Conditioned Brownian motion in planar domains [1]

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Abstract. We give an upper bound for the Green functions of conditioned Brownian motion in planar domains. A corollary is the conditional gauge theorem in bounded planar domains.

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1. Introduction. Let D be a domain in $\mathbb{R}^2$ that has a Green function $g_D(x,y)$. If x, y, and z are in the same component of D, let

$$g_z(x,y) = \frac{g_D(x,y)g_D(y,z)}{g_D(x,z)}. \tag{1.1}$$

If x, y, z are not all in the same component, we set $g_z(x,y) = 0$. If $z \in D$, then $g_z(x,y)$ is the Green function of Brownian motion conditioned to hit z before exiting D. That is, if $z \in D$, then $g_D(\cdot, z)$ is harmonic in $D - \{z\}$, and $g_z(x,y)$ is the Green function of Brownian motion in $D - \{z\}$, h-path transformed by the function $g_D(\cdot, z)$. (See [Do] for further information on h-path transforms.)

Our main result is the following.

Theorem 1.1. *Suppose D is a bounded domain in $\mathbb{R}^2$. Then there exists a constant c_1 such that*

$$g_z(x,y) \le c_1\big(1 + \log^+(1/|x-y|) + \log^+(1/|y-z|)\big). \tag{1.2}$$

Here $\log^+ x = 0 \vee \log x$ and c_1 depends only on the diameter of D.

Our motivation in proving Theorem 1.1 is to obtain the conditional gauge theorem in arbitrary bounded planar domains. To state our result, we introduce some notation. Let $B(x,r)$ denote the open ball about x of radius r. A function $q : \mathbb{R}^2 \to \mathbb{R}$ is in the Kato class if

$$\limsup_{\varepsilon\to 0} \sup_x \int_D |q(y)| 1_{B(x,\varepsilon)}(y)(1 + \log^+(1/|x-y|))\, dy = 0. \tag{1.3}$$

Let Δ_1 be the minimal Martin boundary of D (see [Do]). The conditional gauge is the function

$$E_q(x,z) = \mathbb{E}_z^x \exp\Big(\int_0^{\tau_D} q(X_s)ds\Big),$$

where $x \in D$, $z \in \Delta_1$, $\mathbb{E}_z^x$ is the expectation of Brownian motion h-path transformed to go to z before exiting D, τ_D is the time to exit D, and X_t is the trajectory of the process.

Theorem 1.2. (Conditional gauge) *If $E_q(x_0, z_0) < \infty$ for some pair (x_0, z_0) with $x_0 \in D$, $z_0 \in \Delta_1$, it is finite for all pairs (x,z) with $x \in D$, $z \in \Delta_1$ and there exists c_2 such that*

$$c_2^{-1} \le E_q(x,z)/E_q(x_0,z_0) \le c_2, \qquad x \in D, z \in \Delta_1.$$

The conditional gauge theorem is of great importance in the study of the Schrödinger operator $(1/2)\Delta u + qu$. See [CFZ] for a good discussion of the history and uses of the conditional gauge theorem and a proof in the case of Lipschitz domains in $\mathbb{R}^d$, $d \ge 3$. See also [CZ]. In the case of $\mathbb{R}^2$, the conditional gauge theorem has been previously established for bounded Jordan domains in [Zh], for simply connected and certain other domains in

[McC], and for bounded domains satisfying a uniform capacity condition in [Cr]. Theorem 1.2 makes no assumptions on the domain D other than being bounded.

As will be apparent from our proof of Theorem 1.1, requiring the domain to be bounded is too strong. Let $\mathrm{Cap}_{B(x,r)}(E)$ denote the capacity of the set E with respect to the Green potential $g_{B(x,r)}$. (For definitions, see (1.8) below or [PS].)

Theorem 1.3. *Suppose D is a domain in $\mathbb{R}^2$ satisfying*

(a) D has a Green function.

(b) There exists c_3 such that if $y \in D$, there exists $r > 0$ (depending on y) such that

$$\mathrm{Cap}_{B(y,20r)}(D^c \cap B(y,r)) \geq c_3. \tag{1.4}$$

Let

$$h_y(x) = \sup\{g_D(w,y) : |w-y| = 3|x-y|/2000\}.$$

Then there exists c_4 such that

$$g_z(x,y) \leq c_4(1 + h_y(x) + h_y(z)), \qquad x,y,z \in D. \tag{1.5}$$

Proposition 2.4 below is one of the main estimates needed for the proof of Theorem 1.1. One may look at this proposition as a special case of the following more general statement, which has some interest of its own.

Suppose that D_n are connected planar domains. Fix some $R > 0$, $\alpha \in (0,1)$ and $\rho \in (0, \alpha R/2)$. Let h_n be the harmonic function in $B(x,\rho) \cap D_n$ with boundary values 1 on $\partial B(x,\rho) \cap D_n$ and 0 elsewhere. Note that $\mathbb{P}^x_{h_n}$ is the distribution of Brownian motion starting from x and conditioned to exit $B(x,\rho)$ before hitting D_n^c. We will write $\widetilde{\mathbb{P}}^x_{h_n}$ to denote the $\mathbb{P}^x_{h_n}$-distribution of $X_t - x$, i.e., the process shifted so that it starts from the origin. Let $\widetilde{\mathbb{P}}^x$ denote the distribution of Brownian motion starting from 0 and stopped upon hitting $\partial B(0,\rho)$. Our next result is concerned with weak convergence and for this reason it will be more convenient to assume that conditioned processes are stopped rather than killed at their lifetime. Note that with $\mathbb{P}^x_{h_n}$-probability 1, the left limit of the process is well defined at its lifetime.

Theorem 1.4. *Let $a_n = \mathrm{Cap}_{B(0,R)}(D_n^c \cap B(0,\alpha R))$. If $a_n \to 0$ then $\widetilde{\mathbb{P}}^x_{h_n}$ converges weakly to $\widetilde{\mathbb{P}}^x$. The convergence is uniform in the following sense. Given an open set $A \subseteq C[0,\infty)$ and $\beta > 0$, there exist $a_0 > 0$ such that if $a_n < a_0$ then $\widetilde{\mathbb{P}}^x_{h_n}(A) \geq \widetilde{\mathbb{P}}^x(A) - \beta$ for all domains D_n and all $x \in D_n \cap B(0,\alpha R/2)$.*

The point of Theorem 1.4 is that the convergence holds for all points $x \in D_n \cap B(0,\alpha R/2)$. If a_n is small then for "most" x, the probability $\mathbb{P}^x(\tau_{B(0,\alpha R)} < \tau_{D_n})$ is very close to 1 and for such x the conclusion of Theorem 1.4 follows immediately. However, if

$\mathrm{Cap}_{B(0,R)}(D_n^c \cap B(0,\alpha R)) > 0$ then there are necessarily some points $x \in D_n \cap B(0,\alpha R)$ such that $\mathbb{P}^x(\tau_{B(x,\rho)} < \tau_{D_n})$ is arbitrarily small. For these points, conditioning on the event $\{\tau_{B(x,\rho)} < \tau_{D_n}\}$ could conceivably change the distribution of the process; we show that this is not the case.

In the remainder of this section we introduce some further notation and present a brief summary of some facts about capacities. In the next section we prove four propositions about capacities and hitting probabilities. The techniques used here may be of independent interest. In Section 3 we prove Theorems 1.1, 1.2, and 1.3. We also discuss some extensions and lack of extensions to Theorems 1.2 and 1.3. Theorem 1.4 is proved in Section 4. An easy example supplied in Section 4 shows that Theorem 1.4 is false in dimensions greater than 2.

If U is a domain, g_U will denote the Green function and G_U the corresponding Green potential. For the ball $B(0,R)$ we have the formula

$$g_{B(0,R)}(x,y) = \frac{1}{\pi} \log\Big(\frac{|x|\,|y - R^2x/|x|^2|}{R|x-y|}\Big), \qquad x,y \in B(0,R). \tag{1.6}$$

If $x, y \in B(0,R)^c$, then $g_{B(0,R)^c}(x,y)$ is also given by the right-hand side of (1.6); see [PS], p. 114. We let $T_E = T(E)$ and $\tau_E = \tau(E)$ be the first hit to E and first exit from E, respectively. That is

$$T_E = \inf\{t : X_t \in E\}, \qquad \tau_E = \inf\{t : X_t \notin E\}.$$

E^r will denote the set of points that are regular for E. This means $x \in E^r$ if and only if $\mathbb{P}^x(T_E = 0) = 1$. If E is a set in a domain U, the capacitary or equilibrium measure for E is a measure μ concentrated on E^r such that $G_U\mu \le 1$ on U and $G_U\mu = 1$ on E^r. We have the formula

$$G_U\mu(x) = \mathbb{P}^x(T_E < \tau_U). \tag{1.7}$$

The capacity of E with respect to the Green potential G_U is defined to be

$$\mathrm{Cap}_U(E) = \mu(U) = \mu(E^r). \tag{1.8}$$

(We will not be using the notion of logarithmic capacity in this paper.) For proofs and further details, see [PS].

We will use $\mathbb{B}$ to denote the ball $B(0,2000)$.

There exists c_5 such that if I is a line segment of length at most 1 and λ is linear measure on I (i.e., one dimensional Lebesgue measure), then

$$G_{\mathbb{B}}\lambda(x) \le c_5, \qquad x \in I. \tag{1.9}$$

To see this, by (1.6) and symmetry considerations, $G_{\mathbb{B}}\lambda(x)$ will be largest if $x = 0$ and I is of length 1 and centered at the origin. (1.9) then follows from the finiteness of $\int_{-1/2}^{1/2} \log(1/r)dr$.

2. Capacity. In this section we prove a number of propositions concerning capacity. We suppose D is an open connected domain in $\mathbb{R}^2$, but not necessarily simply connected.

Proposition 2.1. *Let $\delta > 0$, $\alpha \in (0,1)$, and suppose*

$$\mathrm{Cap}_{\mathbb{B}}(D^c \cap \{B(0,110) - B(0,10\alpha)\}) \geq \delta.$$

There exists c_1 depending only on α such that if $x \in \partial B(0,200)$, then

$$\mathbb{P}^x(\tau_D < T_{B(0,\alpha)}) \geq c_1\delta. \tag{2.1}$$

Proof. Let $A = D^c \cap \{B(0,110) - B(0,10\alpha)\}$. We show first there exists c_2 such that

$$\mathrm{Cap}_{B(0,\alpha)^c}(A) \geq c_2^{-1}\delta. \tag{2.2}$$

Let μ be the capacitary measure of A with respect to $g_{\mathbb{B}}$. From the explicit formulas (1.6) for $g_{\mathbb{B}}(x,y)$ and $g_{B(0,\alpha)^c}(x,y)$, we conclude that there exists c_2 such that $g_{B(0,\alpha)^c}(x,y) \leq c_2 g_{\mathbb{B}}(x,y)$ if $x, y \in B(0,111) - B(0,9\alpha)$. Hence

$$\begin{aligned} G_{B(0,\alpha)^c}\mu(x) &= \int g_{B(0,\alpha)^c}(x,y)\mu(dy) \leq c_2 \int g_{\mathbb{B}}(x,y)\mu(dy) \\ &= c_2 G_{\mathbb{B}}\mu(x) \leq c_2 \end{aligned}$$

for $x \in \overline{A}$. By the maximum principle ([PS], p. 163), $G_{B(0,\alpha)^c}\mu \leq c_2$ on $B(0,\alpha)^c$. Consequently $c_2^{-1}\mu$ is a measure concentrated on A^r whose $G_{B(0,\alpha)^c}$ potential is bounded by 1 on $B(0,\alpha)^c$. It follows that

$$\mathrm{Cap}_{B(0,\alpha)^c}(A) \geq c_2^{-1}\mu(A^r) \geq c_2^{-1}\delta,$$

which is (2.2).

Next, let ν be the capacitary measure of A with respect to $g_{B(0,\alpha)^c}$. Let $\sigma(dz)$ be surface measure on $\partial B(0,200)$, normalized to have total mass 1. By rotational symmetry, $G_{B(0,\alpha)^c}\sigma = c_3$ on $\partial B(0,200)$ for some constant c_3. $G_{B(0,\alpha)^c}\sigma$ is harmonic off the support of σ and 0 on $\partial B(0,\alpha)$. So there exists c_4 such that if $x \in B(0,111) - B(0,9\alpha)$, then

$$G_{B(0,\alpha)^c}\sigma(x) = c_3\mathbb{P}^x(T_{\partial B(0,100)} < T_{\partial B(0,\alpha)}) \geq c_3c_4. \tag{2.3}$$

By the symmetry of $g_{B(0,\alpha)^c}(x,y)$ in x and y and Fubini's theorem,

$$\int G_{B(0,\alpha)^c}\nu(y)\sigma(dy) = \int G_{B(0,\alpha)^c}\sigma(x)\nu(dx) \geq c_3c_4\nu(\overline{A}) \geq (c_3c_4)(c_2^{-1}\delta). \tag{2.4}$$

Finally, by the Harnack inequality, there exists c_5 such that if $x, z \in \partial B(0,200)$,

$$\mathbb{P}^x(T_A < T_{B(0,\alpha)}) \geq c_5\mathbb{P}^z(T_A < T_{B(0,\alpha)}). \tag{2.5}$$

Integrating over z with respect to σ,

$$\begin{aligned}\mathbb{P}^x(\tau_D < T_{B(0,\alpha)}) \geq \mathbb{P}^x(T_A < T_{B(0,\alpha)}) &\geq c_5\int \mathbb{P}^z(T_A < T_{B(0,\alpha)})\sigma(dz)\\ &= c_5\int G_{B(0,\alpha)^c}\nu(z)\sigma(dz).\end{aligned}$$

Combining with (2.4) proves (2.1) with $c_1 = c_2^{-1}c_3c_4c_5$. □

Let

$$\begin{aligned} S &= \{re^{i\theta} : -\pi/2 < \theta < 2\pi, 6+\theta/\pi < r < 7+\theta/\pi\},\\ L &= \{re^{i\theta} : \theta = 2\pi, 8 < r < 9\},\\ R &= \{re^{i\theta} : 2\pi - 1/32 < \theta < 2\pi + 1/32, 5 < r < 10\}, \qquad \text{and}\\ W &= \{re^{i\theta} : 2\pi - 1/32 < \theta < 2\pi + 1/32, r = 5\}.\end{aligned}$$

Let x_0 be the point $6e^{-i\pi/4}$. Let γ be linear measure (i.e., one dimensional Lebesgue measure) on L.

Proposition 2.2. *There exist ε and c_6 such that if $\mathrm{Cap}_{\mathbb{B}}(D^c \cap R) \leq \varepsilon$ and*

$$B = \{x \in L : \mathbb{P}^x(T_W < T_{D^c\cup(\partial R - W)}) \geq c_6\}, \tag{2.6}$$

then $\gamma(B) \geq 1/2$.

Proof. Let $x' = 17/2$. By the support theorem for Brownian motion, there exists c_7 such that

$$\mathbb{P}^{x'}(T_W < T_{\partial R - W}) \geq c_7. \tag{2.7}$$

By Harnack's inequality, there exists c_8 such that if $x \in L$,

$$\mathbb{P}^x(T_W < T_{\partial R - W}) \geq c_8\mathbb{P}^{x'}(T_W < T_{\partial R - W}). \tag{2.8}$$

We will show

$$\gamma\{x \in L : \mathbb{P}^x(\tau_D \leq T_{\partial R}) > c_7c_8/2\} < 1/2. \tag{2.9}$$

That this proves the proposition with $c_6 = c_7c_8/2$ can be seen as follows: Recall that $\gamma(L) = 1$. If

$$\mathbb{P}^x(\tau_D \le T_{\partial R}) < c_7c_8/2,$$

then

$$\mathbb{P}^x(T_W < T_{D^c\cup(\partial R-W)}) \ge \mathbb{P}^x(T_W < T_{\partial R-W}) - \mathbb{P}^x(\tau_D \le T_{\partial R}) \ge c_7c_8/2,$$

or $x \in B$. So it suffices to show (2.9).

Let $E = D^c \cap R$. Let μ be the capacitary measure of E with respect to $g_{\mathbb{B}}$. Let

$$H(x) = \mathbb{P}^x(T_E < \tau_{\mathbb{B}}) = G_{\mathbb{B}}\mu(x).$$

Let $C = \{x \in L : H(x) \ge c_7c_8/2\}$. Then

$$\int_C G_{\mathbb{B}}\mu(x)\gamma(dx) \ge (c_7c_8/2)\gamma(C).$$

On the other hand,

$$\int_C G_{\mathbb{B}}\mu(x)\gamma(dx) \le \int G_{\mathbb{B}}\mu(x)\gamma(dx) = \int G_{\mathbb{B}}\gamma(x)\mu(dx).$$

By (1.9) there exists c_9 such that if $x \in L$,

$$G_{\mathbb{B}}\gamma(x) \le c_9.$$

By the maximum principle, $G_{\mathbb{B}}\gamma$ is bounded by c_9 on $\mathbb{B}$. Hence

$$(c_7c_8/2)\gamma(C) \le c_9 \int \mu(dx) = c_9\mu(\overline{E}) = c_9\mathrm{Cap}_{\mathbb{B}}(E) \le c_9\varepsilon.$$

If we choose ε less than $c_7c_8/4c_9$, then $\gamma(C) < 1/2$. Finally, note that $\{\tau_D \le T_{\partial R}\} \subseteq \{T_E < \tau_{\mathbb{B}}\}$, and so if $x \in L$ and $\mathbb{P}^x(\tau_D \le T_{\partial R}) > c_7c_8/2$, then $x \in C$. □

Proposition 2.3. *There exist ε and c_{10} such that if B is any subset of L with $\gamma(B) \ge 1/2$ and $\mathrm{Cap}_{\mathbb{B}}(S \cap D^c) \le \varepsilon$, then*

$$\mathbb{P}^{x_0}(X_{\tau_S} \in B, \tau_S < \tau_D) \ge c_{10}\mathbb{P}^{x_0}(\tau_S < \tau_D). \tag{2.10}$$

Proof. Let $E = D^c \cap S$. Let $H(x) = \mathbb{P}^x(T_E < \tau_{\mathbb{B}})$. S is a Lipschitz domain and so surface measure and harmonic measure are mutually absolutely continuous ([Da]). There exists c_{11} such that

$$\mathbb{P}^{x_0}(X_{\tau_S} \in B) \ge c_{11}. \tag{2.11}$$

Combining this with Harnack's inequality, there exists c_{12} such that if $x \in B(x_0, 1/300)$, then

$$\mathbb{P}^x(X_{\tau_S} \in B) \geq c_{12}. \tag{2.12}$$

Our first goal is to show that there exists $u \in [1/900, 2/900]$ such that

$$H \leq c_{12}/2 \qquad \text{on } \partial B(x_0, u). \tag{2.13}$$

Suppose (2.13) does not hold. Let μ be the capacitary measure of E with respect to $g_{\mathbb{B}}$. For every $u \in [1/900, 2/900]$, there exists $z_u \in \partial B(x_0, u)$ such that $H(z_u) > c_{12}/2$. Since $H(x) = G_{\mathbb{B}}\mu(x)$ is lower semicontinuous, there exists a radial line segment containing z_u on which $H > c_{12}/2$ (radial with respect to the center x_0). By compactness, there exist finitely many radial line segments such that $H > c_{12}/2$ on each one and for each $u \in [1/900, 2/900]$, $\partial B(x_0, u)$ intersects at least one of the radial line segments. By taking finite intersections, we can find radial line segments $I_1, \ldots, I_N$ (not necessarily open intervals) that are disjoint, on each of which $H \geq c_{12}/2$, each is contained in $\overline{B(x_0, 2/900)} - B(x_0, 1/900)$, and for each $u \in [1/900, 2/900]$, $\partial B(x_0, u)$ intersects exactly one I_i. Let $I = \cup_{i=1}^N I_i$ and let λ be arc length measure (i.e., linear measure) on I.

Since $H \geq c_{12}/2$ on I,

$$(c_{12}/2)(1/900) \leq \int_I G_{\mathbb{B}}\mu(x)\lambda(dx) = \int G_{\mathbb{B}}\lambda(y)\mu(dy). \tag{2.14}$$

For any fixed $y \in \overline{B(x_0, 2/900)} - B(x_0, 1/900)$, let I_y consist of a single radial line segment passing through y, that is, $I_y = x_0 + \{(r, \theta_0) : 1/900 \leq r \leq 2/900\}$, where $\theta_0 = \arg(y - x_0)$. By (1.9) there exists c_{13} such that if λ_y is linear measure on I_y, $G_{\mathbb{B}}\lambda_y(y) \leq c_{13}$. On $\partial B(x_0, v)$, $g_{\mathbb{B}}(y, z)$ is largest when $z \in I_y$. Thus

$$\begin{aligned} G_{\mathbb{B}}\lambda(y) &= \int_I g_{\mathbb{B}}(y, z)\lambda(dz) \leq \int_{I_y} g_{\mathbb{B}}(y, z)\lambda_y(dz) \\ &= G_{\mathbb{B}}\lambda_y(y) \leq c_{13}. \end{aligned}$$

Thus $G_{\mathbb{B}}\lambda(y) \leq c_{13}$ for $y \in I$. By the maximum principle, $G_{\mathbb{B}}\lambda \leq c_{13}$ for all $y \in \mathbb{B}$. Hence the right hand side of (2.14) is bounded by

$$c_{13}\mu(\overline{E}) = c_{13}\mathrm{Cap}_{\mathbb{B}}(E) \leq c_{13}\varepsilon.$$

If we take $\varepsilon \leq c_{12}/3600c_{13}$, then (2.13) is proved.

Next, suppose $x \in \partial B(x_0, u)$. Combining (2.13) and (2.12),

$$\mathbb{P}^x(X_{\tau_S} \in B, \tau_S < T_E) \geq c_{12}/2. \tag{2.15}$$

By the strong Markov property,

$$\mathbb{P}^{x_0}(X_{\tau_S} \in B, \tau_S < \tau_D) = \mathbb{E}^{x_0}\Big[\mathbb{P}^{X_{T(\partial B(x_0,u))}}(X_{\tau_S} \in B, \tau_S < \tau_D); T(\partial B(x_0,u)) < T_E\Big]$$
$$\geq (c_{12}/2)\mathbb{P}^{x_0}(T(\partial B(x_0,u)) < T_E). \tag{2.16}$$

Another application of the strong Markov property yields

$$\mathbb{P}^{x_0}(\tau_S < \tau_D) \leq \mathbb{P}^{x_0}(T(\partial B(x_0,u)) < T_E). \tag{2.17}$$

Combining (2.16) and (2.17) proves (2.10) with $c_{10} = c_{12}/2$. □

Let us say that "X_t makes a a loop around $B(0,R)$" if the graph $\{X_s : 0 \leq s \leq \tau_D \wedge T(\partial B(0,5R/4)) \wedge T(\partial B(0,5R/2))\}$ contains a closed curve with $B(0,R)$ contained in its interior.

Proposition 2.4. *There exist* ε *and* c_{14} *such that if* $\mathrm{Cap}_{\mathbb{B}}(D^c \cap B(0,100)) < \varepsilon$, *then*

$$\mathbb{P}^{x_0}(X_t \textit{ makes a loop around } B(0,4)) \geq c_{14}\mathbb{P}^{x_0}(T(\partial B(0,4)) < \tau_D). \tag{2.18}$$

Proof. Using Proposition 2.2, there exist ε and c_6 such that if $B \subseteq L$ is defined by (2.6), $\gamma(B) \geq 1/2$ and

$$\mathbb{P}^x(T_W < T_{D^c \cup (\partial R - W)}) \geq c_6$$

if $x \in B$. Taking ε smaller if necessary, we use Proposition 2.3 to see that there exists c_{10} such that (2.10) holds. If X_t starts at x_0, exits S in B before hitting D^c, and then exits R in W before hitting D^c, then X_t must make a loop around $B(0,4)$ before hitting time τ_D. Hence the left hand side of (2.18) is bigger than

$$\mathbb{P}^{x_0}(X_{\tau_S} \in B, \tau_S < \tau_D, X_{\tau_R} \circ \theta_{\tau_S} \in W, \tau_R \circ \theta_{\tau_S} < \tau_D \circ \theta_{\tau_S}).$$

By the strong Markov property, the choice of B, and (2.10), this is equal to

$$\mathbb{E}^{x_0}\Big[\mathbb{P}^{X_{\tau(S)}}(X_{\tau_R} \in W, \tau_R < \tau_D); X_{\tau_S} \in B, \tau_S < \tau_D\Big]$$
$$\geq c_6\mathbb{P}^{x_0}(X_{\tau_S} \in B, \tau_S < \tau_D)$$
$$\geq c_6 c_{10}\mathbb{P}^{x_0}(\tau_S < \tau_D)$$
$$\geq c_6 c_{10}\mathbb{P}^{x_0}(T(\partial B(0,4)) < \tau_D).$$

□

3. Green functions. About each point $y \in D$ we will determine a radius r such that $B(y, 2000r)$ contains at least some of D^c but not too much. Let

$$\eta = \mathrm{Cap}_{\mathbb{B}}(B(0,100) - B(0,50)).$$

Lemma 3.1. *Let $y \in D$ and $\varepsilon < \eta$. There exists $\alpha \in (0,1)$ (depending on ε but not y) and $r > 0$ (depending on ε and y) such that*

$$\mathrm{Cap}_{B(y,2000r)}(D^c \cap B(y,100r)) < \varepsilon \tag{3.1}$$

and

$$\mathrm{Cap}_{B(y,2000r)}(D^c \cap \{B(y,110r) - B(y,10\alpha r)\} > \varepsilon/2. \tag{3.2}$$

Proof. Fix y and ε and let

$$A_y = \{s : \mathrm{Cap}_{B(y,2000s)}(D^c \cap B(y,100s)) \geq \varepsilon\}.$$

This set is not empty, for if s is sufficiently large, $D \subseteq B(y,50s)$. Then

$$\mathrm{Cap}_{B(y,2000s)}(D^c \cap B(y,100s)) \geq \mathrm{Cap}_{B(y,2000s)}(B(y,100s) - B(y,50s)) = \eta$$

by scaling and translation invariance. Thus for s sufficiently large, $s \in A_y$.

On the other hand, since D is open, for s sufficiently small, $D^c \cap B(y,100s) = \emptyset$. If $s_0 = \inf A_y$, then $s_0 > 0$.

Choose $r \in (21s_0/22, s_0)$. Since $r < s_0$, $r \notin A_y$, and (3.1) holds. Choose $t \in (s_0, 21s_0/20) \cap A_y$. Then $\mathrm{Cap}_{B(y,2000t)}(D^c \cap B(y,100t)) > \varepsilon$. By our choice of t, $B(y,110r) \supseteq B(y,100t)$. Then $\mathrm{Cap}_{B(y,2000t)}(D^c \cap B(y,110r)) \geq \varepsilon$. Let μ be the capacitary measure of $D^c \cap B(y,110r)$ with respect to $g_{B(y,2000t)}$. Since $t > r$, then $g_{B(y,2000t)} \geq g_{B(y,2000r)}$ pointwise, and hence $G_{B(y,2000r)}\mu \leq 1$ in $B(y,2000r)$. It follows that

$$\mathrm{Cap}_{B(y,2000r)}(D^c \cap B(y,110r)) \geq \mu(D^c \cap B(y,110r)) \geq \varepsilon. \tag{3.3}$$

Finally, choose $\alpha < 1$ small so that $\mathrm{Cap}_{\mathbb{B}}(B(0,10\alpha)) < \varepsilon/2$. By scaling and translation invariance, $\mathrm{Cap}_{B(y,2000r)}(B(y,10\alpha r)) < \varepsilon/2$. (3.2) follows from this and (3.3). □

Lemma 3.2. *Let $y \in D$, $\varepsilon < \eta$, and let α and r be chosen as in Lemma 3.1 so that (3.1) and (3.2) hold. Suppose $w \in \partial B(y,3r)$. Then there exists c_2, depending on ε and α, but not y or r, such that*

$$g_D(w,y) \leq c_2. \tag{3.4}$$

Proof. Let $E = D \cup B(y,2\alpha r)$. Since $D \subseteq E$, it suffices to obtain the bound

$$g_E(w,y) \leq c_2, \qquad w \in \partial B(y,3r). \tag{3.5}$$

We use a renewal argument. Let

$$A = \sup_{v \in \partial B(y,200r)} \mathbb{E}^v \int_0^{\tau_E} 1_{B(y,\alpha r)}(X_s)ds,$$
$$\rho = \sup_{v \in \partial B(y,200r)} \mathbb{P}^v(T_{B(y,\alpha r)} < \tau_E).$$

By Proposition 2.1 and (3.2), $\rho < 1$. By the strong Markov property, if $v \in \partial B(y, 200r)$,

$$\begin{aligned}\mathbb{E}^v \int_0^{\tau_E} 1_{B(y,\alpha r)}(X_s)ds &= \mathbb{E}^v \Big[\mathbb{E}^{X_{T(B(y,\alpha r))}} \int_0^{\tau_E} 1_{B(y,\alpha r)}(X_s)ds; T_{B(y,\alpha r)} < \tau_E\Big] \\ &\le \mathbb{E}^v \Big[\mathbb{E}^{X_{T(B(y,\alpha r))}} \int_0^{\tau_{B(y,200r)}} 1_{B(y,\alpha r)}(X_s)ds\Big] \\ &\quad + \mathbb{E}^v \Big[\mathbb{E}^{X_{T(B(y,\alpha r))}} \int_{\tau_{B(y,200r)}}^{\tau_E} 1_{B(y,\alpha r)}(X_s)ds; T_{B(y,\alpha r)} < \tau_E\Big]. \end{aligned} \tag{3.6}$$

The first term on the right hand side of (3.6) is bounded by $c_3 r^2$ for some constant c_3 since $\sup_{z\in\mathbb{R}^2} \mathbb{E}^z \tau_{B(y,200r)} \le c_3 r^2$. If $u \in \partial B(y, \alpha r)$, by the strong Markov property

$$\mathbb{E}^u \int_{\tau_{B(y,200r)}}^{\tau_E} 1_{B(y,\alpha r)}(X_s)ds = \mathbb{E}^u \Big[\mathbb{E}^{X_{\tau(B(y,200r))}} \int_0^{\tau_E} 1_{B(y,\alpha r)}(X_s)ds\Big] \le A. \tag{3.7}$$

Hence the second term on the right hand side of (3.6) is bounded by

$$A\mathbb{P}^v(T_{B(y,\alpha r)} < \tau_E) \le \rho A.$$

Substituting these bounds in (3.6) and then taking the sup over $v \in \partial B(y, 200r)$,

$$A \le c_3 r^2 + \rho A.$$

A is finite, since

$$\mathbb{E}^v \tau_E \le \mathbb{E}^v \tau_{B(0,M)} < \infty. \tag{3.8}$$

Hence

$$A \le c_3 r^2/(1-\rho). \tag{3.9}$$

Since $w \in \partial B(y, 3r)$, $B(y, 2\alpha r) \subseteq E$, and $g_E(w, \cdot)$ is harmonic in E, by the Harnack inequality there exists c_4 such that

$$g_E(w, y) \le c_4 g_E(w, z), \qquad z \in B(y, \alpha r).$$

Integrating over $B(y, \alpha r)$,

$$g_E(w, y) \le c_4 |B(y, \alpha r)|^{-1} \int_{B(y,\alpha r)} g_E(w, z)dz = c_5 r^{-2} \mathbb{E}^w \int_0^{\tau_E} 1_{B(y,\alpha r)}(X_s)ds.$$

As in (3.6),

$$\mathbb{E}^w \int_0^{\tau_E} 1_{B(y,\alpha r)}(X_s)ds \le \mathbb{E}^w \tau_{B(y,200r)} + \mathbb{E}^w \mathbb{E}^{X_{\tau(B(y,200r))}} \int_0^{\tau_E} 1_{B(y,\alpha r)}(X_s)ds \le c_3 r^2 + A,$$

hence

$$g_E(w,z) \le c_5 r^{-2}[c_3 r^2 + A].$$

With (3.9) this proves (3.5). □

Proof of Theorem 1.1. Let x, y, z be distinct points in D. Fix y. Choose ε less than η and sufficiently small so that the conclusion of Proposition 2.4 holds. Choose r as in Lemma 3.1.

Suppose for now that $x, z \notin B(y, 2000r)$. We want an upper bound on $g_z(x,y)$, and by the maximum principle for functions that are harmonic with respect to h-path transformed Brownian motion (see [Do]), it suffices to get an upper bound on $g_z(w,y)$, $w \in \partial B(y, 6r)$. Let $\lambda = g_D(y,z)$. Since $g_D(\cdot, z)$ is lower semicontinuous, the set $\{v : g_D(v,z) > \lambda/2\}$ is open. Since $g_D(\cdot, z)$ is harmonic in $D - \{z\}$, y must be in the same component of $\{v : g_D(v,z) > \lambda/2\}$ as z by the maximum principle. Therefore there exists a curve C contained in D and connecting y and z such that $g_D(v,z) \ge \lambda/2$ for all $v \in C$.

Fix $w \in \partial B(y, 6r)$, and let $a = \mathbb{P}^w(T_{\partial B(y,4r)} < \tau_D)$. By Proposition 2.4, translation invariance, rotation invariance, and scaling, there exists c_6 such that

$$\mathbb{P}^w(X_t \text{ makes a loop around } B(y,4r)\ ) \ge c_6 a.$$

If X_t makes a loop around $B(y,4r)$, the path of X_t must intersect C, and thus

$$\mathbb{P}^w(T_C < \tau_D \wedge T_{B(y,4r)}) \ge c_6 a.$$

By the strong Markov property and the fact that $g_D(\cdot, z)$ is harmonic in $D - \{z\}$,

$$g_D(w,z) \ge \mathbb{E}^w[g_D(X_{T_C}, z); T_C < \tau_D \wedge T_{B(y,4r)}] \ge (c_6 a)\lambda/2. \tag{3.10}$$

Now let us look at $g_D(w,y)$. By the strong Markov property and Lemma 3.2,

$$g_D(w,y) \le \mathbb{E}^w[g_D(X_{T(B(y,3r))}, y); T_{B(y,3r)} < \tau_D] \le c_2 \mathbb{P}^w(T_{B(y,4r)} < \tau_D) \le c_2 a. \tag{3.11}$$

Substituting in the formula for $g_z(w,y)$,

$$g_z(w,y) = \frac{g_D(w,y) g_D(y,z)}{g_D(w,z)} \le \frac{(c_2 a)\lambda}{c_6 a \lambda/2} = \frac{2c_2}{c_6}, \tag{3.12}$$

as desired.

It remains to consider the case when x or z is in $B(y, 2000r)$. Suppose $|x-y| \le |z-y|$ and $|x-y| \le 2000r$. Let us set $r' = |x-y|/2000$. By the construction of Lemma 3.1, (3.1) still holds with r' in place of r. The argument we have just given in the first part of the proof still holds, except that we cannot use Lemma 3.2 and we need a substitute for (3.11).

Choose M so that $D \subseteq B(0,M)$. If $v \in \partial B(y, 3r')$,

$$g_D(v,y) \le g_{B(0,M)}(v,y) \le c_7(1+\log^+(1/|v-y|)) \le c_8(1+\log^+(1/r')) \tag{3.13}$$

for some constants c_7 and c_8 by the formula for $g_{B(0,M)}$. Since $r' = |x-y|/2000$,

$$g_D(v,z) \le c_9(1+\log^+(1/|x-y|)). \tag{3.14}$$

As in (3.11), if $w \in \partial B(y, 6r')$,

$$g_D(w,y) \le c_9 a(1+\log^+(1/|x-y|)). \tag{3.15}$$

If we use (3.15) in place of (3.11), we get

$$g_z(w,y) \le \frac{2c_9(1+\log^+(1/|x-y|))}{c_6}. \tag{3.16}$$

Finally, suppose $|z-y| < |x-y|$. Since $g_z(x,y) = g_x(z,y)$, we simply reverse the roles of x and z.

Checking where the constants come from, we see that they all depend only on the diameter of D. □

Remark. We learned the idea of using loops to estimate occupation times from [Dv].

Proof of Theorem 1.2. Let $x_0 \in D$. If z is in the Martin boundary of D, there exist $z_n \in D$ converging to some point $z_0 \in \partial D$ such that $g_D(x,z_n)/g_D(x_0,z_n) \to M(x,z)$, the Martin kernel with pole at z, and the convergence is uniform over x in compact subsets of $D - \{x_0\}$. If we replace z in (1.1) by z_n, divide the numerator and denominator by $g_D(x_0,z_n)$, and then let $n \to \infty$, we get from (1.2) that

$$\frac{g_D(x,y)M(y,z)}{M(x,z)} \le c_{10}(1+\log^+(1/|x-y|)+\log^+(1/|z_0-y|)).$$

The argument of [Cr], Theorem 6, now applies and gives Theorem 1.2. □

Proof of Theorem 1.3. We obtain Theorem 1.3 by examining where in the proof of Theorem 1.1 we used the fact that D was a bounded domain. There are three places. First, in the proof of Lemma 3.1 we needed to know that the set A_y was nonempty. This is implied by (1.4). Second, let $y \in D$. Since D has a Green function, it is easy to see that

$E = D \cup B(y, 2\alpha r)$ does also, provided $\alpha \in (0,1)$ and r is chosen as in Lemma 3.1. Since the Green function is locally integrable, a compactness argument shows that

$$\sup_{v \in \partial B(y,200r)} \int_0^{\tau_E} 1_{B(y,\alpha r)}(X_s) ds = \sup_{v \in \partial B(y,200r)} \int_{B(y,\alpha r)} g_E(v,z) dz < \infty.$$

This substitutes for (3.8). Finally, in place of (3.13) we use the definition of $h_y(x)$. $\square$

Remarks. 1. If D is contained in an infinite strip, then D satisfies the hypotheses of Theorem 1.3. It is not hard, using a renewal argument, to see there exists c_{11} such that

$$h_y(x) \leq c_{11}(1 + \log^+(1/|x-y|)).$$

(1.4) is proved the same way as in Lemma 3.1, and so (1.2) holds for domains contained inside strips.

2. The half plane is an example of a domain where Theorem 1.3 (a) and (b) hold but (1.2) does not. To see this, set $z = i$, $x_N = Ni$, $y_N = 1/2 + Ni$, and let $N \to \infty$.

3. One might ask whether in Theorem 1.2 one can weaken the conditions on q. For example, suppose D is bounded and suppose that we only require that

$$\limsup_{\varepsilon \to 0} \sup_x \int_D |q(y)| 1_{B(x,\varepsilon)}(y) g_D(x,y) dy = 0.$$

Does the conclusion of Theorem 1.2 still hold? (This is suggested by some results in [CFZ] and [McC].) It turns out the answer is no. The counterexample is rather lengthy and we do not present it here.

4. Weak convergence.

Proof of Theorem 1.4. We drop the subscript n throughout the proof. Suppose $b < 1$ and $r \in (0, \rho/6)$. Let $K = D^c \cap B(0, \alpha R)$ and

$$N = \{y \in B(x,3r) : \mathbb{P}^x(\tau_{B(0,R)} < T_K) > b\}.$$

Let $y_0 = (0, (1+\alpha)R/2)$. It follows from (1.6) that $g_{B(0,R)}(y_0, y) < c_1 < \infty$ for all $y \in B(0, \alpha R)$. Let μ be the capacitary measure for K in $B(0,R)$. Note that the mass of μ is equal to a. Hence

$$\mathbb{P}^{y_0}(T_K < \tau_{B(0,R)}) = G_{B(0,R)}\mu(y_0) \leq ac_1. \tag{4.1}$$

There exists c_2 such that if $A \subseteq B(0, \alpha R)$, then

$$\mathbb{P}^{y_0}(T_A < \tau_{B(0,R)}) \geq \mathbb{P}^{y_0}(X_{R^2 \wedge \tau(B(0,r))} \in A) > c_2|A|,$$

where $|A|$ stands for the area of A. This, (4.1), the strong Markov property applied at T_{N^c} and the definition of N imply that

$$|N^c|c_2 < \mathbb{P}^{y_0}(T_{N^c} < \tau_{B(0,R)}) \leq \mathbb{P}^{y_0}(T_K < \tau_{B(0,R)})/(1-b) \leq ac_1/(1-b).$$

Thus $|N^c| < ac_1/c_2(1-b)$. Fix b and r for the moment. If $\delta > 0$, we may choose a so small that

$$|N \cap B(x, 3r)| > (1-\delta)|B(x, 3r)| \tag{4.2}$$

and $N \cap (B(x, 2r) - B(x, r)) \neq \emptyset$.

Suppose that $b_1 < 1$ and let

$$V = \{y \in B(x, 2r) - B(x, r) : \mathbb{P}_h^x(T_N < \tau_{B(x,3r)}) > b_1\}.$$

We will argue that for a fixed b_1 and suitable choice of a, b and δ, every continuous path Γ starting at x and ending at a point of $\partial B(x, 2r)$ must intersect V. Suppose to the contrary that there is a path $\Gamma \subseteq V^c$. Choose any $y_1 \in N \cap (B(x, 2r) - B(x, r))$ (recall that this set is non-empty). In view of (4.2),

$$\mathbb{P}^{y_1}(T_N < \tau_{B(x,3r)}) > c_3$$

where $c_3 < 1$ may be chosen as close to 1 as we like provided δ is taken sufficiently small. The last inequality and the fact that $y_1 \in N$ imply in view of the definition of N that

$$\mathbb{P}_h^{y_1}(T_N < \tau_{B(x,3r)}) > c_3 - (1-b) = c_4. \tag{4.3}$$

Here $c_4 < 1$ can be taken as close to 1 as we like if b is chosen close to 1 and δ small. The probability that a Brownian path starting from y_1 will make a closed loop around $B(x, r)$ before leaving $B(x, 2r) - B(x, r)$ is greater than $c_5 > 0$. The $\mathbb{P}_h^{y_1}$-probability of the same event is greater than or equal to $c_5 - (1-b) = c_6$ and we may assume that $c_6 > 0$ by choosing a sufficiently large b. A path containing a closed loop around $B(x, r)$ within $B(x, 2r) - B(x, r)$ must intersect Γ, and therefore it must intersect V^c. By the strong Markov property applied at the hitting time of V^c,

$$\mathbb{P}_h^{y_1}(T_{\partial B(x,3r)} < T_N) \geq c_6(1-b_1). \tag{4.4}$$

If c_4 is sufficiently close to 1 then $c_6(1-b_1) > 1-c_4$, and therefore (4.3) and (4.4) contradict each other. This proves that every continuous path Γ starting at x and ending at a point of $\partial B(x, 2r)$ must intersect V.

The last remark implies that every trajectory of the $\mathbb{P}_h^x$-process must hit V and so by the strong Markov property it hits N before hitting $\partial B(x, 3r)$ with probability greater than

b_1. The process $\{X_s, s \leq T_N\}$ under $\mathbb{P}^x_h$ conditioned by $\{T_N < \tau_{B(x,3r)}\}$ is a conditioned Brownian motion in a subset of $B(x,r)$, so by the results of [CM] its expected lifetime is bounded by $c_7 r^2$. If we choose r sufficiently small, we can make this expected lifetime arbitrarily small. Note that the diameter of the trace of $\{X_t, t \leq T_N\}$ is bounded by $6r$ provided the event $\{T_N < \tau_{B(x,3r)}\}$ holds. The estimates of the diameter and T_N do not depend on the shape of D or x — they only depend on r. The process $\{X_s - x, s \geq T_N\}$ under $\mathbb{P}^x_h$ is an h-process in $D - x$ starting from a point of $N - x$. It is easy to see that for any open set $\widetilde{A} \subseteq C[0,\infty)$ and $\gamma > 0$, this process will take values in $\widetilde{A}$ with probability greater than $\widetilde{\mathbb{P}}^x(\widetilde{A}) - \gamma$ if we choose a suitably small r and large b in the definition of N.

One can easily prove the following remark about weak convergence. Suppose $A \subseteq C[0,\infty)$ is an open set and $\beta > 0$. Then one can find $\gamma > 0$ with the following properties. Let A_γ be the family of all functions in A whose Skorohod distance from A^c is greater than γ. Assume that for some process Y and a random time S, the trajectory of the process $\{Y(s), s \geq S\}$ is in A_γ with probability greater than $\widetilde{\mathbb{P}}^x(A_\gamma) - \gamma$. Moreover, assume that $S < \gamma$ and $\max_{s \leq S} |Y(s) - Y(0)| < \gamma$ with probability greater than $1 - \gamma$. Then the probability that $\{Y(s), s \geq 0\}$ is in A is greater than $\widetilde{\mathbb{P}}^x(A) - \beta$. This lemma applies to $X_t - x$ under $\mathbb{P}^x_h$ when $a \to 0$ according to the facts listed in the previous paragraph (T_N plays the role of S and A_γ plays the role of $\widetilde{A}$). This completes the proof. □

Theorem 1.4 may be used to give an alternative proof of Proposition 2.4.

Proof of Proposition 2.4. The event $B = \{T(\partial B(0,4)) < \tau_{B(x_0,30)}\}$ is a closed set in $C[0,\infty)$.

In this proof, "X_t makes a a loop around $B(0,R)$" will refer to the event "the graph $\{X_s : 0 \leq s \leq T_{\partial B(0,5R/4)} \wedge T_{\partial B(0,5R/2)}\}$ contains a closed curve with $B(0,R)$ contained in its interior" (we do not require that the loop is made before hitting D^c). It is easy to see that this event contains an open event $A \subseteq C[0,\infty)$ with $\mathbb{P}^{x_0}(A) > 0$.

Let $\mathbb{P}^{x_0}_h$ denote the distribution $\mathbb{P}^{x_0}$ conditioned by $\{T_{\partial B(x_0,30)} < \tau_D\}$. Since A is open and B is closed, Theorem 1.4 shows that for a fixed $\beta > 0$, we may find $\varepsilon > 0$ sufficiently small so that $\mathrm{Cap}_{\mathbb{B}}(D^c \cap B(0,100)) < \varepsilon$ implies that $\mathbb{P}^{x_0}_h(A) \geq \mathbb{P}^{x_0}(A) - \beta$ and $\mathbb{P}^{x_0}(B) \geq \mathbb{P}^{x_0}_h(B) - \beta$.

Note that $B_1 = \{T(B(0,3)) < \tau_{B(x_0,30)}\}$ is an open set in $C[0,\infty)$ and $B_1 \subseteq B$. For β sufficiently small we have

$$\mathbb{P}^{x_0}_h(B) \geq \mathbb{P}^{x_0}_h(B_1) \geq \mathbb{P}^{x_0}(B_1) - \beta > 0.$$

Hence, it is possible to choose $c_1 > 0$ such that $\mathbb{P}^{x_0}(A) \geq c_1 \mathbb{P}^{x_0}(B)$ and $c_2, \beta > 0$ so that $c_1(\mathbb{P}^{x_0}_h(B) - \beta) - \beta \geq c_2 \mathbb{P}^{x_0}_h(B)$ (we may have to adjust ε).

Then

$$
\begin{aligned}
&\mathbb{P}^{x_0}(X_t \text{ makes a loop around } B(0,4) \text{ before } \tau_D)\\
&\quad \geq \mathbb{P}^{x_0}(X_t \text{ makes a loop around } B(0,4) \text{ and } T_{\partial B(x_0,30)} < \tau_D)\\
&\quad = \mathbb{P}_h^{x_0}(X_t \text{ makes a loop around } B(0,4))\mathbb{P}^{x_0}(T_{\partial B(x_0,30)} < \tau_D)\\
&\quad \geq \mathbb{P}_h^{x_0}(A)\mathbb{P}^{x_0}(T_{\partial B(x_0,30)} < \tau_D)\\
&\quad \geq (\mathbb{P}^{x_0}(A) - \beta)\mathbb{P}^{x_0}(T_{\partial B(x_0,30)} < \tau_D)\\
&\quad \geq (c_1\mathbb{P}^{x_0}(B) - \beta)\mathbb{P}^{x_0}(T_{\partial B(x_0,30)} < \tau_D)\\
&\quad \geq (c_1(\mathbb{P}_h^{x_0}(B) - \beta) - \beta)\mathbb{P}^{x_0}(T_{\partial B(x_0,30)} < \tau_D)\\
&\quad \geq c_2\mathbb{P}_h^{x_0}(B)\mathbb{P}^{x_0}(T_{\partial B(x_0,30)} < \tau_D)\\
&\quad = c_2\mathbb{P}_h^{x_0}(T(\partial B(0,4)) < \tau_{B(x_0,30)})\mathbb{P}^{x_0}(T_{\partial B(x_0,30)} < \tau_D)\\
&\quad = c_2\mathbb{P}^{x_0}(T(\partial B(0,4)) < \tau_{B(x_0,30)}, T_{\partial B(x_0,30)} < \tau_D)\\
&\quad \geq c_2\mathbb{P}^{x_0}(T(\partial B(0,4)) < \tau_D).
\end{aligned}
$$

□

Example 4.1. Let $D = D(\varepsilon) = K^c$ where

$$K = \{x = (x_1, x_2, x_3) \in R^3 : |x_1| \leq 1, x_2^2 + x_3^2 = \varepsilon\}.$$

The capacity of K in $B(0,20)$ can be made arbitrarily close to 0 by taking ε small enough. However, Brownian motion starting from 0 and conditioned to hit $\partial B(0,2)$ before hitting D^c does not converge in distribution to standard Brownian motion as ε goes to 0 because it has to travel a distance 1 within a very thin tube. This shows that Theorem 4.1 cannot be generalized to higher dimensions.

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