

The heat equation in time dependent domains with Neumann boundary conditions

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Abstract

We study the heat equation in domains in \mathbf{R}^n with insulated *fast* moving boundaries. We prove existence and uniqueness theorems in the case that the boundary moves at speeds that are square integrable.

1 Introduction

In this paper, and its two companion papers,[?] and [?], we study the heat equation in domains in \mathbf{R}^n with insulated *fast* moving boundaries. In such a domain, the insulated moving boundary will tend to collect heat energy and the temperature will rise, while the medium will cause that energy to diffuse away from the boundary and thus lower the temperature. If the boundary moves fast enough ($\sim \frac{1}{\sqrt{T-t}}$), singularities can develop. These *heat atoms* are described, via the study of a reflecting Brownian motion, in [?].

We show below that no such singularities develop if the boundary moves at speeds that are square integrable. Although the description of this problem is fairly direct in terms of reflecting Brownian motion, even the formulation of a partial differential equation and boundary condition requires a bit of thought. We show how to do this, find an appropriate weak formulation, and then prove existence and uniqueness in the case of a C^2 boundary

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moving with an L^2 velocity.

The analytic literature on the heat equation with moving and free boundaries is enormous, but, to the best of our knowledge, there doesn't seem to be any literature which treats, or can be easily adapted to treat these fast moving insulated boundaries.

In the formulation and solution of a problem with a general moving boundary, a coordinate invariant approach simplifies matters, and eliminates the need to frequently write cumbersome transformation formulas. We formulate our problem in a purely geometric fashion, using coordinates occasionally to accommodate the reader who is not comfortable with this notation.

As far as we can tell, the heat equation in this sort of time-dependent domain is not a mathematical description of any physical heat dissipation phenomenon. A physical boundary that would collect heat energy would also collect mass, changing the diffusivity and, more importantly, moving heat by convection.

2 The Initial Boundary Value Problem

The flow of heat in a medium with density (actually, density times specific heat) ρ and diffusivity κ is modeled by

$$(1) \quad \frac{\partial}{\partial t} \int_{\omega} \rho u = \int_{\partial\omega} \kappa du$$

The temperature is denoted by u and the left hand side of equation (1) is the heat energy in the region ω .

The rate of change in heat energy for a static (in time) region is equal to the flux of heat across the boundary, which we take to be the (anisotropic) diffusivity κ times the differential, du , of the temperature.

If ω is not static, (1) becomes

$$(2) \quad \frac{\partial}{\partial t} \int_{\omega} \rho u = \int_{\partial\omega} (\kappa du + v \lrcorner \rho u)$$

The second term on the right represents the flow of energy due to the motion of ω . The vector field v denotes the velocity of the boundary (i.e.

each point on the boundary of ω evolves according to the differential equation $x' = v(x, t)$. The expression $v \lrcorner \rho$ denotes the interior product \lrcorner of the vector field v and the differential n-form ρ . If we introduce any riemannian metric and let dV , dS , and ν denote the volume element, the element of surface area on $\partial\omega$ and the normal vector, respectively, then

$$(3) \quad \rho = \rho_D dV = \rho_D d\nu \wedge dS$$

$$(4) \quad v \lrcorner \rho \Big|_{\partial\omega} = \nu \cdot v dS$$

where ρ_D is the density of the form ρ . We wish to describe the flow of heat in a moving domain $\Omega_t \subset \mathbf{R}^n$. Our domain Ω_t has an insulated boundary, thus no heat energy will flow through any part of $\partial\Omega_t$. Let $\omega \subset \mathbf{R}^n$ be a (possibly moving) domain. Then

$$(5) \quad \frac{\partial}{\partial t} \int_{\omega \cap \Omega_t} \rho u = \int_{\partial(\omega \cap \Omega_t) \setminus \partial\Omega_t} (\kappa du + v \lrcorner \rho u)$$

Carrying out the differentiation on the left hand side gives:

$$\begin{aligned} & \int_{\omega \cap \Omega_t} \frac{\partial}{\partial t} (\rho u) + \int_{\partial(\omega \cap \Omega_t)} v \lrcorner \rho u \\ &= \int_{\partial(\omega \cap \Omega_t)} (\kappa du + v \lrcorner \rho u) - \int_{\partial(\omega \cap \Omega_t) \cap \partial\Omega_t} (\kappa du + v \lrcorner \rho u) \\ & \int_{\omega \cap \Omega_t} \frac{\partial}{\partial t} (\rho u) = \int_{\partial(\omega \cap \Omega_t)} \kappa du - \int_{\partial(\omega \cap \Omega_t) \cap \partial\Omega_t} (\kappa du + v \lrcorner \rho u) \\ & \int_{\omega \cap \Omega_t} \frac{\partial}{\partial t} (\rho u) = \int_{\omega \cap \Omega_t} d\kappa du - \int_{\partial(\omega \cap \Omega_t) \cap \partial\Omega_t} (\kappa du + v \lrcorner \rho u) \end{aligned}$$

Since ω is arbitrary, we may use the standard calculus of variations argument (first choose $\omega \subset\subset \Omega_t$ and then choose ω to contain a neighborhood of a subset of Ω_t) to derive the partial differential equation and the boundary condition. Adding the initial condition then give the full initial boundary value problem below:

$$(6) \quad \begin{aligned} & \frac{\partial}{\partial t} (\rho u) = d\kappa du \\ & (\kappa du + v \lrcorner \rho u) \Big|_{\partial\Omega_t} = 0 \\ & u \Big|_{t=0} = u_0(x) \end{aligned}$$

When we make our existence proof, it will be convenient to work in a fixed (in time) domain. For this reason, we describe Ω_t as the diffeomorphic image of a fixed domain Ω_0 . Such a diffeomorphism may be generated by the flow of any time dependent vector field, $v(x, t)$, which is an extension to \mathbf{R}^n of the vector which describes the velocity of $\partial\Omega_t$, i.e.

$$(7) \quad \frac{\partial\Psi}{\partial t} = v(t, \Psi)$$

$$(8) \quad \Psi(0, \cdot) = I$$

We say that Ω_t is the push-forward of Ω_0 by $\Psi(t, \cdot)$, which we write as

$$(9) \quad \Omega_t = \Psi_*(\Omega_0)$$

Instead of (??), we shall derive equations for $w(t, x) = u(t, \Psi(t, x))$. To accomplish this, we return to (??) and choose

$$(10) \quad \omega = \Psi_*\omega_0$$

where $\omega_0 \subset \mathbf{R}^n$ does not depend on t . Now (??) becomes

$$(11) \quad \begin{aligned} \frac{\partial}{\partial t} \int_{\Psi_*(\omega_0 \cap \Omega_0)} \rho u &= \int_{\Psi_*(\partial(\omega_0 \cap \Omega_0) \setminus \partial\Omega_0)} (\kappa du + v \lrcorner \rho u) \\ \frac{\partial}{\partial t} \int_{\Psi_*(\omega_0 \cap \Omega_0)} \Psi^{-1*}(\rho_0 w) &= \int_{\Psi_*(\partial(\omega_0 \cap \Omega_0) \setminus \partial\Omega_0)} \Psi^{-1*}(\kappa_0 dw + v_0 \lrcorner \rho_0 w) \\ \frac{\partial}{\partial t} \int_{\omega_0 \cap \Omega_0} \rho_0 w &= \int_{\partial(\omega_0 \cap \Omega_0) \setminus \partial\Omega_0} (\kappa_0 dw + v_0 \lrcorner \rho_0 w) \end{aligned}$$

where w , ρ_0 , κ_0 , and v_0 are the pull-backs of w , ρ , κ , and v by $\Psi(t, \cdot)$. In the formulas below – $D\Psi$ is the jacobian matrix of Ψ , viewed as a mapping from \mathbf{R}^n to itself for each fixed t . The explicit formula for each pullback is listed below:

$$(12) \quad \begin{aligned} w &= \Psi^*(u) = u(t, \Psi(t, x)) \\ \rho_0 &= \Psi^*(\rho) = \det(D\Psi) \rho_D(t, \Psi) dx_1 \dots dx_n \\ \kappa_0 &= \Psi^*(\kappa) = \frac{D\Psi^T \kappa D\Psi}{\det(D\Psi)} \\ v_0 &= \Psi_*^{-1} v = (D\Psi)^{-1} v(t, \Psi) \end{aligned}$$

As the region $\omega_0 \cap \Omega_0$ is not changing with time, we may move the time differentiation under the integral in (??), so that

$$\begin{aligned}
\int_{\omega_0 \cap \Omega_0} \frac{\partial}{\partial t}(\rho_0 w) &= \int_{\partial(\omega_0 \cap \Omega_0) \setminus \partial \Omega_0} (\kappa_0 dw + v_0 \lrcorner \rho_0 w) \\
&= \int_{\partial(\omega_0 \cap \Omega_0)} (\kappa_0 dw + v_0 \lrcorner \rho_0 w) - \int_{\partial(\omega_0 \cap \Omega_0) \cap \partial \Omega_0} (\kappa_0 dw + v_0 \lrcorner \rho_0 w) \\
&= \int_{\omega_0 \cap \Omega_0} d(\kappa_0 dw + v_0 \lrcorner \rho_0 w) - \int_{\partial(\omega_0 \cap \Omega_0) \cap \partial \Omega_0} (\kappa_0 dw + v_0 \lrcorner \rho_0 w)
\end{aligned}$$

which gives us the initial boundary value problem:

$$\begin{aligned}
(13) \quad \frac{\partial}{\partial t}(\rho_0 w) &= d(\kappa_0 dw + v_0 \lrcorner \rho_0 w) \\
(\kappa_0 dw + v_0 \lrcorner \rho_0 w) \Big|_{\partial \Omega_0} &= 0 \\
w \Big|_{t=0} &= w_0
\end{aligned}$$

In euclidean coordinates, (??) reads

$$\begin{aligned}
(14) \quad \frac{\partial}{\partial t}(\rho_0 w) &= \frac{\partial}{\partial x_i} \left(\kappa_0^{ij} \frac{\partial w}{\partial x_j} + v_0^i \rho_0 w \right) \\
\nu_i (\kappa_0^{ij} \frac{\partial w}{\partial x_j} + v_0^i \rho_0 w) \Big|_{\partial \Omega_0} &= 0 \\
w \Big|_{t=0} &= w_0
\end{aligned}$$

3 A Weak Formulation in Ω_0

Our existence proof will require that we solve an inhomogeneous version of (??),

$$\begin{aligned}
(15) \quad \frac{\partial}{\partial t}(\rho_0 w) - d(\kappa_0 dw + v_0 \lrcorner \rho_0 w) &= F_0 \\
\kappa_0 dw + v_0 \lrcorner \rho_0 w \Big|_{\partial \Omega} &= f_0 \\
w \Big|_{t=0} &= w_0
\end{aligned}$$

Let ϕ be any function, $\rho_0(t, \cdot)$ any n - form, and $\eta(t, \cdot)$ any $(n - 1)$ - form for each t . We define

$$\begin{aligned}\langle \phi, \rho_0 \rangle &:= \int_0^\tau \left(\int_\Omega \phi \rho_0 \right) dt \\ \langle \phi, \rho_0 \rangle_{t=0} &:= \int_\Omega \phi(0, \cdot) \rho_0(0, \cdot) \\ \langle \phi, \alpha \rangle_{\partial\Omega} &:= \left(\int_0^\tau \int_{\partial\Omega} \phi(0, \cdot) \eta(0, \cdot) \right) dt\end{aligned}$$

Proposition 1 *A smooth function w satisfies (??), if and only if , for all smooth ϕ*

$$(16) \quad \langle \phi, \frac{\partial}{\partial t}(\rho_0 w) \rangle + \langle d\phi, v_0 \lrcorner \rho_0 w \rangle - \langle \phi, w \rangle_{t=0} \\ = \langle \phi, F_0 \rangle - \langle \phi, f_0 \rangle_{\partial\Omega} + \langle \phi, w_0 \rangle_{t=0}$$

Proof

If w satisfies (??), the initial condition guarantees that the the last terms on the right and left hand sides of (??) are equal. If we multiply the differential equation by ϕ , integrate both sides and integrate by parts in x , we obtain the equality of the rest.

If we begin with (??), then we proceed in the calculus of variations fashion. First choose only ϕ which vanish at $t = 0$ and on $\partial\Omega_t$ and integrate by parts to conclude that the differential equation holds. Next choose ϕ which vanish at $t = 0$ but not on $\partial\Omega_t$ to obtain the boundary conditions. Finally choose ϕ which do not vanish on $t = 0$ to see that the initial condition holds. □

We shall define some Hilbert and Banach spaces below. We recall first the Sobolev space

$$(17) \quad H^1(\Omega_0) = \left\{ w \mid w \in L^2(\Omega_0), dw \in L^2(\Omega_0) \right\} \\ \|w\|_{H^1}^2 = \|w\|_{L^2}^2 + \|dw\|_{L^2}^2$$

We recall that multiplication by functions with bounded derivatives is a bounded operator on H^1 and its dual, H^{1*} . The proof is just the Liebnitz rule and duality.

Lemma 2 For any $\rho_0 \in W^{1,\infty}$

$$\begin{aligned}\|\rho_0 w\|_{H^1} &\leq \|\rho_0\|_{W^{1,\infty}} \|w\|_{H^1} \\ \|\rho_0 w\|_{H^{1*}} &\leq \|\rho_0\|_{W^{1,\infty}} \|w\|_{H^{1*}}\end{aligned}$$

In the rest of this section, we will consider functions of t and x as functions of time which take values in Banach spaces which are functions of x . For example, we think of a function in L^2 of t and x as belonging to $L^2(L^2)$, with the norm $\|f\|_{L^2(L^2)}^2 = \int \|f\|_{L^2(dx)}^2 dt$. Specifically, we will work with the following two spaces and their duals.

$$\begin{aligned}\mathcal{B} &= \{w \mid w \in C(L^2(\Omega)), dw \in L^2(L^2(\Omega))\} \\ \|u\|_{\mathcal{B}}^2 &= \|u\|_{L^\infty(L^2)}^2 + \|du\|_{L^2(L^2)}^2\end{aligned}$$

$$(18) \quad \mathcal{H} = \{w \mid w \in L^2(H^1(\Omega)), \frac{\partial w}{\partial t} \in L^2(H^{1*}(\Omega))\}$$

$$(19) \quad \|u\|_{\mathcal{H}}^2 = \|u\|_{L^2(H^1)}^2 + \left\| \frac{\partial u}{\partial t} \right\|_{L^2(H^{1*})}^2$$

We shall work on the time interval $[0, \tau]$; the next lemma gives some simple relationships between our norms.

Lemma 3

$$(20) \quad \|w\|_{L^\infty(L^2)}^2 \leq \frac{2}{\tau} \|w\|_{L^2(L^2)}^2 + \|w\|_{L^2(H^1)} \left\| \frac{\partial w}{\partial t} \right\|_{L^2(H^{1*})} \leq (1 + 2/\tau) \|w\|_{\mathcal{H}}^2$$

$$(21) \quad \|w\|_{\mathcal{B}} \leq (1 + 2/\tau) \|w\|_{\mathcal{H}}$$

$$(22) \quad \|w\|_{L^2(H^1)} \leq (1 + \sqrt{\tau}) \|w\|_{\mathcal{B}}$$

Proof

$$\begin{aligned}w^2(t_*, x) &= \frac{1}{t_*} \int_0^{t_*} \frac{\partial}{\partial t} (tw^2) \\ &= \frac{1}{t_*} \int_0^{t_*} w^2 + 2 \int_0^{t_*} \frac{t}{t_*} w \frac{\partial w}{\partial t} \\ \int_{\Omega} w^2(t, x) dx &\leq \frac{1}{t_*} \|w\|_{L^2(L^2)}^2 + 2 \|w\|_{L^2(H^1)} \left\| \frac{\partial w}{\partial t} \right\|_{L^2(H^{1*})}\end{aligned}$$

We may assume that $t_* > \tau/2$, otherwise we repeat the computation above, integrating instead from t_* and τ and replacing t by $t - \tau$. This

establishes (??) and (??) is its immediate consequence. Finally, (??) follows from the fact that the L^2 norm is bounded by the L^∞ norm times the square root of the length of the interval. \square

The following simple lemma and its corollary allow us to formulate the weak version of the boundary value problem as an equation among elements of \mathcal{B}^* .

Lemma 4 *Let $\chi(s)$ be a smooth cutoff, equal to 1 at $s = 0$ and 0 in a neighborhood of $s = 1$, then*

$$(23) \quad \langle u, v \rangle_{t=0} = \left\langle \frac{\partial u}{\partial t}, \chi(t)v \right\rangle + \left\langle u, \frac{\partial}{\partial t} \chi(t)v \right\rangle$$

Let ν be a normal coordinate in a neighborhood of $\partial\Omega$ and ϵ small enough that $\chi(\nu/\epsilon)$ is supported in that neighborhood, then

$$(24) \quad \langle u, v \rangle_{\partial\Omega} = \left\langle \frac{\partial u}{\partial \nu}, \chi(\nu/\epsilon)v \right\rangle + \left\langle u, \frac{\partial}{\partial \nu} \chi(\nu/\epsilon)v \right\rangle$$

Proof

This is just the fundamental theorem of calculus. Namely,

$$uv \Big|_{t=0} = \int_0^\tau \frac{\partial}{\partial t} (u\chi v) dt$$

The second assertion is similar with the normal coordinate ν replacing the time coordinate. \square

Corollary 5 *Let F, f , and w_0 be smooth functions, then each term in*

$$\mathcal{F} := \langle \cdot, F \rangle - \langle \cdot, f \rangle_{\partial\Omega} + \langle \cdot, w_0 \rangle_{t=0}$$

*is an element of \mathcal{B}^**

Proof

That the first term is an element of \mathcal{B}^* is obvious. To see that the second is, note that $\chi(\nu/\epsilon)f$ is well defined on and is zero outside a tubular neighborhood of $\partial\Omega$. In particular, for $w \in \mathcal{B}$

$$\begin{aligned} |\langle w, f \rangle_{\partial\Omega}| &= \left\langle \frac{\partial w}{\partial \nu}, \chi f \right\rangle + \left\langle w\nu, \frac{\partial \chi}{\partial \nu} f \right\rangle \\ &\leq \|w\|_{L^2(H^1)} \|\chi f\|_{L^2(L^2)} + \|w\|_{L^2(L^2)} \left\| \frac{\partial \chi}{\partial \nu} f \right\|_{L^2(L^2)} \\ &\leq \|w\|_{L^2(H^1)} \|\chi\|_{L^2(d\nu)} \|f\|_{L^2(\partial\Omega)} + \|w\|_{L^2(L^2)} \left\| \frac{\partial \chi}{\partial \nu} \right\|_{L^2(d\nu)} \|f\|_{L^2(\partial\Omega)} \end{aligned}$$

A similar argument, using (??), instead of (??), shows that $\langle \cdot, w_0 \rangle_{t=0} \in \mathcal{B}^*$ \square

We give three (easily seen to be) equivalent definitions of the mapping T below so that, following proposition ??, the weak formulation of our heat equation is exactly $Tw = \mathcal{F}$.

$$\begin{aligned}
(25) \quad Tw &= \langle \cdot, \frac{\partial}{\partial t}(\rho_0 w) \rangle + \langle d \cdot, \kappa_0 dw + v_0 \lrcorner \rho_0 w \rangle + \langle \cdot, \rho_0 w \rangle_{t=0} \\
Tw &= \langle \cdot, \rho_0 \frac{\partial w}{\partial t} \rangle + \langle d \cdot, \kappa_0 dw \rangle + \langle d \cdot, v_0 \lrcorner \rho_0 w \rangle + \langle \cdot, \frac{\partial \rho_0}{\partial t} w \rangle + \langle \cdot, \rho_0 w \rangle_{t=0} \\
Tw &= -\langle \frac{\partial}{\partial t} \cdot, \rho_0 w \rangle + \langle d \cdot, \kappa_0 dw \rangle + \langle d \cdot, v_0 \lrcorner \rho_0 w \rangle + \langle \cdot, \frac{\partial \rho_0}{\partial t} w \rangle + \langle \cdot, \rho_0 w \rangle_{t=\tau}
\end{aligned}$$

Proposition 6 T is a bounded operator mapping

$$\begin{aligned}
T : \mathcal{H} &\rightarrow \mathcal{B}^* \\
T : \mathcal{B} &\rightarrow \mathcal{H}^*
\end{aligned}$$

$$\begin{aligned}
(26) \quad \|T\|_{\mathcal{H}, \mathcal{B}^*}, \|T\|_{\mathcal{B}, \mathcal{H}^*} &\leq \|\rho_0\|_{L^\infty(W^{1,\infty})} \\
&\quad + \|\frac{\partial \rho_0}{\partial t}\|_{L^2(L^\infty)} + \|\kappa_0\|_{L^\infty(L^\infty)} + \|v_0 \lrcorner \rho_0\|_{L^2(L^\infty)}
\end{aligned}$$

Proof

For ϕ in $L^2(H^1)$

$$(27) \quad |\langle \phi, \rho_0 \frac{\partial w}{\partial t} \rangle| \leq \|\phi\|_{L^2(H^1)} \|\rho_0 \frac{\partial w}{\partial t}\|_{L^2(H^{1*})}$$

$$(28) \quad \leq \|\phi\|_{L^2(H^1)} \|\rho_0\|_{L^2(W^{1,\infty})} \|\frac{\partial w}{\partial t}\|_{L^2(H^{1*})}$$

$$(29) \quad |\langle d\phi, \kappa_0 dw \rangle| \leq \|\phi\|_{L^2(H^1)} \|\kappa_0\|_{L^\infty(L^\infty)} \|dw\|_{L^2(L^2)}$$

$$(30) \quad |\langle d\phi, v_0 \lrcorner \rho_0 dw \rangle| \leq \|\phi\|_{L^2(H^1)} \|v_0 \lrcorner \rho_0 dw\|_{L^2(L^2)}$$

$$(31) \quad \leq \|\phi\|_{L^2(H^1)} \|v_0 \lrcorner \rho_0\|_{L^2(L^\infty)} \|dw\|_{L^2(L^2)}$$

$$(32) \quad |\langle \phi, \frac{\partial \rho_0}{\partial t} w \rangle| \leq \|\phi\|_{L^2(L^2)} \|\frac{\partial \rho_0}{\partial t} w\|_{L^2(L^2)}$$

$$(33) \quad \leq \|\phi\|_{L^2(L^2)} \|\frac{\partial \rho_0}{\partial t}\|_{L^2(L^\infty)} \|w\|_{L^\infty(L^2)}$$

$$(34) \quad |\langle \phi, \rho_0 w \rangle|_{t=0,\tau} \leq \|\phi\|_{L^\infty(L^2)} \|\rho_0\|_{L^\infty(L^\infty)} \|w\|_{L^\infty(L^2)}$$

$$(35) \quad |\langle \frac{\partial}{\partial t} \phi, \rho_0 w \rangle| \leq \|\frac{\partial}{\partial t} \phi\|_{L^2(H^{1*})} \|\rho_0 w\|_{L^2(H^1)}$$

$$(36) \quad \leq \|\frac{\partial}{\partial t} \phi\|_{L^2(H^{1*})} \|\rho_0\|_{L^\infty(W^{1,\infty})} \|w\|_{L^2(H^1)}$$

□

Proposition 7 (Coercivity Estimates) For $w \in \mathcal{H}$

$$(37) \quad \|w\|_{\mathcal{B}}^2 \leq K_1 \|Tw\|_{\mathcal{B}^*}^2$$

$$(38) \quad \|w\|_{\mathcal{H}}^2 \leq K_2 \|Tw\|_{\mathcal{B}^*}^2$$

where the constant K_1 depends only on $\|\frac{\partial \rho_0}{\partial t}\|_{L^1(L^\infty)}$, $\|v_0 \lrcorner \rho_0\|_{L^2(L^\infty)}$, $\|\kappa_0\|_{L^\infty(L^\infty)}$, and the (strictly positive) infima of ρ_0 and κ_0 . The constant K_2 also depends on $\|D\rho_0\|_{L^\infty(L^\infty)}$.

Proof

For convenience, we assume that $\rho_0 \geq 1$ and $\kappa_0 \geq I$. As $Tw \in \mathcal{H}^*$ and $w \in \mathcal{H}$, we substitute w into (??), setting $\tilde{T}w = Tw - \langle \cdot, w \rangle_{t=0}$.

$$\langle \cdot, \rho_0 \frac{\partial w}{\partial t} \rangle + \langle d \cdot, \kappa_0 dw \rangle = -\langle d \cdot, v_0 \lrcorner \rho_0 w \rangle - \langle \cdot, \frac{\partial \rho_0}{\partial t} w \rangle + \langle \cdot, \tilde{T}w \rangle$$

obtaining

$$\langle w, \rho_0 \frac{\partial w}{\partial t} \rangle + \langle dw, \kappa_0 dw \rangle = -\langle dw, v_0 \lrcorner \rho_0 w \rangle - \langle w, \frac{\partial \rho_0}{\partial t} w \rangle + \langle w, \tilde{T}w \rangle$$

$$(39) \quad \begin{aligned} \frac{1}{2} \langle w, \rho_0 w \rangle_{t=\tau} + \langle dw, \kappa_0 dw \rangle - \frac{1}{2} \langle w, \rho_0 w \rangle_{t=0} \\ = -\langle dw, v_0 \lrcorner \rho_0 w \rangle - \frac{1}{2} \langle w, \frac{\partial \rho_0}{\partial t} w \rangle + \langle w, \tilde{T}w \rangle \end{aligned}$$

so that

$$(40) \quad \begin{aligned} \frac{1}{2} \|w(\tau, \cdot)\|_{L^2}^2 + \|dw\|_{L^2(L^2)}^2 - \frac{1}{2} \|w(0, \cdot)\|_{L^2}^2 \leq \|w\|_{L^2(H^1)} \|v_0 \lrcorner \rho_0\|_{L^2(L^\infty)} \|w\|_{L^\infty(L^2)} \\ + \frac{1}{2} \|\frac{\partial \rho_0}{\partial t}\|_{L^1(L^\infty)} \|w\|_{L^\infty(L^2)}^2 + \|w\|_{L^2(H^1)} \|\tilde{T}w\|_{L^2(H^1*)} \end{aligned}$$

$$\begin{aligned}
\frac{1}{2} \|w(\tau, \cdot)\|_{L^2}^2 + \|dw\|_{L^2(L^2)}^2 &\leq \frac{1}{2} \|w(0, \cdot)\|_{L^2}^2 \\
&+ \|w\|_{L^2(H^1)} \|v_0 \lrcorner \rho_0\|_{L^2(L^\infty)} \|w\|_{L^\infty(L^2)} \\
&+ \frac{1}{2} \left\| \frac{\partial \rho_0}{\partial t} \right\|_{L^1(L^\infty)} \|w\|_{L^\infty(L^2)}^2 \\
&+ \|w\|_{L^2(H^1)} \|Tw\|_{L^2(H^{1*})} \\
&\leq \frac{1}{2} \|w(0, \cdot)\|_{L^2}^2 \\
&+ (\|v_0 \lrcorner \rho_0\|_{L^2(L^\infty)} \epsilon_1 + \epsilon_2) \|w\|_{L^2(H^1)}^2 + \\
&\left(\frac{\|v_0 \lrcorner \rho_0\|_{L^2(L^\infty)}}{\epsilon_1} + \frac{1}{2} \left\| \frac{\partial \rho_0}{\partial t} \right\|_{L^1(L^\infty)} \right) \|w\|_{L^\infty(L^2)}^2 \\
&+ \frac{1}{\epsilon_2} \|\tilde{T}w\|_{L^2(H^{1*})}^2
\end{aligned}$$

We choose $\epsilon_1 = \frac{1}{4\|v_0 \lrcorner \rho_0\|_{L^2(L^\infty)}}$ and $\epsilon_2 = \frac{1}{4}$ to obtain

$$\begin{aligned}
(41) \quad \frac{1}{2} \|w(\tau, \cdot)\|_{L^2}^2 + \frac{1}{2} \|dw\|_{L^2(L^2)}^2 &\leq \frac{1}{2} \|w(0, \cdot)\|_{L^2}^2 \\
&+ \left(\frac{\tau}{2} + 4\|v_0 \lrcorner \rho_0\|_{L^2(L^\infty)}^2 + \frac{1}{2} \left\| \frac{\partial \rho_0}{\partial t} \right\|_{L^1(L^\infty)} \right) \|w\|_{L^\infty(L^2)}^2 + 4\|\tilde{T}w\|_{L^2(H^{1*})}^2
\end{aligned}$$

We discard the second term on the left and take the sup over $\tau \leq \tau_*$, noticing that every term except the first on the left attains its sup at $\tau = \tau_*$.

$$\begin{aligned}
\left(\frac{1}{2} - \frac{\tau}{2} - 4\|v_0 \lrcorner \rho_0\|_{L^2(L^\infty)}^2 - \frac{1}{2} \left\| \frac{\partial \rho_0}{\partial t} \right\|_{L^1(L^\infty)} \right) \|w\|_{L^\infty(L^2)}^2 \\
\leq \frac{1}{2} \|w(0, \cdot)\|_{L^2}^2 + 4\|\tilde{T}w\|_{L^2(H^{1*})}^2
\end{aligned}$$

Provided that we choose $\tau = \tau_1$ small enough to guarantee that

$$A_1 := \frac{\tau_1}{2} + 4\|v_0 \lrcorner \rho_0\|_{L^2(L^\infty)}^2 + \frac{1}{2} \left\| \frac{\partial \rho_0}{\partial t} \right\|_{L^1(L^\infty)} < \frac{1}{2}$$

We have

$$(42) \quad \|w\|_{L^\infty((0, \tau), L^2)}^2 \leq \frac{1}{1 - A} \left(\|w(0, \cdot)\|_{L^2}^2 + 8\|\tilde{T}w\|_{L^2((0, \tau), H^{1*})}^2 \right)$$

We can apply this estimate repeatedly with the interval $(0, \tau_1)$ replaced by (τ_j, τ_{j+1}) , noting that $\sum \|\tilde{T}w\|_{L^2((\tau_j, \tau_{j+1}), H^{1*})}^2 = \|\tilde{T}w\|_{L^2((0, \tau_N), H^{1*})}^2$

$$\begin{aligned}
\|w\|_{L^\infty((0,\tau_N),L^2)}^2 &\leq \prod_{j=1}^N \left(\frac{1}{1-A_j} \right) \left(\|w(0,\cdot)\|_{L^2}^2 + 8\|\tilde{T}w\|_{L^2((0,\tau_N),H^{1*})}^2 \right) \\
&\leq \prod_{j=1}^N \left(\frac{1}{1-A_j} \right) 8\|Tw\|_{L^2((0,\tau_N),H^{1*})}^2
\end{aligned}$$

where

$$A_j = \frac{\tau_j - \tau_{j-1}}{2} + \int_{\tau_j}^{\tau_{j+1}} \left(4\|v_0 \lrcorner \rho_0\|_{L^\infty}^2 + \frac{1}{2} \left\| \frac{\partial \rho_0}{\partial t} \right\|_{L^\infty} \right) dt$$

and

$$\sum_{j=1}^N A_j = \frac{\tau_N}{2} + \int_0^{\tau_N} \left(4\|v_0 \lrcorner \rho_0\|_{L^\infty}^2 + \frac{1}{2} \left\| \frac{\partial \rho_0}{\partial t} \right\|_{L^\infty} \right) dt$$

Hence, on passing to the limit as the intervals become small, we have

$$(43) \quad \|w\|_{L^\infty(L^2)}^2 \leq 8e^{\left(\frac{\tau}{2} + 4\|v_0 \lrcorner \rho_0\|_{L^2(L^\infty)}^2 + \frac{1}{2} \left\| \frac{\partial \rho_0}{\partial t} \right\|_{L^1(L^\infty)}\right)} \|Tw\|_{L^2(H^{1*})}^2$$

If we now return to (??), reorganize, and insert (??), we have

$$\begin{aligned}
(44) \quad \|dw\|_{L^2(L^2)}^2 &\leq \left(\frac{\tau}{2} + 4\|v_0 \lrcorner \rho_0\|_{L^2(L^\infty)}^2 + \frac{1}{2} \left\| \frac{\partial \rho_0}{\partial t} \right\|_{L^1(L^\infty)} \right) \|w\|_{L^\infty(L^2)}^2 \\
&\quad + 8\|Tw\|_{L^2(H^{1*})}^2 \\
&\leq \|Tw\|_{L^2(H^{1*})}^2 \times \\
&\quad \left[8 + \left(\frac{\tau}{2} + 4\|v_0 \lrcorner \rho_0\|_{L^2(L^\infty)}^2 + \frac{1}{2} \left\| \frac{\partial \rho_0}{\partial t} \right\|_{L^1(L^\infty)} \right) e^{\left(\frac{\tau}{2} + 4\|v_0 \lrcorner \rho_0\|_{L^2(L^\infty)}^2 + \frac{1}{2} \left\| \frac{\partial \rho_0}{\partial t} \right\|_{L^1(L^\infty)}\right)} \right]
\end{aligned}$$

Lastly, we return to our original weak formulation (??),

$$\left\langle \cdot, \rho_0 \frac{\partial w}{\partial t} \right\rangle = - \left\langle \cdot, Tw \right\rangle - \left\langle d\cdot, \kappa_0 dw \right\rangle - \left\langle d\cdot, v_0 \lrcorner \rho_0 w \right\rangle - \left\langle \cdot, \frac{\partial \rho_0}{\partial t} w \right\rangle - \left\langle \cdot, \rho_0 w \right\rangle_{t=0}$$

choose a smooth ϕ vanishing at $t = 0$, and apply the above to $\frac{\phi}{\rho_0}$

$$\left\langle \phi, \frac{\partial w}{\partial t} \right\rangle = - \left\langle \frac{\phi}{\rho_0}, Tw \right\rangle - \left\langle d\frac{\phi}{\rho_0}, \kappa_0 dw \right\rangle - \left\langle d\frac{\phi}{\rho_0}, v_0 \lrcorner \rho_0 w \right\rangle - \left\langle \frac{\phi}{\rho_0}, \frac{\partial \rho_0}{\partial t} w \right\rangle$$

recalling that $\rho_0 \geq 1$

$$\begin{aligned}
|\langle \phi, \frac{\partial w}{\partial t} \rangle| &\leq \|\phi\|_{L^2(H^1)} \|\rho_0\|_{L^\infty(W^{1,\infty})} \|Tw\|_{L^2(H^{1*})} \\
&\quad + \|\phi\|_{L^2(H^1)} \|\rho_0\|_{L^\infty(W^{1,\infty})} \|\kappa_0\|_{L^\infty(L^\infty)} \|dw\|_{L^2(L^2)} \\
&\quad + \|\phi\|_{L^2(H^1)} \|\rho_0\|_{L^\infty(W^{1,\infty})} \|v_0 \lrcorner \rho_0\|_{L^2(L^\infty)} \|w\|_{L^\infty(L^2)} \\
&\quad + \|\phi\|_{L^2(L^2)} \|\frac{\partial \rho_0}{\partial t}\|_{L^2(L^\infty)} \|w\|_{L^\infty(L^2)}
\end{aligned}$$

which, combined with (??) and (??), yields the necessary estimate for $\|\frac{\partial w}{\partial t}\|_{L^2(H^{1*})}$ \square

Theorem 8 *Suppose that*

1. ρ_0, κ_0 are bounded above and below
2. $v_0 \lrcorner \rho_0, \frac{\partial \rho_0}{\partial t}$ are bounded in $L^2(L^\infty)$
3. $D\rho_0$ is bounded in $L^\infty(L^\infty)$

Then, for every $\mathcal{F} \in \mathcal{B}^$, there exists a unique $w \in \mathcal{H}$ satisfying*

$$\begin{aligned}
Tw &= \mathcal{F} \\
\|w\|_{\mathcal{B}} &\leq K_2 \|\mathcal{F}\|_{\mathcal{B}^*} \\
\|w\|_{\mathcal{H}} &\leq K_3 \|\mathcal{F}\|_{\mathcal{B}^*}
\end{aligned}$$

where the constant K_3 depend on all the bounds in items 1 – 3 above, and the constant K_2 depends only on the bounds in items 1 and 2.

Proof

Uniqueness and the estimates follow from Proposition ???. To prove existence, we introduce a one parameter family of operators, T^λ . Let

$$\begin{aligned}
\rho_0^\lambda &:= (1 - \lambda)\rho_0 + \lambda 1 \\
\kappa_0^\lambda &:= (1 - \lambda)\kappa_0 + \lambda I \\
(v_0 \lrcorner \rho_0)^\lambda &:= (1 - \lambda)v_0 \lrcorner \rho_0
\end{aligned}$$

and define

$$(45) \quad T_\lambda w = \langle \cdot, \frac{\partial}{\partial t}(\rho_0^\lambda w) \rangle + \langle d\cdot, \kappa_0^\lambda dw + (v_0 \lrcorner \rho_0)^\lambda w \rangle - \frac{1}{2} \langle \cdot, \rho_0^\lambda w \rangle_{t=0}$$

It is straightforward to check that

$$(46) \quad \|T_{\lambda_0} - T_{\lambda_1}\|_{\mathcal{H}, L^2(H^{1*})} \leq |\lambda_0 - \lambda_1| K$$

where

$$(47) \quad K = \left(\|1 - \rho_0\|_{L^\infty(W^{1,\infty})} + \left\| \frac{\partial}{\partial t} \rho_0 \right\|_{L^2(L^\infty)} + \|1 - \kappa_0\|_{L^\infty(L^\infty)} + \|v_0 \lrcorner \rho_0\|_{L^2(L^\infty)} \right)$$

is independent of λ .

When $\lambda = 0$,

$$(48) \quad T_0 w = \langle \cdot, \frac{\partial}{\partial t} w \rangle + \langle d \cdot, dw \rangle + \langle \cdot, w \rangle_{t=0}$$

so that

$$(49) \quad T_0 w = \mathcal{F} = \langle \cdot, F \rangle + \langle \cdot, f \rangle_{\partial\Omega} + \langle \cdot, w_0 \rangle_{t=0}$$

is the weak formulation of the constant coefficient heat equation with Neumann boundary conditions, i.e.

$$(50) \quad \frac{\partial w}{\partial t} - \Delta w = F$$

$$(51) \quad w \Big|_{\partial\Omega} = f$$

$$(52) \quad w \Big|_{t=0} = w_0$$

so that we have existence for smooth (F, f, w_0) from semigroup theory. But such (F, f, w_0) 's are dense in B^* , so that the coercivity estimate (??) gives the existence of a bounded T_0^{-1} from \mathcal{B}^* to \mathcal{H} .

But the formula

$$(53) \quad T_{\lambda_1}^{-1} = \left(I - T_{\lambda_0}^{-1} (T_{\lambda_0} - T_{\lambda_1}) \right) T_{\lambda_0}^{-1}$$

implies that, if T_{λ_0} is invertible, then so is T_{λ_1} , as long as

$$(54) \quad \|T_{\lambda_0}^{-1} (T_{\lambda_0} - T_{\lambda_1})\| < 1$$

while (??) and (??) guarantee (??) as long as $|\lambda_0 - \lambda_1|$ is smaller than a single uniform constant. Hence T_λ is invertible for all λ between zero and one.

4 A Weak Formulation in Ω_t

We have produced a solution w to a weak heat equation on our fixed domain. We expect $u = w(t, \Psi(x, t))$ to solve a weak formulation of (??) on our original moving domain. The following propositions indicate that this is the case. The proofs are analogous to the corresponding results in the fixed domain.

Proposition 9 *A smooth function u satisfies (??), if and only if, for all smooth ϕ ,*

$$(55) \quad \langle \phi, \frac{\partial}{\partial t}(\rho u) \rangle + \langle d\phi, \kappa dw + v \lrcorner \rho u \rangle + \langle \phi, d(v \lrcorner \rho u) \rangle - \langle \phi, u \rangle_{t=0} \\ = -\langle \phi, u_0 \rangle_{t=0}$$

If we define T so that $\langle \phi, Tu \rangle$ is the left hand side of (??), then

Proposition 10 *T is a bounded operator mapping*

$$T : \mathcal{H} \rightarrow \mathcal{B}^* \\ T : \mathcal{B} \rightarrow \mathcal{H}^*$$

$$\|T\|_{\mathcal{H}, \mathcal{B}^*}, \|T\|_{\mathcal{B}, \mathcal{H}^*} \leq \\ \|\rho\|_{L^\infty(W^{1, \infty})} + \|\frac{\partial \rho}{\partial t}\|_{L^2(L^\infty)} + \|\kappa\|_{L^\infty(L^\infty)} + \|v \lrcorner \rho\|_{L^2(L^\infty)} + \|d(v \lrcorner \rho)\|_{L^2(L^\infty)}$$

Proposition 11 (Coercivity Estimates)

$$(56) \quad \|w\|_{\mathcal{B}}^2 \leq K_1 \|Tw\|_{\mathcal{B}^*}^2$$

$$(57) \quad \|w\|_{\mathcal{H}}^2 \leq K_2 \|Tw\|_{\mathcal{B}^*}^2$$

where the constant K_1 depends only on $\|\frac{\partial \rho}{\partial t}\|_{L^1(L^\infty)}$, $\|v \lrcorner \rho\|_{L^2(L^\infty)}$, $\|d(v \lrcorner \rho)\|_{L^2(L^\infty)}$, $\|\kappa\|_{L^\infty(L^\infty)}$, and the (strictly positive) infima of ρ and κ . The constant K_2 also depends on $\|d\rho\|_{L^\infty(L^\infty)}$.

In the theorem below, $D\Psi$ denotes the jacobian of Ψ , and $v = \frac{\partial \Psi}{\partial t}(t, \Psi^{-1}(t, \cdot))$.

Theorem 12 *Suppose that*

1. $\rho, \kappa, D\Psi, D\Psi^{-1}$ are bounded above and below

2. $v \lrcorner \rho, d(v \lrcorner \rho), \frac{\partial \rho}{\partial t}$ are bounded in $L^2(L^\infty)$

3. $D\rho, D^2\Psi$ are bounded in $L^\infty(L^\infty)$

then, For every $\mathcal{F} \in \mathcal{B}^*$, there exists a unique $w \in \mathcal{H}$ satisfying

$$\begin{aligned} Tu &= \mathcal{F} \\ \|u\|_{\mathcal{B}} &\leq K_2 \|\mathcal{F}\|_{\mathcal{B}^*} \\ \|u\|_{\mathcal{H}} &\leq K_3 \|\mathcal{F}\|_{\mathcal{B}^*} \end{aligned}$$

where the constant K_3 depend on all the bounds in items 1 – 3 above, and the constant K_2 depends only on the bounds in items 1 and 2.

Proof

It is an easy matter to check that ρ_0, v_0, κ_0 given by (??) satisfy the hypotheses of Theorem ?? and that the natural pullback of an $\mathcal{F} \in \mathcal{B}^*$ also belongs to the corresponding \mathcal{B}^* . Thus we can produce a weak solution w to (??). Then $u = w(t, \Psi(t, x))$ must belong to \mathcal{H} and satisfy (??). \square

5 The Constant Coefficient Heat Equation

For the constant coefficient heat equation, (ρ is the euclidean volume form and κ is the euclidean star operator. In particular, $(\frac{\partial \rho}{\partial t} = 0)$. In this section, $Tu = \mathcal{F}$ is the weak formulation of

$$(58) \quad \begin{aligned} \frac{\partial}{\partial t} w - \Delta w &= F \\ \left(\frac{\partial w}{\partial \nu} + \nu \cdot \nabla w \right) \Big|_{\partial \Omega_t} &= f \\ w \Big|_{t=0} &= w_0 \end{aligned}$$

and theorem ?? implies

Theorem 13 *Suppose that $\partial \Omega_t$ is given as the solution to the equation $G(x, t) = 0$ and that*

1. $|\frac{\partial G}{\partial x}|$ is bounded from above and below (in $L^\infty(L^\infty)$)
2. $\frac{\partial G}{\partial t}$ and $\frac{\partial^2 G}{\partial t \partial x}$ are bounded in $L^2(L^\infty)$

3. $\frac{\partial^2 G}{\partial x^2}$ is bounded in $L^\infty(L^\infty)$

then, For every $\mathcal{F} \in \mathcal{B}^*$, there exists a unique $w \in \mathcal{H}$ satisfying

$$\begin{aligned} Tu &= \mathcal{F} \\ \|u\|_{\mathcal{B}} &\leq K_2 \|\mathcal{F}\|_{\mathcal{B}^*} \\ \|u\|_{\mathcal{H}} &\leq K_3 \|\mathcal{F}\|_{\mathcal{B}^*} \end{aligned}$$

where the constant K_3 depend on all the bounds in items 1 – 3 above, and the constant K_2 depends only on the bounds in items 1 and 2.

Proof

In order to apply Theorem ?? all we need check is that there is a C^2 -diffeomorphism Ψ mapping Ω_0 to Ω_t , and that its first and second derivatives depend only on the corresponding derivatives of G . It suffices to construct Ψ for small t and then repeat the construction.

We will construct Ψ to be the identity except in a neighborhood of the boundary $\partial\Omega_0$, where the tubular neighborhood theorem asserts that a neighborhood $N(\partial\Omega_0)$ is C^2 -diffeomorphic to $\partial\Omega_0 \times [-\varepsilon, \varepsilon]$, and in which $\partial\Omega_0$ has coordinates $(m_0, 0)$. The implicit function theorem, together with the continuity of $\frac{\partial G}{\partial x}$ with respect to t , implies that, in this coordinate system, $\partial\Omega_t$ has coordinates $(m_0, g(t, m_0))$, with $g \in C^2$. If ϕ is a smooth cutoff, equal to one in $\partial\Omega_0 \times [-\varepsilon/2, \varepsilon/2]$ and 0 near the boundary of the tube, then

$$\Psi(m_0, s) = (m_0, s - \phi(s)g(t, s))$$

provides the needed diffeomorphism. \square

In one dimension, our region is an interval $(\gamma_1(t), \gamma_2(t))$ and we may choose

$$\Psi(t, x) = \frac{\gamma_2}{\gamma_2 - \gamma_1} (x - \gamma_1)$$

When we apply Theorem ?? in this case, $D^2\Psi = 0$, so that we need only assume that the interval does not shrink to a point and that $(\gamma_1(t), \gamma_2(t))$ have square integrable derivatives.

Theorem 14 *Suppose that $\gamma_2 - \gamma_1$ is bounded from below and that $\gamma_1'(t)$ and $\gamma_2'(t)$ are bounded in L^2 then, for every $\mathcal{F} \in \mathcal{B}^*$, there exists a unique $w \in \mathcal{H}$ satisfying*

$$\begin{aligned} Tu &= \mathcal{F} \\ \|u\|_{\mathcal{B}} &\leq K_2 \|Tu\|_{\mathcal{B}^*} \end{aligned}$$

where the constant K_2 depends only on the bounds above.

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