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Abdalla Dali Nimer

The Geometry of uniform measures

Abdalla Dali Nimer

A dissertation
submitted in partial fulfillment of the
requirements for the degree of

Doctor of Philosophy

University of Washington

2017

Reading Committee:

Tatiana Toro, Chair

Steffen Rohde

C. Robin Graham

Program Authorized to Offer Degree:
Department of Mathematics

University of Washington

Abstract

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Abdalla Dali Nimer

Chair of the Supervisory Committee:
Professor Tatiana Toro
Department of Mathematics

Uniform measures have played a fundamental role in geometric measure theory since they naturally appear as tangent objects. They were first studied in the groundbreaking work of Preiss where he proved that a Radon measure is n -rectifiable if and only if the n -density at almost every point of its support is positive and finite (see [P]). However, very little is understood about them: for instance the only known n -uniform measures not supported on an affine n -plane were constructed by Preiss in 1987.

In this thesis, we prove that the Hausdorff dimension of the singular set of any n -uniform measure is at most $n - 3$. Then we characterize 3-uniform measures with dilation invariant support and construct an infinite family of 3-uniform measures all distinct and non-isometric, one of which is the Preiss cone.

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ACKNOWLEDGMENTS

First and foremost, I would like to thank my advisor Tatiana Toro. I would not be the mathematician or person I am today if not for her continual support, her extensive knowledge and brilliance and her incomparable generosity. More than an advisor, I think of her as a mentor and a guide and consider myself incredibly lucky to have crossed her path.

I would also like to thank Elena Erosheva, Steffen Rohde and Robin Graham for agreeing to be on the committee and taking the time to read my thesis, and John Lee for being part of my general exam committee.

Many of the ideas developed in this thesis originate from David Preiss' remarkable work. I am very grateful for having had the chance to visit him for a quarter and owe him special thanks for the time he has given me.

Last but not least, I thank my mother Hana, my late father Said and my siblings Maissam and Rakan for their unfailing love and support.

Chapter 1

INTRODUCTION

1.1 Introduction to n -uniform measures

The notion of n -uniform measure arose from the study of the relation between the rectifiability of a measure and its density. In [Be1], [Be2] and [Be3], Besicovitch proved the following theorem.

Theorem 1.1.1 (Besicovitch). *If $E \subset \mathbb{R}^2$ is such that*

$$\lim_{r \rightarrow 0} \frac{\mathcal{H}^1(B(x, r) \cap E)}{2r} = 1 \quad (1.1.1)$$

for \mathcal{H}^1 almost every $x \in E$, then \mathcal{H}^1 almost all of E can be covered by a countable union of continuously differentiable curves.

This led to the natural question of whether this result generalizes to higher dimensional euclidean spaces and higher densities. To answer that question for $d = 3$ and $n = 2$, Marstrand introduced a seminal notion of weak tangent planes in [Mar], which was then extended by Mattila in [Mat1] to general n, d .

Theorem 1.1.2 (Marstrand-Mattila). *Let $0 < n \leq d < \infty$ be integers and ω_n denote the Lebesgue measure of the unit ball in \mathbb{R}^n . If $E \subset \mathbb{R}^d$ is such that*

$$\lim_{r \rightarrow 0} \frac{\mathcal{H}^n(B(x, r) \cap E)}{\omega_n r^n} = 1 \quad (1.1.2)$$

for \mathcal{H}^n almost every $x \in E$, then \mathcal{H}^n almost all of E can be covered by a countable union of continuously differentiable n -submanifolds.

The main idea of the proof of this theorem is that it is enough to prove that the set E has weak tangent planes at almost every point to deduce rectifiability: this means that

at any small enough scale r , the set E is close in measure to some plane V_r . In [P], Preiss formalized and extended the notion of weak tangent planes to the more general idea of tangent measures, and proved the following remarkable theorem relating the density of a measure and its rectifiability (See Definition 2.1.11).

Theorem 1.1.3 (Preiss). *Let $0 < n \leq d < \infty$ be integers, and Φ be a Radon measure in \mathbb{R}^d . There exists $\tilde{\omega} = \tilde{\omega}(n, d)$ that satisfies the following property. If Φ is such that*

$$0 < \limsup_{r \rightarrow 0} \frac{\Phi(B(x, r))}{r^n} \leq (1 + \tilde{\omega}) \liminf_{r \rightarrow 0} \frac{\Phi(B(x, r))}{r^n} < \infty \quad (1.1.3)$$

for Φ -almost every $x \in \mathbb{R}^d$, then $\Phi \lll \mathcal{H}^n$ and Φ -almost all of \mathbb{R}^d can be covered by a countable union of continuously differentiable n -submanifolds.

One of the main insights in the proof of this theorem is that the “closer” a measure is to having n -density, the closer its tangents are to being n -uniform, i.e. the measure of a ball of radius r centered at any point of the support is cr^n for some absolute positive constant c . In particular, if a measure has n -density at almost every point, then its tangents are n -uniform measures at almost every point. Having weak tangent planes at a point translates to having all tangent measures flat at this point, i.e. at almost every point of the support, every tangent measure is of the form $c\mathcal{H}^n \llcorner V$ for some n -subspace V . Notice that if V is an affine n -subspace of \mathbb{R}^d , then the measure \mathcal{H}^n is n -uniform. Indeed, it is trivial that for every $p \in V$, we have for every $r > 0$

$$\mathcal{H}^n(B(p, r) \cap V) = \omega_n r^n.$$

One is therefore tempted to make the following conjecture:

Conjecture 1.1.4. *Every n -uniform measure in \mathbb{R}^d is flat.*

This conjecture is true for $n = 1, 2, d$. If that conjecture were true in general, then Preiss’ theorem would follow from the (relatively easy to prove) fact that tangents are n -uniform at almost every point. However, in [P], Preiss showed that it is not.

Theorem 1.1.5. (*Preiss*) Let $C \subset \mathbb{R}^4$ be the following cone:

$$C = \{(x, y, z, t) \in \mathbb{R}^4; x^2 + y^2 + z^2 = t^2\}. \quad (1.1.4)$$

Then, for every $p \in C$ and $r > 0$, we have:

$$\mathcal{H}^3(B(p, r) \cap C) = \frac{4}{3}\pi r^3. \quad (1.1.5)$$

Therefore, to prove Theorem 1.1.3, one needs a better description of the geometry of n -uniform measures. Marstrand had already proven that at almost every point of the support of a measure Φ having positive and finite density, there exists at least one flat tangent measure. Preiss' approach to solving this problem was to prove a connectedness theorem for tangent measures. Roughly speaking, he proved on one hand that the set of tangent measures at a point is connected in the metric of weak convergence of measures, and on the other that flat and non-flat measures are in distinct connected components. This implies that the existence of one flat measure in the set of tangent measures at a point forces all tangent measures at this point to be flat.

Since n -uniform measures appear as tangents to a large class of measures, any improvement in their description leads to a better understanding of the regular and singular sets of the measures they are tangent to. For instance, the results on the connectedness of the set of tangent measures and the separation of flat and non-flat measures had far reaching consequences in different areas of analysis. (See [KPT], [PTT], [DKT].) Very little is known about n -uniform measures. Kirchheim and Preiss proved in [KiP] that the support of such a measure is an analytic variety and Tolsa showed in [T] that it is uniformly rectifiable. However, since the only two known examples of n -uniform measures, up to taking cartesian products, were the flat measures and the cone C , it is difficult to make sense of the "separation between flat and non-flat" result as it describes a landscape that is essentially unknown. One of the main contributions of this thesis is a construction of new examples.

1.2 Structure of the thesis

The thesis is divided into two further chapters. We briefly present their content, as a proper introduction is contained within each individual chapter.

In Chapter 2, we study the singular set of an n -uniform measure in \mathbb{R}^d . Let μ be an n -uniform measure in \mathbb{R}^d . At every point x of its support Σ , there exists a unique tangent measure ν^x up to normalization. Moreover, this tangent measure is conical: this means that for every Borel set A , and every $r > 0$, $\nu^x(rA) = r^n \nu^x(A)$. Whenever $\nu^x = c\mathcal{H}^n \llcorner V_x$ for some n -subspace V_x , we say x is a regular or flat point of μ . Every non-flat point is called singular. We can therefore decompose Σ into its regular part \mathcal{R}_μ which is an open set and its singular part \mathcal{S}_μ which is a closed set. Our aim is to prove a bound on the Hausdorff dimension of \mathcal{S}_μ .

Our main result is that the Hausdorff dimension of \mathcal{S}_μ is at most $n - 3$. Moreover, this result is sharp: Preiss' example (1.1.4) allows one to construct n -uniform measures with $\mathcal{S}_\mu = \mathbb{R}^{n-3}$.

This result is proven in two steps: we first consider the case of a 3-uniform conical measure and prove that it is a $C^{1,\alpha}$ submanifold away from the origin. The second step is an inductive dimension reduction argument on the singular set of general n -uniform measures. The dimension reduction argument assumes that one's object of study has the property that its singularities are "stable" under blow-ups. We therefore have to prove such a stability result for n -uniform measures. This follows from a generalization of the connectedness of tangent measures result of Preiss: it will force singularities to blow up to singularities.

In Chapter 3, we answer the question of the existence of non flat 3-uniform measures different from the cone C constructed by Preiss, in the affirmative. To do this, we build on a result from Chapter 2 describing the spherical component of a conical 3-uniform measure to be a bounded 2-uniform measure up to $r = 2$. We use this to give a characterization of conical 3-uniform measures via their spherical component: the latter need to be a union of 2-spheres. The problem of constructing 3-uniform measures is thus reduced to the combina-

torial problem of finding configurations of 2-spheres that satisfy certain conditions in terms of their centers and relative angles. The chapter ends with a construction of a family of such examples.

Chapter 2

A SHARP BOUND ON THE HAUSDORFF DIMENSION OF THE SINGULAR SET OF N -UNIFORM MEASURES

This chapter gives a sharp bound on the Hausdorff dimension of the singular set of an n -uniform measure in \mathbb{R}^d , where a point is called singular if its tangent is not n -flat: we prove that the Hausdorff dimension of the singular set of an n -uniform measure is at most $n - 3$. This bound effectively proves that the case of Preiss' 3-uniform cone is in fact the worst in terms of the dimension of its singular set.

A measure Φ is n -rectifiable if it is absolutely continuous to \mathcal{H}^n and there exists a countable collection of C^1 n -manifolds $\{M_j\}_j$ such that $\Phi(\mathbb{R}^d \setminus \bigcup_j M_j) = 0$. In [P], Preiss proved the following remarkable theorem relating the rectifiability of a measure to its density.

Theorem 2.0.1 ([P]). *A Radon measure Φ of \mathbb{R}^d is n -rectifiable if and only if it satisfies the following property:*

$$\text{The density } \Theta^n(x) = \lim_{r \rightarrow 0} \frac{\Phi(B(x, r))}{\omega_n r^n} \text{ exists and is positive and finite for } \Phi - \text{a.e. } x \in \mathbb{R}^d. \quad (2.0.1)$$

To prove this theorem, Preiss studies the geometry of n -uniform measures which appear as tangents (blow-ups) to measures satisfying (2.0.1). A measure μ is said to be n -uniform if there exists a constant $c > 0$ such that for any x in the support of μ and any radius $r > 0$, we have:

$$\mu(B(x, r)) = cr^n. \quad (2.0.2)$$

In [P], Preiss also provides a classification of the cases $n = 1, 2$ in \mathbb{R}^d for any d . In these cases, μ is n -Hausdorff measure restricted to a line or a plane respectively.

Interestingly, flat measures are not the only examples of uniform measures. Indeed, in [KoP], Kowalski and Preiss proved that μ is $(d-1)$ -uniform in \mathbb{R}^d if and only if $\mu = c\mathcal{H}^{d-1}\llcorner V$ where V is a $(d-1)$ -plane, or $d \geq 4$ and there exists an orthonormal system of coordinates in which $\mu = \mathcal{H}^{d-1}\llcorner(C \times W)$ where W is a $(d-4)$ -plane and C is the Preiss cone. The classification for $n \geq 3$ and codimension $d-n \geq 2$ remains an open question.

Kirchheim and Preiss later proved in [KiP] that the support Σ of a uniformly distributed measure (of which n -uniform measures are an example) is a real analytic variety, namely the intersection of countably many zero sets of analytic functions. An application of the stratification theorem for real analytic varieties implies that Σ must be a countable union of real analytic manifolds and the singular set has Hausdorff dimension at most $(n-1)$.

We investigate the Hausdorff dimension of the singular set \mathcal{S}_μ of an n -uniform measure μ . Our main result is the following theorem.

Theorem 2.0.2. *Let μ be an n -uniform measure in \mathbb{R}^d , $3 \leq n \leq d$ and denote the support of μ by Σ . Then Σ can be written as a disjoint union*

$$\Sigma = \mathcal{R}_\mu \cup \mathcal{S}_\mu \tag{2.0.3}$$

where \mathcal{S}_μ the singular set is a closed set,

$$\dim_{\mathcal{H}}(\mathcal{S}_\mu) \leq n-3, \tag{2.0.4}$$

and \mathcal{R}_μ is a $C^{1,\alpha}$ submanifold of dimension n in \mathbb{R}^d . Here $\dim_{\mathcal{H}}$ denotes the Hausdorff dimension.

In the cases $n=3, d=3$, it is a standard result that the only 3-uniform measure (up to normalization) is 3-Lebesgue measure. In this case, the bound is obvious. To see that this bound cannot be improved, let $n \geq 3, d > n$ and consider the measure μ defined as:

$$\mu = \mathcal{H}^n \llcorner M, \tag{2.0.5}$$

where M is the set

$$M = \{(x_1, \dots, x_d); x_4^2 = x_1^2 + x_2^2 + x_3^2 \text{ and } x_{n+1} = \dots = x_d = 0\}. \tag{2.0.6}$$

By [KoP], since products of uniform measures are uniform, μ is n -uniform. Moreover, the singular set of μ is \mathbb{R}^{n-3} which has Hausdorff dimension $n - 3$.

This theorem is first proven for the base case $n = 3$. The crux of this argument is a theorem stating that conical 3-uniform measures have at most one singularity at the origin. We then prove that singular sets of n -uniform measure behave nicely under blow-ups. This allows us to adapt Federer's dimension reduction argument to generalize our base case to any dimension n .

We briefly discuss the steps of our proof. In the first section, we limit our investigation to n -uniform conical measures. A conical uniform measure is a measure that is dilation invariant up to appropriate scaling. We first obtain a polar decomposition for such a measure. By polar decomposition, we mean that the measure decomposes into a spherical and a radial component. Using this decomposition, we isolate the spherical component of a 3-uniform measure and prove that it is locally 2-uniform. This allows us to deduce that every point of the spherical component is flat from which the following theorem follows.

Theorem 2.0.3. *Let ν be a conical 3-uniform measure in \mathbb{R}^d and let Σ be its support. Then there exists $\gamma > 0$ such that $\Sigma \setminus \{0\}$ is a $C^{1,\gamma}$ submanifold of dimension 3 .*

In the case where ν is a conical n -uniform measure, for general n , the spherical component turns out to be uniformly distributed. This means that there exists a function $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that for any point x of its support, and any positive radius $r > 0$

$$\nu(B(x, r)) = \phi(r).$$

We use this result to show that for a conical n -uniform measure, Kirchheim and Preiss' result can be improved to an algebraic variety that is symmetric with respect to the origin.

Theorem 2.0.4. *Let ν be a conical n -uniform measure in \mathbb{R}^d and Σ its support. Then Σ is an algebraic variety and*

$$\Sigma = -\Sigma. \tag{2.0.7}$$

In the second section, we first start by proving a lemma about the connectedness of blow-ups along a sequence of points. This connectedness is expressed in terms of a measure's distance from flat measures. To this effect, we use a positive functional F defined on Radon measures satisfying $F(\mu) = 0$ if and only if μ is flat .

Lemma 2.0.5. *There exists $\epsilon_0 > 0$ such that the following holds. Let μ be an n -uniform measure in \mathbb{R}^d , $(x_k)_k \subset \text{supp}(\mu)$ and $(\tau_k)_k, (\sigma_k)_k$ sequences of positive numbers decreasing to 0. We also assume that $\sigma_k < \tau_k$ and that there exist n -uniform measures α and β such that:*

$$\mu_{x_k, \tau_k} \rightharpoonup \alpha \text{ and } \mu_{x_k, \sigma_k} \rightharpoonup \beta.$$

Then:

$$F(\alpha) < \epsilon_0 \implies F(\beta) < \epsilon_0.$$

We use this lemma to deduce a theorem about the convergence of singular sets. Roughly speaking, blow-ups preserve singularity.

Theorem 2.0.6. *Let μ be an n -uniform measure in \mathbb{R}^d , $x_0 \in \text{supp}(\mu)$, $(x_j)_j \subset \mathcal{S}_\mu$, $(r_j)_j$ any sequence of positive numbers decreasing to 0. Also assume that $y_j = \frac{x_j - x_0}{r_j} \in \overline{B(0, 1)}$, $y_j \rightarrow y$.*

Then

$$y \in \mathcal{S}_\nu,$$

where ν is the tangent to μ at x_0 with appropriate normalization.

2.1 Preliminaries

Let us start by defining Hausdorff measure and the concepts of upper and lower density. Though these definitions are standard, we make them to keep track of the constants, especially in the second section.

Definition 2.1.1. *We define ω_s to be the constant:*

$$\omega_s = \frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2} + 1)},$$

so that in particular ω_m is the volume of the unit ball $B^m(0, 1)$ when $m \in \mathbb{N}$. For $\delta \in (0, \infty]$, define \mathcal{H}_δ^s , $s \leq d$, to be the measure in \mathbb{R}^d defined in the following way. If $A \subset \mathbb{R}^d$

$$\mathcal{H}_\delta^s(A) = \omega_s \inf \sum_j \left(\frac{\text{diam}(E_j)}{2} \right)^s,$$

where the infimum is taken over all countable coverings $\{E_j\}_j$ of A such that $\text{diam}(E_j) < \delta$.

Then define s -Hausdorff measure \mathcal{H}^s to be

$$\begin{aligned} \mathcal{H}^s(A) &= \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^s(A), \\ &= \sup_{\delta > 0} \mathcal{H}_\delta^s(A). \end{aligned} \tag{2.1.1}$$

It is a standard result of measure theory that \mathcal{H}^s is a Borel regular measure on \mathbb{R}^d .

Definition 2.1.2. Let Φ be a measure on \mathbb{R}^d , $x \in \mathbb{R}^d$. We define the lower density $\theta_*^s(\Phi, x)$ and upper density $\theta^{*,s}(\Phi, x)$ of Φ at x to be

$$\begin{aligned} \theta_*^s(\Phi, x) &= \liminf_{r \rightarrow 0} \frac{\Phi(B(x, r))}{\omega_s r^s} \\ \theta^{*,s}(\Phi, x) &= \limsup_{r \rightarrow 0} \frac{\Phi(B(x, r))}{\omega_s r^s}. \end{aligned} \tag{2.1.2}$$

If the limsup and the liminf coincide, we call their common value the density of Φ at x and denote it by $\theta^s(\Phi, x)$.

We will need the two following theorems in Section 3 of the paper. The co-area formula will allow us to decompose a conical uniform measure into a spherical and a radial component. As for the area formula, it will be used to compute the measure of a ball by the spherical component.

Theorem 2.1.3 ([S]). [The area formula] Let $f : \mathbb{R}^m \rightarrow \mathbb{R}^d$ be a 1-1 C^1 function where $m < d$. Then, for any Borel set $A \subset \mathbb{R}^m$, we have:

$$\int_A Jf(x) d\mathcal{L}^m(x) = \mathcal{H}^m(f(A)) \tag{2.1.3}$$

where

$$Jf(x) = \sqrt{\det((df(x))^* \circ df(x))}, \quad (2.1.4)$$

and $(df(x))^*$ is the adjoint of $df(x)$.

The co-area formula can be viewed as a more general form of Fubini's theorem. To state it, we first need to define a notion of gradients for Lipschitz functions whose domain is a rectifiable set.

Let M be an n -rectifiable set in \mathbb{R}^d (in particular $n \leq d$) that is, M can be written as a countable union of C^1 manifolds $\{N_i\}$ up to a set of \mathcal{H}^n -measure zero. Let $f : M \rightarrow \mathbb{R}^m$ with $f = (f_1, \dots, f_m)$ be a Lipschitz function. Then by Rademacher's theorem, f is almost everywhere differentiable (the same holds for each f_i). With this in mind, for $x \in N_j$ for some j (in particular this is true for \mathcal{H}^n almost every $x \in M$), $T_x M = T_x N_j$ the tangent plane at x as a point of N_j . At \mathcal{H}^n -almost every point of M , we can define the gradient $\nabla^M f_i = \nabla^{N_j} f_i$ of f_i and the linear map $d^M f(x) : T_x M \rightarrow \mathbb{R}^m$ in the following way:

$$d^M f(x)(\tau) = \sum_{j=1}^m \langle \tau, \nabla^M f_j(x) \rangle e_j, \quad (2.1.5)$$

where $\{e_j\}$ is an orthonormal basis of \mathbb{R}^m .

Theorem 2.1.4 ([S]). *[The co-area formula] Let $M \subset \mathbb{R}^d$ be an n -rectifiable set and $f : M \rightarrow \mathbb{R}^m$, $m < n \leq d$ a Lipschitz function. Then for any non-negative Borel function $g : M \rightarrow \mathbb{R}$, we have:*

$$\int_M g(x) J_M^* f(x) d\mathcal{H}^n(x) = \int_{\mathbb{R}^m} \int_{f^{-1}(y) \cap M} g(z) d\mathcal{H}^{n-m}(z) d\mathcal{L}^m(y), \quad (2.1.6)$$

where

$$J_M^* f(x) = \sqrt{\det(d^M f(x) \circ (d^M f(x))^*)}. \quad (2.1.7)$$

When studying the convergence of Radon measures, it is often very useful to metrize the space of Radon measures. We start by defining the notion of the support of a measure.

Definition 2.1.5. Let μ be a measure in \mathbb{R}^d . We define the support of μ to be

$$\text{supp}(\mu) = \{x \in \mathbb{R}^d; \mu(B(x, r)) > 0, \text{ for all } r > 0\}. \quad (2.1.8)$$

Note that the support of a measure is a closed subset of \mathbb{R}^d .

We can define weak convergence for a sequence of Radon measures.

Definition 2.1.6. Let $\Phi, \Phi_j, j > 0$ be Radon measures in \mathbb{R}^d . We say that Φ_j converges weakly to Φ if for every $f \in C_c(\mathbb{R}^d)$, the following holds:

$$\int f(z)d\Phi_j(z) \rightarrow \int f(z)d\Phi(z). \quad (2.1.9)$$

We denote it by $\Phi_j \rightharpoonup \Phi$.

The results in this section appear in this form in [Mat1].

Theorem 2.1.7. Let Φ_j be a sequence of Radon measures on \mathbb{R}^d . Then $\Phi_j \rightharpoonup \Phi$, if and only if for any K compact subset of \mathbb{R}^d and any G open subset of \mathbb{R}^d the following hold:

1. $\Phi(K) \geq \limsup \Phi_j(K)$.
2. $\Phi(G) \leq \liminf \Phi_j(G)$.

Theorem 2.1.8. Let Φ_j be a sequence of Radon measures on \mathbb{R}^d such that

$$\sup_j (\Phi_j(K)) < \infty,$$

for all compact sets $K \subset \mathbb{R}^d$. Then there is a weakly convergent subsequence of Φ_j .

We now want to define a metric on the space of Radon measures.

Definition 2.1.9. Let $0 < r < \infty$. We denote by $\mathcal{L}(r)$ the set of all non-negative Lipschitz functions f on \mathbb{R}^d with $\text{spt}(f) \subset B(r)$ and with $\text{Lip}(f) \leq 1$. For Radon measures Φ and Ψ on \mathbb{R}^d , set

$$F_r(\Phi, \Psi) = \sup \left\{ \left| \int f d\Phi - \int f d\Psi \right| : f \in \mathcal{L}(r) \right\}.$$

We also define \mathcal{F} to be

$$\mathcal{F}(\Phi, \Psi) = \sum_k 2^{-k} F_k(\Phi, \Psi).$$

It is easily seen that F_r satisfies the triangle inequality for each $r > 0$ and that \mathcal{F} is a metric.

Proposition 2.1.10. *Let Φ, Φ_k be Radon measures on \mathbb{R}^d . Then the following are equivalent:*

1. $\Phi_j \rightarrow \Phi$.
2. $\lim \mathcal{F}(\Phi_j, \Phi) \rightarrow 0$
3. For all $r > 0$, $\lim_{j \rightarrow \infty} F_r(\Phi_j, \Phi) = 0$.

Let μ be a Radon measure on \mathbb{R}^d and Σ its support. For $a \in \mathbb{R}^d$, $r > 0$, define $T_{a,r}$ to be the following homothety that blows up $B(a, r)$ to $B(0, 1)$:

$$T_{a,r}(x) = \frac{x - a}{r}.$$

We define the image $T_{a,r}[\mu]$ of μ under $T_{a,r}$ to be the following measure:

$$\begin{aligned} T_{a,r}[\mu](A) &= \mu(T_{a,r}^{-1}(A)), \\ &= \mu(rA + a), \quad A \subset \mathbb{R}^d. \end{aligned}$$

Definition 2.1.11 ([P]). *We say that ν is a tangent measure of μ at a point $x_0 \in \mathbb{R}^d$ if ν is a non-zero Radon measure on \mathbb{R}^n and if there exist sequences (r_i) and (c_i) of positive numbers such that $r_i \downarrow 0$ and:*

$$c_i T_{x_0, r_i}[\mu] \rightarrow \nu \text{ as } i \rightarrow \infty, \tag{2.1.10}$$

where the convergence in (2.1.10) is the weak convergence of measures. We write $\nu \in \text{Tan}(\mu, x_0)$.

Remark 2.1.1. By Remark 14.4.3 in [Mat2], if

$$0 < \Theta_*^n(\mu, x_0) \leq \Theta^{*n}(\mu, x_0) < \infty, \tag{2.1.11}$$

and if $\nu \in \text{Tan}(\mu, x_0)$, then we can choose (r_i) such that:

$$r_i^{-n} T_{x_0, r_i}[\mu] \rightarrow c\nu \text{ as } i \rightarrow \infty, \quad (2.1.12)$$

for some $c > 0$. In the setting of this paper, (2.1.11) will always hold and we will only use (2.1.12) when talking about tangent measures.

Definition 2.1.12. *A measure on \mathbb{R}^d is called n -flat if it is equal to $c\mathcal{H}^n \llcorner V$, where V is an n -plane, and $0 < c < \infty$.*

Let μ be a Radon measure on \mathbb{R}^d and x_0 be a point in the support Σ of μ . We will call x_0 a flat (or regular) point of Σ if there exists an n -plane V such that

$$\text{Tan}(\mu, x_0) = \{c\mathcal{H}^n \llcorner V ; c > 0\}. \quad (2.1.13)$$

Any point of Σ that is not flat will be called a singular (or non-flat) point.

Definition 2.1.13. *Let μ be a Radon measure in \mathbb{R}^d .*

- *We say μ is uniformly distributed if there exists a positive function $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that:*

$$\mu(B(x, r)) = \phi(r), \text{ for all } x \in \Sigma, r > 0.$$

We call ϕ the distribution function of μ .

- *If there exists $c > 0$ such that $\phi(r) = cr^n$, we say μ is n -uniform.*
- *If μ is an n -uniform measure such that $T_{0,r}[\mu] = r^n \mu$ for all $r > 0$, we call it a conical n -uniform measure.*

In [[P], Theorem 3.11], Preiss showed that if μ is an n -uniform measure, there exists a unique n -uniform measure λ such that:

$$r^{-n} T_{x,r}[\mu] \rightarrow \lambda, \text{ as } r \rightarrow \infty, \quad (2.1.14)$$

for all $x \in \mathbb{R}^d$. λ is called the tangent measure of μ at ∞ .

The following theorem describes a basic but essential property of uniformly distributed measures: how radial functions integrate against them.

Theorem 2.1.14. *Let μ be a uniformly distributed measure on \mathbb{R}^d and f be a non-negative Borel function on \mathbb{R}_+ . For all $z, y \in \text{supp}(\mu)$, we have:*

$$\int f(|x - z|)d\mu(x) = \int f(|x - y|)d\mu(x).$$

Proof. This is a simple application of Fubini's theorem. Indeed, if $f = \alpha\chi_I$, where $\alpha \geq 0$ and $I = (c, d)$ is an interval

$$\begin{aligned} \int f(|x - z|)d\mu(x) &= \alpha \int_0^1 \mu(\{x; \chi_I(|x - z|) \geq t\})dt, \\ &= \alpha (\mu(B(z, d) \cap B(z, d)^c)), \\ &= \alpha (\mu(B(y, d) \cap B(y, c)^c)), \text{ since } \mu(B(z, r)) = \mu(B(y, r)) \text{ for all } r \\ &= \int f(|x - y|)d\mu(x). \end{aligned}$$

The result follows for general non-negative Borel functions by linearity of the integral and density of step functions. \square

In [P], Preiss introduced the following k -forms which were essential to understand the structure of uniform measures.

Definition 2.1.15 (3.4.(1), [P]). *For μ uniformly distributed measure in \mathbb{R}^d , $s > 0$ and $k \in \mathbb{N}$, define the following symmetric k -linear form $b_{k,s} \in \odot^k \mathbb{R}^d$:*

$$b_{k,s}(u_1 \odot \dots \odot u_k) = (2s)^k (I(s)k!)^{-1} \int \langle z, u_1 \rangle \dots \langle z, u_k \rangle e^{-s|z|^2} d\mu(z), \quad (2.1.15)$$

where

$$I(s) = \int e^{-s|z|^2} d\mu(z).$$

We will quote a theorem by Preiss describing Taylor expansions of those forms, and two consequences of this expansion.

Theorem 2.1.16 (3.6, [P]). *Let μ be a uniformly distributed measure in \mathbb{R}^d .*

1. *There are symmetric forms $b_k^{(j)} \in \odot^k \mathbb{R}^d$ such that:*

$$(a) \quad b_{k,s} = \sum_{j=1}^q s^j \frac{b_k^{(j)}}{j!} + o(s^q) \text{ as } s \downarrow 0 \text{ for every } k = 1, 2, \dots \text{ and every } q = 1, 2, \dots$$

$$(b) \quad b_k^{(i)} = 0 \text{ whenever } 2i < k.$$

$$(c) \quad \sum_{k=1}^{2q} b_k^{(q)}(x^k) = |x|^{2q} \text{ for every } q = 1, 2, \dots \text{ and every } x \in \Sigma.$$

Moreover, the forms $b_k^{(j)}$ are uniquely determined by (1a).

2. *There are symmetric forms $\hat{b}_k^{(j)} \in \odot^k \mathbb{R}^d$ such that:*

$$(a) \quad s^{-k} b_{k,s} = \sum_{j=1}^q s^{-j} \frac{\hat{b}_k^{(j)}}{j!} + o(s^{-q}) \text{ as } s \uparrow \infty, \text{ for every } k = 1, 2, \dots \text{ and every } q = 1, 2, \dots, \text{ and}$$

$$(b) \quad \hat{b}_k^{(i)} = 0 \text{ whenever } k > 2i.$$

Moreover, the forms $\hat{b}_k^{(j)}$ are determined uniquely by (2a).

If μ is assumed to be conical, one gets the following improvement on Theorem 2.1.14 and Theorem 2.1.16.

Theorem 2.1.17 (3.10,[P]). *Let μ be a uniformly distributed conical measure. Then there exists n such that μ is n -uniform and:*

- *if $x \in \Sigma$ and $\lambda > 0$, then $\lambda x \in \Sigma$, where Σ is the support of μ .*
- *whenever $u \in \Sigma$, $e \in \mathbb{R}^n$, $|u| = |e|$ and f is a non-negative Borel function on \mathbb{R}^2 then:*

$$\int_{\mathbb{R}^d} f(|z|^2, \langle z, u \rangle) d\mu(z) = C \int_{\mathbb{R}^n} f(|x|^2, \langle x, e \rangle) d\mathcal{L}^n(x). \quad (2.1.16)$$

- *For every $s > 0$ and $k = 1, 2, \dots$, we have*

$$b_{2k-1,s} = 0 \text{ and } b_{2k,s} = \frac{s^k}{k!} b_{2k}^k. \quad (2.1.17)$$

- If Σ denotes the support of μ we have:

$$\Sigma \subset \bigcap_{k>0} \{x \in \mathbb{R}^d; b_{2^k}^k(x^{2^k}) = |x|^{2^k}\}. \quad (2.1.18)$$

The following theorem is an important consequence of Theorem 2.1.16. Note that the statement here is slightly different from Theorem 3.11 in [P]. Indeed, we restate this theorem on convergence of measures in term of the metric from Definition 2.1.9.

Theorem 2.1.18 (3.11, [P]). *Let μ be an n -uniform measure in \mathbb{R}^d . Then, for every $x \in \Sigma \cup \{\infty\}$, there exists a unique conical n -uniform measure λ_x such that:*

- $Tan(\mu, x) = \{c\lambda_x; c > 0\}$
- $\lim_{r \rightarrow 0} \mathcal{F}(r^{-n}T_{x,r}[\mu], \lambda_x) = 0$ if $x \neq \infty$.
- $\lim_{r \rightarrow \infty} \mathcal{F}(r^{-n}T_{y,r}[\mu], \lambda_\infty) = 0$ for each $y \in \mathbb{R}^d$.

Moreover, for μ -almost every $x \in \Sigma$, λ_x is flat.

We know from Theorem 2.0.1 that an n -uniform measure is n -rectifiable. We can translate this into a corollary on the rectifiability of the support of an n -uniform measure.

Corollary 2.1.19. *Let μ be an n -uniform measure in \mathbb{R}^d with $\Sigma = \text{supp}(\mu)$ and let $c > 0$ be such that for $x \in \Sigma$, $r > 0$*

$$\mu(B(x, r)) = cr^n. \quad (2.1.19)$$

Then Σ is n -rectifiable and

$$\mu = c\omega_n^{-1}\mathcal{H}^n \llcorner \Sigma. \quad (2.1.20)$$

Proof. By Theorem 2.1.30, since μ is n -uniform, Σ is a $C^{1,\alpha}$ n -manifold in the neighborhood of \mathcal{H}^n -almost every point. In particular, denoting the n -density of Σ at x by $\theta^n(\Sigma, x)$, we have:

$$\theta^n(\Sigma, x) = 1, \text{ for } \mathcal{H}^n \text{ almost every } x \in \Sigma. \quad (2.1.21)$$

Let $D(x)$ denote $D(\mu, \mathcal{H}^n, x)$ the Radon-Nikodym derivative of μ with respect to \mathcal{H}^n at x . For $x \in \Sigma$,

$$\begin{aligned} D(x) &= \theta^n(\mu, x)\theta^n(\Sigma, x)^{-1}, \\ &= c\omega_n^{-1}\theta^n(\Sigma, x)^{-1}. \end{aligned}$$

Theorem 2.12 from [Mat2] implies that for all $A \subset \Sigma$

$$\mu(A) = c\omega_n^{-1} \int_{\Sigma} \theta^n(\Sigma, x)^{-1} d\mathcal{H}^n(x). \quad (2.1.22)$$

Combining (2.1.21) and (2.1.22), we get

$$\mu = c\omega_n^{-1}\mathcal{H}^n \llcorner \Sigma. \quad (2.1.23)$$

Now since μ is n -rectifiable by Theorem 2.0.1, there exists an n -rectifiable set M such that:

$$\mu(\mathbb{R}^d \cap M^c) = 0. \quad (2.1.24)$$

Combining (2.1.23) and (2.1.24), we see that there exists a set $N = \Sigma \cap M^c$ of \mathcal{H}^n -measure zero such that:

$$\Sigma = M \cup N.$$

In particular, Σ is n -rectifiable. □

Definition 2.1.20. *Let μ be an n -uniform measure in \mathbb{R}^d , $x_0 \in \text{supp}(\mu) \cup \{\infty\}$. We will call μ^{x_0} the normalized tangent measure to μ at x_0 if $\mu^{x_0} \in \text{Tan}(\mu, x_0)$, and $\mu^{x_0}(B(0, 1)) = \omega_n$.*

One of the most remarkable results in Preiss' paper [P] is a separation between flat and non-flat measures at infinity. We will state a reformulation of this theorem by De Lellis from [Del] which is better adapted to our needs.

Theorem 2.1.21 ([P]). *Let μ be an n -uniform measure in \mathbb{R}^d , ζ its normalized tangent at ∞ (in the sense of Definition (2.1.20)). If $n \geq 3$, then there exists $\epsilon_0 > 0$ (depending only on n and d) such that, if*

$$\min_{V \in G(n,d)} \int_{B(0,1)} \text{dist}^2(z, V) d\zeta(z) \leq \epsilon_0, \quad (2.1.25)$$

then μ is flat.

In particular, if μ is conical and

$$\min_{V \in G(n,d)} \int_{B(0,1)} \text{dist}^2(z, V) d\mu(z) \leq \epsilon_0, \quad (2.1.26)$$

then μ is flat.

[Del] defines certain functionals that measure how far from flat a measure is and behave well under weak convergence.

Definition 2.1.22. *Let $\varphi \in C_c(B(0,2))$, $0 \leq \varphi \leq 1$ and $\varphi = 1$ on $B(0,1)$. We define the functional $F : \mathcal{M}(\mathbb{R}^d) \rightarrow \mathbb{R}$ as*

$$F(\Phi) := \min_{V \in G(n,d)} \int \varphi(z) \text{dist}^2(z, V) d\Phi(z)$$

Lemma 2.1.23 ([Del]). *Let Φ_j, Φ be Radon measures such that $\Phi_j \rightharpoonup \Phi$. Then $F(\Phi_j) \rightarrow F(\Phi)$.*

We can now reformulate Theorem 2.1.21 in terms of the functionals F .

Corollary 2.1.24. *Let μ be an n -uniform measure on \mathbb{R}^d , ζ its normalized tangent at infinity. If $n \geq 3$, there exists $\epsilon_0 > 0$ (depending only on n and d) such that*

$$F(\zeta) \leq \epsilon_0 \implies \mu \text{ is flat.}$$

In particular, if μ is conical and $F(\mu) \leq \epsilon_0$ then μ is flat.

Proof. By definition of φ , we have:

$$\chi_{B(0,1)}(x) \leq \varphi(x),$$

for all $x \in \mathbb{R}^d$. This implies that

$$\min_{V \in G(n,d)} \int_{B(0,1)} \text{dist}^2(z, V) d\zeta(z) \leq F(\zeta),$$

and in particular, if ϵ_0 is the constant from Theorem 2.1.21

$$\begin{aligned} F(\zeta) \leq \epsilon_0 &\implies \min_{V \in G(n,d)} \int_{B(0,1)} \text{dist}^2(z, V) d\zeta(z) \leq \epsilon_0, \\ &\implies \mu \text{ is flat.} \end{aligned}$$

This ends the proof. □

Another result concerning the geometry of supports of uniformly distributed measures was proven in [KiP], with the added condition that their support be bounded. This result states that in this case, the support is in fact an algebraic variety.

Theorem 2.1.25 ([KiP]). *Let μ be a uniformly distributed measure over \mathbb{R}^d with bounded support and let $u \in \Sigma$. Then $x \in \Sigma$ if and only if:*

$$P_k(x) = \int_{\mathbb{R}^d} \langle z - x, z - x \rangle^k - \langle z - u, z - u \rangle^k d\mu(z) = 0, \quad (2.1.27)$$

for every $k \in \mathbb{N}$.

In [DKT] and [PTT], the authors proved that for measures with nice density ratios, Theorem [P] can be improved in the sense that the support is a $C^{1,\beta}$ -manifold in the neighborhood of every flat point, for some $\beta > 0$. Let us start with some definitions. For $x \in \Sigma$ where Σ is a closed set and $r > 0$, set:

$$\theta(x, r) = \frac{1}{r} \inf \left\{ D \left[\Sigma \cap \overline{B(x, r)}, L \cap \overline{B(x, r)} \right] : L \text{ affine n-plane through } x \right\}, \quad (2.1.28)$$

where,

$$D[E, F] = \sup \{ \text{dist}(y, F) : y \in E \} + \sup \{ \text{dist}(y, E) : y \in F \} \quad (2.1.29)$$

denotes the Hausdorff distance between the closed sets E and F .

Definition 2.1.26. Let $\delta > 0$ be given. We say that the closed set $\Sigma \subset \mathbb{R}^d$ is δ -Reifenberg-flat of dimension n if for all compact sets $K \subset \Sigma$ there is a radius $r_K > 0$ such that:

$$\theta(x, r) \leq \delta \text{ for all } x \in K \text{ and } 0 < r \leq r_K. \quad (2.1.30)$$

Definition 2.1.27. We say that the closed set Σ is Reifenberg flat with vanishing constant of dimension n if for every compact subset K of Σ :

$$\lim_{r \rightarrow 0^+} \theta_K(r) = 0 \quad (2.1.31)$$

where

$$\theta_K(r) = \sup_{x \in K} \theta(x, r). \quad (2.1.32)$$

Definition 2.1.28. Let μ be a Radon measure on \mathbb{R}^d .

- We say that μ has n -density ratio locally C^α if, for each compact set $K \subset \Sigma$, there is a constant C_K such that:

$$\left| \frac{\mu(B(x, r))}{\omega_n r^n} - 1 \right| \leq C_K r^\alpha, \quad (2.1.33)$$

for $x \in K$ and $0 < r < 1$.

- If $x \in \Sigma$, $r > 0$ and $t \in (0, 1]$, define the quantity:

$$R_t(x, r) = \frac{\mu(B(x, tr))}{\mu(B(x, r))} - t^n. \quad (2.1.34)$$

We say μ is asymptotically optimally doubling if for each compact set $K \subset \Sigma$, $x \in K$, and $t \in [\frac{1}{2}, 1]$

$$\lim_{r \rightarrow 0^+} \sup_{x \in K} |R_t(x, r)| = 0. \quad (2.1.35)$$

The following results from [PTT] and [L] describe the geometry of the support of a measure based on information on its density ratio.

Theorem 2.1.29 (1.9, [PTT]). *For each $\alpha \in (0, 1]$, there exists $\beta = \beta(\alpha) > 0$ with the following property. Suppose μ is a positive Radon measure supported on $\Sigma \subset \mathbb{R}^d$ and for each compact set $K \subset \Sigma$ there exists a constant C_K such that:*

$$|R_t(x, r)| \leq C_K r^\alpha \text{ for } r \in (0, 1], t \in [\frac{1}{2}, 1] \text{ and } x \in K. \quad (2.1.36)$$

Then:

- *If $n = 1, 2$, Σ is a $C^{1,\beta}$ -submanifold of dimension n in \mathbb{R}^d .*
- *If $n \geq 3$ there exists a constant $\delta(n, d)$ such that if $x_0 \in \Sigma$ and $\Sigma \cap B(x_0, 2R_0)$ is δ -Reifenberg-flat, then $\Sigma \cap B(x_0, R_0)$ is a $C^{1,\beta}$ -submanifold of dimension n in \mathbb{R}^d .*

Theorem 2.1.30 ([PTT], [L]). *For each $\alpha > 0$, there exists $\beta = \beta(\alpha)$ with the following property. If μ is a positive Radon measure supported on $\Sigma \subset \mathbb{R}^d$ whose n -density ratio is locally C^α , then:*

- *(1.10, [PTT]) if $n = 1, 2$, Σ is a $C^{1,\beta}$ submanifold of dimension n in \mathbb{R}^d .*
- *(1.10, [PTT]) if $n \geq 3$, Σ is a $C^{1,\beta}$ submanifold of dimension n in \mathbb{R}^d away from a closed set \mathcal{S} such that $\mathcal{H}^n(\mathcal{S}) = 0$, where $\mathcal{S} = \Sigma \setminus \mathcal{R}$ and $\mathcal{R} = \{x \in \Sigma; \limsup_{r \rightarrow 0} \theta(x, r) = 0\}$.*
- *(1.7, [L]) If $n = 3$, $d = 4$, and x is a non-flat point of Σ , there exists a neighborhood of x which is $C^{1,\beta}$ diffeomorphic to an open piece of the KP-cone C in (1.1.4), containing the singular point 0 .*

Theorem 2.1.31 (III.5.9, [DS]). *Let $\{\mu_j\}_j$ be a sequence of n -uniform measures with constant c , converging weakly to a Radon measure λ . Then for every ball $B \subset \mathbb{R}^d$, we have:*

$$\lim_{j \rightarrow \infty} \left(\sup_{x \in B \cap \text{supp}(\lambda)} \text{dist}(x, \text{supp}\mu_j) \right) = 0$$

and

$$\lim_{j \rightarrow \infty} \left(\sup_{x \in B \cap \text{supp}(\mu_j)} \text{dist}(x, \text{supp}\lambda) \right) = 0.$$

As a corollary of Theorem 2.1.30 and Theorem 2.1.31, we get:

Corollary 2.1.32. *[[PTT]] Let μ be an n -uniform measure in \mathbb{R}^d , let Σ be its support and $x \in \Sigma$ a flat point. Then there exists $R > 0$ depending on x , n , d , μ and β such that $\Sigma \cap B(x, R)$ is a $C^{1,\beta}$ n -submanifold.*

Proof. It is clear from Theorem 2.1.30 that we only need to prove that every flat point of μ is in \mathcal{R} , namely that $\lim_{r \rightarrow 0} \theta(x, r) = 0$ for such an x . But taking μ_j to be μ_{x,r_j} and $\lambda = \mathcal{H}^n \llcorner V$, where V is the tangent plane at x , the result follows directly from Theorem 2.1.31. \square

2.2 Polar decomposition of a conical 3-uniform measure

We first study the case where ν is a conical 3-uniform measure. We start by proving that ν decomposes into a uniformly distributed spherical component and a radial component.

Let ν be a conical n -uniform measure in \mathbb{R}^d , with 0 in its support. Let Σ be the support of ν . In particular, since ν is conical, for any $r > 0$, we have by Theorem 2.1.17

$$\Sigma = r\Sigma.$$

By Theorem 2.1.19, normalizing ν if necessary,

$$\nu = \mathcal{H}^n \llcorner \Sigma.$$

Definition 2.2.1. *Let ν be a conical n -uniform measure in \mathbb{R}^d , with 0 in its support, Σ its support. We define σ to be the spherical component of ν , namely:*

$$\sigma = \mathcal{H}^{n-1} \llcorner (\Sigma \cap S^{d-1}),$$

where $S^{d-1} = \{x \in \mathbb{R}^d; |x| = 1\}$.

Definition 2.2.2. *Let $E \subset S^{d-1}$ and $\rho > 0$. We define ρE the dilate of E by:*

$$\rho E = \{y \in \mathbb{R}^d; \frac{y}{|y|} \in E, |y| = \rho\}.$$

We define E_r^δ , the (r, δ) -neighborhood of E , or δ -neighborhood of rE to be:

$$E_r^\delta = \left\{ y; \frac{y}{|y|} \in E, |y| \in (r(1 - \delta), r(1 + \delta)) \right\}. \quad (2.2.1)$$

Our first goal is to prove a polar decomposition for ν , namely that ν decomposes into its spherical component and a radial component.

Theorem 2.2.3. *Let ν be a conical n -uniform measure in \mathbb{R}^d . Let g be a Borel function on \mathbb{R}^d . Then:*

$$\int g(x) d\nu(x) = \int_0^\infty \rho^{n-1} \int g(\rho x') d\sigma(x') d\rho, \quad (2.2.2)$$

where $\rho = |x|$ and $x' = \frac{x}{|x|}$.

Proof. Let $u : \mathbb{R}^d \rightarrow \mathbb{R}_+$ be the function given by: $u(x) = |x|$. Our first aim is to prove that for any $g = \chi_A$ where $A \subset \mathbb{R}^d$ is a Borel set, we have:

$$\int g(x) d\nu(x) = \int_0^\infty \rho^{n-1} \int g(\rho x') d\sigma(x') d\rho. \quad (2.2.3)$$

Note that if A is a Borel set, $\frac{A}{\rho} \cap S^{d-1}$ being the intersection of the pre-image of A by the dilation homeomorphism, with the Borel set S^{d-1} , is also Borel.

Now, since u is Lipschitz (in fact smooth away from 0), and $J_\Sigma^* u = |\nabla^\Sigma u|$ we can apply the co-area formula (2.1.6) to the rectifiable set Σ , the Lipschitz function u and the Borel function χ_A to get:

$$\int_{A \cap \Sigma} |\nabla^\Sigma u|(y) d\mathcal{H}^n(y) = \int_0^\infty \int_{u^{-1}(\rho)} \chi_{A \cap \Sigma}(y) d\mathcal{H}^{n-1}(y) d\rho. \quad (2.2.4)$$

Note that by Theorem 2.1.30, away from a closed set \mathcal{S} of \mathcal{H}^n measure 0, Σ is a $C^{1,\alpha}$ submanifold of dimension n . Therefore, in (2.2.4), we can define $\nabla^\Sigma u$ to be the gradient in the manifold sense at almost every point.

We first claim that $|\nabla^\Sigma u|(x) = 1$ for almost every $x \in \Sigma$. Let x be a flat point of Σ (namely a point where ν admits a unique flat tangent). We can take $\tau_x = \frac{x}{|x|}$ to be an element of an orthonormal basis of P_x . Indeed, x being a flat point, by Corollary 2.1.32, Σ is a $C^{1,\beta}$ -manifold in a neighborhood of x , and the tangent space at x is P_x . Now consider the curve $\gamma(t) = t\tau_x + x$. Since $x \in \Sigma$ and ν is conical, $\gamma \in \Sigma$, $\gamma(0) = x$ and $\gamma'(0) = \tau_x$. Complete the unit vector $\tau_1 = \tau_x$ to a full orthonormal basis $\{\tau_i\}_{i=1}^n$ of \mathbb{R}^d . We have $\nabla u(x) = \tau_x$. Therefore: $\nabla^{\tau_x} u = \tau_x \cdot \tau_x = 1$ and $\nabla^{\tau_j} u = \tau_x \cdot \tau_j = 0$ for $j > 1$, by construction of the basis. Since almost every point of Σ is flat, this proves that $|\nabla^\Sigma u| = 1$ almost everywhere, proving the claim. Moreover, if $E \subset S^{d-1}$ is a Borel set, since ν is conical and $\Sigma = \frac{\Sigma}{\rho}$, we have:

$$\begin{aligned} \mathcal{H}^{n-1}(\rho E \cap \Sigma) &= \mathcal{H}^{n-1} \left(\rho \left(E \cap \frac{\Sigma}{\rho} \right) \right), \\ &= \rho^{n-1} \mathcal{H}^{n-1} \left(E \cap \frac{\Sigma}{\rho} \right), \\ &= \rho^{n-1} \mathcal{H}^{n-1} (E \cap \Sigma). \end{aligned} \quad (2.2.5)$$

Therefore,

$$\begin{aligned}
\int_{A \cap \Sigma} d\mathcal{H}^n(y) &= \int_{A \cap \Sigma} J^*u(y) d\mathcal{H}^n(y), \\
&= \int_0^\infty \int_{u^{-1}(\rho)} \chi_{A \cap \Sigma}(y) d\mathcal{H}^{n-1}(y) d\rho, \\
&= \int_0^\infty \mathcal{H}^{n-1}(\Sigma \cap A \cap \partial B_\rho) d\rho, \\
&= \int_0^\infty \rho^{n-1} \mathcal{H}^{n-1}\left(\frac{\Sigma}{\rho} \cap \frac{A}{\rho} \cap \partial B_1\right) d\rho, \\
&= \int_0^\infty \rho^{n-1} \mathcal{H}^{n-1}\left(\Sigma \cap \frac{A}{\rho} \cap \partial B_1\right) d\rho, \text{ since } \Sigma \text{ is conical,} \\
&= \int_0^\infty \rho^{n-1} \int \chi_A(\rho z') d\sigma(z') d\rho.
\end{aligned} \tag{2.2.6}$$

Now let g be a non-negative Borel function. Then there exists an increasing sequence of simple functions $\{g_k\}$ converging pointwise to g . In particular, if g_k increase to g pointwise, then G_k increase pointwise to G where $G_k(\rho) = \rho^{n-1} \int g_k(\rho x') d\sigma(x')$ and $G(\rho) = \rho^{n-1} \int g(\rho x') d\sigma(x')$. By the monotone convergence theorem and linearity of the integral, (2.2.2) also holds for non-negative Borel functions. The extension to general Borel functions follows easily. \square

Having proven that ν decomposes into two components, we now study the spherical component σ . By using the polar decomposition, we can prove that σ is uniformly distributed. Of particular interest to us is the case where ν is 3-uniform: the spherical component is then locally 2-uniform. We start with some notations. Denote $\Sigma \cap S^{d-1}$ by Ω . Then $\sigma = \mathcal{H}^{n-1} \llcorner \Omega$.

Let $B_r(x) = \{z \in S^{d-1}; |z - x| < r\}$, and $(B_r(x))_1^\epsilon$ be as in (2.2.1).

Theorem 2.2.4. *Let ν be as in Theorem 2.2.3. Then σ the spherical component of ν is a uniformly distributed measure.*

Proof. Let $x \in \Omega$, $r > 0$. Define the set $N_{\{|x|, r\}} \subset [0, \infty) \times \mathbb{R}$ to be:

$$N_{\{|x|, r\}} = \{(a, b) \in [0, \infty) \times \mathbb{R}; (1 + |x|^2 - r^2)a - 2b < 0\}.$$

Then:

$$\begin{aligned} z \in (B_r(x))_1^\epsilon &\iff \left| \frac{z}{|z|} - x \right|^2 < r^2 \text{ and } |z| \in (1 - \epsilon, 1 + \epsilon) \\ &\iff (1 + |x|^2 - r^2) |z| - 2 \langle z, x \rangle < 0 \text{ and } |z| \in (1 - \epsilon, 1 + \epsilon), \end{aligned}$$

allowing us to rewrite $g(z) = \chi_{(B_r(x))_1^\epsilon}(z)$ in the following way:

$$g(z) = \chi_{N_{\{|x|, r\}}(|z|, \langle z, x \rangle)} \cdot \chi_{(1-\epsilon, 1+\epsilon)}(|z|) = G(|z|, \langle z, x \rangle). \quad (2.2.7)$$

Since ν is a conical uniform measure and g is a function of $|z|$ and $\langle z, x \rangle$ with x in the support of ν , we can apply Theorem 2.1.17 to it. Namely, fix $e \in \mathbb{R}^n$, $|e| = 1$. Then by Theorem 2.1.17 and polar decomposition for Lebesgue measure:

$$\begin{aligned} \nu((B_r(x))_1^\epsilon) &= \int G(|z|, \langle z, x \rangle) d\nu(z) \\ &= \int_{\mathbb{R}^n} G(|z|, \langle z, e \rangle) d\mathcal{L}^n(z), \\ &= \int_0^\infty \left(\int_{\{|y|=\rho\}} \chi_{B_r(e)}(y') \chi_{(1-\epsilon, 1+\epsilon)}(|y|) d\mathcal{H}^{n-1}(y) \right) d\rho, \text{ where } y' = \frac{y}{|y|}, \\ &= \int_0^\infty \rho^{n-1} \int_{S^{n-1}} \chi_{B_r(e)}(y') \chi_{(1-\epsilon, 1+\epsilon)}(\rho) d\sigma^{n-1}(y') d\rho \quad (2.2.8) \\ &= \left(\int_{1-\epsilon}^{1+\epsilon} \rho^{n-1} d\rho \right) (\mathcal{H}^{n-1}(B_r(e) \cap S^{n-1})), \\ &= \frac{(1 + \epsilon)^n - (1 - \epsilon)^n}{n} (\mathcal{H}^{n-1}(B_r(e) \cap S^{n-1})). \end{aligned}$$

where \mathcal{L}^n is n -Lebesgue measure. Dividing by 2ϵ and letting ϵ go to 0 gives:

$$\mathcal{H}^{n-1}(B_r(e) \cap S^{n-1}) = \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \nu((B_r(x))_1^\epsilon). \quad (2.2.9)$$

Note that $\mathcal{H}^{n-1}(B_r(e) \cap S^{n-1})$ does not depend on our choice of e .

On the other hand, by Theorem 2.1.6 and (2.2.5), we get:

$$\begin{aligned}
\nu((B_r(x))_1^\epsilon) &= \int_0^\infty \left(\int_{\Sigma \cap \{|y|=\rho\}} \nu_{B_r(x)} \left(\frac{y}{|y|} \right) \nu_{(1-\epsilon, 1+\epsilon)}(|y|) d\mathcal{H}^{n-1}(y) \right) d\rho, \\
&= \int_{1-\epsilon}^{1+\epsilon} \mathcal{H}^{n-1}(\rho B_r(x) \cap \Sigma) d\rho, \\
&= \int_{1-\epsilon}^{1+\epsilon} \rho^{n-1} \mathcal{H}^{n-1}(B_r(x) \cap \Sigma) d\rho, \\
&= \left(\int_{1-\epsilon}^{1+\epsilon} \rho^{n-1} d\rho \right) (\mathcal{H}^{n-1}(B_r(x) \cap \Sigma)), \\
&= \frac{(1+\epsilon)^n - (1-\epsilon)^n}{n} (\mathcal{H}^{n-1}(B_r(x) \cap \Sigma)).
\end{aligned} \tag{2.2.10}$$

Dividing by 2ϵ and letting ϵ go to 0 gives:

$$\mathcal{H}^{n-1}(B_r(x) \cap \Sigma) = \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \nu((B_r(x))_1^\epsilon). \tag{2.2.11}$$

Combining (2.2.9) and (2.2.11), we get:

$$\sigma(B_r(x)) = \mathcal{H}^{n-1}(B_r(x) \cap \Omega) = \mathcal{H}^{n-1}(B_r(e) \cap S^{n-1}), \text{ for any } x \in \Omega, \text{ and any } e \in S^{n-1}. \tag{2.2.12}$$

In particular, this implies that σ is uniformly distributed. \square

One notable consequence of the above, expressed in the following corollary, is that for $n = 3$, the spherical component is in fact locally 2-uniform.

Corollary 2.2.5. *Suppose ν a 3-uniform conical measure on \mathbb{R}^d . Let σ be its spherical component, and denote the support of σ by Ω . Then there exists a function $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that, for all $x \in \Omega$, for all $r > 0$:*

$$\sigma(B(x, r)) = \phi(r). \tag{2.2.13}$$

Moreover,

$$\phi(r) = \pi r^2 \chi_{(0,2)}(r) + 4\pi \chi_{(2,\infty)}(r). \tag{2.2.14}$$

Proof. (2.2.13) is just a reformulation of Theorem 2.2.4. Let $e = (0, 0, 1)$. We only need to prove that for $r < 2$, we have:

$$\mathcal{H}^2(B_r(e) \cap S^2) = \pi r^2.$$

First, note that $\partial B_r(e) \cap S^2 = \{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 = 1, x^2 + y^2 + (z - 1)^2 = r^2\}$. If $r < \sqrt{2}$, $B_r(e) \cap S^2$ is the portion of the graph of $f(x, y) = \sqrt{1 - (x^2 + y^2)}$ above $z = 1 - \frac{r^2}{2}$.

So we have, by the area formula:

$$\begin{aligned} \mathcal{H}^2(B(e, r) \cap S^2) &= \int_0^{2\pi} \int_0^{\sqrt{1 - (1 - \frac{r^2}{2})^2}} \sqrt{1 + |\nabla f|^2} \rho d\rho d\theta \\ &= 2\pi \int_0^{\sqrt{1 - (1 - \frac{r^2}{2})^2}} \frac{\rho}{\sqrt{1 - \rho^2}} d\rho \\ &= -2\pi \left(\sqrt{1 - \left(1 - \left(1 - \frac{r^2}{2}\right)^2\right)} - 1 \right) \\ &= \pi r^2. \end{aligned}$$

If $\sqrt{2} < r < 2$, $B(e, r)$ and $B(0, 1)$ intersect in $z = 1 - \frac{r^2}{2}$. Moreover, note that the part of S^2 below the plane $z = 1 - \frac{r^2}{2}$ is $B(-e, r')$, where, by applications of Pythagoras' theorem, we have:

$$\begin{aligned} r'^2 &= 1 + \left(2 - \frac{r^2}{2}\right)^2 - \left(\frac{r^2}{2} - 1\right)^2, \\ &= 4 - r^2. \end{aligned}$$

Therefore, by symmetry (since $r' < \sqrt{2}$), we have:

$$\begin{aligned} \mathcal{H}^2(B(e, r) \cap S^2) &= \mathcal{H}^2(S^2) - \mathcal{H}^2(B(-e, r') \cap S^2), \\ &= 4\pi - \pi(4 - r^2), \\ &= \pi r^2. \end{aligned}$$

□

In [KiP], Kircheim and Preiss had proved that the support of a uniformly distributed measure is an analytic variety. We will now deduce from Corollary 2.2.5 that Σ is in fact an algebraic variety.

Recall from (2.1.17) that:

$$b_{2k}^k = k!b_{2k,1} \text{ and } b_{2k-1,1} = 0,$$

where

$$b_{k,1}(x) = 2^k(I(1)(k!))^{-1} \int \langle z, x \rangle^k e^{-|z|^2} d\mu(z) \text{ and } I(s) = \int e^{-s|z|^2} d\nu(z).$$

Applying Theorem 2.2.3 to $b_{k,1}$ and $I(1)$ gives:

$$\begin{aligned} b_{k,1}(x) &= 2^k(I(1)k!)^{-1} \int \langle z, x \rangle^k e^{-|z|^2} d\nu(z), \\ &= 2^k(I(1)k!)^{-1} \int_0^\infty \rho^{n-1} \int \langle \rho z', x \rangle^k e^{-\rho^2} d\sigma(z') d\rho, \\ &= 2^k(\sigma(S^{d-1})k!)^{-1} \frac{\int_0^\infty \rho^{n-1+k} e^{-\rho^2} d\rho}{\int_0^\infty \rho^{n-1} e^{-\rho^2} d\rho} \int \langle z', x \rangle^k d\sigma(z'), \\ &= c(n, k) \int \langle z', x \rangle^k d\sigma(z'), \end{aligned}$$

where $c(n, k) = 2^k(\sigma(S^{d-1})k!)^{-1} \frac{\int_0^\infty \rho^{n-1+k} e^{-\rho^2} d\rho}{\int_0^\infty \rho^{n-1} e^{-\rho^2} d\rho}$. Therefore: $b_{2k}^k(x) = k!c(n, k) \int \langle z', x \rangle^k d\sigma(z')$.

We can now improve Theorem 2.1.17 in the case of the spherical component σ of a conical n -uniform measure ν : indeed, in this case Ω is entirely described by its moments.

Theorem 2.2.6. *Let ν be a conical n -uniform measure, σ its spherical component, $p_{2k}(x) = b_{2k}^k(x)$ and $p_{2k+1}(x) = b_{2k+1,1}(x)$. Moreover, let Ω be the support of σ . Then*

$$p_{2k+1}(x) = 0 \text{ for } k \geq 0, x \in \mathbb{R}^d \tag{2.2.15}$$

and

$$\Omega = \{x; |x| = 1\} \cap \bigcap_{k>0} \{x; p_{2k}(x) = |x|^{2k}\}. \tag{2.2.16}$$

Proof. Call Ω' the right-hand side of (2.2.16). The fact that $\Omega \subset \Omega'$ and (2.2.15) follows from Theorem 2.1.17.

To prove the other inclusion, take any $x \in \mathbb{R}^d$ such that $x \in \Omega'$. Let us rewrite the expressions from Theorem 2.1.25. First, note that since $|x| = 1$:

$$\begin{aligned} \langle z - x, z - x \rangle^l &= (|z|^2 + |x|^2 - 2\langle z, x \rangle)^l \\ &= \sum_0^l \binom{l}{i} (-1)^i (|z|^2 + 1)^{l-i} 2^i \langle z, x \rangle^i. \end{aligned}$$

Moreover, using (2.2.15):

$$\begin{aligned} \int \langle z - x, z - x \rangle^l d\sigma(z) &= \sum_0^l \binom{l}{i} (-1)^i 2^i \int (|z|^2 + 1)^{l-i} \langle z, x \rangle^i d\sigma(z), \\ &= \sum_0^l \binom{l}{i} (-1)^i 2^l \int \langle z, x \rangle^i d\sigma(z), \\ &= \sum_{k:2k \leq l} \binom{l}{2k} 2^l \int \langle z, x \rangle^{2k} d\sigma(z), \\ &= \sum_{k:2k \leq l} \binom{l}{2k} 2^l (k!c(n, 2k))^{-1} p_{2k}(x) \\ &= \sum_{k:2k \leq l} \binom{l}{2k} 2^l k!c(n, 2k)^{-1}, \text{ since } p_{2k}(x) = |x|^{2k} = 1 \text{ by hypothesis.} \end{aligned}$$

Note that we have proved that for all $x \in \Omega'$, $\int \langle z - x, z - x \rangle^l d\sigma(z) = c_l$ where c_l does not depend on the choice of x . In particular, if $u \in \Omega$ is fixed as in Theorem 2.1.25, since $\Omega \subset \Omega'$, we have:

$$\int \langle z - x, z - x \rangle^l - \langle z - u, z - u \rangle^l d\sigma(z) = c_l - c_l = 0. \quad (2.2.17)$$

This proves that $x \in \Omega$ by Theorem 2.1.25. □

As an easy consequence of the above claim, we get:

Corollary 2.2.7. *Let ν be a conical n -uniform measure in \mathbb{R}^d and Σ its support. Then Σ is an algebraic variety and*

$$\Sigma = -\Sigma. \quad (2.2.18)$$

Proof. Let Σ' be $\bigcap_{k>0} \{b_{2k}^k(x) = |x|^{2k}\}$. By (2.1.18), $\Sigma \subset \Sigma'$. Now suppose $x \notin \Sigma$. Then $x \neq 0$ and $\frac{x}{|x|} \notin \Omega$, where Ω is the support of the spherical component of ν . By Theorem 2.2.6, there exists k_0 such that : $p_{2k_0}(\frac{x}{|x|}) \neq 1$.

Multiplying by $|x|^{2k_0}$, we get: $b_{2k}^k(x) \neq |x|^{2k_0}$, and hence $x \notin \Sigma'$.

By Theorem 2.1.17, for a conical measure, $b_{2k}^k(x) = b_{2k,1}(x)$ which is a homogeneous polynomial of even degree. In particular

$$\begin{aligned} x \in \Sigma &\iff b_{2k}^k(x) = |x|^{2k}, \quad k \in \mathbb{N} \\ &\iff b_{2k}^k(-x) = |-x|^{2k}, \quad k \in \mathbb{N} \\ &\iff -x \in \Sigma. \end{aligned}$$

Hence

$$\Sigma = -\Sigma. \tag{2.2.19}$$

□

We will apply Theorem 2.1.29 to deduce that Ω is a $C^{1,\beta}$ manifold for some $\beta > 0$. We then prove that there exists $\gamma > 0$ such that $\Sigma \setminus \{0\}$ is a $C^{1,\gamma}$ -submanifold of dimension 3 in \mathbb{R}^d .

Lemma 2.2.8. *Let ν be a 3-uniform conical measure in \mathbb{R}^d , σ its spherical component and Ω the support of σ . Then there exists $\beta > 0$ such that Ω is a $C^{1,\beta}$ submanifold of dimension 2 in \mathbb{R}^d .*

Proof. According to Theorem 2.1.29, we only need to prove (2.1.36) for σ . Let $r \in (0, 1]$ and $t \in [\frac{1}{2}, 1]$ so that $tr \in (0, 1]$. By Corollary 2.2.5, for any $x \in \Omega$, we have $\frac{\sigma(B(x,tr))}{\sigma(B(x,r))} = \frac{t^2 r^2}{r^2} = t^2$ implying that :

$$|R_t^2(x, r)| = 0, \text{ for } r \in (0, 1], t \in [\frac{1}{2}, 1], \text{ and } x \in \Omega.$$

□

Theorem 2.2.9. *Let ν be a 3-uniform conical measure in \mathbb{R}^d and Σ its support. Then there exists $\gamma > 0$ such that $\Sigma \setminus \{0\}$ is a $C^{1,\gamma}$ submanifold of dimension 3 .*

Proof. By dilation invariance of Σ , it is enough to prove that Σ is a $C^{1,\gamma}$ -manifold in a neighborhood of $x_0 \in \Omega$. Let σ be the spherical component of ν . By Lemma 2.2.8, Ω is a $C^{1,\beta}$ submanifold of dimension 2 in \mathbb{R}^d . In particular, fix $x_0 \in \Omega$. There exists a 2-subspace P_{x_0} of \mathbb{R}^d tangent to Ω at x_0 . Let $\{\tau_1, \tau_2\}$ be an orthonormal basis of P_{x_0} . Since $\Omega \subset S^{d-1}$, $T_{x_0}\Omega \subset T_{x_0}S^{d-1}$ and hence, $x_0 \perp \tau_i$, for $i = 1, 2$. Therefore, letting $\tau_3 = x_0$, we can complete the orthonormal set $\{\tau_i\}_{i=1}^3$ to an orthonormal basis $\{\tau_i\}_{i=1}^d$.

Since Ω is a $C^{1,\beta}$ submanifold of dimension 2, there exists a neighborhood U_0 of x_0 such that $\Omega \cap U_0$ can be written as a $C^{1,\beta}$ graph over P_{x_0} . More specifically, there exist $d - 2$ $C^{1,\beta}$ functions ψ_i on a neighborhood G of $(0, 0)$ in \mathbb{R}^2 such that $\psi_1(0, 0) = 1$, $\psi_i(0, 0) = 0$ for $i > 1$ and:

$$\Omega \cap U_0 = \left\{ x_1\tau_1 + x_2\tau_2 + \sum_{i=1}^{d-2} \psi_i(x_1, x_2)\tau_{i+2}; (x_1, x_2) \in G \right\} \quad (2.2.20)$$

Denote by $\Psi : G \rightarrow \Omega \cap U_0$ the $C^{1,\beta}$ diffeomorphism $\Psi(x_1, x_2) = x_1\tau_1 + x_2\tau_2 + \sum_{i=1}^{d-2} \psi_i(x_1, x_2)\tau_{i+2}$. Let $U = U_0 \cap S^{d-1}$, and $V = U_1^\epsilon$ (where U_1^ϵ is defined as in (2.2.1)) for some $\epsilon < 1$. V is an open neighborhood of x_0 and:

$$y \in \Sigma \cap V \iff y = \lambda y_0, \text{ where } y_0 \in \Omega \cap U_0, \lambda \in (1 - \epsilon, 1 + \epsilon), \quad (2.2.21)$$

$$\iff y = \lambda \Psi(x_1, x_2), \text{ where } \lambda \in (1 - \epsilon, 1 + \epsilon), (x_1, x_2) \in \mathbb{R}^2. \quad (2.2.22)$$

Letting $\Phi : G \times (1 - \epsilon, 1 + \epsilon) \rightarrow \mathbb{R}^d$ be defined as $\Phi((x_1, x_2), \lambda) = \lambda \Psi(x_1, x_2)$, we see that Φ is a C^1 diffeomorphism on $G \times (1 - \epsilon, 1 + \epsilon)$ and :

$$\Phi(G \times (1 - \epsilon, 1 + \epsilon)) = \Sigma \cap V.$$

Hence in the neighborhood of every non-zero point, Σ is a C^1 manifold. Consequently, every non-zero point of Σ is flat. Another application of Theorem 2.1.32 provides us with a $\gamma > 0$ such that Σ is a $C^{1,\gamma}$ submanifold of dimension 3 in a neighborhood of every non-zero point. \square

We obtain the following corollary as a consequence of Theorem 2.2.9.

Corollary 2.2.10. *Let μ be a 3-uniform measure in \mathbb{R}^d . If $x_0 \in \text{supp}(\mu)$, and $\text{Tan}(\mu, x_0) = \{c\nu, c > 0\}$, where ν is normalized so that $\nu(B(0, 1)) = \omega_n$, one of the following statements hold:*

- $\nu = \mathcal{H}^3 \llcorner V_{x_0}$ where V_{x_0} is a 3-dimensional subspace.
- The support of ν is not a plane, and for all $z_0 \in \text{supp}(\nu)$, $z_0 \neq 0$ we have

$$\text{Tan}(\nu, z_0) = \{c\mathcal{H}^3 \llcorner V_{z_0}, c > 0\}$$

where V_{z_0} is a 3-dimensional subspace.

Proof. By Theorem 2.1.18, if $x_0 \in \text{supp}(\mu)$, there exists a unique conical 3-uniform measure ν such that $\text{Tan}(\mu, x_0) = \{c\nu, c > 0\}$. But by Theorem 2.2.9, Theorem 2.1.18 and Corollary 2.1.19, $\nu = c\mathcal{H}^3 \llcorner (\text{supp}(\nu))$ where $\text{supp}(\nu)$ is a 3-dimensional subspace or a $C^{1,\alpha}$ -manifold away from 0. \square

2.3 Dimension reduction of singular sets

In this section, we will use the base case to deduce the Hausdorff dimension of the singular set of any n -uniform measure. We first prove a theorem about the convergence of the set of singularities of a sequence of blowups. Once this theorem is proven, we will have all the tools we need to apply a dimension reduction argument using the base case.

Let us start with some notations. The measure $\mu_{x,r}$ is defined as:

$$\mu_{x,r}(A) = \omega_n(\mu(B(x, r)))^{-1} \mu(rA + x), \quad (2.3.1)$$

for $A \subset \mathbb{R}^d$. In particular if μ is n -uniform and $z \in \text{supp}(\mu)$, then it follows from Theorem 2.1.18 that for any sequence $\eta_j \downarrow 0$

$$\mu_{z,\eta_j} \rightharpoonup \mu^z, \quad (2.3.2)$$

where μ^z is the normalized tangent measure at z as defined in Definition 2.1.20.

The following fact, which is a direct consequence of the definition of the functional F from Definition 2.1.22, will be used often in this section: if Φ is a flat measure, then $F(\Phi) = 0$ and $F(\Phi_{0,C}) = 0$ for any $C > 0$.

Recall Definition 2.1.12.

Definition 2.3.1. *Let μ be an n -uniform measure in \mathbb{R}^d . If x_0 is a non-flat point of $\text{supp}(\mu)$, we call it a singularity of μ . We denote by \mathcal{S}_μ the set of singularities of μ , namely:*

$$\mathcal{S}_\mu = \{x \in \text{supp}(\mu), x \text{ is not a flat point} \}.$$

We start with a lemma which states that under the appropriate conditions, blow-ups along the same sequence of points satisfy some sort of connectedness property.

Lemma 2.3.2. *Let μ be an n -uniform measure in \mathbb{R}^d , $(x_k)_k \subset \text{supp}(\mu)$ and $(\tau_k)_k, (\sigma_k)_k$ sequences of positive numbers decreasing to 0. We also assume that $\sigma_k < \tau_k$ and that there exist n -uniform measures α and β such that:*

$$\mu_{x_k, \tau_k} \rightharpoonup \alpha \text{ and } \mu_{x_k, \sigma_k} \rightharpoonup \beta.$$

Then:

$$F(\alpha) < \epsilon_0 \implies F(\beta) < \epsilon_0,$$

where F is the functional from Definition 2.1.22 and ϵ_0 is the constant from 2.1.24.

Proof. The proof of this lemma is similar to Preiss' proof of Theorem 2.6 in [P]. In particular, it follows closely the proof of Theorem 6.10 in [Del] which is a reformulation of Preiss' theorem for uniform measures.

Assume that $F(\alpha) < \epsilon_0$ and $F(\beta) \geq \epsilon_0$. Then by Theorem 2.1.23 there exists $0 < \kappa < \epsilon_0$ and $k_0 > 0$ so that for $k > k_0$,

$$F(\mu_{x_k, \sigma_k}) > \kappa \text{ and } F(\mu_{x_k, \tau_k}) < \kappa.$$

For $k > 0$, define the function $f_k : (0, \infty) \rightarrow (0, \infty)$ to be

$$f_k(r) = F(\mu_{x_k, r}).$$

If s_j is a sequence of positive numbers converging to some $s_0 > 0$, $\mu_{x_k, s_l} \rightharpoonup \mu_{x_k, s_0}$. Therefore f_k is continuous in r away from 0, for all $k > 0$. So for every $k > k_0$, there exists $\delta_k \in [\sigma_k, \tau_k]$ so that:

$$F(\mu_{x_k, \delta_k}) = \kappa \text{ and } F(\mu_{x_k, r}) \leq \kappa \text{ for } r \in [\delta_k, \tau_k]. \quad (2.3.3)$$

By Theorem 2.1.8, without loss of generality, by passing to a subsequence,

$$\mu_{x_k, \delta_k} \rightharpoonup \xi$$

where ξ is a Radon measure. We claim that ξ is n -uniform and $\xi(B(0, 1)) = \omega_n$. Pick $y \in \text{supp}(\xi)$ and $R > 0$. First note that since the μ_{x_k, δ_k} are n -uniform and all have the same constant ω_n (being normalized), we can apply Theorem 2.1.31 to obtain a sequence y_{k_j} of points in $\text{supp}(\mu_{x_{k_j}, \delta_{k_j}})$ such that $y_{k_j} \rightarrow y$. Without loss of generality, by passing to a subsequence, $y_j \rightarrow y$. Fix $\epsilon > 0$. There exists j_0 such that:

$$j > j_0 \implies |y - y_j| < \frac{\epsilon}{4}.$$

On one hand, we have:

$$\begin{aligned} \xi(B(y, R)) &\leq \liminf \mu_{x_j, \delta_j}(B(y, R)), \\ &\leq \liminf \mu_{x_j, \delta_j}(B(y_j, R + \frac{\epsilon}{4})), \\ &= \omega_n \left(R + \frac{\epsilon}{4} \right)^n. \end{aligned}$$

On the other hand

$$\begin{aligned} \xi(B(y, R)) &\geq \limsup \mu_{x_j, \delta_j}(B(y, R - \frac{\epsilon}{8})), \\ &\geq \limsup \mu_{x_j, \delta_j}(B(y_j, R - \frac{3\epsilon}{8})), \\ &= \omega_n \left(R - \frac{3\epsilon}{8} \right)^n. \end{aligned}$$

Hence, for $y \in \text{supp}(\xi)$ and $R > 0$

$$\omega_n \left(R - \frac{3\epsilon}{8} \right)^n \leq \xi(B(y, R)) \leq \omega_n \left(R + \frac{\epsilon}{4} \right)^n.$$

Since ϵ was chosen arbitrarily,

$$\xi(B(y, R)) = \omega_n R^n,$$

thus proving that ξ is n -uniform.

By Theorem 2.1.23, $F(\xi) = \kappa$. In particular ξ is not flat. We now show that our assumptions imply that ξ is flat at infinity. By Theorem 2.1.21 this is a contradiction with ξ not being flat.

We first claim that $\frac{\tau_k}{\delta_k} \rightarrow \infty$. Assume that $\frac{\tau_k}{\delta_k} \rightarrow C$, $C \geq 1$. Letting $\beta_k = \frac{\tau_k}{\delta_k}$ and writing $\mu_{x_k, \tau_k} = \beta_k^{-n} T_{0, \beta_k}[\mu_{x_k, \delta_k}]$, we have

$$\mu_{x_k, \tau_k} \rightharpoonup \xi_{0, C}$$

since $C \neq 0$, $\mu_{x_k, \delta_k} \rightharpoonup \xi$ and $\xi(B(0, C)) = \omega_n C^n$. But $\mu_{x_k, \tau_k} \rightharpoonup \alpha$ hence $\alpha = \xi_{0, C}$. The fact that α is flat and ξ is not would yield a contradiction.

Now fix $R > 1$. Since $\frac{\tau_k}{\delta_k} \rightarrow \infty$, there exists $k_1 > k_0$ such that for $k > k_1$, we have

$$R\delta_k \in [\delta_k, \tau_k].$$

In particular, since $k_1 > k_0$, if $k > k_1$ we also have, by (2.3.3), $F(\mu_{x_k, R\delta_k}) \leq \kappa$. We deduce that:

$$\limsup_k F(\mu_{x_k, R\delta_k}) \leq \kappa.$$

Since $F_s(\mu_{x_k, R\delta_k}, \xi_{0, R}) = R^{-n-1} F_{Rs}(\mu_{x_k, \delta_k}, \xi)$ for every $s > 0$, $\lim_{k \rightarrow \infty} F_s(\mu_{x_k, R\delta_k}, \xi_{0, R}) = 0$ for every $s > 0$ and hence $\mu_{x_k, R\delta_k} \rightharpoonup \xi_{0, R}$. Consequently, by Theorem 2.1.23:

$$F(\xi_{0, R}) \leq \kappa. \tag{2.3.4}$$

Choosing $R_l \uparrow \infty$, we have $\xi_{0, R_l} \rightharpoonup \psi$ where ψ is the normalized tangent of ξ at ∞ by Theorem 2.1.18. Therefore, by (2.3.4), $F(\psi) \leq \kappa < \epsilon_0$. But by Theorem 2.1.24 this implies that ξ is flat which contradicts $F(\xi) = \kappa$. \square

Remark 2.3.1. 1. I would like to thank Max Engelstein for discussions about the connectedness of cones of pseudo-tangent measures along a subsequence which brought this argument to my attention. It is worth noting that cones of pseudo-tangent sets

(the same holds for measures) are *not* connected in general. A counter-example can be found in Remark 5.5 in [BL]. The interested reader can refer to [P], [KPT] and [BL] for more detailed discussions of cones of tangent measures.

2. Lemma 2.3.2 would imply that pseudo-tangent measures are in fact connected along the same subsequence if not for the condition $\sigma_k < \tau_k$. This apparently technical condition turns out to be necessary for this proof to work. Whether this is just a feature of this specific proof or an actual necessary condition is not clear to the author and seems like an interesting question in its own right.

We can now prove a useful theorem about the behavior of singular sets under blow-ups.

Theorem 2.3.3. *Let μ be an n -uniform measure in \mathbb{R}^d , $x_0 \in \text{supp}(\mu)$, $(x_j)_j \subset \mathcal{S}_\mu$, $(r_j)_j$ any sequence of positive numbers decreasing to 0. Also assume that $y_j = \frac{x_j - x_0}{r_j} \in \overline{B(0, 1)}$, $y_j \rightarrow y$. Then*

$$y \in \mathcal{S}_{\mu^{x_0}},$$

where μ^{x_0} is the normalized tangent at x_0 as defined in Definition 2.1.20.

Proof. Without loss of generality, $x_0 = 0$. Denote μ^{x_0} by ν , and μ^{x_j} by ν_j where μ^{x_j} are the normalized tangents at x_j . Let us start with some remarks.

We first claim that there exists a conical, non-flat n -uniform measure ν^∞ such that a subsequence of $\{\nu_j\}_j$ converges weakly to ν^∞ . Since $x_j \in \mathcal{S}_\mu$, ν_j is non-flat for every $j > 0$. The fact that ν_j is conical and Theorem 2.1.24 then imply that $F(\nu_j) > \epsilon_0$ for all $j > 0$ where F is the functional defined in Definition 2.1.22. Moreover, since $\sup_j(\nu_j(B(0, R))) = \omega_n R^n < \infty$, there exists a Radon measure ν^∞ and a subsequence of ν_j converging to ν^∞ . Without loss of generality, $\nu_j \rightharpoonup \nu^\infty$. Moreover, ν^∞ is n -uniform. The proof of this fact is exactly the same as the proof of the fact that ξ in Lemma 2.3.2 is n -uniform.

By Theorem 2.1.23, $F(\nu_j) \rightarrow F(\nu^\infty)$ and hence $F(\nu^\infty) \geq \epsilon_0$. Moreover, since each ν_j is

conical and $\nu_j \rightharpoonup \nu^\infty$, it follows that for any $r > 0$:

$$\begin{aligned} T_{0,r}[\nu^\infty] &= \lim T_{0,r}[\nu_j], \\ &= r^n \lim \nu_j, \\ &= r^n \nu^\infty. \end{aligned}$$

This proves that ν^∞ is conical.

We also claim that

$$y \in \text{supp}(\nu) \text{ and } \mu_{x_j, r_j} \rightharpoonup \nu_y. \quad (2.3.5)$$

where ν_z denotes $T_{z,1}[\nu]$ whenever $z \in \text{supp}(\nu)$. Indeed, let $\delta > 0$. Then:

$$\begin{aligned} \nu(B(y, \delta)) &\geq \limsup_{i \rightarrow \infty} \mu_{0, r_i} \left(B \left(y, \frac{\delta}{4} \right) \right) \\ &= \limsup_{i \rightarrow \infty} \omega_n (\mu(B(0, r_i)))^{-1} \mu \left(B \left(r_i y, \frac{r_i \delta}{4} \right) \right). \end{aligned}$$

But for i large enough $|y - y_i| \leq \frac{\delta}{8}$ implying that $B(x_i, r_i \frac{\delta}{8}) \subset B(r_i y, \frac{r_i \delta}{4})$. Consequently,

$$\nu(B(y, \delta)) > 0$$

since

$$(\mu(B(0, r_i)))^{-1} \mu \left(B(x_i, r_i \frac{\delta}{8}) \right) = \frac{\delta^n}{8^n}.$$

Let us prove the second part of (2.3.5). Recall Definition 2.1.9.

Fix $R > 0$. Let $\phi \in \mathcal{L}(R)$. Then, on one hand, for j large enough that $|y_j| \leq 2$, we have:

$$\begin{aligned} \left| \int \phi(z) d\mu_{x_j, r_j}(z) - \int \phi(z) dT_{y_j, 1}[\nu](z) \right| &= \left| \int \phi(z - y_j) d\mu_{0, r_j}(z) - \int \phi(z - y_j) d\nu(z) \right|, \\ &\leq F_{R+2}(\mu_{0, r_j}, \nu), \end{aligned} \quad (2.3.6)$$

since $\phi_j(z) = \phi(z - y_j) \in \mathcal{L}(R + 2)$. On the other hand,

$$\begin{aligned} \left| \int \phi(z) dT_{y_j, 1}[\nu](z) - \int \phi(z) dT_{y, 1}[\nu](z) \right| &= \left| \int (\phi(z - y_j) - \phi(z - y)) d\nu(z) \right|, \\ &\leq |y - y_j| \nu(B(0, R + 2)), \end{aligned} \quad (2.3.7)$$

since $Lip(\phi) \leq 1$, ϕ_j and ϕ_y are supported in $B(0, R+2)$ where we define $\phi_y(z) = \phi(z-y)$. This gives, taking the supremum over all $\phi \in \mathcal{L}(R)$:

$$F_R(\mu_{x_j, r_j}, \nu_y) \leq F_{R+2}(\mu_{0, r_j}, \nu) + |y - y_j| \nu(B(0, R+2)),$$

for j large enough. Letting $j \rightarrow \infty$, we get (2.3.5) since R was chosen arbitrarily.

Our proof will now go as follows: we construct sequences of positive numbers σ_k and τ_k decreasing to 0 such that $\mu_{\tilde{x}_k, \sigma_k}$ converges weakly to ν^∞ and $\mu_{\tilde{x}_k, \tau_k}$ converges weakly to α the normalized tangent measure to ν at y . Here, \tilde{x}_k is a subsequence of x_k . We then use Lemma 2.3.2 to deduce that α cannot be flat.

Let us first construct a decreasing sequence $\tilde{\sigma}_j$ such that

$$\frac{\tilde{\sigma}_j}{r_j} \rightarrow 0 \text{ and } \mu_{x_j, \tilde{\sigma}_j} \rightarrow \nu^\infty. \quad (2.3.8)$$

Let $t_j = \frac{1}{j}$. By Theorem 2.1.18, the blow-ups at a point converge to the tangent along *any* sequence going to 0. Moreover this tangent is unique up to normalization. Thus, for every k , we have

$$\mathcal{F}(\mu_{x_k, \frac{1}{j}}, \nu_k) \rightarrow 0,$$

where \mathcal{F} is the metric on Radon measures from Definition 2.1.9. Now construct inductively a decreasing sequence $\{l_k\}_k$ such that, for all $k > 0$

$$l \geq l_k \implies t_l < r_k^2 \text{ and } \mathcal{F}(\mu_{x_k, t_l}, \nu_k) < \frac{1}{2^k}. \quad (2.3.9)$$

Let $\tilde{\sigma}_j = t_{l_j}$ and $\rho_j = \frac{\tilde{\sigma}_j}{r_j}$.

We remark that since $\rho_j \downarrow 0$,

$$(\nu_y)_{0, \rho_j} \rightarrow \alpha \quad (2.3.10)$$

where α is the normalized tangent measure to ν_y at 0. Equivalently, this is the normalized tangent measure to ν at y . Indeed, since $\nu_y = T_{y,1}[\nu]$ and $T_{0, \rho_j} \circ T_{y,1} = T_{y, \rho_j}$, we have

$$\begin{aligned} \rho_j^{-n} T_{0, \rho_j}[\nu_y] &= \rho_j^{-n} T_{0, \rho_j}[T_{y,1}[\nu]], \\ &= \rho_j^{-n} T_{y, \rho_j}[\nu]. \end{aligned}$$

We now construct a sequence $\tilde{\tau}_k$ such that:

$$\mu_{x_{l_k}, \tilde{\tau}_k} \rightharpoonup \alpha,$$

for some subsequence x_{l_k} of x_k .

For every k there exists $l_k > k$, $l_k > l_{k-1}$ such that whenever $l > l_k$

$$F_1(\mu_{x_l, r_l}, \nu_y) < \frac{1}{k} \rho_k^{n+1} \text{ and } \rho_l < \rho_k, \quad (2.3.11)$$

since $\mu_{x_l, r_l} \rightharpoonup \nu_y$ and $\rho_k \rightarrow 0$. Let $\tilde{\tau}_k = r_{l_k} \rho_k$ and $\tilde{x}_k = x_{l_k}$.

We claim that

$$\mu_{\tilde{x}_k, \tilde{\tau}_k} \rightharpoonup \alpha. \quad (2.3.12)$$

Indeed, fix $R > 0$. On one hand, for k large enough that $R\rho_k \leq 1$

$$\begin{aligned} F_R(\mu_{\tilde{x}_k, \tilde{\tau}_k}, \rho_k^{-n} T_{0, \rho_k}[\nu_y]) &= F_R(\rho_k^{-n} T_{0, \rho_k}[\mu_{x_{l_k}, r_{l_k}}], \rho_k^{-n} T_{0, \rho_k}[\nu_y]), \\ &= \rho_k^{-n-1} F_{R\rho_k}(\mu_{x_{l_k}, r_{l_k}}, \nu_y), \\ &\leq \rho_k^{-n-1} F_1(\mu_{x_{l_k}, r_{l_k}}, \nu_y), \\ &< \frac{1}{k}. \end{aligned}$$

The laws of composition used in this calculation are explained in Lemma 2.4 of [B].

On the other hand, $F_R(\rho_k^{-n} T_{0, \rho_k}[\nu_y], \alpha) \rightarrow 0$ by (2.3.10). Since

$$F_R(\mu_{\tilde{x}_k, \tilde{\tau}_k}, \alpha) \leq F_R(\mu_{\tilde{x}_k, \tilde{\tau}_k}, \rho_k^{-n} T_{0, \rho_k}[\nu_y]) + F_R(\rho_k^{-n} T_{0, \rho_k}[\nu_y], \alpha),$$

$F_R(\mu_{\tilde{x}_k, \tilde{\tau}_k}, \alpha) \rightarrow 0$. This proves (2.3.12).

Rename \tilde{x}_k , $\tilde{\sigma}_{l_k}$ and $\tilde{\tau}_k$ to be x_k , σ_k and τ_k . We have proven that:

$$\mu_{x_k, \sigma_k} \rightharpoonup \nu^\infty \text{ and } \mu_{x_k, \tau_k} \rightharpoonup \alpha,$$

with $\sigma_k < \tau_k$ (since $\rho_{l_k} < \rho_k$ by (2.3.11)) and ν^∞ conical and non-flat.

If α were flat, we would have $F(\alpha) = 0 < \epsilon_0$ and $F(\nu^\infty) \geq \epsilon_0$. This contradicts Lemma 2.3.2. Therefore α cannot be flat and $y \in \mathcal{S}_\nu$.

□

Remark 2.3.2. The proof of (2.3.12) is similar to the proof of Lemma 2.6 in [B].

We now use Theorem 2.3.2 to deduce two important corollaries.

Corollary 2.3.4. *Let μ be a 3-uniform measure in \mathbb{R}^d . Then the singular set of μ is discrete. Namely, for every K compact subset of \mathbb{R}^d , $|\mathcal{S}_\mu \cap K| < \infty$. Here, $|A|$ denotes the cardinality of the set $A \subset \mathbb{R}^d$.*

Proof. Assume not. Then there exists K compact subset of \mathbb{R}^d such that $|\mathcal{S}_\mu \cap K| = \infty$. In particular there exists a sequence of points $\{x_j\}_j \subset \mathcal{S}_\mu \cap K$ converging to some $x_\infty \in K$. Moreover, $x_\infty \in \text{supp}(\mu)$ since the support of a measure is a closed set. Let $r_j = |x_j - x_\infty|$ and $y_j = \frac{x_j - x_\infty}{r_j}$. Then by Theorem 2.1.18, $\mu_{x_\infty, r_j} \rightharpoonup \nu$, ν normalized tangent to μ at x_∞ and by compactness, we can assume by passing to a subsequence if necessary that $y_j \rightarrow y \in \partial B(0, 1)$. By (2.3.5), $y \in \text{supp}(\nu)$. Since $y \neq 0$, y must be a flat point of $\text{supp}(\nu)$ by Corollary 2.2.10. This contradicts Theorem 2.3.3. □

Corollary 2.3.5. *Let μ be an n -uniform measure in \mathbb{R}^d , $x_0 \in \text{supp}(\mu)$, ν normalized tangent to μ at x_0 , $\{r_j\}_j$ sequence of positive radii decreasing to 0, $\epsilon > 0$. Then there exists $N = N(\epsilon)$ such that:*

$$n \geq N \implies \frac{\mathcal{S}_\mu - x_0}{r_j} \cap \overline{B(0, 1)} \subset (\mathcal{S}_\nu)_\epsilon \quad (2.3.13)$$

where $(\mathcal{S}_\nu)_\epsilon = \{y \in \mathbb{R}^d; \text{dist}(y, \mathcal{S}_\nu) < \epsilon\}$.

Proof. Suppose not. Then we can construct a subsequence $\{s_j\}$ of $\{r_j\}$ so that:

$$\frac{\mathcal{S}_\mu - x_0}{s_j} \cap \overline{B(0, 1)} \cap (\mathbb{R}^d \setminus (\mathcal{S}_\nu)_\epsilon) \neq \emptyset.$$

Consequently, we can find points $x_j \in \mathcal{S}_\mu$ such that $y_j = \frac{x_j - x_0}{s_j} \in \overline{B(0, 1)}$, $\text{dist}(y_j, \mathcal{S}_\nu) \geq \epsilon$ and $y_j \rightarrow y$. In particular, $y \notin \mathcal{S}_\nu$ since $d(y, \mathcal{S}_\nu) \geq \epsilon$. This contradicts Theorem 2.3.3. □

We are now ready to prove the main theorem of this section.

Theorem 2.3.6. *Let μ be an n -uniform measure in \mathbb{R}^d , $3 \leq n \leq d$ and denote the support of μ by Σ . Then Σ can be written as a disjoint union*

$$\Sigma = \mathcal{R}_\mu \cup \mathcal{S}_\mu \quad (2.3.14)$$

where \mathcal{S}_μ the singular set is a closed set,

$$\dim_{\mathcal{H}}(\mathcal{S}_\mu) \leq n - 3, \quad (2.3.15)$$

and \mathcal{R}_μ is a $C^{1,\alpha}$ submanifold of dimension n in \mathbb{R}^d . Here $\dim_{\mathcal{H}}$ denotes the Hausdorff dimension.

Proof. We start by proving that \mathcal{S}_μ is a closed set and \mathcal{R}_μ is a $C^{1,\alpha}$ submanifold. This is a direct consequence of Theorem 2.1.30. Indeed, in the case where μ is n -uniform, we claim that the condition $\limsup_{r \rightarrow 0} \theta(x, r) = 0$ is equivalent to x is a flat point. Rewrite $\theta(x, r)$ in the following way:

$$\begin{aligned} \theta(x, r) &= \frac{1}{r} \inf \left\{ D \left[\Sigma \cap \overline{B(x, r)}, L \cap \overline{B(x, r)} \right] : L \text{ affine } n\text{-plane through } x \right\}, \\ &= \frac{1}{r} \inf \left\{ D \left[\frac{\Sigma - x}{r} \cap \overline{B(0, 1)}, L \cap \overline{B(0, 1)} \right] : L \text{ } n\text{-plane through } 0 \right\}. \end{aligned}$$

This combined with the fact that there is a unique L_x for every x flat such that $Tan(\mu, x) = \{c\mathcal{H}^n \llcorner L_x ; c > 0\}$ ends the proof of the claim.

We now prove our main claim, namely that

$$\dim_{\mathcal{H}}(\mathcal{S}_\mu) \leq n - 3, \quad (2.3.16)$$

for μ n -uniform.

(2.3.16) holds for $n = 3$ by Theorem 2.2.9. Let $m < d$ and assume (2.3.16) holds for all l -uniform measures in \mathbb{R}^d such that $l < m$. We want to prove that it holds for m -uniform measures.

Let μ be an m -uniform measure. Suppose that $s \in \mathbb{R}_+$ is such that $\mathcal{H}^s(\mathcal{S}_\mu) > 0$.

We start with an overview of the proof. We first prove that there exists a singular point x_0 of the support of μ and a tangent ν of μ at x_0 such that the following holds

$$\mathcal{H}^s(\mathcal{S}_\nu \cap \overline{B(0,1)}) > 0.$$

From that, we deduce that there exists some non-zero singular point ξ of the support of ν such that the tangent λ to ν at ξ satisfies $\mathcal{H}^s(\mathcal{S}_\lambda \cap \overline{B(0,1)}) > 0$. Note that by Theorem 2.1.18, since μ is m -uniform, ν is conical. The advantage of repeating this procedure is that since ν is conical, the support of λ is in fact translation invariant along the vector ξ and so λ can be decomposed into $\mathcal{L}^1 \times \lambda_0$ where \mathcal{L}^1 is 1-Lebesgue measure and λ_0 is $(m-1)$ -uniform. We then apply the induction hypothesis to λ_0 to finish the proof.

We find a singular point x_0 of the support of μ such that the following holds. Let ν be the normalized tangent to μ at x_0 . Then:

$$\mathcal{H}^s(\mathcal{S}_\nu \cap \overline{B(0,1)}) > 0.$$

By Lemma 4.6 in [Mat2],

$$\mathcal{H}^s(\mathcal{S}_\mu) > 0 \iff \mathcal{H}_\infty^s(\mathcal{S}_\mu) > 0.$$

We will use \mathcal{H}_∞^s instead of \mathcal{H}^s to allow a larger choice of coverings. Since $\mathcal{H}_\infty^s(\mathcal{S}_\mu) > 0$, there exists a compact set K such that $\mathcal{H}_\infty^s(\mathcal{S}_\mu \cap K) > 0$. Let $\tilde{\mathcal{S}}_\mu = \mathcal{S}_\mu \cap K$. We have

$$\theta^{s,*}(\mathcal{H}_\infty^s \llcorner \tilde{\mathcal{S}}_\mu, z) \geq 2^{-s}, \quad (2.3.17)$$

for \mathcal{H}^s -almost every $z \in \tilde{\mathcal{S}}_\mu$. This follows from Theorem 3.26 (2), in [S] since $\tilde{\mathcal{S}}_\mu$ is a compact subset of \mathbb{R}^d . In particular, there exists $x_0 \in \tilde{\mathcal{S}}_\mu$ such that:

$$\theta^{s,*}(\mathcal{H}_\infty^s \llcorner \tilde{\mathcal{S}}_\mu, x_0) \geq 2^{-s}, \quad (2.3.18)$$

Consequently, there exists a sequence of radii $\{r_j\}_j$ decreasing to 0 such that:

$$\mathcal{H}_\infty^s \left(\overline{B(0,1)} \cap \frac{\tilde{\mathcal{S}}_\mu - x_0}{r_j} \right) \geq 2^{-s}.$$

Since $r_j \downarrow 0$, $\mu_{x_0, r_j} \rightarrow \nu$ where $\nu = \mu^{x_0}$, the normalized tangent to μ at x_0 . By Theorem 2.3.5, for all $\epsilon > 0$, there exists j_0 such that:

$$\frac{\tilde{\mathcal{S}}_\mu - x_0}{r_j} \cap \overline{B(0, 1)} \subset (\mathcal{S}_\nu)_\epsilon \text{ whenever } j \geq j_0. \quad (2.3.19)$$

Pick $\delta > 0$ and let $\{E_k\}_k$ be a covering of $\tilde{\mathcal{S}}_\nu = \mathcal{S}_\nu \cap \overline{B(0, 1)}$ such that:

$$\mathcal{H}_\infty^s(\tilde{\mathcal{S}}_\nu) > \omega_s 2^{-s} \sum_{k=1}^{\infty} (\text{diam}(E_k))^s - \delta.$$

We can assume that the sets E_k are open (see Theorem 4.4 in [Mat2]). Since $\bigcup E_k$ is open, $\tilde{\mathcal{S}}_\nu$ is compact and $\tilde{\mathcal{S}}_\nu \subset \bigcup E_k$, we can cover $\tilde{\mathcal{S}}_\nu$ with finitely many E_k , $k = 1, \dots, K$. Letting E be the union of this finite cover and ϵ be a number smaller than the minimum of the diameters of the E_k 's in this finite cover, we have:

$$(\mathcal{S}_\nu)_\epsilon \subset E.$$

It follows from (2.3.19) that for j large enough, we have

$$\mathcal{S}_j \subset E,$$

where $\mathcal{S}_j = \frac{\tilde{\mathcal{S}}_\mu - x_0}{r_j} \cap \overline{B(0, 1)}$. Hence, for j large, since $\{E_k\}_{k=1}^K$ covers \mathcal{S}_j

$$\begin{aligned} \mathcal{H}_\infty^s(\mathcal{S}_j) &\leq \omega_s 2^{-s} \sum_{k=1}^K (\text{diam}(E_k))^s, \\ &\leq \mathcal{H}_\infty^s(\tilde{\mathcal{S}}_\nu) + \delta. \end{aligned}$$

Since δ was chosen arbitrarily, we get $\mathcal{H}_\infty^s(\mathcal{S}_j) \leq \mathcal{H}_\infty^s(\tilde{\mathcal{S}}_\nu)$. Letting $j \rightarrow \infty$, we get:

$$2^{-s} \leq \limsup \mathcal{H}_\infty^s(\mathcal{S}_j) \leq \mathcal{H}_\infty^s(\tilde{\mathcal{S}}_\nu).$$

This gives $\mathcal{H}_\infty^s(\mathcal{S}_\nu) \geq \mathcal{H}_\infty^s(\tilde{\mathcal{S}}_\nu) > 0$. The claim is thus proved.

The advantage of ν over μ is that it is conical. Thus if we blow up at a non-zero point of ν , we claim that we obtain a measure that is translation invariant. Since $\mathcal{H}^s(\mathcal{S}_\nu \cap \overline{B(0, 1)}) > 0$, by the same reasoning as for μ , there exists $\xi \neq 0$, $\xi \in \mathcal{S}_\nu \cap \overline{B(0, 1)}$ such that :

$$\theta^{s,*}(\mathcal{H}_\infty^s \llcorner \mathcal{S}_\nu, \xi) \geq 2^{-s}.$$

In particular, there exists a decreasing sequence $\{s_j\}$ such that $\mathcal{H}_\infty^s(\mathcal{S}_\nu \cap \overline{B(\xi, s_j)}) \geq 2^{-s} s_j^s$ and $\nu_{\xi, s_j} \rightharpoonup \lambda$, where $\lambda = \nu^\xi$ is the normalized tangent measure to ν at ξ . Since ν is uniform and ξ is a singular point, λ is a non-flat conical measure. The same procedure as above gives:

$$\mathcal{H}^s(\mathcal{S}_\lambda \cap \overline{B(0, 1)}) > 0. \quad (2.3.20)$$

Let $\Sigma = \text{supp}(\lambda)$. We claim that

$$\Sigma = \mathbb{R}e_1 \oplus A$$

for some A subset of a $(d-1)$ -plane of \mathbb{R}^d such that $\mathcal{H}^{m-1} \llcorner A$ is $(m-1)$ -uniform. We will first prove that

$$T_{t\xi, 1}[\lambda] = \lambda \quad (2.3.21)$$

for any $t > 0$.

Take $t > 0$. Then, on one hand, noting that for $z \in \mathbb{R}^d$:

$$\begin{aligned} T_{(1+t)\xi, s_j}(z) &= \frac{z - \xi - t\xi}{s_j}, \\ &= \frac{\frac{z}{1+t} - \xi}{\frac{s_j}{1+t}}, \\ &= T_{\xi, \frac{s_j}{1+t}} \circ T_{0, 1+t}(z), \end{aligned}$$

we get

$$\begin{aligned} \nu_{(1+t)\xi, s_j} &= s_j^{-m} T_{\xi, \frac{s_j}{1+t}}[T_{0, 1+t}[\nu]], \\ &= s_j^{-m} (1+t)^m T_{\xi, \frac{s_j}{1+t}}[\nu], \text{ since } \nu \text{ is conical} \\ &\rightharpoonup \lambda, \end{aligned} \quad (2.3.22)$$

since the sequence $\frac{s_j}{1+t} \rightarrow 0$ and $s_j^{-m} (1+t)^m T_{\xi, \frac{s_j}{1+t}}[\nu](B(0, 1)) = \lambda(B(0, 1)) = \omega_m$.

On the other hand, we have

$$\begin{aligned} T_{(1+t)\xi, s_j}(z) &= \frac{z - (1+t)\xi}{s_j}, \\ &= \frac{z - (1 + (1 - s_j)t)\xi}{s_j} - t\xi, \\ &= T_{t\xi, 1} \circ T_{(1+(1-s_j)t)\xi, s_j}(z). \end{aligned}$$

We now prove that

$$s_j^{-m} T_{(1+(1-s_j)t)\xi, s_j}[\nu] \rightarrow \lambda. \quad (2.3.23)$$

Let $\phi \in \mathcal{L}(R)$. Then, for j large enough so that $|1 - s_j| \leq 2$ we have:

$$\begin{aligned} & s_j^{-m} \left| \int \phi(z) dT_{(1+(1-s_j)t)\xi, s_j}[\nu](z) - \int \phi(z) dT_{(1+t)\xi, s_j}[\nu](z) \right| \\ & \leq s_j^{-m} \left| \int (\phi(z - (1 + (1 - s_j)t)\xi) - \phi(z - (1 + t)\xi)) dT_{0, s_j}[\nu](z) \right|, \\ & \leq s_j^{-m} \int_{B(0, R+(1+2|t|)|\xi|)} |s_j| |\xi| |t| dT_{0, s_j}[\nu](z), \\ & \leq |s_j| |\xi| |t| \omega_m(R + (1 + 2|t|)|\xi|)^m. \end{aligned}$$

Taking the supremum over all $\phi \in \mathcal{L}(R)$, we get:

$$\begin{aligned} A_j & := F_R(s_j^{-m} T_{(1+(1-s_j)t)\xi, s_j}[\nu], s_j^{-m} T_{(1+t)\xi, s_j}[\nu]), \\ & \leq |s_j| |\xi| |t| \omega_m(R + (1 + 2|t|)|\xi|)^m, \end{aligned} \quad (2.3.24)$$

which goes to 0 as $j \rightarrow \infty$ since $s_j \rightarrow 0$. We have

$$F_R(s_j^{-m} T_{(1+(1-s_j)t)\xi, s_j}[\nu], \lambda) \leq A_j + F_R(s_j^{-m} T_{(1+t)\xi, s_j}[\nu], \lambda). \quad (2.3.25)$$

Since $A_j \rightarrow 0$ by (2.3.24) and, according to (2.3.22), $F_R(s_j^{-m} T_{(1+t)\xi, s_j}[\nu], \lambda) \rightarrow 0$, by using (2.3.25), we prove (2.3.23).

This proves (2.3.21) from which it follows that

$$\Sigma - t\xi = \Sigma \text{ for } t > 0. \quad (2.3.26)$$

Indeed, for $t > 0$,

$$\begin{aligned} z \in \Sigma & \iff \text{For all } r > 0, \lambda(B(z, r)) > 0, \\ & \iff \text{For all } r > 0, T_{t\xi, 1}[\lambda](B(z, r)) > 0, \\ & \iff \text{For all } r > 0, \lambda(B(z + t\xi, r)) > 0, \\ & \iff z \in \Sigma - t\xi. \end{aligned}$$

Adding $t\xi$ on both sides of 2.3.26, we see that

$$\Sigma - t\xi = \Sigma \text{ for } t \in \mathbb{R}. \quad (2.3.27)$$

Let $e_1 = \frac{\xi}{|\xi|}$ and $A = \{x \in \Sigma; x \cdot e_1 = 0\}$. We claim that

$$\Sigma = \mathbb{R}e_1 \oplus A. \quad (2.3.28)$$

On one hand, if $z \in \mathbb{R}e_1 \oplus A$, then there exists $z' \in A$ and $t \in \mathbb{R}$ such that:

$$z = z' + te_1.$$

Since $A \subset \Sigma$ by definition, this implies that $z \in \Sigma + te_1$ and consequently, $z \in \Sigma$ by (2.3.26).

On the other hand, if $z \in \Sigma$, we can write:

$$z = (z - \langle z, e_1 \rangle e_1) + \langle z, e_1 \rangle e_1.$$

Let $t_1 = \langle z, e_1 \rangle$. By (2.3.26), $z - t_1 e_1 \in \Sigma$. Moreover, $\langle z - t_1 e_1, e_1 \rangle = 0$. Therefore, $z - t_1 e_1 \in A$ and $z \in \mathbb{R}e_1 + A$. The uniqueness of such a decomposition follows from the fact that $\mathbb{R}e_1$ and A are orthogonal by construction. This proves (2.3.28).

So there exists $c > 0$ so that $\lambda = c\omega_m^{-1}\mathcal{H}^{m-1}\llcorner(\mathbb{R}e_1 \oplus A)$ by Corollary 2.1.19. By Theorem 3.11 in [KoP], $\lambda_0 = \mathcal{H}^{m-1}\llcorner A$ is $(m-1)$ -uniform.

The final step consists in proving that

$$\mathcal{S}_\lambda \subset \mathbb{R}e_1 \oplus \mathcal{S}_{\lambda_0} \cong \mathbb{R} \times \mathcal{S}_{\lambda_0}. \quad (2.3.29)$$

We start by proving that if $y \in A$ is a $(m-1)$ -flat point of λ_0 , and $t \in \mathbb{R}$, then $te_1 + y \in \Sigma$ is an m -flat point of λ . By Theorem 2.1.30, if y is a flat point of λ_0 , since λ_0 is an $(m-1)$ -uniform measure, there exists a neighborhood U' of y in \mathbb{R}^{d-1} (here \mathbb{R}^{d-1} is identified with the set $\{z \in \mathbb{R}^d; \langle z, e_1 \rangle = 0\}$) such that $A \cap U'$ is a C^1 manifold. More precisely, there exists $(d-m+1)$ C^1 -diffeomorphisms $\{\psi_j\}_j$ from a neighborhood G of \mathbb{R}^{m-1} to \mathbb{R} such that:

$$U' \cap A = \left\{ z_2 e_2 + \dots + z_m e_m + \sum_{i=m+1}^d \psi_j(z_2, \dots, z_m) e_j; (z_1, \dots, z_m) \in G \right\} \quad (2.3.30)$$

where $\{e_j\}_{j=2}^m$ is an orthonormal basis of the tangent plane to $\text{supp}(\lambda_0)$ at y and $\{e_j\}_{j=1}^d$ is a completion of $\{e_j\}_{j=1}^m$ to an orthonormal basis of \mathbb{R}^d . We claim that Σ is a C^1 -manifold in the neighborhood $U = \{se_1 + z'; (s, z) \in (t-1, t+1) \times U'\}$ of $te_1 + y$. Indeed, if $z \in \Sigma \cap U$, then by (2.3.26) and (2.3.30) we can write

$$z = z_1 e_1 + z_2 e_2 + \dots + z_m e_m + \sum_{i=m+1}^d \psi_j(z_2, \dots, z_m) e_j, \quad (2.3.31)$$

where $z_j = \langle z, e_j \rangle$ for $j > 1$ and $z_1 \in (r-1, r+1)$.

We go back to the proof of (2.3.29). Suppose that $\eta \in \mathcal{S}_\lambda$. Then in particular, $\eta \in \Sigma$ and hence $\eta = te_1 + y$ where $t \in \mathbb{R}$, $y \in A$. If y were a flat point of λ_0 , then η would be a flat point of λ . Therefore, $\eta \in \mathbb{R} \times \mathcal{S}_{\lambda_0}$.

We deduce from (2.3.29) that

$$\dim_{\mathcal{H}}(\mathcal{S}_\lambda) \leq \dim_{\mathcal{H}}(\mathcal{S}_{\lambda_0}) + 1. \quad (2.3.32)$$

Note that this inequality holds because \mathcal{S}_λ and \mathcal{S}_{λ_0} are Borel sets and the packing dimension of a line is the same as its Hausdorff dimension. But since $\mathcal{H}^s(\mathcal{S}_\lambda) > 0$,

$$\dim_{\mathcal{H}}(\mathcal{S}_\lambda) \geq s. \quad (2.3.33)$$

Combining (2.3.32) and (2.3.33), we get:

$$s - 1 \leq \dim_{\mathcal{H}}(\mathcal{S}_{\lambda_0}). \quad (2.3.34)$$

On the other hand, \mathcal{S}_{λ_0} being the singular set of an $(m-1)$ -uniform measure, the induction hypothesis implies that $\dim_{\mathcal{H}}(\mathcal{S}_{\lambda_0}) \leq m-4$. Therefore $s \leq m-3$.

We have proven that

$$\mathcal{H}^s(\mathcal{S}_\mu) > 0 \implies s \leq m-3.$$

Therefore,

$$\dim_{\mathcal{H}}(\mathcal{S}_\mu) \leq m-3.$$

□

Chapter 3

CONICAL 3-UNIFORM MEASURES: CHARACTERIZATION & NEW EXAMPLES

In this chapter, we provide a characterization of 3-uniform conical measures in \mathbb{R}^d and describe an infinite family of non-isometric 3-uniform measures. We start by introducing some definitions in order to give precise statements of our results. We say a Radon measure μ in \mathbb{R}^d is uniformly distributed if there exists a real-valued function ϕ so that for every $x \in \text{supp}(\mu)$, and every $r > 0$

$$\mu(B(x, r)) = \phi(r).$$

If there exists $c > 0$ so that

$$\phi(r) = cr^n,$$

we call μ an n -uniform measure. Some obvious examples of n -uniform measures are n -flat measures, i.e. n -Hausdorff measure restricted to an affine n -plane. Indeed, if V is an affine n -plane then for all $x \in V$ and $r > 0$, we have:

$$\mathcal{H}^n(B(x, r) \cap V) = \omega_n r^n,$$

where ω_n denotes the volume of the n -dimensional unit ball. In fact, Preiss proved in [P] that for $n = 1, 2$, the only n -uniform measures in \mathbb{R}^d are the n -flat ones.

In [P], Preiss showed that there exist non-flat n -uniform measures in \mathbb{R}^{n+1} . Moreover, in [KoP], he proved in collaboration with Kowalski that in codimension 1, this measure and flat measures are the only examples of n -uniform measures.

Theorem 3.0.1. *[KoP] Let C be the cone in \mathbb{R}^4 defined by:*

$$C = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4; x_4^2 = x_1^2 + x_2^2 + x_3^2\}. \quad (3.0.1)$$

Then :

- $\mathcal{H}^3 \llcorner C$ is 3-uniform and for all $x \in C$, for all $r > 0$,

$$\mathcal{H}^3(B(x, r) \cap C) = \frac{4}{3}\pi r^3. \quad (3.0.2)$$

- If μ is an n -uniform measure in \mathbb{R}^{n+1} , then either μ is n -flat or, up to isometry, we have:

$$\mu = c\mathcal{H}^n \llcorner (C \times \mathbb{R}^{n-3}). \quad (3.0.3)$$

In higher codimension, there is no such classification result. However, in [KiP], Kirchheim and Preiss proved that the support of an n -uniform measure in any codimension is an analytic variety.

Theorem 3.0.2. (1.4, [KiP]) *Let μ be a uniformly distributed measure over \mathbb{R}^d . Then $\text{supp}(\mu)$ is an analytic variety and there exists an integer $n \in \{0, 1, \dots, d\}$, a constant $c \in (0, \infty)$ and an open subset G of \mathbb{R}^d such that:*

1. $G \cap \text{supp}(\mu)$ is an n -dimensional analytic submanifold of \mathbb{R}^d .
2. $\mathbb{R}^d \setminus G$ is the union of countably many analytic submanifolds of \mathbb{R}^d of dimensions less than n and $\mu(\mathbb{R}^d \setminus G) = \mathcal{H}^n(\mathbb{R}^d \setminus G) = 0$,
3. $\mu(A) = c\mathcal{H}^n(A \cap G \cap \text{supp}(\mu)) = c\mathcal{H}^n(A \cap \text{supp}(\mu))$ for every subset $A \subset \mathbb{R}^d$.

We denote $G \cap \text{supp}(\mu)$ by \mathcal{R} and $\text{supp}(\mu) \setminus G$ by \mathcal{S} and write:

$$\text{supp}(\mu) = \mathcal{R} \cup \mathcal{S}.$$

The only other information on n -uniform measures that appeared since was proved by X. Tolsa. In [T], he proved that n -uniform measures are uniformly rectifiable, a quantified notion of rectifiability.

In this chapter, we show that there are other non-isometric examples of 3-uniform measures in any co-dimension: more specifically, we construct a family of 3-uniform measures

of which $\mathcal{H}^3 \llcorner C$ is a particular example. Moreover, we give a geometric and graph-theoretic characterization of conical 3-uniform measures.

Section 1 consists of preliminaries. It is divided into two subsections, one dealing with results of a geometric and analytic nature and the other with results from discrete mathematics.

We call an n -uniform measure ν conical if for every $A \subset \mathbb{R}^d$, for every $r > 0$, it satisfies

$$\nu(rA) = r^n \nu(A). \quad (3.0.4)$$

Conical measures are of particular interest as they appear as tangent measures of n -uniform measures. One interesting feature of a 3-uniform conical measure is that its spherical component σ is a locally 2-uniform measure (See Theorem 2.2.4). We use this to get a better description of σ . Theorem 3.0.2 says that almost every point of the support of an n -uniform measure is smooth. With this in mind, in [KoP], Kowalski and Preiss start by considering a locally n -uniform measure with smooth support M . Fixing a point x in its support and using the area formula, they write a Taylor expansion for the measure of $B(x, r)$, in terms of r . By equating this expansion with $\omega_n r^n$, they are then able to obtain equations on the curvature of M and its derivatives. In particular they prove that in the case where $n = 2$, the ambient space is \mathbb{R}^3 , and the manifold M is connected, M has to be a piece of a 2-plane or of a 2-sphere. Thus looking at configurations of 2-spheres seems like the natural approach to the problem of describing locally 2-uniform measures. In Section 2, we carry out a similar argument on σ , the spherical component of ν , where the ambient space is \mathbb{R}^d , $d > 3$, to deduce that it is an umbilic manifold. More precisely, we prove that its support is a finite union of disjoint 2-spheres (see Theorem 3.2.1).

In Section 3, we study the configuration of these spheres. Indeed, the fact that σ is locally 2-uniform implies a certain rigidity. In Theorem 3.3.5, we find a sufficient condition for a configuration of 2-spheres in \mathbb{R}^d to be the support of a locally 2-uniform measure. They must have the same radius and be contained in translations of the same linear 3-plane. Moreover, their centers have to be in a specific position: we say they are r -layered (see Definition 3.3.4).

In Theorem 3.3.6, we show that when $d = 5$, the only possible conical 3-uniform measure which is neither flat nor the cone from (1.1.4) is given, up to isometry and normalization, by the following equation:

$$\nu = \mathcal{H}^3 \llcorner (C_1 \cup C_2), \quad (3.0.5)$$

where

$$C_1 = \{x ; x_4 = 0\} \cap \{x ; 3(x_1^2 + x_2^2 + x_3^2) = x_5^2\}, \quad (3.0.6)$$

and

$$C_2 = \{x ; x_4 = 2\sqrt{2}x_5\} \cap \{x ; 3(x_1^2 + x_2^2 + x_3^2) = x_4^2 + x_5^2\}. \quad (3.0.7)$$

We now know that to produce a conical 3– uniform measure, we only need to construct a set of r -layered points which will be the centers of its spherical component's 2-spheres. This condition of being r -layered is the natural geometric condition to consider but it is difficult to work with. Our aim is to find a systematic or algorithmic way of constructing such a set of points. In other words, if we are given an even number $2m$, we want to be able to find the coordinates of all possible well r -layered points in some \mathbb{R}^d .

In Section 4, we invoke graph theory to do such a construction. We construct a graph associated to a configuration of r -layered points and in Lemma 3.3.12, we translate the existence of such a configuration in Euclidean space to a necessary and sufficient condition on the graph. The advantage of this condition is that it is computable, expressed as a bound on the eigenvalues of the Laplacian matrix associated to the graph. We finally prove Theorem 3.3.14 where we describe how to find the coordinates of those centers in the corresponding ambient space and the rank of the linear space generated by the centers. We also show that the number of centers is divisible by 4 .

Finally, in Section 5, we explicitly construct an infinite family of non-isometric 3-uniform measures in Euclidean spaces of different dimensions. To do that, we first construct rectangular parallelotopes whose vertices are r -layered (see Lemma 3.4.1). Using this construction, we produce a family of 3-uniform measures (in Theorem 3.4.2). These configurations of points are highly symmetric. To illustrate the fact that this need not be true in general, we

construct an example with much less symmetry.

3.1 Preliminaries

3.1.1 Geometry and analysis preliminaries

We state two theorems which will be crucial to the description of the geometry of the spherical components. In [KoP], Kowalski and Preiss proved that the curvature of a manifold whose surface measure is locally n -uniform must satisfy the following equation.

Theorem 3.1.1. *[KoP] If a hypersurface $M \subset \mathbb{R}^{n+1}$ of class C^5 is such that for all $x \in M$, there exists $r_0 > 0$ such that for all $r < r_0$,*

$$\mathcal{H}^n(B(x, r) \cap M) = \omega_n r^n, \quad (3.1.1)$$

then we have along M :

$$h^2 = 2\|\vec{h}\|^2 = 2\tau,$$

where \vec{h} denotes the second fundamental form, h the trace of \vec{h} , τ the scalar curvature and $\|\cdot\|$ the norm of a tensor with respect to the Riemannian inner product.

When $n = 2$, this theorem essentially says that all points of the manifold are umbilic. The following is a classical geometry theorem describing umbilic manifolds.

Theorem 3.1.2. *[Sp] For $n \geq 2$, let $M^n \subset \mathbb{R}^d$ be a connected immersed submanifold of \mathbb{R}^d with all points umbilics. Then either M lies in some n -dimensional plane or else M lies in some n -dimensional sphere in some $(n + 1)$ -dimensional plane.*

In [KiP], Kirchheim and Preiss proved that the support of a uniformly distributed measure is an analytic variety. We need the following theorem by Lojasiewicz to describe the geometry of an analytic variety.

Theorem 3.1.3. *[L] Let $\Phi(x_1, \dots, x_d)$ be a real analytic function on \mathbb{R}^d in a neighborhood of the origin. We may assume $\Phi(0, \dots, 0, x_d) \neq 0$. After a rotation of the coordinates*

(x_1, \dots, x_{d-1}) , one has that there exist numbers $\delta_j > 0$, $j = 1, \dots, d$ such that the set Z defined as :

$$Z = \{x = (x_1, \dots, x_d) : |x_j| < \delta_j, \text{ for all } j \text{ and } \Phi(x) = 0\},$$

has a decomposition

$$Z = V^{d-1} \cup \dots \cup V^0. \quad (3.1.2)$$

The set V^0 is either empty or consists of the origin alone. For $1 \leq k \leq d-1$, we may write V^k as a finite, disjoint union of analytic k -submanifolds of \mathbb{R}^d .

Moreover, Z is stratified in the following sense: for each k , the closure of V^k contains all the subsequent V_j 's, i.e. defining Q to be

$$Q = \{x \in \mathbb{R}^d; |x_j| < \delta_j, \text{ for all } j\},$$

we have:

$$V^0 \cup \dots \cup V^{k-1} \subset \overline{Q \cap V^k}. \quad (3.1.3)$$

Corollary 2.2.5 says that the spherical component of a conical 3-uniform measure is locally 2-uniform. The following proves the converse: if Ω is a subset of \mathbb{S}^{d-1} such that $\mathcal{H}^2 \llcorner \Omega$ is locally 2-uniform, and Σ is the cone over Ω then $\mathcal{H}^3 \llcorner \Sigma$ is 3-uniform.

Lemma 3.1.4. *Let Ω be a set in \mathbb{R}^d contained in \mathbb{S}^{d-1} , $\sigma = \mathcal{H}^2 \llcorner \Omega$ and assume that σ satisfies the property that for all $x \in \Omega$, for $r \leq 2$,*

$$\sigma(B(x, r)) = \pi r^2. \quad (3.1.4)$$

Define Σ to be:

$$\Sigma = \left\{ x \in \mathbb{R}^d; \frac{x}{|x|} \in \Omega \right\} \cup \{0\}, \quad (3.1.5)$$

and ν to be $\mathcal{H}^3 \llcorner \Sigma$.

Then for all $x \in \Sigma$, for $r > 0$, we have:

$$\nu(B(x, r)) = \frac{4}{3} \pi r^3. \quad (3.1.6)$$

In particular, ν is 3-uniform.

Proof. We prove that $\nu(B(e, r)) = \frac{4}{3}\pi r^3$, for $e \in \Omega$, $r > 0$. The theorem then follows for any $x \in \Sigma$. Indeed, if $x \in \Sigma$, $x \neq 0$ then $e = \frac{x}{|x|} \in \Omega$. Moreover, by the definition of Σ we have $\frac{\Sigma}{u} = \Sigma$ for any $u > 0$. This gives:

$$\mathcal{H}^3(B(x, r) \cap \Sigma) = \mathcal{H}^3\left(|x| \left(B\left(e, \frac{r}{|x|}\right) \cap \frac{\Sigma}{|x|}\right)\right) = |x|^3 \mathcal{H}^3\left(B\left(e, \frac{r}{|x|}\right) \cap \Sigma\right) = \frac{4}{3}\pi r^3.$$

On the other hand, let $x_i = \frac{e}{i}$ for some $e \in \Omega$ and let $r > 0$. Then since $\chi_{B(x_i, r)}(z) \rightarrow \chi_{B(0, r)}(z)$, for ν -almost every z , we get:

$$\frac{4}{3}\pi r^3 = \lim_{i \rightarrow \infty} \nu(B(x_i, r)) = \nu(B(0, r)).$$

Let us now prove the theorem for $e \in \Omega$. Let $r > 0$ and $g(z) = \chi_{B(e, r)}(z)$. Then, by Lemma 2.2.3,

$$\nu(B(e, r)) = \int_0^\infty \rho^2 \int g(\rho z') d\sigma(z') d\rho = \int_0^\infty \rho^2 \int \chi_{B(\frac{e}{\rho}, \frac{r}{\rho})}(z') d\rho = \int_0^\infty \sigma\left(B\left(\frac{e}{\rho}, \frac{r}{\rho}\right)\right) d\rho.$$

Let us compute $\sigma(B(\frac{e}{\rho}, \frac{r}{\rho}))$.

We first express $B(\frac{e}{\rho}, \frac{r}{\rho}) \cap \mathbb{S}^{d-1}$ as a ball centered on e . Let $z \in B(\frac{e}{\rho}, \frac{r}{\rho}) \cap \mathbb{S}^{d-1}$. Then an easy calculation gives

$$\left|z - \frac{e}{\rho}\right|^2 \leq \left(\frac{r}{\rho}\right)^2 \iff |z - e| \leq \sqrt{\frac{r^2 - (\rho - 1)^2}{\rho}}.$$

Therefore,

$$\sigma\left(B\left(\frac{e}{\rho}, \frac{r}{\rho}\right)\right) = \sigma\left(B\left(e, \sqrt{\frac{r^2 - (\rho - 1)^2}{\rho}}\right)\right) = \pi \cdot \frac{r^2 - (\rho - 1)^2}{\rho}.$$

We now compute $\nu(B(e, r))$. To this effect, we need to consider two cases: when $r \leq 1$ and $r \geq 1$.

If $r \leq 1$, by (2.2.3),

$$\nu(B(e, r)) = \int_{1-r}^{1+r} \rho^2 \sigma\left(B\left(\frac{e}{\rho}, \frac{r}{\rho}\right)\right) d\rho = \pi \int_{1-r}^{1+r} \rho(r^2 - (\rho - 1)^2) d\rho = \frac{4}{3}\pi r^3.$$

In the case where $r \geq 1$, notice that when $\rho \leq r - 1$, $\partial B_\rho \subset B(e, r)$, and when $\rho > r + 1$, $\partial B_\rho \cap B(e, r) = \emptyset$. Therefore, we can write:

$$\nu(B(e, r)) = 4\pi \int_0^{r-1} \rho^2 d\rho + \int_{r-1}^{r+1} (\rho(r^2 - 1) - 2\rho^2 + 2\rho^3) d\rho = \frac{4}{3}\pi r^3.$$

Note that when $\rho \in [r-1, r+1]$, $\frac{r^2 - (\rho-1)^2}{\rho} \leq 4$, justifying the fact that:

$$\sigma \left(B \left(e, \sqrt{\frac{r^2 - (\rho-1)^2}{\rho}} \right) \right) = \pi \frac{r^2 - (\rho-1)^2}{\rho}.$$

□

We also state a theorem due to Archimedes: it says that the surface measure of a 2-sphere is the support of a locally 2-uniform measure. We provide a proof using the area formula.

Lemma 3.1.5 (Archimedes). *Let S be a sphere of radius R in \mathbb{R}^3 . Then for all $u \in S$, for all $\rho \leq 2R$, we have:*

$$\mathcal{H}^2(B(u, \rho) \cap S) = \pi \rho^2. \quad (3.1.7)$$

Proof. Without loss of generality, Hausdorff measure being invariant under isometries and under dilation up to appropriate normalization, we can assume that $S = \mathbb{S}^2$ and $u = (0, 0, 1)$.

We claim that for $e = (0, 0, 1)$ and $r \leq 2$,

$$\mathcal{H}^2(\mathbb{S}^2 \cap B(e, r)) = \pi r^2. \quad (3.1.8)$$

First, note that $\partial B(e, r) \cap S^2 = \{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 = 1, x^2 + y^2 + (z-1)^2 = r^2\}$. If $r < \sqrt{2}$, $B(e, r) \cap S^2$ is the portion of the graph of $f(x, y) = \sqrt{1 - (x^2 + y^2)}$ above $z = 1 - \frac{r^2}{2}$.

So we have, by the area formula:

$$\mathcal{H}^2(B(e, r) \cap S^2) = \int_0^{2\pi} \int_0^{\sqrt{1 - (1 - \frac{r^2}{2})^2}} \sqrt{1 + |\nabla f|^2} \rho d\rho d\theta = \pi r^2.$$

If $\sqrt{2} < r < 2$, $B(e, r)$ and $B(0, 1)$ intersect in $z = 1 - \frac{r^2}{2}$. Moreover, note that the part of S^2 below the plane $z = 1 - \frac{r^2}{2}$ is $B(-e, r')$, where, by applications of Pythagoras' theorem, we have $r'^2 = 4 - r^2$. Therefore, by symmetry (since $r' < \sqrt{2}$), we have:

$$\mathcal{H}^2(B(e, r) \cap S^2) = \mathcal{H}^2(S^2) - \mathcal{H}^2(B(-e, r') \cap S^2) = \pi r^2.$$

This proves (3.1.8).

Therefore, since $\rho \leq 2R$, we have:

$$\mathcal{H}^2(S \cap B(u, \rho)) = \mathcal{H}^2\left(R\left(\mathbb{S}^2 \cap B\left(e, \frac{\rho}{R}\right)\right)\right) = R^2\pi\left(\frac{\rho}{R}\right)^2 = \pi\rho^2.$$

□

3.1.2 Discrete mathematics preliminaries

In Section 4, we need to understand what conditions on a set of distances guarantees their embeddability in Euclidean space. To this end, we use a theorem of embeddability from [B].

Definition 3.1.6. *Let X be a set. We call X a distance space if there exists a distance function $d_X : X \times X \rightarrow Y$, where Y is called the distance set. Typically Y will be taken to be \mathbb{R}_+ .*

We call a distance space (X, d_X) semimetric if d_X has co-domain $\mathbb{R}_+ \cup \{0\}$ and if d_X satisfies for all $p, q \in X$:

- $d_X(p, q) = 0 \iff p = q$,
- $d_X(p, q) = d_X(q, p)$.

We remind the reader of the geodesic distance of two points on a sphere.

Definition 3.1.7. *For two points $x, y \in t\mathbb{S}^m \subset \mathbb{R}^{m+1}$, for some $t > 0$, we define the distance $|\cdot|_{t\mathbb{S}^m}$ to be:*

$$|x - y|_{t\mathbb{S}^m} = t \cdot \arccos\left(\frac{\langle x, y \rangle}{t^2}\right), \quad (3.1.9)$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product.

Theorem 3.1.8. *[B] Let $X = \{p_1, \dots, p_n\}$ be a semimetric space, $t > 0$ and define the $n \times n$ matrix Δ to be:*

$$\Delta = \left(\cos\left(\frac{d_X(p_i, p_j)}{t}\right) \right)_{i,j}. \quad (3.1.10)$$

Then there exist points $\{\xi_i\}_{i=1}^n$ in $t\mathbb{S}^{n-2}$ such that:

$$|\xi_i - \xi_j|_{t\mathbb{S}^{n-2}} = d_X(p_i, p_j) \quad (3.1.11)$$

if and only if $d_X(p_i, p_j) \leq \pi t$, the matrix Δ has rank at most n and all its principal minors are non-negative (or equivalently Δ is positive semidefinite).

An application of this theorem leads to a characterization of a measure ν by a graph associated to it. We give some basic notions of graph theory.

Definition 3.1.9. A graph G consists of:

- A set of vertices $V(G) = \{v_i\}_{i=1}^n$,
- A set of edges $E(G) = \{\{v_i v_j\}\}_{i,j \in J}$ for J a subset of the set of subsets of cardinality two of $\{1, \dots, n\}$.

Definition 3.1.10. 1. We call two edges having a vertex in common adjacent. We say two vertices u and v are adjacent and denote $u \sim v$ if $\{u, v\} \in E(G)$.

2. A weighted graph is a graph to which we associate a weight function $w : E(G) \rightarrow \mathbb{R}_+$.

3. The degree $d(v)$ of a vertex v is defined as $d(v) = \sum_{u \sim v} w(\{u, v\})$.

4. A k -edge coloring of G is a function $c : E(G) \rightarrow \{1, \dots, k\}$ such that $c(e) \neq c(f)$ if e is adjacent to f

Example. An example of a graph which will be used in Section 4 is the complete graph K_n . This graph has n vertices $V(G) = \{v_i\}_{i=1}^n$ and its edges are all the subsets of $V(G)$ of cardinality 2 i.e. $E(G) = \{\{v_i v_j\}\}_{1 \leq i < j \leq n}$.

To each graph are associated two matrices that encode information about its structure: the adjacency matrix and the Laplacian matrix.

Definition 3.1.11. Let G be a weighted graph.

1. The adjacency matrix $A = (A_{ij})_{i,j}$ of G is defined as:

$$A_{ij} = \begin{cases} 0, & \text{if } i = j, \text{ or } \{v_i, v_j\} \notin E(G) \\ w(\{v_i, v_j\}), & \text{if } i \neq j, \{v_i, v_j\} \in E(G). \end{cases} \quad (3.1.12)$$

2. The degree matrix D of G is the diagonal matrix with entries:

$$D_{ii} = d(v_i). \quad (3.1.13)$$

3. The Laplacian $L = (L_{ij})_{i,j}$ of G is defined as

$$L = D - A, \quad (3.1.14)$$

where D is the degree matrix. Its second smallest eigenvalue λ_G is called the spectral gap of L .

4. The normalized Laplacian matrix L_{nor} is defined as

$$L_{nor} = D^{-\frac{1}{2}} L D^{-\frac{1}{2}},$$

i.e

$$(L_{nor})_{ij} = \begin{cases} 1, & \text{if } i = j, \\ -\frac{1}{\sqrt{d(v_i)d(v_j)}}, & \text{if } i \neq j, \{v_i, v_j\} \in E(G), \\ 0, & \text{otherwise.} \end{cases} \quad (3.1.15)$$

The spectral gap of the Laplacian a graph encodes information about the connectedness of a graph. One can interpret it as a quantified version of connectivity in view of the following proposition.

Proposition 3.1.12. [C] *Let G be a weighted graph on n vertices and L its Laplacian. Denote by $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ the eigenvalues of L .*

Then $\lambda_1 = 0$ with eigenvector $e = (1, \dots, 1)$ and

$$\lambda_2 = 0 \implies G \text{ is disconnected.}$$

We call $\lambda_2 =: \lambda_G$ the spectral gap of L .

3.2 The spherical component is a union of 2-Spheres

We now study the geometry of the support of the spherical component σ of the 3-uniform measure ν .

Our aim is to prove that Ω is a finite union of disjoint 2-spheres.

Theorem 3.2.1. *Let ν be a conical 3-uniform measure in \mathbb{R}^d , σ its spherical component and Ω the support of σ . Then*

$$\Omega = \bigcup_{i=1}^M S_i, \tag{3.2.1}$$

where the S_i 's are mutually disjoint 2-spheres.

We start by proving the following intermediate lemma.

Lemma 3.2.2. *Let μ be a 3-uniform measure in \mathbb{R}^d , σ its spherical component and $\text{supp}(\sigma) = \Omega$. Then:*

$$\mathcal{R} \subset \bigcup_{\alpha} S_{\alpha},$$

where the S_{α} 's are 2-spheres and \mathcal{R} is the regular part of Ω as defined in Theorem 3.0.2.

We divide the proof of this lemma into claims which will be proven separately. The setting of the claims is the following: we pick $Q \in \mathcal{R}$. Without loss of generality, by rotating and translating Ω , we can assume that $Q = 0$ and $\Omega \subset \partial B(-p, 0)$ where $p = (0, 0, 1, 0, \dots, 0)$. We can choose a basis $\{e_1, e_2\}$ of $P = T_0\Omega$ satisfying the following: in a neighborhood U of 0, writing \bar{x} the projection of x on P , there exist $d - 2$ real analytic functions z_i of \bar{x} so that

$$\Omega \cap U = \left\{ \bar{x} + \sum_{i=3}^d z_i(\bar{x}) e_i; \bar{x} \in P \cap U \right\}, \quad (3.2.2)$$

and such that $z_i(0) = 0$, $\nabla z_i(0) = 0$ for all i and $\nabla^2 z_4(0) = \text{diag}(\lambda_1, \lambda_2)$.

Claim 1.

$$\nabla^2 z_3(0) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Proof. Indeed since $\Omega \subset \mathbb{S}^{d-1} - p$,

$$x_1^2 + x_2^2 + (z_3 + 1)^2 + z_4^2 + \dots + z_d^2 = 1. \quad (3.2.3)$$

Differentiating (3.2.3) with respect to x_1 then x_2 , and plugging in $z_i(0) = 0$ and $\nabla z_i(0) = 0$, we get:

$$\partial_2 \partial_1 z_3(0) (z_3(0) + 1) = 0, \quad (3.2.4)$$

and hence $\partial_2 \partial_1 z_3(0) = \partial_1 \partial_2 z_3(0) = 0$. Differentiating (3.2.3) twice with respect to x_1 and plugging in $z_i(0) = 0$ and $\nabla z_i(0) = 0$, we get:

$$1 + (z_3(0) + 1) \partial_1 \partial_1 z_3(0) = 0 \quad (3.2.5)$$

and hence $\partial_1 \partial_1 z_3(0) = -1$. Similarly, we get $\partial_2 \partial_2 z_3(0) = -1$. \square

We now write for every $j \geq 5$

$$\nabla^2 z_j(0) = \begin{bmatrix} \mu_{1,j} & m_j \\ m_j & \mu_{2,j} \end{bmatrix} \quad (3.2.6)$$

Denoting by $\rho = \sqrt{x_1^2 + x_2^2}$, we can write the following Taylor expansions for the z_j 's:

$$z_3 = -\frac{1}{2} \rho^2 + O(\rho^3), \quad (3.2.7)$$

$$z_4 = \frac{1}{2} (\lambda_1 x_1^2 + \lambda_2 x_2^2) + O(\rho^3), \quad (3.2.8)$$

$$z_j = \frac{1}{2} (\mu_{1,j} x_1^2 + \mu_{2,j} x_2^2 + 2m_j x_1 x_2) + O(\rho^3). \quad (3.2.9)$$

We will first use the area formula to write a Taylor expansion for $\mathcal{H}^2(B(0, r) \cap \Omega)$ for r small in terms of the λ_j 's, $\mu_{i,j}$'s and m_j 's. We then use the fact that $\mathcal{H}^2 \llcorner \Omega$ is locally 2-uniform to establish relations between the λ_j 's, $\mu_{i,j}$'s and m_j 's. We start by writing the integrand \mathcal{D} appearing in the area formula in terms of the the λ_j 's, $\mu_{i,j}$'s and m_j 's.

Claim 2. For $\bar{x} = (x_1, x_2) \in P \cap U$, we have:

$$\mathcal{D}(\bar{x}) = 1 + \alpha x_1^2 + \beta x_2^2 + \gamma x_1 x_2 + O(\rho^4), \quad (3.2.10)$$

where

$$\alpha = 1 + \lambda_1^2 + \sum_j (\mu_{1,j}^2 + m_j^2), \quad (3.2.11)$$

$$\beta = 1 + \lambda_2^2 + \sum_j (\mu_{2,j}^2 + m_j^2), \quad (3.2.12)$$

$$\gamma = \sum_j 2m_j(\mu_{1,j} + \mu_{2,j}). \quad (3.2.13)$$

Moreover, if we write $x_1 = \rho a_1$ and $x_2 = \rho a_2$ where $a_1 = a_1(\theta) = \cos(\theta)$ and $a_2 = a_2(\theta) = \sin(\theta)$, then (3.2.10) becomes:

$$\mathcal{D}(\rho, \theta) = 1 + \bar{B}(\theta)\rho^2 + O(\rho^4) \quad (3.2.14)$$

where $\bar{B}(\theta) = \alpha a_1^2 + \beta a_2^2 + \gamma a_1 a_2$.

Proof. \mathcal{D} is the sum of the squares of all 2×2 minors of the matrix $Jz(\bar{x})$ which is given (up to a term $O(\rho^3)$ in each entry) by:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -x_1 & -x_2 \\ \lambda_1 x_1 & \lambda_2 x_2 \\ \mu_{1,4} x_1 + m_4 x_2 & \mu_{2,4} x_2 + m_4 x_1 \\ \vdots & \vdots \\ \mu_{1,d} x_1 + m_d x_2 & \mu_{2,d} x_2 + m_d x_1 \end{bmatrix} \quad (3.2.15)$$

If we denote by τ the permutation of 1 and 2 then:

$$\begin{aligned} \mathcal{D}(\bar{x}) = & 1 + \sum_{i=1}^2 x_i^2 + \lambda_i^2 x_i^2 + \left(\sum_{i=1,2} (-1)^{i+1} \lambda_i x_1 x_2 \right)^2 + \sum_{j=4}^d \sum_{i=1}^2 (\mu_{i,j} x_i + m_j x_{\tau(i)})^2 \\ & + \sum_{j=4}^d \left[\left(\sum_{i=1}^2 (-1)^{i+1} \mu_{i,j} x_1 x_2 + m_j x_{\tau(i)}^2 \right)^2 + \left(\sum_{i=1}^2 (-1)^{i+1} (\lambda_i \mu_{\tau(i),j} x_1 x_2 + \lambda_i m_j x_i^2) \right)^2 \right] \\ & + \sum_{4 \leq j < k \leq d} \left(\sum_{i=1}^2 (-1)^{i+1} (\mu_{i,j} x_i + m_j x_{\tau(i)}) (\mu_{2,k} x_{\tau(i)} + m_k x_i) \right)^2 + O(\rho^6). \end{aligned}$$

It is easily seen that the only sums contributing terms of order ρ^2 or lower are the sums on the first line. By expanding the squares, we get (3.2.10) of which (3.2.14) is a direct consequence. \square

Claim 3. For r small enough that $B(0, r) \subset U$, we have:

$$\mathcal{H}^2(B(0, r) \cap \Omega) = \pi r^2 + r^4 \int_0^{2\pi} \left(\frac{\bar{B}(\theta)}{8} - \frac{B(\theta)}{2} \right) d\theta + O(r^6) \quad (3.2.16)$$

where

$$B(\theta) = \sum_{l=3}^d B_l^2(\theta),$$

$$B_3 = \frac{1}{2},$$

$$B_4 = \frac{\lambda_1 a_1^2 + \lambda_2 a_2^2}{2},$$

and

$$B_l = \frac{m_{u_{1,l}} a_1^2 + \mu_{2,l} a_2^2 + 2m_l a_1 a_2}{2}, \text{ for } l \geq 5.$$

Proof. Let $F : \mathbb{R}^2 \rightarrow \mathbb{R}^d$ be the map:

$$F(\bar{x}) = (\bar{x}, z_3(\bar{x}), \dots, z_d(\bar{x})).$$

By the area formula, we have:

$$\mathcal{H}^2(B(0, r) \cap \Omega) = \int_{F^{-1}(B(0, r))} \sqrt{\mathcal{D}(\bar{x})} dA \quad (3.2.17)$$

$$= \int_0^{2\pi} \int_0^{\rho(\theta)} \left(1 + \frac{\bar{B}(\theta)}{2} + O(\rho^4)\right) \rho d\rho d\theta, \quad (3.2.18)$$

$$= \int_0^{2\pi} \left[\frac{\rho^2}{2} + \frac{\bar{B}(\theta)}{8} \rho^4 + O(\rho^6) \right]_0^{\rho(\theta)} d\theta. \quad (3.2.19)$$

We now find $\rho(\theta)$. Note that when $x_1^2 + x_2^2 = \rho(\theta)^2$, we have $F(x_1, x_2) \in \partial B(0, r)$. Hence:

$$\rho(\theta)^2 + \sum_{j=3}^d z_j^2 = r^2. \quad (3.2.20)$$

By (3.2.7), (3.2.8) and (3.2.9), (3.2.20) becomes:

$$\rho^2(\theta) + \sum_{j=3}^d B_j^2(\theta) \rho^4(\theta) = r^2, \quad (3.2.21)$$

$$\rho^2(\theta) + B\rho^4(\theta) = r^2. \quad (3.2.22)$$

Expressing ρ as a power series in terms of r and substituting in (3.2.22), we get:

$$\rho(\theta) = r - \frac{B(\theta)}{2} r^3 + O(r^4), \quad (3.2.23)$$

and consequently

$$\rho^2(\theta) = r^2 - B(\theta)r^4 + O(r^6), \quad (3.2.24)$$

$$\rho^4(\theta) = r^4 + O(r^6).$$

Plugging (3.2.24) in (3.2.19), we get:

$$\mathcal{H}^2(B(0, r) \cap \Omega) = \pi r^2 + r^4 \int_0^{2\pi} \left(\frac{\bar{B}(\theta)}{8} - \frac{B(\theta)}{2} \right) d\theta + O(r^6). \quad (3.2.25)$$

□

Let us express B in term of the λ_i 's, $\mu_{i,j}$'s and m_j 's. We have :

$$\begin{aligned} B &= \sum_{l=3}^d B_l^2, \\ &= \frac{1}{4} + \frac{1}{4} \sum_{i=1}^2 \left(\lambda_i^2 + \sum_{j=5}^d \mu_{i,j}^2 \right) a_i^4 + \frac{1}{4} \left(2\lambda_1\lambda_2 + 2 \sum_{j=5}^d \mu_{1,j}\mu_{2,j} + 4m_j^2 \right) a_1^2 a_2^2 \end{aligned} \quad (3.2.26)$$

$$\begin{aligned} &+ \frac{1}{4} \left(\sum_{j=5}^d m_j \mu_{1,j} \right) a_1^3 a_2 + \frac{1}{4} \left(\sum_j m_j \mu_{2,j} \right) a_1 a_2^3 \\ &= \frac{1}{4} \left(1 + \delta a_1^4 + \epsilon a_2^4 + \iota a_1^2 a_2^2 + \omega a_1^3 a_2 + \kappa a_2^3 a_1 \right), \end{aligned} \quad (3.2.27)$$

where

$$\begin{aligned} \delta &= \left(\lambda_1^2 + \sum_{j=5}^d \mu_{1,j}^2 \right), \\ \epsilon &= \left(\lambda_2^2 + \sum_{j=5}^d \mu_{2,j}^2 \right), \\ \iota &= \left(2\lambda_1\lambda_2 + 2 \sum_{j=5}^d \mu_{1,j}\mu_{2,j} + 4m_j^2 \right), \\ \omega &= \left(\sum_{j=5}^d m_j \mu_{1,j} \right), \end{aligned}$$

and

$$\kappa = \left(\sum_j m_j \mu_{2,j} \right).$$

We now use the fact that $\mathcal{H}^2 \llcorner \Omega$ is locally 2-uniform to deduce a relation between the λ_i 's, $\mu_{i,j}$'s and m_j 's.

Claim 4. *We have:*

$$\lambda_1 = \lambda_2 = \lambda,$$

and for all $j \geq 5$

$$\mu_{1,j} = \mu_{2,j} = \mu_j \text{ and } m_j = 0.$$

Proof. On one hand, by Corollary 2.2.5, we have $\mathcal{H}^2(B(0, r) \cap \Omega) = \pi r^2$. On the other hand, by (3.2.16), we have $\mathcal{H}^2(B(0, r)) = \pi r^2 + r^4 \int_0^{2\pi} \left(\frac{\overline{B}(\theta)}{8} - \frac{B(\theta)}{2} \right) d\theta + O(r^6)$. By equating them we get

$$\int_0^{2\pi} \frac{\overline{B}(\theta)}{8} - \frac{B(\theta)}{2} d\theta = 0. \quad (3.2.28)$$

Rewrite this in term of a_1 and a_2 to get:

$$\frac{\alpha}{8} \int_0^{2\pi} a_1^2 d\theta + \frac{\beta}{8} \int_0^{2\pi} a_2^2 d\theta - \frac{1}{8} \int_0^{2\pi} d\theta - \frac{\delta}{8} \int_0^{2\pi} a_1^4 d\theta - \frac{\epsilon}{8} \int_0^{2\pi} a_2^4 d\theta - \frac{\iota}{8} \int_0^{2\pi} a_1^2 a_2^2 d\theta = 0, \quad (3.2.29)$$

by using the fact that

$$\int_0^{2\pi} \cos(\theta) \sin(\theta) d\theta = \int_0^{2\pi} \cos^3(\theta) \sin(\theta) d\theta = \int_0^{2\pi} \cos(\theta) \sin^3(\theta) d\theta = 0.$$

Moreover, since

$$\begin{aligned} \int_0^{2\pi} \cos^2(\theta) d\theta &= \int_0^{2\pi} \sin^2(\theta) d\theta = \pi \\ \int_0^{2\pi} \cos^4(\theta) d\theta &= \int_0^{2\pi} \sin^4(\theta) d\theta = \frac{3\pi}{4} \\ \int_0^{2\pi} \cos^2(\theta) \sin^2(\theta) d\theta &= \frac{\pi}{4}, \end{aligned}$$

(3.2.29) becomes:

$$4\alpha + 4\beta - 8 - 3\delta - 3\epsilon - \iota = 0.$$

Replacing the letters by their values in terms of the λ_i 's, $\mu_{i,j}$'s and m_j 's gives:

$$(\lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2) + \sum_j (\mu_{1,j}^2 + \mu_{2,j}^2 - 2\mu_{1,j}\mu_{2,j}) + \sum_j 4m_j^2 = 0 \quad (3.2.30)$$

This implies that $\lambda_1 - \lambda_2 = \mu_{1,j} - \mu_{2,j} = m_j = 0$ for all j .

□

We can now prove Lemma 3.2.2

Proof. Write

$$\mathcal{R} = \cup_i M_i, \quad (3.2.31)$$

where each M_i is a connected analytic 2-submanifold of \mathbb{R}^d . Since every point of M_i , $i > 0$, is analytic, it is umbilic and therefore by Theorem 3.1.2 M_i lies in some 2-dimensional sphere S_i (not necessarily distinct). \square

We now finish the proof of Theorem 3.2.1.

Proof. First, we claim that there are only finitely many M_i 's in (3.2.31). Indeed, by Theorem 3.1.3, for every $x \in \Omega$, there exists a neighborhood N_x such that $\Omega \cap N_x$ can be written as :

$$\Omega \cap N_x = V^2 \cup V^1 \cup V^0, \quad (3.2.32)$$

where V^2 is a finite union of analytic 2-submanifolds, V^1 a finite union of analytic 1-submanifolds and V^0 is a finite union of points.

By compactness of Ω , and by taking a finite cover of it by such neighborhoods, we can write Ω as:

$$\Omega = V^2 \cup V^1 \cup V^0, \quad (3.2.33)$$

where V^2 is a finite union of analytic 2-submanifolds, V^1 a finite union of analytic 1-submanifolds and V^0 is a finite union of points.

Calling the analytic 2-submanifolds M_i where each $M_i \subset S_i$ and noting that $V^1 \cup V^0 \subset \overline{V^2}$, we have:

$$\Omega \subset \bigcup_{i=1}^m S_i. \quad (3.2.34)$$

We now proceed to prove that $M_i = S_i$ for all i and $\Omega = \mathcal{R}$.

Suppose that there exists i such that $M_i \neq S_i$, and assume without loss of generality that $i = 1$. Pick $y \in \partial(\Omega \cap S_1)$ (by $\partial(\Omega \cap S_1)$ we mean the boundary in the subspace topology of S_1 in the following). We first claim that $y \in \cup_{i \neq 1} S_i$. Suppose not. Then there exists ϵ such that $B(y, \epsilon) \subset \left(\cup_{i \neq 1} S_i\right)^c$. In particular, by (3.2.34),

$$B(y, \epsilon) \cap \Omega = B(y, \epsilon) \cap \Omega \cap S_1.$$

On the other hand, since $y \in \partial(\Omega \cap S_1)$, there exists a ball $B \subset B(y, \epsilon) \cap \Omega^c \cap S_1$ and consequently, $\mathcal{H}^2(B(y, \epsilon) \cap \Omega^c \cap S_1) > 0$. Thus we have, since $\mathcal{H}^2 \llcorner \Omega$ and $\mathcal{H}^2 \llcorner S_1$ are locally 2-uniform,

$$\begin{aligned} \pi\epsilon^2 &= \mathcal{H}^2(B(y, \epsilon) \cap \Omega) = \mathcal{H}^2(B(y, \epsilon) \cap \Omega \cap S_1), \\ &< \mathcal{H}^2(B(y, \epsilon) \cap \Omega \cap S_1) + \mathcal{H}^2(B(y, \epsilon) \cap \Omega^c \cap S_1), \\ &= \mathcal{H}^2(B(y, \epsilon) \cap S_1) = \pi\epsilon^2, \end{aligned}$$

which yields a contradiction. Hence, $y \in \bigcup_{i \neq 1} S_i$. In other words, for each $y \in \partial(\Omega \cap S_1)$ there exists a finite index set I , $1 \in I$, such that $y \in \Omega \cap \bigcap_{i \in I} S_i$. We now prove that such a set consists of a unique point. Let $e \in \Omega \cap \bigcap_{i \in I} S_i$. By Theorem 3.1.3, for every i , $S_i \cap \Omega = M_i \cup \partial M_i$. In particular, since $e \in S_i \cap \Omega$, there exists a sequence of points $e_l \in M_i$ (possibly all identified with e) such that $e_l \rightarrow e$. But Ω being a $C^{1,\alpha}$ submanifold (Theorem 2.2.8 from Chapter 12), the tangent planes $T_{e_l}\Omega$ converge to $T_e\Omega$. On the other hand, $T_{e_l}S_i$ also converge to T_eS_i . Since $T_{e_l}S_i = T_{e_l}\Omega$, we get $T_e\Omega = T_eS_i$, for all i . In other words the spheres S_i are tangent in e . This implies that $\partial(\Omega \cap S_1)$, being a subset of a finite union of such intersections, is a finite union of points. Therefore, $\Omega \cap S_1$ is a finite union of points. So any sphere S_i such that $M_i \neq S_i$ only intersects Ω in a finite union of points. Since it is clear that two spheres cannot intersect in points from M_i (from a simple density argument), we can exclude a sphere intersecting Ω in a discrete set from our decomposition (3.2.34). This ends the proof that for every i , $M_i = S_i$.

Note that this argument also proves that the spheres are disjoint and that $\Omega = \mathcal{R}$, since $\partial M_i = \emptyset$, for all i .

□

We end this section by proving two simple lemmas about Ω which will be useful in describing the 2-spheres composing it.

Lemma 3.2.3. *For $i > 0$, let r_i be the radius of S_i . Then if $e \in M_i$, we have:*

$$B(e, 2r_i) \cap (\Omega \setminus S_i) = \emptyset \quad (3.2.35)$$

Proof. Let $\rho \in (0, 2r_i)$. Clearly, $r_i < 1$ since S_i is a subset of \mathbb{S}^{d-1} . By Theorem 2.2.5,

$$\sigma(B(e, \rho)) = \pi\rho^2. \quad (3.2.36)$$

On the other hand,

$$\begin{aligned} \sigma(B(e, \rho)) &= \mathcal{H}^2(B(e, \rho) \cap \Omega), \\ &= \mathcal{H}^2(B(e, \rho) \cap S_i) + \mathcal{H}^2(B(e, \rho) \cap (\Omega \setminus S_i)), \\ &= \pi\rho^2 + \mathcal{H}^2(B(e, \rho) \cap (\Omega \setminus S_i)), \text{ by (3.1.7)}. \end{aligned}$$

In particular,

$$\mathcal{H}^2(B(e, \rho) \cap (\Omega \setminus S_i)) = 0 \quad (3.2.37)$$

Assume there exists $x \in B(e, \rho) \cap \Omega \setminus S_i$. Then there exists $\delta > 0$ such that

$$\Omega \cap B(x, \delta) \subset B(e, \rho) \setminus S_i \quad (3.2.38)$$

and consequently

$$\mathcal{H}^2(B(e, \rho) \cap (\Omega \setminus S_i)) > \mathcal{H}^2(\Omega \cap B(x, \delta)) > 0,$$

yielding a contradiction. □

Lemma 3.2.4. *For $i > 0$, $e \in S_i$, there exists $z \in \Omega \setminus S_i$ (not necessarily unique) such that:*

$$|z - e| = 2r_i.$$

In particular, this combined with (3.2.35) implies that $\text{dist}(z, \Omega \setminus S_i) = 2r_i$.

Proof. For $\epsilon > 0$ small enough,

$$\sigma(B(e, 2r_i(1 + \epsilon))) - \sigma(S_i) = 4\pi r_i^2 \epsilon(2 + \epsilon) > 0. \quad (3.2.39)$$

On the other hand,

$$\sigma(B(e, 2r_i(1 + \epsilon))) - \sigma(S_i) = \mathcal{H}^2((\Omega \cap B(e, 2r_i(1 + \epsilon))) \setminus S_i). \quad (3.2.40)$$

In particular, for all $j > 0$, j large enough,

$$\left(\Omega \cap B(e, 2r_i(1 + \frac{1}{j})) \right) \setminus S_i \neq \emptyset,$$

and there exists $z_j \in \left(\Omega \cap B(e, 2r_i(1 + \frac{1}{j})) \right) \setminus S_i$. Passing to a subsequence if necessary, $z_j \rightarrow z$, $z \in \Omega$, $|z - e| = 2r_i$. Moreover, $z \notin S_i$. If it were, then for j large enough, $\text{dist}(z, z_j) < 2r_i$ contradicting 3.2.35. □

3.3 Understanding the configuration of the 2-spheres

We now want to obtain a better description of the spheres that compose the support of Ω . We start with two lemmas of elementary geometry.

Lemma 3.3.1. *Let S be a two 2-dimensional sphere in \mathbb{R}^d such that $S \subset T$, where T is an affine 3-plane. We let $e \in \mathbb{R}^d$ and follow the notation $d(e, S) = D$, $r(S) = \rho$ and $d(e, T) = \delta$. Then, for $D < R$:*

$$B(e, R) \cap S = B(p, x) \cap S, \quad (3.3.1)$$

where $\{p\} = B(e, D) \cap T$ and

$$x^2 = \frac{\rho}{\rho + (D^2 - \delta^2)^{\frac{1}{2}}} (R^2 - D^2).$$

Proof. Let f be such that:

$$B(e, \delta) \cap T = \{f\}.$$

Then:

$$B(e, R) \cap T = B_3(f, \tilde{R}),$$

and

$$B(e, D) \cap T = B_3(f, \tilde{D}),$$

where B_3 denotes the three-dimensional ball in T , $R^2 = \tilde{R}^2 + \delta^2$ and $D^2 = \tilde{D}^2 + \delta^2$. Also note that $B_d(e, R) \cap T \cap S = B_3(f, \tilde{R}) \cap S$ since $S \subset T$.

Let q be the center of S . Then f, p and q are aligned since S and $\partial B_3(E, \tilde{D})$ are tangent at p .

Moreover, $B(f, \tilde{R})$ and S intersect in a circle C . For any $u, v \in C$, $|p - u| = |p - v| = x$. Indeed, since $|f - u| = |f - v| = \tilde{R}$, $|q - u| = |q - v| = \rho$, and f, p, q aligned, p is in the bisecting plane of any two such points. Therefore,

$$B_d(p, x) \cap S = B_3(p, x) \cap S = B_3(f, \tilde{R}) \cap S = B_d(e, R) \cap S.$$

To end the proof, we compute x . Choose $m \in C$ and let n be its projection on the line (fq) . We work in the 2-plane T_2 containing f, q and m . Then $|p - m| = x$, $|q - m| = |p - q| = \rho$, $|f - p| = \tilde{D}$ and $|f - m| = \tilde{R}$. Moreover, we denote $|m - n|$ and $|p - n|$ by l and t respectively. Then, applying Pythagoras' theorem, we get:

$$\rho^2 = l^2 + (\rho - t)^2, \quad (3.3.2)$$

$$x^2 = l^2 + t^2, \quad (3.3.3)$$

$$\tilde{R}^2 = l^2 + (\tilde{D} + t)^2. \quad (3.3.4)$$

Then (3.3.2) becomes $l^2 = 2\rho t - t^2$ and plugging this into (3.3.3) gives

$$x^2 = 2\rho t, \quad (3.3.5)$$

and (3.3.4) becomes

$$t = \frac{\tilde{R}^2 - \tilde{D}^2}{2(\rho + \tilde{D})}.$$

Finally, (3.3.5) gives:

$$x^2 = \frac{\rho}{\rho + \tilde{D}}(\tilde{R}^2 - \tilde{D}^2). \quad (3.3.6)$$

Expressing \tilde{R} and \tilde{D} in terms of R, D and δ ends the proof. \square

Lemma 3.3.2. *Let S be the 2-sphere in \mathbb{S}^{d-1} defined by:*

$$S = \{z \in \mathbb{S}^{d-1}; |z - \xi| = r, z \in V + \xi\},$$

where V is a linear 3-plane. Then for all $z \in \mathbb{R}^d$, if $P_V(z) \neq 0$, denoting the closest point to z and furthest point to z on S by P_S and \bar{P}_S , we have:

$$P_S(z) = r \frac{P_V(z)}{|P_V(z)|} + \xi, \quad (3.3.7)$$

and

$$\bar{P}_S(z) = -r \frac{P_V(z)}{|P_V(z)|} + \xi, \quad (3.3.8)$$

where P_V is the linear projection on V . In particular, if we denote by $D_S(z)$ the distance from z to S and $\bar{D}_S(z)$ the distance between z and the furthest point to z on S , we have:

$$D_S(z) = |z - P_S(z)|,$$

and

$$\bar{D}_S(z) = |z - \bar{P}_S(z)|.$$

Proof. We start by proving that $P_{V+\xi}(z) = P_V(z) + \xi$, where $P_{V+\xi}$ denotes the affine projection on $V + \xi$. First note that ξ is normal to V . Indeed, if e is a unit vector of V , we have $|\xi + re| = |\xi - re| = 1$ since $\xi + re$ and $\xi - re$ are points of $S \subset \mathbb{S}^{d-1}$. This gives

$$\langle \xi, \xi + re \rangle = \langle \xi, \xi - re \rangle,$$

and consequently $\xi \cdot e = 0$.

$P_{V+\xi}(z)$ is the point \tilde{e} that minimize $|z - \tilde{e}|$ for $\tilde{e} \in V + \xi$. Writing $\tilde{e} = e + \xi$, $P_{V+\xi}(z) = e + \xi$ where e minimizes $|z - e - \xi|$, $e \in V$. But, since

$$|z - e - \xi|^2 = |P_V(z) - e|^2 + |P_{V^\perp}(z) - \xi|^2,$$

it is clear that $e = P_V(z)$ is the minimizer we're looking for. This proves that $P_{V+\xi}(z) = P_V(z) + \xi$.

Now if $u \in S \subset V + \xi$ minimizes (resp. maximizes) $|z - u|$, by writing

$$|z - u|^2 = |P_{V+\xi}(z) - u|^2 + |P_{V+\xi}^\perp(z)|^2,$$

we see that u minimizes (resp. maximizes) $|P_{V+\xi}(z) - u| = |P_V(z) - (u - \xi)|$ and consequently u maximizes (resp. minimizes) $\langle P_V(z), u - \xi \rangle$. Therefore, $\frac{u-\xi}{r} = \frac{P_V(z)}{|P_V(z)|}$ (respectively, $\frac{u-\xi}{r} = -\frac{P_V(z)}{|P_V(z)|}$). \square

Using Lemma 3.3.1, Lemma 3.3.2 and the fact that σ is locally 2-uniform, we deduce the following technical lemma which will be our first step towards a description of the spherical component.

Lemma 3.3.3. *Let $\Omega \subset \mathbb{S}^{d-1}$, and $\sigma = \mathcal{H}^2 \llcorner \Omega$. Assume that σ satisfies:*

$$\sigma(B(x, r)) = \pi r^2, \quad (3.3.9)$$

for every $0 \leq r \leq 2$, for every $x \in \Omega$. From Theorems 3.2.1 and 2.2.7 we know that $\Omega = \bigcup_{i=1}^M S_i$ where S_i is a 2-sphere of radius r_{S_i} . Let $\mathfrak{S} = \bigcup_{i=1}^M \{S_i\}$ and fix $z \in \Omega$. Define the integer $m(z)$, the indices $\{i\}_{i=1}^{m(z)}$, the radii $\{R_i(z)\}_{i=1}^{m(z)}$ and the subsets $\{C^i(z)\}_{i=1}^{m(z)}$, $\{C_j^i(z)\}_{0 \leq j \leq i \leq m(z)}$ of \mathfrak{S} inductively in the following manner

- $R_1(z) = 2r_z$ where r_z is the radius of the sphere S_z such that $z \in S_z$.
- $C^0(z) = C_0^0(z) = \{S_z\}$,
- The first layer $C^1(z) = C_1^1(z) = \bigcup \{\{S\}; D_S(z) = R_1(z)\}$ and the contribution of the zero-th to the first layer $C_0^1(z) = \emptyset$,
- If $1 \leq i$, $R_i(z) = \inf \{\overline{D}_S(z); S \in C^{i-1}(z)\}$, and $C_i^i(z) = \bigcup \{\{S\}; D_S(z) = R_i(z)\}$.
- For $0 \leq j \leq i$, the contribution of the j -th layer to the i -th layer

$$C_j^i(z) = \bigcup_{S \in C^j(z)} \{\{S\}; \overline{D}_S(z) > R_i(z)\}.$$

- $C^i(z) = \bigcup_{0 \leq j \leq i} C_j^i(z)$.
- $m(z)$ to be the first integer so that $R_{m(z)} = 2$ and $C_j^{m(z)} = \emptyset$ for all $j \leq m(z)$.

Then, $\Omega = -\Omega$ and for every z , letting

$$c_S(z) = \frac{r_S}{r_S + (D_S(z)^2 - \delta_S(z)^2)^{\frac{1}{2}}},$$

we have for every $0 \leq i \leq m(z)$,

$$4 \sum_{1 \leq j \leq k} \sum_{S \in C^{j-1} \setminus C^j} r_S^2 = \sum_{S \in C^k(z)} c_S(z) D_S(z)^2 \quad (3.3.10)$$

and

$$\sum_{S \in C^k(z)} c_S(z) = 1. \quad (3.3.11)$$

In particular, for every $0 < i < m(z)$, $C^i(z) \neq \emptyset$ and $\Omega = \bigcup_{0 \leq i \leq m(z)} \bigcup_{S \in C^i} S$.

Proof. By Lemma 3.2.1, we know that $\Omega = \bigcup_{i=1}^M S_i$ and $\Omega = -\Omega$. Fix $z \in \Omega$. By Lemmas 3.2.4 and 3.2.35, we know that $C^1(z) \neq \emptyset$. For any i , if $S \in C^i(z)$, then $D_S(z) \leq R_i(z)$ and $\bar{D}_S(z) > R_i(z)$ so that whenever $S \in C^i(z)$ and $R_i(z) < R < R_{i+1}(z)$, we have $S \cap B(z, R) \neq \emptyset$ and $S \cap (B(z, R))^c \neq \emptyset$. Moreover, if $S \in \bigcup_{l \leq i} C^{l-1} \setminus C^l$, then $\bar{D}_S(z) \leq R_i(z)$. Hence, for $R_i(z) < R < R_{i+1}(z)$,

$$B(z, R) \cap \Omega = \left(\bigcup_{l=1}^i \bigcup_{S \in C^{l-1}(z) \setminus C^l(z)} S \right) \cup \left(\bigcup_{S \in C^i(z)} S \cap B(z, R) \right) \quad (3.3.12)$$

Applying \mathcal{H}^2 on both sides, we get from the fact that σ is locally 2-uniform and by Lemma 3.3.1 and (3.1.7),

$$\pi R^2 = \sum_{l=1}^i \sum_{S \in C^{l-1}(z) \setminus C^l(z)} 4\pi r_S^2 + \sum_{S \in C^i(z)} \pi c_S(z) (R^2 - D_S(z)^2). \quad (3.3.13)$$

Differentiating twice with respect to R gives (3.3.11) and plugging (3.3.11) back into (3.3.13) gives (3.3.10). Note that (3.3.11) directly implies that every C^i is non-empty since $c_S(z) > 0$ for every S .

□

We now use this theorem to prove that the support of a locally 2-uniform measure is layered in a sense that will be made precise. Let us start by defining a notion of layering points.

Definition 3.3.4. We call the set $\mathcal{L} \subset S_m$ of permutations $\{l_i\}_{i=0}^{m-1}$ a layering if for each i , l_i has the following properties:

1. $l_0(j) = j$,
2. $l_i(1) = i + 1$,
3. For all $i \neq k$, for all j , $l_i(j) \neq l_k(j)$.
4. $l_i^{-1} = l_i$.

Moreover, if $r > 0$ and $\{\alpha_1, \dots, \alpha_m\}$ is a set of points in \mathbb{R}^d such that:

$$|\alpha_j - \alpha_{l_i(j)}| = 2\sqrt{i}r, \text{ for all } j, i \quad (3.3.14)$$

then we call it an r -layered set of points.

Finally, for $1 \leq i < j \leq m$, we denote by d_{ij} the integer such that:

$$l_{d_{ij}}(i) = j, \quad (3.3.15)$$

and set

$$d_{ii} = 0,$$

for all i . We call the function d such that $d(i, j) = d_{ij}$ the distance function of the layering.

Remark 3.3.1. 1. If $\{\alpha_j\}$ is r -layered by $\{l_i\}$ for some r then for all j ,

$$\{j, l_1(j), \dots, l_{m-1}(j)\} = \{1, \dots, m\}.$$

2. The l_i 's organize the points of P into layers. Let P_j be the sequence:

$$P_j = (\alpha_j, \alpha_{l_1(j)}, \dots, \alpha_{l_{m-1}(j)}).$$

Each P_j is a rearrangement of P_1 "viewed through the lens" of α_j : $\alpha_{l_i(j)}$ is the i -th layer of P_j and is at a distance $2\sqrt{i}r$ from α_j .

Theorem 3.3.5. *Let $\Omega \subset \mathbb{S}^{d-1}$, $\sigma = \mathcal{H}^2 \llcorner \Omega$. Assume Ω is a layered union of spheres i.e $\Omega = (\bigcup_{i=1}^{2m} S_i)$ where:*

1. For $i = 1, \dots, 2m$, S_i is the 2- sphere of radius $r = \frac{1}{\sqrt{2m}}$ and center ξ_i ,
2. For $i = m + 1 \dots 2m$, $S_i = -S_{2m+1-i}$
3. For all $i = 1, \dots, 2m$, $S_i \subset V + \xi_i$ where V is a linear 3-plane such that $\xi_i \in V^\perp$.
4. $\{\xi_i\}_{i=1}^{2m}$ is an r -layered set of points.

Then

$$\sigma(B(x, r)) = \pi r^2, \text{ for } x \in \Omega, 0 \leq r \leq 2. \quad (3.3.16)$$

Proof. First note that conditions 2 and 4 are compatible. Indeed, for $i > m$, we have

$$\begin{aligned} |\xi_1 + \xi_{2m+1-i}|^2 &= 2(|\xi_1|^2 + |\xi_{2m+1-i}|^2) - 2|\xi_1 - \xi_{2m+1-i}|^2, \\ &= 4t^2 + 4(2m - i)r^2, \\ &= 4(2m - 1 - 2m + i)r^2, \\ &= 4(i - 1)r^2, \end{aligned}$$

We first claim that if Ω is a layered union of spheres, then for fixed j , for all $z \in S_j$, for all i we have:

$$D_{S_{l_i(j)}}(z) = 2\sqrt{i}r = \overline{D_{S_{l_{i-1}(j)}}}(z). \quad (3.3.17)$$

We prove it for $j = 1$. The proof for other j 's is exactly similar. First note that by the definition of a layered union of spheres, we have $P_{V^\perp}(z) = \xi_1$ for $z \in S_1$. Moreover $|P_V(z)| = |P_{V+\xi_1}(z) - \xi_1| = r$. Thus

$$\begin{aligned} D_{S_i}(z)^2 &= |z - P_{S_i}(z)|^2, \\ &= |P_V(z) + \xi_1 - P_V(z) - \xi_i|^2, \\ &= |\xi_1 - \xi_i|^2, \\ &= 4(i-1)r^2, \end{aligned}$$

and

$$\begin{aligned} \overline{D_{S_{i-1}}(z)}^2 &= |z - \overline{P_{S_i}(z)}|^2, \\ &= |P_V(z) + \xi_1 + P_V(z) - \xi_{i-1}|^2, \\ &= 4|P_V(z)|^2 + |\xi_1 - \xi_{i-1}|^2, \\ &= 4r^2 + 4(i-2)r^2, \\ &= 4(i-1)r^2. \end{aligned}$$

We now show that (3.3.16) holds. Pick $z \in \Omega$. Without loss of generality, we can assume that $z \in S_1$. Let $0 \leq R \leq 2$. Then there exists i such that $2\sqrt{i}r \leq R \leq 2\sqrt{i+1}r$. If $R = 2\sqrt{i}r$, then by Lemma 3.3.3, $B(z, R) \cap \Omega = \bigcup_{k=1}^{i-1} S_k$ and

$$\begin{aligned} \mathcal{H}^2(B(z, R) \cap \Omega) &= \sum_{k=1}^{i-1} \mathcal{H}^2(S_k), \\ &= \sum_{k=1}^{i-1} \pi 4r_{S_k}^2, \text{ by (3.1.7)} \\ &= \pi \left(4 \sum_{k=1}^{i-1} r^2 \right), \\ &= 4\pi(i-1)r^2 \\ &= \pi R^2. \end{aligned}$$

If $2\sqrt{i}r < R < 2\sqrt{i+1}r$, then $B(z, R) \cap \Omega = \left(\bigcup_{k=1}^{i-1} S_k\right) \cup (S_i \cap B(z, R))$ and

$$\begin{aligned} \mathcal{H}^2(B(z, R) \cap \Omega) &= \sum_{k=1}^{i-1} \mathcal{H}^2(S_k) + \mathcal{H}^2(S_i \cap B(z, R)), \\ &= \pi D_{S_i}(z)^2 + \pi(R^2 - D_{S_i}(z)^2), \text{ by Lemma 3.3.1 and (3.1.7),} \\ &= \pi R^2. \end{aligned}$$

□

Theorem 3.3.6. *Let σ be the spherical component of a conical 3-uniform measure in \mathbb{R}^5 , $\Omega = \text{supp}(\sigma)$. Then Ω is a layered union of spheres.*

Proof. Assume S_1 is the sphere with smallest radius and denote r_{S_1} by r . First note that we can assume $r < \frac{\sqrt{2}}{2}$. Indeed, if $r = \frac{\sqrt{2}}{2}$, then $D_{S_2} = 2r = \sqrt{2}$ which implies that $S_2 = -S_1$ and $\overline{D_{S_2}} = 2$. Therefore, $\Omega = S_1 \cup (-S_1)$ which ends the proof. But $r \leq \frac{\sqrt{2}}{2}$: indeed, on one hand the sum of the squares of the radii is 1 since for any $z \in \Omega$,

$$4\pi = \sigma(B(z, 2)) = 4\pi \sum_{S \in \mathfrak{S}} r_S^2,$$

and on the other hand the fact that $\Omega = -\Omega$ and $\Omega \neq \mathbb{S}^2 \times \{0\}$ implies that there are at least 2 2-spheres in Ω . Hence $r < \frac{\sqrt{2}}{2}$.

Assume that $S_1 = (V_1 + \xi) \cap \mathbb{S}^{d-1}$ where V_1 is a linear 3-plane normal to ξ and ξ is the center of S_1 .

We first claim that for all $z \in S_1$, if the sphere S is in $C^1(z)$, we have:

$$r(S) = r_1.$$

Suppose this were not the case $r := r(S) > r_1$. Denoting $P_S(z)$ by $\zeta \in S$, we have by Claim 3.2.4

$$|\zeta - z| = 2r_1 < 2r,$$

which implies $\text{dist}(\zeta, S_1) < 2r$, contradicting 3.2.4. This proves the claim.

Now set $C^1(1) = \bigcup_{z \in S_1} C^1(z)$ the first layer with respect to S_1 , pick $S \in C^1$ with radius r and center η . We can write $S = V + \eta$, for some linear 3-plane V normal to η . Set $A_S = \{z \in V_1 + \xi; |z - P_S(z)| = 2r\}$. We wish to write an equation for A_S as an object in the 3-space $V_1 + \xi$. Choose orthonormal bases $\{e_i\}_{i=1}^3$ of V_1 and $\langle u, v, w \rangle$ for V and write $\xi = te$. In the following we will denote $\langle z, e_i \rangle$ by z_i and $\langle z, e \rangle$ by z_e .

On one hand, we have

$$\begin{aligned}
|z - P_S(z)|^2 &= 4r^2, \\
|P_V(z)|^2 \left(1 - \frac{r}{|P_V(z)|}\right)^2 + |P_{V^\perp}(z) - \eta|^2 &= 4r^2, \text{ by Lemma 3.3.2,} \\
|P_V(z)|^2 + r^2 - 2r|P_V(z)| + |P_{V^\perp}(z)|^2 + |\eta|^2 - 2\langle \eta, z \rangle &= 4r^2, \\
2 - 2(r|P_V(z)| + \langle \eta, z \rangle) &= 4r^2, \text{ since } |\eta|^2 + r^2 = 1, \text{ and } |z| = 1, \\
r|P_V(z)| = K - \langle \eta, z \rangle, \text{ where } K &= \frac{2 - 4r^2}{2}.
\end{aligned} \tag{3.3.18}$$

We first square and expand the right hand side of (3.3.18). Writing $\langle z, \eta \rangle = \eta_1 z_1 + \eta_2 z_2 + \eta_3 z_3 + t\eta_e$ and expanding the right hand side we get:

$$\begin{aligned}
(K - \langle \eta, z \rangle)^2 &= K^2 + t^2 \eta_e^2 - 2Kt\eta_e + \sum_{i=1}^3 \eta_i^2 z_i^2 + \sum_{1 \leq j < k \leq 3} 2\eta_j \eta_k z_j z_k + \sum_{i=1}^3 (2t\eta_e - 2K)\eta_i z_i,
\end{aligned} \tag{3.3.19}$$

$$= (K + t\eta_e)^2 + \sum_{i=1}^3 \eta_i^2 z_i^2 + \sum_{1 \leq j < k \leq 3} 2\eta_j \eta_k z_j z_k + \sum_{i=1}^3 (2t\eta_e - 2K)\eta_i z_i. \tag{3.3.20}$$

We now square and rewrite the left hand side of (3.3.18). Since $P_V(z) = \langle z, u \rangle u + \langle z, v \rangle v + \langle z, w \rangle w$, we write $|P_V(z)|^2 = \langle z, u \rangle^2 + \langle z, v \rangle^2 + \langle z, w \rangle^2$. Finally we write $\langle z, u \rangle = z_1 u_1 + z_2 u_2 + z_3 u_3$ and expand the square to get:

$$r^2|P_V(z)|^2 = \sum_{i=1}^3 r^2(u_i^2 + v_i^2 + w_i^2)z_i^2 + \sum_{1 \leq j < k \leq 3} 2r^2(u_j u_k + v_j v_k + w_j w_k)z_j z_k \quad (3.3.21)$$

$$+ \sum_{i=1}^3 2tr^2(u_e u_i + v_e v_i + w_e w_i)z_i + r^2 t^2 (u_e^2 + v_e^2 + w_e^2), \quad (3.3.22)$$

$$= \sum_{i=1}^3 r^2|P_V(e_i)|^2 + \sum_{1 \leq j < k \leq 3} 2r^2 \langle P_V(e_i), P_V(e_j) \rangle z_j z_k \quad (3.3.23)$$

$$+ \sum_{i=1}^3 2tr^2 \langle P_V(e_i), P_V(e) \rangle z_i + r^2 t^2 |P_V(e)|^2. \quad (3.3.24)$$

(3.3.18) becomes:

$$\sum_{i=1}^3 a_i z_i^2 + \sum_{1 \leq j < k \leq 3} b_{jk} z_j z_k + \sum_{i=1}^3 c_i z_i = (K - t\eta_e)^2 - r^2 t^2 |P_V(e)|^2, \quad (3.3.25)$$

where

$$a_i = r^2|P_V(e_i)|^2 - \eta_i^2,$$

$$b_{jk} = 2r^2 \langle P_V(e_j), P_V(e_k) \rangle - 2\eta_j \eta_k,$$

and

$$c_i = (2tr^2 \langle P_V(e), P_V(e_i) \rangle - 2t\eta_e + 2K)\eta_i.$$

So each A_S is a quadric in $V_1 + \xi$.

Note that

$$S_1 = \bigcup_{S \in C^1} (A_S \cap S_1).$$

Indeed, if $z \in S_1$, then there exists $S \in C^1$ so that $S \in C^1(z)$. In other words, $dist(z, S) = 2r_1$ or $z \in A_S$. But there are only finitely many S 's in C^1 since Ω is a finite union of spheres. So there exists at least one $A_S \cap S_1$ of dimension 2. But two distinct quadrics can intersect only in a curve or a point if at all. This implies that A_S and S_1 are trivially identified. In other words we can identify the coefficients of the quadric (3.3.25) with the coefficients of the quadric

$$\sum_{i=1}^3 z_i^2 = r_1^2, \quad (3.3.26)$$

up to a multiple $\lambda \in \mathbb{R}$.

We now claim that this implies that $V = V_1$. Indeed since $\dim(V+V_1) \leq 5$, $\dim(V \cap V_1) \geq 1$. We can assume without loss of generality that $e_1 = u$ and $e_1 \in V \cap V_1$. In this case, we get $a_1 = r^2$ since $\eta_1 = 0$, η being normal to V . This implies that $\lambda = 1$ in (3.3.26) and consequently,

$$r^2|P_V(e_i)|^2 - \eta_i^2 = r^2,$$

for $i = 2, 3$.

But P_V being the projection on V ,

$$r^2|P_V(e_i)|^2 - \eta_i^2 < r^2,$$

unless $e_i \in V$ and $\eta_i = 0$. This proves that $V = V_1$

Now write

$$C^1 = C_1^1 \cup C_2^1$$

where

$$C_1^1 = \{S; \dim(A_S \cap S_1) < 2\},$$

and

$$C_2^1 = \{S; A_S \cap S_1 = S_1\}.$$

Since $V := V_1 = V_S$ for every $S \in C_2^1$, we have for $z \in S_1$ and $S \in C_2^1$,

$$D_S^2(z) - \delta_S^2(z) = |z - P_S(z)|^2 - |z - P_{V+\eta}(z)|^2, \quad (3.3.27)$$

$$= |P_V(z) - \frac{r_S}{r_1} P_V(z)|^2 + |P_{V^\perp}(z) - \eta|^2 - |P_{V^\perp}(z) - \eta|^2, \quad (3.3.28)$$

$$= (r_s - r_1)^2. \quad (3.3.29)$$

Thus $(D_S^2(z) - \delta_S^2(z))^{\frac{1}{2}} = r_s - r_1$ since $r_S \geq r_1$ (in fact, for the “first layer” C^1 , $r_S = r_1$, but we assume less so that the proof follows through for other layers) and

$$c_S = \frac{r_S}{2r_S - r} \quad (3.3.30)$$

Note that for $S \in C_2^1$, $c_S(z)$ is independent of z and

$$c_S > \frac{r_S}{2r_S} = \frac{1}{2} \quad (3.3.31)$$

By (3.3.31),

$$\sum_{S \in C^1(z)} c_S > (\#C_2^1) \cdot \frac{1}{2},$$

which implies that C_2^1 contains only one sphere. Call it S_2 .

Now, pick $z \in S_1 \setminus \left(\cup_{S \in C_1^1} A_S \cap S_1 \right)$. By the above, z has only S_2 in its first layer. Therefore, $c_{S_2}(z) = 1$ which implies that $r_{S_2} = r$. We finally deduce that $C_1^1 = \emptyset$. Indeed, suppose that $S \in C_1^1$ and $z \in A_S \cap S_1$. Since S_2 is also in the first layer of z , we have:

$$1 = \sum_{S \in C^1(z)} c_S \geq c_{S_2}(z) + c_S(z) > 1,$$

yielding a contradiction. This proves that $C^1 = \{S_2\}$.

Note that since C^1 is composed of a unique sphere of radius r_1 contained in $V_1 + \xi_i$ we have:

$$D_{S_2} = |P_V(z) + \xi_1 - P_V(z) - \xi_2| = |\xi_1 - \xi_2|.$$

In particular,

$$|\xi_1 - \xi_2| = 2r.$$

Moreover, for all z ,

$$\begin{aligned} \overline{D}_{S_2}(z)^2 &= |z - \overline{P}_s(z)|^2, \\ &= 4|P_V(z)|^2 + |\xi_1 - \xi_2|^2, \\ &= 8r^2. \end{aligned}$$

Now suppose that for some k , and all $i \leq k$

1. There exists a unique sphere S_{i+1} such that for all $z \in S_1$, $C^i(z) = \{S_{i+1}\}$,

2. $S_{i+1} \subset V + \xi_{i+1}$,
3. $r_{S_{i+1}} = r$,
4. For all $z \in S_1$, $D_{S_{i+1}}(z) = \overline{D}_{S_i}(z) = |\xi_1 - \xi_{i+1}| = 2\sqrt{i}r$.

Then repeating the exact same proof as for S_2 , while replacing $D_{S_2} = 2r$ with $D_{S_{i+1}} = 2\sqrt{i}r$, we get the same result for S_{i+2} .

Note that since $\Omega = -\Omega$, we have $\Omega = (\bigcup_{i=1}^m S_i) \cup (\bigcup_{i=1}^m -S_i)$.

Moreover for $i \leq m$, we have

$$\begin{aligned}
|\xi_1 + \xi_i|^2 &= 2(|\xi_1|^2 + |\xi_i|^2) - 2|\xi_1 - \xi_i|^2, \\
&= 4t^2 + 4(i)r^2, \\
&= 4(2m - 1 + i)r^2, \\
&= 4(i - 1)r^2.
\end{aligned}$$

We rename $-S_i$ to be S_{2m+1-i} for $i \leq m$.

We can now prove that $\{\xi_j\}$ is r -layered: choose any j and let $l_i(j)$ be such that $C^i(j) = \{S_{l_i(j)}\}$, where $C^i(j)$ is the i -th layer with respect to S_j . Since the spheres all have same radius the same proof as for S_1 can be repeated to show that for all $z \in S_i$, $D_{S_{l_j(i)}}(z) = 2\sqrt{j}r$ and $|\xi_i - \xi_{l_j(i)}| = D_{S_{l_j(i)}}(z)$. (1) and (2) from Definition 3.3.4 are obvious. We prove that l_i is bijective for every i . Indeed this follows from the fact that for all j ,

$$\bigcup_{i=0}^{2m-1} C^i(j) = \{S_1, \dots, S_{2m}\} = \{S_j, S_{l_1(j)}, \dots, S_{l_{2m-1}(j)}\}.$$

To show (3) from Definition 3.3.4, suppose that there existed j such that $l_i(j) = l_k(j)$. Then $C^i(j) = C^k(j)$ which would in turn imply that $D_{S_{l_i(j)}} = 2\sqrt{i}r = 2\sqrt{k}r = D_{S_{l_k(j)}}$ and $i = k$. Finally

$$|\xi_j - \xi_{l_i(j)}| = 2\sqrt{i}r \implies l_i \circ l_i(j) = j.$$

This proves that the centers are r -layered.

Finally to see that $r = \frac{1}{\sqrt{2m}}$, fix $z \in \text{supp}(\sigma)$ and consider $B(z, 2)$. We have $4\pi = \sigma(B(z, 2)) = \sum_{i=1}^{2m} \mathcal{H}^2(S_i) = 8m\pi r^2$ from which the claim follows. \square

As a consequence we get a classification of conical 3-uniform measures in \mathbb{R}^5 . We first need to prove a lemma stating that a set of layered points is the support of a discrete uniformly distributed measure.

Lemma 3.3.7. *Let $\{\xi_i\}_{i=1}^m \subset \mathbb{R}^d$ be an r -layered set of points. Then for any $c > 0$, the measure*

$$\lambda = c \sum_{i=1}^m \delta_{\{\xi_i\}}$$

is uniformly distributed.

Proof. Fix i . Then, for all $0 \leq j < m - 1$, for $2\sqrt{j}r \leq r \leq 2\sqrt{j+1}r$,

$$\lambda(B(\xi_i, r)) = \lambda(\{\xi_{l_k(i)}\}_{1 \leq k \leq j}), \quad (3.3.32)$$

$$= cj, \quad (3.3.33)$$

and if $r > 2\sqrt{m-1}r$, $\mu(B(\xi_i, r)) = cm$. \square

Theorem 3.3.8. *Let ν be a Radon measure in \mathbb{R}^5 such that for all $r > 0$, $\text{supp}(\nu) = r\text{supp}(\nu)$. Moreover, suppose that $\text{supp}(\nu)$ is not isometric to any affine 3-space nor to the cone in (1.1.4). Then ν is a 3-uniform measure if and only if, there exists $c > 0$ such that, up to isometry,*

$$\nu = c\mathcal{H}^3 \llcorner (C_1 \cup C_2), \quad (3.3.34)$$

where

$$C_1 = \{x; x_4 = 0\} \cap \{x; 3(x_1^2 + x_2^2 + x_3^2) = x_5^2\}, \quad (3.3.35)$$

and

$$C_2 = \{x; x_4 = 2\sqrt{2}x_5\} \cap \{x; 3(x_1^2 + x_2^2 + x_3^2) = x_4^2 + x_5^2\}. \quad (3.3.36)$$

Proof. First, by Theorem 3.3.6, $\Omega = \text{supp}(\nu) \cap \mathbb{S}^4$ is a layered union of $2p$ 2-spheres of same radius $r = \sqrt{\frac{1}{2p}}$, and there exists a linear 3-plane V such that for every sphere S_i of Ω , $S_i \subset V + \xi_i$ where ξ_i is the center of S_i and $|\xi_i| = \sqrt{1 - r^2}$. Moreover, for every i , $\xi_i \in V^\perp$. Since V is 3-dimensional, we can assume without loss of generality that $\{\xi_i\}_i \subset \mathbb{R}^2$. We want to prove that $p = 2$ unless $\text{supp}(\nu)$ is the Kowalski-Preiss cone. By Lemma 3.3.7, if σ is the spherical component of a conical 3-uniform measure, and Ω is its support, then the centers of the 2-spheres in Ω are the support of a discrete uniformly distributed measure on \mathbb{R}^2 , supported on $t\mathbb{S}^1$. By Proposition (2.4) in [KiP] where planar uniformly distributed measures with compact support are classified, these centers are either the vertices of a regular n -gon or the vertices of 2 regular n -gons of same center and same radius. The fact that, in the definition of r -layered points, for a fixed i , ξ_i cannot be equidistant to two other centers implies that the centers are either two antipodal points or two pairs of antipodal points. The first case reduces to the cone (1.1.4). Indeed, up to isometry, we can take the two centers to be $c_1 = (0, 0, 0, \frac{1}{\sqrt{2}}, 0)$ and $c_2 = -c_1$ since $r = \frac{1}{\sqrt{2}}$ implies that $|c_1 - c_2| = \sqrt{2}$. Then, taking the sphere S_1 to be:

$$S_1 = \left\{ (z_1, z_2, z_3, 0, 0) + c_1 ; z_1^2 + z_2^2 + z_3^2 = \frac{1}{2} \right\}, \quad (3.3.37)$$

$S_2 = -S_1$ and $\Omega = S_1 \cup S_2$, it is easily seen that Ω is the spherical component of the cone in (1.1.4).

As for the second case, we have $r = \frac{1}{2}$ and we get a rectangle with width 1 and length $\sqrt{2}$. Viewing the plane as embedded in \mathbb{R}^5 we can find the equation for the support of ν , up to isometry, in the following manner. Choose the centers of the 4 2-spheres $\{S_l\}_{l=1}^4$, each of which has radius $\frac{1}{2}$, to be $c_1 = \left(0, 0, 0, 0, \frac{\sqrt{3}}{2}\right)$, $c_2 = \left(0, 0, 0, \frac{\sqrt{2}}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, $c_3 = -c_1$ and $c_4 = -c_2$. One can easily verify that $|c_1 - c_2| = 1$, the line passing through c_1 and c_2 is parallel to the line passing through c_3 and c_4 and that these two lines are at a distance $\sqrt{2}$ of each other. Moreover suppose the sphere S_l is described by

$$S_l = \left\{ (z_1, z_2, z_3, 0, 0) + c_l, z_1^2 + z_2^2 + z_3^2 = \frac{1}{4} \right\}$$

Let C_1 and C_2 be the cones given by:

$$C_1 = \left\{ x \in \mathbb{R}^5 ; \frac{x}{|x|} \in S_1 \cup S_3 \right\}, \quad (3.3.38)$$

and

$$C_2 = \left\{ x \in \mathbb{R}^5 ; \frac{x}{|x|} \in S_2 \cup S_4 \right\}. \quad (3.3.39)$$

Then denoting the projection onto the i -th coordinate by p_i we have, on one hand:

$$x \in C_1 \iff p_4 \left(\frac{x}{|x|} \right) = 0 \text{ and } p_5 \left(\frac{x}{|x|} \right) = \pm \frac{\sqrt{3}}{2}, \quad (3.3.40)$$

and, on the other hand,

$$x \in C_2 \iff p_4 \left(\frac{x}{|x|} \right)^2 = \pm \frac{\sqrt{2}}{\sqrt{3}} \text{ and } p_5 \left(\frac{x}{|x|} \right) = \pm \frac{1}{2\sqrt{3}}. \quad (3.3.41)$$

This gives:

$$C_1 = \{x ; x_4 = 0\} \cap \{x ; 3(x_1^2 + x_2^2 + x_3^2) = x_5^2\}, \quad (3.3.42)$$

and

$$C_2 = \{x ; x_4 = 2\sqrt{2}x_5\} \cap \{x ; 3(x_1^2 + x_2^2 + x_3^2) = x_4^2 + x_5^2\}. \quad (3.3.43)$$

Then, by Proposition ??, ν is given by

$$\nu = c\mathcal{H}^3 \llcorner (C_1 \cup C_2), \quad (3.3.44)$$

for some $c > 0$.

□

Our aim now is to find a systematic way of producing layerings. To do this we need to define a graph associated to each layering and find conditions on the graph guaranteeing its embeddability in \mathbb{R}^d .

Definition 3.3.9. *Let σ be the spherical component of a 3-uniform conical measure ν in \mathbb{R}^d and let $\{\xi_i\}_{i=1}^{2p}$ be the centers of the spheres in its support, as described in Theorem 3.3.8. We define the graph G_ν of ν to be the weighted graph composed of*

1. the vertices $V(G) = \{v_i\}_{2p}$
2. the edges $E(G) = \{\{v_i, v_j\}\}_{1 \leq i < j \leq 2p}$
3. the weight $w(\{v_i, v_j\}) = d_{ij}$ where d_{ij} are the distance functions that arise from \mathcal{L}_ν .

More generally, we can define the graph associated to an r -layering of points in a similar way.

We start by proving the simple observation that a layering $\{l_i\}_{i=1}^m$ consists of an edge-coloring of the complete graph.

Proposition 3.3.10. *Let $G = K_{2p}$ be the complete graph on the vertices $\{v_i\}_{i=1}^{2p}$.*

Then $\mathcal{L} = \{l_i\}_{i=0}^m$ is a layering if and only if the assignment $c : E(K_{2p}) \rightarrow \{1, \dots, 2p-1\}$ of colors defined by $c(\{v_i, v_j\}) = d_{ij}$ is a $(2p-1)$ coloring of the edges of K_{2p} . We call $G_{\mathcal{L}}$ the graph associated to the layering.

Moreover, if there exist numbers d_{ij} , for $1 \leq i < j \leq 2p$ such that $d_{ij} \in \{1, \dots, 2p-1\}$ for all i, j and the assignment $c(v_i, v_j) = d_{ij}$ defines a $(2p-1)$ edge-coloring of G , define the functions $\mathcal{L} = \{l_k\}_{k=0}^{2p-1}$ in the following manner:

- $l_0(j) = j$ for all $j \in \{1, \dots, 2p-1\}$,
- $l_k(i) = j$ for $k > 0$, where j is the integer such that $d_{ij} = k$.

Then, up to relabeling of the vertices, \mathcal{L} is a layering.

Proof. The proof follows directly from the definition of a layering. Indeed, for c to define an edge-coloring, we only need to prove that $d_{ij} = d_{ik}$ implies that $j = k$. Clearly, if $d_{ij} = d_{ik}$ then $l_{d_{ij}}(i) = j$ and $l_{d_{ij}}(i) = k = j$.

Conversely, if G is as described, we first prove that the functions l_k are well-defined bijections. Pick any $k > 0$ and $1 \leq i \leq 2p$. Since v_i is adjacent to $2p-1$ edges, and c is a $(2p-1)$ coloring, there exists a unique j such that $d_{ij} = k$.

We can relabel the vertices so that $l_k(1) = k + 1$. The fact that $l_k^{-1} = l_k$ is a consequence of the fact that $d_{ij} = d_{ji}$. Finally, suppose that there exists j such $l_k(j) = l_{k'}(j) = i$. Then $k = k' = d_{ij}$. This ends the proof. □

This says that every conical 3-uniform measure gives rise to a $(2p - 1)$ edge-coloring of the complete graph K_{2p} for some p .

We now wish to get results in the other direction. In other words, if a weighted graph is given, what conditions will guarantee that there exists a 3-uniform conical measure associated to it? More precisely, by defining the weighted graph G associated to a $(2p - 1)$ -coloring of K_{2p} (which assigns to each edge the weight $c(\{v_i, v_j\}) = d_{ij}$), what conditions on G guarantee the existence of a conical 3-uniform measure ν such that $G = G_\nu$? By Theorem 3.3.8, every set of $2p$ r -layered points for $r = \sqrt{2p}$ gives rise to a 3-uniform measure. We will use 3.1.8 to find conditions on a set of distances d_{ij} associated to a layering that guarantee its embeddability in Euclidean space.

Definition 3.3.11. Let $\mathcal{L} = \{l_i\}_{i=1}^{2p-1}$ be a layering. We define the matrix $\Delta_{\mathcal{L}}$ associated to the layering to be

$$(\Delta_{\mathcal{L}})_{ij} = \frac{2p - 1 - 2d_{ij}}{2p - 1}.$$

Theorem 3.3.12. Let $p \in \mathbb{N}$, $\mathcal{L} = \{l_i\}_{i=0}^{2p-1}$ be a layering, $r = \frac{1}{\sqrt{2p}}$ and $t = \sqrt{1 - r^2} = \sqrt{\frac{2p-1}{2p}}$. Then there exists an r -layered set of $2p$ points $\{\xi_i\}_{i=1}^{2p}$ in $t\mathbb{S}^{2p-2}$ if and only if the spectral gap λ_G of the Laplacian of the graph $G_{\mathcal{L}}$ associated to the layering satisfies:

$$\lambda_G \geq p(2p - 1) \tag{3.3.45}$$

if and only if

$$\Lambda_G \geq 1$$

where Λ_G is the spectral gap of the normalized Laplacian of G .

Proof. By Theorem 3.1.8, if we take our semi-metric space to be $\{\xi_i\}_{i=1}^{2p}$ with the distance set $\left\{ \frac{\sqrt{2p-1}}{\sqrt{2p}} \arccos \left(\frac{2p-1-2d_{ij}}{n} \right) \right\}$, there exist points $\{\xi_i\}_{i=1}^{2p} \subset \mathbb{R}^{2p-1}$, $|\xi_i| = t$ with distance set $|\xi_i - \xi_j|_{tS} = \frac{\sqrt{2p-1}}{\sqrt{2p}} \arccos \left(\frac{2p-1-2d_{ij}}{n} \right) = \bar{d}_{ij}$ if and only if the matrix Δ given by:

$$\Delta_{ij} = \cos \left(\frac{\bar{d}_{ij}}{t} \right) = \frac{2p-1-2d_{ij}}{2p-1} \quad (3.3.46)$$

is positive semi-definite.

Note that for this choice of \bar{d}_{ij} , if we find points $\{\xi_i\}_{i=1}^{2p}$ with the prescribed distance set, their euclidean distance will be:

$$\begin{aligned} |\xi_i - \xi_j|^2 &= |\xi_i|^2 + |\xi_j|^2 - 2 \langle \xi_i, \xi_j \rangle, \\ &= 2 \cdot \frac{2p-1}{2p} - 2 \cdot \frac{2p-1}{2p} \cos \left(\frac{\bar{d}_{ij}}{t} \right), \\ &= 2 \cdot \frac{2p-1}{2p} - 2 \cdot \frac{2p-1}{2p} \cdot \frac{2p-1-d_{ij}}{2p-1}, \\ &= 4 \cdot d_{ij} \cdot \frac{1}{2p}. \end{aligned}$$

We will first rewrite the matrix Δ in terms of the Laplacian of G and the fact that Δ is positive semi-definite will then allow us to deduce the lower bound on λ_G . Denote the Laplacian of G by L . For $i \neq j$,

$$\Delta_{ij} = 1 - \frac{2}{2p-1} d_{ij} = 1 + \frac{2}{2p-1} L_{ij} \quad (3.3.47)$$

and for $i = j$,

$$\Delta_{ii} = 1 = 1 + \frac{2}{2p-1} \cdot \frac{2p(2p-1)}{2} - 2p = 1 + \frac{2}{2p-1} L_{ii} - 2p. \quad (3.3.48)$$

Therefore,

$$\Delta_{ij} = 1 - 2p\delta_{ij} + \frac{2}{2p-1} L_{ij},$$

where δ_{ij} is the Kronecker symbol. This follows from the fact that each vertex of G has degree $\frac{2p(2p-1)}{2}$. Indeed, each v_i has $2p-1$ edges adjacent to it, all of distinct weight between 1 and $2p-1$. So $d(v_i) = \sum_{i=1}^{2p-1} i = \frac{2p(2p-1)}{2}$.

This implies that

$$\Delta = J - 2pI_{2p} + \frac{2}{2p-1}L, \quad (3.3.49)$$

where J is the matrix whose entries are all 1 and I_{2p} is the identity matrix.

J has eigenvalues $2p$ and 0 , the vector $e_1 = (1, \dots, 1)$ is a common eigenvector of J for the eigenvalue $2p$ and of L for the eigenvalue 0 . Hence we can choose e_1 to be a common eigenvector corresponding to the 0 eigenvalue for L . Let e be an eigenvector of L orthogonal to e_1 and λ the corresponding eigenvalue. Since e is orthogonal to e_1 ,

$$\begin{aligned} \Delta \cdot e &= J \cdot e - 2p e + \lambda \frac{2}{2p-1} e, \\ &= \left(\frac{2\lambda}{2p-1} - 2p \right) e. \end{aligned}$$

Hence, Δ is positive semi-definite if and only if $\frac{2\lambda}{2p-1} - 2p \geq 0$ if and only if $\lambda \geq p(2p-1)$. In particular, if λ_G is the second smallest eigenvalue of L , Δ is positive semi-definite if and only if $\lambda_G \geq p(2p-1)$. The bound on Λ_G follows from the fact the vertices of G all have the same degree $p(p-1)$ which implies $L_{nor} = \frac{1}{p(2p-1)}L$.

□

The fact that the matrix Δ from the proof of Theorem 3.3.8 is positive semi-definite encodes information on the geometry of the set of points it describes. We start with a lemma.

Lemma 3.3.13. *If $\{\xi_i\}_{i=1}^{2p}$ is an r -layered set of points, and $\{l_i\}$ the functions layering it, then for $j = 1, \dots, 2p$ we have:*

1. $l_{2p-1}(j) = 2p + 1 - j$
2. $l_i \circ l_{2p-1} = l_{2p-1} \circ l_i = l_{2p-1-i}$

Proof. To prove 1., note that $|\xi_j - \xi_{l_{2p-1}(j)}| = 2\sqrt{2p-1}r = 2\frac{\sqrt{2p-1}}{\sqrt{2p}} = 2t$. So ξ_j and $\xi_{l_{2p-1}(j)}$ are antipodal points. Now pick j . Since ξ_j and $\xi_{l_{2p-1}(j)}$ are antipodal, we have:

$$|\xi_1 - \xi_j|^2 + |\xi_1 - \xi_{l_{2p-1}(j)}|^2 = |\xi_{l_{2p-1}(j)} - \xi_j|^2,$$

which implies, after dividing by $4r^2$, that

$$j - 1 + l_{2p-1}(j) - 1 = 2p - 1$$

since $l_i(j) = j + 1$ for all j . This proves 1. Now to prove 2. and 3., consider the rectangle formed by $\xi_j, \xi_{l_i(j)}, \xi_{l_{2p-1}(j)}, \xi_{l_{2p-1} \circ l_i(j)}$. We have

$$|\xi_j - \xi_{l_i(j)}|^2 + |\xi_j - \xi_{l_{2p-1} \circ l_i(j)}|^2 = 2p - 1.$$

This implies that $i + |\xi_j - \xi_{l_{2p-1} \circ l_i(j)}|^2 = 2p - 1$ and

$$l_{2p-1} \circ l_i = l_{2p-1-i}. \quad (3.3.50)$$

Applying l_i to the left in (3.3.50), we get:

$$l_{2p-1} = l_{2p-1-i} \circ l_i. \quad (3.3.51)$$

We obtain the other identities similarly. □

Theorem 3.3.14. *Let $\{l_i\}_{i=0}^{2p-1}$ be a layering and let Δ be the matrix $\Delta = J - 2pI_{2p} + \frac{2}{2p-1}L$ where J is the matrix with 1 in all its entries, I_{2p} is the identity matrix and L is the Laplacian of the graph associated to the layering. Then, if Δ is positive semi definite, there exists a matrix A of rank at most p such that:*

$$\Delta = A^T A \quad (3.3.52)$$

and the columns $\{\xi_i\}_{i=1}^{2p}$ of A form a set of r -layered points in $t\mathbb{S}^{2p-1}$ where $r = \sqrt{\frac{1}{2p}}$ and $t = \sqrt{\frac{2p-1}{2p}}$. Moreover p must be even.

Proof. Since Δ is positive semi-definite, there exists a set of r -layered points $\{\xi_i\}$ in $t\mathbb{S}^{2p-2}$ by Theorem 3.3.8.

We prove that p is even. Consider the sets $A_j = \{j, l_1(j), l_{2p-1}(j), l_{2p-2}(j)\}$. We claim that for $j \neq k$, either $A_j = A_k$ or $A_j \cap A_k = \emptyset$. Suppose that $A_j \cap A_k \neq \emptyset$ and let s be in the intersection. Notice that by Lemma 3.3.13 if $s \in A_j \cap A_k$, then $l_1(s), l_{2p-1}(s), l_{2p-2}(s)$ are all

in $A_j \cap A_k$. Since those elements are all distinct, $A_j = A_k$. Therefore these sets partition $\{1, \dots, 2p\}$ which implies that 4 divides $2p$ and p is even.

To prove that Δ has rank at most p , we rewrite it in a more convenient way. Let $\{e_j\}$ be an orthonormal basis of \mathbb{R}^{2p} . Define for each $i = 0, \dots, 2p-1$ the permutation matrix A_i defined by

$$A_i(e_j) = e_{l_i(j)}. \quad (3.3.53)$$

We claim that Δ can be written as:

$$\Delta = \sum_{i=0}^{p-1} \frac{2p-1-2i}{2p-1} (A_i - A_{2p-1-i}). \quad (3.3.54)$$

First note that $\frac{2p-1-2(2p-1-i)}{2p-1} = -\frac{2p-1-2i}{2p-1}$. Now the image of e_j by the matrix on the right of 3.3.53 is:

$$\begin{aligned} \sum_{i=0}^{p-1} \frac{2p-1-2i}{2p-1} (A_i \cdot e_j - A_{2p-1-i} \cdot e_j) &= \sum_{i=0}^{p-1} \frac{2p-1-2i}{2p-1} e_{l_i(j)} + \frac{2p-1-2(2p-1-i)}{2p-1} e_{l_{2p-1-i}(j)}, \\ &= \sum_{i=0}^{2p-1} \frac{2p-1-2d_{j,l_i(j)}}{2p-1} e_{l_i(j)}, \\ &= \sum_{k=0}^{2p-1} \Delta_{jk} e_k, \end{aligned}$$

proving the claim.

Consider the orthogonal basis $\{u_i\}_{i=1}^{2p}$ defined in the following way:

$$u_j = \begin{cases} e_j + e_{2p+1-j}, & j \leq p \\ e_j - e_{2p+1-j}, & j \geq p+1. \end{cases}$$

We claim that $\Delta u_j = 0$ for $j \leq p$ and $\Delta u_j \in \text{span}\{u_{p+1}, \dots, u_{2p}\}$ for $j \geq p+1$.

Indeed, for $j \leq p$,

$$\begin{aligned}
\Delta u_j &= \sum_{i=0}^{p-1} \frac{2p-1-2i}{2p-1} (A_i \cdot u_j - A_{2p-1-i} \cdot u_j), \\
&= \sum_{i=0}^{p-1} \frac{2p-1-2i}{2p-1} (A_i \cdot e_j + A_i \cdot e_{2p+1-j} - A_{2p-1-i} \cdot e_j - A_{2p-1-i} \cdot e_{2p+1-j}), \\
&= \sum_{i=0}^{p-1} \frac{2p-1-2i}{2p-1} (e_{l_i(j)} + e_{l_i(2p+1-j)} - e_{l_{2p-1-i}(j)} - e_{l_{2p-1-i}(2p+1-j)}), \\
&= \sum_{i=0}^{p-1} \frac{2p-1-2i}{2p-1} (e_{l_i(j)} + e_{l_{2p-1-i}(j)} - e_{l_{2p-1-i}(j)} - e_{l_i(j)}), \text{ by Lemma 3.3.50,} \\
&= 0
\end{aligned}$$

On the other hand, for $j \geq p+1$:

$$\begin{aligned}
\Delta u_j &= \sum_{i=0}^{p-1} \frac{2p-1-2i}{2p-1} (A_i \cdot u_j - A_{2p-1-i} \cdot u_j), \\
&= \sum_{i=0}^{p-1} \frac{2p-1-2i}{2p-1} (A_i \cdot e_j - A_i \cdot e_{2p+1-j} - A_{2p-1-i} \cdot e_j + A_{2p-1-i} \cdot e_{2p+1-j}), \\
&= \sum_{i=0}^{p-1} \frac{2p-1-2i}{2p-1} (e_{l_i(j)} - e_{l_i(2p+1-j)} - e_{l_{2p-1-i}(j)} + e_{l_{2p-1-i}(2p+1-j)}), \\
&= \sum_{i=0}^{p-1} \frac{2p-1-2i}{2p-1} (e_{l_i(j)} - e_{l_{2p-1-i}(j)} - e_{l_{2p-1-i}(j)} + e_{l_i(j)}), \text{ by Lemma 3.3.50,} \\
&= 2 \sum_{i=0}^{p-1} \frac{2p-1-2i}{2p-1} u_{\min(l_i(j), l_{2p-1-i}(j))}
\end{aligned}$$

This proves that Δ has rank at most p .

Finally, we describe how to find the corresponding r -layered points. Since $\Delta_{ij} = \langle \xi_i, \xi_j \rangle$ for the points whose existence is guaranteed by Theorem 3.1.8, if we find a matrix A with columns x_i such that

$$\Delta = A^\top A,$$

then $\Delta_{ij} = \langle x_i, x_j \rangle$ and we can set $\xi_i = x_i$. To find such a matrix, we diagonalize Δ . Since it is symmetric, there exists an orthogonal matrix P and a diagonal matrix D so that:

$\Delta = PDP^\top$. Since Δ is positive semi-definite, all the entries of D are non-negative. Denoting by $D^{\frac{1}{2}}$ the diagonal matrix with entries the square roots of the entries of D , we can write:

$$\Delta = PD^{\frac{1}{2}}D^{\frac{1}{2}}P^\top.$$

Choose A^\top to be $PD^{\frac{1}{2}}$. By Theorem [7.2.10] in [HJ], A and Δ have the same rank, which ends the proof. \square

We can now put those results together in the following theorem.

Theorem 3.3.15. *Let \mathfrak{G} be the set of weighted graphs G satisfying:*

- $G = K_{4p}$, $p \in \mathbb{N}$,
- G is weighted by $w : E(G) \rightarrow \{1, \dots, 4p - 1\}$ and the assignment of labels corresponding to w is an edge-coloring of G ,
- The second smallest eigenvalue λ_G of the (non-normalized) Laplacian of G satisfies:

$$\lambda_G \geq \frac{4p(4p - 1)}{2}.$$

For every graph $G \in \mathfrak{G}$, $|V(G)| = 4p$, let \mathcal{L} be the layering associated to it. Construct the set of points $\{\xi_i\}_{i=1}^{4p} \subset \mathbb{R}^{4p-1}$ associated to \mathcal{L} , set $c_i = (0, 0, 0, \xi_i)$ for $i = 1, \dots, 4p$ and define S_i to be the 2-sphere of radius $r = \sqrt{\frac{1}{4p}}$ centered at c_i , such that $S_i = (V + c_i) \cap \mathbb{S}^{4p+1}$ where $V = \mathbb{R}^3 \times \{0\}$. Setting $\Omega = \bigcup_{i=1}^{4p} S_i$ and

$$\Sigma = \left\{ x \in \mathbb{R}^{4p+2} ; \frac{x}{|x|} \in \Omega \right\} \cup \{0\},$$

and $\nu = \mathcal{H}^3 \llcorner \Sigma$, we have for all $x \in \Sigma$, $r > 0$,

$$\nu(B(x, r)) = \frac{4\pi}{3} r^3.$$

In particular, ν is 3-uniform.

Proof. The theorem is a direct consequence of Theorems 3.3.8, 3.3.14 and 3.3.12. \square

3.4 Examples of 3 uniform measures

We already know an example of an Ω that is locally 2-uniform. Indeed, if C is the KP-cone described in 1.1.4, take Ω to be its spherical component. Then by Theorem 2.2.5, Ω is locally 2-uniform. The following two results state that there exist other non-isometric examples.

Lemma 3.4.1. *Let $r = 2^{-\frac{n+1}{2}}$, $n = 0, 1, \dots$. Construct the rectangular parallelotope R_{n+1} in \mathbb{R}^{n+1} inductively in the following manner. Let α_1 be the origin and α_2 be any point such that $|\alpha_2| = 2r$. Assume the rectangular parallelotope R_k with vertices $\alpha_1, \dots, \alpha_{2^k}$ has been constructed and is contained in an affine k -plane L_k . Let γ_k be a vector normal to L_k such that $|\gamma_k| = 2\sqrt{2^k}r$. Set $\alpha_{2^{k+i}} = \alpha_i + \gamma_k$ for $i = 1, \dots, 2^k$.*

Then the vertices of R_{n+1} are r -layered and translating R_{n+1} , we can assume that its vertices are contained in $\partial B(0, t)$ where $t = \sqrt{1 - r^2}$.

Proof. Let α_1 be the origin and α_2 be any point such that $|\alpha_2| = 2r$, and define l_1 to be the permutation of 1 and 2. Moreover, let L_1 be the line passing through α_1 and α_2 . We construct the set inductively. Assume that for $1 \leq k \leq n$, we have constructed 2^k points $\{\alpha_l\}_{l=1}^{2^k}$, an affine k -plane L_k and an index set J_k such that:

- $\{\alpha_l\}_{l=1}^{2^k}$ are the vertices of a rectangular parallelotope R_k of edges $\{[\alpha_i \alpha_j]\}_{(i,j) \in J_k}$ contained in L_k
- The main diagonal of R_k has length $2\sqrt{2^k - 1}r$.

Let γ_k be a vector normal to L_k such that $|\gamma_k| = 2\sqrt{2^k}r$ and set $\alpha_{2^k+m} = \alpha_m + \gamma_k$ for $m = 1, \dots, 2^k$.

We first define the action of the existing layering functions on the newly constructed points in the following manner: for $i \in \{1, 2, \dots, 2^k\}$, define

$$l_i(2^k + m) = 2^k + l_i(m), \text{ for } m \in \{1, 2, \dots, 2^k\}. \quad (3.4.1)$$

Since $|\alpha_{2^k+l_i(m)} - \alpha_{2^k+m}| = |\alpha_m - \alpha_{l_i(m)}| = 2\sqrt{i}r$, then (3.3.14) from Definition 3.3.4 is satisfied. Moreover, $l_i^2(2^k + m) = l_i(2^k + l_i(m)) = 2^k + l_i^2(m) = 2^k + l_i^2(m)$. So $l_i^{-1} = l_i$.

We also claim that l_i is bijective on $\{1, 2, \dots, 2^{k+1}\}$. Indeed, on one hand on $\{1, \dots, k\}$ and

$$l_i(\{2^k + 1, 2^k + 2, \dots, 2^{k+1}\}) \subset \{2^k + 1, 2^k + 2, \dots, 2^{k+1}\}$$

,

On the other hand, for $m, m' \in \{1, 2, \dots, 2^k\}$,

$$\begin{aligned} l_i(2^k + m) = l_i(2^k + m') &\implies 2^k + l_i(m) = 2^k + l_i(m'), \\ &\implies m = m'. \end{aligned}$$

It is clear that 1 and 2 from Definition 3.3.4 are satisfied. To see that 3 is satisfied, suppose there exists j, i and p such that $l_i(2^k + j) = l_p(2^k + j)$. Then $2^k + l_i(j) = 2^k + l_p(j)$ implying that $i = p$. Since l_i is bijective on $\{1, \dots, 2^k\}$ by definition this proves the claim.

We now define the 2^k new layering functions. First define l_{2^k} in the following manner:

$$l_{2^k}(m) = \begin{cases} m + 2^k & \text{if } m \in \{1, 2, \dots, 2^k\} \\ m - 2^k & \text{if } m \in \{2^k + 1, 2^k + 2, \dots, 2^{k+1}\} \end{cases}$$

Clearly, l_{2^k} is a permutation, and satisfies (2), (4) and (3.3.14) from Definition 3.3.4. We claim that l_{2^k} also satisfies 3. Indeed suppose that there existed $i < 2^k$ and j such that $l_{2^k}(j) = l_i(j)$. Then either $j \leq 2^k$ in which case $l_i(j) \leq 2^k$ and $l_{2^k}(j) > 2^k$ yielding a contradiction. Similarly, if $j > 2^k$, $l_i(j) > 2^k$ while $l_{2^k}(j) \leq 2^k$. This proves the claim.

If $i < 2^k$, the fact that $l_i|_J$ and $l_{2^k}|_J$ have disjoint ranges for $J = \{1, \dots, 2^k\}$ or $J = \{2^k + 1, \dots, 2^{k+1}\}$ proves that $l_i(j) \neq l_{2^k}(j)$ for all j .

We now define the remaining layering functions $\{l_{2^k+i}\}$ in the following manner:

$$l_{2^k+i}(m) = \begin{cases} 2^k + l_i(m) & \text{if } m \in \{1, 2, \dots, 2^k\} \\ l_i^{-1}(m - 2^k) & \text{if } m \in \{2^k + 1, 2^k + 2, \dots, 2^{k+1}\} \end{cases}$$

If $m, i \in \{1, 2, \dots, 2^k\}$,

$$\begin{aligned}
|\alpha_m - \alpha_{2^k+l_i(m)}|^2 &= |\alpha_m - \alpha_{l_i(m)} - \gamma_k|^2, \\
&= |\alpha_m - \alpha_{l_i(m)}|^2 + |\gamma_k|^2, \text{ since } \gamma_k \perp (\alpha_m - \alpha_{l_i(m)}), \\
&= 4ir^2 + 4 \cdot 2^k r^2, \\
&= 4(2^k + i)r^2.
\end{aligned}$$

So (3.3.14) is satisfied. We claim that (3) is also satisfied. Indeed, on one hand, if $i < 2^k$, $p < 2^k$, the fact that $l_i(j) \neq l_{2^k+p}(j)$ for all j follows similarly as for l_{2^k} and l_i . On the other hand,

$$l_{2^k+i}(j) = l_{2^k}(j) \implies 2^k + l_i(j) = 2^k + j \implies j = 0.$$

It is easily seen that,

$$l_{2^k+i}(\{1, 2, \dots, 2^k\}) = \{2^k + 1, 2^k + 2, \dots, 2^{k+1}\}$$

and

$$l_{2^k+i}(\{2^k + 1, 2^k + 2, \dots, 2^{k+1}\}) = \{1, 2, \dots, 2^k\},$$

by definition of l_{2^k+i} and the bijectivity of l_i (and l_i^{-1}) on its domain. Therefore, l_{2^k+i} is bijective.

Finally for $j \leq 2^k$,

$$l_{2^k+i}^2(j) = l_{2^k+i}(2^k + l_i(j)) = 2^k + l_i^2(m) = 2^k + m.$$

A similar argument shows that $l_{2^k+i}^2(j) = j$ if $j > 2^k$. Therefore 4 is satisfied.

This proves that $\{\alpha_1, \dots, \alpha_{2^{k+1}}\}$ is an r -layered set.

Note that $\{\alpha_i\}_{i=1}^{2^{k+1}}$ are the vertices of a rectangular parallelotope contained in the $(k+1)$ -affine space L_{k+1} spanned by L_k and γ_k . This parallelotope has R_k as one of its faces, all the edges of R_k , $\{[\alpha_{i+2^k}\alpha_{j+2^k}]\}_{(i,j) \in J_k}$ and $\{[\alpha_i\alpha_{i+2^k}]\}_{1 \leq i \leq 2^k}$ as its edges. Moreover, the main diagonal of R_{k+1} has length $|\alpha_1 - \alpha_{2^{k+1}}| = 2\sqrt{2^{k+1} - 1}r$. By induction, repeating this process for $k = n$, we get 2^{n+1} points forming a rectangular parallelotope R_{n+1} in \mathbb{R}^{n+1} with main

diagonal having length $2\sqrt{2^{n+1}-1}r = 2t$. This implies that R_{n+1} is inscribed in a sphere of radius t . By translating, we can assume that R_{n+1} is inscribed in $\partial B_t(0)$.

Finally, note that $|\alpha_i - \alpha_{l_{2^{n+1}-1}(i)}| = 2t$. Since α_i and $\alpha_{l_{2^{n+1}-1}(i)}$ are in $\partial B_t(0)$ they must be antipodal points. Therefore, $\alpha_i = -\alpha_{l_{2^{n+1}-1}(i)}$ \square

This allows us to construct a locally 2-uniform measure in \mathbb{R}^{n+4} . More precisely,

Corollary 3.4.2. *Let $n \geq 1$, $r = \frac{1}{2^{n+1}}$, $t = \sqrt{1-r^2}$, and $\{\alpha_1, \dots, \alpha_{2^{n+1}}\}$ be an r -layered set as in Lemma 3.4.1, such that $|\alpha_j| = t$, for $j = 1, \dots, 2^{n+1}$. Define the points c_i in \mathbb{R}^{n+4} to be*

$$c_i = (0, 0, 0, \alpha_i)$$

and the corresponding 2-spheres S_i as:

$$S_i = \{z \in \mathbb{R}^{n+4}; z = (z_1, z_2, z_3, \alpha_i), z_1^2 + z_2^2 + z_3^2 = r^2.\} \quad (3.4.2)$$

In particular, for each i , $S_i \subset \mathbb{S}^{n+3}$. Let Ω be the set

$$\Omega = \left(\bigcup_{i=1}^{2^{n+1}} S_i \right), \quad (3.4.3)$$

and σ the measure

$$\sigma = \mathcal{H}^2 \llcorner \Omega. \quad (3.4.4)$$

Then for all $x \in \Omega$, for $r \leq 2$, we have:

$$\sigma(B(x, r)) = \pi r^2. \quad (3.4.5)$$

Proof. This follows directly from the fact that the c_j are r -layered and Theorem 3.3.5. \square

Using Corollary 3.1.4 we obtain the following theorem.

Theorem 3.4.3. *Let R_{n+1} be the parallelotope from Lemma 3.4.1, $n \geq 0$. For every $l = 1, \dots, 2^{n+1}$ set the point $c_l \in \mathbb{R}^{n+4}$ to be:*

$$c_l = (0, 0, 0, \alpha_l). \quad (3.4.6)$$

Let V be a linear 3-plane in \mathbb{R}^{n+4} , S_l be the 2-sphere

$$S_l = (V + c_l) \cap \mathbb{S}^{n+3},$$

centered at c_l , Ω be the set

$$\Omega = \bigcup_{l=1}^{2^{n+1}} S_l, \quad (3.4.7)$$

and Σ be the set

$$\Sigma = \left\{ x \in \mathbb{R}^{n+4}; \frac{x}{|x|} \in \Omega \right\} \cup \{0\}. \quad (3.4.8)$$

Then $\nu = \mathcal{H}^3 \llcorner \Sigma$ is a 3-uniform measure and for any $x \in \Sigma$, $r > 0$,

$$\nu(B(x, r)) = \frac{4}{3} \pi r^3, \quad (3.4.9)$$

Proof. This is a direct consequence of Corollary 3.4.2 and Lemma 3.1.4. \square

It is interesting to note however that these are not the only examples. To illustrate this, we give another example with less symmetry.

Example. Consider the graph whose adjacency matrix is given by

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 0 & 3 & 5 & 2 & 4 & 7 & 6 \\ 2 & 3 & 0 & 1 & 6 & 7 & 4 & 5 \\ 3 & 5 & 1 & 0 & 7 & 6 & 2 & 4 \\ 4 & 5 & 6 & 7 & 0 & 1 & 2 & 3 \\ 2 & 4 & 7 & 6 & 1 & 0 & 3 & 5 \\ 6 & 7 & 4 & 5 & 2 & 3 & 0 & 1 \\ 7 & 6 & 2 & 4 & 3 & 5 & 1 & 0 \end{bmatrix} \quad (3.4.10)$$

One can easily verify that the entries d_{ij} of the matrix A give an edge coloring of K_8 . Moreover the matrix 7Δ , where Δ is as in Definition 3.3.11, is given by:

$$\begin{bmatrix}
 7 & 5 & 3 & 1 & -1 & -3 & -5 & -7 \\
 5 & 7 & 1 & -3 & 3 & -1 & -7 & -5 \\
 3 & 1 & 7 & 5 & -5 & -7 & -1 & -3 \\
 1 & -3 & 5 & 7 & -7 & -5 & 3 & -1 \\
 -1 & -3 & -5 & -7 & 7 & 5 & 3 & 1 \\
 3 & -1 & -7 & -5 & 5 & 7 & 1 & -3 \\
 -5 & -7 & -1 & -3 & 3 & 1 & 7 & 5 \\
 -7 & -5 & 3 & -1 & 1 & -3 & 5 & 7
 \end{bmatrix} \tag{3.4.11}$$

which has 4 positive eigenvalues and 4 null eigenvalues. This means that there exist points $\{\xi_i\}_{i=1}^8$ with prescribed distances $|\xi_i - \xi_j| = \sqrt{\frac{d_{ij}}{2}}$, embedded in the sphere tS^3 where $t = \sqrt{\frac{7}{8}}$, which are well $\frac{1}{\sqrt{8}}$ -layered. This configuration is composed of the tetrahedron given by $\{\xi_i\}_{i=1}^4$ and its antipode on tS^3 . Of course, one can construct a 3-uniform measure in \mathbb{R}^7 using these points as in Theorem 3.3.5.

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