Essays on Contract Theory and Mechanism Design

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Abstract

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My dissertation investigates optimal contracts for experimentation and a matching problem for the runway slot allocation. The first chapter of my dissertation examines the role of monitoring in experimentation where agents may observe success privately. In the benchmark model without monitoring, private observability of success is inconsequential as the agent never wants to delay announcing success. However, with monitoring of the agent's effort, private observability of success plays a role in choosing the optimal time for monitoring. When success is observed publicly, the optimal time for a principal to hire a monitor is at the start of the relationship. On the contrary, if the agent observes success privately, and the discount factor is high enough, monitoring is performed during the final period. The second chapter discusses optimal contracts for both experimentation and production. It can be optimal to pay a rent after failure and over experimentation can be optimal. Over production can occur in the exploitation phase. The third chapter considers a financially significant matching problem that emerges when inclement weather conditions strike an airport and runway slots must be reallocated.

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DEDICATION

to my family

Chapter 1

EXPERIMENTATION, PRIVATE OBSERVABILITY OF SUCCESS, AND THE TIMING OF MONITORING

1.1 Introduction and Related Literature

Consider a venture capitalist (principal) who provides funds to an entrepreneur (agent) hoping to succeed in a risky but lucrative project. Both parties are initially unsure whether the project is ”good”, i.e., if it is possible to implement it successfully at all. The entrepreneur is asked to experiment with the project a certain period of time. The principal needs to determine how long the experimentation will last. She faces two problems: the effort of the agent may not be observable (moral hazard) and the result of each experiment may not be publicly observed.

The principal may therefore not have the correct information about the viability of the project as experimentation proceeds. Consider first the case when success is observed publicly. Suppose the agent secretly shirks and success is not achieved. The principal uses the observed failure to update his beliefs on project quality and becomes more pessimistic from that period on. The agent, in contrast, knowing that the experiment was not successful because he was shirking, will not update his beliefs regarding the project quality. Furthermore, if success is privately observed then the agent might postpone announcing success when the project is successfully implemented. This report makes the principal more pessimistic while the agent knows the project is certainly good.

Another example is a pharmaceutical company that employs a research organization to carry out clinical trials on a group of volunteers to demonstrate the effectiveness of a new drug. If the company does not to observe the agent testing the drug directly, it may doubt whether the agent is exerting effort. If the agent chooses to shirk, he simply can report that the drug was tested unsuccessfully. If the company remains unaware of this falsehood, it will adjust its beliefs about the drug’s quality accordingly, becoming more pessimistic. Even if the agent discovers the drug is performing well, he may delay announcing success in favor of personal gain.

In this paper, we ask whether the principal can benefit from hiring a monitor in this environment. In a dynamic relationship, collecting information during every period is prohibitively costly, raising the question of *when* monitoring is most effective. First, we derive the optimal contract without monitoring that determines the length of the principal-agent relationship and solves the agent's moral hazard, ensuring that he works properly during every period of the relationship and announces achieved success promptly. Second, we study the benefits of monitoring of the agent's effort. The optimal contract then becomes contingent on the monitor's reports. Finally, we examine how private observability of success influences the structure of the optimal contract and the optimal timing of monitoring.

To answer these questions, we use a simple two-armed bandit model[1]: The agent can "pull the risky arm" by exerting effort toward implementing the project (achieve success), or he can "pull the safe arm" and shirk. While pulling the risky arm is costly, it allows the project to be implemented successfully if it is good. Pulling the safe arm, however, yields zero return, regardless of project quality.

Our results show that, in the benchmark case without monitoring, the principal commits to terminating the relationship if the agent does not succeed by a certain period. The agent is rewarded only if the project is implemented successfully and receives nothing otherwise. The nominal value of the reward increases to account for rising pessimism as the project does not succeed over time. In particular, the agent is rewarded for earlier success, as the discounted payments are decreasing over time. So even if the agent observes success privately, he will never delay announcing success and the optimal contract is unchanged. Surprisingly, with monitoring of the agent's effort, private observability of success plays a role in choosing the optimal time for monitoring. We show that in the case in which success is observed publicly, the principal should monitor the agent at the beginning of their relationship. If the agent observes success privately, the optimal time for monitoring is affected by patience. For high enough discount factor, monitoring

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[1] See Keller et al. (2005) for more details.

should be performed during the final period.

Consider the first-best scenario, in which effort and all the information the agent learned are observed publicly. Since each attempt to implement the project is costly, and the principal becomes more pessimistic with every period in which success is not announced, the first-best solution is characterized by a stopping rule. The agent is allowed to attempt to implement the project for several periods only.

In the second-best case, the principal faces two problems: the agent chooses effort level privately (moral hazard), and, in addition, success may not be publicly observed. Suppose success is observed publicly. In the case the agent secretly shirks and success is not achieved, the principal becomes relatively more pessimistic from that period on. She then would adjust the agent's reward to induce him to exert effort accordingly: After every period the agent does not succeed, the principal supplies him with larger payments to encourage him to continue working in the next period. The agent, in contrast, understands that the observed failures are uninformative, and at the beginning of the next period would hold the same beliefs as in the previous one.

Besides, private observability of success may exacerbate the problem. In some settings, the principal can observe success easily. In the example of the pharmaceutical company, the clinical research organization may have difficulty hiding a revolutionary drug's success. However, success might be much more difficult to ascertain when information gathering does not involve extreme outcomes.

Consider the agent's incentive to announce privately observed success at a certain period of the relationship. This decision is affected not only by the payment tied to success or failure in this particular period, as determined by the optimal contract, but also by payments in all subsequent periods of the relationship. For example, if the discounted value of the promised future reward exceeds the current value, then the agent will postpone an announcement. We show that under the optimal contract, if the agent postpones an announcement of success, the discounted value of his reward will only decrease. Thus, private observability does not worsen the problem, as the agent cannot benefit from hiding success; however, it becomes a crucial factor in defining the optimal time for monitoring.

Given the optimal contract, the agent receives a strictly positive rent, and, as a result, the project is terminated inefficiently early. One way the principal can alleviate this inefficiency is by hiring a monitor who can observe the effort level chosen by the agent. The principal can avoid paying the promised reward since the moral hazard problem vanishes when the monitor is hired. The principal values monitoring as it mitigates the rent paid to the agent and, consequently, allows the relationship to be extended.

The principal benefits from monitoring as she pays the agent a smaller reward in case he succeeds during the monitoring period - we refer to this as the *static effect*. Moreover, recall that the agent can shirk and attempt to implement the project during later periods. As a result, his incentives to work during each period, except for the final one, depend not only on the payment determined by the contract and success of that particular period, but also on the payments for all subsequent periods.[2] The effects of future monitoring reflect in the earlier periods, as monitoring acts as a threat that causes the agent to perceive shirking as less attractive. Thus the *dynamic effect* emerges: The principal can diminish all of the agent's rewards in all periods before he is monitored.

We demonstrate that the dominating effect depends on whether the agent observes success privately or not. The optimal timing of monitoring is governed by the sum of the two effects. Since without monitoring, the agent is rewarded for earlier success, the expected reward is strictly decreasing. This implies that the benefit from the static effect is strictly decreasing in monitoring timing as well. With the dynamic effect, the principal benefits even if the agent does not succeed at the period of monitoring: For earlier periods of monitoring, the benefit from the dynamic effect is strictly increasing, as it allows paying less in several periods. However, as time passes without success, both parties become sufficiently pessimistic, and the benefit from the dynamic effect decreases eventually.

Consider first the case in which success is observed publicly. Because monitoring eradicates the moral hazard problem during the monitoring period, the principal benefits from it only if the agent succeeds in this specific period, which in turn is possible only if

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[2] Halac et al. (2016) discuss this dynamic agency effect in their model with moral hazard and adverse selection.

the project is good. In earlier periods, the benefit from the static effect increases while the benefit from the dynamic effect goes up, whereas benefits from both effects decline toward the end of the relationship. The benefit from the static effect decreases faster than the benefit from the dynamic effect increases - this result is at the heart of our analysis. Recall that the benefit from the dynamic effect increases only in earlier periods: Because the principal is promising to pay less with each period, the chance that the agent will shirk grows smaller. The principal saves more in these early periods by opting to monitor later due to the dynamic effect. The optimal contract, in contrast, mitigates the moral hazard problem for every period, not only for those in which the benefit from the dynamic effect is increasing, as it induces the agent to exert effort as long as the relationship lasts. Thus, when success is observed publicly, the static effect dominates, and monitoring is optimal during the first period.

When the agent observes success privately, the benefit from the static effect becomes smaller because the principal now still pays some rent to the agent during the monitoring period. Recall that when the agent announces success, he takes into account not only the payment tied to success in this precise period but also payments for success in all subsequent periods of the relationship: If the discounted value of the promised reward in the next period exceeds the current reward, then the agent will postpone an announcement. Given that the optimal contract without monitoring includes a decreasing discounted reward value, the principal can decrease the reward in one period up to the discounted value of the reward in the following period at most. As the discount factor increases, the benefit from the static effect decreases for all periods except the final one. Our paper contributes to the literature on incentives for experimentation. Most studies model experimentation using a two-armed bandit model with a risky arm that might yield exponentially distributed payoffs and a safe arm that offers a safe payoff. The literature on incentives for experimentation could be divided into two parts, depending on who is initially the owner of the project.

A group of papers considered an entrepreneur who owns a project and is raising the funds necessary to implement a project in a competitive market. In settings with private learning and moral hazard, Bergemann and Hege (1998) considered the provision of venture capital in a dynamic agency model. The optimal share contract operates on the provision that if the entrepreneur succeeds, he conveys a part of the project to the investor. In Bergemann and Hege (1998), the share in the earlier periods can rise or fall, but the agent receives the expected value of the project. In our paper, on the other hand, a nominal reward for success is always (weakly) increasing, and the agent receives a positive rent. Bergemann and Hege (2005) study built on their Bergemann and Hege (1998) study, with one crucially distinct feature - the time horizon is infinite, and the funding decision is renegotiated each period.

Another group of papers considered a principal who owns a research idea but lacks the decisive skills necessary to implement it and must hire an agent in order to do so. Halac et al. (2016) considered the challenges of creating a contract for a project of uncertain feasibility with adverse selection and moral hazard. The optimal contract involves paying the agent initially and penalizing him progressively if success has not been observed. Our research differs from Halac et al. (2016) in that we assume the probability of success is known, but the agent is protected by limited liability. The agent thus cannot be penalized for failure; instead, we show how bonus contracts and optimal monitoring discourage the agent from hiding success.

Gerardi and Maestri (2012) analyzed how an agent can be incentivized to obtain and announce information over time. The authors assumed that the principal observes the state of nature with some time lag, and, with the optimal contract, he can reward or punish the agent after comparing the agent's report with the revealed state. In this study, the agent is rewarded only if his report matches the true state, whereas in our study, the true state is learned only if the project is successfully implemented. Mason and Valimaki (2015) considered an infinitely lasting relationship in a model without learning and with moral hazard and continuous effort. They demonstrated that the agent's wage declines over time; however, they did not study monitoring.

The only researchers who have studied optimal monitoring time sets in optimal contracts for experimentation are Bergemann and Hege (1998). In this study, when success is publicly observed, monitoring is optimal toward the end of the project, whereas in our model, the monitor is hired optimally at the beginning of the relationship. Our paper complements Bergemann and Hege (2005) result in that it highlights the pivotal role of market structure on the optimal timing of monitoring. In addition, our paper extends Bergemann and Hege (2005) result, as we show that when the agent observes success privately, patience influences the optimal timing of monitoring, which was not explored in their paper. As the discount factor increases, the principal must promise identical rewards for success at every period except the final one, making monitoring more valuable at the end of the relationship and supporting Bergemann and Hege (2005) result.

Most of these papers assumed project success would be publicly observed.[3] We, in contrast, assume that the agent could observe success privately. We also assume that information is hard - that is, the agent can either postpone announcing a successful implementation or hide it completely by destroying evidence.

We argue that our findings are not only theoretical, but also empirically significant. Following our example of the pharmaceutical company, it is widely acknowledged that on-site clinical trial monitoring is a source of significant inefficiency in the conduct of clinical trials, and that current monitoring activities do not always lead to increased quality in clinical trials.

1.2 Model

A principal owns a valuable idea that could result in a lucrative project, but he lacks the decisive skills needed to implement it. He hires an agent protected by limited liability to perform the project. Both parties initially are uncertain about the project's quality; that is, the common prior on the project being "good" is $\beta_0 \in (0,1)$.[4] If the project is good, then it can be implemented successfully with a known positive probability, in which case it will yield a fixed return of $V > 0$, which is commonly known at the beginning of the relationship. To implement the good project, the agent must exert effort that is assumed to be subject to a binary choice: $e \in \{0,1\}$. If the project is bad, then it will yield zero,

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[3] Halac et al. (2016) discuss the robustness of an optimal contract to project success being privately observed by the agent.

[4] It is important that β_0 is strictly positive and strictly less than one. Otherwise, no additional information arrives as the relationship proceeds; in this case, there is no learning regarding the quality of the project, and the problem simplifies to standard dynamic moral hazard.

regardless of effort.[5] Exerting effort costs $c > 0$ per period.

The agent's ability, λ, which is the probability of achieving success given that the project is good, conditional on exerting effort, is common knowledge during contracting. Finally, we assume that the effort choice is not observable and that the agent can postpone announcement of or hide a successful implementation.

An important feature of this model is learning project quality. When the agent does not succeed, despite exerting effort, he updates[6] his beliefs regarding the quality of the project using Bayes' rule and becomes more pessimistic. Denoting by $\tilde{\beta}_t$, the updated belief of the agent that the project is good at the beginning of period t after $t-1$ failures, we present:

$$\tilde{\beta}_t = \frac{\tilde{\beta}_{t-1}(1-\lambda)}{\tilde{\beta}_{t-1}(1-\lambda)+1-\tilde{\beta}_{t-1}}, \text{ which simplifies to } \tilde{\beta}_t = \frac{\beta_0(1-\lambda)^{t-1}}{\beta_0(1-\lambda)^{t-1}+1-\beta_0}.$$

Since the agent chooses effort level privately, both parties do not share the same beliefs necessarily as their relationship evolves. The principal becomes more pessimistic every period the agent does not announce success. However, if the agent secretly shirks he becomes relatively more optimistic from that period on. Consider a hypothetical scenario in Figure 1.1 below.

Given the parameters, the bold line reflects the evolution of beliefs if the agent continues exerting effort for 10 periods. Suppose the agent secretly shirks at $t = 5$, but reports that success has not been achieved, despite exerting effort. The principal would use this report to update his beliefs and become relatively more pessimistic from period $t = 6$ on. The agent, in contrast, would understand that the reported failure was uninformative, and at the beginning of period $t = 6$ would have the same beliefs as in the previous period. Importantly, this difference in beliefs following one deviation at $t = 6$ would carry into all future periods until the relationship ends.

The optimal contract has to take into account four crucial features of the relationship between the principal and the agent: First, the results of each period affects the relationship; with each failure the agent reports, the principal becomes more pessimistic. Second,

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[5] We refer to an implementation of the project as "success" and to lack of success as "failure."

[6] A failure is more informative if beliefs are close to $\frac{1}{2}$, while beliefs change slowly when parties are relatively certain about the quality of the project. See Bergemann and Hege (1998) for more details.

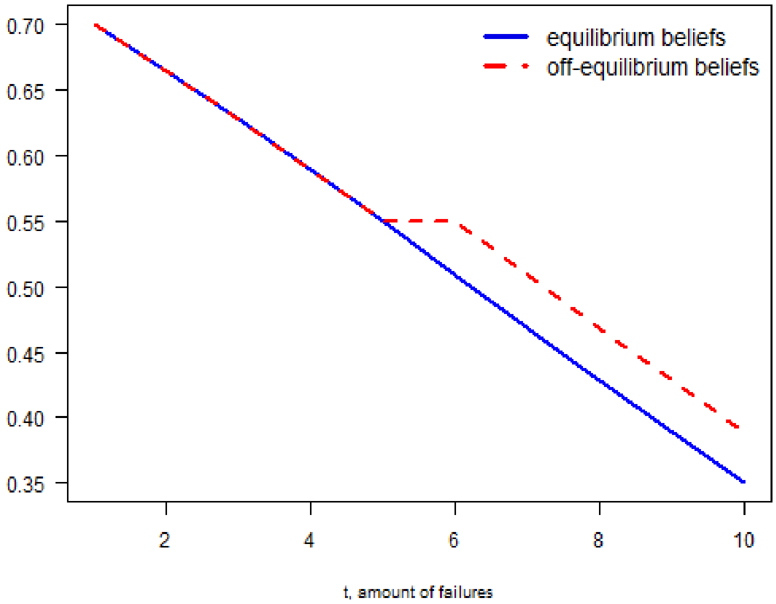

Figure 1.1: Learning the Quality of the Project with $\lambda = 0.15$ and $\beta_0 = 0.7$

during each period, the agent chooses privately whether to exert the effort necessary for the project to succeed. Third, the agent is protected by limited liability, so the principal cannot sell the project to the agent. Finally, the agent observes successful project implementation privately. As a result, the payment structure must ensure that the agent neither postpones nor hides the announcement of a successful implementation.

Both parties are risk neutral and share a common discount factor $\delta \in (0, 1]$. An optimal contract must specify how many failures the principal will tolerate and a sequence of transfers as a function of the agent's reports,[7] which in this case is whether or not the agent succeeded. All transfers are from the principal to the agent.

The contract is given by $\varpi = (T, \{b_t\}_{t=1}^{T}, \{w_t\}_{t=1}^{T})$, where $T \in N$ is the duration of the relationship, b_t is the payment to the agent in case he reports success at period $1 \leqslant t \leqslant T$ and w_t is the payment to the agent conditional on reporting failures from the beginning of the relationships up to period $1 \leqslant t \leqslant T$.

Under certain circumstances, the agent can postpone or even hide success. To un-

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[7] Because success is observed privately, both parties do not necessarily share the same history as the relationship evolves.

derstand how this fact matters and if it affects the structure of the optimal contract, consider the agent's incentive to announce that he successfully completed the project at $t < T$. This decision is affected not only by the payment tied to success or failure in this particular period, as determined by the optimal contract, but, in addition, by payments in all subsequent periods of the planning stage. For example, if the discounted value of the promised reward for success in the future exceeds the current value, then the agent will postpone an announcement; if the agent is rewarded for consecutive failures, he would benefit from hiding success completely.

To prevent the agent from hiding success, the optimal contract will have to satisfy the following incentive compatibility constraint for every period $1 \leqslant t \leqslant T$:

(IC)
$$b_t \geqslant w_t + \delta b_{t+1} \text{ for } t = 1, ..., T-1,$$
$$b_t \geqslant w_t \text{ for } t = 1, ..., T.$$

Given the optimal contract and effort levels the agent chooses, we can specify the agent's expected utility and the principal's expected profit. The agent's expected utility from accepting contract ϖ at time zero while exerting an effort profile $\vec{e}$ and reporting each project truthfully as a failure or success is:

$$\begin{gathered} U(\varpi, \vec{e}) = (1-\beta_0) \textstyle\sum_{t=1}^{T} \delta^t (w_t - e_t c) \\ +\beta_0 \textstyle\sum_{t=1}^{T} \delta^t (\prod_{s=1}^{t-1} (1-\lambda e_s))(e_t(\lambda b_t - c) + (1-\lambda e_t) w_t), \end{gathered}$$

where $\vec{e} = (e_1, ..., e_T)$ is an effort profile with $e_t \in \{0, 1\}$[8] for $1 \leqslant t \leqslant T$.

First, the agent has a chance to succeed during the relationship; this occurs only if both the project is good, which is true with probability β_0, and if the agent is exerting effort. Conditional on the project being good, the relationship lasts for an arbitrary period, $t \leqslant T$, with probability $\prod_{s=1}^{t-1}(1-\lambda e_s)$. If the agent exerts effort at period t, his expected payoff at this period is:

$$\lambda b_t + (1-\lambda) w_t - c,$$

whereas in case he shirks, the agent receives only w_t, as defined by the contract. Second, if the project is bad, which happens with probability $1 - \beta_0$, the agent never succeeds, regardless of effort profile.

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[8] We refer to $e_t = 1$ as "work" and to $e_t = 0$ as "shirk".

The principal's expected profit from offering contract ϖ at time zero if the agent exerts an effort profile $\vec{e}$ and reports failures and project success truthfully is:

$$\pi(\varpi,\vec{e}) = -(1-\beta_0)\sum_{t=1}^{T}\delta^t w_t$$
$$+\beta_0\sum_{t=1}^{T}\delta^t(\prod_{s=1}^{t-1}(1-\lambda e_s))(e_t\lambda(V-b_t)-(1-\lambda e_t)w_t).$$

The optimal contract will have to satisfy the following moral hazard constraint at each period for all possible histories and all possible effort paths in the future:

(MH) $$\vec{1}\in\arg\max_{\vec{e}}U(\varpi,\vec{e}).$$

Given that the *MH* constraint is satisfied, the principal's expected profit from offering contract ϖ at time zero becomes:

$$\pi(\varpi,\vec{1}) = -(1-\beta_0)\sum_{t=1}^{T}\delta^t w_t + \beta_0\sum_{t=1}^{T}\delta^t(1-\lambda)^{t-1}(\lambda(V-b_t)-(1-\lambda)w_t).$$

Consider the first-best case: The principal observes the effort choice and project outcome. As the relationship proceeds, if success is not being achieved, then every period the marginal benefit, $\lambda\tilde{\beta}_t V$, is the expected value of the project and takes into account both probability of success and current beliefs. Since beliefs are declining as time goes on without success, the marginal benefit decreases strictly. The marginal cost-of-effort, c, is constant. As a result, the first-best solution is characterized by stopping time $T\in N$, such that the agent is allowed to exert effort up until that date only, as follows:

$$T^{FB} = \arg\max_t\{\lambda\tilde{\beta}_t V \geqslant c\}^{9}.$$

Consider the example in Figure 1.2 below, where $\lambda = 0.15$, $\beta_0 = 0.7$, $V = 20$ and $c = 1$, and where the agent starts with $MB_1 = 2.1$ and continues experimenting with the project for ten periods at most.

When the agent chooses effort level privately, the optimal contract must ensure the agent works in every period, which is guaranteed by the *MH* constraint. Since the agent is protected by limited liability, he cannot pay the principal, as the *LL* constraint reflects. In addition, the *IC* constraint ensures the agent neither postpones success nor hides it. The principal's optimization problem in this case becomes the following[10]:

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[9] Recall that $\tilde{\beta}_t$ are beliefs evolved as a result of the agent exerting effort in all periods until t.

[10] We assume that V is high enough, and it is optimal for the principal when the agent exerts effort in every period.

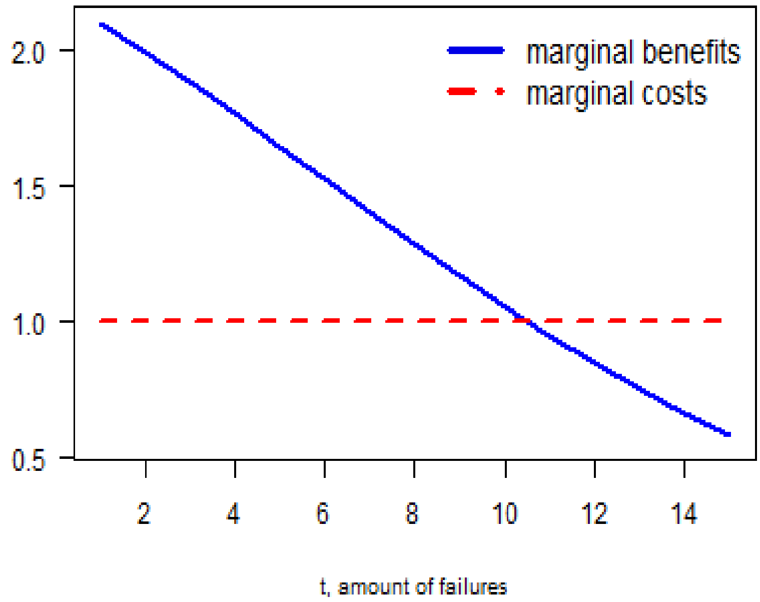

Figure 1.2: The First-Best Benchmark with $\lambda = 0.15$ and $\beta_0 = 0.7$

$$[P^{SB}] \ \max_{\varpi} \pi(\varpi, \vec{1}) \text{ subject to}$$

(MH) $$\vec{1} \in \arg\max_{\vec{e}} U(\varpi, \vec{e}),$$

(IC) $$b_t \geqslant w_t + \delta b_{t+1} \text{ for } t = 1, ..., T-1,$$

$$b_t \geqslant w_t \text{ for } t = 1, ..., T,$$

(LL) $$b_t, \, w_t \geqslant 0 \text{ for } t = 1, ..., T.$$

In this model, the moral hazard problem in each period translates into asymmetric information regarding beliefs about the project's quality in all consecutive periods. Before we present a detailed solution to the principal's optimization problem, consider the agent's incentives to deviate at period $t \leqslant T$, assuming that the agent was behaving $1 \leqslant s < t$ without success in all prior periods and will work $t < s \leqslant T$ in all subsequent periods. In case the agent decides to shirk at the beginning of period $t \leqslant T$, his continuation value from the relationship is:

$$U_t(\varpi, (0, 1, ..., 1)) = w_t + (1 - \tilde{\beta}_t) \sum_{s=t+1}^{T} \delta^{s-t}(w_s - c)$$
$$+ \tilde{\beta}_t \sum_{s=t+1}^{T} \delta^{s-t}(1-\lambda)^{s-t-1}(\lambda b_s + (1-\lambda) w_s - c).$$

Note that if the agent follows this one-period deviation, he gets only w_t at period t, since he fails for sure. If the project is good, which, based on history, is true with current beliefs $\tilde{\beta}_t$, then the agent has a chance to succeed in all future periods $s > t$ until the relationship is terminated. If the project is bad, which is true with probability $1 - \tilde{\beta}_t$, the agent will receive w_s in all future periods $s > t$ despite exerting effort.

In contrast, if the agent decides to work at period t, his continuation value from the relationship is:

$$U_t(\varpi, (1, 1, ..., 1)) = -c + \lambda\tilde{\beta}_t b_t + (1 - \lambda\tilde{\beta}_t)w_t + (1 - \tilde{\beta}_t)\textstyle\sum_{s=t+1}^{T} \delta^{s-t}(w_s - c)$$
$$+\tilde{\beta}_t \textstyle\sum_{s=t+1}^{T} \delta^{s-t}(1 - \lambda)^{s-t}(\lambda b_s + (1 - \lambda)w_s - c).$$

Notice that at period t, the agent has a chance to succeed and receive b_t. This occurs either if the project is good or with probability $\lambda\tilde{\beta}_t$. In case the agent is unlucky, with probability $1 - \lambda\tilde{\beta}_t$, he gets w_t, despite exerting high effort. As in the case where the agent deviates, if the project is bad, the agent will receive w_s for all future periods.

When the agent deviates at period t, he knows that failure at this period should not change beliefs and make parties more pessimistic regarding the project's quality. However, if this deviation is not observed by the principal, she will consider a failure reported at period t as a signal that the project is more likely to be bad. Importantly, this difference in beliefs reverberates into all future periods. Thus, in this model, the moral hazard problem in each period translates into asymmetrical information regarding beliefs about the project's quality in all consecutive periods.

Combining the two continuation values, the moral hazard constraint at period t (assuming that the agent was behaving in all prior periods $s < t$ and will work in all subsequent periods $s > t$) becomes the following:

$$(MH_t)\ b_t - w_t \geq \frac{c}{\lambda\tilde{\beta}_t} + \textstyle\sum_{s=t+1}^{T} \delta^{s-t}(1 - \lambda)^{s-t-1}(\lambda b_s + (1 - \lambda)w_s - c).$$

When the agent chooses his effort level privately, he receives a strictly positive rent. The agent could shirk and report that the project failed. The principal can motivate the agent to exert effort by paying a higher reward for success and a lower one for failure. The gap between these payments must be wide enough for the agent to believe it is in his best interest to exert all his efforts after taking into account current beliefs of the project's quality and probability of success. If the agent and principal share the same beliefs about the project's quality, a standard moral hazard problem takes place within each period[11]. Instead, the agent receives a positive moral hazard rent. Since the principal benefits only if the project is released to market, it gains advantage by awarding little to the agent if failure, and, given the limited liability constraint, the agent is paid nothing if he fails overall.

Moreover, if the agent and the principal do not hold common beliefs, the former receives additional reward (the learning rent). If the agent deviates from project goals at one period, his chance to succeed in all future periods remains. Although the agent will not receive anything if he deviates from his duties during a particular period, he becomes relatively more optimistic than the principal for all future periods. That means a deviation at one period carries into all future periods by creating asymmetric beliefs among parties. In some sense, the agent is relatively more patient than the principal in all periods except the last. During this final period, the agent cannot benefit from shirking, since he will not gain from this, and his rent is contingent on the combination of moral hazard and limited liability only. Because of the positive rent the agent receives, the project could be terminated inefficiently early.

Proposition 1.1. The agent receives a positive reward if the project is implemented successfully and nothing otherwise. In particular,

$$w_t = 0 \text{ and } b_t = \frac{c}{\lambda\tilde{\beta}_t} + c\sum_{s=1}^{T-t}\delta^s\frac{1-\beta_0}{\beta_0(1-\lambda)^{t+s-1}} \text{ for } 1 \le t \le T^{SB} \text{ with the following properties:}$$

- if $\delta = 1$, b_t is *constant*[12];
- if $0 < \delta < 1$, b_t is strictly *increasing* whereas $\delta^t b_t$ is strictly *decreasing*.

Moreover, the project is terminated inefficiently early; that is, $T^{SB} \leqslant T^{FB}$.

Proof: See Appendix *A.1*.

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[11] To minimize risk, the principal ideally would sell the project to the agent; however, this is not feasible because the agent is protected by limited liability.

[12] Note that when $\delta = 1$ the optimal contract is unique up to payoff-irrelevant alteration.

Our results show that the agent's nominal reward is weakly increasing while the discounted value of the reward is weakly decreasing in time. For $\delta = 1$, similar reward structure holds in Halac et al. (2016) who argue that in the case of no discounting, the principal can be restricted to use constant bonus contracts. When the discount factor is less than one, the agent's reward is strictly increasing while the expected reward is strictly decreasing. This resembles the payment scheme in Gerardi and Maestri (2012) when the agent's report matches the true state observed by the principal. In contrast, in Bergemann and Hege (1998), the agent's reward for earlier success can rise or fall and even becomes strictly decreasing for high enough discount factor.

An immediate and perhaps fascinating conclusion from Proposition 1.1 is that the agent's private observability of success does not exacerbate the problem; that is, the agent will never postpone an announcement of success even if the *IC* constraint was not taken into account directly when solving the principal's optimization problem. The reason is that the optimal contract makes the value of the discounted reward strictly decreasing. This result plays a key role in our analysis, as we will demonstrate that private observability of success becomes critical when it comes to optimal monitoring timing.

To demonstrate the intuition behind Proposition 1.1, we clarify the dynamics of the moral hazard and learning rents. The first component is always increasing, since the agent becomes more pessimistic as time proceeds without success, and his motivation to exert himself becomes costlier. The second component, however, is non-monotonic and depends on the discount factor. Consider a case in which both the agent and the principal are patient (the discount factor equals one). Under these circumstances, the principal can wait for success indefinitely. Without loss of generality, the principal can offer a contract with constant nominal reward and a deterministic deadline that will ensure the agent will exert effort in every period. Since the moral hazard rent is increasing strictly and the nominal reward is constant, the learning rent decreases strictly. If the principal is patient, the agent benefits less from deviating from project goals since the fixed deadline gives him a smaller horizon to benefit from asymmetric beliefs.

However, if parties to a contract are impatient (the discount factor is less than one), the learning rent becomes non-monotonic. Since $c\sum_{s=1}^{T-t}\delta^s\frac{1-\beta_0}{\beta_0(1-\lambda)^{t+s-1}} = \frac{c\delta(1-\beta_0)}{\beta_0(1-\lambda-\delta)}\frac{(1-\lambda)^{T-t}-\delta^{T-t}}{(1-\lambda)^{T-1}}$, the learning rent is increasing at period $1 \leqslant t \leqslant T$ if and only if

$$\text{either } \delta < 1-\lambda \text{ and } (\tfrac{1-\lambda}{\delta})^{T-t}\ln\delta > \ln(1-\lambda) \text{ or}$$
$$\delta > 1-\lambda \text{ and } (\tfrac{1-\lambda}{\delta})^{T-t}\ln\delta < \ln(1-\lambda).$$

This means that except for the final period, if the agent shirks and does not incur the cost-of-effort, he will have an additional attempt to successfully implement the project and receive a reward. This weakens the agent's incentives to work during each period. The principal, however, benefits if the project is successfully implemented only, and it cannot wait indefinitely, as later success are discounted. This allows the agent to be relatively more patient than the principal. By this logic, the learning rent is increasing. However, the later the agent deviates, the fewer periods remain to exploit the difference in beliefs and, as a result, the learning rent eventually decreases before vanishing completely during the final period. Since the agent has more incentives to deviate at the beginning of the relationship, the optimal contract makes the discounted reward strictly decreasing.

The agent's incentive to announce success at a certain period of the relationship is affected by the payment tied to success or failure in this particular period and, in addition, by payments in all subsequent periods of the relationship. For instance, if the current value of the reward is smaller than the discounted value of the promised future reward, the agent will postpone an announcement of success. We show that the optimal contract makes the discounted reward strictly decreasing; this feature prevents private observability of success from exacerbating the contracting environment.

We present a particular example to better demonstrate the decomposition of the two rents in Figure 1.3 below. Suppose $\lambda = 0.15$, $\beta_0 = 0.7$, $\delta = 1$ and $c = 1$. Note that when there is no discounting, the nominal value of b_t is constant, whereas the discounted value of the reward is decreasing, as suggested by Proposition 1.1. In this case, $(\frac{1-\lambda}{\delta})^{T-t}\ln(\delta) > \ln(1-\lambda)$ and $\delta > 1-\lambda$, and the learning rent is decreasing for all periods.

Now consider an example with discounting, as depicted in Figure 1.4. Suppose that $\lambda = 0.15$, $\beta_0 = 0.7$, $\delta = 0.9$ and $c = 1$. In this case, $\delta > 1-\lambda$ and $(\frac{1-\lambda}{\delta})^{T-t}\ln(\delta) < \ln(1-\lambda)$ for $t \leqslant 12$, and the learning rent is increasing for these periods and decreasing thereafter.

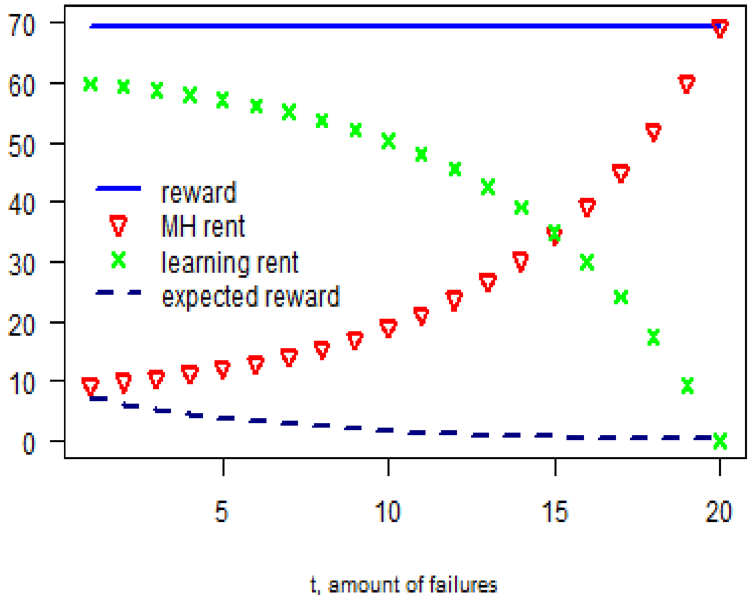

Figure 1.3: The Optimal Contract with $\delta = 1$

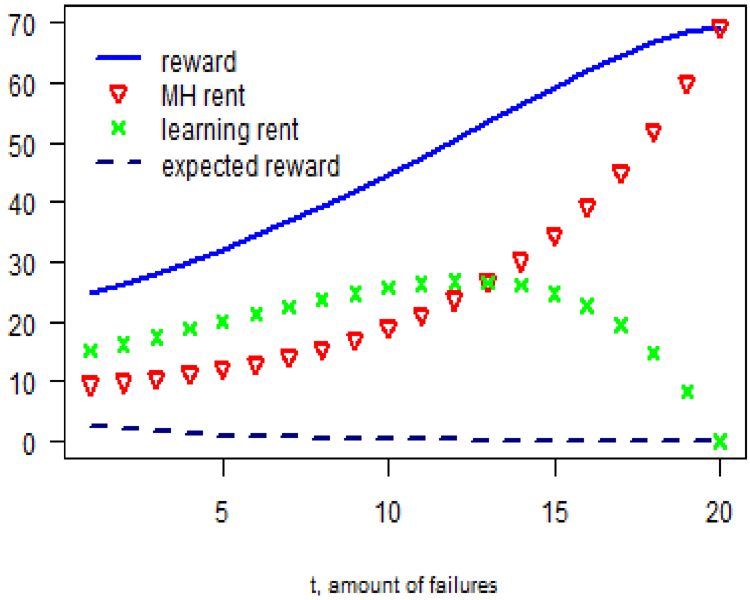

Figure 1.4: The Optimal Contract with $\delta = 0.9$

As in the previous case, the moral hazard rent is increasing strictly to account for the agent's increasing pessimism.

1.2.1 Monitoring

Given the optimal contract described in the previous section, the agent receives a strictly positive rent, and the project consequently is terminated inefficiently early. One way the principal can alleviate this inefficiency is by hiring a monitor who assumedly can observe the effort level the agent chooses perfectly. In reality, monitoring is widely used in contracting for experimentation. For example, when running clinical trials, the pharmaceutical company hires an independent data safety monitoring board (DSMB) consisting of experts in the relevant clinical discipline. The DSMB members schedule several meetings as the trials proceed and advise on the conduct of the trial and the integrity of the data. They also evaluate interim analyses and judge efficacy and net clinical effects.

The principal values monitoring because it mitigates the rent paid to the agent and allows the relationship to be extended. For simplicity we assume that monitoring allows a perfect assessment of the agent's effort; however, our results could be extended easily to account for noisy monitoring. In addition, we assume that monitoring costs $\gamma > 0$ per period, the salary of the monitor.

The benefit of hiring a monitor is that for the period the monitor is hired, the principal can promise to pay less since the moral hazard problem is alleviated, inducing the static effect. Recall that without monitoring, the agent is rewarded more for earlier success and, consequently, the expected reward strictly decreases in the optimal contract. This influences the static effect, which strictly decreases in time. In addition, the dynamic effect from monitoring emerges, which reduces the learning rent the agent receives in all periods prior to monitoring. The prospect of future monitoring acts as a threat and makes the agent less likely to shirk in the earlier periods since the benefit of doing so is smaller. As demonstrated previously, the learning rent is non-monotonic, which makes the dynamic effect non-monotonic, as well.

Thus, the optimal time for monitoring is governed by the sum of the two aforementioned effects. We demonstrate that the dominating effect depends on whether the agent observes success privately or not. This is an important result, since without monitoring, private observability does not play any role in the optimal contract. However, it is crucial in determining the optimal monitoring timing.

1.2.2 Success is publicly observed

We begin the analysis of the optimal timing of monitoring with an example, where the relationship lasts exogenously for two periods ($T = 2$), and the principal can perfectly observe the effort level during one period only when success is publicly observed. The principal's optimization problem (assuming it is optimal when the agent exerts effort in every period) is:

$$\max_{\varpi} \pi(\varpi, \vec{1}) \text{ subject to}$$

(MH) $$\vec{1} \in \arg\max_{\vec{e}} U(\varpi, \vec{e}),$$

(LL) $$b_1,\, b_2,\, w_1,\, w_2 \geqslant 0.$$

First, suppose the principal chooses to monitor the agent at the beginning of the relationship. In this case, the *MH* constraint could be replaced by:

$$(MH_2)\ \lambda\tilde{\beta} b_2 + (1-\lambda\tilde{\beta})w_2 - c \geqslant w_2,$$

which ensures that the agent behaves at $t = 2$, given that success has not been achieved at period $t = 1$. Then, a solution to the optimization problem with monitoring involves $b_2 = \frac{c}{\lambda\tilde{\beta}}$ and $w_2 = 0$, where $\tilde{\beta} = \dfrac{\beta_0(1-\lambda)}{\beta_0(1-\lambda)+1-\beta_0}$. The principal's expected profit in this case is the following:

$$\begin{aligned}\pi_{m=1} &= \beta_0 \textstyle\sum_{t=1}^{2} \delta^t (1-\lambda)^{t-1}\lambda V - \delta^2\beta_0(1-\lambda)\lambda b_2 - \gamma \\ &= \beta_0 \textstyle\sum_{t=1}^{2} \delta^t (1-\lambda)^{t-1}\lambda V - \delta^2 c(1-\lambda\beta_0) - \gamma.\end{aligned}$$

Second, suppose the principal hires the monitor at $t = 2$. In this case, the MH constraint could be replaced by:

$$(MH_1)\ \lambda\beta_0 b_1 + (1-\lambda\beta_0)w_1 - c \geqslant w_1,$$

which ensures that the agent behaves at $t = 1$, given that he will exert effort at $t = 2$. Then, the solution to the optimization problem involves $w_1 = b_2 = w_2 = 0$ and $b_1 = \frac{c}{\lambda\beta_0}$. The principal's expected profit is the following:

$$\begin{aligned}\pi_{m=2} &= \beta_0 \textstyle\sum_{t=1}^{2} \delta^t (1-\lambda)^{t-1}\lambda V - \delta\beta_0\lambda b_1 - \gamma \\ &= \beta_0 \textstyle\sum_{t=1}^{2} \delta^t (1-\lambda)^{t-1}\lambda V - \delta c - \gamma.\end{aligned}$$

Since $-\delta^2 c(1-\lambda\beta_0) > -\delta c$, it is optimal to monitor at $t = 1$. The intuition is that if monitoring occurs at $t = 1$, the principal expects to pay a reward at $t = 2$, conditional on the agent failing at $t = 1$, despite exerting effort as reflected by $(1-\lambda\beta_0)$ in the principal's expected profit $\pi_{m=1}$. The example above is straightforward but does not capture the main intuition fully, as when the relationship lasts for two periods and the monitor is hired, the agent does not receive any learning rent. We will demonstrate, however, that the result of this example extends to a general setting where the duration of the contract is long enough that the agent is granted a strictly positive learning rent.

Suppose now the principal can monitor the agent perfectly at any period $m \leqslant T^M_{Public}$, where T^M_{Public} is the duration of the contract with monitoring when success is publicly observed. We would like to understand all the benefits from monitoring in this case. First, the principal can avoid paying b_m since the moral hazard problem at period m vanishes. This static effect is at the heart of our analysis, as it will be playing an important role when success is observed privately. Since the static effect alleviates the moral hazard problem at the period of monitoring, the principal benefits from it only if the agent succeeds at period m, which in turn is possible only if the project is good. Thus, the static effect is:

$$SE_m = \delta^m Prob(\text{success at } m) b_m = \delta^m \beta_0 (1-\lambda)^{m-1}\lambda b_m.$$

Second, recall from the MH_m constraint that if the principal decreases a reward for success, b_m, he can scale down all the rewards in all the preceding periods, $1 \leqslant s < m$. This effect, which we call the dynamic effect, will be shown to play an auxiliary role in the environment we consider. Importantly, unlike with the static effect, the principal benefits from the dynamic effect even if the agent does not succeed at some period m. Since the agent always, except for the final period, has a chance to succeed in the later periods, the future rewards make it costlier to ensure the agent behaves at the beginning of the relationship. Intuitively, the promise of future monitoring echoes into the earlier periods, as it acts as a threat and it changes the agent's options if he decides to shirk in the earlier periods.

The dynamic effect is defined as:

$$DE_m = \sum_{t=1}^{m-1} \delta^t Prob(\text{success at } t < m)[\text{nominal decrease in } b_t].$$

What is a nominal decrease in b_t for $t < m$ that is possible because of monitoring that will occur at period m? Under the optimal contract all the MH_t constraints are binding:

$$(MH_t)\ b_t - w_t = \frac{c}{\lambda\tilde{\beta}_t} + \sum_{s=t+1}^{T} \delta^{s-t}(1-\lambda)^{s-t-1}(\lambda b_s + (1-\lambda)w_s - c),$$

and, given that the agent receives nothing if the project does not succeed, we have

$$b_t = \frac{c}{\lambda\tilde{\beta}_t} + \sum_{s=t+1}^{T} \delta^{s-t}(1-\lambda)^{s-t-1}(\lambda b_s - c).$$

Consequently, a nominal decrease in b_t for $t < m$ due to monitoring at period $1 \leqslant m \leqslant T$ is:

$$\delta^{m-t}(1-\lambda)^{m-t-1}\lambda b_m.$$

Finally, the dynamic effect becomes:

$$DE_m = \delta^m \beta_0 \lambda^2 (1-\lambda)^{m-2}(m-1)b_m.$$

Thus, the *total effect* of monitoring at period m, $TE_m = DE_m + SE_m$, combines the benefit of paying less at period m, which is decreasing in time, and the benefit of scaling down all the rewards in previous periods, which is non-monotonic. It turns out that the former effect is dominant, as stated in Proposition 1.2.

To understand how monitoring changes rewards consider a numerical example. Suppose $\lambda = 0.15$, $\beta_0 = 0.7$, $\delta = 0.9$ and $c = 1$. Assume monitoring occurs at period $m = 10$. Since a nominal decrease in b_t for $t < m$ due to monitoring is $\delta^{m-t}(1-\lambda)^{m-t-1}\lambda b_m$, the principal now makes rewards for success smaller for periods $t = 1, ..., 9$, as reflected in Figure 1.5 below.

Proposition 1.2. When success cannot be hidden, monitoring is optimal at the beginning of the relationship. Moreover, the project is terminated inefficiently early:

$$T^{SB} \leq T^{M}_{Public} \leq T^{FB}.$$

Proof: See Appendix *A.2*.

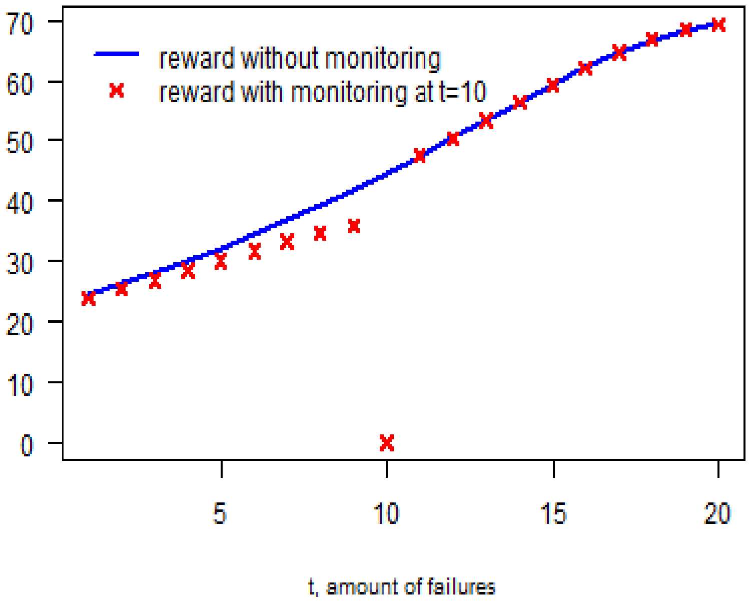

Figure 1.5: Rewards with Monitoring at $t = 10$ when Success is Public

Why does the static effect dominate when success is publicly observed? First, the static effect eradicates the moral hazard problem at the period of monitoring, and this mitigates the moral hazard rent,[13] which was shown to be strictly increasing. What does the dynamic effect accomplish? It mitigates the learning rent, as it makes shirking a less attractive option for the agent. The dynamic effect increases only during earlier periods: Monitoring influences these periods, but as parties to a contract become increasingly pessimistic, the dynamic effect decreases. Under the optimal contract, however, the payment structure optimally mitigates the learning problem for every period of the relationship, not just those in which the dynamic effect is increasing, as this induces the agent to exert effort throughout the length of the relationship. That is why when success is publicly observed, the static effect dominates.

We illustrate this section with a numerical example. Suppose $\lambda = 0.15$, $\beta_0 = 0.7$, $\delta = 0.9$ and $c = 1$. The static effect, $SE_t = \delta^t \beta_0 (1-\lambda)^{t-1} \lambda b_t$, is strictly decreasing, as reflected in Figure 1.6 below. The dynamic effect, $DE_t = \delta^t \beta_0 \lambda^2 (1-\lambda)^{t-2} (t-1) b_t$,

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[13] The principal benefits from the static effect only if the agent succeeds at the period when he is monitored, which in turn is possible only if the project is good.

is non-monotonic. For early periods $t \leqslant 6$, the agent has to be paid more to behave, since if he deviates once, he can leverage the fact that he is relatively more optimistic until the deadline. However, as time goes by without success, both parties become more pessimistic, and since the expected value of b_t goes down, the dynamic effect diminishes, as well.

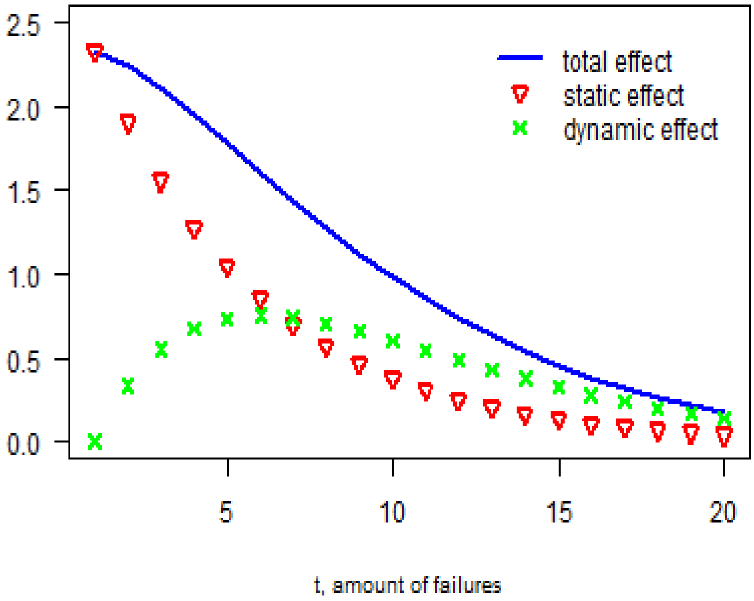

Figure 1.6: Effect from Monitoring when Success is Public $\delta = 0.9$

1.2.3 Success is privately observed

Suppose the agent can postpone an announcement of a successful implementation. We will denote $T^M_{Private}$ as the duration of the contract with monitoring when success is observed privately. The principal can still benefit from hiring a monitor; however, now the modified reward structure must ensure the agent does not have incentives to postpone or hide success. This is where the additional *IC* constraints:

(IC)
$$b_t \geqslant w_t + \delta b_{t+1} \text{ for } t = 1, ..., T-1,$$

$$b_t \geqslant w_t \text{ for } t = 1, ..., T,$$

become relevant and, as we will demonstrate, will be binding for the period when monitoring is implemented.

How can the principal benefit from monitoring when the agent observes success privately? First, as with public observability, the principal can pay less during the monitoring period because the moral hazard problem at period m vanishes. The static effect, however, is different. The reason is that when the agent announces success he takes into account not only the payment tied to success in this precise period but also payments tied to success in all future periods: in case the discounted value of the promised reward in the very next period exceeds the current reward, then the agent will postpone an announcement. For example, if the principal sets $b_m = 0$, then in the case the agent succeeds at this exact period, he will postpone an announcement until the later period as then he gets a positive reward. Given that the optimal contract without monitoring exhibits a decreasing discounted reward value, the principal can decrease the reward in one period at most up to the discounted value of the reward in the following period only. As the discount factor increases, the static effect becomes smaller for all periods except the very last one. Thus, the static effect now has to be modified and becomes: $SE_m = \delta^m \beta_0 (1-\lambda)^{m-1} \lambda (b_m - \delta b_{m+1})$, where $b_{T^M_{Private}+1} = 0$.

In addition, the dynamic effect, has to be modified to take into account the *IC* constraint, as well.

As in the previous case, we will first consider an example where the relationship lasts for two periods ($T = 2$), and the principal can perfectly observe the effort level during one period only when the agent privately observes success. The principal's optimization problem is:

$$\max_{\varpi} \pi(\varpi, \vec{1}) \text{ subject to}$$

(MH) $$\vec{1} \in \arg\max_{\vec{e}} U(\varpi, \vec{e}),$$

(IC) $$b_1 \geqslant w_1 + \delta b_2,$$

$$b_1 \geqslant w_1,$$

$$b_2 \geqslant w_2,$$

(LL) $$b_1,\ b_2,\ w_1,\ w_2 \geqslant 0.$$

First, suppose the principal monitors the agent at $t = 1$. In this case, the *MH* constraint could be replaced by:

$$(MH_2) \qquad \lambda\tilde{\beta}b_2 + (1-\lambda\tilde{\beta})w_2 - c \geqslant w_2,$$

$$\textbf{(IC)} \qquad b_1 \geqslant w_1 + \delta b_2,$$

that ensures that the agent behaves at $t = 2$, given that success has not been achieved at period $t = 1$ and, in addition, that he does not postpone an announcement of success from the first period. Then, a solution to the optimization problem involves $w_1 = w_2 = 0$, $b_2 = \frac{c}{\lambda\tilde{\beta}}$ and $b_1 = \delta b_2 = \delta\frac{c}{\lambda\tilde{\beta}}$ where $\tilde{\beta} = \dfrac{\beta_0(1-\lambda)}{\beta_0(1-\lambda)+1-\beta_0}$. The principal's expected profit in this case becomes:

$$\begin{aligned}\pi_{m=1} &= \beta_0\textstyle\sum_{t=1}^{2}\delta^t(1-\lambda)^{t-1}\lambda V - \delta^2 c(1-\lambda\beta_0) - \delta^2\lambda\beta_0\frac{c}{\lambda\tilde{\beta}} - \gamma\\ &= \beta_0\textstyle\sum_{t=1}^{2}\delta^t(1-\lambda)^{t-1}\lambda V - \delta^2 c(1-\lambda\beta_0+\frac{\beta_0}{\tilde{\beta}}) - \gamma.\end{aligned}$$

Second, suppose the principal hires a monitor at the second period. In this case, the MH constraint could be replaced solely by:

$$(MH_1)\ \lambda\beta_0 b_1 + (1-\lambda\beta_0)w_1 - c \geqslant w_1,$$

which ensures that the agent behaves at $t = 1$, given that he will exert effort at $t = 2$. Note that in this case, the principal does not have to pay anything at the final period, since the agent cannot benefit from hiding his early success.[14] The solution to the optimization problem involves $w_2 = 0$ and $b_1 = \frac{c}{\lambda\beta_0}$. The principal's expected profit is the following:

$$\pi_{m=2} = \beta_0\textstyle\sum_{t=1}^{2}\delta^t(1-\lambda)^{t-1}\lambda V - \delta\beta_0\lambda b_1 - \gamma = \beta_0\sum_{t=1}^{2}\delta^t(1-\lambda)^{t-1}\lambda V - \delta c - \gamma.$$

It is optimal to monitor at $t = 2$ if $\delta^2 c(1-\lambda\beta_0+\frac{\beta_0}{\tilde{\beta}}) > \delta c$ or, equivalently, when:

$$\delta > \tfrac{1-\lambda}{(2-\lambda)(1-\lambda\beta_0)},$$

whereas if $\delta < \frac{1-\lambda}{(2-\lambda)(1-\lambda\beta_0)}$, monitoring is performed optimally at the beginning of the relationship. The intuition is straightforward: If monitoring occurs at $t = 1$, the principal has to pay a reward to ensure the agent does not postpone announcing success, whereas if monitoring occurs at the final period, no positive reward is needed.

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[14] We assume that if the agent is indifferent between announcing success and postponing this announcement, he would choose the former always.

We will demonstrate that when the agent observes success privately, monitoring at the end of the relationship is always optimal when the discount factor is large enough. First, we showed that the dynamic effect is proportional to the change in the reward promised for success during the monitoring period. This is because with private observability, the reduction in the learning rent is smaller and decreases further as the discount increases. Since the agent cannot benefit from hiding success at the final period, the principal can fully eliminate the moral hazard problem in this period. As the discount factor increases enough, the benefit of a smaller reward at the final period increases due to the static effect. We summarize results in Proposition 1.3.

To illustrate how monitoring changes rewards when success is private consider a numerical example. Suppose $\lambda = 0.15$, $\beta_0 = 0.7$, $\delta = 0.86$ and $c = 1$. Assume monitoring occurs at period $m = 6$: the principal now makes rewards for success smaller for periods $t = 1, ..., 6$, as reflected in Figure 1.7 below.

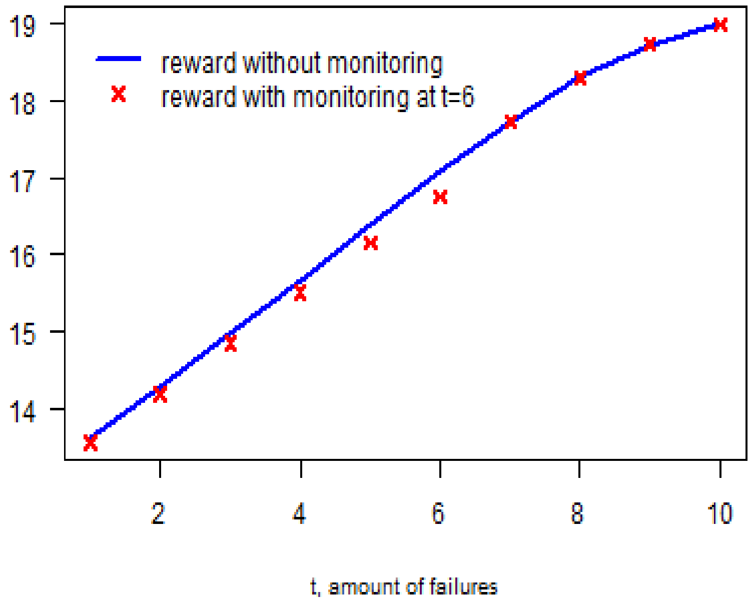

Figure 1.7: Rewards with Monitoring at $t = 6$ when Success is Private with $\delta = 0.86$

As can be seen, the benefit from monitoring are decreasing in the discount factor. For example, when $\delta = 1$ all the *IC* constraints are binding and monitoring at $t = 6$ becomes completely ineffectual, as reflected in Figure 1.8 below.

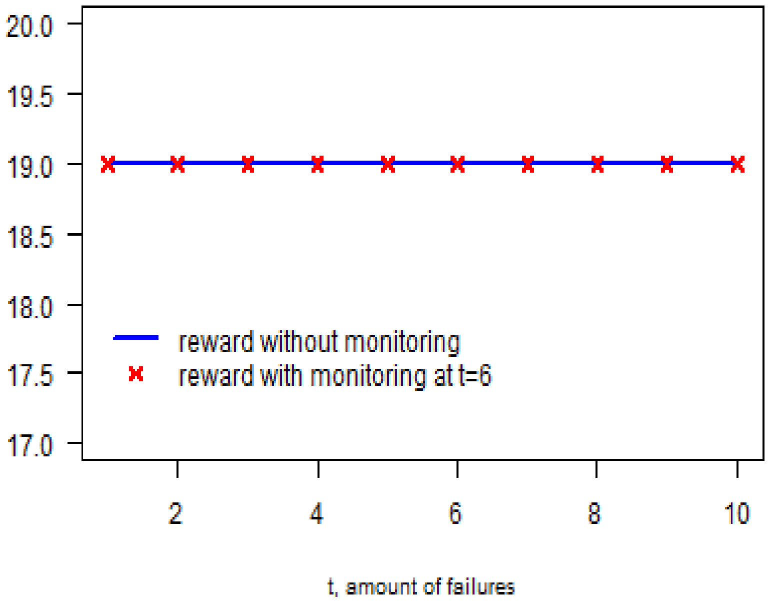

Figure 1.8: Rewards with Monitoring at $t = 6$ when Success is Private with $\delta = 1$

Since now rewards for periods before monitoring are smaller a question arises whether the agent will postpone announcement of success during later periods. Importantly, a nominal decrease in rewards due to the dynamic effect is strictly increasing in time; the agent cannot benefit from postponing early success and announcing it at some period before he is monitored. However, the question remains if the agent will announce success at some period during or after monitoring occurs. We will show that with the modified reward structure the agent will not postpone announcement of success during any later period of the contract.

Proposition 1.3. When the agent observes success privately the optimal time for monitoring is affected by patience. If the discount factor is high enough, monitoring is used optimally at the end of the relationship. The project is terminated inefficiently early:

$$T^{SB} \leqslant T^{M}_{Private} \leqslant T^{M}_{Public} \leqslant T^{FB}.$$

Proof: See Appendix *A.3.*

Consider an example in Figure 1.9 where the discount factor is not high enough for the monitoring to occur optimally at the end of the relationship.

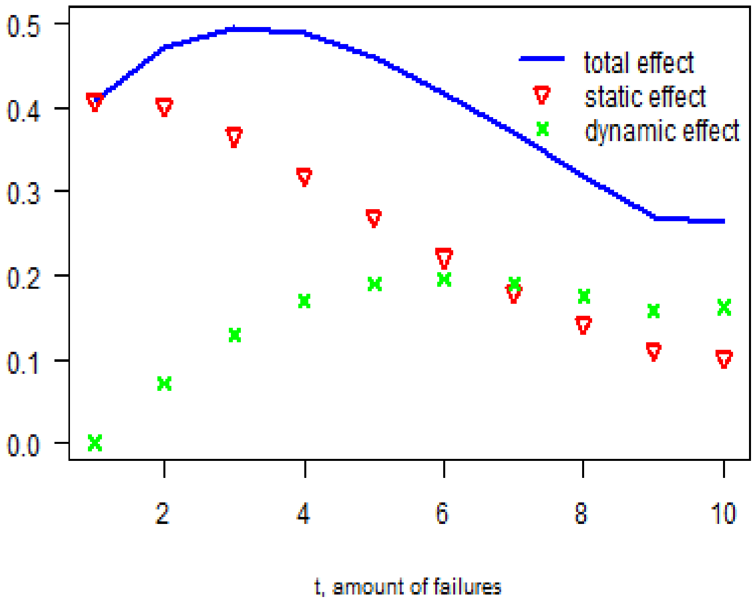

Figure 1.9: Effect from Monitoring when Success is Private with $\delta = 0.86$

When $\delta = 0.86$, monitoring is employed optimally at the third period. However, if we increase the discount factor up to $\delta = 0.92$, monitoring will be employed optimally toward the end of the relationship. In this case, both the static and dynamic effects are non-monotonic, as Figure 1.10 demonstrates below.

In Proposition 1.3, we proved that when the discount factor is high enough, the monitor is hired optimally at the end of the relationship. Specifically, this implies that when $\delta = 1$, monitoring is optimal at the end of the relationship. However, for a smaller discount factor, optimal timing of monitoring can occur toward the beginning of the relationship. If the discount factor is small enough, monitoring can even be implemented optimally at the start of the relationship.

Throughout the paper we assumed that the monitor is hired for one period only. Nevertheless, our results could be extended easily to examine a case in which it is optimal to hire the monitor for several periods. Consider first the case in which success is publicly observed. Since we proved in Proposition 1.1 that the total effect of monitoring is strictly decreasing, monitoring is optimal in earlier periods of the relationships. This result is intuitive and supported by real-life observations: The typical financing cycle of a start-up

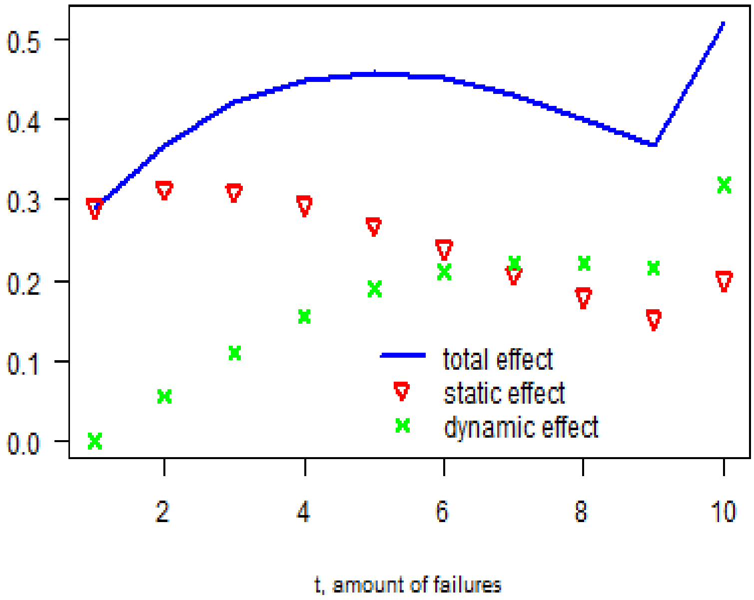

Figure 1.10: Effect from Monitoring when Success is Private with $\delta = 0.92$

firm in its earlier stages include relationship financing, in which the entrepreneur and the investor share common beliefs regarding the quality of the project because, for example, the entrepreneur spends time on-site, which is a form of monitoring. In later periods of a start-up firm's financing, parties to a contract shift to an arm's-length relationship, and the investor commits to halt funding if the project is not successfully implemented by a specific deadline.

When success is privately observed and the monitor is hired for several periods, the periods of monitoring might be not adjacent; that is, monitoring may occur during some periods, suspended for several periods, then reintroduced during later periods.

The results of our research have empirical implications. In the case that hiding success from the principal is prohibitively costly, if the agent formed the original idea for the project, and investors are competing to fund the project, monitoring should be performed at the end of the relationship. However, if the agent is hired by the owner of the project, monitoring is employed optimally at the beginning of the relationship. If success is enormously costly to observe, monitoring optimally is performed toward the end of the relationship if parties to the contract are patient enough. Thus, we emphasize the pivotal role of private observability and market structure on the optimal monitoring timing.

1.3 Conclusion

This paper examines optimal contracts and the role of monitoring for experimentation in settings with moral hazard and private observability of success. In the benchmark case without monitoring, we have found that the agent is rewarded only if he succeeds; otherwise, he receives nothing. The nominal value of the payment is strictly increasing to account for increasing pessimism, whereas the discounted values of the optimal rewards decrease over time. Thus, the agent's private observation of success is irrelevant since he will never postpone an announcement of success.

Nevertheless, we demonstrate that private observability factors into the optimal time for monitoring. When success is impossible to hide from the principal, monitoring at the beginning of the relationship improves the efficiency of financial contracting. This contrasts Bergemann and Hege (1998) results, which demonstrated that monitoring is optimal toward the end of the project. The authors considered a version of our model in which the agent is the owner of the project and raises funds in a competitive market. In their model, the agent's reward for earlier periods can rise or fall with at most one extremum, whereas in our paper, a nominal value of the reward for success is always increasing.

At the same time, when the agent observes success privately, patience influences the optimal time for monitoring. When the discount factor increases, the immediate effect of monitoring decreases, as the principal must pay a high enough reward to prevent the agent from hiding success. As the discount factor increases enough, the principal must promise almost identical rewards for success at every period except the final one, making monitoring more valuable at the end of the relationship and supporting Bergemann and Hege (1998) result.

Throughout the paper, we assumed that the monitor is honest and does not collude with the agent. An interesting avenue for future research is to study the optimal time for monitoring when the monitor could be colluding with the agent. For example, a pharmaceutical company hires an agent who may considering hiding the results of the clinical trials and reselling them to a rival company. Since the agent's reward is decreasing, this feature could postpone the optimal time for monitoring to later periods.

Chapter 2

EXPERIMENTATION, ENDOGENOUS ASYMMETRY OF INFORMATION, AND OPTIMAL CONTRACTS

2.1 Introduction

Before[1] embarking on a large project, a principal would typically like to get more information about its profitability to determine how much resources to allocate to the project. For instance, when an oil company explores new areas for oil fields, it performs seismic surveys and exploration drills to figure out the amount of oil it can expect from the areas[2]. While the oil company experiments with different potential sites, it also diverts resources and delays the production of oil. This creates a trade-off between experimentation and exploitation which is typical of two-armed bandit problems[3]. The principal is like a gambler trying to decide which arm of different slot machines to pull in a sequence of trials.

An additional complexity arises if the experiments are performed by an agent with private information about the quality of the experimentation. In this case, the experimentation process itself can then create asymmetric information about the profitability of the project. For instance, if the agent has private information about his ability to perform the experiments or if his effort cannot be observed, then he may become better informed than the principal about the profitability of the project. This asymmetry of information has important implications for the optimal decisions in the exploitation phase following experimentation.

Consider the oil field example. Exploration drills will demonstrate the profitability of

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[1] This Chapter is a joint work with Fahad Khalil and Jacques Lawarree.

[2] Other applications are the testing of new drugs, the adoption of new technologies or products, the identification of new investment opportunities, the evaluation of the state of the economy, consumer search, etc.

[3] See Keller et al. (2005).

the oil field. However, if the agent is not very skilled at experimenting (or, alternatively, shirks during the experimentation), a poor result from the exploration well only provides weak evidence of a poor project. However, the principal, unaware of the agent's skill (or his shirking), may become more pessimistic than the agent. A new trade-off appears for the principal. More experiments may provide more information about the profitability of the well (particularly if the experiments are successful) but can also increase asymmetric information when the experiments are not successful. Because of this asymmetry of information, when production ultimately starts, the principal may not allocate the right amount of resources to the exploitation of the field.

In this paper, we present a model where the principal hires an agent to implement a project with uncertain cost: production cost can be high or low. Before production takes place, the principal asks the agent to gather information about the actual production cost. This is the experimentation phase. We assume first that the information gathering takes the form of looking for good news, i.e., low cost of production. When experimentation succeeds, it is publicly revealed that the cost is low. This assumes that the agent cannot hide success, an assumption we relax later in the paper. The agent is privately informed about his ability to perform the experiment. A high-ability agent has a high likelihood of finding low cost if this is indeed the case. Failure to detect low cost during the experimentation phase can create asymmetric information between the principal and the agent when the later has to produce. If a low ability agent claims to have high ability and experimentation fails, the principal is now more pessimistic than the low type agent. The latter knows that he is not very skilled at experimenting, so his failure does not indicate strongly that the project is costly.

A key contribution of our model is to combine experimentation and the scale of production. At the end of the experimentation stage, there is a non-trivial decision regarding the scale of output. This decision will affect the experimentation stage and vice versa. Even if the experimentation stage does not lead to success, a lot of information is still learned, and it could be profitably used during the production stage at the end of the experimentation stage. Our paper highlights a significant trade-off: if the principal asks the agent to experiment longer, there is a greater chance to succeed and fine-tune the size of the project; however, experimentation is costly since it leads to asymmetric information, and production has to be postponed.

The asymmetric information created by experimentation has interesting consequences on the optimal contract, particularly on the length of the experimentation period, on the existence of a rent and the timing of its payment and, finally, on the output. First, we show that length of the experimentation period could be shorter or longer than the first best length while most models of experimentation find under experimentation relative to the first best. The reason is that the driving factor for the rent, and therefore the length of the experimentation phase, is the difference in expected costs between the types after experimentation fails. When the agent lies about his type, his expected cost after failure is different from the principal's expected cost. As a result, the agent's informational rent is positively related to the difference in the expected cost of the two different types. We show that this difference is non-monotonic in time so that the principal may sometimes benefit from increasing the length of the experimentation phase.

Second, we show that it is possible that both types can get a rent and that the rent may be paid after failure of the experimentation phase. Our model displays a standard incentive problem where the low type wants to pretend to be a high type. As we explained before, when the low type pretends to be a high type and experimentation fails, the principal is more pessimistic than the agent and she would overcompensate the lying low type agent in the production phase. To deter lying, the principal must pay a reward or rent to the low type.

Why the high type may command rent is more interesting. The reason is related to the dynamics of the problem restricting the principal's choice of when to offer the rent to the low type. In this model, using time for screening the types turns out to be complex as, even under full information, it is possible to have either the high type or the low type experiment more. Intuitively, there are two main effects that determine the length of the experimentation phase. The first one is that the efficiency in searching should lead to a longer experimentation for each type. The second effect is that the high type's effectiveness should lead to pessimistic beliefs when successive failures occur and an earlier termination of the experimentation phase. In general, it is not possible to say whether the high type or the low type experiments more.

The stochastic structure of the dynamic problem helps to pin down the optimal payments to the two types, and which can also lead to rent for the high type. If a high type pretends to be a low type, he faces a gamble. He will be under-compensated at the production stage if experimentation fails as he is relatively more pessimistic compared to the principal. However, the rent given to the low type can be attractive to the high type. Suppose the principal rewards early success for the low type. Such a scheme may look attractive for the high type who is more likely to succeed early in the experimentation phase. But an alternative scheme, such as rewarding late failure by the low type, may also become attractive for the high type during a long experimentation phase because successive failures convince the high type that the project is high cost and experimentation is likely to fail. We show that the dynamic nature of the learning by two types who learn at different speeds can force the principal to give a rent to both types.

As indicated by the above arguments regarding the relative likelihood of success, we find that the timing of the payments matter. When the principal must pay a rent to the high type, it will be as a reward for early success since the low type is less likely to succeed early. If the principal wants to reward a low type for success, it has to be very late in the experimentation stage.

Remarkably, we also show that it may be optimal for the principal to reward the low type for *failure*. When the optimal length of the experimentation period is short, the low type is more likely to fail than the high type and rewarding failure becomes a useful tool to screen the types. Moreover, even if we relax the assumption that the agent cannot hide success, this result survives. If the agent could hide success, he can guarantee apparent failure in the experimentation phase. In such a case, preventing the agent from hiding success requires additional ex post moral hazard constraints that impose additional costs to the principal, but rewarding failure can still be optimal due to the same argument based on relative likelihood of success and failure.

Third, unless experimentation succeeds, the principal will use the choice of output to screen the two types during the production phase. Since the low type always gets a rent, we expect and find that the output of the high type is distorted downward as in a standard second best contract. However, when the high type also commands a rent, the output of the low type is now distorted *upward*. The reason is that a higher required output from the low type is very costly for the high type when experimentation fails. In such a case, the high type being more pessimistic has higher expected costs than the low type and a higher output is more costly to produce.

Our paper is related to the growing literature on contracting for experimentation following Bergemann and Hege (1998), Bergemann and Hege (2005). Most of that literature has a different focus as it studies a moral hazard model and do not consider adverse selection[4]. There are some recent exceptions in Halac et al. (2016). This latter paper also studies an experimentation phase with adverse selection on the speed of learning. They assume that the agent can shirk during the experimentation phase. As a consequence, the asymmetric learning affects the bonus that needs to be paid to induce the agent to work. They show that, without an additional moral hazard constraint, the first best can be reached. In our model, we impose limited liability instead of a moral hazard constraint. In contrast to Halac et al. (2016), we add an exploitation (production) phase following the experimentation phase, and production takes place under asymmetric information if experimentation fails. Unlike Halac et al. (2016), we find over-experimentation relative to the first best because the difference in expected production cost can decrease in time after a succession of failures.

Our paper is also related to the literature on principal-agent contracts with endogenous information before production. All these papers consider one shot models, where an agent exerts effort that increases the precision of the signal of the state (relevant for a production decision). By modeling effort as experimentation, we introduce a dynamic learning aspect, and especially the possibility of learning with asymmetric speeds. Importantly, in our model, the principal can determine the degree of asymmetric information by choosing the length of the experimentation phase. The dynamic optimization problem also leads to the possibility of incentives for each type pretending to be the other.

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[4] See also Gerardi and Maestri (2012).

2.2 The Model (Learning good news)

A principal hires an agent to implement a project of a variable size. The marginal cost of the project, c, or the state of nature, is initially unknown, but it is common knowledge that the cost can be low, $\underline{c}$, with probability $\beta_0 \in (0,1)$, or high, $\bar{c}$, with probability $1-\beta_0$. Both the principal and agent are risk neutral and have a common discount factor $\delta \in (0,1]$. Before the actual production (or *exploitation stage*) occurs, the agent can gather information regarding the production cost (*experimentation stage*). If during the *experimentation* stage the exact value of the marginal cost has not been observed, the agent will be asked to produce based on the expected cost.[5]

The experimentation stage takes place over a maximum of T periods (determined by the principal) when the agent gathers information about the cost of the project. In the base model, we assume that information gathering takes the form of looking for good news. If the cost is actually low, the agent learns whether the cost is low with probability λ in each period $t \leqslant T$, $t \in N$.[6] If the agent learns that the cost is low (*good news*) in a period t, we will say that the experimentation was successful.[7] The experimentation stage then stops and production takes place based on $c = \underline{c}$. If the agent fails to learn that the cost is low in a period t, we will say that the experimentation failed in that period. Then, experimentation resumes if $t < T$, but both the agent and the principal become more *pessimistic* about the likelihood of the cost being low. We say that experimentation was not successful if experimentation fails all T times.

We assume that the agent is privately informed about his experimentation skill represented by λ. Therefore, the principal faces an adverse selection problem. As we will see next, this implies that the principal and agent may update their beliefs differently during the experimentation stage. With probability ν (resp. $1-\nu$), the agent has high (resp. low) skills in learning the cost, i.e., the agent is high (resp. low) type, $\theta = H$ (resp. $\theta = L$). Thus, we define the learning parameter with the type superscript:

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[5] We assume that the agent will learn the exact cost later but the production decision has to be made in advance.

[6] We assume that in the first-best case it is always optimal to experiment at least once.

[7] We assume that the agent cannot hide the evidence of the cost being low.

$$\lambda^\theta = Pr(\text{type } \theta \text{ learns } c = \underline{c} | c = \underline{c}),$$

where $0 < \lambda^L < \lambda^H < 1$.[8] If experimentation fails in a period, different types of agent form different beliefs about the cost of the project. Denoting by β_t^θ, the updated belief of agent θ that the cost is actually low at the beginning of period t after $t-1$ failures, for $t > 1$, we have $\beta_t^\theta = \dfrac{\beta_{t-1}^\theta(1-\lambda^\theta)}{\beta_{t-1}^\theta(1-\lambda^\theta)+1-\beta_{t-1}^\theta}$, which can be re-written in terms of β_0 as follows:

$$\beta_t^\theta = \frac{\beta_0(1-\lambda^\theta)^{t-1}}{\beta_0(1-\lambda^\theta)^{t-1}+1-\beta_0}.$$

The agent θ's expected cost is then

$$c_t^\theta = \beta_t^\theta \underline{c} + (1-\beta_t^\theta)\overline{c}.$$

After each failure, β_t^θ falls, agent θ becomes more pessimistic about the true cost being low, and the expected cost rises. Two aspects of learning are worth noting. First, as time goes by without learning that the cost is low, the expected cost becomes higher due to Bayesian updating and converges to $\overline{c}$. Second, for the same number of failures during the experimentation stage, they become *relatively more pessimistic* when the agent is a high type than a low type. An example of how the expected cost c_t^θ converges to $\overline{c}$ is presented in Figure 2.1 below.

For future use, we also note that the difference in the expected cost, $\triangle c_t = c_t^H - c_t^L > 0$ for $t \in N$ is a *non-monotonic* function of time, and both c_t^H and c_t^L approach $\overline{c}$ proceeding indefinitely with time without success. Furthermore, there exists a unique time period $t_\triangle$ such that $\triangle c_t$ achieves the highest value at this time period; in particular

$$t_\triangle = \arg\max_{1 \leqslant t \leqslant T} \tfrac{(1-\lambda^H)^t-(1-\lambda^L)^t}{(1-\beta_0+\beta_0(1-\lambda^H)^t)(1-\beta_0+\beta_0(1-\lambda^L)^t)}.$$

Finally, in each period, experimentation costs $\gamma > 0$ and we assume that this cost γ is paid by the principal at the end of each period.

After the experimentation stage ends, production takes place in the production stage. The principal values the project according to a function $V(q)$, where $q > 0$ is the size of the project. The function $V(\cdot)$ is strictly concave, twice differentiable on $(0, +\infty)$,

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[8] If $\lambda^\theta = 1$, the first failure would be a perfect signal regarding the project quality.

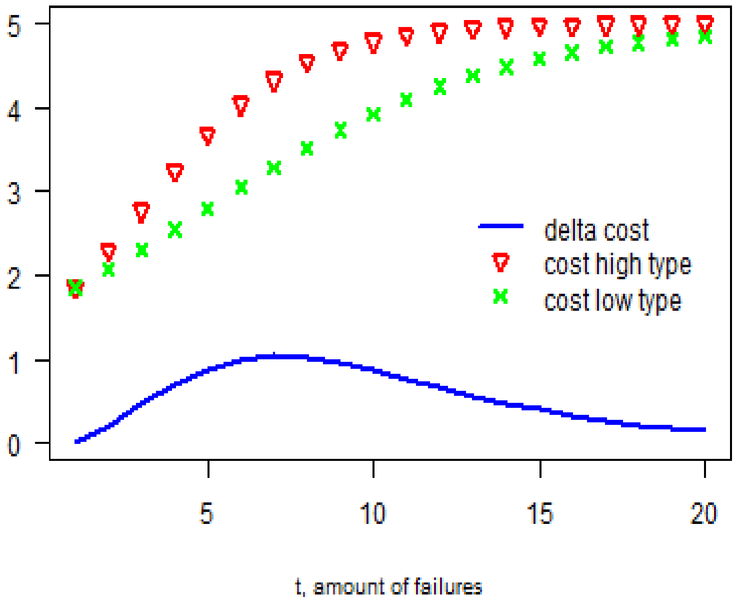

Figure 2.1: The Expected Cost

and satisfies the Inada conditions. The size of the project and the payment to the agent are determined by the contract offered by the principal before the experimentation stage takes place.

Before the experimentation stage takes place, the principal offers the agent a menu of dynamic contracts. A contract specifies, for each type of agent, the length of the experimentation stage, the size of the project, and a transfer as a function of the publicly observable history, which, in this case, is whether or not the agent succeeded while experimenting.

Relying on the revelation principle, we use a direct truthful mechanism, where the agent is asked to announce his type, denoted by $\hat{\theta}$. A contract is defined formally by $\varpi^{\hat{\theta}} = (T^{\hat{\theta}}, \{w_t^{\hat{\theta}}(\underline{c}), q_t^{\hat{\theta}}(\underline{c})\}_{t=1}^{T^{\hat{\theta}}}, \{w^{\hat{\theta}}(c_{T^{\hat{\theta}}}^{\hat{\theta}}), q^{\hat{\theta}}(c_{T^{\hat{\theta}}}^{\hat{\theta}})\})$, where $T^{\hat{\theta}} \in N$ is the maximum duration of the experimentation stage for the announced type $\hat{\theta}$, $w_t^{\hat{\theta}}(\underline{c})$ and $q_t^{\hat{\theta}}(\underline{c})$ are the agent's wage and the output produced if he observed $c = \underline{c}$ in period $t \leqslant T^{\hat{\theta}}$ and $w^{\hat{\theta}}(c_{T^{\hat{\theta}}}^{\hat{\theta}})$ and $q^{\hat{\theta}}(c_{T^{\hat{\theta}}}^{\hat{\theta}})$ are the agent's wage and the output produced if the agent fails $T^{\hat{\theta}}$ consecutive times.

An agent of type θ, announcing his type as $\hat{\theta}$, receives expected utility $U^{\theta}(\varpi^{\hat{\theta}})$ at time zero from a contract $\varpi^{\hat{\theta}}$:

$$\begin{aligned} U^{\theta}(\varpi^{\hat{\theta}}) &= \beta_0 \textstyle\sum_{t=1}^{T^{\hat{\theta}}} \delta^t (1-\lambda^{\theta})^{t-1}\lambda^{\theta}(w_t^{\hat{\theta}}(\underline{c}) - \underline{c}q_t^{\hat{\theta}}(\underline{c})) \\ &+\delta^{T^{\hat{\theta}}}(1-\beta_0+\beta_0(1-\lambda^{\theta})^{T^{\hat{\theta}}})(w^{\hat{\theta}}(c^{\hat{\theta}}_{T^{\hat{\theta}}}) - c^{\theta}_{T^{\hat{\theta}}} q^{\hat{\theta}}(c^{\hat{\theta}}_{T^{\hat{\theta}}})). \end{aligned}$$

Conditional on the actual cost being low, which happens with probability β_0, the probability of succeeding for the first time in period $t \leqslant T^{\hat{\theta}}$ is given by $(1-\lambda^{\theta})^{t-1}\lambda^{\theta}$. If the agent succeeds, he will produce $q_t^{\hat{\theta}}(\underline{c})$ and will be paid $w_t^{\hat{\theta}}(\underline{c})$ by the principal. In addition, it is possible that the agent never observes the low cost ($c = \underline{c}$). This is the case either if the cost is actually high ($c = \overline{c}$), which happens with probability $1-\beta_0$, or, if the agent fails $T^{\hat{\theta}}$ times despite $c = \underline{c}$, which happens with probability $\beta_0(1-\lambda^{\theta})^{T^{\hat{\theta}}}$. In this case, the agent produces $q^{\hat{\theta}}(c^{\hat{\theta}}_{T^{\hat{\theta}}})$ and will be paid $w^{\hat{\theta}}(c^{\hat{\theta}}_{T^{\hat{\theta}}})$.

The optimal contract will have to satisfy the following incentive compatibility constraint for all θ and $\hat{\theta}$:

(IC)
$$U^{\theta}(\varpi^{\theta}) \geqslant U^{\theta}(\varpi^{\hat{\theta}}).$$

The principal's expected payoff at time zero from a contract ϖ^{θ} offered to the agent of type θ is

$$\begin{aligned} \pi^{\theta}(\varpi^{\theta}) &= \beta_0 \textstyle\sum_{t=1}^{T^{\theta}} \delta^t (1-\lambda^{\theta})^{t-1}\lambda^{\theta}(V(q_t^{\theta}(\underline{c})) - w_t^{\theta}(\underline{c}) - \frac{\sum_{s=1}^{t}\delta^s}{\delta^t}\gamma) \\ &+\delta^{T^{\theta}}(1-\beta_0+\beta_0(1-\lambda^{\theta})^{T^{\theta}})(V(q^{\theta}(c^{\theta}_{T^{\theta}})) - w^{\theta}(c^{\theta}_{T^{\theta}}) - \frac{\sum_{s=1}^{T^{\theta}}\delta^s}{\delta^{T^{\theta}}}\gamma). \end{aligned}$$

Thus, the principal's objective function is:

$$\nu\pi^{H}(\varpi^{H}) + (1-\nu)\pi^{L}(\varpi^{L}).$$

To summarize, the timing is as follows:

1. The agent learns his type θ.
2. The principal offers a contract to the agent. In case the agent rejects the contract, the game is over and both parties get payoffs normalized to zero; if the agent accepts the contract, the game proceeds to the experimentation stage with duration as specified in the contract.
3. The experimentation stage begins.

4. If the agent learns that $c = \underline{c}$, the experimentation stage stops and the production stage starts with output and transfers as specified in the contract.

   In case no success is observed during the experimentation stage, the production stage starts with output and transfers as specified in the contract.

2.2.1 The First Best Benchmark

Suppose the agent's type θ is common knowledge before the principal offers the contract. The first-best solution is found by maximizing the principal's profit such that $U^\theta(\varpi^\theta) \geqslant 0$. Since the expected cost is rising as long as success is not obtained, the first-best solution is characterized by a termination date T^θ_{FB} such that the agent of type θ is allowed to experiment only up until that date and not any longer:

$$\begin{gathered} T^\theta_{FB} \in \arg\max_{T^\theta}\{\pi^\theta(\varpi^\theta) = \beta_0 \textstyle\sum_{t=1}^{T^\theta} \delta^t (1-\lambda^\theta)^{t-1}\lambda^\theta (V(q^\theta_t(\underline{c})) - w^\theta_t(\underline{c}) - \frac{\sum_{s=1}^{t}\delta^s}{\delta^t}\gamma) \\ +\delta^{T^\theta}(1-\beta_0+\beta_0(1-\lambda^\theta)^{T^\theta})(V(q^\theta(c^\theta_{T^\theta})) - w^\theta(c^\theta_{T^\theta}) - \frac{\sum_{s=1}^{T^\theta}\delta^s}{\delta^{T^\theta}}\gamma)\}. \end{gathered}$$

Note that to simplify the notation we will call P^θ_T, the probability that an agent of type θ does not succeed during the T periods of the experimentation phase:

$$P^\theta_T \equiv 1 - \beta_0 + \beta_0(1-\lambda^\theta)^T.$$

Note that T^θ_{FB} is bounded and it is the highest t^θ such that

$$\begin{gathered} \delta\beta^\theta_{t^\theta}\lambda^\theta[V(q^\theta_{t^\theta}(\underline{c})) - \underline{c}q^\theta_{t^\theta}(\underline{c})] + \delta(1-\beta^\theta_{t^\theta}\lambda^\theta)[V(q^\theta(c^\theta_{t^\theta})) - c^\theta_{t^\theta}q^\theta(c^\theta_{t^\theta})] \geqslant \\ \gamma + [V(q^\theta(c^\theta_{t^\theta-1})) - c^\theta_{t^\theta-1}q^\theta(c^\theta_{t^\theta-1})]. \end{gathered}$$

The intuition is that, by extending the experimentation stage by one additional period, the agent of type θ can learn that $c = \underline{c}$ with probability $\beta^\theta_{t^\theta}\lambda^\theta$. If the agent succeeds, the efficient output will be produced such that $V'(q^\theta_{t^\theta}(\underline{c})) = \underline{c}$ for any t^θ. The transfer covers the actual cost and no rent is given to the agent. In case the agent fails, the efficient output based on the current *expected* cost, such that $V'(q^\theta(c^\theta_{t^\theta})) = c^\theta_{t^\theta}$ for any t^θ.[9] The transfer covers the expected cost and no expected rent is given to the agent. For example,

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[9] Note that our definition of the first-best stopping time slightly differs from that of Halac et al. (2016). In their paper the discount factor δ does not affect the first-best stopping time. In contrast, we are following the traditional tradeoff between "exploration" and "exploitation" from the literature on experimentation.

if $\lambda^L = 0.2$, $\lambda^H = 0.22$, $\underline{c} = 0.5$, $\overline{c} = 20$, $\beta_0 = 0.4$, $\delta = 0.9$, $\gamma = 2$, and $V = 10\sqrt{q}$, then the first-best termination date for the high type agent is $T^H_{FB} = 4$, whereas it is optimal to allow the low type agent to experiment for at most three periods, $T^L_{FB} = 3$.

Note that the first-best termination date of the experimentation stage T^θ_{FB} is a *non-monotonic* function of the agent's type. This non-monotonicity is a result of two countervailing forces. In any given period of the experimentation stage, the high type is more likely to learn $c = \underline{c}$ (conditional on the actual cost being low) since $\lambda^H > \lambda^L$. This suggests that the principal should allow the high type to experiment longer. However, at the same time, the high type agent becomes relatively more pessimistic with repeated failures. This is seen in Figure 2.2 below, where the *conditional* probability of success for the high type becomes smaller than that for the low type at some point. Given these two countervailing forces, the first-best stopping time for the high type agent can be shorter or longer than that of the type L agent depending on the parameters of the problem. For example, if we now change λ^H to 0.4 and β_0 to 0.5 then the low type agent is allowed to experiment longer, that is, $T^H_{FB} = 4 < T^L_{FB} = 7$.

In Figure 2.2, we depict the conditional probability of success, $\beta_0(1 - \lambda^\theta)^{t-1}\lambda^\theta$, for both types. For any parameters these curves cross exactly once as can be seen in the figure. Then, for any $t \leqslant 3$ the high type has a higher conditional probability of success than the low type (conditional on the project being good) since $\lambda^H > \lambda^L$. However, for $t > 3$ the low type is more likely to succeed since the type H agent is now much more pessimistic.

2.2.2 Asymmetric information

Assume now that the agent privately knows this type. To understand the role of beliefs in generating rent in the production phase, we start with a benchmark without an experimentation phase but with asymmetric information about expected cost of production. The principal can only screen the agents with the output and payments. We obtain a standard second best contract, where the hidden parameter is the expected marginal cost. Suppose in this case, a type θ agent's belief is β^θ implies that the expected cost at the production stage is $c^\theta = \beta^\theta \underline{c} + (1 - \beta^\theta)\overline{c}$, with $c^H > c^L$ (since the high type is relatively

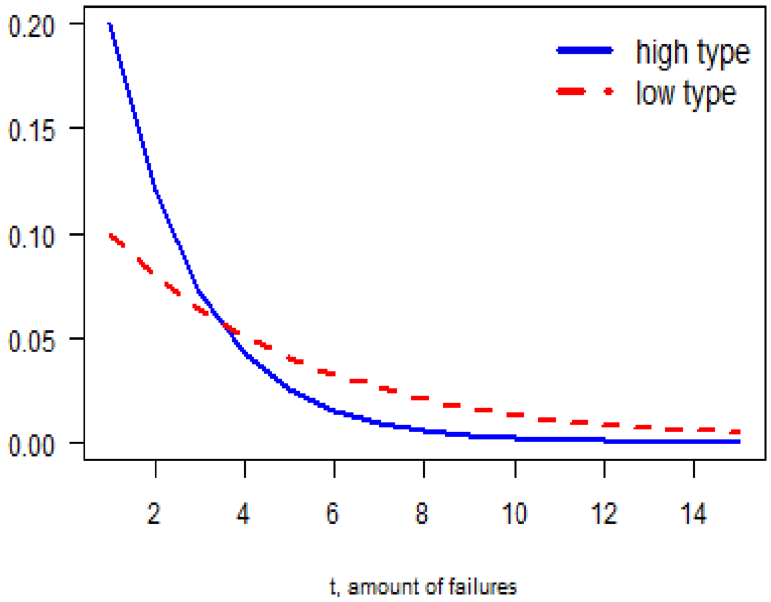

Figure 2.2: Probability of Success with $\beta_0 = 0.5$, $\lambda^H = 0.4$, $\lambda^L = 0.2$

more pessimistic). As a result, the principal's optimization problem now is:

$$\max_{q^H(c^H),w^H(c^H),q^L(c^L),w^L(c^L)} \nu[V(q^H(c^H)) - w^H(c^H)] + (1-\nu)[V(q^L(c^L)) - w^L(c^L)] \text{ s.t.}$$

(IR^H) $$w^H(c^H) - c^H q^H(c^H) \geqslant 0,$$

(IR^L) $$w^L(c^L) - c^L q^L(c^L) \geqslant 0,$$

$(IC^{H,L})$ $$w^H(c^H) - c^H q^H(c^H) \geqslant w^L(c^L) - c^H q^L(c^L),$$

$(IC^{L,H})$ $$w^L(c^L) - c^L q^L(c^L) \geqslant w^H(c^H) - c^L q^H(c^H).$$

It can be easily shown that the optimal contract resembles a standard second-best contract with adverse selection. In particular, (IR^H) and $(IC^{L,H})$ constraints are binding, the low type gets a positive informational rent and produces the first-best output: $V'(q^L(c^L)) = c^L$. The high type gets zero rent and his output is distorted as follows: $V'(q^H(c^H)) = c^H + \frac{(1-\nu)}{\nu}(c^H - c^L)$. As in a standard adverse selection model $q^H_{SB}(c^H) < q^H_{FB}(c^H) < q^L_{FB}(c^L) = q^L_{SB}(c^L)$.

We now return to our main case where an experimentation phase precedes production. Recall that asymmetric information arises in our setting because the two types learn asymmetrically in the experimentation phase, and not because there is any inherent difference in their ability to implement the project. Private information can exist only if experimentation fails. If an agent experiences success before the terminal date, T^θ, the true cost $c = \underline{c}$ is revealed and the agent earns zero rent. Thus, it is only if there is failure during the entire experimentation stage that an agent may benefit from his private information as the principal and the agent could have asymmetric beliefs about expected cost.

In this problem, it is the low type who has an incentive to claim to be a high type: since a high type must be given his expected cost following failure, a low type will have to be given a rent to truthfully report his type as his expected cost is lower, that is, $c^L_{T^H} < c^H_{T^H}$.

Before presenting the (IC) constraints again, we introduce some notation by defining y^θ_t and x^θ as follows:

$$y^\theta_t \equiv w^\theta_t(\underline{c}) - \underline{c}q^\theta_t(\underline{c}) \text{ for } 1 \leqslant t \leqslant T^\theta,$$
$$x^\theta \equiv w^\theta(c^\theta_{T^\theta}) - c^\theta_{T^\theta} q^\theta(c^\theta_{T^\theta}).$$

In other words, y^θ_t is the rent payment to the θ type who succeeds in period t and x^θ is the rent payment to the θ type who failed during the entire experimentation stage. Also recall that $P^\theta_T = 1 - \beta_0 + \beta_0(1-\lambda^\theta)^T$ is the probability that the agent of type θ does not succeed during the T periods of the experimentation phase. The two incentive constraints are given next.

$$(IC^{L,H})\ \beta_0 \textstyle\sum_{t=1}^{T^L} \delta^t (1-\lambda^L)^{t-1}\lambda^L y^L_t + \delta^{T^L} P^L_{T^L} x^L \geqslant$$
$$\beta_0 \textstyle\sum_{t=1}^{T^H} \delta^t (1-\lambda^L)^{t-1}\lambda^L y^H_t + \delta^{T^H} P^L_{T^H}[x^H + \triangle c_{T^H} q^H(c^H_{T^H})],$$
$$(IC^{H,L})\ \beta_0 \textstyle\sum_{t=1}^{T^H} \delta^t (1-\lambda^H)^{t-1}\lambda^H y^H_t + \delta^{T^H} P^H_{T^H} x^H \geqslant$$
$$\beta_0 \textstyle\sum_{t=1}^{T^L} \delta^t (1-\lambda^H)^{t-1}\lambda^H y^L_t + \delta^{T^L} P^H_{T^L}[x^L - \triangle c_{T^L} q^L(c^L_{T^L})].$$

We denote the rent to the low type by ρ_L, which is the right hand side of $(IC^{L,H})$ constraint. We will see later that it is also possible that the $(IC^{H,L})$ becomes binding in this problem. We denote the rent to the high type by ρ_H, which is the right hand side of $(IC^{H,L})$.

Finally, we assume the agent must be paid his expected production costs whether experimentation succeeds or fails. To account for this, we impose conditions that will require that the agent be paid his expected production cost whether experimentation fails or succeeds. We will call them the ex post (i.e., after experimentation) (IR) constraints:

$$(IRS_t^{\theta}) \qquad y_t^{\theta} \geqslant 0 \text{ for } t \leqslant T^{\theta},$$

$$(IRF_{T^{\theta}}^{\theta}) \qquad x^{\theta} \geqslant 0,$$

where the S and F are to denote success and failure.

Note that they imply that the ex ante (IR) is redundant, i.e., we have $U^{\theta}(\varpi^{\theta}) \geqslant 0$. In contrast, if the principal only had to satisfy an ex ante participation constraint $U^{\theta}(\varpi^{\theta}) \geqslant 0$, the principal can use the fact that the high type is relatively more likely to succeed (conditional on $c = \underline{c}$) to screen the agent without distorting the duration of the experimentation stage. In other words, since success during the experimentation stage is a random event that is correlated with the agent's type, we can apply well-known ideas from mechanisms la Cremer-McLean that says the principal can still receive the first best profit. We explain this next in the context of our model.

To implement the first best, the principal has to counter the incentive of the low type to pretend to be a high type. Relative to the first best payments, the principal can change the payments to the high type only. She can increase the payment in case of success and lower it otherwise, while keeping the high type at zero expected rent, his level of utility in the first best contract. This payment scheme will lower the rent of the *low* type who is less likely to succeed in the experimentation phase. By choosing the appropriate transfers, the principal can obtain the first best level of profit. While this scheme ensures a zero expected rent for each type, it also implies a large positive ex post rent in case of success and a large negative ex post rent in case of failure. When such schemes are allowed, the first best can be reached.

We now return to the problem with ex post (IR) constraints, and the optimal contract is derived in Appendix A. The principal has three tools to screen the agent the length of the experimentation period, the timing of the payments and the output for each type. We examine them below.

We already know from the first best contract that the length of the experimentation period is non-monotonic in types. So, in general, we cannot say whether the low type or high type experiments longer. This result extends to the second best and we cannot say, in general, whether T^L is larger or smaller than T^H.

Moreover, each type may under- or over-experiment compared to the first best.[10] The reason is that the driving factor for the rent and therefore the length of the experimentation phase is the difference in expected costs between the types after experimentation fails. When the agent lies about his type, his expected cost after failure is different from principal's expected cost. As a result, the agent's informational rent is positively related to the difference in the expected cost different types have. As we saw earlier in Figure 2.2, the difference in the expected cost, $\triangle c_t = c_t^H - c_t^L > 0$ for $t \in N$, is a *non-monotonic* function of time. Because of that, the principal will sometimes benefit from increasing the length of the experimentation phase.

One important aspect of the length of the experimentation period is whether T^L is larger or smaller than a critical value $\hat{T}^L$. This critical value determines which type is more likely to succeed or fail during the experimentation phase. In any period $t < \hat{T}^L$, the high type, who chooses the contract designed for the low type, is relatively more likely to succeed than fail compared to the low type. For $t > \hat{T}^L$, the opposite is true. This feature plays an important role in structuring the optimal contract. The critical value $\hat{T}^L$ determines whether the principal will choose to reward success or failure in the optimal contract. We now turn to this issue and provide the precise derivation of $\hat{T}^L$.

In the dynamic optimization problem, the principal has to determine the optimal payment plan for each agent, especially when to pay the rent ρ_L to the low type as the $(IC^{L,H})$ is binding. If $(IC^{H,L})$ is not binding, we show in case 1 of Proposition 2.1 that the principal can use any combination of y_t^L and x^L: there is no restriction on how the principal pays ρ_L to the low type as long as $\beta_0 \sum_{t=1}^{T^L} \delta^t (1-\lambda^L)^{t-1} \lambda^L y_t^L + \delta^{T^L} P_{T^L}^L x^L = \delta^{T^H} P_{T^H}^L \triangle c_{T^H} q^H(c_{T^H}^H)$.

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[10] One exception: when $(IC^{L,H})$ is not binding, the length of the experimentation period is identical to the first best for the low type.

If the $(IC^{H,L})$ is binding, the high type has an incentive to claim to be a low type and we are in case 2 of proposition 2.1. This is the more interesting case and it allows us to characterize the stochastic structure of the dynamic problem. In particular, we will explain the derivation of the critical cut-off period $\hat{T}^L$ mentioned above.

We show in Appendix *B* that the principal will not commit to pay rent after success in two different periods. This allows us to simplify the problem as $y_j^L > 0$ for at most one $j = t$. Given this, the principal must decide whether to pay the rent after success in some period t during the experimentation phase $(y_j^L > 0)$ or at the end following failure $(x^L > 0)$.

If she pays the rent after success in some period j, such that $1 \leqslant t \leqslant T^L$, then

$$y_j^L = \frac{\rho_L}{\beta_0 \delta^j (1-\lambda^L)^{j-1} \lambda^L},$$

and $y_t^L = 0$ for $t \neq j$, and $x^L = 0$. If she pays after failure, then

$$x^L = \frac{\rho_L}{\delta^{T^L} P^H_{T^L}},$$

and $y_t^L = 0$ for $t \leqslant T^L$, where $P_t^\theta = 1 - \beta_0 + \beta_0(1-\lambda^\theta)^t$.

A key insight of this paper is that, paying the rent to the low type early or late has different incentive effects as the relative likelihood of reaching different periods is different for each type. In general, the principal will have to pay close attention to when she pays the rent to the low type. A concern is not to encourage the high type from pretending to be low in order to claim the rent ρ_L. We will argue below that misreporting results is a gamble for the high type: he has a chance to obtain the low-type's rent ρ_L, but he will incur an expected loss in the production phase if he fails during the experimentation phase. If this gamble results in a negative expected return, the high type has no incentive to pretend to be the low type. The principal would then pay zero rent to the high type $(y_t^H = x^H = 0)$, while paying ρ_L to the low type using any combination of y_t^L and x^L as long as she does not violate the $(IC^{H,L})$. In sum, if the principal does not have to worry about the rent to the low type encouraging the high type to misreport, it does not matter if the rent is paid after success or after failure. This is case 1 in Proposition 2.1.

To illustrate how the timing of payments matters, suppose that the principal chooses to reward the agent for an early success, say success in period 1 only: $\rho_L = y_1^L > 0$. This payment scheme should look attractive to the high type who is more likely to succeed in period 1 than the low type. This explains why, under certain parameters, the high type incentive constraint ($IC^{H,L}$) may become binding. We call this situation case 2 in Proposition 2.1. The principal then must modify the payment scheme.

A key finding is that the principal will discourage the high type from mimicking the low type by rewarding early failure or late success. To understand the nature of restrictions imposed by dynamic learning by different types at different speeds, we compare the relative incentive effects on the *high* type of rewarding the *low* type after success or after failure. If the principal rewards the low type only after failure, the high type's expected utility from misreporting (i.e., the *RHS* of the ($IC^{H,L}$) constraint) is:

$$U_F^H(\varpi^L) = \delta^{T^L} P^H_{T^L}[w^L(c^L_{T^L}) - c^L_{T^L} q^L(c^L_{T^L}) - \triangle c_{T^L} q^L(c^L_{T^L})] = \\ \frac{\delta^{T^L} P^H_{T^L}}{\delta^{T^L} P^L_{T^L}} \rho_L - \delta^{T^L} P^H_{T^L} \triangle c_{T^L} q^L(c^L_{T^L}).$$

If the principal rewards the low type only after success in some period j, with $1 \leqslant j \leqslant T^L$, the high type's expected utility from misreporting (i.e., the *RHS* of the ($IC^{H,L}$) constraint) is:

$$U_S^H(\varpi^L) = \beta_0 \delta^j (1-\lambda^H)^{j-1} \lambda^H (w^L_j(\underline{c}) - \underline{c} q^L_j(\underline{c})) - \delta^{T^L} P^H_{T^L} \triangle c_{T^L} q^L(c^L_{T^L}) \\ = \frac{\beta_0 \delta^j (1-\lambda^H)^{j-1} \lambda^H}{\beta_0 \delta^j (1-\lambda^L)^{j-1} \lambda^L} \rho_L - \delta^{T^L} P^H_{T^L} \triangle c_{T^L} q^L(c^L_{T^L}).$$

Comparing these two expressions, we can study the nature of the gamble for the high type when he misreports and get some insight on optimal payment schemes. The first term in each expression is the expected gain from obtaining ρ_L by misreporting, and the second is the possibility of incurring a loss if experimentation fails.

The principal will reward the low type after success or failure depending on which yields a lower *RHS* of ($IC^{H,L}$) constraint. If the low type is rewarded for success in period j, then it is important to look at the relative probability of success of a high type in period j:

$$\frac{\beta_0 \delta^j (1-\lambda^H)^{j-1} \lambda^H}{\beta_0 \delta^j (1-\lambda^L)^{j-1} \lambda^L},$$

which is coefficient of ρ_L in $U_S^H(\varpi^L)$. If the low type is rewarded only after failure, then it is important to look at the relative probability of failure conditional on reaching T^L:

$$\frac{\delta^{T^L} P^H_{T^L}}{\delta^{T^L} P^L_{T^L}},$$

which is the coefficient of ρ_L in $U^H_F(\varpi^L)$. The relative probability of success for a high type decreases with j, while the relative probability of failure for a high type is constant given T^L. Thus, there is a j such that the *RHS* of $(IC^{H,L})$ under success or failure equal each other, which is achieved when the two coefficients of ρ_L are equal to each other. Defining $\hat{T}^L (= j)$ by setting the two coefficients equal, we have

$$\frac{(1-\lambda^H)^{\hat{T}^L-1}\lambda^H}{(1-\lambda^L)^{\hat{T}^L-1}\lambda^L} \equiv \frac{\delta^{T^L} P^H_{T^L}}{\delta^{T^L} P^L_{T^L}}.$$

Thus, if the principal wants to reward the low type after success, it will only be optimal if the experimentation stage lasts long enough. If $T^L < \hat{T}^L$, then, $\frac{(1-\lambda^H)^{\hat{T}^L-1}\lambda^H}{(1-\lambda^L)^{\hat{T}^L-1}\lambda^L} > \frac{\delta^{T^L} P^H_{T^L}}{\delta^{T^L} P^L_{T^L}}$ for all j and the high type will have an advantage over the low type in obtaining any reward given after success. To provide the rent ρ_L to the low type, the principal can do no better than relying on rewarding failure. If $T^L > \hat{T}^L$, the principal does not have to reward the low type for failure and can reward success. Indeed, she can pay the reward at T^L as the *RHS* of $(IC^{H,L})$ is smallest.

Again, depending on the value of the parameters, it is possible that such scheme may not be enough to prevent the high type from obtaining a rent when mimicking the low type. We show that the principal will pay the high type's rent by rewarding success in the first period only. Intuitively, this is the period when success is most likely to come from a high type than a low type.

It is unusual to have both types earning a rent and we will provide some intuition for that here using an example. Recall that misreporting for the high type represents a gamble but that is not the case for the low type. So, we focus on why the principal optimally chooses to reward both types rather than give rent only to the low type. To get some intuition, consider the following example. If $V(q) = 3.5\sqrt{q}$, $\beta_0 = 0.7$, $\underline{c} = 0.1$, $\overline{c} = 10$, $\delta = 0.9$, $\gamma = 1$, $\lambda^L = 0.14$, $\lambda^H = 0.35$ then $\hat{T}^L = 5$ and the principal optimally chooses $T^H = 8$, $T^L = 12$ and grant rent only to the low type. However, if we increase only the parameter $\lambda^H = 0.82$, then $\hat{T}^L = 3$ and the principal optimally chooses $T^H = 3$, $T^L = 11$ and grants rent to both types.

Intuitively, when the high type is very efficient in learning the true cost ($\lambda^H = 0.82$), by rewarding the low type for success the principal makes it more likely that the high type will receive this reward. Thus, ideally the principal would like to reward the low type for a sequence of failures. So the principal wants to reward failure for as many periods as possible. However, as T^L increases, the high type is more confident that the project is bad and failure is more likely. Rewarding failure now becomes attractive for the high type. This happens once $T^L > \hat{T}^L$. At that point, the principal no longer reduces the rent by rewarding failure and must switch to reward success from the low type.

Finally, since the experimentation is followed by a production phase, the principal can use the choice of output to screen the types and limit the rent to both types. If the experimentation was successful, there is no asymmetric information and no reason to distort the output. Both types produce the first best output. If the experimentation failed to reveal the cost, asymmetric information will induce the principal to distort the output to limit the rent. This is a familiar result in contract theory. In a standard second best contract a la Baron-Myerson, the type who receives rent produces the first best level of output while the type with no rent under-produces relative to the first best.

When only the low type's incentive constraint binds, the optimal contract requires, as in Baron-Myerson, that the low type produces the first best output while the high type under-produces relative to the first best. When the low type claims to be high, he is asked to produce a lower output and his rent per unit is lower.

The dynamic feature of our model, however, creates a possibility that the high type's incentive constraint also binds. To limit the rent of the high type, the principal will then increase the output of the low type and requires over-production relative to the first best. To understand the intuition behind this result, recall that the rent of the high type mimicking the low type has two components. The first component is the rent promised to the low type after failure in the experimentation phase. The second component is negative and comes from the higher expected cost of producing the output required from the low type $q^L(c^L_{T^L})$. By making this output higher, the principal can strengthen the negative component and lower the rent of the high type.

It is worth noting our very unusual result where the low type over-produces relative to the first best output while the high type under-produces. This is a direct result of our dynamic model where both types can earn rent. We summarize our results in in the following proposition:

Proposition 2.1.

1. In the optimal contract, each type may under-experiment or over-experiment relative to the first best.
2. The low type always receives a rent and $(IC^{L,H})$ always binds.
3. In case 1, when $(IC^{H,L})$ is slack, the principal has no restriction when paying the rent to the low type (whether rewarding success or failure) beside what $(IC^{L,H})$ requires.
4. In case 2, when $(IC^{H,L})$ binds, the principal must reward early failure and late success when paying the rent to the low type. If, in addition, the high type receives a rent, the principal must pay this rent by rewarding success in the very first period.
5. The high type under-produces relative to the first best output. The low type over-produces if the high type receives a rent and produces at the first best level otherwise.

Proof: See Appendix *B*.

2.3 Conclusion

In this paper, we have studied the interaction between exploration and exploitation where the length of the experimentation phase determines the degree of asymmetric information at the production phase. This interaction affects the optimal project scale after success as well as after failure. While success in experimentation typically resolves uncertainty in a two-armed bandit model, a lot of learning still occurs after successive failures, and that also determines the scale of the project, which is novel to the literature. While there has been much recent attention on studying incentives for experimentation in two-armed bandit settings, details of the exploitation phase are typically suppressed to focus on incentives for exploration. In reality, each phase impacts the other in interesting ways and our paper is a step towards studying this interaction.

There is also a significant literature on endogenous information in contract theory but typically relying on static models of learning. By modeling experimentation in a dynamic setting, we have endogenized the degree of asymmetry of information in a principal agent model and also related it to the length of the learning phase.

Two conflicting forces of expertise complicates the use of time as a screening device - an efficient experimenter is more effective at learning good news but will also become pessimistic after successive failures. Beside the length of the experimentation phase, the principal can screen the agent with the timing of payment and the size of the output required in the exploitation phase. The stochastic structure of the dynamic setting determines how the principal optimally chooses the payment scheme for agents who learn at different rates.

Interestingly, both incentive constraints can be binding, which is novel in this literature. The standard incentive problem for this model is for the low type to claim to be a high type. Then, he will be less pessimistic after failure than the principal, who will overcompensate him for the expected cost if she believes his claim. The low type will command an information rent. A high type on the other hand faces a gamble if he pretends to be a low type. There is a standard negative rent after failure as he would be more pessimistic than the principal, who would under compensate him in the production stage. But, the high type has a chance to obtain the rent promised to a low type, and this can make the high type's incentive constraint to also be binding. This restricts the principal as to when she can pay the low-type his rent.

By analyzing the stochastic structure of the dynamic problem, we clarify how the principal can rely on the relative likelihood ratios of success and failure of the two types in order to screen them. The rent to a high type should come after early success and to the low type for late success. If the experimentation phase is not long enough, the principal has no recourse but to pay the low type's rent after failure, which is another novel result. While our main section relies on publicly observed success and experimenting for "good news", we show that our main insights survive if the agent can hide success or if we changed to model to learn "bad news". Without an exploitation stage with a scalable project size after failure, there would be under experimentation relative to the first best.

With a scalable project size, we find a new result that over-experimentation can be also optimal. Over production can occur in the exploitation phase. Analogous results are obtained whether we consider a good news or a bad news model of experimentation.

Chapter 3

PROPERTY RIGHTS IN RUNWAY SLOTS ALLOCATION

3.1 Introduction

Landing[1] and takeoff slots provide the rights to use airport runways at specific times. Allocation of these slots affects industry competition and airline profits, which in turn influences the amount passengers pay for flights. Although slots are assigned to scheduled flights in advance, unpredictable conditions such as changes in weather may lead to the reallocation of slots among carriers. This paper examines slot allocation challenges and proposes a mechanism that improves recognition of economically relevant property: respecting airlines' property rights.

Around the world, the airline industry suffers from inclement weather conditions. Conditions such as high winds and thunderstorms are responsible for 40 percent of the total amount of time flights are delayed. They limit the rate at which planes can take off and land on runways safely, and in most countries, they cause the airport's landing schedule to be modified for safety issues, as prescribed by law. In the United States, the Federal Aviation Administration (FAA) is responsible for mitigating the negative consequences of weather change. It reduces the allowed arrival rates of flights at an affected airport, meaning some flights are postponed or canceled. To alter the schedule efficiently, the FAA may elicit some of the airline's private information, such as the earliest feasible departure time from the origin airport for each flight. This opens the door to a matching problem under asymmetric information that is a subject of this paper.

Decades ago, the FAA used the Grover-Jack algorithm to manage this problem. This algorithm assigned arrival slots based on feasible departure times *reported* by the airlines themselves. If, for example, a plane was experiencing mechanical issues, under Grover-Jack, the delay was added to the weather delay, penalizing the airline twice. This is

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[1] This Chapter is a joint work with Ken C. Ho.

known as the “double penalty” problem, and it led to airlines intentionally withholding information about mechanical delays and other issues. This sometimes led to landing slots remaining unused when they could have been used to land other airlines’ flights. Eventually, the FAA and industry experts agreed the Grover-Jack needed improvement.

The FAA has developed a new scheme that assigns arrival slots based on *original* schedules. It currently is employing the GDP, a two-step procedure. The first stage of the GDP involves the ration-by-schedule (RBS) algorithm, which assigns the arrival slots among the participating airlines on a first-come, first-serve basis, according to the originally scheduled times of arrival (unlike the self-reported times used in Grover-Jack) For example, if the airport could land one flight every minute initially, then, after the GDP has been implemented, it can land one flight every two minutes. The first 30 flights from the initial schedule would be assigned newly created thirty 2-minute slots.[2]

The second step of the GDP involves the compression algorithm, which is an exchange mechanism that allows one company that cannot use a slot assigned during RBS to either reassign this slot to a later flight it operates or to exchange it with another airline.[3] This trading feature gives compression a presumable advantage over the Grover-Jack mechanism. However, GDP prescribes how the slots must be exchanged: If an airline chooses not to use a slot, it can be used only by the earliest flight of another company.

We argue that a solution to the runway slot allocation problem should be mindful of property rights. Airlines buy[4] runway slots for significant amounts of money, and in exchange, the airport grants airlines the right to use landing slots at specified times. Both the government and the airlines have agreed that purchased slots should be treated as property. A report funded by the European Commission suggests that airlines should be allowed to buy and sell their runway slots as they please, just like any other asset.

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[2] According to RBS, the other 30 flights initially scheduled for the second 30-minute period would be delayed to an undetermined time period.

[3] When an airline has decided a slot cannot be used, it trades it for the best feasible substitute, usually owned by the airline that took the initial slot. The airline that gives up the initial slot is better off because the one it accepts typically is earlier than the original. Airlines are assumed to prefer scheduling flights as early as possible when feasible.

[4] We cautiously use word “buy,” as a slot is not a physical object, but constitutes the right to use a runway.

In the United States, slots are treated as property under certain circumstances. For example, on June 9, 2015, United Airlines exchanged its takeoff and landing slots at JFK International Airport with Delta Air Lines for slots at Newark. This deal was approved by the regulator.

While the current RBS procedure assigns flights based on their initial schedules, we propose a mechanism that takes into account initial slot assignment through a different method. If an airline initially owns adjacent slots that constitute a new slot after the arrival rate is lessened, our method designates this new slot as belonging to the airline. The slot is free for any flight the airline chooses, and it may choose to give up the slot for a better alternative. The airline also may exchange other slots it owns for newly available ones; the more slots an airline initially owns, the more chances it has to pick its most desired slots.

Throughout the paper we assume that each airline has a priority ranking system for its flights such that all flights can be ranked strictly according to importance. We also assume that for each flight, the airline prefers to assign the earliest feasible slot, where feasibility is defined as enough time for the flight take off from the origin airport and arrive at its destination. Thus, each airline is assumed to have preferences for sets of slots determined by lexicographic preferences over flights and the earliest feasible arrival times.

In this paper, we propose a mechanism that mitigates the negative effects of inclement weather conditions and respects carriers' property rights over slots. Unlike the current GDP procedure, which does not improve carrier circumstances any more than RBS does, we take the initial slot allocation as a starting point. We define adjacent slots initially owned by the same airline as constituting a new slot the airline is endowed with after the arrival rate is reduced: The airline is free to either use this slot itself or exchange it with any other airline. For initial slots that become part of a newly created slot, we indicate that airlines jointly own a group of indivisible objects. If this is the case, we will reallocate these slots using a hybrid mechanism that combines serial dictatorship with endogenous priorities and the top-trading cycle (TTC) procedure.

When a persistent overload due to weather is predicted for the next several hours at an airport, the FAA announces that the arrival rate must be reduced. This results in a shortage of slots. During the length of time in which access restrictions are in force, the earliest feasible arrival times for initially scheduled flights may change due to delays at other airports or technical reasons. Only carriers know if some flights have to be postponed or even canceled. Our proposed mechanism starts with airlines reporting their private information to a central clearinghouse. First, airlines are asked to report a ranking profile of their initially scheduled flights during the announced time period. Second, each carrier reports a vector of earliest feasible arrival times for each flight.

After all the airlines report their private information, the central clearing house assigns newly created slots. In case there is a slot that can be used by only one airline (possibly by several flights), this slot is permanently assigned to that airline and the flight that was ranked first according to the submitted preferences. If there is a flight that can use several slots feasibly, the flight is assigned to the earliest one. In the case that one flight is assigned and some slots continue to be demanded by one airline only, the previous step is repeated. If airlines report their preferences and the earliest feasible arrival time honestly, which is the case in equilibrium, then all such slots will be assigned to carriers.

At this point of the algorithm, the only remaining slots are those that are demanded by more than one airline. We then distinguish two types of slots. If a new slot consists of several initial adjacent slots owned by the same airline, it will be designated as property of the airline. The second type is assumed as jointly owned by all participating carriers, and positive probability may be assigned to any airline. If each slot has an assigned owner, a variant of the TTC can be used to determine to whom these slot should be assigned. If none have an assigned owner, a modification of the deferred acceptance (DA) algorithm or a form of a serial dictatorship can be implemented instead. Because our approach combines both the slots that are jointly owned and those owned by individual carriers, we use a hybrid mechanism that employs features of TTC and serial dictatorship with endogenous order.

Each remaining slot will be allocated via a lottery that proceeds as follows. Each airline receives a specific amount of "tickets" that is equal to the amount of initially owned slots. The chances to win a lottery for each airline is the ratio of the amount of lottery tickets it owns to the total amount of initially scheduled unassigned flights. The winner of a lottery points to a slot that is matches the earliest feasible arrival time of the winner?s highest-ranked flight. Two alternatives can then occur: either the slot will already belong to another carrier, or it will not. In the latter case, the slot would be assigned to the winner of the lottery and its highest-ranked flight. In the former case, the original owner of the slot is allowed to pick its preferred slot. If there is no conflict, the two slots are assigned to each airline. A cycle of several airlines may form at this step. Each airline that gets a slot at this point is sacrificing an amount of tickets equal to the ratio of the new arrival rate to the initial one. The total amount of tickets is reduced based on the amount of newly assigned flights. In addition to respecting property rights, our mechanism satisfies relevant variables: It is feasible and efficient. Moreover, it provides an outcome from the core and is strategy-proof.

3.2 Model

There is a finite set of airlines $A = \{a_1, a_2, ..., a_{|A|}\}$ and a finite set of controlled flights $F^o = \cup_{a \in A} F^o_a = \{f_0, f_1, ..., f_{|F^o|-1}\}$, where the F^o_a denotes all flights owned by airline a.[5] Some flights might be canceled during or before the start of GDP; we use $F \subseteq F^o$ to denote the set of non-canceled flights and $F_a \subseteq F^o_a$ to denotes all flights owned by airline a that are not canceled. There is a set of original slots $S^o = \{s^o_0, s^o_1, ..., s^o_{|L|-1}\}$, where the length of each slot is normalized to one unit of time. Note that $|F^o|$ of the $|L|$ original slots were owned by some airlines. Let the set of available GDP slots be $S = \{s_0, s_1, ...\}$, where the length of each slot is $l > 1$ unit of time.[6] For $n = 0, 1, ...$, slot s_n is the time interval $[nl, (n+1)l]$. We use s_n to represent nl on the time line.[7]

There is an earliest feasible arrival time $e_f \in S$ for each flight $f \in F$; each flight f can

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[5] A flight is "controlled" means it is included in the GDP program. Note that we do not used the term "affected" since there are more flights affected by the launch of GDP in different airports.

[6] Note that available GDP slots do not have to be adjacent since there are exempted flights in GDPs (for examples, international and airborne flights). The number of arrivals an airport can accept each hour is called the Airport Acceptance Rate/Airport Arrival Rate (AAR); if 1 unit of time is 1 minute, then $l = \frac{60}{AAR}$.

[7] We model one runway, but the model can be extended to a problem with several runways, in which there will be multiple identical slots available at a time.

be feasibly assigned to slot n only if $e_f \leq s_n$.[8] Let $e = (e_f)_{a \in F} \in \mathbb{R}_+^{|F|}$ be the vector of all earliest feasible arrival times. A **landing schedule** is an injective function $\Pi : F \to S$. A **partial landing schedule** $\Pi_a : F_a \to \Phi(a)$ for each a. A landing schedule Π is **feasible** if $\forall f \in F$, $\Pi(f) \geq e_f$. A landing schedule Π is **non-wasteful** if $\nexists f \in F$ and $s \in S$ such that $\Pi^{-1}(s) = \emptyset$ and $e_f \leq s < \Pi(f)$.

$\Phi : A \to 2^S$ is a **slot ownership function** if $a \neq a' \implies \Phi(a) \cap \Phi(a') = \emptyset$, where $\Phi(a)$ is the set of slots owned by airline a. Let $\Pi_a : F_a \to \Phi(a)$ be a partial landing schedule for a. Φ is consistent with Π if $\forall a \in A$, $\forall f \in F_a$, $\Pi(f) \in \Phi(a)$. A pair (Π, Φ) satisfying this consistency condition is an **assignment**. The original assignment (Π^o, Φ^o) is assumed to be consistent, so given the initial landing schedule $\Pi^o : F^o \to S^o$,we can derive the initial slot ownership function $\Phi^o : A \to 2^{S^o}$. Let $S_a = \{s_n \in S | s_n = [nl, (n+1)l] \subseteq \cup_{s_n^o \in \Phi^o(a)} [n, n+1]\}$ be the set of available GDP slots that their time intervals are entirely owned by airline a. Let S_A be the sets of S_a $\forall a \in A$.

We assume each airline company $a \in A$ has an importance ranking of flights that it owns. Formally, let R_a be a strict total order over F_a; we interpret R_a as a's importance ranking. If $f \in F_a$ is more important than $f' \in F_a$, we write $f R_a f'$. Given F_a and R_a, let $e_a = (e_{f_{a,1}}, e_{f_{a,2}}, ..., e_{f_{a,|F_a|}}) \in \mathbb{R}^{|F_a|}$ be a vector of earliest feasible arrival times such that $f_{a,i} R_a f_{a,i+1}$ for $i \in \{1, ..., |F_a|\}$. Let $R = (R_a)_{a \in A}$ be the importance ranking profile.

Airline a's preference over landing schedules is induced by R_a and e_a. All else being equal, airline a prefers flight $f \in F_a$ with e_f to land as early as possible. Given a landing schedule Π, we define the **delay** for f by $d_f(\Pi) = s_f - e_f$, where s_f is the slot assigned to f, and if $s_f = \emptyset$ or $s_f < e_f$, the we set $d_f(\Pi) = M$, where M is a sufficient large positive number such that $M > s - e_f$ for all $s \in S$ and $e_f \in T$.[9]

For all feasible landing schedules Π and Π', airline a lexicographically prefers Π to Π' if and only if the first non-zero coordinate of $x_a = (x_1, x_2, ..., x_{|F_a|})$ is positive, where for $i \in \{1, ..., |F_a|\}$ and $f_{a,i} R_a f_{a,i+1}$, $x_i = d_{f_{a,i}}(\Pi') - d_{f_{a,i}}(\Pi)$, and we write $\Pi \succ_a \Pi'$. Conversely, if the first non-zero coordinate of x_a is negative, it prefers Π to Π'. If Π is

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[8] Following the literature, we call e_f the earliest feasible arrival time for f , but strictly speaking, e_f is the earliest feasible arrival slot for f.

[9] Note that this delay is respected to e_f but not its original slot s_f^o.

feasible but Π' infeasible for a, it prefers Π to Π'. If airline a is indifferent between Π and Π', we write $\Pi \sim_a \Pi'$; this will happen when (i) both of them are infeasible, and (ii) all coordinates of x_a equal to 0, which will only happen when $\Pi_a = \Pi'_a$. Since airlines only care their only flights, we will also use $\succsim_a$ to compare partial landing schedules for a. Let $\succsim = (\succsim_a)_{a \in A}$ be the preference profile of all airlines.

An **instance** of an airport slots allocation problem is a 6-tuple $I = (S, A, F, R, e, \Phi^o)$.

A mechanism φ is a function that maps an instance I to a landing schedule Π; formally, $\varphi : I \to \Pi$. Our mechanism includes an endogenous random variable R_S. Therefore, φ is a multi-valued function. But since our results hold for any realization of $\varphi(I)$, from now on, for brevity, we will use $\varphi(I)$ to denote a particular realization.

Let $\varphi_f(I)$ be the outcome for $f \in F$, and $\varphi_a(I) \equiv \cup_{f \in F_a} \varphi_f(I)$ be the outcome for $a \in A$. Since the other parameters are fixed, we often write $\varphi(\boldsymbol{R}, \boldsymbol{e})$ to represents $\varphi(I)$.

Our mechanism employs the following *endogenous trading cycle algorithm* using $(\boldsymbol{R}, \boldsymbol{e})$ as inputs. The algorithm consists with a main stage and a supplemental stage.

Main Stage:

Step 0 **- allocation of non-scarce resources:**

Set $F^{0-0} = F$ and $e = e^{0-0}$.

(i) Use the smallest element of e^{0-0} to identify the earliest slot $s \in S$ that is demanded, eliminates all slots that are earlier than s, denote the resulting set S^{0-0}.

(ii) If the earliest slot only can be used by one flight $f \in F^{0-0}$, assign the slot to this flight. Denote the resulting sets S^{0-1}, F^{0-1} and e^{0-1} respectively.

(iii) Repeat (i) and (ii) until the earliest slot is demanded by more than one flights. In general, denote the intermediate sets S^{0-t},F^{0-t} and e^{0-t} for $t = 2, ...$

Denote the resulting sets S^1, F^1 and A^1, where A^1 includes airlines with flights in F^1. In general, S^k, F^k and A^k for $k = 1, 2, ...$ only include slots, flights and airlines in step k. Remove flights $F \setminus F^1$ from R accordingly; denote the resulting ranking profile R^1. Let F_a^N be the flights of airline a that are assigned a slot in this step, and let F_a^1 be the remaining flights of airline a that are not assigned a slot. In general, F_a^k for $k = 1, 2, ...$ only includes flights of airline a in step k.

Step 1 **- allocation of scare resources:**

According to Φ^o, construct S_a for each $a \in A$. If $S_a \cap S^1$ is non-empty, order $S_a \cap S^1$ in increasing order, assign the earliest slot to a's highest ranked flight in F_a^1 that can feasibly use it. Assign the next earliest slot to its remaining highest ranked flight that can feasibly use it. Repeat until there is no more slot.

The price of a lottery ticket is 1 controlled flight. The budgets for each airline $a \in A^1$ is $b_a^1 = |F_a^o| - |F_a^N|$.

Run a lottery such that each airline a's chance to win is

$$\frac{b_a^1}{\sum_{a \in A^1} b_a^1}.$$

Let $a(i)$ be the ith important flight of airline a according to R^1.[10] If a wins the lottery in step 1, put $a(1)$ in the 1st place of R_S , where R_S is the order of picking a slot.

Assign $a(1)$ the earliest slot in S^1 such that a's highest ranked flight in F_a^1 that can feasibly use it.

(i) If it picks an empty slot s', let $S^2 = S^1 \setminus s'$ and goes to the next step.

(ii) If it is demanding some slot s' that is endowed to some other airline b, and b has remaining flight in F^1, insert $b(1)$ before $a(1)$, that is, let $R_S : b(1), a(1)$. If b has no remaining flight, let $S^2 = S^1 \setminus s'$ and goes to the next step. If a cycle forms, it is formed by the remaining most important flights and slots that owned by their airlines. Each of these flights requests a slot that owned by the airline which is next in the cycle. For each $a \in A$, let s_a be some slot in $S_a \cap S^1$. A cycle in step 1 is an ordered list $(s_a, a(1), s_b, b(1), ..., s_z, z(1))$ of slots and flights where $a(1)$ demands s_b,...,$z(1)$ demands s_a. Remove all flights in the cycle by assigning them the slots they demand. Let $S^2 = S^1 \setminus \{s_a, ..., s_z\}$ and goes to the next step.

(iii) If such flight already has a slot s endowed to it and its want to use it, let $S^2 = S^1 \setminus s$ and insert $a(2)$ after $a(1)$, that is, let $R_S : a(1), a(2)$.

An airline could get at most 1 slot in a cycle, but it might get more from (iii). For each airline a that gets m slots in this step, reduce its budget to $b_a^2 = |b_a^1| - m$. Denote the resulting sets F^2, A^2 and F_a^2 for each $a \in A$.

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[10] We use $a(i)$ but not $f_{a,i}$ since ith important flight in R_a^1 might be different from ith important flight in R_a.

Step $n \geq 2$ - run a lottery such that each airline a's chance to win is

$$\frac{b_a^n}{\sum_{a \in A^n} b_a^n}.$$

Let $a(n-1)$ be the last flight of airline a in R_S. Assign $a(n)$ the earliest slot in S^n such that a's highest ranked flight in F_a^n that can feasibly use it.

(i) If it picks an empty slot s', let $S^{n+1} = S^n \setminus s'$ and goes to the next step.

(ii) If it is demanding some slot that is endowed to some other airline b,and b has remaining flight in F^n, insert $b(n)$ before $a(n)$, that is, let $R_S : ..., b(n), a(n)$. If b has no remaining flight, let $S^{n+1} = S^n \setminus s'$ and goes to the next step. If a cycle forms, a cycle in step n is an ordered list $(s_a, a(n), s_b, b(n), ..., s_z, z(n))$, where for each $a \in A$, s_a is some slot in $S_a \cap S^n$. Remove all flights in the cycle by assigning them the slots they demand. Let $S^{n+1} = S^n \setminus \{s_a, ..., s_z\}$ and goes to the next step.

(iii) If such flight already has a slot s endowed to it and its want to use it, let $S^{n+1} = S^n \setminus s$ and insert $a(n+1)$ after $a(n)$, that is, let $R_S : ..., a(n), a(n+1)$.

For each airline a that gets m slots in this step, reduce its budget to $b_a^{n+1} = |b_a^n| - m$. Denote the resulting sets F^{n+1}, A^{n+1} and F_a^{n+1} for each $a \in A$.

The main stage terminates when the set of remaining flights F^k becomes empty.

Supplemental Stage:

Let V^1 be the set of vacant slots and A_S^1 be the set of airlines with positive budgets $b_a^{S:1}$ for $a \in A_S^1$.

Step $n \geq 1$: If the earliest vacant slot was endowed to some a with positive budget, assign it to a. Otherwise, run a lottery such that each airline a's chance to win the earliest vacant slot is

$$\frac{b_a^{S:n}}{\sum_{a \in A_S^1} b_a^{S:n}}.$$

Denote the resulting sets V^{n+1}, A_S^{n+1}. Update $b_a^{S:n+1} = b_a^{S:n} - 1$ for the winner a and $b_{a'}^{S:n+1} = b_{a'}^{S:n}$ for $a' \in A_S^{n+1} \setminus \{a\}$. Repeat until all budgets are zero.

This algorithm will stop in finite steps. Indeed, it might stop right after step 0. Note that there will always be feasible slots for flights, yet they might be late and outside the GDP time window. The initial budget for airline a to compete scare resources is b_a^1, which includes the flights that are canceled, so $b_a^1 \geq |F_a^1|$. This will provide no incentive for airline a to hide its cancellations. An airline a will stop participating in the next main stage lottery once it has no remaining flight even if it still has positive remaining budget. This budget, which is the number of a's canceled flights, will become the initial budget in the supplemental step, and it is denoted by $b_a^{S:1}$ in the algorithm. V^1 includes vacant slots that are not assigned in both step 0 and step 1.

In a typical YRMH-IGYT mechanism, a cycle is formed by exclusively existing tenants and their houses. By contrast, a cycle in our mechanism forms by flights and slots that are not necessarily endowed to these flights.

A mechanism φ is *feasible* if for any instance I, $\varphi(I)$ is feasible. A mechanism φ is *non-wasteful* if for any instance I, $\varphi(I)$ is non-wasteful.

A landing schedule Π is **Pareto efficient** if $\nexists \Pi'$ such that (i) $\forall a \in A$, $\Pi' \succsim_a \Pi$, and (ii) $\exists a \in A$, $\Pi' \succ_a \Pi$. A mechanism φ is *Pareto efficient* if for any instance I, $\varphi(I)$ is Pareto efficient.

A mechanism φ is **strategy-proof** if there does not exist $(\mathbb{R}, e)$, an airline a with $\widehat{R_a, e_a}$ such that $\varphi_a(\widehat{R_a, e_a}, (R, e)_{-a}) \succsim_a \varphi_a(R, e)$.

Let $\Pi_a^{S'}$ be a partial landing scheduled for a such that $\Pi_a^{S'} \succsim_a \Pi'_a$ for all Π'_a given F_a and $S' \subseteq S$.

A mechanism φ is **individually rational** if for any instance I, $\forall a \in A$, $\varphi_a(I) \succsim_a \Pi_a^{S_a}$.

A landing schedule Π is in the **core** if $\nexists \Pi'$ and $A' \subseteq A$ such that (i) $\forall f \in \cup_{a \in A'} F_a$, $\Pi'(f) \in S_{A'}$, and (ii) $\forall a \in A'$, $\Pi' \succ_a \Pi$. A mechanism φ is a **core-selecting** if for any instance I, $\varphi(I)$ is in the core.[11]

A mechanism φ **respects property rights** if $\forall s \in S_a$, in the first step that s is being assigned, a has the right to (i) use it, (ii) trade it for a better alternative.

Theorem 3.1: The endogenous trading cycle mechanism is feasible, non-wasteful, individually rational, Pareto efficient, respects property rights.

Proof: See Appendix C.

We provide an example to illustrate how the proposed mechanism works. Suppose the length of time during which restrictions on access to the airport are in force equals to

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[11] As in SV 2013, the core with weak domination (strong core in their paper), could be empty. An airline with 2 flights canceled but has an endowed slot that is demanded by 2 other airlines.

10 initial slot durations, $L = 10$. Suppose the FAA requires new arrival rate to be twice smaller than the initial one, that is, $l = 2$, so we have twelve newly created slots twice longer than initial ones. There are three airlines that we refer to as A, B and C and a set of flights for each of them, $F_A = \{f_3, f_4, f_7, f_{10}\}$, $F_B = \{f_2, f_5, f_6\}$ and $F_C = \{f_1, f_8, f_9\}$,respectively.

Flights f_1 and f_2 were canceled. In addition, for each of the remaining flights there is an earliest feasible arrival time (nominated in units of new slots) $\vec{e}_f = (\times, \times, 1, 1, 1, 2, 2, 0, 4, 6)$ for each flight so that each flight can be feasibly assigned to slot s_n only if $e_f = n \leq s_n$ for $n = 0, 1, 2, ...$ See Table 3.1 below for the initial schedule and slot assignment.

Table 3.1: Initial Assignment

Flight	1	2	3	4	5	6	7	8	9	10
Airline	C	B	A	A	B	B	A	C	C	A
e_f	×	×	1	1	1	2	2	0	4	6

After the new arrival rate is determined there are 10 newly created slots, $s_0, s_1, ..., s_{11}$, to be allocated. Assume that priorities of each airline are as follows:

$\triangleright_A$: f_3, f_7, f_4, f_{10};

$\triangleright_B$: f_2, f_6, f_5;

$\triangleright_C$: f_1, f_9, f_8.

According to our mechanism slot s_1 becomes endowment of airline A; slot s_2 belongs to airline B.

Step 0. Given that slot s_0 can be feasibly used only by f_8. Thus, slot s_0 is permanently assigned to f_8.

The main steps of the algorithm and final allocation of slots are presented below:

As we explained before, GDP starts with the RBS procedure that rations slots based on the *initial* schedule as reflected in Table 3.5 below.

The Compression algorithm allows the airlines to exchange slots they cannot use. If

Table 3.2: Step 0

Slot	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9
Airline	C	A	B							
Flight	f_8									

Table 3.3: Steps of the Algorithm

Step	$\vec{Pr}_{win}$	Winner	Slots assigned	Slots left
1	$(\frac{4}{9}, \frac{3}{9}, \frac{2}{9})$	A	f_3 to s_1	s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9
2	$(\frac{3}{8}, \frac{3}{8}, \frac{2}{8})$	A	f_6 to s_2, f_7 to s_3	s_4, s_5, s_6, s_7, s_8, s_9
3	$(\frac{2}{6}, \frac{2}{6}, \frac{2}{6})$	A	f_4 to s_4	s_5, s_6, s_7, s_8, s_9
4	$(\frac{1}{5}, \frac{2}{5}, \frac{2}{5})$	C	f_9 to s_5	s_6, s_7, s_8, s_9
5	$(\frac{1}{3}, \frac{2}{3}, \frac{0}{3})$	B	f_7 to s_7, f_{11} to s_{10}	s_7, s_8, s_9
6	$(\frac{1}{1}, \frac{0}{1}, \frac{0}{1})$	A	f_{10} to s_7	s_8, s_9
S:1	$(\frac{0}{2}, \frac{1}{2}, \frac{1}{2})$	B	$\emptyset$ to s_8	s_9
S:2	$(\frac{0}{1}, \frac{0}{1}, \frac{1}{1})$	C	$\emptyset$ to s_9	

Table 3.4: Outcome of the Algorithm

Slot	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9
Airline	C	A	B	A	A	C	B	A	B	C
Flight	f_8	f_3	f_6	f_7	f_4	f_9	f_5	f_{10}	$\emptyset$	$\emptyset$

an airline has a slot that cannot be used by a flight assigned to this slot during RBS, the airline may assign a later flight to this slot. Crucially, it has to pick the *earliest*

Table 3.5: Outcome of the RBS

Slot	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9
Airline	C	B	A	A	B	B	A	C	C	A
Flight	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}

feasible flight according to RBS. For example, Company C that owns slot s_0 had to cancel flight f_1. According to the Compression algorithm, the airline has to assign not the most preferred flight but the earliest flight that this company owns that can use it! It is clear that the Compression will not accommodative the preference of the airline as it implicitly assumed the earlier the better regardless of flights. If an airline cannot use a slot by itself, it has to exchange it with another one. In particular, an airline that owns the earliest flight that can feasibly use the slot will be allowed to take it. In exchange, the first airline will get a slot that the flight of the second carrier was assigned to. Thus, in the Compression algorithm, the airline itself does not choose whom to trade with. In our example, the outcome of the Compression algorithm becomes the following:

Table 3.6: Outcome of Compression

Slot	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9
Airline	C	B	A	A	B	A	C	A	B	C
Flight	f_8	f_5	f_3	f_4	f_6	f_7	f_9	f_{10}	$\emptyset$	$\emptyset$

We compare the outcome of our proposed mechanism and Compression below.

3.3 Conclusion

This paper examines the runway slot allocation problem that emerges when inclement weather conditions strike an airport. In such conditions, airports have to modify their

Table 3.7: Comparison of Compression and Our Mechanism

Slot	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9
Compression	f_8	f_5	f_3	f_4	f_6	f_7	f_9	f_{10}	$\emptyset$	$\emptyset$
Our Mechanism	f_8	f_3	f_6	f_7	f_4	f_9	f_5	f_{10}	$\emptyset$	$\emptyset$

initial schedules, as each flight will require more time to land. In addition, some flights may be canceled, and the earliest feasible arrival time may change for others. This creates challenges in matching flights with newly created slots. We argue that the current GDP program does not respect property rights, as it assigns flights to slots based on airlines' initial schedules. This implies that airlines are not free to use the runway slots they have paid for.

We propose a mechanism that respects property rights. It allows airlines to use a newly created slot if it initially paid for the time interval. First, airlines are required to submit a priority ranking of their flights along with the earliest feasible arrival times for each flight that is not canceled. Second, each slot is assigned through a lottery in which the airlines? chances of winning depend on the amount of initially owned slots. The winner of a lottery can point to any slot that has not yet been assigned; if the winner prefers a slot owned by another airline, a trading cycle takes place.

This mechanism satisfies factors besides property rights. It is feasible in that it assigns flights only to slots they can use. It is efficient, as no slot is left unassigned when a flight can use it feasibly. It is also strategy-proof and provides an outcome from the core.

To emphasize the importance of property rights, we assume that airlines have lexicographic preferences over flights. Although this simplifies analysis, it does not allow the study of full trade-offs or of sacrificing less costly flights to move more important flights to earlier slots. While this extension is practical and relevant, it is likely to cause certain irregularities, as the literature on many-to-one matching with general preferences has demonstrated. In future research, we plan to conduct a detailed analysis of this problem.

BIBLIOGRAPHY

Bergemann, D. and Hege, U. (1998). Venture capital financing, moral hazard, and learning. *Journal of Banking and Finance*, 22(6-8):703–735.

Bergemann, D. and Hege, U. (2005). The financing of innovation: learning and stopping. *Rand Journal of Economics*, 22(6-8):703–735.

De Chiara, A. and Livio, L. (2016). The threat of corruption and the optimal supervisory task. *Journal of Economic Behavior & Organization*, 133:172–186.

Gerardi, D. and Maestri, L. (2012). A principal-agent model of sequential testing. *Theoretical Economics*, 7:425–463.

Halac, M., Kartik, N., and Liu, Q. (2016). Optimal contracts for experimentation. *Review of Economic Studies*, 83(3):1040–1091.

Horner, J. and Samuelson, L. (2005). Incentives for experimenting agents. *Rand Journal of Economics*, 44(4):632–663.

Keller, G., Rady, S., and Cripps, M. (2005). Strategic experimentation with exponential bandits. *Econometrica*, 73:39–68.

Khalil, F. and Lawarre, J. (1995). Collusive auditors. *The American Economic Review*, 85(2):442–446.

Kofman, F. and Lawarre, J. (1993). Strategic experimentation with exponential bandits. *Econometrica*, 61(3):629–656.

Mason, R. and Valimaki, J. (2015). Getting it done: dynamic incentives to complete a project. *Journal of the European Economic Association*, 13(1):62–97.

Ravikumar, B. and Zhang, Y. (2012). Optimal auditing and insurance in a dynamic model of tax compliance. *Theoretical Economics*, 7:241–282.

Appendix A

TECHNICAL DETAILS , CHAPTER 1

A.1 Proof of Proposition 1.1

The principal's optimization problem is the following:

$$[P^{SB}] \ \max_{\varpi} \pi(\varpi, \vec{1}) \text{ subject to}$$

(MH) $$\vec{1} \in \arg\max_{\vec{e}} U(\varpi, \vec{e}),$$

(IC) $$b_t \geqslant w_t + \delta b_{t+1} \text{ for } t = 1, ..., T-1,$$

$$b_t \geqslant w_t \text{ for } t = 1, ..., T,$$

(LL) $$b_t,\, w_t \geqslant 0 \text{ for } t = 1, ..., T.$$

We first solve an auxiliary problem $P1.1$, where the global MH constraint is replaced by a sequence of MH_t constraints for $t = 1, ..., T$ that ensure the agent does not want to deviate at period t, given that he was behaving in all prior periods $s < t$ and will work in all subsequent periods $s > t$. In addition, $P1.1$ ignores the IC constraint, which will demonstrate automatic satisfaction.

The optimization problem $P1.1$ is:

$$[P1] \ \max_{\varpi} \pi(\varpi, \vec{1}) \text{ subject to}$$

(MH_t) $b_t - w_t \geqslant \dfrac{c}{\lambda\tilde{\beta}_t} + \sum_{s=t+1}^{T} \delta^{s-t}(1-\lambda)^{s-t-1}(\lambda b_s + (1-\lambda)w_s - c)$ for $t = 1, ..., T$,
(LL) $b_t,\, w_t \geqslant 0$ for $t = 1, ..., T$.

Lemma 1. The following payment sequence solves $P1.1$:

$$w_t = 0 \text{ and } b_t = \frac{c}{\lambda\tilde{\beta}_t} + c\sum_{s=1}^{T-t} \delta^s \frac{1-\beta_0}{\beta_0(1-\lambda)^{t+s-1}} \text{ for } 1 \leqslant t \leqslant T.$$

Proof: Note that increasing w_t makes it more difficult to satisfy the MH_t constraints and lessens the objective function. As a result, the optimal solution must have $w_t = 0$ for $1 \leqslant t \leqslant T$, and the problem can be rewritten as:

$$[P1] \ \max_{\varpi} \pi(\varpi, \vec{1}) \text{ subject to}$$

(MH_t) $b_t \geqslant \frac{c}{\lambda\tilde{\beta}_t} + \sum_{s=t+1}^{T} \delta^{s-t}(1-\lambda)^{s-t-1}(\lambda b_s - c)$ for $t = 1, ..., T$,
(LL) $b_t \geqslant 0$ for $t = 1, ..., T$.

The auxiliary problem $P1$ has the Lagrangian:

$$L = \beta_0 \sum_{t=1}^{T} \delta^t (1-\lambda)^{t-1} \lambda (V - b_t)$$
$$+ \sum_{t=1}^{T} \mu_t \Big(b_t - \frac{c}{\lambda\tilde{\beta}_t} - \sum_{s=t+1}^{T} \delta^{s-t}(1-\lambda)^{s-t-1}(\lambda b_s - c)\Big) + \sum_{t=1}^{T} \xi_t b_t.$$

The Kuhn-Tucker conditions for the optimization problem are:

$[b_t] : -\beta_0 \delta^t (1-\lambda)^{t-1}\lambda + \mu_t - \sum_{j=1}^{t-1} \mu_j \delta^{t-j}(1-\lambda)^{t-j-1}\lambda + \xi_t = 0$ for $1 \leqslant t \leqslant T$,

complemented by the constraints of the problem and the corresponding complementary slackness conditions.

If $\xi_t > 0$ for some $1 \leqslant t \leqslant T$, then MH_t would be violated and, as a result, we must have $\xi_t = 0$ for $t = 1, ..., T$.

Consider the first-order conditions with respect to b_t:

$t = 1$: $-\beta_0\delta\lambda + \mu_1 = 0 \Longrightarrow \mu_1 = \beta_0\delta\lambda$;

$t = 2$: $-\beta_0\delta^2\lambda(1-\lambda) + \mu_2 - \mu_1\delta\lambda = 0 \Longrightarrow \mu_2 = \beta_0\delta^2\lambda > 0$;

repeating this procedure until the final period T, we have:

$t = T$: $\mu_T = \beta_0\delta^T\lambda > 0$.

Thus, all MH_t constraints must be binding.

First, we will prove that with $b_t = \frac{c}{\lambda\tilde{\beta}_t} + c\sum_{s=1}^{T-t} \delta^s \frac{1-\beta_0}{\beta_0(1-\lambda)^{t+s-1}}$, it is the case that:

$$b_t = \frac{c}{\lambda\tilde{\beta}_t} + \sum_{s=t+1}^{T} \delta^{s-t}(1-\lambda)^{s-t-1}(\lambda b_s - c) \text{ for } 1 \leqslant t \leqslant T.$$

For $t = T$, the equality $b_t = \frac{c}{\lambda\tilde{\beta}_T}$ trivially follows from the proposed formula itself. For any $t < T$, assume that MH_s holds for $s = t+1$; that is:

$$b_{t+1} = \frac{c}{\lambda\tilde{\beta}_{t+1}} + \sum_{s=t+2}^{T} \delta^{s-t}(1-\lambda)^{s-t-1}(\lambda b_s - c).$$

We need to show that $b_t = \frac{c}{\lambda\tilde{\beta}_t} + \sum_{s=t+1}^{T} \delta^{s-t}(1-\lambda)^{s-t-1}(\lambda b_s - c)$, or, using the line above:

$$b_t = \frac{c}{\lambda\tilde{\beta}_t} + \delta(\lambda b_{t+1} - c) + \sum_{s=t+2}^{T} \delta^{s-t}(1-\lambda)^{s-t-1}(\lambda b_s - c)$$
$$= \frac{c}{\lambda\tilde{\beta}_t} + \delta(\lambda b_{t+1} - c) + \Big(b_{t+1} - \frac{c}{\lambda\tilde{\beta}_{t+1}}\Big) = \frac{c}{\lambda\tilde{\beta}_t} - \frac{c}{\lambda\tilde{\beta}_{t+1}} - \delta c - (1+\delta\lambda)b_{t+1}.$$

Since $b_{t+1} = \dfrac{c}{\lambda\tilde{\beta}_{t+1}} + \frac{c(1-\beta_0)}{\beta_0(1-\lambda)^t}\frac{\delta}{(1-\delta-\lambda)}(1-(\frac{\delta}{1-\lambda})^{T-t-1})$ and $b_t = \dfrac{c}{\lambda\tilde{\beta}_t} + \frac{c(1-\beta_0)}{\beta_0(1-\lambda)^{t-1}}\frac{\delta}{(1-\delta-\lambda)}(1-(\frac{\delta}{1-\lambda})^{T-t})$, it suffices to show that:

$$\begin{aligned} &\frac{c}{\lambda\tilde{\beta}_t} + \tfrac{c(1-\beta_0)}{\beta_0(1-\lambda)^{t-1}}\tfrac{\delta}{(1-\delta-\lambda)}(1 - (\tfrac{\delta}{1-\lambda})^{T-t}) \\ = \frac{c}{\lambda\tilde{\beta}_t} - \frac{c}{\lambda\tilde{\beta}_{t+1}} - \delta c - (1+\delta\lambda)(&\frac{c}{\lambda\tilde{\beta}_{t+1}} + \tfrac{c(1-\beta_0)}{\beta_0(1-\lambda)^t}\tfrac{\delta}{(1-\delta-\lambda)}(1 - (\tfrac{\delta}{1-\lambda})^{T-t-1})), \end{aligned}$$

which is easily verified for any $1 \leqslant t \leqslant T$. *Q.E.D.*

We will demonstrate that with the proposed solution, any period is optimal for the agent to work in, regardless of previous effort history profile. Consider the final period T. Note that if the agent deviates and shirks his duties at some arbitrary period $t < T$, he only can be more optimistic at period T. Thus, for any history of prior effort, the current belief β_T can be higher only than $\tilde{\beta}_T$. Now MH_T $\lambda\tilde{\beta}_T b_T = c$ is satisfied since $\tilde{\beta}_T \geqslant \beta_T$ and $\lambda\beta_T b_T \geqslant c$. Next, assume that working in any period is optimal for the agent, regardless of the previous effort history profile at period $t+1 \leqslant T$. Consider period t as any history of prior effort with current beliefs β_t. Since we already showed that $b_t = \dfrac{c}{\lambda\tilde{\beta}_t} + \sum_{s=t+1}^{T} \delta^{s-t}(1-\lambda)^{s-t-1}(\lambda b_s - c)$, for any $\beta_t \geqslant \tilde{\beta}_t$ it is apparent that $b_t \geqslant \dfrac{c}{\lambda\beta_t} + \sum_{s=t+1}^{T} \delta^{s-t}(1-\lambda)^{s-t-1}(\lambda b_s - c)$, and working is optimal for the agent.

Finally, it can be shown by induction that any reward profile that makes every MH_t constraint binding must coincide with $b_t = \dfrac{c}{\lambda\tilde{\beta}_t} + c\sum_{s=1}^{T-t}\delta^s \frac{1-\beta_0}{\beta_0(1-\lambda)^{t+s-1}}$ for $1 \leqslant t \leqslant T$. Moreover, since we already proved that working for the agent is optimal in any period, regardless of the previous effort history profile, this also ensures that the agent would find it optimal to work in period t for any possible effort profile before t.

Recall that when solving $P1$, we ignored the IC constraint. Since $w_t = 0$ and $b_t > 0$, the proposed solution obviously satisfies $b_t \geqslant w_t$ for $t = 1, ..., T$. Thus the final condition we must check is $b_t \geqslant w_t + b_{t+1}$ for $t = 1, ..., T-1$.

Given that $\sum_{s=1}^{T-t} \frac{\delta^{s-t}}{(1-\lambda)^{s-1}} = \frac{1-(\frac{\delta}{1-\lambda})^{T-t}}{1-\frac{\delta}{1-\lambda}}$, by performing some algebra, one could verify that:

$$b_t = \frac{c}{\lambda\tilde{\beta}_t} + c\delta\tfrac{1-\beta_0}{\beta_0}\tfrac{1-(\frac{\delta}{1-\lambda})^{T-t}}{(1-\lambda)^{t-1}(1-\lambda-\delta)} = \frac{c}{\lambda\tilde{\beta}_t} + c\delta\tfrac{1-\beta_0}{\beta_0}\tfrac{(1-\lambda)^{T-t}-(1-\delta)^{T-t}}{(1-\lambda)^{T-1}(1-\lambda-\delta)}.$$

Then, it follows that $b_t \geqslant \delta b_{t+1}$ if and only if:

$$\tfrac{c}{\lambda}(\tfrac{\beta_0(1-\lambda)^{t-1}+1-\beta_0}{\beta_0(1-\lambda)^{t-1}} - \delta\tfrac{\beta_0(1-\lambda)^t+1-\beta_0}{\beta_0(1-\lambda)^t}) \geqslant \tfrac{c\delta(1-\beta_0)(\delta(1-\lambda)^{T-t-1}-\delta^{1+T-t-1}-(1-\lambda)^{T-t}+\delta^{T-t})}{\beta_0(1-\lambda)^{T-1}(1-\lambda-\delta)},$$

$$\tfrac{c(\beta_0(1-\lambda)^t(1-\delta)+(1-\beta_0)(1-\lambda-\delta))}{\lambda\beta_0(1-\lambda)^t} \geqslant -\tfrac{\delta c(1-\beta_0)}{\beta_0(1-\lambda)^t},$$

which holds as an equality if $\delta = 1$ (and as a strict inequality as long as $\delta < 1$) for any t.

Thus, $w_t = 0$ and $b_t = \dfrac{c}{\lambda\tilde{\beta}_t} + c\sum_{s=1}^{T-t}\delta^s \tfrac{1-\beta_0}{\beta_0(1-\lambda)^{t+s-1}}$ for $1 \leqslant t \leqslant T$ is a solution to the principal's optimization problem.

Finally, since $\lambda\tilde{\beta}_t b_t > c$, for all $1 \leqslant t \leqslant T-1$, the project is terminated inefficiently early, $T^{SB} \leqslant T^{FB}$. *Q.E.D.*

A.2 Proof of Proposition 1.2

We will call T^M_{Public} the duration of the contract when the principal performs monitoring and success is observed publicly. The total benefit, TE_m, from monitoring is a sum of the static and dynamic effects. The static effect from monitoring at period m, SE_m is:

$$SE_m = \delta^m\beta_0(1-\lambda)^{m-1}\lambda b_m,$$

where $b_m = \dfrac{c}{\lambda\tilde{\beta}_m} + c\sum_{s=1}^{T-m}\delta^s\tfrac{(1-\beta_0)}{\beta_0(1-\lambda)^{m+s-1}}$.

The dynamic effect is:

$$DE_m = \textstyle\sum_{t=1}^{m-1}\delta^t Prob(\text{success at } t < m)[\text{nominal decrease in } b_t].$$

First, we need to define a nominal decrease in b_t for $t < m$ that is possible because of monitoring that will occur at period m. Recall that the optimal payment structure makes all MH_t constraint binding or, equivalently,

$$b_t = \frac{c}{\lambda\tilde{\beta}_t} + \textstyle\sum_{s=t+1}^{T}\delta^{s-t}(1-\lambda)^{s-t-1}(\lambda b_s - c),$$

and by decreasing a reward in a certain period m (in the right-hand side), the principal can decrease a reward in the left-hand side. Thus, a nominal decrease in b_t for $t < m$ is:

$$\delta^{m-t}(1-\lambda)^{m-t-1}\lambda b_m.$$

As a result, the dynamic effect becomes:

$$DE_m = \textstyle\sum_{t=1}^{m-1}\delta^t(\beta_0(1-\lambda)^{t-1}\lambda)\delta^{m-t}(1-\lambda)^{m-t-1}\lambda b_m = \delta^m\beta_0\lambda^2(1-\lambda)^{m-2}(m-1)b_m.$$

We can then calculate the total effect from monitoring at period m:

$$\begin{aligned} TE_m &= \delta^m \beta_0 (1-\lambda)^{m-1} \lambda b_m + \delta^m \beta_0 \lambda^2 (1-\lambda)^{m-2} (m-1) b_m \\ &= \delta^m \beta_0 (1-\lambda)^{m-2} \lambda (1-\lambda+\lambda(m-1)) b_m. \end{aligned}$$

We will prove that TE_m is strictly decreasing in m. First, recall form Proposition 1 that b_t was chosen optimally such that $b_t \geqslant \delta b_{t+1}$ for $t = 1, ..., T-1$, and, as a result, $\delta^m b_m$ is decreasing in m.

It suffices to show that $\varphi(m) = (1-\lambda)^{m-2}\lambda(1-\lambda+\lambda(m-1))$ is decreasing in m as well. Notice that $\varphi(1) = \varphi(2) = 1$.

Because $\frac{d\varphi(m)}{dm} = (1-\lambda)^{m-2}(\lambda + (1-\lambda+\lambda(m-1))ln(1-\lambda))$, it is sufficient to show that $f(\lambda, m) = \lambda + (1-\lambda+\lambda(m-1))ln(1-\lambda)$ is negative for $m > 1$. Note that $\frac{\partial f}{\partial m} = \lambda ln(1-\lambda) < 0$ for any m and $f(\lambda, 2) = \lambda + ln(1-\lambda)$. Consider $g(\lambda) = f(\lambda, 2) = \lambda + ln(1-\lambda)$, with $\frac{\partial g}{\partial \lambda} = -\frac{\lambda}{1-\lambda}$ being negative for all $0 < \lambda < 1$ and $\lim_{\lambda \to +0} g(\lambda) = 0$. Thus, $f(\lambda, m)$ is negative, and, as a result, $\varphi(m)$ is decreasing in m.

Since TE_m is decreasing in m, monitoring is implemented optimally at the very first period. Given that the principal can promise paying less, he can use these funds to extend the duration of the relationship up to T^M_{Public} and, as a result, $T^{SB} \leqslant T^M_{Public}$. Since the agent still receives a positive rent, the project is still terminated inefficiently early: $T^M_{Public} \leqslant T^{FB}$. *Q.E.D.*

A.3 Proof of Proposition 1.3

We will call $T^M_{Private}$ the duration of the contract when the agent observes success privately and the principal performs monitoring. As in the case when success is publicly observed, the total effect, TE_m, of monitoring is a sum of the static and dynamic effects. However, for this case, we redefine both effects to account for the possibility of hiding success. In particular, the additional IC constraints become relevant:

$$b_t \geqslant w_t + \delta b_{t+1} \text{ for } t \leqslant T^M_{Private} - 1,$$

$$b_t \geqslant w_t \text{ for } t \leqslant T^M_{Private}.$$

Since we proved that all $w_t = 0$, the second constraint will not affect either of the two effects, whereas the first constraint will limit the amount of money the principal will save by monitoring, except during the final period of the relationship. The static effect of monitoring at period m, SE_m is:

$$SE_m = \delta^m \beta_0 (1-\lambda)^{m-1} \lambda (b_m - \delta b_{m+1}).^{[1]}$$

To simplify notation, we will define function η_m as follows:

$$\eta_m = \begin{cases} b_m - \delta b_{m+1} & 1 \leqslant m < T^M_{Private} \\ b_m & m = T^M_{Private} \end{cases}.$$

Thus, $SE_m = \delta^m \beta_0 (1-\lambda)^{m-1} \lambda \eta_m$ for $t = 1, ..., T^M_{Private}$.

The dynamic effect is then:

$$DE_m = \textstyle\sum_{t=1}^{m-1} \delta^t (\beta_0 (1-\lambda)^{t-1} \lambda) \delta^{m-t} (1-\lambda)^{m-t-1} \lambda \eta_m = \delta^m \beta_0 \lambda^2 (1-\lambda)^{m-2} (m-1) \eta_m.$$

Thus, the total effect becomes

$$TE_m = \delta^m \beta_0 (1-\lambda)^{m-2} \lambda (1 - \lambda + \lambda(m-1)) \eta_m = \delta^m \beta_0 \lambda \varphi(m) \eta_m.$$

Before we define the optimal timing of monitoring, consider agent's incentives to postpone announcement of success. Since now rewards for periods before monitoring are smaller some of the *IC* constraints might be violated. First, a nominal decrease in a reward in period t due to the dynamic effect, $\delta^{m-t}(1-\lambda)^{m-t-1}\lambda\eta_m$, is increasing in time; that is, $\frac{\delta^m (1-\lambda)^m \lambda \eta_m}{\delta^t (1-\lambda)^{t+1}}$ is strictly increasing in t. Recall from Proposition 1.1 that rewards without monitoring were (weakly) decreasing in time. Given this, rewards before monitoring period will be strictly decreasing in time as well and, as a result, the agent cannot benefit from postponing early success and announcing it at some period before he is monitored.

In addition, the discounted value of rewards for success after the period when the monitor is hired are decreasing as well by Proposition 1.1 since they are not modified. However, the agent might postpone an announcement till the period of monitoring. Given that monitoring occurs at period m the reward for success at this period is δb_{m+1} so that the agent does not postpone success in case it is achieved exactly at period m. Thus, it suffices to guarantee that

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[1] For convenience, we will say that $b_{T^M_{Private}+1} = 0$.

$$b_{m-1} - \delta^{m-t}(1-\lambda)^{m-t-1}\lambda\eta_m \geqslant \delta^2 b_{m+1} \text{ for } t = m-1,$$

which can be rewritten as

$$b_{m-1} - \delta\lambda\eta_m \geqslant \delta(b_m - \eta_m),$$

and will be shown to be satisfied for any m in Lemma 2.

Lemma 2. $b_{m-1} - \delta\lambda\eta_m \geqslant \delta(b_m - \eta_m)$ for $1 \leqslant m \leqslant T^M_{Private}$.

Proof: Recall that from Proposition 1.1, it follows that b_m is increasing strictly, whereas δb_{m+1} is decreasing strictly. In particular, for $t = 1, ..., T^M_{Private} - 1$:

$$\begin{gathered}\eta_m = b_m - \delta b_{m+1} = \\ \tfrac{c}{\lambda}\big(\tfrac{\beta_0(1-\lambda)^{m-1}+1-\beta_0}{\beta_0(1-\lambda)^{m-1}} - \delta\tfrac{\beta_0(1-\lambda)^{m}+1-\beta_0}{\beta_0(1-\lambda)^{m}}\big) - \tfrac{c\delta(1-\beta_0)(\delta(1-\lambda)^{T-m-1}-\delta^{1+T-m-1}-(1-\lambda)^{T-m}+\delta^{T-m})}{\beta_0(1-\lambda)^{T-1}(1-\lambda-\delta)} \\ = \tfrac{c(\beta_0(1-\lambda)^m(1-\delta)+(1-\beta_0)(1-\lambda-\delta))}{\lambda\beta_0(1-\lambda)^m} + \tfrac{\delta c(1-\beta_0)}{\beta_0(1-\lambda)^m} = \tfrac{c(1-\delta)(\beta_0(1-\lambda)^{m-1}+1-\beta_0)}{\lambda\beta_0(1-\lambda)^{m-1}}.\end{gathered}$$

As a result, η_m evolves as follows:

$$\eta_m = \begin{cases} (1-\delta)\tfrac{c(\beta_0(1-\lambda)^{m-1}+1-\beta_0)}{\lambda\beta_0(1-\lambda)^{m-1}} & 1 \leqslant m < T^M_{Private} \\ \tfrac{c(\beta_0(1-\lambda)^{m-1}+1-\beta_0)}{\lambda\beta_0(1-\lambda)^{m-1}} & m = T^M_{Private} \end{cases}.$$

Note that $b_{m-1} - \delta\lambda\eta_m \geqslant \delta(b_m - \eta_m)$ can be rewritten using the definition of η_m as follows:

$$\eta_{m-1} \geqslant \delta(1-\lambda)\eta_m,$$

and simplifying we have

$$\begin{gathered}(1-\delta)\tfrac{c(\beta_0(1-\lambda)^{m-2}+1-\beta_0)}{\lambda\beta_0(1-\lambda)^{m-2}} \geqslant \delta(1-\lambda)(1-\delta)\tfrac{c(\beta_0(1-\lambda)^{m-1}+1-\beta_0)}{\lambda\beta_0(1-\lambda)^{m-1}}, \\ \beta_0(1-\lambda)^{m-2} + 1 - \beta_0 \geqslant \delta(\beta_0(1-\lambda)^{m-1} + 1 - \beta_0), \\ \beta_0(1-\lambda)^{m-2}(1-\delta(1-\lambda)) + (1-\delta)(1-\beta_0) \geqslant 0.\end{gathered}$$

which holds as an equality in case $\delta = 1$ (and as a strict inequality as long as $\delta < 1$) for any period of monitoring $1 \leqslant m \leqslant T^M_{Private}$. *Q.E.D.*

First, consider the case when $\delta = 1$. From Proposition 1, it follows that optimal b_m is constant over time, and, as a result, $\eta_m = 0$ for $t \leqslant T^M_{Private} - 1$. For the final period, however, $\eta_{T^M_{Private}} = b_{T^M_{Private}} > 0$. Clearly, monitoring at the final period is optimal.

Consider now the case for $\delta < 1$.

$$TE_m = \delta^m \beta_0 (1-\lambda)^{m-2} \lambda (1-\lambda+\lambda(m-1)) \eta_m$$
$$= \begin{cases} (1-\delta) c^{\frac{\delta^m (\beta_0 (1-\lambda)^{m-1} + 1 - \beta_0)(1-\lambda+\lambda(m-1))}{1-\lambda}} & 1 \leqslant m < T^M_{Private} \\ c^{\frac{\delta^m (\beta_0 (1-\lambda)^{m-1} + 1 - \beta_0)(1-\lambda+\lambda(m-1))}{1-\lambda}} & m = T^M_{Private} \end{cases}.$$

We define time period $1 \leqslant j \leqslant T^M_{Private}$ as follows:

$$j \in \arg\max_{1 \leqslant s < T^M_{Private}} c^{\frac{\delta^s (\beta_0 (1-\lambda)^{s-1} + 1 - \beta_0)(1-\lambda+\lambda(s-1))}{1-\lambda}}.[2]$$

We will show that with a high enough discount factor, monitoring at the final period is always optimal. First, there is a (unique) $\tilde{\delta}$, such that:

$$(1-\tilde{\delta}) c^{\frac{\delta^j (\beta_0 (1-\lambda)^{j-1} + 1 - \beta_0)(1-\lambda+\lambda(j-1))}{1-\lambda}} = c^{\frac{\delta^{T^M_{Private}} (\beta_0 (1-\lambda)^{T^M_{Private}-1} + 1 - \beta_0)(1-\lambda+\lambda(T^M_{Private}-1))}{1-\lambda}}$$
$$\text{or, equivalently, } \tilde{\delta} = 1 - \frac{c^{\frac{\delta^{T^M_{Private}} (\beta_0 (1-\lambda)^{T^M_{Private}-1} + 1 - \beta_0)(1-\lambda+\lambda(T^M_{Private}-1))}{1-\lambda}}}{c^{\frac{\delta^j (\beta_0 (1-\lambda)^{j-1} + 1 - \beta_0)(1-\lambda+\lambda(j-1))}{1-\lambda}}} \leqslant 1.$$

For any $\delta > \tilde{\delta}$ the total effect of monitoring at period m, TE_m achieves its highest value at the final period $T^M_{Private}$, whereas when $\delta < \tilde{\delta}$ monitoring is implemented at period j, which, in general, is not the final period. As the discount factor increases, both the static and dynamic effects diminish, and monitoring is implemented optimally at the final period. Since the principal promises paying less, the money that he saves could be used to extend the duration of the relationship beyond $T^M_{Private}$. In comparison with the case in which success is observed publicly, the principal saves less money; however, the second-best outcome is improved marginally: $T^{SB} \leqslant T^M_{Private} \leqslant T^M_{Public} \leqslant T^{FB}$. *Q.E.D.*

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[2] Since $T^M_{Private} < \infty$, a time period j is well defined, although it might be not unique due to the non-monotonicity of $\delta^m (\beta_0 (1-\lambda)^{m-1} + 1 - \beta_0)(1-\lambda+\lambda(m-1))$ in m.

Appendix B

TECHNICAL DETAILS, CHAPTER 2

Proof of Proposition 2.1. The principal's optimization problem is to choose contracts ϖ^H and ϖ^L to[1]

$$\begin{gathered}\max\{\nu[\beta_0 \textstyle\sum_{t=1}^{T^H} \delta^t(1-\lambda^H)^{t-1}\lambda^H(V(q_t^H(\underline{c})) - y_t^H - \underline{c}q_t^H(\underline{c}) - \mathit{\Gamma}_t)\\ +\delta^{T^H}P^H_{T^H}(V(q^H(c^H_{T^H})) - x^H - c^H_{T^H}q^H(c^H_{T^H}) - \mathit{\Gamma}_{T^H})]\\ +(1-\nu)[\beta_0 \textstyle\sum_{t=1}^{T^L} \delta^t(1-\lambda^L)^{t-1}\lambda^L(V(q_t^L(\underline{c})) - y_t^L - \underline{c}q_t^L(\underline{c}) - \mathit{\Gamma}_t)\\ +\delta^{T^L}P^L_{T^L}(V(q^L(c^L_{T^L})) - x^L - c^L_{T^L}q^L(c^L_{T^L}) - \mathit{\Gamma}_{T^L})]\}\text{ s.t.}\end{gathered}$$

$$\begin{gathered}(IC^{L,H})\ \beta_0 \textstyle\sum_{t=1}^{T^L} \delta^t(1-\lambda^L)^{t-1}\lambda^L y_t^L + \delta^{T^L}P^L_{T^L}x^L \geqslant\\ \beta_0 \textstyle\sum_{t=1}^{T^H} \delta^t(1-\lambda^L)^{t-1}\lambda^L y_t^H + \delta^{T^H}P^L_{T^H}[x^H + \triangle c_{T^H}q^H(c^H_{T^H})],\\ (IC^{H,L})\ \beta_0 \textstyle\sum_{t=1}^{T^H} \delta^t(1-\lambda^H)^{t-1}\lambda^H y_t^H + \delta^{T^H}P^H_{T^H}x^H \geqslant\\ \beta_0 \textstyle\sum_{t=1}^{T^L} \delta^t(1-\lambda^H)^{t-1}\lambda^H y_t^L + \delta^{T^L}P^H_{T^L}[x^L - \triangle c_{T^L}q^L(c^L_{T^L})].\end{gathered}$$

$$(IRS_t^H) \qquad y_t^H \geqslant 0 \text{ for } t \leqslant T^H,$$

$$(IRS_t^L) \qquad y_t^L \geqslant 0 \text{ for } t \leqslant T^L,$$

$$(IRF^H_{T^H}) \qquad x^H \geqslant 0,$$

$$(IRF^L_{T^L}) \qquad x^L \geqslant 0.$$

It turns out in this case the solution to the problem crucially depends on whether the $(IC^{H,L})$ constraint is binding or not; we explore each case separately in what follows.

Case 1

The $(IC^{H,L})$ constraint is not binding.

In this case the high type does not receive any rent and it immediately follows that $x^H = 0$ and $y_t^H = 0$ for $1 \leqslant t \leqslant T^H$. The constraint $(IC^{L,H})$ must be binding. If the

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[1] Where $\mathit{\Gamma}_t = \frac{\sum_{s=1}^{t}\delta^s}{\delta^t}\gamma$.

$(IC^{L,H})$ constraint was not binding, it would be possible to decrease the payment to the low type without violating any other constraint and improve the value of the objective function. In addition, the $(IC^{H,L})$ constraint simplifies to

$$(IC^{H,L}) \qquad \beta_0 \sum_{t=1}^{T^L} \delta^t (1-\lambda^H)^{t-1} \lambda^H y_t^L + \delta^{T^L} P_{T^L}^H x^L \leqslant \delta^{T^L} P_{T^L}^H \triangle c_{T^L} q^L(c_{T^L}^L).$$

Using the binding constraint $(IC^{L,H})$ to define informational rent of the low type as $\rho^L(T^H) = \delta^{T^H} P_{T^H}^L \triangle c_{T^H} q^H(c_{T^H}^H)$ and to replace x^L in the objective function, the principal's optimization problem is to choose T^H, $\{q_t^H(\underline{c})\}_{t=1}^{T^H}$, $q^H(c_{T^H}^H)$, T^L, $\{y_t^L\}_{t=1}^{T^L}$, $\{q_t^L(\underline{c})\}_{t=1}^{T^L}$, and $q^L(c_{T^L}^L)$

$$\begin{gathered}
\max\{\nu[\beta_0 \sum_{t=1}^{T^H} \delta^t (1-\lambda^H)^{t-1} \lambda^H (V(q_t^H(\underline{c})) - \underline{c} q_t^H(\underline{c}) - \Gamma_t) \\
+\delta^{T^H} P_{T^H}^H (V(q^H(c_{T^H}^H)) - c_{T^H}^H q^H(c_{T^H}^H) - \Gamma_{T^H})] \\
+(1-\nu)[\beta_0 \sum_{t=1}^{T^L} \delta^t (1-\lambda^L)^{t-1} \lambda^L (V(q_t^L(\underline{c})) - \underline{c} q_t^L(\underline{c}) - \Gamma_t) \\
+\delta^{T^L} P_{T^L}^L (V(q^L(c_{T^L}^L)) - c_{T^L}^L q^L(c_{T^L}^L) - \Gamma_{T^L})] - (1-\nu)\rho^L(T^H)\} \text{ s.t.}
\end{gathered}$$

$$(IC^{H,L}) \qquad \begin{gathered} \beta_0 \sum_{t=1}^{T^L} \delta^t (1-\lambda^H)^{t-1} \lambda^H y_t^L + \\ P_{T^L}^H \Big(\frac{\delta^{T^H} P_{T^H}^L \triangle c_{T^H} q^H(c_{T^H}^H) - \beta_0 \sum_{t=1}^{T^L} \delta^t (1-\lambda^L)^{t-1} \lambda^L y_t^L}{P_{T^L}^L}\Big) \leqslant \\ \delta^{T^L} P_{T^L}^H \triangle c_{T^L} q^L(c_{T^L}^L), \end{gathered}$$

$$(IRS_t^L) \qquad y_t^L \geqslant 0 \text{ for } t \leqslant T^L,$$

$$(IRF_{T^L}^L) \qquad \delta^{T^H} P_{T^H}^L \triangle c_{T^H} q^H(c_{T^H}^H) - \beta_0 \sum_{t=1}^{T^L} \delta^t (1-\lambda^L)^{t-1} \lambda^L y_t^L \geqslant 0.$$

Labeling $\{\alpha_t^L\}_{t=1}^{T^L}$, ξ^H, α^L as the Lagrange multipliers of the constraints associated with (IRS_t^L), $(IC^{H,L})$ and $(IRF_{T^L}^L)$ respectively, the optimal contract has the following Lagrangian:

$$\begin{gathered}
\mathcal{L}(T^H, \{q_t^H(\underline{c})\}_{t=1}^{T^H}, q^H(c_{T^H}^H), T^L, \{y_t^L\}_{t=1}^{T^L}, \{q_t^L(\underline{c})\}_{t=1}^{T^L}, q^L(c_{T^L}^L), \{\alpha_t^L\}_{t=1}^{T^L}, \xi^H, \alpha^L) \\
= \nu[\beta_0 \sum_{t=1}^{T^H} \delta^t (1-\lambda^H)^{t-1} \lambda^H (V(q_t^H(\underline{c})) - \underline{c} q_t^H(\underline{c}) - \Gamma_t) \\
+\delta^{T^H} P_{T^H}^H (V(q^H(c_{T^H}^H)) - c_{T^H}^H q^H(c_{T^H}^H) - \Gamma_{T^H})] \\
+(1-\nu)[\beta_0 \sum_{t=1}^{T^L} \delta^t (1-\lambda^L)^{t-1} \lambda^L (V(q_t^L(\underline{c})) - \underline{c} q_t^L(\underline{c}) - \Gamma_t) \\
+\delta^{T^L} P_{T^L}^L (V(q^L(c_{T^L}^L)) - c_{T^L}^L q^L(c_{T^L}^L) - \Gamma_{T^L})] - (1-\nu)\rho^L(T^H) \\
+\xi^H (\delta^{T^L} P_{T^L}^H \triangle c_{T^L} q^L(c_{T^L}^L) - \frac{\delta^{T^H} P_{T^L}^H P_{T^H}^L \triangle c_{T^H} q^H(c_{T^H}^H)}{P_{T^L}^L} \\
+\beta_0 \sum_{t=1}^{T^L} \delta^t (\frac{P_{T^L}^H}{P_{T^L}^L} (1-\lambda^L)^{t-1} \lambda^L - (1-\lambda^H)^{t-1} \lambda^H) y_t^L) + \sum_{t=1}^{T^L} \alpha_t^L y_t^L
\end{gathered}$$

$$+\alpha^L(\delta^{T^H} P^L_{T^H} \triangle c_{T^H} q^H(c^H_{T^H}) - \beta_0 \textstyle\sum_{t=1}^{T^L} \delta^t(1-\lambda^L)^{t-1}\lambda^L y^L_t).$$

The Inada conditions give us interior solutions for $q^H_t(\underline{c})$, $q^H(c^H_{T^H})$, $q^L_t(\underline{c})$, and $q^L(c^L_{T^L})$. We also assumed that $T^L > 0$ and $T^H > 0$. Therefore, the Kuhn-Tucker conditions for the optimization problem are:

$\frac{\partial\mathcal{L}}{\partial T^H} = 0$; $\frac{\partial\mathcal{L}}{\partial q^H_t(\underline{c})} = 0$ for $1 \leqslant t \leqslant T^H$; $\frac{\partial\mathcal{L}}{\partial q^H(c^H_{T^H})} = 0$;
$\frac{\partial\mathcal{L}}{\partial T^L} = 0$; $\frac{\partial\mathcal{L}}{\partial q^L_t(\underline{c})} = 0$ for $1 \leqslant t \leqslant T^L$; $\frac{\partial\mathcal{L}}{\partial q^L(c^L_{T^L})} = 0$;
$\frac{\partial\mathcal{L}}{\partial y^L_t} = 0$ for $1 \leqslant t \leqslant T^L$;
$\frac{\partial\mathcal{L}}{\partial \alpha^L_t} \geqslant 0$; $\alpha^L_t \geqslant 0$; $\alpha^L_t \frac{\partial\mathcal{L}}{\partial \alpha^L_t} = 0$ for $1 \leqslant t \leqslant T^L$;
$\frac{\partial\mathcal{L}}{\partial \xi^H} \geqslant 0$; $\xi^H \geqslant 0$; $\xi^H \frac{\partial\mathcal{L}}{\partial \xi^H} = 0$;
$\frac{\partial\mathcal{L}}{\partial \alpha^L} \geqslant 0$; $\alpha^L \geqslant 0$; $\alpha^L \frac{\partial\mathcal{L}}{\partial \alpha^L} = 0$;

which include the constraints from the problem themselves and corresponding complementary slackness conditions.

Since the $(IC^{H,L})$ constraint is not binding, $\xi^H = 0$.

For now, let T^L and T^H to be the optimal duration of the experimentation stage for the low and high types, respectively. We will determine both variables and specify if each of them is above or below the first-best level later.

If $\xi^H = 0$, the claim below shows that there are no restrictions in choosing $\{y^L_t\}_{t=1}^{T^L}$ except those imposed by the $(IC^{L,H})$ constraint. In other words, when the $(IC^{H,L})$ constraint is not binding, the principal can choose any combinations of payments to the low type $(x^L, \{y^L_t\}_{t=1}^{T^L})$ such that the $(IC^{L,H})$ constraint is binding.

Claim 1: $\xi^H = 0 \Rightarrow \alpha^L = 0$ and α^L_t for all $t \leqslant T^L$.

Proof: We can rewrite the Kuhn-Tucker conditions as follows:

$\frac{\partial\mathcal{L}}{\partial y^L_t} = \alpha^L_t - \alpha^L \beta_0 \delta^t (1-\lambda^L)^{t-1}\lambda^L = 0$ for $1 \leqslant t \leqslant T^L$;
$\frac{\partial\mathcal{L}}{\partial \alpha^L_t} = y^L_t \geqslant 0$; $\alpha^L_t \geqslant 0$; $\alpha^L_t y^L_t = 0$ for $1 \leqslant t \leqslant T^L$;

Suppose to the contrary that $\alpha^L > 0$. Then,

$$\delta^{T^H} P^L_{T^H} \triangle c_{T^H} q^H(c^H_{T^H}) - \beta_0 \textstyle\sum_{t=1}^{T^L} \delta^t(1-\lambda^L)^{t-1}\lambda^L y^L_t = 0,$$

and there must exist $y^L_s > 0$ for some $1 \leqslant s \leqslant T^L$. Then, we have $\alpha^L_s = 0$, which leads to a contradiction since $\frac{\partial\mathcal{L}}{\partial y^L_t} = 0$ cannot be satisfied unless $\alpha^L = 0$. Suppose to the contrary that α_s^L for some $1 \leqslant s \leqslant T^L$. Then, $\alpha^L > 0$, which leads to a contradiction as we have just shown above. *Q.E.D.*

Claim 1 allows us to prove that the duration of the experimentation stage for the low type is not distorted. The multipliers $\xi^H = \alpha^L = 0$ and α_t^L for all $t \leqslant T^L$ by claim 1, and the low type's rent $\rho^L(T^H) = \delta^{T^H} P^L_{T^H} \triangle c_{T^H} q^H(c^H_{T^H})$ is not affected by T^L. Therefore, the F.O.C. with respect to T^L is identical to that under first best:

$$\tfrac{\partial \mathcal{L}}{\partial T^L} = \tfrac{\partial E_\theta \pi^{FB}(\varpi^\theta)}{\partial T^L} = 0,$$

or, equivalently, $T^L_{SB} = T^L_{FB}$ when $(IC^{H,L})$ is not binding.

However, since the low type's information rent depends on T^H, there will be a distortion in the duration of the experimentation stage for the high type:

$$\tfrac{\partial \mathcal{L}}{\partial T^H} = \tfrac{\partial \{E_\theta \pi^{FB}(\varpi^\theta) - (1-\nu)\rho^L(T^H)\}}{\partial T^H} = 0,$$

Since the informational rent of the low-type agent, $\rho^L(T^H) = \delta^{T^H} P^L_{T^H} \triangle c_{T^H} q^H(c^H_{T^H})$, is non-monotonic in T^H, it is not possible, in general, to determine whether $T^H_{SB} > T^H_{FB}$ or $T^H_{SB} < T^H_{FB}$.

To characterize the optimal output choices, consider now the following Kuhn-Tucker conditions for the optimization problem:

$[q_t^H(\underline{c})]$ $\nu\beta_0\delta^t(1-\lambda^H)^{t-1}\lambda^H(V'(q_t^H(\underline{c})) - \underline{c}) = 0$ for $1 \leqslant t \leqslant T^H$;
$[q_t^L(\underline{c})]$ $(1-\nu)\beta_0\delta^t(1-\lambda^L)^{t-1}\lambda^L(V'(q_t^L(\underline{c})) - \underline{c}) = 0$ for $1 \leqslant t \leqslant T^L$;
$[q^H(c^H_{T^H})]$ $\nu\delta^{T^H} P^H_{T^H}(V'(q^H(c^H_{T^H})) - c^H_{T^H}) - (1-\nu)\delta^{T^H} P^L_{T^H} \triangle c_{T^H} = 0$;
$[q^L(c^L_{T^L})]$ $(1-\nu)\delta^{T^L} P^L_{T^L}(V'(q^L(c^L_{T^L})) - c^L_{T^L} = 0)$;

. The conditions above imply that there is no distortion in the output relative to the first-best level after success has been observed by either type; that is $V'(q_t^\theta(\underline{c})) = \underline{c}$ for $\theta \in \{H, L\}$. There is no distortion in the output relative to the first-best level after T^L failures have been reported by the low type either; that is

$$q^L_{SB}(c^L_{T^L}) = q^L_{FB}(c^L_{T^L}).$$

However, $V'(q^H(c^H_{T^H})) = c^H_{T^H} + \frac{(1-\nu)P^L_{T^H}}{\nu P^H_{T^H}} \triangle c_{T^H}$, which given that function $V(\cdot)$ is increasing and concave implies underproduction by the high type after T^H failures have been reported by the high type:

$$q^H_{SB}(c^H_{T^H}) < q^H_{FB}(c^H_{T^H}).$$

Case 2

The $(IC^{H,L})$ constraint is binding.

We will show that when the $(IC^{H,L})$ becomes binding there are certain restrictions on the payment structure to the low type. In particular, we will show that when the duration of the experimentation stage for the low type is long enough, it is optimal to reward later success. In contrast, when T^L is sufficiently small, the low type is rewarded for failure.

Case 2.1: The high type does not receive any rent.

Consider first the case when the high type does not receive any rent; it immediately follows that $x^H = 0$ and $y_t^H = 0$ for $1 \leqslant t \leqslant T^H$. The constraint $(IC^{L,H})$ must be binding. As in the previous case, let T^L and T^H to be the optimal duration of the experimentation stage for the low and high types, respectively.

We can rewrite the Kuhn-Tucker conditions as follows:

$\frac{\partial \mathcal{L}}{\partial y_t^L} = \xi^H \beta_0 \delta^t (\frac{P_{T^L}^H}{P_{T^L}^L}(1-\lambda^L)^{t-1}\lambda^L - (1-\lambda^H)^{t-1}\lambda^H) + \alpha_t^L - \alpha^L \beta_0 \delta^t (1-\lambda^L)^{t-1}\lambda^L = 0$ for $1 \leqslant t \leqslant T^L$;

$\frac{\partial \mathcal{L}}{\partial \alpha_t^L} = y_t^L \geqslant 0$; $\alpha_t^L \geqslant 0$; $\alpha_t^L y_t^L = 0$ for $1 \leqslant t \leqslant T^L$;

The first set of conditions can be rewritten as

$$\xi^H \beta_0 \delta^t f_2(t) + \alpha_t^L - \alpha^L \beta_0 \delta^t (1-\lambda^L)^{t-1}\lambda^L = 0 \text{ for } 1 \leqslant t \leqslant T^L,$$

where $f_2(t)$ is defined as

$$f_2(t, T^L) = \frac{P_{T^L}^H}{P_{T^L}^L}(1-\lambda^L)^{t-1}\lambda^L - (1-\lambda^H)^{t-1}\lambda^H.$$

Claim 2.1.1: There exists a unique $\hat{T}^L > 1$, such that $f_2(\hat{T}^L) = 0$, and

$$f_2(t, T^L) = \begin{cases} < 0, & t < \hat{T}^L \\ > 0, & t > \hat{T}^L \end{cases}$$

Proof: Recall that P_T^θ is defined as

$$P_T^\theta = 1 - \beta_0 + \beta_0(1-\lambda^\theta)^T,$$

and is equal to the probability that the agent of type does not succeed during the experimentation stage which lasted for $T \in N$ periods. Then $\frac{P^H_{T^L}}{P^L_{T^L}}$ is a ratio of probability that the high type does not succeed to probability that the low type does not succeed for T^L periods during the experimentation stage. At the same time, $\beta_0(1-\lambda^\theta)^{t-1}\lambda^\theta$ is probability that the agent of type θ succeeds at period $t \leqslant T^L$ of the experimentation stage and $\frac{\beta_0(1-\lambda^H)^{t-1}\lambda^H}{\beta_0(1-\lambda^L)^{t-1}\lambda^L} = \frac{(1-\lambda^H)^{t-1}\lambda^H}{(1-\lambda^L)^{t-1}\lambda^L}$ is a ratio of probabilities of success at period t by two types. As a result, we can rewrite $f_2(t, T^L) > 0$ as

$$\frac{1-\beta_0+\beta_0(1-\lambda^H)^{T^L}}{1-\beta_0+\beta_0(1-\lambda^L)^{T^L}} > \frac{(1-\lambda^H)^{t-1}\lambda^H}{(1-\lambda^L)^{t-1}\lambda^L} \text{ for } 1 \leqslant t \leqslant T^L \text{ or, equivalently,}$$
$$\frac{1-\beta_0+\beta_0(1-\lambda^H)^{T^L}}{(1-\lambda^H)^{t-1}\lambda^H} > \frac{1-\beta_0+\beta_0(1-\lambda^L)^{T^L}}{(1-\lambda^L)^{t-1}\lambda^L} \text{ for } 1 \leqslant t \leqslant T^L,$$

where $\frac{1-\beta_0+\beta_0(1-\lambda^\theta)^{T^L}}{(1-\lambda^\theta)^{t-1}\lambda^\theta}$ can be interpreted as a likelihood ratio.

We will say that when $f_2(t, T^L) > 0$ (< 0) the high type is relatively more likely to fail (succeed) than the low type during the experimentation stage if he chooses a contract designed for the low type.

There exists a unique time period $\hat{T}^L(T^L, \lambda^L, \lambda^H, \beta_0)$ such that $f_2(\hat{T}^L, T^L) = 0$ defined as

$$\hat{T}^L \equiv \hat{T}^L(T^L, \lambda^L, \lambda^H, \beta_0) = 1 + \frac{ln(\frac{P^H_{T^L}\lambda^L}{P^L_{T^L}\lambda^H})}{ln(\frac{1-\lambda^H}{1-\lambda^L})},$$

where uniqueness follows from $\frac{(1-\lambda^H)^{t-1}\lambda^H}{(1-\lambda^L)^{t-1}\lambda^L}$ being strictly decreasing in t and $\frac{\lambda^H}{\lambda^H} > 1 > \frac{P^H_{T^L}}{P^L_{T^L}}$.[2] In addition, for $t < \hat{T}^L$ it follows that $f_2(t, T^L) < 0$ and, as a result, the high type is relatively more likely to succeed than the low type whereas for $t > \hat{T}^L$ the opposite is true. This feature plays an important role in structuring an optimal contract and should appear intuitive. If the high type accepts a contract designed for the low type, he has a different probability of success in every period $t \leqslant T^L$ conditional on not succeeding in periods before and the project being good. For early periods of the information gathering stage the high type is more likely to succeed. At the same time, as time goes without

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[2] To explain, $f_2(t, T^L) = 0$ if and only if $\frac{1-\beta_0+\beta_0(1-\lambda^H)^{T^L}}{1-\beta_0+\beta_0(1-\lambda^L)^{T^L}} = \frac{(1-\lambda^H)^{t-1}\lambda^H}{(1-\lambda^L)^{t-1}\lambda^L}$. Given that the right hand side of the equation above is strictly decreasing since $\frac{1-\lambda^H}{1-\lambda^L} < 1$ and if evaluated at $t = 1$ is equal to $\frac{\lambda^H}{\lambda^L}$. Since $\frac{1-\beta_0+\beta_0(1-\lambda^H)^{T^L}}{1-\beta_0+\beta_0(1-\lambda^L)^{T^L}} < 1$ and $\frac{\lambda^H}{\lambda^L} > 1$ the uniqueness immediately follows. So $\hat{T}^L$ satisfies $\frac{P^H_{T^L}}{P^L_{T^L}} = \frac{(1-\lambda^H)^{\hat{T}^L-1}\lambda^H}{(1-\lambda^L)^{\hat{T}^L-1}\lambda^L}$.

success the high type gets much more pessimistic and conditional probability of success, $\beta_0(1-\lambda^H)^{t-1}\lambda^H$, becomes smaller than that of the low type. *Q.E.D.*

Note that $\hat{T}^L$ is uniquely defined as we already proved that $f_2(t, T^L)$ has a unique solution. When $T^L < \hat{T}^L$ we showed that $f_2(t, T^L) < 0$ for all $t \leqslant T^L$, whereas if $T^L > \hat{T}^L$ then for periods $T^L \geqslant t \geqslant \hat{T}^L$ function $f_2(t)$ becomes strictly positive. Intuitively, if the experimentation stage for the low type is relatively short then the high type is more likely to succeed before the implementation stage begins. However, as the experimentation stage for the low type gets longer the high type becomes more pessimistic as time goes without success and closer to the end of the information gathering stage he becomes relatively less likely to succeed.

Now we can formally prove that depending on the value of T^L the low type is rewarded either for *late* success or failures.

Claim 2.1.2: $\alpha^L = 0 \Longleftrightarrow T^L \leqslant \hat{T}^L$, and it is optimal to set $x^L > 0$ and $y_t^L = 0$ for all $t \leqslant T^L$. [If $T^L < \hat{T}^L$, then $x^L > 0$ and $y_t^L = 0$ for all $t \leqslant T^L$ is uniquely optimal, but if $T^L = \hat{T}^L$, then both $x^L > 0$ and $y^L_{T^L} > 0$ can be optimal.][3]

Proof: Suppose $\alpha^L = 0$, then it must be that $\xi^H \beta_0 \delta^t f_2(t) + \alpha_t^L$ for $1 \leqslant t \leqslant T^L$. Since we have $\alpha_t^L \geqslant 0$ for all $t \leqslant T^L$, we must have $f_2(t, T^L) \leqslant 0$, and $T^L \leqslant \hat{T}^L$ by the properties of $f_2(t, T^L)$ in the claim 2.1.1 above. It is optimal to set

$x^L = \delta^{T^H - T^L} \frac{P^L_{T^H}}{P^L_{T^L}} \triangle c_{T^H} q^H(c^H_{T^H})$ and $y_t^L = 0$ for all $t \leqslant T^L$.[4] *Q.E.D.*

Claim 2.1.3: $\alpha^L > 0 \Longleftrightarrow T^L > \hat{T}^L$, and it is optimal to set $x^L = 0$ and $y_t^L > 0$ for some $t > \hat{T}^L$.

Proof: If $\alpha^L > 0$, then it must be that

$$\delta^{T^H} P^L_{T^H} \triangle c_{T^H} q^H(c^H_{T^H}) - \beta_0 \sum_{t=1}^{T^L} \delta^t (1-\lambda^L)^{t-1} \lambda^L y_t^L = 0,$$

and there must exist $y_k^L > 0$ for some $1 \leqslant k \leqslant T^L$, and we have $\xi^H \beta_0 \delta^k f_2(k) + \alpha_k^L - \alpha^L \beta_0 \delta^k (1-\lambda^L)^{k-1} \lambda^L = 0$, with α_k^L, which implies $f_2(k) > 0$. From claim 2.1.1, this is only possible if $k > \hat{T}^L$, which in turn implies that $T^L > \hat{T}^L$. Thus, for $t \leqslant \hat{T}^L$ it must be

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[3] $\alpha_t^L > 0$ for all $t < \hat{T}^L$ since $f_2 < 0$.

[4] If $T^L < \hat{T}^L$, then $x^L > 0$ and $y_t^L = 0$ for all $t \leqslant T^L$ is uniquely optimal, but if $T^L = \hat{T}^L$, then both $x^L > 0$ and $y^L_{T^L} > 0$ can be optimal.

that all $\alpha_t^L > 0$, and for $t > \hat{T}^L$ it follows that α_t^L might be zero. It is optimal to choose $x^L = 0$ and $y_t^L > 0$ for some $t > \hat{T}^L$. *Q.E.D.*

In summary, when $T^L > \hat{T}^L$, the low type receives rent after success ($y_t^L > 0$) only if $t > \hat{T}^L$. When $T^L \leqslant \hat{T}^L$ the low type receives his rent only through x^L, i.e., when he does not succeed in the experimentation stage.

Since the Lagrange multipliers ξ^H, α^L and α_k^L can be positive, we can now have distortion in both T^L and T^H. In particular, the F.O.C. with respect to T^L becomes:

$$\frac{\partial \mathcal{L}}{\partial T^L} = \frac{\partial \{E_\theta \pi_{FB}(\varpi^\theta)\}}{\partial T^L} +$$
$$\frac{\partial\{\xi^H(\delta^{T^L} P^H_{T^L} \triangle c_{T^L} q^L(c^L_{T^L}) - \frac{\delta^{T^H} P^H_{T^L} P^L_{T^H} \triangle c_{T^H} q^H(c^H_{T^H})}{P^L_{T^L}} + \beta_0 \sum_{t=1}^{T^L} \delta^t (\frac{P^H_{T^L}}{P^L_{T^L}} (1-\lambda^L)^{t-1}\lambda^L - (1-\lambda^H)^{t-1}\lambda^H) y_t^L)\}}{\partial T^L} +$$
$$\frac{\partial\{\sum_{t=1}^{T^L} \alpha_t^L y_t^L + \alpha^L(\delta^{T^H} P^L_{T^H} \triangle c_{T^H} q^H(c^H_{T^H}) - \beta_0 \sum_{t=1}^{T^L} \delta^t (1-\lambda^L)^{t-1}\lambda^L y_t^L)\}}{\partial T^L} = 0,$$

Moreover, since the high type's information rent depends on T^H, there will be a distortion in the duration of the experimentation stage for the high type:

$$\frac{\partial \mathcal{L}}{\partial T^H} = \frac{\partial \{E_\theta \pi_{FB}(\varpi^\theta) - (1-\nu)\rho^L(T^H)\}}{\partial T^H} +$$
$$\frac{\partial\{\xi^H(\delta^{T^L} P^H_{T^L} \triangle c_{T^L} q^L(c^L_{T^L}) - \frac{\delta^{T^H} P^H_{T^L} P^L_{T^H} \triangle c_{T^H} q^H(c^H_{T^H})}{P^L_{T^L}} + \beta_0 \sum_{t=1}^{T^L} \delta^t (\frac{P^H_{T^L}}{P^L_{T^L}} (1-\lambda^L)^{t-1}\lambda^L - (1-\lambda^H)^{t-1}\lambda^H) y_t^L)\}}{\partial T^H} +$$
$$\frac{\partial \alpha^L\{(\delta^{T^H} P^L_{T^H} \triangle c_{T^H} q^H(c^H_{T^H}) - \beta_0 \sum_{t=1}^{T^L} \delta^t (1-\lambda^L)^{t-1}\lambda^L y_t^L)\}}{\partial T^H} = 0,$$

It is not possible, in general, to determine whether $T^H_{SB} > T^H_{FB}$ or $T^H_{SB} < T^H_{FB}$ and $T^L_{SB} > T^L_{FB}$ or $T^L_{SB} < T^L_{FB}$.

To characterize the optimal output choices, we now consider the following Kuhn-Tucker conditions for the optimization problem:

$[q_t^H(\underline{c})]$ $\nu\beta_0\delta^t(1-\lambda^H)^{t-1}\lambda^H(V'(q_t^H(\underline{c})) - \underline{c}) = 0$ for $1 \leqslant s \leqslant T^H$;

$[q_t^L(\underline{c})]$ $(1-\nu)\beta_0\delta^t(1-\lambda^L)^{t-1}\lambda^L(V'(q_t^L(\underline{c})) - \underline{c}) = 0$ for $1 \leqslant s \leqslant T^L$;

$[q^H(c^H_{T^H})]$ $\nu\delta^{T^H} P^H_{T^H}(V'(q^H(c^H_{T^H})) - c^H_{T^H}) - (1-\nu)\delta^{T^H} P^L_{T^H} \triangle c_{T^H} - \xi^H \frac{\delta^{T^H} P^H_{T^L} P^L_{T^H} \triangle c_{T^H}}{P^L_{T^L}} +$ $\alpha^L \delta^{T^H} P^L_{T^H} \triangle c_{T^H} = 0$;

$[q^L(c^L_{T^L})]$ $(1-\nu)\delta^{T^L} P^L_{T^L}(V'(q^L(c^L_{T^L})) - c^L_{T^L}) + \xi^H \delta^{T^L} P^H_{T^L} \triangle c_{T^L} = 0$;

The first two conditions above imply that there is no distortion in the output relative to the first-best level after success has been observed by either type; that is $V'(q_t^\theta(\underline{c})) = \underline{c}$ for $\theta \in \{H, L\}$. However, unlike in case 1, there is distortion in the output relative to the first-best level after T^L failures have been reported by the low type; that is

$$(1-\nu)P^L_{T^L}[V'(q^L(c^L_{T^L}))-c^L_{T^L}]=-\xi^H P^H_{T^L}\triangle c_{T^L},$$

which given that function $V(\cdot)$ is increasing and concave and ξ^H being strictly positive implies overproduction by the low type after T^L failures have been reported by the low type:

$$q^L_{SB}(c^L_{T^L})>q^L_{FB}(c^L_{T^L}).$$

We can simplify the Kuhn-Tucker condition for $q^H(c^H_{T^H}$ in the following way:

$$\nu\delta^{T^H}P^H_{T^H}(V'(q^H(c^H_{T^H}))-c^H_{T^H})=(1-\nu)\delta^{T^H}P^L_{T^H}\triangle c_{T^H}+\frac{\delta^{T^H}P^L_{T^H}\triangle c_{T^H}}{P^L_{T^L}}(\xi^H P^H_{T^L}-\alpha^L P^L_{T^L}).$$

We show next that $\xi^H P^H_{T^L}-\alpha^L P^L_{T^L}>0$, which implies there is underproduction in $q^H(c^H_{T^H})$ after a high type fails.

Recall that for $T^L_{SB}<\hat{T}^L$ the low type receives his rent only through x^L, i.e., when he does not succeed in the experimentation stage. This means that $\alpha^L=0$ for this case and, given that function $V(\cdot)$ is increasing and concave and ξ^H being strictly positive implies underproduction by the high type after T^H failures have been reported by the high type:

$$q^H_{SB}(c^H_{T^H})<q^H_{FB}(c^H_{T^H}).$$

When $T^L_{SB}>\hat{T}^L$ the low type receives rent after success ($y^L_t>0$) only if $t>\hat{T}^L$. Let's call this exact period of time when the low type is promised a rent $\bar{t}$, thus, $y^L_{\bar{t}}>0$ for $T^L_{SB}>\bar{t}>\hat{T}^L$. Since $y^L_{\bar{t}}>0$ from the Kuhn-Tucker conditions for the optimization problem we have

$$\xi^H\beta_0\delta^{\bar{t}}f_2(\bar{t})-\alpha^L\beta_0\delta^{\bar{t}}(1-\lambda^L)^{\bar{t}-1}\lambda^L=0,$$

where $\alpha^L>0$ and $f_2(\bar{t})>0$.

We can rewrite the equation above to get the following:

$$\xi^H=\alpha^L\frac{(1-\lambda^L)^{\bar{t}-1}\lambda^L}{f_2(\bar{t})},$$

Thus, to show that $\xi^H P^H_{T^L}-\alpha^L P^L_{T^L}>0$, we replace ξ^H to have

$$\alpha^L(\frac{(1-\lambda^L)^{\bar{t}-1}\lambda^L}{f_2(\bar{t})}P^H_{T^L}-P^L_{T^L})>0,$$

or,

$$\frac{P^H_{T^L}}{P^L_{T^L}}(1-\lambda^L)^{\bar{t}-1}\lambda^L > f_2(\bar{t}),$$

which is true since $f_2(\bar{t}) = \frac{P^H_{T^L}}{P^L_{T^L}}(1-\lambda^L)^{\bar{t}-1}\lambda^L - (1-\lambda^H)^{\bar{t}-1}\lambda^H$.

As a result, $\xi^H P^H_{T^L} - \alpha^L P^L_{T^L} > 0$ and given that function $V(\cdot)$ is increasing and concave and ξ^H and α^L being strictly positive the optimal contract exhibits underproduction after T^H failures have been reported by the high type.

Given that $q^L_{SB}(c^L_{T^L}) > q^L_{FB}(c^L_{T^L})$, the value of the objective function is smaller in Case 1. At the same time, the principal can relax the $(IC^{H,L})$ constraint by postponing the reward to the low type, y^L_t, to a later period; this is feasible only if $t < T^L$. Thus, the optimal contract has $x^L > 0$ and $y^L_t = 0$ for all $t \leqslant T^L$ in case $T^L \leqslant \hat{T}^L$. In case $T^L > \hat{T}^L$, it is optimal to set $x^L = 0$ and $y^L_t > 0$ for some $t > \hat{T}^L$. We will see that this is indeed true when both types receive strictly positive informational rent in Case 2.2.

Case 2.2: The high type gets positive rent.

Consider now the case when both types receive a strictly positive informational rent (assume $P^H_{T^H}P^L_{T^L} - P^H_{T^L}P^L_{T^H} \neq 0$.[5])

The constraint $(IC^{L,H})$ must be binding. If the $(IC^{L,H})$ constraint was not binding, it would be possible to decrease the payment to the low type without violating any other constraint and improve the value of the objective function. In addition, the $(IC^{H,L})$ constraint is binding as well by the same logic.

Using the binding $(IC^{L,H})$ and $(IC^{H,L})$ constraints we express x^L and x^H as follows:

$$x^H(\{y^H_t\}^{T^H}_{t=1}, \{y^L_t\}^{T^L}_{t=1}, T^H, T^L, q^H(c^H_{T^H}), q^L(c^L_{T^L})) = \frac{\beta_0 \sum_{t=1}^{T^H} \delta^t [P^H_{T^L}(1-\lambda^L)^{t-1}\lambda^L - P^L_{T^L}(1-\lambda^H)^{t-1}\lambda^H] y^H_t}{\delta^{T^H}(P^H_{T^H}P^L_{T^L} - P^H_{T^L}P^L_{T^H})}$$

---

[5] Note that the incentive compatibility constraints may be rewritten as:

$$\begin{gathered}
x^H \delta^{T^H}(P^H_{T^H}P^L_{T^L} - P^H_{T^L}P^L_{T^H}) = \beta_0 \sum_{t=1}^{T^H} \delta^t [P^H_{T^L}(1-\lambda^L)^{t-1}\lambda^L - P^L_{T^L}(1-\lambda^H)^{t-1}\lambda^H] y^H_t \\
+\beta_0 \sum_{t=1}^{T^L} \delta^t [P^L_{T^L}(1-\lambda^H)^{t-1}\lambda^H - P^H_{T^L}(1-\lambda^L)^{t-1}\lambda^L] y^L_t \\
+P^H_{T^L}(\delta^{T^H} P^L_{T^H} \triangle c_{T^H} q^H(c^H_{T^H}) - \delta^{T^L} P^L_{T^L} \triangle c_{T^L} q^L(c^L_{T^L})), \\
\text{and } x^L \delta^{T^L}(P^H_{T^H}P^L_{T^L} - P^H_{T^L}P^L_{T^H}) = \beta_0 \sum_{t=1}^{T^H} \delta^t [P^H_{T^H}(1-\lambda^L)^{t-1}\lambda^L - P^L_{T^H}(1-\lambda^H)^{t-1}\lambda^H] y^H_t \\
+\beta_0 \sum_{t=1}^{T^L} \delta^t [P^L_{T^H}(1-\lambda^H)^{t-1}\lambda^H - P^H_{T^H}(1-\lambda^L)^{t-1}\lambda^L] y^L_t \\
+P^L_{T^H}(\delta^{T^H} P^H_{T^H} \triangle c_{T^H} q^H(c^H_{T^H}) - \delta^{T^L} P^H_{T^L} \triangle c_{T^L} q^L(c^L_{T^L})).
\end{gathered}$$

In case $P^H_{T^H}P^L_{T^L} - P^H_{T^L}P^L_{T^H} = 0$, x^H and x^L become impossible to use as screening variables. Intuitively, it implies that the likelihood ratio of reaching the last period of the experimentation stage is the same for both types.

$$+\frac{\beta_0 \sum_{t=1}^{T^L} \delta^t [P^L_{T^L}(1-\lambda^H)^{t-1}\lambda^H - P^H_{T^L}(1-\lambda^L)^{t-1}\lambda^L] y^L_t}{\delta^{T^H}(P^H_{T^H}P^L_{T^L} - P^H_{T^L}P^L_{T^H})}$$
$$+\frac{P^H_{T^L}(\delta^{T^H} P^L_{T^H} \triangle c_{T^H} q^H(c^H_{T^H}) - \delta^{T^L} P^L_{T^L} \triangle c_{T^L} q^L(c^L_{T^L}))}{\delta^{T^H}(P^H_{T^H}P^L_{T^L} - P^H_{T^L}P^L_{T^H})};$$
$$x^L(\{y^H_t\}_{t=1}^{T^H}, \{y^L_t\}_{t=1}^{T^L}, T^H, T^L, q^H(c^H_{T^H}), q^L(c^L_{T^L})) =$$
$$\frac{\beta_0 \sum_{t=1}^{T^H} \delta^t [P^H_{T^H}(1-\lambda^L)^{t-1}\lambda^L - P^L_{T^H}(1-\lambda^H)^{t-1}\lambda^H] y^H_t}{\delta^{T^L}(P^H_{T^H}P^L_{T^L} - P^H_{T^L}P^L_{T^H})}$$
$$+\frac{\beta_0 \sum_{t=1}^{T^L} \delta^t [P^L_{T^H}(1-\lambda^H)^{t-1}\lambda^H - P^H_{T^H}(1-\lambda^L)^{t-1}\lambda^L] y^L_t}{\delta^{T^L}(P^H_{T^H}P^L_{T^L} - P^H_{T^L}P^L_{T^H})}$$
$$+\frac{P^L_{T^H}(\delta^{T^H} P^H_{T^H} \triangle c_{T^H} q^H(c^H_{T^H}) - \delta^{T^L} P^H_{T^L} \triangle c_{T^L} q^L(c^L_{T^L}))}{\delta^{T^L}(P^H_{T^H}P^L_{T^L} - P^H_{T^L}P^L_{T^H})}.$$

The principal's optimization problem is to choose T^H, $\{q^H_t(\underline{c})\}_{t=1}^{T^H}$, $q^H(c^H_{T^H})$, $\{y^H_t\}_{t=1}^{T^H}$, T^L, $\{y^L_t\}_{t=1}^{T^L}$, $\{q^L_t(\underline{c})\}_{t=1}^{T^L}$, and $q^L(c^L_{T^L})$ to

$$\max\{\nu[\beta_0 \textstyle\sum_{t=1}^{T^H} \delta^t(1-\lambda^H)^{t-1}\lambda^H(V(q^H_t(\underline{c})) - y^H_t - \underline{c}q^H_t(\underline{c}) - \mathit{\Gamma}_t) + \delta^{T^H} P^H_{T^H}(V(q^H(c^H_{T^H})) -$$
$$x^H(\{y^H_t\}_{t=1}^{T^H}, \{y^L_t\}_{t=1}^{T^L}, T^H, T^L, q^H(c^H_{T^H}), q^L(c^L_{T^L})) - c^H_{T^H}q^H(c^H_{T^H}) - \mathit{\Gamma}_{T^H})]$$
$$+(1-\nu)[\beta_0 \textstyle\sum_{t=1}^{T^L} \delta^t(1-\lambda^L)^{t-1}\lambda^L(V(q^L_t(\underline{c})) - y^L_t - \underline{c}q^L_t(\underline{c}) - \mathit{\Gamma}_t) + \delta^{T^L} P^L_{T^L}(V(q^L(c^L_{T^L})) -$$
$$x^L(\{y^H_t\}_{t=1}^{T^H}, \{y^L_t\}_{t=1}^{T^L}, T^H, T^L, q^H(c^H_{T^H}), q^L(c^L_{T^L})) - c^L_{T^L}q^L(c^L_{T^L}) - \mathit{\Gamma}_{T^L})]\} \text{ s.t.}$$

(IRS^H_t) $\quad y^H_t \geqslant 0$ for $t \leqslant T^H$,

(IRS^L_t) $\quad y^L_t \geqslant 0$ for $t \leqslant T^L$,

$(IRF^H_{T^H})$ $\quad x^H \geqslant 0$,

$(IRF^L_{T^L})$ $\quad x^L \geqslant 0$.

Labeling $\{\alpha^H_t\}_{t=1}^{T^H}$, $\{\alpha^L_t\}_{t=1}^{T^L}$, ξ^H, ξ^L as the Lagrange multipliers of the constraints associated with (IRS^H_t), (IRS^L_t), $(IRF^H_{T^H})$ and $(IRF^L_{T^L})$ respectively, the optimal contract has the following Lagrangian:

$$\mathcal{L}(T^H, \{q^H_t(\underline{c})\}_{t=1}^{T^H}, q^H(c^H_{T^H}), \{y^H_t\}_{t=1}^{T^H}, T^L, \{y^L_t\}_{t=1}^{T^L}, \{q^L_t(\underline{c})\}_{t=1}^{T^L}, q^L(c^L_{T^L}), \{\alpha^H_t\}_{t=1}^{T^H}, \{\alpha^L_t\}_{t=1}^{T^L}, \xi^H, \xi^L) =$$
$$\nu[\beta_0 \textstyle\sum_{t=1}^{T^H} \delta^t(1-\lambda^H)^{t-1}\lambda^H(V(q^H_t(\underline{c})) - y^H_t - \underline{c}q^H_t(\underline{c}) - \mathit{\Gamma}_t) + \delta^{T^H} P^H_{T^H}(V(q^H(c^H_{T^H})) -$$
$$x^H(\{y^H_t\}_{t=1}^{T^H}, \{y^L_t\}_{t=1}^{T^L}, T^H, T^L, q^H(c^H_{T^H}), q^L(c^L_{T^L})) - c^H_{T^H}q^H(c^H_{T^H}) - \mathit{\Gamma}_{T^H})]$$
$$+(1-\nu)[\beta_0 \textstyle\sum_{t=1}^{T^L} \delta^t(1-\lambda^L)^{t-1}\lambda^L(V(q^L_t(\underline{c})) - y^L_t - \underline{c}q^L_t(\underline{c}) - \mathit{\Gamma}_t) + \delta^{T^L} P^L_{T^L}(V(q^L(c^L_{T^L})) -$$
$$x^L(\{y^H_t\}_{t=1}^{T^H}, \{y^L_t\}_{t=1}^{T^L}, T^H, T^L, q^H(c^H_{T^H}), q^L(c^L_{T^L})) - c^L_{T^L}q^L(c^L_{T^L}) - \mathit{\Gamma}_{T^L})]$$
$$\textstyle\sum_{t=1}^{T^H} \alpha^H_t y^H_t + \sum_{t=1}^{T^L} \alpha^L_t y^L_t$$

$$\begin{aligned}&+\xi^H x^H(\{y_t^H\}_{t=1}^{T^H}, \{y_t^L\}_{t=1}^{T^L}, T^H, T^L, q^H(c_{T^H}^H), q^L(c_{T^L}^L))\\ &+\xi^L x^L(\{y_t^H\}_{t=1}^{T^H}, \{y_t^L\}_{t=1}^{T^L}, T^H, T^L, q^H(c_{T^H}^H), q^L(c_{T^L}^L)).\end{aligned}$$

The Inada conditions give us interior solutions for $q_t^H(\underline{c})$, $q^H(c_{T^H}^H)$, $q_t^L(\underline{c})$, and $q^L(c_{T^L}^L)$. We also assumed that $T^L > 0$ and $T^H > 0$. Therefore, the Kuhn-Tucker conditions for the optimization problem are:

$\frac{\partial \mathcal{L}}{\partial T^H} = 0$; $\frac{\partial \mathcal{L}}{\partial q_t^H(\underline{c})} = 0$ for $1 \leqslant t \leqslant T^H$; $\frac{\partial \mathcal{L}}{\partial q^H(c_{T^H}^H)} = 0$;
$\frac{\partial \mathcal{L}}{\partial T^L} = 0$; $\frac{\partial \mathcal{L}}{\partial q_t^L(\underline{c})} = 0$ for $1 \leqslant t \leqslant T^L$; $\frac{\partial \mathcal{L}}{\partial q^L(c_{T^L}^L)} = 0$;
$\frac{\partial \mathcal{L}}{\partial y_t^H} = 0$ for $1 \leqslant t \leqslant T^H$;
$\frac{\partial \mathcal{L}}{\partial \alpha_t^H} \geqslant 0$; $\alpha_t^H \geqslant$; $\alpha_t^H \frac{\partial \mathcal{L}}{\partial \alpha_t^H} = 0$ for $1 \leqslant t \leqslant T^H$;
$\frac{\partial \mathcal{L}}{\partial \xi^H} \geqslant 0$; $\xi^H \geqslant 0$; $\xi^H \frac{\partial \mathcal{L}}{\partial \xi^H} = 0$;
$\frac{\partial \mathcal{L}}{\partial y_t^L} = 0$ for $1 \leqslant t \leqslant T^L$;
$\frac{\partial \mathcal{L}}{\partial \alpha_t^L} \geqslant 0$; $\alpha_t^L \geqslant$; $\alpha_t^L \frac{\partial \mathcal{L}}{\partial \alpha_t^L} = 0$ for $1 \leqslant t \leqslant T^L$;
$\frac{\partial \mathcal{L}}{\partial \xi^L} \geqslant 0$; $\xi^L \geqslant 0$; $\xi^L \frac{\partial \mathcal{L}}{\partial \xi^L} = 0$;

which include the constraints from the problem themselves and corresponding complementary slackness conditions.

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial y_t^H} &= -\nu\{\beta_0\delta^t(1-\lambda^H)^{t-1}\lambda^H + \delta^{T^H} P_{T^H}^H(\frac{\beta_0\delta^t(P_{T^L}^H(1-\lambda^L)^{t-1}\lambda^L - P_{T^L}^L(1-\lambda^H)^{t-1}\lambda^H)}{\delta^{T^H}(P_{T^H}^H P_{T^L}^L - P_{T^L}^H P_{T^H}^L)})\}\\ &\quad -(1-\nu)\delta^{T^L} P_{T^L}^L \frac{\beta_0\delta^t(P_{T^H}^H(1-\lambda^L)^{t-1}\lambda^L - P_{T^H}^L(1-\lambda^H)^{t-1}\lambda^H)}{\delta^{T^L}(P_{T^H}^H P_{T^L}^L - P_{T^L}^H P_{T^H}^L)} + \alpha_t^H\\ &\quad +\xi^H \frac{\beta_0\delta^t(P_{T^L}^H(1-\lambda^L)^{t-1}\lambda^L - P_{T^L}^L(1-\lambda^H)^{t-1}\lambda^H)}{\delta^{T^H}(P_{T^H}^H P_{T^L}^L - P_{T^L}^H P_{T^H}^L)} + \xi^L \frac{\beta_0\delta^t(P_{T^H}^H(1-\lambda^L)^{t-1}\lambda^L - P_{T^H}^L(1-\lambda^H)^{t-1}\lambda^H)}{\delta^{T^L}(P_{T^H}^H P_{T^L}^L - P_{T^L}^H P_{T^H}^L)};\\ \frac{\partial \mathcal{L}}{\partial y_t^L} &= -(1-\nu)\{\beta_0\delta^t(1-\lambda^L)^{t-1}\lambda^L + \delta^{T^L} P_{T^L}^L(\frac{\beta_0\delta^t(P_{T^H}^L(1-\lambda^H)^{t-1}\lambda^H - P_{T^H}^H(1-\lambda^L)^{t-1}\lambda^L)}{\delta^{T^L}(P_{T^H}^H P_{T^L}^L - P_{T^L}^H P_{T^H}^L)})\}\\ &\quad -\nu\delta^{T^H} P_{T^H}^H \frac{\beta_0\delta^t(P_{T^L}^L(1-\lambda^H)^{t-1}\lambda^H - P_{T^L}^H(1-\lambda^L)^{t-1}\lambda^L)}{\delta^{T^H}(P_{T^H}^H P_{T^L}^L - P_{T^L}^H P_{T^H}^L)} + \alpha_t^L\\ &\quad +\xi^H \frac{\beta_0\delta^t(P_{T^L}^L(1-\lambda^H)^{t-1}\lambda^H - P_{T^L}^H(1-\lambda^L)^{t-1}\lambda^L)}{\delta^{T^H}(P_{T^H}^H P_{T^L}^L - P_{T^L}^H P_{T^H}^L)} + \xi^L \frac{\beta_0\delta^t(P_{T^H}^L(1-\lambda^H)^{t-1}\lambda^H - P_{T^H}^H(1-\lambda^L)^{t-1}\lambda^L)}{\delta^{T^L}(P_{T^H}^H P_{T^L}^L - P_{T^L}^H P_{T^H}^L)}.\end{aligned}$$

We can rewrite the Kuhn-Tucker conditions above as follows:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial y_t^H} &= \frac{\beta_0\delta^t}{\psi}[P_{T^H}^H f_1(t,T^H)[\nu P_{T^L}^H + (1-\nu)P_{T^L}^L - \frac{\xi^L}{\delta^{T^L}}] + \frac{\xi^H}{\delta^{T^H}} P_{T^L}^L f_2(t,T^L) + \frac{\alpha_t^H\psi}{\beta_0\delta^t}];\\ \frac{\partial \mathcal{L}}{\partial y_t^L} &= \frac{\beta_0\delta^t}{\psi}[P_{T^L}^L f_2(t,T^L)[\nu P_{T^H}^H + (1-\nu)P_{T^H}^L - \frac{\xi^H}{\delta^{T^L}}] + \frac{\xi^L}{\delta^{T^L}} P_{T^H}^H f_1(t,T^H) + \frac{\alpha_t^L\psi}{\beta_0\delta^t}],\end{aligned}$$

where $\psi = \psi(T^H, T^L) = P_{T^H}^H P_{T^L}^L - P_{T^L}^H P_{T^H}^L = 0$, $f_1(t,T^H) = \frac{P_{T^H}^L}{P_{T^H}^H}(1-\lambda^H)^{t-1}\lambda^H - (1-\lambda^L)^{t-1}\lambda^L$.[6]

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[6] $f_2(t,T^L) = \frac{P_{T^L}^H}{P_{T^L}^L}(1-\lambda^L)^{t-1}\lambda^L - (1-\lambda^H)^{t-1}\lambda^H$ as it was defined in Case 2.1.

Let T^L and T^H to be the optimal duration of the experimentation stage for the low and high types, respectively. We will determine both variables and specify if each of them is above or below the first-best level later for each case we consider further.

Now we will show that there exists a duration of the experimentation stage for the high type such that the low type is relatively more likely to fail if he lies for any period smaller than this threshold level.

Claim 2.2: There exists a unique $\hat{T}^H > 1$, such that $f_1(\hat{T}^H, T^H) = 0$, and

$$f_1(t, T^H) = \begin{cases} > 0, & t < \hat{T}^H \\ < 0, & t > \hat{T}^H \end{cases}$$

Proof: Recall that P^θ_T is defined as

$$P^\theta_T = 1 - \beta_0 + \beta_0(1 - \lambda^\theta)^T,$$

and is equal to the probability that the agent of type does not succeed during the experimentation stage which lasted for $T \in N$ periods. Then $\frac{P^L_{T^H}}{P^H_{T^H}}$ is a ratio of probability that the low type does not succeed to probability that the high type does not succeed for T^H periods during the experimentation stage. At the same time, $\beta_0(1 - \lambda^\theta)^{t-1}\lambda^\theta$ is probability that the agent of type θ succeeds at period $t \leqslant T^H$ of the experimentation stage and $\frac{\beta_0(1-\lambda^L)^{t-1}\lambda^L}{\beta_0(1-\lambda^H)^{t-1}\lambda^H} = \frac{(1-\lambda^L)^{t-1}\lambda^L}{(1-\lambda^H)^{t-1}\lambda^H}$ is a ratio of probabilities of success at period t by two types. As a result, we can rewrite $f_1(t, T^H) > 0$ as

$$\frac{1-\beta_0+\beta_0(1-\lambda^L)^{T^H}}{1-\beta_0+\beta_0(1-\lambda^H)^{T^H}} > \frac{(1-\lambda^L)^{t-1}\lambda^L}{(1-\lambda^H)^{t-1}\lambda^H} \text{ for } 1 \leqslant t \leqslant T^H \text{ or, equivalently,}$$
$$\frac{1-\beta_0+\beta_0(1-\lambda^L)^{T^H}}{(1-\lambda^L)^{t-1}\lambda^L} > \frac{1-\beta_0+\beta_0(1-\lambda^H)^{T^H}}{(1-\lambda^H)^{t-1}\lambda^H} \text{ for } 1 \leqslant t \leqslant T^H.$$

We will say that when $f_1(t, T^H) > 0$ (< 0) the low type is relatively more likely to fail (succeed) than the high type during the experimentation stage if he chooses a contract designed for the high type.

There exists a unique time period $\hat{T}^H(T^H, \lambda^L, \lambda^H, \beta_0)$ such that $f_1(\hat{T}^H, T^H) = 0$ defined as

$$\hat{T}^H \equiv \hat{T}^H(T^H, \lambda^L, \lambda^H, \beta_0) = 1 + \frac{ln(\frac{P^L_{T^H}\lambda^H}{P^H_{T^H}\lambda^L})}{ln(\frac{1-\lambda^L}{1-\lambda^H})},$$

where uniqueness follows from $\frac{(1-\lambda^L)^{t-1}\lambda^L}{(1-\lambda^H)^{t-1}\lambda^H}$ being strictly increasing in t and $\frac{\lambda^L}{\lambda^H} < 1 < \frac{P^L_{T^H}}{P^H_{T^H}}$.[7] In addition, for $t < \hat{T}^H$ it follows that $f_1(t, T^H) > 0$ and, as a result, the high type is relatively more likely to succeed than the low type whereas for $t > \hat{T}^H$ the opposite is true. *Q.E.D.*

Note that $\hat{T}^H$ is uniquely defined as we already proved that $f_1(t, T^H)$ has a unique solution. When $T^H < \hat{T}^H$ we showed that $f_1(t, T^H) > 0$ for all $t \leqslant T^H$, whereas if $T^H > \hat{T}^H$ then for periods $T^H \geqslant t \geqslant \hat{T}^H$ function $f_1(t)$ becomes strictly negative.

In what follows, we will find a solution to the principal's optimization problem. Since there are several possible combinations of feasible payments schemes, we explore them case by case depending on which participation constraints are binding.

Case 2.2.1: Suppose that $\xi^H = \xi^L = 0$, i.e., the $(IRF^H_{T^H})$ and $(IRF^L_{T^L})$ constraints are not binding. We can rewrite the Kuhn-Tucker conditions as follows:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial y^H_t} &= \frac{\beta_0 \delta^t}{\psi}[P^H_{T^H} f_1(t, T^H)[\nu P^H_{T^L} + (1-\nu) P^L_{T^L}] + \frac{\alpha^H_t \psi}{\beta_0 \delta^t}] \text{ for } 1 \leqslant t \leqslant T^H; \\
\frac{\partial \mathcal{L}}{\partial y^L_t} &= \frac{\beta_0 \delta^t}{\psi}[P^L_{T^L} f_2(t, T^L)[\nu P^H_{T^H} + (1-\nu) P^L_{T^H}] + \frac{\alpha^L_t \psi}{\beta_0 \delta^t}] \text{ for } 1 \leqslant t \leqslant T^L.
\end{aligned}$$

Since $f_1(t, T^H)$ is strictly positive for all $t < \hat{T}^H$ from $P^H_{T^H} f_1(t, T^H)[\nu P^H_{T^L} + (1 - \nu) P^L_{T^L}] = -\frac{\alpha^H_t \psi}{\beta_0 \delta^t}$ it must be that $\alpha^H_t > 0$ for all $t < \hat{T}^H$ and $\psi < 0$. In addition, since $f_2(t, T^L)$ is strictly negative for $t < \hat{T}^L$ from $P^L_{T^L} f_2(t, T^L)[\nu P^H_{T^H} + (1-\nu) P^L_{T^H}] = -\frac{\alpha^L_t \psi}{\beta_0 \delta^t}$ it must be that that $\alpha^L_t > 0$ for $t < \hat{T}^L$ and $\psi > 0$, which leads to a contradiction.[8]

Case 2.2.2: Suppose that $\xi^H = 0$ and $\xi^L > 0$, i.e., the $(IRF^H_{T^H})$ constraint is not binding but $(IRF^L_{T^L})$ constraint is binding. We can rewrite the Kuhn-Tucker conditions as follows:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial y^H_t} &= \frac{\beta_0 \delta^t}{\psi}[P^H_{T^H} f_1(t, T^H)[\nu P^H_{T^L} + (1-\nu) P^L_{T^L} - \frac{\xi^L}{\delta^{T^L}}] + \frac{\alpha^H_t \psi}{\beta_0 \delta^t}] \text{ for } 1 \leqslant t \leqslant T^H; \\
\frac{\partial \mathcal{L}}{\partial y^L_t} &= \frac{\beta_0 \delta^t}{\psi}[P^L_{T^L} f_2(t, T^L)[\nu P^H_{T^H} + (1-\nu) P^L_{T^H}] + \frac{\xi^L}{\delta^{T^L}} P^H_{T^H} f_1(t, T^H) + \frac{\alpha^L_t \psi}{\beta_0 \delta^t}] \text{ for } 1 \leqslant t \leqslant T^L.
\end{aligned}$$

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[7] To explain, $f_1(t, T^H) = 0$ if and only if $\frac{1-\beta_0+\beta_0(1-\lambda^L)^{T^H}}{1-\beta_0+\beta_0(1-\lambda^H)^{T^H}} = \frac{(1-\lambda^L)^{t-1}\lambda^L}{(1-\lambda^H)^{t-1}\lambda^H}$. Given that the right hand side of the equation above is strictly increasing since $\frac{1-\lambda^L}{1-\lambda^H} > 1$ and if evaluated at $t = 1$ is equal to $\frac{\lambda^L}{\lambda^H}$. Since $\frac{1-\beta_0+\beta_0(1-\lambda^L)^{T^H}}{1-\beta_0+\beta_0(1-\lambda^H)^{T^H}} > 1$ and $\frac{\lambda^L}{\lambda^H} < 1$ the uniqueness immediately follows. So $\hat{T}^H$ satisfies $\frac{P^L_{T^H}}{P^H_{T^H}} = \frac{(1-\lambda^L)^{\hat{T}^H-1}\lambda^L}{(1-\lambda^H)^{\hat{T}^H-1}\lambda^H}$.

[8] If there was a solution with $\xi^H = \xi^L = 0$ then with necessity it would be possible only if T^H and T^L are such that it holds simultaneously $P^H_{T^H} P^L_{T^L} - P^H_{T^L} P^L_{T^H} > 0$ and $P^H_{T^H} P^L_{T^L} - P^H_{T^L} P^L_{T^H} < 0$, since the two conditions are mutually exclusive the conclusion immediately follows. Note that we assumed so far that $\psi \neq 0$; we study $\psi = 0$ in details later in Case 2.3.

Suppose $\frac{\xi^L}{\delta^{T^L}} = \nu P^H_{T^L} + (1-\nu) P^L_{T^L}$. Then $P^L_{T^L} f_2(t, T^L)[\nu P^H_{T^H} + (1-\nu) P^L_{T^H}] + \frac{\xi^L}{\delta^{T^L}} P^H_{T^H} f_1(t, T^H) = (P^H_{T^L}(1-\lambda^L)^{t-1}\lambda^L - P^L_{T^L}(1-\lambda^H)^{t-1}\lambda^H)[\nu P^H_{T^H} + (1-\nu) P^L_{T^H}] + (P^L_{T^H}(1-\lambda^H)^{t-1}\lambda^H - P^H_{T^H}(1-\lambda^L)^{t-1}\lambda^L)[\nu P^H_{T^L} + (1-\nu) P^L_{T^L}] = -\psi((1-\nu)(1-\lambda^L)^{t-1}\lambda^L + \nu(1-\lambda^H)^{t-1}\lambda^H)$, which implies that $-\psi((1-\nu)(1-\lambda^L)^{t-1}\lambda^L + \nu(1-\lambda^H)^{t-1}\lambda^H) + \frac{\alpha^L_t \psi}{\beta_0 \delta^t} = 0$ for $1 \leqslant t \leqslant T^L$.

Thus, $\frac{\alpha^L_t}{\beta_0 \delta^t} = (1-\nu)(1-\lambda^L)^{t-1}\lambda^L + \nu(1-\lambda^H)^{t-1}\lambda^H > 0$ for $1 \leqslant t \leqslant T^L$, which leads to a contradiction since then $x^L = y^L_t = 0$ for $1 \leqslant t \leqslant T^L$ which implies that the low type does not receive any rent.

If α^H_s for some $1 \leqslant s \leqslant T^H$ then $P^H_{T^H} f_1(s, T^H)[\nu P^H_{T^L} + (1-\nu) P^L_{T^L} - \frac{\xi^L}{\delta^{T^L}}] = 0$, which implies that $\frac{\xi^L}{\delta^{T^L}} = \nu P^H_{T^L} + (1-\nu) P^L_{T^L}$.[9] Since we rule out this possibility it immediately follows that all $\alpha^H_t > 0$ for all $1 \leqslant t \leqslant T^H$ which implies that $y^H_t = 0$ for $1 \leqslant t \leqslant T^H$.

Lemma 1. There exists at most one $j \leqslant T^L$ such that $y^L_j > 0$.

Proof: Assume to the contrary that there are two distinct periods $1 \leqslant k, m \leqslant T^L$ such that $k \neq m$ and $y^L_k, y^L_m > 0$. Then from the Kuhn-Tucker conditions it follows that

$$P^L_{T^L} f_2(k, T^L)[\nu P^H_{T^H} + (1-\nu) P^L_{T^H}] + \frac{\xi^L}{\delta^{T^L}} P^H_{T^H} f_1(k, T^H) = 0,$$

and, in addition, $P^L_{T^L} f_2(m, T^L)[\nu P^H_{T^H} + (1-\nu) P^L_{T^H}] + \frac{\xi^L}{\delta^{T^L}} P^H_{T^H} f_1(m, T^H) = 0$.

Thus, $\frac{f_2(m,T^L)}{f_1(m,T^H)} = \frac{f_2(k,T^L)}{f_1(k,T^H)}$, which can be rewritten as follows:

$$(P^H_{T^L}(1-\lambda^L)^{m-1}\lambda^L - P^L_{T^L}(1-\lambda^H)^{m-1}\lambda^H)(P^L_{T^H}(1-\lambda^H)^{k-1}\lambda^H - P^H_{T^H}(1-\lambda^L)^{k-1}\lambda^L) =$$
$$(P^H_{T^L}(1-\lambda^L)^{k-1}\lambda^L - P^L_{T^L}(1-\lambda^H)^{k-1}\lambda^H)(P^L_{T^H}(1-\lambda^H)^{m-1}\lambda^H - P^H_{T^H}(1-\lambda^L)^{m-1}\lambda^L),$$
$$\psi[(1-\lambda^H)^{k-1}(1-\lambda^L)^{m-1} - (1-\lambda^H)^{m-1}(1-\lambda^L)^{k-1}] = 0,$$
$$(1-\lambda^L)^{m-k}(1-\lambda^H)^{k-m} = 1$$
$$\left(\tfrac{1-\lambda^L}{1-\lambda^H}\right)^{(m-k)} = 1,$$

which implies that $m = k$ and we have a contradiction. *Q.E.D.*

Finally, from $P^H_{T^H} f_1(t, T^H)[\nu P^H_{T^L} + (1-\nu) P^L_{T^L} - \frac{\xi^L}{\delta^{T^L}}] = -\frac{\alpha^H_t \psi}{\beta_0 \delta^t}$ we conclude that $T^H \leqslant \hat{T}^H$[10] and either $\psi > 0$ and $\frac{\xi^L}{\delta^{T^L}} > \nu P^H_{T^L} + (1-\nu) P^L_{T^L}$ or $\psi < 0$ and $\frac{\xi^L}{\delta^{T^L}} < \nu P^H_{T^L} + (1-\nu) P^L_{T^L}$.

Case 2.2.2.1: $T^H \leqslant \hat{T}^H$, $\psi > 0$, $\frac{\xi^L}{\delta^{T^L}} > \nu P^H_{T^L} + (1-\nu) P^L_{T^L}$, $\xi^H = 0$, $\alpha^H_t > 0$ for $1 \leqslant t \leqslant T^H$.

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[9] If $s = \hat{T}^H$, then both $x^H > 0$ and $y^H_{\hat{T}^H} > 0$ can be optimal.

[10] If $T^H > \hat{T}^H$ then there would be a contradiction similar to Case 2.2.1.

We know that there exists only one time period $1 \leqslant j \leqslant T^L$ such that $y_j^L > 0$ ($\alpha_j^L = 0$). This implies that

$$P_{T^L}^L f_2(j,T^L)[\nu P_{T^H}^H + (1-\nu)P_{T^H}^L] + \tfrac{\xi^L}{\delta^{T^L}} P_{T^H}^H f_1(j,T^H) = 0$$
$$\text{and } P_{T^L}^L f_2(t,T^L)[\nu P_{T^H}^H + (1-\nu)P_{T^H}^L] + \tfrac{\xi^L}{\delta^{T^L}} P_{T^H}^H f_1(t,T^H) = -\tfrac{\alpha_t^H \psi}{\beta_0 \delta^t} < 0 \text{ for } 1 \leqslant t \neq j \leqslant T^L.$$

Alternatively, $f_2(t,T^L) < \frac{f_1(t,T^H)}{f_1(j,T^H)} f_2(j,T^L)$ for $1 \leqslant t \neq j \leqslant T^L$.

If $f_1(j,T^H) > 0$ ($j < \hat{T}^H$) then

$$(P_{T^L}^H(1-\lambda^L)^{t-1}\lambda^L - P_{T^L}^L(1-\lambda^H)^{t-1}\lambda^H)(P_{T^H}^L(1-\lambda^H)^{j-1}\lambda^H - P_{T^H}^H(1-\lambda^L)^{j-1}\lambda^L) <$$
$$(P_{T^L}^H(1-\lambda^L)^{j-1}\lambda^L - P_{T^L}^L(1-\lambda^H)^{j-1}\lambda^H)(P_{T^H}^L(1-\lambda^H)^{t-1}\lambda^H - P_{T^H}^H(1-\lambda^L)^{t-1}\lambda^L).$$
$$\psi[(1-\lambda^H)^{t-1}(1-\lambda^L)^{j-1} - (1-\lambda^H)^{j-1}(1-\lambda^L)^{t-1}] < 0 \text{ for } 1 \leqslant t \neq j \leqslant T^L.$$

Thus, $\psi[1 - (\frac{1-\lambda^L}{1-\lambda^H})^{(t-j)}] < 0$, which implies that $t > j$ for all $1 \leqslant t \neq j \leqslant T^L$ or, equivalently, $j = 1$.

If $f_1(j,T^H) < 0$ ($j > \hat{T}^H$) then the opposite must be true and $t < j$ for all $1 \leqslant t \neq j \leqslant T^L$ or, equivalently, $j = T^L$.

For $j > \hat{T}^H$, $f_1(j,T^H) < 0$ and it follows that

$$P_{T^L}^L f_2(t,T^L)[\nu P_{T^H}^H + (1-\nu)P_{T^H}^L] + \tfrac{\xi^L}{\delta^{T^L}} P_{T^H}^H f_1(t,T^H) <$$
$$-\psi((1-\nu)(1-\lambda^L)^{t-1}\lambda^L + \nu(1-\lambda^H)^{t-1}\lambda^H) < 0,$$

which implies that $y_j^L > 0$ is only possible for $j < \hat{T}^H$. Thus this case is only possible if $j = 1$.

Case 2.2.2.2: $T^H \leqslant \hat{T}^H$, $\psi < 0$, $\frac{\xi^L}{\delta^{T^L}} < \nu P_{T^L}^H + (1-\nu)P_{T^L}^L$, $\xi^H = 0$, $\alpha_t^H > 0$ for $1 \leqslant t \leqslant T^H$.

As in the previous case there exists only one time period $1 \leqslant s \leqslant T^L$ such that $y_s^L > 0$ ($\alpha_s^L = 0$).

This implies that

$$P_{T^L}^L f_2(s,T^L)[\nu P_{T^H}^H + (1-\nu)P_{T^H}^L] + \tfrac{\xi^L}{\delta^{T^L}} P_{T^H}^H f_1(s,T^H) = 0, \text{ and}$$
$$P_{T^L}^L f_2(t,T^L)[\nu P_{T^H}^H + (1-\nu)P_{T^H}^L] + \tfrac{\xi^L}{\delta^{T^L}} P_{T^H}^H f_1(t,T^H) = -\tfrac{\alpha_t^H \psi}{\beta_0 \delta^t} > 0 \text{ for } 1 \leqslant t \neq s \leqslant T^L.$$

Alternatively, $f_2(t,T^L) > \frac{f_1(t,T^H)}{f_1(s,T^H)} f_2(s,T^L)$ for $1 \leqslant t \neq s \leqslant T^L$.

If $f_1(s,T^H) > 0$ ($s < \hat{T}^H$) then

$$(P^H_{T^L}(1-\lambda^L)^{t-1}\lambda^L - P^L_{T^L}(1-\lambda^H)^{t-1}\lambda^H)(P^L_{T^H}(1-\lambda^H)^{s-1}\lambda^H - P^H_{T^H}(1-\lambda^L)^{s-1}\lambda^L) >$$
$$(P^H_{T^L}(1-\lambda^L)^{s-1}\lambda^L - P^L_{T^L}(1-\lambda^H)^{s-1}\lambda^H)(P^L_{T^H}(1-\lambda^H)^{t-1}\lambda^H - P^H_{T^H}(1-\lambda^L)^{t-1}\lambda^L).$$

$\psi[1-(\frac{1-\lambda^L}{1-\lambda^H})^{(s-t)}] < 0$, which implies that $t > j$ for all $1 \leqslant t \neq s \leqslant T^L$ or, equivalently, $s = 1$.

If $f_1(s, T^H) < 0$ $(s > \hat{T}^H)$ then the opposite must be true and $t < s$ for all $1 \leqslant t \neq s \leqslant T^L$ or, equivalently, $s = T^L$.

For $t > \hat{T}^H$ it follows that

$$P^L_{T^L} f_2(t, T^L)[\nu P^H_{T^H} + (1-\nu)P^L_{T^H}] + \frac{\xi^L}{\delta^{T^L}} P^H_{T^H} f_1(t, T^H) >$$
$$-\psi((1-\nu)(1-\lambda^L)^{t-1}\lambda^L + \nu(1-\lambda^H)^{t-1}\lambda^H) > 0,$$

which implies that $y^L_s > 0$ is only possible for $s < \hat{T}^H$. Thus this case is only possible if $s = 1$.

For both cases we just considered, we have

$$x^H = \frac{\beta_0 \delta P^L_{T^L}(-f_2(1,T^L))y^L_1}{\delta^{T^H}\psi} + \frac{P^H_{T^L}(\delta^{T^H} P^L_{T^H} \triangle c_{T^H} q^H(c^H_{T^H}) - \delta^{T^L} P^L_{T^L} \triangle c_{T^L} q^L(c^L_{T^L}))}{\delta^{T^H}\psi} > 0;$$
$$x^L = \frac{\beta_0 \delta P^H_{T^H} f_1(1,T^H)y^L_1}{\delta^{T^L}\psi} + \frac{P^L_{T^H}(\delta^{T^H} P^H_{T^H} \triangle c_{T^H} q^H(c^H_{T^H}) - \delta^{T^L} P^H_{T^L} \triangle c_{T^L} q^L(c^L_{T^L}))}{\delta^{T^L}\psi} = 0.$$

Note that case 2.2 is only possible if $\delta^{T^H} P^L_{T^H} \triangle c_{T^H} q^H(c^H_{T^H}) - \delta^{T^L} P^L_{T^L} \triangle c_{T^L} q^L(c^L_{T^L}) > 0$ which together with $x^H > 0$ implies that $\psi = P^H_{T^H} P^L_{T^L} - P^H_{T^L} P^L_{T^H} > 0$. Since $f_1(1, T^H) > 0$, $x^L = 0$ is possible only if $\delta^{T^H} P^H_{T^H} \triangle c_{T^H} q^H(c^H_{T^H}) - \delta^{T^L} P^H_{T^L} \triangle c_{T^L} q^L(c^L_{T^L}) < 0$.

However,

$$\delta^{T^H} P^L_{T^H} \triangle c_{T^H} q^H(c^H_{T^H L}) > \delta^{T^L} P^L_{T^L} \triangle c_{T^L} q^L(c^L_{T^L}),$$
$$\delta^{T^H} P^H_{T^H} \triangle c_{T^H} q^H(c^H_{T^H}) > \delta^{T^L} \frac{P^H_{T^H}}{P^L_{T^H}} P^L_{T^L} \triangle c_{T^L} q^L(c^L_{T^L}).$$

Note that $P^H_{T^H} P^L_{T^L} - P^H_{T^L} P^L_{T^H} > 0$ implies $\frac{P^H_{T^H}}{P^L_{T^H}} P^L_{T^L} > P^H_{T^L}$ and then $\delta^{T^H} P^H_{T^H} \triangle c_{T^H} q^H(c^H_{T^H}) > \delta^{T^L} P^H_{T^L} \triangle c_{T^L} q^L(c^L_{T^L})$, which leads to a contradiction.

Case 2.2.3: Suppose that $\xi^H > 0$ and $\xi^L = 0$, i.e., the $(IRF^H_{T^H})$ constraint is binding but $(IRF^L_{T^L})$ constraint is binding.

Claim 2.2.3. $\xi^H > 0$ and $\xi^L = 0 \Longleftrightarrow T^L \leqslant \hat{T}^L$, $\alpha^L_t > 0$ for $t \leqslant T^L$ (it is optimal to set $x^L > 0$, $y^L_t = 0$ for $t \leqslant T^L$) and $\alpha^H_t > 0$ for all $t > 1$ and $\alpha^H_1 = 0$ (it is optimal to set $x^H = 0$, $y^H_t = 0$ for all $t > 1$ and $y^H_1 > 0$).

Proof: We can rewrite the Kuhn-Tucker conditions as follows:

$$\frac{\partial\mathcal{L}}{\partial y_t^H} = \frac{\beta_0\delta^t}{\psi}[P^H_{T^H} f_1(t,T^H)[\nu P^H_{T^L} + (1-\nu)P^L_{T^L}] + \frac{\xi^H}{\delta^{T^H}} P^L_{T^L} f_2(t,T^L) + \frac{\alpha^H_t\psi}{\beta_0\delta^t}] \text{ for } 1 \leqslant t \leqslant T^H;$$
$$\frac{\partial\mathcal{L}}{\partial y_t^L} = \frac{\beta_0\delta^t}{\psi}[P^L_{T^L} f_2(t,T^L)[\nu P^H_{T^H} + (1-\nu)P^L_{T^H} - \frac{\xi^H}{\delta^{T^L}}] + \frac{\alpha^L_t\psi}{\beta_0\delta^t}] \text{ for } 1 \leqslant t \leqslant T^L.$$

Suppose $\frac{\xi^H}{\delta^{T^H}} = \nu P^H_{T^H} + (1-\nu)P^L_{T^H}$. Then $P^H_{T^H} f_1(t,T^H)[\nu P^H_{T^L} + (1-\nu)P^L_{T^L}] + \frac{\xi^H}{\delta^{T^H}} P^L_{T^L} f_2(t,T^L) = (P^L_{T^H}(1-\lambda^H)^{t-1}\lambda^H - P^H_{T^H}(1-\lambda^L)^{t-1}\lambda^L)[\nu P^H_{T^L} + (1-\nu)P^L_{T^L}] + (P^H_{T^L}(1-\lambda^L)^{t-1}\lambda^L - P^L_{T^L}(1-\lambda^H)^{t-1}\lambda^H)[\nu P^H_{T^H} + (1-\nu)P^L_{T^H}] = -\psi((1-\nu)(1-\lambda^L)^{t-1}\lambda^L + \nu(1-\lambda^H)^{t-1}\lambda^H)$, which implies that $-\psi((1-\nu)(1-\lambda^L)^{t-1}\lambda^L + \nu(1-\lambda^H)^{t-1}\lambda^H) + \frac{\alpha^H_t\psi}{\beta_0\delta^t} = 0$ for $1 \leqslant t \leqslant T^H$.

Then $\frac{\alpha^H_t}{\beta_0\delta^t} = (1-\nu)(1-\lambda^L)^{t-1}\lambda^L + \nu(1-\lambda^H)^{t-1}\lambda^H > 0$ for $1 \leqslant t \leqslant T^H$, which leads to a contradiction since then $x^H = y^H_t = 0$ for $1 \leqslant t \leqslant T^H$ which implies that the high type does not receive any rent. If α^L_s for some $1 \leqslant s \leqslant T^L$ then $P^L_{T^L} f_2(t,T^L)[\nu P^H_{T^H} + (1-\nu)P^L_{T^H} - \frac{\xi^H}{\delta^{T^H}}]$, which implies that $\frac{\xi^H}{\delta^{T^H}} = \nu P^H_{T^H} + (1-\nu)P^L_{T^H}$.[11] Since we already rule out this possibility it immediately follows that all $\alpha^L_t > 0$ for all $1 \leqslant t \leqslant T^L$ which implies that $y^L_t = 0$ for $1 \leqslant t \leqslant T^L$.

Lemma 2. There exists at most one $j \leqslant T^H$ such that $y^H_j > 0$.

Proof: Assume to the contrary that there are two distinct periods $1 \leqslant k, m \leqslant T^H$ such that $k \neq m$ and $y^H_k, y^H_k > 0$. Then from the Kuhn-Tucker conditions it follows that

$$P^H_{T^H} f_1(k,T^H)[\nu P^H_{T^L} + (1-\nu)P^L_{T^L}] + \frac{\xi^H}{\delta^{T^H}} P^L_{T^L} f_2(k,T^L) = 0, \text{ and}$$
$$P^H_{T^H} f_1(m,T^H)[\nu P^H_{T^L} + (1-\nu)P^L_{T^L}] + \frac{\xi^H}{\delta^{T^H}} P^L_{T^L} f_2(m,T^L) = 0.$$

Thus, $\frac{f_2(m,T^L)}{f_1(m,T^H)} = \frac{f_2(k,T^L)}{f_1(k,T^H)} \Longleftrightarrow (\frac{1-\lambda^L}{1-\lambda^H})^{(m-k)} = 1$, which implies that $m = k$ and leads to a contradiction. *Q.E.D.*

Finally, from $P^L_{T^L} f_2(t,T^L)[\nu P^H_{T^H} + (1-\nu)P^L_{T^H} - \frac{\xi^H}{\delta^{T^H}}] = -\frac{\alpha^L_t\psi}{\beta_0\delta^t}$ for $1 \leqslant t \leqslant T^L$ and we conclude that $T^L \leqslant \hat{T}^L$[12] and either $\psi > 0$ and $\frac{\xi^H}{\delta^{T^H}} < \nu P^H_{T^H} + (1-\nu)P^L_{T^H}$ or $\psi < 0$ and $\frac{\xi^H}{\delta^{T^H}} > \nu P^H_{T^H} + (1-\nu)P^L_{T^H}$.

Case 2.2.3.1: $T^L \leqslant \hat{T}^L$, $\psi > 0$, $\frac{\xi^H}{\delta^{T^H}} < \nu P^H_{T^H} + (1-\nu)P^L_{T^H}$, $\xi^L = 0$, $\alpha^L_t > 0$ for $1 \leqslant t \leqslant T^L$.

There exists only one time period $1 \leqslant j \leqslant T^H$ such that $y^H_j > 0$ ($\alpha^H_j = 0$). This implies that

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[11] If $t = \hat{T}^L$, then both $x^L > 0$ and $y^L_{\hat{T}^L} > 0$ can be optimal.

[12] If $T^L > \hat{T}^L$ then there would be a contradiction similar to Case 2.2.2.

$$P^H_{T^H} f_1(j,T^H)[\nu P^H_{T^L} + (1-\nu)P^L_{T^L}] + \frac{\xi^H}{\delta^{T^H}} P^L_{T^L} f_2(j,T^L) = 0 \text{ and}$$
$$P^H_{T^H} f_1(t,T^H)[\nu P^H_{T^L} + (1-\nu)P^L_{T^L}] + \frac{\xi^H}{\delta^{T^H}} P^L_{T^L} f_2(t,T^L) = -\frac{\alpha^H_t \psi}{\beta_0 \delta^t} < 0 \text{ for } 1 \leqslant t \neq j \leqslant T^H.$$

Alternatively, $f_1(t,T^H) < \frac{f_1(j,T^H)}{f_2(j,T^L)} f_2(t,T^L)$ for $1 \leqslant t \neq j \leqslant T^H$.

If $f_2(j,T^L) > 0$ ($j > \hat{T}^L$) then $f_1(t,T^H) f_2(j,T^L) < f_1(j,T^H) f_1(t,T^H) f_2(t,T^L)$,

$$(P^H_{T^L}(1-\lambda^L)^{j-1}\lambda^L - P^L_{T^L}(1-\lambda^H)^{j-1}\lambda^H)(P^L_{T^H}(1-\lambda^H)^{t-1}\lambda^H - P^H_{T^H}(1-\lambda^L)^{t-1}\lambda^L) <$$
$$(P^H_{T^L}(1-\lambda^L)^{t-1}\lambda^L - P^L_{T^L}(1-\lambda^H)^{t-1}\lambda^H)(P^L_{T^H}(1-\lambda^H)^{j-1}\lambda^H - P^H_{T^H}(1-\lambda^L)^{j-1}\lambda^L),$$
$$\psi[1 - (\tfrac{1-\lambda^L}{1-\lambda^H})^{(j-t)}] < 0, \text{ which implies that } t < j \text{ for all } 1 \leqslant t \neq j \leqslant T^H \text{ or, equivalently,}$$
$$j = T^H.$$

If $f_2(j,T^L) < 0$ ($j < \hat{T}^L$) then the opposite must be true and $t > j$ for all $1 \leqslant t \neq j \leqslant T^H$ or, equivalently, $j = 1$.

For $t > \hat{T}^L$ it follows that

$$P^H_{T^H} f_1(t,T^H)[\nu P^H_{T^L} + (1-\nu)P^L_{T^L}] + \frac{\xi^H}{\delta^{T^H}} P^L_{T^L} f_2(t,T^L) <$$
$$-\psi((1-\nu)(1-\lambda^L)^{t-1}\lambda^L + \nu(1-\lambda^H)^{t-1}\lambda^H) < 0,$$

which implies that $y^H_j > 0$ is only possible for $j < \hat{T}^L$ and we have $j = 1$.

Case 2.2.3.2: $T^L \leqslant \hat{T}^L$, $\psi < 0$, $\frac{\xi^H}{\delta^{T^H}} > \nu P^H_{T^H} + (1-\nu)P^L_{T^H}$, $\xi^L = 0$, $\alpha^L_t > 0$ for $1 \leqslant t \leqslant T^L$.

There exists only one time period $1 \leqslant j \leqslant T^H$ such that $y^H_j > 0$ ($\alpha^H_j = 0$). This implies that

$$P^H_{T^H} f_1(j,T^H)[\nu P^H_{T^L} + (1-\nu)P^L_{T^L}] + \frac{\xi^H}{\delta^{T^H}} P^L_{T^L} f_2(j,T^L) = 0 \text{ and}$$
$$P^H_{T^H} f_1(t,T^H)[\nu P^H_{T^L} + (1-\nu)P^L_{T^L}] + \frac{\xi^H}{\delta^{T^H}} P^L_{T^L} f_2(t,T^L) = -\frac{\alpha^H_t \psi}{\beta_0 \delta^t} > 0 \text{ for } 1 \leqslant t \neq j \leqslant T^H.$$

Alternatively, $f_1(t,T^H) > \frac{f_1(j,T^H)}{f_2(j,T^L)} f_2(t,T^L)$ for $1 \leqslant t \neq j \leqslant T^H$.

If $f_2(j,T^L) > 0$ ($j > \hat{T}^L$) then $\psi[1 - (\frac{1-\lambda^L}{1-\lambda^H})^(t-j)] < 0$, which implies that $t < j$ for all $1 \leqslant t \neq j \leqslant T^H$ or, equivalently, $j = T^H$.

If $f_2(j,T^L) < 0$ ($j < \hat{T}^L$) then the opposite must be true and $t > j$ for all $1 \leqslant t \neq j \leqslant T^H$ or, equivalently, $j = 1$.

For $t > \hat{T}^L$ it follows that

$$P^H_{T^H} f_1(t,T^H)[\nu P^H_{T^L} + (1-\nu)P^L_{T^L}] + \frac{\xi^H}{\delta^{T^H}} P^L_{T^L} f_2(t,T^L) >$$
$$-\psi((1-\nu)(1-\lambda^L)^{t-1}\lambda^L + \nu(1-\lambda^H)^{t-1}\lambda^H) > 0,$$

which implies that $y_j^H > 0$ is only possible for $j < \hat{T}^L$ and we have $j = 1$. *Q.E.D.*

From the binding incentive compatibility constraints, we derive

$$x^H = \frac{\beta_0 \delta P^L_{T^L} f_2(1,T^L) y_1^H}{\delta^{T^H} \psi} + \frac{P^H_{T^L}(\delta^{T^H} P^L_{T^H} \triangle c_{T^H} q^H(c^H_{T^H}) - \delta^{T^L} P^L_{T^L} \triangle c_{T^L} q^L(c^L_{T^L}))}{\delta^{T^H} \psi} = 0;$$

$$x^L = \frac{\beta_0 \delta P^H_{T^H}(-f_1(1,T^H)) y_1^H}{\delta^{T^L} \psi} + \frac{P^L_{T^H}(\delta^{T^H} P^H_{T^H} \triangle c_{T^H} q^H(c^H_{T^H}) - \delta^{T^L} P^H_{T^L} \triangle c_{T^L} q^L(c^L_{T^L}))}{\delta^{T^L} \psi} > 0.$$

Solving for the optimal payments and information rents we have

$$y_1^H = \frac{P^H_{T^L}(\delta^{T^H} P^L_{T^H} \triangle c_{T^H} q^H(c^H_{T^H}) - \delta^{T^L} P^L_{T^L} \triangle c_{T^L} q^L(c^L_{T^L}))}{\beta_0 \delta P^L_{T^L} f_2(1,T^L)};$$

$$x^L = \frac{\delta^{T^L} \lambda^L P^H_{T^L} \triangle c_{T^L} q^L(c^L_{T^L}) - \delta^{T^H} \lambda^H P^L_{T^H} \triangle c_{T^H} q^H(c^H_{T^H})}{\delta^{T^L} P^L_{T^L} f_2(1,T^L)};$$

$$\rho^L = \delta^{T^L} P^L_{T^L} x^L = \frac{\delta^{T^L} \lambda^L P^H_{T^L} \triangle c_{T^L} q^L(c^L_{T^L}) - \delta^{T^H} \lambda^H P^L_{T^H} \triangle c_{T^H} q^H(c^H_{T^H})}{f_2(1,T^L)};$$

$$\rho^H = \beta_0 \delta \lambda^H y_1^H = \frac{P^H_{T^L} \lambda^H (\delta^{T^H} P^L_{T^H} \triangle c_{T^H} q^H(c^H_{T^H}) - \delta^{T^L} P^L_{T^L} \triangle c_{T^L} q^L(c^L_{T^L}))}{P^L_{T^L} f_2(1,T^L)}.$$

Since the Lagrange multipliers $\xi^L = 0$, $\xi^H > 0$, α_t^L for $t \leqslant T^L$, $\alpha_t^H > 0$ for all $t > 1$ and $\alpha_1^H = 0$ we can now have distortion in both T^L and T^H. In particular, the F.O.C. with respect to T^L becomes:

$$\frac{\partial \mathcal{L}}{\partial T^L} = \frac{\nu[\beta_0 \sum_{t=1}^{T^H} \delta^t (1-\lambda^H)^{t-1} \lambda^H (V(q_t^H(\underline{c})) - y_t^H - \underline{c} q_t^H(\underline{c}) - \mathit{\Gamma}_t) +}{\partial T^L} +$$
$$\frac{\delta^{T^H} P^H_{T^H}(V(q^H(c^H_{T^H})) - x^H(\{y_t^H\}_{t=1}^{T^H}, \{y_t^L\}_{t=1}^{T^L}, T^H, T^L, q^H(c^H_{T^H}), q^L(c^L_{T^L})) - c^H_{T^H} q^H(c^H_{T^H}) - \mathit{\Gamma}_{T^H})]}{\partial T^L} +$$
$$\frac{(1-\nu)[\beta_0 \sum_{t=1}^{T^L} \delta^t (1-\lambda^L)^{t-1} \lambda^L (V(q_t^L(\underline{c})) - y_t^L - \underline{c} q_t^L(\underline{c}) - \mathit{\Gamma}_t)}{\partial T^L} +$$
$$\frac{+\delta^{T^L} P^L_{T^L}(V(q^L(c^L_{T^L})) - x^L(\{y_t^H\}_{t=1}^{T^H}, \{y_t^L\}_{t=1}^{T^L}, T^H, T^L, q^H(c^H_{T^H}), q^L(c^L_{T^L})) - c^L_{T^L} q^L(c^L_{T^L}) - \mathit{\Gamma}_{T^L})]}{\partial T^L} +$$
$$\frac{\sum_{t=2}^{T^H} \alpha_t^H y_t^H + \sum_{t=1}^{T^L} \alpha_t^L y_t^L + \xi^H x^H(\{y_t^H\}_{t=1}^{T^H}, \{y_t^L\}_{t=1}^{T^L}, T^H, T^L, q^H(c^H_{T^H}), q^L(c^L_{T^L}))}{\partial T^L} = 0.$$

In addition, there will be a distortion in the duration of the experimentation stage for the high type:

$$\frac{\partial \mathcal{L}}{\partial T^H} = \frac{\nu[\beta_0 \sum_{t=1}^{T^H} \delta^t (1-\lambda^H)^{t-1} \lambda^H (V(q_t^H(\underline{c})) - y_t^H - \underline{c} q_t^H(\underline{c}) - \mathit{\Gamma}_t) +}{\partial T^H} +$$
$$\frac{\delta^{T^H} P^H_{T^H}(V(q^H(c^H_{T^H})) - x^H(\{y_t^H\}_{t=1}^{T^H}, \{y_t^L\}_{t=1}^{T^L}, T^H, T^L, q^H(c^H_{T^H}), q^L(c^L_{T^L})) - c^H_{T^H} q^H(c^H_{T^H}) - \mathit{\Gamma}_{T^H})]}{\partial T^H} +$$
$$\frac{(1-\nu)[\beta_0 \sum_{t=1}^{T^L} \delta^t (1-\lambda^L)^{t-1} \lambda^L (V(q_t^L(\underline{c})) - y_t^L - \underline{c} q_t^L(\underline{c}) - \mathit{\Gamma}_t)}{\partial T^H} +$$
$$\frac{+\delta^{T^L} P^L_{T^L}(V(q^L(c^L_{T^L})) - x^L(\{y_t^H\}_{t=1}^{T^H}, \{y_t^L\}_{t=1}^{T^L}, T^H, T^L, q^H(c^H_{T^H}), q^L(c^L_{T^L})) - c^L_{T^L} q^L(c^L_{T^L}) - \mathit{\Gamma}_{T^L})]}{\partial T^H} +$$
$$\frac{\sum_{t=2}^{T^H} \alpha_t^H y_t^H + \sum_{t=1}^{T^L} \alpha_t^L y_t^L + \xi^H x^H(\{y_t^H\}_{t=1}^{T^H}, \{y_t^L\}_{t=1}^{T^L}, T^H, T^L, q^H(c^H_{T^H}), q^L(c^L_{T^L}))}{\partial T^H} = 0.$$

It is not possible, in general, to determine whether $T^H_{SB} > T^H_{FB}$ or $T^H_{SB} < T^H_{FB}$ and $T^L_{SB} > T^L_{FB}$ or $T^L_{SB} < T^L_{FB}$.

From the Kuhn-Tucker conditions for the optimization problem we have

$$P^H_{T^H} f_1(1, T^H)\mathbb{E}_\theta P^\theta_{T^L} + \frac{\xi^H}{\delta^{T^H}} P^L_{T^L} f_2(1, T^L) = 0,$$
$$\frac{\xi^H}{\delta^{T^H}} = -\frac{P^H_{T^H} f_1(1,T^H)\mathbb{E}_\theta P^\theta_{T^L}}{P^L_{T^L} f_2(1,T^L)} = -\frac{P^L_{T^H}\lambda^H - P^H_{T^H}\lambda^L}{P^H_{T^L}\lambda^L - P^L_{T^L}\lambda^H}\mathbb{E}_\theta P^\theta_{T^L}$$

To characterize the optimal output choices, we now consider the following Kuhn-Tucker conditions for the optimization problem:

$[q^H_t(\underline{c})]$ $\nu\beta_0\delta^t(1-\lambda^H)^{t-1}\lambda^H(V'(q^H_t(\underline{c})) - \underline{c}) = 0$ for $1 \leqslant s \leqslant T^H$;

$[q^L_t(\underline{c})]$ $(1-\nu)\beta_0\delta^t(1-\lambda^L)^{t-1}\lambda^L(V'(q^L_t(\underline{c})) - \underline{c}) = 0$ for $1 \leqslant s \leqslant T^L$;

$[q^H(c^H_{T^H})]$ $\nu\delta^{T^H} P^H_{T^H}(V'(q^H(c^H_{T^H})) - c^H_{T^H} - \frac{\delta^{T^H} P^L_{T^H} P^H_{T^L} \triangle c_{T^H}}{\delta^{T^H}\psi}) - (1-\nu)\delta^{T^L} P^L_{T^L}(-\frac{\delta^{T^H} P^L_{T^H} P^H_{T^H} \triangle c_{T^H}}{\delta^{T^L}\psi}) + \xi^H \frac{\delta^{T^H} P^L_{T^H} P^H_{T^L} \triangle c_{T^H}}{\delta^{T^H}\psi} = 0$;

$[q^L(c^L_{T^L})]$ $\nu\delta^{T^H} P^H_{T^H}(\frac{\delta^{T^L} P^L_{T^L} P^H_{T^L} \triangle c_{T^L}}{\delta^{T^H}\psi}) + (1-\nu)\delta^{T^L} P^L_{T^L}(V'(q^L(c^L_{T^L})) - c^L_{T^L} + \frac{\delta^{T^L} P^L_{T^H} P^H_{T^L} \triangle c_{T^L}}{\delta^{T^L}\psi}) + \xi^H(-\frac{\delta^{T^L} P^L_{T^L} P^H_{T^L} \triangle c_{T^L}}{\delta^{T^H}\psi}) = 0$;

The first two conditions above imply that there is no distortion in the output relative to the first-best level after success has been observed by either type; that is $V'(q^\theta_t(\underline{c})) = \underline{c}$ for $\theta \in \{H, L\}$. However, there is distortion in the output relative to the first-best level after T^L failures have been reported by the low type; that is

$$(1-\nu)[V'(q^L(c^L_{T^L})) - c^L_{T^L}] + \frac{P^H_{T^L}\triangle c_{T^L}}{\delta^{T^H}\psi}(\mathbb{E}_\theta P^\theta_{T^H} - \frac{\xi^H}{\delta^{T^H}}) = 0.$$

Given that

$$\mathbb{E}_\theta P^\theta_{T^H} - \frac{\xi^H}{\delta^{T^H}} = \frac{(\nu P^H_{T^H} + (1-\nu)P^L_{T^H})(P^H_{T^L}\lambda^L - P^L_{T^L}\lambda^H) + (\nu P^H_{T^L} + (1-\nu)P^L_{T^L})(P^L_{T^H}\lambda^H - P^H_{T^H}\lambda^L)}{P^H_{T^L}\lambda^L - P^L_{T^L}\lambda^H} = \frac{\psi\mathbb{E}_\theta\lambda^\theta}{P^H_{T^L}\lambda^L - P^L_{T^L}\lambda^H},$$

we can rewrite the Kuhn-Tucker condition as follows

$$(1-\nu)[V'(q^L(c^L_{T^L})) - c^L_{T^L}] = -\frac{\mathbb{E}_\theta\lambda^\theta P^H_{T^L}\triangle c_{T^L}}{P^L_{T^L}\lambda^H - P^H_{T^L}\lambda^L},$$

which given that function $V(\cdot)$ is increasing and concave implies overproduction by the low type after T^L failures have been reported by the low type:

$$q^L_{SB}(c^L_{T^L}) > q^L_{FB}(c^L_{T^L}).$$

We can simplify the Kuhn-Tucker condition for $q^H(c^H_{T^H}$ in the following way:

$$\nu\delta^{T^H} P^H_{T^H}(V'(q^H(c^H_{T^H})) - c^H_{T^H}) = \frac{\delta^{T^H} P^L_{T^H} P^H_{T^H} \triangle c_{T^H} \mathbb{E}_\theta P^\theta_{T^L}}{\psi} - \xi^H \frac{P^L_{T^H} P^H_{T^L} \triangle c_{T^H}}{\psi}$$
$$= \frac{\delta^{T^H} P^L_{T^H} \triangle c_{T^H} \mathbb{E}_\theta P^\theta_{T^L}}{\psi}\left(P^H_{T^H} + P^H_{T^L} \frac{P^L_{T^H}\lambda^H - P^H_{T^H}\lambda^L}{P^H_{T^L}\lambda^L - P^L_{T^L}\lambda^H}\right) = \frac{P^L_{T^H} \triangle c_{T^H} \mathbb{E}_\theta P^\theta_{T^L}}{P^H_{T^L}\lambda^L - P^L_{T^L}\lambda^H},$$

that is, $\nu\delta^{T^H} P^H_{T^H}(V'(q^H(c^H_{T^H})) - c^H_{T^H}) = \frac{P^L_{T^H} \triangle c_{T^H} \mathbb{E}_\theta P^\theta_{T^L}}{P^H_{T^L}\lambda^L - P^L_{T^L}\lambda^H}$.

Given that function $V(\cdot)$ is increasing and concave the optimal contract exhibits underproduction after T^H failures have been reported by the high type in this case.

Case 2.2.4: Suppose that $\xi^H > 0$ and $\xi^L > 0$, i.e., both the $(IRF^H_{T^H})$ and $(IRF^L_{T^L})$ constraints are binding.

Claim 2.2.4. $\xi^H > 0$ and $\xi^L > 0 \iff T^L > \hat{T}^L$, $\alpha^L_t > 0$ for $t < T^L$, $\alpha^L_{T^L} = 0$ (it is optimal to set $x^L = 0$, $y^L_t = 0$ for $t < T^L$, $y^L_{T^L} > 0$) and $\alpha^H_t > 0$ for all $t > 1$ and $\alpha^H_1 = 0$ (it is optimal to set $x^H = 0$, $y^H_t = 0$ for all $t > 1$ and $y^H_1 > 0$).

Proof: We can rewrite the Kuhn-Tucker conditions above as follows:

$$\frac{\partial \mathcal{L}}{\partial y^H_t} = \frac{\beta_0\delta^t}{\psi}[P^H_{T^H} f_1(t, T^H)[\nu P^H_{T^L} + (1-\nu)P^L_{T^L} - \frac{\xi^L}{\delta^{T^L}}] + \frac{\xi^H}{\delta^{T^H}} P^L_{T^L} f_2(t, T^L) + \frac{\alpha^H_t \psi}{\beta_0\delta^t}] \text{ for } 1 \leqslant t \leqslant T^H;$$

$$\frac{\partial \mathcal{L}}{\partial y^L_t} = \frac{\beta_0\delta^t}{\psi}[P^L_{T^L} f_2(t, T^L)[\nu P^H_{T^H} + (1-\nu)P^L_{T^H} - \frac{\xi^H}{\delta^{T^L}}] + \frac{\xi^L}{\delta^{T^L}} P^H_{T^H} f_1(t, T^H) + \frac{\alpha^L_t \psi}{\beta_0\delta^t}] \text{ for } 1 \leqslant t \leqslant T^L.$$

Suppose $\frac{\xi^H}{\delta^{T^H}} = \nu P^H_{T^H} + (1-\nu)P^L_{T^H}$. Then $\frac{\xi^L}{\delta^{T^L}} P^H_{T^H} f_1(t, T^H) + \frac{\alpha^L_t \psi}{\beta_0\delta^t} = 0$ for $1 \leqslant t \leqslant T^L$ and $\alpha^L_t > 0$[13] for $1 \leqslant t \leqslant T^L$ which leads to a contradiction since then $x^L = y^L_t = 0$ for $1 \leqslant t \leqslant T^L$ which implies that the low type does not receive any rent.

Suppose $\frac{\xi^L}{\delta^{T^L}} = \nu P^H_{T^L} + (1-\nu)P^L_{T^L}$. Then $\frac{\xi^H}{\delta^{T^H}} P^L_{T^L} f_2(t, T^L) + \frac{\alpha^H_t \psi}{\beta_0\delta^t} = 0$ for $1 \leqslant t \leqslant T^H$ and $\alpha^H_t > 0$[14] for $1 \leqslant t \leqslant T^H$ which leads to a contradiction since then $x^H = y^H_t = 0$ for $1 \leqslant t \leqslant T^H$ which implies that the high type does not receive any rent.

Lemma 3. There exists at most one $j \leqslant T^L$ such that $y^L_j > 0$ and at most one $s \leqslant T^H$ such that $y^H_s > 0$.

Proof: Assume to the contrary that there are two distinct periods $1 \leqslant k, m \leqslant T^L$ such that $k \neq m$ and $y^L_k, y^L_m > 0$. Then from the Kuhn-Tucker conditions it follows that

$$P^L_{T^L} f_2(k, T^L)[\nu P^H_{T^H} + (1-\nu)P^L_{T^H} - \frac{\xi^H}{\delta^{T^L}}] + \frac{\xi^L}{\delta^{T^L}} P^H_{T^H} f_1(k, T^H) = 0, \text{ and}$$
$$P^L_{T^L} f_2(m, T^L)[\nu P^H_{T^H} + (1-\nu)P^L_{T^H} - \frac{\xi^H}{\delta^{T^L}}] + \frac{\xi^L}{\delta^{T^L}} P^H_{T^H} f_1(m, T^H) = 0.$$

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[13] If $t = \hat{T}^L$, then $y^L_{\hat{T}^L} > 0$ can be optimal.

[14] If $t = \hat{T}^H$, then $y^H_{\hat{T}^H} > 0$ can be optimal.

Thus, $\frac{f_2(m,T^L)}{f_1(m,T^H)} = \frac{f_2(k,T^L)}{f_1(k,T^H)} \Longleftrightarrow (\frac{1-\lambda^L}{1-\lambda^H})^{(m-k)} = 1$, which implies that $m = k$ and leads to a contradiction.

Assume to the contrary that there are two distinct periods $1 \leqslant s, p \leqslant T^H$ such that $s \neq p$ and $y_s^H, y_p^H > 0$. Then from the Kuhn-Tucker conditions it follows that

$$P_{T^H}^H f_1(s,T^H)[\nu P_{T^L}^H + (1-\nu)P_{T^L}^L - \tfrac{\xi^L}{\delta^{T^L}}] + \tfrac{\xi^H}{\delta^{T^H}} P_{T^L}^L f_2(s,T^L), \text{ and}$$
$$P_{T^H}^H f_1(p,T^H)[\nu P_{T^L}^H + (1-\nu)P_{T^L}^L - \tfrac{\xi^L}{\delta^{T^L}}] + \tfrac{\xi^H}{\delta^{T^H}} P_{T^L}^L f_2(p,T^L).$$

Thus, $(\frac{1-\lambda^L}{1-\lambda^H})^{(s-p)} = 1$, which implies that $s = p$ and leads to a contradiction. *Q.E.D.*

Lemma 4: Both types are promised a positive rent at an *extreme* time period, i.e. only at *the last* or *the first* period of the experimentation stage.

Proof: Since there exists only one time period $1 \leqslant j \leqslant T^L$ such that $y_j^L > 0$ ($\alpha_j^L = 0$) it follows that

$$P_{T^L}^L f_2(j,T^L)[\nu P_{T^H}^H + (1-\nu)P_{T^H}^L - \tfrac{\xi^H}{\delta^{T^L}}] + \tfrac{\xi^L}{\delta^{T^L}} P_{T^H}^H f_1(j,T^H) = 0, \text{ and}$$
$$P_{T^L}^L f_2(t,T^L)[\nu P_{T^H}^H + (1-\nu)P_{T^H}^L - \tfrac{\xi^H}{\delta^{T^L}}] + \tfrac{\xi^L}{\delta^{T^L}} P_{T^H}^H f_1(t,T^H) = -\tfrac{\alpha_t^L \psi}{\beta_0 \delta^t} < \text{ for } 1 \leqslant t \neq j \leqslant T^L.$$

Alternatively, $\frac{\xi^L}{\delta^{T^L}}[f_1(t,T^H) - \frac{f_2(t,T^L)}{f_2(j,T^L)} f_1(j,T^H)] = -\frac{\alpha_t^L \psi}{\beta_0 \delta^t P_{T^H}^H}$ for $1 \leqslant t \neq j \leqslant T^L$.

Suppose $\psi > 0$. Then $f_1(t,T^H) - \frac{f_2(t,T^L)}{f_2(j,T^L)} f_1(j,T^H) < 0$ for $1 \leqslant t \neq j \leqslant T^L$.

If $f_2(j,T^L) > 0$ ($j > \hat{T}^L$) then $\psi[1 - (\frac{1-\lambda^L}{1-\lambda^H})^{(j-t)}] < 0$, which implies that $t < j$ for all $1 \leqslant t \neq j \leqslant T^L$, which implies that $j = T^L > \hat{T}^L$. If $f_2(j,T^L) < 0$ ($j < \hat{T}^L$) then $\psi[1 - (\frac{1-\lambda^L}{1-\lambda^H})^{(j-t)}] > 0$, which implies that $t > j$ for all $1 \leqslant t \neq j \leqslant T^L$, which implies that $j = 1$.

Suppose $\psi < 0$. Then $f_1(t,T^H) - \frac{f_2(t,T^L)}{f_2(j,T^L)} f_1(j,T^H) > 0$ for $1 \leqslant t \neq j \leqslant T^L$.

If $f_2(j,T^L) > 0$ ($j > \hat{T}^L$) then $\psi[1 - (\frac{1-\lambda^L}{1-\lambda^H})^{(j-t)}] > 0$, which implies that $t < j$ for all $1 \leqslant t \neq j \leqslant T^L$, which implies that $j = T^L > \hat{T}^L$. If $f_2(j,T^L) < 0$ ($j < \hat{T}^L$) then $\psi[1 - (\frac{1-\lambda^L}{1-\lambda^H})^{(j-t)}] < 0$, which implies that $t > j$ for all $1 \leqslant t \neq j \leqslant T^L$, which implies that $j = 1$.

Since there exists only one time period $1 \leqslant s \leqslant T^H$ such that $y_s^H > 0$ ($\alpha_s^H = 0$) it follows that

$$P_{T^H}^H f_1(s,T^H)[\nu P_{T^L}^H + (1-\nu)P_{T^L}^L - \tfrac{\xi^L}{\delta^{T^L}}] + \tfrac{\xi^H}{\delta^{T^H}} P_{T^L}^L f_2(t,T^L) = 0,$$
$$P_{T^H}^H f_1(t,T^H)[\nu P_{T^L}^H + (1-\nu)P_{T^L}^L - \tfrac{\xi^L}{\delta^{T^L}}] + \tfrac{\xi^H}{\delta^{T^H}} P_{T^L}^L f_2(t,T^L) = -\tfrac{\alpha_t^H \psi}{\beta_0 \delta^t} < 0 \text{ for}$$
$$1 \leqslant s \neq t \leqslant T^H.$$

Alternatively, $\frac{\xi^H}{\delta^{T^H}}[f_2(t,T^L) - \frac{f_1(t,T^H)}{f_1(s,T^H)} f_2(s,T^L)] = -\frac{\alpha_t^H \psi}{\beta_0 \delta^t}$ for $1 \leqslant t \neq s \leqslant T^H$.

Suppose $\psi > 0$. Then $f_2(t,T^L) - \frac{f_1(t,T^H)}{f_1(s,T^H)} f_2(s,T^L) < 0$ for $1 \leqslant t \neq s \leqslant T^H$.

If $f_1(s,T^H) > 0$ $(s < \hat{T}^H)$ then $\psi[1 - (\frac{1-\lambda^L}{1-\lambda^H})^{(t-s)}] < 0$, which implies that $t > s$ for all $1 \leqslant t \neq s \leqslant T^H$, which implies that $s = 1$. If $f_1(s,T^H) < 0$ $(s > \hat{T}^H)$ then $\psi[1 - (\frac{1-\lambda^L}{1-\lambda^H})^{(t-s)}] > 0$, which implies that $t < s$ for all $1 \leqslant t \neq s \leqslant T^H$, which implies that $s = T^H > \hat{T}^H$.

Suppose $\psi < 0$. Then $f_2(t,T^L) - \frac{f_1(t,T^H)}{f_1(s,T^H)} f_2(s,T^L) > 0$ for $1 \leqslant t \neq s \leqslant T^H$.

If $f_1(s,T^H) > 0$ $(s < \hat{T}^H)$ then $\psi[1 - (\frac{1-\lambda^L}{1-\lambda^H})^{(t-s)}] > 0$, which implies that $t > s$ for all $1 \leqslant t \neq s \leqslant T^H$, which implies that $s = 1$. If $f_1(s,T^H) < 0$ $(s > \hat{T}^H)$ then $\psi[1 - (\frac{1-\lambda^L}{1-\lambda^H})^{(t-s)}] < 0$, which implies that $t < s$ for all $1 \leqslant t \neq s \leqslant T^H$, which implies that $s = T^H > \hat{T}^H$. *Q.E.D.*

The Lagrange multipliers are uniquely defined as follows:

$$\begin{aligned} \frac{\xi^L}{\delta^{T^L}} &= \frac{\psi[\nu(1-\lambda^H)^{s-1}\lambda^H + (1-\nu)(1-\lambda^L)^{s-1}\lambda^L]}{P^H_{T^H}[f_1(j,T^H)f_2(s,T^L) - f_1(s,T^H)f_2(j,T^L)]} > 0; \\ \frac{\xi^H}{\delta^{T^L}} &= \frac{\psi[\nu(1-\lambda^H)^{j-1}\lambda^H + (1-\nu)(1-\lambda^L)^{j-1}\lambda^L]}{P^L_{T^L}[f_1(j,T^H)f_2(s,T^L) - f_1(s,T^H)f_2(j,T^L)]} > 0. \end{aligned}$$

If $s = T^H > \hat{T}^H$ then the optimal contract involves

$$\begin{aligned} x^H &= \frac{\beta_0 \delta^{T^H} P^L_{T^L} f_2(T^H,T^L) y^H_{T^H} - \beta_0 \delta P^L_{T^L} f_2(1,T^L) y^L_1}{\delta^{T^H}(P^H_{T^H} P^L_{T^L} - P^H_{T^L} P^L_{T^H})} \\ &+ \frac{P^H_{T^L}(\delta^{T^H} P^L_{T^H} \triangle c_{T^H} q^H(c^H_{T^H}) - \delta^{T^L} P^L_{T^L} \triangle c_{T^L} q^L(c^L_{T^L}))}{\delta^{T^H}(P^H_{T^H} P^L_{T^L} - P^H_{T^L} P^L_{T^H})}; \\ x^L &= \frac{\beta_0 \delta^{T^H} P^H_{T^H} f_1(T^H,T^H) y^H_{T^H} - \beta_0 \delta^{T^H} [P^H_{T^H} f_1(T^H,T^H)] y^H_{T^H}}{\delta^{T^L}(P^H_{T^H} P^L_{T^L} - P^H_{T^L} P^L_{T^H})} \\ &+ \frac{P^L_{T^H}(\delta^{T^H} P^H_{T^H} \triangle c_{T^H} q^H(c^H_{T^H}) - \delta^{T^L} P^H_{T^L} \triangle c_{T^L} q^L(c^L_{T^L}))}{\delta^{T^L}(P^H_{T^H} P^L_{T^L} - P^H_{T^L} P^L_{T^H})}. \end{aligned}$$

Since $f_2(1,T^L) < 0$ from $\frac{\xi^L}{\delta^{T^L}} > 0$ it follows that $\psi > 0$ which implies that $P^L_{T^H}(\delta^{T^H} P^H_{T^H} \triangle c_{T^H} q^H(c^H_{T^H}) > \delta^{T^L} P^H_{T^L} \triangle c_{T^L} q^L(c^L_{T^L}))$. However, then we have a contradiction since $-f_1(T^H,T^H)$ and $-f_1(1,T^H)$ imply that $x^L > 0$. As a result, $s = 1$.

If $T^L > \hat{T}^L$ then the optimal contract involves

$$\begin{aligned} x^H &= \frac{\beta_0 \delta^{T^H} P^L_{T^L} f_2(1,T^L) y^H_1 - \beta_0 \delta^{T^L} P^L_{T^L} f_2(T^L,T^L) y^L_{T^L}}{\delta^{T^H}(P^H_{T^H} P^L_{T^L} - P^H_{T^L} P^L_{T^H})} \\ &+ \frac{P^H_{T^L}(\delta^{T^H} P^L_{T^H} \triangle c_{T^H} q^H(c^H_{T^H}) - \delta^{T^L} P^L_{T^L} \triangle c_{T^L} q^L(c^L_{T^L}))}{\delta^{T^H}(P^H_{T^H} P^L_{T^L} - P^H_{T^L} P^L_{T^H})}; \\ x^L &= \frac{\beta_0 \delta^{T^L} P^H_{T^H} f_1(T^L,T^H) y^L_{T^L} - \beta_0 \delta P^H_{T^H} f_1(1,T^H) y^H_1}{\delta^{T^L}(P^H_{T^H} P^L_{T^L} - P^H_{T^L} P^L_{T^H})} \end{aligned}$$

$$+\frac{P^L_{T^H}(\delta^{T^H}P^H_{T^H}\triangle c_{T^H}q^H(c^H_{T^H})-\delta^{T^L}P^H_{T^L}\triangle c_{T^L}q^L(c^L_{T^L}))}{\delta^{T^L}(P^H_{T^H}P^L_{T^L}-P^H_{T^L}P^L_{T^H})};$$

Solving the system above we have:

$$y^H_1=\frac{\delta^{T^H}P^L_{T^H}\triangle c_{T^H}q^H(c^H_{T^H})(1-\lambda^H)^{T^L-1})\lambda^H-\delta^{T^L}P^H_{T^L}\triangle c_{T^L}q^L(c^L_{T^L})(1-\lambda^L)^{T^L-1}\lambda^L}{\beta_0\delta\lambda^L\lambda^H((1-\lambda^L)^{T^L-1}-(1-\lambda^H)^{T^L-1})};$$

$$y^L_{T^L}=\frac{\delta^{T^H}P^L_{T^H}\triangle c_{T^H}q^H(c^H_{T^H})\lambda^H-\delta^{T^L}P^H_{T^L}\triangle c_{T^L}q^L(c^L_{T^L})\lambda^L}{\beta_0\delta\lambda^L\lambda^H((1-\lambda^L)^{T^L-1}-(1-\lambda^H)^{T^L-1})}.$$

Since the Lagrange multipliers $\xi^L>0$, $\xi^H>0$, α^L_t for $t\leqslant T^L$, $\alpha^H_t>0$ for all $t>1$ and $\alpha^H_1=0$ we can now have distortion in both T^L and T^H. In particular, the F.O.C. with respect to T^L becomes:

$$0=\frac{\partial\mathcal{L}}{\partial T^L}=\frac{\nu[\beta_0\sum_{t=1}^{T^H}\delta^t(1-\lambda^H)^{t-1}\lambda^H(V(q^H_t(\underline{c}))-y^H_t-\underline{c}q^H_t(\underline{c})-\mathit{\Gamma}_t)+}{\partial T^L}+$$
$$\frac{\delta^{T^H}P^H_{T^H}(V(q^H(c^H_{T^H}))-x^H(\{y^H_t\}_{t=1}^{T^H},\{y^L_t\}_{t=1}^{T^L},T^H,T^L,q^H(c^H_{T^H}),q^L(c^L_{T^L}))-c^H_{T^H}q^H(c^H_{T^H})-\mathit{\Gamma}_{T^H})]}{\partial T^L}+$$
$$\frac{(1-\nu)[\beta_0\sum_{t=1}^{T^L}\delta^t(1-\lambda^L)^{t-1}\lambda^L(V(q^L_t(\underline{c}))-y^L_t-\underline{c}q^L_t(\underline{c})-\mathit{\Gamma}_t)}{\partial T^L}+$$
$$\frac{+\delta^{T^L}P^L_{T^L}(V(q^L(c^L_{T^L}))-x^L(\{y^H_t\}_{t=1}^{T^H},\{y^L_t\}_{t=1}^{T^L},T^H,T^L,q^H(c^H_{T^H}),q^L(c^L_{T^L}))-c^L_{T^L}q^L(c^L_{T^L})-\mathit{\Gamma}_{T^L})]}{\partial T^L}+$$
$$\frac{\sum_{t=2}^{T^H}\alpha^H_t y^H_t+\sum_{t=1}^{T^L-1}\alpha^L_t y^L_t+\xi^H x^H(\{y^H_t\}_{t=1}^{T^H},\{y^L_t\}_{t=1}^{T^L},T^H,T^L,q^H(c^H_{T^H}),q^L(c^L_{T^L}))+\xi^L x^L(\{y^H_t\}_{t=1}^{T^H},\{y^L_t\}_{t=1}^{T^L},T^H,T^L,q^H(c^H_{T^H}),q^L(c^L_{T^L}))}{\partial T^L}.$$

In addition, there will be a distortion in the duration of the experimentation stage for the high type:

$$0=\frac{\partial\mathcal{L}}{\partial T^H}=\frac{\nu[\beta_0\sum_{t=1}^{T^H}\delta^t(1-\lambda^H)^{t-1}\lambda^H(V(q^H_t(\underline{c}))-y^H_t-\underline{c}q^H_t(\underline{c})-\mathit{\Gamma}_t)+}{\partial T^H}+$$
$$\frac{\delta^{T^H}P^H_{T^H}(V(q^H(c^H_{T^H}))-x^H(\{y^H_t\}_{t=1}^{T^H},\{y^L_t\}_{t=1}^{T^L},T^H,T^L,q^H(c^H_{T^H}),q^L(c^L_{T^L}))-c^H_{T^H}q^H(c^H_{T^H})-\mathit{\Gamma}_{T^H})]}{\partial T^H}+$$
$$\frac{(1-\nu)[\beta_0\sum_{t=1}^{T^L}\delta^t(1-\lambda^L)^{t-1}\lambda^L(V(q^L_t(\underline{c}))-y^L_t-\underline{c}q^L_t(\underline{c})-\mathit{\Gamma}_t)}{\partial T^H}+$$
$$\frac{+\delta^{T^L}P^L_{T^L}(V(q^L(c^L_{T^L}))-x^L(\{y^H_t\}_{t=1}^{T^H},\{y^L_t\}_{t=1}^{T^L},T^H,T^L,q^H(c^H_{T^H}),q^L(c^L_{T^L}))-c^L_{T^L}q^L(c^L_{T^L})-\mathit{\Gamma}_{T^L})]}{\partial T^H}+$$
$$\frac{\sum_{t=2}^{T^H}\alpha^H_t y^H_t+\sum_{t=1}^{T^L}\alpha^L_t y^L_t+\xi^H x^H(\{y^H_t\}_{t=1}^{T^H},\{y^L_t\}_{t=1}^{T^L},T^H,T^L,q^H(c^H_{T^H}),q^L(c^L_{T^L}))+\xi^L x^L(\{y^H_t\}_{t=1}^{T^H},\{y^L_t\}_{t=1}^{T^L},T^H,T^L,q^H(c^H_{T^H}),q^L(c^L_{T^L}))}{\partial T^H}.$$

It is not possible, in general, to determine whether $T^H_{SB}>T^H_{FB}$ or $T^H_{SB}<T^H_{FB}$ and $T^L_{SB}>T^L_{FB}$ or $T^L_{SB}<T^L_{FB}$.

Given that

$$f_1(T^L,T^H)f_2(1,T^L)-f_1(1,T^H)f_2(T^L,T^L)=\frac{\lambda^L\lambda^H}{P^L_{T^L}P^H_{T^H}}\psi((1-\lambda^L)^{T^L-1}-(1-\lambda^H)^{T^L-1}),$$

we can rewrite

$$\frac{\xi^L}{\delta^{T^L}}=\frac{\mathbb{E}_\theta\lambda^\theta P^L_{T^L}f_2(T^L,T^L)}{\lambda^L\lambda^H((1-\lambda^L)^{T^L-1}-(1-\lambda^H)^{T^L-1})}\text{ and }\frac{\xi^L}{\delta^{T^L}}=\frac{\mathbb{E}_{(1-\lambda^\theta)^\theta\theta}\lambda^\theta P^H_{T^H}f_1(1,T^H)}{\lambda^L\lambda^H((1-\lambda^L)^{T^L-1}-(1-\lambda^H)^{T^L-1})}.$$

To characterize the optimal output choices, we now consider the following Kuhn-Tucker conditions for the optimization problem:

$[q_t^H(\underline{c})]$ $\nu\beta_0\delta^t(1-\lambda^H)^{t-1}\lambda^H(V'(q_t^H(\underline{c}))-\underline{c})=0$ for $1\leqslant s\leqslant T^H$;

$[q_t^L(\underline{c})]$ $(1-\nu)\beta_0\delta^t(1-\lambda^L)^{t-1}\lambda^L(V'(q_t^L(\underline{c}))-\underline{c})=0$ for $1\leqslant s\leqslant T^L$;

$[q^H(c^H_{T^H})]$ $\nu\delta^{T^H}P^H_{T^H}(V'(q^H(c^H_{T^H}))-c^H_{T^H}-\frac{\delta^{T^H}P^L_{T^H}P^H_{T^L}\triangle c_{T^H}}{\delta^{T^H}\psi})-(1-\nu)\delta^{T^L}P^L_{T^L}(\frac{\delta^{T^H}P^L_{T^H}P^H_{T^H}\triangle c_{T^H}}{\delta^{T^L}\psi})+$
$\xi^H\frac{\delta^{T^H}P^L_{T^H}P^H_{T^L}\triangle c_{T^H}}{\delta^{T^H}\psi}+\xi^L\frac{\delta^{T^H}P^L_{T^H}P^H_{T^H}\triangle c_{T^H}}{\delta^{T^L}\psi}=0$;

$[q^L(c^L_{T^L})]$ $\nu\delta^{T^H}P^H_{T^H}\frac{\delta^{T^L}P^L_{T^L}P^H_{T^L}\triangle c_{T^L}}{\delta^{T^H}\psi}+(1-\nu)\delta^{T^L}P^L_{T^L}(V'(q^L(c^L_{T^L}))-c^L_{T^L}+\frac{\delta^{T^L}P^L_{T^H}P^H_{T^L}\triangle c_{T^L}}{\delta^{T^L}\psi})+$
$\xi^H(-\frac{\delta^{T^L}P^L_{T^L}P^H_{T^L}\triangle c_{T^L}}{\delta^{T^H}\psi})+\xi^L(-\frac{\delta^{T^L}P^L_{T^H}P^H_{T^L}\triangle c_{T^L}}{\delta^{T^L}\psi})=0$;

The first two conditions above imply that there is no distortion in the output relative to the first-best level after success has been observed by either type; that is $V'(q_t^\theta(\underline{c}))=\underline{c}$ for $\theta\in\{H,L\}$.

However, there is distortion in the output relative to the first-best level after T^L failures have been reported by the low type; that is

$$(1-\nu)P^L_{T^L}[V'(q^L(c^L_{T^L}))-c^L_{T^L}]+\frac{P^L_{T^L}P^H_{T^L}\triangle c_{T^L}}{\psi}\mathbb{E}_\theta P^\theta_{T^H}=\frac{P^H_{T^L}\triangle c_{T^L}}{\psi}(\frac{\xi^H}{\delta^{T^H}}P^L_{T^L}+\frac{\xi^L}{\delta^{T^L}}P^L_{T^H});$$
$$(1-\nu)P^L_{T^L}[V'(q^L(c^L_{T^L}))-c^L_{T^L}]=\frac{P^H_{T^L}\triangle c_{T^L}}{\psi}(\frac{\xi^H}{\delta^{T^H}}P^L_{T^L}+\frac{\xi^L}{\delta^{T^L}}P^L_{T^H}-P^L_{T^L}\mathbb{E}_\theta P^\theta_{T^H});$$

Given that $\frac{\xi^H}{\delta^{T^H}}P^L_{T^L}+\frac{\xi^L}{\delta^{T^L}}P^L_{T^H}-P^L_{T^L}\mathbb{E}_\theta P^\theta_{T^H}=-\frac{\psi(1-\lambda^L)^{T^L-1}\mathbb{E}_\theta\lambda^\theta}{\lambda^H((1-\lambda^L)^{T^L-1}-(1-\lambda^H)^{T^L-1})}$ we conclude

$$[V'(q^L(c^L_{T^L}))-c^L_{T^L}]=-\frac{P^H_{T^L}(1-\lambda^L)^{T^L-1}\mathbb{E}_\theta\lambda^\theta}{(1-\nu)P^L_{T^L}\lambda^H((1-\lambda^L)^{T^L-1}-(1-\lambda^H)^{T^L-1})}\triangle c_{T^L},$$

and given that function $V(\cdot)$ is increasing and concave the optimal contract exhibits overproduction after T^L failures have been reported by the low type in this case:

$$q^L_{SB}(c^L_{T^L})>q^L_{FB}(c^L_{T^L}).$$

In addition, there is distortion in the output relative to the first-best level after T^H failures have been reported by the low type; that is

$$\nu P^H_{T^H}[V'(q^H(c^H_{T^H}))-c^H_{T^H}]-\frac{P^H_{T^H}P^L_{T^H}\triangle c_{T^H}\mathbb{E}_\theta P^\theta_{T^H}}{\psi}+\frac{P^L_{T^H}\triangle c_{T^L}}{\psi}(\frac{\xi^H}{\delta^{T^H}}P^H_{T^L}+\frac{\xi^L}{\delta^{T^L}}P^H_{T^H});$$
$$\nu P^H_{T^H}[V'(q^H(c^H_{T^H}))-c^H_{T^H}]=\frac{P^L_{T^H}\triangle c_{T^H}}{\psi}(P^H_{T^H}\mathbb{E}_\theta P^\theta_{T^L}-\frac{\xi^H}{\delta^{T^H}}P^H_{T^L}-\frac{\xi^L}{\delta^{T^L}}P^H_{T^H});$$

Given that $P^H_{T^H}\mathbb{E}_\theta P^\theta_{T^L}-\frac{\xi^H}{\delta^{T^H}}P^H_{T^L}-\frac{\xi^L}{\delta^{T^L}}P^H_{T^H}=\frac{\psi\mathbb{E}_\theta(1-\lambda^\theta)^{T^L-1}\lambda^\theta}{\lambda^L((1-\lambda^L)^{T^L-1}-(1-\lambda^H)^{T^L-1})}$ we conclude

$$V'(q^H(c^H_{T^H}))-c^H_{T^H}=\frac{P^L_{T^H}\mathbb{E}_\theta(1-\lambda^\theta)^{T^L-1}\lambda^\theta}{\nu P^H_{T^H}\lambda^L((1-\lambda^L)^{T^L-1}-(1-\lambda^H)^{T^L-1})}\triangle c_{T^H},$$

Given that function $V(\cdot)$ is increasing and concave the optimal contract exhibits under-production after T^H failures have been reported by the high type in this case: $q^H_{SB}(c^H_{T^H}) < q^H_{FB}(c^H_{T^H})$.

Case 2.3: Both types receive a strictly positive informational rent ($P^H_{T^H}P^L_{T^L} - P^H_{T^L}P^L_{T^H} = 0$.) The constraint $(IC^{L,H})$ must be binding. If the $(IC^{L,H})$ constraint was not binding, it would be possible to decrease the payment to the low type without violating any other constraint and improve the value of the objective function. In addition, the $(IC^{H,L})$ constraint is binding as well by the same logic. From the binding $(IC^{L,H})$ and $(IC^{H,L})$ constraints we have:

$$\begin{gathered}
\beta_0 \textstyle\sum_{t=1}^{T^H} \delta^t [P^H_{T^L}(1-\lambda^L)^{t-1}\lambda^L - P^L_{T^L}(1-\lambda^H)^{t-1}\lambda^H] y^H_t \\
+\beta_0 \textstyle\sum_{t=1}^{T^L} \delta^t [P^L_{T^L}(1-\lambda^H)^{t-1}\lambda^H - P^H_{T^L}(1-\lambda^L)^{t-1}\lambda^L] y^L_t \\
+P^H_{T^L}(\delta^{T^H} P^L_{T^H} \triangle c_{T^H} q^H(c^H_{T^H}) - \delta^{T^L} P^L_{T^L} \triangle c_{T^L} q^L(c^L_{T^L})) = 0, \text{ and} \\
\beta_0 \textstyle\sum_{t=1}^{T^H} \delta^t [P^H_{T^H}(1-\lambda^L)^{t-1}\lambda^L - P^L_{T^H}(1-\lambda^H)^{t-1}\lambda^H] y^H_t \\
+\beta_0 \textstyle\sum_{t=1}^{T^L} \delta^t [P^L_{T^H}(1-\lambda^H)^{t-1}\lambda^H - P^H_{T^H}(1-\lambda^L)^{t-1}\lambda^L] y^L_t \\
+P^L_{T^H}(\delta^{T^H} P^H_{T^H} \triangle c_{T^H} q^H(c^H_{T^H}) - \delta^{T^L} P^H_{T^L} \triangle c_{T^L} q^L(c^L_{T^L})) = 0.
\end{gathered}$$

The principal's optimization problem is to choose T^H, $\{q^H_t(\underline{c})\}_{t=1}^{T^H}$, $q^H(c^H_{T^H})$, x^H, $\{y^H_t\}_{t=1}^{T^H}$, T^L, $\{y^L_t\}_{t=1}^{T^L}$, x^L, $\{q^L_t(\underline{c})\}_{t=1}^{T^L}$, and $q^L(c^L_{T^L})$ to

$$\begin{gathered}
\max\{\nu[\beta_0 \textstyle\sum_{t=1}^{T^H} \delta^t (1-\lambda^H)^{t-1}\lambda^H (V(q^H_t(\underline{c})) - y^H_t - \underline{c} q^H_t(\underline{c}) - \mathit{\Gamma}_t) \\
+\delta^{T^H} P^H_{T^H}(V(q^H(c^H_{T^H})) - x^H - c^H_{T^H} q^H(c^H_{T^H}) - \mathit{\Gamma}_{T^H})] \\
+(1-\nu)[\beta_0 \textstyle\sum_{t=1}^{T^L} \delta^t (1-\lambda^L)^{t-1}\lambda^L (V(q^L_t(\underline{c})) - y^L_t - \underline{c} q^L_t(\underline{c}) - \mathit{\Gamma}_t) \\
+\delta^{T^L} P^L_{T^L}(V(q^L(c^L_{T^L})) - x^L - c^L_{T^L} q^L(c^L_{T^L}) - \mathit{\Gamma}_{T^L})]\} \text{ s.t.}
\end{gathered}$$

$$\begin{gathered}
(IC^{L,H})\ \beta_0 \textstyle\sum_{t=1}^{T^H} \delta^t [P^H_{T^L}(1-\lambda^L)^{t-1}\lambda^L - P^L_{T^L}(1-\lambda^H)^{t-1}\lambda^H] y^H_t \\
+\beta_0 \textstyle\sum_{t=1}^{T^L} \delta^t [P^L_{T^L}(1-\lambda^H)^{t-1}\lambda^H - P^H_{T^L}(1-\lambda^L)^{t-1}\lambda^L] y^L_t \\
+P^H_{T^L}(\delta^{T^H} P^L_{T^H} \triangle c_{T^H} q^H(c^H_{T^H}) - \delta^{T^L} P^L_{T^L} \triangle c_{T^L} q^L(c^L_{T^L})) = 0 \\
(IC^{H,L})\ \beta_0 \textstyle\sum_{t=1}^{T^H} \delta^t [P^H_{T^H}(1-\lambda^L)^{t-1}\lambda^L - P^L_{T^H}(1-\lambda^H)^{t-1}\lambda^H] y^H_t \\
+\beta_0 \textstyle\sum_{t=1}^{T^L} \delta^t [P^L_{T^H}(1-\lambda^H)^{t-1}\lambda^H - P^H_{T^H}(1-\lambda^L)^{t-1}\lambda^L] y^L_t \\
+P^L_{T^H}(\delta^{T^H} P^H_{T^H} \triangle c_{T^H} q^H(c^H_{T^H}) - \delta^{T^L} P^H_{T^L} \triangle c_{T^L} q^L(c^L_{T^L})) = 0
\end{gathered}$$

(IRS^H_t) $\qquad$ $y^H_t \geqslant 0$ for $t \leqslant T^H$,

(IRS_t^L) $\qquad y_t^L \geqslant 0$ for $t \leqslant T^L$,

$(IRF_{T^H}^H)$ $\qquad x^H \geqslant 0,$

$(IRF_{T^L}^L)$ $\qquad x^L \geqslant 0.$

Labeling α^L, α^H, $\{\alpha_t^H\}_{t=1}^{T^H}$, $\{\alpha_t^L\}_{t=1}^{T^L}$, ξ^H, ξ^L as the Lagrange multipliers of the constraints associated with $(IC^{L,H})$,$(IC^{H,L})$, (IRS_t^H), (IRS_t^L), $(IRF_{T^H}^H)$ and $(IRF_{T^L}^L)$ respectively, the optimal contract has the following Lagrangian:

$$
\begin{gathered}
\mathcal{L}(T^H, \{q_t^H(\underline{c})\}_{t=1}^{T^H}, q^H(c_{T^H}^H), \{y_t^H\}_{t=1}^{T^H}, T^L, \{y_t^L\}_{t=1}^{T^L}, \{q_t^L(\underline{c})\}_{t=1}^{T^L}, q^L(c_{T^L}^L), \alpha^L, \alpha^H, \{\alpha_t^H\}_{t=1}^{T^H}, \{\alpha_t^L\}_{t=1}^{T^L}, \xi^H, \xi^L) = \\
\nu[\beta_0 \textstyle\sum_{t=1}^{T^H} \delta^t (1-\lambda^H)^{t-1} \lambda^H (V(q_t^H(\underline{c})) - y_t^H - \underline{c} q_t^H(\underline{c}) - \mathit{\Gamma}_t) \\
+\delta^{T^H} P_{T^H}^H (V(q^H(c_{T^H}^H)) - x^H - c_{T^H}^H q^H(c_{T^H}^H) - \mathit{\Gamma}_{T^H})] \\
+(1-\nu)[\beta_0 \textstyle\sum_{t=1}^{T^L} \delta^t (1-\lambda^L)^{t-1} \lambda^L (V(q_t^L(\underline{c})) - y_t^L - \underline{c} q_t^L(\underline{c}) - \mathit{\Gamma}_t) \\
+\delta^{T^L} P_{T^L}^L (V(q^L(c_{T^L}^L)) - x^L - c_{T^L}^L q^L(c_{T^L}^L) - \mathit{\Gamma}_{T^L})] \\
+\alpha^L(\beta_0 \textstyle\sum_{t=1}^{T^H} \delta^t [P_{T^L}^H (1-\lambda^L)^{t-1} \lambda^L - P_{T^L}^L (1-\lambda^H)^{t-1} \lambda^H] y_t^H \\
+\beta_0 \textstyle\sum_{t=1}^{T^L} \delta^t [P_{T^L}^L (1-\lambda^H)^{t-1} \lambda^H - P_{T^L}^H (1-\lambda^L)^{t-1} \lambda^L] y_t^L \\
+P_{T^L}^H (\delta^{T^H} P_{T^H}^L \bigtriangleup c_{T^H} q^H(c_{T^H}^H) - \delta^{T^L} P_{T^L}^L \bigtriangleup c_{T^L} q^L(c_{T^L}^L)) \\
+\alpha^H(\beta_0 \textstyle\sum_{t=1}^{T^H} \delta^t [P_{T^H}^H (1-\lambda^L)^{t-1} \lambda^L - P_{T^H}^L (1-\lambda^H)^{t-1} \lambda^H] y_t^H \\
+\beta_0 \textstyle\sum_{t=1}^{T^L} \delta^t [P_{T^H}^L (1-\lambda^H)^{t-1} \lambda^H - P_{T^H}^H (1-\lambda^L)^{t-1} \lambda^L] y_t^L \\
+P_{T^H}^L (\delta^{T^H} P_{T^H}^H \bigtriangleup c_{T^H} q^H(c_{T^H}^H) - \delta^{T^L} P_{T^L}^H \bigtriangleup c_{T^L} q^L(c_{T^L}^L))) \textstyle\sum_{t=1}^{T^H} \alpha_t^H y_t^H + \textstyle\sum_{t=1}^{T^L} \alpha_t^L y_t^L \\
+\xi^H x^H + \xi^L x^L.
\end{gathered}
$$

The Inada conditions give us interior solutions for $q_t^H(\underline{c})$, $q^H(c_{T^H}^H)$, $q_t^L(\underline{c})$, and $q^L(c_{T^L}^L)$. We also assumed that $T^L > 0$ and $T^H > 0$. Therefore, the Kuhn-Tucker conditions for the optimization problem are:

$\frac{\partial \mathcal{L}}{\partial T^H} = 0$; $\frac{\partial \mathcal{L}}{\partial q_t^H(\underline{c})} = 0$ for $1 \leqslant t \leqslant T^H$; $\frac{\partial \mathcal{L}}{\partial q^H(c_{T^H}^H)} = 0$;
$\frac{\partial \mathcal{L}}{\partial T^L} = 0$; $\frac{\partial \mathcal{L}}{\partial q_t^L(\underline{c})} = 0$ for $1 \leqslant t \leqslant T^L$; $\frac{\partial \mathcal{L}}{\partial q^L(c_{T^L}^L)} = 0$;
$\frac{\partial \mathcal{L}}{\partial y_t^H} = 0$ for $1 \leqslant t \leqslant T^H$;
$\frac{\partial \mathcal{L}}{\partial \alpha_t^H} \geqslant 0$; $\alpha_t^H \geqslant 0$;
$\frac{\partial \mathcal{L}}{\partial \alpha^H} = 0$; $\alpha^H > 0$; $\frac{\partial \mathcal{L}}{\partial x^H} = 0$;
$\alpha_t^H \frac{\partial \mathcal{L}}{\partial \alpha_t^H} = 0$ for $1 \leqslant t \leqslant T^H$; $\frac{\partial \mathcal{L}}{\partial \xi^H} \geqslant 0$; $\xi^H \geqslant 0$;
$\xi^H \frac{\partial \mathcal{L}}{\partial \xi^H} = 0$;

$\frac{\partial \mathcal{L}}{\partial y^L_t} = 0$ for $1 \leqslant t \leqslant T^L$;

$\frac{\partial \mathcal{L}}{\partial \alpha^L_t} \geqslant 0$; $\alpha^L_t \geqslant 0$;

$\alpha^L_t \frac{\partial \mathcal{L}}{\partial \alpha^L_t} = 0$ for $1 \leqslant t \leqslant T^L$;

$\frac{\partial \mathcal{L}}{\partial \xi^L} \geqslant 0$; $\xi^L \geqslant 0$;

$\xi^L \frac{\partial \mathcal{L}}{\partial \xi^L} = 0$;

$\frac{\partial \mathcal{L}}{\partial \alpha^L} = 0$; $\alpha^L > 0$; $\frac{\partial \mathcal{L}}{\partial x^L} = 0$;

which include the constraints from the problem themselves and corresponding complementary slackness conditions.

Claim 2.3. $P^H_{T^H} P^L_{T^L} - P^H_{T^L} P^L_{T^H} = 0 \Longleftrightarrow \xi^H > 0, \xi^L > 0, \alpha^H_t > 0$ for $t \leqslant min\{\hat{T}^H, \hat{T}^H\}$ and $\alpha^L_t > 0$ for $t \geqslant max\{\hat{T}^H, \hat{T}^H\}$ (it is optimal to set $x^L = x^H = 0$, $y^H_t = 0$ for $t \leqslant min\{\hat{T}^H, \hat{T}^L\}$ and $y^L_t = 0$ for $t \geqslant max\{\hat{T}^H, \hat{T}^L\}$).

Proof: We can rewrite the Kuhn-Tucker conditions above as follows:

$\frac{\partial \mathcal{L}}{\partial x^H} = -\nu \delta^{T^H} P^H_{T^H} + \xi^H = 0$, which implies that $\xi^H > 0$ and, as a result, $x^H = 0$;

$\frac{\partial \mathcal{L}}{\partial x^L} = -(1-\nu) \delta^{T^L} P^L_{T^L} + \xi^L = 0$, which implies that $\xi^L > 0$ and, as a result, $x^L = 0$;

$\frac{\partial \mathcal{L}}{\partial y^H_t} = -\nu \beta_0 \delta^t (1-\lambda^H)^{t-1} \lambda^H + \alpha^L \beta_0 \delta^t [P^H_{T^L} (1-\lambda^L)^{t-1} \lambda^L - P^L_{T^L} (1-\lambda^H)^{t-1} \lambda^H] + \alpha^H \beta_0 \delta^t [P^H_{T^H} (1-\lambda^L)^{t-1} \lambda^L - P^L_{T^H} (1-\lambda^H)^{t-1} \lambda^H] + \alpha^H_t = 0$ for $1 \leqslant t \leqslant T^H$;

$\frac{\partial \mathcal{L}}{\partial y^L_t} = -(1-\nu) \beta_0 \delta^t (1-\lambda^L)^{t-1} \lambda^L + \alpha^L \beta_0 \delta^t [P^L_{T^L} (1-\lambda^H)^{t-1} \lambda^H - P^H_{T^L} (1-\lambda^L)^{t-1} \lambda^L] + \alpha^H \beta_0 \delta^t [P^L_{T^H} (1-\lambda^H)^{t-1} \lambda^H - P^H_{T^H} (1-\lambda^L)^{t-1} \lambda^L] + \alpha^L_t = 0$ for $1 \leqslant t \leqslant T^L$.

Since $-\nu (1-\lambda^H)^{t-1} \lambda^H + \alpha^L P^L_{T^L} f_2(t, T^L) - \alpha^H P^H_{T^H} f_1(t, T^H) + \frac{\alpha^H_t}{\beta_0 \delta^t} = 0$ for $1 \leqslant t \leqslant T^H$ we immediately conclude that $\alpha^H_t > 0$ for $t \leqslant min\{\hat{T}^H, \hat{T}^L\}$ as $-\nu (1-\lambda^H)^{t-1} \lambda^H + \alpha^L P^L_{T^L} f_2(t, T^L) - \alpha^H P^H_{T^H} f_1(t, T^H) < 0$.

In addition, since $-(1-\nu)(1-\lambda^L)^{t-1} \lambda^L - \alpha^L P^L_{T^L} f_2(t, T^L) + \alpha^H P^H_{T^H} f_1(t, T^H) + \frac{\alpha^L_t}{\beta_0 \delta^t} = 0$ for $1 \leqslant t \leqslant T^L$ we immediately conclude that $\alpha^L_t > 0$ for $t \geqslant max\{\hat{T}^H, \hat{T}^L\}$ as $-(1-\nu)(1-\lambda^L)^{t-1} \lambda^L - \alpha^L P^L_{T^L} f_2(t, T^L) + \alpha^H P^H_{T^H} f_1(t, T^H) < 0$. *Q.E.D.*

Appendix C

TECHNICAL DETAILS, CHAPTER 3

Proof of Theorem 3.1. Feasibility: An each step, no flight gets an infeasible slot.

Non-wastefulness: This is by construction of the mechanism. If a flight $\exists f \in F$ such that $s \in V^1$ with $e_f < s$, then it must be the case $\varphi_f(I) < s$.

Individual rationality: In the main round , in the pre-competition stage, each $f \in F_a$ that can feasibly use a slot in $S_a \setminus S^1$ will be assigned to such slot; in the main stage, all slots in $S_a \cap S^1$ will be endowed to a's flights in a fashion that favors more important flights. So $\Pi_a^{S_a}$ for each $a \in A$ is constructed. Then at each step that an endowed slot is being assigned, an airline can either use that slot on that flight or trade it for a better slot for its remaining most important flight. Let $x_i = d_{f_{a,i}}(\Pi_a^{S_a}) - d_{f_{a,i}}(\varphi_a(I))$ for $i \in \{1, ..., |F_a|\}$ and $f_{a,i} R_a f_{a,i+1}$. When an airline use an endowed slot, $x_i = 0$ for some i, and when the first time an airline (i) trades an endowed slot with some other slot or (ii) pick an empty slot, $x_i > 0$, which would be the first non-zero coordinate of $x_a = (x_1, x_2, ..., x_{|F_a|})$. Therefore,$\forall a \in A$, $\varphi_a(I) \succsim_a \Pi_a^{S_a}$.

Pareto efficiency:

Flights that leave at the pre-competition stage are already getting the earliest slot they can get, and no slot in $S \setminus S^1$ can be used to make flights leave at later steps better off.

Consider the main stage, any flight that leaves at step 1 is assigned its top choice that is available and cannot be made better off. Any flight that leaves at step 2 is assigned its top choice that is available among those slots remaining at Step 2 and since slots are distinct time intervals, it cannot be made better off without hurting some flight who left at Step 1. Proceeding in a similar way, no flight can be made better off without hurting some flight that left at an earlier step.

Moreover, for an airline, a flight left at an earlier step is more important than a flight left later, so it can not make itself better off as well. Therefore, φ is Pareto efficient.

Respects property rights:

In each step an endowed slot is being assigned, if it is not assigned to the airline, that means the airline trades it for some better slot. In particular, in the pre-competition stage, the airline trades an infeasible slot for some feasible slot. In the main stage, it trades it for a better slot for its remaining most important flight. In the supplemental stage, an airline will not get its endowed slot only if it has zero budget at that step, which means it traded it for some better slot already.

Core-selecting:

Suppose $\exists \Pi'$ and $A' \subseteq A$ such that (i) $\forall f \in \cup_{a \in A'} F_a$, $\Pi'(f) \in S_{A'}$, and (ii) $\forall a \in A'$, $\Pi' \succ_a \varphi(I)$. Therefore, $\forall a \in A'$, the first non-zero coordinate of $x_a = (x_1, x_2, ..., x_{|F_a|})$ is positive, where for $i \in \{1, ..., |F_a|\}$ and $f_{a,i} R_a f_{a,i+1}$, $x_i = d_{f_{a,i}}(\varphi(I)) - d_{f_{a,i}}(\Pi')$.

Consider $f_{a,i}$ where x_i is the first non-zero coordinate of x_a for $a \in A'$. Note that $\Pi'(f_{a,i})$ is earlier than $\varphi_{f_{a,i}}(I)$, $\Pi'(f_{a,i}) \in S_{A'}$, $\Pi'(f_{a,i})$ is not available when $f_{a,i}$ is picking a slot in φ. There is a $\Pi'(f_{a,i})$ for each $a \in A'$; let S_T be the collection of $\Pi'(f_{a,i})$ for all $a \in A'$. S_T is the set of slots that make airlines in A' prefer Π'.

(i) If a is the owner and $\Pi'(f_{a,i})$ is used by some $f_{a,j}$ in φ, then it must be $f_{a,j} R_a f_{a,i}$. Since x_i is the first non-zero coordinate, then $x_j = 0$, i.e., $f_{a,j}$ is getting the same slot under Π', a contradiction.

The same argument applies to all airline in A'. Therefore, $\forall a \in A'$, $\Pi'(f_{a,i})$ is coming from some airline $a' \in A'$ with $a' \neq a$.

Let $s_a \in S_T \subseteq S_a \cap S^1$ endowed to airline $a \in A'$ be the first slot being assigned to some f in φ. a will pick a slot for its highest ranked remaining flight $f_{a,j}$ before s_a is assigned. At this step, all slots in S_T are available (otherwise it contradicts the way we pick s_a).

(ii) If $f_{a,i} = f_{a,j}$, a picks $\varphi_{f_{a,i}}(I)$ but not $\Pi'(f_{a,i})$, a contradiction.

(iii) If $f_{a,i} R_a f_{a,j}$, this means $\Pi'(f_{a,i})$ is still available after $f_{a,i}$ picked a slot in φ, a contradiction.

(iv) If $f_{a,j} R_a f_{a,i}$, we have $\varphi_{f_{a,j}}(I) = \Pi'(f_{a,j}) \in S_{A'}$.

If $\varphi_{f_{a,j}}(I) \in S_a$, then a will pick another slot. Let the last flight of a gets a slot be

$f_{a,k}$. If s_a is picked by some flight of a at this step, then this flight must be higher ranked than $f_{a,i}$ because $\Pi'(f_{a,i})$ is still available (a result from (i), $\Pi'(f_{a,i}) \notin S_a$). Then s_a will be used by a under Π' but not another airline, a contradiction.

So s_a is not used by a and $f_{a,k}$ is higher ranked than $f_{a,i}$. That means airline a trades s_a for $\varphi_{f_{a,k}}(I) \in S_{A'}$ from some airline $b \in A'$. *Let $\varphi_{f_{b,j}}(I)$ be the slot obtained by b in this trade. Because all slots in S_T are still available, the flight $f_{b,j}$ is higher ranked than $f_{b,i}$, so $\varphi_{f_{b,j}}(I) = \Pi'(f_{b,j}) \in S_{A'}$. If $\varphi_{f_{b,j}}(I) \in S_a$ we have a cycle.

If $\varphi_{f_{b,j}}(I) \in S_c$, $c \in A'$ will be the next airline in line of the trade, and the same argument * applies. Because none of the airline in line gets a slot outside $S_{A'}$ and A' is finite, there must exist a cycle contains exclusively airlines in A'. Let $z \in A'$ be the airlines gets s_a for $f_{z,j}$, again, since all slots in S_T are still available, $f_{z,j}$ is higher ranked than $f_{z,i}$ and therefore $\varphi_{f_{z,j}}(I) = \Pi'(f_{z,j}) = s_a$. This contradicts the fact s_a makes some airline prefecr Π' to $\varphi(I)$.

Strategyproofness:

We consider each stage to see if an airline a can manipulate its outcome by misreporting R_a and e_a such that the final landing schedule $\varphi(\widehat{R_a, e_a}, (R,e)_{-a})$ will make a weakly better off, that is, $\varphi(\widehat{R_a, e_a}, (R,e)_{-a}) \succsim_a \varphi_a(R,e))$.

Let $x_a = (x_1, x_2, ..., x_{|F_a|})$ be a vector such that for $i \in \{1, ..., |F_a|\}$ and $f_{a,i} R_a f_{a,i+1}$, $x_i = d_{f_{a,i}}(\varphi(R,e))) - d_{f_{a,i}}(\varphi(\widehat{R_a, e_a}, (R,e)_{-a}))$. Let x_j be the first non-zero coordinate. If there is no such x_j, then $\varphi(\widehat{R_a, e_a}, (R,e)_{-a}) \sim_a \varphi_a(R,e))$, and we are done. Suppose not. This implies $e_{f_{a,j}} \leq \varphi_f(\widehat{R_a, e_a}, (R,e)_{-a}) < \varphi_f(R,e)$.

In the supplemental stage, if $f_{a,j}$ wants a slot that is being assigned in this stage, it will get that slot in the main stage of φ, a contradiction

In the pre-competition stage, R is not used. The only way that $f_{a,j}$ can get a slot (that is feasible for $f_{a,j}$ and is not being assigned in this stage under φ) is when a is able to make $e_{f_{a,j}} \leq s$ can be only used by $f_{a,j}$.

Note that $\varphi_{f_{a,k}}(\widehat{R_a, e_a}, (R,e)_{-a}) = \varphi_{f_{a,k}}(R,e)$ for all $f_{a,k} R_a f_{a,j}$. So this can happen only when the earliest slot $s' \in S_1$ is demanded by a with $f_{a,x}, f_{a,x'}, ...$when $s' = \varphi_{f_{a,x}}(R,e)$ for $f_{a,x} R_a f_{a,j}$ and $f_{a,j} R_a f_{a,x'}, ...$ or by a with $f_{a,x'}, f_{a,x''}, ...$ for $f_{a,j} R_a f_{a,x'}, f_{a,x''}, ...$ and at most one airline b with $f_{b,x}$, then in the in the prior case, a can misreport (infeasible or later times) $\widehat{e_{f_{a,x'}}, e_{f_{a,x''}}}, \ldots$ to give that slot to $f_{a,k}$ (otherwise, this slot will not go to $f_{a,k}$ until the main stage), and in the latter case, a can also misreport $\widehat{e_{f_{a,x'}}, e_{f_{a,x''}}}, \ldots$ to give that slot to $f_{b,x}$; again, the next demanded slot is demanded by a with $f_{a,y}, f_{a,y'}, \ldots$when $s' = \varphi_{f_{a,y}}(R, e)$ for $f_{a,y} R_a f_{a,j}$ and $f_{a,j} R_a f_{a,y'}, \ldots$ or by a with $f_{a,y'}, f_{a,y''}, \ldots$ for $f_{a,j} R_a f_{a,y'}, f_{a,y''}, \ldots$ and at most one airline c with $f_{c,y} \ldots$.

$f_{a,j}$ will get s from φ's main stage because the only competitors $f_{b,x}, f_{c,x} \ldots$ will always try to get a slot earlier than s and flights $f_{a,x'}, \ldots, f_{a,y'}, \ldots$ will not compete with them before $f_{a,j}$ gets a slot. So this contradicts x_j is the first non-zero coordinate.

Now consider the main stage, we want $a(j)$ to pick a slot earlier such that $x_j > 0$. Again, $\varphi_{f_{a,k}}(\widehat{R_a, e_a}, (R, e)_{-a}) = \varphi_{f_{a,k}}(R, e)$ for all $f_{a,k} R_a f_{a,j}$.

$a(k) > a(j)$ obviously cannot help $a(j)$ to pick a better slot.

The only way a can put $a(j)$ into R_S is to let $a(j-1)$ pick an endowed slot, but such slot must be the same as $\varphi_{f_{a,j-1}}(R, e)$, the $a(j)$ will get the slot it can get from φ, a contradiction. *Q.E.D.*