Compact Moduli of Surfaces in Three-Dimensional Projective Space

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Abstract

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The main goal of this paper is to construct a compactification of the moduli space of degree $d \geq 5$ hypersurfaces in $\mathbb{P}^3$, i.e. a parameter space whose interior points correspond to (equivalence classes of) smooth hypersurfaces in $\mathbb{P}^3$ and whose boundary points correspond to degenerations of such hypersurfaces. Following a trail blazed by numerous others (see, for example [18], [3], [12]), we consider a hypersurface D in $\mathbb{P}^3$ as a pair $(\mathbb{P}^3, D)$ satisfying certain properties. We find a modular compactification of such pairs and use their properties to classify the pairs on the boundary to the moduli space.

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Chapter 1

INTRODUCTION

There is a well-known compactification $\overline{\mathcal{M}}_g$ of the moduli space of smooth curves of genus g and the compacitification is modular in the sense that the boundary points correspond to stable curves. The existence of such a moduli space sparks many related questions.

An obvious generalization would be to study the moduli space of higher dimensional smooth varieties. This has been a topic of significant study with many contributors, but compactifying such a moduli space is a challenge. First, one has to understand the higher dimensional analog of stability for curves and impose the 'right' conditions on degenerations of families of smooth varieties with certain invariants. Things are further complicated when techniques that are used in studying the moduli space of curves, such as Geometric Invariant Theory, do not work in higher dimensions.

Perhaps in a different direction, one could study a more specific problem than the moduli space of genus g curves. For example, is there a modular compactification of the moduli space of plane curves of fixed degree? Using the Hilbert scheme, one can find a compactification of the space parameterizing smooth degree d curves in $\mathbb{P}^2$, but the boundary does not have a good modular interpretation. For instance, there are points on the boundary that correspond to several different limits of families of plane curves. In [12], Hacking was able to re-frame this problem and find a good compactification using pairs. Instead of studying curves C, Hacking worked with pairs $(\mathbb{P}^2, C)$ and certain allowable degenerations. Remembering the embedding of C in $\mathbb{P}^2$ and extracting certain properties enabled him to find a compactification with a modular interpretation.

Hacking was not the first to consider moduli of pairs and his work followed related work by Kollár, Shepherd-Barron, Alexeev, and others. Pairs are also natural to study as a generalization of $\overline{\mathcal{M}}_g$ to $\overline{\mathcal{M}}_{g,n}$, parameterizing pairs $(C, \sum p_i)$ of curves with n marked points.

This paper stems from the natural generalization of Hacking's work: find a good compactification of the moduli space of surfaces S in $\mathbb{P}^3$ by studying pairs (X, D) that arise as limits of pairs $(\mathbb{P}^3, S)$. To find a natural polarization, or ample line bundle, on these pairs, we consider surfaces of degree $d \geq 5$ so that $K_{\mathbb{P}^3} + S$ is ample. In fact, we will only consider what we call H-ϵ stable pairs which, among other things, must have semi-log canonical singularities. On the interior of the moduli space, the surfaces S appearing are all of general type and could be studied in many ways, but as in Hacking's work, studying these pairs has many advantages. Remembering the embedding of S into $\mathbb{P}^3$ allows us to not only have a modular compactification of a space parameterizing $(\mathbb{P}^3, S)$ but also to classify the pairs appearing on the boundary of the moduli space.

A main result is that the class of pairs defined does actually give a compactification of the moduli space of pairs $(\mathbb{P}^3, S)$.

Theorem 1.0.1. *For odd degree d, the moduli space of three-dimensional H-ϵ stable pairs of degree d is a proper Deligne-Mumford stack.*

The oddness of degree d is perhaps surprising. While this result is expected, it does not follow from recent results on moduli of pairs. The main issue is that the class of pairs defined below is not obviously a bounded family. If boundedness was immediately known, [2] would imply that the moduli space is a Deligne-Mumford stack and properness would follow from a relatively standard argument using the Minimal Model Program.

Hacon, McKernan, and Xu recently proved a strong result about boundedness of families of certain pairs (X, D). However, it requires the coefficients of the divisors appearing in D belong to a DCC set. Here, in the definition of H-ϵ stable pair, one requires that $(X, (\frac{4}{d}+\epsilon)D)$

is slc for ϵ sufficiently small. However, ϵ is not bounded from below, so results on boundedness like those in [14] do not apply. If ϵ was required to belong to a DCC set, [14, Theorem 1.1] would apply to show the given pairs belong to a bounded family.

This motivates two possible avenues of exploration: attempt to bound ϵ in a DCC set or attempt to find boundedness in another way. Choosing a DCC set may seem like an obvious way to proceed, but in the proof of properness of the stack, one may have to shrink ϵ to maintain control of the singularities of the pair. Therefore, we could not directly see a way to restrict ϵ in a DCC set and still find a compact moduli space.

A seemingly unrelated goal of this project was to classify the singular pairs appearing on the boundary of the moduli space. In working on this problem, the classification results gave enough control on the singularities of the boundary of the moduli space for odd degree d to apply an older result of Hacon, McKernan, and Xu showing this is a bounded family. In other words, regardless of what set ϵ lives in, the classification results for odd degree d actually imply boundedness.

One should note that this theorem is not false for even degree d, just not known. If H-ϵ stable pairs of even degree can be shown to be bounded, then Theorem 1.0.1 is true for all H-ϵ stable pairs.

Therefore, the following theorems serve two purposes: explicit classification of singular threefolds appearing in the moduli space and a means to achieve boundedness without carefully studying the numbers ϵ that appear in H-ϵ stable pairs. Classification is of interest even in the absence of the boundedness implication.

The first result is about ambient spaces X with mild singularities. We show that, if X appearing in an H-ϵ stable pair of odd degree has canonical singularities, then X is actually $\mathbb{P}^3$. If this is the case, it implies that $D \in \left| -\frac{d}{4}K_{\mathbb{P}^3}\right|$ with $(\mathbb{P}^3, \frac{4}{d}D)$ log terminal, so X determines D.

Theorem 1.0.2. *Given a three-dimensional H-ϵ stable pair (X, D) of odd degree d, if X has* *canonical singularities, then* $X \cong \mathbb{P}^3$ *and* $D \in |\mathcal{O}_{\mathbb{P}^3}(d)|$ *such that* $(X, \frac{4}{d}D)$ *is log terminal.*

There certainly are examples of H-ϵ stable pairs of even degree with canonical singularities that are not isomorphic to $\mathbb{P}^3$, brought to the author's attention by Paul Hacking. However, in the odd degree case, the boundary is very special.

The next result follows from the study of more complicated singularities. In order to guarantee a compact moduli space, we consider semi log canonical (slc) pairs. But, we show in Section 5 that pairs with strictly slc singularities can only appear in the moduli space of pairs with even degree d. Therefore, for odd degree, we only need to consider semi log terminal pairs to construct the moduli space.

Theorem 1.0.3. *Given a three-dimensional H-*ϵ *stable pair of odd degree d,* (X, D) *is semi log terminal. In other words, no strictly semi log canonical degenerations of* $\mathbb{P}^3$ *appear on the boundary of the moduli space.*

This has interesting consequences in the proof of properness. If we have a family of H-ϵ stable pairs over a punctured curve, to show properness, we complete the family over the curve, resolve the singularities, and eventually take a log canonical model. In general, the log canonical model of a log terminal pair is always log canonical. However, this result implies that the log canonical model actually has milder singularities and is log terminal.

A map of this paper. We begin with preliminary notions needed to define H-ϵ stable pairs (Chapter 2).

In Chapter 3, we define H-ϵ stable pairs and use the existence of minimal models to prove that a family of pairs over a punctured curve can be extended in an essentially unique way, justifying the definition.

In Chapter 4, we prove a number of technical lemmas about extremal contractions in the minimal model program, classify varieties with strictly log canonical singularities, and build up the necessary machinery to prove Theorem 1.0.3. Using a careful study of extremal contractions in the minimal model program, we generalize [16, Main Theorem] to show that certain strictly lc Fano varieties with a finite number of lc singular points have the structure of a cone over an exceptional divisor with discrepancy -1:

Theorem 1.0.4. *Let X be a projective variety with a finite number of strictly log canonical singularities $\{p_1, \ldots, p_n\}$ and $-K_X$ ample. If $a(E, X) \in \{-1, \mathbb{R}^{\geq 0}\}$ for every exceptional divisor E over X with $\mathrm{center}_X(E) \subset \{p_1, \ldots, p_n\}$, then X is a cone over a numerically Calabi-Yau variety.*

We prove a number of related results on the structure of slc Fano varieties. We will see in Chapter 5 how this has implications to boundedness of odd degree pairs. We also discuss canonical and log terminal Fano threefolds as a step toward classifying H-ϵ stable pairs.

Finally, in Chapter 5, we classify strictly slc pairs appearing as H-ϵ stable pairs (in particular, for odd d, there are none) and further analyze the moduli space in question, proving that it is a Deligne-Mumford stack. Appendix A is a study of necessary conditions for weighted projective spaces to admit smoothings to $\mathbb{P}^n$ and details on weighted blowups, a first step toward classifying the (semi-) log terminal pairs on the boundary.

Chapter 2

BACKGROUND

2.1 Singularities

Singular varieties appear naturally in many contexts and are of particular importance in moduli problems. Therefore, we begin with an introduction to singularities.

Remark 2.1.1. Due to the nature of the moduli problem at hand, we restrict ourselves to studying varieties over $\mathbb{C}$.

Definition 2.1.2. An irreducible variety X is normal if, for each point $x \in X$, the local ring $\mathcal{O}_x$ is integrally closed in $K(X)$.

Given any variety, we can form the unique normalization X^ν of X, together with a natural map $X^\nu \to X$. It is characterized by the property that any map $f : Y \to X$ from a normal variety to X factors through X^ν. Geometrically, normality means the following.

Theorem 2.1.3 (Serre)**.** *A variety X is normal if and only if the set of singular points of X has codimension at least 2 and, for every point $x \in X$ with codim $x \geq 2$, depth $\mathcal{O}_x \geq 2$.*

For instance, this implies that normal curves are smooth, and the normalization of any curve is a desingularization of it.

On smooth varieties, we have the canonical sheaf ω_X and, because of the equivalence of Weil divisors and invertible sheaves, we can write $\omega_X = \mathcal{O}(K_X)$ for some Weil divisor K_X, and we call any such divisor a canonical divisor.

On normal varieties, the singular locus has codimension ≥ 2, so we can easily define ω_X. If X is normal, then $\omega_{X-\mathrm{Sing}(X)}$ corresponds to a Weil divisor class $K_{X-\mathrm{Sing}(X)}$. By normality, the class groups of X and $X - \mathrm{Sing}(X)$ are isomorphic, so we can extend this to a Weil divisor class K_X on X. Note that $\mathcal{O}(K_X)$ may not be an invertible sheaf, so K_X may not be Cartier.

Definition 2.1.4. We will say that a normal variety X is $\mathbb{Q}$-Gorenstein if some multiple of K_X is Cartier.

Using the above definitions, we can define a 'measure' of how singular a variety is, called the discrepancy.

If X is a normal, $\mathbb{Q}$-Gorenstein variety, let m be an integer such that mK_X is Cartier. Suppose Y is a normal variety and $f : Y \to X$ a proper, birational morphism and E an irreducible exceptional divisor. If $e \subset E$ is a general point and $\{y_i\}$ a local coordinate system for E at e (so locally $E = V(y_1)$), then for a local generator l of $\mathcal{O}(mK_X)$,

$$f^*(l) = uy_1^r(dy_1 \wedge \cdots \wedge dy_n)^{\otimes m}$$

where u is a unit.

One can check that the rational number $a(E, X) := r/m$ does not depend on m or Y, leading to the following definition.

Definition 2.1.5. Given a normal, $\mathbb{Q}$-Gorenstein variety X, the number $a(E, X)$ defined above is called the discrepancy of E with respect to X.

If K_Y is Cartier (for instance, if $f : Y \to X$ is a resolution of singularities), we can restate this. Let $Z = \mathrm{Ex}(f)$, the exceptional locus of f. Because $f\big|_{Y-Z} : Y - Z \to X - f(Z)$ is an isomorphism, and $f(Z)$ has codimension ≥ 2 in X, we must have the Cartier divisor $mK_Y - f^*(mK_X)$ supported on Z. Hence, if we denote the f-exceptional divisors E_i, we can write

$$mK_Y \sim f^*(mK_X) + \sum ma_i(E_i, X)E_i$$

or, dividing by m and using numerical equivalence,

$$K_Y \equiv f^*(K_X) + \sum a_i(E_i, X)E_i.$$

We can also define a discrepancy for pairs. A pair (X, D) is a variety X with a $D = \sum a_i D_i$ a formal linear combination of prime divisors.

Definition 2.1.6. Let (X, D) be a pair where X is a normal variety and $D = \sum a_i D_i$ is a $\mathbb{Q}$-linear combination of prime divisors such that $K_X + D$ is $\mathbb{Q}$-Cartier. Let $f : Y \to X$ be a birational morphism from a smooth variety Y with exceptional divisors E_i and strict transform $f_*^{-1} D = \sum a_i f_*^{-1} D_i$. Then, as before, we can write

$$m(K_Y + f_*^{-1} D) \sim f^*(m(K_X + D)) + \sum m a_i(E_i, X, D) E_i$$

or

$$K_Y + f_*^{-1} D \sim f^*(K_X + D) + \sum a_i(E_i, X, D) E_i$$

with $a_i(E_i, X, D) \in \mathbb{Q}$.

The rational number $a_i(E_i, X, D)$ is called the discrepancy of E_i with respect to the pair (X, D). If F is any divisor on X, we define $a(F, X, D) = -\mathrm{coeff}_F D$, so $a(D_i, X, D) = -a_i$ and $a(F, X, D) = 0$ for $F \neq D_i$.

Using the discrepancy, one can define the type of singularities of a pair (X, D).

Definition 2.1.7. Let (X, D) be a pair where X is a normal variety and D is a sum of distinct prime divisors $D = \sum a_i D_i$, $a_i \leq 1$. Assume that $K_X + D$ is $\mathbb{Q}$-Cartier. Then, (X, D) is

$$\left.\begin{array}{l} \text{terminal} \\ \text{canonical} \\ \text{klt} \\ \text{plt} \\ \text{lc} \end{array}\right\} \text{if } a(E, X, D) \left\{\begin{array}{l} > 0 \text{ for all exceptional } E \\ \geq 0 \text{ for all exceptional } E \\ > -1 \text{ for all } E \\ > -1 \text{ for all exceptional } E \\ \geq -1 \text{ for all exceptional } E \end{array}\right.$$

Above, klt stands for Kawamata log terminal, plt for purely log terminal, and lc for log canonical. Note that the difference between klt and plt comes from the divisor D (and the notions coincide if $D = 0$). Also, to determine if the singularities of a pair (X, D)

are terminal, canonical, klt, or lc, it is sufficient to check the coefficients of the exceptional divisors in a single log resolution by [19, Corollaries 2.12, 2.13].

There is one other class of singularities that we will be concerned with: divisorial log terminal or dlt singularities.

Definition 2.1.8. A pair (X, D) where X is normal, $D = \sum a_i D_i$ where $0 \leq a_i \leq 1$, and $K_X + D$ $\mathbb{Q}$-Cartier is dlt if there exists a log resolution $f : Y \to X$ such that $a(E, X, D) > -1$ for every exceptional divisor $E \subset Y$.

Note that this definition only requires *one* log resolution such that $a(E, X, D) > -1$, not all of them. For example, the pair $(\mathbb{P}^2, D)$ where $D = L_1 + L_2$, the sum of two lines that intersect transversally, is dlt because the identity map is a log resolution. However, it is not true that $a(E, X, D) > -1$ for every exceptional divisor; if Y is the blow up of $\mathbb{P}^2$ at $L_1 \cap L_2$ with exceptional divisor E, then $a(E, X, D) = -1$.

We give an example of computing discrepancies below.

Example 2.1.9. Let C be a rational normal curve of degree n and let X be the cone over C. We compute the singularities of X as follows. First, X has a resolution of singularities $f : Y \to X$ where Y is the blow up of the vertex. In this case, the exceptional divisor E is isomorphic to the curve C, and $E^2 = \deg \mathcal{O}_E(E) = \deg \mathcal{N}_{E/Y} = -n$. Since E is the only exceptional divisor, we can write

$$K_Y = f^* K_X + aE$$

for some $a \in \mathbb{Q}$. Because Y and E are smooth, we can use the adjunction formula to write

$$K_E = (K_Y + E)\big|_E.$$

Combining these formulas, we have

$$K_E = (f^* K_X + (a+1)E)\big|_E.$$

Taking the degree of both sides, since $E \cong \mathbb{P}^1$ we get

$$-2 = \deg K_E = f^*K_X \cdot E + (a+1)E^2 = -n(a+1),$$

since E is contracted by f. Solving for a, we find that $a = -1 + \frac{2}{n}$. Therefore, the cone over the conic is canonical, but the cone over any higher degree rational normal curve is log terminal.

In order to study moduli spaces, we also have to consider non-normal varieties. They arise naturally in moduli problems as degenerations of smooth varieties (for instance, the nodal cubic is a non-normal degeneration of a family of elliptic curves). However, we must restrict ourselves to a certain class of non-normal varieties; we'd at least like to have a notion of canonical divisor. We define semi log canonical pairs (X, D) below. In some sense, this amounts to requiring that we can make sense of K_X and that the normalization is log canonical.

Definition 2.1.10. A variety X is demi-normal [19, Definition 5.1] if X is S_2 and its codimension 1 points are either regular points or double normal crossing points (nodes). Note that this is a natural weaking of normality, where X is S_2 and its codimension 1 points are regular.

Remark 2.1.11. On any projective scheme X over k, the dualizing sheaf ω_X is S_2 [21, Corollary 5.69] and, if X is demi-normal, is locally free in codimension 1 [19, Aside 5.93], so corresponds to a Weil divisor class. We will define K_X, the canonical divisor, to be an Weil divisor in this class. If X is normal, this is the usual canonical divisor class described above.

Definition 2.1.12. A pair (X, D) is semi log canonical, slc, (respectively semi log terminal, slt), if

- X is demi-normal.
- $K_X + D$ is $\mathbb{Q}$-Cartier.

- If $\nu : X^\nu \to X$ is the normalization of X, Δ^ν the conductor, and D^ν the preimage of D, then $(X^\nu, \Delta^\nu + D^\nu)$ is log canonical (respectively, log terminal). (Note this makes sense because $K_{X^\nu} + \Delta^\nu + D^\nu \sim \nu^*(K_X + D)$ [19, 5.7.5].)

2.2 The Minimal Model Program

The minimal model program is, in part, a way to produce a "simplest" member of a given birational equivalence class of algebraic varieties. For curves, this is very easy. We will see in the next section that this program allows us to complete families of stable pairs.

The minimal model program is relatively simple in dimension ≤ 2.

Theorem 2.2.1. *In each birational equivalence class of curves, there exists exactly one smooth projective curve.*

In other words, a smooth, projective curve already is the simplest member of its birational equivalence class. If you happen to have a singular curve, its normalization produces a smooth, projective one.

For smooth surfaces, the complexity increases slightly because there are many smooth, birational, non-isomorphic surfaces. Consider the simple example of $\mathbb{P}^2$ and $\mathbb{P}^2$ blown up at a point–they're birational, but somehow $\mathbb{P}^2$ is simpler than the blow up. This leads to a naive idea to create a minimal surface: take a surface, and "blow down" all the things that were blow ups, and hope that gives you a minimal model. The surprising fact is that this actually works!

Definition 2.2.2. A smooth rational curve C on a surface S is called a (-1)-curve if $C^2 = -1$.

Remark 2.2.3. The exceptional divisor of a blow up of a point on a smooth surface is a (-1)-curve.

Theorem 2.2.4 (Castelnuovo)**.** *If C is a (-1)-cure on a smooth surface S', then there exists a smooth surface S and a morphism $f : S' \to S$ that is the blow up of S at a point such that C is the exceptional divisor.*

This theorem tells us that we can always "blow down" (-1)-curves on surfaces, and this process must terminate because each blow down decreases the Picard number of the surface. The end result will be the "simplest" member of the equivalence class.

Definition 2.2.5. A minimal surface is a smooth projective surface that does not contain any (-1) curves.

We can use Castelnuovo's theorem to blow down (-1) curves to always find a minimal surface in a birational equivalence class.

Example 2.2.6. Even though we can produce such a surface, it may not be unique. For example, consider the blow up of $\mathbb{P}^2$ at two points: if we blow down each of the exceptional divisors, we'll get a $\mathbb{P}^2$, but if instead we blow down the strict transform of the line joining the two points, we'll get $\mathbb{P}^1 \times \mathbb{P}^1$.

Now, we'd like to have an analog of Castelnuovo's theorem in higher dimensions but we cannot extend the notion of (-1) curves that way. However, we can re-interpret (-1) curves on surfaces in a way that generalizes to higher dimensions.

Definition 2.2.7. On a smooth, projective variety X, we say the canonical divisor K_X is **nef** if $K_X \cdot C \geq 0$ for every curve $C \subset X$.

Proposition 2.2.8. *If X is a smooth, projective surface of general type, there exists a (-1) curve on X if and only if K_X is not nef.*

Proof. (Sketch.) If C is a (-1) curve, then by adjunction, $K_C = (K_X + C)\big|_C$, hence $-2 = \deg K_C = (K_X + C) \cdot C = K_X \cdot C - 1$, so $K_X \cdot C = -1$ and K_X is not nef.

If K_X is not nef, then there exists a curve B such that $K_X \cdot B < 0$, and by Bend-and-break, this implies there is in fact a rational curve C such that $K_X \cdot C < 0$. Again, using adjunction, if $K_X \cdot C = -1$, then C is a (-1)-curve. If $K_X \cdot C \leq -2$, then $C^2 \geq 0$, which implies that we can deform C, hence X is uniruled and not of general type.

□

With this proposition as motivation, we will change our definition of minimal surfaces to those where K_S is nef, and find minimal surfaces by contracting K_S-negative curves.

Definition 2.2.9. A smooth, projective surface S is minimal if K_S is nef.

Note that this definition excludes $\mathbb{P}^2$ and ruled surfaces. Because the K_S-negative curves are precisely the (-1) curves on surfaces of general type, Castelnuovo's theorem allows us to contract them and produce a smooth, minimal model of S.

With this in mind, we'd like to define minimal models of higher dimensional varieties as those where K_X is nef and preform a similar contraction procedure to take an arbitrary smooth variety Y and produce a minimal model. In order to do this, there are a few questions to be addressed: can we always contract K_X-negative curves? does the contraction process terminate? does the contraction process introduce singularities?

The answers to these questions come in various degrees of difficulty. We'll start with the cone theorem and contraction theorem, which answer the first question affirmatively.

Definition 2.2.10. Given a proper variety X, the effective cone $NE(X)$ is the collection of effective 1 cycles on X modulo numerical equivalence. We also often consider the closure $\overline{NE}(X)$.

Example 2.2.11. On $\mathbb{P}^n$, every curve is numerically equivalent to dL, where L is a line, so $\overline{NE}(X)$ is a one dimensional ray.

In general, it is very difficult to determine the cone $\overline{NE}(X)$, but the following theorem helps us to partially understand its structure.

Theorem 2.2.12. *(Cone theorem.) Let X be a smooth, projective variety. Then, there exists a countable set of rational curves $C_i \subset X$ such that $0 \leq -(C_i \cdot K_X) \leq \dim X + 1$ and*

$$\overline{NE}(X) = \overline{NE}(X)_{K_X \geq 0} + \sum \mathbb{R}_{\geq 0}[C_i]$$

where $\overline{NE}(X)_{K_X \geq 0}$ represents the 1 cycles C such that $C \cdot K_X \geq 0$ and $[C_i]$ is the class of C_i in $NE(X)$.

If we could contract the K_X-negative curves C_i in the above theorem, while maintaining control of the singularities on X, we'd end up with a minimal variety. This motivates the contraction theorem.

Theorem 2.2.13. *(Contraction theorem.) Let X be a smooth, projective variety. Then, there exists a unique contraction morphism $c_F : X \to Z$ of any K_X-negative extremal face $F \subset \overline{NE}(X)$ such that $(c_F)_* \mathcal{O}_X = \mathcal{O}_Z$ and an irreducible curve $C \subset X$ is mapped to a point if and only if $[C] \in F$. Here, Z is a projective variety with terminal singularities.*

We'd like to use this theorem to contract the K_X-negative extremal rays in the effective cone, but there are multiple issues. The contraction process may introduce singularities, and, as stated, neither the cone nor contraction theorems apply to singular varieties. We also need to determine if the contraction process terminates and yields a minimal model. Finally, we'll be using the minimal model program on pairs (X, D), so we'd like to have these theorems in this setting, as well.

Fortunately, both the cone theorem and contraction theorem can be generalized to singular varieties, and pairs, which will be useful to us in later sections.

Theorem 2.2.14. *(Cone theorem, 2.) Let (X, D) be a projective, $\mathbb{Q}$-factorial dlt pair with D effective. Then, there exists a countable set of rational curves $C_i \subset X$ such that $0 \leq -(C_i \cdot (K_X + D)) \leq 2 \dim X$ and*

$$\overline{NE}(X) = \overline{NE}(X)_{K_X + D \geq 0} + \sum \mathbb{R}_{\geq 0}[C_i].$$

Proof. See [21, Theorem 3.7 1]. □

Theorem 2.2.15. *(Contraction theorem, 2.) Let (X, D) be a projective, $\mathbb{Q}$-factorial dlt pair with D effective. Then, there exists a unique contraction morphism to a projective variety Z $c_F : X \to Z$ of any $K_X + D$-negative extremal face $F \subset \overline{NE}(X)$ such that $(c_F)_*\mathcal{O}_X = \mathcal{O}_Z$ and an irreducible curve $C \subset X$ is mapped to a point if and only if $[C] \in F$.*

Proof. See [21, Theorem 3.7 3]. □

As in the case of surfaces, each contraction does decrease the Picard number of the variety so we hope we are done. However, there is no control over the singularities of Z in the second version of the contraction theorem.

There are three possibilities for a given contraction morphism: one is a *fiber type contraction* where $\dim X > \dim Y$. This happens, for instance, if $X = \mathbb{P}^n$ and $D = 0$. The problem is then (in principle) reduced to studying the lower dimensional variety Y and the fibers of the contraction.

The next possibility is a *divisorial contraction* where c_F is birational and $\mathrm{Ex}(c_F)$ is a divisor on X. In this case, $\rho(Z) < \rho(X)$ so Z is "simpler" than X, which is what we saw contracting (-1)-curves on surfaces of general type. In this case, the Z remains $\mathbb{Q}$-factorial and $(Z, (c_F)_*D)$ is dlt, so we can continue to apply the contraction theorem.

However, unlike the case of surfaces, there exist *small contractions*, i.e. birational contraction morphisms as above such that codim $\mathrm{Ex}(c_F) \geq 2$. In this case, the resulting variety Z is not even $\mathbb{Q}$-factorial: no multiple of K_Z is Cartier. If it were, then for some m, both mK_X and mK_Z are Cartier, hence mK_X and $(c_F)^*mK_Z$ are Cartier divisors that agree outside the codimension 2 exceptional set, hence they agree on X. However, c_F is a contraction map, so for a curve C such that $[C] \in F$, $mK_X \cdot C < 0$ but $(c_F)^*mK_Z \cdot C = 0$.

Therefore, the singularities of Z are too bad for us to handle, so we cannot continue to contract K-negative rays on Z. Therefore, we do something else: starting with X, we remove the offending exceptional set $E = \mathrm{Ex}(c_F)$ and compactify the result by adding in a new subvariety E^+ in an operation called a flip.

Definition 2.2.16. Given a normal scheme X, a $\mathbb{Q}$-divisor D such that $K_X + D$ is $\mathbb{Q}$-Cartier, and a proper birational morphism $f : X \to Y$ to a normal scheme Y with codim $\mathrm{Ex}(f) \geq 2$ and $-(K_X + D)$ f-ample, a **flip** of f is a normal scheme X^+ and a proper birational morphism $f^+ : X^+ \to Y$ such that

1. $K_{X^+} + D^+$ is $\mathbb{Q}$-Cartier, where D^+ is the birational transform of D,
2. $K_{X^+} + D^+$ is f^+-ample, and
3. codim $\mathrm{Ex}(f^+) \geq 2$.

This is illustrated in the following diagram:

$$\begin{array}{ccc} X & \overset{\phi}{\dashrightarrow} & X^+ \\ {\scriptstyle -(K_X+D) \text{ is } f\text{-ample}} \searrow & & \swarrow {\scriptstyle (K_{X^+}+D^+) \text{ is } f^+\text{-ample}} \\ & Y & \end{array}$$

The rational map $\phi : X \dashrightarrow X^+$ is also called a flip.

It is not obvious that flips exist, terminate, or make anything better. They are mostly a mystery. However, the following implies that we can at least continue the minimal model program (contracting K-negative rays) after a flip:

Proposition 2.2.17. *Let (X, D) be a projective, $\mathbb{Q}$-factorial dlt pair, and $\phi : X \dashrightarrow X^+$ a flip of f. Then, (X^+, D^+) is dlt and $\mathbb{Q}$-factorial, and $\rho(X^+) = \rho(X)$.*

Until recently, existence of flips was an open problem. Termination of flips is still open in dimension ≥ 4, but in many cases, the existence of minimal models (below) has been proven.

Definition 2.2.18. Given a pair (X, D) over an algebraically closed field of characteristic 0 with X normal, D a boundary, and $f : X \to S$, (X, D) is an f-**minimal model** if it is $\mathbb{Q}$-factorial and dlt and $K_X + D$ is f-nef.

Theorem 2.2.19. *(see [19, 1.30.5]) If $f : X \to S$ is a proper, dominant morphism between normal, irreducible schemes and D an effective divisor such that (X, D) is klt and $K_{X_{gen}} + D_{gen}$ or D_{gen} is big, where X_{gen} is the generic fiber of f, then (X, D) has an f-minimal model.*

Theorem 2.2.20. *(see [19, 1.30.8]) Assume that $f : X \to S$ is projective and there is an effective divisor D' such that $(X, D + D')$ is dlt and $K_X + D + D' \sim_{\mathbb{Q},f} 0$. Then, (X, D) has an f-minimal model.*

To briefly summarize the minimal model program (when $f : X \to k$ is the structure map), we proceed as outlined below.

- Starting from a projective, $\mathbb{Q}$-factorial dlt pair (X, D), we'd like to produce a birational minimal model (X', D'). If $K_X + D$ is nef, we are done.

- If not, first use the cone and contraction theorem to contract a $K_X + D$ negative extremal ray. If the contraction morphism $f : X \to Z$ is birational, produce a new pair (X_1, D_1) by either setting $X_1 = Z$, $D_1 = f_*D$, if f is a divisorial contraction, or by setting $X_1 = X^+$, $D_1 = D^+$ if f is a small contraction. If the contraction morphism is a fiber type contraction, we stop and set $X' = X$ and $D' = D$.

- Repeat the process as necessary, constructing pairs (X_i, D_i), until we stop with a fiber type contraction or a true minimal model, where $K_{X_i} + D_i$ is nef.

Therefore, using the minimal model program, we can start with any variety or pair (with controlled singularities), and produce a minimal model. We can take this one step further to produce a canonical model, as defined below.

Definition 2.2.21. The canonical model (X^c, D^c) of an lc pair (X, D) is the variety

$$X^c = \operatorname{Proj} \oplus_{m \geq 0} H^0(X, m(K_X + \lfloor D \rfloor))$$

together with the birational transform D^c of D.

Definition 2.2.22. If $f : X \to S$ is a proper morphism, the relative canonical model (X^c, D^c) of an lc pair (X, D) is the variety

$$X^c = \operatorname{Proj} \oplus_{m\geq 0} f_*(X, m(K_X + \lfloor D \rfloor))$$

together with the birational transform D^c of D.

If (X, D) is a minimal model (respectively, an f-minimal model), then one can show that $K_{X^c} + D^c$ is ample (respectively, f-ample).

Theorem 2.2.23. *The relative canonical model is unique.*

Proof. See [21, Theorem 3.52]. □

Once we have constructed the canonical model, ampleness of $K_{X^c} + D^c$ and uniqueness of the canonical model are the essential tools to show certain moduli problems admit a proper moduli space.

Chapter 3

H-ϵ STABLE PAIRS

3.1 Definition and Motivation

We are interested in studying the moduli space of hypersurfaces S (of a fixed degree) in $\mathbb{P}^3$. As motivated in the introduction, instead of stuyding moduli of such S directly, we study moduli of pairs (X, D) where X is a degeneration of $\mathbb{P}^3$ and D is a degeneration of S.

The idea of studying moduli of pairs is not new: Mumford's study of moduli of curves with marked points is one example. Increasing the complexity, one could ask about the moduli space of degree d plane curves and about a compactification of said moduli space. This was studied in different ways by many authors, and a meaningful compactification was constructed in [12] by considering moduli of pairs (X, D) where X was a slc surface that smoothed to $\mathbb{P}^2$ and D was a divisor such that $dK_X + 3D \cong 0$ and $K_X + (\frac{3}{d} + \epsilon)D$ was ample for some (and hence all) ϵ sufficiently small. He was able to show that, for d not a multiple of 3, this moduli stack is proper, separated, and smooth. He was also able to explicitly determine the surfaces X (and thus the divisors D) appearing on the boundary of the moduli space.

This was a variant on another construction of compact moduli of such pairs (see [18] and [3]), where the moduli space with some fixed $\epsilon \in \mathbb{Q}$ was considered.

Now, consider the direct generalization of [12]: a compactification of the moduli space of degree d hypersurfaces in $\mathbb{P}^3$. Unfortunately, much of the work in [12] relies on the existing classification of surface singularities, but the approach of recasting the problem in terms of pairs (X, D) has some advantages. To this extent (motivated by [12]), we present the following definitions.

Definition 3.1.1. A pair (X, D), where X is a threefold and D is an effective $\mathbb{Q}$-Cartier divisor, is said to be ϵ-semistable in the sense of Hacking, or H-ϵ semistable, of degree d ($d \in \mathbb{N}, d \geq 4$) if

- X is normal and log terminal.
- The pair $(X, \frac{4}{d}D)$ is log canonical.
- $dK_X + 4D$ is linearly equivalent to zero.
- There is a deformation $(\mathcal{X}, \mathcal{D})/T$ of (X, D) over the germ of a curve such that the general fiber $\mathcal{X}_t \cong \mathbb{P}^3$ and the divisors $K_{\mathcal{X}/T}$ and $\mathcal{D}$ are $\mathbb{Q}$-Cartier.

Studying H-ϵ semistable has some advantages; in particular, the threefold X is required to be normal, facilitating a simpler study of the divisor D. However, they have one distinct disadvantage: the moduli space of H-ϵ semistable pairs is not separated. There are example of families of log smooth pairs with more than one semistable limit. Therefore, we will primarily concern ourselves with H-ϵ stable pairs, defined below. The moduli space of H-ϵ stable pairs is separated; limits are unique in an appropriate sense (Theorem 3.2.3).

Definition 3.1.2. A pair (X, D), where X is a threefold and D is an effective $\mathbb{Q}$-Cartier divisor, is said to be ϵ-stable in the sense of Hacking, or H-ϵ stable, of degree d if

- The pair $(X, (\frac{4}{d} + \epsilon)D)$ is semi log canonical and the divisor $K_X + (\frac{4}{d} + \epsilon)D$ is ample for some $\epsilon > 0$.
- The divisor $dK_X + 4D$ is linearly equivalent to zero.
- There is a deformation $(\mathcal{X}, \mathcal{D})/T$ of (X, D) over the germ of a curve such that the general fiber $\mathcal{X}_t \cong \mathbb{P}^3$ and the divisors $K_{\mathcal{X}/T}$ and $\mathcal{D}$ are $\mathbb{Q}$-Cartier.

Considering H-ϵ stable pairs has an advantage over semistable pairs, some of which are detailed in the following trivial lemma. Also, it gives a separatedness condition on the moduli space (Theorem 3.2.5 below).

Lemma 3.1.3. *If (X, D) is an H-ϵ stable pair, the following hold:*

(a) K_X is anti-ample.

(b) D is ample.

(c) Both K_X and D are $\mathbb{Q}$-Cartier.

(d) If X is strictly slc, the strictly slc locus of X is not contained in the support of D.

Remark 3.1.4. In the definitions above, we require that the threefold X has a smoothing to $\mathbb{P}^3$. This is useful (and used below), but there may be pairs (X, D) satisfying the other conditions of the definitions such that X does not smooth to $\mathbb{P}^3$, indicating that the moduli space is not irreducible, and those threefolds would be of interest themselves.

3.2 Limits of H-ϵ Stable Pairs Exist

First, we prove that we can extend families of H-ϵ semistable pairs (in a not necessarily unique way). To do this, we first have a lemma and technical definition.

Lemma 3.2.1. *Let $\mathcal{X}/T$ be a flat family of projective varieties over the germ of a curve such that the general fiber is normal. Let $\mathcal{X}^\times/T^\times$ be the restriction of the family to the punctured curve $T^\times = T - 0$. If $\mathcal{B}$ is a relatively nef $\mathbb{Q}$-Cartier divisor $\mathcal{X}$ such that $\mathcal{B}\big|_{\mathcal{X}^\times} \sim 0$, then $\mathcal{B} \sim 0$ in $\mathrm{Cl}(\mathcal{X}/T)$.*

Proof. Let $X_1, X_2, \ldots, X_n$ be the irreducible components of $X = \mathcal{X}_0$, the fiber over the closed point. Then, there is an exact sequence

$$0 \to \mathbb{Z}X \to \oplus\mathbb{Z}X_i \to \mathrm{Cl}(\mathcal{X}/T) \to \mathrm{Cl}(\mathcal{X}^\times/T^\times) \to 0.$$

Since $\mathcal{B}\big|_{\mathcal{X}^\times} \sim 0$, we can write $\mathcal{B} \sim \sum a_i X_i$ for $a_i \in \mathbb{Z}$, arranged so that $a_1 \leq a_2 \leq \cdots \leq a_n$. Because $X \sim 0$ in $\mathrm{Cl}(\mathcal{X}/T)$, we can assume $a_i \leq 0$ for all i and $a_1 = 0$. Assume to the contrary that there exists an i such that $0 = a_1 = a_2 = \cdots = a_{i-1} > a_i \geq \cdots \geq a_n$. For each $j \leq i-1$, and any curve $C \subset X_j$ with $C \not\subset X_k$ for $k \neq j$, we have $X_k \cdot C \geq 0$. Therefore,

$$\mathcal{B} \cdot C = a_i X_i \cdot C + \cdots + a_n X_n \cdot C \leq 0.$$

But, since $\mathcal{B}$ is relatively nef, this implies $X_l \cdot C = 0$ for $i \leq l \leq n$. However, if there exists an l such that $X_l \cap X_j \neq \emptyset$, then, choosing any curve $C \subset X_j$ that intersects (but is not contained in) X_l, we must have $X_l \cdot C > 0$, as it counts the number of points in the intersection. Since X is connected, we must have $X_l \cap X_j \neq \emptyset$ for some l, j such that $i \leq l \leq n$ and $j \leq i-1$. Therefore, we have a contradiction, so $0 = a_1 = a_2 = \cdots = a_n$ and $B \sim 0$. □

Definition 3.2.2. Let $(\mathcal{X}, \mathcal{D})/T$ be a pair consisting of a normal variety $\mathcal{X}$ and an effective Weil divisor $\mathcal{D}$, flat over the DVR T. Let X be the fiber over the closed point. A semistable log resolution of $(\mathcal{X}, \mathcal{D})$ is a proper birational morphism $f : \mathcal{Y} \to \mathcal{X}$ such that Y is smooth, $\mathrm{Ex}(f)$ is a divisor, $g_*^{-1}X$ is reduced, and $\mathrm{Ex}(f) \cup g^{-1} \operatorname{Supp} \mathcal{D} \cup g_*^{-1}(X)$ is a simple normal crossing divisor.

Semistable log resolutions exist (possibly after finite surjective base change) by [21, Theorem 7.17].

Using existence of resolutions and the previous lemma, we will first show that families of log smooth pairs $(\mathbb{P}^3, D)$ over a punctured curve can be completed to a family of semistable pairs, Theorem 3.1.1, not necessarily in a unique way. See Theorem 3.2.6 for a summary of this process.

Theorem 3.2.3. *Let $0 \in T$ be the germ of a curve and write $T^\times = T - 0$. Let $\mathcal{D}^\times \subset \mathbb{P}^3 \times T^\times$ be a family of smooth hypersurfaces over $T^\times$ of degree $d \geq 4$. Then, there exists a finite surjective base change $S \to T$ and a family $(\mathcal{X}, \mathcal{D})/S$ of H-ϵ semistable pairs extending the pullback of the family $(\mathbb{P}^3 \times T^\times, \mathcal{D}^\times)/T^\times$ such that the divisors $K_{\mathcal{X}}$ and $\mathcal{D}$ are $\mathbb{Q}$-Cartier.*

Proof. Complete $(\mathbb{P}^3 \times T^\times, \mathcal{D}^\times)$ to a flat family $(\mathbb{P}^3 \times T, \mathcal{D})$ over T. Possibly after base change, which we supress in the notation, there is a semistable log resolution $\pi : (\mathcal{X}_1, \mathcal{D}_1) \to (\mathbb{P}^3 \times T, \mathcal{D})$ which is an isomorphism over $T^\times$.

Now, run a $K_{\mathcal{X}_1} + \frac{4}{d}\mathcal{D}_1$ MMP over T. Let $(\mathcal{X}_2, \mathcal{D}_2)/T$ denote the end product. Since it is the end product of an MMP, $(\mathcal{X}_2, X + \frac{4}{d}\mathcal{D}_2)$ is dlt and $\mathbb{Q}$-factorial. Also, $K_{\mathcal{X}_2} + \frac{4}{d}\mathcal{D}_2$ is relatively nef. Because $(\mathcal{X}_1^\times, \mathcal{D}_1^\times) \cong (\mathbb{P}^3 \times T^\times, \mathcal{D}^\times) \cong (\mathcal{X}_2^\times, \mathcal{D}_2^\times)$, and $(K_{\mathcal{X}_1} + \frac{4}{d}\mathcal{D}_1)|_{\mathcal{X}_\infty{}^\times} \sim 0$, the divisor $K_{\mathcal{X}_2} + \frac{4}{d}\mathcal{D}_2$ vanishes on $\mathcal{X}_2^\times$, hence by Theorem 3.2.1, $dK_{\mathcal{X}_2} + 4\mathcal{D}_2 \sim 0$.

Next, run a $K_{\mathcal{X}_2}$ MMP over T. This ends in a fibration $(\mathcal{X}, \mathcal{D})/T$, and $(\mathcal{X}, \mathcal{D})$ is the required completion of $(\mathbb{P}^3 \times T^\times, \mathcal{D}^\times)$, as we verify below. First, because it is the total space of an MMP fibration and the general fiber is $\mathbb{P}^3$, $\mathcal{X}/T$ is a Mori fiber space, $(\mathcal{X}, X)$ is dlt, and $\mathcal{X}$ is $\mathbb{Q}$-factorial. This implies that $K_{\mathcal{X}}$ and $\mathcal{D}$ are $\mathbb{Q}$-Cartier. Also, because $\rho(\mathcal{X}/T) = 1$, from the exact sequence used in Theorem 3.2.1, tensoring with $\mathbb{Q}$ implies X is irreducible and therefore normal and log terminal [21, Proposition 5.51]. Finally, because $(\mathcal{X}_2, X_2 + \frac{4}{d}\mathcal{D}_2)$ was dlt and $dK_{\mathcal{X}_2} + 4\mathcal{D}_2 \sim 0$, we have that $(\mathcal{X}, X + \frac{4}{d}\mathcal{D})$ is log canonical and $dK_{\mathcal{X}} + \frac{4}{d}\mathcal{D} \sim 0$. Then, by adjunction, $(X, \frac{4}{d}D)$ is log canonical and $dK_X + 4D \sim 0$. □

Ultimately, this shows that families of log smooth pairs $(\mathbb{P}^3, D)$ over a punctured curve can be completed to a 'well behaved one': we can find a normal variety X and a divisor D_X such that $(X, \frac{4}{d}D_X)$ is log canonical that complete the family. However, there may be more than one such limit, even with the requirement that the canonical divisor of the family is $\mathbb{Q}$-Cartier. The problem arises precisely with pairs $(\mathbb{P}^3, D)$ such that the semistable limit $(X, \frac{4}{d}D_X)$ is strictly log canonical. However, if we instead work with H-ϵ stable pairs (Theorem 3.1.2), we can modify these semistable limits to a unique limit, although we have to sacrifice normality of X.

Example 3.2.4. As an example of non-uniqueness, we consider the problem in one dimension less: curves $C \subset \mathbb{P}^2$ such that $(\mathbb{P}^2, \frac{3}{d}C)$ is log canonical. If $d = 5$ and C is a quintic curve with an A_9 singularity locally of the form $x^2 + y^{10}$, a computation shows $(\mathbb{P}^2, \frac{3}{5}C)$ is a 2-dimensional

H-ϵ semistable pair, and in particular, is log canonical. The log canonical threshold of the pair $(\mathbb{P}^2, C)$ is $\frac{3}{5}$, so for any $\epsilon > 0$, $(\mathbb{P}^2, (\frac{3}{5} + \epsilon)C)$ is not log canonical. Therefore, $(\mathbb{P}^2, C)$ is a strictly semistable pair. However, the pair $(\mathbb{P}(1,1,4), C')$, where C' has an A_9 singularity, also has log canonical threshold $\frac{3}{5}$, and $\mathbb{P}(1,1,4)$ admits a $\mathbb{Q}$-Gorenstein smoothing to $\mathbb{P}^2$, hence $(\mathbb{P}(1,1,4), C')$ is another strictly semistable pair. However, these can both be obtained as limits of the **same** family $(\mathbb{P}^2_{t\neq 0}, \mathcal{C}_{t\neq 0})$ of quintic curves degenerating to a curve C with an A_9 singularity, so the semistable limit is not unique. However, the stable replacement of both pairs is $(\mathbb{P}(1,1,5) \cup X_6, \overline{C})$ (see [12, Section 11] for a description of X_6). This is summarized below.

We consider $\pi : \mathbb{P}(1,1,5) \cup X \to \mathbb{P}^2$, obtained by performing the $(5,1)$ weighted blowup of the A_9 singularity in the fiber $(\mathbb{P}^2_{t=0}, C)$. Then the strict transform of the curve $L_x := \{x = 0\} \subset \mathbb{P}^2$ becomes $\pi^* L_x = \hat{L}_x + 5E$. By the projection formula, we see that $\hat{L}_x^2 = -1$, and so is a contractible curve. Contracting this curve (which is $K_{\mathcal{X}}$-trivial, where $\mathcal{X}$ is the family of surfaces) gives $g : \mathbb{P}(1,1,5) \cup X \to \mathbb{P}(1,1,5) \cup X_6$. In particular, the image of X is a surface Y such that $\rho(Y) = 1$ and Y has a unique singularity of type $\frac{1}{5}(1,4)$ and so $Y \cong X_6$. Furthermore, because the contraction g was the contraction of a $K_{\mathcal{X}}$-trivial curve in the family of surfaces and because $\mathbb{P}(1,1,5) \cup X_6$ admits a $\mathbb{Q}$-Gorenstein smoothing to $\mathbb{P}^2$, $(\mathbb{P}(1,1,5) \cup X_6, \overline{C})$ is the stable replacement of $(\mathbb{P}^2, C)$, where $\overline{C}$ denotes the image of C on $\mathbb{P}(1,1,5) \cup X_6$.

$$\begin{array}{ccc} & \mathbb{P}(1,1,5) \cup X & \\ \swarrow^{\pi} & & \searrow^{g} \\ \mathbb{P}^2 & & \mathbb{P}(1,1,5) \cup X_6 \end{array}$$

We note that we can complete this diagram (performing the flop of $\hat{L}_x$), and the output is a curve C' with an A_9 singularity inside $\mathbb{P}(1,1,4)$.

$$\begin{array}{ccccccc} & & \mathbb{P}(1,1,5)\cup X & \overset{f}{\dashrightarrow} & X^+\cup\mathbb{P}(1,4,5) & & \\ & \swarrow^{\pi} & & \searrow^{g} \quad \swarrow^{h} & & \searrow^{\psi} & \\ \mathbb{P}^2 & & & \mathbb{P}(1,1,5)\cup X_6 & & & \mathbb{P}(1,1,4) \end{array}$$

Finally, ψ can be realized as the $(5,1)$ blowup of the curve C' with an A_9 singularity through the unique singularity of type $\frac{1}{4}(1,1)$ inside $\mathbb{P}(1,1,4)$. This shows that the stable replacement of $(\mathbb{P}(1,1,4), C')$ is also $\mathbb{P}(1,1,5)\cup X_6$. Furthermore, this proves that both $(\mathbb{P}^2, C)$ and $(\mathbb{P}(1,1,4), C')$ are the semistable limits of the same family: take any smoothing $(\mathcal{X}, \mathcal{C})_t$ of $(\mathbb{P}^2, C)$; performing the process above (a weighted blowup followed by a flop and a contraction) shows we can replace $(\mathbb{P}^2, C)$ by $(\mathbb{P}(1,1,4), C')$ without changing the general fiber of the family.

Although this example is only in dimension 2, by taking appropriate cones over these surfaces, one could potentially construct two families of three-dimensional H-ϵ semistable pairs with isomorphic general fiber but different central fiber.

Therefore, to get a separated moduli space, it is necessary to work with stable (non-normal) pairs. Below, we will prove that families of H-ϵ stable pairs over a punctured curve can be extended (possibly after base change) in a unique way. Again, see Theorem 3.2.6 for a summary of this process.

Theorem 3.2.5. *Let $0 \in T$ be the germ of a curve and write $T^\times = T - 0$. Let $\mathcal{D}^\times \subset \mathbb{P}^3 \times T^\times$ be a family of smooth hypersurfaces over $T^\times$ of degree $d \geq 5$. Then, there exists a finite surjective base change $T' \to T$ and a family $(\mathcal{X}, \mathcal{D})/T'$ of H-ϵ stable pairs extending the pullback of the family $(\mathbb{P}^3 \times T^\times, \mathcal{D}^\times)/T^\times$ such that the divisors $K_{\mathcal{X}}$ and $\mathcal{D}$ are $\mathbb{Q}$-Cartier. The family is unique in the following sense: any two such families become isomorphic after a further finite surjective base change.*

Proof. As constructed in the proof of Theorem 3.2.3, let $(\mathcal{X}_1, \mathcal{D}_1)$ be a family of H-ϵ semistable pairs extending $(\mathbb{P}^3 \times T^\times, \mathcal{D}^\times)$. By construction, $(\mathcal{X}_1, X_1 + \frac{4}{d}\mathcal{D}_1)$ is log canonical and the pair

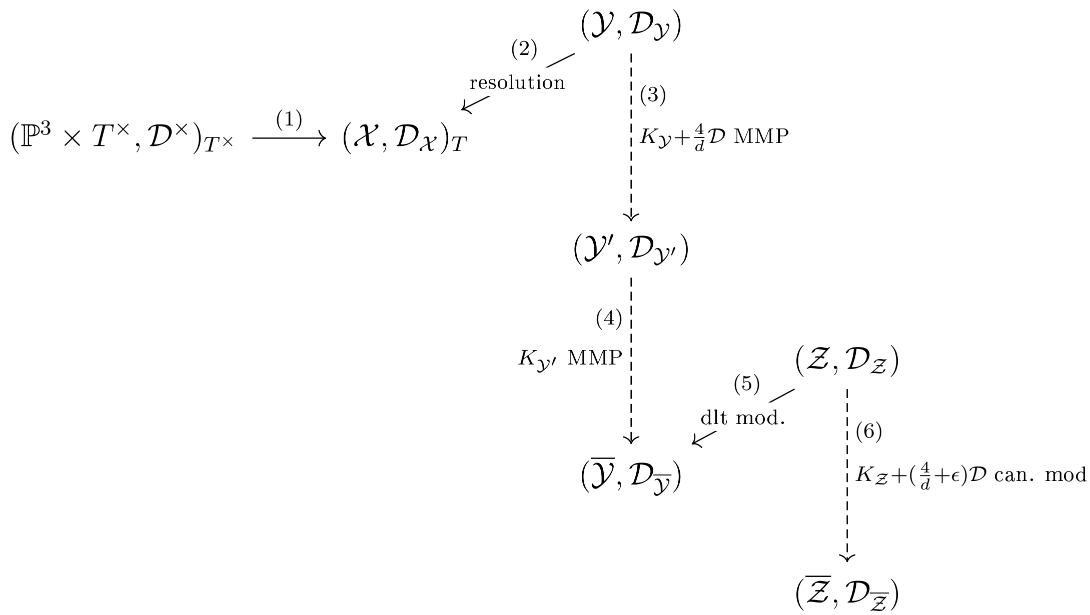
$(\mathcal{X}_1, X_1)$ is dlt, as verified in the proof of Theorem 3.2.3. We can find a minimal dlt model $\pi : (\mathcal{X}_2, \mathcal{D}_2) \to (\mathcal{X}_1, \mathcal{D}_1)$ such that $dK_{\mathcal{X}_2} + 4\mathcal{D}_2 = \pi^*(dK_{\mathcal{X}_1} + 4\mathcal{D}_1)$ and $(\mathcal{X}_2, X_2 + \frac{4}{d}\mathcal{D}_2)$ is dlt [20, Theorem 3.1]. Hence, $(\mathcal{X}_2, X_2 + (\frac{4}{d} + \epsilon)\mathcal{D}_2)$ is dlt for ϵ sufficiently small [21, Corollary 2.39].

Now, let $(\mathcal{X}, \mathcal{D})$ be the $K_{\mathcal{X}_2} + (\frac{4}{d} + \epsilon)\mathcal{D}_2$ canonical model. Then, $(\mathcal{X}, X + (\frac{4}{d} + \epsilon)\mathcal{D})$ is log canonical, $K_{\mathcal{X}} + (\frac{4}{d} + \epsilon)\mathcal{D}$ is $\mathbb{Q}$-Cartier, $dK_{\mathcal{X}} + 4\mathcal{D} \sim 0$, and $K_{\mathcal{X}} + X + (\frac{4}{d} + \epsilon)\mathcal{D}$ is relatively ample. Therefore, by adjunction, $(X, (\frac{4}{d} + \epsilon)D)$ is slc, $dK_X + 4D \sim 0$, and $K_X + (\frac{4}{d} + \epsilon)D$ is ample.

Further note that $K_{\mathcal{X}}$ and $\mathcal{D}$ are $\mathbb{Q}$-Cartier, since $K_{\mathcal{X}} + (\frac{4}{d} + \epsilon)\mathcal{D}$ is $\mathbb{Q}$-Cartier and $dK_{\mathcal{X}} + 4\mathcal{D} \sim 0$, hence Cartier.

Finally, to see uniqueness, observe that any two such families have a common semistable log resolution (possibly after base change, suppressed in the notation). Then, because each family is slc and the divisor $K_{\mathcal{X}} + (\frac{4}{d} + \epsilon)\mathcal{D}$ is ample for all ϵ sufficiently small, each family is a log canonical model of the resolution. Because log canonical models are unique, this implies (possibly after base change), the limits are unique. □

Remark 3.2.6. The diagram below summarizes the proof of properness (existence of unique limits).



1. Complete the family of pairs over a punctured curve.

2. After base change (suppressed in the notation/diagram), there exists a semistable log resolution, changing only the central fiber.

3. Given the resolution, run a $K_{\mathcal{Y}} + \frac{4}{d}\mathcal{D}_{\mathcal{Y}}$ MMP over T.

4. Next, run a $K_{\mathcal{Y}'}$ MMP over T. This additional minimal model program ensures the special fiber is irreducible and both K and D are $\mathbb{Q}$-Cartier.

5. Take a minimal dlt modification [20, Definition 1.9] of $(\overline{\mathcal{Y}}, \frac{4}{d}\mathcal{D}_{\overline{\mathcal{Y}}})$, extracting any -1 divisors over the strictly log canonical locus. Because $(\mathcal{Z}, \frac{4}{d}\mathcal{D}_{\mathcal{Z}})$ is dlt and the singularities behave in a particular way, for ϵ sufficiently small, $(\mathcal{Z}, (\frac{4}{d} + \epsilon)\mathcal{D}_{\mathcal{Z}})$ is dlt.

6. Finally, take the $K_{\mathcal{Z}} + (\frac{4}{d} + \epsilon)\mathcal{D}_{\mathcal{Z}}$ canonical model. Uniqueness follows.

Chapter 4

CLASSIFICATION

There are many other ingredients in the study of these moduli spaces. For fixed ϵ, the family of H-ϵ stable pairs is bounded [14, Theorem 1.1], so we can embed all H-ϵ stable pairs into a large projective space and (hope to) use the Hilbert scheme to construct a moduli space. However, we do not know the families are bounded as ϵ tends to 0. Therefore, it is of interest to try to bound the families of H-ϵ stable pairs in another way. In Hacking's work, there is a more elementary way to show boundedness in terms of the degree d as in [12, Theorem 4.5]. However, this uses the existing classification of slc surfaces, so to generalize for threefolds, would probably involve an explicit understanding of all slc threefolds.

Instead, we try to classify only the threefolds appearing in H-ϵ stable pairs. It would suffice to classify the singular Fano threefolds appearing there, since the ample divisor D must be in a linear system determined by a multiple of K_X. Towards this result, we are able to classify the strictly log canonical threefolds in this moduli problem.

Classification is of interest for many reasons. As discussed in the introduction, if the varieties X in the moduli problem are at worst semi-log terminal, boundedness is known by a result of Hacon, McKernan, and Xu [13, Corollary 1.7]. The condition on singularities and the fact that $dK_X + 4D \sim 0$ imply that these pairs are actually ϵ-log terminal (meaning the discrepancy is greater than or equal to $-1 + \epsilon$ for some fixed $\epsilon > 0$). Therefore, existing results apply and show that the moduli problem is bounded. There is also the following result of de Fernex and Fusi, summarized nicely in a paper by Totaro, that implies these log terminal varieties are rational.

Theorem 4.0.1 (Totaro, [28])**.** *Rationality specializes in families of complex klt varieties of* *dimension at most 3.*

Therefore, if X is a log terminal degeneration of $\mathbb{P}^3$, it is rational.

A classification of rational, log terminal varieties that admit a smoothings to $\mathbb{P}^3$ is not currently known. One necessary criterion is that $(-K_X)^3 = 64$ (see below), so certainly not all rational Fano varieties are eligible. This is an open question that, if solved, would contribute to a complete classfication result of all pairs on the boundary of the moduli space of H-ϵ stable pairs. In light of Theorem 4.1.12, this is the only case that remains. We should point out that such a classification is known in dimension 2 (log terminal surfaces that smooth to $\mathbb{P}^2$) [24], but the proof relies heavily on machinery only in place for surfaces, so cannot be generalized in its current state to the case of threefolds.

Proposition 4.0.2. *If $f : \mathcal{X} \to C$ is a flat family of n-dimensional projective varieties over a pointed curve $0 \in C$. Assume that $K_{\mathcal{X}/C}$ is $\mathbb{Q}$-Cartier. Assume that the general fiber X_t is smooth and the special fiber X_0 is normal. Then, $(K_{X_0})^n = (K_{X_t})^n$. In particular, if $X_t \cong \mathbb{P}^3$, $(K_{X_0})^3 = -64$.*

Proof. Let l be an integer such that $lK_{\mathcal{X}}$ is Cartier. Then, for any $t \in C$, $\mathcal{O}_{X_t}(lK_{X_t}) \cong \omega_{X_t}^{[l]} \cong (\omega_{\mathcal{X}}^{[l]})_t$. By definition, $(lK_{X_t})^n$ is the coefficient of $m_1 m_2 \ldots m_n$ in $\chi(X_t, \mathcal{O}_{X_t}((m_1 + m_2 + \cdots + m_n)lK_{X_t}))$. Because f is flat, this polynomial is constant, so $(lK_{X_t})^n = l^n K_{X_t}^n$ is constant. Therefore, $K_{X_t}^n$ is constant, as desired. □

Remark 4.0.3. The assumption that $K_{\mathcal{X}}$ is $\mathbb{Q}$-Cartier is essential; see, for example, [21, Example 7.61].

It is also relatively easy to construct a non-rational degeneration of $\mathbb{P}^3$, as shown by the following example. Any such example is at least log canonical, in light of Theorem 4.0.1.

Example 4.0.4. Given a projectively normal variety $V \subset \mathbb{P}^N$, there is a standard degeneration of of V to a cone over its hyperplane section [21, 7.61]. Thus, taking the 4-uple embedding $\mathbb{P}^3 \hookrightarrow \mathbb{P}^{34}$, the general hyperplane section of the image corresponds to a K3 surface in $\mathbb{P}^3$, which has trivial canonical divisor. The cone X over such a surface S is log canonical: let Y be the blow up of X at the vertex, $f : Y \to X$. Then, f is birational with exceptional divisor isomorphic to S, so $K_Y \sim f^*K_X + aS$. By adjunction, $K_S \sim (K_Y + S)\big|_S$, so $0 \sim K_S \sim (f^*K_X + (a+1)S)\big|_S$. Given any curve $C \subset S$, $0 = K_S \cdot C = (a+1)S\big|_S \cdot C$, hence $a = -1$ and X is log canonical by definition. A calculation shows that $-K_Y - S$ is nef and 0 exactly on curves contained in the exceptional locus S, so $-K_X$ is ample. However, for X to occur as a threefold in a pair (X, D) on the boundary of the moduli space above, we must have $-\frac{d}{4}K_X \equiv D$. By the discussion above, $K_X \cdot C \in \mathbb{Z}$ for any curve $C \subset X$, and a calculation shows $K_X \cdot \Gamma = -1$ for a ruling of the cone. Since the singularity of X is strictly log canonical, in order for $(X, (\frac{4}{d} + \epsilon)D)$ to also be log canonical, D must miss the singularity of X. Hence, D is contained in the smooth locus of X and is therefore Cartier, so $D \cdot C \in \mathbb{Z}$, which implies $\frac{d}{4} \in \mathbb{Z}$. Therefore, for d not divisible by 4, X cannot occur as a boundary threefold.

In light of this and the comment on boundedness above, we first focus on the strictly log canonical threefolds appearing in the moduli problem. The main result is that, for odd degree d, there are none.

4.1 Strictly Log Canonical Fano Threefolds

The inspiration for classification of the strictly log canonical threefolds in this moduli problem is the following theorem:

Theorem 4.1.1 (Ishii, [16])**.** *If X is a normal, Gorenstein variety of dimension n with K_X anti-ample and with finite (non-empty) irrational locus, then X is a cone over a variety S with canonical singularities and $K_S \sim 0$.*

Proof. See [16]. □

If X is a normal, Gorenstein variety with K_X anti-ample, the log canonical locus coincides with the irrational locus [21, Corollary 5.24]. Therefore, this theorem implies that if a threefold X has a finite (non-empty) lc locus, it is either a cone over a K3 surface or two dimensional Abelian variety.

The following is a generalization of this result.

Theorem 4.1.2. *Let X be a projective variety with a finite number of strictly log canonical singularities $\{p_1, \ldots, p_n\}$ and $-K_X$ ample. If $a(E, X) \in \{-1, \mathbb{R}^{\geq 0}\}$ for every exceptional divisor E over X with center$_X(E) \subset \{p_1, \ldots, p_n\}$, then X is a cone over a variety Z with $K_Z \equiv 0$.*

The extra hypotheses in this result arise from removing the Gorenstein hypotheses in Theorem 4.1.1. In order to ensure X is a cone, there needs to be a certain extremal ray in the cone of curves. In some sense, this is just a special case of the later result, Theorem 4.1.12.

Before getting to the proof, we provide a few technical lemmas. In all cases, we consider dlt pairs (X, D) and study properties of various K_X-negative and $K_X + D$-negative contractions. The motivating idea is to study contractions that happen 'over' D. Divisorial contractions that are $K_X + D$-negative and D-positive must have a certain structure, as explained below. We begin by discussing the negativity of K_X in certain K_X-negative contractions. Namely, the next lemma shows that K_X can't be 'too' negative on fibers.

Lemma 4.1.3. *Let X be a normal projective variety such that K_X is $\mathbb{Q}$-Cartier. If $\phi : X \to Y$ is a contraction of a K_X-negative extremal ray with fibers of dimension at most 1, then each fiber F is a chain of $\mathbb{P}^1$s whose configuration is a tree such that $-1 \leq K_X \cdot C < 0$ for each irreducible component C of F.*

Proof. By assumption, $R^2\phi_*\mathcal{F} = 0$ for any coherent sheaf $\mathcal{F}$ on X. By Grauert - Riemenschneider vanishing, $R^1\phi_*\omega_X = 0$, and by [17], because $-K_X$ is ϕ-ample, we have $R^1\phi_*\mathcal{O}_X = 0$. Then, consider any sheaf of ideals J such that $\mathcal{O}_X/J$ is supported on a fiber F of ϕ:

$$0 \to J \to \mathcal{O}_X \to \mathcal{O}_X/J \to 0.$$

Pushing forward to Y, we see that $R^1\phi_*\mathcal{O}_X/J = H^1(F, \mathcal{O}_X/J|_F) = 0$. Similarly, we know that $H^1(F, \omega_X/J\omega_X|_F) = 0$ so $H^1(F, (\omega_X/J\omega_X)|_F/T) = 0$, where $T \subset (\omega_X/J\omega_X)|_F$ denotes torsion. Taking J to be the ideal of F, we see that F is a chain of $\mathbb{P}^1$s whose configuration is a tree. Then, consider an irreducible component $C \subset F$ and the sheaf $\omega_X \otimes \mathcal{O}_C/T$, where T is the torsion in $\omega_X \otimes \mathcal{O}_C$. This is a torsion-free sheaf on $\mathbb{P}^1$, so must be a vector bundle of the form $\omega_X \otimes \mathcal{O}_C/T \cong \oplus \mathcal{O}_{\mathbb{P}^1}(a_i)$. The vanishing of H^1 given above implies that $a_i \geq -1$ for each i. If m is an integer such that $\omega_X^{[m]}$ is Cartier, we must have that $\omega_X^{[m]} \otimes \mathcal{O}_C = \mathcal{O}_{\mathbb{P}^1}(b)$ is a negative degree line bundle. But, there is a nonzero morphism $(\omega_X \otimes \mathcal{O}_C/T)^{\otimes m} \to \omega_X^{[m]} \otimes \mathcal{O}_C$, and $a_i \geq -1$ implies that $b \geq -1$. Therefore, $-1 \leq K_X \cdot C < 0$. □

The previous lemma bounds the negativity of K_X. Morally, this hints that if $K_X \cdot C \geq -1$ for contracted curves, and $C \cap D \neq \emptyset$, that should force $(K_X + D) \cdot C \geq 0$. Certainly this could be false if X was highly singular and $D \cdot C \notin \mathbb{Z}$, but with a few restrictions on the singularities, we can apply the lemma to our advantage.

We begin with an observation about these contractions.

Lemma 4.1.4. *If (X, D) is dlt and D is an effective prime divisor that is Cartier in codimension 2, then any $K_X + D$-negative extremal divisorial contraction is an isomorphism on D if and only if the exceptional divisor does not intersect D.*

Proof. Let $\phi : X \to Y$ be the given contraction. Because ϕ is $K_X + D$ negative and divisorial, the negativity lemma implies

$$\phi^*(K_Y + D') = K_X + D - aE$$

where $D' = \phi_* D$, E is the exceptional divisor, and $a > 0$. Because D is Cartier in codimension 2, $K_X + D|_D = K_D$, so restricting this to D gives

$$\phi^*(K_{D'} + Diff_{D'}(0)) = K_D - aE|_D$$

where $Diff_{D'}(0)$ is the correction term to the adjunction formula needed if D' is not Cartier in codimension 2. This correction term is effective, and $aE|_D$ is antieffective, so $\phi|_D : D \to D'$ is an isomorphism if and only if $E|_D = 0$. This also shows that $Diff_{D'}(0) = 0$. □

From this observation and Theorem 4.1.3, we can draw a number of corollaries. Namely, if we have a 'nice' contraction that is an isomorphism on D, because the fibers are well behaved, this will force the map to be a fibration.

Corollary 4.1.5. *If (X, D) is dlt and D is an effective, prime divisor that is Cartier in codimension 2, then any $K_X + D$-negative, D-positive extremal contraction that contracts a divisor but contracts no curves in D is a Fano fiber contraction $X \to D$.*

Proof. Let $\pi : X \to Y$ be the contraction. If a divisor is contracted, then the morphism is either a divisorial contraction or Fano fiber contraction onto a variety with strictly lower dimension. If no curves in D are contracted, the induced map $D \to \pi(D)$ is finite, but $(Y, \pi_* D)$ is dlt by [21], hence $\pi_* D$ is normal. Furthermore, because no curves in D are contracted, the fibers have dimension at most 1. But, if π was divisorial, Theorem 4.1.3 implies $K_X \cdot C \geq -1$ for C contracted by π. However, because D is Cartier in codimension 2, for a general fiber C, $D \cdot C \in \mathbb{Z}$, hence $(K_X + D) \cdot C \geq 0$. Therefore, the contraction must be a fibration with general fiber $\mathbb{P}^1$. In this case, for general fiber C, $K_X \cdot C = -2$, so we must have $D \cdot C = 1$, so $\pi|_D : D \to \pi(D)$ is generically of degree 1. Therefore, by Zariski's Main Theorem, and because D is prime, $\pi_* D$ must be isomorphic to D and $\pi : X \to Y$ is a Fano fiber contraction and $Y \cong D$. In particular, X is almost a $\mathbb{P}^1$-bundle over D (the general fiber is $\mathbb{P}^1$) and the fiber D is a section of this almost-bundle. □

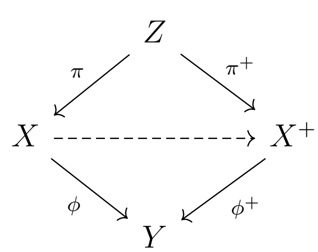
Corollary 4.1.6. *If X is a terminal variety and (X, D) is dlt for some effective prime divisor D where $-D|_D$ is nef, then any $K_X + D$-negative D-positive contraction gives a Fano fibration $X \to D$.*

Proof. If X is terminal, the singular set has codimension at least 3 in X, hence D is Cartier in codimension 2. If $-D|_D$ is nef, then any D-positive contraction contracts no curves in D, so by Theorem 4.1.5, the contraction of such a ray gives X the structure of an almost-$\mathbb{P}^1$-bundle over D, or precisely, a Fano fibration $X \to D$. □

We should point out that Theorem 4.1.3 does not require the contraction be divisorial; it could be a small contraction and the result would still hold. Although small contractions behave remarkably differently than divisorial contractions, we can still ask about small contractions that enjoy many of the same properties as those above. In particular, the next lemma shows that small contractions with $K_X + D$-negative and D-positive properties cannot exist.

Lemma 4.1.7. *If X is a terminal variety and (X, D) is a canonical pair such that D is an effective prime divisor, then the contraction of a $K_X + D$-negative, D-positive extremal ray R that contracts no curves in D cannot be a small contraction.*

Proof. Assume the small contraction exists. Because this is a K_X-negative contraction, we consider the flip of ϕ as in the following diagram, where Z is the blow up of the exceptional locus.



Note that the fiber of the contraction $\phi : X \to Y$ is not contained in D, by assumption. Because every exceptional divisor E has nonnegative discrepancy $a(E, X, D)$, if $D_Z = \pi_*^{-1} D$, we have

$$K_Z + D_Z = \pi^*(K_X + D) + \sum a_i E_i$$

where $a_i \geq 0$ for each i. Restricting to D, because D is Cartier in codimension 2, we get

$$K_{D_Z} = \pi|_D^*(K_D) + \sum a_i E_i|_{D_Z}.$$

But, by Lemma 3.38 in [21], flips can only improve singularities, so

$$\pi^{+*}(K_{X^+} + D^+) = \pi^*(K_X + D) - \sum c_i E_i$$

where $c_i \geq 0$. Then, restricting to D we see that

$$\pi^{+*}(K_{D^+} + Diff_{D^+}(0)) = \pi^*(K_D) - \sum c_i E_i|_{D_Z}.$$

Substituting, we see that

$$\pi^{+*}(K_{D^+} + Diff_{D^+}(0)) = K_{D_Z} - \sum a_i E_i|_{D_Z} - \sum c_i E_i|_{D_Z}.$$

However, $\pi|_{D_Z}$ was the resolution of the rational map $D \dashrightarrow D^+$, so $\pi^+|_{D_Z} : D_Z \to D^+$ is an isomorphism, which is a contradiction.

□

We can tie the previous lemmas together in the following result, seemingly technical but the key idea in the proof of Theorem 4.1.2.

Lemma 4.1.8. *Let X be a terminal variety and (X, D) a canonical pair with D an effective integral divisor such that $K_X|_D$ is nef. If the class of a $K_X + D$-negative extremal ray R contains a curve C such that $C \cap D$ is finite and non-empty, and the contraction of R has fiber dimension at most 1, then it must be a Fano fibration $X \to Y$ such that the general fiber is isomorphic to $\mathbb{P}^1$ and $Y \cong D$.*

Proof. Because terminal varieties are singular only in codimension ≥ 3, the assumption on (X, D) implies that D is Cartier in codimension 2.

By Theorem 4.1.7, the contraction of R cannot be a small contraction. However, any curve $C \subset D$ has $K_X \cdot C \geq 0$, hence the contraction $\phi : X \to Y$ of a $K_X + D$-negative D-positive extremal ray cannot contract any curves in D. Therefore, $\phi|_D : D \to \phi(D)$ is a finite morphism. Note that, for general C contracted by ϕ, $D \cdot C > 0$ and $D \cdot C \in \mathbb{Z}$ because D is Cartier in codimension 2. Assume for contradiction that ϕ was a divisorial contraction. Then, because no curves in D are contracted, the fibers of ϕ have dimension at most one. If this is the case, Theorem 4.1.3 implies that $K_X \cdot C \geq -1$. However, this means $(K_X + D) \cdot C \geq 0$, contradicting our assumption.

Therefore, we must have $\phi : X \to Y$ a Fano fiber contraction of relative dimension 1. This implies the general fiber of ϕ is isomorphic to $\mathbb{P}^1$, as desired. To see that $Y \cong D$, note that the map $\phi|_D : D \to \phi(D)$ is finite but, for general fiber C of ϕ, $K_X \cdot C = -2$, so in order for ϕ to have been a $K_X + D$ negative contraction, we must have $D \cdot C = 1$. Therefore, $\phi|_D$ is finite and generically of degree 1, so by Zariski's Main Theorem, $\phi|_D : D \to \phi(D) = Y$ is an isomorphism.

□

We are now ready to prove Theorem 4.1.2.

Proof. Taking a minimal $\mathbb{Q}$-factorial dlt model of X, there is a $\mathbb{Q}$-factorial variety Y and a morphism $\pi : Y \to X$ extracting all divisors E_i with discrepancy $a(E_i, X) = -1$ such that K_Y is relatively nef. Let $E = \sum E_i$ and observe that $K_Y + E = \pi^* K_X$. Because $-K_X$ is ample and $K_Y + E$ is trivial on E and negative on all curves not contained in E, there must exist a $K_Y + E$ negative, E positive extremal ray R in $\overline{NE(Y)}$.

Let $\phi : Y \to S$ be the contraction of R. By assumption, the pair (Y, E) is canonical along E (since $a(F, Y, E) = a(F, X)$ for any exceptional divisor F over X), so Lemma 4.12 applies and $\phi : Y \to S$ is a fiber contraction of relative dimension 1. Note that, for a general fiber F of ϕ, $K_Y \cdot F = -2$ and $[F] \in R$, so $(K_Y + E) \cdot F < 0$. Choosing an appropriate fiber that misses the singular points of Y, one sees that $E_i \cdot F \in \mathbb{Z}$ for each i because F is contained in the smooth locus of Y. Therefore, because $K_Y \cdot F = -2$ and $(K_Y + E) \cdot F < 0$, there is only one exceptional divisor $E_0 = E$. Because (Y, E) is dlt, E is normal and ϕ contracts no curves in E, hence $S \cong E$, giving $\phi : Y \to S$ the structure of a $\mathbb{P}^1$ bundle. However, E is contractible $\pi : Y \to X$, we see that X is cone over E (where 'cone' is interpreted as the contraction of a section of a $\mathbb{P}^1$-bundle over E). We can further characterize E by observing that $(K_Y + E)|_E = K_E$, hence K_E is numerically trivial.

□

Since one cannot guarantee that the exceptional divisors over a variety are in the set given in Theorem 4.1.2, we first make an easy observation, whose proof is the same as that above.

Proposition 4.1.9. *Let X be a projective variety with a finite number of strictly log canonical singularities $\{p_1, \ldots, p_n\}$ and $-K_X$ ample. Consider a minimal dlt modification $\pi : Y \to X$ extracting the -1 divisors of X, so $K_Y + E = \pi^*(K_X)$. If there exists an extremal ray $R \in \overline{NE(Y)}$ such that a curve $C \not\subset E$, $[C] \in R$, intersects E at a smooth point of Y, then X is a cone over a numerically Calabi-Yau variety.*

To remove the restrictions on the discrepancies in Theorem 4.1.2, we would like to say there always exists a ray as in Theorem 4.1.9. However, it is not obvious why this is true or clear that it should be ture. Instead, we include various generalizations of the result Theorem 4.1.2.

We pause to remark that many standard examples of log canonical singularites have resolutions where an exceptional divisor is not rational or ruled, related to the fact that log canonical singularities do not have to be rational singularities. So, one might expect that -1 exceptional divisors over a log canonical singularity are often not rational or ruled. If that is the case, the following result characterizes these singularities.

Theorem 4.1.10. *If X is a projective 3-dimensional variety with a finite number of strictly log canonical singularities and $-K_X$ ample such that at least one exceptional divisor E over X with discrepancy $a(E, X) = -1$ is not rational or ruled, then there is only one such E and X is birational to a $\mathbb{P}^1$ bundle over E.*

Proof. We proceed in a similar fashion to that of the previous proof.

There is a $\mathbb{Q}$-factorial variety Y and a morphism $\pi : Y \to X$ extracting all divisors Δ_i with discrepancy $a(\Delta_i, X) \leq 0$ such that K_Y is relatively nef. Let $E = \sum \Delta_j$ be the sum over divisors Δ_j with discrepancy -1 and $F = \sum -a(\Delta_k, X)\Delta_k$ be the sum over divisors with discrepancy larger than -1. By construction of Y (which is terminal, hence has finitely many singular points), these effective divisors are Cartier in codimension 2, $\pi^*(K_X) = K_Y + E + F$, and for any curves $C \subset \operatorname{Supp}(E + F)$, $K_Y \cdot C \geq 0$. By assumption on X, the general curve through E has negative K_X-degree.

We would like to find an E positive and $K_Y + E$ negative extremal ray in the cone of curves. If so, we can proceed as in the proof of Theorem 4.1.2 to conclude that the contraction of such a ray gives a Fano fibration $\phi : Y \to E$ (and E consists of only one component). Because a general curve through E doesn't intersect F, $F = 0$. Therefore, the log canonical locus in X consists of a single point $x \in X$, and for a general fiber l of ϕ, $\pi^* K_X \cdot l = -1$ and $E \cdot l = 1$.

In general, by the construction of Y, there must exist $K_Y + E + F$-negative and $E + F$-positive extremal rays. If we cannot find a ray that is E-positive, because K_Y is nef relative to $\pi : Y \to X$, we must have every K_Y-negative and $K_Y + E$-negative extremal ray be E-trivial. We proceed by running an MMP on Y, contracting K_Y-negative rays. (Note this is an MMP on Y, not on the pair $(Y, E + F)$). At any point in time, if we reach an intermediate variety Y' with a $K_{Y'} + E'$-negative E-positive extremal ray R in $N\bar{E}(X)$, the MMP terminates with the contraction of R if the fiber dimension is at most 1. This follows from Theorem 4.1.4 and Theorem 4.1.7.

Assume we do not find such a ray. Because X was a Fano threefold, the MMP must terminate with a Fano fibration $f : Y' \to S$ such that $\dim S < 3$. We claim that the only components of E that could be contracted by an MMP are rational or ruled. If $\phi : Y' \to Y''$ is a divisorial contraction of a component Δ of E', because (Y, E) is canonical (and hence (Y', E') is canonical), Δ is a canonical, rationally connected surface. Because such surfaces are rational, the result follows. If no component of Δ is contracted until the termination of the MMP $f : Y' \to S$, there are a few cases to consider. Either Y' is a terminal Fano variety of Picard rank 1, and because $K_{Y'} + \Delta$ is negative, Δ is a smooth Fano surface, hence rational. If Y' has Picard rank 2 and C is a curve, if Δ is a fiber of f', again it is smooth, Fano, and rational. If instead $f'|_\Delta : \Delta \to C$ is surjective, Δ is a ruled surface. Finally, if $\dim S = 2$, $\dim f'(\Delta) = 0$ implies Δ is Fano and therefore rational. If $\dim f'(\Delta) = 1$, Δ is again a smooth ruled surface, and we are left only with the desired result, $S \cong \Delta$.

Note this implies that E has at most one non-rational or ruled component.

□

Note that, for a log canonical variety, there is certainly no need for such a non-rational or ruled exceptional divisor to exist in the resolution. In fact, even for surfaces, there are easy examples of log canonical singularities whose resolution graph consists only of rational curves. For classification purposes, we would like to also characterize these log canonical threefolds. Because we are starting with a Fano variety, if we run a standard minimal model program, it should terminate in a Fano fibration: we can never change the general curve from a K-negative curve to a K-nonnegative curve. So, taking a modification $X' \to X$ extracting the -1-divisors, a run of the MMP on X' will terminate in a fibration $X'' \to Z$, where Z has dimension $0, 1$, or 2. In the study of moduli of pairs (X, D) where these varieties appear as X, we would like to understand the structure of the fibration $X'' \to Z$. In particular, in Chapter 5, an understanding of these fibrations will illuminate the requirement that d be odd in Theorem 1.0.3. Therefore, it will be beneficial to understand the the termination of the MMP in these cases, which is the content of the following result.

Theorem 4.1.11. *If X is a projective 3-dimensional variety with a finite number of strictly log canonical singularities and $-K_X$ ample such that every exceptional divisor E over X with discrepancy $a(E, X) = -1$ is rational or ruled, then X is birational to a threefold Y admitting a fibration $Y \to S$ such that one of the following holds: $S \cong E_0$ for some component E_0 of E and the general fiber is $\mathbb{P}^1$ or $\dim S = 1$ where the general fiber is a smooth del Pezzo surface* *and E_0 is ruled over S. In each case E, is ample with respect to the fibration.*

Proof. Consider a $\mathbb{Q}$-factorial dlt modification of X: a morphism $\pi : X' \to X$ extracting the -1 divisors over the strictly log canonical singular points. Note that $K_{X'} + E = \pi^* K_X$. From the previous discussion, we know that a run of the minimal model program on X' terminates in a fibration. The content of this theorem then has two parts: (1) we can choose a minimal model program so the fibration is of relative dimension 1 or 2 and (2) the divisor E_0 is ample with respect to that fibration.

Because X' is a threefold, any run of the minimal model program terminates, so we have the flexibility to choose desirable extremal rays to contract at each step.

Certainly $\rho(X') > 1$ and, if $\rho(X') = 2$, one extremal ray in $\overline{NE(X')}$ must correspond to the morphism $\pi : X' \to X$. Also, there must exist a $K_{X'} + E$-negative, E-positive extremal ray; if not, because every curve in E is E-negative by construction, it would imply $-E$ is nef. Therefore, the other extremal ray in $\overline{NE(X')}$ must be one of these. Denote this extremal ray by R. We claim that the contraction of R must be a fibration $\phi : X' \to E$ with general fiber $\mathbb{P}^1$. If the contraction of R was birational, we have the following picture.

$$\begin{array}{ccc} X' & \xrightarrow{\pi} & X \\ \downarrow{\scriptstyle \phi} & & \\ Y & & \end{array}$$

Here Y is a threefold. Let $C \subset X'$ be a curve contracted by ϕ, and assume first that ϕ is a divisorial contraction, contracting the divisor D.

We pause to make the following claim.

Claim. If S is a smooth surface with distinct contractible curves C_1 and C_2, then there is a curve C_0 such that $C_0 \cdot C_1 = 0$ and $C_0 \cdot C_2 = 0$.

In this context, either the intersection matrix for C_1 and C_2 is negative definite or the image of $\bar{C}_1$ and $\bar{C}_2$ have self intersection 0. In the first case, the union $C_1 \cup C_2$ is contractible, so there exist many such curves C_0 (contract $C_1 \cup C_2$ and take the preimage of any curve missing the image of $C_1 \cup C_2$). In the second case, arrange indices so, after the contraction of C_1, $\bar{C}_2$ has self-intersection 0. Then, from the study of the cone of curves, there must exist a C_2-trivial extremal ray. Therefore, there is either a fibration contracting $\bar{C}_2$ as one of the fibers, and the preimage of any other fiber has the desired property, or a birational morphism contracting a curve that does not intersect C_2.

Returning to the problem at hand, consider a smooth surface S on X containing C that is not contracted by ϕ. Then, C is contractible on S by assumption, and because $\rho(Y) = 1$ and X' and Y are $\mathbb{Q}$-factorial, $S \cap E$ must be non empty. (If it were empty, E would still be contractible, so $\rho(Y)$ must have been strictly larger than 1). Therefore, not only is C contractible on S, but $S \cap E = C'$ is another contractible curve. Therefore, by the previous claim, there is a third curve C_0 such that $C_0 \cdot C = 0$ and $C_0 \cdot C' = 0$. However, by construction, C_0 does not meet D nor does it meet E.

Now, we can use intersection theory to arrive at a contradiction. Because $\rho(X) = 2$, these were the only possible contractions. Let π be the contraction of the ray R_1 and ϕ the contraction of the ray R_2. By the negativity lemma and by hypothesis, $D \cdot R_1 = \alpha_1 > 0$ and $E \cdot R_1 = -\beta_1 < 0$. Similarly, $D \cdot R_2 = -\alpha_2 < 0$ and $E \cdot R_2 = \beta_2 > 0$. For there to exist a curve that does not meet E or D, we must have $\alpha_1 = r\alpha_2$ and $\beta_1 = r\beta_2$ for some $r \in \mathbb{Q}$. However, this means there are no curves that are strictly D-trivial and E-positive (or E-trivial and D-positive), contradicting the fact that E and D were separately contractible.

If instead ϕ were a small contraction, we can make a similar argument, taking D to be a surface that contains the curve contracted by ϕ.

Therefore, $\phi : X' \to Y$ must be a fibration. Because every curve in E is $K_{X'}$-positive, no curve in E can be contracted, and by the same argument in the proof of Theorem 4.1.4, E has only one component and $Y \cong E$.

In the general case, if $\rho(X')$ is unknown, we can run a $K_{X'}$ minimal model program on

X'. Because X' is generically K'_X-negative and $K_{X'} + E$ is antinef, there must always exist a K-negative, extremal ray to contract. Furthermore, we can take these extremal rays to be E-positive: if there were no E-positive rays, $-E$ would be nef, contradicting the effectivity of E. Also, because the only $K_{X'}$-positive curves on X' were contained in E, if a curve is $K_{X'}$-positive, it must be non-positive on E. This even holds after some number of flips: the only possible curves to flip would be $K_{X'}$-negative and E-positive, hence after the flip would be $K_{X'}$-positive and E-negative.

By the previous lemmas, if any contraction of a $K_{X'} + E$-negative, E-positive extremal ray has one dimensional fibers and at least one contracted curve intersects E at a point where $E \cdot C \in Z$, the minimal model program terminates in a fibration over E, where E is ample with respect to the fibration. To prove the theorem, assume the minimal model program terminates in a fibration of relative dimension 2 or 3.

If the fibration has relative dimension 2 over a curve C, because the final step was the contraction of a $K_{X'} + E$-negative, E-positive extremal ray, E is ample over C. Therefore, assume a run of the minimal model program terminates in a fibration $Z \to \operatorname{Spec} k$, where Z is a Fano threefold of Picard number 1.

To reach this point, we must have arrived at a threefold X'' that has $\rho(X'') = 2$ and then contracted a divisor in X'' to get the variety Z. Considering the threefold X'', we can make essentially the same argument as above (in the case $\rho(X') = 2$) to conclude that contracting the other extremal ray in $\overline{NE(X'')}$ must have given us the desired fibration.

Because $\overline{NE(X'')}$ has two extremal rays and at least one is $K_{X''}$-negative, assuming both correspond to contractions as in

$$\begin{array}{ccc} X'' & \xrightarrow{f_1} & Z_1 \\ \downarrow{\scriptstyle f_2} & & \\ Z_2 & & \end{array}$$

then we claim that one contraction is the desired fibration. Assume the contrary. If both are small contractions, because we are considering a threefold and termination of flips is known, we could perform some number of flips to put ourselves in the situation of at least one contraction being divisorial. If they are both divisorial, replacing D and E with the divisors contracted by f_1 and f_2 in the above argument, we reach a contradiction.

Therefore, two cases remain: either both extremal rays are contractible yet one corresponds to a divisorial contraction and the other to a small contraction, or only one ray is contractible. We address these separately.

In the first case, we can make the same argument as that above and deduce a contradiction. In the second case, let R be the non-contractible extremal ray. Note that we were put into this situation by a run of the minimal model program on X', where (X', E) was dlt and each contraction has been $K_{X'} + E$-negative and E-positive. Then, (X'', E'') is still dlt. Therefore, the only potential curves that could correspond to a non-contractible extremal ray R are curves that are $K_{X''}$-positive and $K_{X''} + E''$-positive. The only way these curves could have appeared is if there was a small contraction and a flip was needed to continue the MMP. However, if the previous step of the MMP was that flip, the R is certainly still contractible by definition of flip. If, at each point of the MMP between flip creating the curves in R and the current variety X'', the birational modifications happened away from the curves in R, again, they would still be contractible.

Therefore, we can assume that the curves C in R were flipped from another curve and either a divisorial contraction or another flip happened, where these modifications intersected C. Diagrammatically, we are in the following situation:

$$X' \dashrightarrow X^- \dashrightarrow X^+ \xrightarrow{f} X'' \xrightarrow{f_1} Z_1$$

For simplicity, assume $X^- \dashrightarrow X^+$ is the flip creating the curve C, followed by a divisorial contraction $f : X^+ \to X''$. Suppose f is the contraction of the divisor D. Choose a surface S such that $S \cdot C < 0$ (for example, the image of a surface S^- on X^- that C^- intersected).

Contracted curves in D intersect D negatively and S positively, C intersects D positively and S negatively, and because D is contractible, there must exist many D-trivial curves. Then, we can do an intersection theory argument to show that, after the contraction of D, C must actually become $K_{X''} + E''$-negative, contradicting our assumption that it was not contractible.

In summary, if a run of the minimal model program on X' creates a variety X'' with $\rho(X) = 2$, then the extremal rays give

$$\begin{array}{ccc} X'' & \xrightarrow{f_1} & Z_1 \\ \downarrow{\scriptstyle f_2} & & \\ Z_2 & & \end{array}$$

where one of Z_1 or Z_2 is a fibration of relative dimension 1 or 2, as desired.

□

Although there have been various generalizations of Ishii's result, we summarize them here.

Theorem 4.1.12. *If X is a projective 3-dimensional, strictly log canonical, normal variety and $-K_X$ is ample, then X is birational to a terminal threefold Y admitting a fibration $Y \to S$ such that one of the following holds.*

(i) $S \cong E$ for some exceptional divisor E over X with discrepancy -1 and the general fiber is $\mathbb{P}^1$.

(ii) S is a curve where the general fiber is a smooth del Pezzo surface and E is birationally ruled over S.

Note that the power of this result is not about the termination in a Fano fibration, but about the end behavior of E, an exceptional divisor over X with discrepancy -1. The conclusion is that E is relatively ample with respect to the morphism $Y \to S$, so $E \cdot C$ is positive for generic C, and the fibration implies that K_Y is negative, but not too negative. This observation is the crucial part of the proof of Theorem 1.0.3.

We can also point out that this classification is effective: there exist examples of both (i) and (ii) occurring. For instance, the cone over a K3 surface, Theorem 4.0.4, is an example with one -1 exceptional divisor E whose dlt model (which happens to be a resolution of singularities) is a $\mathbb{P}^1$ bundle over E.

For examples of threefolds of type (ii), many appear in [4]. For convenience, we sketch [4, Example 6.1] here. Details can be found in the original paper. Consider $X' = \mathbb{P}(\mathcal{O}_C^{\oplus 2} \oplus \mathcal{L})$, where C is a smooth, genus 1 curve, and $\mathcal{L}$ is an ample line bundle on C. If $E \cong C \times \mathbb{P}^1$ is the divisor defined by the quotient $\mathcal{O}_C^{\oplus 2} \oplus \mathcal{L} \to \mathcal{O}_C^{\oplus 2}$, then there is a birational morphism $X' \to X$ contracting E onto a $\mathbb{P}^1$. A computation shows that X is Gorenstein, Fano, and log canonical along the image of E. This fits into part (ii) of the above result because X', the dlt model (and resolution) of X was defined as a $\mathbb{P}^2$ bundle over C.

Next, we would like to extend the above theorem to the case of varieties X with $-K_X$ ample and a higher dimensional locus of log canonical singularities. We continue to restrict ourselves to the case of threefolds.

Using the same set-up $\pi : Y \to X$ and studying the same contractions, we would like to show that the contraction of a K_Y-negative, E-positive extremal ray cannot be birational. This would imply the existence of a Fano fiber contraction $\phi : Y \to S$ such that S has lower dimension. Again, if the contraction is F-positive, the MMP terminates immediately with a fiber structure of relative dimension 1 or 2.

Theorem 4.1.13. *Let X be a normal, projective threefold with a strictly log canonical singularly along a curve $C \in X$, at worst log terminal singularities away from C, and $-K_X$ ample. If $Y \to X$ is the extraction of the -1 divisors E, there is a rational map $\phi : Y \to S$ with a Fano fiber structure such that $S \cong E_0$ with general fiber $\mathbb{P}^1$ or S is a curve and E_0 is* *birationally ruled over S.*

Proof. This is a consequence of the proof of the previous result, and is proved in the same way. The only change is that, in the dlt modification $X' \to X$, we only have that $K_{X'} + E$ is trivial on contracted curves, not on all curves in E. However, this does not affect the proof.

□

Lastly, we can immediately generalize this result to the case of non-normal slc varieties with anti-ample canonical sheaf and a strictly log canonical singularity. This is equivalent to studying the case of a pair (X, Δ) where $-(K_X + \Delta)$ is ample and the 1-dimensional locus of log canonical singularities intersects Δ. Because the locus of log canonical singularities must intersect Δ [1] but cannot be contained in Δ, X must in fact have a log canonical singularity along a curve. Then, a a dlt modification X' of X and minimal model program on X' gives the same conclusion, where Δ is considered as a component of E.

In Chapter 5, we use these results to further analyze the moduli space of H-ϵ stable pairs presented above.

Although the focus thus far has been on strictly log canonical varieties X with anti-ample canonical class, we also study the purely log terminal varieties. Explicitly understanding these varieties would be useful in classifying the singular varieties on the boundary of the moduli space of H-ϵ stable pairs. In that vein, we will first focus on the canonical threefolds appearing in the moduli problem.

4.2 Canonical Fano Threefolds

Much is known about canonical threefolds in general, and a standard reference is [27]. In the Fano case, particularly when X is Gorenstein, such threefolds can be classified by invariants like K_X^3 and the Fano index.

If X has at worst canonical singularities, the Fletcher-Reid plurigenus formula [27, Theorem 10.2] gives the plurigenera of X in terms of K_X^3, $\chi(\mathcal{O}_X)$, and coefficients c_P determined by a basket of singularities for X. In the proof of the theorem, Reid shows that the coefficients c_P can be computed in terms of the finitely many points Q_i such that $K_{X'}$ is not Cartier at Q_i, where $X' \to X$ is a crepant partial resolution such that X' has only terminal singularities.

In [10, Theorem 1.1], Fletcher shows the plurigenus formula is exact, meaning that any two canonical threefolds with the same plurigenera have the same K_X^3, $\chi(\mathcal{O}_X)$, and basket of singularities. The contribution from the singularities is nonzero precisely when there are points Q_i such that $K_{X'}$ is not Cartier at Q_i. In our case, because X is a flat degeneration of $\mathbb{P}^3$, the plurigenera of X and $\mathbb{P}^3$ are the same, so this inversion of the plurigenus formula implies that X' must be a terminal Gorenstein variety. Because $X' \to X$ is any crepant partial resolution such that X' has only terminal singularities, $K_{X'}^3 = -64$ and we can take X' to be $\mathbb{Q}$-factorial.

Although there are potentially many canonical degenerations of $\mathbb{P}^3$, there are not many terminal degenerations. Namely, there is only $\mathbb{P}^3$.

Theorem 4.2.1. *If X is a terminal variety that admits a smoothing to $\mathbb{P}^3$, then $X \cong \mathbb{P}^3$.*

Proof. The Fletcher-Reid plurigenus formula shows that if X is not Gorenstein, it does not admit a smoothing to $\mathbb{P}^3$, so it suffices to consider Gorenstein threefolds X. In this case, [6, Theorem 2.1] implies that the Fano index of X, the maximal integer r such that $K_X \sim -rH$ for $\mathcal{O}(H) \in \mathrm{Pic}(X)$, is equal to that of $\mathbb{P}^3$. Therefore, the Fano index of X is 4. Then, [6, Theorem 3.1] says that, because the Fano index is maximal, $X \cong \mathbb{P}^3$. $\square$

There do exist non-trivial canonical degenerations of $\mathbb{P}^3$. The following is an example of such a variety, pointed out by Hacking.

Example 4.2.2. First, observe that the standard embedding of the quadric surface $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ is an element of the linear system $\mathcal{O}_{\mathbb{P}^3}(2)$. Then, let Z be the image of the Veronese embedding of $\mathbb{P}^3 \hookrightarrow \mathbb{P}^9$. There is a standard degeneration from Z to the cone over a hyperplane section of Z by taking the cone over Z (see, for example, [21, Example 7.61]). In this case, the hyperplane section of Z corresponds to an element of $\mathcal{O}_{\mathbb{P}^3}(2)$, and is the $\mathcal{O}(2,2)$ embedding of the quadric surface in $\mathbb{P}^8$. A computation shows that the cone over this is indeed Gorenstein–it is the cone over the anti-canonical embedding of $\mathbb{P}^1 \times \mathbb{P}^1$. A check shows that this has canonical singularities; for details see [19, Lemma 3.1]. Therefore, this gives an example of a flat degeneration of $\mathbb{P}^3$ to a Gorenstein, strictly canonical variety.

However, one should note that, in the moduli problem at hand, this particular example can only appear on the boundary of the moduli space for even degree d. To see this, we will use the relationship $dK_X + 4D \sim 0$. There are two cases to consider.

If D misses the singular point of X, consider the strict transform D' in the resolution $\pi : X' \to X$ obtained by blowing up the singular point. The singularity is canonical and $\pi^* K_X = K_{X'}$. However, X' is the projectivization of a vector bundle over $\mathbb{P}^1 \times \mathbb{P}^1$ and admits a morphism $X' \to \mathbb{P}^1 \times \mathbb{P}^1$ contracting the fibers. Because each fiber $F \cong \mathbb{P}^1$, $K_{X'} \cdot F = -2$. This implies $K_X \cdot \pi(F) = -2$. Because D misses the singular point of X, $D \cdot \pi(F) \in \mathbb{Z}$, so the relationship $dK_X + 4D \sim 0$ implies $D \cdot \pi(F) = -\frac{d}{4} K_X \cdot \pi(F) = -\frac{d}{2}$. Therefore, d must be even.

In the second case, assume D contains the singular point of X. In order for (X, D) to be an H-ϵ stable pair, both K_X and D must be $\mathbb{Q}$-Cartier. Therefore, by the study of divisors on cones and [19, Proposition 3.14], D must be the cone over a divisor in the linear system $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(n, n)$. Therefore, the relationship $dK_X + 4D \sim 0$ implies that $d(-2, -2) + 4(n, n) \sim 0$, or $n = -\frac{d}{2}$. Again, this implies that d must be even.

The following theorem shows this behavior is typical. If X is a canonical degeneration of $\mathbb{P}^3$, the degree d must be even.

Theorem 4.2.3. *For odd degree d, if X is a canonical threefold appearing in an H-ϵ-stable pair (X, D) of degree d, then $X \cong \mathbb{P}^3$.*

If X has only terminal singularities, Theorem 4.2.1 implies the result. If X has canonical singularities, consider a crepant partial resolution $X' \to X$ such that X' is terminal and $\mathbb{Q}$-factorial. Before giving the proof, we give a sketch of the argument.

By [10], if $K_{X'}$ is not Cartier, there is a nonzero contribution to a basket of singularities on X, so X is not isomorphic to $\mathbb{P}^3$. It then suffices to consider the case where X' is a terminal, $\mathbb{Q}$-factorial Gorenstein variety with $-K_{X'}$ nef.

Running a minimal model program on X', if it terminates in a morphism $X' \dashrightarrow Y \to \operatorname{Spec} k$, then Y must be a terminal Fano threefold with $\rho(Y) = 1$. Studying the pseudo-index of Y as in [6] and combining this with the fact that $K_{X'}^3 = -64$ would imply that X' itself must have been $\mathbb{P}^3$, so $X \cong \mathbb{P}^3$. If a run of the minimal model program on X' terminates in a morphism $X' \dashrightarrow Y \to C$, where C is a curve, the generic fiber of $Y \to C$ must be a smooth del Pezzo surface, so there are sufficiently general curves $L \subset X'$ such that $K_{X'} \cdot L = -3$ or -2, and if the termination is in a surface W, there are sufficiently general curves $L \subset X'$ such that $K_{X'} \cdot L = -2$. If any of these curves miss the exceptional divisors of the partial resolution $\pi : X' \to X$, then $D \cdot \pi(L) \in \mathbb{Z}$, and we can argue as in the example above to show that d must be even. Similarly, we can reach the same conclusion if D does not pass through the strictly canonical singularities of X.

The remaining case is when D contains the strictly canonical singularities of X and the general fiber L intersects the exceptional divisors of $\pi : X' \to X$, because it is not obvious that $D \cdot \pi(L) \in \mathbb{Z}$. Because of this, the proof is rather technical.

First, let us recall results of Cutkosky on contractions of extremal rays on terminal, $\mathbb{Q}$-factorial Gorenstein threefolds.

Lemma 4.2.4. *[7, Lemma 2] Suppose that X is a terminal, $\mathbb{Q}$-factorial Gorenstein threefold. Then, X is factorial.*

Lemma 4.2.5. *[7, Lemma 3] Suppose that X is a terminal, $\mathbb{Q}$-factorial Gorenstein threefold and $\phi : X \to Y$ is the contraction of a K_X-negative extremal ray with at most one dimensional fibers. Then, Y is factorial. In particular, Y is a terminal, $\mathbb{Q}$-factorial Gorenstein threefold,* *and ϕ is cannot be a small contraction.*

Theorem 4.2.6. *[7, Theorem 4] Suppose that X is a terminal, $\mathbb{Q}$-factorial Gorenstein threefold and $\phi : X \to Y$ is a birational contraction of a surface $W \subset X$ to a curve $C \subset Y$. Then, Y is smooth near C.*

Theorem 4.2.7. *[7, Theorem 5] Suppose that X is a terminal, $\mathbb{Q}$-factorial Gorenstein threefold and $\phi : X \to Y$ is a birational contraction of a surface $W \subset X$ to a point $p \subset Y$. Then, one of the four cases below occur:*

(i) *Y is nonsingular near p, $W \cong \mathbb{P}^2$, and $\mathcal{O}_W(W) \cong \mathcal{O}_{\mathbb{P}^2}(-1)$.*

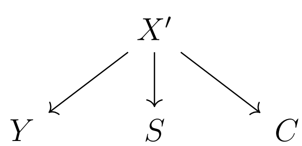
(ii) *$W \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathcal{O}_W(W) \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1)$.*

(iii) *W is isomorphic to a reduced, irreducible singular quadric surface D in $\mathbb{P}^3$ and W satisfies $\mathcal{O}_W(W) \cong \mathcal{O}_{\mathbb{P}^3}(-1) \otimes \mathcal{O}_D$.*

(iv) *Y is singular at p, $W \cong \mathbb{P}^2$, and W satisfies $\mathcal{O}_W(W) \cong \mathcal{O}_{\mathbb{P}^2}(-2)$.*

Now we can prove the Theorem 4.2.3.

Proof. Let us begin with the simplest case: no component of the locus of canonical singularities is contained in D. Then, the contraction of a $K_{X'}$ negative extremal ray must be birational $X' \to Y$ or a Fano fibration $X' \to S$ or $X' \to C$, where $\dim S = 2$, $\dim C = 1$:



Because $-K_{X'}$ is nef and non-trivial, the contraction cannot be $X' \to \operatorname{Spec} k$. Also, by Theorem 4.2.5, $X' \to Y$ is necessarily a divisorial contraction. Therefore, in every case, the generic curve contracted has $K_{X'} \cdot C = -1, -2$, or -3, so the image of C on X has $K_X \cdot \pi(C) = -1, -2$, or -3. Because D does not contain the locus of canonical singularities, for a sufficiently generic curve C, $D \cdot \pi(C) \in \mathbb{Z}$. Therefore, the relationship $dK_X + 4D \sim 0$ implies d is even.

If a component Δ of the locus of canonical singularities is contained in D, we can separate into two cases: either Δ is one- or zero-dimensional.

Case 1. $\dim \mathbf{\Delta} = \mathbf{1}$.

If Δ is one-dimensional, then consider the partial resolution $\pi : X' \to X$. Because X has only canonical singularities, the fibers of π must be chains of rational curves. We can study the pullback $\pi^* D$: in particular, $\pi^* D = \tilde{D} + \sum a_i F_i$, where $\tilde{D}$ is the strict transform of D and $F = \bigcup F_i$ is the fiber over Δ. For $C \subset F_0$ contracted by π, $K_{X'} \cdot C = 0$ and $F \cdot C < 0$. However, $-2 = K_{F_0} \cdot C = (K_{X'} + F_0) \cdot C$, so there can be at most one component F_i meeting C with $F_i \cdot C = 1$. Therefore, either there is no such F_i and

$$0 = \pi^* D \cdot C = \tilde{D} \cdot C + \sum a_i F_i \cdot C = n + a_0(-2)$$

so $a_0 \in \mathbb{Z}[1/2]$ or there is some F_i that meets C and a contracted curve $C' \subset F_i$ meeting F_0 such that

$$0 = \pi^* D \cdot C = \tilde{D} \cdot C + \sum a_i F_i \cdot C = n + a_0(-2) + a_i(1)$$

$$0 = \pi^* D \cdot C' = \tilde{D} \cdot C' + \sum a_i F_i \cdot C' = 0 + a_0(1) + a_i(-2)$$

so $a_0, a_i \in \mathbb{Z}[1/3]$.

This shows for generic curves in X meeting D, the intersection with D is in $\mathbb{Z}[\frac{1}{6}]$.

With this in mind, now contract a $K_{X'}$ negative extremal ray on X'. As above, we have the following options:

$$\begin{array}{ccccc} & & X' & & \\ & \overset{A}{\swarrow} & \downarrow{\scriptstyle B} & \overset{C}{\searrow} & \\ Y & & S & & C \end{array}$$

Case A.

Assume first $X' \to Y$ is divisorial.

If $X' \to Y$ is divisorial and with at most one dimensional fibers, the generic fiber C has $K_{X'} \cdot C = -1$, so the image in X has $K_X \cdot \pi(C) = -1$. Because $D \cdot \pi(C) \in \mathbb{Z}[\frac{1}{6}]$, the relationship $dK_X + 4D \sim 0$ implies d must be even. If $X' \to Y$ is divisorial but contracts a surface to a point, if any case other than (i) occurs as in Theorem 4.2.7, we still find a generic curve C in the fiber with $K_{X'} \cdot C = -1$.

If case (i) occurs, the threefold Y is still terminal, $\mathbb{Q}$-factorial, and Gorenstein, so we can contract a new K_Y negative extremal ray and repeat. If at any point our contraction one of the cases $(ii), (iii)$, or (iv), by the same argument above, we are done. If we perform a divisorial contraction with at most one-dimensional fibers, again the output is terminal, $\mathbb{Q}$-factorial and Gorenstein, so we can continue. Therefore, it suffices to analyze the possible fibrations that arise as minimal models of a terminal, $\mathbb{Q}$-factorial, Gorenstein variety X' where, at each step of the minimal model program, the resulting variety is also terminal, $\mathbb{Q}$-factorial, and Gorenstein.

However, after some number of divisorial contractions, we reach the point of a fibration, then the divisorial contractions were blow ups of some point(s) on the fibration. Therefore, either the general fiber of the fibration doesn't intersect F, or after blowing up, a fiber of the divisorial contraction doesn't intersect F. Therefore, its image on X has $D \cdot C \in \mathbb{Z}$. Arguing as above implies d is even.

Therefore, the only two cases that remain to be studied are if the only possible $K_{X'}$ negative contraction yields a fibration.

Case B.

If $\phi : X' \to S$ is a fibration with general fiber $\cong \mathbb{P}^1$, either there are F-trivial fibers C or F is relatively ample. In the first case, $K_X \cdot \pi(C) = -2$ and $D \cdot \pi(C) \in \mathbb{Z}$, so d be even. By [], S must be smooth and X' must be a conic bundle over S. If $X' \to S$ has any singular fibers, then there exist curves C such that $K_{X'} \cdot C = -1$, and we argue as before to show d must be even. Therefore, we may assume every fiber is smooth and $X' \to S$ is a smooth $\mathbb{P}^1$-bundle over a smooth surface S. Furthermore, by [6][Lemma 2.5], $-K_S$ is big and nef. Because F is relatively ample, the induced morphism $F \to S$ must be finite.

However, F is contractible on X', so we have a diagram

$$\begin{array}{ccc} X' & \xrightarrow{\pi} & X \\ \Big\downarrow{\scriptstyle \phi} & & \\ S & & \end{array}$$

Consider a smooth curve $C \subset S$ such that $Z = \phi^{-1}(C)$ contains a contracted curve in F. Because every fiber of ϕ is $\mathbb{P}^1$, Z is a ruled surface over C and because $F \to S$ is finite, $F|_Z$ is a multisection of $\phi|_Z : Z \to C$. However, this multisection is contractible in Z to a surface $\overline{Z} \subset X$. The intersection theory on ruled surfaces implies that $F|_Z$ is actually a section.

This is true for any such Z, so the degree of $\phi|_F : F \to S$ must be 1, hence $S \cong F$ and F is a section of ϕ. Because F is contracted to a curve via $\pi : X' \to X$, $S \cong F$ must be a ruled surface over C with $-K_S$ big and nef, so S must be $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{F}_1$, or $\mathbb{F}_2$.

Because each of these surfaces have $\rho(S) = 2$, it follows that $\rho(X) = 3$. However, we are assuming $\phi : X' \to S$ is the only possible $K_{X'}$-negative extremal contraction. Because $-K_{X'}$ is nef, this means that there must be at least two $K_{X'}$-trivial extremal rays. One such is the contraction of F to the curve C. Because the only curves in X' that are $K_{X'}$-trivial are contained in F, there must be another curve $C \subset F$ that is $K_{X'}$-trivial, a contradiction. Therefore, such an X' cannot be a terminal model of a Fano threefold X that is canonical along a curve. (However, it can (and does!) appear if X were only canonical at a point, see Theorem 4.2.2.)

Case C.

If $\phi : X' \to C$ is a fibration with general fiber a smooth del Pezzo surface and $C \cong \mathbb{P}^1$, we can first note that if the general fiber is a surface other than $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$, there exist curves C with $K_{X'} \cdot C = -1$, so we argue as before to conclude d is even. Similarly, if the fiber is $\mathbb{P}^2$, there exist curves C with $K_{X'} \cdot C = -3$, and again we can conclude d is even. Therefore, it suffices to analyze the case when the general fiber is $\mathbb{P}^1 \times \mathbb{P}^1$. Because $\rho(C) = 1$ and ϕ was an extremal contraction, $\rho(X') = 2$. Things already seem a bit suspicious: it is likely that there are divisors D_1 and D_2 whose restriction to each fiber are the different rulings. Those are not linearly equivalent nor are they linearly equivalent to the general fiber F, hence it seems like we must have $\rho(X') \geq 3$, a contradiction.

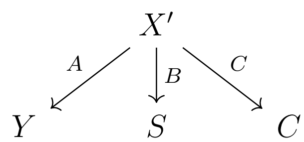
Rigorously, suppose X' does exist. If F was contained in a fiber of ϕ, then there exist many F-trivial curves with $K_{X'} \cdot C = -2$, and on X, $D \cdot \pi(C) \in \mathbb{Z}$. As usual, we consider the relation $dK_X + 4D \sim 0$, so find that d must be even.

Now consider the case that F is ϕ-ample, so $\phi|_F : F \to C$ gives F the structure of a ruled surface over C and contracts only $K_{X'}$-negative curves. Because $\pi|_F$ also contracts F to a curve but contracts only $K_{X'}$-trivial curves, F must have the structure of a product, so $F \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Let $\Gamma \cong \mathbb{P}^1 \times \mathbb{P}^1$ be a fiber of ϕ. We claim that F, Γ, and $K_{X'}$ are non-linearly equivalent divisors, so we must have $\rho(X') \geq 3$, a contradiction. To see the claim, note that $\Gamma|_F$ must be a ruling of F, so $\Gamma|_F \in |\mathcal{O}_F(1,0)|$. Next, observe that $K_{X'}|_F$ is negative on the fibers contracted by ϕ and trivial on the fibers contracted by π. However, these are the two rulings of F, so $K_{X'}|_F \in |\mathcal{O}_F(-2,0)|$. Furthermore, $K_{X'}$ and Γ are certainly not linearly equivalent. Finally, consider $F|_F$. On the fibers of F contracted by π, by the negativity lemma, this must be negative, so $F|_F \in |\mathcal{O}_F(a,-b)|$ for $b > 0$. Therefore, F cannot be linearly equivalent to any linear combination of Γ and $K_{X'}$, so $\rho(X') \geq 3$, so X' cannot exist.

Case 2. $\dim \mathbf{\Delta} = \mathbf{0}$.

Lastly, suppose the locus of log canonical singularities is a point contained in D. We can study the same contractions:



In this case, if F is the exceptional locus of the map $\pi : X' \to X$, the curves in F are all $K_{X'}$ -trivial, so cannot be contracted by a $K_{X'}$-negative contraction. Therefore, the third arrow (Case C) $X' \to C$ is not possible.

Case A.

Because the curves in F are all $K_{X'}$ -trivial, the only possible K-negative divisorial contraction over F is $X' \to Y$ that has at most one dimensional fibers. However, then Y would be terminal, Gorenstein, and $\mathbb{Q}$-factorial, so we can continue the minimal model program on Y. Much of this argument is the same as Case A above. If divisorial contractions happen first, there will exist curves with $K_{X'} \cdot C$ equal to $-1, -2$, or -3 that don't intersect F, and (invoking factoriality of X'), $D \cdot \pi(C) \in \mathbb{Z}$, and we can conclude d is even.

Case B.

The remaining case is if the only $K_{X'}$ negative contraction is a fibration $X' \to S$, and as in Case B above, we can assume every fiber is smooth and isomorphic to $\mathbb{P}^1$.

Therefore, we find ourselves in the situation where $\phi : X' \to S$ is a smooth $\mathbb{P}^1$-bundle over a smooth surface S and by [6, Lemma 2.5], $-K_S$ is big and nef. Exactly as above, we can conclude $S \cong F$ and F is a section of ϕ.

Briefly turning our attention to the map $\pi : X' \to X$, because F is contracted to a point by π, every curve in F is $K_{X'}$-trivial and F-negative. By adjunction, for $C \in F$, $(K_{X'} + F) \cdot C = K_F \cdot C$, fo $F \cdot C = K_F \cdot C$. Therefore, not only is $-K_S = -K_F$ big and nef, but it is ample, so $S = F$ is a Fano surface. If there are any -1 curves on F, taking the intersection product with $\pi * D = \tilde{D} + aF$ implies $a \in \mathbb{Z}$, so for any curve C on X, $D \cdot C \in \mathbb{Z}$. Therefore, for a fiber of ϕ with $K_{X'} \cdot C = -2$, we find that d must be even. Similarly, if $F \cong \mathbb{P}^2$, we find lines with $F \cdot C = -3$, so $D \cdot C \in \mathbb{Z}[1/3]$ and the same conclusion holds.

Therefore, the only remaining case is if $F \cong \mathbb{P}^1 \times \mathbb{P}^1$:

$$\begin{array}{ccc} X' & \xrightarrow{\pi} & X \\ \downarrow{\scriptstyle \phi} & & \\ \mathbb{P}^1 \times \mathbb{P}^1 & & \end{array}$$

and $\pi : X' \to X$ contracts F. Then, X is locally isomorphic to the cone over the anticanonically embdedded $\mathbb{P}^1 \times \mathbb{P}^1$, Theorem 4.2.2, so again d must be even.

□

Ultimately, the odd degree pairs are behaving in a very special way: oddness of the degree is forcing constraints on the threefolds X that can appear. Summarizing the previous two sections, no log canonical threefolds X can appear and the only canonical threefold allowed is $\mathbb{P}^3$. That leads us to the following section.

4.3 Log Terminal Degenerations of $\mathbb{P}^3$

To completely classify the ambient space in the odd degree case, it remains to understand log terminal threefolds X that are degenerations of $\mathbb{P}^3$. We will approach this in general (not only in the odd case). One should note that, although we've focused our attention only on the ambient threefolds X, it is 'enough' to classify only these threefolds: if we know X, we can determine all possible D by varying $D \in |-\frac{d}{4}K_X|$.

There are natural log terminal varieties to consider: weighted projective space, Appendix A. We pause to briefly summarize the case in dimension 2, due to Manetti and Hacking.

Theorem 4.3.1 (Manetti)**.** *If X is a normal, log terminal degeneration of $\mathbb{P}^2$ such that the total space is $\mathbb{Q}$-Gorenstein, then $X \cong \mathbb{P}(p^2, q^2, r^2)$ or a smoothing of such a space, where*

$$3pqr = p^2 + q^2 + r^2.$$

Futhermore, all such varieties admit a $\mathbb{Q}$-Gorenstein smoothing to $\mathbb{P}^2$.

In addition to the theorem, we can describe all solutions with an infinite graph:

Theorem 4.3.2 (Hacking)**.** *All solutions to*

$$3pqr = p^2 + q^2 + r^2$$

can be obtained by starting with the obvious solution $(1, 1, 1)$ and performing a sequence of mutations: if (p, q, r) is a solution, then $(p, q, 3pq - r)$ is a solution.

One could hope for such an analog in the three dimensional case, although that seems out of reach: the proof of this theorem heavily relies on the classification of (all) surface singularities. However, there are partial results, using properties of weighted projective spaces, summarized in Appendix A.

Proposition 4.3.3. *If $\mathbb{P}(a, b, c, d)$ admits a $\mathbb{Q}$-Gorenstein smoothing to $\mathbb{P}^3$, then*

$$64abcd|(a + b + c + d)^3.$$

Proof. In order for $X = \mathbb{P}(a, b, c, d)$ to have a $\mathbb{Q}$-Gorenstein smoothing, $K_X^3 = K_{\mathbb{P}^3}^3 = -64$. But, $\mathcal{O}(K_X) = \mathcal{O}(-a - b - c - d)$, and

$$K_X^3 = \frac{(-a - b - c - d)^3}{abcd}.$$

Therefore, we must have

$$64abcd = (a + b + c + d)^3.$$

However, because the weighted projective space $\mathbb{P}(a, b, c, d)$ can potentially be normalized by dividing by a common factor of a, b, c, d, the proposition follows. □

Remark 4.3.4. This formula is distinctly different from the two dimensional version; indeed the previous version has been simplified from this form. Analyzing the square of the canonical divisor would say $X = \mathbb{P}(a, b, c)$ could only smooth to $\mathbb{P}^2$ if

$$9abc = (a + b + c)^2.$$

Taking square roots of both sides and applying a little bit of number theory implies that a, b, and c have to be perfect squares. Setting $a = p^2, b = q^2, c = r^2$ gives the above version.

Remark 4.3.5. Although having such a formula seems like an easy test to determine smoothability, the divisibility complicates matters. For example, $\mathbb{P}(1, 1, 1, 9)$ (the cone over the degree 9 embedding of $\mathbb{P}^2$) admits a smoothing to $\mathbb{P}^3$, however $(1, 1, 1, 9)$ is not a solution to the equation

$$64abcd = (a + b + c + d)^3.$$

But, a multiple of it is: $\mathbb{P}(3, 3, 3, 27)$ is isomorphic to $\mathbb{P}(1, 1, 1, 9)$, and $(3, 3, 3, 27)$ does satisfy the equation. However, $(1, 1, 1, 9)$ does satisfy the relation

$$64abcd|(a + b + c + d)^3.$$

One could make the immediate generalization to n-dimensional weighted projective spaces:

Proposition 4.3.6. *If $\mathbb{P}(a_0, a_1, \ldots, a_n)$ admits a $\mathbb{Q}$-Gorenstein smoothing to $\mathbb{P}^n$, then*

$$(n + 1)^n \Pi a_i | (\sum a_i)^n.$$

The proof is the same as above.

With an equation like this, one immediately questions if there are infinitely many solutions, or if there is a procedure for obtaining solutions as in the two dimensional case. There is certainly an infinite family of solutions, which geometrically makes sense. If we have a degeneration of $\mathbb{P}^2$ to such a weighted projective space, it should induce a degeneration of $\mathbb{P}^3$ to some sort of cone over that weighted projective space. This is the content of the following propositions, stated in the three-dimensional case and in general.

Proposition 4.3.7. *If $\mathbb{P}(a, b, c)$ admits a smoothing to $\mathbb{P}^2$ (so $a = p^2, b = q^2, c = r^2$ in the previous theorem), then $d = \sqrt{abc} \in \mathbb{Z}$ and $P(a, b, c, d)$ satisfies the condition*

$$64abcd|(a + b + c + d)^3.$$

Proof. Because a, b, c are perfect squares, we have $d \in \mathbb{Z}$. But, because $\mathbb{P}(a, b, c)$ admits a smoothing to $\mathbb{P}^2$,

$$3d = a + b + c$$

so

$$(a + b + c + d)^3 = (4d)^3 = 64d^3 = 64abcd.$$

□

Proposition 4.3.8. *If $\mathbb{P}(a_0, a_1, \ldots, a_n)$ satisfies*

$$(n + 1)^n \Pi a_i = (\sum a_i)^n,$$

then $b = (\Pi a_i)^{1/n} \in \mathbb{Z}$ and $\mathbb{P}(a_0, a_1, \ldots, a_n, b)$ satisfies

$$(n + 2)^{n+1} b \Pi a_i = (b + \sum a_i)^{n+1}.$$

The proof is the same as above.

These propositions are weaker than we might like, though. They only imply that these weighted projective spaces satisfy the necessary conditions to admit a smoothing to $\mathbb{P}^n$, not that it is sufficient.

At least in the three dimensional case, to understand if these weighted projective spaces could smooth to $\mathbb{P}^3$, one would have to study the versal deformation theory of these cyclic quotient singularities. It is enough to work locally: by standard cohomology calculations for weighted projective space, local to global deformations are unobstructed, as $H^2(X, \mathcal{T}_X) = 0$.

Studying this deformation theory could lead to a complete classification of the boundary threefolds, at least in the odd degree case.

Chapter 5

THE MODULI SPACE

5.1 Singularities and Boundedness

In the consideration of moduli of pairs (X, D), we can use the classification in the previous section to show that there are no such strictly slc varieties with d odd.

First note that if (X, D) is an H-ϵ stable pair of degree d and $(X^\nu, \Delta + D^\nu)$ the normalization, then the locus of log canonical singularities of X^ν cannot be contained in $\operatorname{Supp} \Delta + D^\nu$. Also, if X has a finite number of log canonical singularities, by [1][Theorem 17.4], X must be normal and have only one such singularity. Using this observation and the work from Chapter 4, we can prove the following theorem.

Theorem 5.1.1. *If (X, D) is an H-ϵ stable pair of degree d, and d is odd, then $(X, \frac{4}{d}D)$ is semi log terminal.*

Proof. In the definition of H-ϵ stable pair, we require that $(X, (\frac{4}{d} + \epsilon)D)$ is slc for all ϵ sufficiently small. If $(X, (\frac{4}{d})D)$ is strictly slc, then X must be strictly slc and the locus of log canonical singularities cannot be contained in $\operatorname{Supp} D$. Therefore, we can use the classification of strictly log canonical varieties from the previous section to analyze the possible H-ϵ strictly slc pairs.

First consider the case of normal varieties. If the locus of log canonical singularities is finite (and hence a single point), then either X is a cone as in Theorem 4.1.2 or birational to a fibration $Y \to S$ as in Theorem 4.1.12. In the first case, we can choose a ruling l of X (an image of one of the $\mathbb{P}^1$s contracted from $Y \to S$ on X) sufficiently generally so l intersects

D where D is a Cartier divisor. Then, $D \cdot C \in \mathbb{Z}$, but $D \cdot C \equiv \frac{d}{4} K_X \cdot C = \frac{d}{4}$, hence we must have $4|d$.

In the second case, we consider the possible fibrations. If $Y \to S$ is a fibration of relative dimension 1, we have $S \cong E$ where E is an exceptional divisor over X with discrepancy $a(E, X) = -1$. Choosing a sufficiently general fiber $l \cong \mathbb{P}^1$, it intersects E at one point and $K_Y \cdot l = -2$, so its image l' in X has $K_X \cdot l' = -1$. Choosing l to instersect D where it is Cartier gives the same conclusion as above: we must have $4|d$.

If $Y \to S$ is a fibration of relative dimension 2, S is a curve and the image of E is S. In the proof, one sees that every curve in Y intersects E, hence every curve in a fiber L of the fibration intersects $E \cap L$. Because the general fiber is a smooth Fano surface, this implies that the general fiber $L \cong \mathbb{P}^2$ and, for l a line in a general fiber, $K_Y \cdot l = K_L \cdot l = -3$. Because $E \cdot l > 0$ and $(K_Y + E) \cdot l < 0$, we have that $K_X \cdot l' = -1$ or -2. In either case, choosing l to intersect D where it is Cartier implies that $4|d$ or $2|d$.

If the locus of log canonical singularities is one-dimensional, then we can use Theorem 4.1.13 and analyze the cases of the resulting fibration $Y \to S$ exactly as above.

Next, we turn our attention to non-normal slc pairs satisfying the above conditions. Because $(X, (\frac{4}{d} + \epsilon)D)$ is slc, the locus of strictly log canonical singularities cannot be contained in the non-normal locus. Let $\nu : X^\nu \to X$ be the normalization with conductor Δ so $K_{X^\nu} + \Delta + D^\nu \sim \nu^*(K_X + D)$. Because $K_{X^\nu} + \Delta$ is anti-ample, by [1, Theorem 17.4], the strictly log canonical locus must intersect Δ but not be contained in it. By Theorem 4.1.13 and the argument above, replacing K_X with $K_{X^\nu} + \Delta$, we get the same conclusion that we must have $2|d$.

Therefore, for odd degree d, the moduli space of H-ϵ stable pairs (X, D) as above contains no slc pairs. In other words, only pairs (X, D) such that $(X, (\frac{4}{d} + \epsilon)D)$ is slt can occur in the moduli space.

□

This has a number of interesting consequences. First, a corollary of Theorem 4.0.1:

Corollary 5.1.2. *For d odd, the varieties X occurring as a degeneration of $\mathbb{P}^3$ in an H-ϵ stable pair are rational.*

As mentioned at the beginning of Chapter 4, we are also interested in the boundedness of families of H-ϵ stable pairs. For fixed ϵ, the family of H-ϵ stable pairs is bounded, but allowing ϵ to be arbitrary allows one to show that families of H-ϵ stable pairs over a punctured base can be completed in a unique way (Theorem 3.2.5). However, in light of Theorem 5.1.1, we can say something about boundedness. The following theorem is a special case of [13, Corollary 1.7] for threefolds.

Theorem 5.1.3. *The set of H-ϵ stable pairs (X, D) of fixed odd degree d form a bounded family.*

Proof. Restricting to the normal case, by Theorem 3.1, $(X, \frac{4}{d}D)$ is klt, and because of the assumption that $dK_X + 4D \sim 0$, the pairs $(X, \frac{4}{d}D)$ are ϵ-log terminal because $dK_X + 4D \sim 0$ is linear equivalence (not just numerical). Then, $-K_X$ is ample and $K_X + \frac{4}{d}D$ is numericaly trivial by assumption, hence by [13, Corollary 1.7], form a bounded family.

We can restrict ourselves to the normal case because the non-normal pairs are in bijection with normal pairs and a certain involution as in [19, Theorem 5.13].

□

5.2 Algebraicity

Using recent work, we can further analyze the structure of the moduli space of H-ϵ stable pairs. We study *Kollár families* of pairs, $\mathbb{Q}$-Gorenstein families where $\omega_{\mathcal{X}}^{[n]}$ commutes with base change for all n.

There seem to be more than one avenue to show that the moduli space of H-ϵ stable pairs is an algebraic stack. Using [12, c.f. Theorem 4.4] and following his work, one can show that the moduli space is indeed an algebraic stack.

Definition 5.2.1. Let $p \in X$ be a germ of an slc variety. Define the index of p in X to be the minimal $N > 0$ such that NK_X is Cartier. Let $Z \to X$ be the canonical covering $Z = \mathrm{Spec}_X \mathcal{O} \oplus \mathcal{O}(K_X) \oplus \cdots \oplus \mathcal{O}((N-1)K_X)$, a μ_N quotient (cf. [27]). A deformation $\mathcal{X}/S$ of X is $\mathbb{Q}$-Gorenstein if there is a μ_N-equivariant deformation $\mathcal{Z}/S$ of Z whose quotient is $\mathcal{X}/S$.

Definition 5.2.2. Let $\mathcal{X}/S$ be a flat family of slc varieties. We say that $\mathcal{X}$ is weakly $\mathbb{Q}$-Gorenstein if, for some $N > 0$, $\omega_{\mathcal{X}/S}^{[N]}$ is invertible. The minimal such N is called the index of $\mathcal{X}$.

The following lemmas show that $\mathbb{Q}$-Gorenstein implies weakly $\mathbb{Q}$-Gorenstein and that the conditions are equivalent if the general fiber is canonical and the base is a curve.

Lemma 5.2.3. *Let $p \in X$ be a germ of an slc variety. A $\mathbb{Q}$-Gorenstein deformation $\mathcal{X}/S$ of X of index N is weakly $\mathbb{Q}$-Gorenstein of index N.*

Proof. This follows directly from [12, Lemma 3.3]. □

Lemma 5.2.4. *Let $\mathcal{X}/T$ be a flat family of slc varieties over the germ of a curve. If the general fiber has canonical singularities and $K_{\mathcal{X}}$ is $\mathbb{Q}$-Cartier, then $\mathcal{X}/S$ is a $\mathbb{Q}$-Gorenstein deformation of $\mathcal{X}_0$.*

Proof. Using the stronger inversion of adjunction result in [25, Lemma 2.10], the proof of Lemma 3.4 in [12] applies directly. □

Many properties of $\mathbb{Q}$-Gorenstein deformations are collected in [12, Section 3]. By considering the canonical covering of higher index, we can show that studying deformations of the pair (X, D) amounts to studying deformations of X because the presence of the divisor D does not add any further obstructions. This follows by taking the canonical covering Z

corresponding to $4NK_X$, multiplying the index by 4: the relationship $dK_X + 4D \sim 0$ implies that D_Z is Cartier on Z. We will call such a deformation a $4-\mathbb{Q}$-Gorenstein deformation.

Theorem 5.2.5. *Let $(\mathcal{X}, \mathcal{D})/A$ be a $\mathbb{Q}$-Gorenstein family H-ϵ stable pairs. Let $A' \to A$ be an infinitesimal extension and $\mathcal{X}' \to A'$ a $4-\mathbb{Q}$-Gorenstein deformation of $\mathcal{X}/A$. Then, there exists a $\mathbb{Q}$-Gorenstein deformation $(\mathcal{X}', \mathcal{D}')/A'$ of $(\mathcal{X}, \mathcal{D})/A$.*

Proof. Using the following lemma in place of Lemma 3.14 in the proof of Theorem 3.12 in [12], the same proof holds. □

Lemma 5.2.6. *Let (X, D) be an H-ϵ stable pair. Then, $H^1(X, \mathcal{O}_X(D)) = 0$.*

Proof. By Serre duality, $H^1(X, \mathcal{O}_X(D)) = H^2(X, \mathcal{O}_X(K_X - D))^\vee$, and $-(K_X - D)$ is ample. If X is log terminal, this follows from Kodaira vanishing. If X is log canonical, we can use a version of Kodaira vanishing in[11, Theorem 1.2] to get the same conclusion. If X is not normal, we can use an even stronger version of Kodaira vanishing in [23, Corollary 1.3] to conclude the same thing. This result assumes that X is Cohen-Macaulay, but this is automatic by [20, Corollary 7.13] for X in an H-ϵ stable pair because X necessarily admits a smoothing to $\mathbb{P}^3$.

□

Next, in the definition of H-ϵ-stable pairs, we require that (X, D) has a smoothing to $(\mathbb{P}^3, S)$. Therefore, we are interested only in certain 'smoothable' deformations of (X, D), made precise below.

Definition 5.2.7. Let $(X, D)/\mathbb{C}$ be an H-ϵ stable pair of degree d. Let $(\mathcal{X}^u, \mathcal{D}^u)/S_0$ be a versal $\mathbb{Q}$-Gorenstein deformation of (X, D), where S_0 is finite type over $\mathbb{C}$. Let $S_1 \subset S_0$ be the open subscheme where the fibers of $\mathcal{X}^u$ over S_0 are isomorphic to $\mathbb{P}^3$ and S_2 the (scheme-theoretic) closure of S_1 in S_0. A $\mathbb{Q}$-Gorenstein deformation of (X, D) is said to be smoothable if it can be obtained by pullback from the deformation $(\mathcal{X}^u, \mathcal{D}^u) \times_{S_0} S_2 \to 0 \in S_2$.

For odd degree d, the necessity of this definition is an interesting question. Do there exist deformations where $S_2 \subsetneq S_0$? In the case of plane curves in $\mathbb{P}^2$, Hacking was able to show the appropriate analog of this definition is actually vacuous. The obstruction to smoothing is known to be a certain cohomology group, although computation of that cohomology group over all H-ϵ stable pairs does not seem feasible.

In any case, we are finally able to define a moduli functor.

Definition 5.2.8. Let Sch be the category of noetherian schemes over $\mathbb{C}$. For $d \in \mathbb{N}$, we define the stack $\mathcal{M}_d \to Sch$ by

$$\mathcal{M}_d(S) = \left\{ (\mathcal{X}, \mathcal{D})/S \;\middle|\; \begin{array}{c} (\mathcal{X}, \mathcal{D})/S \text{ is a } \mathbb{Q}\text{-Gorenstein smoothable family} \\ \text{of H-}\epsilon \text{ stable pairs of degree d} \end{array} \right\}$$

In [12], it is shown that this $\mathbb{Q}$-Gorenstein deformation condition is equivalent to requiring the Kollár condition on families.

Using the deformation theory in [12, Section 3] and Theorem 3.2.5, we deduce the following theorem.

Theorem 5.2.9. *The moduli space of H-ϵ stable pairs of odd degree d is a proper Deligne-Mumford stack.*

Remark 5.2.10. We pause here to address what is known for the even degree d case. At present, the structure of the proof is:

H-ϵ stable pairs of odd degree have at worst slt singularities

$\Downarrow$

H-ϵ stable pairs of odd degree at ϵ-slt

$\Downarrow$

H-ϵ stable pairs of odd degree form a bounded family

$$\Downarrow (*)$$

$\mathcal{M}_d$ is a proper Deligne Mumford stack

The proof of properness does not rely on oddness of degree; in fact, the only part where oddness of degree is necessary is in the proof of boundedness. Therefore, if we were able to show boundedness in another way for even degree pairs, the rest of the proof is already complete and we would know $\mathcal{M}_d$ is a proper DM stack for all degree d.

One could define an alternative moduli functor via the work of Abramovich and Hasset [2].

We can consider the substack of the algebraic stack $\mathcal{K}^{\omega}_{\text{slc}}$ (cf. [2, Section 5]) satisfying the locally closed condition $dK_X + 4D \sim 0$ [22, Lemma 5.8]. This condition is algebraic, so we could define a variant $\mathcal{M}'_d$ of $\mathcal{M}_d$ as this substack. However, it is not clear is the presence of the divisor D has an effect on the structure of this stack.

The moduli stack is worth much further investigation and many questions remain. In [12], it was shown that the moduli space of H-ϵ stable pair (representing curves in $\mathbb{P}^2$) was actually smooth for d not divisible by 3. One could ask the same question for H-ϵ stable pairs in this context, or at least how many connected components appear. If the moduli space is not smooth, it suggests the presence of pairs (X, D) that satisfy the definition of H-ϵ stable pairs but do not admit smoothings to $\mathbb{P}^3$ which would be of interest themselves. At that point, it would be of interest to remove the smoothing requirement in the definition of H-ϵ stable pair.

In the future, we hope to explicitly determine the boundary for H-ϵ stable pairs of degree 5. As indicated above, this is equivalent to classifying the possible log terminal degenerations of $\mathbb{P}^3$ that can appear on the boundary of such pairs.

BIBLIOGRAPHY

[1] *Flips and abundance for algebraic threefolds.* Société Mathématique de France, Paris, 1992. Papers from the Second Summer Seminar on Algebraic Geometry held at the University of Utah, Salt Lake City, Utah, August 1991, Astérisque No. 211 (1992) (1992).

[2] Dan Abramovich and Brendan Hassett. Stable varieties with a twist. In *Classification of algebraic varieties*, EMS Ser. Congr. Rep., pages 1–38. Eur. Math. Soc., Zürich, 2011.

[3] Valery Alexeev. Moduli spaces $M_{g,n}(W)$ for surfaces. In *Higher-dimensional complex varieties (Trento, 1994)*, pages 1–22. de Gruyter, Berlin, 1996.

[4] Mauro C. Beltrametti, Andreas Höring, and Carla Novelli. Fano varieties with small non-klt locus. *Int. Math. Res. Not. IMRN*, (11):3094–3120, 2015.

[5] Caucher Birkar, Paolo Cascini, Christopher D. Hacon, and James McKernan. Existence of minimal models for varieties of log general type. *J. Amer. Math. Soc.*, 23(2):405–468, 2010.

[6] Cinzia Casagrande, Priska Jahnke, and Ivo Radloff. On the Picard number of almost Fano threefolds with pseudo-index > 1. *Internat. J. Math.*, 19(2):173–191, 2008.

[7] Steven Cutkosky. Elementary contractions of Gorenstein threefolds. *Math. Ann.*, 280(3):521–525, 1988.

[8] Olivier Debarre. *Higher-dimensional algebraic geometry.* Universitext. Springer-Verlag, New York, 2001.

[9] Igor Dolgachev. Weighted projective varieties. In *Group actions and vector fields (Vancouver, B.C., 1981)*, volume 956 of *Lecture Notes in Math.*, pages 34–71. Springer, Berlin, 1982.

[10] A. R. Fletcher. Inverting Reid's exact plurigenera formula. *Math. Ann.*, 284(4):617–629, 1989.

[11] Osamu Fujino. Kodaira vanishing theorem for log-canonical and semi-log-canonical pairs. *Proc. Japan Acad. Ser. A Math. Sci.*, 91(8):112–117, 2015.

[12] Paul Hacking. Compact moduli of plane curves. *Duke Math. J.*, 124(2):213–257, 2004.

[13] Christopher D. Hacon, James McKernan, and Chenyang Xu. ACC for log canonical thresholds. *Ann. of Math. (2)*, 180(2):523–571, 2014.

[14] Christopher D. Hacon, James McKernan, and Chenyang Xu. Boundedness of moduli of varieties of general type, 2014. arXiv:1412.1186v1 [math.AG].

[15] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[16] Shihoko Ishii. Quasi-Gorenstein Fano 3-folds with isolated nonrational loci. *Compositio Math.*, 77(3):335–341, 1991.

[17] Yujiro Kawamata, Katsumi Matsuda, and Kenji Matsuki. Introduction to the minimal model problem. In *Algebraic geometry, Sendai, 1985*, volume 10 of *Adv. Stud. Pure Math.*, pages 283–360. North-Holland, Amsterdam, 1987.

[18] J. Kollár and N. I. Shepherd-Barron. Threefolds and deformations of surface singularities. *Invent. Math.*, 91(2):299–338, 1988.

[19] János Kollár. *Singularities of the minimal model program*, volume 200 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2013. With a collaboration of Sándor Kovács.

[20] János Kollár and Sándor J. Kovács. Log canonical singularities are Du Bois. *J. Amer. Math. Soc.*, 23(3):791–813, 2010.

[21] János Kollár and Shigefumi Mori. *Birational geometry of algebraic varieties*, volume 134 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1998.

[22] Sándor J. Kovács. Young person's guide to moduli of higher dimensional varieties. In *Algebraic geometry—Seattle 2005. Part 2*, volume 80 of *Proc. Sympos. Pure Math.*, pages 711–743. Amer. Math. Soc., Providence, RI, 2009.

[23] Sándor J. Kovács, Karl Schwede, and Karen E. Smith. The canonical sheaf of Du Bois singularities. *Adv. Math.*, 224(4):1618–1640, 2010.

[24] Marco Manetti. Normal degenerations of the complex projective plane. *J. Reine Angew. Math.*, 419:89–118, 1991.

[25] Zsolt Patakfalvi. Fibered stable varieties. *Trans. Amer. Math. Soc.*, 368(3):1837–1869, 2016.

[26] Julie Rana. A boundary divisor in the moduli spaces of stable quintic surfaces. *Internat. J. Math.*, 28(4):1750021, 61, 2017.

[27] Miles Reid. Young person's guide to canonical singularities. In *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, volume 46 of *Proc. Sympos. Pure Math.*, pages 345–414. Amer. Math. Soc., Providence, RI, 1987.

[28] Burt Totaro. Rationality does not specialise among terminal varieties. *Math. Proc. Cambridge Philos. Soc.*, 161(1):13–15, 2016.

Appendix A

WEIGHTED PROJECTIVE SPACE

We begin by recalling the classic set-up of weighted projective space. Given a set of positive integers $Q = \{q_0, q_1, \ldots, q_n\}$, consider the polynomial ring $S(Q) = \mathbb{C}[x_0, x_1, \ldots, x_n]$ where each monomial x_i has weight q_i. Define $\mathbb{P}(q_0, q_1, \ldots, q_n) = \operatorname{Proj} S(Q)$.

Certainly $\mathbb{P}(1, 1, \ldots, 1) = \mathbb{P}^n$, but in general, $\mathbb{P}(q_0, q_1, \ldots, q_n)$ is a singular projective variety. Consider the weighted projective space $\mathbb{P}(1, 1, n)$. The coordinate ring $S(Q) = k[x, y, z]$ has monomials x and y of weight 1 and z of weight n. Then, the degree n monomials are $x^n, x^{n-1}y, \ldots, xy^{n-1}, y^n, z$. The inclusion $R = k[x^n, x^{n-1}y, \ldots, xy^{n-1}, y^n, z] \to k[x, y, z]$ induces an isomorphism of varieties $\mathbb{P}(1, 1, n) \cong \operatorname{Proj} R$, but $\operatorname{Proj} R$ is the cone over the rational normal curve of degree n.

We could alternatively define weighted projective space as a quotient of affine space. If $\mathbb{A}^{n+1}$ has coordinates $x_0, x_1, \ldots, x_n$, then the weighted projective space $\mathbb{P}(q_0, q_1, \ldots, q_n)$ is $\mathbb{A}^{n+1} - \{0\}/\sigma$, where $\sigma(x_0, x_1, \ldots, x_n) = (\zeta_{q_0} x_0, \zeta_{q_1} x_1, \ldots, \zeta_{q_n} x_n)$, with ζ_{q_i} a q_i^{th} root of unity. Using this description, we can see the standard affine open subsets: for each variable x_i, $D(x_i) = \mathbb{A}^n/\sigma_i$, where the coordinates on $\mathbb{A}^n$ are $(y_0, y_1, \ldots, \hat{y}_i, \ldots, y_n) = (x_0/x_i, x_1/x_i, \ldots, x_n/x_i)$, and $\sigma(y_0, \ldots, y_n) = (\zeta_{q_i}^{q_0} y_0, \ldots, \zeta_{q_i}^{q_n} y_n)$.

From this description, we see that if $q_i = 1$ for any i, $D(x_i) = \mathbb{A}^n$.

We pause to recall more facts about weighted projective space.

(i) Weighted projective spaces have cyclic quotient singularities, so are at worst log terminal varieties.

(ii) Weighted projective spaces are $\mathbb{Q}$-Fano varieties with Picard number 1.

(iii) The hyperplane section $\mathcal{O}(H)$ need not be a Cartier divisor, however, intersection theory is well understood: for example, if $\mathcal{O}(d_1)$ and $\mathcal{O}(d_2)$ are two divisors on $\mathbb{P}(q_0, q_1, q_2)$,

$$\mathcal{O}(d_1) \cdot \mathcal{O}(d_2) = \frac{d_1 d_2}{q_0 q_1 q_2}.$$

(iv) The canonical divisor K_X on the weighted projective space $X = \mathbb{P}(q_0, q_1, \ldots, q_n)$ is given by $\mathcal{O}(K_X) = O(-\sum q_i)$. The top intersection of the canonical divisor is

$$\mathcal{O}(K_X)^n = \frac{(-\sum q_i)^n}{\Pi q_i}.$$

In the context of this paper, weighted projective spaces give natural degenerations of $\mathbb{P}^3$. We can also use weighted projective spaces to form weighted blow ups, a standard tool in constructing stable limits of families.

Definition A.0.1. The blow up of $\mathbb{A}^n$ with well-formed weights $(a_1, \ldots, a_n)$ is the closure of the graph of $\phi : \mathbb{A}^n \dashrightarrow \mathbb{P}(a_1, \ldots, a_n) \subset \mathbb{A}^n \times \mathbb{P}(a_1, \ldots, a_n)$, where $\phi : \mathbb{A}^n \dashrightarrow \mathbb{P}(a_1, \ldots, a_n)$ is the map $(x_1, \ldots, x_n) \to [x_1 : \cdots : x_n]$.

Alternatively, one could say the weighted blow up of $\mathbb{A}^n$ is covered by affine charts

$$U_1 = \mathbb{A}^n/\sigma, \quad \sigma(y_1, \ldots, y_n) = (\zeta_{a_i}^{-a_1} y_1, \ldots, \zeta_{a_i} y_i, \ldots, \zeta_{a_i}^{-a_n} y_n),$$

where ζ_{a_i} is an i^{th} root of unity. In the i^{th} chart, these coordinates are related to the coordinates on $\mathbb{A}^n$ by $x_i = y_i^{a_i}$ and $x_j = y_j y_i^{a_j}$ for $j \neq i$.

If $n = 2$, any weighted projective space $\mathbb{P}(a_1, a_2) \cong \mathbb{P}^1$, so the weighted blow up takes a particularly nice form. In fact, it can be realized as a standard blow up of a non-reduced ideal. For the benefit of the reader, we work out the following claim.

Proposition A.0.2. *Assume p and q are relatively prime. The (p, q)-blow up of $\mathbb{A}^2_{x,y}$ is the normalization of the blow up of the non-reduced ideal (x^q, y^p).*

Proof. The standard blow up of the ideal (x^q, y^p) is given by $V(x^q v - y^p u) \subset \mathbb{A}^2_{x,y} \times \mathbb{P}^1_{u,v}$. This has two affine charts corresponding to the non-vanishing of u and v.

Let $V_1 = D(u) = V(x^q v - y^p) \subset \mathbb{A}^3$, with affine coordinate ring $R_1 = k[x, y, v]/(x^q v - y^p)$. Let $V_2 = D(v) = V(x^q - y^p u) \subset \mathbb{A}^3$, with affine coordinate ring $R_2 = k[x, y, u]/(x^q - y^p u)$.

To show that this is the (p, q) weighted blow up, we consider the charts in the definition above. The weighted blow up is covered by $U_1 = \mathbb{A}^2_{r,s}/\sigma$, where $\sigma(r, s) = (\zeta_p r, \zeta_p^{-q} s)$, and $U_2 = \mathbb{A}^2_{w,z}/\tau$, where $\tau(w, z) = (\zeta_q^{-p} w, \zeta_q z)$. These coordinates are related to x and y on charts by $x = r^p$, $y = sr^q$, and $x = wv^p$, $y = v^q$.

In fact, these equations tell us that V_1 is the normalization of U_1. The affine coordinate ring of V_1 is the ring of invariants $k[r, s]^\sigma = k[r^p, s^p, sr^q, \ldots] \subset k[r, s]$. Now consider the map from R_1 to $k[r, s]^\sigma$ induced by these coordinates:

$$\begin{aligned} k[x, y, v]/(x^q v - y^p) &\to k[r, s]^\sigma \\ x &\mapsto r^p \\ y &\mapsto sr^q \\ v &\mapsto s^p. \end{aligned}$$

A check shows any element in $k[r, s]^\sigma$ is algebraic over R_1, and $k[r, s]^\sigma$ is algebraically closed, so V_1 is the normalization of U_1. Similarly, the same holds for U_2.

Therefore, the weighted blow up is the normalization of a standard (non-weighted) blow up.

□

One can work out similar details for $n > 2$ or for weighted blow-ups of cyclic quotient singularities.