nu$. We wish to compute $\overline{K}(X, Y)$, the second fundamental form of Q with respect to $\bar{g}$.

A straightforward computation shows that for any vector fields X, Y, we have

$$\overline{\nabla}_X Y = \nabla_X Y + r^{-1}\left[dr(X)Y + dr(Y)X - \langle X, Y \rangle_{\overline{g}}\, \mathrm{grad}_{\overline{g}}\, r\right].$$

For $q \in Q$ and $X, Y \in TQ|_q$, it follows (taking extensions where necessary) that

$$\begin{aligned} \overline{K}(X, Y) &= -\langle \overline{\nabla}_X (r^{-1}\nu), Y \rangle_{\overline{g}} \\ &= -r^{-1}\left\langle \nabla_X \nu + dr(X)\overline{\nu} + dr(\overline{\nu})X - \langle X, \overline{\nu} \rangle\, \mathrm{grad}_{\bar{g}}\, r - dr(X)\overline{\nu}, Y \right\rangle_{\overline{g}} \\ &= r^{-1}\left(r^2 K(X, Y) - dr(\overline{\nu})\langle X, Y \rangle_{\overline{g}}\right). \end{aligned} \tag{6.2}$$

Therefore,

$$\overline{K} = \frac{\lambda - dr(\overline{\nu})}{r}\overline{g}(X, Y) \tag{6.3}$$

is equivalent to $K_Q = \lambda g$.

Thus, we see that for $r \neq 0$, Q is $\bar{g}$-umbilic with possibly non-constant umbilic coefficient. But we also see that, for $\overline{K}$ to remain smooth up to the boundary – which it surely must, since $\bar{g}$ is a smooth metric – we must have

$$dr(\overline{\nu}) \xrightarrow[r \to 0]{} \lambda.$$

In particular, since ∂_r is the inward-pointing unit $\bar{g}$-normal at M, which we denote by $\overline{\nu}_M$, we find that (denoting $\overline{\nu} = \overline{\nu}_Q$ for clarity)

$$\left\langle \overline{\nu}_Q, \overline{\nu}_M \right\rangle_{\overline{g}} = \lambda, \qquad (r = 0), \tag{6.4}$$

is equivalent to $K = \lambda g + O(r^{-1})$. Since $\cos(\theta) = -\langle \bar{\nu}_Q, \bar{\nu}_M \rangle$, our claim is established. ∎

As mentioned, the above result actually holds even for admissible metrics. We now obtain a result that in general does not. We will henceforth in this section assume that inner products are with respect to $\overline{g}$ if not otherwise specified.

Proposition 6.5. *Let X, M, Q, g, λ, and K_Q be as in Proposition 6.4. Let $\bar{\nu}_M$ be the X-inward $\bar{g}$-unit normal to M, and let $\bar{\nu}_S$ be the M-inward h-unit normal to S in M. Moreover, let $\overline{R}$ be the curvature tensor of $\bar{g}$. Write $\bar{K}_S = \eta h$, where $\bar{K}_S$ is the second fundamental form of S in M with respect to h and $\bar{\nu}_S$, and $\eta \in C^\infty(S)$; we can write this by Corollary 6.3. Then for any $Z \in TS$,*

$$Z(\eta) = -\overline{R}(\bar{\nu}_M, Z, \bar{\nu}_M, \bar{\nu}_S). \tag{6.5}$$

Proof. Let ν, K and $\overline{\nu}$ be as in the previous proof. We will assume that (6.4) holds, and we multiply through by r in (6.3) to obtain

$$r\overline{K}(X, Y) = (\lambda - \langle \overline{\nu}_Q, \partial_r \rangle)\overline{g}(X, Y). \tag{6.6}$$

This equation holds on Q for any $X, Y \in C^\infty(TQ)$ if and only if $K = \lambda g$. We will repeatedly differentiate it covariantly to obtain new equations.

Before proceeding, we write Q locally as the graph of a function,

$$x^n = \varphi(r, x^1, \ldots, x^{n-1}).$$

Notice that $\varphi(0, x^1, \ldots, x^{n-1}) \equiv 0$ by our choice of coordinates on M. We next define a local frame on Q by

$$E_\alpha = \frac{\partial}{\partial x^\alpha} + \frac{\partial \varphi}{\partial x^\alpha} \frac{\partial}{\partial x^n}.$$

In particular, $\{E_s\}_{s=1}^{n-1}$ is also a local frame for S at $r = 0$, in fact the coordinate frame.

Define $f \in C^\infty([0, \varepsilon) \times V)$ by $f = \varphi - x^n$. Then f vanishes precisely on Q, and we may write $\overline{\nu} = \frac{-\operatorname{grad}_{\overline{g}} f}{|\operatorname{grad}_{\overline{g}} f|}$. Using this and the fact that, at $r = 0$, the normal $\overline{\nu}$ may be written as

$$\bar{\nu} = \lambda \partial_r + \sqrt{1 - \lambda^2} \partial_{x^n} \tag{6.7}$$

by (6.4), it is straightforward to show that

$$\left. \frac{\partial \varphi}{\partial r} \right|_{r=0} = \frac{-\lambda}{\sqrt{1 - \lambda^2}}. \tag{6.8}$$

We intend to apply $\overline{\nabla}^Q_{E_0}$, the Levi-Civita connection of $\bar{g}|_{TQ}$, to both sides of (6.6). Doing this once, and utilizing the metric property of the Levi-Civita connection, we obtain

$$\overline{K}_{\alpha\beta} + r\overline{\nabla}^Q_{E_0}\overline{K}_{\alpha\beta} = -E_0\left(\langle\partial_r, \overline{\nu}\rangle\right)\overline{g}_{\alpha\beta}, \tag{6.9}$$

which should hold for all r, along Q. Taking $r = 0$, we get

$$\overline{K}_{\alpha\beta} = -E_0(\langle\partial_r, \overline{\nu}\rangle)\overline{g}_{\alpha\beta} \tag{6.10}$$

Now

$$E_0(\langle\partial_r, \overline{\nu}\rangle) = \langle\overline{\nabla}_{E_0}\partial_r, \overline{\nu}\rangle + \langle\partial_r, \overline{\nabla}_{E_0}\overline{\nu}\rangle \tag{6.11}$$

(where $\overline{\nabla}$ is still the Levi-Civita connection for $\bar{g}$ on TX). Since $E_0 = \partial_r + \frac{\partial\varphi}{\partial r}\partial_{x^n}$, we have $\partial_r = E_0 - \frac{\partial\varphi}{\partial r}\partial_{x^n}$. Hence,

$$\begin{aligned}\langle\partial_r, \overline{\nabla}_{E_0}\overline{\nu}\rangle &= \langle E_0, \overline{\nabla}_{E_0}\overline{\nu}\rangle - \frac{\partial\varphi}{\partial r}\langle\partial_{x^n}, \overline{\nabla}_{E_0}\overline{\nu}\rangle \\ &= -\overline{K}(E_0, E_0) - \frac{\partial\varphi}{\partial r}\langle\partial_{x^n}, \overline{\nabla}_{E_0}\overline{\nu}\rangle.\end{aligned} \tag{6.12}$$

Moreover,

$$\begin{aligned}\langle\overline{\nabla}_{E_0}\partial_r, \overline{\nu}\rangle &= \langle\overline{\nabla}_{E_0}E_0, \overline{\nu}\rangle - \langle\overline{\nabla}_{E_0}\left(\frac{\partial\varphi}{\partial r}\partial_{x^n}\right), \overline{\nu}\rangle \\ &= \overline{K}(E_0, E_0) - E_0\left(\frac{\partial\varphi}{\partial r}\right)\langle\partial_{x^n}, \overline{\nu}\rangle - \frac{\partial\varphi}{\partial r}\langle\overline{\nabla}_{E_0}\partial_{x^n}, \overline{\nu}\rangle \\ &= \overline{K}(E_0, E_0) - \frac{\partial^2\varphi}{\partial r^2}\langle\partial_{x^n}, \overline{\nu}\rangle - \frac{\partial\varphi}{\partial r}\langle\overline{\nabla}_{E_0}\partial_{x^n}, \overline{\nu}\rangle.\end{aligned} \tag{6.13}$$

From (6.11),(6.12), and (6.13), we get

$$E_0\langle\partial_r, \overline{\nu}\rangle = -\left(\frac{\partial^2\varphi}{\partial r^2}\langle\partial_{x^n}, \overline{\nu}\rangle + \frac{\partial\varphi}{\partial r}E_0\langle\partial_{x^n}, \overline{\nu}\rangle\right). \tag{6.14}$$

Now $1 \equiv |\overline{\nu}|^2_{\overline{g}} = \overline{g}^{ij}\langle\overline{\nu}, \partial_{x^i}\rangle\langle\overline{\nu}, \partial_{x^j}\rangle$. We apply E_0 to this equation, take $r = 0$, and use the facts that $TS \subset TQ$, so that $\overline{\nu} \perp_{\overline{g}} TS$; that $\partial_{x^n}(|\partial_{x^n}|_{\overline{g}}) = 0$ on S; and that, because g is Einstein, $\partial_r\overline{g}|_{r=0} = 0$ to conclude that

$$\langle\partial_{x^n}, \overline{\nu}\rangle E_0\langle\partial_{x^n}, \overline{\nu}\rangle + \langle\partial_r, \overline{\nu}\rangle E_0\langle\partial_r, \overline{\nu}\rangle = 0. \tag{6.15}$$

Combining this with (6.4), (6.8), (6.10), and (6.14), we finally obtain

$$\overline{K}_{\alpha\beta} = (1-\lambda^2)^{3/2}\left(\frac{\partial^2\varphi}{\partial r^2}\right)\overline{g}_{\alpha\beta} \qquad (r=0). \tag{6.16}$$

By applying Lemma 6.2, the fact that M is totally geodesic, and (6.7), we may conclude that

$$\overline{K}_S = (1-\lambda^2)\left(\frac{\partial^2\varphi}{\partial r^2}\right)\overline{g}|_{TS}. \tag{6.17}$$

We have thus expressed the second-order term of the development of Q in terms of the geometry of S in M.

We can use the foregoing computations to rewrite (6.9) as

$$\overline{K}_{\alpha\beta} + r\overline{\nabla}^Q_{E_0}\overline{K}_{\alpha\beta} = \left(\left(\frac{\partial^2\varphi}{\partial r^2}\right)\langle\partial_{x^n},\overline{\nu}\rangle + \left(\frac{\partial\varphi}{\partial r}\right)E_0\langle\partial_{x^n},\overline{\nu}\rangle\right)\overline{g}_{\alpha\beta}.$$

To this, we now apply $\overline{\nabla}^Q_{E_0}$ once again. We obtain

$$\begin{aligned} 2\overline{\nabla}^Q_{E_0}\overline{K}_{\alpha\beta} + r(\overline{\nabla}^Q_{E_0})^2\overline{K}_{\alpha\beta} = \Bigg[&\left(\frac{\partial^3\varphi}{\partial r^3}\right)\langle\partial_{x^n},\overline{\nu}\rangle + 2\left(\frac{\partial^2\varphi}{\partial r^2}\right)E_0\langle\partial_{x^n},\overline{\nu}\rangle \\ &+ \left(\frac{\partial\varphi}{\partial r}\right)E_0^2\langle\partial_{x^n},\overline{\nu}\rangle\Bigg]\overline{g}_{\alpha\beta}. \end{aligned} \tag{6.18}$$

We take $r=0$ and utilize equation (6.15) and find in particular that, at $r=0$,

$$2\overline{\nabla}^Q_{E_0}\overline{K}_{\alpha\beta} = -E_0^2(\langle\partial_r,\overline{\nu}\rangle)\bar{g}_{\alpha\beta}. \tag{6.19}$$

We next wish to apply the Codazzi-Mainardi equation (see, e.g., [Spi99], Theorem III.1.11). Codazzi states that

$$\langle\overline{R}(E_\gamma,E_\alpha)E_\beta,\overline{\nu}\rangle = \left(\overline{\nabla}^Q_{E_\gamma}\overline{K}\right)(E_\alpha,E_\beta) - \left(\overline{\nabla}^Q_{E_\alpha}\overline{K}\right)\left(E_\gamma,E_\beta\right).$$

Hence, taking $\gamma=0$,

$$\overline{\nabla}^Q_{E_0}\overline{K}_{\alpha\beta} = \overline{\nabla}^Q_{E_\alpha}K_{0\beta} + \langle\overline{R}(E_0,E_\alpha)E_\beta,\overline{\nu}\rangle. \tag{6.20}$$

We next take $\alpha = s$ (i.e., $1\le\alpha\le n-1$) and $\beta=0$. Hence, we have

$$\overline{\nabla}^Q_{E_0}\overline{K}_{s0} = \overline{\nabla}^Q_{E_s}\overline{K}_{00} + \langle R(E_0,E_s)E_0,\overline{\nu}\rangle. \tag{6.21}$$

But along S, $\overline{K}$ is known by (6.16), and we find that

$$\overline{\nabla}^{Q}_{E_s}\overline{K}_{00} = (1-\lambda^2)^{3/2}\left(\frac{\partial^3\varphi}{\partial r^2\partial x^s}\right)\overline{g}_{00}. \tag{6.22}$$

At $r = 0$, $\overline{g}_{00} = \frac{1}{1-\lambda^2}$ (recall that this is expressed in the $\{E_\alpha\}$ frame, not the coordinate frame). This allows us to conclude, via (6.21) and (6.22), that

$$\overline{\nabla}^{Q}_{E_0}\overline{K}_{s0} = \sqrt{1-\lambda^2}\left(\frac{\partial^3\varphi}{\partial r^2\partial x^s}\right) + \langle\overline{R}(E_0, E_s)E_0, \overline{\nu}\rangle.$$

We can now substitute $\overline{\nu}$ and the definitions of E_α in terms of the coordinate frame to compute this curvature term in the coordinate basis. (Every appearance of $\overline{R}$ in index notation will be with reference to local coordinates, not the frame on Q). We also utilize that, because g is Einstein, M is totally geodesic in X with respect to $\overline{g}$, and so again by Codazzi, $\langle\overline{R}(\partial_{x^n}, \partial_{x^s})\partial_r, \partial_{x^n}\rangle = 0$. Carrying this straightforward computation out at $r = 0$, we find that

$$\langle\overline{R}(E_0, E_s)E_0, \overline{\nu}\rangle = \frac{1}{\sqrt{1-\lambda^2}}\overline{R}_{0s0n}.$$

Applying these computations to (6.19) and noting that $\overline{g}_{s0} = 0$ at $r = 0$, we conclude that

$$\frac{\partial^3\varphi}{\partial r^2\partial x^s} = -\frac{1}{1-\lambda^2}\overline{R}_{0s0n}. \tag{6.23}$$

In conjunction with (6.17), this yields the claim. ■

Remark. Because Euclidean space is flat, this is precisely what we needed to ensure that, in the $n = 2$ hyperbolic case, only circles and lines can occur at the corner boundary: if g is the hyperbolic metric on $\mathbb{H}^3$, then one choice for the compactified metric $\bar{g}$ is the Euclidean metric itself. In this case, $\overline{R} = 0$, so by (6.5), S has a constant umbilic coefficient in $(M, \bar{g}|_{TM})$. But the only umbilic hypersurfaces in Euclidean 2-space with constant umbilic coefficients are circles and straight lines. Notice that in terms purely of the local boundary geometry, this restriction occurs at higher order for $n = 2$ than for $n \geq 3$, where it is implied by Corollary 6.3.

The arguments of the preceding proposition can be extended to yield further conditions on $\bar{g}$ and Q, and perhaps S. For example, let us return to (6.20), this time with $\alpha = s$ and $\beta = t$

$(1 \le s, t \le n-1)$. The term $\overline{\nabla}^Q_{E_s}\overline{K}_{0t}$ can be evaluated using our earlier calculations. We get

$$\overline{\nabla}^Q_{E_s}\overline{K}_{0t} = \overline{\nabla}_{E_s}\left[(1-\lambda^2)^{3/2}\left(\frac{\partial^2\varphi}{\partial r^2}\right)\right]\overline{g}_{0t} = 0.$$

Hence, at $r = 0$,

$$\overline{\nabla}^Q_{E_0}\overline{K}_{st} = \overline{R}(E_0, E_s, E_t, \overline{\nu}).$$

Expanding the right-hand side in the coordinate frame again yields

$$\overline{\nabla}^Q_{E_0}\overline{K}_{st} = \lambda(\overline{R}_{0st0} - \overline{R}_{nstn}). \tag{6.24}$$

By (6.19), $\operatorname{tf}_{\bar g}\overline{\nabla}^Q_{E_0}\overline{K}_{st} = 0$; thus we conclude that along S,

$$\lambda\, \operatorname{tf}_{\bar g|_{TS}}\left(\overline{R}_{0st0} - \overline{R}_{nstn}\right) = 0.$$

This represents an additional φ-independent condition on $\bar g$.

We can generalize this process to see the structure of the conditions that would arise as we differentiated more. Covariantly differentiating (6.6) $k-1$ times by E_0 yields

$$k(\overline{\nabla}^Q_{E_0})^{k-1}\overline{K}_{\alpha\beta} = E_0^k(\langle\partial_r, \overline{\nu}_Q\rangle)\bar g_{\alpha\beta}.$$

At each step, taking $\alpha = \beta = 0$ or $\alpha = s, \beta = t$ yields new constraints on $\frac{\partial^k\varphi}{\partial r^k}|_{r=0}$ and also on $\bar g$, its curvature tensor, and its derivatives. Taking $\alpha = 0, \beta = s$ yields additional constraints on lower-order (that is, already-developed) r-derivatives of φ, possibly also putting further constraints on $\bar g$, as in (6.24). In this way, a vastly overdetermined system of equations for $\bar g$ and φ would be produced. Actually carrying these computations out to higher order becomes quickly unwieldly; in any case, we have already developed extremely restrictive constraints on a smooth solution. We therefore turn now to studying formal existence of normal-form metrics on the blowup. As we will see, this is precisely the right relaxation of smoothness to allow relatively unique general solutions.

6.2 Formal Existence

We now consider formal existence in greater generality. We will take as our data a smooth manifold M^n with boundary S, equipped with an asymptotically hyperbolic conformal class $[h]$; and a constant $\lambda \in (-1, 1)$. Ideally, we would like to realize M as the infinite boundary of an appropriate cornered space (X, M, Q), and then to construct an Einstein CAH metric g satisfying $\mathrm{Ric}(g) + ng = 0$ to as high an order as possible at the corner, and satisfying $K_Q = \lambda g|_{TQ}$ along a finite boundary Q. We know by the previous section that this problem cannot be solved if we take g to be smooth. We thus look for solutions on a blowup space, relaxing the requirement that they be smooth on X.

Motivated by Proposition 6.4 and Theorem 1.3 and by our need to break the gauge in the Einstein equations, we take $\widetilde{X} = [0, \theta_0] \times M$, where $\theta_0 = \cos^{-1}(-\lambda)$. We look for metrics in the normal form (1.4). We then take $\widetilde{S} = [0, \theta_0]_\theta \times S$, $\widetilde{Q} = \{\theta_0\} \times M$, and $\widetilde{M} = \{0\} \times M$. We look for a metric g on $\widetilde{X} \setminus (\widetilde{M} \cup \widetilde{S})$, of the form

$$g = \csc^2(\theta) \left[d\theta^2 + h_\theta \right], \tag{6.25}$$

where h_θ is a smooth one-parameter family of AH metrics on M with $h_0 \in [h]$, and satisfying the Einstein equation to as high an order as possible at $\widetilde{S}$ and the equation $K_{\widetilde{Q}} = \lambda g|_{T\widetilde{Q}}$ along $\widetilde{Q}$. Our goal is to prove Theorem 1.5.

Note that there is no loss of generality in our choice of $\widetilde{X}$: although a general cornered space (X, M, Q) might have boundary components M and Q of differing topology, our construction is formal and at $\widetilde{S}$, where the topology is determined by $S = \partial M$ alone.

Throughout this section, it will be convenient to work explicitly with sections of the 0-edge bundle ${}^{0e}T^*\widetilde{X}$ and its tensor products, as well as the 0-bundle ${}^{0}T^*M$. We let $\{x^\mu\}$ be local coordinates near a point p of M, so that (θ, x^μ) gives a coordinate system on $\widetilde{X}$ near $[0, \theta_0] \times \{p\}$. We define a frame for ${}^{0e}T^*\widetilde{X}$ given by

$$\begin{aligned} \omega^0 &= \frac{d\theta}{\sin(\theta)} \\ \omega^\mu &= \frac{dx^\mu}{\rho \sin(\theta)} \end{aligned}$$

(where $\rho = x^n$).

It will be useful to compute the umbilic condition in normal form.

Lemma 6.6. *For a metric g on $\widetilde{X}$ in the form (6.25), let $K_{\widetilde{Q}}$ be the second fundamental form of $\widetilde{Q} \setminus \widetilde{S}$, and $\lambda = -\cos(\theta_0)$. Then the condition $K_{\widetilde{Q}} = \lambda g|_{T\widetilde{Q}}$ is equivalent to the condition*

$$\partial_\theta h_\theta|_{\theta=\theta_0} = 0.$$

Proof. Plainly, the inward-pointing normal to $\widetilde{Q}$ is given by $\nu = -\sin(\theta_0)\frac{\partial}{\partial\theta}$. By Weingarten's equation,

$$\begin{aligned} K_{\mu\nu} &= -g_{k\nu}\nabla_\mu \nu^k \\ &= \sin(\theta_0)\Gamma_{\mu 0\nu}, \end{aligned}$$

where $\Gamma_{\mu 0\nu} = \frac{1}{2}(\partial_\mu g_{0\nu} + \partial_\theta g_{\mu\nu} - \partial_\nu g_{0\nu}) = \frac{1}{2}\partial_\theta g_{\mu\nu}$. Since $g_{\mu\nu} = \csc^2(\theta)h_{\mu\nu}$ and $\partial_\theta(\csc^2(\theta)) = -2\csc^2(\theta)\cot(\theta)$, the result follows. ∎

As discussed in the introduction, to prove Theorem 1.5 we will choose a conformal representative $h \in [h]$; we will take it to be in AH normal form. Then h_0 in (6.25), as discussed, will be of the form χh, where χ is a function to be determined. This motivates the following proposition, which we will use to prove the theorem. Recall that $\mathcal{M}(\theta_0, M)$ is defined in Section 3.2.

Proposition 6.7. *Let $n \geq 2$, and suppose S is a smooth manifold of dimension $n-1$. Given a one-parameter family of metrics k_ρ on S, there exists a one-parameter family of smooth AH metrics $\{h_\theta : 0 \leq \theta \leq \theta_0\}$ on $S \times [0, \varepsilon)_\rho$ and a function $\chi \in C^\infty(S \times [0, \varepsilon))$, with $\bar{h}_\theta = \rho^2 h_\theta$ and χ unique mod $O(\rho^n)$, such that $h_\theta \in \mathcal{M}(\theta_0, S \times [0, \varepsilon))$, and*

(a) $\chi|_{\rho=0} = 1$;

(b) $\partial_\theta \bar{h}_\theta|_{\theta=\theta_0} = 0$ *for all* ρ;

(c) $\partial_\theta \bar{h}_\theta|_{\rho=0} = 0$ *for all* θ;

(d) $\bar{h}_0 = \chi(d\rho^2 + k_\rho)$; *and*

(e) if g is the metric on $[0,\theta_0]\times S\times[0,\varepsilon)$ given by

$$g=\csc^2(\theta)[d\theta^2+h_\theta], \tag{6.26}$$

then $\mathrm{Ric}(g)+ng=O_g(\rho^n)$.

Moreover, $\bar{h}_\theta$ is even in θ to order n.

We next prove that, assuming Proposition 6.7 is true, Theorem 1.5 follows.

Proof of Theorem 1.5 using Proposition 6.7. Let $h\in[h]$, and let $\psi : S\times[0,\varepsilon)\to W\subseteq M$ be a diffeomorphism with a neighborhood W of S in M such that $\psi^*h=\frac{d\rho^2+k_\rho}{\rho^2}$, with k_ρ a one-parameter family of metrics on S. Then ψ induces a diffeomorphism $id\times\psi : [0,\theta_0]\times S\times[0,\varepsilon)\to\widetilde{X}=[0,\theta_0]\times M$.

By Proposition 6.7, there exists a one-parameter family $\tilde{h}_\theta\in\mathcal{M}(\theta_0,S\times[0,\varepsilon))$ of AH metrics on $S\times[0,\varepsilon)$, and a smooth function $\chi\in C^\infty(S\times[0,\varepsilon))$ satisfying conditions (a) - (d), and such that $\tilde{g}=\csc^2(\theta)(d\theta^2+\tilde{h}_\theta)$ is Einstein mod $O_g(\rho^n)$; moreover, both $\bar{\tilde{h}}_\theta$ and χ are uniquely defined mod $O(\rho^n)$.

Let $h_\theta=(\psi^{-1})^*\tilde{h}_\theta\in\mathcal{M}(\theta_0,W)$, and $g=((id\times\psi)^{-1})^*\tilde{g}$; then it is clear that $g=\csc^2(\theta)(d\theta^2+h_\theta)$, that $h_0=((\psi^{-1})^*\chi)h\in[h]$, and that $\mathrm{Ric}(g)+ng=O_g(\rho^n)$. Moreover, by condition (b) in Proposition 6.7, we conclude that $\partial_\theta h_\theta|_{\theta=\theta_0}=0$. By Lemma 6.6, and because $\cos(\theta_0)=-\lambda$, it follows that along $\widetilde{Q}\setminus\widetilde{S}$, we have $K_{\widetilde{Q}}=\lambda g|_{\widetilde{Q}}$. Thus, existence is established.

For uniqueness, suppose now that we have another one-parameter family h'_θ of AH metrics on $\widetilde{M}$ such that $g'=\csc^2(\theta)[d\theta^2+h'_\theta]$ is also Einstein mod $O_g(\rho^n)$, $h'_0\in[h]$, and $K'_{\widetilde{Q}}=\lambda g'|_{\widetilde{Q}}$. Suppose also that $\partial_\theta(\rho^2h_\theta)|_{\widetilde{S}}=0$. Now let $\tilde{h}'_\theta=\psi^*h'_\theta$, and $\tilde{g}'=(\mathrm{id}\times\psi)^*g'$. Notice that if we write $h'_0=\Omega^2h_0$, then $\Omega|_S=1$. Then it is easy to see that $\tilde{g}'$ and $\tilde{h}'_\theta$ satisfy conditions (a) - (e) of Proposition 6.7. Thus, $\tilde{g}-\tilde{g}'=O_{\tilde{g}}(\rho^n)$. Pushing forward again, we may conclude that $g-g'=O_g(\rho^n)$. ■

We now begin working toward a proof of Proposition 6.7. Since we will have no further cause to refer to the setting of Theorem 1.5, for the remainder of this section we will for convenience let

$\widetilde{X} = [0, \theta_0] \times S \times [0, \varepsilon)$, let $\widetilde{M} = \{0\} \times S \times [0, \varepsilon)$, let $M = S \times [0, \varepsilon)$, let $\widetilde{Q} = \{\theta_0\} \times S \times [0, \varepsilon)$, and let $\widetilde{S} = [0, \theta_0] \times S \times \{0\}$.

We begin by computing the following.

Lemma 6.8. *Let S be a manifold of dimension $n-1$, let $\theta_0 \in (0, \pi)$, and let g be a metric in the normal form (6.26) on $\widetilde{X} = [0, \theta_0] \times S \times [0, \varepsilon)_\rho$. Set $E = \operatorname{Ric}(g) + ng$. Then $E = \hat{E}_{ij} dx^i dx^j = E_{ij} \omega^i \omega^j$, where*

$$E_{00} = -\frac{1}{2} \sin^2(\theta) \bar{h}^{\mu\nu} \partial_\theta^2 \bar{h}_{\mu\nu} + \frac{1}{2} \sin(\theta) \cos(\theta) \bar{h}^{\mu\nu} \partial_\theta \bar{h}_{\mu\nu} + \frac{1}{4} \sin^2(\theta) \left| \partial_\theta \bar{h} \right|_{\bar{h}}^2 \tag{6.27}$$

$$E_{0\sigma} = \frac{1}{2} \sin^2(\theta) \left[\bar{h}^{\mu\nu} \partial_\theta (\bar{h}_{\mu\nu}) \rho_\sigma - n \rho^\mu \partial_\theta \bar{h}_{\sigma\mu} + \rho \left(\nabla^\mu (\partial_\theta \bar{h}_{\sigma\mu}) - \nabla_\sigma \left(\left(\partial_\theta \bar{h} \right)_\mu^\mu \right) \right) \right] \tag{6.28}$$

$$\begin{aligned} E_{\mu\nu} = &-\frac{1}{2} \sin^2(\theta) \partial_\theta^2 \bar{h}_{\mu\nu} + \frac{n-1}{2} \sin(\theta) \cos(\theta) \partial_\theta \bar{h}_{\mu\nu} + \frac{1}{2} \sin^2(\theta) \bar{h}^{\eta\lambda} \partial_\theta (\bar{h}_{\mu\eta}) \partial_\theta (\bar{h}_{\nu\lambda}) \\ &- \frac{1}{4} \sin^2(\theta) \bar{h}^{\eta\lambda} \partial_\theta (\bar{h}_{\eta\lambda}) \partial_\theta (\bar{h}_{\mu\nu}) + \frac{1}{2} \sin(\theta) \cos(\theta) \bar{h}^{\eta\lambda} \partial_\theta (\bar{h}_{\eta\lambda}) \bar{h}_{\mu\nu} \\ &+ (1-n) \sin^2(\theta) \left(|d\rho|_{\bar{h}}^2 - 1 \right) \bar{h}_{\mu\nu} + (n-2) \rho \sin^2(\theta) \nabla_\mu \rho_\nu + \rho \sin^2(\theta) \nabla^\eta \rho_\eta \bar{h}_{\mu\nu} \\ &+ \rho^2 \sin^2(\theta) \operatorname{Ric}(\bar{h})_{\mu\nu}. \end{aligned} \tag{6.29}$$

Here indices are raised and covariant derivatives taken with respect to $\bar{h} = \rho^2 h$, and $\bar{h} = \bar{h}_{\mu\nu} dx^\mu dx^\nu$.

Proof. Using the form (6.26) of the metric, we compute the Christoffel symbols as follows in coordinates:

$$\begin{aligned} &\Gamma_{000} = -\csc^2(\theta) \cot(\theta) && \Gamma_{\mu\nu 0} = \rho^{-2} \csc^2(\theta) \cot(\theta) \bar{h}_{\mu\nu} - \tfrac{1}{2} \rho^{-2} \csc^2(\theta) \partial_\theta \bar{h}_{\mu\nu} \\ &\Gamma_{0\mu 0} = 0 && \Gamma_{0\mu\sigma} = -\rho^{-2} \csc^2(\theta) \cot(\theta) \bar{h}_{\mu\sigma} + \tfrac{1}{2} \rho^{-2} \csc^2(\theta) \partial_\theta \bar{h}_{\mu\sigma} \\ &\Gamma_{00\sigma} = 0 && \Gamma_{\mu\nu\sigma} = -2\rho^{-3} \csc^2(\theta) \bar{h}_{\sigma(\mu} \rho_{\nu)} + \rho^{-3} \csc^2(\theta) \bar{h}_{\mu\nu} \rho_\sigma \\ & && \qquad + \rho^{-2} \csc^2(\theta) \overline{\Gamma}_{\mu\nu\sigma}, \end{aligned} \tag{6.30}$$

where $\overline{\Gamma}$ is the Christoffel symbol of $\bar{h}$. The result now follows from a tedious but straightforward computation using the equation

$$R_{ij} = \frac{1}{2} g^{kl} \left(\partial_{il}^2 g_{jk} + \partial_{jk}^2 g_{il} - \partial_{kl}^2 g_{ij} - \partial_{ij}^2 g_{kl} \right) + g^{kl} g^{pq} \left(\Gamma_{ilp} \Gamma_{jkq} - \Gamma_{ijp} \Gamma_{klq} \right). \tag{6.31}$$

■

We state the following, which will be of use later.

Lemma 6.9. *If* $h_\theta \in \mathcal{H}_{n,l}(\theta_0, M, S^2({}^0T^*M))$ *for some* $l \geq 0$ *and* $\partial_\theta \bar{h}_\theta|_{\rho=0} = 0$*, then for each* $j \geq 0$*,* $\partial_\rho^j E(g)|_{\rho=0} \in \oplus_{i=0}^{m_j}(\theta^n \log(\theta))^i C^\infty(\widetilde{S}, S^2({}^{0e}T^*\widetilde{X}))$ *for some finite* $m_j \geq 0$*. Moreover, if* $\bar{h}_\theta$ *is even in* θ *through order* $k \geq 2$*, then* $E(g)$ *is even through the same order as a section of* $S^2({}^{0e}T^*\widetilde{X})$*.*

Proof. We sketch the proof for E_{00} in (6.27). The evenness claim is clear, since the number of factors of $\sin(\theta)$ is the same as the number of derivatives with respect to θ in each term. For the first claim, inspect each term and notice that at each finite power of ρ, there is a bounded power of $\theta^n \log(\theta)$ by the hypothesis on h_θ. The powers of $\theta^n \log(\theta)$ in $\bar{h}^{-1}$ are bounded at each finite order in ρ due to the hypothesis that $\partial_\theta \bar{h}_\theta|_{\rho=0} = 0$. The proof for the other components of E is similar. ■

Our approach to proving Proposition 6.7 will be to construct the metric term by term in powers of ρ by solving the indicial equation, just as for the scalar Laplacian. There are two complications compared to that case: because the operator is nonlinear and acts on sections of a 0-edge bundle, the definition of the indicial operator is more involved and depends on the metric; and because the indicial operator acts differently on different parts of the isotypic decomposition of the metric tensor, there are in effect really several indicial operators. This also occurs in the usual AH case – see e.g. [GL91]. In that case, however, the various parts of the indicial operator are all algebraic, not differential operators.

At each order, we will have to solve a regular singular system of ODEs given by the indicial operators. Because we have gauge-broken the Einstein equations by requiring the metric to be in normal form, the system is overdetermined – we have $\frac{n(n+1)}{2}$ unknowns, but $\frac{(n+1)(n+2)}{2}$ equations. We will therefore follow the usual expedient of using the Bianchi identities to show that the extra equations are automatically satisfied once we have determined the solution using $\frac{n(n+1)}{2}$ equations. It is by the Bianchi equations, as we will see, that χ will be uniquely determined at each order.

Because of the form of metric (6.26), we will be interested in perturbations

$$g \mapsto g + \rho^\gamma \varphi,$$

where $\gamma > 0$ and $\varphi = \varphi_{\mu\nu}\omega^\mu\omega^\nu = \rho^{-2}\csc^2(\theta)\varphi_{\mu\nu}dx^\mu dx^\nu$ is a section of the bundle $\mathcal{T} = \left\{\eta \in S^2({}^{0e}T^*\widetilde{X}) : \sin(\theta)\frac{\partial}{\partial\theta} \lrcorner\, \eta = 0\right\}$. A section σ of T can be identified with a one-parameter family σ_θ $(0 \le \theta \le \theta_0)$ of sections of $S^2({}^0T^*\widetilde{M})$ over $\widetilde{M}$. We will also refer to $\bar{\varphi} = \rho^2\sin^2(\theta)\varphi = \varphi_{\mu\nu}dx^\mu dx^\nu$, which is a section of $S^2T^*\widetilde{X}$ with the property that $\frac{\partial}{\partial\theta} \lrcorner\, \bar{\varphi} = 0$. Fix a metric $g \in C^\infty(\widetilde{X}, S^2({}^{0e}T^*\widetilde{X}))$. We now define the indicial operator $I^\gamma : C^\infty(\widetilde{S}, \mathcal{T}) \to C^\infty(\widetilde{S}, S^2({}^{0e}T^*\widetilde{X}))$, depending on g, as follows. Let $\varphi \in C^\infty(\widetilde{S}, \mathcal{T})$ be a section, and let $\tilde{\varphi} \in C^\infty(\widetilde{X}, \mathcal{T})$ be any smooth extension of φ to $\widetilde{X}$. Then define $I^\gamma(\varphi)$ by

$$I^\gamma(\varphi) = \rho^{-\gamma}\left(E(g + \rho^\gamma\tilde{\varphi}) - E(g)\right)|_{\rho=0},$$

where the restriction to $\rho = 0$ is taken as a section of $S^2({}^{0e}T^*\widetilde{X})$. The definition is independent of the extension $\tilde{\varphi}$ chosen. As in the scalar case, I^γ is an ordinary differential operator acting in θ.

Proposition 6.10. *The indicial operator I^γ for a metric g in the normal form (6.26) has the form $I^\gamma(\varphi) = I^\gamma_{ij}(\varphi)\omega^i\omega^j$, where*

$$2I^\gamma_{00}(\varphi) = -\sin^2(\theta)\bar{h}^{\mu\nu}\partial^2_\theta\varphi_{\mu\nu} + \sin(\theta)\cos(\theta)\bar{h}^{\mu\nu}\partial_\theta\varphi_{\mu\nu} \tag{6.32}$$

$$2I^\gamma_{0\sigma}(\varphi) = \sin^2(\theta)\left[(\gamma - n)\,\rho^\mu\partial_\theta\varphi_{\mu\sigma} - (\gamma - 1)\,\rho_\sigma\bar{h}^{\mu\nu}\partial_\theta\varphi_{\mu\nu}\right] \tag{6.33}$$

$$\begin{aligned} 2I^\gamma_{nn}(\varphi) = &-\sin^2(\theta)\partial^2_\theta\varphi_{nn} + (n-1)\sin(\theta)\cos(\theta)\partial_\theta\varphi_{nn} \\ &+ \sin(\theta)\cos(\theta)\partial_\theta(\bar{h}^{\mu\nu}\varphi_{\mu\nu}) + (\gamma-2)(\gamma+1-n)\sin^2(\theta)\varphi_{nn} \\ &+ \gamma(2-\gamma)\sin^2(\theta)\bar{h}^{\mu\nu}\varphi_{\mu\nu} \end{aligned} \tag{6.34}$$

$$\begin{aligned} 2\bar{h}^{\mu\nu}I^\gamma_{\mu\nu}(\varphi) = &-\sin^2(\theta)\partial^2_\theta(\bar{h}^{\mu\nu}\varphi_{\mu\nu}) + (2n-1)\sin(\theta)\cos(\theta)\partial_\theta(\bar{h}^{\mu\nu}\varphi_{\mu\nu}) \\ &+ 2(\gamma-n)(\gamma+1-n)\sin^2(\theta)\varphi_{nn} - 2\gamma(\gamma-n)\sin^2(\theta)\bar{h}^{\mu\nu}\varphi_{\mu\nu} \end{aligned} \tag{6.35}$$

$$2I^\gamma_{sn}(\varphi) = -\sin^2(\theta)\partial^2_\theta\varphi_{sn} + (n-1)\sin(\theta)\cos(\theta)\partial_\theta\varphi_{sn} \tag{6.36}$$

$$2\mathring{I}^\gamma_{st}(\varphi) = -\sin^2(\theta)\partial^2_\theta\mathring{\varphi}_{st} + (n-1)\sin(\theta)\cos(\theta)\partial_\theta\mathring{\varphi}_{st} - \gamma(\gamma+1-n)\sin^2(\theta)\mathring{\varphi}_{st}, \tag{6.37}$$

where $\mathring{\varphi}_{st} = \varphi_{st} - \frac{1}{n-1}\bar{h}^{rq}\varphi_{rq}\bar{h}_{st}$, and similarly for $\mathring{I}_{st}$.

Note that on each fiber F of $\widetilde{S}$, I restricts to an operator $I : C^\infty(F, \mathcal{T}) \to C^\infty(F, S^2({}^{0e}T^*\widetilde{X}))$.

Proof. We set $\hat{g} = g + \rho^\gamma \varphi_{\mu\nu}\omega^\mu\omega^\nu$. Writing $\hat{g}$ in the form (6.26), we see that the change $g \mapsto \hat{g}$ is equivalent to the change $\bar{h}_{\mu\nu}dx^\mu dx^\nu \mapsto (\bar{h}_{\mu\nu} + \rho^\gamma\varphi_{\mu\nu})dx^\mu dx^\nu$. We use this expression in equations (6.27) - (6.29) to compute I^γ_{00}, $I^\gamma_{0\sigma}$, and $I^\gamma_{\mu\nu}$, using the formula $I^\gamma(\varphi) = \rho^{-\gamma}\left[E(g+\rho^\gamma\tilde{\varphi}) - E(g)\right]|_{\rho=0}$. We then specialize $I^\gamma_{\mu\nu}$ with various choices of μ and ν to obtain the result. We will carry out only the computation for I_{00}; the rest are similar.

Because $\partial_\theta \bar{h}_\theta|_{\rho=0} = 0$, then in particular $\partial_\theta\bar{h}_\theta = O(\rho)$ and $\partial_\theta \bar{h}^{-1} = O(\rho)$. It follows that the effect of the perturbation on the inverse metric $\bar{h}^{\mu\nu}$ may be ignored, as may the last term in (6.27). Formula (6.32) then follows immediately. ■

Before proving Proposition 6.7, we define some notation that will be useful. Notice that, by the product structure of $\widetilde{X}$, there is a natural decomposition ${}^{0e}T\widetilde{X} \approx {}^{0e}T[0,\theta_0] \oplus {}^{0e}TS \oplus {}^{0e}T[0,\varepsilon)$, where the summands on the right have their obvious meanings. Similarly, there is a natural decomposition ${}^0T\widetilde{M} \approx {}^0TS \oplus {}^0T[0,\varepsilon)$. For a section $T \in C^\infty(\widetilde{X}, S^2({}^{0e}T^*\widetilde{X}))$, we let $T|_{TS}$ be the restriction of T to the middle factor in the above three-way decomposition. Now if k is a metric on S, then $\rho^2 k$ is a metric on ${}^0TS$. For $T \in C^\infty(\widetilde{X}, S^2({}^{0e}T^*\widetilde{X}))$ and k a metric on S, we will define the notation

$$\operatorname{tf}_k T := \operatorname{tf}_{\rho^2 k}(T|_{TS}),$$

so that $\operatorname{tf}_k T$ is a section in $\csc^2(\theta)C^\infty(\widetilde{X}, S^2({}^0TS))$. In components, this takes the usual form

$$(\operatorname{tf}_k T)_{st} = T_{st} - \frac{1}{n-1}k^{pq}T_{pq}k_{st}.$$

Similarly, if T is a section of $S^2({}^0T^*M)$, then $\operatorname{tf}_k T$ will refer to $\operatorname{tf}_k(T|_{TS})$. We will also use the notation $\operatorname{tf}_k T$ in its usual sense when T is an ordinary symmetric two-tensor.

Proof of Proposition 6.7. We will construct a solution order-by-order in ρ. At each step, we will solve the regular singular system of ODEs given by the operators (6.32) - (6.37). As mentioned earlier, we will actually use only some of the equations to solve for φ, and will show that the others are satisfied by our solution using the Bianchi identity.

We are determining h_θ in (6.26); although we will work instead with $\bar{h}_\theta = \rho^2 h_\theta$. Our boundary condition at $\theta = \theta_0$ is that $\partial_\theta \bar{h}_\theta|_{\theta=\theta_0} = 0$. At $\widetilde{M}$, our boundary condition is that $\bar{h}_0 = \chi(d\rho^2 + k_\rho)$, where χ is as-yet an undetermined function.

We assume for now that $n > 2$.

We define $\bar{h}_\theta^{(0)} = d\rho^2 + k_\rho$, and $g^{(0)} = \csc^2(\theta)(d\theta^2 + \rho^{-2}\bar{h}^{(0)})$. It is straightforward to show using Lemma 6.8 that $E^{(0)} := \mathrm{Ric}(g^{(0)}) + ng^{(0)} = O_g(\rho)$ – only terms on the last two lines of (6.29) are nonvanishing. We similarly define $\chi^{(0)} \in C^\infty(M)$ by $\chi^{(0)} \equiv 1$.

We will now proceed by induction. Let $1 \leq \gamma \leq n-1$, and suppose for purpose of induction that we have a metric $g^{(\gamma-1)} = \csc^2(\theta)\left[d\theta^2 + \rho^{-2}\bar{h}_\theta^{(\gamma-1)}\right]$ and a smooth function $\chi^{(\gamma-1)} \in C^\infty(\widetilde{M})$ such that

(i) $\partial_\theta \bar{h}_\theta^{(\gamma-1)}|_{\rho=0} = 0$;

(ii) $\bar{h}_\theta^{(\gamma-1)} \in \mathscr{H}_{n,l}(\theta_0, M, S^2T^*M)$ for some l;

(iii) $\chi \in C^\infty(M)$;

(iv) $\partial_\theta \bar{h}_\theta^{(\gamma-1)}|_{\theta=\theta_0} = 0$;

(v) $\chi^{(\gamma-1)}|_{\rho=0} = 1$;

(vi) $\bar{h}_\theta^{(\gamma-1)}|_{\theta=0} = \chi^{(\gamma-1)}(d\rho^2 + k_\rho)$;

(vii) $E^{(\gamma-1)} := E(g^{(\gamma-1)}) = O_g(\rho^\gamma)$;

(viii) $\bar{h}_\theta^{(\gamma-1)}$ is even in θ through order n;

(ix) $\rho^{-\gamma}\,\mathrm{tf}_{k_0}\, E^{(\gamma-1)}|_{\rho=0} \in \mathscr{B}_{n,m}(\theta_0, S, S^2({}^0T^*M))$ for some m; and

(x) $\chi^{(\gamma-1)}$ and $\bar{h}_\theta^{(\gamma-1)}$ satisfying conditions (i) - (vii) are uniquely defined modulo $O(\rho^\gamma)$.

We wish to show that we can construct a function $\chi^{(\gamma)}$ and a family of metrics $\bar{h}^{(\gamma)}$ such that these conditions are all satisfied with γ everywhere replaced by $\gamma + 1$. (Conditions (viii) - (ix) will be used at several points in the induction step.) Put differently, we wish to show that we may uniquely define perturbations $\varphi^{(\gamma)} \in \csc^2(\theta)\mathcal{H}_{n,l'}(\theta_0, S, S^2({}^0T^*M))$ (for some l') and $\psi^{(\gamma)} \in C^\infty(S)$ such that, taking $\bar{h}^{(\gamma)} = \bar{h}^{(\gamma-1)} + \rho^\gamma \bar{\varphi}^{(\gamma)}$ and $\chi^{(\gamma)} = \chi^{(\gamma-1)} + \rho^\gamma \psi^{(\gamma)}$, the desired conditions are satisfied. Actually, condition (vi) will be satisfied only through order γ, due to the impact on higher-order terms on the right-hand side of changing χ at order γ; we will restore (vi), however, without affecting any other conditions by adding one more perturbation that is independent of θ. We will henceforth refer just to φ and ψ, leaving the (γ) implicit.

Define $f = \rho^{-\gamma} E^{(\gamma-1)}|_{\rho=0} \in C^{n-1}(\widetilde{S}, S^2({}^{0e}T^*\widetilde{X})|_{\widetilde{S}})$. It is easy to see that we will have completed the induction if we can find φ and ψ such that

(1) $I^\gamma(\varphi) = -f$, where I^γ is the indicial operator defined above;

(2) $\varphi_{nn}|_{\theta=0} = \psi$;

(3) $\bar{h}^{\mu\nu}\varphi_{\mu\nu}|_{\theta=0} = n\psi$;

(4) $\varphi_{ns}|_{\theta=0} = 0$;

(5) $\mathring{\varphi}_{st}|_{\theta=0} = 0$;

(6) $\partial_\theta \bar{\varphi}|_{\theta=\theta_0} = 0$;

(7) $\bar{\varphi}$ is even in θ through order n;

(8) $\bar{\varphi} \in \mathcal{H}_{n,l'}(\theta_0, S, S^2T^*M)$ for some l';

(9) $\psi \in C^\infty(\widetilde{M})$; and

(10) $\rho^{-(\gamma+1)} \operatorname{tf}_{k_0} E\left(g^{(\gamma)}\right)\big|_{\rho=0} \in \mathcal{B}_{n,m'}(\theta_0, S, S^2({}^0T^*M))$ for some m',

with φ and χ determined uniquely by conditions (1) - (6). Notice by the induction hypothesis and Lemma 6.9 that f is even in θ at least through order n.

Fix $x \in \widetilde{M} \cap \widetilde{S} \approx S$, which we regard as determining a fiber. We will determine φ and ψ on the fiber $[0, \theta_0] \times \{x\} \times \{0\}$. Since our constructions will all depend smoothly on x, we suppress it when convenient and write φ as a function of θ alone. We first regard $\psi(x)$ as a free parameter and show that, for any choice of $\psi(x)$, $\varphi(x, \theta)$ is uniquely determined. Thus, for now regard $\psi(x)$ as given, and $\chi^{(\gamma)} = \chi^{(\gamma-1)} + \rho^\gamma \psi$.

We first determine $\mathring{\varphi}_{st} = \mathrm{tf}_{k_0} \bar{\varphi}$. Our boundary condition at $\theta = \theta_0$ is, as noted above, $\partial_\theta \bar{\varphi}|_{\theta=\theta_0} = 0$. Our boundary condition at $\theta = 0$ is $\mathring{\varphi}_{st} = 0$. Now we wish to solve $\mathring{I}_{st}(\varphi) = -\mathring{f}_{st}$. By (6.37), $\mathring{I}$ acts as a scalar on $\mathring{\varphi}$. Moreover, $\mathring{I}$ is merely $\frac{1}{2}$ times the indicial operator of the scalar Laplacian, by Lemma 5.2. Now by (ix), $\mathring{f}_{st} \in \mathscr{B}_{n,m}(\theta_0, S, S^2({}^0T^*M))$, and so by Propositions 5.5 and 5.8, the equation $\mathring{I}_{st}(\varphi) = -\mathring{f}_{st}$ has a unique solution $\mathring{\varphi}_{st}$ in $\mathscr{A}_{n,m}(\theta_0, S, S^2({}^0T^*M))$. By the form (5.9) of the Green's operator, we also may conclude that $\mathring{\varphi}_{st}$ is even to order n in θ.

Now suppose n is odd. Because $\bar{h}$ is even to order n in θ, in particular $\partial_\theta \bar{h} = O(\theta)$. It then easily follows from (6.29) that $\mathring{f}_{st}$ is in fact even to order $n + 2$. Then, by Proposition 5.8, it follows that $\mathring{\bar{\varphi}}$ is smooth and contains no logarithmic terms. Whether n is even or odd, then, $\mathring{\varphi}_{st} \in \mathscr{H}_{n,m}(\theta_0, S, S^2T^*M)$.

We next determine the trace $\bar{h}^{\mu\nu}\varphi_{\mu\nu}$, which for convenience we denote $\ell(\theta)$ for the remainder of this proof. Because of the overdetermined nature of our system, it would appear *a priori* possible to use either I^γ_{00} or $\bar{h}^{\mu\nu} I^\gamma_{\mu\nu}$ to do this. However, I^γ_{00} is simpler because it involves only the trace of φ, whereas $\bar{h}^{\mu\nu} I^\gamma_{\mu\nu}$ involves both the trace and φ_{nn}. (Because $\bar{h}$ at $\rho = 0$ is independent of θ, we may regard I^γ_{00} as giving a differential equation for ℓ.) We thus proceed with I^γ_{00}. As usual, we wish to solve the equation $I^\gamma_{00}(\varphi) = -f_{00}$, subject to the conditions $\ell(0) = n\psi(x)$ and $\ell'(\theta_0) = 0$. We claim that $f_{00} = O(\theta^4)$: by the induction hypothesis, $\partial_\theta \bar{h}^{(\gamma-1)} = O(\theta)$, so the last term in (6.27) is $O(\theta^4)$. The other two terms may be written $-\sin^2(\theta)(\bar{h}^{\mu\nu}\partial_\theta^2 \bar{h}_{\mu\nu} - \cot(\theta)\bar{h}^{\mu\nu}\partial_\theta \bar{h}_{\mu\nu})$. But since $\bar{h}^{(\gamma-1)}$ is even in θ to order n, the term in parenthesis is $O(\theta^2)$, as claimed. Moreover, since $\partial_\theta \bar{h}_\theta$ is even through order n in θ, it follows from (6.27) that f_{00} is as well. It is easy to verify that a solution to the equation $I^{\gamma}_{00}\ell = -f_{00}$ satisfying $\ell(0) = n\psi(x)$ and $\ell'(\theta_0) = 0$ is given by

$$\begin{aligned}\ell(\theta) =& 2\left(\cos(\theta)\int_{\theta}^{\theta_0}\csc^3(\phi) f_{00}(\phi)d\phi + \int_0^{\theta}\cot(\phi)\csc^2(\phi) f_{00}(\phi)d\phi\right.\\ &\left.- \int_0^{\theta_0}\csc^3(\phi) f_{00}(\phi)d\phi\right) + n\psi(x).\end{aligned}$$

The solution is easily shown to be unique: the homogeneous equation is linear, with general solution $a\cos(\theta) + b$. Given the requirements that $\ell(0) = 0 = \ell'(\theta_0)$, we may deduce that $a = 0 = b$. Furthermore, since f_{00} is even in θ through order n, the solution $\ell(\theta)$ is as well. This may be easily verified by differentiating the above solution formula, obtaining $\ell'(\theta) = -\sin(\theta)\int_{\theta}^{\theta_0}\csc^3(\phi) f_{00}(\phi)d\phi$. Similarly, Lemma 6.9 implies that $f_{00} \in \mathcal{H}_{n,l'}(\theta_0, S)$ for some l', and since $\int \theta^p \log(\theta)^q d\theta = O(\theta^{p+1}\log(\theta)^q)$, we conclude that $\ell(\theta) \in \mathcal{H}_{n,l'}(\theta_0, S)$ as well.

Next we wish to determine φ_{nn}, for which we use I^{γ}_{0n}. We get the equation

$$(\gamma - n)\partial_\theta\varphi_{nn} = (\gamma - 1)\ell'(\theta) - f_{0n}(\theta),$$

with conditions $\varphi_{nn}(0) = \psi(x)$ and $\varphi'_{nn}(\theta_0) = 0$. Now $\ell'(\theta_0) = 0$ by construction. Moreover, by (6.28) and our inductive hypothesis – according to which $\partial_\theta\bar{h}^{(\gamma-1)}|_{\theta=\theta_0} = 0$ – we see that $f_{0n}(\theta_0) = 0$ as well. Thus, the right hand side vanishes at $\theta = \theta_0$, and our boundary condition at θ_0 is satisfied automatically. We can therefore integrate to uniquely determine φ_{nn} subject to the condition that $\varphi_{nn}(0) = \chi(x)$. By construction of $\ell(\theta)$ and by (6.28), the right-hand side of the above equation is odd through order $n-1$; therefore, $\varphi_{nn}(\theta)$ is even through order n. A similar argument as for the trace also shows that $\varphi_{nn} \in \mathcal{H}_{n,l'}(\theta_0, S)$.

We have only to determine φ_{ns}, which is to say, $\left(\frac{\partial}{\partial\rho} \lrcorner \bar{\varphi}\right)|_{TS}$. To do this, we use I_{0s}. We get the equation

$$(\gamma - n)\partial_\theta\varphi_{ns} = -f_{0s}(\theta).$$

As in the previous case, the boundary condition at θ_0 is automatically satisfied, and we can integrate to get a unique solution satisfying our conditions; parity is preserved as desired, and $\varphi_{ns} \in \mathcal{H}_{n,m}(\theta_0, S, T^*M)$.

We have determined φ, and thus have constructed a $g^{(\gamma)}$ so that E^{γ}_{00}, $E^{\gamma}_{0\sigma}$, and $\mathring{\hat{E}}^{\gamma}_{st} = O_g(\rho^{\gamma+1})$. However, it remains to analyze $\bar{h}^{\mu\nu}E^{\gamma}_{\mu\nu}$, E^{γ}_{nn}, and E^{γ}_{ns}, since their corresponding indicial operators were not used in our construction. (We will henceforth omit the (γ) from E for clarity.) For this, we will use the contracted Bianchi identities, which state that $2\nabla^i E_{ij} = \nabla_j E^i_i$; or working now in the coordinate frame, that

$$0 = B_i := 2g^{jk}\partial_k \hat{E}_{ij} - g^{jk}\partial_i \hat{E}_{jk} - 2g^{jk}g^{ql}\Gamma_{jkq}\hat{E}_{il}.$$

We apply this to $g^{(\gamma)}$ using our earlier computations (6.30) of Christoffel symbols. Still working in the coordinate frame, we find

$$\begin{aligned} B_0 =& \sin^2(\theta)\partial_\theta \hat{E}_{00} + 2\rho^2\sin^2(\theta)\bar{h}^{\mu\nu}\partial_\nu \hat{E}_{0\mu} - \rho^2\sin^2(\theta)\bar{h}^{\mu\nu}\partial_\theta \hat{E}_{\mu\nu} \\ &+ 2(1-n)\sin(\theta)\cos(\theta)\hat{E}_{00} + \sin^2(\theta)\bar{h}^{\mu\nu}\partial_\theta(\bar{h}_{\mu\nu})\hat{E}_{00} + 2(2-n)\rho\sin^2(\theta)\rho^\lambda \hat{E}_{0\lambda} \\ &- 2\rho^2\sin^2(\theta)\bar{h}^{\mu\nu}\bar{h}^{\eta\lambda}\bar{\Gamma}_{\mu\nu\eta}\hat{E}_{0\lambda}; \\ B_\sigma =& 2\sin^2(\theta)\partial_\theta \hat{E}_{0\sigma} - \sin^2(\theta)\partial_\sigma \hat{E}_{00} + 2\rho^2\sin^2(\theta)\bar{h}^{\mu\nu}\partial_\nu \hat{E}_{\mu\sigma} \\ &- \rho^2\sin^2(\theta)\bar{h}^{\mu\nu}\partial_\sigma \hat{E}_{\mu\nu} + 2(1-n)\sin(\theta)\cos(\theta)\hat{E}_{0\sigma} + \sin^2(\theta)\bar{h}^{\mu\nu}\partial_\theta(\bar{h}_{\mu\nu})\hat{E}_{0\sigma} \\ &+ 2(2-n)\rho\sin^2(\theta)\rho^\lambda \hat{E}_{\sigma\lambda} - 2\rho^2\sin^2(\theta)\bar{h}^{\mu\nu}\bar{h}^{\eta\lambda}\bar{\Gamma}_{\mu\nu\eta}\hat{E}_{\sigma\lambda}, \end{aligned}$$

where $\bar{\Gamma}$ is the Christoffel symbol of $\bar{h}$.

We evaluate B_0 mod $O(\rho^{\gamma+1})$, using the fact that we already know the following:

$$\begin{array}{lll} \hat{E}_{00} = O(\rho^{\gamma+1}) & \hat{E}_{0\mu} = O(\rho^{\gamma}) & \mathring{\hat{E}}_{st} = O(\rho^{\gamma-1}) \\ \hat{E}_{ns} = O(\rho^{\gamma-2}) & \hat{E}_{nn} = O(\rho^{\gamma-2}) & \bar{h}^{\mu\nu}\hat{E}_{\mu\nu} = O(\rho^{\gamma-2}). \end{array}$$

The first row are all $O_g(\rho^{\gamma+1})$, as desired, but the second row are one order lower. Putting these into the equation for B_0 and setting it equal to 0 yields

$$\bar{h}^{\mu\nu}\partial_\theta \hat{E}_{\mu\nu} = O(\rho^{\gamma-1}).$$

This says that $\rho^{1-\gamma}\bar{h}^{\mu\nu}\hat{E}_{\mu\nu}|_{\rho=0}$ is a constant, say $\frac{c}{2}$. We need $c = 0$. By definition of the indicial operator, the equation $\rho^{1-\gamma}\bar{h}^{\mu\nu}\hat{E}_{\mu\nu}|_{\rho=0} = \frac{c}{2}$ is equivalent to saying $2\bar{h}^{\mu\nu}I_{\mu\nu}(\varphi) + 2\bar{h}^{\mu\nu}f_{\mu\nu} = c$. The left-hand side of this latter equation, of course, is already determined up to choice of ψ, since

φ is. Notice in (6.35) that $\bar{h}^{\mu\nu} I_{\mu\nu}$ depends on φ_{nn} and $\bar{h}^{\mu\nu}\varphi_{\mu\nu}$, which in turn we have determined using the operators (6.33) and (6.32), respectively. Neither of these operators has a zeroth-order part, and so adding δ to ψ adds δ to φ_{nn} and $n\delta$ to $\bar{h}^{\mu\nu}\varphi_{\mu\nu}$, by our boundary conditions (2) and (3). Now using equation (6.35), but shifting it to the coordinate frame, we see that adding δ to ψ adds $2(1-n)(\gamma-n)(\gamma+1)\delta$ to $2I_{\mu\nu}(\varphi)$. Thus, since $\gamma \neq n$, there is a unique choice of $\psi(x)$ such that $c = 0$; and so we find that $\chi^{(\gamma)} = \chi^{(\gamma-1)} + \rho^{\gamma}\psi$ is uniquely determined up through order γ so that $\bar{h}^{\mu\nu} E_{\mu\nu} = O_g(\rho^{\gamma})$; and there remains no further freedom in our system.

It remains to analyze $\hat{E}_{n\sigma}$. We next look at B_s. We find that

$$2\rho\partial_\rho \hat{E}_{ns} + 2(2-n)\hat{E}_{ns} = O(\rho^{\gamma-1}).$$

Now write $\hat{E}_{ns} = \xi_s \rho^{\gamma-2}$, which we may do by the above computations. Putting this into our equation, we find

$$(\nu - n)\xi_s = O(\rho^{\gamma-1}),$$

as desired.

Before proceeding to $\hat{E}_{nn}$, we note that we can write $\hat{E}_{\mu\nu} = \alpha\rho^{\gamma-2}\rho_\mu\rho_\nu + 2\xi_{(\mu}\rho_{\nu)} + \frac{1}{n}\ell\bar{h}_{\mu\nu} + \eta_{\mu\nu}$, where $\rho^\mu\xi_\mu = 0 = \bar{h}^{\mu\nu}\eta_{\mu\nu}$ and $0 = \rho^\mu\eta_{\mu\nu}$, and finally $\eta_{\mu\nu} = \eta_{(\mu\nu)}$. Then every term here except $\alpha\rho^{\gamma-2}$ is $O(\rho^{\gamma-1})$ by our earlier analysis. Now using the Bianchi identity $B_n = 0$, we find

$$2\rho\partial_\rho \hat{E}_{nn} + 2(2-n)\hat{E}_{nn} = O(\rho^{\gamma-1}).$$

Since $\hat{E}_{nn} = \alpha\rho^{\gamma-2}$, we find that $(\gamma - n)\alpha = O(\rho)$, and thus, since $\gamma \neq n$, we conclude that $\hat{E}_{nn} = O(\rho^{\gamma-1})$.

Thus, $E(g^{(\gamma)}) = O_g(\rho^{\gamma+1})$. Now $\bar{h}_\theta^{(\gamma)}$ lies in $\mathcal{H}_{n,l'}(\theta_0, S, S^2T^*M)$ for some l', is even to order n in θ, and satisfies our boundary conditions. Also it is unique subject to these conditions. It remains only to show that (10) obtains. This is trivial if n is odd; so let n be even.

Consider equation (6.29). Let v be any term on the right hand side except for the first two and except for

$$w = \frac{1}{2}\sin(\theta)\cos(\theta)\bar{h}^{\eta\lambda}\partial_\theta(\bar{h}_{\eta\lambda})\bar{h}_{\mu\nu};$$

then since $\bar{h}_\theta \in \mathcal{A}_{n,l'}(\theta_0, S, S^2T^*M)$, it follows easily that for every $j \geq 0$, we have

$$\partial_\rho^j v|_{\rho=0} \in \mathcal{B}_{n,m'}(\theta_0, S, S^2({}^0T^*M)) \text{ (some } m'\text{)}.$$

For example, take $v = \frac{1}{2}\sin^2(\theta)\bar{h}^{\eta\lambda}\partial_\theta(\bar{h}_{\mu\eta})\partial_\theta(\bar{h}_{\nu\lambda})$. The lowest order at which $\log(\theta)$ can appear in $\partial_\theta\bar{h}_\theta$ is at power θ^{n-1}. Since there is a factor of $\sin^2(\theta)$, $\log(\theta)$ therefore does not appear in v before order θ^{n+1} (and in fact θ^{n+2}, since $\partial_\theta\bar{h}_\theta = O(\theta)$ as well). Similarly, for $i > 1$, $\log(\theta)^i$ never appears before order θ^{in}, due to the factor of $\sin^2(\theta)$ and the hypothesis that $\bar{h}_\theta \in \mathcal{A}_{n,l'}(\theta_0, S, S^2T^*M)$. The remaining terms are similar. Likewise, if v is the *sum* of the first two terms, we have the same result, because n is an indicial root at $\theta = 0$ of the operator $-\sin^2(\theta)\partial_\theta^2 + (n-1)\sin(\theta)\cos(\theta)\partial_\theta$. Now $0 = \partial_\rho^\gamma\left(E_{\mu\nu}^{(\gamma)}\right)|_{\rho=0}$; since the only term in (6.29) that might contribute a term of the form $\theta^n\log(\theta)$ is w, we may conclude that, in fact, w does not contribute such a term for $\bar{h}^{(\gamma)}$ (as there is nothing to cancel it out). Since $\partial_\theta\bar{h}_{\eta\lambda}^{(\gamma)} = O(\theta)$, and thus the $\theta^n\log(\theta)$ terms that might be present in $\bar{h}^{\eta\lambda}$ and $\bar{h}_{\mu\nu}^{(\gamma)}$ cannot contribute a $\log(\theta)$ to w at order θ^n, we conclude that $\partial_\rho^j(\sin(\theta)\cos(\theta)\bar{h}^{\eta\lambda}\partial_\theta\bar{h}_{\eta\lambda}^{(\gamma)})|_{\rho=0} \in \mathcal{B}_{n,m'}(\theta_0, S)$ for $0 \leq j \leq \gamma$.

Now consider $\operatorname{tf}_{k_0}(\rho^{-(\gamma+1)}E^{(\gamma)})|_{\rho=0}$. As we have seen by analyzing (6.29), every term lies in $\mathcal{B}_{n,m'}(\theta_0, S, S^2({}^0T^*M))$ except possibly

$$\begin{aligned}\operatorname{tf}_{k_0}(\rho^{-(\gamma+1)}w)|_{\rho=0} &= \frac{1}{(\gamma+1)!}\operatorname{tf}_{k_0}\partial_\rho^{\gamma+1}w|_{\rho=0} \\ &= \frac{1}{2(\gamma+1)!}\operatorname{tf}_{k_0}\partial_\rho^{\gamma+1}(\sin(\theta)\cos(\theta)\bar{h}^{\eta\lambda}\partial_\theta(\bar{h}_{\eta\lambda})\bar{h}_{\mu\nu})|_{\rho=0}.\end{aligned}$$

But since $\operatorname{tf}_{k_0}\bar{h}_{st} = 0$, this term vanishes unless at least one factor of ∂_ρ falls on $\bar{h}_{\mu\nu}$. This leaves at most γ derivatives to fall on $\sin(\theta)\cos(\theta)\bar{h}^{\eta\lambda}\partial_\theta(\bar{h}_{\eta\lambda})$; and as just seen, for any $j \leq \gamma$, $\partial_\rho^j(\sin(\theta)\cos(\theta)\bar{h}^{\eta\lambda}\partial_\theta\bar{h}_{\eta\lambda})|_{\rho=0} \in \mathcal{B}_{n,m'}(\theta_0, S)$. We therefore may conclude that

$$\operatorname{tf}_{k_0}\rho^{-(\gamma+1)}E^{(\gamma)}|_{\rho=0} \in \mathcal{B}_{n,m'}(\theta_0, S, S^2({}^0T^*M)).$$

Thus, $\bar{h}_\theta^{(\gamma)}$ satisfies all our desired conditions except (vi). As mentioned earlier, however, this is easily fixed. Set $b = \chi^{(\gamma)}(d\rho^2 + k_\rho) - \bar{h}_0^{(\gamma)} \in C^\infty(\widetilde{M}, S^2T^*\widetilde{M})$, and extend it to a section in $C^\infty(\widetilde{X}, S^2T^*\widetilde{M})$ by making it constant in θ. Now replace $\bar{h}^{(\gamma)}$ by $\bar{h}^{(\gamma)} + b$. Since b is independent of θ, this obtains condition (vi) without compromising our other conditions.

And so by induction, we may construct $g^{(n-1)}$ such that $E(g^{(n-1)}) = O_g(\rho^n)$, satisfying the desired boundary condition at $\widetilde{M}$ to order ρ^n and to infinite order at $\widetilde{Q}$. This completes the proof for $n > 2$.

If $n = 2$, the above proof needs slight modification. At the first step, we define $\bar{h}^{(0)}_\theta = d\rho^2 + k_0$, which is constant in θ and also in ρ; and also define $\chi^{(0)} = 1$. It follows that $E^{(0)} = O_g(\rho^2)$, since the only term in the Einstein equations that does not vanish is the Ricci term in (6.29). We need only solve one more equation, the first-order perturbation, to be done. We set $\bar{h}^{(1)}_\theta = \bar{h}^{(0)}_\theta + \rho\bar{\varphi}$. The equation we wish to solve is $I^1(\varphi) = 0$, with boundary conditions $\partial_\theta\bar{\varphi}|_{\theta=\theta_0} = 0$ and $\bar{\varphi}|_{\theta=0} = \chi^{(1)}\partial_\rho\bar{h}$, where $\chi^{(1)} = \chi^{(0)} + \rho\psi$.

Now $E^{(0)}_{0\sigma} \equiv 0$ and $E^{(0)}_{00} \equiv 0$, so the above analysis of equations (6.32) and (6.33) goes through without problem; this determines $\bar{h}^{\mu\nu}\varphi_{\mu\nu}$, φ_{nn}, and φ_{ns}. It remains to determine $\mathring{\varphi}_{st}$. But notice that when $n = 2$ and $\gamma = 1$, any constant is a solution to $\mathring{I}_{st}(\varphi) = 0$; so we may simply set $\mathring{\varphi}_{st} = \chi \operatorname{tf}_{\bar{h}} \partial_\rho\bar{h}|_{\rho=0}$. We have thus determined $g^{(1)}$, subject to the freedom in ψ; the Bianchi analysis goes through as before, determining ψ. ■

Notice that it is clear from the above proof that the Taylor coefficients of $\bar{h}_\theta$ in ρ are, through order $n-1$, universal functions of θ and of k_0, the derivatives of k_ρ at $\rho = 0$, and their tangential derivatives. Moreover, the jth Taylor coefficient function depends only of $\partial^i_\rho k_\rho|_{\rho=0}$ for $j \leq i$.

If $\theta_0 > \frac{\pi}{2}$, then by Proposition 5.5 there will be an indicial root γ_0 for $\mathring{I}^\gamma_{st}$ between n and $n-1$, and uniqueness in Proposition 6.7 will not be quite to order n without the requirement of smoothness; we will expect additional solutions with leading asymptotics at order ρ^{γ_0}.

We will now focus on the proof of Theorem 1.6. Suppose that $\theta_0 = \frac{\pi}{2}$. By Propositions 5.7, 5.8, and 6.10, we know that n is an indicial root, and that we can solve the tracefree part of the Einstein equations to order $O_g(\rho^{n+1})$ by a smooth perturbation only if the tracefree tangential part of $\rho^{-n}E(g^{(n-1)})|_{\rho=0}$ is orthogonal to $w_0 = \sin^n(\theta)$ with respect to the measure $\sin^{-(n+1)}(\theta)d\theta$. This suggests a way to define the obstruction tensor promised in Theorem 1.6. Suppose M^n is a manifold with boundary S, and equipped with a metric τ. Near S, we can uniquely define a diffeomorphism $\eta : S \times [0, \varepsilon)_\rho \hookrightarrow M$ so that $\eta^*\tau = d\rho^2 + k_\rho$, and so that $\eta|_{S\times\{0\}} = \mathrm{id}_S$. Then by

Proposition 6.7, there is a metric g in the normal form (6.26) on $\mathring{\widetilde{X}}$, where $\widetilde{X} = \left[0, \frac{\pi}{2}\right] \times S \times [0, \varepsilon)$, and a function $\chi \in C^\infty(S \times [0, \varepsilon))$ such that (a) - (e) hold. Now notice that for any section ${}^0T \in C^\infty(\widetilde{S}, S^2({}^{0e}T^*\widetilde{X}))$ satisfying $T = O_g(\sin^2(\theta))$, we can get a well-defined corresponding section $T \in C^\infty(\widetilde{S}, S^2T^*\widetilde{X})$ by setting $T = \rho^2({}^0T)$. Now observe that $E(g) \in C^\infty(S^2({}^{0e}T^*\widetilde{X}))$, with $E(g) = O_g(\rho^n)$. In particular, $\rho^{-n}\, \mathrm{tf}_{k_0}\, E(g)|_{\rho=0} \in C^\infty(\widetilde{S}, S^2({}^{0e}T^*\widetilde{X}))$. Moreover, $E(g) = O_g(\sin^2(\theta))$. This follows easily from equations (6.27) - (6.29), remembering that $\partial_\theta \bar{h}_\theta = O(\theta)$ by evenness in θ. Thus, $\rho^2\left[\rho^{-n}\, \mathrm{tf}_{k_0}\, E(g)|_{\rho=0}\right] \in C^\infty(\widetilde{S}, S^2T^*\widetilde{X})$. For shorthand, we write $\rho^{2-n}\, \mathrm{tf}_{k_0}\, E(g)|_{\rho=0} \in C^\infty(\widetilde{S}, S^2T^*\widetilde{X})$. Thus, we define a smooth symmetric tracefree tensor $\mathcal{K}(\tau)$ on S by

$$\begin{aligned} \mathcal{K}(\tau) &= \left\langle \rho^{2-n}\, \mathrm{tf}_{k_0}\, E(g)|_{\rho=0}, w_0 \right\rangle_{\sin^{-(n+1)}(\theta)d\theta} \\ &= \rho^{2-n} \int_0^{\frac{\pi}{2}} \csc(\theta)\mathring{E}(g)d\theta \Bigg|_{\rho=0} \in S^2T^*S, \end{aligned} \tag{6.38}$$

where $\mathring{E}$ here refers to $\mathrm{tf}_{k_0}\, E$.

Proof of Theorem 1.6. We first must show that $\mathcal{K}(\tau)$ is well defined. First, the integral (6.38) converges, since $E_{\mu\nu} = O(\theta)$ by (6.29). Next, although $\bar{h}_\theta$ is only determined mod $O(\rho^n)$, perturbations of the form $\bar{h}_\theta \mapsto \bar{h}_\theta + \rho^n\bar{\varphi}$ satisfying $\bar{\varphi}|_{\theta=0} = 0$ and $\partial_\theta\bar{\varphi}|_{\theta=\theta_0} = 0$ leave $\left\langle \mathrm{tf}_{k_0}\, \rho^{2-n} E(g)|_{\widetilde{S},TS}, w_0 \right\rangle_{\sin^{-(n+1)}(\theta)d\theta}$ unchanged, since n is an indicial root of $\mathring{I}^n_{st}$ and, by Propositions 5.7, 5.8, and 6.10, the image of $\mathring{I}^n$ is orthogonal to $\sin^n(\theta)$. Thus, $\mathcal{K}(\tau)$ is well defined.

Next, we must show that the conformal transformation law holds. Suppose $\hat{\tau} = \Omega^2\tau$, where $\Omega \in C^\infty(M)$. Let $\hat{\eta} : S \times [0, \varepsilon)_{\hat{\rho}} \hookrightarrow M$ be a diffeomorphism onto a neighborhood of S so that $\hat{\eta}^*\hat{\tau} = d\hat{\rho}^2 + \hat{k}_{\hat{\rho}}$ and so that $\hat{\eta}|_{S\times\{0\}} = \mathrm{id}_S$. Let $\hat{h}_\theta$ and $\hat{\chi}$ satisfy (a) - (e) in Proposition 6.7, in particular with $\bar{\hat{h}}_{\theta_0} = \hat{\chi}(d\hat{\rho}^2 + \hat{k}_{\hat{\rho}})$. Similarly, we take $\hat{g} = \csc^2(\theta)[d\theta^2 + \hat{h}_\theta]$.

Set $\tilde{h}_\theta = \hat{\eta}^*(\eta^{-1})^*h_\theta$, as well as $\tilde{\chi} = \hat{\eta}^*(\eta^{-1})^*\chi$ and $\tilde{\rho} = \hat{\eta}^*(\eta^{-1})^*\rho$. Similarly set $\widehat{\Omega} = \hat{\eta}^*\Omega$. Now plainly

$$\tilde{\rho}^2\tilde{h}_0 = \tilde{\chi}\hat{\eta}^*\tau = \widehat{\Omega}^{-2}\tilde{\chi}(d\hat{\rho}^2 + \hat{k}_{\hat{\rho}}) = \widehat{\Omega}^{-2}\tilde{\chi}\hat{\chi}^{-1}\hat{\rho}^2\hat{h}_0.$$

This implies that $\tilde{h}_0 = \frac{\hat{\rho}^2\tilde{\chi}}{\tilde{\rho}^2\widehat{\Omega}^2\hat{\chi}}\hat{h}_0$. Since $\tilde{h}_0$ and $\hat{h}_0$ are both AH metrics on $S \times [0, \varepsilon)$, we must therefore have

$$\left.\frac{\hat\rho}{\tilde\rho}\right|_{\hat\rho=0} = \hat\Omega|_{\hat\rho=0}.$$

Now

$$\hat\rho^2\tilde h_0 = \frac{\hat\rho^2}{\tilde\rho^2}\tilde\rho^2\tilde h_0 = \frac{\hat\rho^2}{\tilde\rho^2}\hat\Omega^{-2}\tilde\chi(d\hat\rho^2 + \hat k_{\hat\rho}),$$

where $\left.\frac{\hat\rho^2}{\tilde\rho^2}\hat\Omega^{-2}\tilde\chi\right|_{\hat\rho=0} = 1$. Thus, by the uniqueness statement in Proposition 6.7, $\tilde h_\theta = \hat h_\theta$ mod $O(\hat\rho^n)$. Set $\tilde g = \csc^2(\theta)[d\theta^2 + \tilde h_\theta] = \hat\eta^*(\eta^{-1})^* g$. It then follows from the discussion in the first paragraph of this proof and the fact that $\eta|_{\rho=0} = \hat\eta|_{\hat\rho=0} = \mathrm{id}$ that

$$\begin{aligned}
\mathcal{K}(\tau) &= \left\langle \rho^{2-n}\,\mathrm{tf}_{k_0}\,E(g)|_{\rho=0}, w_0\right\rangle_{\sin^{-(n+1)}(\theta)d\theta} \\
&= \left\langle \tilde\rho^{2-n}\,\mathrm{tf}_{\hat k_0}\,E(\tilde g)|_{\rho=0}, w_0\right\rangle_{\sin^{-(n+1)}(\theta)d\theta} \\
&= \left\langle \tilde\rho^{2-n}\,\mathrm{tf}_{\hat k_0}\,E(\hat g)|_{\rho=0}, w_0\right\rangle_{\sin^{-(n+1)}(\theta)d\theta} \\
&= \left.\frac{\hat\rho^{n-2}}{\tilde\rho^{n-2}}\left\langle \hat\rho^{2-n}\,\mathrm{tf}_{\hat k_0}\,E(\hat g)|_{TS}, w_0\right\rangle_{\sin^{-(n+1)}(\theta)d\theta}\right|_{\rho=0} \\
&= \hat\Omega|_S^{n-2}\mathcal{K}(\hat\tau) \\
&= \Omega|_S^{n-2}\mathcal{K}(\hat\tau),
\end{aligned}$$

which is the desired result.

We need finally to show that $\mathcal{K}$ is generically nontrivial. We will do this by showing that

$$\mathcal{K}(\tau) = c\,\mathrm{tf}_{k_0}\,\partial_\rho^n k_\rho|_{\rho=0} + \mathcal{K}'(\tau), \tag{6.39}$$

where $c \neq 0$ and $\mathcal{K}'(\tau)$ depends only on $\partial_\rho^j k_\rho|_{\rho=0}$ for $j < n$. To proceed, let $\kappa = \partial_\rho^n k_\rho|_{\rho=0}$, and extend it to be a section in $C^\infty(\widetilde S, S^2T^*S)$ by taking it to be constant in θ. Define $\bar\varphi \in \mathcal{H}_n(\theta_0, S, S^2T^*S)$ by $\partial_\rho^n \bar h_{st} = \kappa_{st} + \partial_\rho^n((\chi - 1)k_\rho)|_{\rho=0} + \bar\varphi$, where the second term, like κ, is extended to be constant in θ. Notice that the second term depends only on $\partial_\rho^j k_\rho|_{\rho=0}$ for $j < n$. Plainly, we have $\bar\varphi|_{\theta=0} = 0$ and $\partial_\theta\bar\varphi|_{\theta=\theta_0} = 0$. It follows then that the inner product $\langle \rho^{2-n}\,\mathrm{tf}_{k_0}\,E|_{\rho=0}, w_0\rangle$ is independent of $\bar\varphi$, by the discussion in the first paragraph of this proof. We wish to find the coefficient of $\mathring{k}_{st}$ in $\langle \rho^{2-n}\,\mathrm{tf}_{k_0}\,E, w_0\rangle$, and in particular to show that it is nonvanishing.

Consider now the last term of $E_{\mu\nu}$ in (6.29), which is $u_{\mu\nu} = \rho^2 \sin^2(\theta)\,\mathrm{Ric}(\bar{h})_{\mu\nu}$. Recall that the expression for $\mathrm{Ric}(\bar{h})_{\mu\nu}$ is

$$\mathrm{Ric}(\bar{h})_{\mu\nu} = \frac{1}{2}\bar{h}^{\eta\lambda}(\partial^2_{\mu\lambda}\bar{h}_{\nu\eta}+\partial^2_{\nu\eta}\bar{h}_{\mu\lambda}-\partial^2_{\eta\lambda}\bar{h}_{\mu\nu}-\partial^2_{\mu\nu}\bar{h}_{\eta\lambda})+\bar{h}^{\eta\lambda}\bar{h}^{\sigma\tau}(\Gamma_{\mu\lambda\sigma}\Gamma_{\nu\eta\tau}-\Gamma_{\mu\nu\sigma}\Gamma_{\eta\lambda\tau}) \quad (6.40)$$

Now because $\bar{h}_\theta|_{\rho=0} = d\rho^2 + k_0$, we see that the third term contributes to u a term of the form $-\frac{1}{2(n-2)!}\sin^2(\theta)\rho^n\partial^n_\rho\bar{h}_{\mu\nu}$. No other term of (6.40) contributes a multiple of $\partial^n_\rho\bar{h}_{\mu\nu}$ at order ρ^n. Thus, in particular, u contributes a term of the form $-\frac{1}{2(n-2)!}\sin^2(\theta)\rho^n\mathring{\kappa}_{st}$ (as well as terms involving $\mathring{\varphi}_{st}$ and lower orders of k_ρ) to $\mathring{E}_{st}$.

Consider next the second-last term of (6.29), which is $\rho\sin^2(\theta)\nabla^\eta\rho_\eta\bar{h}_{\mu\nu}$. Because of the factor of ρ, it does not make any contribution of the form $\rho^n\mathring{\kappa}_{st}$ to $\mathring{E}_{st}$.

Next consider the third-last term, $v_{\mu\nu} = (n-2)\rho\sin^2(\theta)\nabla_\mu\rho_\nu$. It is easy to compute that this takes the form

$$v_{\mu\nu} = \frac{n-2}{2}\sin^2(\theta)\rho(\partial_\rho\bar{h}_{\mu\nu} - 2\partial_{(\mu}\bar{h}_{\nu)n}) + O(\rho).$$

The third-last term thus contributes a term of the form $\frac{n-2}{2(n-1)!}\sin^2(\theta)\rho^n\mathring{\kappa}_{st}$ to $\mathring{E}_{st}$.

We claim that no other term of (6.29) contributes a term involving κ to E at order ρ^n. The first two terms do not, because κ does not depend on θ. The next three terms do not because $\partial_\theta\bar{h} = O(\rho)$; and the next, because $|d\rho|^2_{\bar{h}} - 1 = O(\rho)$. Thus, the only contributions of $\mathring{\kappa}_{st}$ to $\mathring{E}_{st}$ are those already calculated from the seventh and ninth terms; their sum is $\frac{-1}{2(n-1)!}\rho^n\sin^2(\theta)\mathring{\kappa}_{st}$. But it is plain that the coefficient of ρ^n here is not orthogonal to $\csc(\theta)$; indeed,

$$\begin{aligned}\frac{-1}{2(n-1)!}\left\langle\sin^2(\theta)\mathring{\kappa}_{st}, w_0\right\rangle_{\sin^{-(n+1)}(\theta)d\theta} &= \frac{-1}{(n-1)!}\int_0^{\frac{\pi}{2}}\sin(\theta)\mathring{\kappa}_{st}\,d\theta \\ &= \frac{-1}{2(n-1)!}\mathring{\kappa}_{st}.\end{aligned}$$

Thus, (6.39) holds with $c = \frac{-1}{2(n-1)!}$. Now τ (thus k_ρ) may be changed at order ρ^n independently of any lower orders; and we have seen above that changing φ does not alter the inner product $\langle\rho^{2-n}\mathring{E}|_{\rho=0}, w_0\rangle_{\sin^{-(n+1)}(\theta)d\theta}$. Thus, for generic choices of $\mathrm{tf}_{k_0}\,\partial^n_\rho k_\rho|_{\rho=0} = \mathring{\kappa}$, $\mathcal{K}(\tau)$ is nonvanishing. ■

Chapter 7

FUTURE DIRECTIONS

The preceding work puts focus on several natural lines of inquiry.

Theorem 1.4 suggests that a polyhomogeneity result should obtain for harmonic functions (and perhaps eigenfunctions of the Laplacian) on CAH spaces with inhomgeneous Dirichlet conditions at M and the homogeneous Neumann condition at Q. Proving such a theorem, in addition to its inherent interest, will be a valuable first step to developing tools for doing geometric analysis on 0-edge spaces. Appropriate function spaces will have to be developed on the blowup to handle conditions at all three boundary faces at once, and then bootstrapping techniques adapted. One could also pursue a pseudodifferential calculus for these spaces, which was the approach used in the edge case in [Maz91]. (See also [Hua06].) Obvious generalizations of this problem would then be to study different boundary conditions and their effects, as well as more general elliptic 0-edge operators.

Having studied regularity, it will be interesting to pursue also questions of existence and uniqueness of scalar harmonic functions with different choices of boundary values at M and Q. These questions should use many of the same tools developed for regularity.

A related problem is to pose and then study the scalar scattering problem on a CAH Einstein space, analogous to the scattering problem studied in [GZ03]. That paper also relates the scattering matrix on an AH space to interesting geometric invariants of the boundary, such as at the Q-curvature. It will be interesting to see what, if any such relationships hold at the corner of a CAH Einstein space. Another significant application of scattering in the AH setting has been in studying the fractional Laplacian, which may be defined holographically in terms of a scattering problem ([CdMG11]). Thus, one could hope to start with a manifold M with boundary, realize it as the infinite face of a cornered space X, put a boundary condition (say the totally geodesic one) on Q, and then use a scattering problem for the (formally) Einstein metric on X to give a new definition of the fractional Laplacian on a manifold with boundary. This could be useful, for example, in studying Dirichlet problems for the fractional Laplacian.

The problems suggested for the scalar Laplacian largely have much more challenging analogies for the Einstein operator, which on the other hand should be more interesting. Much as the Fefferman-Graham formal existence result (Theorem 2.5,[FG85]) suggested a regularity result (Theorem 2.6,[CDLS05]) so our formal existence result, Theorem 1.5, suggests a regularity theorem of polyhomogeneity type for CAH Einstein metrics satisfying the CMC umbilic condition $K_Q = \lambda g|_{TQ}$ at Q. Due to the analysis carried out in this thesis of the tracefree tangential part of the Einstein operator, we may expect that the regularity behavior of Einstein metrics will depend crucially on the value of λ (and thus of $\theta_0 = \cos^{-1}(-\lambda)$). If $\theta_0 = \frac{\pi}{2}$, for example, corresponding to the totally geodesic condition $\lambda = 0$, we expect that Theorem 1.5 can be extended to powers beyond n by the introduction of logarithmic terms as in Theorem 1.4, and that a polyhomogeneity result will follow in turn. (It follows from work not included in this thesis that there is no obstruction to infinite-order formal existence other than the obstruction at each order $n + 2k$ to solving the trace-free tangential part of the indicial equation.) For other values of λ, more complicated behaviors can be expected, including non-integer powers of the radial coordinate.

Attacking this problem will require significant analysis of operators on spaces of 0-edge tensor fields, analogous to that carried out in the usual case in [Lee06], but complicated by the presence, on the blowup, of three boundary faces instead of one and boundary conditions at two faces. The related fact, already seen in this thesis, that indicial operators are regular singular ordinary differential operators as opposed to algebraic ones will also again change the analysis. Since the Laplacian is the principal part of the trace-free part of the gauge-broken Einstein operator, the analysis carried out in the scalar case will be very useful in this work.

Additional questions include studying regularity for different boundary conditions on Q; see [And08b].

Studying existence of Einstein metrics has proved to be an extremely challenging problem even in the usual AH setting, still open in most cases, and there is every reason to expect it to be even more challenging with a corner. A classic result in the AH setting is the perturbation result ([GL91]) on the sphere, wherein existence is shown for perturbations of the conformal infinity $[h]$ of the sphere. This was generalized to other manifolds in [Biq00] and [Lee06]. Given the machinery that could be developed in studying the regularity problem, it may be possible to reproduce the AH perturbation result in the CAH setting. Another such problem, which has no analogue in the non-cornered AH setting, is to study existence under perturbations of λ, given a value for which an Einstein metric exists.

Olivier Biquard has pointed out to us that a global perturbation existence result in the case of totally geodesic finite boundary Q should follow from low-regularity versions of the usual AH perturbation existence theorem ([GL91],[Biq00],[Lee06]). Specifically, suppose we are given a cornered space (X, M, Q) with a CAH Einstein metric g and conformal infinity $[h]$ on M, and such that Q is totally geodesic. We may double X across Q, and extend g to a C^1 metric on $2X$ because Q is totally geodesic. The doubled infinite boundary, $2M$, is a manifold without boundary, and has a double conformal infinity $2[h]$, which likewise will only be C^1. And so, by the usual perturbation result, but with low regularity requirements, it would follow that any sufficiently small perturbation of $2[h]$ will lead to a new Einstein metric g' on $2X$. Restricting this new metric to X then produces a perturbation result on X. There are, of course, many details to work through in working out this argument fully.

There have been some very interesting rigidity theorems for AH Einstein manifolds, such as in [Qin03] and [ST05]. Generalizing these results to CAH spaces gives another set of interesting problems.

A quite novel feature of CAH spaces, and specifically of their blowups, is that they contain a corner that is the intersection of two faces with edge structures. (Such a situation has been considered before only in [Hua06], to our knowledge.) Many of the novel features that arise in studying the aforementioned problems will be manifestations of this fact. A fruitful path to pursue after some of the above questions have been studied may therefore be to generalize the tools and results that are developed to more general spaces with intersecting edge structures. Examples would include AH spaces with corners of higher codimension, or the setting of [MW14] with the addition of boundary knots (which is a situation the authors of that paper have begun studying).

In addition to these questions in geometric analysis, the work in this thesis raises several more geometric questions. It would be interesting to understand more about the properties of the conformal hypersurface invariant identified as an obstruction in Theorem 1.6. Other conformal hypersurface invariants have recently arisen in the study of the singular Yamabe problem (see [GW15] and [Gra16]), of renormalized area of minimal surfaces ([GR]), and of the the conformal Gauss map ([QWZ16]). Where this obstruction falls among them, however, is not yet completely clear.

Another invariant object which we have not studied in depth so far is in the setting of Theorem 1.5 with a given, fixed value of λ. The induced metric h_0 on M, or rather its $(n-1)$-jet at S, is invariantly defined. A natural question is – does this lead to any interesting conformal invariants? Although it is not proved in this thesis, we have done work that shows that uniqueness fails in general at order n in the expansion, even if $\lambda \neq 0$. In case $\lambda \in (-1, 1)$ is such that no indicial roots are integers, we would like to understand whether there are any additional natural conditions that would restore uniqueness and determine an infinite invariant jet for h_0. This would be a way of singling out a specific representative in a conformal class (to infinite order), something we can currently do only through order $n-1$.

A further geometric question that merits study is the generalization to the cornered setting of the renormalized volume, and its relation, if any, to the normal form we have given in this paper.

Finally, the definition given in this paper of admissible metrics g on a blowup $b : \widetilde{X} \to X$ depends on the existence of a smooth metric g_+ on X such that $g = b^* g_+ + \mathcal{L}$. It would be interesting to know whether an intrinsic definition can be given that relies only on the structure of $\widetilde{X}$ as a 0-edge space, perhaps with some additional invariant structure arising from b. Examples of such conditions can be found in [GK12].

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