Permutation diagrams in symmetric function theory and Schubert calculus

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Abstract

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A fundamental invariant of a permutation is its inversion set, or diagram. Natural machinery in the representation theory of symmetric groups produces a symmetric function from any finite subset of $\mathbb{N}^2$, namely its *generalized Schur function*. When this subset is a permutation diagram, one obtains the Stanley symmetric function of a permutation. By exploiting this connection to representation theory, we obtain results on the interaction of pattern avoidance with the theory of Stanley symmetric functions. In particular, we show that for any positive integer k, the permutations whose Stanley symmetric function has at most k (classical) Schur function terms are exactly those which avoid a finite set of patterns.

The cohomology ring of a Grassmannian is a quotient of the ring of symmetric functions, and Liu has given a class of subvarieties—the diagram varieties—whose cohomology classes are conjecturally represented by the generalized Schur functions. We give a counterexample to this conjecture. On the other hand, we use a degeneration of Coskun's rank varieties to show that Liu's conjecture does give an upper bound on the classes of diagram varieties of permutation diagrams. We also show that the cohomology class of any rank variety is represented by a Stanley symmetric function, using Knutson–Lam–Speyer's work on positroid varieties.

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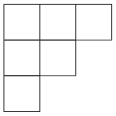
Chapter 1

INTRODUCTION

A *diagram* is a finite subset of $\mathbb{N}^2$, frequently drawn as a collection of boxes. Diagrams are ubiquitous in algebraic combinatorics; for instance, the assignments of a positive integer to each box of



satisfying various rules naturally count:

- The dimension of an irreducible representation of the symmetric group $\mathfrak{S}_6$.
- The dimension of an irreducible $\mathrm{GL}(N, \mathbb{C})$-subrepresentation inside $(\mathbb{C}^N)^{\otimes 6}$.
- The determinant
$$\det \begin{bmatrix} \binom{2+N}{3} & \binom{3+N}{4} & \binom{4+N}{5} \\ \binom{N}{1} & \binom{1+N}{2} & \binom{2+N}{3} \\ 0 & 1 & \binom{N}{1} \end{bmatrix}.$$
- The products of adjacent transpositions $(i\ i+1)$ which equal the reverse permutation 4321 and have minimal length (six).
- The degree as a projective variety of the set of 3-dimensional subspaces of $\mathbb{C}^6$ having reduced row echelon form
$$\begin{bmatrix} 1 & * & 0 & * & 0 & * \\ 0 & 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & * \end{bmatrix},$$

as a subvariety of the Grassmannian (here $*$ represents an arbitrary complex number).

- The permutations in $\mathfrak{S}_6$ equivalent to 645123 under the transitive closure of the subword relations $\cdots xzy \cdots \sim \cdots zxy \cdots$ and $\cdots yxz \cdots \sim \cdots yzx \cdots$ when $x < y < z$ (this is known as *Knuth equivalence*).

A permutation $w \in \mathfrak{S}_n$ is characterized by its inversions—pairs of indices $i < j$ with $w(i) > w(j)$—and one can think of the set of these pairs as a diagram. These permutation diagrams form a central thread of this thesis, in the sense that we will investigate various algebraic and geometric objects associated to them.

We begin by discussing three closely related rings: the ring of representations of symmetric groups, the ring of symmetric functions, and the cohomology ring of a Grassmannian. More details about these objects and the connections between them can be found in [9].

1 Representations of $\mathfrak{S}_n$

Let R_n be the $\mathbb{Q}$-vector space with basis consisting of the isomorphism classes $[V]$ of finite-dimensional complex $\mathfrak{S}_n$-representations, modulo the relations $[V \oplus W] = [V] + [W]$. We will define[1] a multiplication which makes $\bigoplus_{n=0}^{\infty} R_n$ into a graded commutative algebra.

Given groups $H \subseteq G$ and a representation V of H, we define a G-representation $\operatorname{Ind}_H^G V$, called the *induction* of V from H to G, as follows. The set $G \times V$ is a vector space with operations $(g, v) + (g, w) = (g, v + w)$ and $c(g, v) = (g, cv)$, and is a representation of G via the action $g'(g, v) = (g'g, v)$. To account for the pre-existing action of H, we take $G \times V$ modulo the relations $(g, hv) = (gh^{-1}, v)$ for $h \in H$, and

---

[1] Notice this is *not* the same as the representation ring of a single $\mathfrak{S}_n$, which is just R_n equipped with tensor product as multiplication.

this quotient space is $\operatorname{Ind}_H^G V$. More concisely,

$$\operatorname{Ind}_H^G V = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V,$$

where $\mathbb{C}[G]$ is the complex group ring of G.

Now define a bilinear map $\circ : R_m \times R_n \to R_{m+n}$ by

$$[V] \circ [W] = [\operatorname{Ind}_{\mathfrak{S}_m \times \mathfrak{S}_n}^{\mathfrak{S}_{m+n}} V \otimes W].$$

This is called the *induction product*. Let R be the graded ring $\bigoplus_{n=0}^{\infty} R_n$.

A *diagram* is a finite subset of $\mathbb{N} \times \mathbb{N}$, whose elements will be called *cells*. We draw diagrams in matrix coordinates, so that $(1,1)$ is in the upper left, and use $\circ$ and $\cdot$ respectively for elements and non-elements of D:

$$\{(1,1),(1,2),(2,2),(2,3)\} = \begin{array}{ccc} \circ & \circ & \cdot \\ \cdot & \circ & \circ \end{array}$$

We now construct a family of complex $\mathfrak{S}_n$-modules indexed by diagrams D with n cells, which will include the irreducible representations. For general diagrams D, these modules were introduced by James and Peel [11].

A *bijective filling* of an n-cell diagram D is a bijection $T : D \to [n]$, where $[n]$ denotes $\{1, 2, \ldots, n\}$. Given a bijective filling T, define two subgroups of $\mathfrak{S}_n$:

$$\begin{aligned}
\operatorname{Row}(T) &= \{\sigma \in \mathfrak{S}_n : \text{whenever } a, b \in [n] \text{ are in the same row of } T\text{, so are } \sigma(a), \sigma(b)\},\\
\operatorname{Col}(T) &= \{\sigma \in \mathfrak{S}_n : \text{whenever } a, b \in [n] \text{ are in the same column of } T\text{, so are } \sigma(a), \sigma(b)\}.
\end{aligned}$$

The *Young symmetrizer* of T is

$$y_T = \sum_{\substack{p \in \operatorname{Row}(T) \\ q \in \operatorname{Col}(T)}} \operatorname{sgn}(q) q p,$$

an element of the group algebra $\mathbb{C}[\mathfrak{S}_n]$.

Definition 1.1.1 ([11])**.** The *Specht module* of a diagram D is the (left) $\mathfrak{S}_n$-module

$$S^D = \mathbb{C}[\mathfrak{S}_n]y_T$$

for a fixed choice of bijective filling T of D.

Notice that the choice of bijective filling does not affect the isomorphism type of S^D, because replacing T by σT has the effect of conjugating $\mathrm{Row}(T)$, $\mathrm{Col}(T)$, and y_D by σ.

A *partition* is a weakly decreasing finite sequence of positive integers $\lambda = (\lambda_1 \geq \cdots \geq \lambda_\ell)$. We write $\ell(\lambda)$ for the length of λ, and $|\lambda|$ for $\sum_i \lambda_i$. We also sometimes say that λ *is a partition of* n to mean $|\lambda| = n$. When unambiguous, we write partitions in the form 321 rather than $(3, 2, 1)$. The *Young diagram* of λ is

$$\{(i, j) : 1 \leq i \leq \ell, 1 \leq j \leq \lambda_i\};$$

This definition follows the English convention for Young diagrams, so that, for example, the diagram of 4221 is

$$\begin{array}{cccc} \circ & \circ & \circ & \circ \\ \circ & \circ & \cdot & \cdot \\ \circ & \circ & \cdot & \cdot \\ \circ & \cdot & \cdot & \cdot \end{array}$$

The modules S^λ form a complete, irredundant set of irreducible complex representations for $\mathfrak{S}_n$ as λ ranges over partitions of n.

Example 1.1.2. The Specht module $S^{(n)}$ is the trivial representation of $\mathfrak{S}_n$, while $S^{(1^n)}$ is the one-dimensional representation where $w \in \mathfrak{S}_n$ acts as multiplication by its sign; here 1^n is the partition with n 1's. For a non-partition example, take $D = \{(i, i) : 1 \leq i \leq n\}$. Then S^D is the regular representation of $\mathfrak{S}_n$.

The algebraic properties of S^D frequently capture some interesting combinatorics associated with the diagram D. If λ is a partition, then $\dim S^\lambda$ is the number of standard tableaux of shape λ; these are the bijective fillings of λ which increase down columns and rightward across rows. This is also the case for the skew shape λ/μ, which is the diagram $\lambda \setminus \mu$ when $\mu \subseteq \lambda$ (as diagrams). For instance, if $\lambda = 3321$ and $\mu = 21$, so

$$\lambda/\mu = \begin{array}{ccc} \cdot & \cdot & \circ \\ \cdot & \circ & \circ \\ \circ & \circ & \cdot \\ \circ & \cdot & \cdot \end{array},$$

then $\dim S^{\lambda/\mu}$ is the number of permutations $w_1w_2w_3w_4w_5w_6$ satisfying $w_1 < w_2 > w_3 < w_4 > w_5 < w_6$, the *alternating permutations* in $\mathfrak{S}_6$.

The diagrams of greatest interest for us will be the permutation diagrams.

Definition 1.1.3. The *permutation diagram* (or just diagram) of $w \in \mathfrak{S}_n$ is

$$D(w) = \{(i,j) : 1 \leq j < w(i), 1 \leq i < w^{-1}(j)\}.$$

Example 1.1.4. We will usually write permutations in one-line notation, so that $w = 31524$ sends 1 to 3, 2 to 1, etc. When we need cycle notation, we will use parentheses: $31524 = (1\,3\,5\,4\,2)$. The diagram of this permutation is

$$D(31524) = \begin{array}{ccccc} \circ & \circ & \times & \cdot & \cdot \\ \times & \cdot & \cdot & \cdot & \cdot \\ \cdot & \circ & \cdot & \circ & \times \\ \cdot & \times & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \times & \cdot \end{array}.$$

Here we have decorated the diagram with $\times$ at the positions $(i, w(i))$, though these are not actually members of $D(w)$; the diagram $D(w)$ contains exactly those cells in $[5] \times [5]$ which do not lie directly below or right of an $\times$.

The objects counted by $\dim S^{D(w)}$ turn out to be the *reduced words* of w: the minimal-length products of adjacent transpositions $(i\ i+1)$ which equal w. We will be interested in a more refined object associated to w and $S^{D(w)}$, namely the Stanley symmetric function F_w, but first we must discuss symmetric functions in general.

2 *Symmetric functions*

Definition 1.2.1. A formal power series $f \in \mathbb{Q}[[x_1, x_2, \ldots]]$ is a *symmetric function* if it has finite degree and

$$f(x_1, x_2, \ldots) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots)$$

for every permutation σ of $\mathbb{N}$ fixing all but finitely many points. The symmetric functions form a subring of $\mathbb{Q}[[x_1, x_2, \ldots]]$ graded by degree, which we denote by Λ, with nth graded piece Λ_n.

Remark 1.2.2. The assumption of finite degree rules out, for example, $\sum_{n=1}^{\infty} \sum_{i=0}^{\infty} x_n^i$. This ensures that Λ is graded.

The dimension of Λ_n is the number of partitions of n. Indeed, if m_λ is the sum of all monomials $x_1^{a_1} x_2^{a_2} \cdots$ whose exponent vector $(a_1, a_2, \ldots)$ is a permutation of λ (up to adding 0's), then the m_λ form a basis of Λ_n as λ ranges over partitions of n. The ring of symmetric functions is in fact isomorphic to the ring of representations R from Section 1, and we sketch the correspondence now.

Definition 1.2.3. The *kth homogeneous symmetric function* h_k is the sum of all degree k monomials. The *kth elementary symmetric function* e_k is the sum of all squarefree degree k monomials. The *kth power sum symmetric function* p_k is $\sum_i x_i^k$. For a partition λ, we then define

$$\begin{aligned} h_\lambda &= h_{\lambda_1} \cdots h_{\lambda_\ell}, \\ e_\lambda &= e_{\lambda_1} \cdots e_{\lambda_\ell}, \text{ and} \\ p_\lambda &= p_{\lambda_1} \cdots p_{\lambda_\ell}. \end{aligned}$$

Given $\lambda \vdash n$ with length ℓ, the *Young subgroup* $\mathfrak{S}_\lambda$ of $\mathfrak{S}_n$ is the group of permutations which, for each $0 \leq i \leq \ell - 1$, maps

$$\{\lambda_1 + \cdots + \lambda_i + j : 1 \leq j \leq \lambda_{i+1}\}$$

onto itself. Consider the (right) action of $\mathfrak{S}_\lambda$ on words of length n on the alphabet $\mathbb{N}$, where the weight of a word $a = a_1 \cdots a_n$ is $x^a = x_{a_1} \cdots x_{a_n}$. The words $w = w^1 \cdots w^\ell$ where each w^i is weakly increasing with length λ_i are orbit representatives for this action, and the weight generating function of such words is simply h_λ.

On the other hand, Pólya's enumeration theorem says that this weight generating function is equal to

$$\frac{1}{|\mathfrak{S}_\lambda|} \sum_{\sigma \in \mathfrak{S}_\lambda} p_{\mathrm{cyc}(\sigma)}, \tag{1.1}$$

where $\mathrm{cyc}(\sigma)$ is the cycle type[2] of σ [27, Section 7.24]. It is not hard to show that (1.1) is equal to

$$\sum_{\mu \vdash n} \psi^\lambda(\mu) \frac{p_\mu}{z_\mu},$$

where $\psi^\lambda(\mu)$ is the number of (right) cosets $\mathfrak{S}_n / \mathfrak{S}_\lambda$ fixed by a permutation with cycle type μ, and z_μ is the size of the centralizer of a permutation with cycle type μ. Notice that ψ^λ is the character of $\mathrm{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n} \mathbb{C}$, where $\mathbb{C}$ is the trivial representation, because this is the permutation representation of $\mathfrak{S}_n$ on cosets $\mathfrak{S}_n / \mathfrak{S}_\lambda$.

Definition 1.2.4. Suppose V is an $\mathfrak{S}_n$-module with character χ. The *Frobenius characteristic* of V is the symmetric function

$$\mathrm{ch}(V) = \sum_{\mu \vdash n} \chi(\mu) \frac{p_\mu}{z_\mu},$$

where $\chi(\mu)$ is the value of χ on any permutation with cycle type μ.

---

[2] If σ is the product of disjoint cycles with lengths $\lambda_1 \geq \cdots \geq \lambda_\ell$, the partition λ is called the cycle type of σ.

Thus the identity $h_\lambda = \sum_{\mu \vdash n} \psi^\lambda(\mu) z_\mu^{-1} p_\mu$ says that $\operatorname{ch}(\operatorname{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n} \mathbb{C}) = h_\lambda$. Similarly, if sgn_k is the one-dimensional sign representation of $\mathfrak{S}_k$ and $\operatorname{sgn}_\lambda = \operatorname{sgn}_{\lambda_1} \otimes \cdots \otimes \operatorname{sgn}_{\lambda_\ell}$, one can show that $\operatorname{ch}(\operatorname{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n} \operatorname{sgn}_\lambda) = e_\lambda$ by running the above argument with the action of $\mathfrak{S}_\lambda$ on words over $[n]$ replaced with a signed action.

The characters ψ^λ are linearly independent because $\psi^\lambda(\mu) = 0$ if $\mu \not\leq \lambda$ in dominance[3], while $\psi^\lambda(\mu) \neq 0$ (one can find a definition of dominance order before Lemma). Since $\dim R_n = p(n)$, the classes $[\operatorname{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n} \mathbb{C}]$ form a basis of R, the representation ring of symmetric groups constructed in Section 1. Likewise, it is a classical fact that the h_λ form a basis for Λ as λ ranges over all partitions. Therefore the Frobenius characteristic defines a graded linear isomorphism $R \to \Lambda$. Finally, one can show that $\operatorname{Ind}_{K_1}^G V \otimes \operatorname{Ind}_{K_2}^G W \simeq \operatorname{Ind}_{K_1 \times K_2}^G (V \otimes W)$ when K_1 and K_2 are subgroups of G with $K_1 \times K_2 \simeq K_1 K_2$ via the product map. This implies the following very useful fact.

Theorem 1.2.5. *The Frobenius characteristic $R \to \Lambda$ is an isomorphism of graded algebras.*

Definition 1.2.6. The *Schur functions* s_λ are the images of the irreducible Specht modules S^λ under the Frobenius characteristic. More generally, the Schur function of any diagram D is defined to be $s_D = \operatorname{ch}(S^D)$.

Example 1.2.7. Example 1.1.2 mentioned that $S^{(n)}$ is the trivial representation and $S^{(1^n)}$ is the sign representation. Hence $s_n = h_n$ and $s_{1^n} = e_n$. The class of the regular representation of $\mathfrak{S}_n$ in R is $[S^{(1)}]^n$, with Frobenius characteristic $s_1^n = e_{1^n} = h_{1^n}$.

The irreducible $\mathfrak{S}_n$-modules for varying n form a basis of R, so the Schur functions s_λ form a basis of Λ. Decomposing an $\mathfrak{S}_n$-module V into irreducibles is equivalent to writing $\operatorname{ch}(V)$ as a linear combination of the basis of Schur functions, in the sense that $V \simeq \bigoplus_\lambda a_\lambda S^\lambda$ if and only if $\operatorname{ch}(V) = \sum_\lambda a_\lambda s_\lambda$.

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[3] The *dominance order* on partitions of a fixed n is defined by $\lambda \leq \mu$ if $\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i$ for i, padding λ and μ with 0's to make them the same length if necessary.

Following Stanley [28], we define a family of symmetric functions which play a central role in this thesis. Write s_i for the adjacent transposition $(i\ i+1)$. Then $\mathfrak{S}_n$ is generated by $s_1, \ldots, s_{n-1}$. A *reduced word* for $w \in \mathfrak{S}_n$ is a minimal-length word $a = a_1 \cdots a_\ell$ such that $w = s_{a_1} \cdots s_{a_\ell}$. Let $\operatorname{Red}(w)$ be the set of reduced words for w. Given a word a of length ℓ, call an increasing word $1 \leq i_1 \leq \cdots \leq i_\ell$ *unbounded-compatible* for a if $a_j < a_{j+1}$ implies $i_j < i_{j+1}$ for each j.

Definition 1.2.8 ([28])**.** Given a permutation $w \in \mathfrak{S}_n$, its *Stanley symmetric function* is

$$F_w = \sum_{a \in \operatorname{Red}(w)} \sum_i x_{i_1} \cdots x_{i_\ell},$$

where i runs over unbounded-compatible words for a. Here $\ell = \ell(w)$ is the length of any reduced word of w.

For a proof that F_w is symmetric, see [28, Theorem 2.1].

Warning. Our F_w is Stanley's $F_{w^{-1}}$. The reason for this is Theorem 3.1.6 below.

Example 1.2.9. Take $w = 2143$. Then $\operatorname{Red}(w) = \{13, 31\}$, and

$$F_{2143} = \sum_{i_1 < i_2} x_{i_1} x_{i_2} + \sum_{i_1 \leq i_2} x_{i_1} x_{i_2} = e_2 + h_2 = s_{11} + s_2.$$

Although we will not focus on it, one motivation for working with Stanley symmetric functions is to enumerate reduced words. Let $[x^\alpha]f$ denote the coefficient of the monomial x^α in a formal power series f. Notice that

$$\dim \operatorname{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n} \mathbb{C} = [\mathfrak{S}_n : \mathfrak{S}_\lambda] = \binom{n}{\lambda_1, \ldots, \lambda_\ell} = [x_1 x_2 \cdots x_n] h_\lambda,$$

so by linearity, $\dim V = [x_1 x_2 \cdots x_n] \operatorname{ch}([V])$ for any $\mathfrak{S}_n$-module V. In particular, $\dim S^\lambda = f^\lambda = [x_1 \cdots x_n] s_\lambda$ is the number of standard tableaux of shape λ. On the other hand, $[x_1 \cdots x_{\ell(w)}] F_w$ is $\# \operatorname{Red}(w)$. Therefore if we write

$$F_w = \sum_\lambda a_\lambda s_\lambda,$$

then $\# \operatorname{Red}(w) = \sum_\lambda a_\lambda f^\lambda$. The f^λ are easily computable by the hook-length formula [27, Corollary 7.21.6], and it turns out that expanding F_w into Schur functions is reasonably tractable as well, thanks to a recurrence we will see in Section 1.

The starting point of our results on Stanley symmetric functions is a result of Stanley characterizing those w for which F_w is a single Schur function.

Definition 1.2.10. The *flatten map* fl sends a tuple $x = (x_1, \ldots, x_n)$ of distinct real numbers to the unique permutation $w = fl(x) \in \mathfrak{S}_n$ such that $w(i) < w(j)$ if and only if $x_i < x_j$. A permutation $w \in \mathfrak{S}_n$ *contains* a permutation $v \in \mathfrak{S}_k$ if there is $1 \leq i_1 < \cdots < i_k \leq n$ such that $fl(w(i_1) \cdots w(i_k)) = v$. If w does not contain v, we say it *avoids* v. In either case, the smaller permutation w is frequently called a *pattern*.

Theorem 1.2.11 (Corollary 4.2, [28])**.** *The Stanley symmetric function F_w is a single Schur function s_λ if and only if w avoids* 2143.

Remark 1.2.12. What Stanley actually shows is that $F_w = s_\lambda$ if and only if $\operatorname{shape}(w)^t = \operatorname{shape}(w^{-1})$, where superscript t denotes conjugation, and that in this case $\lambda = \operatorname{shape}(w)$. Here $\operatorname{shape}(w)$ is the partition obtained by sorting $\operatorname{code}(w) = (c_1, c_2, \ldots, c_n)$, where $c_i = \#\{j > i : w(j) < w(i)\}$. It is not hard to show this condition is equivalent to avoiding 2143.

Stanley symmetric functions are nonnegative integer combinations of Schur functions—a direct corollary of Theorem 3.1.6 below—and so there is a multiset $M(w)$ of partitions such that $F_w = \sum_{\lambda \in M(w)} s_\lambda$. In this language, Theorem 1.2.11 shows that $\#M(w) = 1$ if and only if w avoids 2143. Permutations which avoid 2143 are often called *vexillary*, which explains the next definition.

Definition 1.2.13. Let k be a positive integer. We say that w is *k-vexillary* if $\#M(w) \leq k$.

Our first main result is a pattern avoidance characterization of the k-vexillary permutations.

Theorem (Theorem 3.3.10)**.** *For any $k \geq 1$, there is a finite set of patterns $V_k \subseteq \bigcup_{i=1}^{4k} \mathfrak{S}_i$ such that w is k-vexillary if and only if w avoids every member of V_k.*

Part of Theorem 3.3.10 is implied by a result giving a more refined relationship between $M(w)$ and $M(v)$ when w contains v.

Theorem (Theorem 3.2.1)**.** *Suppose w and v are permutations with w containing v. There is an injection $\iota : M(v) \hookrightarrow M(w)$ such that*

(i) $\lambda \subseteq \iota(\lambda)$ whenever $\lambda \in M(v)$, and

(ii) if λ appears with multiplicity m in $M(v)$, then $\iota(\lambda)$ appears with multiplicity at least m in $M(w)$.

There is a combinatorial interpretation of the partitions $M(w)$ in terms of reduced words, due to Edelman–Greene [5] which makes counting these partitions more natural that it might at first appear. A *semistandard tableau* of shape λ is a filling of the diagram of λ with positive integers, not necessarily bijective, which is strictly increasing down columns and weakly increasing rightward across rows.

Theorem 1.2.14 ([5])**.** *Given a permutation w, there is a set $\mathcal{EG}(w)$ of semistandard Young tableaux and a bijection*

$$\mathrm{Red}(w) \leftrightarrow \{(P, Q) : P \in \mathcal{EG}(w), Q \textit{ a standard tableau of the same shape as } P\}$$

The tableaux $\mathcal{EG}(w)$ are those semistandard tableaux whose column words—the words obtained by reading up columns starting with the leftmost—are reduced words for w. The transposed shapes of these tableaux give the Schur function expansion

$$F_w = \sum_{P \in \mathcal{EG}(w)} s_{\mathrm{shape}(P)^t},$$

where λ^t is the conjugate of λ. Hence, $M(w) = \{\mathrm{shape}(T)^t : T \in \mathcal{EG}(w)\}$. Define the permutation statistic $EG(w) = \#\mathcal{EG}(w) = \#M(w)$, which we call the *Edelman–Greene number*.

Our proofs of Theorems 3.3.10 and 3.2.1 go via the representation theory of $\mathfrak{S}_n$. The starting point is the following theorem, whose history is somewhat convoluted but seems to have been first written down by Reiner and Shimozono [23, Theorem 33], though Kraśkiewicz and Pragacz proved the analogous result for representations of a Borel subgroup of GL_N rather than S_n. For completeness, we will give a proof in Chapter 3.

Theorem 1.2.15. *The Frobenius characteristic of the Specht module $S^{D(w)}$ is F_w.*

3 Schubert calculus

Let $\mathrm{Gr}(k, n)$ be the Grassmannian variety of k-dimensional subspaces of $\mathbb{C}^n$. Fix a basis $e_1, \ldots, e_n$ of $\mathbb{C}^n$. Each $V \in \mathrm{Gr}(k, n)$ is the rowspan of a unique matrix A in reduced row echelon form, where we use the convention that the pivot entry in a row is the *last* nonzero entry, and that the pivots in rows $1, \ldots, k$ occur in columns $i_1 < \cdots < i_k$. Alternatively, we could put A into opposite reduced row echelon form, by which we mean that the pivot entry in a row is the *first* nonzero entry.

Definition 1.3.1. Given a k-subset $I \subseteq [n]$, the *Schubert cell* Z_I° is the set of k-planes in $\mathrm{Gr}(k, n)$ whose reduced row echelon form has pivots in columns I. The *opposite Schubert cell* Ω_I° is the set of k-planes in $\mathrm{Gr}(k, n)$ whose opposite reduced row echelon form has pivots in columns I. The *Schubert variety* Z_I (resp. *opposite Schubert variety* Ω_I) is the closure of Z_I° (resp. Ω_I°).

In a matrix in reduced row echelon form with pivot columns $I = \{i_1 < \cdots < i_k\}$, the zeros outside of columns I form a Young diagram λ. Specifically, λ is the partition $(n - k + 1 - i_1, n - k + 2 - i_2, \ldots, n - i_k)$. This gives a bijection between k-subsets of $[n]$ and partitions contained in the rectangle $[k]\times[n-k]$, and we will usually index Schubert varieties by partitions rather than subsets. The same goes for opposite Schubert varieties.

Example 1.3.2. The Schubert cell $Z^\circ_{22} = Z^\circ_{\{3,4,7,8\}} \subseteq \mathrm{Gr}(4,8)$ consists of the rowspans of matrices

$$\begin{bmatrix} * & * & 1 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & 1 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & * & * & 1 & 0 \\ * & * & 0 & 0 & * & * & 0 & 1 \end{bmatrix}.$$

The opposite Schubert cell $\Omega^\circ_{22} = \Omega^\circ_{\{3,4,7,8\}} \subseteq \mathrm{Gr}(4,8)$ consists of the rowspans of matrices

$$\begin{bmatrix} 0 & 0 & 1 & 0 & * & * & 0 & 0 \\ 0 & 0 & 0 & 1 & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Notice that Z°_λ is isomorphic to affine space of dimension $k(n-k)-|\lambda|$, so Z_λ is an irreducible subvariety of $\mathrm{Gr}(k,n)$ of codimension $|\lambda|$. A Schubert variety Z_λ and an opposite Schubert variety Ω_μ always intersect transversely, and their intersection $Z_\lambda \cap \Omega_\mu$ is an irreducible variety called a *Richardson variety.*

In the classical topology, Schubert cells form a CW decomposition of $\mathrm{Gr}(k,n)$ having only cells of even real dimension. Hence, their cohomology classes, called *Schubert classes*, form a linear basis of the cohomology ring $H^*(\mathrm{Gr}(k,n),\mathbb{Z})$. Let $\sigma_\lambda = [Z_\lambda]$ be the cohomology class of Z_λ. The surprise is now that the Schubert classes multiply essentially just as Schur functions do.

Theorem 1.3.3. *The linear map $\phi : \Lambda \to H^*(\mathrm{Gr}(k,n),\mathbb{Z})$ defined by*

$$\phi(s_\lambda) = \begin{cases} \sigma_\lambda & \textit{if } \lambda \subseteq [k]\times[n-k] \\ 0 & \textit{otherwise} \end{cases}$$

is a surjective ring homomorphism.

The study of Schubert varieties and the cohomology ring of Grassmannians dates back, in spirit, to the enumerative geometry of Schubert and others in the 19th century. For example, consider the problem of determining how many lines in $\mathbb{CP}^3$ intersect four fixed lines in general position. Equivalently, how many planes (through the origin) in $\mathbb{C}^4$ nontrivially intersect four fixed planes in general position? The Schubert variety $Z_{(1)} = Z_{\{2,4\}} \subseteq \mathrm{Gr}(2,4)$ is the set of planes in $\mathbb{C}^4$ which intersect the plane $\langle e_1, e_2 \rangle$ nontrivially, where e_1, e_2 are standard basis vectors. The answer to the enumerative question is then the number of points in the intersection of four generic translates of $Z_{(1)}$, and the calculation $\sigma_{(1)}^4 = 2\sigma_{(2,2)}$ shows that this answer is 2.

Every subvariety X of $\mathrm{Gr}(k,n)$ has an associated cohomology class, which one can think of as the Poincaré dual of a simplicial homology class obtained by triangulating X in a sufficiently nice way; see [9, Appendix B] for another approach. In light of Theorem 1.3.3, given a symmetric function f it is natural to ask for interesting subvarieties of $\mathrm{Gr}(k,n)$ with cohomology class $\phi(f)$. In his thesis, Liu conjecturally identifies a collection of subvarieties whose classes should correspond to Schur functions of diagrams [19].

Definition 1.3.4. If $D \subseteq [k] \times [n-k]$ is a diagram, define

$$X_D^\circ = \{\text{rowspan}[A \mid I_k] : A \in M_{k,n-k}, A_{ij} = 0 \text{ if } (i,j) \in D\}.$$

Here $M_{k,n-k}$ is the set of $k \times (n-k)$ complex matrices and I_k is the $k \times k$ identity matrix. The *diagram variety* of D is $X_D = \overline{X_D^\circ}$.

Just as with Schubert varieties, X_D is an irreducible variety of codimension $|D|$. Indeed, if λ is a partition then X_λ and Z_λ are related by permuting columns. In particular, $[X_\lambda] = \sigma_\lambda$.

Conjecture 1.3.5 (Conjecture 1, [19])**.** The cohomology class of X_D is $\phi(s_D)$.

Given a diagram $D \subseteq [k] \times [n-k]$, let $D^\vee = ([k] \times [n-k]) \setminus D)$. The *degree* of a codimension d subvariety X of $\mathrm{Gr}(k,n)$ is the integer $\deg(X)$ defined by $[X]\sigma_1^{k(n-k)-d} = \deg(X)\sigma_{(k^{n-k})}$. Under the Plücker embedding, this gives the usual notion of the degree of a subvariety of projective space, namely the number of points in the intersection of X with a generic d-dimensional linear subspace. One can check using Pieri's rule that $\deg(\sigma_\lambda) = f^{\lambda^\vee}$, the number of standard Young tableaux of shape $\lambda^\vee$. Since this is also $\dim S^{\lambda^\vee}$, comparing degrees gives the following weaker version of Conjecture 1.3.5.

Conjecture 1.3.6 (Conjecture 2, [19])**.** The degree of X_D is $\dim S^{D^\vee}$.

Liu proved Conjecture 1.3.5 or 1.3.6 for several interesting classes of diagrams. However, it turns out that Conjectures 1.3.5 and 1.3.6 are both false in general.

Theorem 1.3.7. *If $D = \{(1,1),(2,2),(3,3),(4,4)\}$, $k = 4$ and $n = 8$, then Conjectures 1.3.5 and 1.3.6 fail. Specifically, $\phi(s_D) - [X_D] = \sigma_{22}$.*

This diagram variety is small enough that one can find its degree by computer and compare it to $\dim S^{D^\vee}$. In Section 2 we will give a more concrete way to see where the discrepancy comes from.

Let $\mathrm{Fl}(k_1,\ldots,k_\ell;n)$ be the variety of flags $V_1 \subseteq \cdots \subseteq V_\ell \subseteq \mathbb{C}^n$ with $\dim V_i = k_i$. Such a flag variety also has a CW decomposition into Schubert cells (or into opposite Schubert cells), which are now indexed by permutations of $[n]$ with descent set contained in $\{k_1,\ldots,k_\ell\}$. As in the Grassmannian, a Richardson variety is the intersection of a Schubert variety and an opposite Schubert variety. Coskun [1] defines *rank varieties* in $\mathrm{Gr}(k,n)$ in terms of conditions on $\dim(V \cap S)$ for various subspaces S, and shows that they are exactly the varieties obtained by projecting Richardson varieties from $\mathrm{Fl}(1,\ldots,k;n)$ to $\mathrm{Gr}(k,n)$.

Theorem (Theorem 4.3.1)**.** *For any rank variety Σ, there is a Stanley symmetric function F_w with $[\Sigma] = \phi([F_w])$.*

Theorem 4.3.1 is inspired by and closely related to Knutson–Lam–Speyer's work on *positroid varieties* Π_f [12], extending work of Postnikov [22]. These subvarieties of a Grassmannian are indexed by certain affine permutations $f : \mathbb{Z} \to \mathbb{Z}$, and Knutson–Lam–Speyer show that they are exactly the projections of Richardson varieties in $\mathrm{Fl}(1, \ldots, n; n)$ to $\mathrm{Gr}(k, n)$. In particular, every rank variety is a positroid variety.

They also show that the cohomology class $[\Pi_f]$ is represented by the affine Stanley symmetric function $\tilde{F}_f$, a generalization of the ordinary Stanley symmetric function due to Lam (see Definition 4.2.5). However, even when Π_f is a rank variety it need not be the case that $\tilde{F}_f$ is equal to any ordinary Stanley symmetric function F_w. Thus, the content of Theorem 4.3.1 is that there is nevertheless some permutation w such that $\phi(F_w) = \phi(\tilde{F}_f)$ when Π_f is a rank variety.

Although Conjecture 1.3.5 is false in general, and even for permutation diagrams, its prediction nevertheless gives an upper bound on the class $[X_{D(w)}]$.

Theorem (Theorem 5.3.3). *For any permutation $w \in \mathfrak{S}_n$, the class $\phi(F_w) - [X_{D(w)}]$ is a nonnegative linear combination of Schubert classes.*

We prove Theorem 5.3.3 by writing down a rank variety Σ with $[\Sigma] = \phi(F_w)$ and then finding a degeneration from Σ to a variety containing $X_{D(w)}$ as a top-dimensional component.

4 Outline

In Chapter 2, we set up the tools we will need for working with Specht modules, and prove results relating the Specht module of a diagram D to the Specht module of a subdiagram of D. We connect these results in Chapter 3 to Stanley symmetric functions and permutation patterns, and prove Theorems 3.3.10 and 3.2.1. We also give a necessary condition for F_w to be a multiplicity-free sum of Schur functions in terms of pattern avoidance. Chapters 2 and 3, as well as Theorem 5.1.3, are based on joint work with Sara Billey [3].

In Chapter 4, we make precise the relationship between rank varieties and positroid varieties, and use it to prove Theorem 4.3.1. Finally, in Chapter 5, we discuss the diagram varieties $X_{D(w)^\vee}$ and $X_{D(w)}$. This includes a proof of Conjecture 1.3.5 in the case that $D = D(w)^\vee$ and F_w is multiplicity free, a byproduct of the techniques in Chapters 2 and 3. On the other hand, we give a counterexample to Conjecture 1.3.5 for a particular $D = D(w)$. We also prove Theorem 5.3.3.

Chapter 2

SPECHT MODULES OF DIAGRAMS AND SUBDIAGRAMS

Recall the definition of a Specht module for a diagram D from Section 1: if T is a bijective filling of D, and $\mathrm{Row}(T)$ (resp. $\mathrm{Col}(T)$) is the subgroup of $\mathfrak{S}_{|D|}$ stabilizing the rows (resp. columns) of T, then

$$S^D = \mathbb{C}[\mathfrak{S}_{|D|}] \sum_{\substack{q \in \mathrm{Col}(T) \\ p \in \mathrm{Row}(T)}} \mathrm{sgn}(q) qp.$$

Although the case where D is a partition is classical, James and Peel introduced these modules for general diagrams and provided an important partial decomposition result, Theorem 2.0.2 [11]. It is an open problem to find a reasonable combinatorial algorithm for decomposing S^D into irreducibles for a general diagram D. Reiner and Shimozono do so in [25] for *percent-avoiding* diagrams D—those with the property that if $(i_1, j_1), (i_2, j_2) \in D$ with $i_1 > i_2$ and $j_1 < j_2$, then at least one of (i_1, j_2) and (i_2, j_1) is in D. This includes the class of skew shapes and permutation diagrams. In a different direction, Liu [18] views a bipartite graph as a diagram D by saying $(i, j) \in D$ when a white vertex i and black vertex j are connected by an edge. He then obtains a recursive decomposition rule for S^D when D corresponds to a forest.

We have two main goals in this chapter. The first is to provide a partial decomposition result for Specht modules which simultaneously generalizes Theorem 2.0.2 and the classical Pieri rule. This result, Theorem 2.1.1, is well-suited to understanding the structure of $S^{D(w)}$. Our second goal is to give a connection between S^D and the Specht module of a subdiagram of D, in the following sense.

Definition 2.0.1. A subset D' of a diagram D is a *subdiagram* if it is the intersection of some rows and columns with D. That is, there are sets $U, V \subseteq \mathbb{N}$ such that $D' = (U \times V) \cap D$.

In pursuing these goals, we will rely heavily on certain transformations of diagrams, which we call *James–Peel moves.* Given two positive integers a, b, let $R_{a\to b}D$ be the diagram which contains a cell (i, j) if and only if one of the following cases holds:

- $i \neq a, b$ and $(i, j) \in D$;
- $i = b$ and either $(a, j) \in D$ or $(b, j) \in D$; or
- $i = a$ and both $(a, j), (b, j) \in D$.

That is, $R_{a\to b}D$ is obtained by moving cells in row a to row b if the appropriate position is empty. Similarly, we define $C_{c\to d}D$ by moving cells of D in column c to column d if possible. For example,

$$D = \begin{array}{cccc} \cdot & \circ & \cdot & \circ \\ \cdot & \cdot & \circ & \circ \end{array} \qquad R_{2\to 1}D = \begin{array}{cccc} \cdot & \circ & \circ & \circ \\ \cdot & \cdot & \cdot & \circ \end{array}$$

We also define $R_{a\to b}T$ and $C_{c\to d}T$ for a filling T, in the same way. From here through the proof of Theorem 2.1.1, we always view S^D, $S^{R_{a\to b}D}$, and $S^{C_{c\to d}D}$ as the specific left ideals in $\mathbb{C}[S_{|D|}]$ generated by y_T, $y_{R_{a\to b}T}$, and $y_{C_{c\to d}T}$ for a fixed filling T of D. Here y_T is the Young symmetrizer

$$y_T = \sum_{\substack{q\in \mathrm{Col}(T) \\ p\in \mathrm{Row}(T)}} \mathrm{sgn}(q)qp. \tag{2.1}$$

Theorem 2.0.2. *[11, Theorem 2.4] Consider cells $(i_1, j_1), (i_2, j_2)$ of the diagram D such that $(i_1, j_2), (i_2, j_1) \notin D$. Let $D_R = R_{i_1\to i_2}D$ and $D_C = C_{j_1\to j_2}D$. Then there is a surjective homomorphism $\phi : S^D \twoheadrightarrow S^{D_R}$ with $S^{D_C} \subseteq \ker\phi$.*

One immediate consequence of Theorem 2.0.2 is the converse to the statement that if D is a partition then S^D is irreducible (over $\mathbb{C}$): if S^D is irreducible, then D is equivalent to a partition. Indeed, otherwise D contains a pair of cells $(i_1, j_1), (i_2, j_2)$ satisfying the hypothesis of Theorem 2.0.2, which then gives two distinct, proper, nontrivial submodules of S^D. Theorem 2.0.2 and Theorem 2.1.1 are both independent of the base field, but for our application of Specht modules in Section 3 it will be sufficient to work over $\mathbb{C}$. In that case, exact sequences split, and Theorem 2.0.2 yields an inclusion $S^{D_R} \oplus S^{D_C} \hookrightarrow S^D$.

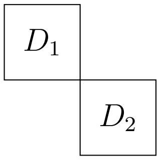
Remark 2.0.3. For arbitrary a, b, c, and d we always get a surjection $S^D \twoheadrightarrow S^{R_{a\to b}D}$, and an injection $S^{C_{c\to d}D} \hookrightarrow S^D$. Indeed, suppose we fix rows a and b. If there are c and d such that (a, c) and (b, d) satisfy the hypotheses of Theorem 2.0.2, then there is a surjection $S^D \twoheadrightarrow S^{R_{a\to b}D}$ as claimed. If not, then one of rows a and b contains the other, so applying $R_{a\to b}$ to D simply switches rows a and b. In this case, $S^D \simeq S^{R_{a\to b}D}$.

1 *The subdiagram Pieri rule*

Given two diagrams D_1 and D_2 with $D_1 \subseteq [r] \times [c]$, let

$$D_1 \cdot D_2 = D_1 \cup \{(i + r, j + c) : (i, j) \in D_2\}.$$

Graphically, $D_1 \cdot D_2$ is the diagram



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The classical Pieri rule gives a decomposition of the product $s_\lambda s_k$ for any partition λ and positive integer k. A *horizontal strip* is a diagram in which every column contains at most one cell. Pieri's rule then states that

$$s_\lambda s_k = \sum_\mu s_\mu,$$

where μ runs over all partitions of $|\lambda| + k$ such that μ/λ is a horizontal strip. This can also be viewed as a decomposition rule for a Specht module over $\mathbb{C}$:

$$S^{\lambda\cdot(k)} \simeq \bigoplus_{\substack{\mu\vdash|\lambda|+k \\ \mu/\lambda \text{ a horizontal strip}}} S^{\mu}.$$

The minimal diagram to which Theorem 2.0.2 applies is $(1)\cdot(1) = \begin{smallmatrix}\circ & \cdot \\ \cdot & \circ\end{smallmatrix}$, with the result being that $S^{11} \oplus S^{2} \hookrightarrow S^{(1)\cdot(1)}$. Pieri's rule says that this injection is an isomorphism. We can therefore think of Theorem 2.0.2 as beginning with James–Peel moves on $(1)\cdot(1)$ realizing Pieri's rule, and allowing us to apply those same moves to any diagram whenever we have $(1)\cdot(1)$ as a subdiagram (except that the isomorphism $S^{11} \oplus S^{2} \simeq S^{(1)\cdot(1)}$ is replaced by an injection, in general).

The next theorem works in the same way, except that the subdiagram can have the form $\lambda\cdot(k)$ for any integer k and partition λ—hence the title of this section. That said, we will only state and prove the theorem explicitly in the case where the subdiagram has the form $(p-1, p-2, \ldots, 1)\cdot(1)$. This is the only case we will need, and it is somewhat awkward to state the theorem in full generality, so we will be content with giving an example of the more general version later (Example 2.2.6).

Theorem 2.1.1. *Let* δ_p *denote the so-called* staircase *partition* $(p-1, p-2, \ldots, 1)$. *Suppose* D *contains* $\delta_p \cdot (1)$ *as a subdiagram in rows* $1, \ldots, p$ *and columns* $1, \ldots, p$. *There is a filtration*

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_p = S^D$$

of S^D *by* $S_{|D|}$*-submodules such that for each* $1 \le j \le p$, *there is a surjection*

$$M_j/M_{j-1} \twoheadrightarrow S^{R_{p\to p-j+1}C_{p\to j}D}.$$

As with Theorem 2.0.2, Theorem 2.1.1 holds over any field, but we will mostly use the corollary that over $\mathbb{C}$ we obtain an injection

$$\bigoplus_{j=1}^{p} S^{R_{p\to p-j+1}C_{p\to j}D} \hookrightarrow S^D.$$

To simplify indexing, we have assumed that $\delta_p \cdot (1)$ occurs as a subdiagram in rows $1, \ldots, p$ and columns $1, \ldots, p$. This incurs no loss of generality, since equivalent diagrams have isomorphic Specht modules.

For the proof of Theorem 2.1.1, we will need to know more about the homomorphism ϕ in Theorem 2.0.2, and how these homomorphisms interact for various choices of James–Peel moves. Given (i_1, j_1) and (i_2, j_2) as in Theorem 2.0.2, write $T_R = R_{i_1 \to i_2} T$ and $T_C = C_{j_1 \to j_2} T$. Let Y and Z be sets of coset representatives in $\mathrm{Col}(T_C)$ and $\mathrm{Row}(T_R)$, respectively, such that

$$\mathrm{Col}(T_C) = Y(\mathrm{Col}(T_C) \cap \mathrm{Col}(T)) \quad \text{and} \quad \mathrm{Row}(T_R) = (\mathrm{Row}(T_R) \cap \mathrm{Row}(T))Z.$$

Define ϕ to be right multiplication by $\sum_{\pi \in Z} \pi$. Then the following identities using Young symmetrizers (2.1) imply Theorem 2.0.2:

(a) $y_T \sum_{\pi \in Z} \pi = y_{T_R}$ (implies $\phi(S^D) = S^{D_R}$)

(b) $\sum_{\pi \in Y} \mathrm{sgn}(\pi)\pi \cdot y_T = y_{T_C}$ (implies $S^{D_C} \subseteq S^D$)

(c) $y_{T_C} \sum_{\pi \in Z} \pi = 0$ (implies $S^{D_C} \subseteq \ker \phi$).

Lemma 2.1.2. *Suppose* $R_{a \to b} C_{c \to d} D = C_{c \to d} R_{a \to b} D$. *Let*

$$\phi : S^D \twoheadrightarrow S^{R_{a \to b} D} \quad \textit{and}$$
$$\phi' : S^{C_{c \to d} D} \twoheadrightarrow S^{R_{a \to b} C_{c \to d} D}$$

be the surjections constructed above. Then $\phi' = \phi|_{S^{C_{c \to d} D}}$.

Proof. Fix a filling T of D and take sets of coset representatives Z and Z' with

$$\mathrm{Row}(R_{a \to b} T) = (\mathrm{Row}(R_{a \to b} T) \cap \mathrm{Row}(T))Z \quad \text{and}$$
$$\mathrm{Row}(R_{a \to b} C_{c \to d} T) = (\mathrm{Row}(R_{a \to b} C_{c \to d} T) \cap \mathrm{Row}(C_{c \to d} T))Z',$$

so that ϕ and ϕ' are right multiplication by $\sum_{\pi \in Z} \pi$ and $\sum_{\pi \in Z'} \pi$ respectively.

Applying a move $C_{c \to d}$ to a filling does not affect its row group, so

$$\begin{aligned}
\mathrm{Row}(R_{a \to b}T) = \mathrm{Row}(C_{c \to d}R_{a \to b}T) &= \mathrm{Row}(R_{a \to b}C_{c \to d}T) \\
&= (\mathrm{Row}(R_{a \to b}C_{c \to d}T) \cap \mathrm{Row}(C_{c \to d}T))Z' \\
&= (\mathrm{Row}(C_{c \to d}R_{a \to b}T) \cap \mathrm{Row}(C_{c \to d}T))Z' \\
&= (\mathrm{Row}(R_{a \to b}T) \cap \mathrm{Row}(T))Z'.
\end{aligned}$$

Thus we can take $Z' = Z$. □

We now restate and prove Theorem 2.1.1.

Theorem (Theorem 2.1.1)**.** *Suppose D contains $\delta_p \cdot (1)$ as a subdiagram in rows $1, \ldots, p$ and columns $1, \ldots, p$. There is a filtration*

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_p = S^D$$

of S^D by $S_{|D|}$-submodules such that for each $1 \le j \le p$, there is a surjection

$$M_j / M_{j-1} \twoheadrightarrow S^{R_{p \to p-j+1} C_{p \to j} D}.$$

Proof. Let $F_j = C_{p \to j} D$ and $G_j = R_{p \to p-j+1} C_{p \to j} D$. Define

$$M_j = \sum_{i=1}^{j} S^{F_i} \subseteq S^D,$$

with the containment by Theorem 2.0.2.

For each j, consider the two surjections

$$\begin{aligned}
&\phi_j : S^D \twoheadrightarrow S^{R_{p \to p-j+1} D} \quad \text{and} \\
&\theta_j : S^{F_j} \twoheadrightarrow S^{G_j}
\end{aligned}$$

given by Theorem 2.0.2. We have $R_{p \to p-j+1} C_{p \to j} D = C_{p \to j} R_{p \to p-j+1} D$. Indeed, this commutation property depends only on the subdiagram of D in rows $p - j + 1, p$ and columns j, p. By hypothesis this subdiagram is

$$\begin{array}{cc} \cdot & \cdot \\ \cdot & \circ \end{array}$$

and either order of James–Peel moves results in the subdiagram

$$\begin{array}{ccc} \circ & \cdot & \\ & & \cdot \\ \cdot & \cdot & \end{array}$$

Therefore, Lemma 2.1.2 says that $\theta_j = \phi_j|_{S^{F_j}}$.

If $1 \leq i < j$, then $(i, p-j+1), (p,p) \in D$ and $(i,p), (p, p-j+1) \notin D$, so Theorem 2.0.2 implies that $S^{F_i} \subseteq \ker \phi_j$, hence $M_{j-1} \subseteq \ker \phi_j$. Thus, $S^{F_j} \cap M_{j-1} \subseteq S^{F_j} \cap \ker \phi_j = \ker \theta_j$, so θ_j descends to a surjection

$$S^{F_j}/(S^{F_j} \cap M_{j-1}) \twoheadrightarrow S^{G_j}.$$

Since there is a canonical isomorphism

$$M_j/M_{j-1} \simeq S^{F_j}/(S^{F_j} \cap M_{j-1})$$

given by $m + M_{j-1} \mapsto m + S^{F_j} \cap M_{j-1}$ where $m \in S^{F_j}$, we are done. $\square$

Remark 2.1.3. Theorem 2.0.2 and Theorem 2.1.1 lead to the existence of *Specht series* for certain Specht modules. A Specht series for an S_n-module M is a filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_N = M$ where each quotient M_{i+1}/M_i is isomorphic to a (classical) Specht module S^λ. Over $\mathbb{C}$ these are just composition series, but in general they are coarser, since Specht modules are indecomposable but not necessarily irreducible in finite characteristic. James and Peel showed that skew representations $S^{\lambda/\mu}$ have Specht series [11], and Theorem 3.1.5 below constructs a Specht series for $S^{D(w)}$.

Example 2.1.4. Take

$$D = D(4261735) = \begin{array}{ccccc} \circ & \circ & \circ & \cdot & \cdot \\ \circ & \cdot & \cdot & \cdot & \cdot \\ \circ & \cdot & \circ & \cdot & \circ \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \circ & \cdot & \circ \end{array}$$

where we have omitted the last empty rows and columns. The subdiagram in rows $1, 2, 5$ and columns $1, 2, 5$ is $(2, 1) \cdot (1)$. Apply Theorem 2.1.1 to this subdiagram. The three diagrams constructed are

$$R_{5\to1}D = \begin{array}{ccccc} \circ & \circ & \circ & \cdot & \circ \\ \circ & \cdot & \cdot & \cdot & \cdot \\ \circ & \cdot & \circ & \cdot & \circ \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \circ & \cdot & \cdot \end{array} \quad R_{5\to2}C_{5\to2}D = \begin{array}{ccc} \circ & \circ & \circ \\ \circ & \circ & \circ \\ \circ & \circ & \circ \end{array} \quad C_{5\to1}D = \begin{array}{ccccc} \circ & \circ & \circ & \cdot & \cdot \\ \circ & \cdot & \cdot & \cdot & \cdot \\ \circ & \cdot & \circ & \cdot & \circ \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \circ & \cdot & \circ & \cdot & \cdot \end{array}$$

Theorem 2.1.1 now says $S^{(3,3,3)} \oplus S^{R_{5\to1}D} \oplus S^{C_{5\to1}D} \hookrightarrow S^D$. Applying Theorem 2.0.2 to the cells $(2, 1)$ and $(5, 3)$ in $R_{5\to1}D$ gives $S^{(4,3,2)} \oplus S^{(4,3,1,1)} \hookrightarrow S^{R_{5\to1}D}$. Using the cells $(1, 2)$ and $(3, 5)$ in $C_{5\to1}D$, Theorem 2.0.2 gives $S^{(4,2,2,1)} \oplus S^{(3,3,2,1)} \hookrightarrow S^{C_{5\to1}D}$. In fact, all these inclusions turn out to be isomorphisms:

$$S^D \simeq S^{(3,3,3)} \oplus S^{(4,3,2)} \oplus S^{(4,3,1,1)} \oplus S^{(4,2,2,1)} \oplus S^{(3,3,2,1)}.$$

2 *James–Peel trees*

James–Peel moves and Theorem 2.1.1 present one possible way to decompose a Specht module into irreducibles. In general it is not known if an arbitrary Specht module can be decomposed by finding some appropriate tree of James–Peel moves, as the inclusion in Theorem 2.1.1 may not be an isomorphism. The way we prove Theorem 3.2.1 is to find such a decomposition for the case of permutation diagrams. James–Peel moves will be useful for us because they are well-behaved with respect to subdiagram inclusion, and pattern inclusion for permutations corresponds to subdiagram inclusion on the level of permutation diagrams.

To be more precise about this, we make the following definition.

Definition 2.2.1. A *James–Peel tree* for a diagram D is a rooted tree $\mathcal{T}$ with vertices labeled by diagrams and edges labeled by sequences of James–Peel moves, satisfying the following conditions:

- The root of $\mathcal{T}$ is D.
- If B is a child of A with a sequence $\mathbf{JP}$ of James–Peel moves labeling the edge A—B, then $B = \mathbf{JP}(A)$.
- If A has more than one child, these children arise as a result of applying Theorem 2.1.1 to A. That is, A contains $\delta_p \cdot (1)$ as a subdiagram in rows $i_1 < \cdots < i_p$ and columns $j_1 < \cdots < j_p$, and each edge leading down from A is labeled $R_{i_p \to i_{p-k+1}} C_{j_p \to j_k}$ for some distinct values $1 \le k \le p$ (not all such k need appear).

See Examples 2.2.6 and 2.2.12 later in this section for examples of James–Peel trees. Note that the vertex labels are completely determined by the root and the edge labels. When a vertex is labeled by a permutation diagram $D(w)$, sometimes we will refer to it simply as w.

Theorem 2.1.1 and Remark 2.0.3 immediately imply the following lemma.

Lemma 2.2.2. *If D has a James–Peel tree $\mathcal{T}$ with leaves $A_1, \ldots, A_m$, then there is an injection $\bigoplus_i S^{A_i} \hookrightarrow S^D$ of $\mathbb{C}[S_{|D|}]$-modules.*

Definition 2.2.3. A James–Peel tree $\mathcal{T}$ for D is *complete* if its leaves $A_1, \ldots, A_m$ are equivalent to Young diagrams of partitions, and if $S^D \simeq \bigoplus_i S^{A_i}$.

In [11], an algorithm is given which constructs a complete James–Peel tree when D is a skew shape. More generally, Reiner and Shimozono [24] construct a complete James–Peel tree for any *column-convex* diagram: a diagram D for which $(a, j), (b, j) \in D$ with $a < b$ implies $(i, j) \in D$ for all $a < i < b$. In the next section we construct a complete James–Peel tree for the diagram of a permutation, so it is worth noting that neither of these classes of diagrams contains the other. For example, $D(37154826)$ is not equivalent to any column-convex or row-convex diagram, while the column-convex diagram

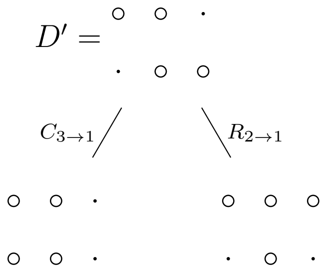
$$\begin{matrix} \circ & \circ & \cdot & \cdot \\ \circ & \cdot & \circ & \cdot \\ \circ & \cdot & \cdot & \circ \end{matrix}$$

is not equivalent to the diagram of any permutation. The James–Peel trees constructed in [11] and [24] are binary trees based on moves from Theorem 2.0.2. Our James–Peel trees use the more general Theorem 2.1.1, and do not need to be binary; a vertex can have an arbitrary number of children.

Definition 2.2.4. Given a James–Peel tree $\mathcal{T}'$ for a subdiagram D' of D, the *induced* James–Peel tree $\mathcal{T}$ for D is defined as follows. Start with $\mathcal{T}$ an unlabeled tree isomorphic to $\mathcal{T}'$, with $\phi : \mathcal{T}' \to \mathcal{T}$ an isomorphism. Give each edge $\phi(A_1)$—$\phi(A_2)$ of $\mathcal{T}$ the same label as the edge A_1—A_2 of $\mathcal{T}'$. Label the root $\phi(D')$ of $\mathcal{T}$ with D, and label the rest of the vertices according to the James–Peel moves labeling the edges in $\mathcal{T}$.

Observe that the first two conditions of Definition 2.2.1 clearly hold for $\mathcal{T}$ as constructed. The subdiagram of D' needed in the third condition for $\mathcal{T}'$ and D' works just as well for $\mathcal{T}$ and D, when viewed as a subdiagram of D. Thus, $\mathcal{T}$ is a James–Peel tree for D.

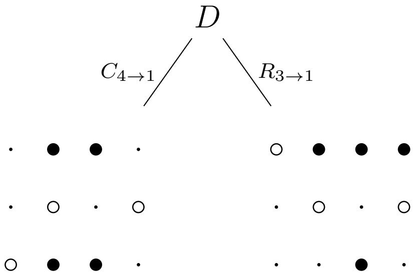
Example 2.2.5. Let $\mathcal{T}'$ be the following (complete) James–Peel tree:



Let D be the diagram

$$\begin{array}{cccc} \cdot & \bullet & \bullet & \cdot \\ \cdot & \circ & \cdot & \circ \\ \circ & \cdot & \bullet & \bullet \end{array}$$

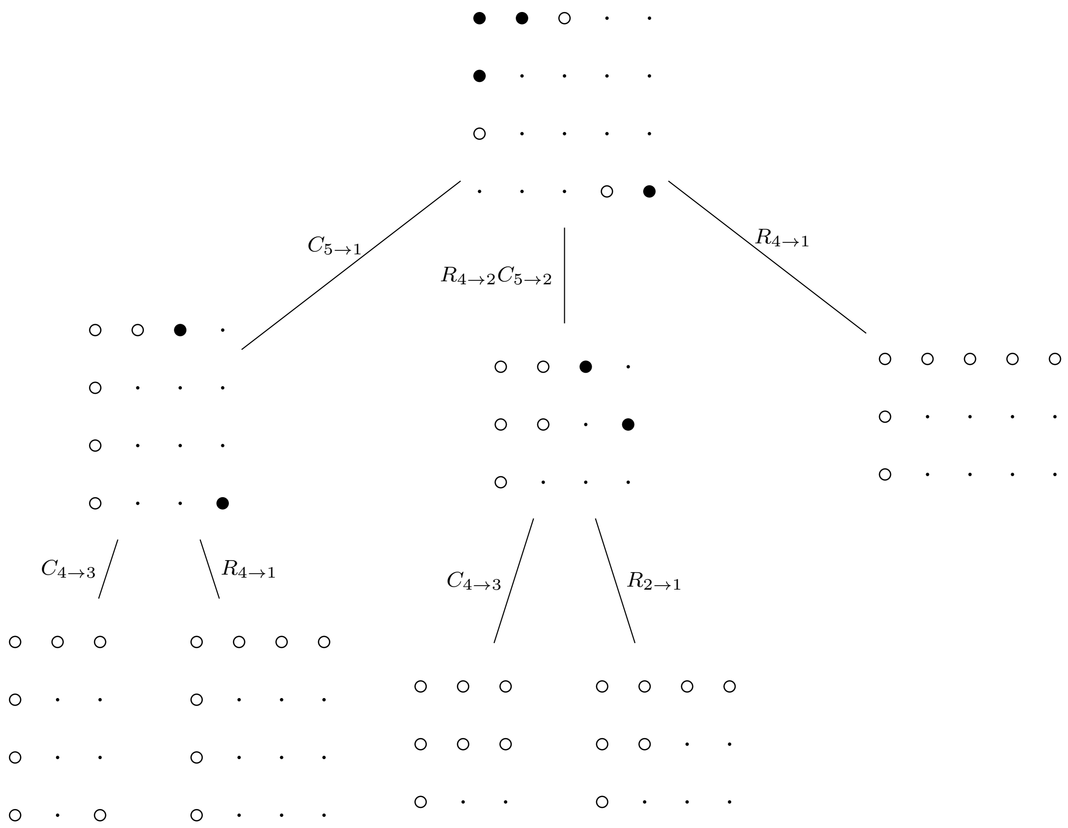
which contains D' as a subdiagram in rows $1, 3$ and columns $2, 3, 4$, as marked by the darkened cells. The James–Peel tree for D induced by $\mathcal{T}'$ is



Notice that the diagram at any node of the induced tree contains the diagram at the corresponding node of $\mathcal{T}'$ as a subdiagram in rows $1, 3$ and columns $2, 3, 4$.

The notion of an induced James–Peel tree provides a convenient way to discuss the generalization of Theorem 2.1.1 from the case of a subdiagram $\delta_p \cdot (1)$ to that of any subdiagram $\lambda \cdot (k)$ with λ a partition. Specifically, we can write down a complete James–Peel tree for $\lambda \cdot (k)$, and then consider the induced James–Peel tree for D when D contains $\lambda \cdot (k)$ as a subdiagram.

Example 2.2.6. We give a complete James–Peel tree for $\lambda = (3, 1, 1)$, $k = 2$. In each non-leaf vertex, we have darkened the cells to which Theorem 2.1.1 is being applied.



Suppose D contains D' as a subdiagram, and that D' has a complete James–Peel tree. Then the number of irreducibles, counting multiplicity, in the decomposition of S^D is at least the corresponding number for $S^{D'}$. This follows immediately from the existence of the induced James–Peel tree for D'. We would like a more refined correspondence, however, so we now discuss two canonical submodules of S^D.

Definition 2.2.7. Given a diagram D contained in $[m] \times [n]$, define

$$D^{\max} = (R_{m\to1} \cdots R_{2\to1})(R_{m\to2} \cdots R_{3\to2}) \cdots (R_{m\to m-1})D \quad \text{and}$$
$$D^{\min} = (C_{n\to1} \cdots C_{2\to1})(C_{n\to2} \cdots C_{3\to2}) \cdots (C_{n\to n-1})D.$$

These diagrams are both equivalent to partitions—an identification we will freely make—and satisfy $S^{D^{\max}}, S^{D^{\min}} \hookrightarrow S^D$ by Remark 2.0.3. Specifically, take the row lengths of D (the number of cells in each row) and sort them in decreasing order; the resulting partition is equivalent to $D^{\min}$. Likewise, the partition obtained by sorting the column lengths of D is equivalent to $(D^{\max})^t$. This second description implies the following lemma.

Lemma 2.2.8. *If* $\mathbf{I}$ *is any sequence of James–Peel row moves, then* $D^{\max} = (\mathbf{I}D)^{\max}$. *Likewise, if* $\mathbf{J}$ *is a sequence of column moves, then* $D^{\min} = (\mathbf{J}D)^{\min}$.

The partitions $D^{\min}$ and $D^{\max}$ play a special role in the structure of S^D: they give bounds for the possible partitions λ such that $S^\lambda \hookrightarrow S^D$. The *dominance order* on partitions of a fixed n is defined by $\lambda \leq \mu$ if, for each $1 \leq i \leq \max(\ell(\lambda), \ell(\mu))$,

$$\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i.$$

In this definition we take $\lambda_i = 0$ if $i > \ell(\lambda)$ and the same for μ. In the case $D = D(w)$, the next lemma is Theorem 4.1 from [28].

Lemma 2.2.9. *Let* D *be any diagram and* λ *a partition. If* $S^\lambda \hookrightarrow S^D$*, then* $D^{\min} \leq \lambda \leq D^{\max}$ *in dominance order. Also,* $S^{D^{\min}}$ *and* $S^{D^{\max}}$ *appear in* S^D *with multiplicity one.*

Proof. We will prove the part of the statement referring to $D^{\min}$, with the proof for $D^{\max}$ being analogous. Induct on $\ell(D^{\min})$, the number of non-empty rows of D. The lemma is obvious when D is a single row. Let H be a nonempty row of D with minimal length, and set $E = D \setminus H$. Then D is the image of $E \cdot H$ under James–Peel moves—push H to its original row, then each cell individually to its original column—so $S^D \hookrightarrow S^{E \cdot H}$.

Suppose $S^\mu \hookrightarrow S^D \hookrightarrow S^{E\cdot H}$. Since $s_{E\cdot H} = s_E s_H$, Pieri's rule says μ is obtained from some λ with $S^\lambda \hookrightarrow S^E$ by adding $|H|$ cells to it, no two in the same column. By construction, $D^{\min}$ is obtained by appending a row of length $|H|$ to $E^{\min}$. The induction hypothesis gives $E^{\min} \leq \lambda$. To show $D^{\min} \leq \mu$, let α and β be the partitions equivalent to $D^{\min}$ and $E^{\min}$. Observe that for all $1 \le m \le \ell(E^{\min})$,

$$\sum_{i=1}^{m} \alpha_i = \sum_{i=1}^{m} \beta_i \le \sum_{i=1}^{m} \lambda_i \le \sum_{i=1}^{m} \mu_i, \tag{2.2}$$

while if $m = \ell(E^{\min}) + 1 = \ell(D^{\min})$, both sides of (2.2) are equal to $|D|$, since $\ell(\mu) \le \ell(\lambda) + 1 \le \ell(E^{\min}) + 1$.

By induction, $S^{E^{\min}}$ appears with multiplicity one in S^E. Since Pieri's rule is multiplicity-free, $S^{D^{\min}}$ appears with multiplicity one in $S^{E \cdot H}$, hence with multiplicity at most one in S^D. Furthermore, $S^{D^{\min}}$ does actually appear in S^D, because $D^{\min}$ is the image of D under James–Peel moves, and so it appears with multiplicity one. $\square$

Let $\operatorname{shapes}(D)$ denote the multiset of partitions of $n = |D|$ such that $S^D \simeq \bigoplus_{\lambda \in \operatorname{shapes}(D)} S^\lambda$. Given partitions λ and μ, let $\lambda + \mu$ be the partition $(\lambda_1 + \mu_1, \lambda_2 + \mu_2, \ldots)$, padding λ or μ with 0's as necessary. Let $\lambda \cup \mu$ be the partition whose parts are the multiset union of the parts of λ and of μ.

Lemma 2.2.10. *Suppose D_1 and D_2 are subdiagrams of D, each with a complete James–Peel tree, and such that $D_1 = (U \times V) \cap D$ and $D_2 = (U^c \times V^c) \cap D$, where $U^c = \mathbb{N} \setminus U$. Let $F_1 = (U^c \times V) \cap D$ and $F_2 = (U \times V^c) \cap D$. Then there is a well-defined injection $\iota : \operatorname{shapes}(D_1) \times \operatorname{shapes}(D_2) \to \operatorname{shapes}(D)$ given by*

$$\iota(\lambda, \mu) = (\lambda \cup F_1^{\min}) + (F_2^{\max} \cup \mu).$$

Proof. Without loss of generality we can assume $U = \{1, 2, \ldots, i\}$ and $V = \{1, 2, \ldots, j\}$ by permuting rows and columns of D if necessary. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be complete James–Peel trees for D_1 and D_2. Then we can further assume that the leaves of $\mathcal{T}_1$ and $\mathcal{T}_2$ are all partition diagrams by doing extra James–Peel moves to sort the rows and columns.

Let $\mathcal{T}$ be the James–Peel tree for D induced from $\mathcal{T}_1$, with $\phi : \mathcal{T}_1 \to \mathcal{T}$ an isomorphism as in Definition 2.2.4. Since D contains $D_1 = \phi^{-1}(D)$ as a subdiagram, each vertex A of $\mathcal{T}$ contains $\phi^{-1}(A)$ as a subdiagram. In particular, each leaf B of $\mathcal{T}$ has the block form

$$B = \begin{array}{|c|c|}\hline \lambda & F_2' \\ \hline F_1' & D_2 \\ \hline \end{array}\,, \tag{2.3}$$

where λ is the shape of $\phi^{-1}(B)$, F_1' is the image of F_1 under moves $C_{c\to d}$, and F_2' the image of F_2 under moves $R_{a\to b}$.

Using the block form (2.3), we next add a single child to each leaf B of $\mathcal{T}$. By Lemma 2.2.8, $(F_1')^{\min} = (F_1)^{\min}$ and $(F_2')^{\max} = (F_2)^{\max}$. Thus, there is a sequence $\mathbf{I}_B$ of upward row moves involving only rows in U, and a sequence $\mathbf{J}_B$ of leftward column moves involving only columns in V, such that $\mathbf{J}_B(F_1') = F_1^{\min}$ and $\mathbf{I}_B(F_2') = F_2^{\max}$. Since the upper-left block is a partition diagram, it is unaffected by the James–Peel moves $\mathbf{I}_B$ and $\mathbf{J}_B$. Since no cell of D_2 lies in a row in U or a column in V, the moves $\mathbf{I}_B$ and $\mathbf{J}_B$ do not change D_2 either. Thus, we can define

$$\tilde{B} = \mathbf{I}_B\mathbf{J}_B(B) = \begin{array}{|c|c|}\hline \lambda & F_2^{\max} \\ \hline F_1^{\min} & D_2 \\ \hline \end{array}\,.$$

To each leaf B of $\mathcal{T}$, attach the child $\tilde{B}$ via an edge labeled $\mathbf{I}_B\mathbf{J}_B$. Note, the result is still a James–Peel tree for D. We will abuse notation and again call this tree $\mathcal{T}$.

We modify $\mathcal{T}$ one more time by augmenting each leaf with an induced tree for D_2. Specifically, to each leaf $\tilde{B}$ of $\mathcal{T}$, attach the James–Peel tree for $\tilde{B}$ induced by $\mathcal{T}_2$. As above, each leaf C of the new tree descending from $\tilde{B}$ now has block form

$$C = \begin{array}{|c|c|}\hline \lambda & F_2'' \\ \hline F_1'' & \mu \\ \hline \end{array}\,, \tag{2.4}$$

where (λ, μ) is a pair of shapes in $\mathrm{shapes}(D_1)\times\mathrm{shapes}(D_2)$, F_1'' is the result of applying row moves to $F_1^{\min}$, and F_2'' is the result of applying column moves to $F_2^{\max}$. Notice that the upper-right and lower-left block of $\tilde{B}$ and C are equivalent, since both are equivalent to partitions. The upper-left block of $\tilde{B}$ and C are exactly the same since the induced tree for $\tilde{B}$ does not touch the first i rows and j columns. Again, we abuse notation by calling this tree $\mathcal{T}$.

Finally, we modify $\mathcal{T}$ once again so the leaves all have block form with four partition shapes. Assume C is a descendant of $\tilde{B}$ with block diagonal shapes λ, μ as in (2.4) above. Let $\mathbf{I}_C$ be the sequence of upward James–Peel row moves needed to sort the rows of F_1'' into the partition shape F_1^{min}, and let $\mathbf{J}_C$ be the sequence of leftward column moves needed to sort the columns of F_2'' into the partition shape F_2^{max}. Note that such moves will not change the shapes λ and μ when applied to C, since they are partitions. Thus, one can define

$$\tilde{C} = \mathbf{I}_C \mathbf{J}_C(C) = \begin{array}{|c|c|} \hline \lambda & F_2^{\text{max}} \\ \hline F_1^{\text{min}} & \mu \\ \hline \end{array}\,, \tag{2.5}$$

with all four subdiagrams equal to honest left- and top-justified Young diagrams. For each leaf C of $\mathcal{T}$, attach $\tilde{C} = \mathbf{I}_C \mathbf{J}_C(C)$ as a child.

Observe that the resulting tree $\mathcal{T}$ is a James–Peel tree for D, and the leaves are in bijection with the multiset $\text{shapes}(D_1) \times \text{shapes}(D_2)$. One can also see that if a leaf $\tilde{C}$ of $\mathcal{T}$ has diagonal shapes λ and μ, then

$$\tilde{C}^{\text{max}} = (\lambda \cup F_1^{\text{min}}) + (F_2^{\text{max}} \cup \mu),$$

and the shape of $\tilde{C}^{\text{max}}$ only depends on λ, μ, F_1, and F_2, and not on B or C. Thus, we define $\iota(\lambda, \mu)$ to be the partition of shape $\tilde{C}^{\text{max}}$ which is in $\text{shapes}(D)$ by Lemma 2.2.9 and Lemma 2.2.2. This gives a well-defined injection of multisets

$$\iota : \text{shapes}(D_1) \times \text{shapes}(D_2) \to \text{shapes}(D)$$

as intended. □

For the most part we will only need a simpler version of this lemma.

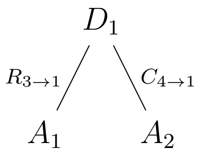
Corollary 2.2.11. Suppose D has a subdiagram D' with a complete James–Peel tree. There is an injection $\iota : \text{shapes}(D') \hookrightarrow \text{shapes}(D)$ such that $\lambda \subseteq \iota(\lambda)$. Moreover,

$\iota(\lambda)$ depends only on λ: if λ appears k times in shapes(D'), then $\iota(\lambda)$ appears at least k times in shapes(D).

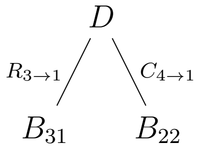
Example 2.2.12. Take the diagram $D = D(316524)$. Rows 5 and 6, and column 6, are empty, so we omit them in the following picture:

$$D = \begin{array}{cccc|c} \circ & \circ & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \circ & \cdot & \circ & \circ \\ \hline \cdot & \circ & \cdot & \circ & \cdot \end{array}.$$

Let D_1 be the subdiagram on rows 1, 2, 3 and columns 1, 2, 3, 4. A complete James–Peel tree $\mathcal{T}_1$ for D_1 is



where $A_1 \simeq (3,1)$ and $A_2 \simeq (2,2)$. The James–Peel tree $\mathcal{T}_1$ for D_1 induces the following James–Peel tree $\mathcal{T}$ for D



where $B_{31} = R_{3\to1}D$ and $B_{22} = C_{4\to1}D$ with

$$B_{31} = \begin{array}{cccc|c} \circ & \circ & \cdot & \circ & \circ \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \circ & \cdot & \cdot & \cdot \\ \hline \cdot & \circ & \cdot & \circ & \cdot \end{array} \quad \text{and} \quad B_{22} = \begin{array}{cccc|c} \circ & \circ & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \circ & \circ & \cdot & \cdot & \circ \\ \hline \circ & \circ & \cdot & \cdot & \cdot \end{array}.$$

Following the proof of Lemma 2.2.10, we next apply leftward column moves to the lower left subdiagrams F_1' to get $(F_1')^{\min}$, and upward row moves to the upper right subdiagrams (F_2') to get $(F_2')^{\max}$:

$$\tilde{B}_{31} = C_{4\to 1}B_{31} = \begin{array}{cccc|c} \circ & \circ & \cdot & \circ & \circ \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \circ & \cdot & \cdot & \cdot \\ \hline \circ & \circ & \cdot & \cdot & \cdot \end{array} \quad \text{and} \quad \tilde{B}_{22} = R_{3\to 1}B_{22} = \begin{array}{cccc|c} \circ & \circ & \cdot & \cdot & \circ \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \circ & \circ & \cdot & \cdot & \cdot \\ \hline \circ & \circ & \cdot & \cdot & \cdot \end{array} .$$

At this point we would apply James–Peel moves to the lower-right subdiagram, but it is empty so there are no moves to do. Finally, we apply leftward column moves and rightward row moves to make all four subdiagrams into Young diagrams (up to trailing empty rows and columns):

$$\tilde{C}_{31,\varnothing} = R_{3\to 2}C_{4\to 3}C_{2\to 1}\tilde{B}_{31} = \begin{array}{cccc|c} \circ & \circ & \circ & \cdot & \circ \\ \circ & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \circ & \circ & \cdot & \cdot & \cdot \end{array} \quad \text{and} \quad \tilde{C}_{22,\varnothing} = R_{3\to 2}\tilde{B}_{22} = \begin{array}{cccc|c} \circ & \circ & \cdot & \cdot & \circ \\ \circ & \circ & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \circ & \circ & \cdot & \cdot & \cdot \end{array} .$$

Now, taking unions before additions as the order of operations:

$$(\tilde{C}_{31,\varnothing})^{\max} = (3,1) \cup (2) + (1) \cup \varnothing = (3,2,1) + (1) = (4,2,1) \quad \text{and}$$
$$(\tilde{C}_{22,\varnothing})^{\max} = (2,2) \cup (2) + (1) \cup \varnothing = (2,2,2) + (1) = (3,2,2).$$

For the injection $\iota : \text{shapes}(D_1) \hookrightarrow \text{shapes}(D)$ we therefore take $\iota(3,1) = (4,2,1)$ and $\iota(2,2) = (3,2,2)$. Indeed, $s_{D_1} = s_{32/1} = s_{31} + s_{22}$, while $s_D = s_{322} + s_{331} + s_{421}$.

Chapter 3

STANLEY SYMMETRIC FUNCTIONS AND PERMUTATION PATTERNS

In this chapter we apply the results in Chapter 2 to Stanley symmetric functions, and prove Theorems 3.2.1 and 3.3.10. We begin by defining Schubert polynomials, from which Stanley symmetric functions can be obtained by a limiting procedure. Schubert polynomials also lead to an important recurrence for Stanley symmetric functions.

1 *The Lascoux–Schützenberger tree*

Given any sequence $a = (a_1, \ldots, a_\ell)$ of positive integers, an *unbounded-compatible sequence* for a is a sequence $1 \le i_1 \le \cdots \le i_\ell$ such that if $a_j < a_{j+1}$, then $i_j < i_{j+1}$. An unbounded-compatible sequence i for a is a *compatible sequence* if it satisfies $i_j \le a_j$ for each j. Let $C(a)$ be the set of compatible sequences for a, and $C_\infty(a)$ the set of unbounded-compatible sequences.

The Stanley symmetric function F_w can now be viewed as the weight generating function for unbounded-compatible sequences for all reduced words of w, namely

$$F_w = \sum_{a \in \mathrm{Red}(w)} \sum_{i \in C_\infty(a)} x^i.$$

Here x^i means $x_{i_1} \cdots x_{i_\ell}$ when $i = (i_1, \ldots, i_\ell)$.

Definition 3.1.1. The *Schubert polynomial* of a permutation w is

$$\mathfrak{S}_w = \sum_{a \in \mathrm{Red}(w)} \sum_{i \in C(a)} x^i.$$

Unlike the Stanley symmetric function, Schubert polynomials are honest polynomials rather than power series.

Schubert polynomials were introduced by Lascoux and Schützenberger using a different definition [17], and are important in Schubert calculus, where they give particularly nice representatives for the cohomology classes of Schubert varieties in a full flag variety. It is a theorem of Billey–Jockusch–Stanley [2] that our definition is equivalent. Our definition of Schubert polynomials makes clear that F_w is a *stable Schubert polynomial*, in the following sense. Given a permutation $w \in \mathfrak{S}_n$, let $1^m \times w$ be the permutation in $\mathfrak{S}_{n+m}$ fixing $1, \ldots, m$ and sending $i + m$ to $w(i) + m$ for $1 \leq i \leq n$. Then

$$F_w = \lim_{m \to \infty} \mathfrak{S}_{1^m \times w},$$

in the sense that for any fixed d, there is m for which $F_w(x_1, \ldots, x_d) = \mathfrak{S}_{1^m \times w}(x_1, \ldots, x_d)$.

Stanley symmetric functions can be decomposed into Schur functions using a recursion introduced in [16, 17]. Given a permutation w, let r be maximal with $w(r) > w(r+1)$. Then let $s > r$ be maximal with $w(s) < w(r)$. Let t_{ij} denote the transposition $(i\,j)$, and define

$$T(w) = \{wt_{rs}t_{rj} : \ell(wt_{rs}t_{rj}) = \ell(w) \text{ for some } j\};$$

or, if the set on the right-hand side is empty, then set $T(w) = T(1 \times w)$ (which in this case will equal $\{wt_{r+1,s+1}t_{r+1,1}\}$). The members of $T(w)$ are called *transitions* of w. The *Lascoux–Schützenberger tree* (*L–S tree* for short) is the finite rooted tree of permutations with root w where the set of children of a vertex v is

$$\begin{cases} \varnothing & v \text{ is vexillary (avoids 2143)} \\ T(v) & \text{otherwise} \end{cases}$$

The finiteness of this tree is not immediately obvious [16], and we include a short proof in Remark 3.3.16. More on the Lascoux–Schützenberger tree and its relationship to Schubert polynomials and Stanley symmetric functions can be found in [21].

Remark 3.1.2. Since $F_w = F_{1\times w}$, our results in this chapter will be unaltered if we replace w by $1\times w$. The L–S tree is finite, so there is some m such that in constructing the tree for $1^m \times w$, we never need to make the replacement of v by $1 \times v$. Thus we will ignore this possible step in what follows.

Monk's rule for Schubert polynomials expresses $\mathfrak{S}_{s_i}\mathfrak{S}_w$ as a linear combination of Schubert polynomials, and after applying a bit of algebra to it one can pass to the stable limit to obtain a recurrence for Stanley symmetric functions in terms of the L–S tree:

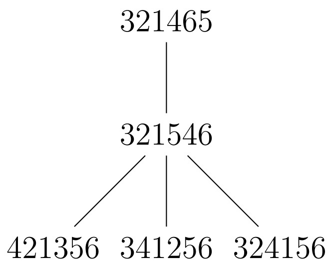
$$F_w = \sum_{v\in T(w)} F_v. \tag{3.1}$$

The *code* of a permutation $w \in \mathfrak{S}_n$ is the composition $\mathrm{code}(w) = (c_1, \dots, c_n)$ where $c_i = \#\{j > i : w(j) < w(i)\}$. The *shape* of w is the partition obtained by sorting $\mathrm{code}(w)$ into decreasing order. Stanley showed that if w is vexillary then $F_w = s_{\mathrm{shape}(w)}$ [28]. The recursion (3.1) therefore provides an algorithm for expanding F_w into Schur functions:

$$F_w = \sum_v s_{\mathrm{shape}(v)}$$

where v runs over the leaves of the L–S tree.

Example 3.1.3. The L–S tree of 321465 is



Hence $F_{321465} = F_{321546} = F_{421356} + F_{341256} + F_{324156}$. We have

$$\mathrm{code}(421356) = (3, 1, 0, 0, 0, 0)$$

$$\mathrm{code}(341256) = (2, 2, 0, 0, 0, 0)$$

$$\mathrm{code}(324156) = (2, 1, 1, 0, 0, 0),$$

so $F_{321465} = s_{31} + s_{22} + s_{211}$.

The next lemma and theorem show that, upon replacing permutations with their diagrams, the Lascoux–Schützenberger tree is a James–Peel tree for $D(w)$.

Lemma 3.1.4. *Given a permutation w, as above let r be the maximal descent of w and $s > r$ maximal such that $w(r) > w(s)$. Take $w' = wt_{rs}t_{rj} \in T(w)$. Then*

$$D(w') = R_{r\to j}C_{w(s)\to w(j)}D(w) = C_{w(s)\to w(j)}R_{r\to j}D(w).$$

Proof. We will show that the change in passing from $D(w)$ to $D(w')$ is described by Figure 3.1. That is, we move the cells in each shaded region of $D(w)$ in Figure 3.1 into the corresponding (formerly cell-free) shaded region of $D(w')$, and also move the cell $(r, w(s))$, denoted by $\circ$ above, to $(j, w(j))$. Here only the region $[j, r] \times [w(j), w(s)]$ together with row s and column $w(r)$ have been drawn. We will show that the rest of the diagram remains unchanged. We use an empty box like $\boxed{}$ to indicate a segment of a row or column containing no cells. As usual, we place an $\times$ at coordinates $(i, w(i))$ in $D(w)$ and $(i, w'(i))$ in $D(w')$. The condition that $wt_{rs}t_{rj}$ is a maximal transition says that there is no $\times$ in the rectangle $(j, r) \times (w(j), w(s))$.

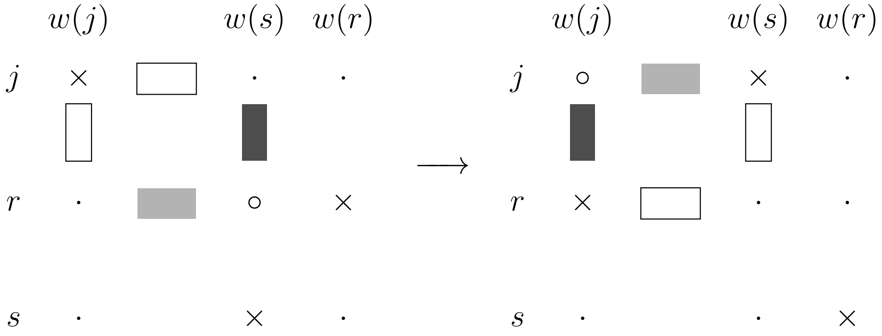


Figure 3.1: Effect of a maximal transition on permutation diagrams

Consider row by row the effect on diagrams of passing from w to w'. It is clear that rows $k < j$ of $D(w')$ match those of $D(w)$. Rows $k > s$ also match; indeed, they are all empty.

In row j, by passing from $D(w)$ to $D(w')$, we could only gain cells. Specifically, a cell is gained in column $w(k)$ if and only if the following equivalent conditions hold:

- $w(j) < w(k) < w(s)$ and $k > j$, or $w(k) = w(j)$;
- $w(j) < w(k) < w(s)$ and $k > r$, or $w(k) = w(j)$;
- $(r, w(k)) \in D(w)$ and $w(j) < w(k)$, or $w(k) = w(j)$.

On the other hand, in row r, we could only lose cells. A cell is lost in column $w(k)$ if and only if the following equivalent conditions hold:

- $w(j) < w(k) < w(r)$ and $k > r$;
- $w(j) < w(k) < w(s)$ and $k > r$, or $w(k) = w(s)$;
- $(r, w(k)) \in D(w)$ and $w(j) < w(k)$, or $w(k) = w(s)$.

Thus, the effect of passing from w to w' on rows j and r is to move all cells in row r between columns $w(j)$ and $w(s)$ up to row j, and to move $(r, w(s))$ to $(j, w(j))$.

Now say $j < k < r$. The only column in which a cell could be gained in row k is column $w(j)$, which happens if and only if the following equivalent conditions hold:

- $w(k) > w(j)$;
- $w(k) > w(s)$;
- $(k, w(s)) \in D(w)$.

Conversely, if there is a cell in row k and column $w(s)$ of $D(w)$, there is no such cell in $D(w')$.

We now have that within the region $[j, r] \times [w(j), w(s)]$, one does obtain $D(w')$ from $D(w)$ by performing the indicated James–Peel moves. To show that in fact $D(w') = R_{r\to j} C_{w(s)\to w(j)} D(w)$, we must show that these James–Peel moves do not move any cells outside of $[j, r] \times [w(j), w(s)]$. That is:

(i) If $(k, w(s)) \in D(w)$ for $k < j$ or $k > r$, then $(k, w(j)) \in D(w)$.

(ii) If $(r, w(k)) \in D(w)$ for $w(k) < w(j)$ or $w(k) > w(s)$, then $(j, w(k)) \in D(w)$.

For (i), rows $k > r$ are empty, so assume $k < j$ and $(k, w(s)) \in D(w)$. Then $w(k) > w(s) > w(j)$ and $k < j$ give $(k, w(j)) \in D(w)$. For (ii), $(r, w(s))$ is the rightmost cell in row r by the choice of r and s, so assume $w(k) < w(j)$ and $(r, w(k)) \in D(w)$. Then $(j, w(k)) \in D(w)$, because $j < r < k$ and $w(k) < w(j)$. □

Let JP(w) be the tree of diagrams obtained by replacing permutations in the L–S tree with their diagrams. Label the edges according to Lemma 3.1.4, so that the edge from $D(w)$ to $D(w')$ for $w' = wt_{rs}t_{rj} \in T(w)$ is labeled $R_{r\to j} C_{w(s)\to w(j)}$.

Theorem 3.1.5. *The tree* JP(w) *is a James–Peel tree.*

Proof. Induct on the Lascoux–Schützenberger tree of w. If w is vexillary, the tree with one vertex $D(w)$ and no edges is a complete James–Peel tree for $D(w)$. Otherwise, let $v^1, \dots, v^p$ be the transitions of w, say $v^i = wt_{rs}t_{rj_i}$ where $s > r > j_1 > \cdots > j_p$. Then $w(j_1) < \cdots < w(j_p) < w(s) < w(r)$, so $fl(w(j_p) \cdots w(j_1) w(r) w(s)) = p \cdots 1 (p+2)(p+1)$, and $D(p \cdots 1 (p+2)(p+1))$ is exactly $(p-1, \dots, 1) \cdot (1)$ after removing an empty row and column. Thus, $D(w)$ contains $(p-1, \dots, 1) \cdot 1$ as a subdiagram in rows $\{j_p, \dots, j_2, r\}$ and columns $\{w(j_1), \dots, w(j_{p-1}), w(s)\}$.

Let

$$D_i = \begin{cases} R_{r\to j_i} C_{w(s)\to w(j_i)} D(w) & \text{if } 1 < i < p, \\ C_{w(s)\to w(j_1)} D(w) & \text{if } i = 1, \text{ and} \\ R_{r\to j_p} D(w) & \text{if } i = p. \end{cases}$$

The diagrams D_i are exactly those produced by Theorem 2.1.1, and $\bigoplus_{i=1}^{p} S^{D_i} \hookrightarrow S^{D(w)}$.

Let $\mathcal{T}$ be the James–Peel tree with root $D(w)$ and children D_i, where $D(w)$ is connected to D_i by an edge labeled with the appropriate James–Peel move(s). Next, connect D_1 to a child $E_1 = R_{r \to j_1} D_1$ by an edge labeled $R_{r \to j_1}$, and D_p to a child $E_p = C_{w(s) \to w(j_p)} D_p$ by an edge labeled $C_{w(s) \to w(j_p)}$. The leaves $E_1, D_2, \ldots, D_{p-1}, E_p$ of $\mathcal{T}$ are now exactly $D(v^1), \ldots, D(v^p)$ by Lemma 3.1.4. By induction on the L–S tree, each $D(v^i)$ has a complete James–Peel tree; attach it to the leaf $D(v^i)$ of $\mathcal{T}$. $\square$

Theorem 3.1.6 (Reiner–Shimozono [23], Kraśkiewicz–Pragacz [14])**.** *The Frobenius characteristic of the Specht module $S^{D(w)}$ is F_w.*

Proof. Let V_w be the $\mathfrak{S}_{\ell(w)}$-module corresponding to the Schur-positive symmetric function F_w. By construction,

$$V_w \simeq \bigoplus_{\lambda \in \text{leaves}(\text{JP}(w))} S^{\lambda},$$

since the right side is also the direct sum over the shapes of the leaves of the L–S tree for w. Since $\text{JP}(w)$ is a James–Peel tree for $D(w)$, this shows that $V_w \hookrightarrow S^{D(w)}$.

On the other hand, $\dim V_w$ is the coefficient of $x_1 \cdots x_\ell$ in F_w, which is $|\text{Red}(w)|$. Theorem 4.1 of [13] shows that this is also the dimension of $S^{D(w)}$. $\square$

Corollary 3.1.7. The James–Peel tree $\text{JP}(w)$ for $D(w)$ is complete.

Remark 3.1.8. One can also define *Schur modules* and *flagged Schur modules* of diagrams, as in [25]. These are $\text{GL}(N)$- and B-modules, respectively, with $B \subseteq \text{GL}(N)$ a Borel subgroup, and correspond to Specht modules by Schur–Weyl duality. Kraśkiewicz and Pragacz proved that the character of the flagged Schur module of $D(w)$ is the Schubert polynomial $\mathfrak{S}_w$, and their proof uses essentially the techniques of Theorems 3.1.5, 3.1.4, and 2.1.1, although in different language [14]. Theorem 3.1.6 can also be proved from this point of view, using the connection between Schubert polynomials and Stanley symmetric functions.

2 *k-vexillary and multiplicity-free permutations*

With the connection between Stanley symmetric functions and Specht modules in place, our first main results follow. Recall that $M(w)$ is the multiset of partitions such that $F_w = \sum_{\lambda \in M(w)} s_\lambda$.

Theorem 3.2.1. *Suppose w and v are permutations with w containing v. There is an injection $\iota : M(v) \hookrightarrow M(w)$ such that*

(i) $\lambda \subseteq \iota(\lambda)$ whenever $\lambda \in M(v)$, and

(ii) if λ appears with multiplicity m in $M(v)$, then $\iota(\lambda)$ appears with multiplicity at least m in $M(w)$.

Proof. Suppose w contains v as a pattern in positions $i_1, \ldots, i_p$. Then $D(w)$ contains $D(v)$ as a subdiagram in rows $i_1, \ldots, i_p$ and columns $w(i_1), \ldots, w(i_p)$. Now apply Corollary 2.2.11 using the complete James–Peel tree $JP(v)$. $\square$

Definition 3.2.2. Recall that $w \in \mathfrak{S}_n$ is *k-vexillary* if $\#M(w) \leq k$. We will say that w is *multiplicity-free* if no partition appears in $M(w)$ more than once.

Corollary 3.2.3. If a permutation w is k-vexillary and v is a pattern in w, then v is k-vexillary.

Corollary 3.2.4. If a permutation w is multiplicity-free and v is a pattern in w, then v is multiplicity-free. More generally, if $\langle F_w, s_\lambda \rangle \leq k$ for all λ then $\langle F_v, s_\mu \rangle \leq k$ for all μ. Here $\langle \cdot, \cdot \rangle$ is the Hall inner product on symmetric functions, defined by declaring the Schur functions to be an orthonormal basis.

Remark 3.2.5. Theorem 3.2.1 shows the existence of an injection $\mathcal{EG}(v) \hookrightarrow \mathcal{EG}(w)$ which respects inclusion of shapes for v a pattern contained in w, but an explicit map on tableaux is lacking. The Edelman–Greene correspondence shows that this is equivalent to an injection $\mathrm{Red}(v) \hookrightarrow \mathrm{Red}(w)$ which is an inclusion on the shapes of

Edelman–Greene insertion tableaux. Tenner's characterization of vexillary permutations yields an explicit injection in the case where v is vexillary [29].

So far we have only used Corollary 2.2.11, but the full strength of Lemma 2.2.10 yields another interesting result. Recall the notation $EG(w) = \#M(w)$.

Theorem 3.2.6. *Let $w \in \mathfrak{S}_n$ be a permutation and $I \subseteq [n]$. If u_1 is the subsequence of w in positions I, and u_2 the subsequence in positions $[n] \setminus I$, then*

$$EG(w) \geq EG(fl(u_1)) \cdot EG(fl(u_2)).$$

Let EG_n be the statistic EG thought of as a random variable on $\mathfrak{S}_n$. Theorem 3.2.6 gives an exponential lower bound on the expected value of the statistic EG on $\mathfrak{S}_n$.

Theorem 3.2.7. *For a fixed n_0 and any n, $\mathbb{E}[EG_n] \geq \mathbb{E}[EG_{n_0}]^{\lfloor n/n_0 \rfloor}$. In particular, $\mathbb{E}[EG_n] \geq (0.072)(1.299)^n$.*

Proof. As w runs over $\mathfrak{S}_{a+b}$, every pair of permutations in $\mathfrak{S}_a \times \mathfrak{S}_b$ is obtained $\binom{a+b}{a}$ times as the pair

$$(fl(w(1) \cdots w(a)), fl(w(a+1) \cdots w(a+b))).$$

Hence Theorem 3.2.6 implies

$$\sum_{w \in \mathfrak{S}_{a+b}} EG(w) \geq \binom{a+b}{a} \sum_{u_1 \in \mathfrak{S}_a} EG(u_1) \sum_{u_2 \in \mathfrak{S}_b} EG(u_2),$$

or $\mathbb{E}[EG_{a+b}] \geq \mathbb{E}[EG_a]\mathbb{E}[EG_b]$. This proves the first claim. The second is obtained by taking $n_0 = 11$ and calculating $\mathbb{E}[EG_{11}]$ explicitly, then using $\lfloor n/11 \rfloor \geq n/11 - 10/11$. $\square$

3 Reduced James–Peel trees

We have shown that the property of being k-vexillary respects pattern containment, and our next goal is to show that in fact it is characterized by the avoidance of a finite set of patterns for any k. The key step is to remove some inessential moves from the James–Peel tree for $D(w)$, namely those which only permute rows or columns.

If D is an arbitrary diagram, and σ and τ are permutations, let $(\sigma, \tau)D$ be the diagram $\{(\sigma(i), \tau(j)) : (i,j) \in D\}$. Given a James–Peel tree $\mathcal{T}$ for D, let $(\sigma,\tau)\mathcal{T}$ denote the James–Peel tree for $(\sigma,\tau)D$ obtained by replacing every James–Peel move $R_{x \to y}$ labeling an edge of $\mathcal{T}$ by $R_{\sigma(x) \to \sigma(y)}$, and every move $C_{x \to y}$ by $C_{\tau(x) \to \tau(y)}$, and relabeling vertices accordingly. Whenever a move labeling an edge e of a James–Peel tree just permutes rows or columns, we can eliminate that move from the tree at the cost of relabeling rows and columns of James–Peel moves below e, as follows.

Definition 3.3.1. Given a James–Peel tree $\mathcal{T}$ of a diagram D, define the *reduced James–Peel tree* $\operatorname{red}(\mathcal{T})$ of D inductively:

- If D has no children in $\mathcal{T}$, then $\operatorname{red}(\mathcal{T}) = \mathcal{T}$.
- If D has just one child F, and $D = (\sigma, \tau)F$ for some $\sigma, \tau \in \mathfrak{S}_\infty$, then let $\mathcal{T}_1$ be the subtree of $\mathcal{T}$ below F with root F. Then $\operatorname{red}(\mathcal{T}) = (\sigma,\tau)\operatorname{red}(\mathcal{T}_1)$.
- If D has at least two children $F_1, F_2, \ldots, F_p$ or D has one child F_1 not equivalent to D, then let $\mathcal{T}_i$ be the subtree of $\mathcal{T}$ below F_i with root F_i. Then $\operatorname{red}(\mathcal{T})$ is $\mathcal{T}$ with each $\mathcal{T}_i$ replaced by $\operatorname{red}(\mathcal{T}_i)$.

Definition 3.3.2. A rooted tree is *bushy* if every non-leaf vertex has at least two children.

Lemma 3.3.3. *If $\mathcal{T}$ is a complete James–Peel tree for D, then $\operatorname{red}(\mathcal{T})$ is a complete James–Peel tree for D. Furthermore, $\operatorname{red}(\mathcal{T})$ is bushy.*

Proof. Note that $\operatorname{red}(\mathcal{T})$ is still a James–Peel tree for D. Because equivalent diagrams have isomorphic Specht modules, if $\mathcal{T}$ is complete then so is $\operatorname{red}(\mathcal{T})$.

Next, for any vertex A of $\mathcal{T}$, the subtree of $\mathcal{T}$ below A is itself a complete James–Peel tree for A. In particular, S^A is determined by the leaves below it. Therefore, if A has only a single child B in $\mathcal{T}$, then S^A and S^B are isomorphic.

Now suppose $\mathcal{T}$ is not bushy. The only way this can happen is if $\mathcal{T}$ has a vertex A with only one child B, but A and B are not equivalent. There is a James–Peel move relating A and B (or a sequence of moves, but we can consider them one at a time), say $B = R_{a\to b}A$. If one of rows a and b of A is contained in the other, then $R_{a\to b}A$ is simply A with those two rows interchanged, so rows a or b are not comparable under inclusion since A and B are not equivalent. There are cells $(a, j_1), (b, j_2) \in A$ with $(a, j_2), (b, j_1) \notin A$. By Theorem 2.0.2, $S^B \oplus S^{C_{j_1\to j_2}A} \hookrightarrow S^A$. As $S^{C_{j_1\to j_2}A} \neq 0$, S^B is not isomorphic to S^A. This contradicts the previous paragraph, so $\mathcal{T}$ must be bushy. ☐

Lemma 3.3.4. *The number of edges in a bushy tree with k leaves is at most $2k - 2$.*

Proof. This follows by induction on the number of leaves. ☐

Recall JP(w) is the James–Peel tree for $D(w)$ constructed in Theorem 3.1.5, and let $\mathrm{RJP}(w) = \mathrm{red}(\mathrm{JP}(w))$. The vertex $D(w)$ and its children in the tree JP(w) are shown in Figure 3.2. Here the $v^i = vt_{rs}t_{rj_i}$ are the transitions of v, with $j_1 > \cdots > j_p$.

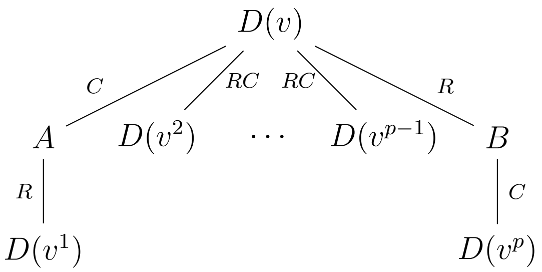


Figure 3.2: Part of the James–Peel tree JP(w).

The rows of $D(v)$ involved in row moves are $r, j_1, \ldots, j_p$, and the columns involved in column moves are $v(s), v(j_1), \cdots, v(j_p)$. In Figure 3.2, we have elongated two of the paths for the proofs to come.

Lemma 3.3.5. *Suppose v has transitions $v^1, \ldots, v^p$ as above. Then $D(v^1)$ is equivalent to $C_{v(s)\to v(j_1)}D(v)$, and $D(v^p)$ is equivalent to $R_{r\to j_p}D(v)$.*

Proof. By Lemma 3.1.4,

$$D(v^1) = C_{v(s)\to v(j_1)}R_{r\to j_1}D(v)$$

and

$$D(v^p) = R_{r\to j_p}C_{v(s)\to v(j_p)}D(v).$$

It suffices to check that column $v(s)$ of $D(v)$ contains column $v(j_p)$, and that row r contains row j_1. Suppose the first of these fails, meaning that there is $(i, v(j_p)) \in D(v)$ with $(i, v(s)) \notin D(v)$. Choose the maximal such i. Then $vt_{rs}t_{ri}$ is a transition of v, which is impossible since $i < j_p$. The argument for the row containment is analogous. $\square$

Thus, upon passing to $\mathrm{RJP}(w)$, the edges A—$D(v^1)$ and B—$D(v^p)$ are contracted. For a diagram D, write $[D]$ for the equivalence class of diagrams containing D. We use this notation below when we have a diagram equivalent to D but do not need to specify exactly what the diagram is. The vertex $D(w)$ and its children in the tree $\mathrm{RJP}(w)$ are shown in Figure 3.3.

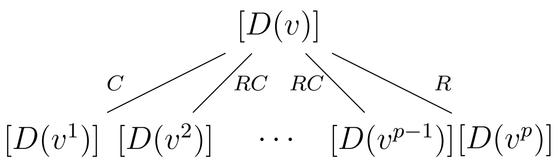


Figure 3.3: Part of the reduced James–Peel tree $\mathrm{RJP}(w)$.

We omit the proof that both moves on the edges labeled RC survive in $\mathrm{RJP}(w)$, as it is unimportant for our purposes: to use the number of James–Peel moves performed in the graph $\mathrm{RJP}(w)$ to obtain an upper bound on $EG(w)$. We will therefore speak of R-edges, C-edges, and RC-edges of $\mathrm{RJP}(w)$, each non-leaf vertex having exactly one R-edge and one C-edge leading to children.

Now suppose $\mathcal{T}$ is a subtree of $\mathrm{RJP}(w)$ with the same root. Let $R(\mathcal{T})$ be the union of $\{a, b\}$ over all $R_{a\to b}$ appearing in $\mathcal{T}$, and $C(\mathcal{T})$ the union of $\{c, d\}$ over all $C_{c\to d}$ appearing in $\mathcal{T}$. Write $R(\mathcal{T}) \cup w^{-1}C(\mathcal{T}) = \{i_1 < \cdots < i_r\}$, and define the permutation associated to this tree

$$w_{\mathcal{T}} = fl(w(i_1)\cdots w(i_r)).$$

Remark 3.3.6. In Remark 3.1.2 we noted that, for convenience, w could be replaced by $1^m \times w$ to remove the necessity of sometimes replacing v by $1 \times v$ in the L–S tree. The definition of $w_{\mathcal{T}}$ above is then an abuse of notation, since we are really taking a subsequence of $1^m \times w$. However, rows and columns $1, \ldots, m$ of $D(1^m \times w)$ are empty, so are not affected by the James–Peel moves in $\mathrm{RJP}(w)$ or $\mathcal{T}$. This means that the subsequence defining $w_{\mathcal{T}}$ occurs entirely after the mth position of $1^m \times w$, so we are free to shift it down by m and consider it as a subsequence of w. This applies also to Theorems 3.3.9 and 3.3.10 below.

We would like to bound the number of letters of $w_{\mathcal{T}}$ in terms of the number of leaves of $\mathcal{T}$. Such a bound depends on the sizes of $R(\mathcal{T})$ and $C(\mathcal{T})$, so the following definition is convenient to get good bounds.

Definition 3.3.7. A subtree $\mathcal{T}$ of $\mathrm{RJP}(w)$ with root $D(w)$ is *colorful* if each non-leaf vertex of $\mathcal{T}$ has at least the two children corresponding to its R-edge and its C-edge. Thus, colorful implies bushy.

Lemma 3.3.8. *Let $\mathcal{T}$ be a subtree of* $\mathrm{RJP}(w)$ *rooted at $D(w)$ with k leaves. Then $k \leq EG(w_{\mathcal{T}}) \leq EG(w)$. If $\mathcal{T}$ is colorful, then $w_{\mathcal{T}} \in \mathfrak{S}_m$ for some $m \leq 4k - 4$.*

Proof. Up to relabeling rows and columns to account for flattening, the tree $\mathcal{T}$ is a (not necessarily complete) James–Peel tree for $D(w_{\mathcal{T}})$, so $k \leq EG(w_{\mathcal{T}})$. Theorem 3.2.1 implies $EG(w_{\mathcal{T}}) \leq EG(w)$.

Suppose $\mathcal{T}$ is colorful. The number of letters in $w_{\mathcal{T}}$ is at most $\#R(\mathcal{T}) + \#C(\mathcal{T})$. Consider the vertex indexed by $D(v)$ in the full tree $\mathrm{JP}(w)$. Say $v^i = vt_{rs}t_{rj_i}$ are the transitions of v, with $j_1 > \cdots > j_p$. The rows of $D(v)$ involved in row moves are $r, j_1, \ldots, j_p$, and the columns involved in column moves are $v(s), v(j_1), \cdots, v(j_p)$. However, $R_{r\to j_1}D(v) \simeq D(v)$ and $C_{v(s)\to v(j_p)}D(v) \simeq D(v)$ by Lemma 3.3.5, so these edges are contracted in the reduced tree, so row j_1 and column $v(j_p)$ will not contribute to $R(\mathcal{T})$ and $C(\mathcal{T})$ respectively. Thus, if a vertex F (which is equivalent to some $D(v)$) of $\mathcal{T}$ has p children, then the edges leading down from F contribute at most p elements to each of $R(\mathcal{T})$ and $C(\mathcal{T})$. Summing over all vertices,

$$\begin{aligned}
\#R(\mathcal{T}) + \#C(\mathcal{T}) &\leq 2\deg(D(w)) + \sum_{\substack{F\in\mathcal{T}\\ F\neq D(w)}} 2(\deg(F)-1)\\
&= 2\left[\sum_{F\in\mathcal{T}} \deg(F)\right] - 2\#V(\mathcal{T}) + 2\\
&= 4\#E(\mathcal{T}) - 2(\#V(\mathcal{T})-1)\\
&= 2\#E(\mathcal{T})\\
&\leq 4k-4,
\end{aligned}$$

with the last inequality by Lemma 3.3.4. ☐

In particular, taking $\mathcal{T} = \mathrm{RJP}(w)$ in Lemma 3.3.8 gives the following theorem.

Theorem 3.3.9. *Any permutation w contains a pattern $v \in \mathfrak{S}_m$ such that $EG(w) = EG(v)$, for some $m \leq 4 \cdot EG(w) - 4$.*

More generally, Lemma 3.3.8 lets us show that k-vexillarity is characterized by avoiding a *finite* set of patterns.

Theorem 3.3.10. *Let w be a permutation with $EG(w) > k$. Then w contains a pattern $v \in \mathfrak{S}_m$ such that $EG(v) > k$, for some $m \leq 4k$.*

Proof. By Lemma 3.3.8, it suffices to exhibit a colorful subtree of $\mathrm{RJP}(w)$ rooted at $D(w)$ with $k+1$ leaves. Construct such a tree $\mathcal{T}$ as follows. First take $\mathcal{T}$ to have only the vertex $D(w)$. Add the two children of $D(w)$ corresponding to the R-edge and the C-edge. Continue adding the remaining children of $D(w)$ until $\mathcal{T}$ has $k+1$ leaves, or until all children have been added. If all children of $D(w)$ have been added and $\mathcal{T}$ has fewer than $k+1$ leaves, then since $\mathrm{RJP}(w)$ has at least $k+1$ leaves, there is a leaf F of $\mathcal{T}$ with at least two children. Now repeat this process starting with F in place of $D(w)$. Iterating, eventually $\mathcal{T}$ will have $k+1$ leaves, and is colorful by construction. $\square$

Theorem 3.3.11. *Let V_k be the set of non-k-vexillary members of $\mathfrak{S}_m$ for $1 \leq m \leq 4k$. Then a permutation w is k-vexillary if and only if it avoids all patterns in V_k.*

For $k = 2$, we can explicitly find all non-2-vexillary patterns in $\mathfrak{S}_m$ for $1 \leq m \leq 8$ and eliminate those containing a smaller non-2-vexillary pattern to find a minimal list.

Theorem 3.3.12. *A permutation w is 2-vexillary if and only if it avoids all of the following 35 patterns.*

21543
32154
214365
214635
215364
231564
241365
241635
312645
314265
315264
426153
2547163
4265173
5173264
5271436
5276143
5472163
25476183
26481537
26487153
26581437
26587143
51736284
51763284
54726183
54762183
61832547
61837254
61873254
64821537
64872153
65821437
65827143
65872143

This process is also feasible for $k = 3$ or 4, in which case we need to look at non-3-vexillary (resp. non-4-vexillary) patterns up through $\mathfrak{S}_{12}$ (resp. $\mathfrak{S}_{16}$). In both cases we find that the bound in Theorem 3.3.11 is not sharp.

Theorem 3.3.13. *A permutation w is 3-vexillary if and only if it avoids a list of 91 patterns in $\mathfrak{S}_6 \cup \mathfrak{S}_7 \cup \mathfrak{S}_8$, and 4-vexillary if and only if it avoids a list of 2346 patterns in $\bigcup_{i=6}^{12} \mathfrak{S}_i$. For the full lists of patterns, see*

`http://www.math.washington.edu/~billey/papers/k.vex.html`.

If one wants to compute or bound $EG(w)$, the Lascoux–Schützenberger tree is almost certainly more efficient than using our pattern characterizations. However, pattern characterizations lend themselves nicely to comparison. As an example, we give a sense in which the essential set of a 3-vexillary permutation is relatively simple.

In [8], Fulton defined the *essential set* of a permutation w, $\mathrm{Ess}(w)$, to be the set of southeast corners of the connected components of the diagram $D(w)$. He showed that the rank conditions for the Schubert variety indexed by w need only be checked at cells in the essential set. See [26, Prop.4.6] for an alternative description of the essential set as the set of minimal bigrassmannian elements not below w in Bruhat order (a permutation is *bigrassmannian* if it and its inverse both have at most one descent).

Fulton also showed how to characterize vexillary permutations by their essential sets. The *SW-NE order* on $\mathbb{N}^2$ is the partial order defined by $(i_1, j_1) \leq (i_2, j_2)$ if $i_1 \geq i_2$ and $j_1 \leq j_2$ (one should think of matrix coordinates here). Fulton showed that w is vexillary if and only if $\mathrm{Ess}(w)$ is a chain in the NW-SE order. Thus, the essential set lies along a lattice path going from the southwest corner of the diagram to the northeast using only north and east steps. Equivalently, $\mathrm{Ess}(w)$ has no antichain of size 2 when w is vexillary.

One can characterize permutations whose essential set consists of two non-intersecting such lattice paths in terms of pattern avoidance.

Lemma 3.3.14. *The essential set* $\mathrm{Ess}(w)$ *has no antichain of size* 3 *in NW-SE order* *if and only if w avoids the following 25 patterns.*

214365
2416375
2417365
3152746
3152764
3251746
3251764
4216375
4216735
35172846
35172864
35182746
35182764
35271846
35271864
35281746
35281764
53172846
53172864
53182746
53182764
53271846
53271864
53281746
53281764

Corollary 3.3.15. If a permutation is 3-vexillary, then its essential set has no antichain of size 3.

Proof. None of the patterns in Lemma 3.3.14 are 3-vexillary, so this follows from Corollary 3.2.3. ☐

Remark 3.3.16. The essential set can be used to give a short proof that the Lascoux-Schützenberger tree is finite. The L–S tree can contain only finitely many w with more than one maximal transition, for example because $F_w = \sum_v F_v$ for v running over transitions of w, and the coefficient of $x_1 \cdots x_\ell$ in F_w is always positive. Hence it suffices to show that there are only finitely many w in the tree with exactly one maximal transition.

Suppose w has exactly one maximal transition $v = wt_{rs}t_{rj}$, where r is the largest index of a non-empty row in $\mathrm{Ess}(w)$. Then j must be $r-1$ and $w(j)$ must be $w(s)-1$, since otherwise there would be either no maximal transitions or more than one. Lemma 3.1.4 shows that $D(v) = D(w) \setminus \{(r, w(s))\} \cup \{(r-1, w(s)-1)\}$, and one can then check that likewise $\mathrm{Ess}(v) = \mathrm{Ess}(w) \setminus \{(r, w(s))\} \cup \{(r-1, w(s)-1)\}$. The same argument holds if w must be replaced by $1 \times w$ in the algorithm. Thus, in passing from w to v, the rightmost element of the lowest non-empty row of the essential set moves either leftward or upward.

We have shown that k-vexillary permutations and multiplicity-free permutations both respect pattern containment, and that the k-vexillary permutations are characterized by avoiding a finite set of patterns, so it is natural to wonder whether the multiplicity-free permutations are as well. The following conjecture has been tested through $\mathfrak{S}_{12}$, and as noted, one direction follows from Corollary 3.2.4.

Conjecture 3.3.17. The set of multiplicity-free permutations is closed under taking patterns, and the minimal patterns all occur in $\mathfrak{S}_n$ for $n \leq 11$.

For the minimal list of 189 patterns up to $\mathfrak{S}_{11}$, see

`http://www.math.washington.edu/~billey/papers/k.vex.html`.

Chapter 4

RANK VARIETIES AND POSITROID VARIETIES

We begin by discussing a family of subvarieties of the Grassmannian introduced by Coskun [4], the *rank varieties*. These are defined by conditions on intersections with subspaces of the form $\langle e_a, e_{a+1}, \dots, e_b \rangle$, where $e_1, \dots, e_n$ is a basis of $\mathbb{C}^n$. Allowing the indices in these interval subspaces to wrap around modulo n gives a larger class of varieties called *positroid varieties*, due to Knutson–Lam–Speyer [12]. Rank varieties are indexed by sets of intervals, while positroid varieties are indexed by certain affine permutations (among other objects), and we work out exactly which positroid varieties are rank varieties. The main result of this chapter is that the cohomology class of any rank variety is represented by a Stanley symmetric function.

1 *Rank varieties*

Definition 4.1.1 ([1])**.** A *rank set* is a finite set of intervals $M = \{[a_1, b_1], \dots, [a_k, b_k]\}$ with $a_i \leq b_i$ positive integers, where all a_i are distinct and all b_i are distinct. If S is a set of positive integers, let $S(M)$ denote the set of intervals $S' \in M$ such that $S' \subseteq S$.

Suppose M is a rank set with $b \leq n$ for all $[a, b] \in M$, and $\#M = k$. Coskun [4] defines a closed subvariety Σ_M of $\mathrm{Gr}(k, n)$ as the closure of the locus

$$\{V \in \mathrm{Gr}(k,n) : \dim(V \cap \langle S \cap T \rangle) = \#(S \cap T)(M) \text{ for } S, T \in M\}.$$

Here $\#(S \cap T)(M)$ is the number of intervals of M contained in $S \cap T$, as defined above. Σ_M is called a *rank variety.*

Theorem 4.1.2 ([4], Lemma 3.29)**.** *The rank variety Σ_M is irreducible of dimension $\sum_{S \in M}(\#S - \#S(M))$.*

The variety Σ_M has a useful interpretation in coordinates, stated without proof in [1]. Fix a basis $e_1, \dots, e_n$ of $\mathbb{C}^n$.

Lemma 4.1.3. *Let U_M be the locus of k-planes with a basis $\{v_S : S \in M\}$, indexed by the intervals in M, such that the coefficient of e_i in v_S is nonzero if and only if $i \in S$. Then $\overline{U_M} = \Sigma_M$.*

Example 4.1.4. If $M = \{[1,2], [3,4], [2,5]\}$ and $n = 5$, then

$$U_M = \left\{ \text{rowspan} \begin{pmatrix} * & * & 0 & 0 & 0 \\ 0 & 0 & * & * & 0 \\ 0 & * & * & * & * \end{pmatrix} : \text{every } * \text{ nonzero} \right\} = \left\{ \text{rowspan} \begin{pmatrix} * & 1 & 0 & 0 & 0 \\ 0 & 0 & * & 1 & 0 \\ 0 & * & * & 0 & 1 \end{pmatrix} \right\}.$$

Proof of Lemma 4.1.3. Clearly $U_M \subseteq \Sigma_M$. We will show that $\dim U_M \geq \dim \Sigma_M$, which is enough by the irreducibility of Σ_M.

Let $N_i = M_i \setminus \{b_j : M_j \subseteq M_i, j \neq i\}$. Let A be the open subset of $\bigoplus_{i=1}^k \langle N_i \rangle$ consisting of those $(v_1, \dots, v_k)$ such that

- $\dim\langle v_1, \dots, v_k \rangle = k$,
- if $p \in N_i$, the coefficient of e_p in v_i is nonzero, and
- the coefficient of e_{b_i} in v_i is 1.

If $(v_1, \dots, v_k) \in A$, then $\langle v_1, \dots, v_k \rangle \in U_M$. By Theorem 4.1.2, $\dim \Sigma_M = \dim A$. Therefore it is enough to show that the map $A \to U_M$, $(v_1, \dots, v_k) \mapsto \langle v_1, \dots, v_k \rangle$ is injective.

Suppose that $\langle v_1, \dots, v_k \rangle = \langle w_1, \dots, w_k \rangle$, with $(v_1, \dots, v_k)$ and $(w_1, \dots, w_k)$ in A. Write

$$v_i = \gamma_1 w_1 + \cdots + \gamma_k w_k. \tag{4.1}$$

Because the a_i are all distinct, and the b_i are all distinct, $\gamma_j \neq 0$ forces $M_j \subseteq M_i$. Suppose there is $\gamma_j \neq 0$ with $j \neq i$, and choose such a j making b_j maximal. By maximality, the only summand in (4.1) with nonzero e_{b_j} coefficient is $\gamma_j w_j$. But the coefficient of e_{b_j} in v_i is 0, since $b_j \notin N_i$. This contradiction means that $\gamma_j = 0$ unless $j = i$, and then we must have $\gamma_i = 1$. That is, $v_i = w_i$. □

2 Positroid varieties

Definition 4.2.1. An *affine permutation* is a bijection $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$f(i+n) = f(i) + n$$

for all i and some fixed n. Write $\breve{\mathfrak{S}}_n$ for the set of affine permutations with a particular n.

Note that the image of any set $\{a, a+1, \ldots, a+n-1\}$ completely determines an affine permutation. Call such an image a *window*. We will write affine permutations in one-line notation as the image of $[n]$: 14825 fixes 1, sends 3 to 8, 7 to 9, etc. Members of any window are all distinct modulo n, so $\sum_{i=1}^{n} f(i) \equiv n(n+1)/2 \pmod{n}$. Let $\mathrm{av}(f)$ be the integer $\frac{1}{n}\sum_{i=1}^{n} f(i) - i$.

Warning. Affine permutations are usually required to satisfy $\mathrm{av}(f) = 0$, which ours need not. However, for a fixed k, affine permutations in $\breve{\mathfrak{S}}_n$ satisfying $\mathrm{av}(f) = k$ are in bijection with those satisfying $\mathrm{av}(f) = 0$ by subtracting k from each entry in a window. When we refer to constructions on affine permutations that require a Coxeter group structure (e.g. length, reduced words, Stanley symmetric functions), we are implicitly using this isomorphism to transport that structure from the "usual" affine permutation group $\{f \in \breve{\mathfrak{S}}_n : \mathrm{av}(f) = 0\}$.

The *length* $\ell(f)$ of an affine permutation $f \in \breve{\mathfrak{S}}_n$ is the number of inversions $i < j$, $f(i) > f(j)$, provided that we regard any two inversions $i < j$ and $i + pn < j + pn$ as equivalent.

Definition 4.2.2. An affine permutation $f \in \breve{\mathfrak{S}}_n$ is *bounded* if $i \leq f(i) \leq i+n$ for all i. Let $\mathrm{Bound}(k,n)$ denote the set of bounded affine permutations in $\breve{\mathfrak{S}}_n$ with $\mathrm{av}(f) = k$.

Any affine permutation f has a permutation matrix, the $\mathbb{Z} \times \mathbb{Z}$ matrix A with $A_{i,f(i)} = 1$ and all other entries 0. For any $i, j \in \mathbb{Z}$, let

$$[i,j](f) = \{p < i : f(p) > j\}. \tag{4.2}$$

Thus, $\#[i,j](f)$ is the number of 1's strictly northeast of (i,j) in the permutation matrix of f, in matrix coordinates.

We will abuse notation by writing $\langle X \rangle$ both for the span of the vectors in X, if $X \subseteq \mathbb{C}^n$, and for the span of vectors e_i with $i \in X$, if $X \subseteq [n]$. If $X \subseteq [n]$, let $\mathrm{Proj}_X : \mathbb{C}^n \to \langle X \rangle$ be the projection which fixes those basis vectors e_i with $i \in X$ and sends the rest to 0. For integers $i \leq j$, write $[i,j]$ for $\{i, i+1, \ldots, j\}$. We interpret indices of basis vectors modulo n, so that $\langle [i,j] \rangle \subseteq \mathbb{C}^n$ even if i, j fail to lie in $[1,n]$.

Definition 4.2.3 ([12])**.** Given a bounded affine permutation $f \in \mathrm{Bound}(k,n)$, the *positroid variety* $\Pi_f \subseteq \mathrm{Gr}(k,n)$ is the closure of

$$\{V \in \mathrm{Gr}(k,n) : \dim(\mathrm{Proj}_{[i,j]} V) = k - \#[i,j](f) \text{ for all } i \leq j\}.$$

Theorem 4.2.4 ([12], Theorem 5.9)**.** *The positroid variety $\Pi_f \subseteq \mathrm{Gr}(k,n)$ is irreducible of codimension $\ell(f)$.*

Knutson–Lam–Speyer also computed the cohomology class of Π_f in terms of affine Stanley symmetric functions. These are a class of symmetric functions indexed by affine permutations introduced in [15], and we now give a definition.

Let $\breve{\mathfrak{S}}_n^k$ be the set of affine permutations with $\mathrm{av}(f) = k$. Then $\breve{\mathfrak{S}}_n^0$ is a Coxeter group with simple generators $s_0, \ldots, s_{n-1}$, where s_i interchanges $i + np$ and $i + 1 + np$ for every p. A *reduced word* for $f \in \breve{\mathfrak{S}}_n^0$ is a word $a_1 \cdots a_\ell$ in the alphabet $[0, n-1]$ with $s_{a_1} \cdots s_{a_\ell} = f$ and such that ℓ is minimal with this property. Let $\mathrm{Red}(f)$ denote the set of reduced words for f. A reduced word $a = a_1 \cdots a_\ell$ is *cyclically decreasing* if all the a_i are distinct, and if whenever j and $j+1$ appear in a (modulo n), $j+1$ precedes j. An affine permutation is cyclically decreasing if it has a cyclically decreasing reduced word. For a partition λ, let m_λ be the monomial symmetric function indexed by λ.

Definition 4.2.5. The *affine Stanley symmetric function* of $f \in \breve{\mathfrak{S}}_n^0$ is

$$\tilde{F}_f = \sum_{\lambda \vdash \ell(f)} c_{f,\lambda} m_\lambda,$$

where $c_{f,\lambda}$ is the number of factorizations $f^1 \cdots f^{\ell(\lambda)} = f$ where $\ell(f) = \sum_i \ell(f_i)$, each f_i is cyclically decreasing, and $\ell(f_i) = \lambda_i$.

As mentioned above, subtracting k from each entry of a window for $f \in \breve{\mathfrak{S}}_n^k$ gives an isomorphism $\breve{\mathfrak{S}}_n^k \to \breve{\mathfrak{S}}_n^0$, which we use to extend the definition of $\tilde{F}_f$ to all of $\breve{\mathfrak{S}}_n$.

Theorem 4.2.6 ([12], Theorem 7.1)**.** *For $f \in \mathrm{Bound}(k, n)$, the cohomology class $[\Pi_f]$ is $\phi(\tilde{F}_f)$.*

The ordinary Stanley symetric functions indexed by members of $\mathfrak{S}_n$ are a special case of affine Stanley symmetric functions. To be precise, we can view $w \in \mathfrak{S}_n$ as the affine permutation in $\breve{\mathfrak{S}}_n^0$ sending $i + pn$ to $w(i) + pn$ for $1 \leq i \leq n$. Then $F_w = \tilde{F}_w$.

Being defined by rank conditions on intersections with interval subspaces, rank varieties should be special instances of positroid varieties. Say $M = \{[a_1, b_1], \ldots, [a_k, b_k]\}$ is a rank set with $b_1 < \cdots < b_k \leq n$. Define

$$\{c_1 < \cdots < c_{n-k}\} = [n] \setminus \{a_1, \ldots, a_k\} \text{ and}$$
$$\{d_1 < \cdots < d_{n-k}\} = [n] \setminus \{b_1, \ldots, b_k\}.$$

Let f_M be the affine permutation mapping b_i to $a_i + n$ and d_i to c_i. Then f_M is bounded because $a_i \leq b_i$, which implies $d_i \leq c_i$. This provides a bijection between rank sets for $\mathrm{Gr}(k, n)$ and members f of $\mathrm{Bound}(k, n)$ such that the subsequence of $f(1) \cdots f(n)$ with entries in $[n]$ is increasing.

Example 4.2.7. Take $M = \{[1,1],[3,4],[2,5]\}$ and $n = 5$ as above. Then $(b_1, b_2, b_3) = (1,4,5)$ and $(a_1, a_2, a_3) = (1,3,2)$, so $(d_1, d_2) = (2,3)$ while $(c_1, c_2) = (4,5)$. Hence $f_M = 64587$.

Theorem 4.2.8. *The rank variety* Σ_M *is the positroid variety* Π_{f_M}*.*

Proof. First we show that $\Pi_{f_M} \subseteq \Sigma_M$. To do this we check that for any interval $[r,s] \subseteq [n]$,

$$\#[r,s](M) = \#[s+1, n+r-1](f_M).$$

The following are equivalent:

- $[a_p, b_p] \in [r,s](M)$,
- $b_p \leq s$ and $a_p \geq r$,
- $b_p < s+1$ and $f_M(b_p) > n+r-1$,
- $b_p \in [s+1, n+r-1](M)$.

This shows that $\#[r,s](M)$ is the number of elements of $[s+1, n+r-1](M)$ of the form b_p. But in fact every $q \in [s+1, n+r-1](M)$ is some b_p, because $f_M(q) > n+r-1 \geq n$ and $1 \leq q \leq s \leq n$ imply $q \in \{b_1, \ldots, b_p\}$.

Now say V is in the open subset of Π_{f_M} where

$$\dim \operatorname{Proj}_{[s+1,n+r-1]} V = k - \#[s+1, n+r-1](f_M)$$

for all $r \leq s$. If $[r,s] = [a_i, b_i] \cap [a_j, b_j]$ for some i and j, then

$$\begin{aligned} \dim(V \cap \langle [r,s] \rangle) &= k - \dim \operatorname{Proj}_{[s+1,n+r-1]} V \\ &= \#[s+1, n+r-1](f_M) \\ &= \#[r,s](M). \end{aligned}$$

That is, V is in Σ_M.

Both Π_{f_M} and Σ_M are irreducible, so equality will follow if we show they have the same codimension $\ell(f_M)$. By Theorem 4.1.2,

$$\begin{aligned}
\operatorname{codim}\Sigma_M &= k(n-k) - \sum_{S\in M}(\#S - \#S(M)) \\
&= k(n-k) - \sum_{i=1}^{k}(b_i - a_i) + \sum_{S\in M}(\#S(M)-1) \\
&= k(n-k) - \sum_{i=1}^{n} i + \sum_{i=1}^{n-k} d_i + \sum_{i=1}^{n} i - \sum_{i=1}^{n-k} f_M(d_i) + \sum_{S\in M}(\#S(M)-1) \\
&= \sum_{i=1}^{n-k}(k + d_i - f_M(d_i)) + \sum_{S\in M}(\#S(M)-1).
\end{aligned}$$

Inversions of f_M come in three types:

(1) $b_j < b_i$ with $f_M(b_j) > f_M(b_i)$,

(2) $b_j < d_i$ (with $f_M(b_j) > n \geq f_M(d_i)$ automatically), and

(3) $d_i < b_j < d_i + n$ with $f_M(b_j) > f_M(d_i) + n$.

In particular, there are no inversions just among the entries $f_M(d_i)$.

For fixed i, inversions of type (1) correspond to elements of $[a_i, b_i](M) \setminus \{[a_i, b_i]\}$. Indeed, $[a_j, b_j] \subseteq [a_i, b_i]$ if and only if $b_j \leq b_i$ and

$$f_M(b_j) - n \geq a_j \geq a_i = f_M(b_i) - n.$$

Hence the total number of inversions of this type is $\sum_{S\in M}(\#S(M)-1)$.

It remains to show that $\sum_{i=1}^{n-k}(k + d_i - f_M(d_i))$ counts the inversions of types (2) and (3). Alternatively, we can show that of the pairs (b_j, d_i) for fixed i, exactly $f_M(d_i) - d_i$ of them are not inversions. Let A be the set of values $f_M(b_j) - n$ for (b_j, d_i) a noninversion with i fixed, i.e. for $b_j > d_i$ and $f_M(b_j) - n < f_M(d_i)$. Then

$$A = [1, f_M(d_i) - 1] \setminus (\{f_M(d_1), \ldots, f_M(d_{n-k})\} \cup \{f_M(b_j) - n : b_j < d_i\}).$$

Say $g \in S_n$ is the ordinary permutation with $g(d_p) = f_M(d_p)$ and $g(b_p) = f_M(b_p) - n$. We have $f_M(d_1) < f_M(d_2) < \cdots < f_M(d_i)$, so

$$\begin{aligned} A &= [1, f_M(d_i) - 1] \setminus (\{g(d_1), \ldots, g(d_{i-1})\} \cup \{g(b_j) : b_j < d_i\}) \\ &= [1, f_M(d_i) - 1] \setminus g([1, d_i - 1]). \end{aligned}$$

Since f_M is bounded, $b_j < d_i$ implies $g(b_j) < d_i \leq f_M(d_i)$. This together with $f_M(d_1) < f_M(d_2) < \cdots$ show that $g([1, d_i - 1]) \subseteq [1, f_M(d_i) - 1]$, so A has size $f_M(d_i) - d_i$. □

3 *Cohomology classes of rank varieties*

Coskun gives a recursive rule to calculate the cohomology class of a rank variety [4]. Since rank varieties are positroid varieties, Theorem 4.2.6 gives a more direct answer, namely that $[\Sigma_M]$ is $\phi(\tilde{F}_{f_M})$. The goal of this section is to show that $[\Sigma_M]$ is actually represented by an ordinary Stanley symmetric function.

Theorem 4.3.1. *For any rank variety $\Sigma_M \subseteq \mathrm{Gr}(k, n)$, there is a permutation w such that $[\Sigma_M] = \phi(F_w)$.*

We will not prove this theorem by showing that $\tilde{F}_{f_M}$ is an ordinary Stanley symmetric function, since this is not true in general, as the next example shows.

Example 4.3.2. Let $M = \{[1, 1], [3, 3]\}$ with $\Sigma_M \subseteq \mathrm{Gr}(2, 4)$. Then $f_M = 5274$, but $\tilde{F}_{5274} = s_{22} + s_{31} - s_4$ is not equal to any F_w, since ordinary Stanley symmetric functions are Schur-positive. On the other hand, $F_{31524} = s_{22} + s_{31}$, and $\phi(F_{31524}) = \sigma_{22} = \phi(\tilde{F}_{5274})$.

Instead, our strategy will be to replace a rank set M with a new rank set M' in such a way that the truth of Theorem 4.3.1 for $\Sigma_{M'}$ implies it for Σ_M, and so that after enough replacements we end up with a rank variety $\Sigma_N = \Pi_f$ where $\tilde{F}_f$ is obviously an ordinary Stanley symmetric function.

Specifically, if M is a rank set, define a new rank set

$$\kappa(M) = \{[a, b+1] : [a, b] \in M\}.$$

We call the operation κ *stretching.* Say that M is *stretched* if whenever $[a, b], [a', b'] \in M$, we have $b > a'$. For any M, there is an m so that $\kappa^m(M)$ is stretched. Given a rank variety Σ_M in $\mathrm{Gr}(k, n)$, we always interpret $\Sigma_{\kappa M}$ as a subvariety of $\mathrm{Gr}(k, n+1)$. Let τ be the affine permutation $\tau(i) = i + 1$.

Lemma 4.3.3. *Suppose M is stretched and $\Sigma_M \subseteq \mathrm{Gr}(k, n)$. Let b be minimal such that $[a, b] \in M$ for some a, and let $y = f_M(b-1)$. Then $\tau^{-y+1} f_M \tau^{b-2}$ restricted to $[n]$ is an ordinary permutation $w \in \mathfrak{S}_n$, and $[\Sigma_M] = \phi(F_w)$.*

Proof. We first check that f_M maps $[b-1, n+b-2]$ into $[y, y+n-1]$, which shows that $\tau^{-y+1} f_M \tau^{b-2}$ is an ordinary permutation on $[n]$. Suppose $i \in [b-1, n+b-2]$.

- If $n < i \leq n+b-2$, then $y \leq n < i \leq f_M(i)$ since f_M is bounded. On the other side, $f_M(i) < f_M(n+b-1) = y+n$, because the entries $f_M(n+1), \ldots, f_M(n+b-1)$ come in increasing order by the choice of b.

- If $b-1 \leq i \leq n$ and $f_M(i) > n$, then certainly $y \leq n \leq f_M(i)$. On the other side, $f_M(i) - n$ is the left endpoint of an interval in M, while b is the right endpoint, so $f_M(i) - n < b$ since M is stretched. Hence $f_M(i) \leq b-1+n \leq f_M(b-1)+n = y+n$. But we cannot have $f_M(i) = y+n$ since $f_M^{-1}(y+n) = b-1+n > n$ ($b = 1$ is impossible because M is stretched), so $f_M(i) \leq y+n-1$.

- If $b-1 \leq i \leq n$ and $f_M(i) \leq n$, then $f_M(i) \leq y+n-1$ is clear. On the other side, the entries of f_M lying in $[n]$ come in increasing order, so $y = f_M(b-1) \leq f_M(i)$.

Let w be the restriction of $\tau^{-y+1} f_M \tau^{b-2}$ to $[n]$. By definition,

$$\tilde{F}_{\tau^{-y+1} f_M \tau^{b-2}} = F_w.$$

The isomorphisms $\breve{\mathfrak{S}}_n^r \to \breve{\mathfrak{S}}_n^0$ are given by left multiplying by τ^r, so $\tilde{F}_{\tau f} = \tilde{F}_f$ for any f. On the other hand, $\tau^{-1} s_i \tau = s_{i-1}$, and so conjugation by τ gives a bijection between the cyclically decreasing factorizations of f and those of $\tau^{-1} f \tau$ which preserves the lengths of the factors. This means $\tilde{F}_{\tau^{-1} f \tau} = \tilde{F}_f$, so

$$\tilde{F}_{f\tau} = \tilde{F}_{\tau(\tau^{-1} f \tau)} = \tilde{F}_{\tau^{-1} f \tau} = \tilde{F}_f.$$

Now we are done by Theorem 4.2.6, since $\tilde{F}_{f_M} = \tilde{F}_{\tau^{-y+1} f_M \tau^{b-2}} = F_w$.

□

We now give a precise relationship between the classes of Σ_M and $\Sigma_{\kappa M}$. Given a map $\iota : \mathrm{Gr}(k,n) \to \mathrm{Gr}(k,n+1)$, we get a pullback

$$\iota^* : H^p(\mathrm{Gr}(k,n+1),\mathbb{Z}) \to H^p(\mathrm{Gr}(k,n),\mathbb{Z})$$

and a pushforward

$$\iota_* : H^p(\mathrm{Gr}(k,n),\mathbb{Z}) \to H^{2k+p}(\mathrm{Gr}(k,n+1),\mathbb{Z}),$$

the latter obtained from the pushforward on homology via Poincaré duality. These two maps on cohomology are related by the projection formula

$$\iota_*(\beta \iota^*(\alpha)) = \alpha \iota_*(\beta).$$

More information about the cohomology class of a variety and the pullback and pushforward can be found in [9, Appendix B].

Theorem 4.3.4. *Let* $\iota : \mathrm{Gr}(k,n) \hookrightarrow \mathrm{Gr}(k,n+1)$ *be the inclusion induced by an inclusion* $\mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1}$*. Then* $\iota^*[\Sigma_{\kappa M}] = [\Sigma_M]$.

Before proving Theorem 4.3.4, we explicitly describe an algorithm which, given a rank variety $\Sigma_M \subseteq \mathrm{Gr}(k,n)$, produces a permutation w_M such that $[\Sigma_M] = \phi(F_{w_M})$.

Step 1. Choose m such that $\kappa^m M$ is stretched, the minimal (positive) such m being

$$m = \max(0, 1 + \max_{S,T}(\min(S) - \max(T))).$$

Step 2. Find b minimal such that $[a, b] \in \kappa^m M$, and set $y = f_{\kappa^m M}(b - 1)$.

Step 3. Define $w_M \in \mathfrak{S}_n$ by $w_M(i) = f_{\kappa^m M}(b - 2 + i) - y + 1$.

The correctness of this algorithm follows from Lemma 4.3.3 and Theorem 4.3.4; for more details, see the proof of Theorem 4.3.1 below.

Example 4.3.5. Let $M = \{[1, 3], [3, 6], [4, 5]\}$, so $\Sigma_M \subseteq \mathrm{Gr}(3, 6)$ has a dense open subset consisting of rowspans of matrices with the form

$$\begin{bmatrix} * & * & * & 0 & 0 & 0 \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & 0 \end{bmatrix}$$

The minimal m such that $\kappa^m M$ is stretched is $m = 2$, and $\kappa^2 M = \{[1, 5], [3, 8], [4, 7]\}$. We have $f_{\kappa^2 M} = 256798(12)(11)$. The minimal b with $[a, b] \in \kappa^2 M$ is $b = 5$, and then $y = f_{\kappa^2 M}(b - 1) = 7$.

Write out more entries of $f_{\kappa^2 M}$, demarcating the window from 1 to n with vertical bars and denoting negative numbers with horizontal bars:

$$f_{\kappa^2 M} = \cdots \bar{6}\bar{3}\bar{2}\bar{1}1043 \big| 256798(12)(11) \big| (10)(13)(14)(15)(17)(16)(18)(19) \cdots$$

Shift the window so that its leftmost entry is in position $b - 1 = 4$:

$$f_{\kappa^2 M} = \cdots \bar{6}\bar{3}\bar{2}\bar{1}1043256 \big| 798(12)(11)(10)(13)(14) \big| (15)(17)(16)(18)(19) \cdots$$

The entries of the window now fill out exactly the interval $[7, 14]$. That is, $\tau^{-6} f_{\kappa^2 M} \tau^3$ restricted to $[8]$ is an ordinary permutation, namely $w_M = 13265478$.

Theorem 4.3.4 will be straightforward after the next two lemmas. Say $M = \{[a_1, b_1], \dots, [a_k, b_k]\}$ with $b_1 < \cdots < b_k$. Define $h : \mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1}$ by

$$h(e_i) = \begin{cases} e_{b_j+1} - e_{b_{j+1}+1} & \text{if } i = b_j + 1 \text{ for } j < k \\ e_i & \text{otherwise.} \end{cases}$$

Let $\iota : \mathrm{Gr}(k, n) \hookrightarrow \mathrm{Gr}(k, n+1)$ be the inclusion induced by h.

Lemma 4.3.6. *With ι as above, $\Sigma_{\kappa M} \cap \iota(\mathrm{Gr}(k, n)) = \iota(\Sigma_M)$.*

Proof. We must show that

$$\dim(V \cap \langle S \cap T\rangle) \geq \#(S \cap T)(M)$$

for all $S, T \in M$ is equivalent to

$$\dim(h(V) \cap \langle \kappa S \cap \kappa T\rangle) \geq \#(\kappa S \cap \kappa T)(\kappa M)$$

for all $S, T \in M$. In fact, it will turn out that for each i, j, the left (resp. right) side of the first inequality is equal to the left (resp. right) side of the second.

It is not hard to check that $[a, b] \subseteq S \cap T$ (with $a \leq b$) if and only if $[a, b+1] \subseteq \kappa S \cap \kappa T$, and so $\#(S \cap T)(M) = \#(\kappa S \cap \kappa T)(\kappa M)$. Since h is injective,

$$\dim(V \cap \langle S \cap T\rangle) = \dim(h(V) \cap h(\langle S \cap T\rangle)),$$

so we will be done if we show that $\mathrm{im}(h) \cap \langle \kappa S \cap \kappa T\rangle = h(\langle S \cap T\rangle)$.

Suppose $p \in S \cap T$, and write $S = [a_i, b_i]$, $T = [a_j, b_j]$. If $p \neq b_q + 1$ for any $q < k$, then $h(e_p) = e_p \in \langle \kappa S \cap \kappa T\rangle$, so assume $p = b_q + 1$. Then $b_q + 1 \leq \min(b_i, b_j)$. Since $b_1 < b_2 < \cdots$, this means $b_{q+1} \leq \min(b_i, b_j)$. Therefore $b_{q+1} + 1 \in \kappa S \cap \kappa T$, so $h(e_{b_q+1}) = e_{b_q+1} - e_{b_{q+1}+1} \in \langle \kappa S \cap \kappa T\rangle$. Hence

$$h(\langle S \cap T\rangle) \subseteq \langle \kappa S \cap \kappa T\rangle \cap \mathrm{im}\, h. \tag{4.3}$$

Let $\alpha \in (\mathbb{C}^{n+1})^*$ be the linear functional defined by $\alpha(\sum_p c_p e_p) = c_{b_1+1} + \cdots + c_{b_k+1}$. Then $\ker\alpha = \operatorname{im} h$. In particular, $e_{b_i+1} \notin \operatorname{im} h$ since $\alpha(e_{b_i+1}) = 1$. If $\kappa S \cap \kappa T$ is nonempty, then it contains $b_i + 1$ or $b_j + 1$, so $\langle \kappa S \cap \kappa T \rangle \cap \operatorname{im} h$ is properly contained in $\langle \kappa S \cap \kappa T \rangle$. But then

$$\dim(\langle \kappa S \cap \kappa T \rangle \cap \operatorname{im} h) < \dim\langle \kappa S \cap \kappa T \rangle = 1 + \dim h(\langle S \cap T \rangle),$$

so the containment (4.3) must be an equality. □

Write $\mathfrak{p}$ for the space of linear maps $\mathbb{C}^n \to \mathbb{C}^n$ sending $\langle e_1, \dots, e_k \rangle$ into itself. The tangent space to $\operatorname{Gr}(k,n)$ at $\langle e_1, \dots, e_k \rangle$ is $\mathfrak{gl}_n/\mathfrak{p} \simeq \operatorname{Hom}(\langle e_1, \dots, e_k \rangle, \mathbb{C}^n/\langle e_1, \dots, e_k \rangle)$. More generally, the tangent space to $\operatorname{Gr}(k,n)$ at V is $\operatorname{Hom}(V, \mathbb{C}^n/V)$. With this identification, the differential of the quotient map $q : \mathrm{GL}_n(\mathbb{C}) \twoheadrightarrow \operatorname{Gr}(k,n)$ sending A to the span of its first k rows is

$$dq_A(\phi) = \pi_{q(A)} \circ \phi \circ A^{-1}|_{q(A)}, \tag{4.4}$$

where $\phi \in \mathfrak{gl}_n$ and $\pi_V : \mathbb{C}^n \twoheadrightarrow \mathbb{C}^n/V$ is the quotient map. Because of our convention of writing members of $\operatorname{Gr}(k,n)$ as row spans, A^{-1} should be interpreted as the linear transformation sending e_i to the ith *row* of A^{-1}.

Lemma 4.3.7. *The intersection $\Sigma_{\kappa M} \cap \iota(\operatorname{Gr}(k,n))$ is generically transverse.*

Proof. Let α be as in the proof of Lemma 4.3.6. The tangent space to $\iota(\operatorname{Gr}(k,n)) = \operatorname{Gr}(k, \ker\alpha)$ at V is $\operatorname{Hom}(V, (\ker\alpha)/V)$. As $V \subseteq \ker\alpha$, the functional α descends to $\tilde{\alpha} \in (\mathbb{C}^{n+1}/V)^*$, so we can also write the tangent space $T_V \operatorname{Gr}(k, \ker\alpha)$ as $\{\phi \in \operatorname{Hom}(V, \mathbb{C}^{n+1}/V) : \tilde{\alpha}\phi = 0\}$.

Say the first k rows of $A \in \mathrm{GL}_{n+1}(\mathbb{C})$ are $v_1, \dots, v_k$. Define a spanning set $\{\phi_{ij} : i \leq k\}$ for $T_{q(A)} \operatorname{Gr}(k, n+1)$ by

$$\phi_{ij}(v_p) = \begin{cases} e_j + V & \text{if } p = i, \\ 0 & \text{otherwise.} \end{cases}.$$

Define

$$Z = \{A \in \mathrm{GL}_{n+1}(\mathbb{C}) : A_{ij} = 0 \text{ if } i \leq k \text{ and } j \notin [a_i, b_i + 1]\}.$$

Then $q(Z)$ contains a dense open subset U of $\Sigma_{\kappa M}$, where $q : \mathrm{GL}_{n+1}(\mathbb{C}) \twoheadrightarrow \mathrm{Gr}(k, n{+}1)$ sends A to the span of its first k rows.

Say $\psi_{ij} \in T_A Z$ is the map sending e_i to e_j and all other e_p's to 0, for $i \leq k$ and $j \in [a_i, b_i + 1]$. Using equation (4.4), $dq_A(\psi_{ij}) = \phi_{ij}$. Therefore if $V \in U \subseteq \Sigma_{\kappa M}$, then the tangent space $T_V \Sigma_{\kappa M}$ contains ϕ_{ij} whenever $j \in [a_i, b_i + 1]$.

If j is not equal to any $b_p + 1$, then $\tilde{\alpha}\phi_{ij} = 0$, so $\phi_{ij} \in T_V \mathrm{Gr}(k, \ker \alpha)$. If $j = b_p + 1$, write

$$\phi_{i,b_p+1} = (\phi_{i,b_p+1} - \phi_{i,b_i+1}) + \phi_{i,b_i+1}.$$

The first summand is in $T_V \mathrm{Gr}(k, \ker \alpha)$, and the second is in $T_V \Sigma_{\kappa M}$. Thus

$$T_V \mathrm{Gr}(k, \ker \alpha) + T_V \Sigma_{\kappa M} = T_V \mathrm{Gr}(k, n + 1),$$

and so $\Sigma_{\kappa M}$ and $\mathrm{Gr}(k, \ker \alpha)$ intersect transversely on U. □

Proof of Theorem 4.3.4. Since ι maps distinct Schubert varieties in $\mathrm{Gr}(k, n)$ onto distinct Schubert varieties in $\mathrm{Gr}(k, n + 1)$, ι_* is injective. Therefore it suffices to show that

$$\iota_* \iota^* [\Sigma_{\kappa M}] = \iota_* [\Sigma_M].$$

The right side here is $[\iota(\Sigma_M)]$ since ι is an embedding. By the projection formula, the left side is $[\Sigma_{\kappa M}][\iota(\mathrm{Gr}(k, n))]$. Lemmas 4.3.6 and 4.3.7 below show that for a suitable choice of ι, $\Sigma_{\kappa M}$ and $\iota(\mathrm{Gr}(k, n))$ intersect generically transversely in $\iota(\Sigma_M)$, so we are done. □

Now we can show that the class of any rank variety is represented by an ordinary Stanley symmetric function.

Proof of Theorem 4.3.1. Having fixed k, write ϕ_n for the map $\phi : \Lambda \to H^*(\mathrm{Gr}(k, n), \mathbb{Z})$, and likewise ι_n for the inclusion $\mathrm{Gr}(k, n) \hookrightarrow \mathrm{Gr}(k, n{+}1)$. The pullback $\iota_n^* : H^*(\mathrm{Gr}(k, n+$

$1), \mathbb{Z}) \to H^*(\mathrm{Gr}(k,n),\mathbb{Z})$ sends σ_λ to σ_λ if $\lambda_1 \leq n$, and to 0 otherwise. Thus the diagram

$$\begin{tikzcd}
H^*(\mathrm{Gr}(k,n),\mathbb{Z}) & H^*(\mathrm{Gr}(k,n+1),\mathbb{Z}) \arrow[l, "\iota_n^*"'] \\
\Lambda \arrow[u, "\phi_n"'] \arrow[r, equal] & \Lambda \arrow[u, "\phi_{n+1}"']
\end{tikzcd}$$

commutes.

Say $\Sigma_M \subseteq \mathrm{Gr}(k,n)$, and take m large enough that $\kappa^m M$ is stretched. Then Lemma 4.3.3 shows that $[\Sigma_{\kappa^m M}] = \phi_{n+m}(F_w)$. By Theorem 4.3.4,

$$\begin{aligned}
[\Sigma_M] &= \iota_n^* \iota_{n+1}^* \cdots \iota_{n+m-1}^* [\Sigma_{\kappa^m M}] \\
&= \iota_n^* \iota_{n+1}^* \cdots \iota_{n+m-1}^* \phi_{n+m}(F_w) \\
&= \phi_n(F_w).
\end{aligned}$$

□

Chapter 5

DIAGRAM VARIETIES OF PERMUTATION DIAGRAMS

In this chapter we consider the diagram varieties $X_{D(w)^\vee}$ and $X_{D(w)}$ for w a permutation. Liu showed that $\deg X_{D(w)^\vee} = \#\operatorname{Red}(w) = \dim S^{D(w)}$, establishing Conjecture 1.3.6 for $D(w)^\vee$ [19]. We will establish the stronger Conjecture 1.3.5 for these diagram varieties in the special case that w is multiplicity-free.

The varieties $X_{D(w)}$ are not so well-behaved, and we will show in this chapter that Conjecture 1.3.5 can fail for them. We will however give a bound on their cohomology classes, showing that $\phi(F_w) - [X_{D(w)}]$ is a nonnegative sum of Schubert classes.

1 *James–Peel moves as degenerations*

Say X is a subvariety of $\operatorname{Gr}(k,n)$. Let $\phi_{t,i\to j}$ be the linear transformation sending e_i to $te_i + (1-t)e_j$. For $t \neq 0$, the varieties $\phi_{t,i\to j}X$ are all isomorphic, so they form a flat family [6, Proposition III-56]. The flat limit $\lim_{t\to 0}\phi_{t,i\to j}X$ then exists as a scheme [10, Proposition 9.8]. The key fact for us is that X and $\lim_{t\to 0}\phi_{t,i\to j}X$ have the same Chow ring class. The process of passing from X to $\lim_{t\to 0}\phi_{t,i\to j}X$ is called a *degeneration*. In the case of the Grassmannian, the canonical map from the Chow ring to the cohomology ring is an isomorphism [7, Chapter 3], so this implies that X and $\lim_{t\to 0}\phi_{t,i\to j}X$ have the same cohomology class.

The James–Peel moves on diagrams from Chapter 2 can be applied just as well to matrices. If A is a matrix, define

$$(C_{i\to j}A)_{pq} = \begin{cases} A_{pi} & \text{if } q = j \text{ and } A_{pj} = 0, \\ 0 & \text{if } q = i \text{ and } A_{pj} = 0, \\ A_{pq} & \text{otherwise.} \end{cases}$$

For example,

$$C_{1\to 2}\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & -7 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \\ 3 & -7 \\ 0 & 0 \end{bmatrix}.$$

One can likewise define row moves $R_{i\to j}$. On the level of matrices, the degenerations $\lim_{t\to 0}\phi_{t,i\to j}X$ correspond to James–Peel column moves, as observed by Liu [19].

Lemma 5.1.1. *Say F is any subset of $[k]\times[n-k]$ and U is the set of k-planes* $\operatorname{rowspan}(A)$ *where $A_{ij} = 0$ whenever $(i,j)\in F$. If* $\operatorname{rowspan} A \in U$ *and $C_{i\to j}A$ has rank k, then* $\operatorname{rowspan} C_{i\to j}A \in \lim_{t\to 0}\phi_{t,i\to j}\overline{U}$.

Proof. Given i, j, define a matrix $A(t)$ by

$$A(t)_{pq} = \begin{cases} t^{-1}a_{pq} & \text{if } q = i \text{ and } a_{pj} \neq 0, \\ a_{pq} - (t^{-1}-1)a_{pi} & \text{if } q = j \text{ and } a_{pj} \neq 0, \\ a_{pq} & \text{otherwise.} \end{cases}$$

Then $\operatorname{rowspan} A(t) \in U$ for $t \neq 0$, and $\lim_{t\to 0}\phi_{t,i\to j}\operatorname{rowspan} A(t) = \operatorname{rowspan} C_{i\to j}A$. □

Lemma 5.1.2 (Proposition 5.3.3, [19])**.** *If D is any diagram, then $[X_D] - [X_{C_{i\to j}D}]$ and $[X_D] - [X_{R_{i\to j}D}]$ are nonnegative sums of Schubert classes.*

Proof. By Lemma 5.1.1, $X_{C_{i\to j}D} \subseteq \lim_{t\to 0}\phi_{t,i\to j}X_D$. Since both sides have the same dimension, $X_{C_{i\to j}D}$ is an irreducible component of $\lim_{t\to 0}\phi_{t,i\to j}X_D$, which has the same cohomology class as X_D. This shows that $[X_D]-[X_{C_{i\to j}D}]$ is a nonnegative sum of Schubert classes.

The corresponding statement for $[X_D]-[X_{R_{i\to j}D}]$ follows by taking transposes. Specifically, the orthogonal complement isomorphism $\mathrm{Gr}(k,n)\to\mathrm{Gr}(n-k,k)$ sends $\mathrm{rowspan}[A\mid I_k]\in X_D$ to $\mathrm{rowspan}[I_{n-k}\mid -A^T]$, so the induced map on cohomology sends $[X_D]$ to $[X_{D^T}]$. Equivalently, the coefficient of σ_λ in $[X_D]$ is the same as the coefficient of σ_{λ^t} in $[X_{D^t}]$. □

What is lacking is the full analogue of Theorem 2.0.2, namely that

$$[X_D]-[X_{C_{j_1\to j_2}D}]-[X_{R_{i_1\to i_2}D}]$$

is a nonnegative sum of Schubert classes when (i_1,j_1) and (i_2,j_2) satisfy the conditions of Theorem 2.0.2. Liu proved this analogue with some further restrictions on (i_1,j_1) and (i_2,j_2) [19, Proposition 5.3.4], but it cannot be true in general, since otherwise our counterexample to Conjecture 1.3.5 in Section 2 below would not be a counterexample. However, none of this is an issue when $[X_D]$ is multiplicity free.

Theorem 5.1.3. *If w is multiplicity-free, then Conjecture 1.3.5 holds for $D(w)^\vee$.*

Proof. Recall the complete James–Peel tree $\mathrm{JP}(w)$ for $D(w)$ from Theorem 3.1.5. Each of its leaves λ is obtained from $D(w)$ by James–Peel moves, or equivalently, $\lambda^\vee$ is obtained from $D(w)^\vee$ by James–Peel moves. By Lemma 5.1.2, $[X_{D(w)^\vee}]$ has a term σ_λ for each such leaf λ, so $[X_{D(w)^\vee}]-\phi(F_w)$ is a nonnegative sum of Schubert classes. Liu showed that $[X_{D(w)^\vee}]$ and $\phi(F_w)$ have the same degree [19, Proposition 5.5.5], so they must be equal. □

2 *A counterexample to Liu's conjecture*

Suppose $D=\{(1,1),(2,2),(3,3),(4,4)\}$, with $k=4$ andr $n=8$. This is a skew diagram $4321/321$. The Specht module S^D is simply the regular representation of S_4, with

$$S^D \simeq S^{1111} \oplus 3S^{211} \oplus 2S^{22} \oplus 3S^{31} \oplus S^4.$$

Magyar has shown that for any diagram E contained in a fixed rectangle, the multiplicity of S^λ in S^E is the multiplicity of $S^{\lambda^\vee}$ in $S^{E^\vee}$ [20]. Hence,

$$S^{D^\vee} \simeq S^{3333} \oplus 3S^{4332} \oplus 2S^{4422} \oplus 3S^{4431} \oplus S^{444},$$

so $\dim S^{D^\vee} = f^{3333} + 3f^{4332} + 2f^{4422} + 3f^{4431} + f^{444} = 24024$.

On the other hand, an explicit calculation in Macaulay2 shows $\deg X_D = 21384$. Therefore Conjectures 1.3.6 and 1.3.5 both fail for D.

The discrepancy in degrees is $24024 - 21384 = 2640 = f^{4422}$, which hints at how to see this discrepancy more explicitly. Given a k-subset I of $[n]$, write p_I for the corresponding Plücker coordinate on $\mathrm{Gr}(k, n)$, so $p_I(A)$ is the minor of A in columns I. Let Y be the scheme determined by the vanishing of the Plücker coordinates $p_{1678}, p_{2578}, p_{3568}, p_{4567}$. These are exactly the Plücker coordinates which vanish on X_D. One can check by computer that $\operatorname{codim} Y = 4$, and so Y is a complete intersection. This implies that

$$[Y] = \sigma_1^4 = \sigma_{1111} + 3\sigma_{211} + 2\sigma_{22} + 3\sigma_{31} + \sigma_4;$$

see [7, Section 5.2.1].

Since the four Plücker coordinates cutting out Y vanish on X_D°, the diagram variety X_D is contained in Y. However, Y has another component, namely the Schubert variety which is the closure of

$$\begin{bmatrix} * & * & 1 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & 1 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & * & * & 1 & 0 \\ * & * & 0 & 0 & * & * & 0 & 1 \end{bmatrix}.$$

This Schubert variety has degree $\dim S^{(22)^\vee} = f^{4422} = 2640$, which is $\deg Y - \deg X_D$. Therefore

$$[X_D] = [Y] - \sigma_{22} = \sigma_{1111} + 3\sigma_{211} + \sigma_{22} + 3\sigma_{31} + \sigma_4.$$

Although $D = \{(1,1),(2,2),(3,3),(4,4)\}$ is not itself a permutation diagram, this counterexample leads directly to one of the form $X_{D(w)}$. Take $w = 21436587$. Then $D(w) = \{(1,1),(3,3),(5,5),(7,7)\}$ can be obtained from D by permuting rows and columns, and viewing D in a larger rectangle. Neither of these operations on diagrams affects s_D or $[X_D]$, identifying the latter with its pullback along an embedding of $\mathrm{Gr}(k,n)$ into $\mathrm{Gr}(k,n+1)$ or $\mathrm{Gr}(k+1,n+1)$. Thus Conjecture 1.3.5 can fail for permutation diagrams.

More counterexamples to Conjecture 1.3.5 can be easily manufactured from this one. One can show that $[X_{D_1 \cdot D_2}] = [X_{D_1}][X_{D_2}]$ and similarly that $s_{D_1 \cdot D_2} = s_{D_1} s_{D_2}$. Therefore if Conjecture 1.3.5 holds for D_1 but not D_2, then it will fail for $D_1 \cdot D_2$.

Remark 5.2.1. One might naturally wonder about the diagram

$$D = \{(1,1),(2,2),(3,3),(4,4),(5,5)\}.$$

However, trying to repeat the analysis above runs into an immediate problem (thanks to Ricky Liu for pointing this out). Namely, the analogue of Y, which is the scheme Z cut out by

$$p_{1789(10)}, p_{2689(10)}, p_{3679(10)}, p_{4678(10)}, p_{56789},$$

no longer has the same codimension (5) as X_D. Indeed, Z contains the codimension 4 Schubert cell

$$\begin{bmatrix} * & * & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & * & * & 1 & 0 & 0 \\ * & * & * & 0 & 0 & * & * & 0 & 1 & 0 \\ * & * & * & 0 & 0 & * & * & 0 & 0 & 1 \end{bmatrix}.$$

3 A bound on $[X_{D(w)}]$

Given a permutation $w \in S_n$, define a rank set $M(w) = \{[w(i), i+n] : 1 \leq i \leq n\}$, so $\Sigma_{M(w)} \subseteq \mathrm{Gr}(n, 2n)$. Then

$$f_{M(w)} = (w \times 12 \cdots n)\tau^{-n},$$

where τ is the affine permutation $\tau(i) = i + 1$. Here, for $w \in S_n$ and $v \in S_m$, $w \times v$ is the permutation in S_{n+m} sending i to $w(i)$ if $i \leq n$ and to $v(i-n)+n$ otherwise. Thus $\tilde{F}_{f_{M(w)}} = \tilde{F}_{w \times 12 \cdots n} = F_{w \times 12 \cdots n} = F_w$. This shows that $[\Sigma_{M(w)}] = \phi(F_w)$.

Define

$$C_w = C_{n+1 \to w(1)} \circ C_{n+2 \to w(2)} \circ \cdots \circ C_{2n \to w(n)}$$

and

$$\phi_{t,w} = \phi_{t,n+1 \to w(1)} \circ \phi_{t,n+2 \to w(2)} \circ \cdots \circ \phi_{t,2n \to w(n)}.$$

We will show that $\lim_{t \to 0} \phi_{t,w} \Sigma_{M(w)}$ contains the diagram variety $X_{D(w)}$ as an irreducible component. First we give an explicit example of this degeneration.

Example 5.3.1. Take $n = 5$ and $w = 24153$, so $M(w) = \{[2,6],[4,7],[1,8],[5,9],[3,10]\}$. Then $\Sigma_{M(w)}$ contains

$$\left\{ \text{rowspan} \begin{pmatrix} 0 & * & * & * & * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & * & * & * & 0 \\ 0 & 0 & * & * & * & * & * & * & * & * \end{pmatrix} \right\}.$$

Here $C_w = C_{6 \to 2} C_{7 \to 4} C_{8 \to 1} C_{9 \to 5} C_{10 \to 3}$, and, for example,

$$C_{7 \to 4} \Sigma_{M(w)} \supseteq \left\{ \text{rowspan} \begin{pmatrix} 0 & * & * & * & * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & 0 & 0 \\ 0 & 0 & 0 & * & * & * & 0 & * & * & 0 \\ 0 & 0 & * & * & * & * & * & * & * & * \end{pmatrix} \right\}.$$

In total,

$$C_w\Sigma_{M(w)} \supseteq \left\{ \text{rowspan} \begin{pmatrix} 0 & \ast & \ast & \ast & \ast & \ast & 0 & 0 & 0 & 0 \\ 0 & \ast & 0 & \ast & \ast & 0 & \ast & 0 & 0 & 0 \\ \ast & \ast & \ast & \ast & \ast & \underline{\ast} & \underline{\ast} & \ast & 0 & 0 \\ \ast & \ast & 0 & \ast & \ast & 0 & 0 & 0 & \ast & 0 \\ \ast & \ast & \ast & \ast & \ast & 0 & \underline{\ast} & 0 & \underline{\ast} & \ast \end{pmatrix} \right\}.$$

Notice that the 0's forced in columns 1 through 5 of these matrices form the diagram $D(24153)$. Columns 6 through 10 do not quite form an identity matrix, but at least on the open subset where this submatrix is invertible, we can clear out the underlined entries below the diagonal. Crucially, whenever $(i, j+n)$ needs to be cleared, row j of $D(24153)$ contains row i, which means this clearing out does not affect the pattern of 0's and $\ast$'s in columns 1 through 5. Hence,

$$C_w\Sigma_{M(w)} \supseteq \left\{ \text{rowspan} \begin{pmatrix} 0 & \ast & \ast & \ast & \ast & 1 & 0 & 0 & 0 & 0 \\ 0 & \ast & 0 & \ast & \ast & 0 & 1 & 0 & 0 & 0 \\ \ast & \ast & \ast & \ast & \ast & 0 & 0 & 1 & 0 & 0 \\ \ast & \ast & 0 & \ast & \ast & 0 & 0 & 0 & 1 & 0 \\ \ast & \ast & \ast & \ast & \ast & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right\} = X^{\circ}_{D(24153)}.$$

By Lemma 5.1.1, this shows that $X_{D(24153)} \subseteq \lim_{t\to 0} \phi_{t,M}\Sigma_{M(w)}$.

Theorem 5.3.2. *The flat limit* $\lim_{t\to 0} \phi_{t,M}\Sigma_{M(w)}$ *contains* $X_{D(w)}$ *as an irreducible component.*

Proof. Let $DM(w)$ be the diagram $\{(i,j) : 1 \leq i \leq n \text{ and } w(i) \leq j \leq i+n\}$. As in Example 5.3.1, it is enough to show that $C_wDM(w)$ has the following two properties:

(a) If $j \leq n$, then $(i,j) \in C_wDM(w)$ if and only if $(i,j) \notin D(w)$.

(b) If $j > n$, then $(i,j) \in C_wDM(w)$ implies row $j-n$ of $D(w)$ contains row i of $D(w)$.

Say $j \leq n$. Then $(i,j) \in C_w DM(w)$ if and only if $j \geq w(i)$ or $w^{-1}(j) + n \leq i + n$, if and only if $(i,j) \notin D(w)$.

Say $j > n$. Then $(i,j) \in C_w DM(w)$ if and only if $j \leq i + n$ and $w(i) \leq w(j-n)$. That is, w has an inversion in positions $j - n < i$ (or they are the same position). It is easy to check that this implies row $j - n$ of $D(w)$ contains row i. $\square$

Since $[\lim_{t \to 0} \phi_{t,M} \Sigma_{M(w)}] = [\Sigma_{M(w)}]$, an immediate corollary is an upper bound on $[X_{D(w)}]$.

Theorem 5.3.3. *$\phi(F_w) - X_{D(w)}$ is a nonnegative combination of Schubert classes.*

However, we emphasize again that this difference of classes can be nonzero, as in the case $w = 21436587$ discussed in Section 2.

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