Pinned Balls, Foldings and Particle Collisions

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Abstract

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This dissertation provides results on three different models related to or inspired by billiards.

- Part I contains proof of the existence of the invariant distribution for a pinned billiard ball model with gravity and the construction of a pinned billiard ball model with a given invariant distribution.

- Part II contains estimates of the convergence rate for the folding model on the unit sphere in $\mathbb{R}^n$. This is a toy model for the coupling approach to the convergence rate in densely packed pinned billiard balls on a torus.

- Part III gives the asymptotic formula for the collision location of two sample particles with fixed initial locations and velocities distributed according to the microcanonical ensemble formula, i.e., independent standard multidimensional normal in $\mathbb{R}^2$ and $\mathbb{R}^3$.

TABLE OF CONTENTS

Page

LIST OF FIGURES

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Chapter 1

PINNED BALLS

1.1 Motivation

The model takes inspiration from physical systems, for example, gas molecules under high pressure. In this context, we treat the particles as fixed balls with the same mass and radius, and their interactions involve fully elastic collisions. For simplicity, we adopt a one-dimensional model, wherein the balls are arranged linearly.

To make the model more realistic and interesting, gravity is introduced, preventing the evolution of velocities from simplifying to mere permutations of the initial values. Additionally, to avert a scenario where velocities approach negative infinity over time, we assume the balls are positioned vertically with the lowest ball on the ground. This arrangement ensures that the velocity at the bottom has the potential to transition to a positive value, contributing to the model's stability.

1.2 Informal review of main results

We will prove that the one dimensional pinned ball model is irreducible, Harris recurrent and has a unique invariant distribution.

We will also construct another model with a predefined invariant distribution.

1.3 Model definition

sec:intro

We will study a model of pinned balls introduced in [1], but in our model, we will add the gravity. The balls are arranged along a vertical line and touch each other. They have pseudo-velocities but they do not move. Pseudo-velocities will evolve like the usual velocities affected by collisions of the balls. Despite their no effect on the position, we will call pseudo-velocities "velocities". Similarly, pseudo-collisions will be called simply "collisions". The collision times will be chosen for each pair by an exogenous Poisson process.

We will label the balls 1 to n, starting from the ball at the top. The velocity of the i-th ball at the time t will be denoted $X_i(t)$, for $i = 1, \ldots, n$ and $t \geq 0$. The collision times will be denoted T_k for $k \geq 1$, with the convention that $T_0 = 0$. We assume that $T_k - T_{k-1}$ are i.i.d. exponential with mean 1. We also need a sequence of random variables U_k, i.i.d., uniform on $\{1, \ldots, n\}$, and independent of $T_k's$.

The velocities are changing due to the constant "gravitational acceleration" $g > 0$ (positive in our notation but pointing down).

The evolution of the process $\mathbf{X}(t) := (X_1(t), \ldots, X_n(t))$ is the following. For every $j = 1, \ldots, n$, $k \geq 0$ and $T_k \leq t < T_{k+1}$, $X_j(t) = X_j(T_k) - g(T - T_k)$.

At any time T_k, the process $\mathbf{X}$ may have a jump. If $U_k = j$ with $j < n$ and $X_j(T_k-) < X_{j+1}(T_k-)$ then we let $X_j(T_k) = X_{j+1}(T_k-)$ and $X_{j+1}(T_k) = X_j(T_k-)$. If $U_k = n$ and $X_n(T_k-) < 0$ then we let $X_n(T_k) = -X_n(T_k-)$. All other velocities remain unchanged at time T_k. If the conditions listed above are not satisfied then $\mathbf{X}(T_k) = \mathbf{X}(T_k-)$.

We will use an alternative description of the state of the process by tracking velocities and pretending that they move from one ball to another ball at the time of the collision.

With probability 1, all $T_k's$ are distinct. But in our proof, we will allow them to be equal. It is easy to see that the following definitions apply in the case when $T_0 \leq T_1 \leq T_2 \ldots$

We will define the velocity process $\mathbf{V}(t) = \{V_1(t), \ldots, V_n(t)\}$ and a function $f(i, t)$ by induction. For every $t \geq 0$, $\{f(i, t)\}_{i=1}^n$ will be a permutation of $\{1, \ldots, n\}$.

We will start with $V_i(0) = X_i(0)$ and $f(i, 0) = i$. Suppose $\mathbf{V}(T_{k-1})$ and $\{f(i, T_{k-1})\}_{i=1}^n$ have been defined. Then for $T_{k-1} < t < T_k$ and $i = 1, \ldots, n$, let $V_i(t) = X_{f(i,T_{k-1})}(t)$ and $f(i, t) = f(i, T_{k-1})$. If $U_k < n$ then at time T_k there exist i_1, i_2 such that $f(i_1, T_{k-1}) = U_k$ and $f(i_2, T_{k-1}) = U_k + 1$. If $X_{U_k}(T_k-) < X_{U_k+1}(T_k-)$, then we let $f(i_1, T_k) = U_k + 1$, $f(i_2, T_k) = U_k$ and $f(i, T_k) = f(i, T_{k-1})$ for all $i \neq i_1, i_2$. If $U_k = n$ or $X_{U_k}(T_k-) \geq X_{U_k+1}(T_k-)$ then we let $f(i, T_k) = f(i, T_{k-1})$ for all $i = 1, \ldots, n$. Finally, we let $V_i(T_k) = X_{f(i,T_k)}(T_k)$.

We will take the filtration $\mathcal{G}(t) = \sigma\left(\{\mathbf{X}(s), \mathbf{V}(s), f(i, s) : 0 \leq s \leq t, 1 \leq i \leq n\}\right)$, and let Leb_k to be the $k-$th dimensional Lebesgue measure.

For most of the time V_i is a linear function with slope $-g$ except that is has jumps at times T_k such that $V_i(T_k-)$ is negative, $U_k = n$ and $f(i, T_k) = n$. At such a time it will

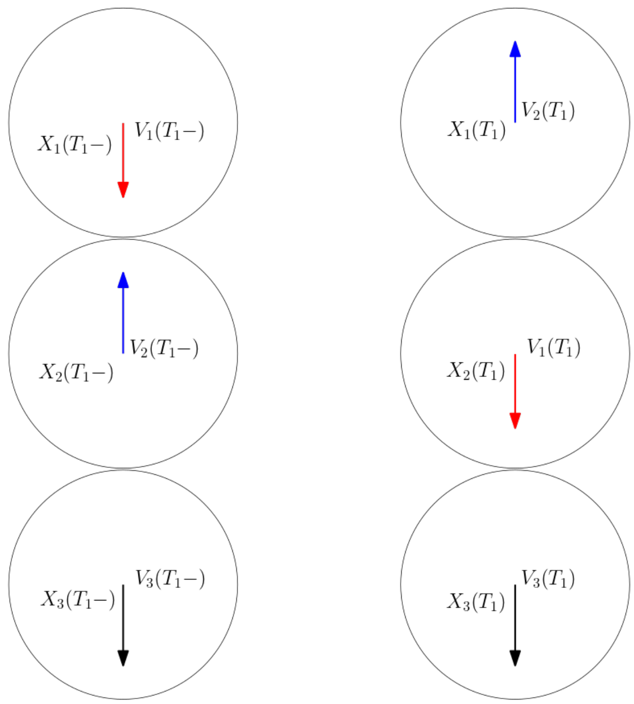

Figure 1.1: The evolution of $\mathbf{X}$ and $\mathbf{V}$ when $n = 3, U_1 = 1$ and $X_1(T_1-) < X_2(T_1-)$.

graph.3ball

jump from $V_i(T_k-)$ to $V_i(T_k) = -V_i(T_k-)$. Hence, for $t_1 < t_2$,

$$V_i(t_2) \geq V_i(t_1) - g(t_2 - t_1). \tag{1.3.1}$$ intro.1

1.4 Neighborhood irreducibility

sec:irr

First we will prove that the process $\mathbf{X}(t)$ is irreducible in the sense that starting from any point it can reach any nonempty open set with positive probability. For convenience, let $\mathbf{g} = (g, g, \cdots, g)$.

irr1 **Definition 1.4.1.** For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, we say $\mathbf{b}$ is accessible from $\mathbf{a}$ by $\mathbf{X}$ (written as $\mathbf{a} \xrightarrow{\mathbf{X}} \mathbf{b}$) if there exist deterministic $t_1, t_2, \ldots, t_m$ and $u_1, \ldots, u_m$ such that when $\mathbf{X}(0) = \mathbf{a}$, $T_i = t_i, U_i = u_i$ for $1 \leq i \leq m$, we have $\mathbf{X}(T_m) - \mathbf{g}t = \mathbf{b}$ for some $t \geq 0$.

irr2 **Remark 1.4.2.** This actually means we can build a trajectory from $\mathbf{a}$ to $\mathbf{b}$. Hence, it is easy to see that $\xrightarrow{\mathbf{X}}$ is transitive, i.e. $\mathbf{a} \xrightarrow{\mathbf{X}} \mathbf{b}$ and $\mathbf{b} \xrightarrow{\mathbf{X}} \mathbf{c}$ imply $\mathbf{a} \xrightarrow{\mathbf{X}} \mathbf{c}$.

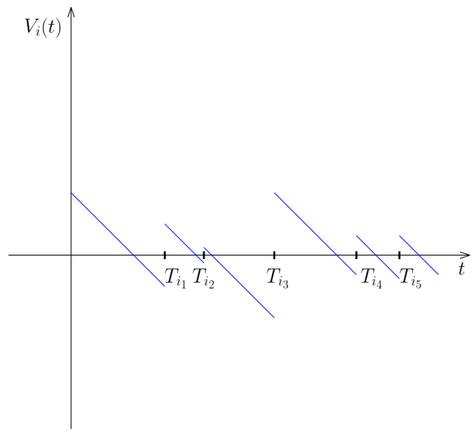

Figure 1.2: Graph of $V_i(t)$.

graph.v

irr3 **Lemma 1.4.3.** *For any* $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n, \mathbf{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$*, we have* $\mathbf{a} \xrightarrow{\mathbf{X}} \mathbf{b}$.

Proof. (i) First we will consider a special case. Assume that $\mathbf{b}$ satisfies: $\{b_i\}_{i=1}^n = \{|a_i|\}_{i=1}^n$ and $b_1 \geq b_2 \geq \cdots \geq b_n \geq 0$.

Let

$$t_k = 0, u_k = q \text{ for } k = pn + q,\ 0 \leq p \leq n-1, 1 \leq q \leq n,$$

$$t_{n^2+k} = 0, u_{n^2+k} = q \quad \text{for } k = p(n-1) + q, 0 \leq p \leq n-2,\ 1 \leq q \leq n-1.$$

We will show that if $\mathbf{X}(0) = \mathbf{a}$, and $T_k = t_k, U_k = u_k$ for $1 \leq k \leq n^2 + (n-1)^2$, then $\mathbf{X}(T_{n^2+(n-1)^2}) = \mathbf{b}$.

It is easy to see that the first $n-1$ collisions will send the smallest velocity to the bottom, i.e.

$$X_n(T_{n-1}) = \min_{1 \leq i \leq n} X_i(T_{n-1}).$$

Combine this with the fact $U_n = n$ to see that $X_n(T_n) = |\min_{1 \leq i \leq n} a_i|$. Hence, at time T_n, there will be at least one nonnegative velocity.

By analogy, every sequence of $n-1$ collisions from $k = pn+1$ to $k = pn+n-1$ will send the smallest velocity to the bottom i.e. $X_n(T_{pn+n-1}) = \min_{1 \leq i \leq n} X_i(T_{pn+n-1})$. The next collision will occur between the n-th ball and the bottom and in either case we will have

$X_n(T_{pn+n}) = |X_n(T_{pn+n-1})|$. Hence, at time $pn+n$ there will be at least $p+1$ nonnegative velocities.

After repeating this for n times all the velocities will be nonnegative, i.e. at time T_{n^2}, $\{X_i(T_{n^2})\}_{i=1}^n = \{|a_i|\}_{i=1}^n$.

We will use the similar scheme $n-1$ times, this time to order the velocities. Specifically, since $T_{n^2+k} = 0, U_{n^2+k} = k$ for $1 \le k \le n-1$, we have

$$X_n(T_{n^2+(n-1)}) = \min_{1\le i\le n} X_i(T_{n^2+(n-1)})$$

After the second round, we obtain

$$X_{n-1}(T_{n^2+2(n-1)}) = \min_{1\le i\le n-1} X_i(T_{n^2+2(n-1)}),$$

$$X_n(T_{n^2+2(n-1)}) = \min_{1\le i\le n} X_i(T_{n^2+2(n-1)}).$$

In general, at time $T_{n^2+a(n-1)}$, $1 \le a \le n-1$,

$$X_i(T_{n^2+a(n-1)}) = \min_{1\le j\le i} X_j(T_{n^2+a(n-1)}),\ for\ \ n-a+1 \le i \le n.$$

In the end, at time $T_{n^2+(n-1)^2}$

$$X_1(T_{n^2+(n-1)^2}) \ge X_2(T_{n^2+(n-1)^2}) \ge \cdots \ge X_n(T_{n^2+(n-1)^2})$$

and

$$\{X_i(T_{n^2+(n-1)^2})\}_{i=1}^n = \{X_i(T_{n^2})\}_{i=1}^n = \{|a_i|\}_{i=1}^n$$

which implies $\mathbf{X}(T_{n^2+(n-1)^2}) = \mathbf{b}$. Therefore, $\mathbf{a} \xrightarrow{\mathbf{X}} \mathbf{b}$.

(ii) Let's consider another special case. Assume that for some $1 \le m \le n$, $\mathbf{a}, \mathbf{b}$ satisfy:

$$a_1 \ge a_2 \ge \cdots \ge a_m > 0,\ a_{m+1} = a_{m+2} = \cdots = a_n = 0,$$

$$b_i = a_i - a_m\ \ for\ 1 \le i \le m-1, b_m = b_{m+1} = \cdots = b_n = 0.$$

Let $\mathbf{X}(0) = \mathbf{a}$. If $m = n$ and $t = a_n/g$, then

$$\mathbf{X}(t) = (a_1 - a_n, a_2 - a_n, \ldots, a_{n-1} - a_n, 0) = \mathbf{b}.$$

Hence, $\mathbf{a} \xrightarrow{\mathbf{X}} \mathbf{b}$.

Now suppose that $1 \le m < n$. Note that

$$\mathbf{X}\left(\frac{a_m}{2g}\right) = \left(a_1 - \frac{a_m}{2}, \dots, a_{m-1} - \frac{a_m}{2}, \frac{a_m}{2}, \underbrace{-\frac{a_m}{2}, \dots, -\frac{a_m}{2}}_{n-m}\right)$$

so

$$\mathbf{a} \xrightarrow{\mathbf{X}} \left(a_1 - \frac{a_m}{2}, \dots, a_{m-1} - \frac{a_m}{2}, \frac{a_m}{2}, \underbrace{-\frac{a_m}{2}, \dots, -\frac{a_m}{2}}_{n-m}\right).$$

By part(i) we have

$$\begin{aligned}&\left(a_1 - \frac{a_m}{2}, \dots, a_{m-1} - \frac{a_m}{2}, \frac{a_m}{2}, -\frac{a_m}{2}, \dots, -\frac{a_m}{2}\right)\\ &\xrightarrow{\mathbf{X}} \left(a_1 - \frac{a_m}{2}, \dots, a_{m-1} - \frac{a_m}{2}, \frac{a_m}{2}, \frac{a_m}{2}, \dots, \frac{a_m}{2}\right).\end{aligned}$$

If $\mathbf{X}(0) = \left(a_1 - \frac{a_m}{2}, \dots, a_{m-1} - \frac{a_m}{2}, \frac{a_m}{2}, \frac{a_m}{2}, \dots, \frac{a_m}{2}\right)$ then

$$\mathbf{X}\left(\frac{a_m}{2g}\right) = (a_1 - a_m, \dots, a_{m-1} - a_m, 0 \dots, 0) = \mathbf{b}$$

So

$$\left(a_1 - \frac{a_m}{2}, \dots, a_{m-1} - \frac{a_m}{2}, \frac{a_m}{2}, \frac{a_m}{2}, \dots, \frac{a_m}{2}\right) \xrightarrow{\mathbf{X}} \mathbf{b}.$$

Hence, by the transitivity we have $\mathbf{a} \xrightarrow{\mathbf{X}} \mathbf{b}$.

(iii) Suppose $\mathbf{b} = \mathbf{0}$.

Let $\{c_i\}_{i=1}^n$ satisfy $\{c_i\}_{i=1}^n = \{|a_i|\}_{i=1}^n$ and $c_1 \ge c_2 \ge \cdots \ge c_n \ge 0$. Then by (i) and (ii) we have

$$\begin{aligned}\mathbf{a} &\xrightarrow{\mathbf{X}} (c_1, c_2, \dots, c_n)\\ &\xrightarrow{\mathbf{X}} (c_1 - c_n, c_2 - c_n, \dots, c_{n-1} - c_n, 0)\\ &\xrightarrow{\mathbf{X}} (c_1 - c_{n-1}, c_2 - c_{n-1}, \dots, c_{n-2} - c_{n-1}, 0, 0)\\ &\dots\\ &\xrightarrow{\mathbf{X}} (c_1 - c_2, 0, \dots, 0)\\ &\xrightarrow{\mathbf{X}} (0, 0, \dots, 0) = \mathbf{b}\end{aligned}$$

(iv) Suppose $\mathbf{a} = 0$. Define a process $\mathbf{X}'(t) := (X_1'(t), \ldots, X_n'(t))$ similar to $\mathbf{X}$ as follows: for every $i = 1, \ldots, n$, $k \geq 0$ and $T_k \leq t < T_{k+1}$, $X_i'(t) = X_i'(T_k) + g(t - T_k)$ (for $\mathbf{X}$ the sign in front of the last term is negative). We will now define "jumps" of $\mathbf{X}'$. If $U_k = j$ with $j < n$ and $X_j'(T_k-) > X_{j+1}'(T_k-)$ then we let $X_j'(T_k) = X_{j+1}'(T_k-)$ and $X_{j+1}'(T_k) = X_j'(T_k-)$. If $U_k = n$ and $X_n'(T_k-) > 0$ then we let $X_n'(T_k) = -X_n'(T_k-)$. All other velocities remain unchanged at time T_k. If the conditions listed above are not satisfied then $\mathbf{X}'(T_k) = \mathbf{X}'(T_k-)$.

Similarly to $\mathbf{a} \xrightarrow{\mathbf{X}} \mathbf{b}$, we say $\mathbf{b}$ is accessible from $\mathbf{a}$ by $\mathbf{X}'$ (written as $\mathbf{a} \xrightarrow{\mathbf{X}'} \mathbf{b}$) if there exist $t_1, t_2, \ldots, t_m$ and $u_1, \ldots, u_m$ such that when $\mathbf{X}'(0) = \mathbf{a}$ and $T_i = t_i, U_i = u_i$ for $1 \leq i \leq m$, then $\mathbf{X}'(t) = \mathbf{b}$ for some $t \geq T_m$.

Clearly $\mathbf{a} \xrightarrow{\mathbf{X}'} \mathbf{b}$ is equivalent to $\mathbf{b} \xrightarrow{\mathbf{X}} \mathbf{a}$. Let $\{c_i\}_{i=1}^n$ satisfy $\{c_i\}_{i=1}^n = \{-|b_i|\}_{i=1}^n$ and $c_1 \leq c_2 \leq \cdots \leq c_n \leq 0$. Following the same argument as in (i), (ii) and (iii) we can show that

$$\begin{aligned}
\mathbf{b} &\xrightarrow{\mathbf{X}'} (c_1, c_2, \ldots, c_n) \\
&\xrightarrow{\mathbf{X}'} (c_1 - c_n, c_2 - c_n, \ldots, c_{n-1} - c_n, 0) \\
&\xrightarrow{\mathbf{X}'} (c_1 - c_{n-1}, c_2 - c_{n-1}, \ldots, c_{n-2} - c_{n-1}, 0, 0) \\
&\cdots \\
&\xrightarrow{\mathbf{X}'} (c_1 - c_2, 0, \ldots, 0) \\
&\xrightarrow{\mathbf{X}'} (0, 0, \ldots, 0) = \mathbf{a}
\end{aligned}$$

Hence, $\mathbf{a} \xrightarrow{\mathbf{X}} \mathbf{b}$.

(v) The general case. By (iii), (iv) and the transitivity of $\xrightarrow{\mathbf{X}}$, we have $\mathbf{a} \xrightarrow{\mathbf{X}} \mathbf{0} \xrightarrow{\mathbf{X}} \mathbf{b}$.

□

Suppose $\mathbf{Y}(t)$ is another process following the same evolution as $\mathbf{X}(t)$ with the same $\{U_k\}_{k=1}^\infty$ but different jump times $\{T_k'\}_{k=1}^\infty$.

irr4 **Lemma 1.4.4.** *For every $k \geq 1$ we have*

$$\|\mathbf{X}(T_k) - \mathbf{Y}(T_k')\|_n \leq \|\mathbf{X}(T_k-) - \mathbf{Y}(T_k'-)\|_n$$

Here $\|\cdot\|_n$ is the Euclidean norm on $\mathbb{R}^n$.

Proof. Suppose $U_k = j$. If both $\mathbf{X}, \mathbf{Y}$ can have jump or neither of them can have jump, then

$$\|\mathbf{X}(T_k) - \mathbf{Y}(T'_k)\|_n = \|\mathbf{X}(T_k-) - \mathbf{Y}(T'_k-)\|_n$$

and we are done.

WLOG, suppose $\mathbf{X}$ can have jump but $\mathbf{Y}$ can not.

If $j = n$, then $X_n(T_k-) < 0, Y_n(T'_k-) \geq 0$.

$$\begin{aligned}\|\mathbf{X}(T_k-) - \mathbf{Y}(T'_k-)\|_n^2 &= \sum_{i=1}^{n-1}(X_i(T_k-) - Y_i(T'_k-))^2 + (X_n(T_k-) - Y_n(T'_k-))^2 \\ &\geq \sum_{i=1}^{n-1}(X_i(T_k) - Y_i(T'_k))^2 + (X_n(T_k) - Y_n(T'_k))^2 \\ &= \|\mathbf{X}(T_k) - \mathbf{Y}(T'_k)\|_n^2\end{aligned}$$

If $j \neq n$, then $X_j(T_k-) < X_{j+1}(T_k-), Y_j(T'_k-) \geq Y_{j+1}(T'_k-)$, and therefore

$$\begin{aligned}&\left(X_j(T_k-) - Y_j(T'_k-)\right)^2 + \left(X_{j+1}(T_k-) - Y_{j+1}(T'_k-)\right)^2 \\ &= X_j(T_k-)^2 + Y_j(T'_k-)^2 + X_{j+1}(T_k-)^2 + Y_{j+1}(T'_k-)^2 \\ &\quad - 2\left(X_j(T_k-)Y_j(T'_k-) + X_{j+1}(T_k-)Y_{j+1}(T'_k-)\right) \\ &\geq X_j(T_k-)^2 + Y_j(T'_k-)^2 + X_{j+1}(T_k-)^2 + Y_{j+1}(T'_k-)^2 \\ &\quad - 2\left(X_j(T_k-)Y_{j+1}(T'_k-) + X_{j+1}(T_k-)Y_j(T'_k-)\right) \\ &= X_{j+1}(T_k)^2 + Y_j(T'_k)^2 + X_j(T_k)^2 + Y_{j+1}(T'_k)^2 \\ &\quad - 2\left(X_{j+1}(T_k)Y_{j+1}(T'_k) + X_j(T_k)Y_j(T'_k)\right) \\ &= \left(X_j(T_k) - Y_j(T'_k)\right)^2 + \left(X_{j+1}(T_k) - Y_{j+1}(T'_k)\right)^2\end{aligned}$$

It is easy to see the lemma follows from the inequality. □

irr5 **Lemma 1.4.5.** *Fix $N < \infty$. If $\sum_{i=1}^{N} |T_i - T'_i| < \delta$ and $\|\mathbf{X}(0) - \mathbf{Y}(0)\|_n < \delta$, then for arbitrary $\{U_i\}_{i=1}^N$ we have*

$$\|\mathbf{X}(T_N) - \mathbf{Y}(T'_N)\|_n \leq \delta + gN\sqrt{n}\delta.$$

Furthermore, for $\max\{T_N, T'_N\} \leq t < \min\{T_{N+1}, T'_{N+1}\}$ (if such t exists) we have

$$\|\mathbf{X}(t) - \mathbf{Y}(t)\|_n \leq \delta + (gN + 1)\sqrt{n}\delta.$$

Proof. For every $0 \le k \le N-1$,

$$\|\mathbf{X}(T_{k+1}-) - \mathbf{Y}(T'_{k+1}-)\|_n^2 = \sum_{i=1}^n (\mathbf{X}_i(T_k) - g(T_{k+1} - T_k) - \mathbf{Y}_i(T'_k) + g(T'_{k+1} - T'_k))^2$$

If $a = T'_{k+1} - T'_k - T_{k+1} + T_k$, then $|a| < \delta$ by assumption, we have

$$\begin{aligned}
\|\mathbf{X}(T_{k+1}-) - \mathbf{Y}(T'_{k+1}-)\|_n^2 &= \sum_{i=1}^n (\mathbf{X}_i(T_k) - \mathbf{Y}_i(T'_k) + ga)^2 \\
&\le \sum_{i=1}^n (\mathbf{X}_i(T_k) - \mathbf{Y}_i(T'_k))^2 + 2g\delta \sum_{i=1}^n |\mathbf{X}_i(T_k) - \mathbf{Y}_i(T'_k)| + ng^2\delta^2 \\
&\le \|\mathbf{X}(T_k) - \mathbf{Y}(T'_k)\|_n^2 + 2g\delta\sqrt{n}\|\mathbf{X}(T_k) - \mathbf{Y}(T'_k)\|_n + ng^2\delta^2 \\
&= (\|\mathbf{X}(T_k) - \mathbf{Y}(T'_k)\|_n + g\delta\sqrt{n})^2
\end{aligned}$$

Apply, lemma 1.4.4

$$\|\mathbf{X}(T_{k+1}) - \mathbf{Y}(T'_{k+1})\|_n \le \|\mathbf{X}(T_{k+1}-) - \mathbf{Y}(T'_{k+1}-)\|_n \le \|\mathbf{X}(T_k) - \mathbf{Y}(T'_k)\|_n + g\delta\sqrt{n}$$

By induction

$$\|\mathbf{X}(T_N) - \mathbf{Y}(T'_N)\|_n \le \|\mathbf{X}(0) - \mathbf{Y}(0)\|_n + g\delta N\sqrt{n} \le \delta + gN\sqrt{n}\delta$$

For $\max\{T_N, T'_N\} \le t < \min\{T_{N+1}, T'_{N+1}\}$

$$\begin{aligned}
\|\mathbf{X}(t) - \mathbf{Y}(t)\|_n &= \|\mathbf{X}(T_N) - \mathbf{g}(t - T_N) - \mathbf{Y}(T'_N) + \mathbf{g}(t - T'_N)\|_n \\
&\le \|\mathbf{X}(T_N) - \mathbf{Y}(T'_N)\|_n + \|(T_N - T'_N)\mathbf{g}\|_n \\
&\le \delta + (gN+1)\sqrt{n}\delta
\end{aligned}$$

□

relation **Remark 1.4.6.** Note in both Theorem 1.4.4 and 1.4.5, we didn't assume any relationship between T_k and T'_k.

Combing the lemmas above together we can get the following theorem:

irr **Theorem 1.4.7.** *(Neighborhood Irreducibility) For any point $\mathbf{x}_0 \in \mathbb{R}^n$, open ball $A \subset \mathbf{R}^n$ and $\varepsilon > 0$,*

$$\mathbb{P}_{x_0}(\inf_{t>0}\{\mathbf{X}(t) \in A\} < \infty) > 0$$

The notation $\mathbb{P}_{x_0}$ means the process starts from x_0.

Proof. Suppose $A = B(\mathbf{x}_1, \varepsilon)$. By Lemma 1.4.3 there exist sequences $t_1, \cdots, t_m$ and $u_1, \cdots, u_m$ such that when $\mathbf{X}(0) = x_0, T_i = t_i, U_i = u_i$ for $1 \le i \le m$ we have $\mathbf{X}(T_m) - \mathbf{g}t = x_1$ for some $t \ge 0$. By applying Lemma 1.4.5 we can see that any $\{T_i, U_i\}_{i=1}^m$ with $U_i = u_i,\ 1 \le i \le m$ and $\sum_{i=1}^m |T_i - t_i| < \frac{\varepsilon}{1+gm\sqrt{n}}$ satisfy

$$\|\mathbf{X}(T_m) - (x_1 + \mathbf{g}t)\|_n \le \varepsilon.$$

Furthermore if T_{m+1} satisfies $T_{m+1} - T_m > t$, we will have

$$\|\mathbf{X}(T_m + t) - x_1\|_n = \|\mathbf{X}(T_m) - \mathbf{g}t - x_1\|_n < \varepsilon.$$

Therefore,

$$\begin{aligned} &\mathbb{P}_{x_0}\left(\inf_{t>0}\{\mathbf{X}(t) \in A\} < \infty\right) \\ &\ge \mathbb{P}\left(\sum_{i=1}^m |T_i - t_i| < \frac{\varepsilon}{1+gm\sqrt{n}}, T_{m+1} - T_m > t\right) \\ &\quad \times \mathbb{P}\left(U_i = u_i,\ \text{for } 1 \le i \le m\right) \\ &> 0 \end{aligned}$$

□

Note that by lemma 1.4.5 slightly changing the start point also won't change the end point a lot, so we can get the following corollary:

balltoball **Corollary 1.4.8.** *For any open balls $A_1, A_2 \in \mathbb{R}^n$ there exists some constant $c > 0$ such that*

$$\mathbb{P}_{\mathbf{x}_0}\left(\inf_{t>0}\{\mathbf{X}(t) \in A_2\} < \infty\right) > c \ \ \textit{for all } \mathbf{x}_0 \in A_1$$

1.5 Existence of invariant distribution

sec:recurrent

Suppose $V_i(0) = 0$ for some i and fix one i with this property. In other words

$$V_i(0) = 0. \tag{1.5.1}$$

assump.recur

exptail **Definition 1.5.1.** We will say that a non-negative random variable X has exponential tail with parameters $(a, c) \in \mathbb{R}^+ \times \mathbb{R}^+$ if the following inequality holds for all $t > 0$

$$\mathbb{P}(X > t) < a \exp(-ct).$$

exptail.sum **Lemma 1.5.2.** *Suppose $\{X_i\}_{i=1}^{\infty}$ are non-negative and have exponential tails with parameters $\{(a_i, c_i)\}_{i=1}^{\infty}$ respectively and ξ is a random variable such that $\xi \le N$, a.s. for some $N < \infty$. Then $X := \sum_{i=1}^{\xi} X_i$ has exponential tail with parameters $\left(\sum_{i=1}^{N} a_i, \min_i c_i/N\right)$.*

Proof. For all $t > 0$ we have

$$\begin{aligned}
\mathbb{P}(X > t) &\le \mathbb{P}\left(\sum_{i=1}^{N} X_i > t\right) \\
&\le \sum_{i=1}^{N} \mathbb{P}(X_i > t/N) \\
&< \sum_{i=1}^{N} a_i \exp(-c_i t/N) \\
&\le \left(\sum_{i=1}^{N} a_i\right) \exp(-\min_i c_i t/N)
\end{aligned}$$

□

Remark 1.5.3. Note that Lemma 1.5.2 does not require any assumptions about dependence or any other relationship between the X_i's, and also ξ doesn't need to be independent from the sequence.

oneside **Lemma 1.5.4.** *Assume $V_i = 0$, for some $1 \le i < n$. If $i < n$ and $X_j(0) \ge X_i(0) = 0$ for every $1 \le j < i$ then*

$$\tau_1 := \inf\{t > 0 : f(i, t) = i + 1\}$$

has exponential tail with parameters $\left(2(2n)^{n-i-1}, \left(\frac{1}{2n}\right)^{n-i}\right)$.

Proof. It follows from our assumption that $X_j(0) \geq X_i(0) = 0$ for every $1 \leq j < i$, that the difference between the velocities of any pair of consecutive balls among the top $n-1$ balls will not change due to gravity alone. This implies that the collision between i-th ball and $(i-1)$-st ball can not happen before τ_1.

We will prove the lemma by induction.

If $i = n-1$ and $V_n(0) > 0$, then τ_1 has exponential tail with parameters $\left(1, \frac{1}{n}\right)$ since

$$\tau_1 = T_{k_1}, \quad \text{where } k_1 = \inf\{k : U_k = n-1\},$$

which has exponential distribution with mean n. Hence in this case, τ_1 has exponential tail with parameters $\left(1, \frac{1}{n}\right)$.

If $i = n-1$ and $V_n(0) \leq 0$, then let $k_2 = \inf\{k : U_k = n\}$ and note that

$$\tau_1 = T_{k_1} \quad \text{where } k_1 = \inf\{k > k_2 : U_k = n-1\}.$$

By Lemma 1.5.2 and the strong Markov property, τ_1 is a random variable with exponential tail with parameters $\left(2, \frac{1}{2n}\right)$. Combining the two cases we see that, for $i = n-1$, τ_1 has exponential tail with parameters $\left(2, \frac{1}{2n}\right)$.

Now suppose the lemma is true when $i > m$ ($m \leq n-2$). We will show it is also true when $i = m$.

Step 1: If $V_{m+1}(0) > V_m(0)$, then

$$\tau_1 = T_{k_1} \quad \text{where } k_1 = \inf\{k : U_k = m\}.$$

In this case τ_1 has the exponential distribution with mean n.

Step 2: If $V_{m+1}(0) \leq V_m(0)$, then let $\tau_2 = \inf\{t > 0 : X_{m+1}(t) > V_m(t)\}$ and note that

$$\tau_1 = T_{k_1} \quad \text{where } k_1 = \inf\{k : T_k > \tau_2, U_k = m\}.$$

By the strong Markov property, $\tau_1 - \tau_2$ has exponential distribution with mean n.

We will now estimate the tail of τ_2. Define $\tilde{f}(k,t)$ as follows: $\tilde{f}(k,t) = j$ if and only if $f(j,t) = k$, and note that $\tilde{f}$ is well-defined since for each $t > 0$, $\{f(k,t)\}_{k=1}^{n}$ is a permutation of $\{1, \ldots, n\}$. Recall that m is fixed, and let

$$\Lambda = \{\tilde{f}(m+1,t) : t < \tau_2\}.$$

Clearly, $|\Lambda| \leq n$.

Step 3: Next we will show that for every $t_1 < t_2 < \tau_2$,

$$X_{m+1}(t_1) + gt_1 \leq X_{m+1}(t_2) + gt_2 \tag{1.5.2}$$ oneside.1

and the equality holds if and only if $\tilde{f}(m+1, s) = \tilde{f}(m+1, t_1)$ for each $t_1 \leq s \leq t_2$. It suffices to prove (1.5.2) when:

(i) $T_k \leq t_1 < t_2 < T_{k+1}$ for some k.

(ii) $T_k \leq t_1 < T_{k+1} \leq t_2 < T_{k+2}$ for some k.

(iii) $T_k \leq t_1 < T_{k+1} < \cdots < T_{k+l} \leq t_2$ for some k and $l \geq 2$

The proof of (i): In this case, X_{m+1} decreases with a constant rate. Hence,

$$X_{m+1}(t_2) + gt_2 = X_{m+1}(T_k) + gT_k = X_{m+1}(t_1) + gt_1$$

and for all $t_1 \leq s \leq t_2$ we have $\tilde{f}(m+1, s) = \tilde{f}(m+1, t_1)$ since there is no jump.

The proof of (ii): Recall that $m \leq n-2$, so in this case, the only possibility for $(m+1)$-st ball to have a collision is to collide with $(m+2)$-nd ball since $X_{m+1} \leq X_m$ before time τ_2. If $U_{k+1} \neq m+1$, X_{m+1} decreases with a constant rate, so in the same way,

$$X_{m+1}(t_2) + gt_2 = X_{m+1}(T_k) + gT_k = X_{m+1}(t_1) + gt_1$$

and for all $t_1 \leq s \leq t_2$ we have $\tilde{f}(m+1, s) = \tilde{f}(m+1, t_1)$. If $U_{k+1} = m+1$, then

$$\begin{aligned}
X_{m+1}(t_2) + gt_2 &= X_{m+1}(T_{k+1}) + gT_{k+1} \\
&= \max\{X_{m+1}(T_{k+1}-), X_{m+2}(T_{k+1}-)\} + gT_{k+1} \\
&= \max\{X_{m+1}(T_k), X_{m+2}(T_k)\} - g(T_{k+1} - T_k) + gT_{k+1} \\
&\geq X_{m+1}(T_k) + gT_k \\
&= X_{m+1}(t_1) + gt_1.
\end{aligned}$$

The equality holds if and only if $X_{m+1}(T_{k+1}-) \geq X_{m+2}(T_{k+1}-)$. In this case $(m+1)$-st ball can not collide with $(m+2)$-nd ball at time T_{k+1}, which implies for all $t_1 \leq s \leq t_2$ we have $\tilde{f}(m+1, s) = \tilde{f}(m+1, t_1)$.

The proof of (iii): Take $t'_1, t'_2, \ldots,$ such that

$$T_{k+1} < t'_1 < T_{k+2} < t'_2 < T_{k+3} < \cdots < t'_{l-1} < T_{k+l}.$$

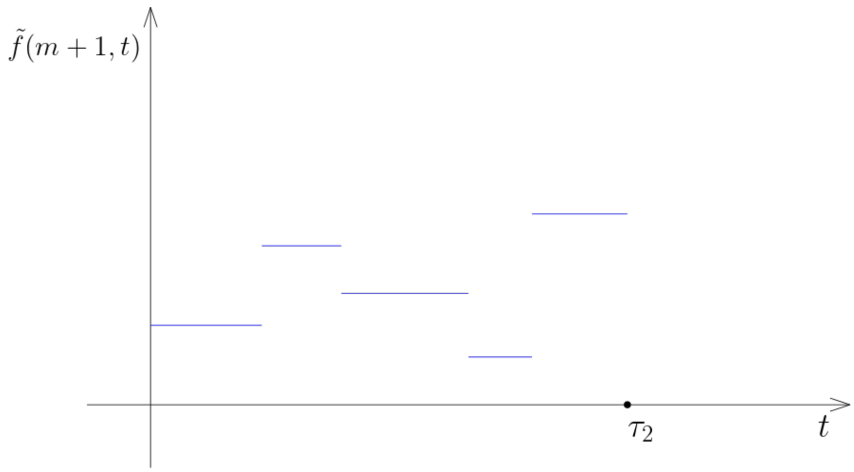

Figure 1.3: The graph of $\tilde{f}(m+1,t), t<\tau_2$. The values on all pictured intervals are distinct.

graph.ftilde

By the result we proved in (ii), we have

$$X_{m+1}(t_1)+gt_1 \le X_{m+1}(t_1')+gt_1' \le \cdots \le X_{m+1}(t_{l-1}')+gt_{l-1}' \le X_{m+1}(t_2)+gt_2$$

and $X_{m+1}(t_1)+gt_1 = X_{m+1}(t_2)+gt_2$ if and only if for all $t_1 \le s \le t_2$ we have $\tilde{f}(m+1,s) = \tilde{f}(m+1,t_1)$.

Step 4: We claim that for all $t_1 < t_2 < \tau_2$ either $\tilde{f}(m+1,t_1) \neq \tilde{f}(m+1,t_2)$ or $\tilde{f}(m+1,s) = \tilde{f}(m+1,t_1)$ for all $t_1 \le s \le t_2$. Suppose $\tilde{f}(m+1,t_1) = \tilde{f}(m+1,t_2)$ but $\tilde{f}(m+1,s)$ is not a constant for $t_1 \le s \le t_2$. By Step 3 we have

$$V_j(t_2) = X_{m+1}(t_2) > X_{m+1}(t_1) - g(t_2-t_1) = V_j(t_1) - g(t_2-t_1)$$

It shows that V_j collide at bottom at some time t_3 between t_1 and t_2. Then $X_{m+1}(t_2) = V_j(t_2) > 0 - g(t_2-t_3) > X_m(0) - gt_2 = X_m(t_2)$ This contradicts with $t_2 < \tau_2$.

Step 5: Let μ denote the Lebesgue measure. We have

$$\begin{aligned}\tau_2 &= \sum_{j\in\Lambda} \mu\left(\{t : \tilde{f}(m+1,t) = j\}\right)\\ &= \sum_{j\in\Lambda} \sup\{t \ge 0 : \tilde{f}(m+1,t) = j\} - \inf\{t \ge 0 : \tilde{f}(m+1,t) = j\}.\end{aligned}$$

Let $j \in \Lambda$ and $\tau_3 = \inf\{t \ge 0 : \tilde{f}(m+1,t) = j\}$. We claim that

$$X_{m+1}(\tau_3) = V_j(\tau_3) \le V_m(\tau_3) \le X_i(\tau_3),\ \forall 1 \le i < m.$$

The first equality comes from the fact $\tilde{f}(m+1,\tau_3) = j$. The second inequality comes from the fact $\tau_3 < \tau_2$. The velocities $V_1, V_2, \ldots, V_{i-1}$ decrease at a constant rate before time τ_1, so the last inequality holds since

$$X_m(\tau_3) = X_m(0) - g\tau_3 \le X_i(0) - g\tau_3 = X_i(\tau_3),\ \forall 1 \le i < m.$$

Then by induction

$$\sup\{t \ge 0 : \tilde{f}(m+1,t) = j\} - \inf\{t \ge 0 : \tilde{f}(m+1,t) = j\}$$

has exponential tail with parameters $\left(2(2n)^{n-m-2}, \left(\frac{1}{2n}\right)^{n-m-1}\right)$. Since $|\Lambda| \le n$, by Lemma 1.5.2, τ_2 has exponential tail with parameters $\left((2n)^{n-m-1}, \frac{1}{n}\cdot\left(\frac{1}{2n}\right)^{n-m-1}\right)$. Combining this with the fact that $\tau_1 - \tau_2$ has exponential tail with parameter $\frac{1}{n}$, we obtain that τ_1 has exponential tail with parameters $\left(2(2n)^{n-m-1}, \left(\frac{1}{2n}\right)^{n-m}\right)$.

Therefore, by induction we proved that τ_1 has exponential tail with parameters $\left(2(2n)^{n-i-1}, (\frac{1}{2n})^{n-i}\right)$ when $i < n$. □

doubleside **Lemma 1.5.5.** *We make one of the following assumptions:*

(1) $V_i(0) = 0$ *for some* $1 \le i < n$.

(2) $V_n(0) = 0$ *and* $V_n(0) \ne \min_{1\le j\le n} V_j(0)$.

Then the time

$$\sigma := \inf\{t > 0 : f(i,t) = i+1 \text{ or } i-1\}$$

has exponential tail with parameters $\left((2n)^n, \frac{1}{n}\left(\frac{1}{2n}\right)^n\right)$.

Proof. Let $k = \arg\min_{1\le j\le i} V_j(0)$.

(i) If $k = i$ then $V_j(0) \ge V_i(0)$ for all $1 \le j < i$ and $i \ne n$ by our assumption. By Lemma 1.5.4 we obtain that σ has exponential tail with parameters $\left(2(2n)^{n-i-1}, (\frac{1}{2n})^{n-i}\right)$. Note that $(\frac{1}{2n})^{n-i} \ge \frac{1}{n}\left(\frac{1}{2n}\right)^n$ and $2(2n)^{n-i-1} \le (2n)^n$, so σ has exponential tail with parameters $\left((2n)^n, \frac{1}{n}\left(\frac{1}{2n}\right)^n\right)$.

(ii) If $k < i$, then let $\sigma' = \inf\{t > 0 : f(k,t) = i\}$. Note that $f(k,\sigma') = i$ which implies $f(i,\sigma') \ne i$, hence $\sigma \le \sigma'$. It suffices to prove that σ' has exponential tail with parameters $\left((2n)^n, \frac{1}{n}\left(\frac{1}{2n}\right)^n\right)$.

The velocity $V_k(0)$ is the smallest among the first k velocities, so $f(k,t)$ can only increase before hitting n. Hence,

$$\sigma' = \sum_{j=k}^{i-1} \left(\sup\{t>0 : f(k,t)=j\} - \inf\{t>0 : f(k,t)=j\}\right).$$

Let $\theta_j = \inf\{t>0 : f(k,t)=j\}$, $j=k,\dots,i-1$. We claim that for $j=k,\dots,i-1$,

$$V_k(\theta_j) \le X_m(\theta_j), \quad \forall 1 \le m < j. \tag{1.5.3}$$ blockabove

We will prove this by induction.

If $j=k$ (hence $\theta_k=0$) then (1.5.3) follows from the fact that $k = \arg\min_{1\le m<i} V_m(0)$.

Now suppose (1.5.3) holds for j. We will prove it also holds for $j+1$.

Since $f(k,t)$ jumped from j to $j+1$ at the time $t=\theta_{j+1}$, we must have had $V_k(\theta_{j+1}-) < X_{j+1}(\theta_{j+1}-)$. By the induction assumption, for $1\le m<j$,

$$V_k(\theta_{j+1}) = V_k(\theta_j) - g(\theta_{j+1}-\theta_j) \le X_m(\theta_j) - g(\theta_{j+1}-\theta_j) = X_m(\theta_{j+1}).$$

Hence, (1.5.3) holds for $j+1$.

It follows from (1.5.3) that the main assumption of Lemma 1.5.4 is satisfied. The lemma and the strong Markov property applied at stopping times θ_j show that

$$\sup\{t>0 : f(k,t)=j\} - \inf\{t>0 : f(k,t)=j\}$$

has exponential tail with parameters $\left(2(2n)^{n-j-1}, (\frac{1}{2n})^{n-j}\right)$ for every $k\le j\le i-1$. Then by Lemma 1.5.2, σ' has exponential tail with parameters $\left(2(i-k)(2n)^{n-k-1}, \frac{1}{i-k}\left(\frac{1}{2n}\right)^{n-k}\right)$. Note that $\frac{1}{i-k}\left(\frac{1}{2n}\right)^{n-k} \ge \frac{1}{n}\left(\frac{1}{2n}\right)^{n}$ and $2(i-k)(2n)^{n-k-1} \le (2n)^n$, so σ' has exponential tail with parameters $\left((2n)^n, \frac{1}{n}\left(\frac{1}{2n}\right)^n\right)$.

□

exptail.thm **Theorem 1.5.6.** *Assume $V_i(0)=0$ (1.5.1). The time $\mathcal{S} := \inf\{t>0 : V_i(t)=0\}$ satisfies for every $t>0$*

$$\exp\left(-\frac{1}{2n}t\right) \le \mathbb{P}(\mathcal{S}>t) \le 8n^2(n+1)^2(2n)^n \exp\left(-\frac{1}{8n^2(n+1)}\left(\frac{1}{2n}\right)^n\right).$$

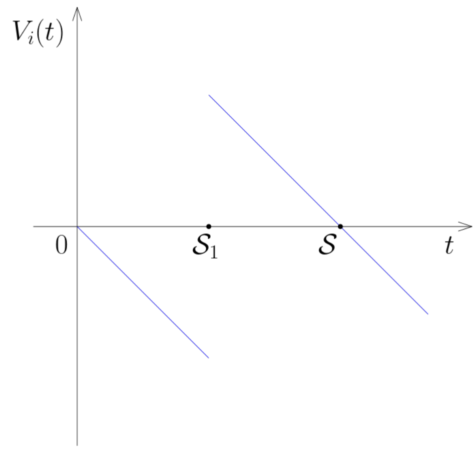

Figure 1.4: Graph of $V_i(t)$ illustrating the relationship between $\mathcal{S}_1$ and $\mathcal{S}$.

graph.s

Proof. Denote $\mathcal{S}_1 = \inf\{t > 0 : V_i(t) > 0\}$, i.e. $\mathcal{S}_1$ is the first time when V_i change to positive, then $\mathcal{S} = 2\mathcal{S}_1$.

(I): Let $K = \sum_{j=1}^{i-1} \mathbb{1}_{\{X_j(0)<0\}}$. We will show that $f(i,t)$ can only jump up at most K times before time $\mathcal{S}_1$. Denote $\Psi(t) = \sum_{j=1}^{f(i,t)-1} \mathbb{1}_{\{X_j(t)<V_i(t)\}}$. Then $\Psi(0) = K$.

For $j \geq 1$, let α_j to be the j-th time before $\mathcal{S}$ when $f(i,t)$ jump up, i.e.

$$\alpha_1 = \inf\{0 < t \leq \mathcal{S}_1 : f(i,t) = f(i,t-) - 1\},$$

$$\alpha_j = \inf\{\alpha_{j-1} < t \leq \mathcal{S}_1 : f(i,t) = f(i,t-) - 1\},\ j > 1.$$

Note that $V_i(t) < 0$ for $t < \mathcal{S}_1$, so there is no collision of V_i with the bottom before time $\mathcal{S}_1$. We claim for any $\alpha_{j-1} \leq t_1 < t_2 < \alpha_j$

$$\Psi(t_1) = \Psi(t_2). \tag{1.5.4}$$ 8.13

Recall that $f(i,t)$ can not decrease when $t \in (\alpha_{j-1}, \alpha_j)$, it suffice to prove (1.5.4) when:

(i) $f(i,t_1) = f(i,t_2)$.

(ii) $f(i,t_2) = f(i,t_1) + 1$.

(iii) $f(i,t_2) = f(i,t_1) + k$ for some $k > 1$.

The proof of (i): In this case each velocity above V_i decreases at a constant rate, hence $\Psi(t_1) = \Psi(t_2)$.

The proof of (ii): Let $\beta = \inf\{t > t_1 : f(i,t) = f(i,t_1) + 1\}$. Since $f(i,t_1) = f(i,t)$ for each $t_1 < t < \beta$ and $f(i,\beta) = f(i,t_2)$, by part (i) we have $\Psi(t_1) = \Psi(\beta-)$ and $\Psi(\beta) = \Psi(t_2)$. Hence we only need to show $\Psi(\beta-) = \Psi(\beta)$. At time β, V_i swapped with the velocity below, so

$$V_i(\beta-) < X_{f(i,\beta-)+1}(\beta-), \;\; f(i,\beta) = f(i,\beta-) + 1, \;\; X_{f(i,\beta)-1}(\beta) = X_{f(i,\beta-)+1}(\beta-).$$

Then

$$\begin{aligned}
\Psi(\beta-) &= \sum_{j=1}^{f(i,\beta-)-1} \mathbb{1}_{\{X_j(\beta-)<V_i(\beta-)\}} \\
&= \sum_{j=1}^{f(i,\beta-)-1} \mathbb{1}_{\{X_j(\beta-)<V_i(\beta-)\}} + \mathbb{1}_{\{X_{f(i,\beta-)+1}(\beta-)<V_i(\beta-)\}} \\
&= \sum_{j=1}^{f(i,\beta-)-1} \mathbb{1}_{\{X_j(\beta)<V_i(\beta)\}} + \mathbb{1}_{\{X_{f(i,\beta-)}(\beta)<V_i(\beta)\}} \\
&= \sum_{j=1}^{f(i,\beta-)} \mathbb{1}_{\{X_j(\beta)<V_i(\beta)\}} \\
&= \sum_{j=1}^{f(i,\beta)-1} \mathbb{1}_{\{X_j(\beta)<V_i(\beta)\}} \\
&= \Psi(\beta).
\end{aligned}$$

The proof of (iii): We can take $t_1 < s_1 < s_2 < \cdots < s_{k-1} < t_2$ such that

$$f(i,s_1) = f(i,t_1) + 1, \;\; f(i,s_2) = f(i,t_1) + 2, \; \ldots, f(i,s_{k-1}) = f(i,t_1) + k - 1.$$

Then by part(ii), we have

$$\Psi(t_1) = \Psi(s_1) = \Psi(s_2) = \cdots = \Psi(s_{k-1}) = \Psi(t_2).$$

Next, we will show for each $\alpha_{j-1} < t_1 < \alpha_j < t_2 < \alpha_{j+1}$, we have $\Psi(t_2) = \Psi(t_1) - 1$.

By (1.5.4) we have $\Psi(t_2) = \Psi(\alpha_j)$ and $\Psi(t_1) = \Psi(\alpha_j-)$. At time α_j, V_i swapped with the velocity above, so

$$V_i(\alpha_j-) > X_{f(i,\alpha_j-)-1}(\alpha_j-), \;\; f(i,\alpha_j) = f(i,\alpha_j-) - 1, \;\; X_{f(i,\alpha_j)+1}(\alpha_j) = X_{f(i,\alpha_j-)-1}(\alpha_j-).$$

Then

$$
\begin{aligned}
\Psi(\alpha_j-)-1 &= \sum_{j=1}^{f(i,\alpha_j-)-1} \mathbb{1}_{\{X_j(\alpha_j-)<V_i(\alpha_j-)\}} - \mathbb{1}_{\{X_{f(i,\alpha_j-)-1}(\alpha_j-)<V_i(\alpha_j-)\}} \\
&= \sum_{j=1}^{f(i,\alpha_j-)-2} \mathbb{1}_{\{X_j(\alpha_j-)<V_i(\alpha_j-)\}} \\
&= \sum_{j=1}^{f(i,\alpha_j)-1} \mathbb{1}_{\{X_j(\alpha_j)<V_i(\alpha_j)\}} \\
&= \Psi(\alpha_j).
\end{aligned}
$$

Recall that $\Psi(0)=K$, so the sequence $\alpha_1,\alpha_2,\ldots$ has the length at most K, i.e. $f(i,t)$ can only jump up at most K times before time $\mathcal{S}_1$.

(**II**): Let $W_0=0$,

$$W_j=\inf\{t\geq R_{j-1}: f(i,t)=n\}, j\geq 1$$

and $N=|\{t<\mathcal{S}_1: f(i,t)=n, f(i,t-)\neq n\}|$ then N is bounded by $K+1$ because there can be at most K positive jumps of $f(i,t)$ before $\mathcal{S}_1$.

We claim that W_j-W_{j-1} has exponential tail with parameters $4K^2(2n)^n$ and $-\frac{1}{2Kn}\left(\frac{1}{2n}\right)^n$ for each $1\leq j\leq N$.

Let $R_0^j=W_{j-1}$ and

$$R_k^{(j)}=\inf\{t\geq R_{k-1}^{(j)}: f(i,t)\neq f(i,t-)\}, k\geq 1.$$

We have

$$W_j-W_{j-1}=\sum_{k=1}^{N_j}\left(R_k^{(j)}-R_{k-1}^{(j)}\right)$$

for some random variable N_j bounded by $2K$ since there can be at most K positive jumps of $f(i,t)$ before $\mathcal{S}_1$. Fix some $1\leq l\leq 2K$, when $N_j=l$, by the Lemma 1.5.5 and the strong Markov property applied at $R_0^{(j)},R_1^{(j)},\cdots,R_{l-1}^{(j)}$, each $R_k^j-R_{k-1}^{(j)}, 1\leq k\leq l$ has exponential tail with parameters $\left((2n)^n, \frac{1}{n}\left(\frac{1}{2n}\right)^n\right)$. Then by Lemma 1.5.2, W_j-W_{j-1} has exponential tail with parameters $\left(l(2n)^n, \frac{1}{nl}\left(\frac{1}{2n}\right)^n\right)$. Hence, for $t > 0$

$$\begin{aligned}
\mathbb{P}\left(W_j - W_{j-1} > t\right) &= \sum_{l=1}^{2K} \mathbb{P}\left(W_j - W_{j-1} > t, N_j = l\right) \\
&< \sum_{l=1}^{2K} l(2n)^n \exp\left(-\frac{1}{nl}\left(\frac{1}{2n}\right)^n t\right) \\
&< 4K^2(2n)^n \exp\left(-\frac{1}{2Kn}\left(\frac{1}{2n}\right)^n t\right).
\end{aligned}$$

Apply Lemma 1.5.2 again we can see $\sum_{j=1}^{N} W_j - W_{j-1}$ has exponential tail with parameters $4K^2(K+1)(2n)^n$ and $-\frac{1}{2K(K+1)n}\left(\frac{1}{2n}\right)^n$.

Fix some $1 \le l \le K+1$, when $N = l$ we have

$$\mathcal{S}_1 = \sum_{j=1}^{N} (W_j - W_{j-1}) + T_{k_l}, \text{where} k_l = \inf\{k : T_k > W_l \ \text{ and } \ U_k = n\}$$

By strong Markov property applied at W_l, the term T_{k_l} has exponential distribution with mean n. Apply Lemma 1.5.2, in this case $\mathcal{S}_1$ has exponential tail with parameters $4K^2(K+1)(2n)^n + 1$ and $-\frac{1}{4K(K+1)n}\left(\frac{1}{2n}\right)^n$. Hence for $t > 0$,

$$\begin{aligned}
\mathbb{P}\left(\mathcal{S}_1 > t\right) &= \sum_{l=1}^{K+1} \mathbb{P}\left(\mathcal{S}_1 > t, N = l\right) \\
&< \sum_{l=1}^{K+1} \left(4K^2(K+1)(2n)^n + 1\right) \exp\left(-\frac{1}{4K(K+1)n}\left(\frac{1}{2n}\right)^n\right) \\
&< 8n^2(n+1)^2(2n)^n \exp\left(-\frac{1}{4n^2(n+1)}\left(\frac{1}{2n}\right)^n\right)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\mathbb{P}\left(\mathcal{S}_1 > t\right) &= \sum_{l=1}^{K+1} \mathbb{P}\left(\mathcal{S}_1 > t, N = l\right) \\
&\ge \sum_{l=1}^{K+1} \mathbb{P}\left(T_{k_l} > t, N = l\right) \\
&= \sum_{l=1}^{K+1} \exp\left(-\frac{1}{n} t\right) \\
&\ge \exp\left(-\frac{1}{n} t\right)
\end{aligned}$$

Therefore,

$$\mathbb{P}\left(\mathcal{S} > t\right) = \mathbb{P}\left(\mathcal{S}_1 > \frac{1}{2}t\right) < 8n^2(n+1)^2(2n)^n \exp\left(-\frac{1}{8n^2(n+1)}\left(\frac{1}{2n}\right)^n\right),$$

$$\mathbb{P}\left(\mathcal{S} > t\right) = \mathbb{P}\left(\mathcal{S}_1 > \frac{1}{2}t\right) > \exp\left(-\frac{1}{2n}t\right).$$

□

Let S be a Hausdorff space endowed with a closed partial ordering $\preceq$ and denote by $\mathcal{B}^{\uparrow}(S)$ the family of increasing Borel sets, that is $B \in \mathcal{B}^{\uparrow}(S)$ iff $s \in B$ and $s \preceq t$ together imply $t \in B$. We shall write $\mu \preceq \nu$ iff $\mu(B) \leq \nu(B)$ for all $B \in \mathcal{B}^{\uparrow}(S)$. The next lemma comes from [7] but we state it with our own notation.

skala93 **Lemma 1.5.7.** *[7, Cor. 7] Let μ, ν be two probability measures on S; then $\mu \preceq \nu$ implies the existence of a measure λ on $S \times S$ with support in $F = \{(s,t) \in S \times S : s \preceq t\}$ such that μ and ν are the marginals of λ.*

outtime.one **Lemma 1.5.8.** *Fix $1 \leq i \leq n$, for every $\varepsilon > 0$, there exists $K > 0$ such that*

$$\limsup_{t\to\infty} \frac{\int_0^t \mathbb{1}_{\{|V_i(u)|>K\}}\, du}{t} < \varepsilon, \quad a.s.$$

Proof. Define $\{S_i\}_{i=0}^{\infty}$ as follows:

$$S_0 = \inf\{t : V_i(t) = 0\}, \quad S_i = \inf\left\{t > \sum_{j=0}^{i-1} S_j : V_i(t) = 0\right\} - \sum_{j=0}^{i-1} S_j \ \text{ for } i \geq 1$$

By the strong Markov property and Theorem 1.5.6 for all $i \geq 1$ we have

$$a_1 \exp(-c_1 t) \leq P(S_i > t) \leq a_2 \exp(-c_2 t), \quad \forall t \geq 0$$

where a_1, a_2, c_1, c_2 are given by Theorem 1.5.6.

Let $b_2 = \frac{\ln(a_2)}{c_2}$, and $Z^{(1)}$ be a shifted exponential random variable with pdf:

$$c_2 \exp\left(-c_2(t-b_2)\right) I_{\{t \geq b_2\}}.$$

Let $Z^{(2)}$ have the exponential distribution with mean $2n$. Then for $t > 0$ we have

$$\mathbb{P}(Z^{(2)} > t) = a_1 \exp(-c_1 t).$$

The space $\mathbb{R}^\infty$ endowed with the distance $d\Big(\{a_j\}_{j=1}^\infty, \{b_j\}_{j=1}^\infty\Big) := \max_j |a_j - b_j|$ is a Hausdorff space. Define a partial order $\preceq$ by

$$\{a_j\}_{j=1}^\infty \preceq \{b_j\}_{j=1}^\infty \Leftrightarrow a_j \leq b_j \ \ \forall j \geq 1$$

Now we claim that there exist sequences $\{S_j^{(1)}\}_{j=1}^\infty$, $\{S_j^{(2)}\}_{j=1}^\infty$, $\{Z_j^{(1)}\}_{j=1}^\infty$ and $\{Z_j^{(2)}\}_{j=1}^\infty$ satisfying:

(1) $\{S_j^{(1)}\}_{j=1}^\infty \stackrel{d}{=} \{S_j^{(2)}\}_{j=1}^\infty \stackrel{d}{=} \{S_j\}_{j=1}^\infty$.

(2) The sequence $\{Z_j^{(1)}\}_{j=1}^\infty$ is a sequence of iid random variables with the same distribution as $Z^{(1)}$. The sequence $\{Z_j^{(2)}\}_{j=1}^\infty$ is a sequence of iid random variables with the same distribution as $Z^{(2)}$.

(3) For every $j \geq 1$, $S_j^{(1)} \leq Z_j^{(1)}$ *a.s* and $S_j^{(2)} \geq Z_j^{(2)}$ *a.s.*

By Lemma 1.5.7, we only need to show:

(I): The set $F = \{(\mathbf{a}, \mathbf{b}) \in \mathbb{R}^\infty \times \mathbb{R}^\infty : \mathbf{a} \preceq \mathbf{b}\}$ is closed in the product topology on $\mathbb{R}^\infty \times \mathbb{R}^\infty$

(II) For any sequence $\{t_j\}_{j=1}^\infty$, we have

$$\mathbb{P}(S_j > t_j, \forall j \geq 1) \leq \mathbb{P}(Z_j^{(1)} > t_j, \forall j \geq 1).$$

(III) For any sequence $\{t_j\}_{j=1}^\infty$, we have

$$\mathbb{P}(S_j > t_j, \forall j \geq 1) \geq \mathbb{P}(Z_j^{(2)} > t_j, \forall j \geq 1).$$

The proof of (I): Suppose a sequence $\{(\mathbf{a}^{(n)}, \mathbf{b}^{(n)})\}$ in F converges to $(\mathbf{a}, \mathbf{b})$ in product topology. Then we have $\lim_{n\to\infty} d(\mathbf{a}^{(n)}, \mathbf{a}) = \lim_{n\to\infty} d(\mathbf{b}^{(n)}, \mathbf{b}) = 0$. Suppose

$$\mathbf{a}^{(n)} = \{a_j^{(n)}\}_{j=1}^\infty, \ \ \mathbf{b}^{(n)} = \{b_j^{(n)}\}_{j=1}^\infty, \ \ \mathbf{a} = \{a_j\}_{j=1}^\infty, \ \ \mathbf{b} = \{b_j\}_{j=1}^\infty$$

For fixed $j \geq 1$, we have

$$|a_j^{(n)} - a_j| \leq d(\mathbf{a}^{(n)}, \mathbf{a}) \to 0, \ \ |b_j^{(n)} - b_j| \leq d(\mathbf{b}^{(n)}, \mathbf{b}) \to 0$$

Hence,

$$a_j = \lim_{n\to\infty} a_j^{(n)} \leq \lim_{n\to\infty} b_j^{(n)} = b_j$$

Since this inequality holds for every $j \geq 1$, we have $\mathbf{a} \preceq \mathbf{b}$ which implies $(\mathbf{a}, \mathbf{b}) \in F$.

The proof of (II): First we will show that $\mathbb{P}\left(S_j > t_j | \sigma(S_j)\right) \leq \mathbb{P}(Z_j^{(1)} > t_j),\ \forall j \geq 1$.
If $t_j \leq b_2$, then $\mathbb{P}(Z_j^{(1)} > t_j) = 1 \geq \mathbb{P}\left(S_j > t_j | \sigma(S_j)\right) \quad a.s.$
If $t_j > b_2$, then

$$\mathbb{P}(Z_j^{(1)} > t_j) = \exp\left(-c_2(t_j - b_2)\right) = \exp(c_2 b_2)\exp(-c_2 t_j) = a_2 \exp(-c_2 t_j).$$

Note our upper bound for S_j is an uniform bound not depend on any information on the past, we have

$$\mathbb{P}\left(S_j > t_j | \sigma(S_j)\right) \leq a_2 \exp(-c_2 t_j) \quad a.s.$$

Combine two cases, we can see that $\mathbb{P}\left(S_j > t_j | \sigma(S_j)\right) \leq \mathbb{P}(Z_j^{(1)} > t_j),\ \forall j \geq 1$. Hence,

$$\begin{aligned}
\mathbb{P}(S_j > t_j, \forall j \geq 1) &= \prod_{j=1}^{\infty} \mathbb{P}\left(S_j > t_j \middle| S_k > t_k, \forall 1 \leq k < j\right) \\
&= \prod_{j=1}^{\infty} \mathbb{P}\left(\mathbb{P}\left(S_j > t_j | \sigma(S_j)\right) \middle| S_k > t_k, \forall 1 \leq k < j\right) \\
&\leq \prod_{j=1}^{\infty} \mathbb{P}\left(\mathbb{P}\left(Z_j^{(1)} > t_j\right) \middle| S_k > t_k, \forall 1 \leq k < j\right) \\
&= \prod_{j=1}^{\infty} \mathbb{P}\left(Z_j^{(1)} > t_j\right) \\
&= \mathbb{P}\left(Z_j^{(1)} > t_j,\ \forall j \geq 1\right)
\end{aligned}$$

The proof of (III): Our argument follows the same reason as in the proof of (II). Since our lower bound for S_j is an uniform bound,

$$\mathbb{P}\left(S_j > t_j | \sigma(S_j)\right) \geq a_1 \exp(-c_1 t_j) = P(Z^{(2)} > t_j) \ \ a.s.$$

Then follow the same way in the proof of (II),

$$\begin{aligned}\mathbb{P}(S_j > t_j, \forall j \geq 1) &= \prod_{j=1}^{\infty} \mathbb{P}\left(S_j > t_j \middle| S_k > t_k, \forall 1 \leq k < j\right) \\ &= \prod_{j=1}^{\infty} \mathbb{P}\left(\mathbb{P}\left(S_j > t_j | \sigma(S_j)\right) \middle| S_k > t_k, \forall 1 \leq k < j\right) \\ &\geq \prod_{j=1}^{\infty} \mathbb{P}\left(\mathbb{P}\left(Z_j^{(2)} > t_j\right) \middle| S_k > t_k, \forall 1 \leq k < j\right) \\ &= \prod_{j=1}^{\infty} \mathbb{P}\left(Z_j^{(2)} > t_j\right) \\ &= \mathbb{P}\left(Z_j^{(2)} > t_j,\ \forall j \geq 1\right)\end{aligned}$$

Since the random variable $Z^{(1)}$ has finite expectation, there exists $K > 0$ such that $E(Z\mathbb{1}_{\{Z>2K\}}) < \frac{n}{4}\varepsilon$. Let $Z^* = \max\{0, Z - 2K\}$ and in the same way let $S_j^* = \max\{0, S_j - 2K\}$, $S_j^{(1)*} = \max\{0, S_j^{(1)} - 2K\}$, $Z_j^{(1)*} = \max\{0, Z_j^{(1)} - 2K\}$ for $j \geq 1$.

We claim

$$\mathbb{P}\left(\limsup_{t\to\infty} \frac{\int_0^t \mathbb{1}_{\{|V_i(t)|>gK\}}\, dt}{t} < \varepsilon\right) = 1$$

For $\sum_{j=0}^{m} S_j \leq t < \sum_{j=0}^{m+1} S_j$ we have

$$\begin{aligned}\frac{\int_0^t \mathbb{1}_{\{|V_i(u)|>gK\}}\, du}{t} &\leq \frac{\int_0^{\sum_{j=0}^{m+1} S_j} \mathbb{1}_{\{|V_i(u)|>gK\}}\, du}{\sum_{j=0}^{m} S_j} \\ &\leq \frac{S_0 + \sum_{j=1}^{m+1} S_j^*}{S_0 + \sum_{j=1}^{m} S_j} \\ &= \frac{\frac{1}{m} S_0 + \frac{1}{m} \sum_{j=1}^{m+1} S_j^*}{\frac{1}{m} S_0 + \frac{1}{m} \sum_{j=1}^{m} S_j}\end{aligned}$$

Hence,

$$\mathbb{P}\left(\limsup_{t\to\infty}\frac{\int_0^t \mathbb{1}_{\{|V_i(t)|>gK\}}\,dt}{t}<\varepsilon\right)\geq \mathbb{P}\left(\limsup_{m\to\infty}\frac{\frac{1}{m}S_0+\frac{1}{m}\sum_{j=1}^{m+1}S_j^*}{\frac{1}{m}S_0+\frac{1}{m}\sum_{j=1}^{m}S_j}<\varepsilon\right)$$

$$=\mathbb{P}\left(\bigcup_{M=1}^{\infty}\bigcap_{m=M}^{\infty}\left\{\frac{\frac{1}{m}S_0+\frac{1}{m}\sum_{j=1}^{m+1}S_j^*}{\frac{1}{m}S_0+\frac{1}{m}\sum_{j=1}^{m}S_j}<\varepsilon\right\}\right)$$

$$=\lim_{M\to\infty}\mathbb{P}\left(\bigcap_{m=M}^{\infty}\left\{\frac{\frac{1}{m}S_0+\frac{1}{m}\sum_{j=1}^{m+1}S_j^*}{\frac{1}{m}S_0+\frac{1}{m}\sum_{j=1}^{m}S_j}<\varepsilon\right\}\right).$$

For each $M\geq 1$ we have

$$\mathbb{P}\left(\bigcap_{m=M}^{\infty}\left\{\frac{1}{m}S_0+\frac{1}{m}\sum_{j=1}^{m+1}S_j^*<\frac{\varepsilon}{m}S_0+\frac{\varepsilon}{m}\sum_{j=1}^{m}S_j\right\}\right)$$

$$\geq\mathbb{P}\left(\left\{\frac{1-\varepsilon}{M}S_0<\frac{n}{2}\varepsilon\right\}\cap\left\{\frac{1}{m}\sum_{j=1}^{m+1}S_j^*<\frac{n}{2}\varepsilon,\ \forall m\geq M\right\}\cap\left\{\frac{1}{m}\sum_{j=1}^{m}S_j>n,\ \forall m\geq M\right\}\right)$$

$$\geq 1-\mathbb{P}\left(S_0\geq\frac{n\varepsilon}{2(1-\varepsilon)}M\right)-\mathbb{P}\left(\sup_{m\geq M}\frac{1}{m}\sum_{j=1}^{m+1}S_j^*\geq\frac{n}{2}\varepsilon\right)-\mathbb{P}\left(\sup_{m\geq M}\frac{1}{m}\sum_{j=1}^{m}S_j\leq n\right)$$

$$=1-\mathbb{P}\left(S_0\geq\frac{n\varepsilon}{2(1-\varepsilon)}M\right)-\mathbb{P}\left(\sup_{m\geq M}\frac{1}{m}\sum_{j=1}^{m+1}S_j^{(1)*}\geq\frac{n}{2}\varepsilon\right)-\mathbb{P}\left(\sup_{m\geq M}\frac{1}{m}\sum_{j=1}^{m}S_j^{(2)}\leq n\right)$$

$$\geq 1-\mathbb{P}\left(S_0\geq\frac{n\varepsilon}{2(1-\varepsilon)}M\right)-\mathbb{P}\left(\sup_{m\geq M}\frac{1}{m}\sum_{j=1}^{m+1}Z_j^{(1)*}\geq\frac{n}{2}\varepsilon\right)-\mathbb{P}\left(\sup_{m\geq M}\frac{1}{m}\sum_{j=1}^{m}Z_j^{(2)}\leq n\right).$$

For the second term, we have $\lim_{M\to\infty}\mathbb{P}\left(S_0\geq\frac{n\varepsilon}{2(1-\varepsilon)}M\right)=0$ since S_0 is finite almost surely.

For the third term, note that by the strong law of large numbers

$$\frac{1}{m}\sum_{j=1}^{m+1}Z_j^{(1)*}\overset{a.s}{\to}\mathbb{E}Z^*=\mathbb{E}(Z\mathbb{1}_{\{Z>2K\}})<\frac{n}{4}\varepsilon.$$

Hence,

$$\lim_{M\to\infty}\mathbb{P}\left(\sup_{m\geq M}\frac{1}{m}\sum_{j=1}^{m+1}Z_j^{(1)*}\geq\frac{n}{2}\varepsilon\right)=\mathbb{P}\left(\bigcap_{M=1}^{\infty}\bigcup_{m=M}^{\infty}\left\{\frac{1}{m}\sum_{j=1}^{m+1}Z_j^{(1)*}\geq\frac{n}{2}\varepsilon\right\}\right)=0.$$

For the forth term, in the same way by the strong law of large numbers we have

$$\frac{1}{m}\sum_{j=1}^{m+1} Z_j^{(2)} \xrightarrow{a.s} \mathbb{E}\left(Z^{(2)}\right) = 2n.$$

Hence,

$$\lim_{M\to\infty} \mathbb{P}\left(\sup_{m\geq M}\frac{1}{m}\sum_{j=1}^{m+1} Z_j^{(2)} \leq n\right) = \mathbb{P}\left(\bigcap_{M=1}^{\infty}\bigcup_{m=M}^{\infty}\left\{\frac{1}{m}\sum_{j=1}^{m+1} Z_j^{(2)} \leq n\right\}\right) = 0.$$

Therefore, we proved

$$\mathbb{P}\left(\limsup_{t\to\infty}\frac{\int_0^t \mathbb{1}_{\{|V_i(u)|>gK\}}\,du}{t} < \varepsilon\right) = 1.$$

□

The following definition comes from [6]

boundpro1 **Definition 1.5.9.** A process Φ is called bounded in probability on average if for each initial condition $x \in X$ and each $\varepsilon > 0$, there exists a compact set C such that

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{P}_x\left(\Phi_u \in C\right)\,du \geq 1-\varepsilon.$$

boundpro **Theorem 1.5.10.** *For every $\varepsilon > 0$, there exists a bounded set C such that*

$$\limsup_{t\to\infty}\frac{\int_0^t \mathbb{1}_{\{\mathbf{X}(u)\notin C\}}\,du}{t} < \varepsilon, \quad a.s,$$

and hence $\mathbf{X}$ is bounded in probability on average.

Proof. By Lemma 1.5.8, there exists $K > 0$ such that for all $1 \leq i \leq n$

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{1}_{\{|V_i(u)|>K\}}\,du < \frac{\varepsilon}{n}, \quad a.s.$$

Take $C = [-K, K]^n$. This set is compact and

$$\begin{aligned}\frac{1}{t}\int_0^t \mathbb{1}_{\{\mathbf{X}(u)\notin C\}}\,du &= \frac{1}{t}\int_0^t \mathbb{1}_{\{\exists\, i:|X_i(u)|>K\}}\,du \\ &= \frac{1}{t}\int_0^t \mathbb{1}_{\{\exists\, i:|V_i(u)|>K\}}\,du \\ &\leq \sum_{i=1}^{n}\frac{1}{t}\int_0^t \mathbb{1}_{\{|V_i(u)|>K\}}\,du\end{aligned}$$

Take the lim sup on the both side to get

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{1}_{\{\mathbf{X}(u)\notin C\}}\,du &\le \limsup_{t\to\infty}\sum_{i=1}^n\frac{1}{t}\int_0^t \mathbb{1}_{\{|V_i(u)|>K\}}\,du\\ &\le \sum_{i=1}^n\limsup_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{1}_{\{|V_i(u)|>K\}}\,du\\ &< \varepsilon \quad a.s.\end{aligned}$$

Hence, by Fubini's Theorem and Fatou's Lemma

$$\begin{aligned}&\liminf_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{P}\left(\mathbf{X}(u)\in C\right)\,du\\ =&\liminf_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{E}\left(\mathbb{1}_{\{\mathbf{X}(u)\in C\}}\right)\,du\\ =&\liminf_{t\to\infty}\mathbb{E}\left(\frac{1}{t}\int_0^t \mathbb{1}_{\{\mathbf{X}(u)\in C\}}\,du\right)\\ \ge&\mathbb{E}\left(\liminf_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{1}_{\{\mathbf{X}(u)\in C\}}\,du\right)\\ =&\mathbb{E}\left(1-\limsup_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{1}_{\{\mathbf{X}(u)\notin C\}}\,du\right)\\ >&1-\varepsilon.\end{aligned}$$

□

feller **Lemma 1.5.11.** *The process* $\mathbf{X}(t)$ *has Feller property, i.e. for every* $t>0$, $P^t(f)$ *takes continuous and bounded function to continuous function.*

Proof. Take $\mathbf{x}=(x_1,\dots,x_n)\in\mathbb{R}^n$, denote

$$E=\left[-\max_i|x_i|-gt-1,\max_i|x_i|+gt+1\right]^n.$$

The number of jump times in $[0,t]$ has Poisson distribution with expectation t, so there exists N such that $P(T_N\le t)<\varepsilon$. Since f is a continuous function and E is a compact set, f is uniformly continuous on E. For any $\varepsilon>0$, there exists $0<\delta<1$ such that

$$|f(\mathbf{y}_1)-f(\mathbf{y}_2)|<\varepsilon,\;\text{ for all } \|\mathbf{y}_1-\mathbf{y}_2\|_n<\delta+(gN+1)\sqrt{n}\delta \text{ and } y_1,y_2\in E$$

If $M=\sup_{\mathbf{y}\in\mathbb{R}^n}f(\mathbf{y})$, then $M<\infty$ since the function f is bounded.

If $\mathbf{X}(t)$ is starting from $\mathbf{x}$, then

$$\mathbf{X}(t) \in \left[-\max_i |x_i| - gt, \max_i |x_i| + gt\right]^n \subset E \quad a.s.$$

For arbitrary $\mathbf{x}' = (x'_1, \cdots, x'_n) \in \mathbb{R}^n$ with $\|\mathbf{x} - \mathbf{x}'\|_n < \delta$, let $\mathbf{X}'(t)$ be the process starting from $\mathbf{x}'$. In the same way we have

$$\mathbf{X}'(t) \in \left[-\max_i |x'_i| - gt, \max_i |x'_i| + gt\right] \subset E \quad a.s.$$

We have

$$\begin{aligned}
|P^t f(\mathbf{x}) - P^t f(\mathbf{x}')| &= \left|\mathbb{E} f\left(\mathbf{X}(t)\right) - \mathbb{E} f\left(\mathbf{X}'(t)\right)\right| \\
&\le \mathbb{E}\left|f\left(\mathbf{X}(t)\right) - f\left(\mathbf{X}'(t)\right)\right| \\
&= \int_{\{T_N \le t\}} \left|f\left(\mathbf{X}(t)\right) - f\left(\mathbf{X}'(t)\right)\right| d\mathbb{P} + \int_{\{T_N > t\}} \left|f\left(\mathbf{X}(t)\right) - f\left(\mathbf{X}'(t)\right)\right| d\mathbb{P}.
\end{aligned}$$

For the first part,

$$\int_{\{T_N \le t\}} \left|f\left(\mathbf{X}(t)\right) - f\left(\mathbf{X}'(t)\right)\right| dP \le 2MP(T_N \le t) < 2M\varepsilon.$$

For the second part, since there are at most $N-1$ jumps during $[0, t]$, by lemma 1.4.5 we have

$$\|\mathbf{X}(t) - \mathbf{X}'(t)\|_n \le \delta + g(N-1)\sqrt{n}\delta, \quad a.s.$$

Hence,

$$\int_{\{T_N > t\}} \left|f\left(\mathbf{X}(t)\right) - f\left(\mathbf{X}'(t)\right)\right| dP \le \int_{\{T_N > t\}} \varepsilon \, dP \le \varepsilon.$$

Therefore, $|P^t f(\mathbf{x}) - P^t f(\mathbf{x}')| \le (2M+1)\varepsilon$ which implies the continuity of $P^t f$. □

strongfeller **Remark 1.5.12.** The strong Feller property does not hold for $\mathbf{X}$. Here is a counterexample: Fix $t > 0$, let $E = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : \max_i x_i \le -gt\}$, $f = \mathbb{1}_E$ and $\tau = T_{k_1}$ where $k_1 = \inf\{k : U_k = n\}$. Then τ has exponential distribution with mean n and

$$P^t f(\mathbf{0}) = \mathbb{E}_{\mathbf{0}}\left(f(\mathbf{X}_t)\right) = \mathbb{P}_0(\mathbf{X}_t \in E) \ge \mathbb{P}(\tau > t) > 0.$$

However, for every $\varepsilon > 0$ we have

$$P^t f\left((\varepsilon, \cdots, \varepsilon)\right) = \mathbb{P}(\mathbf{X}_t \in E) = 0.$$

Hence, $P^t f$ is not a continuous function.

existinvariant **Theorem 1.5.13.** *An invariant probability measure exists for the process* $\mathbf{X}$.

Proof. The following result is in the Theorem 3.1 in [6]: If a process has the weak Feller property and is bounded in probability on average, then an invariant probability measure exists. Hence, our theorem holds. □

onedim **Example 1.5.14.** Consider the case when we only have one ball. The generator for the process is

$$\mathcal{A}f(u) = -gf'(u) + \mathbb{1}_{\{u<0\}}\left(f(-u) - f(u)\right).$$

The adjoint of $\mathcal{A}$ is given by

$$\mathcal{A}^*h(u) = gh'(u) + \mathbb{1}_{\{u>0\}}h(-u) - \mathbb{1}_{\{u<0\}}h(u).$$

In our notation Proposition 9.2 in [4] tells us $h(u)du$ is invariant for the process $\mathbf{X}$ if and only if $\mathcal{A}^*h(u) = 0$.

For $u < 0$,

$$\begin{aligned} &\mathcal{A}^*h(u) = gh'(u) - h(u) = 0 \\ \Longrightarrow &\frac{1}{h}dh = \frac{1}{g}du \\ \Longrightarrow &h(u) = C\exp\left(\frac{u}{g}\right) \end{aligned}$$

For $u > 0$,

$$\begin{aligned} &\mathcal{A}^*h(u) = gh'(u) + h(-u) = gh'(u) + C\exp\left(\frac{u}{g}\right) = 0 \\ \Longrightarrow &h(u) = C\exp\left(-\frac{u}{g}\right) + C_1 \end{aligned}$$

Note that $\int_{\mathbb{R}} h(u)\,du = 1$, so $C_1 = 0$ and $C = \frac{1}{g}$. Therefore, when $n = 1$ the invariant probability measure is a double exponential distribution.

Remark 1.5.15. It seems impossible to derive an explicit formula for the invariant distribution for $n \geq 2$ with the same method.

1.6 *Uniqueness of invariant distribution*

sec:unique

In 1.5 we proved there exists some invariant distribution π. In this section, we will show this π is unique.

The following definition of Harris Recurrence comes from [6, p. 490]

isrecurrence **Definition 1.6.1.** A process $\mathbf{X}(t)$ is called Harris Recurrent if for some $\sigma-$ finite measure ϕ, $\mathbb{P}_{\mathbf{x}_0} \inf_{t>0}\{\mathbf{X}(t) \in A\}) = 1$ whenever $\phi(A) > 0$.

It is shown in [5](*** page) that if $\mathbf{X}$ is a Harris recurrent process than there exists an unique (up to constant multiples) invariant measure. In this way, we only need to prove our process $\mathbf{X}(t)$ is Harris recurrent relative to Lebesgue measure.

Lebirr **Theorem 1.6.2.** *For any Lebesgue positive set A and $\mathbf{x}_0 \in \mathbb{R}^n$ we have*

$$\mathbb{E}_{\mathbf{x}_0}\left(\int_0^\infty \mathbb{1}_{\{\mathbf{X}(t)\in A\}}\, dt\right) > 0$$

Proof. By the inner regularity of Lebesgue measure, there exists a compact positive set contained in A, hence WLOG we may assume A itself is a compact set. Let Leb_k to be the k-dimentional Lebesgue measure.

For each permutation α of $\{1, 2, \cdots, n\}$ and $\varepsilon > 0$, let

$$E_{\alpha,\varepsilon} = \{\mathbf{x} \in \mathbb{R}^n : x_{\alpha(1)} \leq x_{\alpha(2)} - 2\epsilon \leq x_{\alpha(3)} - 3\varepsilon \leq \cdots \leq x_{\alpha(n)} - n\varepsilon\}.$$

For each $\varepsilon > 0$, let

$$G_\varepsilon = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : \min_{1\leq i\leq n} x_i + \max_{1\leq i\leq n} x_i \leq -\varepsilon\}$$

$$H_\varepsilon = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : \min_{1\leq i\leq n} x_i + \max_{1\leq i\leq n} x_i \geq 2\varepsilon\}$$

There exists $\varepsilon > 0$ and a permutation α such that at least one of $A \cap E_{\alpha,2\varepsilon} \cap G_{2\varepsilon}$ and $A \cap E_{\alpha,2\varepsilon} \cap H_{2\epsilon}$ has positive Lebesgue measure. WLOG, we may assume one of $A \in E_{\alpha,2\varepsilon} \cap G_{2\varepsilon}$ or $A \in E_{\alpha,2\varepsilon} \cap H_{2\varepsilon}$ is true.

We can find a sequence of open balls $\{B_i\}_{i=1}^\infty$ such that

(1)$[\cup_\alpha(E_{\alpha,2\varepsilon} \cap G_{2\varepsilon})] \bigcup [\cup_\alpha(E_{\alpha,2\varepsilon} \cap H_{2\varepsilon})] = \bigcup_{i=1}^\infty B_i$

(2) Each B_i is contained in some $E_{\alpha,2\varepsilon} \cap G_{2\varepsilon}$ or $E_{\alpha,2\varepsilon} \cap H_{2\varepsilon}$.

(3) The radius for each B_i is less than 1.

By 1.4.7, we only need to prove that for each B_i there exists some constant $c_i > 0$ such that for any $A \in B_i$ and $\mathbf{a} \in \{-2\varepsilon < a_1 < a_2 < \cdots < a_n < 0\}$ we have

$$\mathbb{E}_{\mathbf{a}} \left(\int_0^\infty \mathbb{1}_{\{\mathbf{X}(t) \in A\}}\, dt \right) \geq c_i \,\mathrm{Leb}_n(A) \tag{1.6.1}$$ cleb

The left hand side is a measure as a function of A, so we only need to prove (1.6.1) for open balls. Suppose $B_i = B(y_0, r)$ and A is any ball $B(y, \delta) \subset B_i$ with $\delta < \frac{\varepsilon}{4}$. The proving (1.6.1) is equivalent to prove

Let $A_1 = B(y, \frac{\delta}{2})$, and $T_1 = \inf\{t > 0 : \mathbf{X}(t) \in A_1\}$. If there is no jump between T_1 and $T_1 + \frac{\delta}{2g}$, the process will be decreasing with a constant speed g. For this reason, it will stay in the bigger ball A for at least $\frac{\delta}{2g}$ time units. Note that $\delta < 1$. Hence,

$$\begin{aligned}
\mathbb{E}_{\mathbf{a}} \left(\int_0^\infty \mathbb{1}_{\{\mathbf{X}(t) \in A\}}\, dt \right) &\geq \frac{\delta}{2g} \mathbb{P}_{\mathbf{a}}\left(T_1 < \infty\right) \mathbb{P}(\text{there is no jump during } (T_1, T_1 + \frac{\delta}{2g})) \\
&= \frac{\delta}{2g} \exp(-\frac{\delta}{2g}) \mathbb{P}_{\mathbf{a}}\left(T_1 < \infty\right) \\
&\geq \frac{\delta}{2g} \exp(-\frac{1}{2g}) \mathbb{P}_{\mathbf{a}}(T_1 < \infty)
\end{aligned}$$

Let K be the hyperplane $\{x_1 + x_2 + \cdots + x_n = b\}$ where $b = \sup\{x_1 + x_2 + \cdots + x_n : \mathbf{x} \in A_1\}$. Let Λ be the projection of A_1 onto K, and $T' = \inf\{t > 0 : \mathbf{X}(t) \in \Lambda\}$. If there is no jump between T' and $T' + \frac{\delta}{2g}$, the process will be decreasing at a constant speed g, it will travel at least $\frac{\delta}{2}$ units of space in a straight line and hence it will hit our ball A_1. In this way, we have

$$\begin{aligned}
\mathbb{P}_{\mathbf{a}}\left(T_1 < \infty\right) &\geq \mathbb{P}_{\mathbf{a}}\left(T' < \infty\right) \mathbb{P}(\text{there is no jump during } (T', T' + \frac{\delta}{2g})) \\
&\geq \exp(-\frac{\delta}{2g}) \mathbb{P}_{\mathbf{a}}(T' < \infty) \\
&\geq \exp(-\frac{1}{2g}) \mathbb{P}_{\mathbf{a}}(T' < \infty)
\end{aligned}$$

Let $\mathbf{n}$ be the unit normal vector to K pointing away from A_1, $A_1' = \{\mathbf{x} + t\mathbf{n} : \mathbf{x} \in \Lambda, t \in [0, \frac{\varepsilon}{4}]\}$ and $T_1' = \inf\{t > 0 : \mathbf{X}(t) \in A_1'\}$. If there is no jump between T_1' and $T_1' + \frac{\epsilon}{g}$, the process will be decreasing at a constant speed g, it will travel at least ϵ units of space in a straight line and hence it will hit the set Λ. In this way, we have

$$\begin{aligned}\mathbb{P}_{\mathbf{a}}\left(T' < \infty\right) &\geq \mathbb{P}_{\mathbf{a}}\left(T_1' < \infty\right) \mathbb{P}\left(\text{there is no jump during } (T_1', T_1' + \frac{\varepsilon}{g})\right) \\ &= \exp(-\frac{\varepsilon}{g}) \mathbb{P}_{\mathbf{a}}\left(T_1' < \infty\right)\end{aligned}$$

Define $B_i' = \{\mathbf{x} + t\mathbf{n} : \mathbf{x} \in B_i, t \in [0, \frac{\varepsilon}{2}]\}$, we claim that $A_1' \subset B_i'$. For $\mathbf{x} = (x_1, \cdots, x_n) \in A_1$, the projection onto K would be $\mathbf{x} + t_1\mathbf{n}$ with $t_1 = \frac{b - \sum_{i=1}^n x_i}{n}$. Then each point in A_1' can be represented as $\mathbf{x} + t_1\mathbf{n} + t_2\mathbf{n}$ with $t_2 \in [0, \frac{\varepsilon}{4}]$. Note that A_1 is an open ball with diameter δ, we have

$$t_1 + t_2 = \frac{b - \sum_{i=1}^n x_i}{n} + t_2 \leq \frac{\delta}{n} + \frac{\varepsilon}{4} < \delta + \frac{\varepsilon}{4} < \frac{\varepsilon}{2}$$

If $B_i \subset E_{\alpha,2\varepsilon} \cap G_{2\varepsilon}$, we will show that $B_i' \subset E_{\alpha,\varepsilon} \cap G_\varepsilon$. Adding $t\mathbf{n}$ will not change the differences between coordinates, so $B_i' \in E_{\alpha,2\varepsilon} \subset E_{\alpha,\varepsilon}$. On the other hand,

$$\begin{aligned}\min_{1\leq i\leq n}(x_i + t) + \max_{1\leq i\leq n}(x_i + t) &= \min_{1\leq i\leq n} x_i + \max_{1\leq i\leq n} x_i + 2t \\ &\leq -2\varepsilon + 2 \cdot \frac{\varepsilon}{2} \\ &= -\varepsilon\end{aligned}$$

Hence, $B_i' \in G_\varepsilon$.

If $B_i \subset E_{\alpha,2\varepsilon} \cap H_{2\varepsilon}$, we can also show that $B_i' \subset E_{\alpha,\varepsilon} \cap H_\varepsilon$. With the same argument, $B_i' \in E_{\alpha,\varepsilon}$ since the differences between coordinates do not change. In addition,

$$\begin{aligned}\min_{1\leq i\leq n}(x_i + t) + \max_{1\leq i\leq n}(x_i + t) &= \min_{1\leq i\leq n} x_i + \max_{1\leq i\leq n} x_i + 2t \\ &\geq 2\varepsilon + 2 \cdot 0 \\ &\geq \varepsilon\end{aligned}$$

Finally we only need to show that $\mathbb{P}_{\mathbf{a}}(T_1' < \infty) \geq c\delta^{n-1}$ for some c only depends on B_i and ε. Since B_i' is a bounded set, there exists $M > 0$ such that $B_i' \subset [-M, M]^n$.

(I) If $B_i \in E_{\mathrm{id},2\varepsilon} \cap G_{2\varepsilon}$.

For $2 \leq k \leq n$, let $b_k = \frac{(2n-k+1)(k-2)}{2} + 1$. We will define b_n's many collisions as following:

$$U_{b_k+i} = n - i, \text{for } 2 \leq k \leq n-1, 0 \leq i \leq b_{k+1} - b_k - 1$$

We claim that

$$\mathbf{X}(T_{b_n}) = (a_1 - gT_{b_n}, -a_n + 2gT_{b_2} - gT_{b_n}, -a_{n-1} + 2gT_{b_3} - gT_{b_n}, \cdots, -a_2 + gT_{b_n})$$

Then for $T_{b_n} < t < T_{b_n+1}$,

$$\mathbf{X}(t) = (a_1 - gt, -a_n + 2gT_{b_2} - gt, -a_{n-1} + 2gT_{b_3} - gt, \cdots, -a_2 + 2gT_{b_n} - gt)$$

Using $X_1, \cdots, X_n$ to represent $T_{b_2}, \cdots, T_{b_n}, t$ we can get

$$\begin{cases} T_{b_2} = \frac{1}{2g}\left((X_2 + a_n) - (X_1 - a_1)\right) \\ T_{b_3} = \frac{1}{2g}\left((X_3 + a_{n-1}) - (X_1 - a_1)\right) \\ \vdots \\ T_{b_n} = \frac{1}{2g}\left((X_n + a_2) - (X_1 - a_1)\right) \\ t - T_{b_n} = -\frac{1}{2g}\left((X_1 - a_1) + (X_n + a_2)\right) \end{cases}$$

Note that for $\mathbf{x} \in A_1' \subset E_{id,\varepsilon} \cap G_\varepsilon$ we have

$$x_1 - a_1 < x_1 + \varepsilon \leq x_2 - \varepsilon < x_2 + a_n < x_2 \leq x_3 - \varepsilon < x_3 + a_{n-1} < \cdots < x_n + a_2$$

$$x_1 + x_n < -\varepsilon < a_1 < a_1 - a_2$$

Hence, A_1' is contained in the range of $\mathbf{X}(t)$.

Define $h, h_1 : \mathbb{R}^n \to \mathbb{R}^{n-1}$ as

$$h(\mathbf{x}) = \left(\frac{1}{2g}\left((x_2 + a_n) - (x_1 - a_1)\right), \cdots, \frac{1}{2g}\left((x_n + a_2) - (x_1 - a_1)\right)\right)$$

$$h_1(\mathbf{x}) = (x_2 - x_1, \cdots, x_n - x_1)$$

Then

$$\begin{aligned} &\mathbb{P}_{\mathbf{a}}(T_1' < \infty) \\ &\geq \left(\frac{1}{n}\right)^{b_n} \int_{\mathbf{x}\in h(A_1')} \int_{s=-\frac{1}{2g}((x_1-a_1)+(x_n+a_2))}^{\infty} \mathbb{P}_{\mathbf{a}}\left(T_{b_n+1} - T_{b_n} \in ds, (T_{b_2}, \cdots, T_{b_n}) \in dx\right) \\ &\overset{*}{\geq} \left(\frac{1}{n}\right)^{b_n} \int_{\mathbf{x}\in h(A_1')} \int_{s=\frac{2M+\varepsilon}{2g}}^{\infty} \mathbb{P}_{\mathbf{a}}\left(T_{b_n+1} - T_{b_n} \in ds\right) \mathbb{P}_{\mathbf{a}}\left((T_{b_2}, \cdots, T_{b_n}) \in dx\right) \\ &\overset{**}{=} \left(\frac{1}{n}\right)^{b_n} e^{-\frac{2M+\varepsilon}{2g}} \int_{\mathbf{x}\in h(A_1')} \mathbb{P}_{\mathbf{a}}\left((T_{b_2}, \cdots, T_{b_n}) \in dx\right) \end{aligned}$$

The inequality $(*)$ comes from the fact that for any $\mathbf{x} \in A_1' \subset B_i' \subset [-M, M]^n$ we have

$$x_1 - a_1 + x_n + a_2 \geq -M + 0 - M - \varepsilon = -2M - \varepsilon$$

The equality $(**)$ comes from the fact that $T_{b_n+1} - T_{b_n}$ has exponential distribution with mean 1. Note that $(T_{b_2}, \cdots, T_{b_n})$ has positive continuous PDF, its PDF will have a positive minimum value on $\bar{B}_i'$. Let m to be this minimum value, then m only depends on B_i and ε. Hence,

$$\begin{aligned}\mathbb{P}_{\mathbf{a}}(T_1' < \infty) &\geq \left(\frac{1}{n}\right)^{b_n} e^{-\frac{2M+\varepsilon}{2g}} \int_{\mathbf{x}\in h(A_1')} \mathbb{P}_{\mathbf{a}}\left((T_{c_2}, \cdots, T_{c_n}) \in dx\right) \\ &\geq m \left(\frac{1}{n}\right)^{b_n} e^{-\frac{2M+\varepsilon}{2g}} \operatorname{Leb}(h(A_1')).\end{aligned}$$

By scaling and translation, $\operatorname{Leb}_{n-1}(h(A_1')) = \left(\frac{1}{2g}\right)^{n-1} Leb_{n-1}(h_1(A_1'))$. Recall that A_1' can be represented as $\mathbf{x} + (t_1+t_2)\mathbf{n}$ with $\mathbf{x} \in A_1, t_1 = \frac{b-\sum_{i=1}^n x_i}{n}, t_2 \in [0, \frac{\varepsilon}{4}]$, so that $h_1(A_1') = h_1(A_1)$. Fix some point $\mathbf{x} = (x_1, \cdots, x_n) \in A_1$, let S to be the slice $\{(x_1, x_2', x_3', \cdots, x_n') : (x_1, x_2', \cdots, x_n') \in A_1\}$, then

$$\operatorname{Leb}_{n-1}(h_1(A_1)) \geq \operatorname{Leb}_{n-1}(h_1(S)) \propto \delta^{n-1}$$

Therefore, $\mathbb{P}_{\mathbf{a}}(T_1' < \infty) \geq c\delta^{n-1}$ with some c only depends on B_i and ε.

(II) If $B_i \in E_{\alpha,2\varepsilon} \cap G_{2\varepsilon}$ for some $\alpha \neq \text{id}$.

Follow the same way in (I), let

$$b_k = \frac{(2n-k+1)(k-2)}{2} + 1, \quad 2 \leq k \leq n$$

.

$$U_{b_k+i} = n - i, \text{ for } 2 \leq k \leq n-1, 0 \leq i \leq b_{k+1} - b_k - 1$$

Then

$$\mathbf{X}(T_{b_n}) = (a_1 - gT_{b_n}, -a_n + 2gT_{b_2} - gT_{b_n}, -a_{n-1} + 2gT_{b_3} - gT_{b_n}, \cdots, -a_2 + gT_{b_n})$$

There exists $\ell > b_n$ and some $u_{b_n+1}, \cdots, u_\ell \in \{1, 2, \cdots, n-1\}$ such that with

$$U_{b_n+1} = u_{b_n+1}, U_{b_n+2} = u_{b_n+2}, \cdots, U_l = u_l$$

Our ℓ here only depends on the permutation α. We have $X_i(T_\ell) = X_{\alpha^{-1}(i)}(T_{b_n}) - g(T_\ell - T_{b_n})$. Define $g_\alpha : \mathbb{R}^n \to \mathbb{R}^n$ to be

$$g_\alpha((x_1, \cdots, x_n)) = (x_\alpha(1), x_\alpha(2), \cdots, x_\alpha(n))$$

Note that reflection preserves measure, so $\text{Leb}_n(g_\alpha(A_1')) = \text{Leb}_n(A_1')$. Then

$$\mathbb{P}_{\mathbf{a}}\left(T_1' < \infty\right) \geq \left(\frac{1}{n}\right)^{\ell} \mathbb{P}_{\mathbf{a}}\left(\mathbf{X}(T_\ell) \in A_1'\right) = \left(\frac{1}{n}\right)^{\ell} \mathbb{P}_{\mathbf{a}}\left(g_\alpha(\mathbf{X}(T_\ell)) \in g_\alpha(A_1')\right)$$

The factor $\left(\frac{1}{n}\right)^{\ell}$ is due to the fact that every jump should happen between specific balls.

Note that $X_{\alpha(i)}(T_\ell) = X_i(T_{b_n}) - g(T_\ell - T_{b_n})$, so

$$g_\alpha(\mathbf{X}(T_\ell)) = (a_1 - gT_\ell, -a_n + 2gT_{b_2} - gT_\ell, \cdots, -a_2 + 2gT_{b_n} - T_\ell)$$

Representing $T_{b_2}, T_{b_3}, \cdots, T_{b_n}, T_l$ with $X_1, \cdots, X_n$ we can get

$$\begin{cases} T_{b_2} = \dfrac{1}{2g}\left((X_2 + a_n) - (X_1 - a_1)\right) \\ T_{b_3} = \dfrac{1}{2g}\left((X_3 + a_{n-1}) - (X_1 - a_1)\right) \\ \vdots \\ T_{b_n} = \dfrac{1}{2g}\left((X_n + a_2) - (X_1 - a_1)\right) \\ T_\ell = -\dfrac{1}{g}(X_1 - a_1) \end{cases}$$

For each $\mathbf{x} \in g_\alpha(A_1') \subset g_\alpha(E_{\alpha,\varepsilon} \cap G_\varepsilon)$ we have

$$x_1 \leq x_2 - 2\varepsilon \leq x_3 - 3\varepsilon \leq \cdots \leq x_n - n\varepsilon$$

$$x_1 + x_n < -\varepsilon < a_1 < a_1 - a_2$$

Hence, $g_\alpha(A_1')$ is contained in the range of $g_\alpha(\mathbf{X}(T_\ell))$. From the formula of $g_\alpha(\mathbf{X}(T_\ell))$, we can see the Jacobian matrix is

$$\frac{\partial\, g_\alpha(\mathbf{X}(T_\ell))}{\partial(T_{b_2}, \cdots, T_{b_n}, T_\ell)} = \begin{pmatrix} 0 & 0 & \cdots & 0 & -g \\ 2g & 0 & \cdots & 0 & -g \\ 0 & 2g & \cdots & 0 & -g \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 2g & -g \end{pmatrix}$$

with determinant $g(2g)^{n-1}$. Hence, $g_\alpha(\mathbf{X}(T_\ell))$ will have a positive continuous PDF, its PDF will have a positive minimum value on $g_\alpha(\bar{B}_i')$. Let m to be this minimum value, then m only depends on B_i and ε. Hence,

$$\mathbb{P}_{\mathbf{a}}(g_\alpha(\mathbf{X}(T_\ell)) \in g_\alpha(A_1')) \geq m\, \text{Leb}_n(g_\alpha(A_1')) = m\, \text{Leb}_n(A_1') \propto \delta^{n-1}$$

(III) If $B_i \subset E_{\mathrm{id},2\varepsilon} \cap H_{2\varepsilon}$.

Follow the same way, let

$$b_k = \frac{(2n-k+1)(k-2)}{2} + 1, \quad 2 \le k \le n$$

.

$$U_{b_k+i} = n - i, \text{ for } 2 \le k \le n-1, 0 \le i \le b_{k+1} - b_k - 1$$

Then

$$\mathbf{X}(T_{b_n}) = (a_1 - gT_{b_n}, -a_n + 2gT_{b_2} - gT_{b_n}, -a_{n-1} + 2gT_{b_3} - gT_{b_n}, \cdots, -a_2 + gT_{b_n})$$

Next, we let $U_{b_n+i} = i$ for $1 \le i \le n$ and $b_{n+1} = b_n + n$, then

$$\mathbf{X}(T_{b_{n+1}}) = (-a_n + 2gT_{b_2} - gT_{b_{n+1}}, -a_{n-1} + 2gT_{b_3} - gT_{b_{n+1}}, \cdots, -a_1 + gT_{b_{n+1}})$$

Using $X_1, \cdots, X_n$ to represent $T_{b_2}, \cdots, T_{b_n}, T_{b_{n+1}}$ we can get

$$\begin{cases} T_{b_2} = \dfrac{1}{2g}\left((X_1 + a_n) + (X_n + a_1)\right) \\ T_{b_3} = \dfrac{1}{2g}\left((X_2 + a_{n-1}) + (X_n + a_1)\right) \\ \quad \vdots \\ T_{b_n} = \dfrac{1}{2g}\left((X_{n-1} + a_2) + (X_n + a_1)\right) \\ T_{b_{n+1}} = \dfrac{1}{g}\left(X_n + a_1\right) \end{cases}$$

Note that for $\mathbf{x} \in A'_1 \subset E_{id,\varepsilon} \cap H_\varepsilon$ we have

$$x_1 + a_n < x_2 - 2\varepsilon < x_2 + a_{n-1} < x_3 - \varepsilon < x_3 + a_{n-2} < \cdots < x_n + a_1$$

$$x_1 + x_n + a_1 + a_n \ge 2\varepsilon + a_1 + a_n > 0$$

Hence, A'_1 is contained in the range of $\mathbf{X}(T_{b_{n+1}})$.

From the formula of $\mathbf{X}(T_{b_{n+1}})$, we can see the Jacobian matrix is

$$\frac{\partial\, \mathbf{X}(T_{b_{n+1}})}{\partial(T_{b_2}, \cdots, T_{b_{n+1}})} = \begin{pmatrix} 2g & 0 & \cdots & 0 & -g \\ 0 & 2g & \cdots & 0 & -g \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 2g & -g \\ 0 & 0 & \cdots & 0 & g \end{pmatrix}$$

with determinant $g(2g)^{n-1}$. Hence, $\mathbf{X}(T_{b_{n+1}})$ will have a positive continuous PDF, its PDF will have a positive minimum value on $\bar{B}'_i$. Let m to be this minimum value, then m only depends on B_i and ε. Hence,

$$\mathbb{P}_{\mathbf{a}}(T'_1 < \infty) \geq \left(\frac{1}{n}\right)^{b_{n+1}} \mathbb{P}_{\mathbf{a}}(\mathbf{X}(T_{b_{n+1}}) \in A'_1) \geq \left(\frac{1}{n}\right)^{b_{n+1}} m\, \mathrm{Leb}_n(A'_1) \propto \delta^{n-1}$$

(IV) If $B_i \subset E_{\alpha,2\varepsilon} \cap H_{2\varepsilon}$, for some $\alpha \neq \mathrm{id}$.

Define $b_2, b_3, \cdots, b_{n+1}$ and $U_i, 1 \leq i \leq b_{n+1}$ as the same in (III), then

$$\mathbf{X}(T_{b_{n+1}}) = (-a_n + 2gT_{b_2} - gT_{b_{n+1}}, -a_{n-1} + 2gT_{b_3} - gT_{b_{n+1}}, \cdots, -a_1 + gT_{b_{n+1}})$$

There exists $\ell' > b_{n+1}$ and some $u_{b_{n+1}+1}, \cdots, u_{\ell'} \in \{1, 2, \cdots, n-1\}$ such that with

$$U_{b_{n+1}+1} = u_{b_{n+1}+1}, U_{b_{n+1}+2} = u_{b_{n+1}+2}, \cdots, U_{\ell'} = u_{\ell'}$$

Our ℓ' here only depends on the permutation α. We have $X_i(T_{\ell'}) = X_{\alpha^{-1}(i)}(T_{b_{n+1}}) - g(T_{\ell'} - T_{b_{n+1}})$. Define $g_\alpha : \mathbb{R}^n \to \mathbb{R}^n$ to be

$$g_\alpha((x_1, \cdots, x_n)) = (x_{\alpha(1)}, x_{\alpha(2)}, \cdots, x_{\alpha(n)})$$

Note that reflection preserves measure, so $\mathrm{Leb}_n(g_\alpha(A'_1) = \mathrm{Leb}_n(A'_1)$. Then

$$\mathbb{P}_{\mathbf{a}}\left(T'_1 < \infty\right) \geq \left(\frac{1}{n}\right)^{\ell'} \mathbb{P}_{\mathbf{a}}\left(\mathbf{X}(T_{\ell'}) \in A'_1\right) = \left(\frac{1}{n}\right)^{\ell'} \mathbb{P}_{\mathbf{a}}\left(g_\alpha(\mathbf{X}(T_{\ell'})) \in g_\alpha(A'_1)\right)$$

Note that $X_{\alpha(i)}(T_{\ell'}) = X_i(T_{b_{n+1}}) - g(T_{\ell'} - T_{b_{n+1}})$, so

$$g_\alpha\left(\mathbf{X}(T_{\ell'})\right) = (-a_n + 2gT_{b_2} - gT_{\ell'}, -a_{n-1} + 2gT_{b_3} - gT_{\ell'}, \cdots, -a_1 + 2gT_{b_{n+1}} - gT_{\ell'})$$

Representing $T_{b_2} - (T_{\ell'} - T_{b_{n+1}}), T_{b_3} - T_{b_2}, \cdots, T_{b_{n+1}} - T_{b_n}$ with $X_1, \cdots, X_n$ we have

$$\begin{cases} T_{b_2} - (T_{\ell'} - T_{b_{n+1}}) = \dfrac{1}{2g}\left((X_1 + a_n) + (X_n + a_1)\right) \\ T_{b_3} - T_{b_2} = \dfrac{1}{2g}\left((X_2 + a_{n-1}) - (X_1 + a_n)\right) \\ T_{b_4} - T_{b_3} = \dfrac{1}{2g}\left((X_3 + a_{n-2}) - (X_2 + a_{n-1})\right) \\ \vdots \\ T_{b_{n+1}} - T_{b_n} = \dfrac{1}{2g}\left((X_n + a_1) - (X_{n-1} + a_2)\right) \end{cases}$$

For each $\mathbf{x} \in g_\alpha(A_1') \subset g_\alpha(E_{\alpha.\varepsilon} \cap H_\varepsilon)$ we have

$$x_1 + a_n < x_2 - 2\varepsilon < x_2 + a_{n-1} < x_3 - \varepsilon < x_3 + a_{n-2} < \cdots < x_n + a_1$$

$$x_1 + x_n + a_1 + a_n \geq 2\varepsilon + a_1 + a_n > 0$$

Hence, $g_\alpha(A_1')$ is contained in the range of $g_\alpha(\mathbf{X}(T_{\ell'}))$. From the formula of $g_\alpha(\mathbf{X}(T_{\ell'}))$, we can see the Jacobian matrix is

$$\frac{\partial\, g_\alpha(\mathbf{X}(T_{\ell'}))}{\partial \left(T_{b_2} - (T_{\ell'} - T_{b_{n+1}}), T_{b_3} - T_{b_2} \cdots, T_{b_{n+1}} - T_{b_n}\right)} = \begin{pmatrix} g & -g & \cdots & -g & -g \\ g & g & \cdots & -g & -g \\ \vdots & \vdots & & \vdots & \vdots \\ g & g & \cdots & g & -g \\ g & g & \cdots & g & g \end{pmatrix}$$

with determinant $2^{n-1}g$. Hence, $g_\alpha\left(\mathbf{X}(T_{\ell'})\right)$ will have a positive continuous PDF, and its PDF will have a positive minimum value on $\bar{B}_i'$. Let m be to be this minimum value, then m only depends on B_i and ε. Hence,

$$\mathbb{P}_{\mathbf{a}}(g_\alpha(\mathbf{X}(T_{\ell'})) \in g_\alpha(A_1')) \geq m\,\mathrm{Leb}_n(g_\alpha(A_1')) = m\,\mathrm{Leb}_n(A_1') \propto \delta^{n-1}$$

□

Lebtopi **Lemma 1.6.3.** *For each Lebesgue positive set A, we also have $\pi(A) > 0$*

Proof. Assume there exists some $A \in \mathbb{R}^n$ such that $\mathrm{Leb}_n(A) > 0$ and $\pi(A) = 0$. Since π is the invariant distribution, for each $t > 0$ we have

$$\int \mathbb{P}_{\mathbf{x}}\left(\mathbf{X}(t) \in A\right) \pi(d\mathbf{x}) = \pi(A) = 0$$

Then

$$\begin{aligned} 0 &= \int_0^\infty \int \mathbb{P}_{\mathbf{x}}\left(\mathbf{X}(t) \in A\right) \pi(d\mathbf{x})\; dt \\ &= \int \int_0^\infty \mathbb{P}_{\mathbf{x}}\left(\mathbf{X}(t) \in A\right) dt\; \pi(d\mathbf{x}) \\ &= \int \int_0^\infty \mathbb{E}_{\mathbf{x}}\left(\mathbb{1}_{\{\mathbf{X}(t) \in A\}}\right) dt\; \pi(d\mathbf{x}) \\ &= \int \mathbb{E}_{\mathbf{x}}\left(\int_0^\infty \mathbb{1}_{\{\mathbf{X}(t) \in A\}}\; dt\right)\; \pi(d\mathbf{x}) \end{aligned}$$

However, by 1.6.2 for each $\mathbf{x} \in \mathbb{R}^n$ we have $\mathbb{E}_{\mathbf{x}} \left(\int_0^\infty \mathbb{1}_{\{\mathbf{X}(t) \in A\}} \, dt \right) > 0$, which leads to a contradiction. □

compactproj **Lemma 1.6.4.** *Suppose h_1 is a map from $\mathbb{R}^n$ to $\mathbb{R}^{n-1}$ that*

$$h_1(\mathbf{x}) = (x_2 - x_1, x_3 - x_1, \cdots, x_n - x_1)$$

Then for each compact set $A \in \mathbb{R}^n$ with $\text{Leb}_n(A) > 0$ we have $\text{Leb}_{n-1}(h_1(A)) > 0$.

Proof. Let proj_1 to be the $1-$th projection map, i.e.

$$\text{proj}_1(\mathbf{x}) = x_1$$

For each $x_1 \in \text{proj}_1(A)$, define $M_{x_1} = \{(x_2, x_3, \cdots, x_n) : (x_1, x_2, x_3, \cdots, x_n) \in A\}$.

We claim that M_{x_1} is closed. Suppose $\{(x_2^{(i)}, \cdots, x_n^{(i)})\}_{i=1}^\infty$ is a sequence in M_{x_1} that converges to $(x_2, \cdots, x_n)$. From the definition of M_{x_1} we can see that $\{(x_1, x_2^{(i)}, \cdots, x_n^{(i)})\}_{i=1}^\infty$ is a sequence in A which converges to $(x_1, x_2, \cdots, x_n)$. Then $(x_1, x_2, \cdots, x_n) \in A$ since A is compact. Hence, $(x_2, \cdots, x_n) \in M_{x_1}$ which implies M_{x_1} is closed.

Recall that $\text{Leb}_n(A) > 0$, there exists $x_1 \in A$ such that $\text{Leb}_{n-1}(M_{x_1}) > 0$. By shifting, we can get

$$\text{Leb}_{n-1}(h_1(A)) \geq \text{Leb}_{n-1}(h_1(M_{x_1})) = \text{Leb}_{n-1}(M_{x_1}) > 0$$

□

Hrecu **Theorem 1.6.5.** *The process $\mathbf{X}$ is Harris recurrent relative to Lebesgue measure i.e. for every point $\mathbf{x}_0 \in \mathbb{R}^n$ and Lebesgue positive set $A \subset \mathbb{R}^n$*

$$\mathbb{P}_{\mathbf{x}_0} \left(\inf_{t>0} \{\mathbf{X}(t) \in A\} \right) = 1$$

Proof. By the inner regularity of Lebesgue measure, A has a compact subset with positive measure. WLOG, we may assume A itself is compact and $A \subset [-M, M]^n$. Let $\tau_A = \inf_{t>0} \{\mathbf{X}(t) \in A\}$. Take an open ball B around point $\mathbf{x}_0$, then $\pi(B) > 0$ with lemma 1.6.3. By Poincaré Recurrence Theorem, starting from $\mathbf{x}_0$ the process will return to B infinitely many times with probability 1. Hence, we only need to prove there exists some $c > 0$ such that

$$\mathbb{P}_{\mathbf{x}} (\tau_A < \infty) > c > 0, \quad \text{for all } \mathbf{x} \in B$$

We will apply the similar method used proving the theorem 1.6.2

For each permutation α of $\{1, 2, \cdots, n\}$ and $\varepsilon > 0$, let

$$E_{\alpha,\varepsilon} = \{\mathbf{x} \in \mathbb{R}^n : x_{\alpha(1)} \leq x_{\alpha(2)} - 2\epsilon \leq x_{\alpha(3)} - 3\varepsilon \leq \cdots \leq x_{\alpha(n)} - n\varepsilon\}.$$

For each $\varepsilon > 0$, let

$$G_\varepsilon = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : \min_{1\leq i\leq n} x_i + \max_{1\leq i\leq n} x_i \leq -\varepsilon\}$$

$$H_\varepsilon = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : \min_{1\leq i\leq n} x_i + \max_{1\leq i\leq n} x_i \geq 2\varepsilon\}$$

There exists $\varepsilon > 0$ and a permutation α such that at least one of $A \cap E_{\alpha,\varepsilon} \cap G_\varepsilon$ and $A \cap E_{\alpha,\varepsilon} \cap H_\epsilon$ has positive Lebesgue measure. WLOG, we may assume one of $A \in E_{\alpha,\varepsilon} \cap G_\varepsilon$ or $A \in E_{\alpha,\varepsilon} \cap H_\varepsilon$ is true.

The set $\{-\varepsilon < x_1 < x_2 < \cdots < x_n < 0\}$ is open, so it contains an open ball B_1. By corollary 1.4.8, there exists some constant $c_1 > 0$ such that

$$\mathbb{P}_{\mathbf{x}}\left(\inf_{t>0}\{\mathbf{X}(t) \in B_1\}\right) > c_1 > 0 \ \text{ for all } \mathbf{x} \in B$$

Hence, we only need to show that there exists some $c > 0$ such that

$$\mathbb{P}_{\mathbf{a}}\left(\tau_A < \infty\right) > c, \ \text{ for all } \mathbf{a} \in B_1$$

(I) If $A \in E_{\mathrm{id},\varepsilon} \cap G_\varepsilon$.

For $2 \leq k \leq n$, let $b_k = \frac{(2n-k+1)(k-2)}{2} + 1$. We will define b_n's many collisions as following:

$$U_{b_k+i} = n - i, \text{for } 2 \leq k \leq n-1, 0 \leq i \leq b_{k+1} - b_k - 1$$

We claim that

$$\mathbf{X}(T_{b_n}) = (a_1 - gT_{b_n}, -a_n + 2gT_{b_2} - gT_{b_n}, -a_{n-1} + 2gT_{b_3} - gT_{b_n}, \cdots, -a_2 + gT_{b_n})$$

Then for $T_{b_n} < t < T_{b_n+1}$,

$$\mathbf{X}(t) = (a_1 - gt, -a_n + 2gT_{b_2} - gt, -a_{n-1} + 2gT_{b_3} - gt, \cdots, -a_2 + 2gT_{b_n} - gt)$$

Using $X_1, \cdots, X_n$ to represent $T_{b_2}, \cdots, T_{b_n}, t$ we can get

$$\begin{cases} T_{b_2} = \frac{1}{2g}\left((X_2 + a_n) - (X_1 - a_1)\right) \\ T_{b_3} = \frac{1}{2g}\left((X_3 + a_{n-1}) - (X_1 - a_1)\right) \\ \vdots \\ T_{b_n} = \frac{1}{2g}\left((X_n + a_2) - (X_1 - a_1)\right) \\ t - T_{b_n} = -\frac{1}{2g}\left((X_1 - a_1) + (X_n + a_2)\right) \end{cases}$$

Note that for $\mathbf{x} \in A \subset E_{id,\varepsilon} \cap G_\varepsilon$ we have

$$x_1 - a_1 < x_1 + \varepsilon \leq x_2 - \varepsilon < x_2 + a_n < x_2 \leq x_3 - \varepsilon < x_3 + a_{n-1} < \cdots < x_n + a_2$$

$$x_1 + x_n < -\varepsilon < a_1 < a_1 - a_2$$

Hence, A is contained in the range of $\mathbf{X}(t)$.

Define $h : \mathbb{R}^n \to \mathbb{R}^{n-1}$ as

$$h(\mathbf{x}) = \left(\frac{1}{2g}\left((x_2 + a_n) - (x_1 - a_1)\right), \cdots, \frac{1}{2g}\left((x_n + a_2) - (x_1 - a_1)\right)\right)$$

Then

$$\begin{aligned} &\mathbb{P}_{\mathbf{a}}(\tau_A < \infty) \\ \geq &\left(\frac{1}{n}\right)^{b_n} \int_{\mathbf{x}\in h(A)} \int_{s=-\frac{1}{2g}((x_1-a_1)+(x_n+a_2))}^{\infty} \mathbb{P}_{\mathbf{a}}\left(T_{b_n+1} - T_{b_n} \in ds, (T_{b_2}, \cdots, T_{b_n}) \in dx\right) \\ \overset{*}{\geq} &\left(\frac{1}{n}\right)^{b_n} \int_{\mathbf{x}\in h(A)} \int_{s=\frac{2M+\varepsilon}{2g}}^{\infty} \mathbb{P}_{\mathbf{a}}\left(T_{b_n+1} - T_{b_n} \in ds\right) \mathbb{P}_{\mathbf{a}}\left((T_{b_2}, \cdots, T_{b_n}) \in dx\right) \\ \overset{**}{=} &\left(\frac{1}{n}\right)^{b_n} e^{-\frac{2M+\varepsilon}{2g}} \int_{\mathbf{x}\in h(A)} \mathbb{P}_{\mathbf{a}}\left((T_{b_2}, \cdots, T_{b_n}) \in dx\right) \end{aligned}$$

The factor $\left(\frac{1}{n}\right)^{b_n}$ is due to the fact that every jump should happen between specific balls. The inequality $(*)$ comes from the fact that for any $\mathbf{x} \in A \subset [-M, M]^n$ we have

$$x_1 - a_1 + x_n + a_2 \geq -M + 0 - M - \varepsilon = -2M - \varepsilon$$

The equality $(**)$ comes from the fact that $T_{b_n+1} - T_{b_n}$ has exponential distribution with mean 1. Note that $(T_{b_2}, \cdots, T_{b_n})$ has positive continuous PDF, its PDF will have a positive minimum value on $h(A)$. Let m_1 to be this minimum value, then

$$\mathbb{P}_{\mathbf{a}}(\tau_A < \infty) \geq \left(\frac{1}{n}\right)^{b_n} e^{-\frac{2M+\varepsilon}{2g}} \int_{\mathbf{x}\in h(A)} \mathbb{P}_{\mathbf{a}}\left((T_{c_2}, \cdots, T_{c_n}) \in dx\right)$$
$$\geq m_1 \left(\frac{1}{n}\right)^{b_n} e^{-\frac{2M+\varepsilon}{2g}} \operatorname{Leb}_{n-1}(h(A)).$$

By shifting, translation and lemma 1.6.4, we have $Leb_{n-1}(h(A)) > 0$.

(II) If $A \in E_{\alpha,\varepsilon} \cap G_\varepsilon$ for some $\alpha \neq \text{id}$.

Follow the same way in (I), let

$$b_k = \frac{(2n-k+1)(k-2)}{2} + 1, \quad 2 \leq k \leq n$$

.

$$U_{b_k+i} = n - i, \text{ for } 2 \leq k \leq n-1, 0 \leq i \leq b_{k+1} - b_k - 1$$

Then

$$\mathbf{X}(T_{b_n}) = (a_1 - gT_{b_n}, -a_n + 2gT_{b_2} - gT_{b_n}, -a_{n-1} + 2gT_{b_3} - gT_{b_n}, \cdots, -a_2 + gT_{b_n})$$

There exists $\ell > b_n$ and some $u_{b_n+1}, \cdots, u_\ell \in \{1, 2, \cdots, n-1\}$ such that with

$$U_{b_n+1} = u_{b_n+1}, U_{b_n+2} = u_{b_n+2}, \cdots, U_l = u_l$$

Our ℓ here only depends on the permutation α. We have $X_i(T_\ell) = X_{\alpha^{-1}(i)}(T_{b_n}) - g(T_\ell - T_{b_n})$. Define $g_\alpha : \mathbb{R}^n \to \mathbb{R}^n$ to be

$$g_\alpha((x_1, \cdots, x_n)) = (x_\alpha(1), x_\alpha(2), \cdots, x_\alpha(n))$$

Note that reflection preserves measure, so $\operatorname{Leb}_n(g_\alpha(A)) = \operatorname{Leb}_n(A)$. Then

$$\mathbb{P}_{\mathbf{a}}\left(\tau_A < \infty\right) \geq \left(\frac{1}{n}\right)^{\ell} \mathbb{P}_{\mathbf{a}}\left(\mathbf{X}(T_\ell) \in A\right) = \left(\frac{1}{n}\right)^{\ell} \mathbb{P}_{\mathbf{a}}\left(g_\alpha(\mathbf{X}(T_\ell)) \in g_\alpha(A)\right)$$

The factor $\left(\frac{1}{n}\right)^{\ell}$ is due to the fact that every jump should happen between specific balls.

Note that $X_{\alpha(i)}(T_\ell) = X_i(T_{b_n}) - g(T_\ell - T_{b_n})$, so

$$g_\alpha(\mathbf{X}(T_\ell)) = (a_1 - gT_\ell, -a_n + 2gT_{b_2} - gT_\ell, \cdots, -a_2 + 2gT_{b_n} - T_\ell)$$

Representing $T_{b_2}, T_{b_3}, \cdots, T_{b_n}, T_l$ with $X_1, \cdots, X_n$ we can get

$$\begin{cases} T_{b_2} = \dfrac{1}{2g}\left((X_2 + a_n) - (X_1 - a_1)\right) \\ T_{b_3} = \dfrac{1}{2g}\left((X_3 + a_{n-1}) - (X_1 - a_1)\right) \\ \quad \vdots \\ T_{b_n} = \dfrac{1}{2g}\left((X_n + a_2) - (X_1 - a_1)\right) \\ T_\ell = -\dfrac{1}{g}(X_1 - a_1) \end{cases}$$

For each $\mathbf{x} \in g_\alpha(A) \subset g_\alpha(E_{\alpha,\varepsilon} \cap G_\varepsilon)$ we have

$$x_1 \leq x_2 - 2\varepsilon \leq x_3 - 3\varepsilon \leq \cdots \leq x_n - n\varepsilon$$

$$x_1 + x_n < -\varepsilon < a_1 < a_1 - a_2$$

Hence, $g_\alpha(A)$ is contained in the range of $g_\alpha(\mathbf{X}(T_\ell))$. From the formula of $g_\alpha(\mathbf{X}(T_\ell))$, we can see the Jacobian matrix is

$$\frac{\partial\, g_\alpha(\mathbf{X}(T_\ell))}{\partial(T_{b_2}, \cdots, T_{b_n}, T_\ell)} = \begin{pmatrix} 0 & 0 & \cdots & 0 & -g \\ 2g & 0 & \cdots & 0 & -g \\ 0 & 2g & \cdots & 0 & -g \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 2g & -g \end{pmatrix}$$

with determinant $g(2g)^{n-1}$. Hence, $g_\alpha(\mathbf{X}(T_\ell))$ will have a positive continuous PDF, its PDF will have a positive minimum value on $g_\alpha(A)$. Let m_2 to be this minimum value, then

$$\mathbb{P}_{\mathbf{a}}(g_\alpha(\mathbf{X}(T_\ell)) \in g_\alpha(A)) \geq m_2 \operatorname{Leb}_n(g_\alpha(A)) = m_2 \operatorname{Leb}_n(A)$$

(III) If $A \subset E_{\mathrm{id},2\varepsilon} \cap H_{2\varepsilon}$.

Follow the same way, let

$$b_k = \frac{(2n-k+1)(k-2)}{2} + 1, \quad 2 \leq k \leq n$$

.

$$U_{b_k+i} = n - i, \;\; \text{for } 2 \leq k \leq n-1, 0 \leq i \leq b_{k+1} - b_k - 1$$

Then

$$\mathbf{X}(T_{b_n}) = (a_1 - gT_{b_n}, -a_n + 2gT_{b_2} - gT_{b_n}, -a_{n-1} + 2gT_{b_3} - gT_{b_n}, \cdots, -a_2 + gT_{b_n})$$

Next, we let $U_{b_n+i} = i$ for $1 \le i \le n$ and $b_{n+1} = b_n + n$, then

$$\mathbf{X}(T_{b_{n+1}}) = (-a_n + 2gT_{b_2} - gT_{b_{n+1}}, -a_{n-1} + 2gT_{b_3} - gT_{b_{n+1}}, \cdots, -a_1 + gT_{b_{n+1}})$$

Using $X_1, \cdots, X_n$ to represent $T_{b_2}, \cdots, T_{b_n}, T_{b_{n+1}}$ we can get

$$\begin{cases} T_{b_2} = \dfrac{1}{2g}\left((X_1 + a_n) + (X_n + a_1)\right) \\ T_{b_3} = \dfrac{1}{2g}\left((X_2 + a_{n-1}) + (X_n + a_1)\right) \\ \quad\vdots \\ T_{b_n} = \dfrac{1}{2g}\left((X_{n-1} + a_2) + (X_n + a_1)\right) \\ T_{b_{n+1}} = \dfrac{1}{g}\left(X_n + a_1\right) \end{cases}$$

Note that for $\mathbf{x} \in A \subset E_{id,\varepsilon} \cap H_\varepsilon$ we have

$$x_1 + a_n < x_2 - 2\varepsilon < x_2 + a_{n-1} < x_3 - \varepsilon < x_3 + a_{n-2} < \cdots < x_n + a_1$$

$$x_1 + x_n + a_1 + a_n \ge 2\varepsilon + a_1 + a_n > 0$$

Hence, A is contained in the range of $\mathbf{X}(T_{b_{n+1}})$.

From the formula of $\mathbf{X}(T_{b_{n+1}})$, we can see the Jacobian matrix is

$$\frac{\partial\, \mathbf{X}(T_{b_{n+1}})}{\partial(T_{b_2}, \cdots, T_{b_{n+1}})} = \begin{pmatrix} 2g & 0 & \cdots & 0 & -g \\ 0 & 2g & \cdots & 0 & -g \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 2g & -g \\ 0 & 0 & \cdots & 0 & g \end{pmatrix}$$

with determinant $g(2g)^{n-1}$. Hence, $\mathbf{X}(T_{b_{n+1}})$ will have a positive continuous PDF, its PDF will have a positive minimum value on A. Let m_3 to be this minimum value, then

$$\mathbb{P}_{\mathbf{a}}(\tau_A < \infty) \ge \left(\frac{1}{n}\right)^{b_{n+1}} \mathbb{P}_{\mathbf{a}}(\mathbf{X}(T_{b_{n+1}}) \in A) \ge \left(\frac{1}{n}\right)^{b_{n+1}} m_3\, \mathrm{Leb}_n(A)$$

(IV) If $A \subset E_{\alpha,2\varepsilon} \cap H_{2\varepsilon}$, for some $\alpha \neq \text{id}$.

Define $b_2, b_3, \cdots, b_{n+1}$ and $U_i, 1 \le i \le b_{n+1}$ as the same in (III), then

$$\mathbf{X}(T_{b_{n+1}}) = (-a_n + 2gT_{b_2} - gT_{b_{n+1}}, -a_{n-1} + 2gT_{b_3} - gT_{b_{n+1}}, \cdots, -a_1 + gT_{b_{n+1}})$$

There exists $\ell' > b_{n+1}$ and some $u_{b_{n+1}+1}, \cdots, u_{\ell'} \in \{1, 2, \cdots, n-1\}$ such that with

$$U_{b_{n+1}+1} = u_{b_{n+1}+1}, U_{b_{n+1}+2} = u_{b_{n+1}+2}, \cdots, U_{\ell'} = u_{\ell'}$$

Our ℓ' here only depends on the permutation α. We have $X_i(T_{\ell'}) = X_{\alpha^{-1}(i)}(T_{b_{n+1}}) - g(T_{\ell'} - T_{b_{n+1}})$. Define $g_\alpha : \mathbb{R}^n \to \mathbb{R}^n$ to be

$$g_\alpha((x_1, \cdots, x_n)) = (x_{\alpha(1)}, x_{\alpha(2)}, \cdots, x_{\alpha(n)})$$

Note that reflection preserves measure, so $\text{Leb}_n(g_\alpha(A) = \text{Leb}_n(A)$. Then

$$\mathbb{P}_{\mathbf{a}}(\tau_A < \infty) \ge \left(\frac{1}{n}\right)^{\ell'} \mathbb{P}_{\mathbf{a}}(\mathbf{X}(T_{\ell'}) \in A) = \left(\frac{1}{n}\right)^{\ell'} \mathbb{P}_{\mathbf{a}}(g_\alpha(\mathbf{X}(T_{\ell'})) \in g_\alpha(A))$$

Note that $X_{\alpha(i)}(T_{\ell'}) = X_i(T_{b_{n+1}}) - g(T_{\ell'} - T_{b_{n+1}})$, so

$$g_\alpha(\mathbf{X}(T_{\ell'})) = (-a_n + 2gT_{b_2} - gT_{\ell'}, -a_{n-1} + 2gT_{b_3} - gT_{\ell'}, \cdots, -a_1 + 2gT_{b_{n+1}} - gT_{\ell'})$$

Representing $T_{b_2} - (T_{\ell'} - T_{b_{n+1}}), T_{b_3} - T_{b_2}, \cdots, T_{b_{n+1}} - T_{b_n}$ with $X_1, \cdots, X_n$ we have

$$\begin{cases} T_{b_2} - (T_{\ell'} - T_{b_{n+1}}) = \dfrac{1}{2g}((X_1 + a_n) + (X_n + a_1)) \\ T_{b_3} - T_{b_2} = \dfrac{1}{2g}((X_2 + a_{n-1}) - (X_1 + a_n)) \\ T_{b_4} - T_{b_3} = \dfrac{1}{2g}((X_3 + a_{n-2}) - (X_2 + a_{n-1})) \\ \vdots \\ T_{b_{n+1}} - T_{b_n} = \dfrac{1}{2g}((X_n + a_1) - (X_{n-1} + a_2)) \end{cases}$$

For each $\mathbf{x} \in g_\alpha(A)$ we have

$$x_1 + a_n < x_2 - 2\varepsilon < x_2 + a_{n-1} < x_3 - \varepsilon < x_3 + a_{n-2} < \cdots < x_n + a_1$$

$$x_1 + x_n + a_1 + a_n \ge 2\varepsilon + a_1 + a_n > 0$$

Hence, $g_\alpha(A)$ is contained in the range of $g_\alpha(\mathbf{X}(T_{\ell'}))$. From the formula of $g_\alpha(\mathbf{X}(T_{\ell'}))$, we can see the Jacobian matrix is

$$\frac{\partial\, g_\alpha(\mathbf{X}(T_{\ell'}))}{\partial\left(T_{b_2}-(T_{\ell'}-T_{b_{n+1}}),T_{b_3}-T_{b_2}\cdots,T_{b_{n+1}}-T_{b_n}\right)}=\begin{pmatrix} g & -g & \cdots & -g & -g \\ g & g & \cdots & -g & -g \\ \vdots & \vdots & & \vdots & \vdots \\ g & g & \cdots & g & -g \\ g & g & \cdots & g & g \end{pmatrix}$$

with determinant $2^{n-1}g$. Hence, $g_\alpha\left(\mathbf{X}(T_{\ell'})\right)$ will have a positive continuous PDF, and its PDF will have a positive minimum value on $g_\alpha(A)$. Let m_4 be to be this minimum value, then

$$\mathbb{P}_{\mathbf{a}}(g_\alpha(\mathbf{X}(T_{\ell'}))\in g_\alpha(A))\geq m_4\,\mathrm{Leb}_n(g_\alpha(A))=m_4\,\mathrm{Leb}_n(A)$$

Combine (I),(II),(III) and (IV) together we can see that for each $\mathbf{a}\in B_1$

$$\mathbb{P}_{\mathbf{a}}(\tau_A<\infty)\geq\min\left\{m_1\left(\frac{1}{n}\right)^{b_n}e^{-\frac{2M+\varepsilon}{2g}}\,\mathrm{Leb}_{n-1}(h(A)),m_2\left(\frac{1}{n}\right)^{\ell}\mathrm{Leb}_n(A),\right.$$
$$\left.m_3\left(\frac{1}{n}\right)^{b_{n+1}}\mathrm{Leb}_n(A),m_4\left(\frac{1}{n}\right)^{\ell'}\mathrm{Leb}_n(A)\right\}>0$$

□

1.7 Model with specific invariant distribution

sec:normaluniform

In this section, instead of the collision with the ground, we will add a very large constant acceleration ng in the upward direction to the ball on the bottom. We can still define the process $\mathbf{V}$ in the same way in section 1.3. Let $\mathcal{S}_n$ be the set of all permutations on $\{1, 2, \cdots, n\}$. Recall the definition for $\mathbf{X}$ and $\mathbf{V}$, at each time t there exists a unique permutation α_t such that

$$V_{\alpha_t(i)}(t) = X_i(t), \text{ for } 1 \le i \le n$$

where $V_{\alpha_t(i)}$ is the $\alpha_t(i)-$th coordinate of $\mathbf{V}$ and X_i is the $i-$th coordinate of $\mathbf{X}$.

For each $\sigma \in \mathcal{S}_n$, define

$$\mathbf{V}^{(\sigma)}(t) = \text{Leb}\left(s \in [0,t] : \alpha_s = \sigma\right), \lambda_\sigma = -\mathbf{g} + ng\mathbf{e}_{\sigma(n)}$$

where $\mathbf{g} = (g, g, \cdots, g)^T$, $\mathbf{e}_{\sigma(n)}$ has 1 for $\sigma(n) - th$ coordinate and 0 for other places. Note that $\|\lambda_\sigma\|_n$ is always a constant, call this constant c_λ.

By the definition of the process, we have

$$\mathbf{V}(t) - \mathbf{V}(0) = -\mathbf{g}t + \sum_{\sigma \in \mathcal{S}_n} ng\mathbf{e}_{\sigma(n)}\mathbf{V}^{(\sigma)}(t) = \sum_{\sigma \in \mathcal{S}_n} (-\mathbf{g} + ng\mathbf{e}_{\sigma(n)})\mathbf{V}^{(\sigma)}(t) = \sum_{\sigma \in \mathcal{S}_n} \lambda_\sigma \mathbf{V}^{(\sigma)}(t).$$

Functions

$$a_{\sigma_1,\sigma_2}(\mathbf{v}) : \mathbb{R}^n \mapsto \mathbb{R}, \quad \sigma_1, \sigma_2 \in \mathcal{S}_n,$$

serve as the intensities for collisions. We will describe the evolution of the process as follows.

Let $(E_k^\sigma)_{\sigma \in \mathcal{S}_n, k \ge 0}$ be a family of i.i.d exponential random variables with mean 1. Let $(T_i)_{i \ge 0}$ be the sequence of collision times, with $T_0 = 0$. Assuming that the process is defined up to time T_i, we recursively define:

$$T_{i+1}^\sigma = \inf_t \left\{ \int_{T_i}^t a_{\alpha(T_i),\sigma}\left(\mathbf{V}(T_i) + \lambda_{\alpha(T_i)}(s - T_i)\right)\, ds \ge E_{i+1}^\sigma \right\},$$

$$T_{i+1} = \min_{\sigma \in \mathcal{S}_n} T_{i+1}^\sigma.$$

Set $\inf \phi = \infty$ for convenience. Then, let

$$\mathbf{V}(s) = \mathbf{V}(T_i) + \lambda_{\alpha(T_i)}(s - T_i), \qquad \text{for } s \in [T_i, T_{i+1}].$$

$$\alpha(s) = \alpha(T_i), \qquad \text{for } s \in [T_i, T_{i+1}).$$

$$\alpha(T_{i+1}) = \arg\min(T_{i+1}^{\sigma} : \sigma \in \mathcal{S}_n).$$

Note that the process α is determined by the sequence $\{\alpha(T_i)\}_{i=0}^{\infty}$. Fix some real numbers v_i, then let $V_i(0) = v_i$. Since our process preserves momentum, $(\alpha, \mathbf{V})$ would be a continuous time Markov process with state space $\mathcal{S}_n \times \mathcal{M}^{n-1}$, where

$$\mathcal{M}^{n-1} = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} v_i\}.$$

Let $(i, i+1)$ be the adjacent transition, i.e. it switches the $i-$th value and the $(i+1)-$th value. Taking the intensities as follows:

$$a_{\sigma_1,\sigma_2}(\mathbf{v}) = \begin{cases} ig(v_{\sigma_1(i+1)} - v_{\sigma_1(i)})^+ & \text{if } \sigma_2 = (i, i+1)\sigma_1 \\ 0 & \text{else} \end{cases}$$

We claim that the process is irreducible in the sense of that there are $\sigma_1 \in \mathcal{S}^n$ and a non-empty set $E_1 \subset \mathcal{M}^{n-1}$ such that

$$\mathbb{P}\left((\alpha, \mathbf{V})(t) \in \sigma_1 \times E_1 \mid \alpha(0) = \sigma, \mathbf{V}(0) = \mathbf{v}\right) > 0$$

for any $(\sigma, \mathbf{v}) \in \mathcal{S}^n \times \mathcal{M}^{n-1}$ and some $t > 0$.

Similar to definition 1.4.1, we will define $\overset{(\alpha,\mathbf{V})}{\longrightarrow}$ as follows.

irrder **Definition 1.7.1.** For $(\beta, \mathbf{a}), (\gamma, \mathbf{b}) \in \mathcal{S}^n \times \mathcal{M}^{n-1}$, we say $(\gamma, \mathbf{b})$ is accessible from $(\beta, \mathbf{a})$ by $(\alpha, \mathbf{V})$ (written as $(\beta, \mathbf{a}) \overset{(\alpha,\mathbf{V})}{\longrightarrow} (\gamma, \mathbf{b})$) if there exist deterministic sequences $\{t_i\}_{i=1}^{m}$ and $\{\alpha_i\}_{i=1}^{m-1}$ such that when $(\alpha(0), \mathbf{V}(0)) = (\beta, \mathbf{a})$, $T_i = t_i$, $\alpha(t_i) = \alpha_i$ for $1 \le i \le m-1$, $T_m = t_m$, $\alpha(t_m) = \gamma$ there exists some $t > t_m$ such that

$$\mathbf{V}(t_m) + \lambda_\gamma(t - T_m) = \mathbf{b}.$$

Suppose $(\alpha', \mathbf{V}')$ is another process following the same evolution as $(\alpha, \mathbf{V})$ with jump times T_i'.

appro **Lemma 1.7.2.** *Fix $N < \infty$. If $\sum_{i=1}^{N} |T_i - T_i'| < \delta$, $\|\mathbf{V}(0) - \mathbf{V}'(0)\|_n < \delta$ and $\alpha(T_i) = \alpha'(T_i')$ for each $1 \le i \le N$, we have*

$$\|\mathbf{V}(T_N) - \mathbf{V}'(T_N')\|_n \le \delta + N c_\lambda \delta.$$

Furthermore, for $\max\{T_N, T'_N\} \le t < \min\{T_{N+1}, T'_{N+1}\}$ *(if such t exists) we have*

$$\|\mathbf{V}(t) - \mathbf{V}'(t)\|_n \le \delta + (N+1)c_\lambda \delta.$$

Proof. For every $1 \le k \le N-1$ we have

$$\begin{aligned}
\|\mathbf{V}(T_{k+1}) - \mathbf{V}'(T'_{k+1})\|_n &= \|\mathbf{V}(T_k) + \lambda_{\alpha(T_k)}(T_{k+1} - T_k) - \mathbf{V}'(T'_k) - \lambda_{\alpha'(T'_k)}(T'_{k+1} - T'_k)\|_n \\
&\le \|\mathbf{V}(T_k) - \mathbf{V}'(T'_k)\|_n + |T_{k+1} - T_k - T'_{k+1} + T'_k| \cdot \|\lambda_{\alpha(T_k)}\|_n \\
&\le \|\mathbf{V}(T_k) - \mathbf{V}'(T'_k)\|_n + \delta \cdot c_\lambda.
\end{aligned}$$

Hence, by induction

$$\|\mathbf{V}(T_N) - \mathbf{V}'(T'_N)\|_n \le \|\mathbf{V}(0) - \mathbf{V}'(0)\|_n + N\delta c_\lambda \le \delta + N c_\lambda \delta.$$

For $\max\{T_N, T'_N\} \le t < \min\{T_{N+1}, T'_{N+1}\}$, we have

$$\begin{aligned}
\|\mathbf{V}(t) - \mathbf{V}'(t)\|_n &= \|\mathbf{V}(T_N) + \lambda_{\alpha(T_N)}(t - T_N) - \mathbf{V}'(T'_N) - \lambda_{\alpha'(T'_N)}(t - T'_N)\|_n \\
&\le \|\mathbf{V}(T_N) - \mathbf{V}'(T'_N)\|_n + |T_N - T'_N| \cdot \|\lambda_{\alpha(T_N)}\|_n \\
&\le \delta + (N+1)c_\lambda \delta.
\end{aligned}$$

□

newirrder **Lemma 1.7.3.** *For any* $\mathbf{a} = (a_1, \cdots, a_n), \mathbf{b} = (b_1, \cdots, b_n) \in \mathcal{M}^{n-1}$*, we have* $\mathbf{a} \overset{(\alpha,\mathbf{V})}{\longrightarrow} \mathbf{b}$.

Proof. Since we allow collisions happens at the time $t = 0$, we can use collisions to sort the initial velocities $a_1, \cdots, a_n$ from large to small, by an argument similar to that in part(i) of the proof of Lemma 1.4.3. Hence, WLOG, we may assume that $a_1 \ge a_2 \ge \cdots \ge a_n$. Let $\bar{a}_n = \frac{1}{n}\sum_{i=1}^n a_i$ and $\mathbf{a}' = (\bar{a}_n, \cdots, \bar{a}_n)$.

(i) We claim that $\mathbf{a} \overset{(\alpha,\mathbf{V})}{\longrightarrow} \mathbf{a}'$. For $1 \le i \le n-1$, let $t_i = (a_1 - a_{n+1-i})/ng$, then $\{t_i\}_{i=1}^{n-1}$ is a sequence of non-negative numbers by our assumption for $\{a_i\}_{i=1}^n$. Let $m_0 = 0$

and $m_j = \sum_{k=1}^{j}(n-k)$ for $1 \le j \le n-1$. It is elementary to check that every integer m between 1 and $\frac{n(n-1)}{2}$ can be represented by a unique pair of (j, i_j) such that $m = m_{j-1}+i_j$, $1 \le j \le n-1$ and $1 \le i_j \le n-j$. We will build a trajectory as follows. We will define T_m and U_m for every $1 \le m \le \frac{n(n-1)}{2}$. Represent m as $m = i_j + m_{j-1}$ for some $1 \le j \le n-1$ and $1 \le i_j \le n-j$, let

$$T_{i_j+m_{j-1}} = \sum_{k=1}^{j} t_k, \quad U_{i_j+m_{j-1}} = n - i_j.$$

We will show that for this trajectory the collisions can happen at positions U_k at time T_k (in the sense that the intensity a is strictly positive) and this trajectory ends at point $\mathbf{a}'$. Note that $T_1 = t_1$. At time T_1- we have

$$\mathbf{X}(T_1-) = (a_1 - gt_1, a_2 - gt_1, \cdots, a_n + (n-1)gt_1).$$

Observe that

$$a_n + (n-1)gt_1 = a_1 - gt_1 \ge a_2 - gt_1 \ge \cdots \ge a_{n-1} - gt_1,$$

we can move the velocity at the bottom to the top. Hence, the trajectory is well defined until time and at that time the velocity is

$$\mathbf{X}(T_{n-1}) = (a_n + (n-1)gt_1, a_1 - gt_1, \cdots, a_{n-1} - gt_1).$$

Suppose the trajectory is well defined until time n_j for some $1 \le j < n-1$ and at the time m_j the velocity satisfies

$$X_i(T_{m_j}) = \begin{cases} a_{n-i+1} - g\sum_{k=1}^{j} t_k + ngt_i & \text{for } 1 \le i \le j \\ a_{i-j} - g\sum_{k=1}^{j} t_k & \text{for } j+1 \le i \le n \end{cases}$$

Then at time $\left(T_{m_j} + t_{j+1}\right)-$, we have

$$X_i(T_{n_j} + t_{j+1}-) = \begin{cases} a_{n-i+1} - g\sum_{k=1}^{j+1} t_k + ngt_i & \text{for } 1 \le i \le j \\ a_{i-j} - g\sum_{k=1}^{j+1} t_k & \text{for } j+1 \le i \le n-1 \\ a_{n-j} - g\sum_{k=1}^{j+1} t_k + ngt_{j+1} & \text{for } i = n \end{cases}$$

Note that

$$a_{n-j} - g\sum_{k=1}^{j+1} t_k + ngt_{j+1} = a_1 - g\sum_{k=1}^{j+1} t_k \geq a_2 - g\sum_{k=1}^{j+1} t_k \cdots \geq a_{n-1-j} - g\sum_{k=1}^{j+1} t_k,$$

we can pop up the velocity at the bottom to the $(j+1)$-th place. Hence, the trajectory is well defined until $m_j + (n-j-1) = n_{j+1}$ and at that time the velocity is

$$X_i(T_{n_{j+1}}) = \begin{cases} a_{n-i+1} - g\sum_{k=1}^{j+1} t_k + ngt_i & \text{for } 1 \leq i \leq j+1 \\ a_{i-j-1} - g\sum_{k=1}^{j+1} t_k & \text{for } j+2 \leq i \leq n \end{cases}$$

Therefore, by induction the trajectory is well defined until $T_{m_{n-1}}$ and at that time

$$X_i(T_{m_{n-1}}) = \begin{cases} a_{n-i+1} - g\sum_{k=1}^{n-1} t_k + ngt_i = \bar{a}_n & \text{for } 1 \leq i \leq n-1 \\ a_1 - g\sum_{k=1}^{n-1} t_k = \bar{a}_n & \text{for} i = n \end{cases}$$

It follows that $\mathbf{a} \overset{(\alpha,\mathbf{V})}{\longrightarrow} \mathbf{a}'$.

(ii) We can define a reverse process using the same idea in the proof of Lemma 1.4.3 (iv). Define a process $\mathbf{X}'(t) := (X_1'(t), \ldots, X_n'(t))$ similar to $\mathbf{X}$ as follows: for every $k \geq 0$ and $T_k \leq t < T_{k+1}$, $X_i'(t) = X_i'(T_k) + g(t - T_k)$ for $i = 1, \ldots, n-1$, and $X_n'(t) = X_n'(T_k) + g(1-n)(t - T_k)$. We will now define "jumps" of $\mathbf{X}'$. If $U_k = j$ with $j < n$ and $X_j'(T_k-) > X_{j+1}'(T_k-)$ then we let $X_j'(T_k) = X_{j+1}'(T_k-)$ and $X_{j+1}'(T_k) = X_j'(T_k-)$. All other velocities remain unchanged at time T_k. If the conditions listed above are not satisfied then $\mathbf{X}'(T_k) = \mathbf{X}'(T_k-)$.

Follow the same argument in (i), $\mathbf{a}'$ is accessible from $\mathbf{a}$ under X'. Then by the reversibility, we have $\mathbf{a}' \overset{(\alpha,\mathbf{V})}{\longrightarrow} \mathbf{a}$

(iii) Combine the results in (ii) and (i) together, we get the desire result.

□

Let $\mathcal{U}(\mathcal{S}_n)$ denotes the uniform distribution on $\mathcal{S}_n$ and $\mathcal{N}_n$ be the $n-$ dimensional standard normal distribution.

ormaluniform **Theorem 1.7.4.** *With following intensities:*

$$a_{\sigma_1,\sigma_2}(\mathbf{v}) = \begin{cases} ig(v_{\sigma_1(i+1)} - v_{\sigma_1(i)}) & \text{if } \sigma_2 = (i \;\; i+1)\sigma_1 \text{ and } v_{\sigma_1(i+1)} > v_{\sigma_1(i)} \\ 0 & \text{else} \end{cases}$$

where $(i \;\; i+1)$ *is the adjacent transposition.(***) Then* $(\alpha, \mathbf{V})$ *has invariant distribution* $\mathcal{U}(\mathcal{S}_n) \times \mathcal{N}_n$.

The proof will refer to the following result [3, Corollary 2.3].

BW **Lemma 1.7.5.** *The invariant distribution for* $(\alpha, \mathbf{V})$ *is* $\mathcal{U}(\mathcal{S}_n) \times \mathcal{N}_n$ *if and only if*

$$\lambda_{\sigma_1} \cdot \mathbf{v} + \sum_{\sigma_2 \in \mathcal{S}_n} a_{\sigma_2,\sigma_1}(\mathbf{v}) - \sum_{\sigma_2 \in \mathcal{S}_n} a_{\sigma_1,\sigma_2}(\mathbf{v}) = 0$$

for all $\sigma_1 \in \mathcal{S}_n$ *and* $\mathbf{v} \in \mathbb{R}^n$.

proof of theorem 1.7.4. Let $A_1 = \{1 \le i \le n-1 : \mathbf{v}_{\sigma_1(i)} < \mathbf{v}_{\sigma_1(i+1)}\}$, $A_2 = \{1 \le i \le n-1 : \mathbf{v}_{\sigma_1(i)} > \mathbf{v}_{\sigma_1(i+1)}\}$, Then for each $1 \le i \le n-1$, it is either $i \in A_1 \cup A_2$ or $\mathbf{v}_{\sigma_1(i)} = \mathbf{v}_{\sigma_1(i+1)}$. Then

$$\begin{aligned}
\sum_{\sigma_2 \in \mathcal{S}_n} a_{\sigma_1,\sigma_2}(\mathbf{v}) - \sum_{\sigma_2 \in \mathcal{S}_n} a_{\sigma_2,\sigma_1}(\mathbf{v}) &= \sum_{i \in A_1} ig\left(\mathbf{v}_{\sigma_1(i+1)} - \mathbf{v}_{\sigma_1(i)}\right) - \sum_{i \in A_2} ig\left(\mathbf{v}_{\sigma_1(i)} - \mathbf{v}_{\sigma_1(i+1)}\right) \\
&= \sum_{i \in A_1} ig\left(\mathbf{v}_{\sigma_1(i+1)} - \mathbf{v}_{\sigma_1(i)}\right) + \sum_{i \in A_2} ig\left(\mathbf{v}_{\sigma_1(i+1)} - \mathbf{v}_{\sigma_1(i)}\right) \\
&= \sum_{i=1}^{n-1} ig\left(\mathbf{v}_{\sigma_1(i+1)} - \mathbf{v}_{\sigma_1(i)}\right) \\
&= -\sum_{i=1}^{n} g\mathbf{v}_{\sigma_1(i)} + ng\mathbf{v}_{\sigma_1(n)} \\
&= \left(-\mathbf{g} + b\mathbf{e}_{\sigma_1(n)}\right) \cdot \mathbf{v} \\
&= \lambda_{\sigma_1} \cdot \mathbf{v}.
\end{aligned}$$

□

Chapter 2

FOLDINGS

2.1 Motivation

The motivation for the folding model in this section comes from the pinned ball model on a torus. Suppose we have n pinned balls with the same radii tightly packed on a d-dimensional torus. Their velocities form a point in $\mathbb{R}^{nd}$. Since collisions preserve energy, the point representing all pseudo-velocities will always stay on a fixed sphere centered at the origin. Each collision could be regarded as a folding on the sphere - see the definition in the next section.

A way to estimate the rate of convergence to the stationary distribution is via couplings. An example of a coupling in our context is to have the same sequence of pairs of balls collide for two copies of the process. This corresponds to a sequence of foldings with respect to hyperplanes in a fixed family. The family has a complicated and unexplored structure. For this reason, in our project, we will consider an i.i.d. sequence of uniformly distributed foldings.

2.2 Informal review of main results

Let d_0 to be the distance before a folding and d_1 to be the distance after the folding. We will show that when n goes to ∞ and d_0 goes to 2, $n(d_0 - d_1)$ converges to the Gamma$\left(\frac{1}{2}, \frac{1}{2}\right)$ distribution. We will also provide the asymptotic distribution of $n\left(1 - \frac{d_1}{d_0}\right)$ when n goes to ∞ and d_0 goes to 0, conditionally on $d_1 < d_0$.

2.3 Model definition

We will consider a random folding in the unit sphere in $\mathbb{R}^n$. To be more specific, let S_n to represent the unit sphere in $\mathbb{R}^n$, i.e.

$$S_n = \{\mathbf{X} \in \mathbb{R}^n : \|\mathbf{X}\| = 1\}, \text{ here } \|\cdot\| \text{ represents the Euclidean norm}$$

Points $\mathbf{X}^{(n)}, \mathbf{Y}^{(n)}$ are two fixed points on S^n. Let $\mathbf{U}^{(n)}$ to be a point uniformly distributed on the sphere S^n, define the folding function as follows:

$$F_{\mathbf{U}^{(n)}}(\mathbf{X}^{(n)}) = \begin{cases} \mathbf{X}^{(n)} & \text{if } \left(\mathbf{U}^{(n)}\right)^T \mathbf{X}^{(n)} \geq 0 \\ \mathbf{X}^{(n)} - 2\left(\left(\mathbf{U}^{(n)}\right)^T \mathbf{X}^{(n)}\right)\mathbf{U}^{(n)} & \text{if } \left(\mathbf{U}^{(n)}\right)^T \mathbf{X}^{(n)} < 0 \end{cases} \tag{2.3.1}$$ `fold`

After folding the point is still on the sphere S^n since

$$\begin{aligned} &\left\|\mathbf{X}^{(n)} - 2\left(\left(\mathbf{U}^{(n)}\right)^T \mathbf{X}^{(n)}\right)\mathbf{U}^{(n)}\right\|^2 \\ =& \left\|\mathbf{X}^{(n)}\right\|^2 + 4\left(\left(\mathbf{U}^{(n)}\right)^T \mathbf{X}^{(n)}\right)^2 \left\|\mathbf{U}^{(n)}\right\|^2 - 4\left(\left(\mathbf{U}^{(n)}\right)^T \mathbf{X}^{(n)}\right)^2 = 1 \end{aligned}$$

Let $d_0 = \left\|\mathbf{X}^{(n)} - \mathbf{Y}^{(n)}\right\|$, i.e. the distance before folding, and $d_1 = \left\|F_{\mathbf{U}^{(n)}}(\mathbf{X}^{(n)}) - F_{\mathbf{U}^{(n)}}(\mathbf{Y}^{(n)})\right\|$, i.e. the distance after the folding. We are interested in the distribution of d_1.

First of all, from the folding function (2.3.1) we can easily see that the distance will not change if both $\mathbf{X}^{(n)}$ and $\mathbf{Y}^{(n)}$ do not fold.

If, both of the points fold, the distance also won't change since follow the same way above we can show that

$$\begin{aligned} &\left\|\mathbf{X}^{(n)} - 2\left(\left(\mathbf{U}^{(n)}\right)^T \mathbf{X}^{(n)}\right)\mathbf{U}^{(n)} - \mathbf{Y}^{(n)} + 2\left(\left(\mathbf{U}^{(m)}\right)^T \mathbf{Y}^{(n)}\right)\mathbf{U}^{(n)}\right\|^2 \\ =& \left\|\mathbf{X}^{(n)} - \mathbf{Y}^{(n)} - 2\left(\mathbf{U}^{(n)}\right)^T \left(\mathbf{X}^{(n)} - \mathbf{Y}^{(n)}\right)\mathbf{U}^{(n)}\right\|^2 \\ =& \left\|\mathbf{X}^{(n)} - \mathbf{Y}^{(n)}\right\|^2 \end{aligned}$$

If, exactly one of the point folds, WLOG, we may assume it is the point $\mathbf{X}^{(n)}$. Then in this case, we have

$$\begin{aligned} d_1^2 =& \left\|\mathbf{X}^{(n)} - 2\left(\left(\mathbf{U}^{(n)}\right)^T \mathbf{X}^{(n)}\right)\mathbf{U}^{(n)} - \mathbf{Y}^{(n)}\right\|^2 \\ =& \left\|\mathbf{X}^{(n)} - \mathbf{Y}^{(n)}\right\|^2 + 4\left(\left(\mathbf{U}^{(n)}\right)^T \mathbf{X}^{(n)}\right)^2 - 4\left(\mathbf{U}^{(n)}\right)^T \mathbf{X}^{(n)} \left(\mathbf{U}^{(n)}\right)^T \left(\mathbf{X}^{(n)} - \mathbf{Y}^{(n)}\right) \\ =& \left\|\mathbf{X}^{(n)} - \mathbf{Y}^{(n)}\right\|^2 + 4\left(\left(\mathbf{U}^{(n)}\right)^T \mathbf{X}^{(n)}\right)\left(\left(\mathbf{U}^{(n)}\right)^T \mathbf{Y}^{(n)}\right) \end{aligned}$$

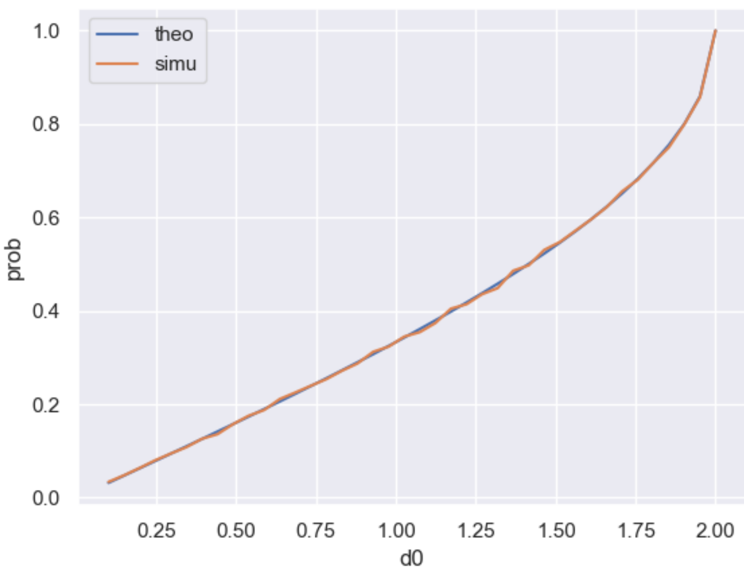

Figure 2.1: The probability of effective folding for different values of d_0 with $n = 5$ and $10,000$ repetitions (orange). The blue line is the theoretical result given in Proposition 2.3.1.

Fig-folding-prob

Therefore, combine the above cases together we can get

$$d_1^2 = d_0^2 \wedge \left(d_0^2 + 4 \left(\left(\mathbf{U}^{(n)} \right)^T \mathbf{X}^{(n)} \right) \left(\left(\mathbf{U}^{(n)} \right)^T \mathbf{Y}^{(n)} \right) \right) \tag{2.3.2}$$

As the above equation suggests, d_1 is always no larger than d_0, i.e. folding will only decrease the distance. By symmetry, it is easy to see that $\mathbb{P}(d_1 < d_2)$ depends on d_0 only. The next proposition gives this possibility.

d1lessd0 **Proposition 2.3.1.** *For every $n \in \mathbb{N}$, we have*

$$\mathbb{P}(d_1 < d_0) = \frac{\arccos\left(1 - \frac{d_0^2}{2}\right)}{\pi} \tag{2.3.3}$$

prob-d1lessd

The simulation result is in Fig. 2.1

Proof. WLOG, we may assume that:

$$\mathbf{X}^{(n)} = \left(\sqrt{1 - \frac{d_0^2}{4}}, \frac{d_0}{2}, 0, \cdots, 0 \right), \quad \mathbf{Y}^{(n)} = \left(\sqrt{1 - \frac{d_0^2}{4}}, -\frac{d_0}{2}, 0, \cdots, 0 \right)$$

Denote $\mathbf{U}^{(n)} = (u_1, u_2, \cdots, u_n)$, then

$$\begin{aligned}
\mathbb{P}(d_1 < d_0) &= \mathbb{P}\left(\left(\left(\mathbf{U}^{(n)}\right)^T \mathbf{X}^{(n)}\right)\left(\left(\mathbf{U}^{(n)}\right)^T \mathbf{Y}^{(n)}\right) < 0\right) \\
&= \mathbb{P}\left(\left(1 - \frac{1}{4}d_0^2\right) u_1^2 - \frac{1}{4}d_0^2 u_2^2 < 0\right) \\
&= \mathbb{P}\left(\frac{u_1^2}{u_2^2} < \frac{d_0^2}{4 - d_0^2}\right) \\
&= \int_0^{\frac{d_0^2}{4-d_0^2}} \frac{1}{\pi\sqrt{x}(1+x)} dx \quad \left(\frac{u_1^2}{u_2^2} \text{ follows } F(1,1) \text{ distribution.}\right) \\
&= \int_0^{\sqrt{\frac{d_0^2}{4-d_0^2}}} \frac{2}{\pi(1+u^2)} du \quad (\text{let } u = \sqrt{x}) \\
&= \frac{2}{\pi} \arctan u \Big|_0^{\sqrt{\frac{d_0^2}{4-d_0^2}}} \\
&= \frac{2}{\pi} \arctan\left(\sqrt{\frac{d_0^2}{4 - d_0^2}}\right) \\
&= \frac{2}{\pi} \arccos \frac{\sqrt{4 - d_0^2}}{2} \\
&= \frac{1}{\pi} \arccos\left(1 - \frac{d_0^2}{2}\right).
\end{aligned}$$

□

2.4 *Main results and proofs*

model

Now we will view our model in another way. The previous section treated the points as fixed while the folding hyper-plane as random. In this section, in the opposite way, we will treat the hyper-plane as fixed and the points as random.

We will still use S_n to represent the unit sphere in $\mathbb{R}^n$, i.e.

$$S_n = \{\mathbf{X} \in \mathbb{R}^n : \|\mathbf{X}\| = 1\}, \text{ here } \|\cdot\| \text{ represents the Euclidean norm}$$

Let $\mathbf{X}^{(n)} = \left(X_1^{(n)}, \cdots, X_n^{(n)}\right)$ and $\mathbf{Y}^{(n)} = \left(Y_1^{(n)}, \cdots, Y_n^{(n)}\right)$ be two points uniformly distributed on S_n with fixed distance $0 \le d_0 \le 2$. For each $1 \le k \le n$, we will use $\mathbf{X}^{(n,k)}$ and $\mathbf{Y}^{(n,k)}$ to represent the first k coordinates of $\mathbf{X}^{(n)}$ and $\mathbf{Y}^{(n)}$, i.e.

$$\mathbf{X}^{(n,k)} = \left(X_1^{(n)}, \cdots, X_k^{(n)}\right), \quad \mathbf{Y}^{(n,k)} = \left(Y_1^{(n)}, \cdots, Y_k^{(n)}\right).$$

We fix $\mathbf{U} = (1, 0, \cdots, 0)$, then

$$d_1^2 = d_0^2 \wedge \left(d_0^2 + 4X_1^{(n)} Y_1^{(n)} \right).$$

Denote $d_{\mathrm{TV}}(P, Q)$ to be the total variation between two distributions P and Q. We will use $\mathcal{L}(\mathbf{W})$ to represent the distribution of $\mathbf{W}$ and $\mathcal{L}(\mathbf{W} \mid \mathbf{V})$ to represent the conditional distribution of $\mathbf{W}$ given $\mathbf{V}$. The following lemma comes from [8] but is stated in our own notation:

onever **Lemma 2.4.1.** *Recall that* $\mathbf{X}^{(n)}$ *is uniformly distributed on the unit sphere* S_n*, and* $\mathbf{X}^{n,k}$ *represents the first* k *coordinates.*

$$d_{TV}\left(\mathcal{L}\left(\sqrt{n}\mathbf{X}^{(n,k)} \right), \mathcal{N}(0, I_k) \right) \le C_{n,k}, \text{ where } C_{n,k} = n^{\frac{1}{2}k}(n-k-2)^{-\frac{1}{2}k} - 1.$$

For fixed k*, the joint distribution of* $\sqrt{n}\mathbf{X}^{(n,k)}$ *converges in total variation to the standard normal distribution on* $\mathbb{R}^k$ *as* $n \to \infty$ *, and all moments of the* $\sqrt{n}x_i^{(n)}$ *converge to the corresponding standard normal moments.*

The next lemma tells us that the conditional distribution $Y^{(n,k)}$given $\mathbf{X}$ depends on the first k coordinates of $\mathbf{X}^{(n)}$ only, which also implies

$$\mathcal{L}\left(\mathbf{Y}^{(n,k)} | \mathbf{X}^{(n)} \right) = \mathcal{L}\left(\mathbf{Y}^{(n,k)} | \mathbf{X}^{(n,k)} \right) \quad \text{for each } 1 \le k \le n.$$

indepent **Lemma 2.4.2.** *For each* $k < n$*, the conditional distribution* $\mathcal{L}\left(\mathbf{Y}^{(n,k)} | \mathbf{X}^{(n)} \right)$ *depends on* $\mathbf{X}^{(n,k)}$ *only, and hence given* $\mathbf{X}^{(n,k)}$*, the random vectors* $\mathbf{Y}^{(n,k)}$ *and* $(X_{k+1}^{(n)}, \cdots, X_n^{(n)})$ *are independent.*

Proof. Suppose $\mathbf{X}_1^{(n)} = (X_{1,1}^{(n)}, \cdots, X_{n,1}^{(n)})$ and $\mathbf{X}_2^{(n)} = (X_{1,2}^{(n)}, \cdots, X_{n,2}^{(n)})$ are two points on the sphere S_n with $X_1^{(n,k)} = X_2^{(n,k)}$, i.e. $X_{i,1}^{(n)} = X_{i,2}^{(n)}$ for each $1 \le i \le k$. We claim that

$$\mathcal{L}\left(\mathbf{Y}^{(n,k)} \mid \mathbf{X}^{(n)} = \mathbf{X}_1^{(n)} \right) = \mathcal{L}\left(\mathbf{Y}^{(n,k)} \mid \mathbf{X}^{(n)} = \mathbf{X}_2^{(n)} \right) \tag{2.4.1}$$ 1

Denote T to be a rotation which sends $\mathbf{X}_1^{(n)}$ to $\mathbf{X}_2^{(n)}$. By the definition of $\mathbf{X}_1^{(n)}$ and $\mathbf{X}_2^{(n)}$ we know that T preserves the first k coordinates. Because of the symmetry of S^n, we have

$$\mathcal{L}\left(\mathbf{Y}^{(n)} \mid \mathbf{X}^{(n)} = \mathbf{X}_1^{(n)} \right) = \mathcal{L}\left(T(\mathbf{Y}^{(n)}) \mid \mathbf{X}^{(n)} = \mathbf{X}_2^{(n)} \right)$$

Since T preserves the first k coordinates, the conditional distribution for the first k coordinates should also be the same, and hence we get the equation (2.4.1). ☐

From now on, we denote

$$\rho := 1 - \frac{d_0^2}{2}, \quad r := \sqrt{1-\rho^2} = \sqrt{d_0^2 - \frac{1}{4}d_0^4}. \tag{2.4.2}$$ formula-r-rho

Define processes $\{\mathbf{W}^{(n,k)}\}_{k=1}^n$ and $\{\mathbf{V}^{(n,k)}\}_{k=1}^n$ as follows:

$$\mathbf{W}^{(n,k)} := \sqrt{n}\mathbf{X}^{(n,k)}, \ \mathbf{V}^{(n,k)} := \sqrt{n-1}\left(\mathbf{Y}^{(n,k)} - \rho\mathbf{X}^{(n,k)}\right)$$

We have the following lemma:

ypart **Lemma 2.4.3.** *For any $x_1 \in [-1,1]$,*

$$d_{TV}\left(\mathcal{L}\left(\mathbf{V}^{(n,1)} \mid \mathbf{W}^{(n,1)} = \sqrt{n}x_1\right), \mathcal{N}(0, r^2(1-x_1^2))\right) \le C_{n-1,1}. \tag{2.4.3}$$ 1result

By convention, when $x_1 = \pm 1$, we let $\mathcal{N}(0, r^2(1-x_1^2))$ to be the degenerate distribution at point 0.

Remark 2.4.4. Actually, $\frac{1}{|r|\sqrt{1-x_1^2}}\phi_1\left(\frac{y_1}{|r|\sqrt{1-x_1^2}}\right) = \frac{1}{\sqrt{2\pi r^2(1-x_1^2)}}\exp\left(-\frac{1}{2r^2(1-x_1^2)}y_1^2\right)$ is the PDF for $\mathcal{N}(0, r^2(1-x_1^2))$.

proof of lemma 2.4.3. By the lemma 2.4.2 we know that for any $x_1, \cdots, x_n$,

$$\mathcal{L}\left(\mathbf{V}^{(n,1)} \mid \mathbf{W}^{(n,1)} = \sqrt{n}x_1\right) = \mathcal{L}\left(\mathbf{V}^{(n,1)} \mid \mathbf{W}^{(n)} = \sqrt{n}(x_1, x_2, \cdots, x_n)\right)$$

Hence, it suffices to show (2.4.3) holds for the point $(x_1, \sqrt{1-x_1^2}, 0, \cdots, 0)$. We will prove it in two steps:

Step 1: First we will show (2.4.3) holds for point $(1, 0, 0, \cdots, 0)$. In this case, the assumptions $\left\|\mathbf{Y}^{(n)} - \mathbf{X}^{(n)}\right\| = d_0$ and $\left\|\mathbf{Y}^{(n)}\right\| = 1$ becomes

$$\begin{cases} (y_1^{(n)} - 1)^2 + (y_2^{(n)})^2 + \cdots + (y_n^{(n)})^2 = d_0^2 \\ (y_1^{(n)})^2 + (y_2^{(n)})^2 + \cdots + (y_n^{(n)})^2 = 1 \end{cases}$$

Solving this equation system we can get

$$y_1^{(n)} = \rho, \ (y_2^{(n)})^2 + \cdots + (y_n^{(n)})^2 = r^2$$

In lemma 2.4.2 we proved that $\mathcal{L}\left(\mathbf{Y}^{(n,k)} \mid \mathbf{X}^{(n,k)}\right) = \mathcal{L}\left(\mathbf{Y}^{(n,k)} \mid \mathbf{X}^{(n)}\right)$. Apply the lemma 2.4.1, for any $k \ge 2$ we have

$$d_{\mathrm{TV}}\left(\mathcal{L}\left(\mathbf{V}^{(n,k)} \mid \mathbf{X}^{(n)}=(1,0,\cdots,0)\right), \frac{1}{r^{k-1}}\prod_{i=2}^{k}\varphi_1\left(\frac{y_i}{r}\right)\delta_{\{y_1=0\}}\right) \leq C_{n-1,k-1}. \tag{2.4.4}$$

2

Here, φ_1 is the PDF for standard normal on $\mathbb{R}$. Note that $\frac{1}{r^{k-1}}\prod_{i=2}^{k}\varphi_1\left(\frac{y_i}{r}\right)\delta_{\{y_1=0\}}$ is actually the joint PDF for the multidimensional normal distribution with mean zero and covariance matrix

$$\Sigma = \begin{pmatrix} 0 & & & \\ & r^2 & & \\ & & \ddots & \\ & & & r^2 \end{pmatrix} \tag{2.4.5}$$

Sigma1

Let $k=2$ in (2.4.4), restrict to the first coordinate and apply lemma 2.4.2 again we can get

$$\begin{aligned} & d_{\mathrm{TV}}\left(\mathcal{L}\left(\mathbf{V}^{(n,1)} \mid \mathbf{W}^{(n,1)}=\sqrt{n}\right), \delta_{\{0\}}\right) \\ =& d_{\mathrm{TV}}\left(\mathcal{L}\left(\mathbf{V}^{(n,1)} \mid \mathbf{X}^{(n)}=(1,0,\cdots,0)\right), \delta_{\{0\}}\right) \leq C_{n-1,1} \end{aligned}$$

Step 2: Now, we will show that the result holds for the point $(x_1, \sqrt{1-x_1^2}, 0, \cdots, 0)$. Define a linear map $T: S_n \to S_n$ as follows:

$$T(\mathbf{u}) = A\mathbf{u} \text{ with } A = \begin{pmatrix} x_1 & -\sqrt{1-x_1^2} & & & \\ \sqrt{1-x_1^2} & x_1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

The rotation T is on S_n and takes point $(1,0,\cdots,0)$ to $(x_1, \sqrt{1-x_1^2}, 0, \cdots, 0)$. Furthermore, for each $2 \leq k \leq n$, the first k coordinates of $T(\mathbf{u})$ only depend on the first k coordinates of $\mathbf{u}$. Then by the symmetry, we have

$$d_{\mathrm{TV}}\left(\mathcal{L}\left(\mathbf{V}^{(n,k)} \mid \mathbf{W}^{(n,k)}\right), \mathcal{N}(0, A_k\Sigma A_k^T)\right) \leq C_{n-1,k-1}, \quad \text{for } k \geq 2.$$

here Σ is given in (2.4.5), A_k and A_k^T means the first k row and k columns of A and A^T. Computing $A_k\Sigma A_k^T$ we have

$$A_k\Sigma A_k^T = \begin{pmatrix} r^2(1-x_1^2) & -r^2x_1\sqrt{1-x_1^2} & & & \\ -r^2x_1\sqrt{1-x_1^2} & r^2x_1^2 & & & \\ & & r^2 & & \\ & & & \ddots & \\ & & & & r^2 \end{pmatrix}$$

Taking $k=2$ and restrict to the first coordinate and apply lemma 2.4.2, we can get

$$d_{\mathrm{TV}}\left(\mathcal{L}\left(\mathbf{V}^{(n,1)}\mid\mathbf{W}^{(n,1)}=\sqrt{n}x_1\right),\mathcal{N}(0,r^2(1-x_1^2))\right)\le C_{n-1,1}$$

□

Using the lemma 2.4.3 we can reach the following theorem:

joint **Theorem 2.4.5.** *Define $W_0^{(n)},V_0^{(n)}$ to be the random variables with joint PDF:*

$$p^{(0)}_{V,W}(v,w)=\begin{cases}\frac{1}{2\pi r\sqrt{1-\frac{w^2}{n}}}\exp\left(-\frac{1}{2}w^2-\frac{1}{2r^2\left(1-\frac{w^2}{n}\right)}v^2\right) & \text{for } |w|<\sqrt{n}.\\ \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}w^2\right)\delta_{\{v=0\}} & \text{for } |w|\ge\sqrt{n}\end{cases}$$

Let Φ_1 be the CDF for standard normal on $\mathbb{R}$. Then

$$d_{TV}\left(\mathcal{L}(\mathbf{V}^{(n,1)},\mathbf{W}^{(n,1)}),\mathcal{L}(V_0^{(n)},W_0^{(n)})\right)\le C_{n,1}+C_{n-1,1}+\Phi(-\sqrt{n}).$$

Proof. Let

$$f(v\mid w)=\frac{1}{\sqrt{2\pi}\cdot r\sqrt{1-\frac{w^2}{n}}}\exp\left(-\frac{1}{2r^2\left(1-\frac{w^2}{n}\right)}v^2\right),$$

Then $\int_{\mathbb{R}}f(v\mid w)dv=1$ for each $|w|<\sqrt{n}$.

Denote $p_{V,W}(v,w)$ to be the joint PDF of $\left(\mathbf{V}^{(n,1)},\mathbf{W}^{(n,1)}\right)$, $p_W(w)$ to be the marginal PDF of $\mathbf{W}^{(n,1)}$. Recall that φ_1 is the PDF for standard normal on $\mathbb{R}$, from lemma 2.4.1 and 2.4.3, we have

$$\int_{\mathbb{R}}|p_W(w)-\varphi_1(w)|\,dw\le 2C_{n,1} \tag{2.4.6}$$ joint.1

$$\int_{\mathbb{R}}\left|\frac{p_{V,W}(v,w)}{p_W(w)}-f(v\mid w)\right|dv\le 2C_{n-1,1},\quad\text{for each } |w|\le\sqrt{n}. \tag{2.4.7}$$ joint.2

First we will verify that $p^{(0)}_{V,W}(v,w)$ is a valid PDF. Note that $p^{(0)}_{V,W}(v,w)=\phi_1(w)f(v\mid w)$ when $|w|<\sqrt{n}$, we have

$$\begin{aligned}\iint p^{(0)}_{V,W}(v,w)dvdw&=\iint_{\{|w|<\sqrt{n}\}}\varphi_1(w)f(v\mid w)dvdw+\int_{\{|w|\ge\sqrt{n}\}}\varphi_1(w)dw\\&=\int_{-\sqrt{n}}^{\sqrt{n}}\varphi_1(w)dw+2\Phi(-\sqrt{n})=\Phi(\sqrt{n})-\Phi(-\sqrt{n})+2\Phi(-\sqrt{n})=1\end{aligned}$$

For the total variation, after computation we can reach

$$
\begin{aligned}
& d_{\mathrm{TV}}\left(\mathcal{L}(\mathbf{V}^{(n,1)}, \mathbf{W}^{(n,1)}), \mathcal{L}(V_0, W_0)\right) \\
=& \frac{1}{2}\iint \left| p_{V,W}(v,w) - p^{(0)}_{V,W}(v,w)\right| dvdw \\
=& \frac{1}{2}\iint_{\{|w|<\sqrt{n}\}} |p_{V,W}(v,w) - \varphi_1(w) f(v \mid w)|\, dvdw + \frac{1}{2}\int_{\{|w|\geq\sqrt{n}\}} \varphi_1(w) dw \\
=& \frac{1}{2}\iint_{\{|w|<\sqrt{n}\}} |p_{V,W}(v,w) - \varphi_1(w) f(v \mid w)|\, dvdw + \Phi(-\sqrt{n})
\end{aligned}
$$

For the first part, using triangular inequality, we can get

$$
\begin{aligned}
& \iint_{\{|w|<\sqrt{n}\}} |p_{V,W}(v,w) - \varphi_1(w) f(v \mid w)|\, dvdw \\
\leq & \iint_{\{|w|<\sqrt{n}\}} \left| \frac{p_{V,W}(v,w)}{p_W(w)} - f(v \mid w)\right| p_W(w) dvdw + \iint_{\{|w|<\sqrt{n}\}} |p_W(w) - \varphi_1(w)|\, f(v \mid w) dvdw \\
\leq & \int_{\{|w|<\sqrt{n}\}} 2C_{n-1,1} p_W(w) dw + \int_{\mathbb{R}} |p_W(w) - \varphi_1(w)| dw \leq 2C_{n-1,1} + 2C_{n,1}
\end{aligned}
$$

Therefore, by combining the above arguments together we got the desired result.

□

TVforXY **Corollary 2.4.6.** *Let* $X_0^{(n)} = \frac{1}{\sqrt{n}} W_0^{(n)}$ *and* $Y_0^{(n)} = \frac{1}{\sqrt{n-1}} V_0 + \rho X_0^{(n)}$*, then* $(X_0^{(n)}, Y_0^{(n)})$ *has joint PDF:*

$$
p^{(0)}_{X,Y}(x,y) = \begin{cases} \frac{\sqrt{n(n-1)}}{2\pi r\sqrt{1-x^2}} \exp\left(-\frac{n}{2}x^2 - \frac{n-1}{2r^2(1-x^2)}(y-\rho x)^2\right) & \text{for } |x|<1. \\ \frac{\sqrt{n}}{\sqrt{2\pi}} \exp\left(-\frac{n}{2}x^2\right) \delta_{\{y=\rho x\}} & \text{for } |x| \geq 1 \end{cases} \tag{2.4.8}
$$

PDFforXY

and

$$
d_{TV}\left(\mathcal{L}(X_1^{(n)}, Y_1^{(n)}), \mathcal{L}(X_0^{(n)}, Y_0^{(n)})\right) \leq C_{n,1} + C_{n-1,1} + \Phi(-\sqrt{n}).
$$

Proof. The joint PDF comes from the change of variable. The upper bound of the total variation comes from the Theorem 2.4.5 and the fact that linear transformation preserves the total variation. □

CDFapprox **Corollary 2.4.7.** *For each $t \in \mathbb{R}$, we have*

$$\left|\mathbb{P}(X_1^{(n)}Y_1^{(n)} < t) - \mathbb{P}(X_0^{(n)}Y_0^{(n)} < t)\right| \le C_{n,1} + C_{n-1,1} + \Phi(-\sqrt{n}).$$

Below is our main approximation result. Recall that r and ρ are defined in (2.4.2).

dmidllecase **Theorem 2.4.8.** *For every $\alpha > 0$ and $0 < d_0 < 2$, when n goes to ∞, we have the following result:*

$$\lim_{n\to\infty} \mathbb{P}\left(X_0^{(n)}Y_0^{(n)} \le -\frac{\alpha}{n}\right) = 2\int_0^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}u^2\right)\Phi\left(\frac{1}{r}\left(\frac{-\alpha}{u} - \rho u\right)\right) du.$$

Proof. For convenience, let $C := \frac{-\alpha}{n}$. From the PDF in (2.4.8), we can see that $X_0^{(n)}Y_0^{(n)}$ degenerates to $X_0^{(n)} \cdot \rho X_0^{(n)} > 0 > C$ when $|X_0^n| > 1$, so

$$\begin{aligned}\mathbb{P}\left(X_0^{(n)}Y_0^{(n)} \le -\frac{\alpha}{n}\right) &= \int_0^1 \int_{-\infty}^{\frac{C}{x}} \frac{\sqrt{n(n-1)}}{2\pi r\sqrt{1-x^2}} \exp\left(-\frac{1}{2}nx^2 - \frac{n-1}{2r^2(1-x^2)}(y-\rho x)^2\right) dydx \\ &+ \int_{-1}^0 \int_{\frac{C}{x}}^{+\infty} \frac{\sqrt{n(n-1)}}{2\pi r\sqrt{1-x^2}} \exp\left(-\frac{1}{2}nx^2 - \frac{n-1}{2r^2(1-x^2)}(y-\rho x)^2\right) dydx\end{aligned}$$

For the first term, note that

$$\int_{\infty}^{\frac{C}{x}} \exp\left(-\frac{n-1}{2r^2(1-x^2)}(y-\rho x)^2\right) = \Phi\left(\frac{\sqrt{n-1}}{r\sqrt{1-x^2}}\left(\frac{C}{x} - \rho x\right)\right) \cdot \frac{r\sqrt{2\pi(1-x^2)}}{\sqrt{n-1}}$$

It follows that

$$\begin{aligned}&\int_0^1 \int_{-\infty}^{\frac{C}{x}} \frac{\sqrt{n(n-1)}}{2\pi r\sqrt{1-x^2}} \exp\left(-\frac{1}{2}nx^2 - \frac{n-1}{2r^2(1-x^2)}(y-\rho x)^2\right) dydx \\ &= \int_0^1 \frac{\sqrt{n}}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}nx^2\right)\Phi\left(\left(\frac{C}{x} - \rho x\right)\frac{\sqrt{n-1}}{r\sqrt{1-x^2}}\right) dx \\ &\overset{u=\sqrt{n}x}{=\!=} \int_0^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right)\Phi\left(\left(-\rho u - \frac{\alpha}{u}\right)\frac{\sqrt{n-1}}{r\sqrt{n-u^2}}\right)\mathbb{1}_{\{u\le\sqrt{n}\}} du \\ &\overset{n\to\infty}{\longrightarrow} \int_0^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right)\Phi\left(\left(-\rho u - \frac{\alpha}{u}\right)\frac{1}{r}\right) du\end{aligned}$$

The last step comes from the bounded convergence theorem and the fact that the integrand uniformly bounded by the integrable function: $\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}u^2\right)$.

In the same way, we can prove that the second term also converges to the same limit, and hence the desired result holds.

$\square$

dgoes2 **Lemma 2.4.9.** *For every $\alpha > 0$, when n goes to ∞ and d_0 goes to 2, we have the following result:*

$$\lim_{\substack{n\to\infty \\ d_0\to 2}} \mathbb{P}\left(X_0^{(n)}Y_0^{(n)} \le -\frac{\alpha}{n}\right) = 2(1-\Phi(\sqrt{\alpha})). \tag{2.4.9}$$ dogoes2resul

Proof. We only need to show that (2.4.9) holds for any sequence $(n, d_n) \to (\infty, 2)$. Let $C = -\frac{\alpha}{n}$. From the proof of Lemma 2.4.8, we have

$$\begin{aligned} &\lim_{\substack{n\to\infty \\ d_n\to 2}} \mathbb{P}\left(X_0^{(n)}Y_0^{(n)} \le -\frac{\alpha}{n}\right) \\ &= \lim_{\substack{n\to\infty \\ d_n\to 2}} 2\int_0^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) \Phi\left(\left(-\rho u - \frac{\alpha}{u}\right)\frac{\sqrt{n-1}}{r\sqrt{n-u^2}}\right) \mathbb{1}_{\{u\le\sqrt{n}\}} du \end{aligned}$$

Note that the integrand is uniformly dominated by the integrable function $\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}u^2\right)$, so we can apply dominated convergence theorem and pass the limit into the integral sign. When n goes to ∞, and that $d_n \to 2$,

$$\left(-\rho u - \frac{\alpha}{u}\right)\frac{\sqrt{n-1}}{r\sqrt{n-u^2}} \to \begin{cases} +\infty & \text{for } u > \sqrt{\alpha} \\ -\infty & \text{for } u < \sqrt{\alpha} \end{cases}$$

Hence,

$$\lim_{\substack{n\to\infty \\ d_n\to 2}} \mathbb{P}\left(X_0^{(n)}Y_0^{(n)} \le -\frac{\alpha}{n}\right) = 2\int_{\sqrt{\alpha}}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}u^2\right) du = 2(1-\Phi(\sqrt{\alpha})).$$

Since the above convergence holds for any sequence $(n, d_n) \to (\infty, 2)$, we have (2.4.9) holds.

□

leftside **Theorem 2.4.10.** *Define $\tilde{d}_1 = n(d_0 - d_1)$, and ξ to be a random variable with Gamma$(\frac{1}{2}, \frac{1}{2})$ distribution, then for every $t > 0$,*

$$\lim_{\substack{n\to\infty \\ d_0\to 2}} \left|\mathbb{P}(\tilde{d}_1 > t) - \mathbb{P}(\xi > t)\right| = 0.$$

i.e. the CDF of $\tilde{d}_1$ converges to the CDF of a Gamma distribution point-wise when $d_0 \to 2$ and $n \to \infty$. Furthermore, since the CDF for Gamma is bounded, continuous and monotone, this convergence is uniform on the whole real line.

Proof. By Corollary 2.4.7,

$$\begin{aligned}
&\left|\mathbb{P}(\tilde{d}_1 > t) - \mathbb{P}(\xi > t)\right| \\
=&\left|\mathbb{P}\left(d_1 < d_0 - \frac{t}{n}\right) - \mathbb{P}(\xi > t)\right| \\
=&\left|\mathbb{P}\left(X_1^{(n)}Y_1^{(n)} < -\frac{d_0 t}{2n} + \frac{t^2}{4n^2}\right) - \mathbb{P}(\xi > t)\right| \\
\leq&\left|\mathbb{P}\left(X_0^{(n)}Y_0^{(n)} < -\frac{d_0 t}{2n} + \frac{t^2}{4n^2}\right) - \mathbb{P}(\xi > t)\right| + C_{n,1} + C_{n-1,1} + \Phi(-\sqrt{n})
\end{aligned}$$

Hence, we only need to show

$$\lim_{\substack{n\to\infty \\ d_0\to 2}} \mathbb{P}\left(X_0^{(n)}Y_0^{(n)} < -\frac{d_0 t}{2n} + \frac{t^2}{4n^2}\right) = \mathbb{P}(\xi > t).$$

Take arbitrary $\delta > 0$ and $N \in \mathbb{N}$. Note that $d_0 \leq 2$, so the inequality $-\frac{d_0 t}{2n} + \frac{t^2}{4n^2} \geq \frac{-t}{n}$ always holds. When $d_0 > 2 - \delta$ and $n > N$, we have

$$\begin{aligned}
&\left|\mathbb{P}\left(X_0^{(n)}Y_0^{(n)} < -\frac{d_0 t}{2n} + \frac{t^2}{4n^2}\right) - \mathbb{P}(\xi > t)\right| \\
\leq&\left|\mathbb{P}\left(X_0^{(n)}Y_0^{(n)} < -\frac{t}{n}\right) - \mathbb{P}(\xi > t)\right| + \mathbb{P}\left(-\frac{t}{n} \leq X_0^{(n)}Y_0^{(n)} < -\frac{d_0 t}{2n} + \frac{t^2}{4n^2}\right) \\
\leq&\left|\mathbb{P}\left(X_0^{(n)}Y_0^{(n)} < -\frac{t}{n}\right) - 2(1-\Phi(\sqrt{t}))\right| + \mathbb{P}\left(-\frac{t}{n} \leq X_0^{(n)}Y_0^{(n)} < -\frac{(2-\delta)t}{2n} + \frac{t^2}{4Nn}\right)
\end{aligned}$$

The first term converges to 0 by Lemma 2.4.9, for the second term, apply 2.4.9 again we can get

$$\lim_{\substack{n\to\infty \\ d_0\to 2}} \mathbb{P}\left(-\frac{t}{n} \leq X_0^{(n)}Y_0^{(n)} < -\frac{(2-\delta)t}{2n} + \frac{t^2}{4Nn}\right) = 2\Phi(\sqrt{t}) - 2\Phi\left(\sqrt{t - \frac{\delta}{2}t - \frac{t^2}{4N}}\right)$$

It follows that

$$\limsup_{\substack{n\to\infty \\ d_0\to 2}} \left|\mathbb{P}\left(X_0^{(n)}Y_0^{(n)} < -\frac{d_0 t}{2n} + \frac{t^2}{4n^2}\right) - \mathbb{P}(\xi > t)\right| \leq 2\Phi(\sqrt{t}) - 2\Phi\left(\sqrt{t - \frac{\delta}{2}t - \frac{t^2}{4N}}\right)$$

This holds for any $\delta > 0$ and $N \in \mathbb{N}$, letting $\delta \to 0$ and $N \to \infty$, we got the desired result.

$\square$

Phi-finite **Lemma 2.4.11.** *For every $a, b > 0$, we have*

$$\int_0^\infty \Phi\left(-\frac{a}{v} - bv\right) dv < +\infty.$$

Proof.

$$\begin{aligned}
\int_0^{+\infty} \Phi\left(-\frac{a}{v} - bv\right) dv &\leq \int_0^{\sqrt{\frac{a}{b}}} \Phi\left(-\frac{a}{v}\right) dv + \int_{\sqrt{\frac{a}{b}}}^\infty \Phi(-bv) dv \\
&= \int_{\sqrt{ab}}^\infty \Phi(-v) \cdot \frac{a}{v^2} dv + \int_{\sqrt{ab}}^\infty \Phi(-v) \cdot \frac{1}{b} dv \\
&\leq \frac{2}{b} \int_{\sqrt{ab}}^\infty \Phi(-v) dv \\
&\leq \frac{2}{b} \int_{\sqrt{ab}}^\infty \frac{1}{v} \cdot \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{v^2}{2}\right) dv \\
&\leq \frac{1}{\sqrt{2\pi}} \cdot \frac{2}{b\sqrt{ab}} \int_0^\infty \exp\left(-\frac{v^2}{2}\right) dv < +\infty.
\end{aligned}$$

$\square$

dgoes0 **Lemma 2.4.12.** *For every $\alpha > 0$, when n goes to ∞ and d_0 goes to 0, we have the following result:*

$$\lim_{\substack{n \to \infty \\ d_0 \to 0}} \frac{1}{d_0} \mathbb{P}\left(X_0^{(n)} Y_0^{(n)} \leq -\frac{\alpha d_0^2}{n}\right) = 2 \int_0^{+\infty} \frac{1}{\sqrt{2\pi}} \Phi\left(-\frac{\alpha}{v} - v\right) dv. \tag{2.4.10}$$ dogoes0resul

Proof. Let $C = -\frac{\alpha d_0^2}{n}$. We only need to show that (2.4.10) holds for any sequence $(n, d_n) \to (\infty, 0)$. Denote $\rho_n := 1 - \frac{d_n^2}{2}, r_n = \sqrt{1 - \rho_n^2}$. Follow the same argument in the proof of lemma 2.4.9, we have

$$\lim_{n \to \infty} \frac{1}{d_n} \mathbb{P}\left(X_0^{(n)} Y_0^{(n)} \leq -\frac{\alpha d_n^2}{n}\right)$$

$$= \lim_{n\to\infty} 2\int_0^{+\infty} \frac{1}{\sqrt{2\pi}d_n} \exp\left(-\frac{1}{2}u^2\right) \Phi\left(\left(\frac{-\alpha d_n^2}{u} - \rho_n u\right) \frac{\sqrt{n-1}}{r_n\sqrt{n-u^2}}\right) \mathbb{1}_{\{0\le u\le\sqrt{n}\}} du$$

The term inside Φ could be rewritten as

$$\left(\frac{-\alpha d_n^2}{u} - \rho_n u\right) \frac{\sqrt{n-1}}{r_n\sqrt{n-u^2}} = \left(\frac{-\alpha d_n}{u} - \frac{\rho_n u}{d_n}\right) \frac{\sqrt{n-1}}{\sqrt{1-\frac{d_n^2}{4}}\cdot\sqrt{n-u^2}}$$

Let $v = \frac{u}{d_n}$, then

$$\lim_{n\to\infty} \frac{1}{d_n}\mathbb{P}\left(X_0^{(n)}Y_0^{(n)} \le -\frac{\alpha d_n^2}{n}\right) \tag{2.4.11}$$

$$= \lim_{n\to\infty} 2\int_0^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{d_n^2 v^2}{2}\right) \Phi\left(\left(\frac{-\alpha}{v} - \rho_n v\right) \frac{\sqrt{n-1}}{\sqrt{1-\frac{d_n^2}{4}}\cdot\sqrt{n-d_n^2v^2}}\right) \mathbb{1}_{\{0\le d_n v\le\sqrt{n}\}} dv \tag{2.4.12}$$

dgoes0-equal1

WLOG, we may assume that $d_n < \frac{1}{2}$ and $n > 2$. Then in this case, we will have $\rho_n > \frac{7}{8}$. Note that for the terms inside the function Φ, we have the follow upper bounds:

$$\frac{\alpha}{v} + \rho_n v \ge \frac{\alpha}{v} + \frac{7}{8}v$$

$$\frac{1}{\sqrt{1-\frac{d_n^2}{4}}} \ge 1$$

$$\frac{\sqrt{n-1}}{\sqrt{n-d_n^2v^2}} \ge \sqrt{\frac{1}{2}}$$

Hence, the integrand in (2.4.12) will be bounded by

$$\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{d_n^2 v^2}{2}\right) \Phi\left(\left(\frac{-\alpha}{v} - \rho_n v\right) \frac{\sqrt{n-1}}{\sqrt{1-\frac{d_n^2}{4}}\cdot\sqrt{n-d_n^2v^2}}\right) \mathbb{1}_{\{0\le d_n v\le\sqrt{n}\}}$$

$$\le \frac{1}{\sqrt{2\pi}}\Phi\left(-\left(\frac{\alpha}{v} + \frac{7}{8}v\right)\frac{1}{\sqrt{2}}\right)$$

By the Lemma 2.4.11, the bound is integrable, and hence we can pass the limit into the integral sign, which leads to:

$$\lim_{n\to\infty}\frac{1}{d_n}\mathbb{P}\left(X_0^{(n)}Y_0^{(n)}\le -\frac{\alpha d_n^2}{n}\right)=\lim_{n\to\infty}2\int_0^{+\infty}\frac{1}{\sqrt{2\pi}}\Phi\left(-\frac{\alpha}{v}-v\right)dv$$

Note that this convergence holds for every sequence $(n, d_n)\to(\infty,0)$, we have (2.4.10) holds.

□

rightside0 **Theorem 2.4.13.** *Define* $\hat{d}_1 = n\left(1-\frac{d_1}{d_0}\right)$ *and:*

$$G(t) := \int_0^{+\infty}\sqrt{2\pi}\Phi\left(-\frac{t}{2v}-v\right)dv. \tag{2.4.13}$$ 10.2

Then for every $t>0$ *we have*

$$|\mathbb{P}(\hat{d}_1 > t \mid \hat{d}_1 > 0) - G(t)| \to 0, \tag{2.4.14}$$ 10.1

when $n\to\infty, d_0\to 0$ *and* $\frac{C_{n,1}+C_{n-1,1}+\Phi(-\sqrt{n})}{d_0}\to 0$.

Furthermore, since $G(t)$ *is bounded, continuous and monotone, this convergence is uniform on the whole real line.*

Proof. (I) First we will show that $1-G(t)$ is a CDF of a continuous random variable. It is easy to see that $\int_0^{+\infty}\sqrt{2\pi}\Phi\left(-\frac{t}{2v}-v\right)dv$ is monotone decreasing. When $t=0$, we have

$$\begin{aligned}\int_0^{+\infty}\sqrt{2\pi}\Phi(-v)\,dv &= \int_0^{+\infty}\int_{-\infty}^{-v}e^{-\frac{1}{2}u^2}dudv\\ &=\int_{-\infty}^{0}\int_0^{-u}e^{-\frac{1}{2}u^2}dvdu\\ &=\int_{-\infty}^{0}-ue^{-\frac{1}{2}u^2}du\\ &=1\end{aligned}$$

When $t\to+\infty$, by then Monotone Convergence Theorem, we have

$$\int_0^{+\infty}\sqrt{2\pi}\Phi\left(-\frac{t}{2v}-v\right)dv\to\int_0^{+\infty}0dv=0.$$

Hence, $1-G(t)$ gives us a CDF.

Next we will show (2.4.13) is differentiable, and hence it is the CDF of a continuous random variable. For each $t > 0$ and $0 < |\Delta_t| < t$, we have

$$\begin{aligned}
&\frac{1}{\Delta_t}\int_0^{+\infty} \sqrt{2\pi}\left(\Phi\left(-\frac{t+\Delta_t}{2v} - v\right) - \Phi\left(-\frac{t}{2v} - v\right)\right) dv \\
&\stackrel{*}{=} -\int_0^{+\infty} \sqrt{2\pi}\varphi\left(-\frac{t+\xi(t,\Delta_t)}{2v} - v\right)\cdot\frac{1}{2v}dv \\
&= -\int_0^{+\infty} \exp\left(-\frac{1}{2}\left(\frac{t+\xi(t,\Delta_t)}{2v} + v\right)^2\right)\cdot\frac{1}{2v}dv
\end{aligned}$$

Here, the step $(*)$ comes from the mean value theorem and $0 \le |\xi(t,\Delta_t)| \le |\Delta_t|$. Then the integral part is dominated by

$$\int_0^{+\infty} \exp\left(-\frac{1}{2}\left(\frac{t}{v}\right)^2\right)\cdot\frac{1}{2v}dv + \int_1^{+\infty}\frac{1}{2v}\exp\left(-\frac{1}{2}v^2\right)dv < +\infty.$$

Hence, by the Dominated Convergence Theorem, the associated PDF is:

$$g(t) := \int_0^{+\infty} \exp\left(-\frac{1}{2}\left(\frac{t}{2v} + v\right)^2\right)\cdot\frac{1}{2v}dv.$$

(II) Second, we will prove (2.4.14). Note that

$$\begin{aligned}
\mathbb{P}(\hat{d}_1 > t \mid \hat{d}_1 > 0) &= \mathbb{P}\left(n\left(1 - \frac{d_1}{d_0}\right) > t\right)\frac{\pi}{\arccos(\rho)} \\
&= \mathbb{P}\left(X_1^{(n)}Y_1^{(n)} < -\frac{td_0^2}{2n} + \frac{t^2d_0^2}{4n^2}\right)\frac{\pi}{\arccos(\rho)}
\end{aligned}$$

So

$$\begin{aligned}
&|\mathbb{P}(\hat{d}_1 > t \mid \hat{d}_1 > 0) - G(t)| \\
\le&\left|\mathbb{P}\left(X_0^{(n)}Y_0^{(n)} < -\frac{td_0^2}{2n}\right)\frac{\pi}{d_0} - G(t)\right| \\
&+\left|\mathbb{P}\left(X_0^{(n)}Y_0^{(n)} < -\frac{td_0^2}{2n}\right)\frac{\pi}{d_0} - \mathbb{P}\left(X_0^{(n)}Y_0^{(n)} < -\frac{td_0^2}{2n} + \frac{t^2d_0^2}{4n^2}\right)\frac{\pi}{d_0}\right|
\end{aligned}$$

$$+\left|\mathbb{P}\left(X_0^{(n)}Y_0^{(n)} < -\frac{td_0^2}{2n}+\frac{t^2d_0^2}{4n^2}\right)\frac{\pi}{d_0} - \mathbb{P}\left(X_1^{(n)}Y_1^{(n)} < -\frac{td_0^2}{2n}+\frac{t^2d_0^2}{4n^2}\right)\frac{\pi}{d_0}\right|$$
$$+\left|\mathbb{P}\left(X_1^{(n)}Y_1^{(n)} < -\frac{td_0^2}{2n}+\frac{t^2d_0^2}{4n^2}\right)\frac{\pi}{d_0} - \mathbb{P}\left(X_1^{(n)}Y_1^{(n)} < -\frac{td_0^2}{2n}+\frac{t^2d_0^2}{4n^2}\right)\frac{\pi}{\arccos(\rho)}\right|$$

The first term converges to 0 by the lemma 2.4.12. The third term is bounded by $\pi(C_{n,1}+C_{n-1,1}+\Phi(-\sqrt{n}))/d_0$ which converges to 0 by assumption. The last term is bounded by $\pi|\frac{1}{d_0}-\frac{1}{\arccos(\rho)}|$ which also converges to 0. So we only need to show the second term also converges to 0.

For each $N>0$, when $n>N$ we have

$$\begin{aligned}
&\left|\mathbb{P}\left(X_0^{(n)}Y_0^{(n)} < -\frac{td_0^2}{2n}\right)\frac{\pi}{d_0} - \mathbb{P}\left(X_0^{(n)}Y_0^{(n)} < -\frac{td_0^2}{2n}+\frac{t^2d_0^2}{4n^2}\right)\frac{\pi}{d_0}\right| \\
\le&\mathbb{P}\left(X_0^{(n)}Y_0^{(n)} < -\frac{td_0^2}{2n}+\frac{t^2d_0^2}{4nN}\right)\frac{\pi}{d_0} - \mathbb{P}\left(X_0^{(n)}Y_0^{(n)} < -\frac{td_0^2}{2n}\right)\frac{\pi}{d_0} \\
\to&G\left(t-\frac{t^2}{2N}\right)-G(t)
\end{aligned}$$

Since this holds for any $N>0$ and when $N\to\infty$, $G\left(t-\frac{t^2}{2N}\right)-G(t)$, we proved that the second term also converges to 0. □

In the above proof, we got the PDF $g(t)$. The next proposition tells us that the rate of growth of this PDF close to 0 is $O(-\log(t))$.

eta-tail **Proposition 2.4.14.** *When $t\to 0$, we have the follow convergence:*

$$\lim_{t\to 0}\frac{g(t)}{-\log t}=\frac{1}{2}.$$

Proof. Note that

$$\begin{aligned}
g(t) &= \int_0^{+\infty}\exp\left(-\frac{1}{2}\left(\frac{t}{2v}+v\right)^2\right)\cdot\frac{1}{2v}dv \\
&= \exp\left(-\frac{t}{2}\right)\int_0^{+\infty}\exp\left(-\frac{t^2}{8v^2}-\frac{1}{2}v^2\right)\frac{1}{2v}dv
\end{aligned} \tag{2.4.15}$$ eta-tail-1

So, we only need to show

$$\lim_{t\to 0}\frac{1}{-\log t}\int_0^{+\infty}\exp\left(-\frac{t^2}{8v^2}-\frac{1}{2}v^2\right)\frac{1}{2v}dv=\frac{1}{2}.$$

We will first analyze the upper bound. Note that

$$\begin{aligned}&\frac{1}{-\log t}\int_0^{+\infty}\exp\left(-\frac{t^2}{8v^2}-\frac{1}{2}v^2\right)\frac{1}{2v}dv\\ \leq&\left(\int_0^1\exp\left(-\frac{t^2}{8v^2}\right)\frac{1}{2v}dv+\int_1^{+\infty}\exp\left(-\frac{1}{2}v^2\right)\cdot\frac{1}{2}dv\right)\frac{1}{-\log t}\end{aligned}$$

For the second term, $\int_1^{+\infty}\exp\left(-\frac{1}{2}v^2\right)\cdot\frac{1}{2}dv<+\infty$, hence

$$\lim_{t\to 0}\int_1^{+\infty}\exp\left(-\frac{1}{2}v^2\right)\cdot\frac{1}{2}dv\cdot\frac{1}{-\log t}=0.$$

For the first term, let $u=\frac{t^2}{8v^2}$ we have

$$\begin{aligned}&\lim_{t\to 0}\int_0^1\exp\left(-\frac{t^2}{8v^2}\right)\frac{1}{2v}dv\cdot\frac{1}{-\log t}\\ =&\lim_{t\to 0}\int_{\frac{t^2}{8}}^{+\infty}\exp(-u)\frac{1}{4u}du\cdot\frac{1}{-\log t}\\ =&\lim_{t\to 0}\frac{\exp\left(-\frac{t^2}{8}\right)\cdot\frac{1}{2t}}{\frac{1}{t}}\quad\text{(L'Hopital Rule)}\\ =&\frac{1}{2}\end{aligned}$$

Hence, we got

$$\limsup_{t\to 0}\frac{1}{-\log t}\int_0^{+\infty}\exp\left(-\frac{t^2}{8v^2}-\frac{1}{2}v^2\right)\frac{1}{2v}dv\leq\frac{1}{2}.$$

Now, we will do the opposite direction. From (2.4.15), we can also get

$$\begin{aligned}&\frac{1}{-\log t}\int_0^{+\infty}\exp\left(-\frac{t^2}{8v^2}-\frac{1}{2}v^2\right)\frac{1}{2v}dv\\ \geq&\left(\int_0^{\sqrt{\frac{t}{2}}}\exp\left(-\frac{t^2}{4v^2}\right)\frac{1}{2v}dv+\int_{\sqrt{\frac{t}{2}}}^{+\infty}\exp\left(-v^2\right)\cdot\frac{1}{2v}dv\right)\frac{1}{-\log t}\end{aligned}$$

For the second term, by L'Hopital Rule we have

$$\begin{aligned}
&\lim_{t\to 0}\int_{\sqrt{\frac{t}{2}}}^{+\infty}\exp\left(-\frac{t^2}{4v^2}\right)\frac{1}{2v}dv\cdot\frac{1}{-\log t}\\
=&\lim_{t\to 0}\frac{\frac{1}{4t}\exp\left(-\frac{t}{2}\right)}{\frac{1}{t}}\\
=&\frac{1}{4}
\end{aligned}$$

For the first term, let $u=\frac{t^2}{4v^2}$, we have

$$\begin{aligned}
&\lim_{t\to 0}\int_{0}^{\sqrt{\frac{t}{2}}}\exp\left(-\frac{t^2}{4v^2}\right)\frac{1}{2v}dv\cdot\frac{1}{-\log t}\\
=&\lim_{t\to 0}\frac{1}{-\log t}\int_{\frac{t}{2}}^{+\infty}\frac{1}{4u}\exp(-u)du\\
=&\lim_{t\to 0}\frac{\frac{1}{4t}\exp\left(-\frac{t}{2}\right)}{\frac{1}{t}}\\
=&\frac{1}{4}
\end{aligned}$$

Hence, we got that

$$\liminf_{t\to 0}\frac{1}{-\log t}\int_0^{+\infty}\exp\left(-\frac{t^2}{8v^2}-\frac{1}{2}v^2\right)\frac{1}{2v}dv\geq\frac{1}{2}$$

Therefore, the desired result holds.

□

Chapter 3

PARTICLE COLLISIONS

3.1 Motivation

The physics "law" known as "equidistribution of energy" and a formula known as the "microcanonical ensemble formula" suggest that velocities of gas molecules in $\mathbb{R}^d$ are i.i.d. standard d-dimensional normal. We will find the collision location for two "randomly" chosen molecules.

3.2 Informal review of main results

Without loss of generality, we assume that two particles are positioned symmetrically with respect to the origin. We will show that the approximate collision location for these particles has a density that is proportional to r^{d-1} and rotation invariant in $\mathbb{R}^d$ for $d = 1, 2, 3$. The asymptotic distribution is a multi-dimensional $t-$distribution for $d \geq 2$. This agrees with the 1$-$dimensional distribution, which happens to be the Cauchy distribution.

3.3 Model definition

.collision.1

Suppose we have two particles, denoted by $\mathbf{P}_1$ and $\mathbf{P}_2$, that are moving balls with a radius $0 < r < 1$ in d-dimensional space $\mathbb{R}^d$. Let $\mathbf{X}_1$ and $\mathbf{X}_2$ represent the initial positions of the centers of $\mathbf{P}_1$ and $\mathbf{P}_2$, respectively, at time 0. Furthermore, let $\mathbf{V}_1$ and $\mathbf{V}_2$ denote the velocities of $\mathbf{P}_1$ and $\mathbf{P}_2$, respectively. Without loss of generality, we may assume that

$$\mathbf{X}_1 = (-1, 0, \cdots, 0) \quad \text{and} \quad \mathbf{X}_2 = (1, 0, \cdots, 0),$$

which means that the two particles are located along the x-axis and are separated by a distance of 2. Then, the centers of the particles at time s can be expressed as $\mathbf{C}_1(s) := \mathbf{X}_1 + s\mathbf{V}_1$ and $\mathbf{C}_2(s) := \mathbf{X}_2 + s\mathbf{V}_2$, respectively.

Denote

$$\mathbf{V}_1 = (v_{1,1}, v_{1,2}, \cdots, v_{1,d}), \quad \mathbf{V}_2 = (v_{2,1}, v_{2,2}, \cdots, v_{2,d}). \tag{3.3.1}$$

def-component

Based on the knowledge from physics, we may assume that

$$\mathbf{V}_1, \mathbf{V}_2 \overset{iid}{\sim} \mathcal{N}(\mathbf{0}, I_d).$$

We are interested in the probability they will collide and also the density function for the collision place for a small radius, i.e. $r \approx 0$.

3.4 Collision probability

sec-col-prob

Denote $E_{r,d}$ to be the event that two particles $\mathbf{P}_1$ and $\mathbf{P}_2$ with radius r will collide in space $\mathbb{R}^d$. Call the collision probability $p_{r,d}$, i.e.

$$p_{r,d} = \mathbb{P}(E_{r,d}). \tag{3.4.1}$$ definition-p

Before computing $p_{r,d}$, we will first introduce the following criterion for collisions:

collision-criterion **Lemma 3.4.1.** *The particles $\mathbf{P}_1$ and $\mathbf{P}_2$ will collide if and only if*

$$\cos\beta \leq -\sqrt{1-r^2},$$

where β is the angle between the vectors $\mathbf{X}_1 - \mathbf{X}_2$ and $\mathbf{V}_1 - \mathbf{V}_2$.

Proof. After time s, the center of $\mathbf{P}_1$ and $\mathbf{P}_2$ will be $X_1 + s\mathbf{V}_1$ and $\mathbf{X}_2 + s\mathbf{V}_2$, so in order to have a collision, we will need

$$\left\|(\mathbf{X}_1 + s\mathbf{V}_1) - (\mathbf{X}_2 + s\mathbf{V}_2)\right\|_2 = 2r,$$

for some $s > 0$. Note that

$$\begin{aligned} &[(\mathbf{X}_1 - \mathbf{X}_2) + s(\mathbf{V}_1 - \mathbf{V}_2)]^T [(\mathbf{X}_1 - \mathbf{X}_2) + s(\mathbf{V}_1 - \mathbf{V}_2)] = 4r^2 \\ \Longrightarrow &\left\|\mathbf{V}_1 - \mathbf{V}_2\right\|_2^2 s^2 + 2(\mathbf{V}_1 - \mathbf{V}_2)^T (\mathbf{X}_1 - \mathbf{X}_2) s + \left\|\mathbf{X}_1 - \mathbf{X}_2\right\|_2^2 - 4r^2 = 0 \end{aligned} \tag{3.4.2}$$ 3.1

In order to make the above quadratic equation has a root in $(0, +\infty)$, first we will need

$$\Delta = 4\left[(\mathbf{V}_1 - \mathbf{V}_2)^T (\mathbf{X}_1 - \mathbf{X}_2)\right]^2 - 4\left\|\mathbf{V}_1 - \mathbf{V}_2\right\|_2^2 \left(\left\|\mathbf{X}_1 - \mathbf{X}_2\right\|_2^2 - 4r^2\right) \geq 0,$$

while implies

$$\begin{aligned} &\left\|\mathbf{V}_1 - \mathbf{V}_2\right\|_2^2 \left\|\mathbf{X}_1 - \mathbf{X}_2\right\|_2^2 \cos^2\beta - \left\|\mathbf{V}_1 - \mathbf{V}_2\right\|_2^2 \left(\left\|\mathbf{X}_1 - \mathbf{X}_2\right\|_2^2 - 4r^2\right) \geq 0 \\ \Longrightarrow &\cos^2\beta - 1 + \frac{4r^2}{\left\|\mathbf{X}_1 - \mathbf{X}_2\right\|_2^2} \geq 0 \end{aligned}$$

Note that $\|\mathbf{X}_1 - \mathbf{X}_2\|_2 = 2$, so

$$\cos^2 \beta \geq 1 - r^2. \tag{3.4.3}$$ 3.2

Let s_1, s_2 be the two roots of (3.4.2), apply Vieta's formula we have

$$\begin{cases} s_1 + s_2 = -\frac{(\mathbf{V}_1-\mathbf{V}_2)^T(\mathbf{X}_1-\mathbf{X}_2)}{\|\mathbf{V}_1-\mathbf{V}_2\|_2^2} = -\frac{\|\mathbf{X}_1-\mathbf{X}_2\|_2}{\|\mathbf{V}_1-\mathbf{V}_2\|_2} \cos\beta \\ s_1 s_2 = \frac{\|\mathbf{X}_1-\mathbf{X}_2\|_2^2-4r^2}{\|\mathbf{V}_1-\mathbf{V}_2\|_2^2} > 0 \end{cases}$$

So to make (3.4.2) have a positive solution, we will need $\cos\beta < 0$.

Therefore, combining this fact with the condition (3.4.3) we will get

$$\cos\beta \leq -\sqrt{1-r^2}.$$

□

Remark 3.4.2. When r is close to 0, in order to have a collision, $\cos\beta$ should be close to -1. This means β is close to π, i.e. $\mathbf{X}_1 - \mathbf{X}_2$ and $\mathbf{V}_1 - \mathbf{V}_2$ almost have the opposite direction. Hence, if $\mathbf{P}_1$ and $\mathbf{P}_2$ were to collide, $\mathbf{V}_1 - \mathbf{V}_2$ should have almost the same direction as the positive $x-$axis.

The case $d = 1$ is a special case since in this case we have

$$\cos\beta = \frac{-(v_{1,1} - v_{2,1})}{|v_{1,1} - v_{2,1}|} = 1 \text{ or } -1.$$

So the condition 3.4.1 reduces to $v_{1,1} > v_{2,1}$. By symmetry, we will have the following result:

ision-prod-1 **Theorem 3.4.3.** *For $d = 1$, $p_{r,1} = \frac{1}{2}$.*

With the lemma above, we can have the following result:

llision-prob **Theorem 3.4.4.** *For $d \geq 2$,*

$$p_{r,d} = \frac{1}{2} F\left(\frac{1}{d-1} \cdot \frac{r^2}{1-r^2}; d-1, 1\right).$$

here $F(\cdot; d-1, 1)$ is the CDF of the $F-$distribution with $d-1$ and 1 degrees of freedom.

Proof. Using lemma 3.4.1, we only need to show

$$\mathbb{P}\left(\cos\beta \le -\sqrt{1-r^2}\right) = \frac{1}{2}F\left(\frac{1}{d-1}\cdot\frac{r^2}{1-r^2}; d-1, 1\right).$$

Recall that

$$\mathbf{X}_1 = (-1, 0, \cdots, 0), \quad \mathbf{X}_2 = (1, 0, \cdots, 0).$$

So we have

$$\begin{aligned}
&\mathbb{P}\left(\cos\beta \le -\sqrt{1-r^2}\right)\\
=&\mathbb{P}\left(\frac{-(v_{1,1}-v_{2,1})}{\|\mathbf{V}_1-\mathbf{V}_2\|_2} \le -\sqrt{1-r^2}\right)\\
=&\mathbb{P}\left(v_{1,1}-v_{2,1} \ge \sqrt{1-r^2}\,\|\mathbf{V}_1-\mathbf{V}_2\|_2\right)\\
=&\mathbb{P}\left((v_{1,1}-v_{2,1})^2 \ge (1-r^2)\sum_{i=1}^{d}(v_{1,i}-v_{2,i})^2, v_{1,1}-v_{2,1}\ge 0\right)\\
=&\mathbb{P}\left(r^2(v_{1,1}-v_{2,1})^2 \ge (1-r^2)\sum_{i=2}^{d}(v_{1,i}-v_{2,i})^2, v_{1,1}-v_{2,1}\ge 0\right)\\
=&\frac{1}{2}\mathbb{P}\left(\frac{\frac{1}{d-1}\sum_{i=2}^{d}(v_{1,i}-v_{2,i})^2}{(v_{1,1}-v_{2,1})^2} \le \frac{1}{d-1}\cdot\frac{r^2}{1-r^2}\right)\\
=&\frac{1}{2}F\left(\frac{1}{d-1}\cdot\frac{r^2}{1-r^2}; d-1, 1\right)
\end{aligned} \tag{3.4.4}$$

3.5

Note that $\{v_{1,i}-v_{2,i}\}_{i=1}^{d}$ follows i.i.d. $\mathcal{N}(0,2)$ distribution, So $\sum_{i=2}^{d}(v_{1,i}-v_{2,i})^2 \perp\!\!\!\perp (v_{1,1}-v_{2,1})^2$ and $sum_{i=2}^{d}(v_{1,i}-v_{2,i})^2 \sim 2\chi_2^{d-1}$, $(v_{1,1}-v_{2,1})^2 \sim 2\chi_2^1$. This gives the last equality in (3.4.4).

□

Using the PDF of $F-$distribution, we can get the following approximation:

llision-prob-approx

Theorem 3.4.5. *For $d \ge 2$, we have*

$$p_{r,d} = \frac{1}{d-1}\cdot\frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)\sqrt{\pi}}r^{d-1} + o(r^{d-1}).$$

Proof. It is well-known that the PDF of $F-$ distribution with $d-1$ and 1 degrees of freedom is:

$$\frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)\sqrt{\pi}}(d-1)^{\frac{d-1}{2}}x^{\frac{d-3}{2}}\left(1+(d-1)x\right)^{-\frac{d}{2}}, x>0.$$

It follows that

$$p_{r,d} = \frac{1}{2}\int_0^{\frac{1}{d-1}\cdot\frac{r^2}{1-r^2}} \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)\sqrt{\pi}} (d-1)^{\frac{d-1}{2}} x^{\frac{d-3}{2}} (1+(d-1)x)^{-\frac{d}{2}}\, dx. \tag{3.4.5}$$ 3.3

By the *L′Hôpital′s* Rule, we have

$$\begin{aligned}
\lim_{r\to 0} \frac{p_{r,d}}{r^{d-1}} &= \lim_{r\to 0} \frac{1}{2}\cdot\frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)\sqrt{\pi}} (d-1)^{\frac{d-1}{2}} \left(\frac{1}{d-1}\cdot\frac{r^2}{1-r^2}\right)^{\frac{d-3}{2}} \\
&\quad \cdot \left(1+\frac{r^2}{1-r^2}\right)^{-\frac{d}{2}} \cdot \frac{1}{d-1}\cdot\frac{2r}{(1-r^2)^2}\cdot\frac{1}{(d-1)r^{d-2}} \\
&= \lim_{r\to 0} \frac{1}{d-1}\cdot\frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)\sqrt{\pi}} \left(1-r^2\right)^{-\frac{1}{2}} \\
&= \frac{1}{d-1}\cdot\frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)\sqrt{\pi}}
\end{aligned}$$

Therefore, the desired result holds. □

prox-special **Remark 3.4.6.** In order to verify and illustrate the last result, we will discuss the special cases $d = 2, 3$.

Substituting $d = 2, 3$, we can get

$$p_{r,2} = \frac{1}{\pi} r + o(r), \quad p_{r,3} = \frac{1}{4} r^2 + o(r^2).$$

Actually, for $p_{r,2}$ and $p_{r,3}$ we can compute their exact values.

If we put $d = 2, 3$ in (3.4.5), we get

$$\begin{aligned}
p_{r,2} &= \frac{1}{2}\int_0^{\frac{r^2}{1-r^2}} \frac{1}{\pi}\cdot\frac{1}{\sqrt{x}\,(1+x)} dx = \frac{1}{\pi}\arctan\sqrt{x}\Big|_0^{\frac{r^2}{1-r^2}} \\
&= \frac{1}{\pi}\arctan\frac{r^2}{1-r^2} = \frac{r}{\pi} + O(r^2)
\end{aligned}$$

$$\begin{aligned}
p_{r,3} &= \frac{1}{2}\int_0^{\frac{r^2}{2(1-r^2)}} (1+2x)^{-\frac{3}{2}} dx = -\frac{1}{2}(1+2x)^{-\frac{1}{2}}\Big|_0^{\frac{r^2}{2(1-r^2)}} \\
&= \frac{1}{2} - \frac{1}{2}\left(1+\frac{r^2}{1-r^2}\right)^{-\frac{1}{2}} = \frac{1-\sqrt{1-r^2}}{2} = \frac{1}{4}r^2 + O(r^4)
\end{aligned}$$

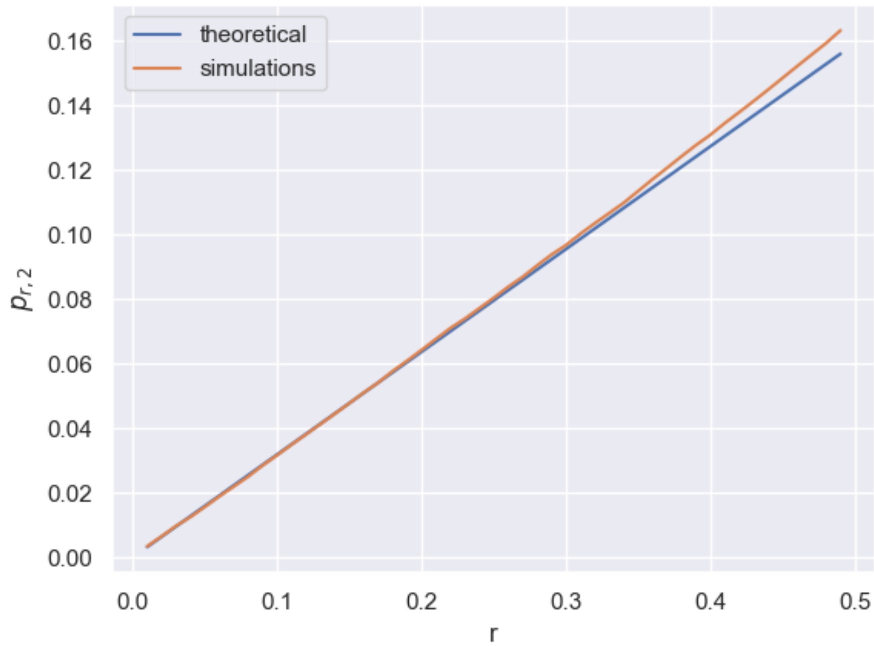

Figure 3.1: Simulation results for $d = 2$ and $100,000$ pairs of particles. The probability of collision as a function of radius. The orange line represents the simulations, and the blue line is r/π, the first term in the theoretical formula.

Fig1

The results of simulations for $d = 2$ with 100000 pairs of particles are shown in Fig. 3.1. The simulation results fit the theoretical formula very well even at a surprisingly large value of $r = 0.3$.

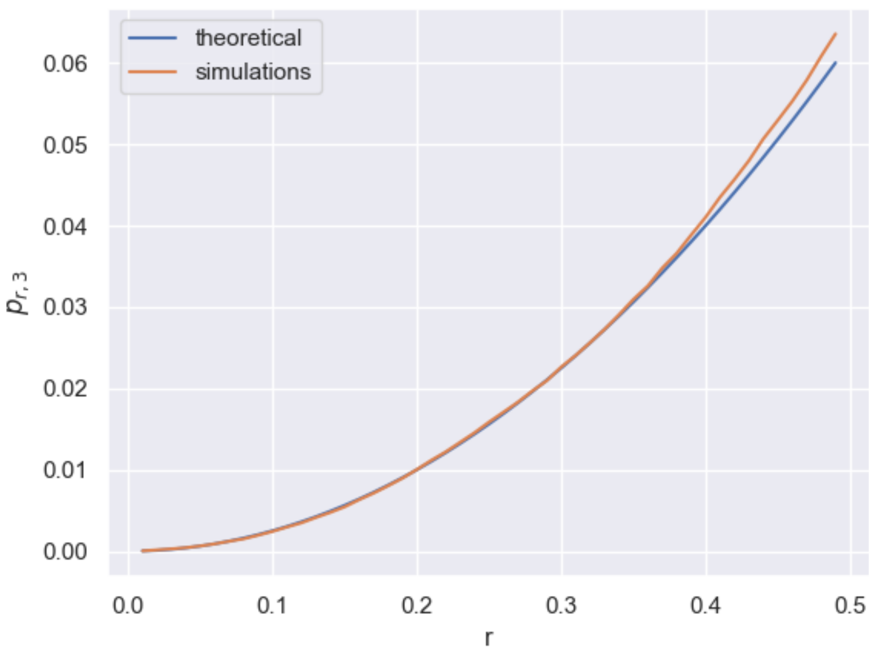

Figure 3.2: Simulation results for $d = 3$ and $100,000$ pairs of particles. The probability of collision as a function of radius. The orange line represents the simulations, and the blue line is $r^2/4$, the first term in the theoretical formula.

Fig2

3.5 Collision density function in dimension 1

sec-coplace-1-dim

We will first discuss the case $d = 1$. Denote T to be the collision time. Recall that $\mathbf{C}_1(T) = \mathbf{X}_1 + T\mathbf{V}_1$ represents the center of particle $\mathbf{P}_1$ at time T. The point $\mathbf{C}_1(T)$ is a good approximation of the collision place, for convenience, we will this point X.

In the special case $d = 1$, we will have the following nice result:

density-1-dim **Theorem 3.5.1.** *For $d = 1$,*

$$\mathbb{P}(X \le x) = \int_{-\infty}^{x} \frac{1-r}{2\pi\left((u+r)^2 + (1-r)^2\right)} du. \tag{3.5.1}$$

formula-1-di

The function $\frac{1}{2\pi(1+u^2)}$ is a defective density since it does not integrate to 1. In view of our definition of $p_{r,d}$ in (3.4.1), we have

$$\int_{-\infty}^{\infty} \frac{1-r}{2\pi\left((u+r)^2 + (1-r)^2\right)} du = p_{r,1} = \frac{1}{2}.$$

Proof. In Section 3.3, we showed that $\mathbf{P}_1$ will collide with $\mathbf{P}_2$ when $v_{1,1} > v_{2,1}$. Then the collision time would be $T = \frac{2-2r}{v_{1,1}-v_{2,1}}$, and the collision place is $-1 + Tv_{1,1}$. So for each $x > 1 - 2r$ we have

$$\begin{aligned}
\mathbb{P}(X \le x) =& \mathbb{P}\left(-1 + \frac{(2-2r)\,v_{1,1}}{v_{1,1} - v_{2,1}} \le x, \quad v_{2,1} < v_{1,1}\right) \\
=& \mathbb{P}\left((2-2r)\,v_{1,1} \le (x+1)\,(v_{1,1} - v_{2,1})\,, \quad v_{2,1} < v_{1,1}\right) \\
=& \mathbb{P}\left(v_{1,1} \ge \frac{x+1}{x-1+2r} \cdot v_{2,1}, \quad v_{2,1} < v_{1,1}\right) \\
=& \int_{-\infty}^{0} \int_{v_{2,1}}^{\infty} \frac{1}{2\pi} \exp\left(-\frac{1}{2}v_{1,1}^2 - \frac{1}{2}v_{2,1}^2\right) dv_{1,1}dv_{2,1} \\
&+ \int_{0}^{\infty} \int_{\frac{x+1}{x-1+2r}v_{2,1}}^{\infty} \frac{1}{2\pi} \exp\left(-\frac{1}{2}v_{1,1}^2 - \frac{1}{2}v_{2,1}^2\right) dv_{1,1}dv_{2,1} \\
=& \int_{-\infty}^{0} \int_{v_{2,1}}^{\infty} \frac{1}{2\pi} \exp\left(-\frac{1}{2}v_{1,1}^2 - \frac{1}{2}v_{2,1}^2\right) dv_{1,1}dv_{2,1} \\
&+ \int_{0}^{\infty} \int_{\frac{x+1}{x-1+2r}}^{\infty} \frac{1}{2\pi} \exp\left(-\frac{1}{2}u^2v_{2,1}^2 - \frac{1}{2}v_{2,1}^2\right) v_{2,1}dudv_{2,1}
\end{aligned}$$

It follows that for $x > 1-2r$,

$$\begin{aligned} f_{r,1}(x) &:= \frac{d}{dx}\mathbb{P}(X \le x) \\ &= \int_0^{\infty} \frac{v_{2,1}}{2\pi} \exp\left(-\frac{1}{2}v_{2,1}^2 - \frac{1}{2}\cdot\left(\frac{x+1}{x-1+2r}\right)^2 v_{2,1}^2\right)\cdot\frac{2-2r}{(x-1+2r)^2}dv_{2,1} \\ &= \frac{1-r}{\pi(x-1+2r)^2}\cdot\frac{1}{1+\left(\frac{x+1}{x-1+2r}\right)^2} \\ &= \frac{1-r}{2\pi\left((x+r)^2+(1-r)^2\right)}. \end{aligned}$$

For the same reason, for $x < 1-2r$ we have

$$\begin{aligned} \mathbb{P}(X \le x) =& \mathbb{P}\left(-1+\frac{(2-2r)\,v_{1,1}}{v_{1,1}-v_{2,1}} \le x, \quad v_{2,1} < v_{1,1}\right) \\ =& \mathbb{P}\left(v_{1,1} \le \frac{x+1}{x-1+2r}\cdot v_{2,1}, \quad v_{2,1} < v_{1,1}\right) \\ =& \int_{-\infty}^{0}\int_{v_{2,1}}^{\frac{x+1}{x-1+2r}v_{2,1}} \frac{1}{2\pi}\exp\left(-\frac{1}{2}v_{1,1}^2 - \frac{1}{2}v_{2,1}^2\right)dv_{1,1}dv_{2,1} \\ =& \int_{-\infty}^{0}\int_{\frac{x+1}{x-1+2r}}^{1} \frac{1}{2\pi}\exp\left(-\frac{1}{2}u^2v_{2,1}^2 - \frac{1}{2}v_{2,1}^2\right)v_{2,1}dudv_{2,1} \end{aligned}$$

and $f_{r,1}(x)$ still has the same formula with

$$f_{r,1}(x) = \frac{1-r}{2\pi\left((x+r)^2+(1-r)^2\right)}.$$

Therefore, combining the two result together, we get equation (3.5.1). □

Remark 3.5.2. The function $\frac{1-r}{\pi\left((x+r)^2+(1-r)^2\right)}$ is the PDF for Cauchy distribution centered at $-r$, and the additional $\frac{1}{2}$ term comes from the fact that in 1 dimensional space, the possibility for having a collision is $\frac{1}{2}$, i.e.

$$\iint_{\mathbb{R}} f_{r,1}(x)dx = p_{r,1} = \frac{1}{2}.$$

Recall that for Cauchy distribution, we have the following well-known result:

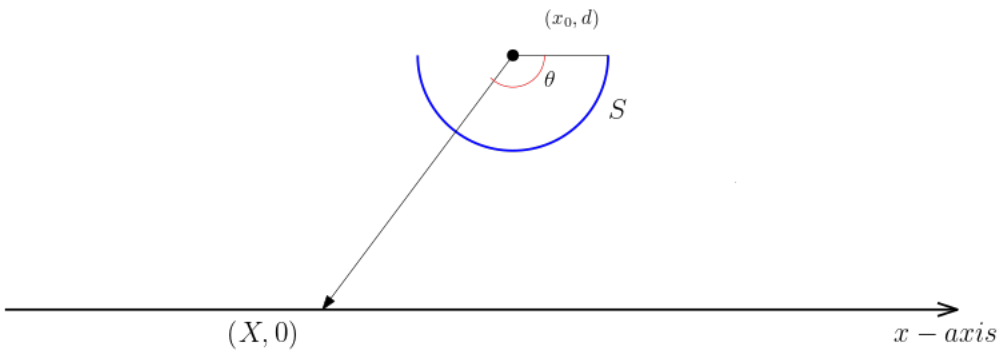

Figure 3.3: Graph for Lemma 3.5.3

Fig3

cauchy **Lemma 3.5.3.** *Suppose a particle starts from a fixed point* (x_0, d) *with the direction uniformly on the semi-circle* S*, see Fig. 3.3, i.e.* θ *follows a uniform distribution on* $(0, \pi)$*. Denote* $(X, 0)$ *to be the point particle hits the* $x-$*axis. Then* X *follows a Cauchy distribution with PDF:*

$$f_X(x) = \frac{1}{\pi} \cdot \frac{d}{(x - x_0)^2 + d^2}.$$

With the above lemma, we can explain why Cauchy distribution occurs in Theorem 3.5.1. Define $W(s) = (\mathbf{X}_1 + s\mathbf{V}_1, \mathbf{X}_2 + s\mathbf{V}_2)$, i.e $W(s)$ is the trajectory of the centers for two particles. The initial location is $W(0) = (-1, 1)$. Note that the particle $\mathbf{P}_1$ is always to the left side of particle $\mathbf{P}_2$, so the collision condition would be:

$$(\mathbf{X}_2 + T\mathbf{V}_2) - (\mathbf{X}_1 + T\mathbf{V}_1) = 2r,$$

which is equivalent to $W(s)$ hitting the line: $\ell : y - x = 2r$. Based on our definition, the random variable X in Theorem 3.5.1 is the $x-$component of $W(T)$. Denote Y to be the $y-$component of $W(T)$.

Since $\mathbf{V}_1, \mathbf{V}_2$ are distributed as i.i.d standard normal distribution, the direction of $W(s)$ is uniform on a circle S, see Fig. 3.4. This explains why we should expect a Cauchy distribution to appear.

To derive the exact formula, consider the transformation:

$$T : (x, y) \mapsto (y + x, y - x).$$

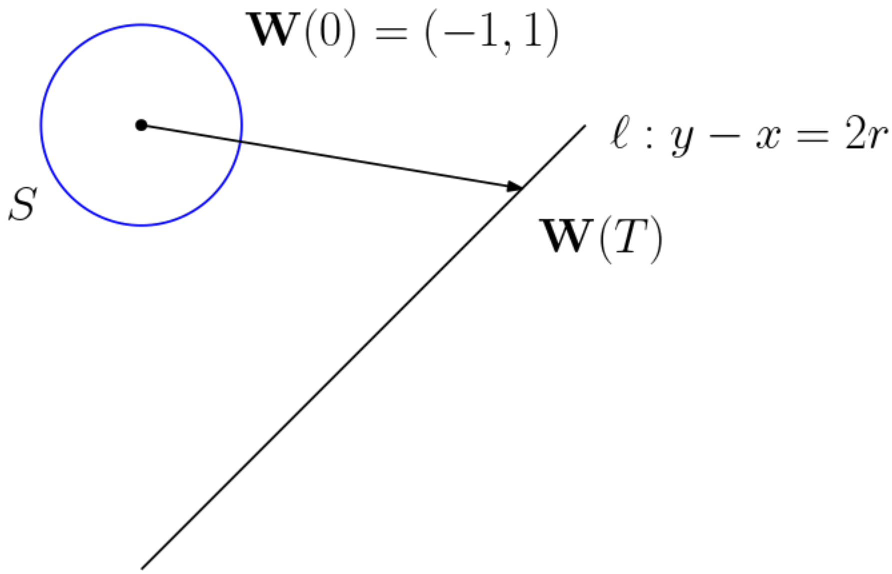

Figure 3.4: A 2−dimensional representation of the trajectories of the centers of two particles moving in 1−dimension.

Fig4

It transforms ℓ to the line $y = 2r$ and $X(0)$ to the point $(0, 2)$. The distance between $(0, 2)$ and the line $y = 2r$ is $2 - 2r$. Applying Lemma 3.5.3 and the fact that trajectory hits the line ℓ with probability $\frac{1}{2}$, we get

$$\begin{aligned}
\mathbb{P}(X \leq x) &= \mathbb{P}(X + Y \leq 2x + 2r) \quad \text{since } Y = X + 2r \\
&= \frac{1}{2}\int_{-\infty}^{2x+2r} \frac{1}{\pi} \cdot \frac{2-2r}{u^2 + (2-2r)^2} du \\
&= \int_{-\infty}^{x} \frac{1}{2\pi} \cdot \frac{1-r}{(u+r)^2 + (1-r)^2} du
\end{aligned}$$

which is the same result as Theorem 3.5.1.

3.6 Collision density function in dimension 2

oplace-2-dim

Let ω_1 be the direction of the velocity $\mathbf{V}_1$, i.e. $\omega_1 = \arctan \frac{v_{1,2}}{v_{1,1}}$. Since $\mathbf{V}_1$ has the standard normal distribution on $\mathbb{R}^2$, we know that ω_1 is independent of the magnitude $\|\mathbf{V}_1\|_2$ and

$$\omega_1 \sim \text{Uniform}[0, 2\pi), \quad \|\mathbf{V}_1\|_2^2 \sim \chi_2^2.$$

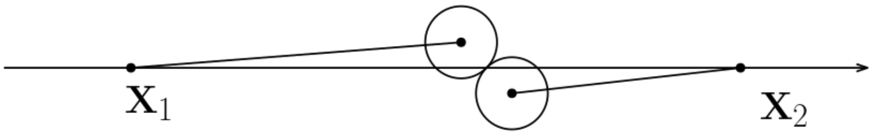

Figure 3.5: An example of trajectories that do not intersect.

trajec-intersec

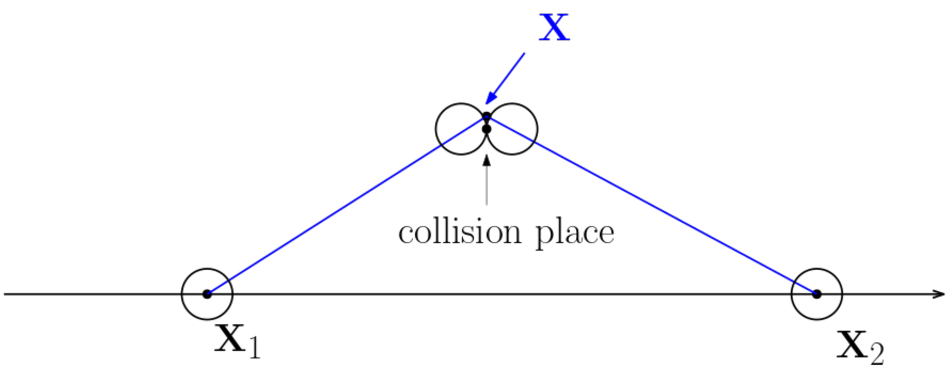

Figure 3.6: Illustration for approximating collision place using the intersection of the trajectories of the centers.

intersection-place

Following the same way, we define $\omega_2 := \arctan \frac{v_{2,2}}{v_{2,1}}$.

Define ℓ_1, ℓ_2 to be the trajectories of particles $\mathbf{P}_1, \mathbf{P}_2$ respectively, i.e.

$$\ell_1 := \{X_1 + s\mathbf{V}_1 : s \geq 0\}, \quad \ell_2 := \{\mathbf{X}_2 + s\mathbf{V}_2 : s \geq 0\}. \tag{3.6.1}$$

Def.trajecto

Note that ℓ_1 and ℓ_2 do not always intersect even when two particles collide. See Fig. 3.5 as an example.

Let $X = \ell_1 \cap \ell_2$ if such an intersection point exists. When radius r is small, this point X is a good approximation for the collision point, see Fig. 3.6.

When X exists, the two particles collide, i.e.

$$\{\ell_1 \cap \ell_2 \neq \phi\} \subseteq E_{r,2}.$$

But the opposite does not hold. The next two lemmas show that the opposite holds with a probability of rate $1 - O(r^2)$.

angle-prob **Lemma 3.6.1.** *Recall that $E_{r,2}$ is the probability for $\mathbf{P}_1$ and $\mathbf{P}_2$ collide in $\mathbb{R}^2$. Let $\eta := \arcsin r$. Let $F_1 = \{|\omega_1 - \pi| \le \eta\} \cup \{0 \le \omega_1 \le \eta\} \cup \{2\pi - \eta \le \omega_1 \le 2\pi\}$, and $F_2 = \{|\omega_2 - \pi| \le \eta\} \cup \{0 \le \omega_2 \le \eta\} \cup \{2\pi - \eta \le \omega_2 \le 2\pi\}$. There exists an absolute constant $C > 0$ such that*

$$\mathbb{P}\left(E_{r,2} \cap (F_1 \cup F_2)\right) \le Cr^2. \tag{3.6.2}$$

angle-prob-result

X-exist **Lemma 3.6.2.**

$$E_{r,2} \cap \{\ell_1 \cap \ell_2 = \phi\} \subseteq E_{r,2} \cap (F_1 \cup F_2). \tag{3.6.3}$$

X-exist-result

Combining this with Lemma 3.6.1, we obtain

$$\mathbb{P}\left(E_{r,2} \cap \{X \text{ does not exist}\}\right) \le Cr^2.$$

Proof of Lemma 3.6.1. we only need to show the follow 6 terms are all Cr^2:

$$\mathbb{P}\left(E_{r,2} \cap \{|\omega_1 - \pi| \le \eta\}\right), \mathbb{P}\left(E_{r,2} \cap \{0 \le \omega_1 \le \eta\}\right), \mathbb{P}\left(E_{r,2} \cap \{2\pi - \eta \le \omega_1 \le 2\pi\}\right)$$

$$\mathbb{P}\left(E_{r,2} \cap \{|\omega_2 - \pi| \le \eta\}\right), \mathbb{P}\left(E_{r,2} \cap \{0 \le \omega_2 \le \eta\}\right), \mathbb{P}\left(E_{r,2} \cap \{2\pi - \eta \le \omega_2 \le 2\pi\}\right)$$

We will first estimate the first term. Conditioning on ω_1 and ω_2, we have

$$\mathbb{P}\left(E_{r,2} \cap \{|\omega_1 - \pi| \le \eta\}\right) = \mathbb{E}\left(\mathbb{P}(E_{r,2} \mid \omega_1, \omega_2) \mathbb{1}_{\{|\omega_1 - \pi| \le \eta\}}\right). \tag{3.6.4}$$

angle-prob-1

Since the whole system is symmetric about the $x-$axis, we only need to consider the case $\pi - \eta \le \omega_1 \le \pi$. Applying Lemma 3.4.1, for the inner term we have

$$\begin{aligned}
&\mathbb{P}(E_{r,2} \mid \omega_1, \omega_2) \\
&= \mathbb{P}\left(\cos\beta \le -\sqrt{1-r^2} \mid \omega_1, \omega_2\right) \\
&= \mathbb{P}\left(\frac{v_{2,1} - v_{1,1}}{\|\mathbf{V}_2 - \mathbf{V}_1\|_2} \le -\sqrt{1-r^2} \mid \omega_1, \omega_2\right) \\
&= \mathbb{P}\left(\left(v_{2,1} - v_{1,1}\right)^2 \ge (1-r^2) \sum_{i=1}^{2} \left(v_{2,i} - v_{1,i}\right)^2, v_{2,1} \le v_{1,1} \mid \omega_1, \omega_2\right)
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{P}\left(r^2 \left(v_{2,1} - v_{1,1}\right)^2 \geq (1-r^2) \left(v_{2,2} - v_{1,2}\right)^2, v_{2,1} \leq v_{1,1} \mid \omega_1, \omega_2\right) \\
&= \mathbb{P}\left(r \left(v_{2,1} - v_{1,1}\right) \leq -\sqrt{1-r^2} \left|v_{2,2} - v_{1,2}\right| \mid \omega_1, \omega_2\right) \\
&= \mathbb{P}\left(r \left(v_{2,1} - v_{1,1}\right) \leq -\sqrt{1-r^2} \left(v_{2,2} - v_{1,2}\right), v_{2,2} \geq v_{1,2} \mid \omega_1, \omega_2\right) \\
&+ \mathbb{P}\left(r \left(v_{2,1} - v_{1,1}\right) \leq -\sqrt{1-r^2} \left(v_{1,2} - v_{2,2}\right), v_{2,2} < v_{1,2} \mid \omega_1, \omega_2\right) \\
&= \mathbb{P}\left(r v_{2,1} + \sqrt{1-r^2} v_{2,2} \leq r v_{1,1} + \sqrt{1-r^2} v_{1,2}, v_{2,2} \geq v_{1,2} \mid \omega_1, \omega_2\right) \qquad (*) \\
&+ \mathbb{P}\left(r v_{2,1} - \sqrt{1-r^2} v_{2,2} \leq r v_{1,1} - \sqrt{1-r^2} v_{1,2}, v_{2,2} < v_{1,2} \mid \omega_1, \omega_2\right) \qquad (**)
\end{aligned}
$$

Recall that

$$
v_{i,1} = \|\mathbf{V}_i\|_2 \cos\omega_i, \quad v_{i,2} = \|\mathbf{V}_i\|_2 \sin\omega_i, \quad \text{for } i = 1, 2,
$$

we obtain

$$
\begin{aligned}
(*) =& \mathbb{P}\Big(\left(r\cos\omega_2 + \sqrt{1-r^2}\sin\omega_2\right) \|\mathbf{V}_2\|_2 \leq \left(r\cos\omega_1 + \sqrt{1-r^2}\sin\omega_1\right) \|\mathbf{V}_1\|_2, \\
& \quad \|\mathbf{V}_2\|_2 \sin\omega_2 \geq \|\mathbf{V}_1\|_2 \sin\omega_1 \mid \omega_1, \omega_2\Big) \\
=& \mathbb{P}\left(\sin(\eta+\omega_2) \|\mathbf{V}_2\|_2 \leq \sin(\eta+\omega_1) \|\mathbf{V}_1\|_2, \|\mathbf{V}_2\|_2 \sin\omega_2 \geq \|\mathbf{V}_1\|_2 \sin\omega_1 \mid \omega_1, \omega_2\right) \\
=& \mathbb{P}\left(\frac{\|\mathbf{V}_1\|_2}{\|\mathbf{V}_2\|_2} \leq \frac{\sin(\eta+\omega_2)}{\sin(\eta+\omega_1)}, \frac{\|\mathbf{V}_1\|_2}{\|\mathbf{V}_2\|_2} \leq \frac{\sin\omega_2}{\sin\omega_1} \mid \omega_1, \omega_2\right)
\end{aligned}
$$

The last step follows from the fact that $\pi - \eta \leq \omega_1 \leq \pi$, and therefore $\sin(\eta + \omega_1) < 0, \sin\omega_1 > 0$.

To make $(*)$ nonzero, we need $\sin(\eta + \omega_2) < 0$ and $\sin\omega_2 > 0$, which implies $\pi - \eta < \omega_2 < \pi$. So we have the follow upper bound

$$
(*) \leq \mathbb{1}_{\{\pi - \eta < \omega_2 < \pi\}}.
$$

In the same way, for $(**)$ we have

$$
(**) = \mathbb{P}\left(\sin(\eta-\omega_2) \|\mathbf{V}_2\|_2 \leq \sin(\eta-\omega_1) \|\mathbf{V}_1\|_2, \|\mathbf{V}_2\|_2 \sin\omega_2 < \|\mathbf{V}_1\|_2 \sin\omega_1 \mid \omega_1, \omega_2\right)
$$

$$=\mathbb{P}\left(\frac{\|\mathbf{V}_1\|_2}{\|\mathbf{V}_2\|_2} \le \frac{\sin(\omega_2-\eta)}{\sin(\omega_1-\eta)}, \frac{\|\mathbf{V}_1\|_2}{\|\mathbf{V}_2\|_2} > \frac{\sin\omega_2}{\sin\omega_1} \mid \omega_1,\omega_2\right)$$

The last step follows from the fact that $\pi-2\eta \le \omega_1-\eta \le \pi-\eta$, and therefore $\sin(\omega_1-\eta) > 0, \sin\omega_1 > 0$.

To make $(**)$ nonzero, we first need $\sin(\omega_2-\eta) > 0$, which implies $\eta < \omega_2 < \pi+\eta$.

We claim that $(**) = 0$ when $\eta < \omega_2 \le \pi-\eta$. It is elementary to check that

$$\sin(\omega_2-\eta)\sin\omega_1 - \sin(\omega_1-\eta)\sin\omega_2 = \sin\eta\sin(\omega_2-\omega_1) < 0.$$

when $\omega_2 \le \pi-\eta \le \omega_1$. This implies

$$\frac{\sin(\omega_2-\eta)}{\sin(\omega_1-\eta)} \le \frac{\sin\omega_2}{\sin\omega_1},$$

which shows $(**)$ equals 0.

Hence, for $(**)$ we have the following upper bound:

$$(**) \le \mathbb{1}_{\{\pi-\eta<\omega_2<\pi+\eta\}}.$$

Combining the two upper bounds together, we obtain

$$\mathbb{P}\left(E_{r,2} \mid \omega_1,\omega_2\right) \le 2\mathbb{1}_{\{|\omega_2-\pi|<\eta\}}.$$

Applying this to Equation (3.6.4), and recalling that $\eta = \arcsin r = O(r)$, we have

$$\mathbb{P}\left(E_{r,2} \cap \{|\omega_1-\pi| \le \eta\}\right) \le \mathbb{E}\left(2\mathbb{1}_{\{|\omega_2-\pi|<\eta\}}\mathbb{1}_{\{|\omega_1-\pi|\le\eta\}}\right) = \frac{2\eta^2}{\pi^2} \le Cr^2.$$

where C is an absolute constant.

One can bound the remaining five terms in a similar way.

□

Proof of Lemma 3.6.2. The event $\ell_1 \cap \ell_2 \ne \phi$ could be written as

$$\mathbf{X}_1 + s_1\mathbf{V}_1 = \mathbf{X}_2 + s_2\mathbf{V}_2, \quad \text{for some } s_1, s_2 \ge 0.$$

which is equivalent to the following equation system having a solution in $\mathbb{R}^+ \times \mathbb{R}^+$

$$\begin{cases} -1 + s_1 v_{1,1} = 1 + s_2 v_{2,1} \\ s_1 v_{1,2} = s_2 v_{2,2} \end{cases} \tag{3.6.5}$$ X-exist-equs

Note that if we release the restriction that the time s is nonnegative in (3.6.1), two trajectories will meet almost surely since ω_1 and ω_2 are continuous random variables, so the above equation system (3.6.5) has a solution in $\mathbb{R}^2$. Furthermore, the solution is unique, call it (s_1, s_2).

Under the condition that two particles collide, the only way for $\ell_1 \cap \ell_2 = \phi$ is one of s_1, s_2 is positive and another one is negative. This implies $v_{2,2} v_{1,2} < 0$.

In the proof of Lemma 3.6.1, we showed that

$$E_{r,2} = \left\{ r\left(v_{2,1} - v_{1,1}\right) \le -\sqrt{1-r^2}\left|v_{2,2} - v_{1,2}\right| \right\}. \tag{3.6.6}$$ X-exist-1

If $v_{2,2} > 0, v_{1,2} < 0$,

$$\begin{aligned} & r\left(v_{2,1} - v_{1,1}\right) \le -\sqrt{1-r^2}\left|v_{2,2} - v_{1,2}\right| \\ \Rightarrow\quad & r\left(v_{2,1} - v_{1,1}\right) \le -\sqrt{1-r^2}\left(v_{2,2} - v_{1,2}\right) \\ \Rightarrow\quad & r v_{2,1} + \sqrt{1-r^2} v_{2,2} \le r v_{1,1} + \sqrt{1-r^2} v_{1,2} \\ \Rightarrow\quad & \sin(\eta + \omega_2) \left\|\mathbf{V}_2\right\|_2 \le \sin(\eta + \omega_1) \left\|\mathbf{V}_1\right\|_2 \end{aligned}$$

Since $v_{2,2} > 0$, we have $0 < \omega_2 < \pi$. If $\omega_2 > \pi - \eta$ then F_2 holds and we are done. If $\omega_2 < \pi - \eta$, then $\sin(\eta + \omega_2) > 0$. It follows that $\sin(\eta + \omega_1)$ should be also positive. Combine this with the assumption $v_{1,2} < 0$, we get $2\pi - \eta < \omega_1 < 2\pi$ and F_1 holds.

Reasoning in the same way, if $v_{2,2} < 0, v_{1,2} > 0$,

$$\begin{aligned} & r\left(v_{2,1} - v_{1,1}\right) \le -\sqrt{1-r^2}\left|v_{2,2} - v_{1,2}\right| \\ \Rightarrow\quad & r\left(v_{2,1} - v_{1,1}\right) \le -\sqrt{1-r^2}\left(v_{1,2} - v_{2,2}\right) \end{aligned}$$

$$\Rightarrow \quad rv_{2,1} - \sqrt{1-r^2}v_{2,2} \leq rv_{1,1} - \sqrt{1-r^2}v_{1,2}$$
$$\Rightarrow \quad \sin(\omega_2 - \eta)\left\|\mathbf{V}_2\right\|_2 \geq \sin(\omega_1 - \eta)\left\|\mathbf{V}_1\right\|_2$$

Since $v_{2,2} < 0$, we have $\pi < \omega_2 < 2\pi$. If $\omega_2 < \pi + \eta$ then F_2 holds and we are done. If $\omega_2 > \pi + \eta$, then $\sin(\omega_2 - \eta) < 0$. It follows that $\sin(\omega_1 - \eta)$ should be also negative. Combining this with the assumption $v_{1,2} > 0$, we get $0 < \omega_1 < \eta$ and F_1 holds.

□

Before continuing to our main result in this section, we first introduce a new coordinate system in $\mathbb{R}^2$, different from the usual $(x, y)-$coordinate system.

Suppose Y is an arbitrary point in $\mathbb{R}^2$. Let α_1 be the angle between $\overrightarrow{\mathbf{X}_1 Y}$ and the positive x-axis, and α_2 be the angle between $\overrightarrow{\mathbf{X}_2 Y}$ and the positive x-axis, see Fig 3.7. Every point in $\mathbb{R}^2$ could be uniquely represented by a pair (α_1, α_2), so (α_1, α_2) forms a coordinate system on $\mathbb{R}^2$. From the graph we can see that α_1 and α_2 either satisfies $\{0 \leq \alpha_1 < \alpha_2 < \pi\}$ or satisfies $\{\pi \leq \alpha_2 < \alpha_1 < 2\pi\}$. So $(\alpha_1, \alpha_2)-$coordinate system has the domain

$$\mathcal{D} := \{(\alpha_1, \alpha_2) : 0 \leq \alpha_1 < \alpha_2 < \pi\} \cup \{(\alpha_1, \alpha_2) : \pi \leq \alpha_2 < \alpha_1 < 2\pi\}.$$

The relation between (x, y) and (α_1, α_2) is given below:

$$\alpha_1 = \arctan \frac{y}{x+1}, \quad \alpha_2 = \arctan \frac{y}{x-1}.$$

So the Jacobian matrix and determinant are:

$$\frac{\partial(\alpha_1, \alpha_2)}{\partial(x, y)} = \begin{bmatrix} -\frac{y}{(x+1)^2 + y^2} & \frac{x+1}{(x+1)^2 + y^2} \\ -\frac{y}{(x-1)^2 + y^2} & \frac{x-1}{(x-1)^2 + y^2} \end{bmatrix},$$
$$\left|\frac{\partial(\alpha_1, \alpha_2)}{\partial(x, y)}\right| = \frac{2|y|}{[(x+1)^2 + y^2]\,[(x-1)^2 + y^2]}.$$

ensity-2-dim **Theorem 3.6.3.** *Consider $0 < \theta < \frac{\pi}{4}$ and a Borel set $A \subset \mathbb{R}^2$ that satisfies*

$$A \subseteq \{(\alpha_1, \alpha_2) : \theta < \alpha_1 < \alpha_2 < \pi - \theta\} \cup \{(\alpha_1, \alpha_2) : \pi + \theta < \alpha_2 < \alpha_1 < 2\pi - \theta\}.$$

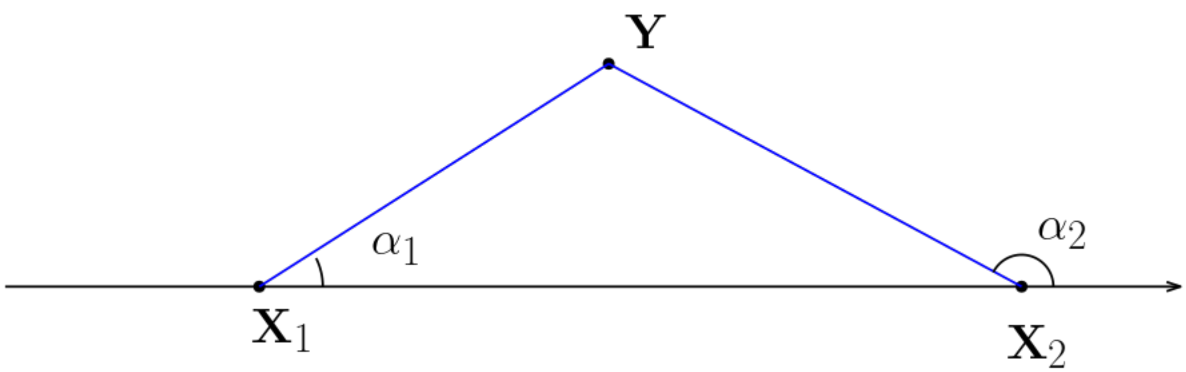

Figure 3.7: Illustration of the alternative (α_1, α_2) coordinate system.

Fig5

(See Fig. 3.8). There exists a constant C depending on θ with order θ^{-6} such that for $\arcsin r < \frac{\theta}{2}$

$$\mathbb{P}(X \in A) = r \iint_A \frac{1}{\pi^2} \cdot \frac{1}{(1 + x^2 + y^2)^2} dxdy + R, \tag{3.6.7}$$

formula-2-di

and $|R|$ is bounded by Cr^2.

Proof. Recalling that $\eta = \arcsin r$, it goes to 0 as $r \to 0$, so WLOG, we can assume $\eta < \theta/2$.

Since collision places are symmetric respect to the $x-$axis, we may assume

$$A = \{(\alpha_1, \alpha_2) : \theta < \alpha_1 < \alpha_2 < \pi - \theta\}.$$

In order to have the two trajectories intersect in A, we need the following two conditions:

- The directions (ω_1, ω_2) are in A.
- The two particles collide. By Lemma 3.4.1 in Section 3.4, this is equivalent to $\cos\beta \leq -\sqrt{1 - r^2}$, where β is the angle between the vectors $X_1 - X_2$ and $\mathbf{V}_1 - \mathbf{V}_2$.

Hence,

$$\begin{aligned} \mathbb{P}(X \in A) &= \mathbb{P}\left(\cos\beta \leq -\sqrt{1 - r^2}, (\omega_1, \omega_2) \in A\right) \\ &= \mathbb{E}\left(\mathbb{P}\left(\cos\beta \leq -\sqrt{1 - r^2} \mid \omega_1, \omega_2\right) \mathbb{1}_{\{A\}}\right). \end{aligned} \tag{3.6.8}$$

density-2-di

For the inner term, follow the same calculation in the proof of Lemma 3.6.1, we have

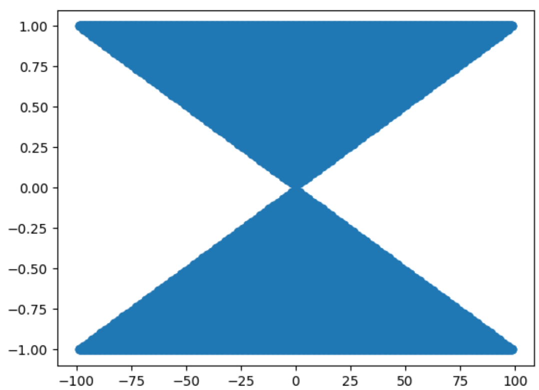

Figure 3.8: Illustration of set A with $\theta = 0.01$ that satisfies the condition in Lemma 3.6.3. Note the aspect ratio of the picture is about $1/100$.

Fig.A

$$
\begin{aligned}
&\mathbb{P}\left(\cos\beta \le -\sqrt{1-r^2} \mid \omega_1, \omega_2\right) \\
=&\mathbb{P}\left(r v_{2,1} + \sqrt{1-r^2} v_{2,2} \le r v_{1,1} + \sqrt{1-r^2} v_{1,2}, v_{2,2} \ge v_{1,2} \mid \omega_1, \omega_2\right) \cdots (*) \\
&+ \mathbb{P}\left(r v_{2,1} - \sqrt{1-r^2} v_{2,2} \le r v_{1,1} - \sqrt{1-r^2} v_{1,2}, v_{2,2} < v_{1,2} \mid \omega_1, \omega_2\right) \cdots (**)
\end{aligned}
$$

We will look at $(*)$ first. Recall that

$$
v_{i,1} = \|\mathbf{V}_i\|_2 \cos\omega_i, \quad v_{i,2} = \|\mathbf{V}_i\|_2 \sin\omega_i, \quad \text{for } i = 1, 2,
$$

we obtain

$$
\begin{aligned}
(*) =&\mathbb{P}\Big(\left(r\cos\omega_2 + \sqrt{1-r^2}\sin\omega_2\right) \|\mathbf{V}_2\|_2 \le \left(r\cos\omega_1 + \sqrt{1-r^2}\sin\omega_1\right) \|\mathbf{V}_1\|_2, \\
&\|\mathbf{V}_2\|_2 \sin\omega_2 \ge \|\mathbf{V}_1\|_2 \sin\omega_1 \mid \omega_1, \omega_2\Big) \\
=&\mathbb{P}\left(\frac{\sin\omega_1}{\sin\omega_2} \le \frac{\|\mathbf{V}_2\|_2}{\|\mathbf{V}_1\|_2} \le \frac{r\cos\omega_1 + \sqrt{1-r^2}\sin\omega_1}{r\cos\omega_2 + \sqrt{1-r^2}\sin\omega_2} \mid \omega_1, \omega_2\right)
\end{aligned}
$$

$$=\mathbb{P}\left(\frac{\sin\omega_1}{\sin\omega_2} \le \frac{\|\mathbf{V}_2\|_2}{\|\mathbf{V}_1\|_2} \le \frac{\sin(\eta+\omega_1)}{\sin(\eta+\omega_2)} \mid \omega_1, \omega_2\right)$$

The second step comes from the fact that

$$\sin\omega_2 > 0, \quad r\cos\omega_2 + \sqrt{1-r^2}\sin\omega_2 = \sin(\eta+\omega_2) > 0.$$

The probability $(*)$ is non-zero because the lower bound is less than the upper bound since

$$\sin\omega_2\sin(\eta+\omega_1) - \sin\omega_1\sin(\eta+\omega_2) = \sin\eta\sin(\omega_2-\omega_1) > 0.$$

Let $g(r;\omega_1,\omega_2) := \left(\frac{r\cos\omega_1+\sqrt{1-r^2}\sin\omega_1}{r\cos\omega_2+\sqrt{1-r^2}\sin\omega_2}\right)^2 = \frac{\sin^2(\eta+\omega_1)}{\sin^2(\eta+\omega_2)}$. Note that $\sin\omega_1, \sin\omega_2$ are both positive and $\frac{\|\mathbf{V}_2\|_2^2}{\|\mathbf{V}_1\|_1}$ follows the F distribution with 2 and 2 degrees of freedom which has PDF $\frac{1}{(1+x)^2}, x > 0$. The term $(*)$ could be regard as the follow function $G(r)$

$$G(r;\omega_1,\omega_2) := (*) = \int_{g(0)}^{g(r;\omega_1,\omega_2)} \frac{1}{(1+x)^2}dx$$

Function G has the following Taylor expansion at point $r = 0$:

$$\begin{aligned}
G(r;\omega_1,\omega_2) &= \frac{1}{(1+g(0))^2}\cdot g'(0)\cdot r + R_1(r;\omega_1,\omega_2) \\
&= 2\left(1+\frac{\sin^2\omega_1}{\sin^2\omega_2}\right)^{-2}\cdot\frac{\sin\omega_1\left(\sin\omega_2\cos\omega_1-\cos\omega_2\sin\omega_1\right)}{\sin^3\omega_2}\cdot r + R_1(r;\omega_1,\omega_2) \\
&= \frac{2\sin\omega_1\sin\omega_2\left(\sin\omega_2\cos\omega_1-\cos\omega_2\sin\omega_1\right)}{\left(\sin^2\omega_1+\sin^2\omega_2\right)^2} r + R_1(r;\omega_1,\omega_2).
\end{aligned}$$

We will show that the remainder term $R_1(r;\omega_1,\omega_2)$ satisfies

$$\iint_A R_1(r;\omega_1,\omega_2)d\omega_1 d\omega_2 = O(r^2).$$

Fix some $0 < \delta < \frac{1}{2}$, by Taylor's inequality, we know that for $0 < r < \delta$

$$|R_1(r;\omega_1,\omega_2)| \le \frac{1}{2}r^2 \sup_{0<r<\delta}\left|G''(r;\omega_1,\omega_2)\right|$$

$$= \frac{1}{2}r^2 \sup_{0<r<\delta} \left| \frac{g''(r;\omega_1,\omega_2)}{(1+g(r;\omega_1,\omega_2))^2} - \frac{2g'(r;\omega_1,\omega_2)^2}{(1+g(r;\omega_1,\omega_2))^3} \right|$$
$$\leq \frac{1}{2}r^2 \sup_{0<r<\delta} \left| \frac{g''(r;\omega_1,\omega_2)}{(1+g(r;\omega_1,\omega_2))^2} \right| + \sup_{0<r<\delta} \left| \frac{2g'(r;\omega_1,\omega_2)^2}{(1+g(r;\omega_1,\omega_2))^3} \right|.$$

For the second term, note that

$$g'(r;\omega_1,\omega_2) = \frac{2\sin(\eta+\omega_1)\sin(\omega_2-\omega_1)}{\sin^3(\eta+\omega_2)} \cdot \frac{1}{\sqrt{1-r^2}}$$

so for $0 < r < \delta$ we have

$$\begin{aligned}
\left| \frac{2g'(r;\omega_1,\omega_2)^2}{(1+g(r;\omega_1,\omega_2))^3} \right| &= \frac{8}{1-r^2} \cdot \frac{\sin^2(\eta+\omega_1)\sin^2(\eta+\omega_2)}{\left(\sin^2(\eta+\omega_1)+\sin^2(\eta+\omega_2)\right)^3} \\
&\leq \frac{8}{1-\delta^2} \cdot \frac{1}{\sin^6(\eta+\omega_1)} \\
&\leq \frac{8}{1-\delta^2} \cdot \frac{1}{\min\{\sin^6(\theta+\eta), \sin^6(\theta-\eta)\}} \\
&\leq \frac{8}{1-\delta^2} \cdot \max\left\{ \frac{1}{\sin^6(\theta+\eta)}, \frac{1}{\sin^6(\theta-\eta)} \right\} \\
&\leq \frac{8}{1-\delta^2} \left(\frac{1}{\sin^6(\theta+\eta)} + \frac{1}{\sin^6(\theta-\eta)} \right) \\
&\leq \frac{8}{(1-\delta^2)^{\frac{3}{2}}} \left(\frac{1}{\sin^6(\theta+\eta)} + \frac{1}{\sin^6(\theta-\eta)} \right).
\end{aligned}$$

For the first term,

$$\begin{aligned}
g''(r;\omega_1,\omega_2) =& 2\sin(\omega_2-\omega_1)\Bigg(\frac{\sin(\eta+\omega_2)\cos(\eta+\omega_1) - 3\sin(\eta+\omega_1)\cos(\eta+\omega_2)}{\sin^4(\eta+\omega_2)(1-r^2)} \\
&+ \frac{\sin(\eta+\omega_1)}{\sin^3(\eta+\omega_2)} \cdot \frac{r}{(1-r^2)^{\frac{3}{2}}} \Bigg)
\end{aligned}$$

so

$$|g''(r;\omega_1,\omega_2)| \leq 2\left(\frac{4}{\sin^4(\eta+\omega_2)(1-\delta^2)} + \frac{1}{\sin^3(\eta+\omega_2)(1-\delta^2)^{\frac{3}{2}}} \right)$$

$$\leq 2 \cdot \frac{5}{(1-\delta^2)^{\frac{3}{2}}} \cdot \frac{1}{\sin^4(\eta+\omega_2)}$$

and for $0 < r < \delta$ we have

$$\begin{aligned}
\left|\frac{g''(r;\omega_1,\omega_2)}{(1+g(r;\omega_1,\omega_2))^2}\right| &\leq \frac{10}{(1-\delta^2)^{\frac{3}{2}}} \cdot \frac{1}{\left(\sin^2(\eta+\omega_1)+\sin^2(\eta+\omega_2)\right)^2}. \\
&\leq \frac{10}{(1-\delta^2)^{\frac{3}{2}}} \cdot \frac{1}{\sin^4(\eta+\omega_2)} \\
&\leq \frac{10}{(1-\delta^2)^{\frac{3}{2}}} \cdot \frac{1}{\sin^6(\eta+\omega_2)} \\
&\leq \frac{10}{(1-\delta^2)^{\frac{3}{2}}} \left(\frac{1}{\sin^6(\theta+\eta)} + \frac{1}{\sin^6(\theta-\eta)}\right)
\end{aligned}$$

Recall that $0 < \theta < \frac{\pi}{4}$, $0 < \eta < \frac{\theta}{2}$ and $0 < \delta < \frac{1}{2}$. It follows that

$$\begin{aligned}
\iint_A R_1(r;\omega_1,\omega_2)d\omega_1 d\omega_2 &\leq r^2 \cdot \frac{8}{(1-\delta^2)^{\frac{3}{2}}} \left(\frac{1}{\sin^6(\theta+\eta)} + \frac{1}{\sin^6(\theta-\eta)}\right) \iint_A d\omega_1 d\omega_2 \\
&\leq r^2 \cdot \frac{8}{(1-\delta^2)^{\frac{3}{2}}} \left(\frac{1}{\sin^6(\theta+\eta)} + \frac{1}{\sin^6(\theta-\eta)}\right) \int_0^\pi \int_0^\pi d\omega_1 d\omega_2 \\
&\leq r^2 \cdot \frac{64}{\sqrt{27}} \left(\frac{1}{\sin^6(\theta)} + \frac{1}{\sin^6(\frac{\theta}{2})}\right) \int_0^\pi \int_0^\pi d\omega_1 d\omega_2 \\
&\leq Cr^2,
\end{aligned}$$

where C depends on θ, and it is of order θ^{-6}.

Now we will look at $(**)$. Follow the same way as in $(*)$, we have

$$\begin{aligned}
(**) &= \mathbb{P}\left(\|\mathbf{V}_2\|_2 \sin(\eta-\omega_2) \leq \|\mathbf{V}_1\|_2 \sin(\eta-\omega_1), \|\mathbf{V}_2\|_2 \sin\omega_2 < \|\mathbf{V}_1\| \sin\omega_1 \mid \omega_1,\omega_2\right) \\
&= \mathbb{P}\left(\frac{\sin(\omega_1-\eta)}{\sin(\omega_2-\eta)} \leq \frac{\|\mathbf{V}_2\|_2}{\|\mathbf{V}_1\|_2} < \frac{\sin\omega_1}{\sin\omega_2}\right). \\
&= \int_{\frac{\sin^2(\omega_1-\eta)}{\sin^2(\omega_2-\eta)}}^{\frac{\sin^2\omega_1}{\sin^2\omega_2}} \frac{1}{(1+x)^2} dx \\
&= \frac{2\sin\omega_1 \sin\omega_2 \left(\sin\omega_2\cos\omega_1 - \cos\omega_2\sin\omega_1\right)}{\left(\sin^2\omega_1+\sin^2\omega_2\right)^2} r + R_2(r;\omega_1,\omega_2).
\end{aligned}$$

Here, $R_2(r;\omega_1,\omega_2)$ is the reminder term which can be proved satisfy

$$\iint_A R_2(r;\omega_1,\omega_2)d\omega_1 d\omega_2 \le C' r^2,$$

where C' is a constant depending on θ with order θ^{-6}.

Plugging into the equation (3.6.8), we get

$$\begin{aligned}\mathbb{P}(X\in A) &= \iint_A \frac{1}{2\pi}\cdot\frac{1}{2\pi}\cdot[(*)+(**)]\,d\omega_1 d\omega_2 \\ &= r\iint_A \frac{\sin\omega_1\sin\omega_2\left(\sin\omega_2\cos\omega_1-\cos\omega_2\sin\omega_1\right)}{\pi^2\left(\sin^2\omega_1+\sin^2\omega_2\right)^2}d\omega_1 d\omega_2+O(r^2)\end{aligned} \tag{3.6.9}$$

dim-2-equ-final

Recall that

$$\omega_1=\arctan\frac{y}{x+1},\quad \omega_2=\arctan\frac{y}{x-1}$$

and when $\theta<\omega_1<\omega_2<\pi-\theta$ the Jacobian matrix is

$$\left|\frac{\partial(\omega_1,\omega_2)}{\partial(x,y)}\right| = \frac{2y}{[(x+1)^2+y^2]\,[(x-1)^2+y^2]},$$

the right hand side of (3.6.9) becomes

$$\mathbb{P}(X\in A) = r\iint_A \frac{1}{\pi^2}\cdot\frac{1}{(1+x^2+y^2)^2}dxdy+O(r^2).$$

□

Remark 3.6.4. Lemma 3.6.1 tells us the probability that X is close to the x-axis is of the order of r^2. Theorem 3.6.3 further tells us that away from the x-axis, the distribution of X has an approximate defective PDF: $\frac{r}{\pi^2(1+x^2+y^2)}$. Taking informally (because this case is not covered by Theorem 3.6.3) $A=\mathbb{R}^2$, we get

$$\begin{aligned}\mathbb{P}(X\in\mathbb{R}^2) &\approx \frac{r}{\pi^2}\iint_{\mathbb{R}^2}\frac{1}{(1+x^2+y^2)^2}dxdy \\ &= \frac{r}{\pi^2}\int_0^\infty\int_0^{2\pi}\frac{\rho}{(1+\rho^2)^2}d\theta d\rho \quad \text{(polar coordinates)} \\ &= \frac{r}{\pi}.\end{aligned}$$

Note that the above informal calculation agrees with our result in Remark 3.4.6.

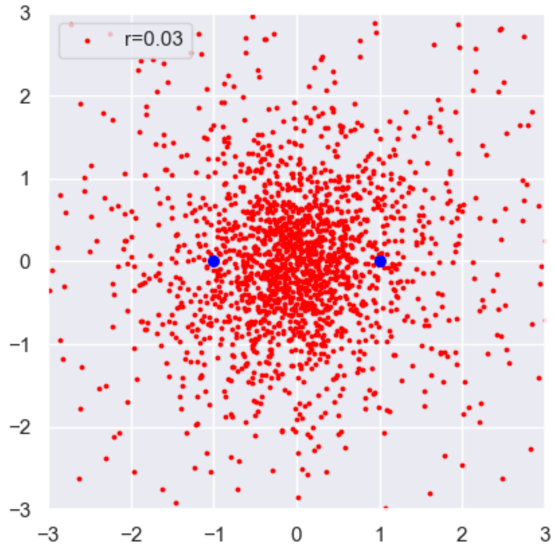

Figure 3.9: Simulation result for $d = 2$, $r = 0.03$ and $2,000$ pairs of particle collisions.

Fig.r003

The defective PDF given in Theorem 3.6.3 is rotation invariant in $\mathbb{R}^2$, this fits our simulation result in Fig. 3.9

Let (ρ, θ) to be the polar coordinates, i.e.

$$\rho = \sqrt{x^2 + y^2}, \quad \theta = \arctan \frac{y}{x}.$$

The approximate defective PDF $\frac{r}{(1+x^2+y^2)^2}$ shows ρ and θ are independent. If we condition on a collision, then θ and ρ have the following distributions:

$$\theta \sim \text{Uniform}(0, 2\pi), \quad \rho \sim CDF : \frac{\rho^2}{1 + \rho^2}, \rho > 0.$$

Fig. 3.10 is the graph for the simulated results.

Similar to Section 3.5, we will use the follow Lemma to explain why our approximated defective PDF is rotation invariant.

uniform-uniform

Lemma 3.6.5. *Suppose a particle starts from a fixed point $X \in \mathbb{R}^d$ with the direction uniformly on the semi-sphere S centered at x. See Fig. 3.11. Denote Y to be the point*

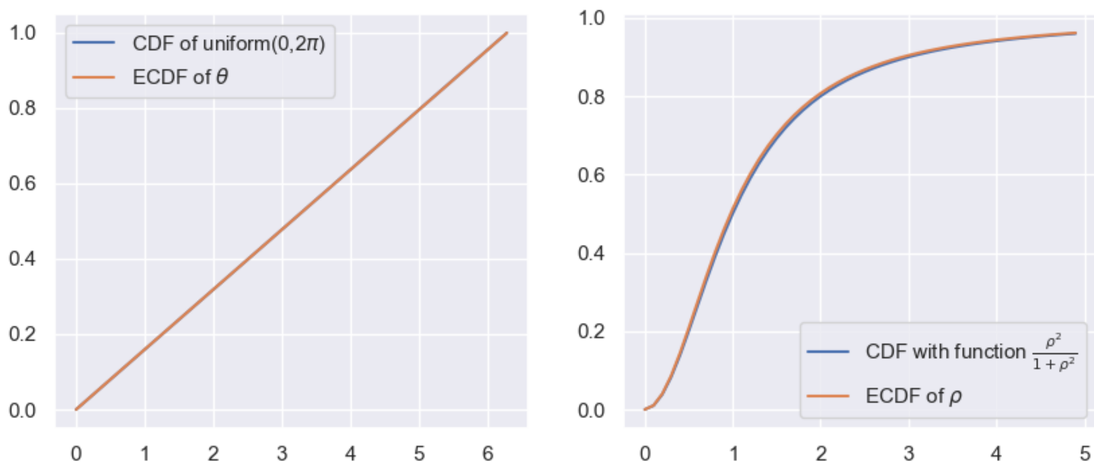

Figure 3.10: Simulation result for $d = 2, r = 0.03$ and $2,000,000$ pairs of particle collisions. The graph on the left represents the empirical CDF of θ v.s. the CDF of uniform distribution on $(0, 2\pi)$. The graph on the right represents the empirical CDF of ρ v.s the CDF $\frac{\rho^2}{1+\rho^2}$.

ig.rho_theta

particle that hits the hyperplane $\mathcal{H}$. Let O be the point on $\mathcal{H}$ such that $\vec{OX}$ orthogonal to $\mathcal{H}$. Then the distribution of Y is rotation invariant with respect to O, i.e. the density function depends on $\|OY\|_2$ only.

Proof. Based on the assumption that the direction is uniform over the sphere, the density function of Y depends only on $\|XY\|_2$. Note that

$$\|XY\|_2^2 = \|OX\|_2^2 + \|OY\|_2^2,$$

so it also depends on $\|OY\|_2$ only. □

We will still use $W(s) = (X_1 + s\mathbf{V}_1, X_2 + s\mathbf{V}_2)$ to represent the trajectory the center for two particles. In the two dimensional case, W is a trajectory in $\mathbb{R}^4$. The event collision equivalents to $W(s)$ hitting the slab:

$$\{(x, y, z, w) : (x - z)^2 + (y - w)^2 \leq 4r^2\}.$$

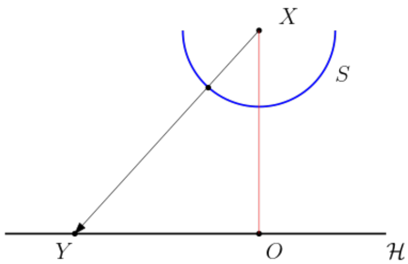

Figure 3.11: Graph for Lemma 3.6.5.

Fig.sec3-5

This slab is a ball around the hyperplane:

$$\mathcal{H} := \{(x, y, z, w) : x = z, y = w\}.$$

For small $r > 0$, informally speaking, we can treat the collision as $W(s)$ hitting $\mathcal{H}$. If we denote $O := (0, 0, 0, 0)$ to be the original point in $\mathbb{R}^4$, then O is a point in $\mathcal{H}$. Note that the vector between $W(0)$ and O is orthogonal to $\mathcal{H}$.

Recall that $\mathbf{V}_1, \mathbf{V}_2$ are distributed as i.i.d standard normal, and the direction of $W(s)$ is uniform on the sphere centered at $W(0)$. Let Y be the hitting point on $\mathcal{H}$, then the collision point in $\mathbb{R}^2$ is the first two coordinates of Y. By Lemma 3.6.5, the distribution of Y depends on $\|OY\|_2$ only. Since Y has the form (x, y, x, y), $\|OY\|_2$ depends on $x^2 + y^2$ only.

3.7 Collision density function in dimension 3

sec-coplace-3-dim

We will use $xyz-$coordinate system for space $\mathbb{R}^3$. We will first define our approximation point $\mathbf{X}$. Recall that ℓ_1, ℓ_2 are the trajectories for $\mathbf{P}_1$ and $\mathbf{P}_2$ respectively. In 3 dimensional space, ℓ_1 and ℓ_2 may not be on the same plane.

Denote $\mathbf{L}_2$ to be the plane containing the x-axis and ℓ_2. Rotate ℓ_1 to $\mathbf{L}_2$ with the same side as $\mathbf{P}_2$, for example, if $\mathbf{L}_2$ is the $xy-$plane, then the rotation will be

$$(x, y, z) \mapsto \left(x, \sqrt{y^2 + z^2}\text{sign}(y_2), 0\right).$$

Call the rotated trajectory ℓ_1'. Denote $\mathbf{X}$ to be the intersection point between ℓ_1' and ℓ_2

(if such point exists). When the radius r is small, $\mathbf{X}$ would be a good approximation for the real collision point.

In this section, we will let ω_1 to be the angle between $\mathbf{V}_1$ and the positive $x-$axis, ω_2 to be the angle between $\mathbf{V}_2$ and the positive $x-$axis.

Recall that $E_{r,3}$ is the event that the particles $\mathbf{P}_1$ and $\mathbf{P}_2$ collide in $\mathbb{R}^3$, and $\eta = \arcsin r$. Similar to the Lemma 3.6.1 and Lemma 3.6.2 in Section 3.6, in $3-$dimensional space, we have the following result:

le-prob-3dim **Lemma 3.7.1.** *There exists an absolute constant $C > 0$ such that*

$$\mathbb{P}\left(E_{r,3} \cap \{X \textit{ does not exist}\}\right) \le Cr^3.$$

Remark 3.7.2. The phrase "X does not exist" means that the two particles collide but the rotated trajectories on the $xy-$plane do not intersect.

Proof. Also WLOG, we all assume the particle $\mathbf{P}_2$ is moving along the $xy-$plane, i.e. $\mathbf{L}_2$ is the $xy-$plane.

Recall that we rotate the particle $\mathbf{P}_1$ to the $xy-$plane with the same side of $\mathbf{P}_2$, and the center for $\mathbf{P}_1$ and $\mathbf{P}_2$ are represented by $\mathbf{C}_1$ and $\mathbf{C}_2$ respectively. Since we assumed $\mathbf{P}_2$ is moving along the $xy-$plane, $\mathbf{C}_1 = (x_1, y_1, z_1)$ will be rotated to $\mathbf{C}_1' = (x_1, \sqrt{y_1^2 + z_1^2}\text{sign}(y_2), 0)$.

In this case, $\mathbf{C}_1'$ has angle ω_1' with the positive $x-$axis, where ω_1' is defined as:

$$\omega_1' = \begin{cases} \omega_1 & \text{if } y_2 > 0 \\ 2\pi - \omega_1 & \text{if } y_2 < 0. \end{cases}$$

By Lemma 3.6.2, we have

$$E_{r,3} \cap \{X \text{ does not exist}\} \subseteq E_{r,3} \cap (F_1 \cup F_2) \tag{3.7.1}$$

Condition-formula

where

$$F_1 = \{|\tan\omega_1| \le \tan\eta\}, \quad F_2 = \{|\tan\omega_2| \le \tan\eta\}.$$

We only need to estimate the term:

$$\mathbb{P}\left(E_{r,3} \cap \{|\omega_1 - \pi| \le \eta\}\right)$$

The rest of the terms can be shown in the same logic.

We will show that if two particles collide, then they still collide after rotation. Computing the distance between two centers we will get

$$\begin{aligned}
\|\mathbf{C}_1' - \mathbf{C}_2\|_2^2 &= (x_1 - x_2)^2 + \left(\operatorname{sign}(y_2)\sqrt{y_1^2 + z_1^2} - y_2\right)^2 \\
&= (x_1 - x_2)^2 + (y_1^2 + z_1^2) + y_2^2 - 2|y_2|\sqrt{y_1^2 + z_1^2} \\
&\leq (x_1 - x_2)^2 + (y_1^2 + z_1^2) + y_2^2 - 2|y_2||y_1| \\
&\leq (x_1 - x_2)^2 + (y_1^2 + z_1^2) + y_2^2 - 2y_1 y_2 \\
&= (x_1 - x_2)^2 + (y_1 - y_2)^2 + z_1^2 \\
&= \|\mathbf{C}_1 - \mathbf{C}_2\|_2^2 \leq (2r)^2
\end{aligned}$$

Follow a similar way as the proof of Lemma 3.6.1, there exists a constant C such that

$$\mathbb{P}(E_{r,3} \mid \omega_1, \omega_2) \leq Cr.$$

Hence,

$$\begin{aligned}
\mathbb{P}\left(E_{r,3} \cap \{|\omega_1 - \pi| \leq \eta\}\right) &\leq Cr \cdot \mathbb{P}(|\omega_1 - \pi| \leq \eta) \\
&= Cr \cdot \frac{1 - \cos\eta}{2} \\
&= Cr \cdot \sin^2\eta \\
&= Cr^3
\end{aligned}$$

□

In Section 3.6, we introduced a coordinate system in 2−dimensional space, which is

$$\alpha_1 = \arctan\frac{\sqrt{y^2 + z^2}}{x + 1}, \quad \alpha_2 = \arctan\frac{\sqrt{y^2 + z^2}}{x - 1}.$$

To determine a point in 3−dimensional space, we will need an additional angle:

$$\gamma := \arctan\frac{z}{y}.$$

Compute the Jacobian matrix we can get

$$\left|\frac{\partial(\alpha_1,\alpha_2,\gamma)}{\partial(x,y,z)}\right| = \frac{2}{((x+1)^2+y^2+z^2)\,((x-1)^2+y^2+z^2)} \tag{3.7.2}$$

eq-3-dim-jacobian

Our main result will be the follow approximation result

ensity-3-dim

Theorem 3.7.3. *Consider* $0<\theta<\frac{\pi}{2}, \delta>0$ *and a Borel set* $A\subset\mathbb{R}^3$ *that satisfies*

$$A \subseteq \{(\alpha_1,\alpha_2,\gamma)\in\mathbb{R}^3 : \theta<\alpha_1<\alpha_2<\pi-\theta, \alpha_2-\alpha_1>\delta, 0\le\gamma<2\pi\}$$

For $r<\min\{\tan^2\theta, \arccos\frac{1}{1+\tan\theta\tan\delta}, \frac{\tan\theta\sin\delta}{4}\}$,

$$\mathbb{P}(X\in A) = r^2\iint_A \frac{1}{\pi^2}\cdot\frac{1}{(1+x^2+y^2+z^2)^3}dxdydz + R, \tag{3.7.3}$$

formula-3-dim

where $|R|$ *is bounded by* Cr^3 *with* C *depending on* θ *and* δ.

Before the proof of Theorem 3.7.3, we will introduce the following lemma

y-3-dim

Lemma 3.7.4. *Suppose a point* $(\alpha_1,\alpha_2,\gamma)$ *in* $\mathbb{R}^3$ *has a coordinate* (x,y,z) *under the* $xyz-$*coordinate system. If there exists some* $0<\theta<\frac{\pi}{2}$ *such that* $\theta<\alpha_1<\alpha_2<\pi-\theta$, *then we have* $\sqrt{y^2+z^2}>\tan\theta$.

Proof. By the definition of α_1,α_2, we have

$$\sqrt{y^2+z^2} = |\tan\alpha_1||x+1| \ge \tan\theta|x+1|,$$

$$\sqrt{y^2+z^2} = |\tan\alpha_2||x-1| \ge \tan\theta|x-1|.$$

The result holds since $\max\{|x+1|,|x-1|\}\ge 1$. □

Next, we will introduce an important term: "effective radius". In our calculations, we will assume the center of particle $\mathbf{P}_2$ is moving in the $xy-$plane. Recall that $\mathbf{T}$ is the collision time. Suppose the center of particle $\mathbf{X}_1$ is $\mathbf{C}_1(T)=(x_1,y_1,z_1)$, then we define the effective radius d to be

$$d=\frac{1}{2}\sqrt{4r^2-z_1^2}.$$

The projections of $\mathbf{V}_1$ and $\mathbf{V}_2$ on $xy-$plane will be called $\mathbf{V}_1'$ and $\mathbf{V}_2'$. We call d "'effective" because two particles with radii d and velocities $\mathbf{V}_1'$, $\mathbf{V}_2'$ moving in $xy-$plane will collide if and only if $\mathbf{P}_1$ and $\mathbf{P}_2$ collide.

Note that $d\le r$.

proof of Theorem 3.7.3. Recall that ω_1, ω_2 are the angles between the positive $x-$ axis and $\mathbf{V}_1, \mathbf{V}_2$ respectively. We further define

$$\gamma_1 := \arctan \frac{v_{1,3}}{v_{1,2}}, \quad \gamma_2 := \arctan \frac{v_{2,3}}{v_{2,2}}.$$

WLOG we may assume that $A = A_1 \times A_2$ where

$$A_1 \subseteq \{(\alpha_1, \alpha_2) : \theta < \alpha_1 < \alpha_2 < \pi - \theta, \alpha_2 - \alpha_1 > \delta\}, \quad A_2 \subseteq \mathbb{R}.$$

Then conditioning on $\omega_1, \omega_2, \gamma_2$ we can get

$$\begin{aligned}\mathbb{P}(X \in A) &= \mathbb{E}\left(\mathbb{P}(E_{r,3} \mid \omega_1, \omega_2, \gamma_2) \mathbb{1}_{\{(\omega_1,\omega_2)\in A_1\}} \mathbb{1}_{\{\gamma_2 \in A_2\}}\right) \\ &= \iint_A \mathbb{P}(E_{r,3} \mid \omega_1, \omega_2, \gamma_2) \cdot \frac{\sin\omega_1}{2} \cdot \frac{\sin\omega_2}{2} \cdot \frac{1}{2\pi} d\omega_1 d\omega_2 d\gamma_2.\end{aligned}$$

Since the whole system is rotation invariant respect to the $x-$axis, when conditioning on γ_2 we can assume $\gamma_2 = 0$, i.e. the center of particle $\mathbf{P}_2$ is moving in the $xy-$plane.

Suppose $\mathbf{X} = (x, y, z)$ and at collision time $\mathbf{C}_1 = (x_1, y_1, z_1)$, then we have $z_1 = y_1 \tan\gamma_1$ and $0 \le z_1 \le 2r$. Recall that $\mathbf{X}$ is formed by rotating $\mathbf{C}_1$ to the plane $\mathbf{L}_2$ and then extending the trajectory. Let $\mathbf{C}_1'$ be the point after rotating $\mathbf{C}_1$ to $\mathbf{L}_2$. In the proof of Lemma 3.7.1, we showed that $\|\mathbf{C}_1' - \mathbf{C}_2\|_2 \le 2r$. Denote ϵ to be the angle formed by $\mathbf{XC}_2$ and $\mathbf{XC}_1'$, then

$$|y - y_1| \le \left\|\mathbf{X} - \mathbf{C}_1'\right\|_2 = \frac{\|\mathbf{C}_1' - \mathbf{C}_2\|_2}{\sin(\omega_2 - \omega_1)} \sin\epsilon \le \frac{2r}{\sin\delta} := Cr.$$

Recall the idea of "effective radius", let $d = \frac{1}{2}\sqrt{4r^2 - z_1^2}$, then

$$\mathbb{P}(E_{r,3} \mid \omega_1, \omega_2, \gamma_2 = 0) = \mathbb{E}_{\gamma_1}\left(\mathbb{P}(E_{d,2} \mid \omega_1, \omega_2, \gamma_1, \gamma_2 = 0)\right).$$

While interpreting the above equation, note that we will consider the velocities to be the projections of $\mathbf{V}_1, \mathbf{V}_2$ on the $xy-$plane. Note the extra factor $\cos\gamma_1$ relative to the $2-$dimensional case, in the formulas given below.

Following the same way in the proof of Theorem 3.6.3, we have

$$\begin{aligned}&\mathbb{P}(E_{d,2} \mid \omega_1, \omega_2, \gamma_1, \gamma_2 = 0) \\ =&\mathbb{P}\left(d\left(\|\mathbf{V}_2\|\cos\omega_2 - \|\mathbf{V}_1\|\cos\omega_1\right) \le -\sqrt{1-d^2}\left|\,\|\mathbf{V}_2\|\sin\omega_2 - \|\mathbf{V}_1\|\sin\omega_1\cos\gamma_1\right|\right)\end{aligned} \tag{3.7.4}$$

eq-3-dim-1

Case I: when $\|\mathbf{V}_2\| \sin\omega_2 \geq \|\mathbf{V}_1\| \sin\omega_1 \cos\gamma_1$. Note that

$$d\cos\omega_2 + \sqrt{1-d^2}\sin\omega_2 = \sin\left(\omega_2 + \eta'\right) > 0, \quad \eta' := \arcsin d < \eta.$$

then the RHS of (3.7.4) becomes

$$\mathbb{P}\left(\frac{\sin\omega_1\cos\gamma_1}{\sin\omega_2} \leq \frac{\|\mathbf{V}_2\|}{\|\mathbf{V}_1\|} \leq \frac{d\cos\omega_1 + \sqrt{1-d^2}\sin\omega_1\cos\gamma_1}{d\cos\omega_2 + \sqrt{1-d^2}\sin\omega_2}\right) \tag{3.7.5}$$ eq-3-dim-2

The random variable $\frac{\|\mathbf{V}_2\|^2}{\|\mathbf{V}_1\|^2}$ has the F distribution with 3 and 3 degrees of freedom which has PDF: $\frac{8}{\pi}\sqrt{x}\,(1+x)^{-3}, x > 0$. So the Taylor expansion for (3.7.5) is

$$\begin{aligned}
&\mathbb{P}\left(\frac{\sin\omega_1\cos\gamma_1}{\sin\omega_2} \leq \frac{\|\mathbf{V}_2\|}{\|\mathbf{V}_1\|} \leq \frac{d\cos\omega_1 + \sqrt{1-d^2}\sin\omega_1\cos\gamma_1}{d\cos\omega_2 + \sqrt{1-d^2}\sin\omega_2}\right) \\
&= \int_{\left(\frac{\sin\omega_1\cos\gamma_1}{\sin\omega_2}\right)^2}^{\left(\frac{d\cos\omega_1+\sqrt{1-d^2}\sin\omega_1\cos\gamma_1}{d\cos\omega_2+\sqrt{1-d^2}\sin\omega_2}\right)^2} \frac{8}{\pi}\sqrt{x}\,(1+x)^{-3}\,dx \qquad (3.7.6) \\
&= \frac{16}{\pi}\cdot\frac{\sin^2\omega_1\sin^2\omega_2\cos^2\gamma_1}{\left(\sin^2\omega_2 + \sin^2\omega_1\cos^2\gamma_1\right)^3}\cdot\left|\sin\omega_2\cos\omega_1 - \sin\omega_1\cos\gamma_1\cos\omega_2\right|\cdot d \\
&\quad + R_1\left(d;\omega_1,\omega_2,\gamma_1\right)
\end{aligned}$$ eq-3-dim-3

Denote $G(d,\omega_1,\omega_2)$ to be the integral (3.7.6). The function G is a C^2 function in the domain $[0,1]\times A_1$. By the Taylor inequality, for the remainder term R_1 we have

$$\begin{aligned}
R_1\left(d;\omega_1,\omega_2\right) &\leq \frac{d^2}{2}\cdot\sup_{0\leq d\leq 1}\left|\frac{\partial^2}{\partial d^2}G\left(d;\omega_1,\omega_2\right)\right| \\
&\leq \frac{d^2}{2}\cdot\sup_{\substack{0\leq d\leq 1\\(\omega_1,\omega_2)\in A_1}}\left|\frac{\partial^2}{\partial d^2}G\left(d,\omega_1,\omega_2\right)\right| = C_1 d^2. \qquad (3.7.7)
\end{aligned}$$ eq-3-dim-4

Note that the supremum in (3.7.7) depends on θ,δ only, and so does C_1.

Case II: when $\|\mathbf{V}_2\|\sin\omega_2 \leq \|\mathbf{V}_1\|\sin\omega_1\cos\gamma_1$. In this case, in the same way, we can show that

$$\begin{aligned}
(3.7.4) \leq &\frac{16}{\pi}\cdot\frac{\sin^2\omega_1\sin^2\omega_2\cos^2\gamma_1}{\left(\sin^2\omega_2 + \sin^2\omega_1\cos^2\gamma_1\right)^3}\cdot\left|\sin\omega_2\cos\omega_1 - \sin\omega_1\cos\gamma_1\cos\omega_2\right|\cdot d \\
&+ R_2\left(d;\omega_1,\omega_2,\gamma_1\right)
\end{aligned}$$

where R_2 is the remainder term with $R_2 \le C_2 d^2$, where C_2 is a constant depends on θ, δ only. It follows that

$$\mathbb{P}\left(E_{r,3} \mid \omega_1, \omega_2, \gamma_2 = 0\right) = \mathbb{E}_{\gamma_1}\left(\mathbb{P}\left(E_{d,2} \mid \omega_1, \omega_2, \gamma_1, \gamma_2 = 0\right)\right) \tag{3.7.8}$$

eq-3-dim-8

$$= \mathbb{E}_{\gamma_1}\left(\frac{32}{\pi} \cdot \frac{\sin^2\omega_1 \sin^2\omega_2 \cos^2\gamma_1}{\left(\sin^2\omega_2 + \sin^2\omega_1\cos^2\gamma_1\right)^3} \cdot \left|\sin\omega_2\cos\omega_1 - \sin\omega_1\cos\gamma_1\cos\omega_2\right| \cdot d\right) \tag{3.7.9}$$

$$+ \mathbb{E}_{\gamma_1}\left(R_1\left(d;\omega_1,\omega_2,\gamma_1\right) + R_2\left(d;\omega_1,\omega_2,\gamma_1\right)\right) \tag{3.7.10}$$

By symmetry, we will only consider the case when $0 \le \gamma_1 < \pi$, and the final answer will multiply by 2. Note that $z_1 = y_1 \tan\gamma_1 < 2r$, we have

$$\gamma_1 \le \tan\gamma_1 \le \frac{2r}{y_1} \le \frac{2r}{y - Cr} \le \frac{2r}{\tan\theta - Cr} \le \frac{4r}{\tan\theta} \le 4\tan\theta.$$

For the second term, we have

$$\begin{aligned} &\mathbb{E}_{\gamma_1}\left(R_1\left(d;\omega_1,\omega_2,\gamma_1\right) + R_2\left(d;\omega_1,\omega_2,\gamma_1\right)\right) \\ \le &\mathbb{E}_{\gamma_1}\left(C_1 r^2 + C_2 r^2\right) \\ \le &\frac{1}{2\pi} \cdot \frac{4\left(C_1 + C_2\right)}{\tan\theta} r^3. \end{aligned}$$

For the first term, define

$$f(\gamma_1,\omega_1,\omega_2) := \frac{32}{\pi} \cdot \frac{\sin^2\omega_1\sin^2\omega_2\cos^2\gamma_1}{\left(\sin^2\omega_2 + \sin^2\omega_1\cos^2\gamma_1\right)^3} \cdot \left|\sin\omega_2\cos\omega_1 - \sin\omega_1\cos\gamma_1\cos\omega_2\right|$$

The function f is a C^2 function of $\gamma_1, \omega_1, \omega_2$ if $\sin\omega_2\cos\omega_1 - \sin\omega_1\cos\gamma_1\cos\omega_2$ is away from 0. This is true since if $\cos\omega_2 \ge 0$ we have

$$\begin{aligned} &\sin\omega_2\cos\omega_1 - \sin\omega_1\cos\gamma_1\cos\omega_2 \\ \ge &\sin\omega_2\cos\omega_1 - \sin\omega_1\cos\omega_2 \\ = &\sin(\omega_2 - \omega_1) > \sin\delta > 0. \end{aligned}$$

If $\cos\omega_2 \le 0$ and $\cos\omega_1 > 0$, then

$$\begin{aligned} &\sin\omega_2\cos\omega_1 - \sin\omega_1\cos\gamma_1\cos\omega_2 \\ \ge &\sin\omega_2\cos\omega_1 > 0. \end{aligned}$$

If $\cos\omega_2 \le 0$ and $\cos\omega_1 \le 0$, then

$$\sin\omega_2\cos\omega_1 - \sin\omega_1\cos\gamma_1\cos\omega_2 > 0 \Leftrightarrow \cos\gamma_1 > \frac{\tan\omega_2}{\tan\omega_1}$$

and this is true since

$$\begin{aligned}\frac{\tan\omega_2}{\tan\omega_1} \le \frac{\tan(\omega_1+\delta)}{\tan\omega_1} &= \frac{1+\frac{\tan\delta}{\tan\omega_1}}{1-\tan\omega_1\tan\delta}\\ &\le \frac{1}{1+\tan\theta\tan\delta} < \cos\gamma_1.\end{aligned}$$

By the mean value theorem,

$$|f(\gamma_1,\omega_1,\omega_2) - f(0,\omega_1,\omega_2)| \le \gamma_1 \cdot \sup_{\substack{0\le\gamma_1\le 4\tan\theta\\(\omega_1,\omega_2)\in A_1}} \left|\frac{\partial}{\partial\gamma_1} f\right| = C_3\gamma_1. \tag{3.7.11}$$ eq-3-dim-5

For the first term, we have

$$\mathbb{E}_{\gamma_1}\left(f\left(\gamma_1,\omega_1,\omega_2\right)d\right) = f\left(0,\omega_1,\omega_2\right)\mathbb{E}_{\gamma_1}\left(d\right) + \mathbb{E}_{\gamma_1}\left(\left[f(\gamma_1,\omega_1,\omega_2) - f(0,\omega_1,\omega_2)\right]d\right)$$

Using the inequality (3.7.11), we have

$$\mathbb{E}_{\gamma_1}\left(|f(\gamma_1,\omega_1,\omega_2) - f(0,\omega_1,\omega_2)|d\right) \le \mathbb{E}_{\gamma_1}\left(C_3\gamma_1 r\right) \le \mathbb{E}_{\gamma_1}\left(C_3\cdot\frac{4r^2}{\tan\theta}\right) \le \frac{C_3}{2\pi}\cdot\frac{16r^3}{\tan^2\theta}.$$

For the first term, note that

$$\frac{1}{2}\sqrt{4r^2 - (y+Cr)^2\tan^2\gamma_1} \le d = \frac{1}{2}\sqrt{4r^2 - y_1^2\tan^2\gamma_1} \le \frac{1}{2}\sqrt{4r^2 - (y-Cr)^2\tan^2\gamma_1},$$

so

$$\mathbb{E}_{\gamma_1}\left(\frac{1}{2}\sqrt{4r^2 - (y+Cr)^2\tan^2\gamma_1}\right) \le \mathbb{E}_{\gamma_1}\left(d\right) \le \mathbb{E}_{\gamma_1}\left(\frac{1}{2}\sqrt{4r^2 - (y-Cr)^2\tan^2\gamma_1}\right) \tag{3.7.12}$$ eq-3-dim-7

For the upper bound, we have using the substitution $u = (y-Cr)\tan\gamma_1$,

$$\begin{aligned}&\left|\mathbb{E}_{\gamma_1}\left(\frac{1}{2}\sqrt{4r^2 - (y-Cr)^2\tan^2\gamma_1}\right) - \frac{r^2}{4y}\right|\\ =&\left|\frac{1}{2\pi}\int_0^{\arctan\frac{2r}{y-Cr}}\frac{1}{2}\sqrt{4r^2-(y-Cr)^2\tan^2\gamma_1}\,d\gamma_1 - \frac{r^2}{4y}\right|\\ =&\left|\frac{1}{4\pi}\int_0^{2r}\sqrt{4r^2-u^2}\cdot\frac{y-Cr}{(y-Cr)^2+u^2}du - \frac{1}{4\pi}\int_0^{2r}\sqrt{4r^2-u^2}\cdot\frac{1}{y}du\right|\end{aligned}$$

$$\leq \frac{1}{4\pi}\int_0^{2r}\sqrt{4r^2-u^2}\left[\left|\frac{y-Cr}{(y-Cr)^2+u^2}-\frac{1}{y-Cr}\right|+\left|\frac{1}{y-Cr}-\frac{1}{y}\right|\right]du$$
$$=\frac{1}{4\pi}\int_0^{2r}\sqrt{4r^2-u^2}\left[\left(\frac{1}{y-Cr}-\frac{y-Cr}{(y-Cr)^2+u^2}\right)+\left(\frac{1}{y-Cr}-\frac{1}{y}\right)\right]du \tag{3.7.13}$$ eq-3-dim-6

Recall that $u \leq 2r, y > \tan\theta, Cr < \frac{\tan\theta}{2}$, and then

$$\frac{1}{y-Cr}-\frac{y-Cr}{(y-Cr)^2+u^2}=\frac{u^2}{(y-Cr)\left[(y-Cr)^2+u^2\right]}$$
$$\leq \frac{u^2}{(y-Cr)^3}\leq\frac{32r^2}{\tan^3\theta}$$

and

$$\frac{1}{y-Cr}-\frac{1}{y}=\frac{Cr}{y\,(y-Cr)}\leq\frac{2Cr}{\tan^2\theta}.$$

It follows that

$$(3.7.13)\leq\left(\frac{32r^2}{\tan^3\theta}+\frac{2Cr}{\tan^2\theta}\right)\frac{1}{4\pi}\int_0^{2r}\sqrt{4r^2-u^2}du$$
$$=\left(\frac{32r^2}{\tan^3\theta}+\frac{2Cr}{\tan^2\theta}\right)\cdot\frac{r^2}{4}.$$

Following the same logic, we can get a similar result for the lower bound in (3.7.12).

Note that $f(0,\omega_1,\omega_2)$ is a continuous function of ω_1,ω_2, so it is bounded over the domain A_1. Recall that above we only considered the case when $0\leq\gamma_1<\pi$. Next we estimate the following term from (3.7.8)

$$\mathbb{P}\left(E_{r,3}\mid\omega_1,\omega_2,\gamma_2=0\right)=f(0,\omega_1,\omega_2)\frac{r^2}{2y}+R_3$$

where R_3 is the remainder term satisfying $|R_3|\leq C_5r^3$ and C_5 depends only on θ,δ.

Recall that $\mathbf{X}_1=(-1,0,0),\mathbf{X}_2=(1,0,0)$ are the initial centers for particles, and $\mathbf{X}=(x,y,z)$ is the approximated collision point. Denote $\rho_1:=\|\mathbf{X}_1-\mathbf{X}\|,\rho_2:=\|\mathbf{X}_2-\mathbf{X}\|,\rho:=\|\mathbf{O}-\mathbf{X}\|$, where $\mathbf{O}$ is the original point.

When $\gamma_2=0$, we have

$$\sin\omega_1=\frac{y}{\rho_1},\quad\cos\omega_1=\frac{x+1}{\rho_1},$$
$$\sin\omega_2=\frac{y}{\rho_2},\quad\cos\omega_1=\frac{x-1}{\rho_2}.$$

so

$$
\begin{aligned}
&\mathbb{P}\left(E_{r,3} \mid \omega_1, \omega_2, \gamma_2 = 0\right) \\
=& f(0, \omega_1, \omega_2)\frac{r^2}{2y} + R_3 \\
=& \frac{32}{\pi} \cdot \frac{\sin^2\omega_1 \sin^2\omega_2}{\left(\sin^2\omega_2 + \sin^2\omega_1\right)^3} \cdot \left|\sin\omega_2\cos\omega_1 - \sin\omega_1\cos\omega_2\right| \cdot \frac{r^2}{2y} + R_3 \\
=& \frac{4\rho_1^3\rho_2^3 r^2}{\pi y^2\left(1+\rho^2\right)^3} + R_3.
\end{aligned}
$$

Note that this is when particle $\mathbf{P}_2$ is on $xy-$plane, now we rotate it to arbitrary γ_2, we will have

$$
\mathbb{P}\left(E_{r,3} \mid \omega_1, \omega_2, \gamma_2\right) = \frac{4\rho_1^3\rho_2^3 r^2}{\pi(y^2+z^2)\left(1+\rho^2\right)^3} + R_3.
$$

Note that $\rho_1 \sin\omega_1 = \rho_2 \sin\omega_2 = \sqrt{y^2+z^2}$ and the Jacobian in (3.7.2) could be written as $\frac{2}{\rho_1^2\rho_2^2}$, so

$$
\begin{aligned}
\mathbb{P}\left(\mathbf{X} \in A\right) =& \iiint_A \left(\frac{4\rho_1^3\rho_2^3 r^2}{\pi(y^2+z^2)\left(1+\rho^2\right)^3} + R_3\right) \frac{\sin\omega_1}{2} \cdot \frac{\sin\omega_2}{2} \cdot \frac{1}{2\pi} d\omega_1 d\omega_2 d\gamma_2 \\
=& \iiint_A \frac{4\rho_1^3\rho_2^3 r^2}{\pi(y^2+z^2)\left(1+\rho^2\right)^3} \cdot \frac{2}{\rho_1^2\rho_2^2} \frac{\sin\omega_1}{2} \cdot \frac{\sin\omega_2}{2} \cdot \frac{1}{2\pi} dx dy dz \\
&+ R_3 \iiint_A \frac{\sin\omega_1}{2} \cdot \frac{\sin\omega_2}{2} \cdot \frac{1}{2\pi} d\omega_1 d\omega_2 d\gamma_2 \\
=& r^2 \iiint_A \frac{1}{\pi^2(1+\rho^2)^3} dx dy dz + R,
\end{aligned}
$$

and $|R|$ is bounded by $C_6 r^3$ for some absolute constant C_6.

$\square$

Remark 3.7.5. The asymptotic defective PDF: $\frac{r^2}{(1+x^2+y^2+z^2)^3}$ is proportional to the PDF of multi-dimensional $t-$distribution.

Remark 3.7.6. Theorem 3.7.3 tells us that away from the x-axis and not too far away, the distribution of X has an approximate defective PDF: $\frac{r^2}{\pi^2(1+x^2+y^2+z^2)^3}$. Speaking informally (because this case is not covered by Theorem 3.7.3) if we take $A = \mathbb{R}^3$, and apply the following polar coordinates in $\mathbb{R}^3$,

$$x = \rho\cos\theta_1, \quad y = \rho\sin\theta_1\cos\theta_2, \quad z = \rho\sin\theta_1\sin\theta_2, \quad \rho > 0,\ \theta_1 \in [0,\pi),\ \theta_2 \in [0,2\pi), \tag{3.7.14}$$

polar-R3

we obtain

$$\begin{aligned}
\mathbb{P}(X \in \mathbb{R}^3) &\approx \frac{r^2}{\pi^2}\iint_{\mathbb{R}^3} \frac{1}{(1+x^2+y^2+z^2)^3} dxdydz \\
&= \frac{r^2}{\pi^2}\int_0^\infty \frac{\rho^2}{(1+\rho^2)^2} d\rho \int_0^\pi \sin\theta_1 d\theta_1 \int_0^{2\pi} d\theta_2 \\
&= \frac{r^2}{\pi^2} \cdot 2 \cdot 2\pi \cdot \int_0^1 \frac{1}{2}\sqrt{u}\sqrt{1-u}du \quad (\text{let } u = \frac{1}{1+\rho^2}) \\
&= \frac{r^2}{\pi^2} \cdot 2 \cdot 2\pi \cdot B\left(\frac{3}{2}, \frac{3}{2}\right) \quad (B \text{ denotes for Beta function}) \\
&= \frac{r^2}{4}.
\end{aligned}$$

In other words, asymptotically for small $r > 0$, the probability of a collision is about $r^2/4$. Note that this agrees with our result in Remark 3.4.6.

The defective PDF given in Theorem 3.7.3 is rotation invariant in $\mathbb{R}^3$.

Let $(\rho, \theta_1, \theta_2)$ to be the polar coordinates defined in (3.7.14). The approximate defective PDF $\frac{r^2}{(1+x^2+y^2+z^2)^3}$ shows ρ, θ_1, θ_2 are independent. If we condition on a collision, then ρ, θ_1, θ_2 have the following distributions:

$$\theta_2 \sim \text{Uniform}(0, 2\pi), \quad \theta_1 \sim PDF: \frac{\sin\theta_1}{2}, \theta_1 > 0, \quad \rho \sim PDF: \frac{16}{\pi} \cdot \frac{\rho^2}{(1+\rho^2)^3}, \rho > 0.$$

Fig. 3.12 is the graph for the simulated results.

3.8 Alternative examples of spherically invariant collision density functions in $\mathbb{R}^3$.

In this section, we will discuss the more general case: two velocities $\mathbf{V}_1, \mathbf{V}_2$ still uniformly distributed over the sphere but for the magnitude, $\|\mathbf{V}_1\|^2, \|\mathbf{V}_2\|^2$ are i.i.d with PDF $p(x), x > 0$ and p is a C^2 function. For the collision density function to be rotation invariant, we have the following condition:

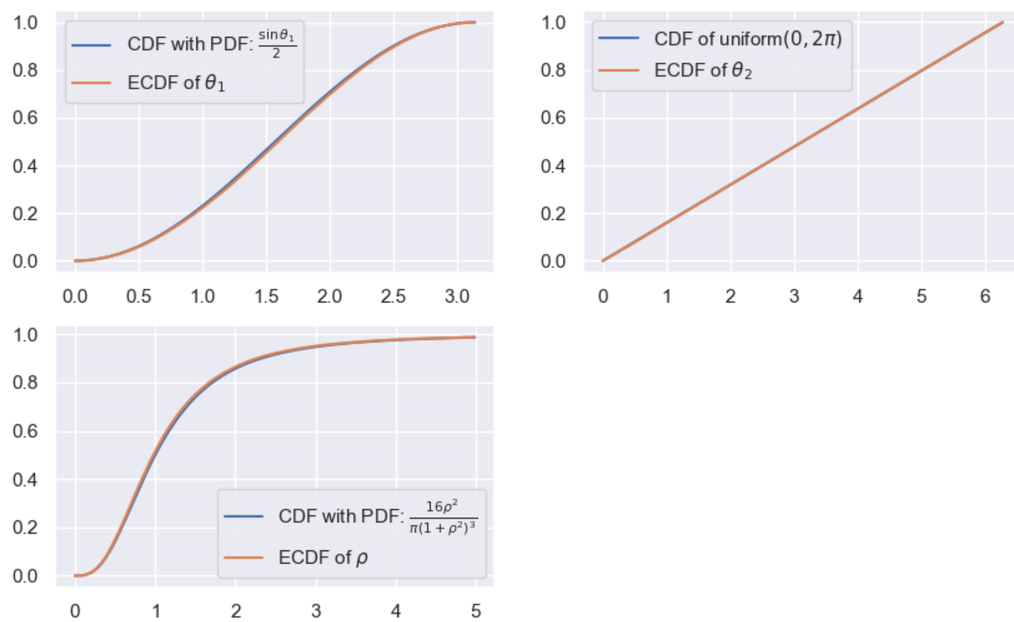

Figure 3.12: The graph on the top left represents the empirical CDF of θ_1 v.s. the CDF with PDF $\frac{\sin\theta_1}{2}$. The graph on the top right represents the empirical CDF of θ_2 v.s. the CDF of uniform distribution on $(0, 2\pi)$. The graph on the bottom left represents the empirical CDF of ρ v.s the CDF with PDF $\frac{16\rho^2}{\pi(1+\rho^2)^3}$.

.rho_theta_3

Theorem 3.8.1. *The density function derived in Theorem 3.7.3 is rotation invariant if and only if there exists some $\psi : \mathbb{R}^+ \mapsto \mathbb{R}^+$ such that for every $u, v > 0$ we have*

$$\int_0^\infty yp\left(\frac{u}{v}y\right)p(y)dy = u^{\frac{1}{2}}v^{\frac{5}{2}}\psi(u+v). \tag{3.8.1}$$ eq-3-dim-11

Proof. The ratio $\frac{\|\mathbf{V}_1\|^2}{\|\mathbf{V}_2\|^2}$ follows PDF:

$$q(x) := \int_0^\infty yp(xy)p(y)dy, x > 0.$$

The equation (3.7.6) becomes

$$\begin{aligned}
&\int_{\left(\frac{\sin\omega_1\cos\gamma_1}{\sin\omega_2}\right)^2}^{\left(\frac{d\cos\omega_1+\sqrt{1-d^2}\sin\omega_1\cos\gamma_1}{d\cos\omega_2+\sqrt{1-d^2}\sin\omega_2}\right)^2} q(x)dx \\
=&q\left(\frac{\sin^2\omega_1\cos^2\gamma_1}{\sin^2\omega_2}\right)\cdot\frac{2\sin\omega_1\cos\gamma_1}{\sin^3\omega_2}\left|\sin\omega_2\cos\omega_1-\sin\omega_1\cos\gamma_1\cos\omega_2\right|\cdot d \\
&+R_1\left(d;\omega_1,\omega_2,\gamma_1\right)
\end{aligned}$$

The other arguments in the proof of Theorem 3.7.3 remain the same, and the formula for the asymptotic density function will be

$$2q\left(\frac{\rho_2^2}{\rho_1^2}\right)\cdot\frac{2\rho_2^3}{\rho_1 y^2}\cdot\frac{2y}{\rho_1\rho_2}\cdot\frac{r^2}{2y}\cdot\frac{\sin\omega_1\sin\omega_2}{8\pi} = q\left(\frac{\rho_2^2}{\rho_1^2}\right)\cdot\frac{r^2}{\pi\rho_1^5\rho_2} \tag{3.8.2}$$ eq-3-dim-12

Note that $\rho_1^2+\rho_2^2 = 2(1+\rho^2)$, so the asymptotic density function is rotation invariant if and only if (3.8.2) is a function of $\rho_1^2+\rho_2^2$.

Suppose (3.8.1) holds, then

$$\begin{aligned}
q\left(\frac{\rho_2^2}{\rho_1^2}\right)\cdot\frac{r^2}{\pi\rho_1^5\rho_2} &= \frac{r^2}{\pi\rho_1^5\rho_2}\int_0^\infty yp\left(\frac{\rho_2^2}{\rho_1^2}y\right)p(y)dy \\
&= \frac{r^2}{\pi\rho_1^5\rho_2}\cdot\rho_2\rho_1^5\psi(\rho_1^2+\rho_2^2) \\
&= \frac{r^2}{\pi}\psi(\rho_1^2+\rho_2^2).
\end{aligned}$$

For the opposite direction, suppose (3.8.2) $= \psi(\rho_1^2+\rho_2^2)$ for some function ψ. Let $u = \rho_2^2, v = \rho_1^2$, then we have

$$\int_0^\infty yp\left(\frac{u}{v}y\right)p(y)dy = q\left(\frac{u}{v}\right)$$
$$= \frac{\pi}{r^2}\rho_1^5\rho_2 \cdot \psi(u+v)$$
$$= \frac{\pi}{r^2}u^{\frac{1}{2}}v^{\frac{5}{2}}\psi(u+v).$$

□

Corollary 3.8.2. *In the Gamma family, i.e.* $p(x) = \frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\exp(-\beta x), x > 0$ *for some* $\alpha, \beta > 0$*, then* (3.8.1) *holds if and only if* $\alpha = \frac{3}{2}$.

Proof. Note that

$$\int_0^\infty yp\left(\frac{u}{v}y\right)p(y)dy$$
$$= \int_0^\infty y \cdot \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \frac{u^{\alpha-1}}{v^{\alpha-1}}y^{\alpha-1}\exp\left(-\beta \cdot \frac{u}{v}y\right) \cdot \frac{\beta^\alpha}{\Gamma(\alpha)}y^{\alpha-1}\exp(-\beta y)dy$$
$$= \frac{\beta^{2\alpha}}{\Gamma(\alpha)^2} \cdot \frac{u^{\alpha-1}}{v^{\alpha-1}}\int_0^\infty y^{2\alpha-1}\exp\left(-\beta\left(\frac{u}{v}+1\right)y\right)dy$$
$$= \frac{\beta^{2\alpha}}{\Gamma(\alpha)^2} \cdot \frac{u^{\alpha-1}}{v^{\alpha-1}} \cdot \frac{\Gamma(2\alpha)}{\left(\beta\left(\frac{u}{v}+1\right)\right)^{2\alpha}} = \frac{\Gamma(2\alpha)}{\Gamma(\alpha)^2}u^{\alpha-1}v^{\alpha+1}(u+v)^{-2\alpha}$$

□

Remark 3.8.3. In this case, $2\beta\|\mathbf{V}_1\|^2$ and $2\beta\|\mathbf{V}_2\|^2$ follow the χ_3^2 distribution.

Corollary 3.8.4. *Note that* $\frac{\|\mathbf{V}_1\|}{\|\mathbf{V}_2\|} = \frac{\|\mathbf{V}_2\|^{-1}}{\|\mathbf{V}_1\|^{-1}}$*, so* (3.8.1) *also holds if* $\|\mathbf{V}_1\|^{-2}, \|\mathbf{V}_2\|^{-2}$ *has PDF:* $\frac{\beta^{\frac{3}{2}}}{\Gamma(\frac{3}{2})}x^{\frac{1}{2}}\exp(-\beta x), x > 0$. *In this case,* $\|\mathbf{V}_1\|^2$ *and* $\|\mathbf{V}_2\|^2$ *follow inverse-Gamma distribution.*

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