©Copyright 2014
Huy V. Tran

The regularity of Loewner curves

Huy V. Tran

A dissertation submitted in partial fulfillment of the
requirements for the degree of

Doctor of Philosophy

University of Washington

2014

Reading Committee:

Steffen Rohde, Chair

Donald E. Marshall

Boris Solomyak

Program Authorized to Offer Degree:
Mathematics

University of Washington

Abstract

The regularity of Loewner curves

Huy V. Tran

Chair of the Supervisory Committee:
Professor Steffen Rohde
Mathematics

The Loewner differential equation, a classical tool that has attracted recent attention due to Schramm-Loewner evolution (SLE), provides a unique way of encoding a simple 2-dimensional curve into a continuous 1-dimensional driving function. In this thesis we study the curve in three cases according to the regularity of driving function: weakly Hölder-1/2, Hölder-1/2 with norm less than 4 and C^α with $\alpha \in (1/2, \infty)$. In the first case, given the existence of the curve we show that the standard algorithm simulating the curve converges in a strong sense. One direct application is for simulating SLE. In the second case, we give another proof of Marshall, Rohde [26] and Lind[22] in which the curve exists and is a quasi-arc. A sufficient condition for the rectifiability of the curve is also given. In the final case, we show that the Loewner curve is in $C^{\alpha+1/2}$. The thesis is a combination of three projects [39], [32] and [21] which are joint work with Joan Lind, Steffen Rohde and Michel Zinsmeister.

TABLE OF CONTENTS

LIST OF FIGURES

ACKNOWLEDGMENTS

First of all I would like to thank everybody whom I have met, discussed or interact with during my time as a graduate student. I have received advice, help, inspiration and learned things from many people to become a better mathematician and person.

I am very thankful to Professor Michel Zinsmeister who met me many years ago in Vietnam. It was my first time speaking English with a (mathematician) foreigner and I was amazed how friendly and humorous he is. My interest in complex analysis came from him and I have became his student since then. I appreciated his hospitality during my time in France when he brought me to interesting conferences and seminars where I saw many famous mathematicians that I had known from textbooks.

I wish to thank my advisor, Professor Steffen Rohde, for countless hours teaching and guiding me. He is like my father, my counselor. Many times I walked into his office with just one or two questions and a miserable feeling about the progress of my work. After hours of discussion I left with more questions but gaining more confidence and with the hope that things would work out as long as I kept working. I always enjoy having discussions with him.

My coauthors Joan Lind, Steffen Rohde, Michel Zinsmeister all played essential parts in the research behind this thesis. I owe thanks to Don Marshall, Elliot Paquette, Fredrik Johansson Viklund, Brent Werness for carefully proofreading many parts in this thesis. If the reader find any mistakes here, that's the author's fault.

During my time as a graduate student I had the privilege of attending the Interaction between Analysis and Geometry programme at the Institute for Pure and Applied Mathematics (IPAM) at UCLA in Los Angeles. My joint work with Joan Lind started when I was there. I most gratefully acknowledge the support of this institution.

I wish also to thank Professors Zhen-Qing Chen, Mathias Drton, Donald Marshall, Steffen Rohde, Boris Solomyak for serving on my supervisory committee.

Finally, I would like to thank my family: my father and mother, my wife Linh Tran, for their continuous, invaluable, and essential support.

DEDICATION

To my parents, my wife Linh Tran, and my son Tien-Minh (Oscar)

Chapter 1
INTRODUCTION

The chordal Loewner differential equation in the upper half plane $\mathbb{H}$

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \lambda_t}, \qquad g_0(z) = z, \tag{1.1}$$

which will be reviewed in Chapter 2, provides a one-to-one correspondence between certain decreasing families of simply connected subdomains $\mathbb{H}\setminus K_t$ of the upper half plane, and real-valued continuous functions λ_t. Initially developed as a tool to study extremal problems in complex analysis [24], it has become an important tool in probability theory, based on Oded Schramm's insight [33] that Brownian motion arises naturally as the driving function λ_t in various settings of random sets K_t.

Consider triples (Ω, a, b) with a simply connected domain Ω and two marked points a, b on the boundary. Suppose that for each triple (Ω, a, b), we consider a random continuous curve $\gamma_{(\Omega,a,b)}$ that goes from a to b in $\overline{\Omega}$. If $\partial\Omega$ is not locally connected, we consider a and b to be prime ends. In this case, the curve $\gamma_{\Omega,a,b}$ is the image of a continuous curve going from -1 to 1 in $\overline{\mathbb{D}}$ under a conformal $\phi : \mathbb{D} \to \Omega$ that sends a and b to -1 and 1 respectively.

Definition 1.0.1. *We say that the family of random continuous curves* $\gamma_{(\Omega,a,b)}$ *is* conformally invariant *if for any* (Ω, a, b) *and any conformal map* $\phi : \Omega \to \mathbb{C}$ *,*

$$\phi \circ \gamma_{(\Omega,a,b)} \text{ has the same law as } \phi_{(\phi(\Omega),\phi(a),\phi(b))}.$$

Definition 1.0.2. *We say that the family of random continuous curves* $\gamma_{(\Omega,a,b)}$ *satisfies the domain Markov property if for every* (Ω, a, b)*, and every* $t > 0$*, the law of the curve* $\gamma[t, \infty)$ *conditionally on* $\gamma[0, t]$ *is the same as the law of* $\gamma_{(\Omega_t,\gamma_t,b)}$*, where* Ω_t *is the connected component of* $\Omega\setminus\gamma_t$ *containing* b *in its boundary.*

In 1999, Schramm introduced a one-parameter family SLE_κ with $\kappa \geq 0$ of measures that satisfy the conformal invariance and Markov domain property. This family of measures is a candidate for the scaling limits of many discrete models in statistical physics. When translated in the upper half-plane $\mathbb{H}$, the SLE_κ is the solution to (1.1) with $\lambda_t = \sqrt{\kappa} B_t$, where B_t is a standard Brownian motion. And thanks to this connection to Brownian motion, SLE can be studied via standard techniques such as stochastic calculus. The SLE_κ is almost surely a curve[1] for all $\kappa \geq 0$ ([31], [17]) and it exhibits phase transitions:

- For $\kappa \in [0, 4]$, SLE_κ is almost surely a simple path contained in $\mathbb{H} \cup \{0\}$.
- For $\kappa \in (4, 8)$, SLE_κ is almost surely a non-simple path.
- For $\kappa \geq 8$, SLE_κ is almost surely a space-filling curve.

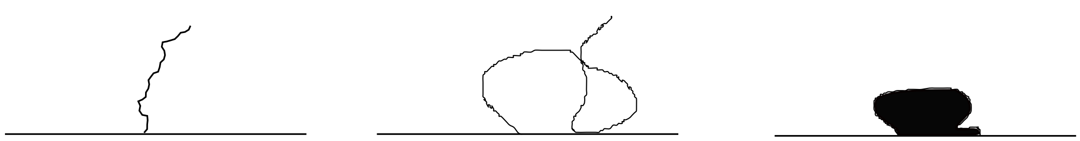


Figure 1.1: Illustration for SLE curves. See Figure 1.2 for simulations of the case $\kappa = 8/3$ and $\kappa = 6$

Its fractal behavior is also known such as Hausdorff dimension [3], Minkowski dimension [30], best Hölder exponent [9] [23], etc.

There are not many studies of the same questions in the deterministic setting, i.e., the Loewner equation with general deterministic driving function. Two projects in this thesis investigate the regularity of the curve given that of the driving function. See Sections 1.2 1.3 in this introductory chapter.

Many models are known to converge to SLE such as interfaces of site percolation on the triangular lattice to SLE_6 [38], loop-erased random walks to SLE_2 [17], harmonic explorer

[1] In other words it is a measure supported on curves.

to SLE_4 [34], contour lines of Gaussian Free Field [35], FK-Ising model to $SLE_{16/3}$ [5], Ising interface to SLE_3[5], etc. It is therefore very desirable to generate pictures of SLE_κ directly to help understand the discrete random paths from those models. This is the subject for the other project which is described below.

1.1 Algorithms simulating Loewner curves and SLE

We are primarily interested in the case when the driving function corresponding to a growing curve. There are so far two methods to directly simulate the Loewner equation. The first method uses the fact that the Loewner equation, in the form (1.1), is a first order ODE, and hence one can numerically solve the equation, for example using Euler's method. Some of the first simulations of the SLE curve were obtained by using this method. This idea to simulate K_t is to determine whether a point z in the upper half plane $\mathbb{H}$ satisfies its existence time $T_z \leq t$ (see Section 2 for precise definitions). One cannot examine all the points in $\mathbb{H}$ so if $T_z \leq t$ one declares that a certain neighborhood of z is in K_t. To calculate the blow-up time T_z one needs to run equation (1.1) until $g_t(z)$ hits $\lambda(t)$. However, there is no general method to solve (1.1) with given driving function. There are a few cases that can be solved explicitly, see [11]. As a result, if γ is the simple curve corresponding to λ then the simulation of $\gamma([0,t]) = K_t$ is a neighborhood of the actual $\gamma([0,t])$. This is often not a good way to visualize the curve γ.

The second method for simulating SLE was suggested by Marshall and Rohde [26]. The algorithm has also been described in [12], [13], where modifications and fast implementations are discussed. The algorithm discretizes the driving function and square-root-interpolates it. As a result the algorithm approximates SLE maps by composing many basic conformal maps, which are easy to compute. One advantage of this algorithm is that it always produces simple and piecewise smooth curves. See Figure 1.2 and detailed description of the algorithm in Chapter 3. In Chapter 3 of this thesis, we will prove:

Theorem 1.1.1. *For $\kappa \neq 8$, let γ^n be the sequence of curves simulated from the second algorithm. Then under the half plane capacity parametrization, the sequence γ^n almost surely converges to SLE_κ in the sup-norm on every finite interval $[0,T]$ with $T > 0$.*

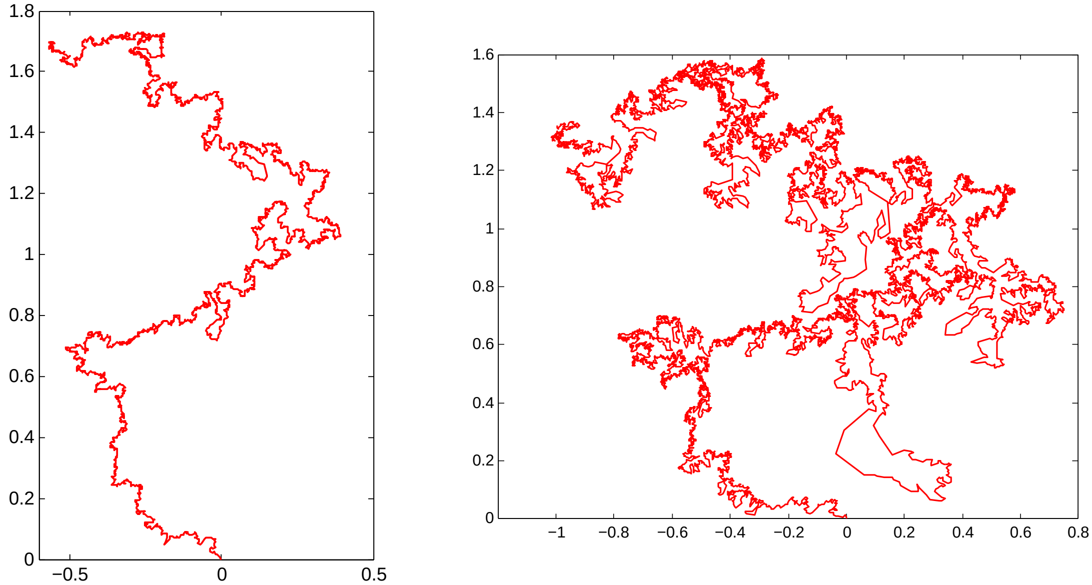


Figure 1.2: Simulations of $\mathrm{SLE}_{8/3}$ (left) and SLE_6 (right) from the same Brownian motion sample with 12800 points

It is known that, for all κ, the sequence γ^n converges to SLE in the context of Carathéodory convergence [14] and Cauchy transforms of probability measures [2]. However these types of convergence relate to Loewner chains rather than curves, see [14, Chapter 4] and respectively [2] for details. For $\kappa \leq 4$, when one views curves as compact sets, the sequence γ^n converges almost surely to SLE_κ in Hausdorff metric [4, Section 7]. A general principle is to set up a theorem for the deterministic Loewner equation and then translate the result into the SLE context. Theorem 1.1.1 will follow from a more general theorem for deterministic curves. In particular, we will show that there is a class of driving functions for which the sup-norm convergence of approximation curves occurs, see Theorem 3.1.2 for the details of the statement. It is shown in [26], [22] and [19] that driving functions whose Hölder-1/2 norms are less than 4 generate simple curves and that the Hilbert space filling curve is generated by a Hölder-1/2 function. Our Theorem 3.1.2 is also applied to these driving functions.

Corollary 1.1.2. *Consider the driving function that generates the Hilbert space filling curve or that has Hölder-1/2 norm less than 4. Then the sequence of curves simulated from the algorithm for this driving function converges uniformly.*

We note that Theorem 3.1.2 also provides the convergence rate of the simulation. The key is to estimate how the curve changes when we modify the driving function since the driving functions of simulated curves converge uniformly. There are two key estimates in the proof of Theorem 3.1.2. One is the boundary behavior of a conformal map and the other is the perturbation of a Loewner chain when there is a small change of its driving function. The latter is a Gronwall-type estimate which appears in [8] and [10]. The two estimates are both related to the growth of the derivative of conformal maps near the boundary which will become part of the assumptions of the main theorem 3.1.2.

We want to remark that the results in this project show the convergence given perfect arithmetic and we haven't tried to estimate round off error.

1.2 The existence of Loewner curves when $||\lambda||_{1/2} < 4$

While it is not hard to see that for simple curves $\gamma \subset \mathbb{H} \cup \{0\}$ with $\gamma_0 = 0$, the family $K_t = \gamma[0,t]$ yields continuous functions λ_t (see Chapter 2 for a precise statement), the converse is not true in general: There are continuous functions λ_t for which the associated hulls K_t are not locally connected, and hence not of the form $\gamma[0,t]$ for some continuous function γ.

In [26] and [22], the continuity problem was treated by viewing the hulls K_t as the result of "conformally welding" two intervals of the real line, and by applying the theory of quasiconformal maps to the welding problem. The main result is

Theorem 1.2.1 ([26], [22])**.** *If the driving function λ has Hölder-1/2 norm less than 4, then the chordal Loewner equation generates a simple curve γ. Moreover, γ is a quasiconformal arc that meets the real line non-tangentially.*

In the stochastic setting of the Schramm-Loewner evolution SLE_κ where the driving term is $\lambda_t = \sqrt{\kappa} B_t$ and B_t is a standard one-dimensional Brownian motion, the conformal welding approach leads to interesting and difficult problems, see [36] and [1] for related deep results. In [31], the almost sure continuity of the SLE_κ hulls was proved by different means: Based on estimates of the expectation $E[|(g_t^{-1})'(z)|^p]$ for suitable exponents p, it was shown that $\lim_{y\to 0} g_t^{-1}(\lambda_t + iy)$ exists, is continuous in t, and that this implies continuity of the hulls.

In Chapter 4 of this thesis, we will adopt the second method to the deterministic setting, and obtain a new and elementary proof of Theorem 1.2.1. The key observation is the following (see Theorem 4.1.1 below): Under the "upward" flow (2.3), the point $z_t = f_t(i) - \lambda_t$ will never leave the cone $\{|x| < cy\}$, where c depends on the Hölder-1/2 norm only. Combined with the integral representation of $\log|f_t'(z)|$ ((4.3) below), this easily gives an estimate for $|(g_t^{-1})'(\lambda_t + iy)|$ that, when integrated, implies existence and continuity of $\lim_{y\to 0} g_t^{-1}(\lambda_t + iy)$. Standard conformal mapping techniques (particularly the Gehring-Hayman inequality) then yield the additional information that the trace is a quasiconformal arc approaching the real line non-tangentially. The constant 4 is sharp as there is a driving function λ with $||\lambda||_{1/2} = 4$ but it does not generate a curve [18].

When applied to the integral representation (4.4) of $\arg f_t'(z)$, our approach also gives a sufficient condition for the driving functions to generate the graph of Lipschitz function:

Theorem 1.2.2. *There exists a constant $C_0 > 0$ such that for every continuous function λ satisfying*

$$\int_0^t \frac{N_{s,t}^\lambda}{(t-s)^{3/2}}\, ds \le C_0 \text{ for all } 0 < t < T \tag{1.2}$$

where $N_{s,t}^\lambda = \sup\{|\lambda_r - \lambda_s| : s \le r \le t\}$, the Loewner equation generates a graph of a Lipschitz function rotated by 90^o.

Theorem 1.2.2 is sharp, as the example $\lambda_t = c\sqrt{1-t}$ $(0 \le t \le 1)$ shows (see [11] or Chapter 2). In this example, the trace is asymptotic to a logarithmic spiral at the tip, hence it is not the graph of any function, and the integral in Theorem 1.2.2 diverges. On the other hand, the straight line of angle $\pi\alpha$ has driving function $\lambda_t = c\sqrt{t}$, where $c = 2(1-2\alpha)/\sqrt{\alpha(1-\alpha)}$, so that every Hölder-1/2 norm can arise from a simple curve. Notice that for $\lambda_t = c\sqrt{t}$, the integral has the finite value $c\sqrt{2}\sinh^{-1}(1)$ for all t.

1.3 Regularity of Loewner curves when $\lambda \in C^\alpha$ with $\alpha > 1/2$

It is natural to ask how properties of the Loewner curve γ correspond to properties of the driving function λ. In Chapter 5 of this paper, we prove the following theorem relating the regularity of λ to the regularity of γ.

Theorem 1.3.1. *Let $\lambda \in C^\beta[0,T]$ for $\beta > 2$. Then the Loewner curve γ is $C^{\beta+\frac{1}{2}}(0,T]$, provided that $\beta + 1/2 \notin \mathbb{N}$.*

See Theorem 5.3.1 for a quantitative version. Theorem 1.3.1 extends the work in [40], where the result was proven for $\beta \in (1/2, 2]$. In Section 5.6, we discuss an example where $\lambda \in C^{3/2}$ but γ fails to be C^2, showing that it is not possible to strengthen Theorem 1.3.1 to say that $\gamma \in C^{n+1}$ when $\lambda \in C^{n+1/2}$. We also address the analytic case:

Theorem 1.3.2. *If λ is real analytic on $[0,T]$, then γ is also real analytic on $(0,T]$.*

Notice that in both of these theorems, the regularity of γ is on the time interval $(0,T]$. With the halfplane-capacity parametrization, it is not possible to extend these results to $t = 0$. To see this, consider the example when the driving function is $\lambda(t) \equiv 0$. Then the corresponding Loewner curve is $\gamma(t) = 2i\sqrt{t}$. Further, with the halfplane-capacity parametrization, $\gamma(t)$ can always be expanded at $t = 0$ in powers of $\sqrt{t}$, as we see in the following theorem.

Theorem 1.3.3. *Assume that $\lambda \in C^{n+\alpha}[0,T]$ for $n \in \mathbb{N}$ and $\alpha \in (0,1]$. Then near $t = 0$,*

$$\gamma(t) = \begin{cases} 2i\sqrt{t} + a_2 t + i\, a_3 t^{3/2} + a_4 t^2 + \cdots + a_{2n} t^n + O(t^{n+\alpha}) & \text{if } \alpha \le 1/2 \\ 2i\sqrt{t} + a_2 t + i\, a_3 t^{3/2} + a_4 t^2 + \cdots + a_{2n} t^n + i\, a_{2n+1} t^{n+1/2} + O(t^{n+\alpha}) & \text{if } \alpha > 1/2 \end{cases}$$

where the real-valued coefficients a_m depend on $\lambda^{(k)}(0)$ for $k = 1, \cdots, \lfloor \frac{m}{2} \rfloor$.

If we make the simple change of parametrization $t = s^2$, then the smoothness extends to $s = 0$.

Theorem 1.3.4. *Let $\Gamma(s) = \gamma(s^2)$ be the reparametrized Loewner curve with driving function λ. If λ is real analytic on $[0,T]$, then Γ is real analytic on $[0,\sqrt{T}]$. If $\lambda \in C^\beta[0,T]$, then curve $\Gamma \in C^{\beta+1/2}[0,\sqrt{T}]$ when $\beta + 1/2 \notin \mathbb{N}$.*

We wish to briefly describe the key tool used in Chapter 5. For $s \in [0,T]$, consider the simple curve $g_s(\gamma(s+u))-\lambda(s)$, which we denote by $\gamma(s,s+u)$, $0 \le u \le T-s$. See Figure 1.3. The curve $\gamma(s,s+u)$ corresponds to the time-shifted driving function $\lambda_s(u) = \lambda(u+s)-\lambda(s)$, $0 \le u \le T-s$. It follows from [40, Theorem 6.2] that under the assumption $\lambda \in C^2([0,T])$, the curve γ is in C^2 and

$$\gamma''(s) = \frac{2\gamma'(s)}{\gamma(s)^2} - 4\gamma'(s)\int_0^s \frac{\partial_s[\gamma(s-u,s)]}{\gamma(s-u,s)^3}du. \tag{1.3}$$

In order to understand the higher differentiability of γ, we need to understand $\gamma(s-u,s)$. Differentiating this function with respect to u, we obtain

$$\partial_u[\gamma(s-u,s)] = \partial_u[g_{s-u}(\gamma(s)) - \lambda(s-u)] = \frac{-2}{\gamma(s-u,s)} + \lambda'(s-u) \text{ for } 0 < u \le s, \tag{1.4}$$

and $\gamma(s-u,s)|_{u=0} = \gamma(s,s) = 0$. We note that the above differential equation does not hold for $u = 0$. This is the reason for us to investigate the following ODE:

$$\begin{aligned} f'(u) &= \frac{-2}{f(u)} + \lambda'(s-u), \qquad 0 \le u \le s, \\ f(0) &= i\varepsilon \in \mathbb{H}. \end{aligned} \tag{1.5}$$

The work in this project depends on a deep understanding of the function $f(u) = f(u,s,\epsilon)$ which is the solution to (1.5). Once we show that $f(u,s,\varepsilon)$ converges uniformly to $\gamma(s-u,s)$ as $\varepsilon \to 0^+$, we can use (1.3) to translate information about f into information about the derivatives of γ.

Remark. Theorem 1.3.1 and Theorem 1.3.2 provide a converse to the results of Earle and Epstein in [7]. Their results (translated from the radial setting to the chordal setting using [25]) state that if any parametrization of γ is C^n, then the halfplane-capacity parametrization of γ is in $C^{n-1}(0,T)$ and $\lambda \in C^{n-1}(0,T)$. They also prove that if γ is real analytic, then λ must be real analytic.

Organization. This thesis is split into four chapters after this one. In Chapter 2 we review Loewner differential equation and give examples. The theorems in Sections 1.1, 1.2 and 1.3 are presented Chapter 3, 4 and 5, respectively.

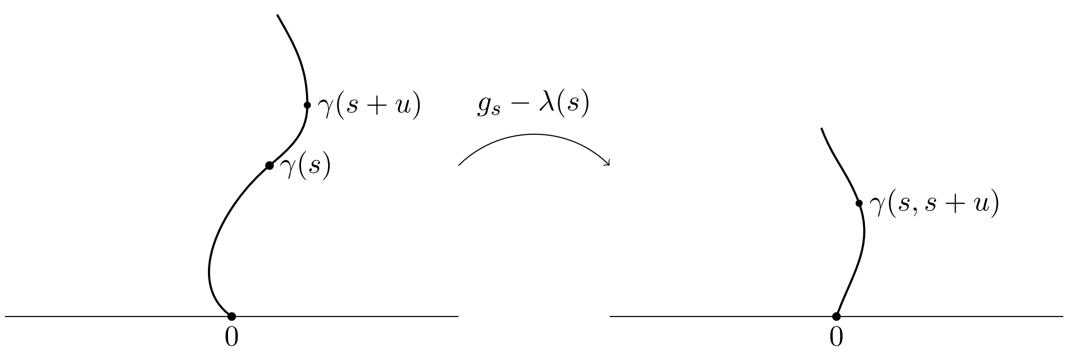


Figure 1.3: The curve $\gamma(s, s + u)$.

Chapter 2

LOEWNER DIFFERENTIAL EQUATION

2.1 Chordal versions

There are many versions of the Loewner equation. The original version is radial Loewner equation in the unit disk $\mathbb{D}$. In this thesis we only focus on the chordal version in the upper half plane $\mathbb{H}$. This version is popular because of its simple form, which was introduced by Schramm [33]. The radial case is introduced in Section 2.3. For more details we refer the reader to [14].

Let $\gamma : [0,T] \to \mathbb{H} \cup \{0\}$ be a simple curve in the upper half-plane $\mathbb{H}$ except that $\gamma_0 \equiv \gamma(0) \in \mathbb{R}$[1]. For each $t \in [0,T]$, by the Riemann mapping theorem there exists a unique conformal map $g_t : \mathbb{H}\backslash\gamma[0,t] \to \mathbb{H}$ satisfying the *hydrodynamic normalization*:

$$g_t(z) = z + \frac{c_t}{z} + \cdots \qquad \text{when } z \to \infty.$$

It can be shown that c_t is a nonnegative, strictly increasing, continuous function and $c_0 = 0$ [14]. Hence one can reparametrize γ according to *half-plane capacity*, which means $c_t = 2t$. Then for each $z \in \mathbb{H}$, the function $t \mapsto g_t(z)$ satisfies the *(downward) chordal Loewner equation*:

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \lambda_t}, \qquad g_0(z) = z, \tag{2.1}$$

where λ is a continuous, real-valued function and $g_t(\gamma_t) = \lambda_t$, see [14, Chapter 4].

Conversely, if one starts with a continuous function $\lambda : [0,T] \to \mathbb{R}$, one can consider the initial value problem for each $z \in \mathbb{H}$:

$$\partial_t g(t,z) = \frac{2}{g(t,z) - \lambda_t}, \qquad g(0,z) = z.$$

For each $z \in \mathbb{H}$ there is a maximal interval for which a solution $g(t,z)$ exists. Let $T_z =$

[1] In this thesis, when writing functions of variable t, we usually let t be a subscript.

$\sup\{s \in [0,T] : g(t,z)$ exists on $[0,s)\}$. It is easy to see that, if $T_z < T$, then

$$\lim_{t\to T_z} g(t,z) = \lambda(T_z).$$

Let $H_t = \{z \in \mathbb{H} : T_z > t\}$ and $g_t(z) = g(t,z)$. Then one can show that the set H_t is simply connected subdomain of $\mathbb{H}$ and $g_t(z)$ is the unique conformal map from H_t onto $\mathbb{H}$ with the following normalization near infinity:

$$g_t(z) = z + \frac{2t}{z} + O\left(\frac{1}{z^2}\right).$$

The *driving function* λ of the *Loewner chain* (g_t) is said to *generate a curve* if there exists a curve γ such that H_t is the unbounded component of $\mathbb{H}\backslash\gamma[0,t]$ for each $t \geq 0$. By [31, Theorem 4.1], this is equivalent to the existence and continuity in $t > 0$ of

$$\gamma_t := \lim_{y\to 0+} g_t^{-1}(\lambda_t + iy). \tag{2.2}$$

By [15, Proposition 2.19] and [9, Proposition 3.11], a very useful and simple criterion for this existence and continuity is the convergence to zero of

$$v(t,\varepsilon) := \int_0^\varepsilon |(g_t^{-1})'(\lambda_t + iy)|dy$$

as $\varepsilon \to 0$, uniformly in $t \in [0,T]$.

Rather than directly working with the Loewner equation (1.1), it is often easier to work with the *upward Loewner equation*:

$$\partial_t f_t(z) = \frac{-2}{f_t(z) - \xi_t}, \qquad f_0(z) = z, \tag{2.3}$$

for $z \in \mathbb{H}$ and real-valued continuous function ξ_t. Since the imaginary part of $f_t(z)$ is strictly increasing, the solution exists uniquely for all time $t \geq 0$. If $(g_t)_{0\leq t\leq T}$ is the solution to (1.1) with driving function λ and if $(f_t)_{0\leq t\leq T}$ is the solution to (2.3) with $\xi_t = \lambda_{T-t}$, then $f_t(z) = g_{T-t}(g_T^{-1}(z))$ and in particular

$$f_T(z) = g_T^{-1}(z).$$

We will frequently use the following two simple properties of the Loewner equation, regarding the translation and concatenation of driving functions:

First, if $a \in \mathbb{R}$ and $\tilde{\xi}_t = \xi_t + a$, then the Loewner chain $(\tilde{f}_t)$ corresponding to $\tilde{\xi}$ is given by

$$\tilde{f}_t(z) = f_t(z-a) + a.$$

Second, let $(f_{1,t})_{0\le t\le T_1}$ (respectively $(f_{2,t})_{0\le t\le T_2}$) be the solution to (2.3) with the driving function ξ_1 defined on $[0, T_1]$ (respectively ξ_2 defined on $[0, T_2]$). Suppose $\xi_1(T_1) = \xi_2(0)$, and define the *concatenation* of ξ_1 and ξ_2 by

$$\xi_t = \begin{cases} \xi_1(t), & t \in [0, T_1], \\ \xi_2(t - T_1), & t \in [T_1, T_1 + T_2]. \end{cases} \tag{2.4}$$

Then the (upward) Loewner solution corresponding to ξ is given by

$$f_t = \begin{cases} f_{1,t}, & t \in [0, T_1], \\ f_{2,t-T_1} \circ f_{1,T_1}, & t \in [T_1, T_1 + T_2]. \end{cases} \tag{2.5}$$

2.2 Examples

There are several driving functions for which we can compute the curve explicitly. Careful computations can be found in [11] and [18]. In this section we describe some of them.

Example 2.2.1. If $\lambda(t) = 0$ on $[0, T]$, then $g_t(z) = \sqrt{z^2 + 4t}$ and $\gamma(t) = 2i\sqrt{t}$ for $t \in [0, T]$.

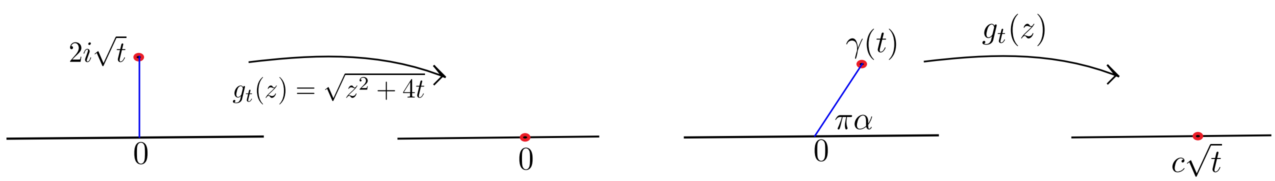


Figure 2.1: Examples 2.2.1 and 2.2.2

Example 2.2.2. Suppose $\lambda(t) = c\sqrt{t}$ on $[0, T]$. Let $\alpha \in (0, 1)$ such that

$$\alpha = \frac{1}{2} - \frac{1}{2}\frac{c}{\sqrt{16 + c^2}},$$

Then if we solve (1.1) we find that

$$\gamma(t) = 2\sqrt{t}\left(\frac{1-\alpha}{\alpha}\right)^{1/2-\alpha} e^{i\pi\alpha}.$$

In these two examples the curve is always inside a fixed cone and the height of the curve is comparable to $\sqrt{t}$. Indeed, we will show in Chapter 4 that if $||\lambda||_{1/2} < 4$ then those properties always hold.

Example 2.2.3. $\lambda(t) = c\sqrt{1-t}$ on $[0,1]$. When $|c| < 4$, $\gamma(t)$ is a curve that is a logarithmic spiral around $\gamma(1) \in \mathbb{H}$. When $|c| \geq 4$, $\gamma(t)$ is a curve that touches the real line at $\gamma(1)$.

Two more examples are presented in Section 5.6. As said in the introduction, the chordal SLE_κ in the upper half-plane is the solution of (1.1) with driving function $\lambda_t = \sqrt{\kappa} B_t$ where B_t is a standard Brownian motion. By abusing notation SLE_κ also stands for the generated curve whose existence is proved in [31].

2.3 Radial Loewner equation

Let μ be a Borel measure on $\partial\mathbb{D}$. Abusing notation we write μ for the corresponding measure on $[0, 2\pi)$, i.e., if $I \subset [0, 2\pi)$, we will write $\mu(I) = \mu\{e^{i\theta} : \theta \in I\}$.

Theorem 2.3.1. *Suppose $\mu_t, t \geq 0$, is a one parameter family of nonnegative Borel measures on $\partial\mathbb{D}$ such that $t \mapsto \mu_t$ is continuous in the weak-** [2] *topology, and for each t, there is an $M_t < \infty$ such that $\sup\{\mu_s(\partial\mathbb{D}) : s \leq t\} < M_t$. For each $z \in \mathbb{D}$, let $g_t(z)$ denote the solution of the initial value problem*

$$\partial_t g_t(z) = g_t(z) \int_0^{2\pi} \frac{e^{i\theta} + g_t(z)}{e^{i\theta} - g_t(z)} \mu_t(d\theta), \quad g_0(z) = z. \tag{2.6}$$

Let T_z be the supremum of all t such that the solution is well defined up to time t with $g_t(z) \in \mathbb{D}$. Let $D_t = \{z : T_z > t\}$. Then g_t is the unique conformal transformation of D_t onto $\mathbb{D}$ such that $g_t(0) = 0$ and $g_t'(0) > 0$. Moreover,

$$\log g_t'(0) = \int_0^t \mu_s([0, 2\pi))\, ds.$$

We will call g_t a *radial Loewner chain* if it satisfies (2.6) with $\mu_t = \delta_{U_t}$ and $t \mapsto U_t$ a continuous function from $[0, \infty)$ to $\mathbb{R}$. We will call either U_t or e^{iU_t} the driving function. In this case, the equation (2.6) becomes

$$\partial_t g_t(z) = g_t(z) \frac{e^{iU_t} + g_t(z)}{e^{iU_t} - g_t(z)}, \quad g_0(z) = z. \tag{2.7}$$

---

[2] It means that for every $f \in C(\partial\mathbb{D})$, the function $t \mapsto \int f d\mu_t$ is continuous as a function of t.

Figure 2.2: Chordal Loewner chain (left) and radial Loewner chain

The radial SLE_κ in $\mathbb{D}$ is (2.7) with $U_t = \sqrt{\kappa}B_t$ and B_t a standard Brownian motion.

The chordal Loewner equation and the radial one are closely related. One can derive one version from the other and vice versa. See [25].

Chapter 3

CONVERGENCE OF AN ALGORITHM SIMULATING LOEWNER CURVES

3.1 Algorithms simulating Loewner equations

We now discuss in more detail the algorithm to simulate Loewner curves. The algorithm is based on two observations. First, fix $s > 0$, and let $(\tilde{g}_t)$ be the solution of the Loewner equation with driving function $\tilde{\lambda}(t) = \lambda(s+t), t \geq 0$. This solution can be obtained by $g_{s+t} \circ g_s^{-1}$. Indeed

$$\partial_t g_{s+t} \circ g_s^{-1}(z) = \frac{2}{g_{s+t} \circ g_s^{-1}(z) - \lambda(s+t)} = \frac{2}{g_{s+t} \circ g_s^{-1}(z) - \tilde{\lambda}(t)},$$

and $g_s \circ g_s^{-1}(z) = z$. By the uniqueness of solution of the equation (1.1), $\tilde{g}_t(z) = g_{s+t} \circ g_s^{-1}(z)$. If we let $\widetilde{K}_t$ be the hull associated with $\tilde{g}_t$ then

$$g_s(K_{s+t}) = \widetilde{K}_t \text{ and } K_{s+t} = K_s \cup g_s^{-1}(\widetilde{K}_t).$$

So in order to compute K_{s+t}, one can compute K_s and g_s^{-1}, by using the information of λ on $[0, s]$, and compute $\widetilde{K}_t$ by using λ on $[s, s+t]$.

The second observation is that when λ is of the form $c\sqrt{t} + d$, for some real constants c and d, one can solve for K_t explicitly. In this case, K_t is a segment in the upper half plane starting at $d \in \mathbb{R}$ that makes an angle $\alpha\pi$ with the positive real axis where

$$\alpha = \frac{1}{2} - \frac{1}{2}\frac{c}{\sqrt{16 + c^2}},$$

and $g_t^{-1}(z + \lambda(t)) = (z + 2\sqrt{t}\sqrt{\frac{\alpha}{1-\alpha}})^{1-\alpha}(z - 2\sqrt{t}\sqrt{\frac{\alpha}{1-\alpha}})^{\alpha} + d$. See [11] for a proof.

We now fix a step $n \geq 1$. Let $t_k = \frac{k}{n}$ for $0 \leq k \leq n$. So $t_0 = 0, t_1, \cdots, t_n = 1$ is a partition of $[0, 1]$. We will solve the Loewner equation with driving functions $\lambda(t + t_k)$ for $0 \leq t \leq \frac{1}{n}$. By the remarks above, one should approximate these driving functions by $c\sqrt{t} + d$ so that one can solve explicitly the Loewner equation. More specifically, we approximate λ by λ^n

such that they attain the same values at t_k's and that λ^n is a scaling and translation of $\sqrt{t}$ on $[t_k, t_{k+1}]$ from $\lambda(t_k)$ to $\lambda(t_{k+1})$. Hence the function λ^n is defined as follows:

$$\lambda^n(t) = \sqrt{n}(\lambda(t_{k+1}) - \lambda(t_k))\sqrt{t - t_k} + \lambda(t_k) \text{ on } [t_k, t_{k+1}].$$

This driving function always produces a simple curve $\gamma^n : [0,1] \to \mathbb{H} \cup \{\lambda(0)\}$.

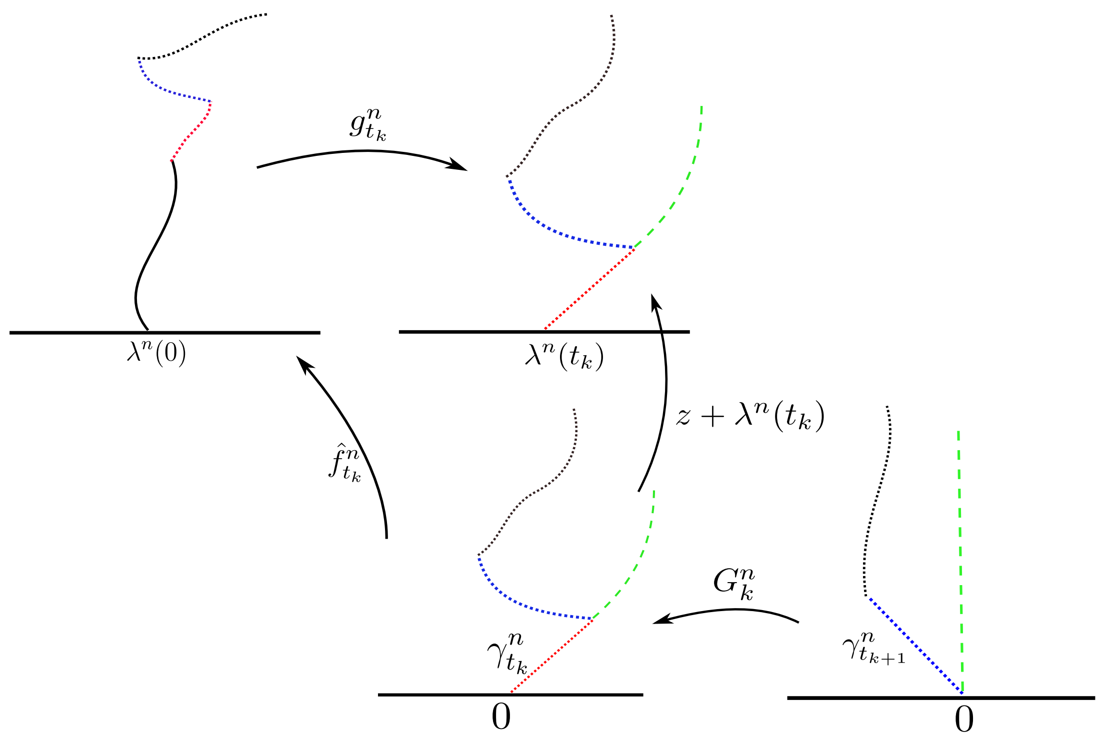


Figure 3.1: At each step k, we compute G^n_k, $\hat{f}^n_{t_k}$ and $\gamma^n_{t_k}$. The k-th sub-arc of the simulation curve γ^n is the image of $\gamma^n_{t_k}$ under $\hat{f}^n_{t_k}$.

Denote by $(g^n_t)_{0 \le t \le 1}$ the Loewner chain corresponding to λ^n. Let f^n_t be the inverse function of g^n_t and $\hat{f}^n_t(z) = f^n_t(z + \lambda^n(t))$. Define

$$G^n_k = (\hat{f}^n_{t_k})^{-1} \circ \hat{f}^n_{t_{k+1}},$$

so that

$$\hat{f}^n_{t_k} = G^n_{k-1} \circ G^n_{k-2} \circ \cdots \circ G^n_0.$$

For each $t \in [0,1]$, let γ_t^n be the image of γ^n under $g_t^n - \lambda^n(t)$, i.e.,

$$\gamma_t^n(s) = g_t^n(\gamma^n(t+s)) - \lambda^n(t) \quad \text{and} \quad \gamma^n(t+s) = \hat{f}_t^n(\gamma_t^n(s)) \quad \text{for } 0 \le s \le 1-t. \tag{3.1}$$

We have chosen λ^n so that $\gamma_{t_k}^n([0, \frac{1}{n}])$ is a segment starting at 0 and that G_k^n has an explicit formula:

$$G_k^n(z) = \left(z + 2\sqrt{\frac{1-\alpha}{n\alpha}}\right)^{1-\alpha} \left(z - 2\sqrt{\frac{\alpha}{n(1-\alpha)}}\right)^{\alpha}$$

where $\alpha = \frac{1}{2} - \frac{1}{2}\frac{\sqrt{n}(\lambda(t_{k+1})-\lambda(t_k))}{\sqrt{16+n(\lambda(t_{k+1})-\lambda(t_k))^2}} \in (0,1)$. See Figure 3.1.

Therefore in order to compute $\gamma^n([0,1])$, we find $\gamma_{t_k}^n([0, \frac{1}{n}])$, $\hat{f}_{t_k}^n$, and then

$$\gamma^n(t) = \hat{f}_{t_k}^n(\gamma_{t_k}^n(t-t_k)) \text{ for } t \in [t_k, t_{k+1}), 0 \le k \le n-1. \tag{3.2}$$

Notice that λ^n converges uniformly to λ on $[0,1]$ since

$$\sup_{t\in[0,1]} |\lambda^n(t) - \lambda(t)| \le 2 \sup_{s,t\in[0,1],|t-s|\le \frac{1}{n}} |\lambda(t) - \lambda(s)|.$$

We mention without proof a geometric property of G_k^n which we will use later.

Lemma 3.1.1. *Consider the conformal map $G(z) = (z+a)^{1-\alpha}(z-b)^{\alpha}$ from $\mathbb{H}$ to $\mathbb{H}$ minus a slit starting at 0, where $a, b > 0$, $\alpha \in (0,1)$ and $\alpha a = (1-\alpha)b$. The point 0 is mapped to the tip of the slit. Then the imaginary part of $G(iy)$ is increasing on $(0,\infty)$. In particular, the image of iy has a larger imaginary part than that of the tip of the slit.*

The long green dashed line of Figure 3.1 illustrates this proposition.

3.1.1 Main results

We shall consider driving functions which have the same regularity as Brownian motion. Without loss of generality, we will prove the convergence in Theorem 1.1.1 only on interval $[0,1]$. A *subpower function* ϕ is a non decreasing function from $[0,\infty)$ to $[0,\infty)$ satisfying:

$$\lim_{x\to\infty} x^{-\nu}\phi(x) = 0 \text{ for all } \nu > 0.$$

If ϕ_1, ϕ_2 are subpower functions then so are $c\phi_1$, ϕ_1^c and $\max(\phi_1, \phi_2)$ for every $c > 0$.

The function λ is called *weakly Hölder-1/2* if there exists a subpower function φ such that

$$\operatorname{osc}(\lambda;\delta) := \sup\{|\lambda(t)-\lambda(s)| : s,t\in[0,1], |t-s|\le\delta\} \le \sqrt{\delta}\varphi(1/\delta) \text{ for all } \delta>0. \quad (3.3)$$

It follows from P.Levy's theorem that the sample paths of Brownian motion are almost surely weakly Hölder-1/2 with subpower function $c\sqrt{\log(\delta)}$, $c>\sqrt{2}$, see [29, Theorem I.2.7].

It is known that if λ is weakly Hölder-1/2 and if there exist $c_0>0$, $y_0>0$ and $0<\beta<1$ so that

$$|\hat{f}_t'(iy)| \le c_0 y^{-\beta} \text{ for all } 0<y\le y_0,\ t\in[0,1], \quad (3.4)$$

where $\hat{f}_t(\cdot) = g_t^{-1}(\lambda(t)+\cdot)$, then $(g_t)_{0\le t\le 1}$ is generated by a curve, see [9, section 3]. This is one of the main ideas to show the existence of SLE curves for $\kappa\ne 8$ ([31]). We note that the Loewner chain of SLE_8 does not satisfy (3.4).

Our main theorem shows that under these hypotheses the algorithm gives the sup-norm convergence of the simulation curves.

Theorem 3.1.2. *Suppose λ is a weakly Hölder-1/2 driving function with a subpower function φ and suppose the condition (3.4) is satisfied. Then the curve γ generated from the Loewner equation can be approximated by the algorithm; that is, there exists a subpower function $\widetilde{\varphi}$ such that for all $n\ge\frac{1}{y_0^2}$ and $t\in[0,1]$,*

$$|\gamma^n(t)-\gamma(t)| \le \frac{\widetilde{\varphi}(n)}{n^{\frac{1}{2}(1-\sqrt{\frac{1+\beta}{2}})}}, \quad (3.5)$$

where γ^n is the curve generated from the algorithm which is explained in section 3.1. The function $\widetilde{\varphi}$ depends on φ, c_0 and β.

A related question is the following: under what additional assumptions, does the uniform convergence of driving functions imply convergence of corresponding curves? *A priori* the convergence of curves occurs in the sense of Carathéodory convergence, see [14], and in the sense of Cauchy transform of probability measures, see [2] for definitions and details. As said in the introduction, these types of convergence do not directly involve the curves. In

[4, Section 7], it is shown that if two driving functions generating simple curves are close in the sup-norm and if one function has the condition (3.4) then the two generated curves are close in Hausdorff distance. One really wants two curves to be close in the sup-norm. Lind, Marshall and Rohde [18] show that if the driving functions have Hölder-1/2 norm less than 4, then the curves converge uniformly. However, the Brownian motion is a.s. not Hölder-1/2. In [37], the authors study sufficient conditions to have uniform convergence of bidirectional paths (the curves and their time-reversals). The paper by Johansson-Viklund [8] uses the tip structure modulus to get another criterion for uniform convergence of curves.

In the rest of this chapter, C stands for absolute constant and ϕ for general subpower functions; c and φ stand for constants and subpower functions that may depend on the assumptions of Theorem 3.1.2. They can change line by line and are indexed when necessary to avoid confusion.

Since we are interested in the same type of driving functions as those in [9, section 3], there are several results from their paper we will use and state here for the convenience of the reader.

Lemma 3.1.3. *[9, Lemma 3.4] Let K be a relatively compact set in $\mathbb{H}$ such that $\mathbb{H}\backslash K$ is simply connected. There exists a constant $C < \infty$ such that*

$$hcap(K) \leq C\, diam(K) height(K),$$

where $height(K) = \sup\{Im\, z : z \in K\}$ and $hcap(K) = \lim_{z\to\infty} z(g_K(z) - z)$ and g_K is the conformal map from $\mathbb{H}\backslash K$ to $\mathbb{H}$ and sends ∞ to ∞.

Lemma 3.1.4. *[40, Lemma 3.1] Suppose γ is the curve generated by a driving function $\lambda(t)$ as in (1.1). Then for all $z \in \gamma([0,t])$,*

$$|Re\ z| \leq \sup_{0\leq s\leq r\leq t} |\lambda(r) - \lambda(s)|$$

and

$$Im\ z \leq 2\sqrt{t}.$$

Lemma 3.1.5. *[9, Proposition 3.8] Let (g_t) be the Loewner chain corresponding to $\lambda(t)$ satisfying (3.3) and (3.4). Then there exists a subpower function φ_1 such that if $0 \leq t \leq$*

$t+s \le 1$ *and* $s \in [0, y^2]$

$$|\gamma(t+s) - \gamma(t)| \le \varphi_1(1/y)[v(t+s,y) + v(t,y)],$$

where

$$v(t,y) = \int_0^y |\hat{f}_t'(ir)| dr \le \frac{c_0}{1-\beta} y^{1-\beta}, \quad 0 < y < y_0,$$

and that

$$|\gamma(t+s) - \gamma(t)| \le \varphi_1(1/y) \frac{2}{1-\beta} y^{1-\beta} \text{ for } 0 \le s \le y^2 \le y_0^2. \tag{3.6}$$

Most of the time we will deal with the behavior of conformal maps near the real and imaginary lines. For every subpower function ϕ, constant $c > 0$ and $n \in \mathbb{N}$, define

$$A_{n,c,\phi} = \{x + iy \in \mathbb{H} : |x| \le \frac{\phi(n)}{\sqrt{n}} \text{ and } \frac{1}{\sqrt{n}\phi(n)} \le y \le \frac{c}{\sqrt{n}}\}.$$

Lemma 3.1.6. *There exist constants* $\alpha > 0$ *and* $c' > 0$ *such that if* z_1 *and* z_2 *are inside the box* $A_{n,c,\phi}$*, and* f *is a conformal map on* $\mathbb{H}$ *then*

$$|f'(z_1)| \le c' \phi(n)^\alpha |f'(i Im\, z_1)| \tag{3.7}$$

and

$$d_{\mathbb{H},hyp}(z_1, z_2) \le c' \log \phi(n) + c'.$$

The constants α *and* c' *depend only on* c*, not on* ϕ *or* n*. The notation* $d_{H,hyp}(z_1, z_2)$ *means the hyperbolic distance between* z_1 *and* z_2 *in a simply connected domain* H*.*

Proof. The proof is similar to [9, Lemma 3.2]. □

Lemma 3.1.7. *[28, Corollary 1.5] (half plane version) If* f *is a conformal map of* $\mathbb{H}$ *into* $\mathbb{C}$ *and if* $z_0, z_1 \in \mathbb{H}$ *then*

$$|f(z_1) - f(z_2)| \le 2\, |(Im\, z_1) f'(z_1)| \exp(4 d_{\mathbb{H},hyp}(z_1, z_2)).$$

3.2 *Proof of Theorem 3.1.2*

3.2.1 Heuristic argument

Let γ_t be the image of γ under $g_t - \lambda(t)$, i.e.

$$\gamma_t(s) = g_t(\gamma(t+s)) - \lambda(t) \text{ and } \gamma(t+s) = \hat{f}_t(\gamma_t(s)).$$

We want to estimate

$$|\gamma(t_{k+1}) - \gamma^n(t_{k+1})| = |\hat{f}_{t_k}(z) - \hat{f}^n_{t_k}(w)| \leq |\hat{f}_{t_k}(z) - \hat{f}_{t_k}(w)| + |\hat{f}_{t_k}(w) - \hat{f}^n_{t_k}(w)| \tag{3.8}$$

where $z = \gamma_{t_k}(\frac{1}{n})$ and $w = \gamma^n_{t_k}(\frac{1}{n})$.

First, z and w are the tips of two curves generated respectively by two driving functions defined on $[0, \frac{1}{n}]$. It follows from Lemma 3.1.4 that $\operatorname{Im} z$ and $\operatorname{Im} w \leq \frac{2}{\sqrt{n}}$.

For the first term in the RHS of (3.8), it follows from Lemma 3.1.7 that

$$|\hat{f}_{t_k}(z) - \hat{f}_{t_k}(w)| \leq 2\,|(\operatorname{Im} z)\hat{f}'_{t_k}(z)| \exp(4d_{\mathbb{H},hyp}(z,w)).$$

Notice that z and w trivially have positive imaginary parts.

If we can show that z and w are in the same box $A_{n,c,\phi}$ then combining with a hypothesis of $\hat{f}'_{t_k}$ on $i\mathbb{R}^+$, we obtain similar inequalities for $|\hat{f}'_{t_k}(z)|$ and $|\hat{f}'_{t_k}(w)|$ from Lemma 3.1.6. Then it follows that

$$2\operatorname{Im} z|\hat{f}'_{t_k}(z)| \exp(4d_{\mathbb{H},hyp}(z,w)) \lesssim (\operatorname{Im} z)^{1-\beta}\phi(n) \lesssim \frac{\phi(n)}{(\sqrt{n})^{1-\beta}} \to 0 \text{ as } n \to \infty,$$

where the notation $f \lesssim g$ means that $f \leq Cg$ for some constant $C > 0$.

Since w is a tip of a straight line generated by a nice driving function, we can show w is in a box $A_{n,c,\phi}$, see (3.11). However, $z = \gamma_{t_k}(\frac{1}{n})$, in the case of SLE, has continuous density on the strip $\{x+iy : x \in \mathbb{R}, 0 \leq y \leq 2\}$. So there might not exist a controllable-sized box $A_{n,c,\phi}$ that contains z. However Lemma 3.2.2 shows the existence of a point in $\gamma_{t_k}([0, \frac{1}{n}]) \cap A_{n,c,\phi}$, and we will use this point instead of $\gamma_{t_k}(\frac{1}{n})$. Then we use the uniform continuity of γ to get back to (3.8).

For the second term in the RHS of (3.8), notice that

$$\hat{f}_{t_k}(w) - \hat{f}^n_{t_k}(w) = f_{t_k}(w + \lambda(t_k)) - f^n_{t_k}(w + \lambda(t_k)).$$

This expression is a perturbation of two solutions of the upward Loewner equation (2.3) with two driving functions $t \mapsto \lambda(t_k - \cdot)$ and $t \mapsto \lambda^n(t_k - \cdot)$.

We will use the following lemma from [10] and the fact that λ and λ^n are close on $[0, t_k]$ and that $|f'_{t_k}(w + \lambda(t_k))|$ is well-controlled.

Lemma 3.2.1. *[10, Lemma 2.3] Let $0 < T < \infty$. Suppose that for $t \in [0,T]$, $f_t^{(1)}$ and $f_t^{(2)}$ satisfy the upward Loewner equation (2.3) with $W_t^{(1)}$ and $W_t^{(2)}$, respectively, as driving terms. Suppose that*

$$\varepsilon = \sup_{s\in[0,T]} |W_s^{(1)} - W_s^{(2)}|.$$

Then if $u = x + iy \in \mathbb{H}$, then

$$|f_T^{(1)}(u) - f_T^{(2)}(u)| \leq \varepsilon \exp\left\{ \frac{1}{2}\left[\log \frac{I_{T,y}|(f_T^{(1)})'(u)|}{y} \log \frac{I_{T,y}|(f_T^{(2)})'(u)|}{y} \right]^{1/2} + \log\log \frac{I_{T,y}}{y} \right\},$$

where $I_{T,y} = \sqrt{4T + y^2}$.

Thus if $|(f_T^{(1)})'(u)| \leq cy^{-\beta}$, then

$$|f_T^{(1)}(u) - f_T^{(2)}(u)| \lesssim \varepsilon y^{-\sqrt{(1+\beta)/2}} \log(I_{T,y}/y).$$

If one can show furthermore

$$\varepsilon \leq \frac{\phi(n)}{\sqrt{n}} \quad \text{and} \quad y = \operatorname{Im} u = \operatorname{Im} w \geq \frac{1}{\phi(n)\sqrt{n}}$$

then

$$|f_T^{(1)}(u) - f_T^{(2)}(u)| \lesssim \frac{\phi(n)^{c''}}{n^{\frac{1}{2}(1-\sqrt{\frac{1+\beta}{2}})}} \to 0.$$

From here, we only have an estimate for $|\gamma(t) - \gamma^n(t)|$ when $t = t_k$. To have an estimate on the whole interval we notice that

$$\gamma^n([t_{k+1}, t_{k+2}]) = \hat{f}_{t_k}^n(\gamma_{t_k}^n[\frac{1}{n}, \frac{2}{n}]) = G_k^n(\gamma_{t_{k+1}}^n[0, \frac{1}{n}]).$$

It follows from a property of G_k^n (Lemma 3.1.1), that every point in $\gamma_{t_k}^n([\frac{1}{n}, \frac{2}{n}])$ is in a box $A_{n,c,\phi}$ and hence we can apply the same argument for (3.8) with $\gamma^n(t_{k+1})$ being replaced by any $\gamma^n(t)$, $t_{k+1} \leq t \leq t_{k+2}$.

Now we will go into the details of the proof.

3.2.2 Proof of Theorem 3.1.2

Fix an arbitrary interval $I = [t_k, t_{k+2}]$, $0 \leq k \leq n-2$. Denote $\gamma_k = \gamma_{t_k}$, $\gamma_k^n = \gamma_{t_k}^n$.

We will estimate $|\gamma(s+t_k)-\gamma^n(r+t_k)|$ for all $r\in[\frac{1}{n},\frac{2}{n}]$ and with a specific s chosen later. Combining with the uniform continuity of γ, we will have an estimate for

$$|\gamma(r+t_k)-\gamma^n(r+t_k)| \quad \text{with all } r\in[\frac{1}{n},\frac{2}{n}].$$

From now on, we will choose n so that $\frac{1}{n}\le y_0^2$. Denote $z=\gamma_k(s), w=\gamma_k^n(r)$. By the triangle inequality,

$$|\gamma(s+t_k)-\gamma^n(r+t_k)| = |\hat{f}_{t_k}(z)-\hat{f}^n_{t_k}(w)| \le |\hat{f}_{t_k}(z)-\hat{f}_{t_k}(w)| + |\hat{f}_{t_k}(w)-\hat{f}^n_{t_k}(w)|. \quad (3.9)$$

The first term in the RHS of (3.9). It follows from Lemma 3.1.7 that

$$|\hat{f}_{t_k}(z)-\hat{f}_{t_k}(w)| \le (2\mathrm{Im}\, z)|\hat{f}'_{t_k}(z)|\exp(4d_{\mathbb{H},hyp}(z,w)). \quad (3.10)$$

The next lemma shows the existence of a point in $\gamma_k([0,\frac{2}{n}])\cap A_{n,c,\phi}$.

Lemma 3.2.2. *There exists a subpower function ϕ depending only on φ, c_0 and β of Theorem 3.1.2 such that for $n\ge 1$ and $0\le k\le n-1$, there exists $s\in[0,\frac{2}{n}]$ such that $\gamma_k(s)\in A_{n,2\sqrt{2},\phi}$.*

Proof. Since $\eta := \gamma_k([0,\frac{2}{n}])$ is the curve generated by the Loewner equation (1.1) with driving function $\lambda(t_k+.)-\lambda(t_k)$ on $[0,\frac{2}{n}]$ and since λ is weakly Hölder-1/2, it follows from Lemma 3.1.4 that

$$|\mathrm{Re}\,\gamma_k(s)| \le \sqrt{\frac{2}{n}}\varphi(\frac{n}{2}) =: \frac{\varphi_2(n)}{\sqrt{n}} \quad \text{and} \quad \mathrm{Im}\,\gamma_k(s)\le \frac{2\sqrt{2}}{\sqrt{n}}$$

for all $s\in[0,\frac{2}{n}]$.

This implies

$$\mathrm{diam}(\eta)\le \mathrm{height}(\eta)+\mathrm{width}(\eta)\le \frac{2\sqrt{2}}{\sqrt{n}}+\frac{2\varphi_2(n)}{\sqrt{n}}=\frac{\varphi_3(n)}{\sqrt{n}}.$$

It follows from Lemma 3.1.3 that

$$\frac{2}{n}=\mathrm{hcap}(\eta)\le C\ \mathrm{diam}(\eta)\ \mathrm{height}(\eta),$$

so,

$$\mathrm{height}(\eta)\ge \frac{1}{\sqrt{n}\varphi_4(n)}.$$

The lemma follows by choosing a highest point and letting $\phi := \varphi_5 := \max(\varphi_4,\varphi_2)$. □

With this specific point $\gamma_k(s)$, we can use the inequality (3.7) in Lemma 3.1.6. To have a bound for $\exp(4d_{\mathbb{H},hyp}(z,w))$ one needs to show that $w = \gamma_k^n(r)$ is also in $A_{n,2\sqrt{2},\varphi_5}$. By the same argument in the above lemma, since $\gamma_k^n([0,\frac{1}{n}])$ and $\gamma_{k+1}^n([0,\frac{1}{n}])$ are line segments,

$$\gamma_k^n(\frac{1}{n}) \text{ and } \gamma_{k+1}^n(\frac{1}{n}) \in A_{n,2\sqrt{2},\varphi_5}. \tag{3.11}$$

Lemma 3.2.3. *There exists a subpower function ϕ depending only on φ, c_0 and β such that for all n, k and $r \in [\frac{1}{n}, \frac{2}{n}]$, $\gamma_k^n(r)$ is in the box $A_{n,2\sqrt{2},\phi}$.*

Proof. Notice that $\gamma_k^n(r)$ is the tip of the Loewner curve generated by the driving function $t \mapsto \lambda^n(t+t_k) - \lambda^n(t_k)$, $t \in [0,r]$. It follows from Lemma 3.1.4 that

$$|\mathrm{Re}\,\gamma_k^n(r)| \le \sup\{|\lambda^n(t+t_k) - \lambda^n(t_k)|, t \in [0,r]\} \le \sqrt{\frac{2}{n}}\varphi(\frac{n}{2}) = \frac{\varphi_2(n)}{\sqrt{n}}. \tag{3.12}$$

and

$$\mathrm{Im}\,\gamma_k^n(r) \le 2\sqrt{r} \le 2\sqrt{\frac{2}{n}}.$$

Now the rest of the proof is to find a lower bound for $\mathrm{Im}\,\gamma_k^n(r)$. Fix $r \in [\frac{1}{n}, \frac{2}{n}]$. Let $x + iy := \gamma_{k+1}^n(r - \frac{1}{n})$, $G := (\hat{f}_{t_k}^n)^{-1} \circ \hat{f}_{t_{k+1}}^n$.

Since $\gamma_{k+1}^n([0,\frac{1}{n}])$ is a line segment with the tip $\gamma_{k+1}^n(1/n)$ in $A_{n,2\sqrt{2},\varphi_5}$, by Lemma 3.1.6,

$$d_{\mathbb{H},hyp}(x+iy, iy) \le C\log\varphi_5(n) + C$$

and

$$d_{\mathbb{H},hyp}(x+iy, iy) = d_{\mathbb{H}\setminus\gamma_k^n[0,\frac{1}{n}],hyp}(\gamma_k^n(r), G(iy)) \ge d_{\mathbb{H},hyp}(\gamma_k^n(r), G(iy)).$$

It follows from (3.11) and Lemma 3.1.1 that

$$\mathrm{Im}\,\gamma_k^n(r) \ge \frac{\mathrm{Im}\,G(iy)}{C\varphi_5(n)^C} \ge \frac{\mathrm{Im}\,\gamma_k^n(1/n)}{C\varphi_5(n)^C} \ge \frac{1}{C\sqrt{n}\varphi_5(n)^{C+1}} = \frac{1}{\sqrt{n}\varphi_6(n)}.$$

So $\gamma_k^n(r)$ and $\gamma_k(s)$ are both in $A_{n,2\sqrt{2},\varphi_7}$ with $\varphi_7 = \max(\varphi_5, \varphi_6)$. □

We now apply Lemmas 3.1.6, 3.1.7 and the assumption (3.4) and obtain

$$\begin{aligned} |\hat{f}_{t_k}(z) - \hat{f}_{t_k}(w)| &\le (2\mathrm{Im}\,z)|\hat{f}'_{t_k}(z)|\exp(4d_{\mathbb{H},hyp}(z,w)) \\ &\le C(\mathrm{Im}\,z)\varphi_7(n)^\alpha|\hat{f}'_{t_k}(i\mathrm{Im}\,z)|\exp(C\log\varphi_7(n) + C) \\ &\le (\mathrm{Im}\,z)^{1-\beta}Cc_0\varphi_7(n)^\alpha\exp(C\log\varphi_7(n) + C) \\ &\le (\frac{2\sqrt{2}}{\sqrt{n}})^{1-\beta}Cc_0\varphi_7(n)^\alpha\exp(C\log\varphi_7(n) + C) = \frac{\varphi_8(n)}{\sqrt{n}^{1-\beta}}. \end{aligned} \tag{3.13}$$

The second term in the RHS of (3.9).

Let $u = x + iy := w + \lambda(t_k)$. Since $\lambda(t_k) = \lambda^n(t_k)$,

$$\hat{f}_{t_k}(w) - \hat{f}^n_{t_k}(w) = f_{t_k}(u) - f^n_{t_k}(u).$$

Applying Lemma 3.2.1, we get

$$|f_{t_k}(u) - f^n_{t_k}(u)| \le \varepsilon \exp\left\{\frac{1}{2}\left[\log\frac{I_{t_k,y}|f'_{t_k}(u)|}{y}\log\frac{I_{t_k,y}|(f^n_{t_k})'(u)|}{y}\right]^{1/2} + \log\log\frac{I_{t_k,y}}{y}\right\},$$

where $I_{t_k,y} = \sqrt{4t_k + y^2}$ and $\varepsilon = \sup_{t\in[0,t_k]}|\lambda(t) - \lambda^n(t)| \le \frac{2\varphi(n)}{\sqrt{n}}$.

Since $y = \operatorname{Im} u = \operatorname{Im} w \in [\frac{1}{\sqrt{n}\varphi_7(n)}, \frac{2\sqrt{2}}{\sqrt{n}}]$,

$$\frac{I_{t_k,y}}{y} \le 2\sqrt{2}\sqrt{n}\varphi_7(n).$$

Since $f'_{t_k}(u) = \hat{f}'_{t_k}(w)$, it follows from (3.4) that

$$|f'_{t_k}(u)| \le \frac{c_0}{y^\beta} \le c_0\varphi_7(n)^\beta\sqrt{n}^\beta.$$

Also

$$|(f^n_{t_k})'(u)| \le C(y^{-1} + 1) \le 2C\varphi_7(n)\sqrt{n},$$

where the first inequality holds for all hydrodynamic normalized conformal maps of $\mathbb{H}$.

It follows that

$$\begin{aligned}|\hat{f}_{t_k}(w) - \hat{f}^n_{t_k}(w)| &\le \frac{2\varphi(n)}{\sqrt{n}}\exp\left\{\sqrt{\frac{1+\beta}{2}}\log(c\varphi_7(n)\sqrt{n}) + \log\log 2\sqrt{2n}\varphi_7(n)\right\} \\ &=: \frac{\varphi_9(n)}{\sqrt{n}^{1-\sqrt{\frac{1+\beta}{2}}}}. \qquad (3.14)\end{aligned}$$

End of the proof of Theorem 3.1.2.

It follows from (3.9), (3.13) and (3.14) that

$$|\gamma(s + t_k) - \gamma^n(r + t_k)| \le \frac{\varphi_8(n)}{\sqrt{n}^{1-\beta}} + \frac{\varphi_9(n)}{(\sqrt{n})^{1-\sqrt{\frac{1+\beta}{2}}}} =: \frac{\varphi_{10}(n)}{(\sqrt{n})^{1-\sqrt{\frac{1+\beta}{2}}}}$$

for all $r \in [\frac{1}{n}, \frac{2}{n}]$.

Using the uniform continuity (3.6) of γ, we obtain

$$|\gamma(r) - \gamma^n(r)| \le \frac{\varphi_{11}(n)}{(\sqrt{n})^{1-\sqrt{\frac{1+\beta}{2}}}}$$

for all $r \in [t_{k+1}, t_{k+2}]$ and $0 \leq k \leq n-2$, hence for all $r \in [0,1]$. $\square$

Proof of Corollary 1.1.2.

Under the assumption that the Hölder-1/2 norm is less than 4 or that the curve is the Hilbert space-filling curve, it was shown in [26], [22] and [19] that the unbounded complements of the hulls generated by the driving function are John domains. Therefore, it follows from [28, Chapter 5] that the condition (3.4) is satisfied. $\square$

3.3 Applications

3.3.1 Variants of the algorithm

Variant 1. The conclusion of Theorem 3.1.2 still holds for every λ^n that satisfies:

$$|\lambda^n(t_k) - \lambda(t_k)| \leq \frac{\varphi(n)}{\sqrt{n}} \tag{3.15}$$

and

$$\lambda^n(t) = \sqrt{n}(\lambda^n(t_k) - \lambda^n(t_k))\sqrt{t - t_k} + \lambda^n(t_k) \text{ on } [t_k, t_{k+1}]. \tag{3.16}$$

Indeed, the main inequality (3.9) has a slightly change

$$\begin{aligned} |\gamma(s+t_k) - \gamma^n(r+t_k)| = |\hat{f}_{t_k}(z) - \hat{f}^n_{t_k}(w)| &\leq |\hat{f}_{t_k}(z) - \hat{f}_{t_k}(w + \lambda^n(t_k) - \lambda(t_k))| \\ &+ |f_{t_k}(w + \lambda^n(t_k)) - f^n_{t_k}(w + \lambda^n(t_k))|. \end{aligned} \tag{3.17}$$

We can see that z and $w + \lambda^n(t_k) - \lambda(t_k)$ are still in the same box $A_{n,c,\phi}$. Hence the same argument follows.

Variant 2. Lemma 3.1.1 and the property (3.11), hence Lemma 3.2.3, are still true if instead of square-root-interpolating λ^n on $[t_k, t_{k+1}]$, we consider any function interpolating between $\lambda^n(t_k)$ and $\lambda^n(t_{k+1})$ such that when we run the Loewner equation for $0 \leq t \leq \frac{1}{n}$, the resulting $\gamma^n_{t_k}$ has non decreasing imaginary part on $[0, \frac{1}{n}]$.

In particular, linear-interpolating λ^n

$$\lambda^n(t) = \lambda^n(t_k) + n(\lambda^n(t_{k+1}) - \lambda^n(t_k))(t - t_k) \text{ on } [t_k, t_{k+1}]$$

also give the same conclusion as in Theorem 3.1.2 (see [11] for linear driving functions).

Variant 3. Instead of using tilted slits on each small interval one can use vertical slits [13]. In this case, λ^n is a step function:

$$\lambda^n(t) = \lambda(t_k) \text{ for } t \in [t_k, t_{k+1})$$

and

$$\gamma^n_{t_k}(t) = \lambda(t_k) + 2i\sqrt{t} \text{ on } [0, \frac{1}{n}).$$

However γ^n defined by (3.2) is not a curve. We can do as follows. We compute γ^n at discrete points $t = t_0, t_1, \cdots, t_n$. Then connect them with a straight line in that order. As plotted in [13], it is almost impossible to distinguish this curve and the one from the (main) algorithm. Indeed, the same proof of Theorem 3.1.2 is carried over for this algorithm and the same conclusion of this theorem holds.

3.3.2 Speed of convergence to SLE_κ

We can estimate the speed of convergence of the algorithm to $\gamma^\kappa := \mathrm{SLE}_\kappa$ with $\kappa \neq 8$.

Corollary 3.3.1. *There exist constants $c_1, c_2, c_3, c_4 > 0$ depending on κ such that*

$$\mathbb{P}\left(||\gamma^\kappa - \gamma^m||_{[0,1],\infty} \le \frac{c_1 (\log m)^{c_2}}{\sqrt{m}^{1-\sqrt{\frac{1+\beta}{2}}}} \textit{ for all } m \ge n \right) \ge 1 - \frac{c_3}{n^{c_4}}.$$

In other words, this corollary implies Theorem 1.1.1.

We will apply Theorem 3.1.2 to the case $\lambda(t) = \sqrt{\kappa} B_t$. It follows from [16, Theorem 3.2.4] that there exist constants c_1 (depending on κ) and c_2 such that

$$\mathbb{P}\left\{ osc(\lambda; \frac{1}{m}) \ge c_1 \sqrt{\frac{\log m}{m}} \text{ for all } m \ge n \right\} \le \frac{c_2}{n^2}. \tag{3.18}$$

Notice that in Theorem 3.1.2, if $\varphi(n) = \sqrt{\log n}$ in (3.3) then from the proof of this theorem, the subpower function in (3.5) is of the form $c(\log n)^{c'}$ for some constants c and c'.

It follows from [9, Proposition 4.2] that there exist constants $\beta' \in (0,1)$, c_3 and $c_4 > 0$ depending on κ such that

$$\sum_{m=n}^{\infty} \sum_{j=1}^{2^{2m}} \mathbb{P}\left\{ |\hat{f}'_{(j-1)2^{-2m}}(i2^{-m})| \ge 2^{\beta' m} \right\} \le \frac{c_3}{2^{nc_4}}.$$

This implies that

$$\mathbb{P}\left\{|\hat{f}'_{(j-1)2^{-2m}}(i2^{-m})| \le 2^{\beta' m} \text{ for all } 1 \le j \le 2^{2m}, m \ge n\right\} \ge 1 - \frac{c_3}{2^{nc_4}}.$$

Thus there exist $c_5 > 0$ and $\beta \in (\beta', 1)$ such that

$$\mathbb{P}\left\{|\hat{f}'_t(iy)| \le \frac{c_5}{y^\beta} \text{ for all } 0 \le y \le 2^{-n}, t \in [0,1]\right\} \ge 1 - \frac{c_3}{2^{nc_4}},$$

or

$$\mathbb{P}\left\{|\hat{f}'_t(iy)| \le \frac{c_5}{y^\beta} \text{ for all } 0 \le y \le \frac{1}{\sqrt{n}}, t \in [0,1]\right\} \ge 1 - \frac{c_3}{n^{c_4/2}}. \tag{3.19}$$

Combining (3.18), (3.19) and Theorem 3.1.2, we get

$$\mathbb{P}\left\{||\gamma^\kappa - \gamma^m||_{[0,1],\infty} \le \frac{c_6(\log m)^{c_7}}{\sqrt{m}^{1-\sqrt{\frac{1+\beta}{2}}}} \text{ for all } m \ge n\right\} \ge 1 - \left(\frac{c_2}{n^2} + \frac{c_3}{n^{c_4/2}}\right)$$

which proves Corollary 3.3.1. □

3.3.3 Random walk algorithm to simulate SLE curves

This algorithm [13, Section 2] is based on the Donsker's invariance theorem: a scaling limit of simple random walk converges in distribution to the Brownian motion.

For fix $\kappa \ge 0$. We choose $a \in (0, \frac{1}{2}]$ such that

$$\kappa = \frac{4(1-2a)^2}{a(1-a)}.$$

Let $f_1(z) = (z+1-a)^{1-a}(z-a)^a$, $f_2(z) = (z+a)^a(z-(1-a))^{1-a}$. For every $i \ge 1$, choose $\phi_i = f_1$ or $\phi_i = f_2$ with equal probability. Then we compute inductively $F_n = F_{n-1} \circ \phi_n$ with $F_0 = id$. The map F_n is conformal from $\mathbb{H}$ to $\mathbb{H}$ minus a slit curve. After rescaling and translating so that this slit curve has the half plane capacity 1, we get a simple curve γ^n. More explicitly, γ^n is generated by λ^n whose formula is

$$\lambda^n(t_k) = \sqrt{\kappa}\frac{S_k}{\sqrt{n}} \text{ for all } t_k,$$

$$\text{and } \lambda^n(t) = \sqrt{n}(\lambda^n(t_{k+1}) - \lambda^n(t_k))\sqrt{t-t_k} + \lambda^n(t_k) \text{ on } [t_k, t_{k+1}],$$

where $S_k = X_1 + \cdots + X_k$, $X_i's$ are iid and $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = \frac{1}{2}$.

By Donsker's invariance theorem, $\lambda^n \xrightarrow{d} \sqrt{\kappa}B|_{[0,1]}$ on $C([0,1],||.||_\infty)$. So $\mathbb{H}\backslash\gamma^n([0,1]) \xrightarrow{d} \mathbb{H}\backslash\gamma^\kappa([0,1])$ in the context of Carathéodory kernel convergence [14] and Cauchy transforms of probability measures [2]. Kennedy [13] raised a question whether γ^n converges in distribution to γ^κ.

We now show that γ^n converges in distribution to γ^κ under the sup-norm of $C([0,1])$ when $\kappa \neq 8$. Indeed, it follows from [16, Theorem 7.1.1] that for each n, we can couple λ^n and the Brownian motion in the same probability space such that

$$\mathbb{P}\{\max_{0\leq j\leq n} |\lambda^n(t_j) - \sqrt{\kappa}B_{t_j}| \geq \frac{C\sqrt{\kappa}\log n}{\sqrt{n}}\} \leq Cn^{-3} \tag{3.20}$$

for some universal constant $C > 0$.

Hence from (3.18), (3.19), (3.20) and the discussion of Variant 1, there exist constants c_8 and c_9 depending on κ such that

$$\mathbb{P}\left\{ ||\gamma^n - \gamma^\kappa||_{[0,1],\infty} \leq \frac{c_8(\log n)^{c_9}}{\sqrt{n}^{1-\sqrt{\frac{1+\beta}{2}}}} \right\} \geq 1 - \frac{c_2}{n^2} - \frac{c_3}{n^{c_4/2}} - \frac{C}{n^3}.$$

This implies that γ^n converges in distribution to γ^κ.

Chapter 4

THE EXISTENCE OF LOEWNER CURVES WHEN $||\lambda||_{1/2} < 4$ AND LIPSCHITZ GRAPHS

This chapter presents joint work with Steffen Rohde and Michel Zinsmeister.

The following notation will be used throughout the rest of the chapter: If $z \in \mathbb{H}$ and $f_t(z)$ the solution to (2.3), we define $x_t := x_t(z, \xi)$ and $y_t := y_t(z, \xi)$ by

$$x_t + iy_t := z_t := f_t(z) - \xi_t.$$

It follows that

$$\partial_t(x_t + \xi_t) = \frac{-2x_t}{x_t^2 + y_t^2} \tag{4.1}$$

and

$$\partial_t y_t = \frac{2y_t}{x_t^2 + y_t^2}. \tag{4.2}$$

The following expressions for $|f_t'(z)|$ and $\arg f_t'(z)$ in terms of x_t and y_t will be used to prove Theorems 1.2.1 and 1.2.2. Since

$$f_t'(z) = e^{\log f_t'(z)} = e^{\int_0^t \partial_s \log f_s'(z)\, ds}$$

and

$$\partial_s \log f_s'(z) = \frac{\partial_s f_s'(z)}{f_s'(z)} = \frac{2}{(f_s(z) - \xi(s))^2},$$

we have

$$|f_t'(z)| = \exp\left(2\int_0^t \frac{x_s^2 - y_s^2}{(x_s^2 + y_s^2)^2} ds\right) = \exp\left(\int_0^t \frac{x_s^2 - y_s^2}{x_s^2 + y_s^2} \cdot \frac{2ds}{x_s^2 + y_s^2}\right) \tag{4.3}$$

and

$$\arg f_t'(z) = -4\int_0^t \frac{x_s y_s}{(x_s^2 + y_s^2)^2}\, ds. \tag{4.4}$$

Finally, we will frequently use the following simple estimate for the oscillation of x_t for general driving functions.

Lemma 4.0.2. *Let ξ be an arbitrary continuous function.*

a) If $x_s \geq 0$ for all $0 \leq s \leq t$, then $x_t \leq x_0 + \xi_0 - \xi_t$.

b) In general, $|x_t| \leq |x_0| + M^{\xi}_{0,t}$ where $M^{\xi}_{0,t} = \sup\{|\xi_r - \xi_t| : r \in [0,t]\}$.

Proof. Since $x_s \geq 0$, the sum $x_t + \xi_t$ is nonincreasing by (4.1), and part a) follows. To prove b), by symmetry we may assume that $x_0 \geq 0$, and we may also assume $|x_t| > x_0$, else b) is trivial. Let $S = \sup\{0 \leq s < t : |x_s| \leq x_0\}$ so that $|x_S| = x_0$.

If $x_t \geq 0$ then $x_t > x_0$ and $x_0 = x_S < x_s$ for $S < s \leq t$. Applying a) with z replaced by $x_S + iy_S$ and ξ replaced by $\xi(\cdot + S)$ we get

$$x_t \leq x_S + \xi_S - \xi_t = x_0 + \xi_S - \xi_t.$$

If $x_t < 0$ then $x_t < -x_0$ and $x_s < x_S = -x_0$ for $S < s \leq t$. Now replacing z by $-(x_S + iy_S)$ and ξ by $-\xi(\cdot + S)$, the claim follows again from a).

□

4.1 Staying in a fixed cone

In this section, we restrict our attention to the upward Loewner equation (2.3) with driving function ξ whose Hölder-1/2 norm satisfies

$$\sigma := ||\xi||_{\frac{1}{2}} = \sup_{s \neq t} \frac{|\xi(t) - \xi(s)|}{|t-s|^{1/2}} < 4.$$

Denote A_c the cone $\{x + iy : |x| \leq cy\}$ and $A_c(v) = v + A_c$ for $v \in \mathbb{R}$. The main result of this section is

Theorem 4.1.1. *There is a constant c_σ such that, if $z_0 = iy$, then $z_t = f_t(z_0 + \xi_0) - \xi_t$ stays in the cone A_{c_σ} for all t. Moreover,*

$$\sqrt{\frac{4t}{1 + c_\sigma^2} + y^2} \leq y_t \leq \sqrt{4t + y^2} \tag{4.5}$$

for all $t \geq 0$, and $c_\sigma \leq \sigma/\sqrt{4 - \sigma^2}$ for $\sigma < 2$.

This theorem easily implies the Hölder continuity of f_t, Corollary 4.1.5 below. The intuition behind the proof of Theorem 4.1.1 is as follows. To first order, $\Delta z_t = \frac{-2}{z_t}\Delta t - \Delta\xi$. Therefore, the larger x_t/y_t is, the stronger $\frac{2}{z_t}\Delta t$ pushes towards the middle of the cone, and dominates $\Delta\xi$ if the Hölder-1/2 norm is small.

We will first show that an upper bound on the growth rate of x_t implies a lower bound on y_t that is comparable to the optimal upper bound $y_t \leq \sqrt{4t + y_0^2}$.

Lemma 4.1.2. *If* $|x_t| < M\sqrt{t}$ *for all* $t \geq Cy_0^2$ *with some* $M < 2$ *and* $C > 0$, *then*

$$y_t^2 \geq Lt \tag{4.6}$$

for all $t \geq 0$, *where* $L = \min(1/C, 4 - M^2) > 0$.

Proof. Since $L \leq 1/C$ we have $Lt \leq t/C \leq y_0^2 < y_t^2$ for $0 < t \leq t_0 := Cy_0^2$. If (4.6) were not true, there would be a minimal $s > t_0$ such that $y_s^2 = Ls$ and $y_t^2 \geq Lt$ on $[0, s]$. It follows from (4.2) that

$$\partial_t \, y_t^2 = \frac{4y_t^2}{x_t^2 + y_t^2} \geq \frac{4Lt}{M^2 t + Lt} = \frac{4L}{M^2 + L} \geq L$$

for all $t_0 \leq t \leq s$, which implies $y_s^2 - y_{t_0}^2 \geq L(s - t_0)$. This contradicts the fact

$$y_s^2 - y_{t_0}^2 = Ls - y_{t_0}^2 < L(s - t_0).$$

□

If $\sigma < 2$, then the assumption $|x_t| < M\sqrt{t}$ of the Lemma is satisfied with $M = \sigma$ by Lemma 4.0.2 and arbitrarily small C, and Theorem 4.1.1 follows easily. The reader who is only interested in a short proof of Theorem 1.2.1 for small Hölder-1/2 norm may thus skip ahead to Corollary 4.1.5. To deal with the case $2 \leq \sigma < 4$, we will show that the trivial bound $|x_t| \leq \sigma\sqrt{t}$ can be improved to an estimate $|x_t| < M\sqrt{t}$ for some $M < 2$ and t large enough, if we assume that z_t stays outside a cone. As a first step, we will show

Lemma 4.1.3. *Let* K *and* M *be finite positive constants. If*

$$Ky_t \leq x_t \leq M\sqrt{t} \quad \textit{for all} \quad t \in [t_0, T],$$

then

$$x_t \leq \left(\sigma - \frac{4K^2}{K^2+1}\frac{1}{M}\right)\sqrt{t} + C \;\; \textit{for all } t \in [t_0, T],$$

where $C = (M + 4K^2/(M(K^2+1)))\sqrt{t_0}$.

Proof. It follows from the differential equation (4.1) for $x_t + \xi_t$ that

$$\begin{aligned} x_t + \xi_t - x_{t_0} - \xi_{t_0} &= \int_{t_0}^t \frac{-2x_s}{x_s^2 + y_s^2} ds = \int_{t_0}^t \frac{-2(\frac{x_s}{y_s})^2}{(\frac{x_s}{y_s})^2 + 1} \frac{1}{x_s} ds \\ &\leq \frac{-2K^2}{K^2+1} \int_{t_0}^t \frac{1}{x_s} ds, \end{aligned}$$

so that

$$x_t \leq x_{t_0} - \frac{2K^2}{K^2+1} \int_{t_0}^t \frac{1}{x_s} ds + \sigma\sqrt{t} \leq M\sqrt{t_0} - \frac{2K^2}{K^2+1} \int_{t_0}^t \frac{1}{M\sqrt{s}} ds + \sigma\sqrt{t}$$

$$= \left(\sigma - \frac{4K^2}{(K^2+1)M}\right)\sqrt{t} + \left(M + \frac{4K^2}{(K^2+1)M}\right)\sqrt{t_0}.$$

□

Lemma 4.1.4. *For every $\sigma < 4$ and $\sigma' > \sigma/2$ there are $K > 0$ and $C > 0$ such that, if $x_0 = Ky_0$ and if $x_t \geq Ky_t$ for all $t \geq 0$, then $|x_t| \leq \sigma'\sqrt{t}$ for all $t \geq Cy_0^2$.*

Proof. Let $M_0 = \sigma$, and $K = \sigma/\sqrt{16 - \sigma^2}$. Recursively define

$$M_{n+1} = \sigma - \frac{4K^2}{(K^2+1)M_n}$$

and notice that $M_n \to \sigma/2$ as $n \to \infty$. Hence there is N such that $M_N < \sigma'$. Because $x_t \leq x_0 + \sigma\sqrt{t}$, for every $M_0' > \sigma$ there is C_0 such that $x_t \leq M_0'\sqrt{t}$ for all $t \in [C_0 y_0^2, T]$. It follows from Lemma 4.1.3 that for every $M_1' > M_1$ there is C_1 such that $x_t \leq M_1'\sqrt{t}$ for all $t \in [C_1 y_0^2, T]$. Similarly, by continuity and N applications of Lemma 4.1.3, for every $M_N' > M_N$ there is C_N such that $x_t \leq M_N'\sqrt{t}$ for all $t \in [C_N y_0^2, T]$. The lemma follows by choosing $M_N' = \sigma'$ and setting $C = C_N$. □

We are now ready to give the

Proof of Theorem 4.1.1. If $\sigma < 2$, we simply apply Lemma 4.1.2 with arbitrarily small C and find that

$$\frac{|x_t|}{y_t} \leq \frac{\sigma\sqrt{t}}{L\sqrt{t}} = \frac{\sigma}{\sqrt{4 - \sigma^2}}$$

for all t so that we can take $c_\sigma = \sigma/\sqrt{4 - \sigma^2}$. In general, fix $\sigma' \in (\sigma/2, 2)$ and let K and C be the constants of Lemma 4.1.4. Since $x_0 = 0$, the points z_t are in the cone A_K for all small t (see the figure below.) If for some t, the point z_t is outside A_K, then we can find an interval $[t_1, t_2]$ containing t so that

$$|x_{t_1}| = Ky_{t_1} \text{ and } |x_s| \geq Ky_s \text{ for all } t_1 \leq s \leq t_2,$$

and without loss of generality we may assume $x_s > 0$ on $[t_1, t_2]$. Replacing ξ by $\tilde{\xi}(\cdot) = \xi(\cdot + t_1) - \xi(t_1)$ on $[0, t_2 - t_1]$, we are now in the situation where $x_0 = Ky_0$ and $x_t \geq Ky_t$ for all $t \in [0, T]$ (where $T = t_2 - t_1$). By Lemma 4.1.4, we can apply Lemma 4.1.2 and obtain

$$\frac{x_t}{y_t} \leq \frac{\sigma' \sqrt{t}}{L\sqrt{t}} = \sigma' \max(C, 1/(4 - \sigma'^2))$$

for $t \geq Cy_0^2$, whereas

$$\frac{x_t}{y_t} \leq \frac{x_0 + \sigma\sqrt{t}}{y_0} \leq K + \sigma\sqrt{C}$$

for $t \leq Cy_0^2$. It follows that z_t never leaves the cone A_c where $c = \max(\sigma' C, \sigma'/(4 - \sigma'^2), K + \sigma\sqrt{C})$.

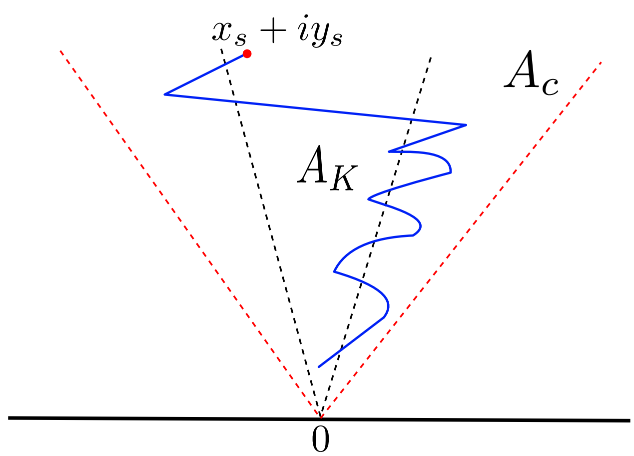


Figure 4.1: A trajectory of $x_t + iy_t$. It never leaves the cone A_c once outside A_K.

Finally, the estimate for y_t follows from $|x_t| \leq cy_t$ and

$$\partial_t y_t^2 = \frac{4y_t^2}{x_t^2 + y_t^2}.$$

□

A simple consequence of Theorem 4.1.1 is the Hölder continuity in bounded subsets of the upper half plane of the solutions f_t to the upward Loewner equation (2.3) with driving functions satisfying $\sigma = ||\xi||_{\frac{1}{2}} < 4$:

Corollary 4.1.5. *If* $\sigma = ||\xi||_{\frac{1}{2}} < 4$, *then*

$$|f_t'(\xi_0 + iy)| \leq (4t + y^2)^{\frac{1-\alpha}{2}} y^{\alpha - 1}$$

for every $y > 0$ *and* $t \in [0, T]$, *where* α *is a constant in* $(0, 1]$ *depending on* σ *only.*

Proof. By (4.2) and (4.3), Theorem 4.1.1 implies that

$$|f_t'(\xi_0 + iy)| \leq \exp\left(\int_0^t \frac{c^2 - 1}{c^2 + 1} \frac{2ds}{x_s^2 + y_s^2}\right) = \left(\frac{y_t}{y}\right)^{\frac{c^2-1}{c^2+1}} \leq (4t + y^2)^{\frac{1-\alpha}{2}} y^{\alpha - 1},$$

where $c = c_\sigma$ and $\alpha = \min\{1 - \frac{c^2-1}{c^2+1}, 1\} \in (0, 1]$. □

Remark 4.1.6. The proof of Theorem 4.1.1 can easily be modified to give the following statement: For every $0 < c_1 < c_2$ there is σ_0 such that, if $z_0 \in A_{c_1}$ and $\sigma \leq \sigma_0$, then $z_t \in A_{c_2}$ for all t. Then (4.5) holds with c_σ replaced by c_2.

Corollary 4.1.7. *There is a constant* σ_0 *such that the following is true: If* $||\xi||_{\frac{1}{2}} \leq \sigma_0$, *if* $0 \leq c \leq 1$ *and* z *is in the cone* $A_c(\xi_0)$, *and if*

$$\int_0^T \frac{M_{0,s}^\xi}{s^{3/2}} ds < \infty,$$

then

$$|\arg f_T'(z)| \leq 8c + 4 \int_0^T \frac{M_{0,s}^\xi}{s^{3/2}} ds.$$

Proof. Let σ_0 be the constant from Remark 4.1.6 with $c_1 = 1$ and $c_2 = \sqrt{3}$. Then if $z_0 \in A_c$ and $c \leq 1$, we have $z_t \in A_{c_2}$ and $y_t \geq \sqrt{y_0^2 + t}$ for all t by (4.5). By (4.4) and Lemma 4.0.2,

$$\begin{aligned} |\arg f_T'(z)| &\leq 4 \int_0^T \frac{|x_s|}{y_s^3} ds \\ &\leq 4 \int_0^T \frac{cy_0 + M_{0,s}^\xi}{(y_0^2 + s)^{3/2}} ds \\ &= 8cy_0 \left(\frac{1}{y_0} - \frac{1}{\sqrt{y_0^2 + T}}\right) + 4 \int_0^T \frac{M_{0,s}^\xi}{s^{3/2}} ds \\ &\leq 8c + 4 \int_0^T \frac{M_{0,s}^\xi}{s^{3/2}} ds. \end{aligned}$$

□

4.2 The proofs of Theorems 1.2.1 and 1.2.2

Throughout this section, we maintain our notation $\sigma = ||\lambda||_{\frac{1}{2}}$, and denote by $\alpha = \alpha_\sigma$ the constant of Corollary 4.1.5. As explained in Chapter 2, in order to show that the Loewner equation generates a curve it suffices to show that

$$v(t,\varepsilon) := \int_0^\varepsilon |(g_t^{-1})'(\lambda_t + iy)|dy$$

goes to zero as $\varepsilon \to 0$, uniformly in $t \in [0,T]$. In our setting, this follows easily from Corollary 4.1.5:

Lemma 4.2.1. *Suppose that $\lambda : [0,T] \to \mathbb{R}$ is Hölder-1/2 continuous with $\sigma < 4$ and (g_t) is the solution to (1.1). Then for every $\varepsilon > 0$ and $0 \le t \le T$,*

$$\int_0^\varepsilon |(g_t^{-1})'(\lambda_t + iy)|\, dy \le \frac{(4t+\varepsilon^2)^{\frac{1-\alpha}{2}}}{\alpha}\varepsilon^\alpha.$$

Proof. Fix $0 \le t \le T$ and $\varepsilon > 0$. Let $\xi(s) = \lambda(t-s)$ for $0 \le s \le t$. Let $(f_s)_{0\le s\le t}$ be the solution to (2.3) with the driving function ξ, so that $g_t^{-1} = f_t$. Hence by Corollary 4.1.5,

$$|(g_t^{-1})'(\lambda_t + iy)| = |f_t'(\xi_0 + iy)| \le (4t+y^2)^{\frac{1-\alpha}{2}} y^{\alpha-1}, \tag{4.7}$$

and the lemma follows by integration. □

Remark 4.2.2. By Proposition 3.9 of [9], we get a quantitative estimate for the modulus of continuity of the trace $\gamma_t := \lim_{y\to 0^+} g_t^{-1}(\lambda_t + iy)$, namely γ is Hölder continuous with exponent $\alpha/2$.

To complete the proof of Theorem 1.2.1, it only remains to show that γ is a simple curve and satisfies the Ahlfors geometric characterization of quasiconformal arcs

$$|\gamma_t - \gamma_s| \le M|\gamma_t - \gamma_r| \tag{4.8}$$

for some constant $M = M_\gamma$ and all $0 \le r \le s \le t \le T$. The key idea is to use the Gehring-Hayman inequality [28, p.72], which says that among all curves in a simply connected plane domain with two fixed end points, the hyperbolic geodesic minimizes the euclidean length, up to a universal multiplicative constant.

Lemma 4.2.3. *If $\sigma < 4$, then γ is a simple curve that stays inside the cone $A_{c_\sigma}(\lambda_0)$ and satisfies* (4.8).

Proof. Again consider the upward Loewner equation (2.3) with the driving function $\xi(s) = \lambda(t-s)$ for $s \in [0,t]$, for fixed $t \in [0,T]$. It follows from Theorem 4.1.1 that for $z = \xi_0 + i\varepsilon = \lambda_t + i\varepsilon$

$$|x_t| \le c_\sigma y_t$$

and

$$\sqrt{\frac{4}{1+c_\sigma^2}t + \varepsilon^2} \le y_t^2 \le \sqrt{4t+\varepsilon^2}.$$

Since $\gamma_t = \lim_{\varepsilon\to 0^+}(g_t^{-1})(\lambda_t + i\varepsilon) = \lim_{\varepsilon\to 0^+}(x_t + iy_t + \xi_t)$, it follows that

$$\frac{2\sqrt{t}}{\sqrt{1+c_\sigma^2}} \le \operatorname{Im}\gamma_t \le 2\sqrt{t}, \tag{4.9}$$

and

$$|\operatorname{Re}\gamma_t - \lambda_0| \le c_\sigma \operatorname{Im}\gamma_t \le 2c_\sigma\sqrt{t}. \tag{4.10}$$

This implies that the curve γ is contained in the cone $A_{c_\sigma}(\lambda_0)$ and meets the real line non-tangentially. It also implies that $\gamma(0,T] \cap \mathbb{R} = \emptyset$, which easily implies that γ is simple (Lemma 4.34 in [14]): Just notice that, if $\gamma_t = \gamma_t'$ for some $t < t'$, then $g_t(\gamma(t,T])$ intersects the real line at λ_t, but that the curve $g_t(\gamma[t,T])$ has driving function $\tilde{\lambda}(t) = \lambda(t+t')$ so that $g_t(\gamma(t,T]) \cap \mathbb{R} = \emptyset$ by the above.

To prove (4.8), fix $0 \le r \le s \le t \le T$, denote $\gamma_r, \gamma_s, \gamma_t$ by u, v, w, and their images under g_r by $u' = \lambda(r), v', w'$. We may assume that the line segment (u,w) is contained in $H_r = \mathbb{H} \setminus \gamma[0,r]$ (else replace u by the point $\hat{u}$ that is closest to w on $(u,w) \cap \gamma[0,r]$, and replace r by $\hat{r} = \gamma^{-1}(\hat{u})$). By (4.9),

$$\operatorname{Im} v' \le 2\sqrt{s-r} \le 2\sqrt{t-r} \le \sqrt{1+c_\sigma^2}\operatorname{Im} w',$$

so that the hyperbolic geodesic $\operatorname{geo}_{\mathbb{H}}(u',v')$ from u' to v' in $\mathbb{H}$ is within bounded hyperbolic distance from $\operatorname{geo}_{\mathbb{H}}(u',w')$. In particular, there is a point $z' = g_r(z)$ on $\operatorname{geo}_{\mathbb{H}}(u',w')$ of bounded hyperbolic distance from v' (where all bounds depend on c_σ only). Denoting ℓ the euclidean length, it follows from the Koebe distortion theorem that

$$|v-w| \le |v-z| + |z-w| \le C\operatorname{dist}(z,\partial H_r) + \ell(\operatorname{geo}_{H_r}(z,w)) \le C\ell(\operatorname{geo}_{H_r}(u,w)),$$

since $z \in \text{geo}_{H_r}(u, v)$. Since the line segment (u, w) is contained in H_r, the Gehring-Hayman inequality implies $\ell(\text{geo}_{H_r}(u, w)) \le C'|u - w|$ and (4.8) follows. This finishes the proof of Lemma 4.2.3 and of Theorem 1.2.1. □

Proof of Theorem 1.2.2. Since we did not assume *a priori* that λ generates a curve, we first observe that

$$||\lambda||_{\frac{1}{2}} \le 3C_0. \tag{4.11}$$

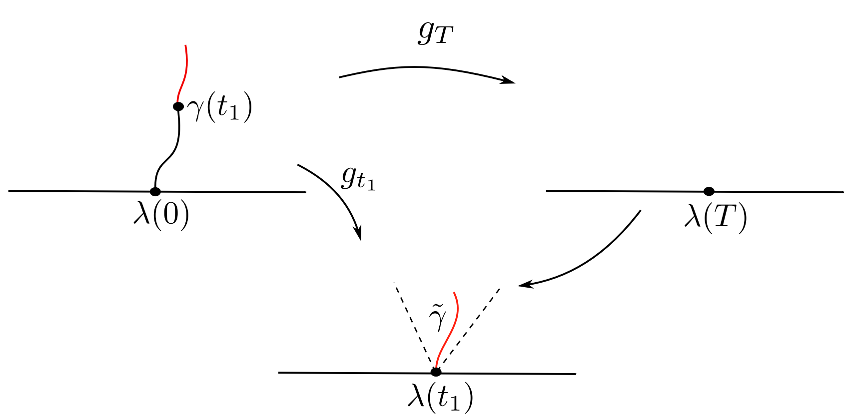
Indeed, since $|\lambda(t_2) - \lambda(t_1)| \le 2N^{\lambda}_{s,t_2}$ for $0 < s \le t_1 < t_2 \le T$, it is not hard to see that λ has a finite Hölder-1/2 norm on every interval $[t_1, t_2]$ inside $(0, T]$. Next, for $t_1 \le s \le r \le t \le t_2$ we have $|\lambda_t - \lambda_s| \le N^{\lambda}_{r,t} + ||\lambda||_{1/2}\sqrt{r - s}$. Integrating both sides of this inequality from s to t with respect to r, dividing by $(t - s)^{3/2}$ and estimating the integral involving N^{λ} by C_0, (4.11) easily follows by choosing s and t appropriately.

If $C_0 < \frac{4}{3}$, Theorem 1.2.1 applies and λ generates a curve γ. We will show that, if C_0 is small enough, then for every pair of points $\gamma(t_1), \gamma(t_2)$ on γ with $0 \le t_1 < t_2 \le T$ we have

$$|\arg(\gamma(t_2) - \gamma(t_1)) - \frac{\pi}{2}| \le C < \frac{\pi}{2}, \tag{4.12}$$

where C depends on C_0 and $\sigma = ||\lambda||_{\frac{1}{2}}$ only. This implies that γ is the graph of a Lipschitz function.

Let $\tilde{\gamma}$ be the image of γ under the map g_{t_1}. This is the curve generated by the driving function $\tilde{\lambda}_t = \lambda_{t+t_1}$, $t \in [0, T - t_1]$, see the figure below. By Lemma 4.2.3 the curve $\tilde{\gamma}$ is



in the cone $A_c(\lambda_{t_1})$, where $c = c_\sigma$ is defined in Section 4.1. With $w = g_{t_1}(\gamma_{t_2}) - \lambda_{t_1}$ we therefore have

$$\begin{aligned} |\arg(\gamma_{t_2} - \gamma_{t_1}) - \frac{\pi}{2}| &= |\arg(w \int_0^1 (g_{t_1}^{-1})'(\lambda_{t_1} + sw)ds) - \frac{\pi}{2}| \\ &\leq \arctan c + \sup_{z \in A_c(\lambda_{t_1})} |\arg(g_{t_1}^{-1})'(z)|. \end{aligned}$$

Applying Corollary 4.1.7 to the driving function $\xi_t = \lambda_{t_1 - t}$ with $t \in [0, t_1]$, assuming C_0 is small enough such that $3C_0 < \sigma_0$ and $c_{3C_0} \leq 1$, we get

$$\begin{aligned} \sup_{z \in A_c(\lambda_{t_1})} |\arg(g_{t_1}^{-1})'(z)| &\leq 8c + 4 \int_0^{t_1} \frac{M_{0,s}^{\xi}}{s^{3/2}} ds \\ &= 8c + 4 \int_0^{t_1} \frac{N_{s,t_1}^{\lambda}}{(t_1 - s)^{3/2}} ds. \end{aligned}$$

Thus

$$|\arg(\gamma_{t_2} - \gamma_{t_1}) - \frac{\pi}{2}| \leq \arctan c + 8c + 4C_0.$$

If $C_0 \to 0$, then $\sigma \to 0$ by (4.11) and therefore $c \to 0$ by Theorem 4.1.1. Thus (4.12) follows if C_0 is sufficiently small, and the theorem is proved. □

Chapter 5

REGULARITY OF LOEWNER CURVES WHEN $\lambda \in C^\alpha$ WITH $\alpha > 1/2$

This chapter presents joint work with Joan Lind. It is organized as follows: Section 5.1 includes initial properties of $f(u, s, \epsilon)$ and some lemmas regarding solutions to a particular class of ODEs. These lemmas will be useful in analyzing f and its partial derivatives, and this is the content of Section 5.2. In Section 5.3, we state and prove a quantitative version of Theorem 1.3.1. Theorem 1.3.2 is proved is Section 5.4, and Theorem 1.3.4 in Section 5.5.1. In Section 5.5.2, we prove Theorem 1.3.3 by constructing a nice curve that well-approximates a given Loewner curve at its base. We conclude in Section 5.6 with two examples.

5.1 Preliminaries

5.1.1 Notation

Let $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. Let I be an interval on the real line. The space $C^0(I)$ consists of all continuous functions on I and $||\phi||_{\infty,I} = \sup_{t\in I} |\phi(t)|$ for $\phi \in C^0(I)$.

Let $\alpha \in (0, 1)$. The function ϕ is in C^α if $||\phi||_{\infty,I} < \infty$ and

$$||\phi||_{C^\alpha} := \sup_{s,t\in I, s\neq t} \frac{|\phi(t) - \phi(s)|}{|t - s|^\alpha} < \infty.$$

Let $n \in \mathbb{N}$, $M > 0$ and $\alpha \in [0, 1]$. The function ϕ is in $C^{n,\alpha}(I; M)$ if $\phi', \cdots, \phi^{(n)}$ exist and are continuous and the following two conditions hold:

$$||\phi^{(k)}||_{\infty,I} \leq M \;\text{ for all } 0 \leq k \leq n,$$
$$\text{and } ||\phi^{(n)}||_{C^\alpha} := \sup_{s,t\in I, s\neq t} \frac{|\phi^{(n)}(t) - \phi^{(n)}(s)|}{|t - s|^\alpha} \leq M.$$

In particular, the n^{th} derivative of functions in $C^{n,1}$ are Lipschitz. A function ϕ is in C^n if $\phi \in C^{n,0}(I; M)$ for some M.

We say that ϕ is weakly $C^1(I)$ if ϕ is in $C^\alpha(I)$ for all $0 < \alpha < 1$, and that ϕ is weakly $C^{n,1}(I)$ if $\phi \in C^n$ and $\phi^{(n)}$ is weakly C^1. The following proposition will be needed in Section 5.5.1.

Proposition 5.1.1. *If a function ϕ belongs to $C^{n,\alpha}(I;M)$ then there exists $c = c(n,M)$ such that for all $t_0, t+t_0 \in I$,*

$$|\phi(t+t_0) - \sum_{k=0}^{n} \frac{1}{k!} t^k \phi^{(k)}(t_0)| \leq ct^{n+\alpha}.$$

The proof follows from the integral form of the remainder of Taylor series.
In this chapter we use C for a universal constant, and c for a constant depending on M, n, T. When constants depend on other factors, we will state this explicitly.

5.1.2 Loewner equation

It was shown that if $\lambda \in C^{1/2}[0,T]$ with $||\lambda||_{C^{1/2}} < 4$ then λ generates a simple quasi-arc γ (previous chapter and [26], [22]). Since we work with $\lambda \in C^\beta$ for $\beta > 2$, we are guaranteed that the corresponding Loewner curve is a simple curve. Further, we can make the assumption that $||\lambda||_{C^{1/2}} \leq 1$ in this chapter. Indeed if $\lambda \in C^\beta$, $\beta > 1/2$ then on small intervals $||\lambda||_{C^{1/2}} \leq 1$. We can apply Theorems 1.3.1 and 1.3.2 on these small intervals, then use the concatenation property of Loewner equation to derive the regularity of γ on $[0,T]$. Henceforth, we assume $||\lambda||_{C^{1/2}} \leq 1$.

We think of (1.5) as a variant of the backward Loewner equation (with $\xi(u) = \lambda(s-u)$ and $f(u) = h_u(i\varepsilon) - \xi(u)$), and our first goal is to understand some basic properties of its solution $f(u) = f(u,s,\epsilon)$, when $(u,s) \in D := \{(u,s) : 0 \leq u \leq s \leq T\}$. Further properties of $f(u,s,\epsilon)$ are in Section 5.2.

Lemma 5.1.2. *Let $\lambda \in C^1([0,T];M)$, and let $0 \leq s \leq T$ and $\varepsilon > 0$. Then the ODE*

$$\begin{aligned} f'(u) &= \frac{-2}{f(u)} + \lambda'(s-u), \qquad 0 \leq u \leq s, \\ f(0) &= i\varepsilon \in \mathbb{H}. \end{aligned}$$

has a unique solution $f(u) = f(u,s,\varepsilon)$, with $0 \leq u \leq s$, satisfying the following properties:
(i) $Im f$ is increasing in u.

(ii) For all $(u,s) \in D = \{(u,s) : 0 \le u \le s \le T\}$

$$\sqrt{3u+\varepsilon^2} \le Im\, f(u,s,\varepsilon) \le \sqrt{4u+\varepsilon^2}$$

$$\text{and} \qquad |Re\, f(u,s,\varepsilon)| \le \sqrt{u} \le \frac{1}{\sqrt{3}} Im\, f(u,s,\varepsilon).$$

(iii) For every $\delta > 0$*, there is* $\varepsilon(\delta) > 0$ *such that*

$$|f(u,s,\varepsilon_1) - f(u,s,\varepsilon_2)| \le \delta \text{ for all } (u,s) \in D \text{ and } \varepsilon_1, \varepsilon_2 \le \varepsilon(\delta).$$

In particular, $f(u,s,\varepsilon)$ *converges uniformly as* $\varepsilon \to 0+$ *to a limit denoted by* $f(u,s)$*. This limit is the family of curves* $\gamma(s-u,s)$ *generated by* λ_s*,* $0 \le s \le T$*.*

(iv) Suppose $\lambda \in C^n([0,T];M)$*, and let* $l+k \le n$ *and* $k \le n-1$*. Then* $\partial_u^l \partial_s^k f$ *exists and is continuous in* $(u,s) \in D$ *for all* $\varepsilon > 0$*.*

(v) If $\lambda \in C^n([0,T];M)$ *and* $1 \le k \le n-1$*, then* $\partial_s^k f(0,s,\varepsilon) = 0$ *for all* $s \in [0,T]$ *and* $\varepsilon > 0$*.*

Proof. The equation (1.5) is of the form:

$$f'(u) = G(f(u),u,s),$$

where $G(z,u,s) = \frac{-2}{z} + \lambda'(s-u)$ is jointly continuous in z,u,s, and Lipschitz in z variable whenever $\mathrm{Im}\, z \ge C > 0$. So the solution exists on some interval containing 0. To show that the solution to (1.5) exists on the whole interval $[0,s]$, it suffices to show that (i) always holds. The idea of $(i)-(iii)$ comes from [32], which contains a study of the Loewner equation when $||\lambda||_{C^{1/2}} < 4$. For the convenience of the reader, we will present the proof here.

Let $x = x(u), y = y(u)$ be real and imaginary parts of $f(u)$. It follows from (1.5) that

$$(x + \lambda(s - \cdot))' = \frac{-2x}{x^2+y^2}, \tag{5.1}$$

$$y' = \frac{2y}{x^2+y^2}. \tag{5.2}$$

In particular, y is increasing and $(y^2)' \le 4$. The former shows (i), and the latter shows that $y \le \sqrt{4u+\varepsilon^2}$.

Now we will show that $|x(u)| \le \sqrt{u}$, for $0 \le u \le s$. Suppose $0 \le x(u)$ and let $u_0 = \sup\{v \in [0,u] : x(v) \le 0\}$. So

$$\partial_v(x(v) + \lambda(s-v)) \le 0 \text{ for } u_0 \le v \le u,$$

and

$$x(u) + \lambda(s-u) \le x(u_0) + \lambda(s-u_0) = \lambda(s-u_0).$$

Hence

$$x(u) \le \lambda(s-u_0) - \lambda(s-u) \le \sqrt{|u_0 - u|} \le \sqrt{u},$$

where the very last inequality follows since $||\lambda||_{1/2} \le 1$. The same argument applies when $x(u) \le 0$, proving that $|x(u)| \le \sqrt{u}$.

Next we will show $y(u) > \sqrt{3u}$ for $0 \le u \le s$. Suppose this is not the case. Then since $y(0) = \varepsilon > 0$, there exists $u_0 \in (0, s]$ such that $y(u_0) = \sqrt{3u_0}$ and $y(u) \ge \sqrt{3u}$ for $u \in [0, u_0]$. It follows from (5.2) that

$$(y^2)' = \frac{4y^2}{x^2 + y^2} \ge \frac{12u}{u + 3u} = 3 \text{ for } 0 \le u \le u_0.$$

So $y(u_0) \ge \sqrt{3u_0 + \varepsilon^2} > \sqrt{3u_0}$. This is a contradiction. Therefore $y(u) > \sqrt{3u}$ and $(y^2)' \ge 3$. These show (ii).

To show (iii), differentiate (1.5) with respect to ε to obtain

$$\partial_u(\partial_\varepsilon f) = \partial_\varepsilon \partial_u f = \frac{2\partial_\varepsilon f}{f^2}.$$

Since $\partial_\varepsilon f(0, s, \varepsilon) = i$,

$$\partial_\varepsilon f(u, s, \varepsilon) = i \exp \int_0^u \frac{2}{f^2(v, s, \varepsilon)}\, dv.$$

This implies

$$\begin{aligned} |\partial_\varepsilon f(u, s, \varepsilon)| &= \exp \int_0^u \operatorname{Re} \frac{2}{f^2(v, s, \varepsilon)}\, dv \\ &= \exp \int_0^u \frac{2(x^2(v) - y^2(v))}{(x^2(v) + y^2(v))^2}\, dv \le 1. \end{aligned}$$

The last inequality comes from (ii). It follows that

$$|f(u, s, \varepsilon) - f(u, s, \varepsilon')| \le |\varepsilon - \varepsilon'|, \text{ for all } 0 \le u \le s \le T,$$

and $f(u,s,\varepsilon)$ converges uniformly in D to a limit, denoted by $f(u,s)$, as $\varepsilon \to 0^+$.

Intuitively the limit $f(u,s)$ is equal to $\gamma(s-u,s)$ since $f(u,s,\varepsilon)$ satisfies the same ODE as $\gamma(s-u,s)$ does, and $\lim_{\varepsilon\to 0^+} f(0,s,\varepsilon) = \gamma(s-u,s)|_{u=0} = 0$. Indeed, from (1.4) and (1.5) we can show that

$$|f(u,s,\varepsilon) - \gamma(s-u,s)| = |f(u_0,s,\varepsilon) - \gamma(s-u_0,s)| \exp \int_{u_0}^{u} \operatorname{Re} \frac{2\,dv}{f(v,s,\varepsilon)\gamma(s-v,s)}, \quad (5.3)$$

with $0 < u_0 \le u \le s \le T$ and $\varepsilon > 0$. Since $\gamma(s-v,s)$ is the tip of a Loewner curve generated by a driving function whose Hölder-1/2 norm is less than 1, then by [40, Lemma 3.1], it satisfies

$$|\operatorname{Re}\gamma(s-v,s)| \le \operatorname{Im}\gamma(s-v,s).$$

This implies that

$$\operatorname{Re} \frac{2}{f(v,s,\varepsilon)\gamma(s-v,s)} \le 0.$$

Let $u_0 \to 0^+$ and then $\varepsilon \to 0^+$ in (5.3) we get $f(u,s) = \gamma(s-u,s)$.

Statement (iv) follows from the standard ODE theory (see [6], for instance) and the fact that G is C^{n-1} in (u,s).

We show (v) by induction. For the base case,

$$\partial_s f(0,s,\varepsilon) = \lim_{\delta\to 0} \frac{f(0,s+\delta,\varepsilon) - f(0,s,\varepsilon)}{\delta} = \lim_{\delta\to 0} \frac{\varepsilon - \varepsilon}{\delta} = 0.$$

Now supppose $\partial_s^k f(0,s,\varepsilon) = 0$ for all $s \in [0,T]$. Then

$$\partial_s^{k+1} f(0,s,\varepsilon) = \lim_{\delta\to 0} \frac{\partial_s^k f(0,s+\delta,\varepsilon) - \partial_s^k f(0,s,\varepsilon)}{\delta} = 0.$$

□

Remark. For convenience, in this paper we only consider $\varepsilon \in (0,1]$. In this case,

$$\sqrt{3u} \le |f(u,s,\varepsilon)| \le \sqrt{Cu+\varepsilon^2} \le C\sqrt{u} + C\varepsilon \le c(T) \text{ for all } 0 \le u,s \le T.$$

Later in Lemma 5.2.2 we will show that $\partial_s^n f$ exists and is continuous in (u,s).

5.1.3 ODE lemmas

The next lemma is one of the key tools to investigate the regularity of $f(u, s, \varepsilon)$.

Lemma 5.1.3. *Consider a complex-valued function X satisfying the initial value problem*

$$X'(u) = P(u)X(u) + Q(u), \qquad X(0) = 0.$$

Suppose $|P(u)| \le -C\,\mathrm{Re}\,P(u)$ and $|Q(u)| \le M_1$ for $0 \le u \le u_0$. Then

$$|X(u)| \le (C+1)M_1 u \text{ for } 0 \le u \le u_0.$$

Proof. Solving the equation, one obtains

$$X(u) = R(u) + e^{-\mu(u)} \int_0^u e^{\mu(v)} P(v) R(v)\, dv,$$

where $\mu(u) = -\int_0^u P(v)\, dv$ and $R(u) = \int_0^u Q(v)\, dv$. Since $|R(u)| \le M_1 u$,

$$\begin{aligned} |X(u)| &\le M_1 u + M_1 u |e^{-\mu(u)}| \int_0^u |e^{\mu(v)}| \cdot |P(v)|\, dv \\ &\le M_1 u + M_1 u e^{-\mathrm{Re}\,\mu(u)} \int_0^u e^{\int_0^v -\mathrm{Re}\,P(w) dw} C(-\mathrm{Re}\,P(v))\, dv \\ &= M_1 u + C M_1 u e^{-\mathrm{Re}\,\mu(u)} \left(e^{-\int_0^u \mathrm{Re}\,P(v)\, dv} - 1 \right) \\ &= M_1 u + C M_1 u e^{-\mathrm{Re}\,\mu(u)} \left(e^{\mathrm{Re}\,\mu(u)} - 1 \right) \\ &\le (C+1) M_1 u. \end{aligned}$$

□

In some cases, we will need a more general version of Lemma 5.1.3.

Lemma 5.1.4. *Let Y be a solution to*

$$Y'(u) = P(u)Y(u) - P(u)Q(u) + R(u), \qquad Y(0) = Q(0)$$

with $|P| \le -C\,\mathrm{Re}\,P$ and $|Q(v) - Q(0)| \le \omega(v)$ on $[0, u_0]$, where ω is a non-decreasing function.

(i) If $|R| \le M_2 u^{\beta-1}$, then

$$|Y(u) - Q(u)| \le (C+1)\omega(u) + (C+1)\frac{M_2}{\beta} u^{\beta}.$$

(ii) If $Y(0) = Q(0) = 0$ *and* $|R| \leq M_2$*, then*

$$|Y(u)| \leq C\omega(u) + (C+1)M_2 u.$$

(iii) More generally,

$$|Y(u) - Q(u)| \leq (C+1)\omega(u) + (C+1)\int_0^u |R(v)|\,dv.$$

Proof. Let $\mu(u) = \int_0^u -P(v)\,dv$ and $S(u) = \int_0^u R(v)\,dv$. We have

$$\begin{aligned} Y(u) &= e^{-\mu(u)}Y(0) + e^{-\mu(u)}\int_0^u e^{\mu(v)}(-PQ+R)\,dv \\ &= Q(0) + e^{-\mu(u)}\int_0^u e^{\mu(v)}(-P)[Q-Q(0)]\,dv + e^{-\mu(u)}\int_0^u e^{\mu(v)}R\,dv \\ &= Q(0) + e^{-\mu(u)}\int_0^u e^{\mu(v)}(-P)[Q-Q(0)]\,dv + S(u) - e^{-\mu(u)}\int_0^u e^{\mu(v)}(-P)S\,dv, \end{aligned}$$

where the last equality follows from an integration by parts. Therefore under the first assumption, $|S(u)| \leq M_2 u^\beta/\beta$ and

$$|Y(u) - Q(u)| \leq |Q(0) - Q(u)| + e^{-\mathrm{Re}\,\mu(u)}\int_0^u e^{\mathrm{Re}\,\mu(v)}C(-\mathrm{Re}\,P)\omega(v)\,dv + |S(u)|$$

$$+e^{-\mathrm{Re}\,\mu(u)}\int_0^u e^{\mathrm{Re}\,\mu(v)}C(-\mathrm{Re}\,P)\frac{M_2}{\beta}u^\beta\,dv$$

$$\leq \omega(u) + C\omega(u) + \frac{M_2}{\beta}u^\beta + C\frac{M_2}{\beta}u^\beta.$$

Under the second assumption,

$$|Y(u)| \leq e^{-\mathrm{Re}\,\mu(u)}\int_0^u e^{\mathrm{Re}\,\mu(v)}C(-\mathrm{Re}\,P)\omega(v)\,dv + |S(u)|$$

$$+e^{-\mathrm{Re}\,\mu(u)}\int_0^u e^{\mathrm{Re}\,\mu(v)}C(-\mathrm{Re}\,P)M_2 u\,dv$$

$$\leq C\omega(u) + M_2 u + CM_2 u.$$

□

5.2 Properties of $f(u, s, \varepsilon)$

In this section, we will prove all important properties of $f(u, s, \varepsilon)$, which are listed in the summary after Lemma 5.2.6. Then we let $\varepsilon \to 0^+$ to get properties of $f(u, s) = \gamma(s - u, s)$. The next two lemmas concern the s-derivatives of f.

Lemma 5.2.1. *Suppose $\lambda \in C^n([0, T]; M)$ with $n \geq 2$. For every $1 \leq k \leq n - 1$, there exists a function $Q_k = Q_k(u, s, \varepsilon)$ such that*

$$\partial_u(\partial_s^k f) = \frac{2}{f^2}\partial_s^k f + Q_k.$$

with $(u, s) \in D$, and $\varepsilon \in (0, 1]$. Moreover there exists constant $c = c(M, n, T) > 0$ so that

$$|\partial_s^k f(u, s, \varepsilon)| \leq cu.$$

Proof. We will prove the lemma by induction. Let $k = 1$ and $n \geq 2$. Fix $s \in [0, T]$ and $\varepsilon \in (0, 1)$, and let $X(u) = \partial_s f(u, s, \varepsilon)$. Then

$$\begin{aligned} X'(u) = \partial_u \partial_s f(u, s, \varepsilon) = \partial_s \partial_u f(u, s, \varepsilon) &= \frac{2}{f^2(u, s, \varepsilon)}\partial_s f(u, s, \varepsilon) + \lambda''(s - u) \\ &= \frac{2}{f^2(u, s, \varepsilon)}X(u) + \lambda''(s - u), \end{aligned}$$

and $X(0) = \partial_s f(0, s, \varepsilon) = 0$. Let $P_s = P_s(u, \varepsilon) = \frac{2}{f^2(u,s,\varepsilon)}$ and $Q_1(u, s, \varepsilon) = \lambda''(s - u)$. Clearly, $|Q_1| \leq M$. We will show that $P_s(\cdot, \varepsilon)$ satisfies the property of P in Lemma 5.1.3. Indeed, let $f(u, s, \varepsilon) = x + iy$. It follows from Lemma 5.1.2(ii) that there exists a constant $C > 0$ such that

$$|P_s(u, \varepsilon)| = \frac{2}{x^2 + y^2} \leq -C\frac{2(x^2 - y^2)}{(x^2 + y^2)^2} = -C \operatorname{Re} P_s(u, \varepsilon).$$

Applying Lemma 5.1.3, we obtain

$$|\partial_s f(u, s, \varepsilon)| \leq cu$$

completing the base case.

Now suppose the lemma holds for $1 \leq k - 1 \leq n - 2$ and $\partial_u(\partial_s^{k-1} f) = P_s \partial_s^{k-1} f + Q_{k-1}$. Then

$$\partial_u \partial_s^k f = \partial_s(\partial_u \partial_s^k f) = P_s \partial_s^k f + Q_k,$$

with $Q_k = \partial_s Q_{k-1} - \frac{4}{f^3}(\partial_s f)(\partial_s^{k-1} f)$. One can show by induction that

$$Q_k = \lambda^{(k+1)}(s-u) + R_k$$

with $R_k(u,s,\varepsilon) = \sum$ terms, where the number of terms is no more than $k-1$ and each term has the form

$$\frac{c}{f^m}\prod_{j=1}^{m-1} \partial_s^{m_j} f,$$

for some $3 \leq m \leq k+1$, and $1 \leq m_j \leq k-1$. This term is by induction no bigger than

$$\frac{c}{u^{m/2}} u^{m-1} = c u^{m/2-1} \leq c(M,k,T)\sqrt{u}.$$

So $|Q_k|$ is bounded by a constant $c = c(M,k,T)$, and hence Lemma 5.1.3 implies that $|\partial_s^k f| \leq cu$. □

Remark. $R_1 = 0$ and R_k satisfies a recursive formula:

$$R_{k+1}(u,s,\varepsilon) = \partial_s R_k(u,s,\varepsilon) - \frac{4}{f(u,s,\varepsilon)^3}(\partial_s f(u,s,\varepsilon))(\partial_s^k f(u,s,\varepsilon)).$$

We have shown that for $1 \leq k \leq n-1$,

$$|R_k| \leq c(M,k,T)\sqrt{u}.$$

Since R_n is only related to $\partial_s^k f$ for $0 \leq k \leq n-1$, we have the same inequality:

$$|R_n| \leq c(M,n,T)\sqrt{u}.$$

Lemma 5.2.2. *Suppose $\lambda \in C^n([0,T];M)$ then $\partial_s^n f(u,s,\varepsilon)$ exists and if $\lambda \in C^{n,\alpha}([0,T];M)$ then*

$$|\partial_s^n f(u,s,\varepsilon)| \leq c u^\alpha,$$

where $c = c(M,n,T)$.

Remark. If $\lambda \in C^n([0,T];M)$ then

$$|\partial_s^n f(u,s,\varepsilon)| \leq c\,\mathrm{osc}(\lambda^{(n)},u,[0,s]) \leq cM.$$

Proof. It follows from the proof of the previous lemma that

$$\begin{aligned}\partial_u(\partial_s^{n-1} f) &= P_s \partial_s^{n-1} f + Q_{n-1} \\ &= P_s \partial_s^{n-1} f + \lambda^{(n)}(s-u) + R_{n-1}.\end{aligned}$$

So

$$\partial_u X = P_s X + Q, \qquad X|_{u=0} = \lambda^{(n-1)}(s),$$

where $X = \partial_s^{n-1} f + \lambda^{(n-1)}(s-u)$ and $Q = -P_s \lambda^{(n-1)}(s-u) + R_{n-1}$. Since Q is C^1 jointly in (u, s), $\partial_s X$ exists and satisfies

$$\partial_u(\partial_s X) = P_s \partial_s X - P_s \lambda^{(n)}(s-u) + R_n.$$

and $\partial_s X|_{u=0} = \lambda^{(n)}(s)$. Hence $\partial_s^n f$ exists and is continuous in (u, s). Since $|R_n| \leq c(M, n, T)$, apply Lemma 5.1.4 (i) with $\omega \equiv M u^\alpha$, $M_2 = c$, and $\beta = 1$ to obtain

$$|\partial_s^n f| = |\partial_s X - \lambda^{(n)}(s-u)| \leq (C+1) M u^\alpha + cu \leq cu^\alpha.$$

□

The next three lemmas concern the oscillation of $\partial_s^k f$ in the variable s. In the proofs, we omit ε from the formulas at times (for ease of reading), but we remind the reader that the functions f, P_s, Q_k, R_k do depend on three variables u, s, ε.

Lemma 5.2.3. *Suppose* $\lambda \in C^{1,\alpha}([0,T]; M)$ *with* $\alpha \in (0, 1]$. *Then*

$$|f(u, s+\delta, \varepsilon) - f(u, s, \varepsilon)| \leq c \min(u\delta^\alpha, \delta u^\alpha),$$

$$|\partial_s f(u, s+\delta, \varepsilon) - \partial_s f(u, s, \varepsilon)| \leq c(1 + \frac{\varepsilon}{\alpha}) \min(u^\alpha, \delta^\alpha)$$

for $0 \leq u \leq s \leq s + \delta \leq T$ *and* $\varepsilon > 0$.

Proof. Since $|\partial_s f(u, s, \varepsilon)| \leq cu^\alpha$ (by Lemma 5.2.2),

$$|f(u, s+\delta, \varepsilon) - f(u, s, \varepsilon)| \leq c\delta u^\alpha.$$

Omitting the parameter ε for convenience, we have

$$\partial_u[f(u, s+\delta) - f(u, s)] = \frac{2}{f(u, s) f(u, s+\delta)}[f(u, s+\delta) - f(u, s)] + \lambda'(s+\delta-u) - \lambda'(s-u),$$

and $f(0, s+\delta) - f(0, s) = 0$. We see that $P := \dfrac{2}{f(u,s)f(u,s+\delta)}$ satisfies

$$|P(u)| \leq -C\mathrm{Re}\, P(u)$$

and that $Q = \lambda'(s+\delta-u) - \lambda'(s-u)$ is bounded by $M\delta^\alpha$. Therefore, Lemma 5.1.3 implies

$$|f(u, s+\delta) - f(u,s)| \leq CMu\delta^\alpha.$$

It remains to prove the last inequality. We have

$$\partial_u[\partial_s f(u, s+\delta) + \lambda'(s+\delta-u)] = P_{s+\delta}\partial_s f(u, s+\delta),$$

and

$$\partial_u[\partial_s f(u,s) + \lambda'(s-u)] = P_s \partial_s f(u,s).$$

So

$$\begin{aligned}
&\partial_u[\partial_s f(u,s+\delta) + \lambda'(s+\delta-u) - \partial_s f(u,s) - \lambda'(s-u)] \\
&\qquad = P_{s+\delta}[\partial_s f(u,s+\delta) + \lambda'(s+\delta-u) - \partial_s f(u,s) - \lambda'(s-u)] \\
&\qquad\qquad - P_{s+\delta}\big(\lambda'(s+\delta-u) - \lambda'(s-u)\big) + (P_{s+\delta} - P_s)\partial_s f(u,s).
\end{aligned}$$

We will apply Lemma 5.1.4 with $Q(u) = \lambda'(s+\delta-u) - \lambda'(s-u)$ and $R(u) = (P_{s+\delta} - P_s)\partial_s f(u,s)$. Note

$$|\lambda'(s+\delta-u) - \lambda'(s-u) - \lambda'(s+\delta) + \lambda'(s)| \leq 2M\min(u^\alpha, \delta^\alpha).$$

Further

$$\begin{aligned}
|P_{s+\delta} - P_s| \cdot |\partial_s f(u,s)| &\leq \frac{c|f(u,s+\delta) - f(u,s)| \cdot |f(u,s) + f(u,s+\delta)|}{u^2} u^\alpha \\
&\leq \frac{cu\delta^\alpha\sqrt{Cu+\varepsilon^2}}{u^2} u^\alpha \\
&\leq c\delta^\alpha u^{\alpha-1/2} + c\delta^\alpha \varepsilon u^{\alpha-1},
\end{aligned}$$

and so

$$\int_0^u |R(v)|\,dv \leq \int_0^u \left(c\delta^\alpha v^{\alpha-1/2} + c\delta^\alpha\varepsilon v^{\alpha-1}\right) dv \leq c\delta^\alpha u^{\alpha+1/2} + c\delta^\alpha \frac{\varepsilon}{\alpha} u^\alpha.$$

Therefore, by Lemma 5.1.4 (iii) with $\omega \equiv 2M\min(u^\alpha, \delta^\alpha)$,

$$|\partial_s f(u, s+\delta) - \partial_s f(u,s)| \le CM\min(u^\alpha, \delta^\alpha) + c\delta^\alpha u^{\alpha+1/2} + c\delta^\alpha \frac{\varepsilon}{\alpha} u^\alpha$$
$$\le c(1+\frac{\varepsilon}{\alpha})\min(u^\alpha, \delta^\alpha).$$

□

Lemma 5.2.4. *Suppose* $\lambda \in C^{n,\alpha}([0,T];M)$ *with* $n \ge 2$ *and* $\alpha \in (0,1]$. *Then*

$$|R_k(u, s+\delta, \varepsilon) - R_k(u,s,\varepsilon)| \le c\delta\sqrt{u} \qquad when \qquad 1 \le k \le n-1,$$

and

$$|\partial_s^k f(u, s+\delta, \varepsilon) - \partial_s^k f(u,s,\varepsilon)| \le cu\delta \qquad when \qquad 1 \le k \le n-2,$$

and

$$|\partial_s^{n-1} f(u, s+\delta, \varepsilon) - \partial_s^{n-1} f(u,s,\varepsilon)| \le c\min(u^\alpha\delta, u\delta^\alpha).$$

Proof. From the Remark following Lemma 5.2.1, we know that $R_1 = 0$, R_k satisfies the recursive formula:

$$R_{k+1} = \partial_s R_k - \frac{4}{f^3}(\partial_s f)(\partial_s^k f),$$

and $|R_k| \le c\sqrt{u}$ for $1 \le k \le n$. Therefore, for $k+1 \le n$, Lemma 5.2.1 implies that

$$|\partial_s R_k| \le |R_{k+1}| + \frac{4}{|f|^3}|\partial_s f| \cdot |\partial_s^k f|$$
$$\le c\sqrt{u}.$$

Thus

$$|R_k(u, s+\delta, \varepsilon) - R_k(u,s,\varepsilon)| \le \int_s^{s+\delta} |\partial_s R_k(u,r,\varepsilon)|dr \le c\delta\sqrt{u},$$

proving the first statement.

When $1 \le k \le n-2$, Lemma 5.2.1 implies that

$$|\partial_s^k f(u, s+\delta, \varepsilon) - \partial_s^k f(u,s,\varepsilon)| \le \int_s^{s+\delta} |\partial_s^{k+1} f(u,r,\varepsilon)|dr \le cu\delta,$$

proving the second statement. From Lemma 5.2.2

$$|\partial_s^{n-1} f(u, s+\delta, \varepsilon) - \partial_s^{n-1} f(u,s,\varepsilon)| \le \int_s^{s+\delta} |\partial_s^n f(u,r,\varepsilon)|dr \le cu^\alpha\delta.$$

To prove the third statement, it remains to show

$$|\partial_s^{n-1} f(u, s+\delta, \epsilon) - \partial_s^{n-1} f(u, s, \epsilon)| \leq c\delta^\alpha u. \tag{5.4}$$

Omitting the parameter ε, we have

$$\begin{aligned}\partial_u[\partial_s^{n-1} f(u, s+\delta) - \partial_s^{n-1} f(u, s)] = P_{s+\delta}[\partial_s^{n-1} f(u, s+\delta) - \partial_s^{n-1} f(u, s)] \\ + (\lambda^{(n)}(s+\delta-u) - \lambda^{(n)}(s-u)) \\ + (P_{s+\delta} - P_s)\partial_s^{n-1} f(u, s) \\ + R_{n-1}(u, s+\delta) - R_{n-1}(u, s).\end{aligned}$$

Since

$$|\lambda^{(n)}(s+\delta-u) - \lambda^{(n)}(s-u)| \leq M\delta^\alpha,$$

and

$$|P_{s+\delta} - P_s| \cdot |\partial_s^{n-1} f(u, s)| \leq \frac{c\delta u C}{u^2} u \leq c\delta \leq c\delta^\alpha,$$

and

$$|R_{n-1}(u, s+\delta) - R_{n-1}(u, s)| \leq c\delta\sqrt{u} \leq c\delta^\alpha,$$

we apply Lemma 5.1.3 with $M_1 = c\delta^\alpha$ to prove (5.4). □

Lemma 5.2.5. *Suppose* $\lambda \in C^{n,\alpha}([0,T]; M)$ *with* $n \geq 2$ *and* $\alpha \in (0, 1]$. *There exists* $c = c(M, n, T)$ *so that*

$$\begin{aligned}|R_{n+1}(u, s, \varepsilon)| &\leq cu^{\alpha-1/2}, \\ |R_n(u, s+\delta, \varepsilon) - R_n(u, s, \varepsilon)| &\leq cu^{\alpha-1/2}\delta, \\ |\partial_s^n f(u, s+\delta, \varepsilon) - \partial_s^n f(u, s, \varepsilon)| &\leq c(1+\frac{\varepsilon}{\alpha}) \min(u^\alpha, \delta^\alpha).\end{aligned}$$

Proof. Let's note that

$$R_n = \sum \frac{c}{f^m} \prod_{j=1}^{m-1} \partial_s^{m_j} f$$

with $3 \leq m \leq n+1$, $1 \leq m_j \leq n-1$, and the number of terms in the sum is no more than $n-1$. Since $\partial_s^n f$ exists, so does R_{n+1}:

$$R_{n+1} = \sum \frac{c}{f^m} \prod_{j=1}^{m-1} \partial_s^{m_j} f,$$

with $3 \leq m \leq n+2$ and $1 \leq m_j \leq n$. We can check that in each product, there is at most one $m_j = n$. Hence

$$|R_{n+1}| \leq cn\frac{u^{m-2}u^{\alpha}}{u^{m/2}} \leq cu^{\alpha+m/2-2} \leq c(M,n,T)u^{\alpha-1/2},$$

and

$$|\partial_s R_n| \leq |R_{n+1}| + \frac{4}{|f|^3}|\partial_s f| \cdot |\partial_s^n f| \leq cu^{\alpha-1/2}.$$

This implies that

$$|R_n(u,s+\delta) - R_n(u,s)| \leq cu^{\alpha-1/2}\delta.$$

It remains to prove the last statement. Now we have

$$\partial_u(\partial_s^n f(u,s+\delta) + \lambda^{(n)}(s+\delta-u)) = P_{s+\delta}\partial_s^n f(u,s+\delta) + R_n(u,s+\delta),$$

and

$$\partial_u(\partial_s^n f(u,s) + \lambda^{(n)}(s-u)) = P_s\partial_s^n f(u,s) + R_n(u,s).$$

Let

$$Y(u) = \partial_s^n f(u,s+\delta) + \lambda^{(n)}(s+\delta-u) - \partial_s^n f(u,s) - \lambda^{(n)}(s-u) \text{ and}$$
$$Q(u) = \lambda^{(n)}(s+\delta-u) - \lambda^{(n)}(s-u).$$

Then

$$\partial_u Y = P_{s+\delta}Y - P_{s+\delta}Q + (P_{s+\delta} - P_s)\partial_s^n f(u,s) + R_n(u,s+\delta) - R_n(u,s).$$

We see that

$$|Q(u) - Q(0)| \leq c\min(u^{\alpha}, \delta^{\alpha}),$$

and

$$|(P_{s+\delta} - P_s)\partial_s^n f(u,s)| \leq \frac{cu\delta\sqrt{Cu+\varepsilon^2}}{u^2}u^{\alpha} \leq c\delta u^{\alpha-1/2} + c\varepsilon\delta u^{\alpha-1}.$$

By Lemma 5.1.4 *(iii)* with $|R(u)| \leq c\delta u^{\alpha-1/2} + c\varepsilon\delta u^{\alpha-1}$,

$$|\partial_s^n f(u,s+\delta,\varepsilon) - \partial_s^n f(u,s,\varepsilon)| = |Y - Q| \leq c\min(u^{\alpha}, \delta^{\alpha}) + c\delta u^{\alpha+1/2} + \frac{c\varepsilon\delta}{\alpha}u^{\alpha}.$$

□

Lemma 5.2.6. *(Boundedness of mixed u and s derivatives.) Suppose $\lambda \in C^n([0,T];M)$. Let $s_0 \in (0,T)$ and $D_0 = \{(u,s) \in D : s_0 \le u\}$. There exists $L_0 = L_0(M,n,T,s_0)$ such that for all $l+k \le n$,*

$$|\partial_u^l \partial_s^k f(u,s,\varepsilon)| \le L_0.$$

In other words, $f \in C^n(D_0; L_0)$ for every $\varepsilon \in (0,1]$.

Proof. The case $l = 0$ and $k \le n$ is proven by Lemmas 5.2.1 and 5.2.2. Consider $k = 0$ and $1 \le l \le n$. We have

$$\partial_u f = \frac{-2}{f} + \lambda'(s-u).$$

This implies that when $u_0 \le u$,

$$|\partial_u f| \le \frac{2}{C\sqrt{u}} + M \le L_0.$$

We can show by induction in l that

$$\partial_u^l f = \frac{2}{f^2}\partial_u^{l-1} f + (-1)^{l-1}\lambda^{(l)}(s-u) + \hat{R}_l,$$

where $\hat{R}_l$ is the sum of a finite number (depending on l) of terms of the form

$$\frac{c}{f^m}\prod_{j=1}^{m-1}\partial_u^{m_j} f$$

with $3 \le m \le l-1$ and $1 \le m_j \le l-2$. Hence by induction $|\partial_u^l f| \le L_0$ for $s_0 \le u \le T$. The other cases $1 \le k \le n-1$ are proved similarly.

□

Summary. We have proved the following properties of $f(u,s,\varepsilon)$:

- $C\sqrt{u+\varepsilon^2} \le |f(u,s,\varepsilon)| \le C'\sqrt{u} + C'\varepsilon$.

- $|\partial_s^k f(u,s,\varepsilon)| \le cu$ for $1 \le k \le n-1$.

- $|\partial_s^n f(u,s,\varepsilon)| \le cu^\alpha$.

- $|\partial_s^k f(u,s+\delta,\varepsilon) - \partial_s^k f(u,s,\varepsilon)| \le cu\delta$ for $1 \le k \le n-2$.

- $|\partial_s^{n-1} f(u,s+\delta,\varepsilon) - \partial_s^{n-1} f(u,s,\varepsilon)| \leq c\min(u\delta^\alpha, u^\alpha\delta)$ if $0 \leq n-1$.

- $|\partial_s^n f(u,s+\delta,\varepsilon) - \partial_s^n f(u,s,\varepsilon)| \leq c(1+\frac{\varepsilon}{\alpha})\min(u^\alpha, \delta^\alpha)$ for $1 \leq n$.

- For every $0 < s_0 < T$, there exists $L_0 = L_0(M,n,T,s_0)$ such that for all $l+k \leq n$, $|\partial_u^l \partial_s^k f(u,s,\varepsilon)| \leq L_0$.

We emphasize that c depends only on M, n, T, not on α and ε. We know from Lemma 5.1.2 that $f(u,s,\varepsilon)$ converges uniformly in D to $f(u,s)$ as $\varepsilon \to 0^+$. For all $l+k = n$, it follows from the proof of previous lemmas that $\partial_u^l \partial_s^k f(u,s,\varepsilon)$ can be expressed in terms of lower derivatives in u and s of $f(u,s,\varepsilon)$. Therefore in $D_0 = \{(u,s) \in D : 0 < s_0 \leq u \leq s \leq T\}$, $\partial_u^l \partial_s^k f(u,s,\varepsilon)$ converges uniformly. This implies that $f(u,s)$ is in $C^n(D_0)$ and satisfies

- $C\sqrt{u} \leq |f(u,s)| \leq C'\sqrt{u}$.

- $|\partial_s^k f(u,s)| \leq cu$ for $1 \leq k \leq n-1$.

- $|\partial_s^n f(u,s)| \leq cu^\alpha$.

- $|\partial_s^k f(u,s+\delta) - \partial_s^k f(u,s)| \leq cu\delta$ for $1 \leq k \leq n-2$.

- $|\partial_s^{n-1} f(u,s+\delta) - \partial_s^{n-1} f(u,s)| \leq c\min(u\delta^\alpha, u^\alpha\delta)$ if $0 \leq n-1$.

- $|\partial_s^n f(u,s+\delta) - \partial_s^n f(u,s)| \leq c\min(u^\alpha, \delta^\alpha)$ for $1 \leq n$.

- For every $0 < s_0 < T$, there exists $L_0 = L_0(M,n,T,s_0)$ such that for all $l+k \leq n$, $|\partial_u^l \partial_s^k f(u,s)| \leq L_0$.

Corollary 5.2.7. *If λ is in $C^{n,\alpha}[0,T]$ with $n \geq 2$ and $\alpha \in (0,1]$ then γ is in $C^n(0,T]$.*

Proof. The previous arguments imply that $\gamma(s-u,s) \in C^n(D_0)$ for every $s_0 \in (0,T)$. Hence $s \mapsto \gamma(0,s) \in C^n(0,T]$. Since $\gamma(s) = \gamma(0,s) + \lambda(0)$, the curve γ is in $C^n(0,T]$. □

5.3 *Smoothness of* γ

The goal of this section is to prove the following:

Theorem 5.3.1. *Suppose* $\lambda \in C^{n,\alpha}([0,T];M)$ *with* $n \geq 2$ *and* $\alpha \in (0,1]$.

(i) If $\alpha < 1/2$ *then* $\gamma \in C^{n,\alpha+1/2}(0,T]$. *And for every* $0 < s_0 < T$, *there exists* $c_0 = c_0(M,n,T,s_0)$ *such that* $\gamma \in C^n([s_0,T];c_0)$ *and*

$$|\gamma^{(n)}(s+\delta) - \gamma^{(n)}(s)| \leq \frac{c_0}{1-2\alpha}\delta^{\alpha+1/2},$$

(ii) If $\alpha = 1/2$ *then* γ *is weakly in* $C^{n,1}(0,T]$. *For every* $0 < s_0 < T$, *there exists* $c_0 = c_0(M,n,T,s_0)$ *such that* $\gamma \in C^n([s_0,T];c_0)$ *and*

$$|\gamma^{(n)}(s+\delta) - \gamma^{(n)}(s)| \leq c_0\delta\left(1 + \log^+ \frac{s}{\delta}\right).$$

(iii) If $\alpha \in (\frac{1}{2}, 1]$ *then* $\gamma \in C^{n+1,\alpha-1/2}(0,T]$. *And there exists* $c_0 = c_0(M,n,T,s_0)$ *such that*

$$|\gamma^{(n+1)}(s+\delta) - \gamma^{(n+1)}(s)| \leq \frac{c_0}{2\alpha-1}\delta^{\alpha-1/2}.$$

Proof. Assume that $\lambda \in C^{n,\alpha}([0,T];M)$ with $n \geq 2$ and $\alpha \in (0,1]$. Fix $s_0 \in (0,T)$ and let $D_0 = \{(u,s) \in D : 0 < s_0 \leq u \leq s \leq T\}$. Recall from [40] that

$$\gamma''(s) = \frac{2\gamma'(s)}{\gamma(s)^2} - 4\gamma'(s)\int_0^s \frac{\partial_s[f(u,s)]}{f(u,s)^3}\,du.$$

We need to show

$$F(s) := \int_0^s \frac{\partial_s f(u,s)}{f(u,s)^3}\,du \text{ is } \begin{cases} \text{in } C^{n-2} & \text{and} \quad F^{(n-2)} \in C^{\alpha+1/2} \text{ when } \alpha \in (0,1/2) \\ \text{in } C^{n-2} & \text{and} \quad F^{(n-2)} \text{ weakly in } C^1 \text{ when } \alpha = 1/2 \\ \text{in } C^{n-1} & \text{and} \quad F^{(n-1)} \in C^{\alpha-1/2} \text{ when } \alpha \in (1/2,1] \end{cases}.$$

Let $F_1(u,s) = \dfrac{\partial_s f(u,s)}{f(u,s)^3}$ and $\hat{R}_1(u,s) = 0$. We define F_k and $\hat{R}_k$ recursively as follows:

$$\begin{aligned} \hat{R}_k &= \partial_s \hat{R}_{k-1} - \frac{3(\partial_s f)(\partial_s^{k-1} f)}{f^4}, \\ F_k &= \partial_s F_{k-1} = \frac{\partial_s^k f}{f^3} + \hat{R}_k. \end{aligned}$$

Let $\hat{F}_k(s) = F_k(s,s)$. Then formally

$$F^{(n-2)}(s) = \hat{F}_1^{(n-3)}(s) + \hat{F}_2^{(n-4)}(s) + \cdots \hat{F}_{n-2}(s) + \int_0^s \left[\frac{\partial_s^{n-1} f(u,s)}{f^3(u,s)} + \hat{R}_{n-1}(u,s)\right] du, \quad (5.5)$$

and

$$F^{(n-1)}(s) = \hat{F}_1^{(n-2)}(s) + \hat{F}_2^{(n-3)}(s) + \cdots \hat{F}_{n-1}(s) + \int_0^s \left[\frac{\partial_s^{n} f(u,s)}{f^3(u,s)} + \hat{R}_{n}(u,s)\right] du. \quad (5.6)$$

We notice that

$$\hat{R}_k = \sum \frac{c}{f^m} \prod_{j=1}^{m-2} (\partial_s^{m_j} f), \quad (5.7)$$

where there are at most $k-1$ terms for the sum, $4 \le m \le k+2$, and $1 \le m_j \le k-1$. Further, when $k \ge 3$ each product contains at most one $m_j = k-1$. Therefore, $\hat{R}_k \in C^{n-(k-1)}(D_0)$, $F_k \in C^{n-k}(D_0)$ and $\hat{F}_k \in C^{n-k}[s_0, T]$. The representation of $\hat{R}_k$ in (5.7) also implies that

$$|\hat{R}_k(u,s)| \le c \text{ for } 1 \le k \le n, \quad (5.8)$$

$$\text{and} \quad |\hat{R}_{n+1}(u,s)| \le \frac{c}{u^{1/2}} \text{ if } \alpha \ge \frac{1}{2}. \quad (5.9)$$

Hence equation (5.5) holds for all $\alpha \in (0,1]$ and equation (5.6) holds when $\alpha \in (1/2, 1]$.

Let

$$I_k(s) := \int_0^s \frac{\partial_s^k f(u,s)}{f(u,s)^3} du \text{ and } IR_k(s) = \int_0^s \hat{R}_k(u,s)\, du.$$

Theorem 5.3.1 will be proven once we show that

- $I_{n-1} + IR_{n-1} \in C^{\alpha+1/2}[s_0, T]$ for $\alpha \in (0, 1/2)$,
- $I_{n-1} + IR_{n-1}$ is weakly $C^1[s_0, T]$ for $\alpha = 1/2$, and
- $I_n + IR_n \in C^{\alpha-1/2}[s_0, T]$ for $\alpha \in (1/2, 1]$,

along with the needed bounds on $|I_k(s+\delta) - I_k(s)|$ and $|IR_k(s+\delta) - IR_k(s)|$. This is the content of the next three lemmas.

□

Lemma 5.3.2. *Suppose* $\lambda \in C^{n,\alpha}([0,T];M)$*, with* $n \geq 2$ *and* $\alpha \in (0, \frac{1}{2}]$*. Then there exists* $c = c(M,n,T)$ *such that for all* $0 < s_0 \leq s \leq s+\delta \leq T$*,*

$$\begin{aligned} |IR_k(s+\delta) - IR_k(s)| &\leq c\delta \text{ for all } 1 \leq k \leq n-1 \text{ and} \\ |IR_n(s+\delta) - IR_n(s)| &\leq c\delta \text{ if } \alpha \geq \frac{1}{2}. \end{aligned}$$

Proof. It follows from the definition of $\hat{R}_k$ and formula (5.8) that for $1 \leq k \leq n-1$,

$$|\hat{R}_k(u, s+\delta) - \hat{R}_k(u,s)| \leq \int_s^{s+\delta} |\partial_v \hat{R}_k(u,v)|\, dv \leq c\delta.$$

Similarly if $\alpha \geq \frac{1}{2}$ equation (5.9) implies

$$|\hat{R}_n(u, s+\delta) - \hat{R}_n(u,s)| \leq \frac{c\delta}{u^{1/2}}.$$

Integrating completes the lemma. □

Lemma 5.3.3. *Suppose* $\lambda \in C^{n,\alpha}([0,T];M)$*, with* $n \geq 2$ *and* $\alpha \in (0, \frac{1}{2}]$*. Then* $I_{n-1} \in C^{\alpha+1/2}[s_0,T]$ *when* $\alpha \in (0,1/2)$ *and* I_{n-1} *is weakly* $C^1[s_0,T]$ *when* $\alpha = 1/2$*. In particular, there exists* $c = c(M,n,T)$ *such that for all* $0 < s_0 \leq s \leq s+\delta \leq T$*,*

$$|I_{n-1}(s+\delta) - I_{n-1}(s)| \leq \begin{cases} c(\frac{1}{1-2\alpha}+1)\delta^{\alpha+1/2} + c(1+\frac{1}{\sqrt{s_0}})\delta & \text{when} \quad 0 < \alpha < \frac{1}{2} \\ c(1+\log^+\frac{s}{\delta}+\frac{1}{\sqrt{s_0}})\delta & \text{when} \quad \alpha = \frac{1}{2} \end{cases}.$$

Proof. We decompose $I_{n-1}(s+\delta) - I_{n-1}(s)$ into the sum of four integrals and bound each integral.

$$\begin{aligned} I_{n-1}(s+\delta) - I_{n-1}(s) &= \int_0^{\delta\wedge s} \frac{\partial_s^{n-1} f(u,s+\delta) - \partial_s^{n-1} f(u,s)}{f(u,s+\delta)^3}\, du \\ &+ \int_{\delta\wedge s}^{s} \frac{\partial_s^{n-1} f(u,s+\delta) - \partial_s^{n-1} f(u,s)}{f(u,s+\delta)^3}\, du \\ &+ \int_0^{s} \frac{\partial_s^{n-1} f(u,s)(f(u,s)^3 - f(u,s+\delta)^3)}{f(u,s)^3 f(u,s+\delta)^3}\, du \\ &+ \int_s^{s+\delta} \frac{\partial_s^{n-1} f(u,s+\delta)}{f(u,s+\delta)^3}\, du. \end{aligned}$$

The first integral:

$$\begin{aligned} \left| \int_0^{\delta\wedge s} \frac{\partial_s^{n-1} f(u,s+\delta) - \partial_s^{n-1} f(u,s)}{f(u,s+\delta)^3}\, du \right| &\leq \int_0^{\delta\wedge s} \frac{cu\delta^\alpha}{u^{3/2}}\, du \\ &= c\delta^\alpha \sqrt{\delta\wedge s} \leq c\delta^{\alpha+1/2}. \end{aligned}$$

The second integral, when $0 < \alpha < 1/2$:

$$\begin{aligned}\left|\int_{\delta\wedge s}^{s} \frac{\partial_s^{n-1} f(u,s+\delta) - \partial_s^{n-1} f(u,s)}{f(u,s+\delta)^3}\,du\right| &\leq \int_{\delta\wedge s}^{s} \frac{cu^\alpha \delta}{u^{3/2}}\,du \\ &\leq \frac{c\delta}{1-2\alpha}(\delta^{\alpha-1/2} - s^{\alpha-1/2}) \\ &\leq \frac{c}{1-2\alpha}\delta^{\alpha+1/2}.\end{aligned}$$

In the case $\alpha = 1/2$, the second integral is bounded by

$$\int_{\delta\wedge s}^{s} c\delta u^{-1}\,du = c\delta \log \frac{s}{s\wedge\delta} = c\delta \log^{+} \frac{s}{\delta}.$$

The third integral:

$$\begin{aligned}\left|\int_0^s \frac{\partial_s^{n-1} f(u,s)(f(u,s)^3 - f(u,s+\delta)^3)}{f(u,s)^3 f(u,s+\delta)^3)}\,du\right| &\leq \int_0^s \frac{cu(u\delta u)}{u^3}\,du \\ &= c\delta s \leq c\delta.\end{aligned}$$

The last integral:

$$\begin{aligned}\left|\int_s^{s+\delta} \frac{\partial_s^{n-1} f(u,s+\delta)}{f(u,s+\delta)^3}\,du\right| \leq \int_s^{s+\delta} \frac{cu}{u^{3/2}}\,du &= c(\sqrt{s+\delta} - \sqrt{s}) \\ &= \frac{c\delta}{\sqrt{s+\delta}+\sqrt{s}} \leq \frac{c}{\sqrt{s_0}}\delta.\end{aligned}$$

□

Lemma 5.3.4. *Suppose $\lambda \in C^{n,\alpha}([0,T];M)$ with $n \geq 2$ and $\alpha \in (\frac{1}{2}, 1]$. Then $I_n \in C^{\alpha-1/2}[s_0,T]$, and there exists $c = c(M,T,n)$ such that for all $0 \leq s \leq s+\delta \leq T$*

$$|I_n(s+\delta) - I_n(s)| \leq \frac{c}{2\alpha-1}\delta^{\alpha-1/2}.$$

Proof. We proceed in a manner similar to the previous proof.

$$\begin{aligned}I_n(s+\delta) - I_n(s) &= \int_0^{\delta\wedge s} \frac{\partial_s^n f(u,s+\delta) - \partial_s^n f(u,s)}{f(u,s+\delta)^3}\,du \\ &+ \int_{\delta\wedge s}^{s} \frac{\partial_s^n f(u,s+\delta) - \partial_s^n f(u,s)}{f(u,s+\delta)^3}\,du \\ &+ \int_0^s \frac{\partial_s^n f(u,s)(f(u,s)^3 - f(u,s+\delta)^3)}{f(u,s)^3 f(u,s+\delta)^3}\,du \\ &+ \int_s^{s+\delta} \frac{\partial_s^n f(u,s+\delta)}{f(u,s+\delta)^3}\,du.\end{aligned}$$

The first integral:

$$\begin{aligned}\left|\int_0^{\delta\wedge s} \frac{\partial_s^n f(u,s+\delta) - \partial_s^n f(u,s)}{f(u,s+\delta)^3}\, du\right| &\le \int_0^{\delta\wedge s} \frac{c\min(u^\alpha,\delta^\alpha)}{u^{3/2}}\, du \\ &\le c\int_0^{\delta\wedge s} u^{\alpha-3/2}\, du \le \frac{c}{2\alpha-1}\delta^{\alpha-1/2}.\end{aligned}$$

The second integral:

$$\begin{aligned}\left|\int_{\delta\wedge s}^{s} \frac{\partial_s^n f(u,s+\delta) - \partial_s^n f(u,s)}{f(u,s+\delta)^3}\, du\right| &\le \int_{s\wedge\delta}^{s} \frac{c\min(u^\alpha,\delta^\alpha)}{u^{3/2}}\, du \\ &\le \int_{s\wedge\delta}^{s} \frac{c\delta^\alpha}{u^{3/2}}\, du \le c\delta^\alpha(\delta^{-1/2} - s^{-1/2}) \le c\delta^{\alpha-1/2}.\end{aligned}$$

The third integral:

$$\begin{aligned}\left|\int_0^{s} \frac{\partial_s^n f(u,s)(f(u,s)^3 - f(u,s+\delta)^3)}{f(u,s)^3 f(u,s+\delta)^3}\, du\right| &\le \int_0^{s} cu^\alpha \frac{u^2\delta}{u^3}\, du \\ &= \int_0^{s} c\delta u^{\alpha-1}\, du = \frac{c\delta}{\alpha}s^\alpha \le c\delta^{\alpha-1/2}.\end{aligned}$$

The last integral:

$$\begin{aligned}\left|\int_s^{s+\delta} \frac{\partial_s^n f(u,s+\delta)}{f(u,s+\delta)^3}\, du\right| &\le \int_s^{s+\delta} \frac{cu^\alpha}{u^{3/2}}\, du = \frac{c}{2\alpha-1}((s+\delta)^{\alpha-1/2} - s^{\alpha-1/2}) \\ &\le \frac{c}{2\alpha-1}\delta^{\alpha-1/2}.\end{aligned}$$

□

5.4 *Real analyticity of* γ

In this section we prove Theorem 1.3.2. There exists $\delta > 0$ such that λ can be extended (complex) analytically to $E = \{z \in \mathbb{C} : d(z,[0,T]) \le \delta\}$. Notice that $f(s,s) = \gamma(0,s) = \gamma(s) - \lambda(0)$ and $f(u,s,\varepsilon)$ converges uniformly to $f(u,s)$ on $D = \{(u,s) : 0 < u \le s, 0 < s \le T\}$. So it suffices to show that $f(u,s,\varepsilon)$ can be extended analytically in the same neighborhood of D (in $\mathbb{C}\times\mathbb{C}$) for all ε. Recall that $G(z,u,s) = \frac{-2}{z} + \lambda'(s-u)$ is analytic in (z,u,s), hence by the dependence of solutions of ODE on parameters (see [6, Theorem 8.1]) the function $f(\cdot,s,\varepsilon)$ in (1.5) exists and is analytic in a neighborhood of $u = 0$ for each $\varepsilon \in (0,1]$ and $s \in E$. The main difficulty is to show this neighborhood is the same for all ε and s.

The outline of this section is as follows: First we show in Lemma 5.4.1 that the equation (1.5) still has solution when s is in the domain

$$E_1 = \{t : 0 < \mathrm{Re}\, t < T + \delta_1, |\mathrm{Im}\, t| < \delta_1\}$$

with δ_1 small enough and not depending on ε. Then in Lemma 5.4.2 we show that one can take complex u-derivatives in (1.5), which means the solutions are extended analytically. Finally by [6, Theorem 8.3] the solutions are analytic in (u, s) on the same domain for all ε.

Let M be an upper bound for the sup-norms of λ' and λ'' on E. As a first step, we will show the following:

Lemma 5.4.1. *There exists $\delta_1 \in (0, \delta)$ depending on δ, M and T such that for every $s \in E_1$ and $\varepsilon \in (0, 1]$, the solution to the equation*

$$\begin{aligned} \partial_u f(u, s, \varepsilon) &= \frac{-2}{f(u, s, \varepsilon)} + \lambda'(s - u), \qquad u \geq 0, \\ f(0, s, \varepsilon) &= i\varepsilon, \end{aligned}$$

exists uniquely for $u \in [0, Re\, s + \delta_1]$. Moreover,

$$\max(\sqrt{2u}, \frac{\varepsilon}{2}) \leq Im\, f(u, s, \varepsilon) \text{ for } 0 \leq u \leq Re\, s + \delta_1.$$

Proof. The solution $f(u, s, \varepsilon)$ exists on a neighborhood of $u = 0$, and it continues to exists as long as it stays above the real line. The uniqueness of this solution comes from standard ODE techniques. To establish the results of the lemma, we will compare $f(u, s, \varepsilon)$ to $f(u, s_0, \varepsilon)$ where $s_0 = \mathrm{Re}\, s$ and

$$\begin{aligned} \partial_u f(u, s_0, \varepsilon) &= \frac{-2}{f(u, s_0, \varepsilon)} + \lambda'(s_0 - u), \qquad u \geq 0, \\ f(0, s_0, \varepsilon) &= i\varepsilon. \end{aligned}$$

It follows from Lemma 5.1.2 (i, ii) that

$$\begin{aligned} \sqrt{3u + \varepsilon^2} &\leq \mathrm{Im}\, f(u, s_0, \varepsilon) \\ \text{and} \quad |\mathrm{Re}\, f(u, s_0, \varepsilon)| &\leq \sqrt{u} \qquad \text{for } 0 \leq u \leq s_0 + \delta_1, \end{aligned}$$

where $\delta_1 < \delta$ will be specified momentarily. By following the same argument in Lemma 5.2.3, we get a bound for the difference of $f(u,s,\varepsilon)$ and $f(u,s_0,\varepsilon)$:

$$|f(u,s,\varepsilon) - f(u,s_0,\varepsilon)| \le CMu|s-s_0| \le CMu\delta_1$$

whenever $0 \le u \le S$ with

$$S = \inf\{0 \le v \le u_0 + \delta_1 : \operatorname{Im} f(v,s,\varepsilon) < \frac{\varepsilon}{3} \text{ or } \frac{|\operatorname{Re} f(v,s,\varepsilon)|}{\operatorname{Im} f(v,s,\varepsilon)} > C_1\},$$

where C_1 is a constant in $(0,1)$ and close to 1. It follows that

$$\operatorname{Im} f(u,s,\varepsilon) \ge \operatorname{Im} f(u,s_0,\varepsilon) - CMu\delta_1 \ge \sqrt{3u+\varepsilon^2} - CMu\delta_1,$$

and

$$|\operatorname{Re} f(u,s,\varepsilon)| \le |\operatorname{Re} f(u,s_0,\varepsilon)| + CMu\delta_1 \le \sqrt{u} + CMu\delta_1.$$

By choosing δ_1 small enough, $\operatorname{Im} f(u,s,\varepsilon) \ge \max(\sqrt{2u}, \varepsilon/2)$ and

$$\frac{|\operatorname{Re} f(u,s,\varepsilon)|}{\operatorname{Im} f(u,s,\varepsilon)} < C_1$$

for all $0 \le u \le S$. It follows that $S = u_0 + \delta_1$ and the lemma follows. □

Now we will show that

Lemma 5.4.2. *For every $\varepsilon \in (0,1]$, $s \in E_1$ and $0 < \tilde{u} < Re\, s + \delta_1$, there exist $r = r(\tilde{u}, M, \delta, T) \in (0, \delta - \delta_1)$ and an analytic extension of $f(\cdot, s, \varepsilon)$ on $B_{\tilde{u}} = \{z \in \mathbb{C} : |z - \tilde{u}| < r\}$ such that*

$$\partial_u f(u,s,\varepsilon) = \frac{-2}{f(u,s,\varepsilon)} + \lambda'(s-u).$$

Proof. We will use the Picard iteration to show that the equation

$$\begin{aligned} g'(u) &= -\frac{2}{g(u)} + \lambda'(s-u), \\ g(\tilde{u}) &= f(\tilde{u}, s, \varepsilon) \end{aligned} \tag{5.10}$$

has a solution on $B_{\tilde{u}} = \{z \in \mathbb{C} : |z - \tilde{u}| < r\}$, where r will be specified later. Indeed for $|u - \tilde{u}| < r$ define $g_0(u) = f(\tilde{u}, s, \varepsilon)$ and

$$g_{n+1}(u) = f(\tilde{u}, s, \varepsilon) + \int_{\tilde{u}}^{u} \frac{-2}{g_n(v)} + \lambda'(s-v)\, dv.$$

We will show by induction on n that g_n is well-defined and analytic in $B_{\tilde{u}}$ and

$$\operatorname{Im} g_n(u) \geq \sqrt{\tilde{u}}.$$

The base case $n = 0$ is clear because of Lemma 5.4.1. Suppose the claim holds for n. The function g_{n+1} is well-defined and analytic in $B_{\tilde{u}}$ since $\frac{1}{g_n}$ is analytic in a simply connected domain. Now

$$\begin{aligned}\operatorname{Im} g_{n+1}(u) &\geq \operatorname{Im} f(\tilde{u}, s, \varepsilon) - |u - \tilde{u}| \max_{v \in B_{\tilde{u}}} \left(\frac{2}{|g_n(v)|} + |\lambda'(s - v)| \right) \\ &\geq \sqrt{2\tilde{u}} - r\left(\frac{2}{\sqrt{\tilde{u}}} + M\right).\end{aligned}$$

The claim holds for $n + 1$ by choosing r small enough depending on $\tilde{u}, M$ and T. We also require that r is small enough so that $2r/\tilde{u} < 1$. Then the sequence g_n converges uniformly in $B_{\tilde{u}}$ since

$$\begin{aligned}|g_{n+1}(u) - g_n(u)| &\leq |u - \tilde{u}| \max_{v \in B_{\tilde{u}}} \frac{2|g_n(v) - g_{n-1}(v)|}{|g_n(v) g_{n-1}(v)|} \\ &\leq \frac{2r}{\tilde{u}} ||g_n - g_{n-1}||_{B_{\tilde{u}}, \infty}.\end{aligned}$$

Let g be the limit. Then this function is analytic and satisfies the differential equation (5.10). In particular $g(u)$ and $f(u, \tilde{u}, \varepsilon)$ solve same initial value problem. Hence they are equal when u is real. In order words, $f(\cdot, s, \varepsilon)$ is extended analytically on $B_{\tilde{u}}$. □

Proof of Theorem 1.3.2. By [6, Theorem 8.3], for every $\varepsilon \in (0, 1]$ the function $f(u, s, \varepsilon)$ is analytic in the domain $\{(u, s) : s \in E_1, u \in B_{\tilde{u}} \text{ for some } \tilde{u} \in (0, \operatorname{Re} s + \delta_1)\}$. It follows that $f(u, s)$ is also analytic in the same domain which contains $\{(s, s) : 0 < s \leq T\}$. Hence $f(s, s)$ and $\gamma(s)$ is real analytic on $(0, T]$. □

5.5 Behavior of γ at $s = 0$

In this section we analyze the behavior of γ at its base, proving Theorem 1.3.4 and Theorem 1.3.3.

5.5.1 *Smoothness of $\gamma(s^2)$ at $s = 0$*

We may extend λ smoothly on $(-\delta, T)$ by the concatenation property of the Loewner equation. Thus, it suffices to show that for fixed $t_0 \in (0, T)$, the curve $\gamma_0(s^2) = g_{t_0}(\gamma(s^2 + t_0))$ is smooth at $s = 0$ provided γ is smooth on $(0, T)$. The idea, illustrated in Figure 5.1, is as follows. Let U be the intersection of $\mathbb{H}$ and a small disk centered at $\lambda(0)$ and let $V = g_{t_0}^{-1}(U)$. Define an analytic branch ϕ of $\sqrt{z - \gamma(t_0)}$ in a neighborhood of $\gamma(t_0)$ such that the branch cut is $\gamma(0, t_0]$. Let $W = \phi(V)$. All we need to check is that for small $\varepsilon > 0$ the images under ϕ of $\gamma(t_0 - \varepsilon, t_0]$ and $\gamma(t_0 + s^2), 0 \le s^2 \le \varepsilon$, are smooth. Finally the smoothness of $\gamma_0(s^2)$ follows immediately from the Schwarz reflection principle through $E = \phi(\gamma(t_0 - \varepsilon, t])$ (in the case γ is analytic) or Kellogg-Warschawski theorem (in the case γ is $C^{n,\alpha}$) for the map $\phi \circ g_{t_0}^{-1}$ from U to W.

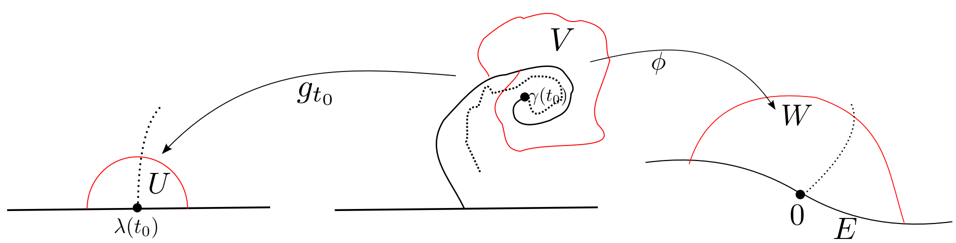


Figure 5.1: Illustration for the proof of Theorem 1.3.4

Proof of Theorem 1.3.4 when λ is analytic. It follows from (1.3) that $\gamma'(t) \neq 0$ for all t. Thus, there exists an (real) analytic function h on $(-\sqrt{\varepsilon}, \sqrt{\varepsilon})$ such that

$$\frac{\gamma(t_0 + s) - \gamma(t_0)}{s} = h(s)^2 \text{ for all } s \in (-\sqrt{\varepsilon}, \sqrt{\varepsilon}) \backslash \{0\}.$$

Let $\phi_1(s) = ish(-s^2)$ and $\phi_2(s) = sh(s^2)$. We see that these two functions are analytic and one-to-one. Moreover,

$$\phi_1(s)^2 = \gamma(t_0 - s^2) - \gamma(t_0) \text{ and}$$

$$\phi_2(s)^2 = \gamma(t_0 + s^2) - \gamma(t_0).$$

Therefore the boundary E of W, which is parametrized by $\phi_1(s)$ near 0, and $\phi(\gamma(t_0 + s^2))$ are analytic. Since the latter map is the image of $\gamma_0(s^2)$ under $\phi \circ g_{t_0}^{-1}$, it follows from the Schwarz reflection principle that $\gamma_0(s^2)$ is analytic at 0. □

Proof of Theorem 1.3.4 when λ is C^β. By Theorem 1.3.4, $\gamma \in C^{n,\alpha}(0, T]$ for appropriate $\alpha \in (0, 1)$. It is not obvious that the function h in the previous case is $C^{n,\alpha}$. Indeed one can find an example of function $\gamma \in C^{n,\alpha}$ but h is not. Now let

$$H(s) = \frac{\gamma(t_0 + s) - \gamma(t_0)}{s} \text{ for } s \in (-\sqrt{\varepsilon}, \sqrt{\varepsilon})\backslash\{0\}, \text{ and } H(0) = \gamma'(t_0).$$

We claim that $H \in C^{n-1,\alpha}(-\sqrt{\varepsilon}, \sqrt{\varepsilon})$. Indeed

$$H^{(n)}(s) = \frac{n!}{s^{n+1}} \sum_{k=0}^{n} \frac{(-1)^k}{k!} s^k \gamma^{(k)}(t_0 + s) - \frac{(-1)^n n!}{s^{n+1}} \gamma(t_0) \text{ for } s \neq 0.$$

Apply Proposition 5.1.1 for functions $\gamma, \gamma', \cdots, \gamma^{(n)}$ to get $|H^{(n)}(s)| \leq cs^{\alpha-1}$ which implies the claim.

Since $\inf_{s\in(-\sqrt{\varepsilon},\sqrt{\varepsilon})} |H(s)| > 0$, it follows from the claim that the function $s \mapsto \sqrt{H(-s^2)}$ is $C^{n-1,\alpha}(-\sqrt{\varepsilon}, \sqrt{\varepsilon})$ for any well-defined square-root function. Let $\phi_1(s)$ be a parametrization near 0 of E such that $\phi_1(s)^2 = \gamma(t_0 - s^2) - \gamma(t_0)$ and $\phi_1(s) = s\sqrt{H(-s^2)}$ for $s \in (-\sqrt{\varepsilon}, \sqrt{\varepsilon})$. Since $\phi_1'(s) = \dfrac{\gamma'(t_0 - s^2)}{\sqrt{H(-s^2)}}$, the function ϕ_1 is $C^{n,\alpha}(-\sqrt{\varepsilon}, \sqrt{\varepsilon})$. The same argument shows that the function $\phi(\gamma(t_0 + s^2))$ is $C^{n,\alpha}[0, \sqrt{\varepsilon})$. Combined with the last two statements, the Kellogg-Warschawski theorem [28, Theorem 3.6] implies that the function $\gamma_0(s^2)$ is $C^{n,\alpha}[0, \sqrt{\varepsilon})$. □

Remark. The proof also shows that if $\lambda \in C^{n,\alpha}([0, T]; M)$ then $\Gamma \in C^{n,\alpha+1/2}([0, T]; c)$ with $c = c(T, M, n, \alpha)$.

5.5.2 *Expansion of γ at $s = 0$*

The goal of this section is to prove Theorem 1.3.3, which illuminates why the s^2 parametrization is a natural parametrization at the base of a Loewner curve γ. To accomplish this, we create a comparison curve $\tilde{\gamma}$ that closely approximates γ near its base and is “nice” at

$s = 0$ (that is, $\tilde{\Gamma}(s) = \tilde{\gamma}(s^2)$ is smooth at $t = 0$.) The properties of the comparison curve are summarized in Proposition 5.5.2 below.

Assume γ is generated by $\lambda \in C^{n,\alpha}[0,T]$. We define $\tilde{\gamma}$ as a perturbation of a vertical slit, as done in Section 4.6 of [14]. Set

$$\phi(z) = z + \sum_{m=2}^{4n+1} \frac{b_m}{2^m} z^m,$$

which is conformal on a neighborhood of the origin. The real-valued coefficients b_m will depend on $\lambda^{(k)}(0)$ as we will describe later. Then define

$$\begin{aligned}\tilde{\gamma}(t) = \phi(2i\sqrt{t}) &= 2i\sqrt{t} + \sum_{m=2}^{4n+1} i^m b_m t^{m/2} \\ &= 2i\sqrt{t} - b_2 t - i\, b_3 t^{3/2} + b_4 t^2 + \cdots + i\, b_{4n+1} t^{2n+1/2}.\end{aligned}$$

Let $g_t : \mathbb{H}\setminus[0, 2i\sqrt{t}] \to \mathbb{H}$ and $\tilde{g}_t : \mathbb{H}\setminus\tilde{\gamma}[0,t] \to \mathbb{H}$ be conformal maps with the hydrodynamic normalization at infinity. Then we set $\phi_t = \tilde{g}_t \circ \phi \circ g_t^{-1}$ and $\tilde{\lambda}(t) = \phi_t(0)$, as illustrated in Figure 5.2. In this form, $\tilde{\gamma}$ and $\tilde{\lambda}$ are not parametrized by halfplane capacity. We will need to reparametrize by $t = t(s)$, which satisfies $t(0) = 0$ and $\frac{dt}{ds} = \phi_t'(0)^{-2}$. Note in particular that $\frac{dt}{ds}\big|_{s=0} = 1$.

Lemma 5.5.1. *Assume ϕ_t, $\tilde{\lambda}$ and $t = t(s)$ are defined as above, and let $k \in \mathbb{N}$. Then there exists $\tilde{T} > 0$, there exist polynomials $p_k(x_1, x_2, \cdots, x_{k+2})$, $q_k(x_1, x_2, \cdots, x_{2k})$ and $r_k(x_1, x_2, \cdots, x_{2k-1})$, and there exist nonzero constants c_k, d_k, e_k so that for $t \in [0, \tilde{T}]$,*

$$\partial_t \phi_t^{(k)}(0) = c_k\, \phi_t^{(k+2)}(0) + p_k\left(\phi_t'(0), \phi_t''(0), \cdots, \phi_t^{(k+1)}(0), \phi_t'(0)^{-1}\right), \tag{5.11}$$

$$\partial_s^k \tilde{\lambda}(t) = d_k\, \phi_t^{(2k)}(0) \cdot \phi_t'(0)^{-2k} + q_k\left(\phi_t'(0), \phi_t''(0), \cdots, \phi_t^{(2k-1)}(0), \phi_t'(0)^{-1}\right), \text{ and} \tag{5.12}$$

$$\partial_s^k t = e_k\, \phi_t^{(2k-1)}(0) \cdot \phi_t'(0)^{-(2k+1)} + r_k\left(\phi_t'(0), \phi_t''(0), \cdots, \phi_t^{(2k-2)}(0), \phi_t'(0)^{-1}\right). \tag{5.13}$$

Further $\tilde{\lambda} \in C^\infty[0, s(\tilde{T})]$ under the halfplane-capacity parametrization.

Proof. Write $\phi_t(z) = \sum_{k=0}^\infty a_k z^k$, keeping in mind that a_k depends on t. Then from Proposition 4.40 in [14],

$$\begin{aligned}\partial_t \phi_t(z) &= 2\left(\frac{\phi_t'(0)^2}{\phi_t(z) - \phi_t(0)} - \frac{\phi_t'(z)}{z}\right) \\ &= -2\, \frac{\sum_{k=0}^\infty (a_1 a_{k+2} + 2a_2 a_{k+1} + \cdots + (k+2)a_{k+2}a_1) z^k}{\sum_{k=0}^\infty a_{k+1} z^k}.\end{aligned} \tag{5.14}$$

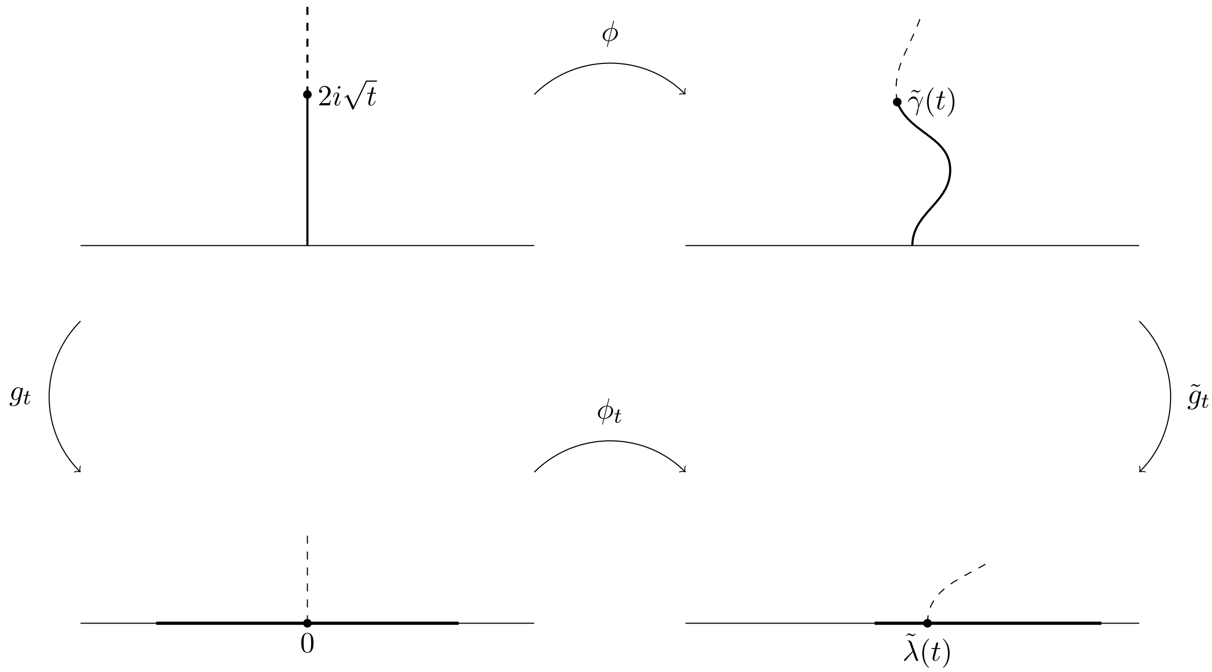


Figure 5.2: The conformal maps $\phi, g_t, \tilde{g}_t, \phi_t$, the comparison curve $\tilde{\gamma}$, and $\tilde{\lambda}$.

Since $a_1 = 1$ when $t = 0$, there exists a neighborhood U of 0 and $\tilde{T} > 0$ so that the denominator is nonzero for $z \in U$ and $t \leq \tilde{T}$. Therefore $\partial_t \phi_t^{(k)}(z)$ is defined for $(z,t) \in U \times [0, \tilde{T}]$. Equation (5.11) follows from (5.14) (with $c_k = -\frac{2(k+3)}{(k+2)(k+1)}$.)

We verify (5.12) inductively. For the base case,

$$\partial_s \tilde{\lambda}(t) = \partial_t \phi_t(0) \cdot \frac{dt}{ds} = -3\, \phi_t''(0) \cdot \phi_t'(0)^{-2}.$$

Assume (5.12) holds for a fixed k. Then

$$\begin{aligned}\partial_s^{k+1} \tilde{\lambda}(t) &= \partial_t \left(d_k\, \phi_t^{(2k)}(0) \cdot \phi_t'(0)^{-2k} + q_k \left(\phi_t'(0), \phi_t''(0), \cdots, \phi_t^{(2k-1)}(0), \phi_t'(0)^{-1} \right) \right) \cdot \phi_t'(0)^{-2} \\ &= d_k\, c_{2k}\, \phi_t^{(2k+2)}(0) \cdot \phi_t'(0)^{-2k-2} + q_{k+1} \left(\phi_t'(0), \phi_t''(0), \cdots, \phi_t^{(2k+1)}(0), \phi_t'(0)^{-1} \right).\end{aligned}$$

We also prove (5.13) inductively. When $k = 1$,

$$\frac{dt}{ds} = \phi_t'(0) \cdot \phi_t'(0)^{-3}.$$

If (5.13) holds for fixed k, then

$$\begin{aligned}\partial_s^{k+1} t &= \frac{d}{dt}\left(e_k\, \phi_t^{(2k-1)}(0)\cdot \phi_t'(0)^{-(2k+1)} + r_k\left(\phi_t'(0), \phi_t''(0), \cdots, \phi_t^{(2k-2)}(0), \phi_t'(0)^{-1}\right)\right)\cdot \phi_t'(0)^{-2} \\ &= e_k\, c_{2k-1}\, \phi_t^{(2k+1)}(0)\cdot \phi_t'(0)^{-(2k+3)} + r_{k+1}\left(\phi_t'(0), \phi_t''(0), \cdots, \phi_t^{(2k)}(0), \phi_t'(0)^{-1}\right).\end{aligned}$$

The last assertion follows from (5.12). □

We are now ready to recursively define the coefficients of ϕ. The coefficient b_m will depend on $\lambda^{(k)}(0)$ for $k = 1, \cdots, \lfloor \frac{m}{2} \rfloor \wedge n$. For even values of m, our choice of b_m will ensure that $\partial_s^k \tilde{\lambda}(0) = \lambda^{(k)}(0)$ for $k \le n$. For odd values of m, we choose b_m so that the t-parametrization of $\tilde{\gamma}$ is close to the halfplane-capacity parametrization.

- Set $b_2 = -\frac{2}{3}\lambda'(0)$. Since $\partial_s \tilde{\lambda}(0) = -\frac{3}{2} b_2$, this implies that $\partial_s \tilde{\lambda}(0) = \lambda'(0)$.

- Set $b_3 = \dfrac{b_2^2}{8}$. This implies that $\frac{d^2 t}{ds^2}\big|_{s=0} = 2b_3 - b_2^2/4 = 0$.

- Assume that $b_2, b_3, \cdots, b_{2k-1}$ have been defined. Then by Lemma 5.5.1,

$$\partial_s^k \tilde{\lambda}(0) = d_k\, \frac{(2k)!}{2^{2k}}\, b_{2k} + q_k\left(1, \frac{1}{2} b_2, \cdots, \frac{(2k-1)!}{2^{2k-1}} b_{2k-1}, 1\right).$$

  If $k \le n$, define b_{2k} so that $\partial_s^k \tilde{\lambda}(0) = \lambda^{(k)}(0)$. If $k > n$, we may define b_{2k} however we like; for instance, we choose b_{2k} so that $\partial_s^k \tilde{\lambda}(0) = 0$.

- Assume that $b_2, b_3, \cdots, b_{2k}$ have been defined. Then by Lemma 5.5.1,

$$\left.\frac{d^{k+1} t}{ds^{k+1}}\right|_{s=0} = e_{k+1}\, \frac{(2k+1)!}{2^{2k+1}}\, b_{2k+1} + r_{k+1}\left(1, \frac{1}{2} b_2, \cdots, \frac{(2k)!}{2^{2k}} b_{2k}, 1\right).$$

  Define b_{2k+1} so that this quantity is zero.

This construction ensures that $\partial_s^k \tilde{\lambda}(0) = \lambda^{(k)}(0)$ for $k \le n$ and that $t = s + O(s^{2n+2})$. The first fact, together with by Theorem 3.3 in [40], implies that $|\gamma(s) - \tilde{\gamma}(t(s))| = O(s^{n+\alpha})$ for s near 0. The second fact implies that under the halfplane-capacity parametrization $\tilde{\gamma}(t(s))$ will have the same coefficients as $\tilde{\gamma}(t)$ for the terms with exponents at most $n + 1/2$. Together, this provides precise information about the expansion of $\gamma(s)$ near $s = 0$. In summary, we have proved the following, which establishes Theorem 1.3.3.

Proposition 5.5.2. *Assume that* $\lambda \in C^{n,\alpha}[0,T]$ *generates the curve* γ. *Then there exists* $\tilde{\lambda} \in C^{\infty}[0,S]$ *that generates a (half-plane capacity parametrized) curve* $\tilde{\gamma} \in C^{\infty}(0,S]$ *with the following properties:*

- $\lambda^{(k)}(0) = \tilde{\lambda}^{(k)}(0)$ *for* $1 \leq k \leq n$.
- $\tilde{\Gamma}(s) = \tilde{\gamma}(s^2)$ *is in* $C^{\infty}[0, \sqrt{S}]$.
- $\tilde{\Gamma}^{(m)}(0)$ *depends on* $\lambda^{(k)}(0)$ *for* $m \leq 2n+1$ *and* $k = 1, \cdots, \lfloor \frac{m}{2} \rfloor$.
- $|\gamma(s) - \tilde{\gamma}(s)| = O(s^{n+\alpha})$.

In particular near $s = 0$, *the curve* γ *has the form*

$$\gamma(s) = \begin{cases} 2i\sqrt{s} + a_2 s + i\, a_3 s^{3/2} + a_4 s^2 + \cdots + a_{2n} s^n + O(s^{n+\alpha}) & \text{if } \alpha \leq 1/2 \\ 2i\sqrt{s} + a_2 s + i\, a_3 s^{3/2} + a_4 s^2 + \cdots + a_{2n} s^n + i\, a_{2n+1} s^{n+1/2} + O(s^{n+\alpha}) & \text{if } \alpha > 1/2 \end{cases}$$

where the real-valued coefficients a_m *depend on* $\lambda^{(k)}(0)$ *for* $k = 1, \cdots, \lfloor \frac{m}{2} \rfloor$.

We note the equations for the first few coefficients:

$$\begin{aligned} a_2 &= \frac{2}{3}\lambda'(0) \\ a_3 &= -\frac{1}{18}\lambda'(0)^2 \\ a_4 &= \frac{4}{15}\lambda''(0) + \frac{1}{135}\lambda'(0)^3 \\ a_5 &= -\frac{1}{15}\lambda''(0)\lambda'(0) + \frac{1}{2160}\lambda'(0)^4 \end{aligned}$$

Coefficients a_2, a_3, a_4 were discovered in [20] by comparison with specific example curves (such as those generated by $c\sqrt{\tau - t}$.)

Along with the tools developed in Sections 5.2 and 5.3, Proposition 5.5.2 could be used to show that if $\Gamma(s) = \gamma(s^2)$, then $\Gamma^{(k)}(0)$ exists and equals $\tilde{\Gamma}^{(k)}(0)$ for $k = 1, \cdots, n+1$.

5.6 *Examples*

In this section we discuss two examples. In the first, the driving function is $C^{3/2}$ and the associated curve is $C^{1,1}$ but not C^2. This example shows that we cannot improve Theorem 1.3.1 to say that the curve is C^{n+1} in the case that the driving function is $C^{n+1/2}$. In the second example, which illustrates the $\alpha = 1$ case of Theorem 5.3.1, the driving function is $C^{0,1}$ and the associated curve is $C^{3/2}$. We describe the needed computational steps to verify these examples, but leave details for the reader.

5.6.1 *Example 1: $\lambda \in C^{3/2}$ and $\gamma \in C^{1,1}$*

This example was communicated to us by Don Marshall.

We will create γ via a sequence of conformal maps, as pictured in Figure 5.3. Let $f_1(z) = z + \frac{1}{z} + c \ln z$, and let $r_{1,2} = \frac{-c \pm \sqrt{c^2+4}}{2}$ be the finite critical points of f_1. Define

$$g(z) = \frac{c\pi}{f_1(z) - f_1(r_1)},$$

which is a conformal map from $\mathbb{H}$ onto the $C^{1,1}$ domain $\mathbb{C} \backslash ((-\infty, 0] \cup$ a circle arc). Finally, set

$$F(z) = i\sqrt{g(z) + 1}.$$

The image of $\mathbb{H}$ under F is a slit half-plane, and we let γ be the resulting slit.

For $t \in [0, 1/4]$, $\gamma(t) = 2i\sqrt{t}$ and $\lambda(t) \equiv 0$. To compute λ and γ for $t > 1/4$, we will need to use the conformal maps, since $\gamma(t) = F(r_2)$ and $\lambda(t) = L^{-1}(r_2)$ for the automorphism L of $\mathbb{H}$ with

$$F(L(z)) = z + 0 + \frac{-2t}{z} + \cdots \quad \text{near infinity.} \tag{5.15}$$

Since L must send ∞ to r_1,

$$L(z) = r_1 + \frac{a}{z-b} = r_1 + \frac{a}{z} + \frac{ab}{z^2} + \frac{ab^2}{z^3} + \frac{ab^3}{z^4} + O(|z|^{-5}) \text{ near infinity,}$$

where $a < 0$ and $b \in \mathbb{R}$. Using this and the Taylor series expansion of $f_1 - f_1(r_1)$ at $z = r_1$, one can compute that

$$f_1(L(z)) - f_1(r_1) = \frac{A}{z^2} + \frac{B}{z^3} + \frac{D}{z^4} + O(1/|z|^5) \text{ near infinity,}$$

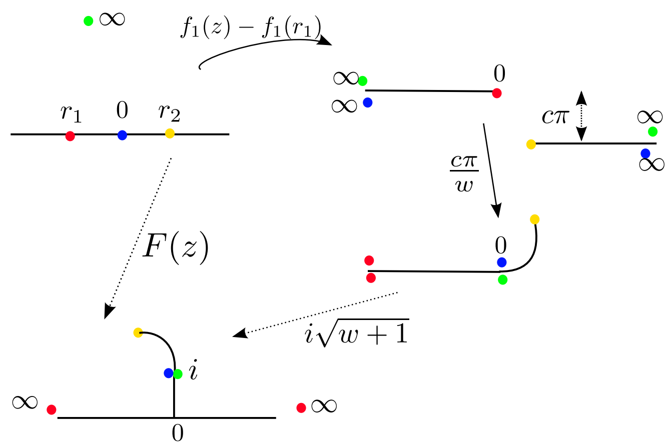


Figure 5.3: Conformal maps used in the construction of γ for Example 1.

with

$$A = \frac{a^2 f_1^{(2)}(r_1)}{2}, \quad B = a^2 b f^{(2)}(r_1) + \frac{a^3 f^{(3)}(r_1)}{6},$$
$$\text{and } D = \frac{3a^2b^2 f^{(2)}(r_1)}{2} + \frac{a^3 b f^{(3)}(r_1)}{2} + \frac{a^4 f^{(4)}(r_1)}{24}.$$

Thus near infinity,

$$\begin{aligned} F(L(z)) &= i\sqrt{\frac{c\pi}{A}z^2 - \frac{c\pi B}{A^2}z - \frac{c\pi D}{A^2} + \frac{c\pi B^2}{A^3} + 1 + O(1/|z|)} \\ &= i\left(-i\sqrt{\frac{c\pi}{|A|}}z - iB\frac{\sqrt{c\pi}}{2|A|^{3/2}} + O(1/|z|)\right) \end{aligned}$$

Note that in choosing the appropriate branch for the square root, we used the fact that $A < 0$. In order to satisfy (5.15), we must have

- $A = -c\pi$, or equivalently, $a = \dfrac{r_1\sqrt{-2\pi c r_1}}{\sqrt{2 - cr_1}}$, and
- $B = 0$, or equivalently, $b = \dfrac{(cr_1 - 3)\sqrt{-2\pi c r_1}}{3(2 - cr_1)^{3/2}}$.

Using these two facts, we expand further and find that at infinity,

$$F(L(z)) = z + 0 - \frac{1}{2}\left(\frac{D}{A} + 1\right)\frac{1}{z} + O(1/|z|^2),$$

which implies that

$$4t = \frac{D}{A} + 1 = \frac{-\pi c r_1(c^2 r_1^2 - 6cr_1 + 6)}{3(2 - cr_1)^3} + 1.$$

Next we compute $\lambda(t)$ for $t > 1/4$:

$$\lambda(t) = L^{-1}(r_2) = b + \frac{a}{r_2 - r_1} = \frac{-2\sqrt{2\pi}(-cr_1)^{3/2}}{3(2 - cr_1)^{3/2}}.$$

Thus with $y = -cr_1$, we have

$$t = \frac{1}{4} + \frac{\pi y(y^2 + 6y + 6)}{12(2 + y)^3} \quad \text{and} \quad \lambda(t) = \frac{-2\sqrt{2\pi} y^{3/2}}{3(2 + y)^{3/2}}.$$

So for $t > 1/4$,

$$\lambda'(t) = \frac{\frac{d\lambda}{dy}}{\frac{dt}{dy}} = \frac{-2\sqrt{2}\sqrt{y}(2 + y)^{3/2}}{\sqrt{\pi}(y + 1)}.$$

Using this, one can show that for $s > t \geq 1/4$,

$$|\lambda'(s) - \lambda'(t)| \leq c\sqrt{y_s - y_t} \leq c'\sqrt{s - t},$$

proving that $\lambda \in C^{3/2}[0, T]$. We also note that away from $t = 1/4$, one can check that $\lambda(t)$ is C^2.

Lastly, for $t \geq 1/4$, $\gamma(t) = F(r_2)$. Using this, one can determine computationally that with the halfplane-capacity parametrization, γ' and γ'' exist on $[1/4, T]$ (by computing, for instance, $\gamma'(t) = \frac{dF(r_2)}{dc} / \frac{dt}{dc}$ and $\gamma'' = \frac{d\gamma'(t)}{dc} / \frac{dt}{dc}$). Further,

$$\lim_{t \searrow 1/4} \gamma'(t) = 2i = \lim_{t \nearrow 1/4} \gamma'(t),$$

but

$$\lim_{t \searrow 1/4} \gamma''(t) = -4i - 16 \neq \lim_{t \nearrow 1/4} \gamma''(t) = -4i.$$

Therefore on the full interval $(0, T]$, γ is $C^{1,1}$ but not C^2.

5.6.2 *Example 2: $\lambda \in C^{0,1}$ and $\gamma \in C^{3/2}$*

Consider the driving function

$$\lambda(t) = \begin{cases} 0 & \text{for} \quad 0 \leq t \leq \frac{1}{4} \\ \frac{3}{2} - \frac{3}{2}\sqrt{1 - 8(t - 1/4)} & \text{for} \quad \frac{1}{4} \leq t < \frac{1}{4} + \frac{1}{10} \end{cases}.$$

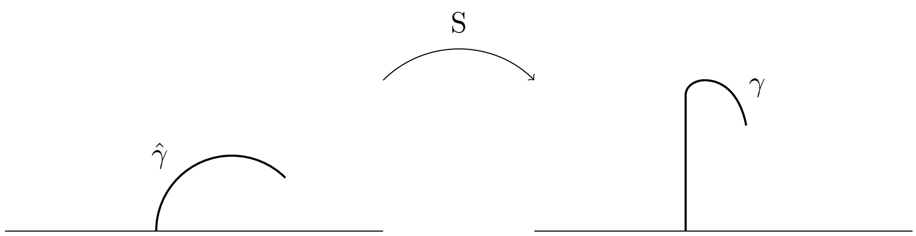


Figure 5.4: The curve γ for Example 2.

There exists $c > 0$ so that

$$|\lambda(t) - \lambda(s)| \leq c|t - s|$$

for all $s, t \in [0, 0.35]$, implying that $\lambda \in C^{0,1}$. However, λ is not in C^1 since λ' is not continuous.

The driving function $\frac{3}{2} - \frac{3}{2}\sqrt{1 - 8s}$, defined on $[0, \frac{1}{8}]$, generates the upper half-circle of radius $\frac{1}{2}$ centered at $\frac{1}{2}$. Let $\hat{\gamma}$ be the portion of this circle generated on the time interval $[0, \frac{1}{10}]$. Then the curve γ generated by λ is the image of $[-1, 1] \cup \hat{\gamma}$ by the map $S(z) = \sqrt{z^2 - 1}$. See Figure 5.4. By Proposition 3.12 in [27], $\gamma \in C^{3/2}$ (and no better) under the arclength parametrization. This is also true under the halfplane-capacity parametrization. Note that $\hat{\gamma}$ is smooth on $(0, \frac{1}{10}]$ (because its driving function is smooth), and near $s = 0$

$$\hat{\gamma}(s) = 2i\sqrt{s} + 4s - 2is^{3/2} + O(s^2)$$

by Theorem 1.3.3. Thus γ is piecewise smooth, and for $t \geq 1/4$

$$\gamma(t) = S(\hat{\gamma}(t - 1/4)) = i + 2i(t - 1/4) + 8(t - 1/4)^{3/2} + O((t - 1/4)^2).$$

From this we can determine that $\gamma \in C^{3/2}(0, 0.35]$ (and no better) under the halfplane-capacity parametrization.

BIBLIOGRAPHY

[1] Kari Astala, Peter Jones, Antti Kupiainen, and Eero Saksman. Random conformal weldings. *Acta Math.*, 207(2):203–254, 2011.

[2] Robert O. Bauer. Discrete Löwner evolution. *Ann. Fac. Sci. Toulouse Math. (6)*, 12(4):433–451, 2003.

[3] Vincent Beffara. The dimension of the SLE curves. *Ann. Probab.*, 36(4):1421–1452, 2008.

[4] Christian Beneš, Fredrik Johansson Viklund, and Michael Kozdron. On the rate of convergence of loop-erased random walk to SLE(2). *Comm. Math. Phys.*, 2012. To appear.

[5] Dmitry Chelkak and Stanislav Smirnov. Universality in the 2D Ising model and conformal invariance of fermionic observables. *Invent. Math.*, 189(3):515–580, 2012.

[6] Earl A. Coddington and Norman Levinson. *Theory of ordinary differential equations.* McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955.

[7] Clifford J. Earle and Adam Lawrence Epstein. Quasiconformal variation of slit domains. *Proc. Amer. Math. Soc.*, 129(11):3363–3372 (electronic), 2001.

[8] Fredrik Johansson Viklund. Convergence rates for loop-erased random walk and other Loewner curves. 2012. Preprint.

[9] Fredrik Johansson Viklund and Gregory F. Lawler. Optimal Hölder exponent for the SLE path. *Duke Math. J.*, 159(3):351–383, 2011.

[10] Fredrik Johansson Viklund, Steffen Rohde, and Carto Wong. On the continuity of SLE_κ in κ. 2012. Preprint.

[11] Wouter Kager, Bernard Nienhuis, and Leo P. Kadanoff. Exact solutions for Loewner evolutions. *J. Statist. Phys.*, 115(3-4):805–822, 2004.

[12] Tom Kennedy. A fast algorithm for simulating the chordal Schramm-Loewner evolution. *J. Stat. Phys.*, 128(5):1125–1137, 2007.

[13] Tom Kennedy. Numerical computations for the Schramm-Loewner evolution. *J. Stat. Phys.*, 137(5-6):839–856, 2009.

[14] Gregory F. Lawler. *Conformally invariant processes in the plane*, volume 114 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 2005.

[15] Gregory F. Lawler. Schramm-Loewner evolution. *Statistical Mechanics, S. Sheffield and T. Spencer, ed., IAS/Park City Mathematical Series, AMS*, pages 231–295, 2009.

[16] Gregory F. Lawler and Vlada Limic. *Random walk: a modern introduction*, volume 123 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 2010.

[17] Gregory F. Lawler, Oded Schramm, and Wendelin Werner. Conformal invariance of planar loop-erased random walks and uniform spanning trees. *Ann. Probab.*, 32(1B):939–995, 2004.

[18] Joan Lind, Donald E. Marshall, and Steffen Rohde. Collisions and spirals of Loewner traces. *Duke Math. J.*, 154(3):527–573, 2010.

[19] Joan Lind and Steffen Rohde. Spacefilling curves and phases of the Loewner equation. 2010. Preprint.

[20] Joan Lind and Steffen Rohde. Loewner curvature. 2013. Preprint.

[21] Joan Lind and Huy Tran. The regularity of Loewner curves. 2014. In preparation.

[22] Joan R. Lind. A sharp condition for the Loewner equation to generate slits. *Ann. Acad. Sci. Fenn. Math.*, 30(1):143–158, 2005.

[23] Joan R. Lind. Hölder regularity of the SLE trace. *Trans. Amer. Math. Soc.*, 360(7):3557–3578, 2008.

[24] Karl Löwner. Untersuchungen über schlichte konforme Abbildungen des Einheitskreises. I. *Math. Ann.*, 89(1-2):103–121, 1923.

[25] Donald E. Marshall. Derivation of chordal Loewner from radial Loewner. 2011. Preprint.

[26] Donald E. Marshall and Steffen Rohde. The Loewner differential equation and slit mappings. *J. Amer. Math. Soc.*, 18(4):763–778 (electronic), 2005.

[27] Donald E. Marshall and Steffen Rohde. Convergence of a variant of the zipper algorithm for conformal mapping. *SIAM J. Numer. Anal.*, 45:2577–2609, 2007.

[28] Ch. Pommerenke. *Boundary behaviour of conformal maps*, volume 299 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992.

[29] Daniel Revuz and Marc Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.

[30] Mohammad A. Rezaei. Hausdorff measure of SLE curves. arXiv:1212.5847.

[31] Steffen Rohde and Oded Schramm. Basic properties of SLE. *Ann. of Math. (2)*, 161(2):883–924, 2005.

[32] Steffen Rohde, Huy Tran, and Michel Zinsmeister. The Loewner equation and Lipschitz graphs. 2013. Preprint.

[33] Oded Schramm. Scaling limits of loop-erased random walks and uniform spanning trees. *Israel J. Math.*, 118:221–288, 2000.

[34] Oded Schramm and Scott Sheffield. Harmonic explorer and its convergence to SLE_4. *Ann. Probab.*, 33(6):2127–2148, 2005.

[35] Oded Schramm and Scott Sheffield. Contour lines of the two-dimensional discrete Gaussian free field. *Acta Math.*, 202(1):21–137, 2009.

[36] Scott Sheffield. Conformal weldings of random surfaces: SLE and the quantum gravity zipper. 2010. arXiv:1012.4797.

[37] Scott Sheffield and Nike Sun. Strong path convergence from Loewner driving function convergence. *Ann. Probab.*, 40(2):578–610, 2012.

[38] Stanislav Smirnov. Critical percolation in the plane: conformal invariance, Cardy's formula, scaling limits. *C. R. Acad. Sci. Paris Sér. I Math.*, 333(3):239–244, 2001.

[39] Huy Tran. Convergence of an algorithm simulating Loewner curves. 2013. Preprint.

[40] Carto Wong. Smoothness of Loewner slits. *Trans. Amer. Math. Soc.*, 366(3):1475–1496, 2014.