Metric Learning for Hermitian Manifolds

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Abstract

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In recent years, manifold learning has emerged as one of the most promising approaches for performing nonparametric dimension reduction. While numerous manifold learning algorithms of varying degrees of complexity have been proposed, these algorithms rarely preserve the intrinsic geometry of the underlying data manifold. In 2013, Perrault-Joncas and Meilă proposed a new approach to addressing this problem: instead of trying to design a manifold embedding algorithm that preserves geometry in the widest possible set of circumstances, their approach was to augment the output of *any* reasonable embedding algorithm with a computation of the pushforward metric, thus allowing us to recover any geometric quantity of interest of the underlying manifold.

In this thesis, we apply this idea to the case of Hermitian manifolds, which are complex manifolds equipped with a "compatible" Riemannian metric. Most manifold embedding algorithms do not preserve the complex geometric structure of the input data set. In this thesis, we propose a complex version of the popular Local Tangent Space Alignment (LTSA) algorithm, which by design outputs a holomorphic embedding of the input data set into a low dimensional space. We then propose two algorithms for computing the pushforward Hermitian metric with respect to this embedding, one is based on the transformation properties of a metric under change of coordinates, while the other is based on the Diffusionmaps Graph Laplacian as in the work of Perrault-Joncas and Meilă.

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Chapter 1

INTRODUCTION

In machine learning, one regularly confronts large high dimensional data sets which nevertheless have an underlying low dimensional structure. For example, consider a set of high resolution grayscale images of the same 3D object taken under fixed lightling conditions by a camera from several different positions and orientations. While the dimensionality of each data point (that is, the number of pixels in each image) may be high, it is clear that the *intrinsic dimension* of the space of all possible images is equal to the number of degrees of freedom of the camera, which is significantly lower.

High dimensional data pose several challenges to data processing, interpretation and analysis (often referred to as the *curse of dimensionality* [2]). Therefore, in such situations, it is reasonable to attempt to find a low dimensional "representation" of the given data set that preserves all of its properties of interest. This process is called dimensionality reduction. One of the most common approaches to dimensionality reduction is to appeal to the *manifold hypothesis*, which assumes that all data points lie on or near a low dimensional smooth manifold $\mathcal{M}$ embedded in a high dimensional space $\mathbb{R}^r$.

If $\mathcal{M}$ is an affine subspace of $\mathbb{R}^r$, the classical method of Principal Component Analysis (PCA) provides a satisfactory way of extracting low dimensional representations of the data. However, the problem becomes significantly more challenging when $\mathcal{M}$ is nonlinear. The usual approach is to try to compute a smooth embedding F of $\mathcal{M}$ into a low dimensional Euclidean space $\mathbb{R}^s$ $(s \ll r)$ and thus obtain a low dimensional representation of the given data set. In recent years, several such algortihms have been developed. Examples include Local Linear Embedding (LLE) [10], Local Tangent Space Alignment (LTSA) [13], Isomap [11], Laplacian Eigenmaps (LE) [1], Diffusion Maps (DM)[3], and Hessian Eigenmaps (HE)

[5], among others. The interested reader may refer to the book [12] for a more detailed look at these manifold embedding algorithms.

Each of these algorithms has its own strengths and weaknesses [6]. The usual drawback is that the embeddings computed by these algorithms generally do not preserve the *geometry* of the underlying manifold, except under very specific circumstances. In other words, the embeddings computed by these algorithms do not in general preserve geometric quantities such as lengths of curves in the manifold, angles between two curves, volumes of subsets of the manifold, etc. Therefore, it is natural to evaluate the embedding algorithms based on how well they preserve the geometry of the original manifold.

Instead of trying to come up with an embedding algorithm that preserves geometry in the widest possible set of circumstances, Perrault-Joncas and Meilă proposed an alternative approach in [9]. They proposed a way to augment the output of *any* reasonable embedding algorithm with additional information that allows us to recover the geometry of the original manifold. Recall that any smooth embedded submanifold $\mathcal{M}$ of $\mathbb{R}^r$ inherits a Riemannian metric g from the standard Euclidean metric on $\mathbb{R}^r$. This Riemannian metric encodes all of the geometric information about the manifold $\mathcal{M}$. The goal is to recover the *Riemannian manifold* $(\mathcal{M}, g)$ from the given data *up to isometry*.

Now suppose we have a manifold embedding algorithm whose output approximates a *smooth embedding* $F : \mathcal{M} \to \mathbb{R}^s$, with $s \ll r$. It is known that most of the manifold embedding algorithms listed above satisfy this condition. Then there exists a unique Riemannian metric F_*g on $F(\mathcal{M})$, called the *pushforward* of g by F, that makes $F : (\mathcal{M}, g) \to (F(\mathcal{M}), F_*g)$ an isometry. Moreover, $h = F_*g$ can be uniquely recovered from the embedding coordinate functions $F = (F^1, \ldots, F^s)$ and the Laplace-Beltrami operator Δ_g on $\mathcal{M}$. In fact, for any $p \in \mathcal{M}$, $h_{F(p)}$ is given by the Moore-Penrose pseudoinverse of the matrix $(\widetilde{h}^{jk})$ given by

$$\widetilde{h}^{jk} = \frac{1}{2}\Delta_g(F^j - F^j(p))(F^k - F^k(p))\Big|_p.$$

Replacing Δ_g by a Graph Laplacian operator allows us to discretize the above equation and compute an approximation to the pushforward metric at each point p in the given data set.

This process, called *metric learning* by the authors in [9], allows us to recover the geometry of the original data manifold $\mathcal{M}$.

The goal of this thesis is to extend these methods to recover the geometry of complex submanifolds of $\mathbb{C}^r$, $r \in \mathbb{N}$. Complex manifolds are smooth manifolds equipped with a smooth atlas for which all transition functions are *holomorphic*. This makes it possible to speak of holomorphic functions and maps on complex manifolds. If in addition, the manifold is equipped with a Riemannian metric that is *compatible* with its complex structure in a certain sense, the metric is called a Hermitian metric and the manifold is called a Hermitian manifold. In particular, every complex submanifold of $\mathbb{C}^r$ inherits a Hermitian metric. A particularly interesting sub-class of Hermitian manifolds is the class of Kähler manifolds. In addition to the complex and Riemannian structures, these manifolds additionally possess a *symplectic* structure and are ubiquitous in fundamental particle physics. We will present a more detailed introduction to complex and Hermitian manifolds in the next chapter.

The problem we consider in this thesis is as follows: Suppose we have a data set $\mathcal{P} = \{z_1, \ldots, z_n\}$ consisting of n points in the high dimensional complex vector space $\mathbb{C}^r$. We hypothesize that $z_1, \ldots, z_n$ lie on a d-dimensional complex submanifold $\mathcal{M}_0$ of $\mathbb{C}^r$. $\mathcal{M}_0$ inherits a Hermitian metric from the standard Hermitian metric on $\mathbb{C}^r$. We also assume that our data points were sampled i.i.d. from a probability distribution that is compactly supported in $\mathcal{M}_0$ and absolutely continuous with respect to the standard Lebesgue measure on $\mathcal{M}_0$. Since the underlying probability distribution is compactly supported, we may assume that our data points come from $\mathcal{M}$, a compact complex submanifold of $\mathcal{M}_0 \subset \mathbb{C}^r$ *with smooth boundary*.

Our goal is to find an embedding of the data set $\mathcal{P}$ into a low dimensional complex vector space $\mathbb{C}^s$, with $d \leq s \ll r$, while "preserving" or recovering the Hermitian structure of the underlying manifold $\mathcal{M}$. We propose a two step solution:

1. First, we use the complex version of the Local Tangent Space Alignment (LTSA) algorithm to construct a mapping $f_n : \mathcal{P} \to \mathbb{C}^s$, with $d \leq s \ll r$, which converges to

a holomorphic embedding $f : \mathcal{M} \to \mathbb{C}^s$ as $n \to \infty$. Thus, the computed embedding preserves the complex manifold structure of $\mathcal{M}$, but not necessarily the Hermitian structure.

2. Next, we compute an approximation h_n to the pushfrward metric on $f(\mathcal{M})$. More precisely, at each of the embedded data points $f(z_j)$, we compute a positive semi-definite Hermitian matrix $h_n(f(z_j))$ that approximates the push-forward Hermitian metric h at $f(z_j)$.

Here's a brief outline of this thesis. In Chapter 2, we recall some of the basic facts about Complex and Hermitian manifolds that are necessary for the rest of the thesis. In Chapter 3, we show that the complex version of the LTSA algorithm can be used to perform Step 1 of the above two step solution, i.e., computing an *approximately holomorphic* embedding of the given data set into $\mathbb{C}^s$. Finally in Chapter 4, we show two ways of computing the pushforward metric on $\mathcal{M}$ with respect to these embedding coordinates, one using the transformation properties of a Hermitian metric under change of coordinates, and the other using a Graph Laplacian operator.

Chapter 2

PRELIMINARIES FROM COMPLEX GEOMETRY

In this chapter, we present a brief overview of some of the basic concepts and results concerning complex and Hermitian manifolds. For more detailed treatments of the subject, the reader can refer to classic texts such as [7, 8], or [4] for a more coordinate-based approach.

Let us begin with the definition of holomorphicity in the context of several complex variables. Given any point $z = (z^1, \ldots, z^d) \in \mathbb{C}^d$, let x^j and y^j denote the real and imaginary parts of z^j $(1 \leq j \leq d)$ respectively. The real cotangent space at any point in $\mathbb{C}^d \cong \mathbb{R}^{2d}$ is spanned by the basis $\{dx^j, dy^j : j = 1, \ldots, d\}$. While dealing with complex-valued covectors, however, it will be more convenient to work with

$$dz^j = dx^j + idy^j, \qquad d\overline{z}^j = dx^j - idy^j, \qquad 1 \leq j \leq d.$$

The set $\{dz^j, d\overline{z}^j : j = 1, \ldots, d\}$ forms a basis for the (complex) vector space of complex-valued covectors. The dual basis for the complexified tangent space is given by

$$\frac{\partial}{\partial z^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j}\right), \qquad \frac{\partial}{\partial \overline{z}^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} + i\frac{\partial}{\partial y^j}\right), \qquad 1 \leq j \leq d.$$

With respect to these bases, the formula for the total differential of a smooth complex valued function on $\mathbb{C}^d$ becomes

$$df = \sum_{j=1}^{d} \left(\frac{\partial f}{\partial z^j} dz^j + \frac{\partial f}{\partial \overline{z}^j} d\overline{z}^j\right).$$

We also define the operators ∂ and $\overline{\partial}$ on smooth functions f by

$$\partial f = \sum_{j=1}^{d} \frac{\partial f}{\partial z^j} dz^j, \qquad \overline{\partial} f = \sum_{j=1}^{d} \frac{\partial f}{\partial \overline{z}^j} d\overline{z}^j \tag{2.1}$$

respectively.

Now let U be an open subset of $\mathbb{C}^d$. We say that a complex valued function $f \in C^\infty(U)$ is *holomorphic* on U if $\overline{\partial} f \equiv 0$, or equivalently, it satisfies the Cauchy-Riemann equations

$$\frac{\partial f}{\partial \overline{z}^j} \equiv 0, \qquad 1 \leq j \leq d. \tag{2.2}$$

As in the case of functions of one complex variabe, it can be shown that f is holomorphic on U iff at every $z_0 \in U$, f has a convergent power series expansion

$$f(z) = f(z_0) + \sum_j a_j(z^j - z_0^j) + \sum_{j,k} a_{jk}(z^j - z_0^j)(z^k - z_0^k) + \cdots \qquad (a_j, a_{jk}, \ldots \in \mathbb{C})$$

in a neighborhood of z_0. For a smooth map $F = (F^1, \ldots, F^{d'}) : U \to \mathbb{C}^{d'}$, we define holomorphicity to mean that each component function F^k is holomorphic in U.

2.1 Complex manifolds

Complex manifolds are topological spaces that locally "look like" open subsets of $\mathbb{C}^d$.

Definition 2.1. A *d-dimensional complex manifold* is a $2d$-dimensional smooth manifold with an open cover $\{U_\alpha\}$ and smooth charts $\varphi_\alpha : U_\alpha \to \mathbb{C}^d$ such that all transition maps $\varphi_\alpha \circ \varphi_\beta^{-1}$ are holomorphic from $\varphi_\beta(U_\alpha \cap U_\beta) \to \mathbb{C}^d$.

Unsurprisingly, the assumption that the transition maps are holomorphic turns out to be much stronger than asuming only that the transition maps in an atlas are all smooth. Complex manifolds are in a sense a lot more "rigid" than smooth manifolds. For example, it follows from Liouville's theorem that a compact, connected complex manifold (without boundary) admits no non-constant holomorphic functions. As a consequence, no compact complex manifold of dimension greater than 0 can be holomorphically embedded into $\mathbb{C}^d$.

Just as the concepts of smooth functions and smooth maps can be extended to smooth manifolds, the concepts of holomorphic functions and maps can be extended to complex manifolds. A smooth function $F : \mathcal{M} \to \mathbb{C}^{d'}$ is said to be holomorphic if for any holomorphic chart (U, φ), $F \circ \varphi^{-1} : \varphi(U) \subset \mathbb{C}^d \to \mathbb{C}^{d'}$ is holomorphic. The definition of holomorphic maps between complex manifolds is similar.

Now, consider a smooth map $f : \mathbb{C}^d \to \mathbb{C}^{d'}$. Writing the k-th component of f as $f^k = u^k + iv^k$, we observe that the Cauchy-Riemann equations (2.2) for f^k are equivalent to

$$\frac{\partial u^k}{\partial x^j} = \frac{\partial v^k}{\partial y^j}, \qquad \frac{\partial u^k}{\partial y^j} = -\frac{\partial v^k}{\partial x^j}, \qquad 1 \leq j \leq d. \tag{2.3}$$

Let us regard $f = (u^1, \ldots, u^{d'}, v^1, \ldots, v^{d'})$ as a smooth map from $\mathbb{R}^{2d} \to \mathbb{R}^{2d'}$ and consider the Jacobian matrix of f with respect to the basis $\{\partial/\partial x^1, \ldots \partial/\partial x^d, \partial/\partial y^1, \ldots, \partial/\partial y^d\}$:

$$Df = \begin{bmatrix} \frac{\partial u^1}{\partial x^1} & \cdots & \frac{\partial u^1}{\partial x^d} & \frac{\partial u^1}{\partial y^1} & \cdots & \frac{\partial u^1}{\partial y^d} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial u^{d'}}{\partial x^1} & \cdots & \frac{\partial u^{d'}}{\partial x^d} & \frac{\partial u^{d'}}{\partial y^1} & \cdots & \frac{\partial u^{d'}}{\partial y^d} \\ \frac{\partial v^1}{\partial x^1} & \cdots & \frac{\partial v^1}{\partial x^d} & \frac{\partial v^1}{\partial y^1} & \cdots & \frac{\partial v^1}{\partial y^d} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial v^{d'}}{\partial x^1} & \cdots & \frac{\partial v^{d'}}{\partial x^d} & \frac{\partial v^{d'}}{\partial y^1} & \cdots & \frac{\partial v^{d'}}{\partial y^d} \end{bmatrix}$$

It can be verified that the equations (2.3) are equivalent to the commutativity condition

$$J_{d'}(Df) = (Df)J_d, \tag{2.4}$$

where for any positive integer m, J_m is the $2m \times 2m$ block diagonal matrix

$$J_m = \begin{bmatrix} 0_m & I_m \\ -I_m & 0_m \end{bmatrix}.$$

Here 0_m and I_m denote the $m \times m$ zero and identity matrices respectively. Notice that J_m satisfies $J_m^2 = -I_{2m}$ and corresponds to the linear map $(z^1, \ldots, z^m) \mapsto i(z^1, \ldots, z^m)$. J_m is called the almost complex structure on $\mathbb{C}^m$, and a smooth map $f : \mathbb{C}^d \to \mathbb{C}^{d'}$ is holomorphic iff it "respects" the complex structures of $\mathbb{C}^d$ and $\mathbb{C}^{d'}$ in the sense of satisfying (2.4). Smooth manifolds equipped with analogous structures are called almost complex manifolds.

Definition 2.2. Let $\mathcal{M}$ be a smooth manifold. An almost complex structure on $\mathcal{M}$ is a smooth $(1,1)$-tensor field J such that at each $p \in \mathcal{M}$, $J_p : T_p\mathcal{M} \to T_p\mathcal{M}$ satsifies $J_p^2 = -\operatorname{Id}$. If $\mathcal{M}$ and $\mathcal{N}$ are almost complex manifolds with almost complex structures J and J' respectively, we say that a smooth map $F : \mathcal{M} \to \mathcal{N}$ is holomorphic if $DF(p) \circ J_p = J'_{F(p)} \circ DF(p)$ at every $p \in \mathcal{M}$.

Every complex m-manifold has a canonical almost complex structure which, in the domain of a holomorphic chart (U, φ), is given by the pullback of the complex structure on $\mathbb{C}^m$ by φ. If (V, ψ) is an overlapping holomorphic chart, the complex structures on $U \cap V$ induced by φ and ψ will be identical since $\psi \circ \varphi^{-1}$ and its inverse are holomorphic. On the other hand, not every almost complex manifold is a complex manifold. In fact, given any $(1,1)$-tensor field A on $\mathcal{M}$, consider its Nijenhuis tensor field N_A defined by

$$N_A(V, W) = [AV, AW] - [V, W] - A[AV, W] - A[V, AW]$$

for all vector fields V, W. It is a deep result of Newlander and Nirenberg (ref. [8], Theorem 2.6.19) that an almost complex structure J on a smooth manifold can be associated with an underlying complex structure if and only $N_J \equiv 0$.

We also have a notion of *submanifolds* of complex manifolds, defined analogously to smooth submanifolds of smooth manifolds.

Definition 2.3. Let $\mathcal{M}$ be a complex manifold and $\mathcal{N}$ a smooth embedded submanifold of $\mathcal{M}$. We say that $\mathcal{N}$ is a complex submanifold of $\mathcal{M}$ if it possesses a complex manifold structure with respect to which the embedding $\mathcal{N} \hookrightarrow \mathcal{M}$ is a holomorphic map.

Using holomorphic versions of the Inverse and Implicit Function Theorems, it can be shown that this is equivalent to the existence of an *adapted holomorphic atlas* for $\mathcal{N}$. To be precise, $\mathcal{N} \hookrightarrow \mathcal{M}$ is a complex submanifold of dimension d' iff at every point $p \in \mathcal{N}$, there exists a holomorphic coordinate chart $(U, (z^1, \ldots, z^d))$ of $\mathcal{M}$ centered at p such that

$$U \cap \mathcal{N} = \{z^{d'+1} = \cdots = z^d = 0\}.$$

Also note that the holomorphicity of the map $\mathcal{N} \hookrightarrow \mathcal{M}$ implies that the almost complex structure on $\mathcal{N}$ is simply the restriction of the almost complex structure of $\mathcal{M}$ to the tangent bundle of $\mathcal{N}$.

2.2 Holomorphic and anti-holomorphic vector bundles

Let $\mathcal{M}$ be a d-dimensional complex manifold and $p \in \mathcal{M}$. Henceforth, we denote by $T_{p,\mathbb{C}}\mathcal{M}$ the complexified tangent space $T_p\mathcal{M} \otimes \mathbb{C}$, and by $T^*_{p,\mathbb{C}}\mathcal{M}$ the complexified cotangent space. Locally in holomorphic coordinates $(z^1, \dots, z^d)$, $T_{p,\mathbb{C}}\mathcal{M}$ (resp., $T^*_{p,\mathbb{C}}\mathcal{M}$) is simply the $\mathbb{C}$-linear span of $\{\partial/\partial z^j, \partial/\partial \overline{z}^j\}$ (resp., $\{dz^j, d\overline{z}^j\}$).

We also extend J to the complexified tangent bundle $T_{\mathbb{C}}\mathcal{M}$ $\mathbb{C}$-linearly. Since $J^2 = -\operatorname{Id}$, J_p has eigenvalues $\pm i$ on $T_{p,\mathbb{C}}\mathcal{M}$. We define

$$\begin{aligned} T^{(1,0)}\mathcal{M} &= \{v \in T_{\mathbb{C}}\mathcal{M} : Jv = +iv\}, \quad \text{and} \\ T^{(0,1)}\mathcal{M} &= \{v \in T_{\mathbb{C}}\mathcal{M} : Jv = -iv\}. \end{aligned}$$

Elements of $T^{(1,0)}\mathcal{M}$ and $T^{(0,1)}\mathcal{M}$ are called holomorphic and anti-holomorphic tangent vectors respectively. In local holomorphic coordinates,

$$T^{(1,0)}_p\mathcal{M} = \operatorname{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial z^j} : 1 \le j \le d\right\}, \qquad T^{(0,1)}_p\mathcal{M} = \operatorname{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial \overline{z}^j} : 1 \le j \le d\right\}.$$

Suppose $\mathcal{M}$ and $\mathcal{N}$ are complex manifolds and $F : \mathcal{M} \to \mathcal{N}$ a holomorphic map. The commutativity condition (2.4) implies that $DF(p)$ maps $T^{(1,0)}_p\mathcal{M} \to T^{(1,0)}_{F(p)}\mathcal{N}$ and $T^{(0,1)}_p\mathcal{M} \to T^{(0,1)}_{F(p)}\mathcal{N}$ respectively. In fact, if $(z^1, \dots, z^d)$ and $(w^1, \dots, w^{d'})$ are holomorphic charts around $p \in \mathcal{M}$ and $F(p) \in \mathcal{N}$, and F has the representation $F = (F^1, \dots, F^{d'})$ in these coordinates, the Jacobian of F with respect to the bases $\{\partial/\partial z^1, \dots, \partial/\partial z^d, \partial/\partial \overline{z}^1, \dots, \partial/\partial \overline{z}^d\}$ and $\{\partial/\partial w^1, \dots, \partial/\partial w^d, , \partial/\partial \overline{w}^1 \dots, \partial/\partial \overline{w}^d\}$ is given by

$$DF = \begin{bmatrix} D_{\mathbb{C}}F & 0 \\ 0 & \overline{D_{\mathbb{C}}F} \end{bmatrix},$$

where

$$D_{\mathbb{C}}F = \begin{bmatrix} \frac{\partial F^1}{\partial z^1} & \cdots & \frac{\partial F^1}{\partial z^d} \\ \vdots & \ddots & \vdots \\ \frac{\partial F^{d'}}{\partial z^1} & \cdots & \frac{\partial F^{d'}}{\partial z^d} \end{bmatrix}.$$

$D_{\mathbb{C}}F$ is called the *complex Jacobian* of F. It is simply the matrix representation of the (complex) linear map $DF(p) : T^{(1,0)}_p\mathcal{M} \to T^{(1,0)}_{F(p)}\mathcal{N}$. Alternatively, $D_{\mathbb{C}}F$ can also be thought of as the matrix of first order coefficients of the power series of F: If F^j $(1 \le j \le d')$ has the power series expansion

$$F^j(z) = F^j(z_0) + \sum_k a^j_k (z^k - z^k_0) + \sum_{k,l} a^j_{kl}(z^k - z^k_0)(z^l - z^l_0) + \cdots$$

around z_0, then

$$(D_{\mathbb{C}}F(z_0))_{jk} = \frac{\partial F^j}{\partial z^k}(z_0) = a^j_k.$$

J also acts on cotangent vectors by $(J\alpha)(v) = \alpha(Jv)$ for all $\alpha \in T^*_{p,\mathbb{C}}\mathcal{M}$, $v \in T_{p,\mathbb{C}}\mathcal{M}$. We also define $T^{*(1,0)}_p\mathcal{M}$ and $T^{*(0,1)}_p\mathcal{M}$ as the eigenspaces of J corresponding to eigenvalues $+i$ and $-i$ respectively. In fact, $T^{*(1,0)}_p\mathcal{M}$ and $T^{*(0,1)}_p\mathcal{M}$ can be naturally identified as the dual vector spaces of $T^{(1,0)}_p\mathcal{M}$ and $T^{(0,1)}_p\mathcal{M}$ respectively. In local holomorphic coordinates (z^j),

$$T^{*(1,0)}_p\mathcal{M} = \operatorname{span}_{\mathbb{C}}\{dz^j\}, \qquad T^{*(0,1)}_p\mathcal{M} = \operatorname{span}_{\mathbb{C}}\{d\overline{z}^j\}.$$

We now move onto complex differential forms. Let $\Omega^k(\mathcal{M}) = C^\infty(\mathcal{M}; \Lambda^k(T^*_{\mathbb{C}}\mathcal{M}))$ denote the space of smooth complex valued k-forms on $\mathcal{M}$. The splitting $T^*_{\mathbb{C}}\mathcal{M} \cong T^{*(1,0)}\mathcal{M} \oplus T^{*(0,1)}\mathcal{M}$ induces a splitting of $\Omega^k(\mathcal{M})$ as

$$\Omega^k(\mathcal{M}) = \bigoplus_{l+m=k} \Omega^{(l,m)}\mathcal{M}$$

where $\Omega^{(l,m)}\mathcal{M}$ is the space of smooth sections of

$$\Lambda^{(l,m)}\mathcal{M} = \coprod_{p\in\mathcal{M}} \Lambda^{(l,m)}_p\mathcal{M} = \coprod_{p\in\mathcal{M}} \left(\bigwedge^l T^{*(1,0)}_p\mathcal{M}\right) \wedge \left(\bigwedge^m T^{*(0,1)}_p\mathcal{M}\right).$$

In local coordinates (z^j), any $\alpha \in \Omega^{(l,m)}\mathcal{M}$ can be written as

$$\alpha = \sum_{|J|=l,|K|=m} \alpha_{JK} dz^J \wedge d\overline{z}^K$$

where $J = (j_1, \dots, j_l), K = (k_1, \dots, k_m)$ are increasing sequences of indices and $dz^J = dz^{j_1} \wedge \cdots \wedge dz^{j_l}$, $d\overline{z}^K = d\overline{z}^{k_1} \wedge \cdots \wedge d\overline{z}^{k_m}$. Finally, observe that the exterior derivative d maps $\Omega^{(l,m)}\mathcal{M}$ to $\Omega^{(l+1,m)}\mathcal{M} \oplus \Omega^{(l,m+1)}\mathcal{M}$. Let $\pi_{(l,m)}$ denote the projection operator of $\Omega^k\mathcal{M}$ onto $\Omega^{(l,m)}\mathcal{M}$.

Definition 2.4. We define the Dolbeault operators $\partial : \Omega^{(l,m)}\mathcal{M} \to \Omega^{(l+1,m)}\mathcal{M}$ and $\overline{\partial} : \Omega^{(l,m)}\mathcal{M} \to \Omega^{(l,m+1)}\mathcal{M}$ by

$$\partial = \pi_{(l+1,m)} \circ d, \qquad \overline{\partial} = \pi_{(l,m+1)} \circ d$$

respectively.

For $f \in \Omega^0\mathcal{M} = C^\infty(\mathcal{M})$ and any local holomorphic chart $(z^1, \ldots, z^d)$, it can be verified that

$$\partial f = \sum_{j=1}^{d} \frac{\partial f}{\partial z^j} dz^j, \qquad \overline{\partial} f = \sum_{j=1}^{d} \frac{\partial f}{\partial \overline{z}^j} d\overline{z}^j.$$

Therefore, this definition of the ∂ and $\overline{\partial}$ operators agrees with the definitions given in (2.1).

Finally, we note that since $d = \partial + \overline{\partial}$ and $d^2 = 0$ on forms of type (l, m), we have $d^2 = (\partial + \overline{\partial})^2 = 0$, which implies the identities

$$\partial^2 = \overline{\partial}^2 = 0, \qquad \partial\overline{\partial} + \overline{\partial}\partial = 0.$$

2.3 Hermitian metrics

We now consider complex manifolds that are equipped with a Riemannian metric. It is natural to assume that the metric is *compatible* with the complex structure of the manifold in the following sense:

Definition 2.5. Let $\mathcal{M}$ be a complex manifold and let J be its canonical almost complex structure. We say that a Riemannian metric g on $\mathcal{M}$ is *compatible* with the almost complex structure J if

$$g(Ju, Jv) = g(u, v) \qquad \text{for all } u, v \in T_p\mathcal{M},\ p \in \mathcal{M}.$$

Observe that any real tensor field on $\mathcal{M}$ can be uniquely extended by complex multilinearity to a tensor field acting on complex tangent and cotangent vectors . A *Hermitian metric* h on $\mathcal{M}$ is a smooth section of $T^*_{\mathbb{C}}\mathcal{M} \otimes T^*_{\mathbb{C}}\mathcal{M}$ given by the complex bi-linear extension of a compatible Riemannian metric g on $\mathcal{M}$. It is easy to see that such an h also satisfies the compatibility condition

$$h(Ju, Jv) = h(u, v) \qquad \text{for all } u, v \in T_{p,\mathbb{C}}\mathcal{M},\ p \in \mathcal{M}.$$

The pair $(\mathcal{M}, h)$ is called a Hermitian manifold.

The primary example of a Hermitian metric is the canonical Euclidean metric on $\mathbb{C}^d \cong \mathbb{R}^{2d}$ given by

$$g_0 = \sum_{j=1}^{d} dx^j \otimes dx^j + \sum_{j=1}^{d} dy^j \otimes dy^j.$$

It can be easily verified that g_0 is compatible with the almost complex structure of $\mathbb{C}^d$. Moreover, the unique complex bi-linear extension of g_0 is given by

$$\begin{aligned} h_0 &= \sum_{j,k=1}^{d} \delta_{jk}(dz^j \otimes d\overline{z}^k + d\overline{z}^k \otimes dz^j) \\ &= \sum_{j,k=1}^{d} 2\delta_{jk} dz^j d\overline{z}^k. \end{aligned}$$

(Here, $\alpha\beta$ denotes the symmetric product $\mathrm{Sym}(\alpha \otimes \beta) = \frac{1}{2}(\alpha \otimes \beta + \beta \otimes \alpha)$.) More generally, let h be a Hermitian metric on a complex manifold $\mathcal{M}$. In any local holomorphic coordinate system $(z^1, \ldots, z^d)$, we may write

$$h = \sum_{j,k} \left(h_{jk} dz^j \otimes dz^k + h_{j\overline{k}} dz^j \otimes d\overline{z}^k + h_{\overline{j}k} d\overline{z}^j \otimes dz^k + h_{\overline{j}\overline{k}} d\overline{z}^j \otimes d\overline{z}^k \right), \tag{2.5}$$

where $h_{jk} = h\left(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k}\right)$, $h_{j\overline{k}} = h\left(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \overline{z}^k}\right)$, etc. The compatibility condition on h implies that

$$\begin{aligned} h_{jk} &= h\left(\tfrac{\partial}{\partial z^j}, \tfrac{\partial}{\partial z^k}\right) = h\left(J\tfrac{\partial}{\partial z^j}, J\tfrac{\partial}{\partial z^k}\right) \\ &= h\left(i\tfrac{\partial}{\partial z^j}, i\tfrac{\partial}{\partial z^k}\right) \\ &= -h\left(\tfrac{\partial}{\partial z^j}, \tfrac{\partial}{\partial z^k}\right) = -h_{jk}, \end{aligned}$$

which implies that $h_{jk} = 0$ for all j, k. Similarly, we also get that all $h_{\overline{j}\overline{k}} = 0$. Moreover, the mixed indices $h_{j\overline{k}}$ and $h_{\overline{j}k}$ satisfy

$$h_{j\overline{k}} = h_{\overline{k}j}, \qquad h_{j\overline{k}} = \overline{h_{\overline{j}k}}.$$

Therefore, (2.5) reduces to

$$\begin{aligned} h &= \sum_{j,k=1}^{d} h_{j\bar{k}}(dz^j \otimes d\bar{z}^k + d\bar{z}^k \otimes dz^j) \\ &= \sum_{j,k=1}^{d} 2h_{j\bar{k}} dz^j d\bar{z}^k \end{aligned}$$

where $(h_{j,\bar{k}}(p))$ is a $d \times d$ Hermitian matrix for every p in the domain of the coordinate chart. Moreover, since the restriction g of h to real tangent vectors is positive definite, it can be shown that the matrix $(h_{j\bar{k}}(p))$ is positive definite as well.

Every Hermitian metric h on $\mathcal{M}$ is canonically associated with a differential 2-form ω defined by

$$\omega(u,v) = h(Ju,v), \qquad u,v \in T_p\mathcal{M},\ p \in \mathcal{M}.$$

This form is usually called the *fundamental form* or the *Hermitian form* associated with h. In local coordinates, ω takes the form

$$\omega = i \sum_{j,k=1}^{d} h_{j\bar{k}} dz^j \wedge d\bar{z}^k.$$

If in particular, $d\omega = 0$, ω is called a *Kähler form* and h is said to be a *Kähler metric.* Notice that if ω is Kähler, it must be closed and non-degenerate, and hence is a *symplectic form* on $\mathcal{M}$. Kähler manifolds form a particularly nice class of Hermitian manifolds on account of this additional structure. They have several applications in theoretical physics, especially in Supersymmetry. For example, the extra unobserved dimensions of spacetime are sometimes assumed to form a Calabi-Yau manifold, which is a special case of a Kähler manifold.

The Hermitian metric h induces a smoothly varying Hermitian inner product on the complexified tangent spaces as follows: For each $p \in \mathcal{M}$ and $u, v \in T_{p,\mathbb{C}}\mathcal{M}$, we define

$$\langle u, v \rangle_h := h_p(u, \bar{v})$$

where $\bar{v}$ denotes the complex conjugate of v. In local coordinates $(z^1, \ldots, z^d)$, if

$$u = \sum_{j=1}^{d} u^j \frac{\partial}{\partial z^j} + u^{\bar{j}} \frac{\partial}{\partial \bar{z}^j}, \qquad v = \sum_{j=1}^{d} v^j \frac{\partial}{\partial z^j} + v^{\bar{j}} \frac{\partial}{\partial \bar{z}^j},$$

then

$$\langle u, v\rangle_h = \sum_{j,k=1}^{d} h_{j\overline{k}}(p)(u^j \cdot \overline{v^k} + u^{\overline{k}} \cdot \overline{v^{\overline{j}}}).$$

Since $(h_{j\overline{k}}(p))$ is a positive definite Hermitian matrix for every $p \in \mathcal{M}$, it follows that $\langle \cdot, \cdot \rangle_h$ is a Hermitian inner product on each $T_{p,\mathbb{C}}\mathcal{M}$. This naturally induces an inner product on complex cotangent vectors, and more generally, on multicovectors. By an abuse of notation, we will use $\langle \cdot, \cdot \rangle_h$ to denote all these inner products. The decomposition

$$\Lambda^k \mathcal{M} = \bigoplus_{k=l+m} \Lambda^{(l,m)} \mathcal{M} \tag{2.6}$$

is orthogonal with respect to this inner product. We extend the Hodge star operator on $\mathcal{M}$ to complex multicovectors so that for $\xi, \eta \in \Lambda^k_p \mathcal{M}$,

$$\langle \xi, \eta \rangle_h = \star(\xi \wedge \star\overline{\eta}).$$

We also obtain an inner product on the space of complex k-forms:

$$(\alpha, \beta) = \int_{\mathcal{M}} \langle \alpha_z, \beta_z \rangle_h \; d\mathrm{Vol}(z)$$

where $d\mathrm{Vol}$ is the Riemannian volume form on $\mathcal{M}$. The orthogonality of the decomposition (2.6) implies that this restricts to an inner product on each $\Omega^{(l,m)}\mathcal{M}$. This allows us to define the adjoints of ∂ and $\overline{\partial}$ as the operators

$$\partial^* : \Omega^{(l+1,m)}\mathcal{M} \to \Omega^{(l,m)}\mathcal{M}, \qquad \overline{\partial}^* : \Omega^{(l,m+1)}\mathcal{M} \to \Omega^{(l,m)}\mathcal{M}$$

that satisfy

$$(\partial^* \alpha, \beta) = (\alpha, \partial\beta), \qquad (\overline{\partial}^* \alpha, \beta) = (\alpha, \overline{\partial}\beta)$$

respectively. In terms of the Hodge star operator, we have the identities

$$\partial^* = - \star \overline{\partial} \star, \qquad \overline{\partial}^* = - \star \partial \star .$$

Finally, recall that on any Riemannian manifold, we have the Hodge Laplacian operator on k-forms defined by

$$\Delta_d = -(dd^* + d^* d)$$

where $d^* = - \star d \star$ is the adjoint of d. On Hermitian manifolds, we have two additional Laplacians:

Definition 2.6. Let $(\mathcal{M}, h)$ be a Hermitian manifold. We define the ∂-Laplacian and $\overline{\partial}$-Laplacian on $\mathcal{M}$ by

$$\begin{aligned} \Delta_\partial &= -(\partial\partial^* + \partial^*\partial) \\ \Delta_{\overline{\partial}} &= -(\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}) \end{aligned}$$

respectively.

In particular, on smooth complex valued functions u on $\mathcal{M}$, we have $\Delta_\partial u = -\partial^*\partial u$ and $\Delta_{\overline{\partial}} u = -\overline{\partial}^*\overline{\partial} u$. In local holomorphic coordinates $(z^1, \ldots, z^d)$, these take the form

$$\Delta_\partial u(z) = \frac{1}{|h|} \sum_{j,k=1}^{d} \frac{\partial}{\partial \overline{z}^k} \left(|h| h^{j\overline{k}} \frac{\partial u}{\partial z^j} \right), \tag{2.7}$$

$$\Delta_{\overline{\partial}} u(z) = \frac{1}{|h|} \sum_{j,k=1}^{d} \frac{\partial}{\partial z^k} \left(|h| \overline{h^{j\overline{k}}} \frac{\partial u}{\partial \overline{z}^j} \right) \tag{2.8}$$

where $(h^{j\overline{k}})$ is the inverse matrix of $(h_{j\overline{k}})$ and $|h| = |\det(h_{j\overline{k}})|$ respectively.

Proposition 2.7. *If $(\mathcal{M}, h)$ is a Kähler manifold, the three Hodge Laplacians defined above are related by*

$$\Delta_\partial = \Delta_{\overline{\partial}} = \frac{1}{2}\Delta_d.$$

On general Hermitian manifolds however, $\Delta_\partial, \Delta_{\overline{\partial}}$ and $\frac{1}{2}\Delta_d$ may differ, but only by first order differential operators. In particular, we have the following result:

Proposition 2.8. *Let $\mathcal{M}$ be a Hermitian manifold. Suppose $\alpha \in \Omega^{(l,m)}\mathcal{M}$ vanishes to second order at $p \in \mathcal{M}$. Then*

$$\Delta_d \alpha(p) = 2\Delta_\partial \alpha(p) = 2\Delta_{\overline{\partial}} \alpha(p).$$

Chapter 3

HOLOMORPHIC EMBEDDING VIA COMPLEX LTSA

Let us recall the setting of our problem: We are given a data set $\mathcal{P} = \{z_1, \ldots, z_n\}$ of n points in $\mathbb{C}^r$. We assume that the data points have been chosen i.i.d. from a d dimensional compact complex manifold with smooth boundary $\mathcal{M} \subset \mathbb{C}^r$ with respect to some probability distribution on $\mathcal{M}$ that is absolutely continuous with respect to the standard Lebesgue measure on $\mathcal{M}$. Our goal is to holomorphically embed $\mathcal{P}$ into a low dimensional space $\mathbb{C}^s$, with $d \leq s \ll r$ and augment it with the pushforward Hermitian metric so as to encode all the complex geometric information of the original data manifold $\mathcal{M}$.

We begin by describing an embedding algorithm that generates a holomorphic embeddingof the data manifold. Our approach is to use the complex version of the LTSA algorithm [13]. Recall that the LTSA algorithm consists of two main steps:

1. Computing a representation of the tangent space of the manifold at each of the data points.

2. Computing the embedding map by *aligning* the tangent spaces in the low dimensional target space.

3.1 Computing the tangent space representations using local data

Consider the data point z_i, $1 \leq i \leq n$. Let $Z_i = [z_{i_1}, \ldots, z_{i_k}]$ be the matrix consisting of the k nearest neighbors of z_i, including itself, in terms of the Euclidean distance in $\mathbb{C}^r$. Since $\mathcal{M}$ is a complex submanifold, it can be locally approximated by an affine *complex-linear subspace* of $\mathbb{C}^r$ with complex dimension d. This leads us to considering the following minimization problem:

$$\min_{z,Q,\theta_j} \sum_{j=1}^{n} \|z_{i_j} - (z + Q\theta_j)\|^2,$$

where $z \in \mathbb{C}^r$, $\theta_j \in \mathbb{C}^d$, and Q is a $r \times d$ complex matrix with orthonormal columns. The columns of Q will form an orthonormal basis for estimated tangent space and θ_j will represent the coordinates of z_{i_j} with respect to this basis. Letting $\Theta = [\theta_1, \dots, \theta_k]$, we can rewrite this minimization problem as

$$\min_{z,Q,\Theta} \|Z_i - (ze^T + Q\Theta)\|_F^2,$$

where $e = (1, \dots, 1)^T \in \mathbb{C}^k$ and $\|\cdot\|_F$ denotes the matrix Frobenius norm. As in the real case, the optimal z, Q and Θ are given by

$$\begin{aligned}
z_i^* &= \frac{1}{k}\sum_{j=1}^{k} z_{i_j} = \frac{1}{k} Z_i e, \\
Q_i &= \text{matrix of } d \text{ left singular eigenvectors corresponding to} \\
&\quad\ \text{the } d \text{ largest singular values of } Z_i\left(I - \tfrac{1}{k}ee^T\right), \quad \text{and} \\
\Theta_i &= Q_i^H Z_i\left(I - \tfrac{1}{k}ee^T\right),
\end{aligned}$$

where Q_i^H is the conjugate transpose of Q_i. Letting $\Theta_i = [\theta_1^{(i)}, \dots, \theta_k^{(i)}]$, we have the decomposition

$$z_{i_j} = z_i^* + Q_i\theta_j^{(i)} + \xi_j^{(i)}, \qquad 1 \le j \le k,$$

where $\xi_j^{(i)}$ are the error terms.

3.2 *Computing the embedding*

Now consider a holomorphic embedding $f : \mathcal{M} \to \mathbb{C}^s$. We are only interested in computing the images of the input data points, i.e.,

$$u_i = f(z_i), \qquad 1 \le i \le n.$$

Fix $i \in \{1, \dots, n\}$. Let $z \in \mathcal{M}$ be a point sufficiently close to z_i^* and let θ denotes its local coordinates with respect to the coordinate chart determined by the orthonormal basis Q_i.

Since f is assumed to be holomorphic, we have the first order Taylor expansion around z_i^*:

$$f(z) = f(z_i^*) + D_{\mathbb{C}} f(z_i^*) Q_i \theta + o(\|z - z_i^*\|),$$

where $D_{\mathbb{C}} F(z_i^*)$ is an $s \times d$ complex matrix representing the complex Jacobian of the holomorphic map f at z_i^*. In particular, we want this approximation to hold for each of the k nearest neighbors z_{i_j} of z_i. Setting $L_i = D_{\mathbb{C}} f(z_i^*) Q_i$ and $u_i^* := f(z_i^*)$, this amounts to requiring that

$$u_{i_j} = u_i^* + L_i \theta_j^{(i)} + \epsilon_j^{(i)}, \qquad 1 \le j \le k,$$

where the error terms $\epsilon_j^{(i)}$ should be small. Note that since we do not know what f is, both u_i^* and L_i are unknown. Define $U_i = [u_{i_1}, \ldots, u_{i_k}]$ and $E_i = [\epsilon_1^{(i)}, \ldots, \epsilon_k^{(i)}]$. We will choose u_i^* and L_i so as to minimize

$$\sum_{j=1}^{k} \|u_{i_j} - u_i^* - L_i \theta_j^{(i)}\|^2 = \|U_i - u_i^* e^T - L_i \Theta_i\|_F^2 = \|E_i\|_F^2.$$

Since the norm of E_i can be reduced by subtracting the mean of the columns of E_i from each of its columns, we will choose $u_i^* = \frac{1}{k} U_i e$, so that the mean of the columns of E_i is zero. It is also easy to see that the optimum L_i is given by

$$L_i = (U_i - u_i^* e^T)\Theta_i^\dagger = U_i \left(I - \frac{1}{k} e e^T\right) \Theta_i^\dagger,$$

where $\Theta_i^\dagger$ denotes the pseudoinverse of Θ_i. Finally, we will choose the global coordinates $u_1, \ldots, u_n$ so as to minimize

$$\begin{aligned} \sum_{i=1}^{n} \|E_i\|_F^2 &= \sum_{i=1}^{n} \left\| U_i \left(I - \frac{1}{k} e e^T\right) - L_i \Theta_i \right\|^2 \\ &= \sum_{i=1}^{n} \left\| U_i \left(I - \frac{1}{k} e e^T\right) \left(I - \Theta_i^\dagger \Theta_i\right) \right\|^2. \end{aligned}$$

Set $U = [u_1, \ldots, u_n]$ and let S_i be the selection matrices with all entries either 0 or 1 so that $U S_i = U_i$ for all i. Then this minimization problem can be recast as

$$\min \|U S W\|_F^2,$$

where $S = [S_1, \ldots, S_n]$ and $W = \mathrm{diag}(W_1, \ldots, W_n)$, with

$$W_i = \left(I - \frac{1}{k}ee^T\right)\left(I - \Theta_i^\dagger \Theta_i\right). \tag{3.1}$$

Now, note that in the absence of additional constraints, $U = 0$ trivially solves this minimization problem. In order to ensure that we get a non-trivial embedding map as the output, we add the constraint that $UU^H = I_s$. This amounts to requiring that the empirical covariance matrix of the coordinate functions $u^1, \ldots, u^s$ of $u = f(z)$ is proportional to the Identity matrix. Writing

$$\|USW\|_F^2 = \mathrm{tr}(USWW^H S^H U^H),$$

we see that the objective function is minimized by taking the columns of U^H to be the eigenvectors corresponding to the smallest eigenvalues of $B := SWW^H S^H$. Since $e = (1, \ldots, 1)^T$ is an eigenvector of B with eigenvalue 0, we choose the columns of U to be the eigenvectors corresponding to the 2nd to the $(s+1)$ st smallest eigenvalues of B. This U gives us our desired output.

The resulting embedding algorithm is summarized in the next page.

Algorithm 1: Complex Local Tangent Space Alignment (CLTSA)

Input: A set of points $\mathcal{P} = \{z_1, \dots, z_n\} \subset \mathbb{C}^r$ sampled i.i.d. without noise from an absolutely continuous probability distribution on a d -dimensional complex submanifold $\mathcal{M}$ of $\mathbb{C}^r$; the dimension of the embedding s; and k, the number of nearest neighbors to be used in local computations.

Step 1 For each $i = 1, \dots, n$,

1. Compute the k nearest neighbors $z_{i_j} (1 \leq j \leq k)$ of z_i, including z_i. Set $Z_i = [x_{i_1}, \dots, x_{i_k}]$.
2. Compute the matrix Q_i by setting its columns equal to the orthonormal eigenvectors corresponding to the d largest eigenvalues of $(Z_i - z_i^* e^T)^H (Z_i - z_i^* e^T)$.
3. Compute $\Theta_i = Q_i^H (Z_i - z_i^* e^T)$.

Step 2 (Computing global embedding coordinates)

1. For each $i = 1, \dots, n$, compute the selection matrix S_i, and $\Theta_i^\dagger$, the Moore-Penrose pseudoinverse of Θ_i.
2. Set $S = [S_1, \dots, S_n]$ and $W = \mathrm{diag}(W_1, \dots, W_n)$, where W_i are given by (3.1).
3. Compute $U = [u_1, \dots, u_n]$, whose rows are conjugate transposes of eigenvectors corresponding to the 2nd to $(s + 1)$ st lowest eigenvalues of $B = SWW^H S^H$.

Output: The embedding coordinates $u_i \in \mathbb{C}^s$ for z_i, $1 \leq i \leq n$.

Chapter 4

ESTIMATING THE PUSHFORWARD METRIC

Having computed a holomorphic embedding f of our data set $\mathcal{P}$ into $\mathbb{C}^s$ (with $d \leq s \ll r$), we now proceed to computing the *pushforward metric* at each $f(z) \in \mathbb{C}^s$, $z \in \mathcal{P}$ that makes f an isometry. Notice that while the pushforward metric is *a-priori* only defined on vectors in $T_q f(\mathcal{M})$, $q \in f(\mathcal{M})$, it can be trivially extended to all tangent vectors in $\mathbb{C}^s$ based at points of $f(\mathcal{M})$ by defining it to be $\equiv 0$ on the normal bundle $Nf(\mathcal{M}) = \coprod_{q \in f(\mathcal{M})} (T_q f(\mathcal{M}))^{\perp}$. Thus, the pushforward metric can be regarded as an $s \times s$ complex positive semidefinite matrix valued function on $f(\mathcal{M})$.

Definition 4.1. Let $(\mathcal{M}, g)$ be a Hermitian manifold and let $f : \mathcal{M} \to \mathbb{C}^s$ be a holomorphic embedding. Then the *pushforward metric* $h = f_* g$ of g by the embedding f is defined by

$$\langle u, v \rangle_{h(f(p))} = \langle Df(p)^{\dagger}(u), Df(p)^{\dagger}(v) \rangle_{g(p)}$$

for all $p \in \mathcal{M}$ and $u, v \in T_{f(p)} f(\mathcal{M}) \oplus T_{f(p)} f(\mathcal{M})^{\perp}$. Here, $Df(p)^{\dagger} : T_{f(p)} f(\mathcal{M}) \oplus T_{f(p)} f(\mathcal{M})^{\perp} \to T_p \mathcal{M}$ is the pseudoinverse of the Jacobian $Df(p)$ of $f : \mathcal{M} \to \mathbb{C}^s$.

Suppose u, v are holomorphic tangent vectors at $q = f(p) \in f(\mathcal{M})$ and $J = D_{\mathbb{C}} f(p)$. If G and H are complex positive definite matrices representing g and h on $T_p^{(1,0)} \mathcal{M}$ and $T_{f(p)}^{(1,0)} f(\mathcal{M})$ respecively, then in matrix notation,

$$u^H J^H H J v = u^H G v$$

which implies

$$H = (J^H)^{\dagger} G J^{\dagger}. \tag{4.1}$$

In particular, when $\mathcal{M} \subset \mathbb{C}^r$ is equipped with the metric inherited from the Euclidean metric on $\mathbb{C}^r$, $G = \Pi^H I_r \Pi$, where I_r is the $r \times r$ identity matrix and Π is the matrix of the projection map onto $T_p\mathcal{M}$. Therefore, we get

$$H(p) = (J^H)^\dagger \Pi(p)^H I_r \Pi(p) J^\dagger.$$

If the embedding f was computed using the Complex LTSA algorithm as proposed in Chapter 3, this immediately suggests a way of computing the pushforward metric. Recall the matrices Q_i and L_i computed in Algorithm 1. For all $i \in \{1, \ldots, n\}$, we had $L_i = D_{\mathbb{C}} f(z_i^*) Q_i$. Using the approximation $J = D_{\mathbb{C}} f(z_i) \approx D_{\mathbb{C}} f(z_i^*)$ we get $J \approx L_i Q_i^H$ and therefore,

$$H(f(z_i)) \approx (L_i^H)^\dagger Q_i^\dagger \Pi(p)^H I_r \Pi(p) (Q_i^H)^\dagger L_i^\dagger.$$

The resulting algorithm is summarized in Algorithm 2

Another possibility is to use the Laplace-Beltrami operator to compute the pushforward metric as in [9].

Proposition 4.2. *Let f be a holomorphic embedding of $\mathcal{M}$ into $\mathbb{C}^s$, and let $D_{\mathbb{C}} f$ denote its complex Jacobian. Then the pushforward metric h at $f(p)$ is given by the pseudoinverse of the matrix $\widetilde{h}$, where*

$$\widetilde{h}^{jk} = \Delta_{\mathcal{M}} \frac{1}{4} (f^j - f^j(p))(\overline{f^k - f^k(p)})|_p$$

where $\Delta_{\mathcal{M}}$ is the Laplace-Beltrami operator on $\mathcal{M}$.

Proof. Choose local holomorphic coordinates $(z^1, \ldots, z^d)$ for $\mathcal{M}$ around p. By eqns. (2.7), (2.8) and Proposition 2.8, we have

$$\begin{aligned}
\Delta_{\mathcal{M}} \frac{1}{4} (f^j - f^j(p))(f^k - f^k(p))|_p &= \Delta_{\partial} \frac{1}{2} (f^j - f^j(p))(f^k - f^k(p))|_p \\
&= \sum_{s,t=1}^{d} h^{s\bar{t}} \frac{\partial}{\partial z^s} (f^j - f^j(p)) \frac{\partial}{\partial \bar{z}^t} (\overline{f^k - f^k(p)}) \Big|_p \\
&= \sum_{s,t=1}^{d} h^{s\bar{t}} \frac{\partial f^j}{\partial z^s} \overline{\frac{\partial f^k}{\partial z^t}} \\
&= (J H^{-1} J^H)_{jk}
\end{aligned}$$

where $J = D_{\mathbb{C}}f(p)$ and $H = (h_{j\bar{k}})$. Letting $\widetilde{H}$ denote the matrix whose entries are given by the left hand side of the above equation, we conclude that $\widetilde{H} = JH^{-1}J^H$. It remains to show that $H = \widetilde{H}^\dagger$, which is equivalent to

$$(J^H)^\dagger HJ^\dagger = (JH^{-1}J^H)^\dagger.$$

This follows from the fact that whenever a matrix A has full column rank and B has full row rank, we have the equality

$$(AB)^\dagger = B^\dagger A^\dagger.$$

Notice that H^{-1} is invertible, J has full column rank, and J^H has full row rank. Therefore, the claim is proved. $\square$

Approximating $\Delta_{\mathcal{M}}$ using a Graph Laplacian, such as the *Diffusion map* Graph Laplacian of Coifman and Lafon [3] allows us to discretize the equation and compute $h(f(z_i))$ for all $i \in \{1, \ldots, n\}$. Note that the Graph Laplacian of Coifman and Lafon is known to converge to the Laplace Beltrami operator under our assumptions on $\mathcal{M}$ and the data set $\mathcal{P}$. The resulting algorithm is summarized in Algorithm 3.

Algorithm 2: Metric Learning using the Jacobian

Input: A set of points $\mathcal{P} = \{z_1, \ldots, z_n\} \subset \mathbb{C}^r$ sampled i.i.d. without noise from an absolutely continuous probability distribution on a d -dimensional complex submanifold $\mathcal{M}$ of $\mathbb{C}^r$; the dimension of the embedding s; and k, the number of nearest neighbors to be used in local computations.

Step 1 Apply CLTSA (Algorithm 1) to the given Input data. For each $i = 1, \ldots, n$, record

1. The matrix Q_i whose columns are the orthonormal eigenvectors corresponding to the d largest eigenvalues of the centered matrix $\left(I - \frac{1}{k}ee^T\right) Z_i^H Z_i \left(I - \frac{1}{k}ee^T\right)$.
2. The matrix Θ_i of local holomorphic coordinates of the k nearest neighbors of z_i computed in Step 1 of CLTSA,
3. The embedding coordinates $U_i = US_i$ computed in Step 2 of LTSA.

Step 2 For each $i = 1, \ldots, n$,

1. Compute $L_i := U_i \left(I - \frac{1}{k}ee^T\right) \Theta_i^\dagger$ and $J_i := L_i Q_i^H$.
2. Estimate the pushforward metric at $u_i = f(z_i)$ given by

$$H(f(z_i)) = (J_i^H)^\dagger \Pi(p)^H I_r \Pi(p) J_i^\dagger.$$

Output: The pushforward Hermitian metric $H(f(z_i))$ at each $f(z_i)$, $1 \le i \le n$.

Algorithm 3: Metric Learning using the Graph Laplacian

Input: A set of points $\mathcal{P} = \{z_1, \ldots, z_n\} \subset \mathbb{C}^r$ sampled i.i.d. without noise from an absolutely continuous probability distribution on a d -dimensional complex submanifold $\mathcal{M}$ of $\mathbb{C}^r$; the dimension of the embedding s; bandwidth parameter $\sqrt{\epsilon}$; and k, the number of nearest neighbors to be used in local computations.

Step 1 Construct the Diffusionmaps Graph Laplacian matrix $\mathcal{L}_{\epsilon,n}$ as described in [3]

Step 2 Obtain the embedding coordinates $f_n(p) = (f_n^1(p), \ldots, f_n^s(p))$ for all $p \in \mathcal{P}$ by

$$[f_n(p)]_{p\in\mathcal{P}} = \mathrm{CLTSA}(\mathcal{P}, d, s, k).$$

Step 3 Calculate the pushforward metric $h_n(p)$ at each $p \in \mathcal{P}$:

1. For each $p \in \mathcal{P}$ and $j, k = 1, \ldots, s$, compute

$$\widetilde{h}_n^{jk} = \frac{1}{4}\mathcal{L}_{\epsilon,n}\left[(f_n^j - f_n^j(p)) \cdot (\overline{f_n^k - f_n^k(p)})\right].$$

2. For each $p \in \mathcal{P}$, form the matrix $\widetilde{h}_n(p) = (\widetilde{h}_n^{jk})$ and compute the embedding metric at p by

$$h_n(p) = (\widetilde{h}_n(p))^\dagger.$$

Output: The pushforward Hermitian metric $h_n(f(p))$ for each $p \in \mathcal{P}$.

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