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Minkowski-type Estimates on the Quantitative Strata of the
Generalized Critical set of Green's functions for Two-Sided NTA
Domains arising from a Free-Boundary Problem for Harmonic
Measure

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Abstract

Minkowski-type Estimates on the Quantitative Strata of the Generalized Critical set of Green's functions for Two-Sided NTA Domains arising from a Free-Boundary Problem for Harmonic Measure

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Mathematics

In this work, we prove three things. The main results are two different results on Minkowski-type estimates on the quantitative strata of the generalized critical set of Green's functions of 2-Sided NTA domains arising from a free-boundary problem for harmonic measure. The first uses simpler techniques and obtains weaker results. The second employs much more complicated machinery and obtains a much stronger result which completely subsumes the results of the first approach. The third result contained in this work is the construction of two families of rectifiable sets which fail to be uniformly rectifiable as dramatically as possible which still retaining nice topological and measure theoretic properties. This third result is independent of the first two, and represents joint work with Max Goering.

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DEDICATION

to the reader, with the following *caveat lector*:

Mathematica est; non legitur.

Chapter 1

INTRODUCTION

This work comprises two lines of inquiry into topics in geometric measure theory. Broadly, geometric measure theory is concerned with the fundamental relationship between the geometric properties of a set and how functions and measures behave upon that set. The first line of inquiry in this work is at the intersection of geometric measure theory, geometric analysis, partial differential equations, and analysis on domains with rough boundaries. This investigation constitutes the main part of this work and comprises Chapters 3 and 4. The second line of inquiry is at intersection of geometric measure theory and the theory of rectifiable sets and comprises Chapter 5.

Chapters 3 and 4 use sophisticated and quantitative versions of Almgren’s “stratification of singularities” theorem. For comparison, a brief outline of Almgren’s “stratification of singularities” theorem is given in Section 1.1.1, below. The improved methods used in Chapters 3 and 4 give quantitative estimates on the fine structure of non-negative harmonic functions which vanish on the boundary of a class of two-sided domains, $\Omega^\pm \subset \mathbb{R}^n$ for $n \geq 3$. These estimates give new insight into both the structure of the boundary, $\partial\Omega^\pm$, and how the critical set approaches the boundary.

Chapter 5 contains joint work with Max Goering (a graduate student at UW). Therein, we construct two examples of rectifiable sets with positive and finite \mathcal{H}^1 -measure which have nowhere locally integrable Jones function (see, Section 1.2 for precise definitions). These examples satisfy different regularity properties: one is connected and one is Ahlfors regular. Both examples generalize to higher dimension and co-dimension. The existence of such sets

has immediate consequences for the characterization of rectifiable sets. In particular, they demonstrates the inability to characterize rectifiability in terms of the finiteness of integrals of the Jones function.

1.1 Introduction for Chapters 3 and 4

The focus of Chapter 3 and 4 is the study of a class of two-phase free boundary problems for harmonic measure. Let $n \geq 3$, $\Omega^+ \subset \mathbb{R}^n$ and $\Omega^- = \overline{\Omega^+}^c$ be unbounded NTA domains (see Definition 2.1.2), ω^\pm their associated harmonic measures, and u^\pm associated Green's functions with poles at infinity. Let $\omega^- \ll \omega^+ \ll \omega^-$ and $h = \frac{d\omega^-}{d\omega^+}$ satisfy $\ln(h) \in C^{0,\alpha}$ for some $0 < \alpha < 1$. In particular, we obtain new results on the structure of the geometric singular set of the boundary, $\partial\Omega^\pm$, and on how the critical set of the Green's functions, u^\pm , approach that boundary.

This problem was introduced without the regularity assumption on ω^\pm by Kenig, Preiss, and Toro [21], with other work under the assumption that $\ln(h) \in VMO(\partial\Omega^\pm)$ by Kenig and Toro [22], Badger [6] [7], and Badger, Engelstein, and Toro [8]. Questions about the structure of the free boundary and the singular set when $\ln(h) \in C^{0,\alpha}$ for $0 < \alpha < 1$ have been addressed by Engelstein [17] and Badger, Engelstein, Toro [9], respectively. In [17], the author shows that under the additional assumption that the boundary is sufficiently flat in the sense of Reifenberg, the boundary is locally $C^{1,\alpha}$. In [9], the authors remove the assumption of flatness and prove that the geometric singular set is contained in countably many $C^{1,\beta}$ submanifolds of the appropriate dimension.

Until recently, almost all work on the two-sided free boundary problem for harmonic measure in higher dimensions has operated under the assumption that Ω^\pm are NTA domains because the NTA conditions allow for scale-invariant estimates of harmonic measure. However, Az-zam, Mouroglou, Tolsa, and Volberg [4] proved, among other things, that if we relax the assumption that the domains are NTA, then $\omega^- \ll \omega^+ \ll \omega^-$ on $G \subset \partial\Omega^\pm$ implies that G

can be decomposed into $G = R \cup B$, where R is $(n - 1)$ -rectifiable and $\omega^\pm(B) = 0$. However, we shall work under the assumption that Ω^\pm are NTA domains.

In this paper, the term *singular set* will refer to the subset of the mutual boundary, $\partial\Omega^\pm$, for which “blow-ups” are not flat. Singularity will be a geometric property. The *critical set* of a Lipschitz function, v , will be the set,

$$\tilde{C}(v) = \{x \in \mathbb{R}^n : \nabla v \text{ exists and } |\nabla v| = 0\}.$$

If $\Omega^+ \subset \mathbb{R}^n$ and $\Omega^- = \overline{\Omega^+}^c$ are unbounded NTA domains, ω^\pm their associated harmonic measures, and u^\pm associated Green’s functions with poles at infinity such that $\omega^- \ll \omega^+ \ll \omega^-$ and $h = \frac{d\omega^-}{d\omega^+}$ satisfies $\ln(h) \in C^{0,\alpha}$ for some $0 < \alpha < 1$, then the geometric singular set of $\partial\Omega^\pm$ is the set for which blow-ups (properly normalized) of $v = u^+ - u^-$ (for special scalings of u^\pm) converge to non-linear homogeneous harmonic polynomials [6]. Thus, philosophically, we should think of the singular set as a part of the critical set. We shall define the *generalized critical set* of v as follows:

$$C(v) = \{x \in \mathbb{R}^n : \text{blow-ups of } v \text{ at } x \text{ are not linear}\} \tag{1.1.1}$$

In this paper, we consider the functions, $v = u^+ - u^-$, for appropriately scaled u^\pm and answer the following question.

Question 1.1.1. How do the strata of the generalized critical set, $C(v)$, sit in space? Since interior estimates blow up as you approach the boundary, even if the singular set of $\partial\Omega^\pm$ is well-behaved, does the critical set oscillate wildly and become “thick” as it approaches the boundary?

Note that this question is about how the singular set sits in the boundary and how the critical set approaches the boundary. Our gauge of how a set “sits in space” or how “thick” it is will be estimates on the volume of its tubular neighborhoods and upper Minkowski dimension.

We shall use the convention that for any $A \subset \mathbb{R}^n$,

$$B_r(A) = \{x \in \mathbb{R}^n : d(A, x) < r\}.$$

Recall that we can define upper Minkowski s -content by

$$\mathcal{M}^{*s}(A) = \limsup_{r \rightarrow 0} \frac{\text{Vol}(B_r(A))}{(2r)^{n-s}} \quad (1.1.2)$$

and upper Minkowski dimension as

$$\overline{\dim}_{\mathcal{M}}(A) = \inf\{s : \mathcal{M}^{*s}(A) = 0\} = \sup\{s : \mathcal{M}^{*s}(A) > 0\}.$$

In this paper, we obtain volume estimates on the tubular neighborhoods of the quantitative stratification of $C(v)$. This quantitative strata is based upon the quantitative strata introduced by Cheeger and Naber in [12] to study the regularity of stationary harmonic maps and minimal currents. Stated roughly, the quantitative stratum, $\mathcal{S}_{\epsilon, r}^k(v)$ is the set of points in \mathbb{R}^n for which at all scales greater than r the function is “ ϵ far away” from homogeneous harmonic polynomials which are translation invariant in more than k directions. See Definition 2.4.9 for rigorous details.

Although not stated in the language of local set approximation (see Badger [7], Badger and Lewis [10]), the tools which we adapt from [24] to answer Question 1.1.1 may be thought of as local set approximation with “extra information.” The quantitative strata implies that when we estimate this set at scales greater than r , we are always approximating by homogeneous harmonic polynomials with zero-sets of the correct dimension. This extra information allows for quantitative refinements of the “Federer dimension-reduction” arguments as developed by Almgren (see, for example, [12], [13]).

The main result of this paper is the following theorem, stated roughly. See Section 2 for definitions.

Theorem 1.1.2. Let $v = u^+ - u^-$ for u^\pm Green's functions of NTA domains $\Omega^+ \subset \mathbb{R}^n$ and $\Omega^- = (\overline{\Omega^+})^c$, respectively whose harmonic measures, ω^\pm , satisfy $\omega^- \ll \omega^+ \ll \omega^-$ and $\|\ln(h)\|_\alpha \leq \Gamma$. Then, for every $0 < \epsilon$ and $k \leq n - 2$, there exists an $0 < r_0$, and a constant, $C < \infty$, such that for every $0 < r < r_0$ and $r < R < 1$,

$$\text{Vol}(B_R(B_{1/4}(0) \cap \mathcal{S}_{\epsilon,r}^k(v))) \leq CR^{n-k}. \quad (1.1.3)$$

Combining this result with an ϵ -regularity theorem due to Engelstein [17] and topological constraints noted in [9], Theorem 1.1.2 implies the following improvement on the structure of $\partial\Omega$.

Corollary 1.1.3. For v as in Theorem 1.1.2, there exists a constant, $C < \infty$ such that,

$$\mathcal{H}^{n-3}(\text{sing}(\partial\Omega^\pm) \cap B_{1/4}(0)) \leq \mathcal{M}^{*,n-3}(\text{sing}(\partial\Omega^\pm) \cap B_{1/2}(0)) \leq C. \quad (1.1.4)$$

This corollary improves previous estimates by [8] which proved that

$$\overline{\dim_{\mathcal{M}}(\text{sing}(\partial\Omega))} \leq n - 3.$$

See 4.1 for full statements, quantification, and dependencies.

We briefly note that because the critical set of u^+ and cu^+ is the same in Ω^+ for any $c \neq 0$, these results naturally extend to any Green's function with pole at infinity for Ω^+ . Since for unbounded NTA domains, non-negative harmonic functions on Ω^+ which vanish on the boundary, $\partial\Omega^\pm$, are uniquely determined by their value at any point $p \in \Omega^+$, every such function is cu^+ for some constant, c . Therefore the above results trivially hold for all non-negative harmonic functions in either Ω^\pm which vanish continuously on $\partial\Omega^\pm$.

1.1.1 Almgren Stratification of Singularities Theorem

The archetype for the arguments presented in Chapters 3 and 4 is due to Almgren. It appears as Corollary 2.27 in his mammoth, 1700-page work [3], but is an elegant and purely geometric result.

In the interest of narrative, we will attempt to sketch briefly a few of the important elements of the argument. We shall omit all of the technicalities related to Q -valued Dirichlet energy minimizers, currents, and varifolds in what follows, as they are tangential to the points we wish to highlight. Instead, we shall (problematically) refer to these objects as “minimal surfaces.” For the purposes of this exposition, the important features of these frameworks is that they provide a class of objects which enjoy the following properties:

1. (Compactness) Sequences of minimal surfaces, Σ_k , with $0 \in \Sigma_k$ have subsequential limits in an appropriate sense.
2. (Blow-ups) If Σ is a minimal surface, $x \in \Sigma$, and we denote the rescaling

$$\Sigma_{x,r} = \frac{1}{r}(\Sigma - x),$$

then compactness immediately gives that for sequences $r_i \rightarrow 0$, there exists a subsequence $r_k \rightarrow 0$ and a cone, \mathcal{C} , such that $\Sigma_{x,r} \rightarrow \mathcal{C}$. We call \mathcal{C} a tangent cone at x . *A priori*, blow-ups are not unique, but all tangent cones to a minimal surface are themselves minimal surfaces.

3. (Monotonicity) If Σ is a minimal surface, $x \in \Sigma$, then for all $0 < r < \text{diam}(\Sigma)$ the density function,

$$\Theta^k(\Sigma, x, r) = \frac{1}{r^k} \mathcal{H}^k(\Sigma \cap B_r(x))$$

is a non-decreasing function of r . This immediately implies that the density, $\Theta^k(\Sigma, x) =$

$\lim_{r \rightarrow 0} \Theta^k(\Sigma, x, r)$, exists for all $x \in \Sigma$.

Roughly speaking, we can classify points in Σ by their tangent behavior. If tangent cones, \mathcal{C} , at x are flat, i.e., a k -plane, then we say x is a regular point. If a blow-up of Σ at x is a cone with singularities, then we say that x is a singular point. Studying the regularity and structure of the singular set is a natural line of inquiry. To investigate the singular set, we need to better understand the tangent cones. Tangent cones, \mathcal{C} , satisfy the following properties:

1. \mathcal{C} is homogeneous of order 0. ($\mathcal{C}_\lambda = \mathcal{C}$) I.e., it is a cone.
2. The density, $\Theta^k(\mathcal{C}, x)$, attains its maximum at 0. By homogeneity, $\Theta^k(\mathcal{C}, 0, r) = \Theta^k(\mathcal{C}, 0)$ for all $r > 0$.
3. The set $V = \{x \in \mathcal{C} : \Theta^k(\mathcal{C}, x) = \Theta^k(\mathcal{C}, 0)\}$ is a linear subspace.
4. The cone \mathcal{C} splits into $\mathcal{C}' \times V$ where \mathcal{C}' is a minimizing cone of dimension $\dim_{\mathcal{H}}(\mathcal{C}) - \dim_{\mathcal{H}}(V)$ contained in V^\perp .
5. \mathcal{C} is invariant under translation along V .

Almgren's idea was to stratify the singular set by the symmetries of tangent cones, where by "number of symmetries" we mean the dimension of V . For $i \in \mathbb{N}$, we define

$$S_i(\Sigma) = \{x \in \Sigma : \max_{\mathcal{C}}(\dim(V)) \leq i\}$$

where the maximum is taken over all tangent cones at x . For this stratification, Almgren proved the following theorem. The reader is encouraged to remember that "minimal surface" is being used loosely to avoid technicalities.

Theorem 1.1.4. (Almgren's stratification of singularities theorem) If $\Sigma \subset \mathbb{R}^n$, a k -dimensional minimal surface, then

$$\dim_{\mathcal{H}}(S_i(\Sigma)) \leq i.$$

Sketch of the Proof. The structure of the argument presented below is the archetype and motivation for the proofs presented in subsequent chapters. We sketch its outline as follows.

First, one needs to show that for all $x \in S_i(\Sigma)$, for all $\delta > 0$, there is an $\epsilon(\delta, x) > 0$ such that for all $0 < \rho \leq \delta$ then in small balls, $B_\rho(x)$, we have the following geometric control:

$$\{y \in B_\rho(x) : \Theta^k(\Sigma, y) \geq \Theta^k(\Sigma, x) - \epsilon\} \subset B_{\delta\rho}(L_{x,\rho}^i)$$

for some affine i -dimensional subspace, $L_{x,\rho}^i$. Since we can do this for *any* $\delta > 0$, in the language of local set approximation, we say that the sets $S_j(\Sigma)$ are unilaterally well-approximated by i -planes. This kind of geometric control is essential to the whole project we wish to undertake, and will be emulated in “quantitative rigidity” results later on. To achieve this kind of result, Almgren essentially relies upon compactness, blow-ups, and monotonicity results mentioned above. A good portion of this work will be dedicated to proving similar results in the context in which we work.

Next, Almgren further stratifies $S_i(\Sigma)$ into countably many pieces, $S_{i,j,q}(\Sigma)$, based upon the behavior of the density function. The stratification is done to give some quantitative control with respect to the previous “quantitative rigidity” result. For $j, q \in \mathbb{N}$, he defines,

$$S_{i,j}(\Sigma) = \{x \in S_i(\Sigma) : \text{quantitative rigidity holds with } \epsilon = j^{-1}\},$$

and further partitions this set into,

$$S_{i,j,q}(\Sigma) = \{x \in S_{i,j}(\Sigma) : \Theta^k(\Sigma, x) \in (\frac{q-1}{j}, \frac{q}{j}]\}.$$

Observe that $S_i(\Sigma) = \cup_{j,q} S_{i,j,q}(\Sigma)$ and that for $y \in S_{i,j,q}(\Sigma)$,

$$S_{i,j,q}(\Sigma) \subset \{x : \Theta^k(\Sigma, x) \geq \Theta^k(\Sigma, y) - j^{-1}\}$$

which, by our geometric control, gives that for for $y \in S_{i,j,q}(\Sigma)$, and radii $0 < \rho < j^{-1}$,

$$S_{i,j,q}(\Sigma) \cap B_\rho(y) \subset B_{\delta\rho}(L_{y,\rho}^i). \quad (1.1.5)$$

The main point is that Almgren achieves a partition of $S_i(\Sigma)$, where he can obtain the type of geometric control in 1.1.5 on each piece, $S_{i,j,q}(\Sigma)$. The dimension bound is then proved by building a sequence of inductively-refined covers.

Roughly speaking, if $y \in S_{i,j,q}(\Sigma)$, we cover the $\delta\rho$ -neighborhood of $L_{y,\delta}^j$ which contains $S_{i,j,q}(\Sigma) \cap B_\rho(y)$ with $C(n)(\delta\rho)^{-j}$ balls of radius $2\delta\rho$. Now, within each of these balls, we have the same geometric control as in 1.1.5. So, iterating the same construction in each of those balls, we build a more refined cover of $S_{i,j,q}(\Sigma) \cap B_\rho(y)$ with $(C(n)\delta^{-j})^2$ balls of radius $2\delta^2\rho$. Iterating this process we build a sequence of covers of $S_{i,j,q}(\Sigma) \cap B_\rho(y)$ with the property that for some explicit constant $\beta = \beta(\delta)$ (choose β such that $\delta^\beta C(n) \leq 1$) the m^{th} -iteration packs as follows

$$\sum_i (\delta^m \rho)^{j+\beta(\delta)} \leq C \rho^{j+\beta(\delta)} < \infty$$

This immediately implies that $\overline{\dim}_{\mathcal{M}}(S_{j,i}(\Sigma)) \leq j + \beta(\delta)$. Recalling that upper Minkowski dimension is not stable under countable unions, but Hausdorff dimension is, we see that summing over i, q , we have

$$\dim_{\mathcal{H}}(S_j(\Sigma)) \leq j + \beta(\delta).$$

To get the desired conclusion, we observe that $\beta(\delta) \rightarrow 0$ as $\delta \rightarrow 0$, and so we may repeat this construction for a sequence of $\delta \rightarrow 0$.

This argument was refined to a quantitative version by Cheeger and Naber in [12]. While an improvement upon the version adumbrated above (notably, allowing for stronger upper

Minkowski $j + \beta(\delta)$ -content estimates), the basic structure is essentially the same. In Chapter 3, we apply Cheeger and Naber’s quantitative Almgren stratification argument to the two-phase free-boundary problem for interior and exterior harmonic measure.

In Chapter 4, we follow a powerful improvement upon these methods due to Naber and Valtorta [24]. The improvement of Naber and Valtorta’s techniques comes from a powerful geometric measure theory result which give packing estimates using the “correct” power. That is, we can bound the sum of the radii of the balls in the covering by,

$$\sum_i (\delta^m \rho)^j \leq C \rho^j < \infty.$$

This allows for much stronger estimates of the form $\overline{\dim}_{\mathcal{M}}(S_j(\Sigma)) \leq j$. See Chapter 4 for more details.

It is interesting to note that in Chapter 3, we shall only use small drops in our monotonic function *across scales* to get geometric control, while in Chapter 4, the idea of controlling the set on which one has small drops (across scales) from the *maximum value* the monotonic function re-emerges as a crucial step towards a more efficient packing estimates.

1.2 Introduction for Chapter 5

A set $E \subset \mathbb{R}^d$ is said to be countably n -rectifiable if there are Lipschitz maps, $f_i : \mathbb{R}^n \rightarrow \mathbb{R}^d$, with $i = 1, 2, \dots$, such that

$$\mathcal{H}^n(\mathbb{R}^d \setminus \cup_i f_i(\mathbb{R}^n)) = 0.$$

In his solution to the Analyst’s Traveling Salesman Problem [20], Peter Jones introduced a local gauge of flatness which has been generalized by David and Semmes [14] to measures and higher dimensions. These families of local gauges of flatness are called the Jones β -numbers, and they have come to dominate the landscape in quantitative techniques relating

rectifiability, potential theory, and boundedness of singular integrals. See, for example, the landmark book [15].

For a set $E \subset \mathbb{R}^d$, $1 \leq p < \infty$, and an integer $1 \leq n \leq d-1$, we write $\mu = \mathcal{H}^n \llcorner E$ and define the Jones β -numbers as follows,

$$\beta_{E;p}^n(x, r) = \left(\inf_{L \subset \mathbb{R}^d \text{ an } n\text{-plane}} \int_{B(x,r)} \left(\frac{\text{dist}(y, L)}{r} \right)^p \frac{d\mu(y)}{r^n} \right)^{\frac{1}{p}}.$$

We also write $\beta_{\mu;p}^n(x, r)$ for $\beta_{E;p}^n(x, r)$, where $\mu = \mathcal{H}^n \llcorner E$ is understood.

In addition to generalizing the Jones β -numbers, [14] also introduced the notion of uniform rectifiability. A set $E \subset \mathbb{R}^d$ is said to be Ahlfors n -regular if there exists $0 < c < C < \infty$ such that $cr^n \leq \mathcal{H}^n(E \cap B(x, r)) \leq Cr^n$ for all $x \in E$ and all $0 < r < \text{diam}(E)$. An n -Ahlfors regular $E \subset \mathbb{R}^d$ is said to be uniformly n -rectifiable if there exist finite constants $\theta, \Lambda > 0$ such that for all $x \in E$ and all $0 < r < \text{diam}(E)$ there is a Lipschitz mapping, $g : B(0, r) \subset \mathbb{R}^n \rightarrow \mathbb{R}^d$, with $\text{Lip}(g) \leq \Lambda$ such that $\mathcal{H}^n(E \cap B(x, r) \cap g(B(0, r))) \geq \theta r^n$.

In [14] the authors show that an n -Ahlfors regular set $E \subset \mathbb{R}^d$, is n -uniformly rectifiable if and only if the Jones β -numbers satisfy the following Carleson condition for some $1 \leq p < \frac{2n}{n-2}$,

$$C_{E;p}^n(x, R) := \int_{B(x,R)} \int_0^R \beta_{E;p}^n(y, r)^2 \frac{dr}{r} d\mu(y) \leq cR^n \quad \text{for all } x \in E, R > 0. \quad (1.2.1)$$

Recently, Tolsa [27] and Azzam and Tolsa [5] show as a special case of their results that E is countably n -rectifiable if and only if,

$$\int_0^1 \beta_{E;2}^n(x, r)^2 \frac{dr}{r} < \infty \quad \text{for } \mathcal{H}^n - a.e. x \in E. \quad (1.2.2)$$

See [26] and [11] for related precedents. We call this integration of the β -numbers over scales the *Jones function* (at x and scale 1).

In this paper, we show that sets which satisfy (1.2.2) can fail to satisfy (1.2.1) as dramatically as possible. We construct two examples of bounded, countably 1-rectifiable sets in \mathbb{R}^2 with positive and finite \mathcal{H}^1 -measure that satisfy various topological and measure-theoretic properties. This is formalized in the following two theorems.

Theorem 1.2.1. *There exists a rectifiable curve (of finite length), $K_0 \subset \mathbb{R}^2$, such that for $\mu = \mathcal{H}^1 \llcorner K_0$, for any $x \in K_0$, and any $\delta > 0$,*

$$\int_{B_\delta(x)} \int_0^\delta \beta_{\mu;2}^1(y, r)^2 \frac{dr}{r} d\mu(y) = \infty.$$

The set K_0 arises from unions of modifications of approximations to snowflake-like sets. Note that by the Analyst's Traveling Salesman theorem [20]

$$\int_{\mathbb{R}^2} \int_0^\infty \beta_{\mu,\infty}^1(y, r)^2 \frac{dr}{r} d\mu(y) < \infty,$$

which prevents K_0 from being upper regular at a generic point.

Theorem 1.2.2. *There is a 1-Ahlfors regular, countably 1-rectifiable set A_0 contained in the unit cube in \mathbb{R}^2 such that for $\mu = \mathcal{H}^1 \llcorner A_0$, for every $x \in A_0$, and for every $\delta > 0$,*

$$\int_{B_\delta(x)} \int_0^\delta \beta_{A_0;2}^1(y, r)^2 \frac{dr}{r} d\mu(y) = \infty.$$

The set A_0 , whose construction was initially motivated by the machinery introduced in [28], is created from scaled unions of approximations to the 4-corner Cantor set. Ultimately the presentation was simpler using the framework of self-similar sets.

Remark 1.2.3. The examples we produce can evidently be used to create higher-dimensional examples by taking Cartesian products with finite intervals. That is, if $A \in \{K_0, A_0\}$ is one of the examples, then you can create an n -dimensional set in \mathbb{R}^d for any positive integer $n < d$ with the same properties as A by considering $E' = A \times [0, 1]^{n-1} \subset \mathbb{R}^{n+1}$, and then obtain $E \subset \mathbb{R}^d$ by embedding E' into the first $(n + 1)$ -dimensions of \mathbb{R}^d .

It's easy to check that the connectivity, countable rectifiability, and/or Ahlfors regularity properties of A are inherited by E . Moreover, it is standard that defining β -numbers over cubes (with sides parallel to the axes in \mathbb{R}^d) instead of balls leads to an equivalent definition of the β -numbers. Consequently it is quick to see that the finiteness of $C_{E;2}^n(x, R)$ is equivalent to the finiteness of $C_{A;2}^1(x', R)$ where x' is the orthogonal projection of $x \in E$ into \mathbb{R}^2 .

Chapter 2

DEFINITIONS AND PRELIMINARIES

2.1 Non-tangentially accessible domains

Non-tangentially accessible (NTA) domains were formally introduced by Jerison and Kenig in [19] to study the boundary behavior of PDEs on non-smooth domains.

Definition 2.1.1. Given a domain $\Omega \subset \mathbb{R}^n$ and two points, $x_1, x_2 \in \Omega$, a *Harnack chain* between x_1 and x_2 is a collection of balls, $\{B_{r_i}(y_i)\}_{i=1}^N$ such that $y_1 = x_1, y_N = x_2, B_{r_i}(y_i) \subset \Omega$ for all i , and $B_{r_i}(y_i) \cap B_{r_{i+1}}(y_{i+1}) \neq \emptyset$ for all $i = 1, \dots, N - 1$.

Definition 2.1.2. A domain $\Omega \subset \mathbb{R}^n$ is a non-tangentially accessible (NTA) domain if there exist constants $M > 1, R_0 > 0$ such that the following holds:

1. Ω satisfies the corkscrew condition. That is, for any $Q \in \partial\Omega$ and $0 < r < R_0$, there exists a point $A_r(Q) \in \Omega$ with the following two properties:

$$|A_r(Q) - Q| < r \quad \text{and} \quad B_{\frac{r}{M}}(A_r(Q)) \subset \Omega.$$

2. $\overline{\Omega}^c$ is also satisfies the corkscrew condition.

3. Ω satisfies the Harnack Chain condition. That is, for any $\epsilon > 0$ and $Q \in \partial\Omega$, if $x_1, x_2 \in \Omega \cap B_{\frac{R_0}{4}}(Q) \setminus B_\epsilon(\partial\Omega)$ and $|x_1 - x_2| \leq 2^k \epsilon$, then there exists a Harnack chain of overlapping balls contained in Ω connecting x_1 and x_2 . Furthermore there are at most Mk such balls in the chain and their diameters can be bounded below by $\frac{1}{M} \min_{1,2} \{dist(x_i, \partial\Omega)\}$.

We say that Ω is a *two-sided NTA domain* if both Ω and $\overline{\Omega}^c$ are NTA domains. For computational ease, we shall only deal with unbounded, two-sided NTA domains, that is, we shall

assume that $R_0 = \infty$. However, the results are essentially local.

We shall use u^\pm to denote the Green's function with pole at infinity corresponding to Ω^\pm , respectively. Recall that u^\pm are unique up to scalar multiplication and that to each u^\pm is associated the harmonic measure ω^\pm , defined by the property that for all $\phi \in C_c^\infty(\mathbb{R}^n)$,

$$\int \Delta \phi u^\pm dx = \int \phi d\omega^\pm.$$

Remark 2.1.3. Observe that if ω^+ is the harmonic measure associated to u^+ , then $c\omega^+$ is the harmonic measure associated to cu^+ for any $c > 0$. Furthermore, for Ω^\pm a pair of two-sided NTA domains with mutual boundary, $\partial\Omega^\pm$, for any $Q \in \partial\Omega^\pm$ and any Green's functions, u^\pm , we can form the function,

$$v^Q(x) = h(Q)u^+(x) - u^-(x).$$

The scaling, $h(Q)u^+$, normalizes the Radon-Nikodym derivative of the harmonic measure associated to $h(Q)u^+$ and u^- at $Q \in \partial\Omega^\pm$. Indeed, $\tilde{h}(Q) = \frac{d\omega^-(Q)}{h(Q)d\omega^+(Q)} = 1$. This has the effect of “aligning” u^- and u^+ at Q . Later, we shall only consider $v = v^0$. As such, these functions will, in general, *not* be “aligned” at $Q \in \partial\Omega^\pm$ for $Q \neq 0$. Cf. Example 2.4.10.

Definition 2.1.4. We define the class $\mathcal{D}(n, \alpha, M_0)$ to be the collection of domains $\Omega^\pm \subset \mathbb{R}^n$ such that Ω^\pm are complementary, unbounded two-sided NTA domains for which $M < M_0$, $\omega^- \ll \omega^+ \ll \omega^-$, the Radon-Nikodym derivative, $h = \frac{d\omega^-}{d\omega^+}$, is such that $\ln(h) \in C^{0,\alpha}(\partial\Omega)$, and $0 \in \partial\Omega^\pm$.

2.2 Almgren Frequency Function

One of the key tools of this paper will be the Almgren frequency function, introduced in [2]. Since we want to capture the behavior of v on more than just a level set, we make the following definitions.

Definition 2.2.1. For any Lipschitz function $v : \mathbb{R}^n \rightarrow \mathbb{R}$, radius $r > 0$, and point $p \in \mathbb{R}^n$, we

define the following quantities: Let

$$H(r, p, v) = \int_{\partial B_r(p)} |v - v(p)|^2 d\sigma$$

$$D(r, p, v) = \int_{B_r(p)} |\nabla v|^2 dx$$

$$N(r, p, v) = r \frac{D(r, p, v)}{H(r, p, v)}.$$

This normalized version of the Almgren frequency function is invariant in the following senses.

Lemma 2.2.2. Let $a, b, c \in \mathbb{R}$ with $a, b \neq 0$. If $w(x) = av(bx) + c$, then

$$N(r, 0, v) = N(b^{-1}r, 0, w)$$

In the harmonic case, the Almgren frequency function is well-studied. If u is harmonic, then $N(r, p, u)$ is monotonically non-decreasing, and $\lim_{r \rightarrow 0} N(r, p, u) = N(0, p, u)$ is the degree of the leading homogeneous harmonic polynomial in the Taylor expansion of u at the point p .

2.3 A class of functions

Because all our results are local, for a function, $f \in C^{0,\alpha}(\mathbb{R}^n)$, we shall use $\|f\|_\alpha$ to denote the local norm,

$$\|f\|_\alpha = \sup_{B_2(0)} |f| + \sup_{x \neq y \in B_2(0)} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

In the subsequent pages, limit-compactness arguments shall be an important tool. Therefore, we introduce the following class of functions.

Definition 2.3.1. Let $\mathcal{A}(\Lambda, \alpha, M_0)$ be the set of functions, $v : \mathbb{R}^n \rightarrow \mathbb{R}$ which have the following properties:

1. $v = v^0 = h(0)u^+ - u^-$ where u^\pm are Green's functions with poles at infinity associated to a two-sided NTA domain, $\Omega^\pm \in \mathcal{D}(n, \alpha, M_0)$ and $h = \frac{d\omega^-}{d\omega^+}$, where ω^\pm are the harmonic measures associated to u^\pm .
2. We make the specific choice of u^\pm such that $h(0) = 1$.
3. $N(1, 0, v) \leq \Lambda$.

Remark 2.3.2. Observe that for any fixed, $\Omega^\pm \in \mathcal{D}(n, \alpha, M_0)$, there is a one-parameter family of associated functions, $v \in \mathcal{A}(\Lambda, \alpha, M_0)$ with $\{v = 0\} = \partial\Omega^\pm$. Indeed, if $v \in \mathcal{A}(\Lambda, \alpha, M_0)$, then $cv \in \mathcal{A}(\Lambda, \alpha, M_0)$ for any $c > 0$. This degree of freedom comes from the non-uniqueness of the Green's function with pole at infinity (see Remark 2.1.3). To avoid degeneracy because of this degree of freedom, in the arguments that follow we must normalize our functions (see Definition 2.4.5).

Remark 2.3.3. In the definition of the class $\mathcal{A}(\Lambda, \alpha, M_0)$ there is no restriction on the magnitude of $\|\ln(h)\|_\alpha$ for the functions, v , we consider. This is done to avoid complications in the compactness arguments that follow. However, in order to obtain results, we must bound $\|\ln(h)\|_\alpha$ by assumption. This bound therefore appears in the statement of our results.

Remark 2.3.4. Recall that for $E \subset \partial\Omega^\pm$,

$$\omega^+(E) = \int \chi_E d\omega^+$$

and

$$\omega^-(E) = \int \chi_E h d\omega^+.$$

Furthermore, if $\ln(h) \in C^{0,\alpha}$ with $\|\ln(h)\|_\alpha \leq \Gamma$, then for all $Q, Q' \in \partial\Omega^\pm$,

$$e^{-\Gamma|Q-Q'|^\alpha} h(Q') \leq h(Q) \leq e^{\Gamma|Q-Q'|^\alpha} h(Q'). \quad (2.3.1)$$

Using Equation 2.3.1 in the above integral equations implies that in any compact set K , if $v \in \mathcal{A}(\Lambda, \alpha, M_0)$ with $\|\ln(h)\|_\alpha \leq \Gamma$, there is a constant, $C(K, \Gamma, \alpha) > 1$ such that for any $E \subset K \cap \partial\Omega^\pm$,

$$C^{-1} \leq \frac{\omega^-(E)}{\omega^+(E)} \leq C$$

Remark 2.3.5. The definition of $\mathcal{A}(\Lambda, \alpha, M_0)$ is symmetric in the sense that if

$$v = h(0)u^+ - u^- \in \mathcal{A}(\Lambda, \alpha, M_0),$$

then

$$\tilde{v} = (-h(0)^{-1})v = h(0)^{-1}u^- - u^+ \in \mathcal{A}(\Lambda, \alpha, M_0).$$

This symmetry comes from relabeling Ω^\pm as Ω^\mp . Multiplication does not change $\partial\Omega^\pm$, nor does it change the Almgren frequency. Also, if $\ln(h) \in C^\alpha$, then $\ln(h^{-1}) \in C^{0,\alpha}$.

For entirely technical reasons, in Sections 3.2, 3.3, 3.4, and 3.5 we shall employ bounds involving ω^- , which ostensibly only hold for $p \in \overline{\Omega^-}$. However, this symmetry allows us to do the arguments again for \tilde{v} to get the same results for $p \in \Omega^+$.

2.4 Rescaling procedures and Symmetry

In order to investigate the fine structures of the generalized critical sets of functions $v \in \mathcal{A}(\Lambda, \alpha, M_0)$, we will need to investigate the behavior of rescalings of v . In fact, we will use two different rescalings of functions v . The first is adapted to the comparison principle (Definition 2.4.1), and the second is adapted to the quantitative stratification introduced by in [12] (Definition 2.4.5).

Definition 2.4.1. Let $\Omega^\pm \in \mathcal{D}(n, \alpha, M_0)$ and $p \in \mathbb{R}^n$ and $d = \text{dist}(p, \partial\Omega^\pm)$. For scales $2d < r$,

we define the rescaling of the function v^Q to scale r at the point p by

$$v_{p,r}^Q(x) = v^Q(rx + p) \frac{r^{n-2}}{\omega^-(B_r(p))}$$

and the corresponding rescaled measure as

$$\omega_{p,r}^\pm(E) = \frac{\omega^\pm(rE + p)}{\omega^\pm(B_r(p))}$$

These rescalings were first introduced for $p = Q \in \partial\Omega^\pm$ by Kenig and Toro in [22]. In this paper, we shall employ the following results by Kenig, Toro, Badger, and Engelstein.

Theorem 2.4.2. ([22], [6], [17]) For $v_{p,r}^Q$ and $\omega_{p,r}^\pm$ as in Definition 2.4.1,

1. Subsequential limits as $r \rightarrow 0$ of the functions $v_{Q,r}^Q$ converge to harmonic polynomials. Furthermore, the degree of these polynomials is bounded, depending only upon the NTA constant, M_0 . [22]
2. Subsequential limits as $r \rightarrow 0$ of the functions $v_{Q,r}^Q$ converge to *homogeneous* harmonic polynomials. Furthermore, the degree of homogeneity is unique along blow-ups. [6]
3. The $v_{Q,r}^Q$ are uniformly locally Lipschitz with Lipschitz constant that only depends upon M_0 . [17]
4. The $\omega_{Q,r}^\pm$ are locally uniformly bounded. [17].

Corollary 2.4.3. For $v_{p,r}^Q$ and $\omega_{p,r}^\pm$ as in Definition 2.4.1, we have that for all points, p , and all admissible radii, r ,

1. The $v_{p,r}^Q$ are uniformly locally Lipschitz with Lipschitz constant that only depends upon M_0 , $\|\ln(h)\|_\alpha$, α , and $|p - Q|$.
2. The $\omega_{p,r}^\pm$ are uniformly locally bounded by constants that only depends upon M_0 .

Proof. Let $Q' \in \partial\Omega^\pm$ be a point such that $|p - Q'| = \text{dist}(p, \partial\Omega^\pm)$. Note that,

$$B_{\frac{1}{2}r}(Q') \subset B_r(p) \subset B_{2r}(Q').$$

To prove (2) we only need to observe that harmonic measure is doubling on NTA domains, with doubling constant which only depends upon M_0 . Therefore, $\omega_{p,r}^\pm$ is comparable to $\omega_{Q',r}^\pm$, and Theorem 2.4.2(4) gives the desired statement.

To see (1), we observe that,

$$\begin{aligned} v^Q(rx + p) \frac{r^{n-2}}{\omega^-(B_{2r}(Q'))} &\leq v^Q(rx + p) \frac{r^{n-2}}{\omega^-(B_r(p))} \\ &= v_{p,r}^Q(x) \\ &\leq v^Q(rx + p) \frac{r^{n-2}}{\omega^-(B_{\frac{1}{2}r}(Q'))}. \end{aligned}$$

By Equation 2.3.1, we have that $v_{p,r}^Q(x) \sim_{\|\ln(h)\|_\alpha, |Q-Q'|} v_{p,r}^{Q'}(x)$. Therefore, by the doubling of harmonic measure on two-sided NTA domains, we have that

$$v_{p,r}^Q \sim_{\|\ln(h)\|_\alpha, |Q-Q'|} v_{p,r}^{Q'}(x) \sim_{M_0} v_{Q',r}^{Q'}(x).$$

Theorem 2.4.2 (3), then gives the desired result. \square

Remark 2.4.4. While subsequential limits as $r \rightarrow 0$ of the functions $v_{Q',r}^Q$ converge to homogeneous harmonic polynomials, it is not true in general that $v_{p,r}^Q$ converges to a homogeneous harmonic polynomial. As $r \rightarrow 0$, the function $v_{p,r}^Q$ will converge to a homogeneous function which agrees with a harmonic polynomial in its positive and negative domains. However, there is no a priori reason for these to “align” and be harmonic across the boundary, cf. Remark 2.1.3 and Example 2.4.10.

Because the functions $v_{p,r}^Q$ are merely Lipschitz, we will often need to work with a mollified

version of them. We will use the convention that $v_\epsilon = v \star \phi_\epsilon$ for $\phi \in C^\infty$ a mollifying function ($\text{spt}(\phi) \subset B_1$ and $\int \phi = 1$).

Definition 2.4.5. Let $f : B_1(0) \rightarrow \mathbb{R}$ be a function in $C(\mathbb{R}^n)$. We define the rescaled function, $T_{x,r}f$ of f at a point $x \in B_{1-r}(0)$ at scale $0 < r < 1$ by

$$T_{x,r}f(y) = \frac{f(x+ry) - f(x)}{(\int_{\partial B_1(0)} (f(x+ry) - f(x))^2 dy)^{1/2}}.$$

We denote the limit as $r \rightarrow 0$ by

$$T_x f(y) = \lim_{r \rightarrow 0} T_{x,r} f(y)$$

Note that the denominator simply normalizes the blow-up. In the case that the denominator is zero, we define $T_{x,r}f = \infty$. The geometry we wish to capture with the blow-ups $T_x f$ is encoded in their translational symmetries.

Definition 2.4.6. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous function. We say f is 0-symmetric if

$$f(x) = cP^+(x) - P^-(x)$$

for some $c > 0$, where P^\pm are the positive and negative parts of a homogeneous harmonic polynomial. We will say that f is k -symmetric if f is 0-symmetric and there exists a k -dimensional subspace, V , such that $f(x+y) = f(x)$ for all $x \in \mathbb{R}^n$ and all $y \in V$.

Remark 2.4.7. The constant, $c > 0$, is there to allow for the function to “hinge” along its zero set. We must allow this kind of “hinging” to accommodate for the “non-alignment” issue in the blow-ups at $Q \in \partial\Omega^\pm \setminus \{0\}$. See Remark 2.4.4.

It is this flexibility in our approximating functions which allows us to get estimates on the *generalized* critical set for a fixed function, $v = v^0$, without rescaling to v^Q for different Q depending on which part of the boundary we are considering.

We now define a quantitative version of symmetry.

Definition 2.4.8. For any $f \in C(\mathbb{R}^n)$, f will be called (k, ϵ, r, p) -symmetric if there exists a k -symmetric function, P , such that,

1. $\int_{\partial B_1(0)} |P|^2 = 1$
2. $\int_{B_1(0)} |T_{p,r}f - P|^2 < \epsilon.$

Sometimes, we shall refer to a function f as being (k, ϵ) -symmetric in the ball $B_r(p)$ to mean f is (k, ϵ, r, p) -symmetric.

This gives a quantitative grasp on how close to being k -symmetric a function is in $B_r(x)$. This quantitative control allows us to define a quantitative stratification.

Definition 2.4.9. Let $v \in \mathcal{A}(\Lambda, \alpha, M_0)$. We denote the (k, ϵ, r) -critical stratum of v by $\mathcal{S}_{\epsilon,r}^k(v)$, and we define it by

$$\mathcal{S}_{\epsilon,r}^k(v) = \{x \in \mathbb{R}^n : v \text{ is not } (k+1, \epsilon, s, x)\text{-symmetric for all } s \geq r\}$$

We shall also use the notation $\mathcal{S}_\epsilon^k(v)$ for $\mathcal{S}_{\epsilon,0}^k(v)$.

It is immediate from the definitions that $\mathcal{S}_{\epsilon,r}^k(v) \subset \mathcal{S}_{\epsilon',r'}^{k'}(v)$ if $k \leq k', \epsilon' \leq \epsilon, r \leq r'$. This in turn implies that we can recover the qualitative stratification

$$\mathcal{S}^k(v) = \{x \in \mathbb{R}^n : T_x v \text{ is not } (k+1)\text{-symmetric}\} = \cup_\eta \cap_r \mathcal{S}_{\eta,r}^k(v).$$

Furthermore, if $x \in \mathcal{S}^k(v)$, then there exists an $0 < \epsilon$ such that $x \in \mathcal{S}_\epsilon^k(v)$. The set, $\mathcal{S}^k(v)$, is the k^{th} stratum of the *generalized* critical set (see Equation 1.1.1 for definition of the generalized critical set). Note that these $\mathcal{S}^k(v)$ are not “singular” strata. Because of “non-alignment,” ∇v may not exist on large parts of $\partial\Omega^\pm$, even where $\partial\Omega^\pm$ is locally $C^{1,\alpha}$. Consider the following illustrative example.

Example 2.4.10. Let $v : \mathbb{R}^4 \rightarrow \mathbb{R}$ be the piece-wise function defined by

$$f(\vec{x}) = \begin{cases} \vec{x} \cdot \vec{e}_n & \text{when } \vec{x} \cdot \vec{e}_n > 0 \\ \vec{x} \cdot \vec{2}e_n & \text{when } \vec{x} \cdot \vec{e}_n \leq 0 \end{cases}$$

This function f is 1-homogeneous and harmonic in its positive and negative parts and singular along the spine, $\{f = 0\}$, in that ∇f does not exist for any $x \in \{f = 0\}$. However, our stratification does not detect this kind of singularity. Indeed, all of the points in the spine $\{f = 0\}$ are in $\mathcal{S}^{n-1}(v)$, which we should think of as the “regular” set, because we could scale the positive part and “align” it to be a linear homogeneous harmonic function. What this stratification does detect is the geometry of the spine.

Theorem 2.4.11. ([8] [9], [17]) For $v \in \mathcal{A}(\Lambda, \alpha, M_0)$, the following hold:

1. $(\mathcal{S}^{n-2}(v) \setminus \mathcal{S}^{n-3}(v)) \cap \partial\Omega^\pm = \emptyset$, ([9], Remark 7.2).
2. There exists an $\epsilon > 0$ such that $\text{sing}(\partial\Omega^\pm) = \mathcal{S}^{n-3}(v) \cap \partial\Omega^\pm \subset \mathcal{S}_\epsilon^{n-2}(v)$, ([17], Theorem 1.1).
3. $\overline{\dim_{\mathcal{M}}(\text{sing}(\partial\Omega^\pm))} \leq n - 3$ ([8], Theorem 7.5).

Remark 2.4.12. At the end of Section 3.2 shall also prove that there exists an $\epsilon > 0$ such that $\text{sing}(\partial\Omega^\pm) = \mathcal{S}^{n-3}(v) \cap \partial\Omega^\pm \subset \mathcal{S}_\epsilon^{n-3}(v)$. This allows us to prove that it satisfies finite upper Minkoski $(n - 3)$ -content, see Corollary 4.1.3.

2.5 The Jones β -numbers and the Discrete Reifenberg Theorem

One of the important tools in the second half of this work will be the Jones β -numbers. For μ a Borel measure, we define $\beta_\mu^k(p, r)^2$ as follows.

Definition 2.5.1.

$$\beta_\mu^k(p, r)^2 = \inf_{L^k} \frac{1}{r^k} \int_{B_r(p)} \frac{\text{dist}(x, L)^2}{r^2} d\mu(x)$$

where the infimum is taken over all affine k -planes.

Recall that the Grassmanian, $G(k, n)$, is compact. Therefore, let $V_\mu^k(p, r)$ denote a k -plane which minimizes the *infimum* in the definition of $\beta_\mu^k(p, r)^2$. Note that this k -plane is not *a priori* unique.

We begin by stating two basic properties of the Jones β -numbers. The first controls how fast the β -numbers can shrink by relating the β -numbers at comparable scales. This property is often called “doubling,” though we have chosen to scale by the number 3. The second property shows how the β -numbers behave under rescaling.

Proposition 2.5.1. *Let $E \subset \mathbb{R}^2$ have $\dim_{\mathcal{H}}(E) = 1$.*

1. *For any ball $B_r(y) \subset B_{3r}(x)$,*

$$\beta_E(y, r)^2 \leq 3\beta_E(x, 3r)^2$$

2. *The β -numbers have the following scaling property. If $E^{z,t} = tE + z$ then $\beta_{E^{z,t}}(x, r)^2 = \beta_E\left(\frac{x-z}{t}, \frac{r}{t}\right)^2$. Consequently, $C_{E^{z,t}}(z, r) = tC_E(0, t^{-1}r)$.*

Lemma 2.5.2. *If $C_1 > 0$ and $\text{diam}\{\text{spt}(\mu)\} = R$, then for all $x \in \text{spt}(\mu)$ there exists some $C = C(C_1, n)$ such that*

$$\int_0^\infty \beta_{\mu;2}^n(x, r)^2 \frac{dr}{r} \leq \int_0^{C_1 R} \beta_{\mu;2}^n(x, r)^2 \frac{dr}{r} + C\beta_{\mu;2}^n(x, R)^2.$$

In particular, if $C_1 \geq 1$ then $C \leq 1/2$ so that

$$\int_0^{C_1 R} \beta_{\mu;2}^n(x, r)^2 \frac{dr}{r} \geq \int_0^\infty \beta_{\mu;2}^n(x, r)^2 \frac{dr}{r} - \frac{1}{2}\beta_{\mu;2}^n(x, R)^2.$$

Proof. Suppose R and μ are as above. Then, for $x \in \text{spt}(\mu)$, $\text{spt}(\mu) \subset B(x, R)$. Hence,

$$\begin{aligned}
\int_{C_1 R}^{\infty} \beta_{\mu;2}^n(x, r)^2 \frac{dr}{r} &= \int_{C_1 R}^{\infty} \left[\inf_L \frac{1}{r^n} \int_{B(x,r)} \left(\frac{\text{dist}(y, L)}{r} \right)^2 d\mu(y) \right] \frac{dr}{r} \\
&\leq \int_{C_1 R}^{\infty} \left[\frac{R}{r} \right]^{n+2} \left[\inf_L \frac{1}{R^n} \int_{B(x,R)} \left(\frac{\text{dist}(y, L)}{R} \right)^2 d\mu(y) \right] \frac{dr}{r} \\
&= R^{n+2} \beta_{\mu;2}^n(x, R)^2 \int_{C_1 R}^{\infty} r^{-(n+3)} dr \\
&= R^{n+2} \beta_{\mu;2}^n(x, R)^2 \frac{(C_1 R)^{-(n+2)}}{n+2} \\
&= \frac{\beta_{\mu;2}^n(x, R)^2}{C_1^{n+2}(n+2)},
\end{aligned}$$

so $C(C_1, n) = \frac{1}{C_1^{n+2}(n+2)}$.

□

The following theorem of Naber and Valtorta is a powerful tool which links the sum of the $\beta_{\mu}^k(p, r)^2$ over all points and scales to packing estimates. Roughly put, it says that if a collection of balls looks very flat with respect to k -planes whenever there are enough of them together, then the whole collection cannot bunch together too much. That is, we get k -dimensional upper density estimates on the collection.

Theorem 2.5.2. (Discrete Reifenberg, [24]) Let $\{B_{\tau_i}(x_i)\}_i$ be a collection of disjoint balls such that for all $i = 1, 2, \dots$ $\tau_i \leq 1$. Let $\epsilon_k > 0$ be fixed. Define a measure,

$$\mu = \sum_i \tau_i^k \delta_{x_i},$$

and suppose that for any $x \in B_2$ and any scale $l \in \{0, 1, 2, \dots\}$, if $B_{r_l}(x) \subset B_2(0)$ and $\mu(B_{r_l}(x)) \geq \epsilon_k r_l^k$ then

$$\sum_{i \geq l} \int_{B_{2r_l}(x)} \beta_{\mu}^k(z, 16r_i)^2 d\mu(z) < r_l^k \delta^2.$$

Then, there exists a $\delta_0 = \delta_0(n, \epsilon_k) > 0$ such that if $\delta \leq \delta_0$,

$$\mu(B_1(0)) = \sum_{i \text{ s.t. } x_i \in B_1(0)} \tau_i^k \leq C(n).$$

Chapter 3

FEDERER DIMENSION REDUCTION I

3.1 Main Results

As a first pass to addressing Question 1.1.1, we have the following result.

Theorem 3.1.1. Let $v \in \mathcal{A}(\Lambda, \alpha, M_0)$ with $\|ln(h)\|_\alpha \leq \Gamma$ and $r > 0$. Then for every $0 < \epsilon$ and $k \leq n - 2$, there exists a $0 < \gamma < 1$ such that for every $N \in \mathbb{N}$

$$\text{Vol}(B_{\frac{1}{4}}(0) \cap B_{\gamma^N}(\mathcal{S}_{\epsilon, r}^k(v))) \leq C(n, k, \Lambda, \alpha, M_0, \epsilon, \Gamma)(\gamma^N)^{n-k-\epsilon}. \quad (3.1.1)$$

As an immediate corollary, we also have that

Corollary 3.1.2. Let $v \in \mathcal{A}(\Lambda, \alpha, M_0)$ for any $0 < \epsilon$

$$\overline{\dim}_{\mathcal{M}}(\mathcal{S}_\epsilon^k(v) \cap B_{\frac{1}{4}}(0)) \leq k \quad (3.1.2)$$

Proof. From the definitions and the previous result, we see that $\overline{\dim}_{\mathcal{M}}(\mathcal{S}_\epsilon^k(v)) \leq k + \epsilon$.

Since we also have a containment, $\mathcal{S}_{\epsilon'}^k(v) \subset \mathcal{S}_\epsilon^k(v)$ if $\epsilon' \leq \epsilon$. Thus, $\overline{\dim}_{\mathcal{M}}(\mathcal{S}_{\epsilon'}^k(v)) \leq k + \epsilon'$ for all $\epsilon' \leq \epsilon$. \square

3.1.1 Outline of the Proof of Theorem 3.1

To prove a result like this, we need to construct a cover and count how many balls we used. Our strategy will be to employ a *quantitative* version of Almgren's stratification of singularities result.

Broadly speaking, we need to get some kind of geometric control which will allow us to cover efficiently. This geometric control comes from two sources, a quantitative rigidity result and a cone-splitting result.

Quantitative rigidity is a way to quantify the fact that the Almgren frequency function is constant on homogeneous harmonic polynomials. Roughly speaking, it says that if the Almgren frequency is *almost* constant, as measured by having small drop,

$$|N(1, 0, v) - N(\gamma, 0, v)| \leq \rho,$$

then v is *almost* a homogeneous harmonic polynomial in $L^2(B_1(0))$.

Then, we will prove a cone-splitting result, which says that if our function, v , is close to being k -symmetric with respect to a k -plane, V , in $B_r(x)$ and close to being 0-symmetric in $B_s(y) \subset B_r(x)$, for some y which is not in V , then v must be $(k + 1)$ -symmetric in $B_r(x)$.

This cone splitting result means that for a given scale, r , set of points in $S_\epsilon^k \cap B_r(x)$ which are close to a homogeneous harmonic polynomial *at scale* r must themselves be close to a k -plane. This is the geometric control we were seeking. The precise formulation of Definition 2.4.9 is essential for proving this result. Both quantitative rigidity and cone-splitting are proven in Section 3.5.

Just as with Theorem 1.1.4 this kind of geometric control comes from having a compact class of functions, detailed knowledge of the blow-ups, and a monotonic function. In our context, the role monotonic quantity is played by the Almgren frequency. Section 3.2 establishes that $\mathcal{A}(\Lambda, \alpha, M_0)$ is sufficiently compact. Section 3.3 is dedicated to generalizing of some of the estimates in [17] to obtain almost monotonicity of the Almgren frequency. With these tools, we obtain the necessary geometric control in balls, $B_r(p)$, where the Almgren frequency has small drop,

$$|N(r, p, v) - N(\gamma r, p, v)| \leq \rho.$$

To control how many times we can have large change in Almgren frequency, we need to show that $N(r, p, v)$ can be bounded uniformly in a ball. Section 3.4 is dedicated to obtaining such a result. With a bound on the total variation of the Almgren frequency, there can only be finitely many scales at which a function, v , has large change in Almgren frequency. By quantitative rigidity, this means there can only be finitely many scales at which a function, v , is not close to being an homogeneous harmonic polynomial. Good geometric control follows at all but finitely many scales.

Just as in the sketch of Theorem 1.1.4, only using Reifenberg-flat control to build our cover gives an extra power in the packing condition ($\beta(\delta)$ in the above sketch). This is inherent to the method. We therefore take care of the errors from the scales at which we have no geometric control by increasing this power slightly. At the cost of increasing the constant, we may take this extra power to be ϵ as in the statement of Theorem 3.1.1. Section 3.6 is devoted to describing the precise construction and its properties. The proof of Theorem 3.1.1 is left to the last two lines of Section 3.6.

In order to push through the calculations, we will view v as a perturbation of a harmonic function. While philosophically, $\|\ln(h)\|_\alpha$ controls how big of a perturbation v is from a harmonic function, actually carrying the error through the calculations is quite technical. For a clear write-up of how this works in the (much simpler) harmonic case, see [13].

3.2 Compactness

The main goal of this section is to show that $\mathcal{A}(\Lambda, \alpha, M_0)$ enjoys sufficient compactness to allow for limit-compactness arguments. Namely, we wish to establish that for any sequence $v_i \in \mathcal{A}(\Lambda, \alpha, M_0)$, we can extract a subsequence which converges to a function v_∞ and that $N(r, p, v_i) \rightarrow N(r, p, v_\infty)$ (Corollary 3.2.11). This requires strong convergence in $W_{loc}^{1,2}(\mathbb{R}^n)$

In our definition of $\mathcal{A}(\Lambda, \alpha, M_0)$, we do not normalize our functions. To avoid degeneracy,

we must state our results for $T_{p,r}v_i$ instead of v_i . In general, $T_{p,r}v \notin \mathcal{A}(\Lambda, \alpha, M_0)$. However, we shall abuse definitions by referring to the following lemmata as a “compactness” results for the class $\mathcal{A}(\Lambda, \alpha, M_0)$.

Throughout this Section, we will be talking about rescalings of v, Ω^\pm , and $\partial\Omega^\pm$ using both rescaling procedures from Definition 2.4.1 and Definition 2.4.5. We shall use $T_{Q,r}$ to denote translated and scaled versions of various objects. For example, for sets this is the usual push-forward.

$$\begin{aligned} T_{Q,r}\Omega^\pm &= \frac{\Omega^\pm - Q}{r} \\ T_{Q,r}\partial\Omega^\pm &= \frac{\partial\Omega^\pm - Q}{r} \end{aligned}$$

However, for the measures, $\omega^\pm, T_{Q,r}\omega^\pm$ will denote the harmonic measures associated to the positive and negative parts of $T_{Q,r}v$. The corkscrew points, $A_R^\pm(Q)$ will always denote the corkscrew point associated to Q at scale R in the domain Ω^\pm . We shall use $T_{Q,r}A_{r'}^\pm(Q')$ will denote the corkscrew point associated to associated to $T_{Q,r}Q' = \frac{Q'-Q}{r} \in T_{Q,r}\partial\Omega^\pm$ at the scale $\frac{r'}{r}$.

Lemma 3.2.1. Let $v \in \mathcal{A}(\Lambda, \alpha, M_0)$ with $\|\ln(h)\|_\alpha \leq \Gamma$. Then for all $Q \in \partial\Omega^\pm \cap B_2(0)$, and all radii $0 < r \leq 2$, the function $T_{Q,r}v$ is locally Lipschitz with uniform constants depending only upon M_0, Γ, α .

Proof. Recall that by Definition 2.4.1,

$$T_{Q,r}v = \frac{r^{n-2}}{\omega^-(B_r(Q))} v(rx + Q).$$

By NTA estimates, for all $0 < r$, $|v(A_r^-(Q))| \sim \frac{\omega^-(B_r(Q))}{r^{n-2}}$ by constants which only depend upon M_0 . Thus, $v_{Q,r}(A_r^-(Q))$ is bounded above and below by constants which only depend

upon M_0 . By Harnack chains and Harnack's Inequality applies to $-v$ in the domain Ω^- , we have $H(1, 0, v_{Q,r}) \geq c(M_0)$. Now, recalling Definition 2.4.5 and the fact that $T_{0,1}v = T_{0,1}(cv)$ for any constant, $c > 0$, we have that $T_{Q,r}v = T_{0,1}v_{Q,r}$. Since we have assumed that $\|\ln(h)\|_\alpha \leq \Gamma$, $Q \in B_2(0)$, and $0 < r \leq 2$, the $v_{Q,r}$ are locally uniformly Lipschitz by Corollary 2.4.3. Thus, $H(1, 0, v_{Q,r}) \geq c(M_0)$ implies that $T_{0,1}v_{Q,r} = T_{Q,r}v$ is also uniformly locally Lipschitz. \square

Lemma 3.2.2. (Local Growth Control) Let $Q \in \partial\Omega^\pm$ and $0 < r < \infty$. Let $v \in \mathcal{A}(\Lambda, \alpha, M_0)$ be such that $\|\ln(h)\|_\alpha \leq \Gamma$. The rescaling $T_{Q,r}v$ satisfies the following minimum growth conditions. For all $0 < \epsilon$, there is a constant, $C(M_0, \alpha, \Gamma, \epsilon, R)$ such that if $p \in B_R(0)$ with $\text{dist}(p, \{T_{Q,r}\partial\Omega^\pm\} \cap B_R(0)) > \epsilon$,

$$|T_{Q,r}v(p)| > C$$

Proof. As in Lemma 3.2.1, $T_{Q,r}v(T_{Q,r}A_r^-(Q))$ is bounded above and below by constants that only depend upon the NTA constant M_0 , Γ , and R . Thus, by Harnack chains between $T_{Q,r}A_r^-(Q)$ and $p \in T_{Q,r}\Omega^- \cap B_R(0)$ such that $\text{dist}(p, \{T_{Q,r}\partial\Omega^\pm\} \cap B_R(0)) > \epsilon$, Harnack's inequality applied to $-T_{Q,r}v$ again, implies that $|T_{Q,r}v(p)| > C$. Note that C only depends upon R, M_0, ϵ .

To get the same inequality for $p \in T_{Q,r}\Omega^+ \cap B_R(0)$, we recall that standard NTA results compare $T_{Q,r}v(T_{Q,r}A_r^+(Q))$ to $T_{Q,r}\omega^+(B_1(0))$. By Remark 2.3.4, $T_{Q,r}\omega^+(B_1(0)) \sim T_{Q,r}\omega^-(B_1(0))$ by constants which only depend upon R, Γ, α , and the NTA constants in the definition of the class $\mathcal{A}(\Lambda, \alpha, M_0)$. Applying the same Harnack chain and Harnack inequality argument gives the rest. \square

Lemma 3.2.3. (Compactness I) Let $\{v_i\}$ be a sequence of functions in $\mathcal{A}(\Lambda, \alpha, M_0)$ such that $\|\ln(h)\|_\alpha \leq \Gamma$. Let $\{Q_i\} \subset \partial\Omega_i^\pm \cap B_1(0)$ and $0 < r_i < 1$. There is a subsequence, $\{v_j\}$, and a Lipschitz function, $v_\infty \in W_{loc}^{1,2}$, such that $T_{Q_j, r_j}v_j \rightarrow v_\infty$ in the following senses:

1. $T_{Q_j, r_j} v_j \rightarrow v_\infty$ in $C_{loc}(\mathbb{R}^n)$
2. $T_{Q_j, r_j} v_j \rightarrow v_\infty$ in $L^2_{loc}(\mathbb{R}^n)$
3. $\nabla T_{Q_j, r_j} v_j \rightarrow \nabla v_\infty$ in $L^2_{loc}(\mathbb{R}^n, \mathbb{R}^n)$

Proof. To see (1), we recall Lemma 3.2.1 and that $T_{Q_i, r_i} v_i(0) = 0$. By Arzela-Ascoli there exists a subsequence that $T_{Q_j, r_j} v_j \rightarrow v_\infty$ in $C_{loc}(\mathbb{R}^n)$. Being uniformly locally Lipschitz and uniformly bounded also implies that the functions $\{T_{Q_j, r_j} v_j\}$ are bounded in $W^{1,2}_{loc}(\mathbb{R}^n)$. By Rellich Compactness, there exists a subsequence $T_{Q_j, r_j} v_j \rightarrow v_\infty$ in $L^2_{loc}(\mathbb{R}^n)$ and $\nabla T_{Q_j, r_j} v_j \rightarrow \nabla v_\infty$ in $L^2_{loc}(\mathbb{R}^n)$. \square

Before we can prove strong convergence $T_{Q_j, r_j} v_j \rightarrow v_\infty$ in $W^{1,2}_{loc}(\mathbb{R}^n)$, we need to control the upper Minkowski dimension of $\{v_\infty = 0\}$.

Lemma 3.2.4. Under the assumptions of Lemma 3.2.3, if $T_{Q_i, r_i} v_i \rightarrow v_\infty$ in $C_{loc}(\mathbb{R}^n)$, then $T_{Q_i, r_i} \partial\Omega^\pm \rightarrow \{v_\infty = 0\}$ locally in the Hausdorff metric on compact subsets.

Proof. We argue by contradiction. Suppose that there exists an $0 < \epsilon$ and radius, $0 < R$, such that we can find a sequence, $T_{Q_i, r_i} v_i$ such that there exists a point, $x_i \in B_R(0) \cap \{T_{Q_i, r_i} v_i = 0\}$, such that $\text{dist}(x_i, \{v_\infty = 0\}) > \epsilon$. Taking a subsequence which converges in $C_{loc}(\mathbb{R}^n)$, we may assume that $x_i \rightarrow x_\infty \in \overline{B_R(0)} \setminus B_\epsilon(\{v_\infty = 0\})$. Now, convergence in $C_{loc}(\mathbb{R}^n)$ implies that $T_{Q_i, r_i} v_i(x_\infty) \rightarrow v_\infty(x_\infty)$. Furthermore, since $T_{Q_i, r_i} v_i$ are uniformly locally Lipschitz, $x_i \rightarrow x_\infty$, and $x_i \in \{T_{Q_i, r_i} v_i = 0\}$, we have that,

$$T_{Q_i, r_i} v_i(x_\infty) \rightarrow 0.$$

This implies $x_\infty \in \{v_\infty = 0\}$, which contradicts our previous assertion, that $x_\infty \in \overline{B_R(0)} \setminus B_\epsilon(\{v_\infty = 0\})$.

The other direction goes similarly. Suppose that we could find a sequence of $T_{Q_i, r_i} v_i \rightarrow v_\infty$

such that there was a point, $x \in \{v_\infty = 0\} \cap B_R(0)$ for which $\text{dist}(x, \{T_{Q_i, r_i} v_i = 0\} \cap B_R(0)) > \epsilon$ for all $i = 1, 2, \dots$. By Lemma 3.2.2, we know that $T_{Q_i, r_i} v_i(x) > C$. This contradicts convergence in $C_{loc}(\mathbb{R}^n)$, however, since $v_\infty(x) = 0$. \square

Lemma 3.2.5. ([22], Theorem 4.1) In general, if $\partial\Omega_i^\pm \in \mathcal{D}(n, \alpha, M_0)$ converge to a closed set, A , locally in the Hausdorff metric on compact subsets, then A divides \mathbb{R}^n into two unbounded, 2-sided NTA domains with NTA constant bounded by $2M_0$.

We must now bound the upper Minkoski dimension of $A = \{v_\infty = 0\}$. We do so crudely, using only that A is the mutual boundary of a pair of two-sided NTA domains. That is, using the machinery of porous sets we are able to prove the following lemma.

Lemma 3.2.6. Let $\Sigma \subset \mathbb{R}^n$ be the mutual boundary of a pair of unbounded two-sided NTA domains with NTA constant $1 < M_0$. Then, there is an $0 < \epsilon = \epsilon(M_0, n)$ such that $\overline{\dim_{\mathcal{M}}}(E) \leq n - \epsilon$.

This is an elementary fact, which seems to be omitted on the literature. We defer the proof to Appendix A. We are may now prove strong convergence.

Lemma 3.2.7. (Strong Compactness I) Let $\{v_i\}$ be a sequence of functions in $\mathcal{A}(\Lambda, \alpha, M_0)$ such that $\|\ln(h_i)\|_\alpha \leq \Gamma$. Let $\{Q_i\} \subset \partial\Omega_i^\pm \cap B_1(0)$ and $0 < r_i < 1$. There is a subsequence, $\{v_j\}$, and a Lipschitz function, $v_\infty \in W_{loc}^{1,2}$, such that $T_{Q_j, r_j} v_j \rightarrow v_\infty$ in the following senses:

1. $T_{Q_j, r_j} v_j \rightarrow v_\infty$ in $C_{loc}(\mathbb{R}^n)$
2. $T_{Q_j, r_j} v_j \rightarrow v_\infty$ in $W_{loc}^{1,2}(\mathbb{R}^n)$

Proof. The only new claim is that $\nabla T_{Q_j, r_j} v_j \rightarrow \nabla v_\infty$ in $L_{loc}^2(\mathbb{R}^n, \mathbb{R}^n)$. By Lemmata 3.2.4, 3.2.5, and 3.2.6, we have that $\overline{\dim_{\mathcal{M}}}(\{v_\infty = 0\}) \leq n - \epsilon$. In particular, then, $\mathcal{H}^n(B_r(\{v_\infty = 0\} \cap B_R(0))) \rightarrow 0$ as $r \rightarrow 0$ (see [23] for fundamental facts about Minkowski content, dimension and Hausdorff measure). Thus, for any $\theta > 0$ we can find an $r(\theta) > 0$ such that $\mathcal{H}^n(B_r(\{v_\infty = 0\} \cap B_R(0))) \leq \theta$. This allows us to estimate,

$$\begin{aligned}
\limsup_{j \rightarrow \infty} D(R, 0, T_{Q_j, r_j} v_j) &= \limsup_{j \rightarrow \infty} \int_{B_R(0)} |\nabla T_{Q_j, r_j} v_j|^2 dx \\
&= \limsup_{j \rightarrow \infty} \left\{ \int_{B_R(0) \cap B_r(\{v_\infty=0\})} |\nabla T_{Q_j, r_j} v_j|^2 dx \right. \\
&\quad \left. + \int_{B_R(0) \setminus B_r(\{v_\infty=0\})} |\nabla T_{Q_j, r_j} v_j|^2 dx \right\} \\
&\leq \lim_{j \rightarrow \infty} \int_{B_R(0) \setminus B_r(\{v_\infty=0\})} |\nabla T_{Q_j, r_j} v_j|^2 dx + C\theta \\
&\leq D(R, 0, v_\infty) + C\theta
\end{aligned}$$

where the penultimate inequality uses that v_j are uniformly Lipschitz, and the last equality follows from $W^{1,2}$ convergence of harmonic functions in the region $B_R(0) \setminus B_r(\{v_\infty = 0\})$. Since $\theta > 0$ was arbitrary, we have that $\lim_{j \rightarrow \infty} D(R, 0, T_{Q_j, r_j} v_j) \leq D(R, 0, v_\infty)$. The other inequality follows from the same trick or from lower semi-continuity. Therefore, we have the equality,

$$\lim_{j \rightarrow \infty} D(R, 0, T_{Q_j, r_j} v_j) = D(R, 0, v_\infty).$$

Thus, we have by weak convergence and norm convergence,

$$\begin{aligned}
\lim_j \|\nabla T_{Q_j, r_j} v_j - \nabla v_\infty\|_{L^2(B_R(0))}^2 &= \lim_j \int_{B_R(0)} |\nabla T_{Q_j, r_j} v_j - \nabla v_\infty|^2 dx \\
&= \lim_j \left(\|\nabla T_{Q_j, r_j} v_j\|_{L^2(B_R(0))}^2 + \|\nabla v_\infty\|_{L^2(B_R(0))}^2 \right. \\
&\quad \left. - 2 \lim_j \langle \nabla T_{Q_j, r_j} v_j, \nabla v_\infty \rangle_{L^2(B_R(0))} \right) \\
&= 2\|\nabla v_\infty\|_{L^2(B_R(0))}^2 - 2\|\nabla v_\infty\|_{L^2(B_R(0))}^2 \\
&= 0
\end{aligned}$$

□

Using Lemma 3.2.7, we now must argue that the rescaling procedures in Definition 2.4.1 and Definition 2.4.5 are roughly equivalent in the class $\mathcal{A}(\Lambda, \alpha, M_0)$ for points *away* from the boundary. To do so, we need to prove a rudimentary growth condition.

Corollary 3.2.8. Let $v \in \mathcal{A}(\Lambda, \alpha, M_0)$, $p \in B_1(0) \cap \Omega^-$ and s be a radius such that $d = \text{dist}(p, \partial\Omega^\pm) < s < 1$. Let $Q \in \partial\Omega^\pm$ be a point such that $|p - Q| = d$, and let $y = T_{Q,2s}p = \frac{p-Q}{2s}$. For any radius, $0 < r < \infty$, we let $x_{\max}(y, r)$ denote the point in $T_{Q,2s}\Omega^- \cap \partial B_r(y)$ which maximizes $|T_{Q,2s}v|$ on $T_{Q,2s}\Omega^- \cap \partial B_r(y)$. We claim that there is a constant, $0 < c = c(M_0) < 1/4$, such that

$$c|T_{Q,2s}v(x_{\max}(y, 1/2))| < |T_{Q,2s}v(x_{\max}(y, 1/2)) - T_{Q,2s}v(y)|. \quad (3.2.1)$$

Proof. First, we note that the right-hand side cannot be zero for non-constant v by the Maximum Principle applied to $T_{Q,2s}\Omega^- \cap B_r(y)$, in which $T_{Q,2s}v$ is a harmonic function. Also, by definition, $T_{Q,2s}v(y)$ and $T_{Q,2s}v(x_{\max}(y, 1/2))$ share the same sign.

Now, suppose that no such constant exists. Then there is a sequence of functions v_i , points p_i , and radii s_i such that

$$2^{-i}|T_{Q_i,2s_i}v_i(x_{\max}(y_i, 1/2))| \geq |T_{Q_i,2s_i}v_i(x_{\max}(y_i, 1/2)) - T_{Q_i,2s_i}v_i(y_i)|,$$

where $Q_i \in \partial\Omega^\pm$ is a point such that $|Q_i - p_i| = d_i$ and $y_i = \frac{p_i - Q_i}{2s_i}$. By Lemma 3.2.3, we can extract a subsequence which converges to a function $T_{Q_i,2s_i} \rightarrow v_\infty$ in the senses of the lemma. Choosing another subsequence, we may also assume that $\lim_{j \rightarrow \infty} x_{\max}(y_j, 1/2) = x$ and $\lim_{j \rightarrow \infty} y_j = y$.

We now argue that $|T_{Q_j,2s_j}v(x_{\max}(y_j, 1/2))| \in [c^{-1}, c]$ for some finite constant, $c = c(M_0)$. To see the lower bound, we note that by the Harnack Chain condition, $|T_{Q_j,2s_j}v(A_1^-(0))| \sim |T_{Q_j,2s_j}v(z)|$ for some $z \in \partial B_{1/2}(y_j) \cap T_{Q_j,2s_j}\Omega^-$ by constants that only depend upon the NTA

constant, M_0 , and that $T_{Q_j, 2s_j} v(A_1^-(0))$ is normalized so that $|T_{Q_j, 2s_j} v(A_1^-(0))| \in [c^{-1}, c]$ for some finite constant $1 < c = c(M_0) < \infty$. Since $x_{\max}(y_i, 1/2)$ was chosen to maximize $|T_{Q_i, 2s_i} v_i|$ on $\partial B_{1/2}(y_i) \cap \Omega^-$,

$$|T_{Q_i, 2s_i} v_i(x_{\max}(y_i, 1/2))| \geq c(M_0).$$

Since the functions, $T_{Q_i, 2s_i} v_i$ are locally Lipschitz uniformly, this implies that there is a constant, depending only upon M_0 , such that $\text{dist}(x_{\max}(y_i, 1/2), \{T_{Q_i, 2s_i} v_i = 0\}) \geq C$. By the Harnack Chain condition, then, $T_{Q_j, 2s_j} v_j(x_{\max}(y_j, 1/2)) \sim T_{Q_j, 2s_j} v_j(A_1^-(0))$. This implies the upper bound,

$$|T_{Q_i, 2s_i} v_i(x_{\max}(y_i, 1/2))| \leq c.$$

Now, we consider the limit function. The boundedness of the $T_{Q_j, 2s_j} v(x_{\max}(y_j, 1/2))$ and the uniform convergence in Lemma 3.2.3 imply that the limit function, v_∞ , satisfies $|v_\infty(x)| \in [c^{-1}, c]$. Furthermore, we claim that $\lim_{j \rightarrow \infty} x_{\max}(y_j, 1/2) = x$ maximizes $|v_\infty|$ on $\Omega_\infty^- \cap \partial B_{1/2}(y)$. If there exists a point, $z \in \Omega_\infty^- \cap \partial B_{1/2}(y)$ such that $|v_\infty(z)| > |v_\infty(x)|$, then there must be points, $z_j \in \Omega_j^- \cap \partial B_{1/2}(y_j)$ such that $T_{Q_j, 2s_j} v_j(z_j) \rightarrow v_\infty(z)$. However, $|T_{Q_j, 2s_j} v_j(z_j)| \leq |T_{Q_j, 2s_j} v_j(x_{\max}(y_j, 1/2))|$ and $\lim_j |T_{Q_j, 2s_j} v_j(x_{\max}(y_j, 1/2))| = |v_\infty(x)| < |v_\infty(z)|$, which is a contradiction. Finally, we claim that by convergence,

$$v_\infty(x) = v_\infty(y).$$

Since $v_\infty = 0$ on $\partial \Omega_\infty$ and $|v_\infty(x)| \in [c^{-1}, c]$, v_∞ must be non-constant. However, as noted at the beginning, this violates the Maximum Principle applied to $\Omega_\infty^- \cap B_{1/2}(y)$, where v_∞ is harmonic. Therefore, such a constant as desired must exist. \square

Remark 3.2.9. If we consider v , instead of $T_{Q, 2s} v$, and abuse notation slightly by letting $x_{\max}(p, r)$ now denote the point in $\Omega^- \cap \partial B_r(p)$ which maximizes $|v|$ on $\Omega^- \cap \partial B_r(p)$, we have a similar result. That is, by unwinding the definition of $T_{q, 2s} v$, we see that Lemma 3.2.8

proves that for all $v \in \mathcal{A}(\Lambda, \alpha, M_0)$, $p \in \Omega^- \cap B_1(0)$ such that $d = \text{dist}(p, \partial\Omega^\pm) < s \leq 1$, we have

$$c|v(x_{\max}(p, s))| < |v(x_{\max}(p, s)) - v(p)|. \quad (3.2.2)$$

for the same constant, $c = c(M_0)$.

The growth control in the Lemma 3.2.8 allows us to control rescalings *away* from $\partial\Omega^\pm$. This gives the following lemma.

Lemma 3.2.10. (Compactness II) Let $\{v_i\}$ be a sequence of functions in $\mathcal{A}(\Lambda, \alpha, M_0)$ such that $\|\ln(h_i)\|_\alpha \leq \Gamma$ and $\{p_i\} \subset B_1(0) \cap \overline{\Omega_i^-}$. Let $d_i = \text{dist}(p_i, \partial\Omega_i^\pm)$ and $d_i < s_i \leq 1$. Then, there is a subsequence, $\{v_j\}$, and a Lipschitz function, $v_\infty \in W_{loc}^{1,2}(\mathbb{R}^n)$, such that $T_{p_j, s_j} v_j \rightarrow v_\infty$ in the following senses:

1. $T_{p_j, s_j} v_j \rightarrow v_\infty$ in $C_{loc}(\mathbb{R}^n)$
2. $T_{p_j, s_j} v_j \rightarrow v_\infty$ in $W_{loc}^{1,2}(\mathbb{R}^n)$

Proof. The key, again, is to show that $T_{p_i, s_i} v_i$ are uniformly locally Lipschitz. For each $p_i \in B_1(0) \cap \Omega_i^-$, we let $Q_i \in \partial\Omega_i^\pm$ be a point such that $|p_i - Q_i| = d_i$. If we denote $y_i = \frac{p_i - Q_i}{2s_i}$, then observe that,

$$T_{p_i, s_i} v_i = T_{y_i, 1/2}(T_{Q_i, 2s_i} v_i).$$

By Lemma 3.2.1, $T_{Q_i, 2s_i} v_i$ are uniformly locally Lipschitz with Lipschitz constant that only depends upon M_0 and Γ . By Lemma 3.2.8, we have that $|T_{Q_i, 2s_i} v_i(x_{\max}(y_i, 1/2))| \in [c^{-1}, c]$ and $|T_{Q_i, 2s_i} v_i(x_{\max}(y_i, 1/2)) - T_{Q_i, 2s_i} v_i(y_i)| > c^{-1}|T_{Q_i, 2s_i} v_i(x_{\max}(y_i, 1/2))|$, where $c = c(M_0)$. Therefore,

$$\frac{1}{\sigma_{n-1}} \int_{\partial B_1(0)} |T_{Q_i, 2s_i} v_i(y_i + \frac{1}{2}x) - T_{Q_i, 2s_i} v_i(y_i)|^2 d\sigma(x) \geq c(M_0).$$

This shows that $T_{p_i, s_i} v_i = T_{y_i, 1/2}(T_{Q_i, 2s_i} v_i)$ are uniformly locally Lipschitz. The rest follows identically as in Lemma 3.2.3 and Lemma 3.2.7. \square

Corollary 3.2.11. Under the hypotheses of Lemma 3.2.10, there exists a subsequence such that

$$N(r, 0, T_{p_j, r_j} v_j) \rightarrow N(r, 0, v_\infty)$$

for all $r \in (0, 2]$.

Proof. This follows from the convergence of the numerator and the denominator. The former is Lemma 3.2.10 (2). The later follows from Lemma 3.2.10 (1). \square

Corollary 3.2.12. Let $v \in \mathcal{A}(\Lambda, \alpha, M_0)$ and $v_\epsilon = v \star \phi_\epsilon$, be a mollification of v . By standard mollification results,

$$v_\epsilon \rightarrow v \text{ in } W_{loc}^{1,2}(\mathbb{R}^n), C_{loc}(\mathbb{R}^n) \text{ as } \epsilon \rightarrow 0.$$

Because of Corollary 3.2.12, we will prove many of our estimates for v_ϵ . Since these estimates hold for all $0 < \epsilon \ll 1$ small enough, by Remark 3.2.12 they hold for v , as well.

3.3 Almost monotonicity

This section is dedicated to extending the following result of Engelstein, [17].

Lemma 3.3.1. ([17]) Let $v \in \mathcal{A}(\Lambda, \alpha, M_0)$ and $Q \in K \subset \subset \partial\Omega^\pm$. There exists a constant, $C \leq \infty$, (which can be taken uniformly over K and $r \in (0, 1]$) such that

$$\liminf_{\epsilon \rightarrow 0} N(r, Q, v_\epsilon^Q) - N(0, Q, v_\epsilon^Q) > -Cr^\alpha.$$

We wish to extend Lemma 3.3.1 in two main ways. First, we need to extend this almost-monotonicity estimate to *any* $p \in B_1(0) \cap \overline{\Omega^-}$. Second, in order to later connect the Almgren frequency to the Jones β -numbers, we will need to estimate the non-degeneracy of $\frac{d}{dr} N(r, p, v_\epsilon)$. Throughout this section, we shall use the notation $(v_\epsilon)_\nu(y) = \nabla v_\epsilon(y) \cdot \nu(y)$, where $\nu(y)$ is the unit normal to $\partial B_r(p)$ at y .

By classical results, (see [17], Section 5.1 for details of the derivation),

$$\begin{aligned}
H(r, p, v_\epsilon)^2 \frac{d}{dr} N(r, p, v_\epsilon) &= 2r \left(\int_{\partial B_r(p)} (v_\epsilon)_\nu^2 d\sigma \int_{\partial B_r(p)} |v_\epsilon - v_\epsilon(p)|^2 d\sigma \right. \\
&\quad - \left. \left[\int_{\partial B_r(p)} (v_\epsilon - v_\epsilon(p))(v_\epsilon)_\nu d\sigma \right]^2 \right) \\
&\quad + 2r \int_{B_r(p)} (v_\epsilon - v_\epsilon(p)) \Delta v_\epsilon dx \int_{\partial B_r(p)} (v_\epsilon - v_\epsilon(p))(v_\epsilon)_\nu d\sigma \\
&\quad - 2H(r, p, v_\epsilon) \int_{B_r(p)} \langle x - p, \nabla v_\epsilon \rangle \Delta v_\epsilon dx
\end{aligned}$$

We decompose $\frac{d}{dr} N(r, p, v_\epsilon) = N'_1(r, p, v_\epsilon) + N'_2(r, p, v_\epsilon)$ as follows.

$$\begin{aligned}
N'_1(r, p, v_\epsilon) &= H(r, p, v_\epsilon)^{-2} 2r \left(\int_{\partial B_r(p)} (v_\epsilon)_\nu^2 d\sigma \int_{\partial B_r(p)} |v_\epsilon - v_\epsilon(p)|^2 d\sigma \right. \\
&\quad \left. - \left[\int_{\partial B_r(p)} (v_\epsilon - v_\epsilon(p))(v_\epsilon)_\nu d\sigma \right]^2 \right).
\end{aligned}$$

Note that by the Cauchy-Schwarz inequality, $N'_1(r, p, v_\epsilon) \geq 0$. We call what remains $N'_2(r, p, v_\epsilon)$.

$$\begin{aligned}
N'_2(r, p, v_\epsilon) &= H(r, p, v_\epsilon)^{-2} \left[2r \int_{B_r(p)} (v_\epsilon - v_\epsilon(p)) \Delta v_\epsilon dx \int_{\partial B_r(p)} (v_\epsilon - v_\epsilon(p))(v_\epsilon)_\nu d\sigma \right. \\
&\quad \left. - 2H(r, p, v_\epsilon) \int_{B_r(p)} \langle x - p, \nabla v_\epsilon \rangle \Delta v_\epsilon dx \right]. \tag{3.3.1}
\end{aligned}$$

Lemma 3.3.2. Let $v \in \mathcal{A}(\Lambda, \alpha, M_0)$, $p \in B_1(0)$ and $0 < r \leq 1$. Let

$$\lambda(p, r, v_\epsilon) = \frac{\int_{\partial B_r(p)} (v_\epsilon(y) - v_\epsilon(p)) \nabla v_\epsilon(y) \cdot (y - p) d\sigma(y)}{H(r, p, v_\epsilon)}.$$

Then, if $C = Lip(v|_{B_2(0)})$,

$$N'_1(r, p, v_\epsilon) \geq \frac{2}{C} \int_{\partial B_r(p)} \frac{|\nabla v_\epsilon \cdot (y - p) - \lambda(p, r, v_\epsilon)(v_\epsilon - v_\epsilon(p))|^2}{|y - p|^{n+2}} d\sigma(y) \tag{3.3.2}$$

Proof. Recall that for the Cauchy-Schwarz Inequality, we have that for $\lambda = \frac{\langle u, v \rangle}{\|v\|^2}$

$$\|v\|^2 \|u - \lambda v\|^2 = \|u\|^2 \|v\|^2 - |\langle u, v \rangle|^2.$$

Choosing $u = \nabla v \cdot (y - p)$ and $v = v - v(p)$, we have

$$N'_1(r, p, v_\epsilon) = H(v, p, v_\epsilon)^{-1} 2r \left(\int_{\partial B_r(p)} |(v_\epsilon)_\nu - \frac{1}{r} \lambda(p, r, v_\epsilon)(v_\epsilon - v_\epsilon(p))|^2 d\sigma \right)$$

Since $C = Lip(v|_{B_2(0)})$, we observe that $H(r, p, v_\epsilon) \leq Cr^{n+1}$. Plugging this into the above equation, we get the desired inequality.

$$N'_1(r, p, v_\epsilon) \geq \frac{2}{C} \int_{\partial B_r(p)} \frac{|\nabla v_\epsilon(y) \cdot (y - p) - \lambda(p, r, v_\epsilon)(v_\epsilon(y) - v_\epsilon(p))|^2}{|y - p|^{n+2}} d\sigma(y)$$

□

Remark 3.3.3. Lemma 3.3.2 is scale invariant in the sense that if $C = Lip(v|_{B_{2r}(p)})$,

$$N'_1(1, 0, T_{p,r}v_\epsilon) \geq \frac{2}{C} \int_{\partial B_1(0)} \frac{|\nabla T_{p,r}v_\epsilon \cdot (y - 0) - \lambda(0, 1, T_{p,r}v_\epsilon)T_{p,r}v_\epsilon|^2}{|y|^{n+2}} d\sigma(y). \quad (3.3.3)$$

Next, we derive an upper bound for $|N'_2(r, p, v_\epsilon)|$. Because we chose to define $h = \frac{d\omega^-}{d\omega^+}$, instead of the reciprocal, we can only state the following lemmata for $p \in \overline{\Omega^-} \cap B_1(0)$. However, the definition of $\mathcal{A}(\Lambda, \alpha, M_0)$ is symmetric in the sense of Remark 2.3.5. We first prove lemmata for $p \in \overline{\Omega^-} \cap B_1(0)$ and shall extend them for all $p \in B_1(0)$ later.

Lemma 3.3.4. Let $v \in \mathcal{A}(\Lambda, \alpha, M_0)$ and let $p \in B_1(0) \cap \overline{\Omega^-}$. Let $d = dist(p, \partial\Omega^\pm)$. Let $Q \in \partial\Omega^\pm$ be such that $|p - Q| = d$. For any $s \geq d$ and $\epsilon \ll s$,

$$\int_{\partial B_s(p)} |v_\epsilon - v(p)|^2 \geq C(M_0) \frac{\omega^-(B_s(Q))^2}{s^{n-3}}. \quad (3.3.4)$$

Proof. We prove this by cases. In [17] Lemma 5.5 Engelstein proves the $p = Q \in \partial\Omega^\pm$ case,

which we now restate. By the corkscrew condition, there exists an $x_0 \in \partial B_s(Q)$ such that $B_{\frac{s}{M_0}}(x_0) \cap \partial\Omega^- = \emptyset$. Thus, by the Harnack chain condition, there exists a slightly smaller ball, $B_{c\frac{s}{M_0}}$ (for any $c \in (\frac{3}{4}, \frac{6}{7})$, say) such that for every $y \in B_{c\frac{s}{M_0}}(x_0)$, if ϵ is small enough, then $v_\epsilon(y) \sim v(y) \sim v(x_0) \sim v(A_s^-(Q))$, all by constants that only depend upon M_0 . By standard NTA estimates, $|v(A_s^-(Q))| \sim \frac{\omega^-(B_s(Q))}{s^{n-2}}$. Since $\partial B_s(Q)$ intersects $B_{c\frac{s}{M_0}}(x_0)$ in a set of size $k|\partial B_s(Q)|$, where $k = k(c, M_0)$, integrating over this set gives the result.

To shift the argument to $p \in B_1(0) \cap \Omega^-$, we recall that by Remark 3.2.9, we have that,

$$\begin{aligned} \int_{\partial B_s(p)} |v_\epsilon - v(p)|^2 &\geq \int_{\partial B_s(p) \cap \Omega^-} |v_\epsilon - v(p)|^2 \\ &\geq c(M_0, n) |v(x_{\max}(p, s))|^2 s^{n-1} \\ &\geq c(M_0, n) |v(A_{2s}^-(Q))|^2 s^{n-1} \\ &\geq c(M_0, n) \frac{\omega^-(B_s(Q))^2}{s^{n-3}}, \end{aligned}$$

where the penultimate inequality comes from the fact that $|v(x_{\max}(p, s))| \sim |v(A_{2s}^-(Q))|$, the proof of which is contained in Lemma 3.2.8. In last inequality, we have used the standard NTA estimate that $|v(A_{2s}^-(Q))| \sim \frac{\omega^-(B_s(Q))}{s^{n-2}}$. \square

Lemma 3.3.5. Let $v \in \mathcal{A}(\Lambda, \alpha, M_0)$, $\|\ln(h)\|_\alpha \leq \Gamma$, and let $p \in B_1(0) \cap \overline{\Omega^-}$. Let $d = \text{dist}(p, \partial\Omega^\pm)$. Let $Q \in \partial\Omega^\pm$ be such that $|p - Q| = d$. For any $s \geq d$,

$$\int_{\partial B_s(p)} |v|^2 \sim_{M_0, \Gamma, \alpha} \int_{\partial B_s(p)} |v - v(p)|^2.$$

Proof. We will show that under these conditions both $\int_{\partial B_s(p)} |v|^2$ and $\int_{\partial B_s(p)} |v - v(p)|^2$ satisfy the same upper and lower bounds. For the later term, we use the lower bound from Lemma 3.3.4.

$$c(n, M_0) \frac{\omega^-(B_s(Q))^2}{s^{n-3}} \leq \int_{\partial B_s(p)} |v - v(p)|^2.$$

To see that $\int_{\partial B_s(p)} |v|^2$ satisfies a similar bound, we shall re-use the notation $x_{\max}(p, s)$ to denote the point in $\partial B_s(p) \cap \Omega^-$ which maximizes $|v|$ on $\partial B_s(p) \cap \Omega^-$. We observe that

$$\begin{aligned} \int_{\partial B_s(p)} |v|^2 &\geq \int_{\partial B_s(p) \cap B_{\delta s}(x_{\max}(p, s))} |v|^2 \\ &\geq C(n, M_0) |v(x_{\max}(p, s))|^2 s^{n-1} \\ &\geq C(n, M_0) \frac{\omega^-(B_s(Q))^2}{s^{n-3}} \end{aligned}$$

where $\delta > 0$ is a proportional constant such that $B_{\delta s}(x_{\max}(p, s)) \subset \Omega^-$. That $\delta = \delta(M_0)$ is proven in Lemma 3.2.8. The last inequality comes from the NTA estimates $|v(x_{\max}(p, s))| \sim |v(A_{2s}^-(Q))| \sim \frac{\omega^-(B_s(Q))}{s^{n-2}}$ in the last lines of Lemma 3.3.4.

Now, we turn to showing that they both satisfy the same upper bound. First, we consider what happens on $\partial B_s(p) \cap \Omega^-$. On this set, observe that trivially, $|v - v(p)| \leq |v(x_{\max}(p, s))|$. Since, as has been noted, $\text{dist}(x_{\max}(p, s), \partial\Omega^\pm) \geq \delta s$, by the Harnack Chain condition, we have that $|v(x_{\max}(p, s))| \sim |v(A_{2s}^-(Q))|$.

Now, we must argue about what happens in $\partial B_s(p) \cap \Omega^+$. First, observe that,

$$\max_{\partial B_s(p) \cap \Omega^+} |v - v(p)| \leq 2 \max\left\{ \max_{\partial B_s(p) \cap \Omega^+} |v|, |v(x_{\max}(p, s))| \right\}.$$

Now, we argue that there is a constant such that $\max_{\partial B_s(p) \cap \Omega^+} |v| \leq C(M_0, \Gamma, \alpha) |v(x_{\max}(p, s))|$. By NTA estimates, $\max_{\partial B_s(p) \cap \Omega^+} |v| \leq C(M_0) \frac{\omega^+(B_{2s}(Q))}{(2s)^{n-2}}$. By Remark 2.3.4, we have that $\frac{\omega^+(B_{2s}(Q))}{(2s)^{n-2}} \leq C(\Gamma, \alpha) \frac{\omega^-(B_{2s}(Q))}{(2s)^{n-2}}$. To finish, we invoke standard NTA estimates to argue that $\frac{\omega^-(B_{2s}(Q))}{(2s)^{n-2}} \sim |v(A_{2s}^-(Q))| \sim |v(x_{\max}(p, s))|$.

Thus, there is a constant, $C(M_0, \Gamma, \alpha)$ such that

$$\begin{aligned} \int_{\partial B_s(p)} |v|^2 &\leq C(M_0, \Gamma, \alpha) \frac{\omega^-(B_s(Q))^2}{s^{n-3}} \\ \int_{\partial B_s(p)} |v - v(p)|^2 &\leq C(M_0, \Gamma, \alpha) \frac{\omega^-(B_s(Q))^2}{s^{n-3}} \end{aligned}$$

□

Lemma 3.3.6. Let $v \in \mathcal{A}(\Lambda, \alpha, M_0)$ with $\|\ln(h)\|_\alpha \leq \Gamma$, and let $p \in B_1(0) \cap \overline{\Omega^-}$. Let $d = \text{dist}(p, \partial\Omega^\pm)$ and $Q \in \partial\Omega^\pm$ such that $|Q - p| = d$. For any $s \geq d$ and $\epsilon \ll s$

$$\left| \int_{B_s(p)} (v_\epsilon - v_\epsilon(p)) \Delta v_\epsilon dx \right| \leq C \|\ln(h)\|_\alpha s^\alpha \frac{\omega^-(B_s(Q))^2}{s^{n-2}} \quad (3.3.5)$$

where the constant $C = C(M_0)$.

Proof. The case $p = Q$ is handled in [9] and [17] (Lemma 5.6 (A)). Let $p \in B_1(0) \cap \overline{\Omega^-}$ with d, Q , and s as above. Observe that while Δv is a measure supported on $\partial\Omega^\pm$, Δv_ϵ is a smooth function.

$$\begin{aligned} \left| \int_{B_s(p)} (v_\epsilon - v_\epsilon(p)) \Delta v_\epsilon dx \right| &= \left| \int_{B_s(p)} (v_\epsilon - v_\epsilon(p))_\epsilon \Delta v dx \right| \\ &\leq \int_{B_s(p)} |(v_\epsilon - v_\epsilon(p))_\epsilon| \left| \frac{h(0)}{h(x)} - 1 \right| d\omega^- \\ &\leq \int_{B_{2s}(Q)} |(v_\epsilon - v_\epsilon(p))_\epsilon| \left| \frac{h(0)}{h(x)} - 1 \right| d\omega^- \\ &\leq \|\ln(h)\|_\alpha (2s)^\alpha \int_{B_{2s}(Q)} |(v_\epsilon - v_\epsilon(p))_\epsilon| d\omega^- \end{aligned}$$

We now repeat the estimates found in [9] Lemmata 4.6 and 4.10. That is, by a change of coordinates, $x = 2sy$, and recalling our definitions

$$v_{Q,2s}(x) = \frac{v(2sx + Q)(2s)^{n-2}}{\omega^-(B_{2s}(Q))}; \quad \omega_{Q,2s}^-(E) = \frac{\omega^-(2sE + Q)}{\omega^-(B_{2s}(Q))}$$

we have

$$\begin{aligned} \int_{B_{2s}(Q)} (v_\epsilon - v_\epsilon(p))_\epsilon d\omega^- &= \frac{\omega^-(B_{2s}(Q))^2}{(2s)^{n-2}} \int_{\mathbb{R}^n} \chi_{B_1}((v_{2s} - v_{2s}(p)) \star \phi_{\epsilon/2s}) \star \phi_{\epsilon/2s} d\omega_{2s}^- \\ &\leq C \frac{\omega^-(B_{2s}(Q))^2}{s^{n-2}} \omega_{2s}^-(B_2(Q)) \\ &\leq C \frac{\omega^-(B_{2s}(Q))^2}{s^{n-2}} \end{aligned}$$

where the last two inequalities are as in [9] Lemma 4.4, because v_{2s} are uniformly locally Lipschitz, $1 + \frac{\epsilon}{2s} < 2$, and $\omega_{2s}^-(B_2(Q))$ is uniformly bounded in s for $Q \in \partial\Omega^\pm \cap B_1(0)$.

The fact that harmonic measure is doubling on NTA domains completes the proof. \square

Remark 3.3.7. Note that Lemma 3.3.4 and Lemma 3.3.6 combined give that for any $p \in B_1(0) \cap \overline{\Omega^-}$ for ϵ small enough,

$$\left| \frac{\int_{B_s(p)} (v_\epsilon - v_\epsilon(p)) \Delta v_\epsilon}{\int_{\partial B_s(p)} (v_\epsilon - v_\epsilon(p))^2} \right| \leq C \|\ln(h)\|_\alpha s^{\alpha-1}$$

where $C = C(M_0, \alpha, \Gamma)$.

In an identical mode of argument, we are able to extend the following lemmata from [9], as well.

Lemma 3.3.8. Let $v \in \mathcal{A}(\Lambda, \alpha, M_0)$ with $\|\ln(h)\|_\alpha \leq \Gamma$, and let $p \in B_1(0) \cap \overline{\Omega^-}$. Let

$d = \text{dist}(p, \partial\Omega^\pm)$ and $Q \in \partial\Omega^\pm$ such that $|p - Q| = d$. For any $s \geq d$ and $\epsilon \ll s$,

$$\left| \int_{B_s(p)} \langle \nabla v_\epsilon, x - p \rangle \Delta v_\epsilon dx \right| \leq C \|\ln(h)\|_\alpha s^\alpha \frac{\omega^-(B_s(Q))^2}{s^{n-2}} \quad (3.3.6)$$

where the constant $C = C(M_0)$.

Proof. Again, the argument for $p = Q$ is in [17]. Let p, Q, d , and s be as above. The argument itself is another use of scaling and NTA properties.

$$\begin{aligned} \left| \int_{B_s(p)} \langle \nabla v_\epsilon, x - p \rangle \Delta v_\epsilon dx \right| &= \left| \int_{B_s(p)} \langle \nabla v_\epsilon, x - p \rangle_\epsilon \Delta v dx \right| \\ &\leq \int_{B_s(p)} |(\langle \nabla v_\epsilon, x - p \rangle)_\epsilon| \left| \frac{h(0)}{h(x)} - 1 \right| d\omega^- \\ &\leq \int_{B_{2s}(Q)} |(\langle \nabla v_\epsilon, x - p \rangle)_\epsilon| \left| \frac{h(0)}{h(x)} - 1 \right| d\omega^- \\ &\leq \|\ln(h)\|_\alpha (2s)^\alpha \int_{B_{2s}(Q)} |(\langle \nabla v_\epsilon, x - p \rangle)_\epsilon| d\omega^- \\ &\leq \|\ln(h)\|_\alpha (2s)^{\alpha+1} \int_{B_{2s}(Q)} |\nabla v_\epsilon|_\epsilon d\omega^- \end{aligned}$$

Now, we wish to use our rescalings, again. Chasing through the change of variables $x = ry + Q$, we see that $\nabla_x v(x) = \frac{1}{r} \nabla_y v(ry + Q) = \frac{\omega^-(B_r(Q))}{r^{n-1}} \nabla_y v_r(y)$. Thus, we calculate that for the change of variables $x = 2sy + Q$,

$$\begin{aligned}
\left| \int_{B_s(p)} \langle \nabla v_\epsilon, x - p \rangle \Delta v_\epsilon dx \right| &\leq \| \ln(h) \|_\alpha (2s)^{\alpha+1} \frac{\omega^-(B_{2s}(0))^2}{(2s)^{n-1}} \int_{B_1(0)} |\nabla v_{Q,2s} \star \phi_{\frac{\epsilon}{2s}}| \star \phi_{\frac{\epsilon}{2s}} d\omega_{2s}^- \\
&\leq \| \ln(h) \|_\alpha C s^{\alpha+1} \frac{\omega^-(B_{2s}(0))^2}{(s)^{n-1}} \omega_{Q,2s}^-(B_2(0)) \\
&\leq \| \ln(h) \|_\alpha C s^\alpha \frac{\omega^-(B_s(0))^2}{s^{n-2}}
\end{aligned}$$

where the last two inequalities are because $v_{Q,r}$ are uniformly locally Lipschitz, $1 + \epsilon/r < 2$, $\omega_{Q,r}^-(B_2(0))$ are uniformly bounded for $Q \in B_1(0)$ and $r < 2$, and the doubling of harmonic measure on NTA domains. \square

Lemma 3.3.9. Let $v \in \mathcal{A}(\Lambda, \alpha, M_0)$ and let $p \in B_1(0)$. Let $d = \text{dist}(p, \partial\Omega^\pm)$ and $Q \in \partial\Omega^\pm$ such that $|p - Q| = d$. For any $s \geq d$ and $\epsilon \ll s$

$$\left| \int_{\partial B_s(p)} (v_\epsilon - v_\epsilon(p))(v_\epsilon)_\nu d\sigma \right| \leq C \frac{\omega^-(B_s(Q))^2}{s^{n-1}} \quad (3.3.7)$$

where the constant $C = C(M_0)$.

Proof. Again, the argument for $p = Q$ is in [17], and the generalization follows using the same techniques detailed above. Namely, the argument itself is another use of scaling, local uniform Lipschitz control, and NTA properties. By the definitions of our blow-ups,

$$\begin{aligned}
\left| \int_{\partial B_s(p)} (v_\epsilon - v_\epsilon(p))(v_\epsilon)_\nu d\sigma \right| &= \\
&\left| \frac{\omega^-(B_{2s}(Q))^2}{(2s)^{2n-3}} (2s)^{n-1} \int_{\partial B_{1/2}(0)} ((v_{p,2s} - v_{p,2s}(0)) \star \phi_{\frac{\epsilon}{2s}}) (\nabla v_{p,2s} \star \phi_{\frac{\epsilon}{2s}}) \cdot \nu d\sigma \right| \\
&\leq C \frac{\omega^-(B_{2s}(Q))^2}{(2s)^{n-2}}
\end{aligned}$$

Doubling of the harmonic measure on NTA domains gives the desired result. \square

Remark 3.3.10. Recalling our expansion of $\frac{d}{dr}N(r, p, v_\epsilon)$ in Equation 3.3.1 and the bounds contained in Lemmata 3.3.4, 3.3.6, 3.3.8, 3.3.9, and Remark 3.3.7 we have that for $v \in \mathcal{A}(\Lambda, \alpha, M_0)$ with $\|\ln(h)\|_\alpha \leq \Gamma$, $\epsilon \ll r$, and $p \in B_1(0) \cap \overline{\Omega^-}$,

$$|N'_2(r, p, v_\epsilon)| \leq C_1 \|\ln(h)\|_\alpha r^{\alpha-1}. \quad (3.3.8)$$

where $C_1 = C(\alpha, M_0, \Gamma)$.

We are now in a position to prove a more general version of Lemma 3.3.1.

Lemma 3.3.11. Let $v \in \mathcal{A}(\Lambda, \alpha, M_0)$ with $\|\ln(h)\|_\alpha \leq \Gamma$, and let $p \in B_1(0) \cap \overline{\Omega^-}$. For any $0 \leq s < S \leq 1$

$$\begin{aligned} \frac{2}{C} \int_{A_{s,S}(p)} \frac{|\nabla T_{0,1}v(y) \cdot (y-p) - \lambda(p, |y-p|, T_{0,1}v)(T_{0,1}v(y) - T_{0,1}v(p))|^2}{|y-p|^{n+2}} dy \\ \leq N(S, p, v) - N(s, p, v) + C_1 \|\ln(h)\|_\alpha S^\alpha \end{aligned} \quad (3.3.9)$$

where $C_1 = C_1(\alpha, M_0, \Gamma)$ and $C(M_0, \Gamma, \alpha) = \text{Lip}(T_{0,1}v|_{B_2(0)})$.

Proof. We begin by normalizing v . Since $N(r, p, v) = N(r, p, cv)$ for any $c \neq 0$, we might as well work with $T_{0,1}v$. Let $d = \text{dist}(p, \partial\Omega^\pm)$. There are two cases: either $d \leq S$ or $S < d$. In the later case, $T_{0,1}v$ is harmonic in $B_S(p)$, and classical results imply the the desired inequality. Therefore, we consider $d \leq S$. By Remark 3.2.12, since $v_\epsilon \rightarrow v$ in $W^{1,2}$ as $\epsilon \rightarrow 0$, we can find an $\epsilon \ll s$ small enough that $|N(s, p, v_\epsilon) - N(s, p, v)| < S^\alpha$ and $|N(S, p, v_\epsilon) - N(S, p, v)| < S^\alpha$. Furthermore, since $T_{0,1}v_\epsilon$ is locally uniformly Lipschitz, $N(r, p, v_\epsilon)$ is continuous in r . Therefore, if $s = 0$, we can find an $0 = s < s_1 < S$ such that $|N(s_1, p, v_\epsilon) - N(s, p, v_\epsilon)| < S^\alpha$.

Thus, we reduce to estimating $N(S, p, T_{0,1}v_\epsilon) - N(s_1, p, T_{0,1}v_\epsilon)$.

$$\begin{aligned} N(S, p, T_{0,1}v_\epsilon) - N(s_1, p, T_{0,1}v_\epsilon) &= \int_{s_1}^S \frac{d}{dr} N(r, p, T_{0,1}v_\epsilon) dr \\ &= \int_{s_1}^S N'_1(r, p, v_\epsilon) dr + \int_{s_1}^S N'_2(r, p, T_{0,1}v_\epsilon) dr \end{aligned}$$

Recalling Remark 3.3.10, for ϵ small enough, we bound,

$$\begin{aligned} \int_{s_1}^S N'_2(r, p, T_{0,1}v_\epsilon) &\geq \int_{s_1}^S -C_1 \|\ln(h)\|_\alpha r^{\alpha-1} \\ &= -C_1 \|\ln(h)\|_\alpha S^\alpha. \end{aligned}$$

By Lemma 3.3.2, we have,

$$\begin{aligned} &\int_{s_1}^S N'_1(r, p, T_{0,1}v_\epsilon) \\ &\geq \int_{s_1}^S \frac{2}{C} \int_{\partial B_r(p)} \frac{|\nabla T_{0,1}v_\epsilon \cdot (y-p) - \lambda(p, |y-p|, T_{0,1}v_\epsilon)(T_{0,1}v_\epsilon - T_{0,1}v_\epsilon(p))|^2}{|y-p|^{n+2}} d\sigma(y) dr \\ &\geq \frac{2}{C} \int_{A_{s_1, S}(p)} \frac{|\nabla T_{0,1}v_\epsilon \cdot (y-p) - \lambda(p, |y-p|, T_{0,1}v_\epsilon)(T_{0,1}v_\epsilon - T_{0,1}v_\epsilon(p))|^2}{|y-p|^{n+2}} dy. \end{aligned}$$

Recalling Corollary 3.2.12 and letting $\epsilon \rightarrow 0$ gives that stated result. \square

Using these estimates it is possible to control the drop across scales from the total drop.

Lemma 3.3.12. If Let $v \in \mathcal{A}(\Lambda, \alpha, M_0)$, with $\|\ln(h)\|_\alpha \leq \Gamma$, and $p \in B_1(0) \cap \overline{\Omega^-}$, then for any $0 \leq r \leq s \leq S \leq R$

$$N(S, p, v) - N(s, p, v) \leq 2C_1 \|\ln(h)\|_\alpha R^\alpha + |N(R, p, v) - N(r, p, v)|$$

Proof. This is essentially a “rays of the sun” argument. To wit,

$$\begin{aligned}
N(S, p, v) - N(s, p, v) &= \int_s^S N'_1(\rho, p, v) + N'_2(\rho, p, v) d\rho \\
&\leq \int_s^S N'_1(\rho, p, v) + |N'_2(\rho, p, v)| d\rho \\
&\leq \int_r^R N'_1(\rho, p, v) + |N'_2(\rho, p, v)| d\rho \\
&\leq 2 \int_r^R |N'_2(\rho, p, v)| d\rho + |N(R, p, v) - N(r, p, v)|
\end{aligned}$$

The bounds in Remark 3.3.10, give the desired statement. \square

3.4 Two Analogs of Harmonic Results

We now turn our attention to proving some analogs of classical harmonic results for the Almgren frequency function for functions $v \in \mathcal{A}(\Lambda, \alpha, M_0)$. The main result of this section is Lemma 3.4.4, which states that we may bound the Almgren frequency uniformly for all points $p \in B_{1/4}(0)$ and all scales $0 < r \leq 1/2$ by a function of $N(1, 0, v) \leq \Lambda$.

For harmonic functions, $u : \tilde{\Omega} \rightarrow \mathbb{R}$, it is a well-known inequality (see [13] Remark 2.6) that for any $p \in \tilde{\Omega}$ and radii $0 < r \leq R$ such that $B_R(p) \subset \tilde{\Omega}$,

$$H(R, p, u) \leq \left(\frac{R}{r}\right)^{(n-1)+2N(R,p,u)} H(r, p, u) \quad (3.4.1)$$

$$H(R, p, u) \geq \left(\frac{R}{r}\right)^{(n-1)+2N(r,p,u)} H(r, p, u) \quad (3.4.2)$$

We generalize this fact for $v \in \mathcal{A}(\Lambda, \alpha, M_0)$.

Lemma 3.4.1. ($H(r, p, v_\epsilon)$ is almost doubling) Let $v \in \mathcal{A}(\Lambda, \alpha, M_0)$ with $\|\ln(h)\|_\alpha \leq \Gamma$, and let $p \in B_{\frac{1}{2}}(0) \cap \overline{\Omega^-}$. Let $d = \text{dist}(p, \partial\Omega^\pm)$ and $Q \in \partial\Omega^\pm$ such that $|p - Q| = d$. For any

$d < s < S \leq 1$ if $\epsilon \ll s$ is sufficiently small,

$$H(S, p, v_\epsilon) \leq \left(\frac{S}{s}\right)^{(n-1)+2(N(S,p,v_\epsilon)+CS^\alpha)} e^{\frac{2C}{\alpha}[S^\alpha-s^\alpha]} H(s, p, v_\epsilon), \quad (3.4.3)$$

where $C = \|\ln(h)\|_\alpha C_1(M_0, \alpha, \Gamma)$.

Proof. First, observe that

$$H'(s, p, v_\epsilon) = \frac{n-1}{r} \int_{\partial B_r(p)} |(v_\epsilon - v_\epsilon(p))|^2 + 2 \int_{B_r(p)} |\nabla v_\epsilon|^2 + 2 \int_{B_r(p)} (v_\epsilon - v_\epsilon(p)) \Delta v_\epsilon.$$

Next, we consider the following identity:

$$\begin{aligned} \ln\left(\frac{H(S, p, v_\epsilon)}{H(s, p, v_\epsilon)}\right) &= \ln(H(S, p, v_\epsilon)) - \ln(H(s, p, v_\epsilon)) \\ &= \int_s^S \frac{H'(r, p, v_\epsilon)}{H(r, p, v_\epsilon)} dr \\ &= \int_s^S \left(\frac{n-1}{r} + \frac{2}{r} N(r, p, v_\epsilon) + 2 \left(\frac{\int_{B_r(p)} v_\epsilon \Delta v_\epsilon}{\int_{\partial B_r(p)} (v_\epsilon - v_\epsilon(p))^2} \right) \right) dr \end{aligned}$$

We bound $N(r, p, v_\epsilon)$ from above using Lemma 3.3.11. We bound the last term using Remark 3.3.7. Plugging in these bounds, we have that for $\epsilon \ll s$,

$$\ln\left(\frac{H(S, p, v_\epsilon)}{H(s, p, v_\epsilon)}\right) \leq [(n-1) + 2(N(S, p, v_\epsilon) + CS^\alpha)] \ln(r)|_s^S + \frac{2C}{\alpha} r^\alpha|_s^S$$

Evaluating and exponentiating gives the desired result. □

Note that the almost monotonicity of the Almgren frequency function also gives the following inequality.

Lemma 3.4.2. Let $v \in \mathcal{A}(\Lambda, \alpha, M_0)$ with $\|\ln(h)\|_\alpha \leq \Gamma$, and let $p \in B_{\frac{1}{2}}(0) \cap \overline{\Omega^-}$. Let $d = \text{dist}(p, \partial\Omega^\pm)$ and $Q \in \partial\Omega^\pm$ such that $|p - Q| = d$. For any $d < s < S \leq 1$ if $\epsilon \ll s$ is sufficiently small,

$$H(S, p, v_\epsilon) \geq \left(\frac{S}{s}\right)^{(n-1)+2(N(s,p,v_\epsilon)-CS^\alpha)} e^{\frac{-2C}{\alpha}[S^\alpha-s^\alpha]} H(s, p, v_\epsilon). \quad (3.4.4)$$

where $C = \|\ln(h)\|_\alpha C_1(M_0, \alpha, \Gamma)$.

The proof is nearly identical, bounding $N(r, p, v_\epsilon)$ from below by Lemma 3.3.11 and Remark 3.3.7.

Remark 3.4.3. Because $H(r, p, v_\epsilon) \rightarrow H(r, p, v)$ as $\epsilon \rightarrow 0$ and $N(r, p, v_\epsilon) \rightarrow N(r, p, v)$ as $\epsilon \rightarrow 0$ (a consequence of Corollary 3.2.12), we have the following inequalities. For all $v \in \mathcal{A}(\Lambda, \alpha, M_0)$ with $\|\ln(h)\|_\alpha \leq \Gamma$, $p \in B_{\frac{1}{2}}(0) \cap \overline{\Omega^-}$, and $d < s < S \leq 1$ where if $d = \text{dist}(p, \partial\Omega^\pm)$.

$$H(S, p, v) \leq \left(\frac{S}{s}\right)^{(n-1)+2(N(S,p,v)+CS^\alpha)} e^{\frac{2C}{\alpha}[S^\alpha-s^\alpha]} H(s, p, v) \quad (3.4.5)$$

$$H(S, p, v) \geq \left(\frac{S}{s}\right)^{(n-1)+2(N(s,p,v)-CS^\alpha)} e^{\frac{-2C}{\alpha}[S^\alpha-s^\alpha]} H(s, p, v). \quad (3.4.6)$$

Now, that we know that $H(s, p, v)$ cannot grow too fast, we come to the important lemma.

This is crucial for the project we wish to undertake.

Lemma 3.4.4. Let $v \in \mathcal{A}(\Lambda, \alpha, M_0)$ as above. There is a function, $C(n, \alpha, \Gamma, \Lambda, M_0)$ such that if $\|\ln(h)\|_\alpha \leq \Gamma$, then for all $p \in B_{\frac{1}{4}}(0) \cap \overline{\Omega^-}$ and all $r \in (0, \frac{1}{2})$

$$N(p, r, v) \leq C(n, \Lambda, \alpha, M_0, \Gamma). \quad (3.4.7)$$

Proof. Recall that by definition of the class $\mathcal{A}(\Lambda, \alpha, M_0)$, $0 \in \partial\Omega$. Thus, for all $p \in B_{\frac{1}{4}}(0) \cap \Omega^-$, $\text{dist}(p, \partial\Omega) = d < \frac{1}{4}$.

We recall that the Almgren frequency function is invariant under rescalings of the function v . Therefore, we normalize our function v by the rescaling $v_{0,1}$ and relabel as v . As remarked in Theorem 2.4.2, v is uniformly locally Lipschitz with a Lipschitz constant which only depends upon the NTA constant M_0 .

Therefore, applying Remark 3.4.3 to $p = 0$ (which implies $v(p) = 0$), letting $r = cR$, and integrating both sides with respect to R from 0 to S , we have that for any $c \in (0, 1)$

$$\begin{aligned} \int_0^S \int_{\partial B_R(0)} |v|^2 d\sigma dR &\leq \int_0^S \left(\frac{1}{c}\right)^{(n-1)+2(N(R,0,v)+CR^\alpha)} e^{\frac{2C}{\alpha}[R^\alpha-(cR)^\alpha]} \int_{\partial B_{cR}(0)} |v|^2 d\sigma dR \\ &\leq \left(\frac{1}{c}\right)^{(n-1)+2(N(S,0,v)+2CS^\alpha)} e^{\frac{2C}{\alpha}S^\alpha} \int_0^S \int_{\partial B_{cR}(0)} |v|^2 d\sigma dR \end{aligned}$$

Thus, we have that for any $c \in (0, 1)$ and any $0 < S \leq 1$

$$\int_{B_S(0)} |v|^2 dV \leq \left(\frac{1}{c}\right)^{2[N(S,0,v)+2CS^\alpha]} \cdot e^{\frac{2C}{\alpha}S^\alpha} \int_{B_{cS}(0)} |v|^2 dV$$

Let $S = 1$ and $c = \frac{1}{16}$. We have that,

$$\int_{B_1(0)} |v|^2 \leq (16)^{2[N(1,0,v)+C]} \cdot e^{\frac{2C}{\alpha}} \int_{B_{\frac{1}{16}}(0)} |v|^2 \quad (3.4.8)$$

We continue underestimating the lefthand side of Equation 3.4.8. For any $p \in B_{\frac{1}{4}}(0) \cap \overline{\Omega^-}$

we have the following,

$$\begin{aligned}
\int_{B_1(0)} |v|^2 &\geq c(n) \int_{B_{\frac{3}{4}}(p)} |v|^2 \\
&\geq c(n) \int_{B_{\frac{3}{4}}(p) \setminus B_{\frac{1}{4}}(p)} |v|^2 \\
&\geq C(n, M_0, \alpha, \Gamma) \int_{B_{\frac{3}{4}}(p) \setminus B_{\frac{1}{4}}(p)} |v - v(p)|^2
\end{aligned}$$

where we have used Lemma 3.3.5, in the last inequality.

Now, we overestimate the righthand side of Equation 3.4.8. For any $p \in B_{\frac{1}{4}}(0) \cap \overline{\Omega^-}$ we have the following:

$$\begin{aligned}
\int_{B_{\frac{1}{16}}(0)} |v|^2 &\leq c(n) \int_{B_{\frac{9}{16}}(p)} |v|^2 \\
&\leq c(n) \left(\int_{B_{\frac{9}{16}}(p) \setminus B_{\frac{1}{4}}(p)} |v|^2 + \int_{B_{\frac{1}{4}}(p)} |v|^2 \right)
\end{aligned}$$

Now, let $x_{max}(p, 1/4) \in \partial B_{\frac{1}{4}}(p) \cap \overline{\Omega^-}$ be the point which maximizes $|v|$ on $\partial B_{\frac{1}{4}}(p) \cap \overline{\Omega^-}$.

Recall from Lemma 3.3.5 that there is a constant, $C(n, \alpha, M_0, \Gamma)$, such that

$$\max_{\partial B_{\frac{1}{4}}(p) \cap \Omega^+} |v| \leq C(n, \alpha, M_0, \Gamma) |v(x_{max}(p, \frac{1}{4}))|.$$

Thus,

$$\begin{aligned} \int_{B_{\frac{1}{4}}(p)} |v|^2 &\leq \omega_n 4^{-n} C(n, \alpha, M_0, \Gamma) |v(x_{\max}(p, \frac{1}{4}))|^2 \\ &\leq C(n, \alpha, M_0, \Gamma) |v(A_{1/4}^-(Q))|^2 \end{aligned}$$

where in the last inequality, $Q \in \partial\Omega^\pm$ such that $|Q - p| = \text{dist}(p, \partial|\Omega^\pm)$ and we have used the Harnack chain condition on $|v|$, which is non-negative and harmonic in Ω^- , to compare $|v(x_{\max}(p, \frac{1}{4}))| \sim |v(A_{1/4}^-(Q))|$ by constants that only depend upon M_0 . In order to do this, we need a uniform lower bound on $\text{dist}(x_{\max}(p, \frac{1}{4}), \partial\Omega^\pm)$. This is contained in Corollary 3.2.8.

Now, by standard NTA estimates, $v(A_{1/4}^-(Q)) \sim_{M_0} \frac{\omega^-(B_{1/4}(Q))}{(1/4)^{n-2}}$. By Remark 2.3.4,

$$\frac{\omega^-(B_{1/4}(Q))}{(1/4)^{n-2}} \sim_{\Gamma, \alpha} \frac{\omega^+(B_{1/4}(Q))}{(1/4)^{n-2}}.$$

And by NTA estimates, again, $\frac{\omega^+(B_{1/4}(Q))}{(1/4)^{n-2}} \sim_{M_0} v(A_{1/4}^+(Q))$. Furthermore, by the corkscrew condition, we know that there is a ball, $B_{\frac{1}{16M_0}}(A_{\frac{1}{16}}^+(Q)) \subset \Omega^+ \cap B_{\frac{9}{16}}(p) \setminus B_{\frac{1}{4}}(p)$. By Harnack chains, then we know that $v(A_{1/4}^+(Q)) \sim_{M_0} v(A_{1/16}^+(Q))$. Thus,

$$\left(\frac{1}{16M_0}\right)^n |v(A_{1/16}^+(Q))|^2 \leq C(M_0) \int_{B_{\frac{9}{16}}(p) \setminus B_{\frac{1}{4}}(p)} |v - v(p)|^2.$$

Hence, we can use Lemma 3.3.5 to continue overestimating the righthand side of Equation

3.4.8, as follows. For any $p \in B_{\frac{1}{4}}(0) \cap \overline{\Omega}^-$ we have,

$$\begin{aligned}
\int_{B_{\frac{1}{16}}(0)} |v|^2 &\leq c(n) \int_{B_{\frac{9}{16}}(p)} |v|^2 \\
&\leq c(n) \left(\int_{B_{\frac{9}{16}}(p) \setminus B_{\frac{1}{4}}(p)} |v|^2 + \int_{B_{\frac{1}{4}}(p)} |v|^2 \right) \\
&\leq c(n) \left(C(n, \alpha, M_0, \Gamma) \int_{B_{\frac{9}{16}}(p) \setminus B_{\frac{1}{4}}(p)} |v - v(p)|^2 + C(n, \alpha, M_0, \Gamma) |v(A_{1/4}^-(Q))|^2 \right) \\
&\leq C(n, \alpha, M_0, \Gamma) \left(\int_{B_{\frac{9}{16}}(p) \setminus B_{\frac{1}{4}}(p)} |v - v(p)|^2 + \int_{B_{\frac{9}{16}}(p) \setminus B_{\frac{1}{4}}(p)} |v - v(p)|^2 \right) \\
&\leq C(n, \alpha, M_0, \Gamma) \left(\int_{B_{\frac{9}{16}}(p) \setminus B_{\frac{1}{4}}(p)} |v - v(p)|^2 \right).
\end{aligned}$$

Putting together these under and over estimates, we have,

$$\int_{B_{\frac{3}{4}}(p)} |v - v(p)|^2 \leq C(n, \alpha, M_0, \Gamma) (16)^{2[N(1,0,v)+C]} \cdot e^{\frac{2C}{\alpha}} \int_{B_{\frac{9}{16}}(p) \setminus B_{\frac{1}{4}}(p)} |v - v(p)|^2. \quad (3.4.9)$$

Now, we wish to bound $\int_{B_{\frac{3}{4}}(p)} |v - v(p)|^2$ from below and $\int_{B_{\frac{9}{16}}(p) \setminus B_{\frac{1}{4}}(p)} |v - v(p)|^2$ from above.

To get the lower bound, we recall that,

$$\frac{d}{dr} \int_{\partial B_r(p)} |v_\epsilon - v_\epsilon(p)|^2 = \frac{n-1}{r} \int_{\partial B_r(p)} |v_\epsilon - v_\epsilon(p)|^2 + 2 \int_{\partial B_r(p)} (v_\epsilon - v_\epsilon(p))(v_\epsilon)_\nu d\sigma.$$

Since the first integrand on the right hand side is positive, we need only bound the second integral. Recall that $v = v_{0,1}$ is Lipschitz with a Lipschitz constant that only depends upon the NTA constant M_0 . Then, for all $p \in B_{\frac{1}{4}}(0)$, $r > d$, and $\epsilon \ll r$,

$$\begin{aligned}
\frac{d}{dr} \int_{\partial B_r(p)} |v_\epsilon - v_\epsilon(p)|^2 &\geq -2 \left| \int_{B_r(p)} (v_\epsilon - v_\epsilon)_\epsilon \Delta v_{0,1} \right| \\
&\geq -2Cr \left| \frac{1}{\omega^-(B_1(0))} (\omega^+(B_r(p)) - \omega^-(B_r(p))) \right| \\
&\geq -2Cr
\end{aligned}$$

where we have used Remark 2.3.4 to obtain the last line. Thus, for any $p \in B_{\frac{1}{4}}(0)$, and $0 < s < S < 1$ we obtain,

$$\begin{aligned}
\int_{\partial B_S(p)} |v_\epsilon - v_\epsilon(p)|^2 - \int_{\partial B_s(p)} |v_\epsilon - v_\epsilon(p)|^2 &= \int_s^S \frac{d}{dr} \int_{\partial B_r(p)} |v_\epsilon - v_\epsilon(p)|^2 dr \\
&\geq \int_s^S -2Cr dr \\
&\geq -C(M_0, \alpha, \Gamma) S^2
\end{aligned}$$

Thus, for all $p \in B_{\frac{1}{4}}(0)$, we may bound $\int_{B_{\frac{3}{4}}(p)} (v_\epsilon - v_\epsilon(p))^2$ from below as follows:

$$\begin{aligned}
\int_{B_{\frac{3}{4}}(p)} |v_\epsilon - v_\epsilon(p)|^2 &\geq \int_{B_{\frac{3}{4}}(p) \setminus B_{\frac{5}{8}}(p)} |v_\epsilon - v_\epsilon(p)|^2 \\
&= \int_{\frac{5}{8}}^{\frac{3}{4}} \int_{\partial B_r(p)} |v_\epsilon - v_\epsilon(p)|^2 dS dr \\
&\geq \int_{\frac{5}{8}}^{\frac{3}{4}} \left(\int_{\partial B_{\frac{5}{8}}(p)} |v_\epsilon - v_\epsilon(p)|^2 dS - C(M_0, \alpha, \Gamma) \right) dr \\
&\geq c \int_{\partial B_{\frac{5}{8}}(p)} |v_\epsilon - v_\epsilon(p)|^2 dS - C(M_0, \alpha, \Gamma).
\end{aligned}$$

To get the upper bound we want, we use the same trick.

$$\begin{aligned}
\int_{B_{\frac{9}{16}}(p) \setminus B_{\frac{1}{4}}(p)} |v_\epsilon - v_\epsilon(p)|^2 &= \int_{\frac{1}{4}}^{\frac{9}{16}} \int_{\partial B_s(p)} |v_\epsilon - v_\epsilon(p)|^2 dS ds \\
&\leq \int_{\frac{1}{4}}^{\frac{9}{16}} \int_{\partial B_{\frac{9}{16}}(p)} |v_\epsilon - v_\epsilon(p)|^2 dS + C(M_0, \alpha, \Gamma) ds \\
&\leq c \int_{\partial B_{\frac{9}{16}}(p)} |v_\epsilon - v_\epsilon(p)|^2 + C(M_0, \alpha, \Gamma).
\end{aligned}$$

Since these inequalities hold for all $\epsilon \ll 1/2$, and by Lemmata 3.2.1 and 3.2.12,

$$\lim_{\epsilon \rightarrow 0} H(r, p, v_\epsilon) = H(r, p, v),$$

they also hold for v . Putting it all together, we plug our above bounds into Equation 3.4.9 and consolidating constants, we obtain the following for all $p \in B_{\frac{1}{4}}(0) \cap \overline{\Omega^-}$,

$$\int_{\partial B_{\frac{5}{8}}(p)} |v - v(p)|^2 dS - C(M_0, \alpha, \Gamma) \leq C(n, \alpha, M_0, \Gamma) (16)^{2[N(1,0,v)+C]} \left(\int_{\partial B_{\frac{9}{16}}(p)} |v - v(p)|^2 + C(M_0, \alpha, \Gamma) \right).$$

Though somewhat messier, we restate the above in the following form for convenience later.

$$\begin{aligned}
\frac{\int_{\partial B_{\frac{5}{8}}(p)} |v - v(p)|^2 dS}{\int_{\partial B_{\frac{9}{16}}(p)} |v - v(p)|^2 dS} &\leq C(n, \alpha, M_0, \Gamma) (16)^{2[N(1,0,v)+C]} + \\
&\frac{C(n, \alpha, M_0, \Gamma) (16)^{2[N(1,0,v)+C]} + C(M_0, \alpha, \Gamma)}{\left(\int_{\partial B_{\frac{9}{16}}(p)} (v - v(p))^2 dS \right)}. \tag{3.4.10}
\end{aligned}$$

Now, we change tack slightly. Observe that,

$$\begin{aligned}
\frac{d}{ds} \ln\left(\frac{1}{s^{n-1}} H(s, p, v_\epsilon)\right) &= \frac{1}{\int_{\partial B_s(p)} (v_\epsilon - v_\epsilon(p))^2} \left[\frac{(1-n)}{s} \int_{\partial B_s(p)} (v_\epsilon - v_\epsilon(p))^2 + \frac{1}{s^{n-1}} H'(s, p, v_\epsilon) \right] \\
&= \frac{1}{\int_{\partial B_s(p)} (v_\epsilon - v_\epsilon(p))^2} \left[2s \int_{B_s(p)} |\nabla v_\epsilon|^2 + 2s \int_{B_s(p)} (v_\epsilon - v_\epsilon(p)) \Delta v_\epsilon \right] \\
&= \frac{2}{s} \left[N(s, p, v_\epsilon) + 2 \left(\frac{\int_{B_s(p)} (v_\epsilon - v_\epsilon(p)) \Delta v_\epsilon}{\int_{\partial B_s(p)} (v_\epsilon - v_\epsilon(p))^2} \right) \right]
\end{aligned}$$

Again, we wish to bound the absolute value of the negative part of the derivative. This amounts to bounding the last term. By Remark 3.3.7, for all $p \in B_{\frac{1}{2}}(0) \cap \overline{\Omega^-}$, $s > d = \text{dist}(p, \partial\Omega^\pm)$, and $\epsilon \ll s$, we have the following.

$$\begin{aligned}
\ln\left(\int_{\partial B_{\frac{5}{8}}(p)} (v_\epsilon - v_\epsilon(p))^2\right) - \ln\left(\int_{\partial B_{\frac{9}{16}}(p)} (v_\epsilon - v_\epsilon(p))^2\right) &= \int_{\frac{1}{2}}^{\frac{9}{16}} \frac{d}{ds} \ln\left(\frac{1}{s^{n-1}} H(s, p, v_\epsilon)\right) ds \\
&\geq \int_{\frac{1}{2}}^{\frac{9}{16}} \frac{2}{s} N(s, p, v_\epsilon) - 2Cs^{\alpha-1} ds \\
&\geq 2\left[N\left(\frac{1}{2}, p, v_\epsilon\right) - C\left(\frac{5}{8}\right)^\alpha\right] \ln(s) \Big|_{\frac{9}{16}}^{\frac{5}{8}} \\
&\quad - \frac{2C}{\alpha} s^\alpha \Big|_{\frac{9}{16}}^{\frac{5}{8}} \\
&\geq 2c\left[N\left(\frac{1}{2}, p, v_\epsilon\right) - C\right] - \frac{2C}{\alpha}
\end{aligned}$$

Since by Lemmata 3.2.1 and 3.2.12 $\lim_{\epsilon \rightarrow 0} H(r, p, v_\epsilon) = H(r, p, v)$ and $\lim_{\epsilon \rightarrow 0} N(r, p, v_\epsilon) = N(r, p, v)$, and the above estimates hold for all $\epsilon \ll 1/2$, the estimates hold for v , as well.

Thus, if we recall Equation 3.4.10, above, we see that

$$\begin{aligned}
2c[N(\frac{1}{2}, p, v) - C] - \frac{2C}{\alpha} &\leq \ln\left(\frac{\int_{\partial B_{\frac{9}{8}}(p)}(v - v(p))^2}{\int_{\partial B_{\frac{9}{16}}(p)}(v - v(p))^2}\right) \\
&\leq \ln\left[\frac{C(n, \alpha, M_0, \Gamma)(16)^{2[N(1,0,v)+C]} + C(M_0, \alpha, \Gamma)}{\int_{\partial B_{\frac{9}{16}}(p)}(v - v(p))^2 dS}\right].
\end{aligned}$$

Now, we clean up the terms inside the logarithm. Recall that by Lemma 3.3.4 and the fact that our $v = v_{0,1}$ with $p \in B_{\frac{1}{4}}(0)$, we have that $\int_{\partial B_{\frac{9}{16}}(p)}(v - v(p))^2 dS > c$. Thus, we have that,

$$\frac{C(n, \alpha, M_0, \Gamma)(16)^{2[N(1,0,v)+C]} + C(M_0, \alpha, \Gamma)}{\int_{\partial B_{\frac{9}{16}}(p)}(v - v(p))^2 dS} \leq C(n, \alpha, M_0, \Gamma)((16)^{2[N(1,0,v)+C]} + 1).$$

Isolating for $N(\frac{1}{2}, p, v)$, now, we have the bound:

$$\begin{aligned}
N(\frac{1}{2}, p, v) &\leq \ln[C(n, \alpha, M_0, \Gamma)(16)^{2[N(1,0,v)+C]} + 1] + \frac{C}{\alpha} \\
&\leq C(n, \alpha, M_0, \Gamma).
\end{aligned}$$

Now, Lemma 3.3.11, gives that for $1/2 > s > d$, if $\epsilon \ll s$, then

$$N(1/2, p, v_\epsilon) + C(1/2)^\alpha > N(s, p, v_\epsilon) \tag{3.4.11}$$

Thus, again taking limits as $\epsilon \rightarrow 0$, $N(s, p, v) < C(n, \alpha, M_0, \Gamma) + C$ for all $s \in (d, 1/2)$.

To push $s \rightarrow 0$ we recall that v is harmonic in $B_d(p)$ and that therefore $N(s, p, v)$ is truly

monotone at scales $0 < s < d$. This completes the proof. \square

We finish this section with a simple corollary.

Corollary 3.4.5. Let $v \in \mathcal{A}(\Lambda, \alpha, M_0)$, such that v satisfies $\|\ln(h)\|_\alpha \leq \Gamma$, and $p \in B_{\frac{1}{4}}(0)$, and $0 < r \leq \frac{1}{18}$. Then, $T_{p,r}v$ is Lipschitz in $B_8(0)$ with constants that only depend upon α, M_0 , and Γ .

Proof. We consider two cases. Suppose that $\text{dist}(p, \partial\Omega^\pm) < 9r$. Let $Q \in \partial\Omega^\pm$ be a point such that $|p - Q| = \text{dist}(p, \partial\Omega^\pm)$. Let $y = T_{Q,10r}p$.

$$\begin{aligned} T_{p,r}v(z) &= T_{y,1/2}T_{Q,2r}v(z) \\ &= \frac{T_{Q,2r}v(y + 1/2z) - T_{Q,2r}v(y)}{(\int_{\partial B_1(0)} (T_{Q,2r}v(y + 1/2z) - T_{Q,2r}v(y))^2 dz)^{1/2}}. \end{aligned}$$

By Lemma 3.2.2, we have that $(\int_{\partial B_1(0)} (T_{Q,2r}v(y + 1/2z) - T_{Q,2r}v(y))^2 dz)^{1/2} \geq C(\alpha, \Gamma, M_0)$.

Thus, we have that

$$|T_{p,r}v(z_1) - T_{p,r}v(z_2)| \leq \frac{|T_{Q,2r}v(y + 1/2z_1) - T_{Q,2r}v(y + 1/2z_2)|}{C}$$

and by Lemma 3.2.1 $T_{Q,2r}v$ are uniformly locally Lipschitz.

If $\text{dist}(p, \partial\Omega^\pm) > 9r$, then $T_{p,r}v$ is harmonic in $B_9(0)$. By Lemma 3.4.4, the functions $T_{p,r}v$ have uniformly bounded energy in $B_9(0)$ and therefore, by elliptic estimates, are uniformly Lipschitz in $B_8(0)$. \square

3.5 Quantitative Rigidity and Cone-Splitting

Throughout the rest of the paper, we shall need to use limit-compactness arguments. The key will be that as $\|\ln(h)\|_\alpha \rightarrow 0$, $v \rightarrow v_\infty$ for some harmonic function v_∞ . We make this rigorous in the following lemma.

Lemma 3.5.1. (Convergence to Harmonic Functions) Let $v_i \in \mathcal{A}(\Lambda, \alpha, M_0)$ with $\|\ln(h_i)\|_\alpha \rightarrow 0$. Assume that $p_i \in B_{\frac{1}{2}}(0) \cap \overline{\Omega}^-$ and $\{r_i\} \subset (0, 1/2]$. Then, there exists a function v_∞ and a subsequence v_j such that $T_{p_j, r_j} v_j \rightarrow v_\infty$ in the senses of Lemma 3.2.10 and v_∞ is harmonic in $B_2(0)$.

Proof. We observe that Lemma 3.4.4 states that $N(2, 0, T_{p_j, r_j} v_j) \leq C$. This directly implies that $\{T_{p_j, r_j} v_j\}_j$ is bounded in $W^{1,2}(B_2(0))$. Note that we may assume that $r_i > \text{dist}(p_i, \partial\Omega_i)$, otherwise $\{T_{p_j, r_j} v_j\}_j$ are harmonic. In this case, harmonic and bounded in $W^{1,2}(B_2(0))$ give the desired result. Assuming that $r_i > \text{dist}(p_i, \partial\Omega_i)$, Lemma 3.2.10 gives a subsequence which converges in the desired senses to a function v_∞ .

We now turn our attention to showing that v_∞ is harmonic. To do this, we will have to investigate the behavior of its mollifications, $v_{\infty, \epsilon} = v_\infty \star \phi_\epsilon$. Observe that by Young's Inequality,

$$\|T_{p_j, r_j} v_{j, \epsilon} - v_{\infty, \epsilon}\|_{L^2(B_2(0))} \leq \|\phi_\epsilon\|_{L^1(B_2(0))} \|T_{p_j, r_j} v_j - v_\infty\|_{L^2(B_2(0))}.$$

Thus, for any $\epsilon > 0$ we have $T_{p_j, r_j} v_{j, \epsilon} \rightarrow v_{\infty, \epsilon}$ as $j \rightarrow \infty$ strongly in $L^2(B_2(0))$. By a similar argument applied to $\nabla T_{p_j, r_j} v_{j, \epsilon}$ we also have that $\nabla T_{p_j, r_j} v_{j, \epsilon} \rightarrow \nabla v_{\infty, \epsilon}$ in $L^2(B_2(0))$ as $j \rightarrow \infty$. Furthermore, by our uniform Lipschitz bounds, $T_{p_j, r_j} v_{j, \epsilon} \rightarrow v_{\infty, \epsilon}$ as $j \rightarrow \infty$ in $C(B_2(0))$, as well.

We will show that for $\epsilon \ll 1$ the function $v_{\infty, \epsilon}$ is harmonic. First, for any test function $\xi \in C_c^\infty(B_2(0))$, we have that

$$\begin{aligned}
\left| \int_{B_2(0)} \xi(\Delta T_{p_j, r_j} v_{j, \epsilon} - \Delta v_{\infty, \epsilon}) dx \right| &= \left| \int_{B_2(0)} \Delta \xi (T_{p_j, r_j} v_{j, \epsilon} - v_{\infty, \epsilon}) dx \right| \\
&\leq \|\Delta \xi\|_{L^2(B_2(0))} \|T_{p_j, r_j} v_{j, \epsilon} - v_{\infty, \epsilon}\|_{L^2(B_2(0))}
\end{aligned}$$

Since, $T_{p_j, r_j} v_{j, \epsilon} \rightarrow v_{\infty, \epsilon}$ strongly in $L^2(B_2(0))$, we have that $\Delta T_{p_j, r_j} v_{j, \epsilon} \rightharpoonup \Delta v_{\infty, \epsilon}$ in $L^2(B_2(0))$.

However, by assumption, we also have that

$$\begin{aligned}
\left| \int_{B_2(0)} \xi \Delta T_{p_j, r_j} v_{j, \epsilon} dx \right| &\leq \int_{B_2(0)} |\xi_\epsilon| |\Delta T_{p_j, r_j} v_{j, \epsilon}| dx \\
&\leq C \left(\max_{B_2(0)} |\xi| \right) \|\ln(h_j)\|_{\alpha} \omega_{r_j, j}^-(B_3(0))
\end{aligned}$$

where we have $\omega_{r_j, j}$ is the harmonic measure associated to $v_{r_j, j}$. Recall that $\omega_{r_j, j}(B_3(0))$ is uniformly bounded [8]. Thus, as $j \rightarrow \infty$, we have that $\Delta T_{p_j, r_j} v_{j, \epsilon} \rightarrow 0$ in $L^2(B_2(0))$, as well. Thus, $\Delta v_{\infty, \epsilon} = 0$ weakly in $L^2(B_2(0))$. Since $v_{\infty, \epsilon} \in C^\infty(B_2(0))$, by classical results, $v_{\infty, \epsilon}$ is harmonic. Note that this holds for all $\epsilon \ll 1$.

Since v_∞ is Lipschitz continuous, $v_{\infty, \epsilon} \rightarrow v_\infty$ in $C(B_R(0))$. Thus, for all $x \in B_R(0)$ we have both that $v_{\infty, \epsilon}(x) \rightarrow v_\infty(x)$ as $\epsilon \rightarrow 0$ and that

$$\int_{B_r(x)} v_{\infty, \epsilon}(y) dy \rightarrow \int_{B_r(x)} v_\infty(y) dy$$

as $\epsilon \rightarrow 0$. Thus, v_∞ must satisfy the Mean Value Property and is therefore harmonic. \square

Remark 3.5.2. By Remark 2.3.5, if $v \in \mathcal{A}(\Lambda, \alpha, M_0)$, then $\tilde{v} = (-h(0)^{-1})v \in \mathcal{A}(\Lambda, \alpha, M_0)$. By applying all the results in Sections 3.2, 3.3, 3.4 and Lemma 3.5.1 to \tilde{v} , we obtain the identical results for $p \in B_{\frac{1}{2}}(0) \cap \overline{\Omega^+}$. Hence, all the estimates in the previous lemmata hold for all $p \in B_{1/2}(0)$.

Now that we have Lemma 3.5.1, we can prove a quantitative rigidity result. Loosely speaking,

it says that if a function $v \in \mathcal{A}(n, \Lambda, \alpha, M_0)$ behaves like a homogeneous harmonic polynomial with respect to the Almgren frequency (in the sense that it has small drop across scales), then it must be close to being a homogeneous harmonic polynomial. This will connect the behavior of the Almgren frequency to our effective stratification.

Lemma 3.5.3. (Quantitative rigidity) Let $v \in \mathcal{A}(\Lambda, \alpha, M_0)$, as above. Let $p \in B_1(0)$. For every $\gamma \in (0, 1)$ and $\delta > 0$, there is an $\rho = \rho(n, \alpha, \gamma, \Lambda, \delta, M_0) > 0$ such that if $\|\ln(h)\|_\alpha \leq \rho$ and

$$|N(1, p, v) - N(\gamma, p, v)| \leq \rho$$

then v is $(0, \delta, 1, p)$ -symmetric.

Proof. We argue by contradiction. Suppose there exists a $\gamma \in (0, 1)$ and a $\delta > 0$ such that there is a sequence of functions, $v_i \in \mathcal{A}(n, \Lambda, \alpha, M_0)$ with $\|\ln(h_i)\|_\alpha \leq 2^{-i}$ for which there exists a point, p_i with

$$|N(1, p_i, v_i) - N(\gamma, p_i, v_i)| \leq 2^{-i}$$

but that no v_i is $(0, \delta, 1, p)$ -symmetric.

By Lemma 3.5.1 we have that there exists a subsequence, such that $T_{p_j, 1}v_j$ converges strongly in $W_{loc}^{1,2}$ to a harmonic function, v_∞ . Therefore $N(r, p, v)$ is monotone increasing. Further, by Corollary 3.2.11, we know that $\lim_{j \rightarrow \infty} N(r, x, T_{p_j, 1}v_j) = N(r, x, v_\infty) \geq 1$. Therefore, we have that,

$$N(1, 0, v_\infty) - N(\gamma, 0, v_\infty) = 0.$$

By classical results, this implies that v_∞ is a homogeneous harmonic polynomial in $B_1(0) \setminus B_\gamma(0)$. By unique continuation, v is a homogeneous harmonic polynomial everywhere. Thus, we arrive at our contradiction, since $T_{p_j, 1}v_j$ were assumed to stay away from all such functions in $L^2(B_1(0))$. \square

Remark 3.5.4. Since $N(r, p, v)$ is scale-invariant, Lemma 4.2.1 is also scale-invariant in the sense that if $N(r, p, v) - N(\gamma r, p, v) \leq \rho$ and $\|\ln(h)\|_\alpha \leq \rho$, then v is $(0, \delta, r, p)$ -symmetric.

We now turn our attention to a “cone-splitting” result about homogeneous harmonic polynomials. Again, our argument boils down to showing that being almost-symmetric and almost-harmonic gets us close enough to import the machinery. First, recall the classical result for homogeneous harmonic polynomials.

Proposition 3.5.5. Let $P : \mathbb{R}^n \rightarrow \mathbb{R}$ be a function which is homogeneous with respect to the origin. Let $k \leq n - 2$. If P is symmetric with respect to some k -dimensional subspace V and P is homogeneous with respect to some point $x \notin V$, then P is $k + 1$ -symmetric with respect to $\text{span}\{x, V\}$.

See [13], (Proposition 2.11), or [18] in the proof of Theorem 4.1.3. We now prove a similar result for our almost-symmetric, almost-harmonic function v .

Lemma 3.5.6. (Cone-splitting) Let $k \leq n - 2$ and $v \in \mathcal{A}(\Lambda, \alpha, M_0)$. For any fixed $\epsilon, \tau > 0$ and any $0 < r \leq 1$, there is a $0 < \delta_0 = \delta_0(n, r, \tau, \epsilon, \Lambda)$ such that for $0 < \delta < \delta_0$, the following holds. If $p \in B_1(0)$ and v is (k, δ, r, p) -symmetric with respect to a k -dimensional subspace V and $(0, \delta, r, x)$ -symmetric for some $x \in B_r(p) \setminus B_{\tau r}(V + p)$, then v is $(k + 1, \epsilon, 1, p)$ -symmetric.

Proof. We argue by limit-compactness. Assume that there exists a $\delta, \tau > 0$ and an $0 < r \leq 1$ for which there exist a sequence of function, $v_i \in \mathcal{A}(\Lambda, \alpha, M_0)$ and points $\{p_i\}$ for which v_i is (k, δ, r, p_i) -symmetric with respect to some V_i and $(0, \delta, r, x_i)$ -symmetric for some $x_i \in B_r(p_i) \setminus B_{\tau r}(V_i + p_i)$, but that all v_i are not $(k + 1, \delta, 1, p_i)$ -symmetric.

We normalize, considering $T_{p_i, r_i} v_i$. By Lemma 3.5.1, there exists a harmonic function, v_∞ such that a subsequence $T_{p_j, r_j} v_j \rightarrow v_\infty$ in the senses of the lemma. Note that v_∞ is non-degenerate. Taking further subsequences, we may reduce to a sequence for which

$$V_j \rightarrow V, \quad p \in \overline{B_1(0)}, \quad x_j \rightarrow x$$

and $x \in \overline{B_r(p)} \setminus B_{\tau r}(V)$.

Since v_∞ is $(k, 0, r, p)$ -symmetric with respect to some V , $(0, 0, r, x)$ -symmetric, and is har-

monic, by the previous proposition, v is $k+1$ -symmetric in $B_r(p)$. Since $v_i \rightarrow v$ in $L^2(B_2(0))$, we have our contradiction. \square

3.6 The Covering and its Properties

The lemmata in the previous section allow us to inductively define a covering with the right packing conditions. Quantitative rigidity allows us to prove a ‘‘Quantitative Differentiation’’ lemma that bounds the number of scales across which the frequency can change by more than some threshold $\rho > 0$. Cone splitting on the other hand, will give us good geometric control of the singular set at scales for which v is close to a homogeneous harmonic polynomial. Together, these things will give us the necessary packing conditions.

First, we describe the covering.

The Construction

Let $v \in \mathcal{A}(n, \Lambda, M_0)$, and let $\epsilon, r > 0$, $k \leq n-2$, and $N \in \mathbb{N}$ be given. We cover $\mathcal{S}_{\epsilon, r}^k(v) \cap B_1(0)$ by balls of radius γ^N . The full cover will actually be a union of collections of balls, $C_{\epsilon, r}^k(T^N)$, indexed by $\{T^N \in \{0, 1\}^N\}$. We will construct these collections, $C_{\epsilon, r}^k(T^N)$, by inductive refinement on N .

We begin by defining an auxiliary quantity. Let

$$\mathcal{D}(v, x, r) = \inf\{\delta_2 > 0 \mid v \text{ is } (0, \delta_2, r, x) \text{ - symmetric}\}.$$

Recall that this is roughly the square of the distance in L^2 from the set of normalized homogeneous harmonic polynomials on $B_r(x)$.

Now, for any $i \in \mathbb{N}$ we can assign to each $x \in B_{\frac{1}{2}}(0)$ an i -tuple, $T^i(x)$, according to the rule

$$(T^i(x))_j = 1 \quad \text{if } \mathcal{D}(v, x, \gamma^j) \geq \delta_0$$

$$(T^i(x))_j = 0 \quad \text{if } \mathcal{D}(v, x, \gamma^j) < \delta_0$$

where δ_0 is a small number to be chosen later. We shall refer to δ_0 as the sorting threshold. We shall use the conventions that $|T^i|$ denote the sum of the entries. Note that there is a partial ordering on the set of these i -tuples. That is, if $k < i$, we can say that $T^k < T^i$ if $(T^k)_j = (T^i)_j$ for all $j \in \{1, 2, \dots, k\}$.

Now, we partition our set according to these i -tuples. For any given i -tuple, $T^i \in \{0, 1\}^i$, we define

$$E(T^i) = \{x \in B_1(0) \mid T^i(x) = T^i\}$$

It follows immediately from the definitions that $E(T^i) \subset E(T^k)$ if and only if $T^k < T^i$.

We now define our covering inductively. For $i = 0$, we let $C_{\epsilon, r}^k(T^0) = B_1(0)$. For $i - 1 < N$, observe that for each non-empty $E(T^{i-1})$, $C_{\epsilon, r}^k(T^{i-1})$ consists of balls of radius γ^{i-1} . Within each such ball, $B_{\gamma^{i-1}}(y)$, partition the set $B_{\gamma^{i-1}}(y) \cap \mathcal{S}_{\epsilon, r}^k(v) \cap E(T^{i-1})$ into the sets $E(T^i)$ for T^i such that $T^{i-1} < T^i$.

For either T^i , then, within each such $B_{\gamma^{i-1}}(y)$ take a minimal covering of $B_{\gamma^{i-1}}(y) \cap \mathcal{S}_{\epsilon, r}^k(v) \cap E(T^{i-1})$ by balls of radius γ^i centered at points in $B_{\gamma^{i-1}}(y) \cap \mathcal{S}_{\epsilon, r}^k(v) \cap E(T^{i-1})$. The union of these balls is our $C_{\epsilon, r}^k(T^i)$.

For some i -tuples, the set $E(T^i)$ may be empty. In this case, we simply allow the corresponding collection of balls, $C_{\epsilon, r}^k(T^i)$, be empty.

Now that we have our sets and covers, let us prove that they have the desired properties.

Lemma 3.6.1. Let $v \in \mathcal{A}(\Lambda, \alpha, M_0)$, $\|\ln(h)\|_\alpha \leq \Gamma$, and $\gamma \in (0, 1)$. For any $\rho > 0$, there exists a constant $D = D(n, \Lambda, \alpha, \Gamma, M_0, \rho, \gamma)$ such that for all $p \in B_{\frac{1}{4}}(0)$,

$$|\{i \in \mathbb{N} \quad : \quad |N(p, \gamma^i, v) - N(p, \gamma^{i+1}, v)| \geq \rho\}| \leq D.$$

Proof. Lemma 3.4.4 gives a function such that for all $p \in B_{\frac{1}{4}}(0)$ and all $r \leq \frac{1}{2}$

$$N(r, p, v) \leq C(n, \Lambda, \alpha, \Gamma, M_0).$$

In the harmonic setting, the Almgren Frequency function is monotonic, and we can bound $\sum_{i=0}^{\infty} |N(\gamma^i, p, v^0) - N(\gamma^{i+1}, p, v^0)| \leq C(n, \Lambda)$. But, in this case, the Almgren Frequency function is only almost monotonic. Thus we must employ a “rays of the sun”-type argument. That is, we bound

$$\begin{aligned} \sum_{i=0}^{\infty} |N(\gamma^i, p, v) - N(\gamma^{i+1}, p, v)| &= \sum_{i=0}^{\infty} (N(\gamma^i, p, v) - N(\gamma^{i+1}, p, v))^+ \\ &\quad + \sum_{i=0}^{\infty} (N(\gamma^i, p, v) - N(\gamma^{i+1}, p, v))^- \\ &\leq C(n, \Lambda, \alpha, \Gamma, M_0) \\ &\quad + 2 \sum_{i=0}^{\infty} (N(\gamma^i, p, v) - N(\gamma^{i+1}, p, v))^- \\ &\leq C(n, \Lambda, \alpha, \Gamma, M_0) \\ &\quad + 2 \sum_{i=0}^{\infty} C_1 \|\ln(h)\|_{\alpha} (\gamma^i)^{\alpha} \\ &\leq C(n, \Lambda, \alpha, \Gamma, M_0) + 2C(\alpha, M_0, \Gamma)C(\gamma, \alpha) \end{aligned}$$

Thus, we have that,

$$\begin{aligned} |\{i \in \mathbb{N} : |N(\gamma^i, p, v) - N(\gamma^{i+1}, p, v)| \geq \rho\}| &\leq \frac{C(n, \Lambda, \alpha, \Gamma, M_0) - 1 + 2C(\alpha, M_0, \Gamma)C(\gamma, \alpha)}{\rho} \\ &= D. \end{aligned}$$

□

As an immediate corollary, we have the following:

Corollary 3.6.2. Let $v \in \mathcal{A}(\Lambda, \alpha, M_0)$. Let $0 < \delta_0$ be the sorting threshold in the construction

above, and let $0 < \gamma < 1$ be the proportion of the scales we consider in the above construction. There is a constant, $\rho = \rho(n, \alpha, \gamma, \Lambda, \delta_0)$, such that if $\|\ln(h)\|_\alpha \leq \rho$ then in the construction described above, there are at most $N^{D(n, \Lambda, \alpha, M_0, \delta_0, \gamma)}$ nonempty sets, $E(T^N)$.

Proof. By quantitative rigidity, Lemma 4.2.1, for any $0 < \delta_0$ there is a $\rho(n, \alpha, \gamma, \Lambda, \delta_0) > 0$ such that if $\|\ln(h)\|_\alpha \leq \rho$ and $|N(\gamma^i, p, v^0) - N_\epsilon(\gamma^i, p, v^0)| \leq \rho$ then v is at least $(0, \delta_0, \gamma^i, p)$ -symmetric. Thus, for any sorting δ_0 in our construction, if we plug the corresponding ρ into Lemma 5.2, we get a $D(n, \Lambda, \alpha, M_0, \delta_0, \gamma)$ such that for all $p \in B_{\frac{1}{4}}(0)$,

$$|T^N(p)| < D.$$

Thus, for any given $N \in \mathbb{N}$, there are at most $\binom{N}{D}$ N -tuples with $|T^N| \leq D$. Since $\binom{N}{D} \leq N^D$, we have the desired claim: $\mathcal{S}_{\epsilon, r}^k \cap B_{\frac{1}{2}}(0)$ is contained in the union of at most N^D nonempty sets, $E(T^N)$, and covered by at most N^D collections of balls, $C_{\epsilon, r}^k(T^N)$. \square

We now prove that this construction satisfies the claimed packing condition.

Lemma 3.6.3. Let $v \in \mathcal{A}(\Lambda, \alpha, M_0)$, $0 < \epsilon, r, k \leq n - 2$, and $0 < \gamma < 1$. We can choose a $0 < \delta_0 = \delta(n, \epsilon, \Lambda)$ such that if $\|\ln(h)\|_\alpha < \rho$, where $\rho = \rho(n, \alpha, \gamma, \Lambda, \delta_0)$ as in Corollary 3.6.2, then for any $N \in \mathbb{N}$ with $\gamma^N \geq r$ there exist constants, $0 < c_1, c_2$ depending only on the ambient dimension, n , such that each collection, $C_{\epsilon, r}^k(T^N)$, consists of at most $(c_1 \gamma^{-n})^D (c_2 \gamma^{-k})^{N-D}$ balls of radius γ^N .

Proof. For the given k, ϵ , choose the sorting constant δ_0 in our construction to be the $\delta_0(n, 1, 1/10, \epsilon, \Lambda)$ which Lemma 3.5.6 guarantees for our choice of $r = 1$ and $\tau = \frac{1}{10}$. That is, let δ_0 be such that if $v \in \mathcal{A}(\Lambda, \alpha, M_0)$ and v is $(k, \delta_0, 1, p)$ -symmetric with respect to a k -dimensional subspace V and $(0, \delta_0, 1, x)$ -symmetric for some $x \in B_1(p) \setminus B_{1/10}(V + p)$, then v is actually $(k + 1, \epsilon, 1, p)$ -symmetric.

Now, for $i \leq N$ and any given N -tuple, T^N , for which $E(T^N)$ is non-empty, let $T^{i-1} < T^N$ and $B_{\gamma^{i-1}}(x) \in C_{\epsilon, r}^k(T^{i-1})$. Consider the set $A = S_{\epsilon, r}^k(v) \cap B_{\gamma^{i-1}}(x) \cap E(T^N)$. We argue by

cases.

Case 1. $(T^N)_i = 0$. That is, v is (k, δ, γ^i, x) -symmetric for some $\delta < \delta_0$. By Lemma 3.5.6, then, if A is not contained in $B_{\gamma^{i-1}}(x) \cap B_{\frac{\gamma^i}{10}}(V^k)$ for some k -dimensional plane V^k , then x would actually be $(k+1, \epsilon, \gamma^i, x)$ -symmetric. This would contradict our definition of $x \in S_{\epsilon, r}^k(v)$.

Thus, we can cover $A \subset B_{\gamma^{i-1}}(x) \cap B_{\frac{\gamma^i}{10}}(V^k)$ by $c_1(n)\gamma^{-k}$ balls of radius γ^i .

Case 2. $(T^N)_i = 1$. In this case, we have no control, so we cover A by $c_2(n)\gamma^{-n}$ balls of radius γ^i .

Carrying this process through our construction, we see by Corollary 3.6.2, for $\|ln(h)\|_\alpha < \rho$, Case 2 can only happen at most D times. Thus, $C_{\epsilon, r}^k(T^N)$ is a collection of at most $(c_1\gamma^{-n})^D(c_2\gamma^{-k})^{N-D}$ balls of radius γ^N , as claimed. \square

3.6.1 Proof of Theorem 3.1.1

In Corollary 3.6.2 and Lemma 3.6.3, we require that $\|ln(h)\|_\alpha$ be small. We now show that we can satisfy this condition.

Lemma 3.6.4. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfy $\|f\|_\alpha = \Gamma$. For every $\rho > 0$, there exists a scale $\kappa(\rho, \Gamma) > 0$ such that the rescaled function $f_{r, y}(x) = f(rx + y)$ has $\|f_{\kappa, y}\|_\alpha < \rho$

Proof. This follows immediately from the definitions.

$$\begin{aligned} |f_{r, y}(x) - f_{r, y}(z)| &= |f(rx + y) - f(rz + y)| \\ &\leq \Gamma |rx - rz|^\alpha \\ &= \Gamma r^\alpha |x - z|^\alpha \end{aligned}$$

Since $r^\alpha \rightarrow 0$ as $r \rightarrow 0$, there exists an $\kappa(\rho, \Gamma) > 0$ such that $\Gamma \kappa(\rho, \Gamma)^\alpha < \rho$. \square

We now prove the desired Minkowski-type bounds on $\mathcal{S}_{\epsilon,r}^k(v)$.

Theorem 3.6.5. Let $v \in \mathcal{A}(\Lambda, \alpha, M_0)$ with $\|\ln(h)\|_\alpha \leq \Gamma$ and $r > 0$. Then, for every $0 < \epsilon$ and $k \leq n - 2$, there exists a $0 < \gamma < 1$ and a $\kappa(\alpha, \Gamma, n, \epsilon, \Lambda) > 0$ such that for every $y \in B_{\frac{1}{4}}(0)$ and $N \in \mathbb{N}$

$$\text{Vol}(B_1(0) \cap B_{\gamma^N}(\mathcal{S}_{\epsilon,r}^k(v_{\kappa,y}))) \leq C(n, k, \Lambda, \alpha, M_0, \epsilon)(\gamma^N)^{n-k-\epsilon}. \quad (3.6.1)$$

Remark 3.6.6. The $\kappa > 0$, above, is simply the scale, at which we begin to be able to apply Lemma 4.12 as in Corollary 3.6.2. Such uniform scale is guaranteed by Lemma 3.6.4. It may be objected that $v_{\kappa,p} \notin \mathcal{A}(\Lambda, \alpha, M_0)$. However, since $y \in B_{\frac{1}{4}}(0)$, Lemma 3.4.4 gives us all the required compactness to push through all the previous lemmata for $v_{\kappa,p}$, without changing the constants. Therefore, we proceed by applying them to the rescaled function $v_{\kappa,p}$.

Proof. Let $\gamma = \gamma(n, \epsilon) = c_2(n)^{\frac{-2}{\epsilon}} < 1$. Our construction gives us a covering of $B_{\frac{1}{2}}(0) \cap \mathcal{S}_{\epsilon,r}^k(v_\kappa)$ by balls of radius γ^N . Doubling the radius of these balls is sufficient, then, to cover $B_{\frac{1}{2}}(0) \cap B_{\gamma^N}(\mathcal{S}_{\epsilon,r}^k(v_\kappa))$.

Thus, we have that,

$$\text{Vol}(B_1(0) \cap B_{\gamma^N}(\mathcal{S}_{\epsilon,r}^k(v_{\kappa,y}))) \leq N^D (c_1 \gamma^{-n})^D (c_2 \gamma^{-k})^{N-D} (\omega_n 2 \gamma^N)^n \quad (3.6.2)$$

Collecting all the γ terms on the right hand side, we see that the exponent is $N(n - k) - D(n - k)$. Recalling that $\gamma = c_2(n)^{\frac{-2}{\epsilon} < 1}$ and D is fixed, we have that there exists a constant $C(n, k, \epsilon, D)$ such that for all $N \in \mathbb{N}$

$$\gamma^{N(n-k)-D(n-k)} \leq C(n, k, \epsilon, D) \gamma^{N(n-k)}$$

Similarly, there exists a constant, $C(D, n)$ such that for all $N \in \mathbb{N}$

$$N^D \leq C(D, n)c_2(n)^N$$

for all $N \in \mathbb{N}$.

Plugging these estimates into our volume estimate above, we have

$$\begin{aligned} \text{Vol}(B_1(0) \cap B_{\gamma^N}(\mathcal{S}_{\epsilon, r}^k(v_{\kappa, y}))) &\leq N^D c_1^D c_2^{N-D} (\omega_n 2)^n \gamma^{N(n-k) - D(n-k)} \\ &\leq C(D, n) c_2(n)^{2N-D} c_1(n)^D (\omega_n 2)^n C(n, k, \epsilon, D) \gamma^{N(n-k)} \\ &\leq C(D, n, \epsilon, k) \gamma^{-N\epsilon} \gamma^{N(n-k)} \\ &\leq C(n, k, \Lambda, \alpha, M_0, \epsilon) (\gamma^N)^{n-k-\epsilon} \end{aligned}$$

□

Thus, the proof of Theorem 3.1.1 reduces to the following argument. Cover $B_{1/4}(0)$ by $c(n)\kappa^{-n}$ balls $B_\kappa(p)$ for $p \in B_{1/4}(0)$. Summing up the bounds in the previous lemma gives the statement.

Chapter 4

FEDERER DIMENSION REDUCTION II**4.1 Main Results and Outline of the Proof**

In this chapter, we strengthen the results of Theorem 3.1.1, by improving the volume bounds on tubular neighborhoods around the $S_{\epsilon,r}^k(v)$. We are able to show the following estimates.

Theorem 4.1.1. Let $v \in \mathcal{A}(\Lambda, \alpha, M_0)$ with $\|\ln(h)\|_\alpha \leq \Gamma$. For every $0 < \epsilon$ and $0 \leq k \leq n - 2$ there is an $0 < r_0(n, \Lambda, \alpha, M_0, \Gamma, \epsilon)$ such that for all $0 < r < r_0$, and any $r \leq R \leq 1$,

$$\text{Vol}(B_R(B_{1/4}(0) \cap \mathcal{S}_{\epsilon,r}^k(v))) \leq C(n, \Lambda, \alpha, M_0, \Gamma, \epsilon)R^{n-k}. \quad (4.1.1)$$

As an immediate corollary, we also have that

Corollary 4.1.2. Let $v \in \mathcal{A}(\Lambda, \alpha, M_0)$ and $0 \leq k \leq n - 2$. For every $0 < \epsilon$,

$$\overline{\dim}_{\mathcal{M}}(\mathcal{S}_\epsilon^k(v)) \leq k, \quad (4.1.2)$$

and there exists a constant such that,

$$\mathcal{M}^{*,k}(\mathcal{S}_\epsilon^k(v) \cap B_{1/4}(0)) \leq C(n, \Lambda, \alpha, M_0, \Gamma, \epsilon). \quad (4.1.3)$$

This follows from Theorem 4.1.1 and the definition of upper Minkowski content (Equation 1.1.2). We are able to strengthen the conclusion of Theorem 4.1.1 when we restrict to considering $\mathcal{S}_{\epsilon,r}^{n-3}(v) \cap \partial\Omega^\pm$.

Corollary 4.1.3. Let $v \in \mathcal{A}(\Lambda, \alpha, M_0)$ with $\|\ln(h)\|_\alpha \leq \Gamma$. Recall that $\text{sing}(\partial\Omega^\pm) = \mathcal{S}^{n-3} \cap \partial\Omega^\pm$, and that there exists an $0 < \epsilon = \epsilon(M_0, \Gamma, \alpha)$ such that $\text{sing}(\partial\Omega^\pm) \subset \mathcal{S}_\epsilon^{n-3}$, (see, Lemma 4.10.1). Thus, there is a constant, $C = C(n, \Lambda, \alpha, M_0, \Gamma) < \infty$ such that,

$$\mathcal{M}^{*,n-3}(\text{sing}(\partial\Omega^\pm) \cap B_{1/4}(0)) \leq C(n, \Lambda, \alpha, M_0, \Gamma) \quad (4.1.4)$$

Proof. This follows immediately from Lemma 4.10.1 and Theorem 4.1.1. \square

Remark 4.1.4. For $0 < \epsilon$ small enough, we have a slightly stronger statement. By Remark 4.10.2, there is an $0 < \epsilon = \epsilon(M_0, \Gamma, \alpha)$ such that for any $0 < r_0$, if $Q \notin \partial\Omega^\pm \cap \mathcal{S}_{\epsilon, r_0}^{n-3}$ then there exists a radius $r_0 \leq r$ such that $\|T_{Q,r}v - L\|_{L^2(B_1(0))}^2 \leq \epsilon$ for an $(n-1)$ -symmetric function, $L =$. Therefore, taking r_0 as in Theorem 4.1.1,

$$\text{Vol}(B_{r_0}(B_{1/4}(0) \cap \mathcal{S}_{\epsilon, r_0}^{n-3}(v))) \leq C(n, \Lambda, \alpha, M_0, \Gamma)r^3,$$

implies that for every $Q \in \partial\Omega^\pm \cap B_{1/4}(0)$ such that $Q \notin \mathcal{S}_{\epsilon, r_0}^{n-3}$, there exists a radius, $r_0 \leq r$ such that $\|T_{Q,r}v - L\|_{L^2(B_1(0))}^2 \leq \epsilon$ for an $n-1$ -symmetric function, $L = a^+\langle x, e^+ \rangle - a^-\langle x, e^- \rangle$ for some $a^\pm \in \mathbb{R}$. These bounds are analogous to the “effective” bounds defined in [13].

Because the generalized critical set is invariant under scalar multiplication, $v \rightarrow cv$, for $c \neq 0$, these results—restricted to $\overline{\Omega^\pm}$ —naturally extend to any Green’s function with pole at infinity for Ω^\pm , respectively. Since for unbounded NTA domains, non-negative harmonic functions on Ω^+ which vanish on the boundary, $\partial\Omega^\pm$, are uniquely determined by their value at any point $p \in \Omega^+$, every such function is cu^\pm for some constant, c . Therefore the above results trivially hold for all such functions.

4.1.1 Outline of the Proof of Theorem 3.1

The main difference between the proof of Theorem 3.1.1 and Theorem 4.1.1 is how we count the balls in our cover. Obtaining geometric control is done in a very similar way, relying

upon only slightly stronger quantitative rigidity results and more information about the non-degeneracy of the almost-monotonicity of the Almgren frequency in this context. It is the overlap incurred by only using Reifenberg-flat control (uniform approximation by k planes) which creates the extra power in the estimates, and a way must be found to count better.

The key to overcoming the overlap problem comes from two sources: first, more information in “bad” balls and second, a better way of counting balls. It turns out that “bad” balls (i.e., ones in which we do not have Reifenberg-flat control) have the following structure: the set of points for which the Almgren frequency is almost constant is contained in the neighborhood of a $(k - 1)$ -dimensional plane, and all points outside this neighborhood have a large drop in their Almgren frequency across scales. See Corollary 4.3.2 for a precise statement. The set which is approximated by a lower-dimensional set enjoys better packing estimates. And, the large drops in Almgren frequency can only happen finitely many times.

The better way of counting balls in our covering comes from the Discrete Reifenberg Theorem (Theorem 2.5.2) from [24]. This is a very powerful result in geometric measure theory which allows one to control how balls pack based upon scale-invariant control of the Jones function. The main challenge is to connect the lower bound on the derivative of the Almgren frequency to the β -numbers centered *away* from $\partial\Omega^\pm$. Section 4.4 is devoted to overcoming this challenge and is one of the main technical achievements of this paper. The crucial insight turns out to be that if you are very close to a 0-symmetric function, and that function is not $(n - 1)$ -symmetric (i.e., piecewise, two-phase linear), then you must be very close to a function with higher-order homogeneity. Higher-order homogeneity means that the Almgren frequency is strictly greater than 1 at the appropriate scales, and this gives us the control we need over the error terms incurred by considering points off the boundary. In Section 4.5, we modify the framework of [24] to accommodate the estimates of Section 4.4. Sections 4.6 and 4.8, construct the covering which proves the theorem according to the program laid out by [24]. They are included for completeness.

4.2 A New Quantitative Rigidity

Recalling Lemma 3.5.1, we must now prove a better Quantitative rigidity result than Lemma 4.2.1. In this version, we drop the absolute values and drop a quantifier by letting $\gamma = \rho$.

Lemma 4.2.1. (Quantitative rigidity) Let $v \in \mathcal{A}(\Lambda, \alpha, M_0)$, as above. Let $p \in B_{\frac{1}{4}}(0) \cap \overline{\Omega}^-$. For every $\delta > 0$, there is an $\gamma = \gamma(n, \Lambda, \alpha, M_0, \delta) > 0$ such that if $\|\ln(h)\|_\alpha \leq \gamma$ and

$$N(1, p, v) - N(\gamma, p, v) \leq \gamma$$

then v is $(0, \delta, 1, p)$ -symmetric.

Proof. We argue by contradiction. Assume that there exists a $\delta > 0$ such that there is a sequence of functions, $v_i \in \mathcal{A}(n, \Lambda, \alpha, M_0)$ with $\|\ln(h_i)\|_\alpha \leq 2^{-i}$ for which there exists a point, p_i with

$$N(1, p_i, v_i) - N(2^{-i}, p_i, v_i) \leq 2^{-i}$$

but that no v_i is $(0, \delta, 1, p_i)$ -symmetric.

By Lemma 3.5.1 we have that there exists a subsequence, such that $T_{p_j, 1}v_j$ converges strongly in $W_{loc}^{1,2}$ to a harmonic function, v_∞ . Therefore $N(r, p, v_\infty)$ is monotone increasing. Further, by Corollary 3.2.11 we know that $\lim_{j \rightarrow \infty} N(r, 0, T_{p_j, 1}v_j) = N(r, 0, v_\infty)$. By Lemma 3.3.12, we have that

$$N(1, 0, v_\infty) - N(0, 0, v_\infty) = 0$$

By classical results, this implies that v_∞ is a homogeneous harmonic polynomial. Thus, we arrive at our contradiction, since $T_{p_j, 1}v_j$ were assumed to stay away from all such functions in $L^2(B_1(0))$. \square

Remark 4.2.2. Since $N(r, p, v)$ is scale-invariant, Lemma 4.2.1 is also scale-invariant in the sense that if $N(r, p, v) - N(\gamma r, p, v) \leq \gamma$ and $\|\ln(h)\|_\alpha \leq \gamma$, then v is $(0, \delta, r, p)$ -symmetric.

Lemma 4.2.3. Let $v \in \mathcal{A}(\Lambda, \alpha, M_0)$ and $0 \leq k \leq n - 2$ and $p \in B_1(0) \cap \overline{\Omega^-}$. For any $0 < \epsilon$, there are constants, $0 < m = m(\epsilon, \Lambda, \alpha, M_0)$ and $0 < \delta_0(\epsilon, \Lambda, \alpha, M_0)$, such that if $\|\ln(h)\|_\alpha \leq \delta_0$ and v is not $(k + 1, \epsilon, 8r, p)$ -symmetric then for all $\rho \in [r, 8r]$,

$$N(\rho, p, v) > 1 + m$$

.

Proof. We argue by contradiction. Suppose that for a given $0 < \epsilon$, no such δ_0 and m exist. That is, suppose there is a sequence of v_i, r_i, ρ_i and p_i such that each v_i is not $(k + 1, \epsilon, 8r_i, p_i)$ -symmetric, but $\|\ln(h_i)\|_\alpha \leq 2^{-i}$ and $N(\rho_i, p_i, v_i) \leq 1 + 2^{-i}$.

We rescale. The functions $T_{p_i, 8r_i} v_i$ converge in the senses of Lemma 3.2.10 to a function v_∞ . Similarly, we may assume that $\frac{\rho_i}{8r_i} \rightarrow \rho \in [1/8, 1]$. Now, since $N(r, p, v)$ is continuous in r , for any $i = 1, 2, \dots$, we have

$$\begin{aligned} N(\rho, 0, v_\infty) &= N(\rho, 0, v_\infty) - N\left(\frac{\rho_i}{8r_i}, 0, v_\infty\right) + N\left(\frac{\rho_i}{8r_i}, 0, v_\infty\right) \\ &\quad + N\left(\frac{\rho_i}{8r_i}, 0, T_{p_i, 8r_i} v_i\right) - N\left(\frac{\rho_i}{8r_i}, 0, T_{p_i, 8r_i} v_i\right) \\ &\leq 1 + 2^{-i} + |N(\rho, 0, v_\infty) - N\left(\frac{\rho_i}{8r_i}, 0, v_\infty\right)| + |N\left(\frac{\rho_i}{8r_i}, 0, v_\infty\right) - N\left(\frac{\rho_i}{8r_i}, 0, T_{p_i, 8r_i} v_i\right)| \end{aligned}$$

Letting $i \rightarrow \infty$, we see that $N(\rho, 0, v_\infty) \leq 1$. By Lemma 3.5.1, v_∞ is harmonic, and therefore $N(0, 0, v_\infty) \geq 1$. Since for harmonic functions, the Almgren frequency is monotonic, $N(r, 0, v_\infty) = 1$ is constant for $r \in [0, \rho]$. By classical results, v_∞ is a homogeneous harmonic polynomial, and $N(1, 0, v_\infty) = 1$ implies that it is linear. This is a contradiction, because by assumption, the $T_{p_i, 8r_i} v_i$ were supposed to stay away from linear homogeneous harmonic polynomials in $L^2(B_1(0))$. Thus, such constants must exist. \square

Remark 4.2.4. By Remark 2.3.5, if $v \in \mathcal{A}(\Lambda, \alpha, M_0)$, then $\tilde{v} = (-h(0)^{-1})v \in \mathcal{A}(\Lambda, \alpha, M_0)$.

By applying all the lemmata in Sections 3.2, 3.3, 3.4, and 3.5 to \tilde{v} , we obtain these results for $p \in B_{\frac{1}{4}}(0) \cap \overline{\Omega^+}$. Hence, all the estimates in the previous lemmata hold for all $p \in B_{1/4}(0)$.

4.3 A Dichotomy

The proof technique in the rest of the paper is an adaptation of techniques developed by Naber and Valtorta in [24]. However, there are two main differences. First, the contexts of previous applications of the [24] framework have substantial differences with our context. [24] deals with stationary varifolds, so their monotonicity formula is the density, $\Theta^n(\Sigma, x, r)$. In [25], the authors consider harmonic maps and their monotonic quantity is the Dirichlet energy. The almost-monotonic quantity used in [16] is the Weiss density, because their blow-ups are all 1-homogeneous. The blow-ups in our context can be homogeneous of order $1, 2, \dots, d(M_0)$. Therefore, we need to use the Almgren frequency, which is constant on harmonic polynomials of all orders of homogeneity. Because of these differences, we cannot apply the statements from previous papers.

Secondly, in [24] the authors are not considering a free boundary problem, and [16] only prove bounds on the geometric singular set of the free boundary, $\partial\Omega^\pm$. We are proving bounds on the generalized critical set of v . This involves some complication, and in this respect Section 4.4 and Section 4.8 contain the important adaptations to the proof technique.

This section is dedicated to proving a lemma that gives us geometric information on the quantitative strata. Roughly, it says that if we can find $(k+1)$ points that are well-separated and the Almgren frequency has very small drop at these points, then the quantitative strata is contained in a neighborhood of the affine k -plane which contains them and we have control on the Almgren frequency for all points in that neighborhood. We shall use the notation $\langle y_0, \dots, y_k \rangle$ to denote the k -dimensional affine linear subspace which passes through y_0, \dots, y_k .

Lemma 4.3.1. Let $v \in \mathcal{A}(\Lambda, \alpha, M_0)$ and $0 < \epsilon$ be fixed. Let $\gamma, \eta', \rho > 0$ be fixed, then there

exist constants, $0 < \eta_0(n, \Lambda, \alpha, E_0, \epsilon, \eta', \gamma, \rho) \ll \rho$ and a $\beta(n, \Lambda, \alpha, E_0, \epsilon, \eta', \rho) < 1$ such that the following holds. If $\eta \leq \eta_0$ and,

1. $E = \sup_{p \in B_1(0)} N(2, p, v) \in [0, E_0]$.
2. There exist points $\{y_0, y_1, \dots, y_k\}$ satisfying $y_i \notin B_\rho(\langle y_0, \dots, y_{i-1}, y_{i+1}, \dots, y_k \rangle)$ and

$$N(\gamma\rho, y_i, v) \geq E - \eta_0$$

for all $i = 0, 1, \dots, k$.

3. $\|\ln(h)\|_\alpha \leq \eta$.

Then, if we denote $\langle y_0, \dots, y_k \rangle = L$, for all $p \in B_\beta(L) \cap B_1(0)$,

$$N(\gamma\rho, p, v) \geq E - \eta'$$

and

$$\mathcal{S}_{\epsilon, \eta_0}^k \cap B_1(0) \subset B_\beta(L).$$

Proof. There are two conclusions. We argue by contradiction for both. Suppose that the first claim fails. That is, assume that there exist constants $\gamma, \rho, \eta' > 0$ for which there exists a sequence of $v_i \in \mathcal{A}(n, \Lambda, \alpha, M_0)$ with $\sup_{p \in B_1(0)} N(2, p, v_i) = E_i \in [0, E_0]$ and points $\{y_{i,j}\}_j$ satisfying (2), above, with $\eta_0 < 2^{-i}$, $\|\ln(h_i)\|_\alpha \leq 2^{-i}$, and a sequence $\beta_i \leq 2^{-i}$ such that for each i , there exists a point $x_i \in B_{\beta_i}(L_i) \cap B_1(0)$ for which $N(\gamma\rho, x_i, v_i) < E - \eta'$.

By Lemma 3.5.1, there exists a subsequence v_j such that $T_{0,1}v_j$ converges to a harmonic function v_∞ . Further, by the compactness of $[0, E_0]$, $\overline{B_1(0)}$, and the Grassmannian we may assume that,

$$E_j \rightarrow E \quad y_{ij} \rightarrow y_i \quad L_j \rightarrow L \quad x_j \rightarrow x_\infty \in \overline{B_1(0)}.$$

Note that the convergence given by Lemma 3.5.1 implies,

$$\sup_{p \in B_1(0)} N(2, p, v_\infty) \leq E \quad N(\gamma\rho, x_\infty, v_\infty) < E - \eta'$$

and

$$N(0, y_i, v_\infty) \geq E$$

for all $j = 0, 1, \dots, k$. Because v_∞ is harmonic, $N(r, p, v_\infty)$ is increasing in r . Therefore, $N(r, y_i, v_\infty) = E$ for all y_i and all $r \in [0, 2]$. Classical results then imply that v_∞ is 0-symmetric in $B_2(y_j)$ for each y_j . Because the y_j are in general position, by Proposition 3.5.5, v_∞ is translation invariant along L in $B_{1+\delta}(0) \subset \cap_j B_2(y_j)$, where $\delta > 0$ depends upon the placement of the $y_j \in \overline{B_1(0)}$. Since $x_\infty \in L$, this implies that $N(0, x_\infty, v_\infty) = E$. But this contradicts $N(\gamma\rho, x_\infty, v_\infty) < E - \eta'$, since $N(r, x_\infty, v_\infty)$ must be increasing in r . This proves the first claim.

Now assume that the second claim fails. That is, fix $\beta > 0$ and assume that there is a sequence of $v_i \in \mathcal{A}(\Lambda, \alpha, M_0)$ with $\sup_{p \in B_1(0)} N(2, p, v_i) = E_i \in [0, E_0]$ and points $\{y_{i,j}\}_j$ satisfying (2), above, with $\|\ln(h_i)\|_\alpha \leq 2^{-i}$ and a sequence of $\eta_i \rightarrow 0$ such that for each i there exists a point $x_i \in \mathcal{S}_{\epsilon, \eta_i}^k(v_i) \cap B_1(0) \setminus B_\beta(L_i)$.

Again, we extract a subsequence, as above. The function v_∞ will be harmonic and k -symmetric in $B_{1+\delta}(0)$. And, $x_i \rightarrow x \in \overline{B_1(0)} \setminus B_\beta(L)$. Note that by our definition of $\mathcal{S}_{\epsilon, \eta_i}^k(v_i)$ and Lemma 3.5.1, $x \in \mathcal{S}_{\epsilon/2}^k(v_\infty)$.

Since v_∞ is k -symmetric in $B_{1+\delta}(0)$, every blow-up at a point in $\overline{B_1(0)}$ will be $(k+1)$ -symmetric. Thus, there must exist a radius, r for which v_∞ is $(k+1, \epsilon/4, r, x)$ -symmetric. This contradicts the conclusion that $x \in \mathcal{S}_{\epsilon/2}^k(v_\infty)$. \square

Consider the following dichotomy: either we can find well-separated $(k+1)$ points, y_{ij} , with very small drop in frequency or we cannot. In the former case, Lemma 4.3.1 implies that the Almgren frequency has small drop on all of $\mathcal{S}_{\epsilon, \eta}^k(v)$ (and we also get good geometric

control). In the later case, the set on which the Almgren frequency has small drop is close to a $(k-1)$ -plane. In this latter case, even though we have no geometric control on $\mathcal{S}_{\epsilon, \eta}^k(v)$, we have very good packing control on the part with small drop in frequency. We make this formal in the following corollary.

Corollary 4.3.2. (Key Dichotomy) Let $\gamma, \rho, \eta' \in (0, 1)$ and $0 < \epsilon$ be fixed. There is an $\eta_0 = \eta_0(n, \Lambda, \alpha, E_0, \epsilon, \eta', \gamma, \rho) \ll \rho$ so that the following holds. For all $v \in \mathcal{A}(\Lambda, \alpha, M_0)$, with $\sup_{p \in B_1(0)} N(2, p, v) \leq E \in [0, E_0]$, if $\eta \leq \eta_0$ and $\|\ln(h)\|_\alpha \leq \eta$, then one of the following possibilities must occur:

1. $N(\gamma\rho, p, v) \geq E - \eta'$ on $S_{\epsilon, \eta_0}^k(v) \cap B_1(0)$, and

$$\mathcal{S}_{\epsilon, \eta_0}^k \cap B_1(0) \subset B_\beta(L).$$

2. There exists a $(k-1)$ -dimensional affine plane, L^{k-1} , such that

$$\{p : N(2\eta, p, v) \geq E - \eta_0\} \cap B_1(0) \subset B_\rho(L^{k-1}).$$

Remark 4.3.3. The former case is simply the conclusion of Lemma 4.3.1. In the later case of the dichotomy, we know that all points in $B_1(0) \setminus B_\rho(L^{k-1})$ must have $N(2\eta, p, v) < E - \eta_0$. Since $N(r, p, v)$ is almost monotonic and uniformly bounded, this can only happen for each p finitely many times.

4.4 Beta numbers

This next section is devoted to proving Lemma 4.4.1, which relates the drop in the Almgren frequency to the Jones β -numbers. Throughout this section, we shall use the notation $A_{r,R}(x) = B_R(x) \setminus B_r(x)$.

Lemma 4.4.1. There exists a constant, $\delta = \delta_0(n, \Lambda, \alpha, M_0, \epsilon) > 0$ such that if $v \in \mathcal{A}(\Lambda, \alpha, M_0)$ with $\|\ln(h)\|_\alpha \leq \delta_0$, then for any $p \in B_{\frac{1}{2}}(0)$ such that if v is $(0, \delta, 8r, p)$ -symmetric, but not

$(k + 1, \epsilon, 8r, p)$ -symmetric and $\|\ln(h)\|_\alpha \leq \delta_0$, then for any finite Borel measure, μ ,

$$\begin{aligned} \beta_{\mu,2}^k(p, r)^2 &\leq \frac{C(n, \Lambda, \alpha, M_0, \epsilon)}{r^k} \int_{B_r(p)} N(8r, y, v) - N(r, y, v) + C_1 \|\ln(h)\|_\alpha r^\alpha d\mu(y) \\ &\quad + C(n, \Lambda, \alpha, M_0, \epsilon) \frac{\mu(B_r(p))}{r^k} r^m \end{aligned} \quad (4.4.1)$$

where $0 < m$ is the constant defined in Lemma 4.2.3 and $C_1 = C_1(n, \Lambda, M_0)$ is the constant from Lemma 3.3.11.

In order to prove this statement for a finite Borel measure whose support is not contained in $\partial\Omega^\pm$, we encounter significant complications. Indeed, if we restrict to finite Borel measures supported on $\partial\Omega^\pm$, the proof techniques of [24] go through verbatim. However, in order to obtain estimates on the generalized critical set, we must do some work. Due to its complicated nature, we break the proof of Lemma 4.4.1 into the subsequent 10 lemmata.

We begin by noting that for any finite Borel measure, μ , and any $B_r(p)$ we can define the μ center of mass, $X = \int_{B_r(p)} x d\mu(x)$, and define a symmetric, non-negative bilinear form,

$$Q(v, w) = \int_{B_r(p)} (v \cdot (y - X))(w \cdot (y - X)) d\mu(y).$$

Let $\vec{v}_1, \dots, \vec{v}_n$ be an orthonormal eigenbasis and $\lambda_1 \geq \dots \geq \lambda_n \geq 0$ their associated eigenvalues. These objects enjoy the following relationships,

$$V_{\mu,2}^k(p, r) = X + \text{span}\{\vec{v}_1, \dots, \vec{v}_k\}, \quad \beta_{\mu,2}^k(x, r)^2 = \frac{\mu(B_r(p))}{r^k} (\lambda_{k+1} + \dots + \lambda_n). \quad (4.4.2)$$

See [24] Definition 7.3 and Lemma 7.4.

Lemma 4.4.2. Let $v \in \mathcal{A}(\Lambda, \alpha, M_0)$. Let μ be a finite Borel measure and Q, λ_i, \vec{v}_i defined as

above. For any i , any z for which $\nabla T_{0,1}v(z)$ is defined, and any scalar $c \in \mathbb{R}$,

$$\begin{aligned} & \lambda_i \frac{1}{r^{n+2}} \int_{A_{3r,4r}(p)} (\vec{v}_i \cdot \nabla T_{0,1}v(z))^2 dz \\ & \leq 5^n \int_{B_r(p)} \int_{A_{2r,7r}(y)} \frac{|c(T_{0,1}v(z) - T_{0,1}v(p)) - \nabla T_{0,1}v(z) \cdot (z - y)|^2}{|z - y|^{n+2}} dz d\mu(y). \end{aligned} \quad (4.4.3)$$

Proof. Observe that by the definition of center of mass,

$$\int_{B_r(p)} \vec{w} \cdot (y - X) d\mu(y) = 0$$

for any $\vec{w} \in \mathbb{R}^n$. Therefore, for any constant $c \in \mathbb{R}$,

$$\begin{aligned} \lambda_i (\vec{v}_i \cdot \nabla T_{0,1}v(z)) &= Q(\vec{v}_i, \nabla T_{0,1}v(z)) \\ &= \int_{B_r(p)} (\vec{v}_i \cdot (y - X)) (\nabla T_{0,1}v(z) \cdot (y - X)) d\mu(y) \\ &= \int_{B_r(p)} (\vec{v}_i \cdot (y - X)) (\nabla T_{0,1}v(z) \cdot (y - X)) d\mu(y) \\ &\quad + \int_{B_r(p)} c(T_{0,1}v(z) - T_{0,1}v(p)) (\vec{v}_i \cdot (y - X)) d\mu(y) \\ &= \int_{B_r(p)} (\vec{v}_i \cdot (y - X)) (c(T_{0,1}v(z) - T_{0,1}v(p)) - \nabla T_{0,1}v(z) \cdot (X - z + z - y)) d\mu(y) \\ &= \int_{B_r(p)} (\vec{v}_i \cdot (y - X)) (c(T_{0,1}v(z) - T_{0,1}v(p)) - \nabla T_{0,1}v(z) \cdot (z - y)) d\mu(y) \\ &\leq \lambda_i^{\frac{1}{2}} \left(\int_{B_r(p)} |c(T_{0,1}v(z) - T_{0,1}v(p)) - \nabla T_{0,1}v(z) \cdot (z - y)|^2 d\mu(y) \right)^{\frac{1}{2}}. \end{aligned}$$

Recalling that $A_{r,R}(p) = B_R(p) \setminus B_r(p)$, we calculate,

$$\begin{aligned}
& \lambda_i \frac{1}{r^{n+2}} \int_{A_{3r,4r}(p)} (\vec{v}_i \cdot \nabla T_{0,1}v(z))^2 dz \leq \\
& \frac{1}{r^{n+2}} \int_{A_{3r,4r}(p)} \int_{B_r(p)} |c(T_{0,1}v(z) - T_{0,1}v(p)) - \nabla T_{0,1}v(z) \cdot (z - y)|^2 d\mu(y) dz \\
& \leq 5^n \int_{B_r(p)} \int_{A_{3r,4r}(p)} \frac{|c(T_{0,1}v(z) - T_{0,1}v(p)) - \nabla T_{0,1}v(z) \cdot (z - y)|^2}{|z - y|^{n+2}} dz d\mu(y) \\
& \leq 5^n \int_{B_r(p)} \int_{A_{2r,7r}(y)} \frac{|c(T_{0,1}v(z) - T_{0,1}v(p)) - \nabla T_{0,1}v(z) \cdot (z - y)|^2}{|z - y|^{n+2}} dz d\mu(y)
\end{aligned}$$

□

Lemma 4.4.3. Let $v \in \mathcal{A}(\Lambda, \alpha, M_0)$, $0 \leq k \leq n - 2$, and $p \in B_1(0)$. Let $0 < \epsilon$ be fixed and $0 < r < 1/2$. There is a $0 < \delta_0(\epsilon, \Lambda, M_0, \alpha)$ such that if $0 < \delta \leq \delta_0$ and v is $(0, \delta)$ -symmetric, but not $(k + 1, \epsilon)$ -symmetric in $B_{8r}(p)$, and $\|\ln(h)\|_\alpha \leq \delta_0$ then there exists a constant, $C = C(n, \Lambda, \alpha, M_0, \epsilon)$, such that for all $\rho \in [r, 8r]$,

$$\int_{\partial B_\rho(p)} (T_{0,1}v(x) - T_{0,1}v(p))^2 d\sigma(x) \leq C\rho^{2(1+\frac{m}{2})} \tag{4.4.4}$$

where $0 < m = m(\epsilon, \Lambda, \alpha, M_0)$ is the constant from Lemma 4.2.3.

Proof. Let $\delta \leq \delta_0$, where $\delta_0 = \delta_0(\epsilon, \Lambda, \alpha, M_0)$ is as in Lemma 4.2.3. Therefore, we have that

$N(\rho, p, v) > 1 + m$. Plugging this into Lemma 3.4.3, we have that, for any $S > \rho$,

$$\begin{aligned} \int_{\partial B_S(p)} (T_{0,1}v(x) - T_{0,1}v(p))^2 d\sigma(x) &= H(S, p, T_{0,1}v) \\ &\geq \left(\frac{S}{\rho}\right)^{(n-1)+2(N(\rho,p,v)-CS^\alpha)} e^{\frac{2C}{\alpha}[S^\alpha-\rho^\alpha]} H(\rho, p, T_{0,1}v) \\ &\geq \left(\frac{S}{\rho}\right)^{(n-1)+2(1+m-CS^\alpha)} e^{\frac{2C}{\alpha}[S^\alpha-\rho^\alpha]} H(\rho, p, T_{0,1}v) \end{aligned}$$

where $C = \|\ln(h)\|_\alpha C(\alpha, M_0, \delta_0)$. Let $0 < r_0 = r_0(\epsilon, \Lambda, \alpha, M_0)$ be small enough that

$$m - Cr_0^\alpha \geq \frac{m}{2}.$$

Thus, if $S = r_0 \geq \rho$ we have by the Lipschitz bounds from Lemma 3.2.1,

$$\begin{aligned} C(\epsilon, \Lambda, \alpha, M_0)r_0^2 &\geq \frac{1}{r_0^{n-1}} H(r_0, p, T_{0,1}v) \\ &\left(\frac{r_0}{\rho}\right)^{2(1+\frac{m}{2})} e^{\frac{2C}{\alpha}[r_0^\alpha-\rho^\alpha]} \frac{1}{\rho^{n-1}} H(\rho, p, T_{0,1}v) \end{aligned}$$

Absorbing all of the constants to one side, we have the following inequality.

$$(\rho)^{2(1+\frac{m}{2})} C(\epsilon, \Lambda, \alpha, M_0) \geq \frac{1}{\rho^{n-1}} H(\rho, p, v).$$

If, on the other hand, $\rho > r_0$, we let $C(\epsilon, \Lambda, \alpha, M_0) = \frac{4C(\epsilon, \Lambda, \alpha, M_0)^2}{r_0^{\frac{m}{2}}}$, where $C(\epsilon, \Lambda, \alpha, M_0)$ is

the uniform Lipschitz constant from Lemma 3.2.1.

$$\int_{\partial B_\rho(p)} (T_{0,1}v(x) - T_{0,1}v(p))^2 d\sigma(x) \leq 4C(\epsilon, \Lambda, \alpha, M_0)^2 \rho^2 \leq \frac{4C(\epsilon, \Lambda, \alpha, M_0)^2}{r_0^{\frac{2m}{2}}} \rho^{2(1+\frac{m}{2})}$$

Taking the maximum of these two constants gives the desired result. \square

Lemma 4.4.4. Let $v \in \mathcal{A}(\Lambda, \alpha, M_0)$ and $0 \leq k \leq n-2$. Let $p \in B_{\frac{1}{4}}(0)$, $0 < r \leq \frac{1}{18}$. Let $0 < \epsilon$ be fixed. There is a $0 < \delta_0(\Lambda, \alpha, M_0, \epsilon)$ such that if $0 < \delta \leq \delta_0$ and v is $(0, \delta)$ -symmetric in $B_{8r}(p)$, but not $(k+1, \epsilon, 8r, p)$ -symmetric, and $\|\ln(h)\|_\alpha \leq \delta_0$, then there is a constant, $C = C(n, \Lambda, \alpha, M_0, \epsilon)$ such that for all $\rho \in [r, 8r]$ and all $y \in B_\rho(p)$,

$$|T_{0,1}v(y) - T_{0,1}v(p)| \leq C\rho^{1+m/2}.$$

Proof. We argue by contradiction. Suppose that there exist a sequence of $v_i \in \mathcal{A}(\Lambda, \alpha, M_0)$, with points, $p_i \in B_1(0)$, and radii, $0 < r_i \leq 1$, such that v_i satisfies $\|\ln(h_i)\|_\alpha \leq 2^{-i}$, v_i is $(0, 2^{-i})$ -symmetric in $B_{8r_i}(p_i)$, and v_i is not $(k+1, \epsilon, 8r_i, p_i)$ -symmetric, but for which there exist $\rho_i \in [r_i, 8r_i]$ and $y_i \in \partial B_{\rho_i}(p_i)$ such that,

$$|T_{0,1}v_i(y_i) - T_{0,1}v_i(p_i)| \geq i\rho_i^{1+m/2}.$$

We consider $T_{p_i, r_i}v_i$. The functions $T_{p_i, r_i}v_i$ are locally uniformly Lipschitz by Lemma 3.2.1, the function v_∞ is locally Lipschitz with the same Lipschitz constants. Further, $v_\infty = T_{0,1}v_\infty$ is not $(k+1, \epsilon, 8, 0)$ -symmetric. However, by assumption,

$$\begin{aligned} |T_{0,1}v_i(y_i) - T_{0,1}v_i(p_i)| &= \left(\int_{\partial B_1(0)} (T_{0,1}v_i(xr_i + p_i) - T_{0,1}v_i(p_i))^2 d\sigma(x) \right)^{\frac{1}{2}} |T_{p_i, r_i}v_i\left(\frac{y_i - p_i}{r_i}\right)| \\ &\geq i\rho_i^{1+m/2}. \end{aligned}$$

Now, by Lemma 4.4.3 for i sufficiently large, we can bound

$$\left(\int_{\partial B_1(0)} (T_{0,1}v_i(xr_i + p_i) - T_{0,1}v_i(p_i))^2 d\sigma(x) \right)^{\frac{1}{2}} \leq C\rho_i^{1+m/2}.$$

Therefore,

$$C|T_{p_i, r_i}v_i\left(\frac{y_i - p_i}{r_i}\right)| \geq i.$$

Since $\frac{y_i - p_i}{r_i} \in \partial B_{\rho_i/r_i}(0)$, and $\rho_i/r_i \in [1, 8]$ this contradicts Corollary 3.4.5 which states that the $T_{p_i, r_i}v_i$ are uniformly Lipschitz in $B_8(0)$. Thus, such a constant must exist. \square

Now we investigate the quantities $\lambda(p, r, v)$. Recall that for $v \in \mathcal{A}(\Lambda, \alpha, M_0)$, $p \in B_1(0)$ and $0 < r \leq 1$ we define,

$$\lambda(p, r, v_\epsilon) = \frac{\int_{\partial B_r(p)} (v_\epsilon(y) - v_\epsilon(p)) \nabla v_\epsilon(y) \cdot (y - p) d\sigma(y)}{H(r, p, v_\epsilon)}$$

$$\lambda(p, r, v) = \lim_{\epsilon \rightarrow 0} \lambda(p, r, v_\epsilon)$$

Note that since v is merely Lipschitz, $\lambda(p, r, v)$ may not be defined directly for all admissible p and r . However, since ∇v is defined in $\mathbb{R}^n \setminus \partial\Omega^\pm$, $\lambda(p, r, v)$ can be defined for $p \in B_1(0)$, $0 < r \leq 1$ such that $\partial B_r(p) \cap \partial\Omega^\pm$ has zero surface measure by its almost everywhere values.

Lemma 4.4.5. There exists a constant, $C(n, D) < \infty$ such that if P is a homogeneous harmonic polynomial of degree $\leq D$ and $H(1, 0, P) = 1$, then for all $p \in B_{\frac{1}{8}}(0)$ and all $r \in [\frac{1}{8}, 1]$

$$|\lambda(p, r, P)| \leq C.$$

Proof. Now, for any homogeneous harmonic polynomial, P , $\lambda(y, r, P)$ is continuous in y and $\overline{B_{\frac{1}{8}}(0)}$ is compact. Furthermore, $\lambda(y, r, P)$ is continuous in r and $[\frac{1}{8}, 1]$ is compact. Therefore, for each homogeneous harmonic polynomial, there exists a constant such that for

all $y \in B_{\frac{1}{8}}(0)$ and all $r \in [\frac{1}{8}, 1]$

$$|\lambda(y, r, P)| \leq C(P).$$

Furthermore, $\lambda(y, r, P)$ is continuous in P , where P ranges among homogeneous harmonic polynomials of degree $\leq D$. More precisely, we note that if P_i is a sequence of homogeneous harmonic polynomials and $P_i \rightarrow P_\infty$ in $L^2(B_2(0))$, then $P_i \rightarrow P$ in $C^\infty(B_{3/2}(0))$. This implies $\lambda(y_i, r_i, P_i) \rightarrow \lambda(y_\infty, r_\infty, P_\infty)$. Since the collection of homogeneous harmonic polynomials of degree $\leq D$ is a finite-dimensional vector space, the collection of homogeneous harmonic polynomials of degree $\leq D$ with $H(1, 0, P) = 1$, is compact. Therefore, we take $C(n, D)$ to be the maximum of $|\lambda(y, r, P)|$ among all P, r, y as described by the hypotheses of the lemma.

□

Lemma 4.4.6. Let $v_i \in \mathcal{A}(\Lambda, \alpha, M_0)$ and $x_i \in B_{1/4}(0)$ and $0 < r_i \leq 1/32$. Let $\rho_i \in [r_i, 8r_i]$. If v_i is $(0, 2^{-i})$ -symmetric in $B_{8r_i}(p_i)$ and $\|\ln(h_i)\|_\alpha < 2^{-i}$, then we may extract a subsequence such that,

$$T_{p_i, 8r_i} v_i \rightarrow v_\infty \quad T_{p_i, 8r_i} y_i \rightarrow y \quad \frac{\rho_i}{r_i} \rightarrow \rho,$$

and

$$\lambda(y_i, \rho_i, v_i) \rightarrow \lambda(y, \rho, v_\infty),$$

where convergence of the functions is in the sense of Lemma 3.5.1.

We defer the proof to Section 6.2. We are now ready to prove a quantitative rigidity result for Lemma 4.4.5.

Lemma 4.4.7. Let $v \in \mathcal{A}(\Lambda, \alpha, M_0)$ and $x \in B_{1/4}(0)$. There exists a constant, $0 < \delta_0(n, \Lambda, \alpha, M_0)$,

such that if $0 < \delta \leq \delta_0$ then for any $0 < r < \frac{1}{4}$, if v is $(0, \delta)$ -symmetric in $B_{8r}(p)$ and $\|\ln(h)\|_\alpha < \delta$, then for all $y \in B_r(p)$ and every $\rho \in [2r, 7r]$ for which $\lambda(y, \rho, v)$ is defined,

$$|\lambda(y, \rho, v)| \leq 2C,$$

where $C = C(n, \Lambda, \alpha, M_0)$ is $C(D)$ from Lemma 4.4.5 with $D = C(n, \Lambda, \alpha, M_0)$ from Lemma 3.4.4.

Proof. We argue by contradiction. Suppose that there is a sequence of $v_i \in \mathcal{A}(\Lambda, \alpha, M_0)$, points, $p_i \in B_{1/4}(0)$, radii, $0 < r_i < \frac{1}{4}$, such that v_i is $(0, 2^{-i})$ -symmetric in $B_{8r_i}(p_i)$ with $\|\ln(h_i)\|_\alpha \leq 2^{-i}$, but for which there exist points, $y_i \in B_{r_i}(p_i)$ and radii, $r'_i \in [r_i, 8r_i]$ for which,

$$|\lambda(y_i, r'_i, v_i)| \geq 2C.$$

We rescale to $T_{p_i, 8r_i}v_i$. By Lemma 3.2.10, we can extract a subsequence which converges to a limit function, v_∞ , in $C_{loc}(\mathbb{R}^n)$ and strongly in $W_{loc}^{1,2}(\mathbb{R}^n)$. By Lemma 3.5.1, v_∞ is harmonic. Because v_∞ is 0-symmetric, it is a homogeneous harmonic polynomial. By Lemma 3.4.4, v_∞ is of degree $\leq C(n, \alpha, M_0, \Lambda)$. Therefore, by Lemma 4.4.5 there is a constant, C , such that for all $y \in B_{\frac{1}{8}}(0)$ and all $r \in [\frac{1}{8}, 1]$

$$|\lambda(y, r, P)| \leq C.$$

Note that by assumption, for each $i = 1, 2, \dots$ there exists a point, $y'_i \in B_{\frac{1}{8}}(0)$ and a radius, $\tilde{r}_i \in [\frac{2}{8}, \frac{7}{8}]$ such that,

$$|\lambda(y'_i, \tilde{r}_i, T_{p_i, 8r_i}v_i)| \geq 2C.$$

Note that because $\overline{B_{\frac{1}{8}}(0)} \times [\frac{2}{8}, \frac{7}{8}]$ is compact, we may assume that $y'_i \rightarrow y_\infty$ and $\tilde{r}_i \rightarrow \tilde{r}$. In order to obtain a contradiction, we now argue that there exists a subsequence such that $\lim_{j \rightarrow \infty} \lambda(y'_i, \tilde{r}_i, T_{p_j, 8r_j} v_j) = \lambda(y_\infty, \tilde{r}, v_\infty)$. However, this is exactly Lemma 4.4.6. Therefore, we have the contradiction we desired. \square

Lemma 4.4.8. Let $v \in \mathcal{A}(\Lambda, \alpha, M_0)$ and $0 \leq k \leq n-2$. Let $p \in B_{1/4}(0)$, $0 < r \leq 1$. Let $0 < \epsilon$ be fixed. There is a $0 < \delta_0(\Lambda, \alpha, M_0, \epsilon)$ such that if v is $(0, \delta)$ -symmetric in $B_{8r}(p)$, but not $(k+1, \epsilon, 8r, p)$ -symmetric, and if $\|\ln(h)\|_\alpha \leq \delta_0$ for any $0 < \delta \leq \delta_0$, then for any $y \in B_r(p)$,

$$\int_{A_{2r, 7r}(y)} \frac{|\lambda(p, 7r, T_{0,1}v)(T_{0,1}v(z) - T_{0,1}v(p)) - \nabla T_{0,1}v(z) \cdot (z - y)|^2}{|z - y|^{n+2}} dz \leq 4 \int_{A_{2r, 7r}(y)} \frac{|\lambda(y, |z - y|, T_{0,1}v)(T_{0,1}v(z) - T_{0,1}v(y)) - \nabla T_{0,1}v(z) \cdot (z - y)|^2}{|z - y|^{n+2}} dz + Cr^m$$

where $C = C(n, \Lambda, \alpha, M_0)$.

Proof. First, we observe that,

$$\begin{aligned} \lambda(p, 7r, T_{0,1}v) &= \lambda(y, |z - y|, T_{0,1}v) + [\lambda(p, 7r, T_{0,1}v) - \lambda(p, r, T_{0,1}v)] \\ &\quad + [\lambda(p, |z - y|, T_{0,1}v) - \lambda(y, |z - y|, T_{0,1}v)] \end{aligned}$$

Making δ_0 small as in Lemma 4.4.7, we have that,

$$|\lambda(p, 7r, T_{0,1}v) - \lambda(y, |z - y|, T_{0,1}v)| \leq 8C.$$

Now, we bound the maximum of the difference. First, we change from the constant, $\lambda(p, 7r, T_{0,1}v)$, to the $\lambda(y, |z - y|, T_{0,1}v)$. Note that,

$$\begin{aligned}
& |\lambda(p, 7r, T_{0,1}v)(T_{0,1}v(z) - T_{0,1}v(p)) - \nabla T_{0,1}v(z) \cdot (z - y)|^2 \\
& \leq 2|(\lambda(p, 7r, T_{0,1}v) - \lambda(y, |z - y|, T_{0,1}v))(T_{0,1}v(z) - T_{0,1}v(p))|^2 \\
& \quad + 2|(\lambda(y, |z - y|, T_{0,1}v))(T_{0,1}v(z) - T_{0,1}v(p)) - \nabla T_{0,1}v(z) \cdot (z - y)|^2.
\end{aligned}$$

Therefore, we estimate,

$$\begin{aligned}
& \int_{A_{2r,7r}(y)} \frac{|\lambda(p, 7r, T_{0,1}v)(T_{0,1}v(z) - T_{0,1}v(p)) - \nabla T_{0,1}v(z) \cdot (z - y)|^2}{|z - y|^{n+2}} dz \\
& \leq \int_{A_{2r,7r}(y)} \frac{2|(\lambda(p, 7r, T_{0,1}v) - \lambda(y, |z - y|, T_{0,1}v))(T_{0,1}v(z) - T_{0,1}v(p))|^2}{|z - y|^{n+2}} dz \\
& \quad + \int_{A_{2r,7r}(y)} \frac{2|(\lambda(y, |z - y|, T_{0,1}v))(T_{0,1}v(z) - T_{0,1}v(p)) - \nabla T_{0,1}v(z) \cdot (z - y)|^2}{|z - y|^{n+2}} dz \\
& \leq \int_{A_{r,8r}(p)} \frac{128|C(T_{0,1}v(z) - T_{0,1}v(p))|^2}{|z - y|^{n+2}} dz \\
& \quad + \int_{A_{2r,7r}(y)} \frac{2|(\lambda(y, |z - y|, T_{0,1}v))(T_{0,1}v(z) - T_{0,1}v(p)) - \nabla T_{0,1}v(z) \cdot (z - y)|^2}{|z - y|^{n+2}} dz
\end{aligned}$$

Now, by Lemma 4.4.4, for $0 < \delta$ small enough, we have that for every $\rho \in [r, 8r]$ and all $z \in B_\rho(p)$, we have the estimate, $|(T_{0,1}v(z) - T_{0,1}v(p))| \leq Cr^{1+\frac{m}{2}}$. Therefore,

$$\begin{aligned}
\int_{A_{r,8r}(p)} \frac{32|C(T_{0,1}v(z) - T_{0,1}v(p))|^2}{|z - y|^{n+2}} dz & \leq \int_{A_{r,8r}(p)} \frac{128C^2(r^{1+\frac{m}{2}})^2}{|z - y|^{n+2}} dz \\
& \leq C(n, \Lambda, \alpha, M_0, \epsilon)r^m
\end{aligned}$$

Secondly, we change from $(T_{0,1}v(z) - T_{0,1}v(p))$ to $(T_{0,1}v(z) - T_{0,1}v(y))$. Note that,

$$\begin{aligned}
& |\lambda(y, |z - y|, T_{0,1}v)(T_{0,1}v(z) - T_{0,1}v(p)) - \nabla T_{0,1}v(z) \cdot (z - y)|^2 \\
&= |\lambda(y, |z - y|, T_{0,1}v)(T_{0,1}v(z) - T_{0,1}v(y) + T_{0,1}v(y) - T_{0,1}v(p)) - \nabla T_{0,1}v(z) \cdot (z - y)|^2 \\
&\leq 2|\lambda(y, |z - y|, T_{0,1}v)(T_{0,1}v(y) - T_{0,1}v(p))|^2 \\
&\quad + 2|\lambda(y, |z - y|, T_{0,1}v)(T_{0,1}v(z) - T_{0,1}v(y)) - \nabla T_{0,1}v(z) \cdot (z - y)|^2
\end{aligned}$$

Therefore, we estimate,

$$\begin{aligned}
& \int_{A_{2r,7r}(y)} \frac{2|\lambda(y, |z - y|, T_{0,1}v)(T_{0,1}v(z) - T_{0,1}v(p)) - \nabla T_{0,1}v(z) \cdot (z - y)|^2}{|z - y|^{n+2}} dz \\
&\leq \int_{A_{2r,7r}(y)} \frac{4|\lambda(y, |z - y|, T_{0,1}v)(T_{0,1}v(y) - T_{0,1}v(p))|^2}{|z - y|^{n+2}} dz \\
&\quad + \int_{A_{2r,7r}(y)} \frac{4|\lambda(y, |z - y|, T_{0,1}v)(T_{0,1}v(z) - T_{0,1}v(y)) - \nabla T_{0,1}v(z) \cdot (z - y)|^2}{|z - y|^{n+2}} dz \\
&\leq \int_{A_{r,8r}(p)} \frac{16|C(T_{0,1}v(y) - T_{0,1}v(p))|^2}{|z - y|^{n+2}} dz \\
&\quad + \int_{A_{r,8r}(y)} \frac{4|\lambda(y, |z - y|, T_{0,1}v)(T_{0,1}v(z) - T_{0,1}v(y)) - \nabla T_{0,1}v(z) \cdot (z - y)|^2}{|z - y|^{n+2}} dz
\end{aligned}$$

Now, we must upper bound $\int_{A_{r,8r}(p)} \frac{4|2C(T_{0,1}v(y) - T_{0,1}v(p))|^2}{|z - y|^{n+2}} dz$. We do so by bounding

$$|T_{0,1}v(y) - T_{0,1}v(p)|^2.$$

By Lemma 4.4.4, for $0 < \delta$ small enough, we have that for every $\rho \in [r, 8r]$ and all $z \in \partial B_\rho(p)$,

$$|(T_{0,1}v(z) - T_{0,1}v(p))| \leq Cr^{1+\frac{m}{2}}.$$

We argue by cases. Suppose that $y \in \partial\Omega^\pm$ or $\partial\Omega^\pm$ does not separate y and p . Then,

$|(T_{0,1}v(y) - T_{0,1}v(p))| \leq |(T_{0,1}v(z) - T_{0,1}v(p))|$ for any $z \in \partial B_\rho(p)$ such that $\partial\Omega^\pm$ separates z and p . If $\partial\Omega^\pm$ does separate p and y , then suppose that $p, y \in \Omega^+$. The Maximum Principle applied to the function $(T_{0,1}v(z) - T_{0,1}v(p))$ in $B_r(p) \cap \Omega^+$ implies that $|T_{0,1}v(y) - T_{0,1}v(p)| \leq Cr^{1+\frac{m}{2}}$. The identical argument for $p, y \in \Omega^-$ shows that the bound holds in that case, as well. Therefore,

$$\begin{aligned} & \int_{A_{r,8r}(p)} \frac{16|C(T_{0,1}v(y) - T_{0,1}v(p))|^2}{|z - y|^{n+2}} dz \\ & \leq \int_{A_{r,8r}(p)} \frac{16C^2(r^{1+\frac{m}{2}})^2}{|z - y|^{n+2}} dz \\ & \leq C(n, \Lambda, \alpha, M_0, \epsilon)r^m \end{aligned}$$

□

Lemma 4.4.9. Fix $0 < \epsilon$. There exists a constant, $\delta = \delta_0(n, \Lambda, \alpha, M_0, \epsilon) > 0$ and a constant, $C(n, \Lambda, \alpha, M_0, \epsilon)$, such that if $v \in \mathcal{A}(\Lambda, \alpha, M_0)$ with $\|\ln(h)\|_\alpha \leq \delta_0$, then for any $p \in B_{\frac{1}{4}}(0)$ and $0 < r \leq \frac{1}{18}$ such that if v is $(0, \delta, 8r, p)$ -symmetric, but not $(k+1, \epsilon, 8r, p)$ -symmetric, then for any orthonormal vectors, $\vec{v}_1, \dots, \vec{v}_{k+1}$,

$$\frac{1}{C} \leq \frac{1}{r^{n+2}} \int_{A_{3r,4r}(x)} \sum_{i=1}^{k+1} (\vec{v}_i \cdot Dv(z))^2 dz.$$

Proof. Since the equation is scale invariant, we argue for $B_1(0)$. Again, we argue by contradiction. Assume that there is a sequence of functions in $v_i \in \mathcal{A}(\Lambda, \alpha, M_0)$ with $\|\ln(h_i)\|_\alpha \leq 2^{-i}$ such that v_j is $(0, 2^{-j}, 8, 0)$ -symmetric, but not $(k+1, \epsilon, 8, 0)$ -symmetric. And, for each i , there exists an orthonormal collection of vectors, $\{\vec{v}_{ij}\}$, such that,

$$\int_{A_{3,4}(0)} \sum_{j=1}^{k+1} (\vec{v}_{ij} \cdot Dv(z))^2 dz \leq 2^{-i}.$$

Again, we use Lemma 3.5.1 to extract a subsequence v_j for which $T_{0,1}v_j$ converge to a harmonic function, v_∞ strongly in $L^2_{loc}(\mathbb{R}^n)$. Similarly, $\{\vec{v}_{ij}\}$ converges to an orthonormal collection $\{\vec{v}_i\}$. Given the assumptions above, v_∞ is also 0-symmetric in $B_8(0)$ and $\nabla v_\infty \cdot \vec{v}_i = 0$ for all $i = 1, \dots, k+1$. Thus, v_∞ is $(k+1)$ -symmetric in $B_8(0)$. But this is our contradiction, since v_j were supposed to stay away from $(k+1)$ -symmetric functions in $L^2(B_1(0))$. \square

4.4.1 The proof of Lemma 4.4.1

Proof. By Lemma 4.4.9 and properties of the β -numbers, we have for $\{\vec{v}_i\}$ the orthonormal basis and λ_i the associated eigenvalues of the quadratic form in Lemma 4.4.2, by (4.4.2),

$$\begin{aligned} \beta_{\mu,2}^k(p,r)^2 &\leq \frac{\mu(B_r(p))}{r^k} n \lambda_{k+1} \\ &\leq \frac{\mu(B_r(p))}{r^k} n C(n, \Lambda, \alpha, M_0, \epsilon) \sum_{i=1}^{k+1} \frac{\lambda_{k+1}}{r^{n+2}} \int_{A_{3r,4r}(p)} (\vec{v}_i \cdot Dv(z))^2 dz. \\ &\leq \frac{\mu(B_r(p))}{r^k} n C(n, \Lambda, \alpha, M_0, \epsilon) \sum_{i=1}^{k+1} \frac{\lambda_i}{r^{n+2}} \int_{A_{3r,4r}(p)} (\vec{v}_i \cdot Dv(z))^2 dz. \end{aligned}$$

We now bound $\frac{\lambda_i}{r^{n+2}} \int_{A_{3r,4r}(p)} (\vec{v}_i \cdot Dv(z))^2 dz$ using Lemma 4.4.2. By (4.4.3),

$$\begin{aligned} &\lambda_i \frac{1}{r^{n+2}} \int_{A_{3r,4r}(p)} (\vec{v}_i \cdot \nabla T_{0,1}v(z))^2 dz \leq \\ &5^n \int_{B_r(p)} \int_{A_{2r,7r}(y)} \frac{|\lambda(p, 7r, T_{0,1}v)(T_{0,1}v(z) - T_{0,1}v(p)) - \nabla T_{0,1}v(z) \cdot (z - y)|^2}{|z - y|^{n+2}} dz d\mu(y). \end{aligned}$$

By Lemma 4.4.7 and Lemma 4.4.8, we have that for $0 < \delta$ sufficiently small, all $y \in B_r(p)$

we can bound,

$$\begin{aligned} & \int_{A_{2r,7r}(y)} \frac{|\lambda(p, 7r, T_{0,1}v)(T_{0,1}v(z) - T_{0,1}v(p)) - \nabla T_{0,1}v(z) \cdot (z - y)|^2}{|z - y|^{n+2}} dz \\ & \leq 4 \int_{A_{r,8r}(y)} \frac{|\lambda(y, |z - y|, T_{0,1}v)(T_{0,1}v(z) - T_{0,1}v(y)) - \nabla T_{0,1}v(z) \cdot (z - y)|^2}{|z - y|^{n+2}} dz + Cr^m \end{aligned}$$

Therefore, collecting constants we have that for δ sufficiently small,

$$\begin{aligned} \beta_{\mu,2}^k(p, r)^2 & \leq \frac{\mu(B_r(p))}{r^k} nC(n, \Lambda, \alpha, M_0, \epsilon)(k+1)5^n \times \\ & \left(\int_{B_r(p)} 4 \int_{A_{r,8r}(y)} \frac{|\lambda(y, |z - y|, T_{0,1}v)(T_{0,1}v(z) - T_{0,1}v(y)) - \nabla T_{0,1}v(z) \cdot (z - y)|^2}{|z - y|^{n+2}} dz + Cr^m d\mu(y) \right) \\ & \leq \frac{C(n, \Lambda, \alpha, M_0, \epsilon)}{r^k} \times \\ & \left(\int_{B_r(p)} \int_{A_{r,8r}(y)} \frac{|\lambda(y, |z - y|, T_{0,1}v)(T_{0,1}v(z) - T_{0,1}v(y)) - \nabla T_{0,1}v(z) \cdot (z - y)|^2}{|z - y|^{n+2}} dz + Cr^m d\mu(y) \right) \end{aligned}$$

Now, using Lemma 3.3.11, we have,

$$\begin{aligned} \beta_{\mu,2}^k(p, r)^2 & \leq \\ & \leq \frac{C(n, \Lambda, \alpha, M_0, \epsilon)}{r^k} \left(\int_{B_r(p)} \frac{C(M_0, \Gamma, \alpha)}{2} (N(8r, y, v) - N(r, y, v) + C_1 \|\ln(h)\|_\alpha r^\alpha) + Cr^m d\mu(y) \right) \\ & \leq \frac{C(n, \Lambda, \alpha, M_0, \epsilon)}{r^k} \int_{B_r(p)} N(8r, y, v) - N(r, y, v) + C_1 \|\ln(h)\|_\alpha r^\alpha d\mu(y) \\ & \quad + C(n, \Lambda, \alpha, M_0, \epsilon) \frac{\mu(B_r(p))}{r^k} r^m \end{aligned}$$

□

4.5 Packing

Now that we have linked the behavior of $N(r, p, v)$ to the β -numbers, we are ready to prove the crucial packing lemma.

Lemma 4.5.1. Fix $0 < \epsilon$, and let $v \in \mathcal{A}(\Lambda, \alpha, M_0)$ satisfy $\|\ln(h)\|_\alpha \leq \eta$ and $\sup_{p \in B_1(0)} N(2, p, v) = E$. There is an $\eta_1(n, \Lambda, \alpha, M_0, \epsilon) > 0$ such that if $\eta \leq \eta_1$, then for any $r > 0$ if $\{B_{2r_p}(p)\}$ is a collection of disjoint balls satisfying

$$N(\eta r_p, p, v) \geq E - \eta_1, \quad p \in S_{\eta_1, r}^k, \quad r \leq r_p \leq 1, \quad (4.5.1)$$

we have the following packing condition,

$$\sum_p r_p^k \leq C_2(n, \Lambda, \alpha, M_0, \epsilon). \quad (4.5.2)$$

Proof. Choose $\delta_0(n, \Lambda, \alpha, M_0, \epsilon)$ as in Lemma 4.4.1, and $\gamma(n, \Lambda, \alpha, M_0, \delta_0)$ as in Lemma 4.2.1. Note that we may assume without loss of generality that $\eta_1 \leq 1$, and so for $C_1(\alpha, M_0, 1)$ the constant in Lemma 3.3.11, let,

$$\eta_1 \leq \frac{\min\{\delta_0, \gamma\}}{2C_1 + 1}.$$

We will employ the convention that $r_i = 2^{-i}$.

For each $i \in \mathbb{N}$, define the truncated measure,

$$\mu_i = \sum_{r_p \leq r_i} r_p^k \delta_p$$

We will write $\beta_i(x, r) = \beta_{\mu_i, 2}^k(x, r)$. Observe that β_i enjoy the following properties. First,

because the balls are disjoint, for all $j \geq i$,

$$\beta_i(x, r_j) = \begin{cases} \beta_j(x, r_j) & x \in \text{supp}(\mu_j) \\ 0 & \text{otherwise} \end{cases}$$

Furthermore, for $r_i \leq 2^{-4}$, recalling Lemma 3.3.12 our assumption of the Almgren frequency gives that $N(16r_i, p, v) - N(r_p, p, v) \leq (2C_1 + 1)\|\ln(h)\|_\alpha$. By Lemma 4.2.1 and Lemma 4.4.1 and our choice of $\eta \leq \eta_1$,

$$\begin{aligned} \beta_{\mu_i, 2}^k(x, r_i)^2 &\leq \frac{C(n, \Lambda, \alpha, \epsilon)}{r_i^k} \int_{B_{r_i}(x)} N(8r_i, y, v) - N(r_i, y, v) + C\|\ln(h)\|_\alpha r_i^\alpha d\mu(y) \\ &\quad + C(n, \Lambda, \alpha, \epsilon) \frac{\mu(B_{r_i}(x))}{r_i^k} r_i^m. \end{aligned}$$

The claim of the lemma is that $\mu_0(B_1(0)) \leq C(n, \Lambda, \alpha, M_0, \epsilon)$. We prove the claim inductively. That is, we shall argue that there is an fixed scale, $0 < R = 2^{-\ell}$, such that for $r_i \leq R$ and all $x \in B_1(0)$,

$$\mu_i(B_{r_i}(x)) \leq C_{DR}(n)r_i^k.$$

Observe that since $r_p \geq r > 0$, for $r_i < r$, the claim is trivially satisfied because $\mu_i = 0$. Assume, then, that the inductive hypothesis holds for all $j \geq i + 1$.

Let $x \in B_1(0)$. We consider $\mu_i(B_{4r_i}(x))$. Observe that we can get a course bound,

$$\mu_j(B_{4r_j}(x)) \leq \Gamma(n)r_j^k, \quad \forall j \geq i - 2, \quad \forall x \in B_1(0),$$

by writing $\mu_j(B_{4r_j}(x)) = \mu_{j+2}(B_{4r_j}(x)) + \sum r_p^k$ where the sum is taken over all $p \in B_{4r_j}(x)$ with $r_{j+2} < r_p \leq r_j$. Since the balls $B_{r_p}(p)$ are disjoint, there is a dimensional constant, $c(n)$, which bounds the number of such points. Thus, we may take $\Gamma(n) = c(n)C_{DR}$.

Now, we calculate,

$$\begin{aligned}
& \sum_{r_j < 2r_i} \int_{B_{2r_i}(x)} \beta_i(z, r_j)^2 d\mu_i(z) = \sum_{r_j < 2r_i} \int_{B_{2r_i}(x)} \beta_j(z, r_j)^2 d\mu_j(z) \\
& \leq C \sum_{r_j < 2r_i} \frac{1}{r_j^k} \int_{B_{2r_i}(x)} \int_{B_{r_j}(z)} N(8r_j, y, v) - N(r_j, y, v) + C_1 \eta r_j^\alpha d\mu_j(y) d\mu_j(z) \\
& \quad + C \sum_{r_j < 2r_i} \int_{B_{2r_i}(x)} \frac{\mu_j(B_{r_j}(z))}{r_j^k} r_j^m d\mu_j(z) \\
& \leq c \sum_{r_j < 2r_i} \int_{B_{2r_i+r_j}(x)} \frac{\mu_j(B_{r_j}(y))}{r_j^k} (N(8r_j, y, v) - N(r_j, y, v) + C_1 \eta r_j^\alpha) d\mu_j(y) \\
& \quad + C \sum_{r_j < 2r_i} \int_{B_{2r_i}(x)} \frac{\mu_j(B_{r_j}(z))}{r_j^k} r_j^m d\mu_j(z) \\
& \leq c\Gamma(n) \int_{B_{4r_i}(x)} \sum_{r_j < 2r_i} (N(8r_j, y, v) - N(r_j, y, v) + C_1 \eta r_j^\alpha) d\mu_j(y) \\
& \quad + C \sum_{r_j < 2r_i} \Gamma(n) \mu_j(B_{4r_i}(x)) r_j^m \\
& \leq c\Gamma(n) \left(\sum_{p \in B_{4r_i}(x) \cap \text{supp}(\mu_i)} r_p^k (N(16r_i, p, v) - N(r_p, p, v) + C_1 \sum_{j \geq i} r_j^\alpha \eta) \right) \\
& \quad + C\Gamma(n) \sum_{r_j < 2r_i} \mu_i(B_{4r_i}(x)) r_j^m \\
& \leq c\Gamma(n) \mu_i(B_{4r_i}(x)) [(2C_1 + 1)\eta + C_1 \sum_{j \geq i} r_j^\alpha \eta] \\
& \quad + C\Gamma(n)^2 \left(\sum_{r_j < 2r_i} r_j^m \right) r_i^k \\
& \leq C(\alpha, M_0) \Gamma(n)^2 \eta r_i^k \\
& \quad + C\Gamma(n)^2 \left(\sum_{j \geq i} 2^{-jm} \right) r_i^k
\end{aligned}$$

Thus, for $\eta \leq \eta_1(n, \Lambda, \alpha, M_0, \epsilon)$ sufficiently small,

$$C\Gamma(n)^2\eta \leq \frac{1}{2}\delta_{DR}.$$

Similarly, there is an $\ell(n, \Lambda, \alpha, M_0, \epsilon)$ such that for all $i \geq \ell$,

$$C\Gamma(n)^2 \left(\sum_{j \geq i} 2^{-jm} \right) \leq \frac{1}{2}\delta_{DR} \quad (4.5.3)$$

In this case, μ_i satisfies the hypotheses of the Discrete Reifenberg Theorem,

$$\sum_{r_j < 2r_i} \int_{B_{2r_i}(x)} \beta_i(z, r_j)^2 d\mu_i(z) \leq \delta_{DR} r_i^k.$$

The Discrete Reifenberg Theorem therefore implies that $\mu_i(B_{r_i}(x)) \leq C_{DR} r_i^k$.

Thus, by induction, the claim holds for all $i \geq \ell$. Where ℓ is the smallest integer such that Equation 4.5.3 holds. Since $\ell = \ell(n, \Lambda, \alpha, M_0, \epsilon)$, we may use a packing argument to obtain estimates at larger scales. That is, $\mu_0(B_1(0)) \leq C_{DR} C(n, \ell)$.

□

4.6 Tree Construction

In this section, we detail two procedures for inductively-refined covering schemes. We will use these covering schemes in the next section to generate the actual cover which proves Theorem 4.1.1. First, we fix our constants.

4.6.1 Fixing Constants and a Definition.

In this section, we fix our constants as follows. Fix $0 < \epsilon$, and let $v \in \mathcal{A}(\Lambda, \alpha, M_0)$. Let $E = \sup_{p \in B_1(0)} N(2, p, v)$ and fix the scale of the covering we wish to construct as $R \in (0, 1]$.

We will let ρ denote the inductive scale at which we will refine our cover. For convenience, we will use the convention $r_i = \rho^{-i}$. Let $\rho < \frac{1}{10}$ be sufficiently small so that

$$2C_2(n, \Lambda, \alpha, M_0, \epsilon)c_2(n)\rho < 1/2.$$

where $C_2(n, \Lambda, \alpha, M_0, \epsilon)$ is as in Lemma 4.5.1 and $c_2(n)$ is a dimensional constant which will be given in the following lemmata.

Let $\delta(n, \Lambda, \alpha, M_0, \epsilon)$ be as in Lemma 4.4.1 and $\gamma(n, \Lambda, \alpha, M_0, \delta)$ as in Lemma 4.2.1. Now, we also let $\eta_1(n, \Lambda, \alpha, M_0, \epsilon)$ be as in Lemma 4.5.1 and

$$\gamma_0 = \eta' = \eta_1/20.$$

Note that while $\gamma_0 \leq \gamma$, Lemma 4.2.1 still holds with γ_0 in place of γ . We then let $\eta = \eta_0(n, \Lambda, \alpha, E + 1, \epsilon, \eta', \gamma_0, \rho)$ as in Corollary 4.3.2. We shall assume that v satisfies

$$\|\ln(h)\|_\alpha \leq \frac{1}{2C_1 + 1}\eta.$$

The sorting principle for our covering comes from Corollary 4.3.2. To formalize this, we make the following definition.

Definition 4.6.1. For $p \in B_2(0)$ and $0 < R < r < 2$, the ball $B_r(p)$ will be called “good” if

$$N(\gamma\rho r, p, v) \geq E - \eta' \quad \text{on} \quad S_{\epsilon, \eta R}^k(v) \cap B_r(p).$$

We will say that $B_r(p)$ is “bad” if it is not good.

Remark 4.6.2. By Corollary 4.3.2, with $E + \eta_0/2$ in place of E , which is admissible by monotonicity and our choice of $\|ln(h)\|_\alpha \leq \frac{1}{2C_1+1}\eta$, in any bad ball $B_r(p)$, there exists a $(k-1)$ -dimensional affine plane, L^{k-1} such that

$$\{N(\gamma\rho r, p, v) \geq E - \eta_0/2\} \cap B_r(p) \subset B_{\rho r}(L^{k-1}).$$

4.6.2 Good trees

Let $x \in B_1(0)$ and $B_{r_A}(x)$ be a good ball for $A \geq 0$. We will detail the inductive construction of a good tree based at $B_{r_A}(x)$. The induction will build a successively refined covering $B_{r_A}(x) \cap S_{\epsilon, \eta R}^k(v)$. We will terminate the process and have a cover which consists of a collection of bad balls with packing estimates and a collection of stop balls whose radii are comparable to R . We shall use the notation \mathcal{G}_i to denote the collection of centers of good balls of scale r_i , \mathcal{B}_i shall denote the collection of centers of bad balls of scale r_i .

Because $B_{r_A}(x)$ is a good ball, at scale $i = A$, we set $\mathcal{G}_A = x$. We let $\mathcal{B}_A = \emptyset$.

Now the inductive step. Suppose that we have constructed our collections of good and bad balls down to scale $j-1 \geq A$. Let $\{z\}_{J_i}$ be a maximal $\frac{2}{5}r_j$ -net in

$$B_{r_A}(x) \cap S_{\epsilon, \eta R}^k(v) \cap B_{r_{j-1}}(\mathcal{G}_{j-1}) \setminus \cup_{i=A}^{j-1} B_{r_i}(\mathcal{B}_i).$$

We then sort these points into \mathcal{G}_j and \mathcal{B}_j depending on whether $B_{r_j}(z)$ is a good ball or a bad ball. If $r_j > R$, we proceed inductively. If $r_j \leq R$, then we stop the procedure. In this case, we let $\mathcal{S} = \mathcal{G}_j \cup \mathcal{B}_j$ and we call this the collection of “stop” balls.

The covering at which we arrive at the end of this process shall be called the “good tree at $B_{r_A}(x)$.” We shall follow [16] and denote this $\mathcal{T}_G = \mathcal{T}_G(B_{r_A}(x))$. We shall call the collection of “bad” ball centers, $\cup_i \mathcal{B}_i$, the “leaves of the tree” and denote this collection by $\mathcal{F}(\mathcal{T}_G)$. We shall denote the collection of “stop” ball centers by $\mathcal{S}(\mathcal{T}_G) = \mathcal{S}$.

For $b \in \mathcal{F}(\mathcal{T}_G)$ we let $r_b = r_i$ for i such that $b \in \mathcal{B}_i$. Similarly, for $s \in \mathcal{S}(\mathcal{T}_G)$, we let $r_s = r_j$ for the terminal j .

Theorem 4.6.3. A good tree, $\mathcal{T}_G(B_{r_A}(x))$, enjoys the following properties:

(A) Tree-leaf packing:

$$\sum_{b \in \mathcal{F}(\mathcal{T}_G)} r_b^k \leq C_2(n, \Lambda, \alpha, M_0, \epsilon) r_A^k$$

(B) Stop ball packing

$$\sum_{s \in \mathcal{S}(\mathcal{T}_G)} r_s^k \leq C_2(n, \Lambda, \alpha, M_0, \epsilon) r_A^k$$

(C) Covering control

$$\mathcal{S}_{\epsilon, \eta R}^k(v) \cap B_{r_A}(x) \subset \bigcup_{s \in \mathcal{S}(\mathcal{T}_G)} B_{r_s}(s) \cup \bigcup_{b \in \mathcal{F}(\mathcal{T}_G)} B_{r_b}(b)$$

(D) Size control: for any $s \in \mathcal{S}(\mathcal{T}_G)$, $\rho R \leq r_s \leq R$.

Proof. First, observe that by construction,

$$\{B_{\frac{r_b}{5}}(b) : b \in \mathcal{F}(\mathcal{T}_G)\} \cup \{B_{\frac{r_s}{5}}(s) : s \in \mathcal{S}(\mathcal{T}_G)\}$$

is pairwise disjoint and centered in the set $\mathcal{S}_{\epsilon, \eta R}^k(v)$. Next, all bad balls and stop balls are centered in a good ball of the previous scale. By our definition of good balls, then, we have for all i

$$N(\gamma r_i, b, v) = N(\gamma \rho r_{i-1}, b, v) \geq E - \eta' \quad \forall b \in \mathcal{B}_i$$

and

$$N(\gamma r_s, s, v) \geq E - \eta' \quad \forall s \in \mathcal{S}(\mathcal{T}_G).$$

Since by monotonicity we have that $\sup_{p \in B_{r_A}(x)} N(2r_A, p, v) \leq E + \eta'$, we can apply Lemma 4.5.1 to $B_{r_A}(x)$ and get the packing estimates, (A), (B).

Covering control follows from our choice of a maximal $\frac{2}{5}r_i$ -net at each scale i . If i is the first scale at which a point, $x \in \mathcal{S}_{\epsilon, \eta R}^k(v)$, was not contained in our inductively refined cover, it would violate the maximality assumption.

The last condition, (D), follows because we stop only if j is the first scale for which $r_j \leq R$. Since we decrease by a factor of ρ at each scale, (D) follows. \square

4.6.3 Bad trees

Let $B_{r_A}(x)$ be a bad ball. Note that for every bad ball, there is a $(k-1)$ -dimensional affine plane, L^{k-1} , associated to it which satisfies the properties elaborated in Corollary 4.3.2. Our construction of bad trees will differ in several respects from our construction of good trees. The idea is still to define an inductively-refined cover at decreasing scales of $B_{r_A}(x) \cap \mathcal{S}_{\epsilon, \eta R}^k(v)$. We shall again sort balls at each step into “good,” “bad,” and “stop” balls. But these balls will play slightly different roles and the “stop” balls will have different radii.

We shall reuse the notation \mathcal{G}_i to denote the collection of centers of good balls of scale r_i , \mathcal{B}_i to denote the collection of centers of bad balls of scale r_i , and \mathcal{S}_i to denote the collection of centers of stop balls of scale r_i .

At scale $i = A$, we set $\mathcal{B}_A = x$, since $B_{r_A}(x)$ is a bad ball, and set $\mathcal{S}_A = \mathcal{G}_A = \emptyset$. Suppose, now that we have constructed good, bad, and stop balls for scale $i-1 \geq A$. If $r_i > R$, then define \mathcal{S}_i to be a maximal $\frac{2}{5}\eta r_{i-1}$ -net in

$$B_{r_A}(x) \cap \mathcal{S}_{\epsilon, \eta R}^k(v) \cap \cup_{b \in \mathcal{B}_{i-1}} B_{r_{i-1}}(b) \setminus B_{2\rho r_{i-1}}(L_b^{k-1}).$$

Note that $\eta \ll \rho$, so $\eta r_{i-1} < r_i$. We then let $\{z\}$ be a maximal $\frac{2}{5}r_i$ -net in

$$B_{r_A}(x) \cap \mathcal{S}_{\epsilon, \eta R}^k(v) \cap \cup_{b \in \mathcal{B}_{i-1}} B_{r_{i-1}}(b) \cap B_{2\rho r_{i-1}}(L_b^{k-1}).$$

We then sort $\{z\}$ into the disjoint union $\mathcal{G}_i \cup \mathcal{B}_i$ depending on whether $B_{r_i}(z)$ is a good ball

or a bad ball.

If $r_i \leq R$, then we terminate the process by defining $\mathcal{G}_i = \mathcal{B}_i = \emptyset$ and letting \mathcal{S}_i be a maximal $\frac{2}{5}\eta r_{i-1}$ -net in

$$B_{r_A}(x) \cap \mathcal{S}_{\epsilon, \eta R}^k(v) \cap B_{r_i}(\mathcal{B}_{i-1}).$$

The covering at which we arrive at the end of this process shall be called the “bad tree at $B_{r_A}(x)$.” We shall follow [16] and denote this $\mathcal{T}_B = \mathcal{T}_B(B_{r_A}(x))$. We shall call the collection of “good” ball centers, $\cup_i \mathcal{G}_i$, the “leaves of the tree” and denote this collection by $\mathcal{F}(\mathcal{T}_B)$. We shall denote the collection of “stop” ball centers by $\mathcal{S}(\mathcal{T}_B) = \cup_i \mathcal{S}_i$.

As before, we shall use the convention that for $g \in \mathcal{F}(\mathcal{T}_B)$ we let $r_g = r_i$ for i such that $g \in \mathcal{G}_i$. However, note that now, if $s \in \mathcal{S}_i \subset \mathcal{S}(\mathcal{T}_B)$, we let $r_s = \eta r_{i-1}$.

Theorem 4.6.4. A bad tree, $\mathcal{T}_B(B_{r_A}(x))$, enjoys the following properties:

(A) Tree-leaf packing:

$$\sum_{g \in \mathcal{F}(\mathcal{T}_B)} r_g^k \leq 2c_2(n) \rho r_A^k$$

(B) Stop ball packing

$$\sum_{s \in \mathcal{S}(\mathcal{T}_B)} r_s^k \leq c(n, \eta) r_A^k$$

(C) Covering control

$$\mathcal{S}_{\epsilon, \eta R}^k(v) \cap B_{r_A}(x) \subset \bigcup_{s \in \mathcal{S}(\mathcal{T}_B)} B_{r_s}(s) \cup \bigcup_{g \in \mathcal{F}(\mathcal{T}_B)} B_{r_g}(g)$$

(D) Size control: for any $s \in \mathcal{S}(\mathcal{T}_B)$, at least one of the following holds:

$$\eta R \leq r_s \leq R \quad \text{or} \quad \sup_{p \in B_{2r_s}(s)} N(2r_s, p, v) \leq E - \eta/2.$$

Proof. Conclusion (C) follows identically as in Theorem 4.6.3. Next, we consider the packing

estimates. Let $r_i > R$. Then, by construction, for any $b \in \mathcal{B}_{i-1}$, we have that

$$\mathcal{G}_i \cup \mathcal{B}_i \cup B_{r_{i-1}}(b) \subset B_{2\rho r_{i-1}}(L_b^{k-1}).$$

Thus, since the points $\mathcal{G}_i \cup \mathcal{B}_i$ are $\frac{2}{5}r_i$ disjoint, we calculate

$$|\mathcal{G}_i \cup \mathcal{B}_i \cup B_{r_{i-1}}(b)| \leq \omega_{k-1} \omega_{n-k+1} (3\rho)^{n-k+1} \frac{1}{\omega_n (\rho/5)^n} \leq c_2(n) \rho^{1-k}.$$

We can push this estimate up the scales as follows,

$$\begin{aligned} |\mathcal{G}_i \cup \mathcal{B}_i \cup B_{r_{i-1}}(b)| r_i^k &\leq c_2(n) \rho^1 |\mathcal{B}_{i-1}| r_{i-1}^k \\ &\leq c_2(n) \rho^1 |\mathcal{B}_{i-1} \cup \mathcal{G}_{i-1}| r_{i-1}^k \\ &\quad \vdots \\ &\leq (c_2 \rho)^{i-A} r_A^k \end{aligned}$$

Summing over all $i \geq A$, then, we have that

$$\sum_{i=A+1}^{\infty} |\mathcal{B}_{i-1} \cup \mathcal{G}_{i-1}| r_i^k \leq \sum_{i=A+1}^{\infty} (c_2 \rho)^{i-A} r_A^k$$

Since we chose $c_2 \rho \leq 1/2$, we have that the sum converges and $\sum_{i=A+1}^{\infty} |\mathcal{B}_{i-1} \cup \mathcal{G}_{i-1}| r_i^k \leq 2c_2 \rho r_A^k$. This proves (A).

To see (B), we observe that for any given scale $i \geq A + 1$, the collection of stop balls, $\{B_{\eta r_{i-1}}(s)\}_{s \in \mathcal{S}_i}$, form a Vitali collection centered in $B_{r_{i-1}}(\mathcal{B}_{i-1})$. Thus, we have that

$$|\{\mathcal{S}_i\}| \leq \frac{10^n}{\eta^n} |\{\mathcal{B}_{i-1}\}|.$$

Since by construction there are no stop balls at the initial scale, A , we compute that

$$\sum_{i=A+1}^{\infty} |\{\mathcal{S}_i\}|(\eta r_{i-1})^k \leq 10^k \eta^{k-n} \sum_{i=A}^{\infty} |\{\mathcal{B}_i\}|r_i^k \leq c(n, \eta)r_A^k$$

This is (B).

We now argue (D). For $s \in \mathcal{S}_i$ where $r_i > R$, by construction $s \in B_{r_{i-1}}(b) \setminus B_{2\rho r_{i-1}}(L^{k-1})$ for some $b \in \mathcal{B}_{i-1}$. By Corollary 4.3.2, the construction, and our choice of $\eta \leq \frac{\rho}{2}$, we have that

$$\sup_{p \in B_{2r_s}(s)} N(2r_s, p, v) \leq \sup_{p \in B_{2\eta r_{i-1}}(s)} N(2\eta r_{i-1}, p, v) \leq E - \eta/2.$$

On the other hand, if $r_i \leq R$, then $r_{i-1} > R$. Thus,

$$R \geq \rho r_{i-1} \geq \eta r_{i-1} = r_s \geq \eta R.$$

This proves (D). □

4.7 The Covering

Assuming that $\|\ln(h)\|_{\alpha} \leq \frac{1}{2C_1}\eta$, for $\eta > 0$ as in Section 10.1, we now wish to build the covering of $\mathcal{S}_{\epsilon, \eta R}^k \cap B_1(0)$ using the tree constructions, above. The idea is that $B_1(0)$ is either a good ball or a bad ball. Therefore, we can construct a tree with $B_1(0)$ as the root. Then in each of the leaves, we construct either good trees or bad trees, depending upon the type of the leaves. Since in each construction, we decrease the size of the leaves by a factor of $\rho < 1/10$, we can continue alternating tree types until the process terminates in finite time.

Explicitly, we let $\mathcal{F}_0 = \{0\}$. and let $B_1(0)$ be the only leaf. We set $\mathcal{S}_0 = \emptyset$. Now, assume that we have defined the leaves and stop balls up to stage $i - 1$. Since by hypothesis, the leaves in \mathcal{F}_i are all good balls or bad balls, if they are good, we define for each $f \in \mathcal{F}_{i-1}$ the

good tree $\mathcal{T}_G(B_{r_f}(f))$. We then set,

$$\mathcal{F}_i = \cup_{f \in \mathcal{F}_{i-1}} \mathcal{F}(\mathcal{T}_G(B_{r_f}(f)))$$

and

$$\mathcal{S}_i = \mathcal{S}_{i-1} \cup \bigcup_{f \in \mathcal{F}_{i-1}} \mathcal{S}(\mathcal{T}_G(B_{r_f}(f)))$$

Since all the leaves of good trees are bad balls, all the leaves of \mathcal{F}_i are bad.

If, on the other hand, leaves of \mathcal{F}_{i-1} are bad, then for each $f \in \mathcal{F}_{i-1}$, we construct a bad tree, $\mathcal{T}_B(B_{r_f}(f))$. In this case, we set

$$\mathcal{F}_i = \cup_{f \in \mathcal{F}_{i-1}} \mathcal{F}(\mathcal{T}_B(B_{r_f}(f)))$$

and

$$\mathcal{S}_i = \mathcal{S}_{i-1} \cup \bigcup_{f \in \mathcal{F}_{i-1}} \mathcal{S}(\mathcal{T}_B(B_{r_f}(f)))$$

Since all the leaves of bad trees are good balls, all the leaves of \mathcal{F}_i are good.

This construction gives the following estimates.

Lemma 4.7.1. For the construction described above, there is an $N \in \mathbb{N}$ such that $\mathcal{F}_N = \emptyset$ with the following properties:

(A) Leaf packing:

$$\sum_{i=0}^{N-1} \sum_{f \in \mathcal{F}_i} r_f^k \leq c(n)$$

(B) Stop ball packing

$$\sum_{s \in \mathcal{S}_N} r_s^k \leq c(n, \epsilon, \Lambda, \alpha, M_0)$$

(C) Covering control

$$\mathcal{S}_{\epsilon, \eta R}^k(v) \cap B_1(0) \subset \cup_{s \in \mathcal{S}_N} B_{r_s}(s)$$

(D) Size control: for any $s \in \mathcal{S}_N$, at least one of the following holds:

$$\eta R \leq r_s \leq R \quad \text{or} \quad \sup_{p \in B_{2r_s}(s)} N(2r_s, p, v) \leq E - \eta/2.$$

Proof. By construction, each of the leaves of a good or bad tree satisfy $r_f \leq r_i$. Thus, there is an i sufficiently large so that $r_i < R$. Thus, N is finite.

To see (A), we use the previous theorems. That is, if the leaves, \mathcal{F}_i , are good, then they are the leaves of bad trees rooted in \mathcal{F}_{i-1} . Thus, we calculate by Theorem 4.6.4,

$$\sum_{f \in \mathcal{F}_i} r_f^k \leq 2c_2(n)\rho \sum_{f' \in \mathcal{F}_{i-1}} r_{f'}^k$$

On the other hand, if the leaves, \mathcal{F}_i , are bad, then they are the leaves of good trees rooted in \mathcal{F}_{i-1} . Thus, we calculate by Theorem 4.6.3,

$$\sum_{f \in \mathcal{F}_i} r_f^k \leq c_1(n) \sum_{f' \in \mathcal{F}_{i-1}} r_{f'}^k$$

Concatenating the estimates, since we alternate between good and bad leaves, we have,

$$\sum_{f \in \mathcal{F}_i} r_f^k \leq c(n)(2c_1(n)c_2(n)\rho)^{i/2}$$

By our choice of ρ , then, $\sum_{f \in \mathcal{F}_i} r_f^k \leq c(n)2^{-i/2}$. The estimate (A) follows immediately.

We now turn our attention to (B). Each stop ball, $s \in \mathcal{S}_N$, is a stop ball coming from a good or a bad tree rooted in one of the leaves of a bad tree or good tree. We have the estimates from Theorems 4.6.3 and 4.6.4, which give bounds packing both leaves and stop

balls. Combining these, we get

$$\begin{aligned} \sum_{s \in \mathcal{S}_N} r_s^k &= \sum_{i=0}^N \sum_{s \in \mathcal{S}_i} r_s^k \\ &\leq \sum_{i=0}^{N-1} \sum_{f \in \mathcal{F}_i} c(n, \eta) r_f^k \\ &\leq .C(n, \eta) \end{aligned}$$

Recalling the dependencies of η gives the desired result.

(C) follows inductively from the analogous covering control in Theorems 4.6.3 and Theorem 4.6.4 applied to each tree constructed. (D) is immediate from these theorems, as well. \square

Corollary 4.7.2. There is an $\eta(n, \epsilon, \Lambda, \alpha, M_0) > 0$ such that if $v \in \mathcal{A}(\Lambda, \alpha, M_0)$ with $\|\ln(h)\|_\alpha \leq \eta$ and $\sup_{p \in B_2(0)} N(2, p, v) \leq E$ then given any $0 < R \leq 1$ there is a collection of balls, $\{B_{r_x}(x)\}_{x \in \mathcal{U}}$ with centers $x \in \mathcal{S}_{\epsilon, \eta R}^k(v) \cap B_1(0)$. Further, $R \leq r_x \leq \frac{1}{10}$ and the collection has the following properties:

(A) Packing:

$$\sum_{x \in \mathcal{U}} r_x^k \leq c(n, \epsilon, \Lambda, \alpha, M_0)$$

(B) Covering control

$$\mathcal{S}_{\epsilon, \eta R}^k(v) \cap B_1(0) \subset \cup_{x \in \mathcal{U}} B_{r_x}(x)$$

(C) Energy drop: For every $x \in \mathcal{U}$, either

$$r_x = R \quad \text{or} \quad \sup_{p \in B_{2r_s}(s)} N(2r_s, p, v) \leq E - \eta/2.$$

This follows immediately from the previous lemma with $\eta \leq \frac{1}{2C_1}\eta_1$, $\mathcal{S}_N = \mathcal{U}$, and setting $r_x = \max\{R, r_s\}$.

Theorem 4.7.3. Let $v \in \mathcal{A}(\Lambda, \alpha, M_0)$. For all $\epsilon > 0$ there exists an $\eta(n, \epsilon, \Lambda, \alpha, M_0) > 0$ such that if $\|\ln(h)\|_\alpha \leq \eta$, then for all $0 < r < 1$ and $k = 1, 2, \dots, n-1$ we can find a collection of balls, $\{B_r(x_i)\}_i$ with the following properties:

1. $\mathcal{S}_{\epsilon, r}^k \cap B_1(0) \subset \cup_i B_r(x_i)$.
2. $|\{x_i\}_i| \leq c(n, \Lambda, \alpha, \epsilon)r^{-k}$

Proof. By the assumptions of the theorem and Lemma 3.4.4, we know that $E \leq C(n, \Lambda, \alpha, M_0)$. Ensuring that $c(n, \Lambda, \alpha, \epsilon)$ is sufficiently large, we may reduce to arguing for $r < \eta$. We now use Corollary 4.8.2, to build the covering \mathcal{U}_1 . If every $r_x = R$, then the packing and covering estimates give the claim directly, since

$$R^{k-n} \text{Vol}(B_R(\mathcal{S}_{\epsilon, \eta R}^k \cap B_1(0))) \leq \omega_n R^{k-n} \sum_{\mathcal{U}_1} (2R)^n = \omega_n 2^n \sum_{\mathcal{U}_1} r_x^k \leq c(n, \Lambda, \epsilon, \alpha)$$

If there exists an $r_x \neq R$, we use Corollary 4.8.2, to build a finite sequence of refined covers, $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \dots$ such that for each for each i , the covering satisfies the following properties:

(A_i) Packing:

$$\sum_{x \in \mathcal{U}_i} r_x^k \leq c(n, \Lambda, \epsilon, \alpha) \left(1 + \sum_{x \in \mathcal{U}_{i-1}} r_x^k\right)$$

(B_i) Covering control

$$\mathcal{S}_{\epsilon, \eta R}^k(v) \cap B_1(0) \subset \cup_{x \in \mathcal{U}_i} B_{r_x}(x)$$

(C_i) Energy drop: For every $x \in \mathcal{U}_i$, either

$$r_x = R \quad \text{or} \quad \sup_{p \in B_{2r_s}(s)} N(2r_s, p, v) \leq E - i\eta/2.$$

(D_i) radius control:

$$\sup_{x \in \mathcal{U}_i} r_x \leq 10^{-i}$$

If we can construct such a sequence of covers, then this process will terminate in finite time, *independent of* R . Recall that blow-ups of v are homogeneous harmonic polynomials. Therefore

$$N(0, p, v) = \lim_{r \rightarrow \infty} N(r, p, v) \geq 1$$

for all $p \in B_1(0)$. By Remark 3.3.10 we have that $N(r, p, v) \geq 1 - C(\alpha, M_0)\eta$ for all $p \in B_{1/2}(0)$. Therefore, we know that once $i > (C(n, \Lambda, \alpha, M_0) + C(\alpha, M_0)\eta - 1)\frac{2}{\eta}$, $r_x = R$ for all $x \in \mathcal{U}_i$. In this case, we will have the claim with a bound of the form

$$R^{k-n} \text{Vol}(B_R(\mathcal{S}_{\epsilon, \eta R}^k \cap B_1(0))) \leq c(n, \Lambda, \epsilon, \alpha)^{(C(n, \Lambda, \alpha, M_0) + C(\alpha, M_0)\eta - 1)\frac{2}{\eta}}$$

Thus, we reduce to inductively constructing the required covers. Suppose we have already constructed \mathcal{U}_{i-1} as desired. For each $x \in \mathcal{U}_{i-1}$ with $r_x > R$, we apply Corollary 4.8.2 at scale $B_{r_x}(x)$ to obtain a new collection of balls, $\mathcal{U}_{i,x}$. From the assumption that $r_x \leq 1/10$ and the way that Holder norms scale, it is clear that v satisfies the hypotheses of Corollary 4.8.2 in $B_{r_x}(x)$. To check packing control, we have that

$$\sum_{y \in \mathcal{U}_{i,x}} r_y^k \leq c(n, \Lambda, \epsilon, \alpha) r_x^k$$

Covering control follow immediately from the statement of Corollary 4.8.2. Similarly, from hypothesis (C_{i-1}) , we have that $\sup_{p \in B_{2r_x}(x)} N(2r_x, p, v) \leq E - (i-1)\eta/2$. Thus, the statement of Corollary 4.8.2 at scale $B_{r_x}(x)$ gives that $\sup_{p \in B_{2r_y}(y)} N(2r_y, p, v) \leq E - (i)\eta/2$ for all $y \in \mathcal{U}_{i,x}$ with $r_y > R$. Radius control follows immediately from the fact that $\sup_{y \in \mathcal{U}_{i,x}} r_y \leq r_x/10 \leq 10^{-i}$.

Thus, if we let

$$\mathcal{U}_i = \{x \in \mathcal{U}_{i-1} | r_x = R\} \cup \bigcup_{\substack{x \in \mathcal{U}_{i-1} \\ r_x > R}} \mathcal{U}_{i,x}$$

then \mathcal{U}_i satisfy the inductive claim. This completes the proof. \square

4.8 The Covering

Assuming that $\|\ln(h)\|_\alpha \leq \frac{1}{2C_1+1}\eta$, for $0 < \eta \leq \eta_0(n, \Lambda, \alpha, E+1, \epsilon, \eta', \gamma_0, \rho)$ as in Section 4.6, we now wish to build the covering of $\mathcal{S}_{\epsilon, \eta R}^k \cap B_1(0)$ using the tree constructions, above. The idea is that $B_1(0)$ is either a good ball or a bad ball. Therefore, we can construct a tree with $B_1(0)$ as the root. Then in each of the leaves, we construct either good trees or bad trees, depending upon the type of the leaves. Since in each construction, we decrease the size of the leaves by a factor of $\rho < 1/10$, we can continue alternating tree types until the process terminates in finite time.

Explicitly, we let $\mathcal{F}_0 = \{0\}$. and let $B_1(0)$ be the only leaf. We set $\mathcal{S}_0 = \emptyset$. Now, assume that we have defined the leaves and stop balls up to stage $i-1$. Since by hypothesis, the leaves in \mathcal{F}_i are all good balls or bad balls, if they are good, we define for each $f \in \mathcal{F}_{i-1}$ the good tree $\mathcal{T}_G(B_{r_f}(f))$. We then set,

$$\mathcal{F}_i = \bigcup_{f \in \mathcal{F}_{i-1}} \mathcal{F}(\mathcal{T}_G(B_{r_f}(f)))$$

and

$$\mathcal{S}_i = \mathcal{S}_{i-1} \cup \bigcup_{f \in \mathcal{F}_{i-1}} \mathcal{S}(\mathcal{T}_G(B_{r_f}(f)))$$

Since all the leaves of good trees are bad balls, all the leaves of \mathcal{F}_i are bad.

If, on the other hand, leaves of \mathcal{F}_{i-1} are bad, then for each $f \in \mathcal{F}_{i-1}$, we construct a bad tree, $\mathcal{T}_B(B_{r_f}(f))$. In this case, we set

$$\mathcal{F}_i = \bigcup_{f \in \mathcal{F}_{i-1}} \mathcal{F}(\mathcal{T}_B(B_{r_f}(f)))$$

and

$$\mathcal{S}_i = \mathcal{S}_{i-1} \cup \bigcup_{f \in \mathcal{F}_{i-1}} \mathcal{S}(\mathcal{T}_B(B_{r_f}(f)))$$

Since all the leaves of bad trees are good balls, all the leaves of \mathcal{F}_i are good.

This construction gives the following estimates.

Lemma 4.8.1. For the construction described above, there is an $N \in \mathbb{N}$ such that $\mathcal{F}_N = \emptyset$ with the following properties:

(A) Leaf packing:

$$\sum_{i=0}^{N-1} \sum_{f \in \mathcal{F}_i} r_f^k \leq c(n)$$

(B) Stop ball packing

$$\sum_{s \in \mathcal{S}_N} r_s^k \leq c(n, \Lambda, \alpha, M_0, \epsilon)$$

(C) Covering control

$$\mathcal{S}_{\epsilon, \eta R}^k(v) \cap B_1(0) \subset \bigcup_{s \in \mathcal{S}_N} B_{r_s}(s)$$

(D) Size control: for any $s \in \mathcal{S}_N$, at least one of the following holds:

$$\eta R \leq r_s \leq R \quad \text{or} \quad \sup_{p \in B_{2r_s}(s)} N(2r_s, p, v) \leq E - \eta/2.$$

Proof. By construction, each of the leaves of a good or bad tree satisfy $r_f \leq r_i$. Thus, there is an i sufficiently large so that $r_i < R$. Thus, N is finite.

To see (A), we use the previous theorems. That is, if the leaves, \mathcal{F}_i , are good, then they are the leaves of bad trees rooted in \mathcal{F}_{i-1} . Thus, we calculate by Theorem 4.6.4,

$$\sum_{f \in \mathcal{F}_i} r_f^k \leq 2c_2(n)\rho \sum_{f' \in \mathcal{F}_{i-1}} r_{f'}^k$$

On the other hand, if the leaves, \mathcal{F}_i , are bad, then they are the leaves of good trees rooted

in \mathcal{F}_{i-1} . Thus, we calculate by Theorem 4.6.3,

$$\sum_{f \in \mathcal{F}_i} r_f^k \leq C_2(n, \Lambda, \alpha, M_0, \epsilon) \sum_{f' \in \mathcal{F}_{i-1}} r_{f'}^k$$

Concatenating the estimates, since we alternate between good and bad leaves, we have,

$$\sum_{f \in \mathcal{F}_i} r_f^k \leq c(n)(2C_2(n, \Lambda, \alpha, M_0, \epsilon)c_2(n)\rho)^{i/2}$$

By our choice of ρ , then, $\sum_{f \in \mathcal{F}_i} r_f^k \leq c(n)2^{-i/2}$. The estimate (A) follows immediately.

We now turn our attention to (B). Each stop ball, $s \in \mathcal{S}_N$, is a stop ball coming from a good or a bad tree rooted in one of the leaves of a bad tree or good tree. We have the estimates from Theorems 4.6.3 and 4.6.4, which give bounds packing both leaves and stop balls. Combining these, we get

$$\begin{aligned} \sum_{s \in \mathcal{S}_N} r_s^k &= \sum_{i=0}^N \sum_{s \in \mathcal{S}_i} r_s^k \\ &\leq \sum_{i=0}^{N-1} \sum_{f \in \mathcal{F}_i} c(n, \eta) r_f^k \\ &\leq C(n, \eta) \end{aligned}$$

Recalling the dependencies of η gives the desired result.

(C) follows inductively from the analogous covering control in Theorems 4.6.3 and Theorem 4.6.4 applied to each tree constructed. (D) is immediate from these theorems, as well. \square

Corollary 4.8.2. Fix $0 < \epsilon$. Let $v \in \mathcal{A}(\Lambda, \alpha, M_0)$ satisfying $\sup_{p \in B_2(0)} N(2, p, v) \leq E$. Fix $0 < \epsilon$. There is an $\eta_0(n, \Lambda, \alpha, M_0, \epsilon, E) > 0$ such that if $0 < \eta \leq \eta_0$ and $\|\ln(h)\|_\alpha \leq \frac{\eta}{2C_1+1}$ then given any $0 < R \leq 1$ there is a collection of balls, $\{B_{r_x}(x)\}_{x \in \mathcal{U}}$ with centers $x \in S_{\epsilon, \eta R}^k(v) \cap B_1(0)$. Further, $R \leq r_x \leq \frac{1}{10}$ and the collection has the following properties:

(A) Packing:

$$\sum_{x \in \mathcal{U}} r_x^k \leq c(n, \Lambda, \alpha, M_0, E, \epsilon)$$

(B) Covering control

$$\mathcal{S}_{\epsilon, \eta R}^k(v) \cap B_1(0) \subset \bigcup_{x \in \mathcal{U}} B_{r_x}(x)$$

(C) Energy drop: For every $x \in \mathcal{U}$, either

$$r_x = R \quad \text{or} \quad \sup_{p \in B_{2r_s}(s)} N(2r_s, p, v) \leq E - \eta_0/2.$$

This follows immediately from the Lemma 4.8.1 with $\eta \leq \eta_1$, $\mathcal{S}_N = \mathcal{U}$, and setting $r_x = \max\{R, r_s\}$.

4.9 Proof of Theorem 4.1.1

Lemma 4.9.1. Let $v \in \mathcal{A}(\Lambda, \alpha, M_0)$ with $\|\ln(h)\|_\alpha \leq \Gamma$. There exists a scale $\kappa(n, \Lambda, \alpha, M_0, \Gamma, \epsilon) > 0$ such for all balls, $B_r(y)$, with $0 < r < \kappa$ and $y \in B_{1/4}(0)$, the function $\tilde{v}(x) = v(rx + y)$ on $B_1(0)$ satisfies the following properties.

$$\begin{aligned} \sup_{p \in B_1(0)} N(2, p, \tilde{v}) &\leq C(n, \Lambda, \alpha, M_0, \Gamma) \\ \|\ln(\tilde{h})\|_{C^{0, \alpha}(B_1(0))} &\leq \frac{\eta_0}{2C_1 + 1} \end{aligned}$$

where $\eta_0 = \eta_0(n, \Lambda, \alpha, C(n, \Lambda, \alpha, M_0, \Gamma) + 1, \eta', \epsilon, \gamma_0, \rho) = \eta_0(n, \Lambda, \alpha, M_0, \Gamma, \epsilon)$ as in Corollary 4.3.2 and $C(n, \Lambda, \alpha, M_0, \Gamma)$ is as in Lemma 3.4.4.

Proof. We know by Lemma 3.4.4 that for any ball, $B_r(y)$, with $0 < r < \frac{1}{2}$ and $y \in B_{1/4}(0)$,

$$\sup_{p \in B_r(y)} N(2r, p, v) \leq C(n, \Lambda, \alpha, M_0, \Gamma).$$

Next we consider the effect of rescaling on Hölder continuous functions.

$$\begin{aligned}
|\tilde{v}(x) - \tilde{v}(z)| &= |v(rx + y) - v(rz + y)| \\
&\leq \Gamma |rx - rz|^\alpha \\
&= \Gamma r^\alpha |x - z|^\alpha
\end{aligned}$$

Since $r^\alpha \rightarrow 0$ as $r \rightarrow 0$, there exists an $\kappa(n, \Lambda, \alpha, M_0, \Gamma, \epsilon) > 0$ such that $\Gamma \kappa^\alpha < \frac{\eta_0}{2C_1+1}$. \square

Theorem 4.9.2. Let $v \in \mathcal{A}(\Lambda, \alpha, M_0)$ with $\|\ln(h)\|_\alpha \leq \Gamma$. For all $\epsilon > 0$ there exists an $\eta_0(n, \Lambda, \alpha, M_0, \Gamma, \epsilon) > 0$ such that for all $0 < R < 1$ and $k = 1, 2, \dots, n-1$ we can find a collection of balls, $\{B_R(x_i)\}_i$ with the following properties:

1. $\mathcal{S}_{\epsilon, \eta_0 R}^k(v) \cap B_{1/4}(0) \subset \cup_i B_R(x_i)$.
2. $|\{x_i\}_i| \leq c(n, \Lambda, \alpha, M_0, \Gamma, \epsilon) R^{-k}$

Proof. Cover $\mathcal{S}_{\epsilon, \eta_0 R}^k(v) \cap B_{1/4}(0)$ by balls $B_\kappa(y_j)$, with $y_j \in B_{1/4}(0)$ such that $B_{1/4}(0) \subset \bigcup_j B_\kappa(y_j)$ for $0 < \kappa(n, \Lambda, \alpha, M_0, \Gamma, \epsilon)$ the constant in Lemma 4.9.1. Note that we need at most $c(n, \Lambda, \alpha, M_0, \Gamma, \epsilon)$ such balls.

We now wish to apply Corollary 4.8.2 to the rescaled functions $\tilde{v}_i(x) = v(\kappa x + y_i)$ in $B_1(0)$. However, a careful reader may object that $\tilde{v}_i \notin \mathcal{A}(n, \Lambda, \alpha)$, since it is possible that $\{0\} \notin \{\tilde{v} = 0\}$. However, by Lemma 3.4.4, we have that $N(r, p, \tilde{v}) \leq C$ for all $0 < r < 2$ and all $p \in B_1(0)$. This, and the local uniformly Lipschitz bound from Lemma 3.2.1 gives us the necessary compactness properties to push through all the previous results for \tilde{v}_i without changing the constants. The geometry of the sets we are considering, *qua* geometry, is invariant under such rescaling. Furthermore, in balls, $B_\kappa(y_i)$, for which \tilde{v}_i is harmonic, the lemmata of this paper simplify and the desired results are already contained in [24].

We now construct the desired covering in $B_1(0)$ for each \tilde{v}_i . Ensuring that $c(n, \Lambda, \alpha, M_0, \Gamma, \epsilon)$ is sufficiently large, we may reduce to arguing for $r < \eta$. We now use Corollary 4.8.2 to build a covering \mathcal{U}_1 . If every $r_x = R$, then the packing and covering estimates give the claim

directly, since

$$R^{k-n} \text{Vol}(B_R(\mathcal{S}_{\epsilon, \eta_0 R}^k(\tilde{v}_i) \cap B_1(0))) \leq \omega_n R^{k-n} \sum_{\mathcal{U}_1} (2R)^n = \omega_n 2^n \sum_{\mathcal{U}_1} r_x^k \leq c(n, \Lambda, \alpha, M_0, \Gamma, \epsilon)$$

If there exists an $r_x \neq R$, we use Corollary 4.8.2, to build a finite sequence of refined covers, $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \dots$ such that for each for each i , the covering satisfies the following properties:

(A_i) Packing:

$$\sum_{x \in \mathcal{U}_i} r_x^k \leq c(n, \Lambda, \alpha, M_0, \Gamma, \epsilon) \left(1 + \sum_{x \in \mathcal{U}_{i-1}} r_x^k\right)$$

(B_i) Covering control

$$\mathcal{S}_{\epsilon, \eta_0 R}^k(\tilde{v}_i) \cap B_1(0) \subset \bigcup_{x \in \mathcal{U}_i} B_{r_x}(x)$$

(C_i) Energy drop: For every $x \in \mathcal{U}_i$, either

$$r_x = R \quad \text{or} \quad \sup_{p \in B_{2r_s}(s)} N(2r_s, p, \tilde{v}_i) \leq C(n, \Lambda, \alpha, M_0, \Gamma) - i \left(\frac{\eta_0}{2}\right).$$

(D_i) radius control:

$$\sup_{x \in \mathcal{U}_i} r_x \leq 10^{-i}$$

If we can construct such a sequence of covers, then we claim that this process will terminate in finite time, *independent of R*. Recall that blow-ups of \tilde{v}_i are homogeneous harmonic polynomials. Therefore

$$N(0, p, \tilde{v}_i) = \lim_{r \rightarrow \infty} N(r, p, \tilde{v}_i) \geq 1$$

for all $p \in \mathbb{R}^n$. By Remark 3.3.10 we have that for all $0 < r \leq 1$,

$$N(r, p, \tilde{v}_i) \geq 1 - C(n, \Lambda, \alpha, M_0, \Gamma, \epsilon),$$

for all $p \in B_1(0)$. Therefore, we know that for i sufficiently large such that,

$$i > (C(n, \Lambda, \alpha, M_0, \Gamma, \epsilon) + C(n, \Lambda, \alpha, M_0, \Gamma, \epsilon) - 1) \frac{2}{\eta_0}$$

it must be the case that $r_x = R$ for all $x \in \mathcal{U}_i$. In this case, we will have the claim with a bound of the form

$$R^{k-n} \text{Vol}(B_R(\mathcal{S}_{\epsilon, \eta_0 R}^k(\tilde{v}_i) \cap B_1(0))) \leq c(n, \Lambda, \alpha, M_0, \Gamma, \epsilon) C(n, \Lambda, \alpha, M_0, \Gamma, \epsilon)$$

Thus, we reduce to inductively constructing the required covers. Suppose we have already constructed \mathcal{U}_{i-1} as desired. For each $x \in \mathcal{U}_{i-1}$ with $r_x > R$, we apply Corollary 4.8.2 at scale $B_{r_x}(x)$ to obtain a new collection of balls, $\mathcal{U}_{i,x}$. From the assumption that $r_x \leq 1/10$ and the way that Hölder norms scale, it is clear that \tilde{v}_i satisfies the hypotheses of Corollary 4.8.2 in $B_{r_x}(x)$ with the same constants. To check packing control, we have that

$$\sum_{y \in \mathcal{U}_{i,x}} r_y^k \leq c(n, \Lambda, \alpha, M_0, \Gamma, \epsilon) r_x^k$$

Covering control follows immediately from the statement of Corollary 4.8.2. Similarly, from hypothesis (C_{i-1}) , we have that $\sup_{p \in B_{2r_x}(x)} N(2r_x, p, \tilde{v}_i) \leq C(n, \Lambda, \alpha, M_0, \Gamma, \epsilon) - (i-1) \frac{\eta_0}{2}$. Thus, the statement of Corollary 4.8.2 at scale $B_{r_x}(x)$ gives that $\sup_{p \in B_{2r_y}(y)} N(2r_y, p, \tilde{v}_i) \leq C(n, \Lambda, \alpha, M_0, \Gamma, \epsilon) - i(\frac{\eta_0}{2})$ for all $y \in \mathcal{U}_{i,x}$ with $r_y > R$. Radius control follows immediately from the fact that $\sup_{y \in \mathcal{U}_{i,x}} r_y \leq r_x/10 \leq 10^{-i}$.

Thus, if we let

$$\mathcal{U}_i = \{x \in \mathcal{U}_{i-1} | r_x = R\} \cup \bigcup_{\substack{x \in \mathcal{U}_{i-1} \\ r_x > R}} \mathcal{U}_{i,x}$$

then \mathcal{U}_i satisfy the inductive claim.

To obtain the cover which proves the theorem, then, we simple scale each covering of

$\mathcal{S}_{\epsilon, \frac{\eta_0 R}{\kappa}}^k(\tilde{v}_i) \cap B_1(0)$ to a covering of $\mathcal{S}_{\epsilon, \eta_0 R}^k(v) \cap B_\kappa(y_i)$ and sum over the $c(n, \Lambda, \alpha, M_0, \Gamma, \epsilon)$ such balls which cover $\mathcal{S}_{\epsilon, \eta_0 R}^k(v) \cap B_{\frac{1}{4}}(0)$. This completes the proof. \square

4.9.1 Proof of Theorem 4.1.1

Proof. By Theorem 4.9.2, we have that,

$$\text{Vol}(B_R(\mathcal{S}_{\epsilon, \eta_0 R}^k(v) \cap B_{1/4}(0))) \leq C(n, \Lambda, \alpha, M_0, \Gamma, \epsilon)R^{n-k}$$

Thus, let $r_0 = \eta_0$ and $r = \eta_0 R'$ for $0 < R' \leq 1$. For any $r \leq R \leq R'$, by containment, we have

$$B_R(\mathcal{S}_{\epsilon, r}^k(v) \cap B_{1/4}(0)) \subset B_{R'}(\mathcal{S}_{\epsilon, r}^k(v) \cap B_{1/4}(0)) \subset \bigcup_i B_{2R'}(x_i).$$

where $\{x_i\}$ are the centers of the balls in the covering constructed in Theorem 4.9.2. Therefore, the estimates in Theorem 4.9.2 give that,

$$\begin{aligned} \text{Vol}(B_R(\mathcal{S}_{\epsilon, r}^k(v) \cap B_{1/4}(0))) &\leq C(n, \Lambda, \alpha, M_0, \Gamma, \epsilon)2^n (R')^{n-k} \\ &\leq C(n, \Lambda, \alpha, M_0, \Gamma, \epsilon)2^n \left(\frac{R}{\eta_0}\right)^{n-k} \\ &\leq C(n, \Lambda, \alpha, M_0, \Gamma, \epsilon)R^{n-k} \end{aligned}$$

by increasing our constant $C(n, \Lambda, \alpha, M_0, \Gamma, \epsilon)$.

For any For any $R' \leq R$, by containment, we have

$$B_R(\mathcal{S}_{\epsilon, r}^k(v) \cap B_{1/4}(0)) \subset \bigcup_i B_{2R}(x_i).$$

where $\{x_i\}$ are the centers of the balls in the covering constructed in Theorem 4.9.2. In this case,

$$\begin{aligned} \text{Vol}(B_R(\mathcal{S}_{\epsilon,r}^k(v) \cap B_{1/4}(0))) &\leq C(n, \Lambda, \alpha, M_0, \Gamma, \epsilon) 2^n (R)^{n-k} \\ &\leq C(n, \Lambda, \alpha, M_0, \Gamma, \epsilon) R^{n-k} \end{aligned}$$

by increasing our constant $C(n, \Lambda, \alpha, M_0, \Gamma, \epsilon)$. This concludes the proof of Theorem 4.1.1. \square

4.10 Proof of Corollary 4.1.3

In this section, we prove that $\text{sing}(\partial\Omega^\pm) \subset \mathcal{S}_\epsilon^{k-3}(v)$ for ϵ small enough.

Lemma 4.10.1. Let $v \in \mathcal{A}(\Lambda, \alpha, M_0)$ with $\|\ln(h)\|_\alpha \leq \Gamma$. Then there exists an $0 < \epsilon = \epsilon(M_0, \alpha, \Gamma)$ such that $\text{sing}(\partial\Omega^\pm) \cap B_{1/4}(0) \subset \mathcal{S}_\epsilon^{n-3}(v)$.

Proof. We must argue that there is an $0 < \epsilon$ such that for all $Q \in \text{sing}(\partial\Omega^\pm) \cap B_1(0)$ and all radii, $0 < r$,

$$\int_{B_1(0)} |T_{Q,r}v - P|^2 dx \geq \epsilon$$

for all $n - 2$ -symmetric functions, P .

If P is $n - 2$ -symmetric, P only depends upon 2 variables. By Complex Analysis all homogeneous harmonic polynomials in 2- dimensions are of the form $q(z) = c(x + iy)^k$. By Theorem 2.4.11, we need only consider $k \geq 2$. Hence, the zero-set, Σ_q , of any q is the union of an even number of infinite rays equidistributed in angle. If we label the connected components of $\mathbb{R}^2 \setminus \Sigma_q$, $\{U_i\}$, we see that by the Maximum Principle, the sign of q must change from one U_i to another, contiguous U_j .

Thus, the zero set of P is $\Sigma_P = \Sigma_q \times \mathbb{R}^{n-2}$ for some homogeneous harmonic polynomial,

$q : \mathbb{R}^2 \rightarrow \mathbb{R}$ of degree ≥ 2 . We label the connected components of $\mathbb{R}^n \setminus \Sigma_q \times \mathbb{R}^{n-2}$, as $\{W_i\}$.

Now, we claim that there is a constant, $0 < c(M_0, \Gamma, \alpha) \leq 1$, such that one of the following statements must hold.

1.

$$\mathcal{H}^n(T_{Q,r}\Omega^- \cap \bigcup_i \{W_i : P > 0 \text{ on } W_i\} \cap B_1(0)) \geq c$$

2.

$$\mathcal{H}^n(T_{Q,r}\Omega^+ \cap \bigcup_i \{W_i : P < 0 \text{ on } W_i\} \cap B_1(0)) \geq c.$$

Note that by Theorem 2.4.2(2), we need only consider P with degree $\leq d(M_0) < \infty$. Reducing to \mathbb{R}^2 , since the rays of Σ_q are equidistributed, for q of degree k the connected components occupy a sector of aperture $\frac{\pi}{k}$. Thus, if $B_{\frac{1}{M_0}}(A_1^\pm(0)) \subset T_{Q,r}\Omega^\pm$, is the ball guaranteed by the corkscrew condition, then for $c = \frac{1}{4M_0^n}$, there exists an integer $k(M_0)$ such that

$$\mathcal{H}^n(B_{\frac{1}{M_0}}(A_1^\pm(0)) \cap \{P \cdot T_{Q,r}v < 0\}) \geq c$$

for all P with degree $\geq k(M_0)$.

For P with degree $\leq k(M_0)$, We argue by contradiction. Suppose that no such constant exists. Then, there would be a sequence of functions, $v_j \in \mathcal{A}(\Lambda, \alpha, M_0)$ with points, $Q_j \in B_{1/4}(0)$ and radii, $0 < r_j \leq 1/2$ and zero sets, Σ_{P_j} for P_j of degree ≥ 2 and $\leq k(M_0)$ such that the scaled and translated mutual boundaries, $T_{Q_j,r_j}\partial\Omega_j^\pm$, satisfy the following property,

$$\begin{aligned} \mathcal{H}^n(T_{Q_j,r_j}\Omega_j^- \cap \bigcup_i \{W_{i,j} : P_j > 0 \text{ on } W_{i,j}\} \cap B_1(0)) &\rightarrow 0 \\ \mathcal{H}^n(T_{Q_j,r_j}\Omega_j^+ \cap \bigcup_i \{W_{i,j} : P_j < 0 \text{ on } W_{i,j}\} \cap B_1(0)) &\rightarrow 0. \end{aligned}$$

By Lemma 3.2.3 there exists a subsequence for which $T_{Q_j,r_j}\partial\Omega_j^\pm$ converge locally in the Hausdorff metric to a limit set, $A \subset \mathbb{R}^n$. By Theorem 3.2.5, A must be the mutual boundary

of a pair of two-sided NTA domains, Ω_∞^\pm with constant $2M_0$. Furthermore, up to scaling and rotation, the number of homogeneous harmonic functions of 2 variables in \mathbb{R}^n with degree ≥ 2 and $\leq k(M_0)$ is finite. Since the space of rotations is compact, we may find a subsequence, Σ_{P_j} , which converge to Σ_{P_∞} , for some $n - 2$ -symmetric P_∞ locally in the Hausdorff metric. This implies that,

$$\mathcal{H}^n(\Omega_\infty^- \cap \bigcup_i \{W_{i,\infty} : P_\infty > 0 \text{ on } W_{i,\infty}\} \cap B_1(0)) = 0 \quad (4.10.1)$$

$$\mathcal{H}^n(\Omega_\infty^+ \cap \bigcup_i \{W_{i,\infty} : P_\infty < 0 \text{ on } W_{i,\infty}\} \cap B_1(0)) = 0. \quad (4.10.2)$$

Indeed, if there were $p \in \bigcup_i \{W_{i,\infty} : P_\infty > 0 \text{ on } W_{i,\infty}\} \cap B_1(0)$ such that $p \in \Omega_\infty^-$, since $W_{i,\infty}$ and Ω^- are open, there would exist a ball $B_\delta(p) \subset \Omega^- \cap W_{i,\infty}$. Therefore, since $\Sigma_{P_j} \rightarrow \Sigma_{P_\infty}$ and $T_{Q_j, r_j} \partial \Omega_j^\pm \rightarrow A$ locally in the Hausdorff metric, for all j sufficiently large, $B_{\frac{1}{2}\delta}(p) \subset W_{i,j} \cap T_{Q_j, r_j} \partial \Omega_j^-$. This is a contradiction. The other equation follows identically.

Now, we claim that $A \cap B_1(0) = \Sigma_{P_\infty} \cap B_1(0)$. Suppose not, then there exists a point, $p \in \Sigma_{P_\infty}$ with $p \notin A$ or there exists a point, $Q \in A$ such that $Q \notin \Sigma_{P_\infty}$. In the former case, suppose the $\text{dist}(p, A) > \delta$. Then, $B_\delta(p)$ must intersect at least 2 contiguous connected components, $W_{i,\infty}, W_{j,\infty}$. Since they are contiguous, the sign of P_∞ must be positive on one and negative on the other. This contradicts Equation 4.10.1. Similarly, if there exists a point, $Q \in A$ such that $Q \notin \Sigma_{P_\infty}$ then there exists a ball $B_\delta(Q)$ which intersects both Ω_∞^\pm but which is contained in a single $W_{i,\infty}$. This also contradicts Equation 4.10.1.

However, if P_∞ is $n - 2$ -symmetric with degree ≥ 2 , then Σ_{P_∞} does not divide \mathbb{R}^n into two connected components. This contradicts our assumption that $A = \Sigma_{P_\infty}$ was the mutual boundary of a pair of two-sided NTA domains with constant $2M_0$. Therefore, such a constant, $0 < c = c(M_0, \Gamma, \alpha)$ must exist.

Without loss of generality, we assume (1) holds. By Lemma 6.1.2 we may find a radius,

$0 < r = r(M_0, \Gamma, \alpha)$ such that $\mathcal{H}^n(B_r(T_{Q,r}\partial\Omega^\pm)) < \frac{1}{20}c(\alpha, M_0, \Gamma)$. Now, consider

$$p \in \bigcup\{W_i : P > 0 \text{ on } W_i\} \cap B_1(0) \setminus B_r(T_{Q,r}\partial\Omega^\pm).$$

By Lemma 3.2.2, $|T_{Q,r}v(p)| \geq c'$ for a constant, $c' = c'(M_0, \Gamma, \alpha)$. Thus,

$$\begin{aligned} \int_{B_1(0)} |T_{Q,r}v - P|^2 dx &\geq \int_{B_1(0) \cap T_{Q,r}\partial\Omega^- \cap \bigcup_i\{W_i : P > 0 \text{ on } W_i\}} |T_{Q,r}v - P|^2 dx \\ &\geq \frac{19}{20}c(\alpha, M_0, \Gamma)c'(\alpha, M_0, \Gamma)^2 \end{aligned}$$

If (2) holds, an identical argument with signs switched proves the claim. \square

Remark 4.10.2. It is clear that the argument above can be modified to show that there is an $0 < \epsilon'$ such that if $Q \in \partial\Omega$ but $Q \notin \mathcal{S}_{\epsilon', r_0}^{n-3}$, then $Q \notin \mathcal{S}_{\epsilon', r_0}^{n-2}$. Indeed, if $Q \notin \mathcal{S}_{\epsilon', r_0}^{n-3}$, then there exists a radius, $r_0 \leq r$, and an $n-2$ -symmetric function, P , such that $\|T_{Q,r}v - P\|_{L^2(B_1(0))}^2 \leq \epsilon'$. However, by taking $\epsilon' < \epsilon(\alpha, M_0, \Gamma)$ in Lemma 4.10.1, we see that P must be $n-1$ -symmetric.

Chapter 5

EXAMPLES OF RECTIFIABLE SETS

5.1 Proof of Theorem 1.2.1

In this chapter, we only consider $E \subset \mathbb{R}^2$ and the β -numbers when $p = 2$. As such, we write β_E, β_μ, C_E , and C_μ in place of $\beta_{E;2}^1, \beta_{\mu;2}^1, C_{E;2}^1$, and $C_{\mu;2}^1$. Moreover, for any set $L \subset \mathbb{R}^2$ we write $B_r(L) = \{x : \text{dist}(x, L) < r\}$ and $B_r = B_r(\{0\})$.

We now construct a 1-rectifiable set which is connected and hence Ahlfors lower-regular for which the Jones function is locally non-integrable. This set will be generated by modifying approximations to the Koch snowflake. This set will not be upper regular.

Let us recall some facts about (and introduce some notation useful for describing) the standard approximation to the Koch snowflake.

Definition 5.1.1. Let $I \subset \mathbb{R}^2$ be a line segment, and fix $0 < \alpha < \pi/2$. Define $P(I)$ as the set which results from the following operation

1. Divide I into three equal subintervals, $I_{\text{left}} \cup I_{\text{center}} \cup I_{\text{right}}$.
2. Over the middle interval, I_{center} , construct an isosceles triangle with angles α and base I_{center} .
3. Delete I_{center} , the base of the isosceles triangle.

We define

$$S(I) = \overline{P(I) \setminus I}, \quad (5.1.1)$$

and call $S(I)$ the *bump*. If q_I is the orthogonal projection onto the line containing I and q_I^\perp is

the orthogonal projection onto I^\perp , then $\text{height}(S(I)) = \text{diam}\{q_I^\perp(S(I))\}$ and $\text{width}(S(I)) = \text{diam}\{\pi_I(S(I))\} = \frac{1}{3}\mathcal{H}^1(I)$.

We shall abuse our notation slightly by saying that for a collection of line segments, E , the set $P(E)$ is obtained by applying P to each maximal line segment contained in E .

Note that if $I = [0, 1] \times \{0\}$ and $\alpha = \frac{\pi}{3}$, the standard approximations to the Koch snowflake are given by $\{P^k(I)\}_{k=1}^\infty$, where P^k denotes applying P iteratively k times. We emphasize a few properties about deformations under the operation P .

Proposition 5.1.1. *For any finite line segment $I \subset \mathbb{R}^2$ and positive integer n ,*

$$\text{height}(S(I)) = \frac{\tan(\alpha)}{6}|I| \tag{5.1.2}$$

$$\mathcal{H}^1(S(I)) = \frac{\sec(\alpha)}{3}|I| \tag{5.1.3}$$

$$\mathcal{H}^1(P^n(E)) = \left(\frac{\sec(\alpha) + 2}{3}\right)^n \mathcal{H}^1(E) \tag{5.1.4}$$

When $\tau = \frac{1}{20} \min\left\{\frac{\tan(\alpha)}{6}, \frac{1}{3}\right\}$, there exists a constant $c_0 = c(\alpha)$ such that for all lines L

$$\mathcal{H}^1(S(I) \setminus B_\tau(L)) \geq c_0 \mathcal{H}^1(S(I)). \tag{5.1.5}$$

Proof. (5.1.2) and (5.1.3) follow from planar geometry. The $n = 1$ case for (5.1.4) follows by adding back in the unchanged intervals I_{left} and I_{right} , which have total length $\frac{2}{3}|I|$. The geometric nature of the definition of P allows us to then iterate this to achieve (5.1.4).

To verify (5.1.5) we proceed by contradiction. Suppose no such constant c_0 exists. Then, there exists a sequence of lines intersecting $S(I)$ such that $\mathcal{H}^1(S(I) \setminus B_\tau(L_i)) < 2^{-i}\mathcal{H}^1(S(I))$. After passing to a subsequence, L_i converge to some line L with the property that $\mathcal{H}^1(S(I) \setminus B_\tau(L)) = 0$. Since $S(I)$ is connected, this implies $S(I) \subset B_{2\tau}(L)$. However, this contradicts the fact that $2\tau \leq \frac{1}{10} \min\{\text{height}(S(I)), \text{width}(S(I))\}$. \square

Definition 5.1.2. Define \mathcal{P}_j to be the set operation defined on line-segments by

$$\mathcal{P}_j(I) = P^{j-1}(S(I)) \cup (I \setminus I_{\text{center}}),$$

recalling the definition of $S(I)$ can be found in (5.1.1). Loosely speaking, for any line segment, I , $\mathcal{P}_j(I)$ is the set that replaces the center of I with a j th approximation of the Koch curve.

Corollary 5.1.2. *For any line segment $I \subset \mathbb{R}^2$ and positive integer n*

$$\mathcal{H}^1(\mathcal{P}_n(I)) = \frac{2}{3}|I| + \left(\frac{\sec(\alpha) + 2}{3}\right)^{n-1} \frac{\sec(\alpha)}{3}|I|. \quad (5.1.6)$$

Moreover, if $\alpha \leq \pi/3$,

$$\text{dist}_{\mathcal{H}}(I, P^n(I)) \leq \frac{\tan(\alpha)}{12}|I|. \quad (5.1.7)$$

Proof. Equations (5.1.3) and (5.1.4) verify (5.1.6). Indeed,

$$\mathcal{H}^1(P^{n-1}(S(I))) = \left(\frac{\sec(\alpha) + 2}{3}\right)^{n-1} \mathcal{H}^1(S(I)) = \left(\frac{\sec(\alpha) + 2}{3}\right)^{n-1} \frac{\sec(\alpha)}{3}|I|.$$

The restriction to $\alpha \leq \pi/3$ ensures the longest line segment of $P^i(I)$ has length at most 3^{-i} . Consequently, (5.1.2) guarantees

$$\text{dist}_{\mathcal{H}}(P^n(I), I) \leq \sum_{i=1}^n \text{dist}_{\mathcal{H}}(P^i(I), P^{i-1}(I)) \leq \sum_{i=1}^n 3^{-i} \text{height}(S(I)) \leq \frac{\tan(\alpha)}{12}|I|.$$

□

Definition 5.1.3. Now, we let n be a natural number to be chosen later and $E_0 = I = [0, 1] \times \{0\}$. We define $E_1 = \mathcal{P}_n(I)$. For $k \geq 2$ inductively define

$$E_k = \mathcal{P}_{kn} \left(\left([0, 3^{-(k-1)}] \times \{0\} \right) \cup \left(\left\{ [3^{-(k-1)}, 1] \times \mathbb{R} \right\} \cap E_{k-1} \right) \right). \quad (5.1.8)$$

Notably, for all integers j the operation \mathcal{P}_j applied to $[0, 3^{-(k-1)}] \times \{0\}$ leaves the segment $[0, 3^{-k}] \times \{0\}$ untouched. Consequently, the sequence of sets $\{E_k\}$ are defined by replacing the “next” triadic interval with a scaled approximation of the Koch snowflake. The fact that each triadic strip $[3^{-k}, 3^{-(k-1)}] \times \mathbb{R}$ is only modified once in the sequence of sets E_k ensures the Hausdorff dimension of the final set remains 1.

Lemma 5.1.3 (Base Set). *Fix $\alpha \leq \pi/3$ and any integer n satisfying¹*

$$3^{-1} \left(\frac{\sec(\alpha) + 2}{3} \right)^n < 1 < 3^{-1} \left(\frac{\sec(\alpha) + 2}{3} \right)^{2n}. \quad (5.1.9)$$

Then the sequence of sets E_k from (5.1.8) converge to a compact and connected Borel set E_∞ in the Hausdorff topology on compact subsets. Furthermore, E_∞ satisfies:

1. $\mathcal{H}^1(E_\infty) < \infty$
2. For all $\delta > 0$, $C_{E_\infty}(0, \delta) = +\infty$.

Proof. The existence of the limiting compact set E_∞ follows from precompactness of sets contained in B_{10} in the Hausdorff distance and (5.1.7) which ensures that $\text{dist}_{\mathcal{H}}(E_{k+1}, E_k) \sim 3^{-k}$. Connected follows since $E_\infty \setminus B_{3^{-k}}(0) = E_k \setminus B_{3^{-k}}(0)$ is connected for each k .

To see that E_∞ has finite length we write $\mathcal{H}^1(E_k) = \mathcal{H}^1(E_k \setminus B_{3^{1-k}}) + \mathcal{H}^1(E_k \cap B_{3^{1-k}})$. Since $E_k \setminus B_{3^{1-k}} = E_{k-1} \setminus B_{3^{1-k}}$ and $\mathcal{H}^1(E_{k-1} \setminus B_{3^{1-k}}) = \mathcal{H}^1(E_{k-1}) - 3^{1-k}$, (5.1.6) implies,

$$\mathcal{H}^1(E_k) - \mathcal{H}^1(E_{k-1}) = 3^{-k} \left[\left(\frac{\sec(\alpha) + 2}{3} \right)^{nk} \sec(\alpha) - 1 \right].$$

Since $\mathcal{H}^1(E) = 1$, iteration yields

$$\mathcal{H}^1(E_k) = 1 + \sum_{i=1}^k 3^{-i} \left[\left(\frac{\sec(\alpha) + 2}{3} \right)^{ni} \sec(\alpha) - 1 \right]. \quad (5.1.10)$$

¹Note that for instance, $\alpha = \pi/3$ and $n \in \{2, 3\}$ satisfies (5.1.9).

In particular, $\lim_{k \rightarrow \infty} \mathcal{H}^1(E_k) < \infty$ whenever n satisfies the lower bound from (5.1.9). Moreover $\mathcal{H}^1(E_\infty) = \lim_{k \rightarrow \infty} \mathcal{H}^1(E_k)$ since for all $j \geq k$,

$$\mathcal{H}^1(E_j \Delta E_k) \leq 2 \sum_{i=k+1}^{\infty} 3^{-i} \left(\frac{\sec(\alpha) + 2}{3} \right)^{ni} \sec(\alpha),$$

which decays to zero as $k \rightarrow \infty$. Hence, (5.1.10) holds for E_∞ and $0 < \mathcal{H}^1(E_\infty) < \infty$.

It only remains to show $C_{E_\infty}(0, \delta) = +\infty$ for all $\delta > 0$. To this end, we first note that when $r = r(n, \alpha) = 3^{-1} \left(\frac{\sec(\alpha) + 2}{3} \right)^n$,

$$\mathcal{H}^1(E_\infty \cap B_{3^{-k}}(0)) = 3^{-k} + \sec(\alpha) \frac{r^{k+1}}{1-r} - \frac{3^{-(k+1)}}{1-3^{-1}}. \quad (5.1.11)$$

Indeed, by (5.1.10) and the trick used to prove (5.1.10)

$$\mathcal{H}^1(E_\infty \cap B_{3^{-k}}(0)) = 3^{-k} + \sum_{i=k+1}^{\infty} 3^{-i} \left[\sec(\alpha) \left(\frac{\sec(\alpha) + 2}{3} \right)^{ni} - 1 \right].$$

Claim: With τ as in Proposition 5.1.1 and $\alpha \leq \pi/3$, there exists a constant c_1 and integer j_0 independent of k such that for any line L , and all k such that $nk - 1 - j_0 \geq 0$,

$$\mathcal{H}^1 \left(\left(E_\infty \setminus B_{\frac{\tau}{2 \cdot 3^k}}(L) \right) \cap B_{3^{-k}} \right) \geq c_1 3^{-k} \left(\frac{\sec(\alpha) + 2}{3} \right)^{nk-1-j_0}. \quad (5.1.12)$$

Proof of Claim. Writing $I' = [0, 1] \times \{0\}$, we will in fact scale by 3^k and show the stronger result that

$$\mathcal{H}^1 \left(\left(\mathcal{P}_{nk}(I') \setminus B_{\frac{\tau}{2 \cdot 3^{j_0}}}(L) \right) \cap B_{3^0} \right) \geq c_1 3^0 \left(\frac{\sec(\alpha) + 2}{3} \right)^{nk-1-j_0} |I'|.$$

To do so, we find a line segment $J \subset S(I') \setminus B_\tau(L)$ such that J has an endpoint in common with one of the two line segments of $S(I')$ and $|J| = 3^{-j_0} \mathcal{H}^1(S(I'))/2$, where j_0 to be

chosen later is independent of L . This specific choice of length and endpoint ensure that $P^{nk-1-j_0}(J) \subset \mathcal{P}_{nk}(I')$. Moreover, the choice of j_0 will both guarantee that $|J|$ is large enough and that $P^{nk-1-j_0}(J)$ remains outside of $B_{\tau/2}(L)$, hence verifying Claim 2.

To find J , we note that the simple shape of $S(I')$ guarantees that $S(I') \setminus B_\tau(L)$ has at most 4 maximal line segments. Hence, there exists a maximal line segment $K_L \subset S(I') \setminus B_\tau(L)$ with $\mathcal{H}^1(K_L) \geq \frac{1}{4}\mathcal{H}^1(S(I') \setminus B_\tau(L))$. If K_L is parallel to L let x_L denote either endpoint of K_L . Otherwise, let x_L denote the unique endpoint of K_L that is not contained in $\overline{B_\tau(L)}$. Define J to be the unique subset of K_L of length $3^{-j_0} \frac{\sec(\alpha)}{6} |I'|$ with endpoint x_L . Now, define j_0 as the smallest integer such that

$$3^{-j_0} < \min \left\{ \frac{c_0}{4}, \left(\frac{\tan(\alpha)}{12} \cdot \frac{\sec(\alpha)}{6} |I'| \right)^{-1} \frac{\tau}{2} \right\},$$

where c_0 is as in Proposition 5.1.1. The first condition ensures that $J \subset K_L$ and (5.1.5) guarantees that the first constraint on j_0 is independent of L and k . The second constraint combined with (5.1.3) and (5.1.7) ensure that $\text{dist}_{\mathcal{H}}(P^{nk-1-j_0}(J), J) \leq \frac{\tau}{2}$. Moreover, choosing j_0 to be the smallest admissible integer, and guarantees that $|J| = 3^{-j_0} \frac{\sec(\alpha)}{6} |I'| \geq c' |I'|$ where c' is independent of L and k . Finally, (5.1.4) completes the proof of the Claim since

$$\mathcal{H}^1(\mathcal{P}_{nk}(I') \setminus B_{\tau/2}(L)) \geq \mathcal{H}^1(P^{nk-1-j_0}(J)) \geq c_1 \left(\frac{\sec(\alpha) + 2}{3} \right)^{nk-1-j_0} |I'|,$$

where c_1 depends only on α .

Whenever $nk - 1 - j_0 \geq 0$, (5.1.12) implies

$$\beta_{E_\infty}(0, 3^{-k})^2 \geq \frac{1}{3^{-k}} \left(\frac{\frac{\tau}{2 \cdot 3^k}}{3^{-k}} \right)^2 \left(c_1 3^{-k} \left(\frac{\sec(\alpha) + 2}{3} \right)^{nk-1-j_0} \right) = c_2 \left(\frac{\sec(\alpha) + 2}{3} \right)^{nk} \quad (5.1.13)$$

Fix $\delta > 0$ and any integer k_δ such that $3^{-k_\delta} < \delta$ and $nk_\delta - 1 - j_0 \geq 0$. Then, with $\mu = \mathcal{H}^1 \llcorner E_\infty$,

repeated applications of Proposition 2.5.1, (5.1.13), and (5.1.11) yield

$$\begin{aligned} \int_{B_\delta(0)} \int_0^\delta \beta_\mu(x, r)^2 \frac{dr}{r} d\mu(x) &\geq \ln(3)3^{-2} \sum_{k=k_\delta}^\infty \mu(B_{3^{-(k+2)}}) \beta_\mu(0, 3^{-(k+2)})^2 \\ &\geq \ln(3)3^{-2} \sum_{k=k_\delta}^\infty \left(3^{-k} + \sec(\alpha) \frac{r^{k+1}}{1-r} - \frac{3^{-(k+1)}}{1-3^{-1}} \right) \left(c_2 \left(\frac{2 + \sec(\alpha)}{3} \right)^{nk} \right). \end{aligned}$$

Due to the lower bound in (5.1.9), this sum diverges if and only if

$$\sum_{k=k_\delta}^\infty \left[\sec(\alpha) \frac{r^{k+1}}{1-r} - \frac{1}{3^{k+1} - 3^k} \right] \left(\frac{2 + \sec(\alpha)}{3} \right)^{nk} = \sum_{k=k_\delta}^\infty \left[\sec(\alpha) \frac{3^k r^{2k+1}}{1-r} - \frac{r^k}{3-1} \right]$$

diverges. Since the lower bound in (5.1.9) ensures $r < 1$, this diverges if and only if $\sum_{k=k_\delta}^\infty (3r^2)^k$ diverges which is equivalent to the upper bound in (5.1.9). \square

We next wish to iterate E_∞ densely “along itself” to create the set from Theorem 1.2.1. When performing this iteration, we only need to take care that the length remains finite. We restate Theorem 1.2.1 for convenience.

Theorem 5.1.4. *There exists a connected set, $K_0 \subset \mathbb{R}^2$ of finite \mathcal{H}^1 -measure such that for any $x \in K_0$ and $\delta > 0$*

$$C_{K_0}(x, \delta) = \infty.$$

Proof. Let $\{r_i\}_{i=1}^\infty$ be a sequence of positive numbers such that $\sum_i r_i \leq 1$. Let $E^{x,r} \subset \mathbb{R}^2$ be the set $E^{x,r} = rE_\infty + x$. We construct K_0 as the union of a countable collection of nested sets $\{\Gamma_i\}$.

Let $\Gamma_0 = E_\infty$. Now, let $\{x_{1,j}\}_{j=1}^{N_1}$ be a maximal 2^{-1-1} -separated collection of points in Γ_0 . Let

$$\Gamma_1 = \Gamma_0 \cup \bigcup_{j=1}^{N_1} E^{x_{1,j}, \frac{r_1}{N_1}}.$$

Suppose that we have defined Γ_{i-1} , some positive integers $\{N_\ell\}_{\ell=1}^{i-1}$ and a collection of points $\{x_{\ell,j} \in \Gamma_{i-2} \mid 1 \leq \ell \leq i-1, 1 \leq j \leq N_\ell\}$ that form a maximal $2^{-(i-1)-1}$ net for Γ_{i-2} . Then choose $N_i \in \mathbb{N}$ and points $\{x_{i,j}\}_{1 \leq j \leq N_i} \subset \Gamma_{i-1}$ so that $\{x_{\ell,j} \in \Gamma_{i-1} \mid 1 \leq \ell \leq i, 1 \leq j \leq N_\ell\}$ is a maximal 2^{-i-1} net in Γ_{i-1} . Then define Γ_i by

$$\Gamma_i = \Gamma_{i-1} \cup \left(\bigcup_{j=1}^{N_i} E^{x_{i,j}, \frac{r_i}{N_i}} \right).$$

We claim that $K_0 = \bigcup_{i=0}^{\infty} \Gamma_i$ is the desired set. First note that since each Γ_i is rectifiable, K_0 is rectifiable. Moreover, $\{x_{i,j}\}_{j=1}^{N_i} \subset \Gamma_{i-1}$ for all i ensures K_0 inherits connectivity from E_∞ . Furthermore, since $\{\Gamma_i\}$ is a nested sequence increasing to K_0 and $\sum_i r_i \leq 1$,

$$\mathcal{H}^1(K_0) = \mathcal{H}^1 \left(E_\infty \cup \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{N_i} E^{x_{i,j}, \frac{r_i}{N_i}} \right) \leq \mathcal{H}^1(E_\infty) \left(1 + \sum_{i=1}^{\infty} r_i \right) \leq 2\mathcal{H}^1(E_\infty).$$

It only remains to show that for $x \in K_0$ and $\delta > 0$ that $C_{K_0}(x, \delta) = \infty$. To this end, fix $x \in K_0$, and $\delta > 0$. By definition of K_0 , there exists ℓ_0 such that $x \in \Gamma_{\ell_0}$. Then, by the net property of the points $\{x_{i,j}\}$, it follows that for $\ell - 1 \geq \ell_0$ large enough that $2^{-\ell-1} < \delta/4$, there exists $i \leq \ell$ with $x_{i,j} \in \Gamma_{\ell-1} \cap B(x, \delta/2) \subset K_0 \cap B(x, \delta/2)$. Writing $\mu = \mathcal{H}^1 \llcorner K_0$ and $\mu_{i,j} = \mathcal{H}^1 \llcorner E^{x_{i,j}, \frac{r_i}{N_i}}$ it follows from monotonicity of the integral that

$$\int_{B_\delta(x)} \int_0^\delta \beta_{K_0;2}(y, r)^2 \frac{dr}{r} d\mu(y) \geq \int_{B_{\delta/2}(x_{i,j})} \int_0^{\delta/2} \beta_{\mu_{i,j};2}(y, r) \frac{dr}{r} d\mu_{i,j}(y), \quad (5.1.14)$$

or equivalently $C_{K_0}(x, \delta) \geq C_{\mu_{i,j}}(x_{i,j}, \delta/2)$. Recalling that $E^{z,t} = tE_\infty + z$, we use (5.1.14), Proposition 2.5.1(2), and Lemma 5.1.3 to conclude

$$C_{K_0}(x, \delta) \geq C_{E^{x_{i,j}, \frac{r_i}{N_i}}} \left(x_{i,j}, \frac{\delta}{2} \right) = \frac{r_i}{N_i} C_{E_\infty} \left(0, \frac{\delta N_i}{2r_i} \right) = \infty.$$

Since $x \in K_0$ and $\delta > 0$ are arbitrary this finishes the proof. \square

The authors thank Matthew Badger for pointing out that $\overline{K_0}$ coincides with the Hausdorff-limit of $\{\Gamma_i\}$. So, Gołab's semi-continuity theorem and Ważewski's theorem ensures $\overline{K_0}$ is a rectifiable curve. See [1] for relevant theorem statements.

5.2 Proof of Theorem 1.2.2

To produce the desired set A_0 , we use approximations of the 4-corner Cantor set to produce a base set that has precise control on the β -numbers at the origin, then we carefully iterate this set “on itself” in order to preserve Ahlfors regularity.

5.2.1 Approximations to the 4-corner Cantor set

Consider the following sequence of approximations to the 4-corner cantor set, by sets of positive and finite \mathcal{H}^1 -measure.

Let $E_0 = [0, 1) \times \{0\}$ and inductively define

$$E_k = \sum_{(i,j) \in \{0,3\}^2} p_{ij} + 2^{-2} E_{k-1} \quad \text{where} \quad p_{ij} = \left(\frac{i}{2^2}, \frac{j}{2^2} \right). \quad (5.2.1)$$

The word similarity is used to refer to any mapping that can be written as a composition of scalings, rotations, reflections, and translations. Throughout the rest of the paper, we say that two sets are similar if one is the image of the other by a similarity. In reality the similarities we discuss can always be written as a scaling and translation, as in (5.2.1).

We let Δ denote the collection of tetradic half-open cubes in \mathbb{R}^2 , that is

$$\Delta = \{[a2^{-2k}, (a+1)2^{-2k}) \times [b2^{-2k}, (b+1)2^{-2k}) \mid a, b, k \in \mathbb{Z}\}.$$

For some $Q \in \Delta$, we let $\ell(Q)$ denote the sidelength of Q . We partition the tetradic cubes into cubes of fixed sidelength by defining $\Delta^i = \{Q \in \Delta \mid \ell(Q) = 2^{-2i}\}$.

In general, for a set $E \subset \mathbb{R}^2$ we the *length of E* and respectively *height of E* by

$$\ell(E) = \text{diam}\{\pi_x(E)\} \quad \text{and} \quad h(E) = \text{diam}\{\pi_y(E)\}$$

where π_x and π_y denote the orthogonal projection onto the horizontal and vertical axes. In particular, for a cube Q , this notion of length coincides with its sidelength.

Definition 5.2.1 (Clusters and sub-clusters). Any set which is similar to any E_k or $E_k \cup [0, 1) \times \{0\}$ for $k \in \mathbb{N}$ will be called a *cluster*.

Moreover, for fixed $k \in \mathbb{N}$, we will call E_k the *0th sub-cluster of E_k* and the 2^{2k} line segments that make up E_k are called the *k th -subclusters of E_k* . For $\ell \in \{1, \dots, k-1\}$, the $2^{2\ell}$ -sets contained in E_k which are similar to $E_{k-\ell}$ are called the *ℓ th sub-clusters of E_k* .

Definition 5.2.2 (Root points). We associate to each cluster and each cube a root point. The *root point* of a cluster E is the lower-most and left-most point in the cluster. Since a sub-cluster is itself a cluster, the notion of a root point extends to sub-clusters. For a cluster E , we let x_E denote its root point. For a tetradic cube $Q \in \Delta$ we let x_Q denote the lower-most and left-most point of Q and call x_Q the *root point of Q* .

Proposition 5.2.1. *For fixed non-negative integer k , the set E_k has the following properties.*

1. *Each E_k is a finite union of 2^{2k} intervals each of length 2^{-2k} . In particular, $\mathcal{H}^1(E_k) = 1$ and E_k is countably 1-rectifiable. Moreover, each connected component I of E_k has $\partial I \subset \ell(I)\mathbb{Z}^2 = 2^{-2k}\mathbb{Z}^2$ and consequently is contained in a line $\mathbb{R} \times \{a2^{-2k}\}$ for some $a \in \mathbb{N}_0$.*

2. If $j \geq 0$ is an integer and if $Q \in \Delta^j$ is such that $Q \cap E_k$ is non-empty, then

$$Q \cap E_k = \begin{cases} x_Q + [0, \ell(Q)) \times \{0\} & j \geq k \\ x_Q + 2^{-2j} E_{k-j} & j \leq k \end{cases} \quad (5.2.2)$$

3. Each E_k is Ahlfors regular with regularity constant independent of k .

4. For $0 \leq j \leq k$ an integer, the j th subcluster of E_k has \mathcal{H}^1 -measure 2^{-2j} .

5. For $1 \leq j \leq k$ an integer, the j th subclusters of E_k are $2 \cdot 2^{-2j}$ -separated horizontally and at least $2 \cdot 2^{-2j}$ -separated vertically. In fact, they are $\left(3 - \frac{3}{4} \sum_{i=1}^{k-j} 2^{-2i}\right) \cdot 2^{-2j}$ -separated vertically.

6. If $J \subset E_k$ is a connected component, then J is a vertical distance of $3 \cdot 2^{-2k}$ from the nearest connected component J' of E_k .

7. There exists a universal constant $c > 0$ such that if $k \geq 2$ and $\mu_k = \mathcal{H}^1 \llcorner E_k$, then for all $x \in E_k$,

$$\int_{6 \cdot 2^{-2k}}^1 \beta_{\mu_k}(x, r)^2 \frac{dr}{r} \geq c(k-2)$$

Proof. (1) follows immediately from (5.2.1) since each $p_{ij} \in 2^{-2}\mathbb{Z}^2$.

To see (2), we first note that the case $j = 0$ is clear for any $k \in \mathbb{N}$. Further, the case $k = 0$ is clear for all $j \in \mathbb{N}$. To proceed inductively suppose that (5.2.2) holds for all $k \in \mathbb{N}$ when $j = \ell - 1$. We will show it holds for all $k \in \mathbb{N}$ when $j = \ell$. Indeed, suppose that $Q \in \Delta^\ell$ has non-empty intersection with E_ℓ . Let x_Q be the root of Q . Choose $p \in \{p_{ij}\}_{(i,j) \in \{0,3\}^2}$ such that $Q \subset p + [0, 2^{-2})^2$. Then, $4(Q \cap E_k - p) = (4Q - 4p) \cap (4E_k - 4p) = \tilde{Q} \cap E_{k-1}$ where $\tilde{Q} := 4Q - 4p \in \Delta^{\ell-1}$. By the inductive assumption,

$$\tilde{Q} \cap E_{k-1} = \begin{cases} x_{\tilde{Q}} + [0, \ell(\tilde{Q})) \times \{0\} & \ell - 1 \geq k - 1 \\ x_{\tilde{Q}} + 2^{-2(i-1)} E_{(\ell-1)-(i-1)} & \ell - 1 \leq k - 1. \end{cases}$$

Translating and scaling this back to what this means about $Q \cap E_k$ verifies the induction.

(3) follows from (1) and (2) since these imply that $\frac{\mathcal{H}^1(Q \cap E_k)}{\ell(Q)} = 1$ for tetradic cubes Q with $\ell(Q) \leq 1$ that intersect E_k . This suffices since any ball contains a tetradic cube of comparable sidelength and is contained in 4^2 tetradic cubes of comparable sidelength.

(4) is equivalent to showing that E_k is made of 2^{2k} intervals, each of length 2^{-2k} .

(5) The horizontal separation is verified by an argument similar to the vertical separation. For the vertical separation, we only verify that the vertical separation is at least $2 \cdot 2^{-2j}$. Indeed, this follows since E_ℓ is contained in the horizontal strips $\mathbb{R} \times [0, 1/4] \cup [3/4, 1]$ for all ℓ . Then, the scaling from (5.2.1) ensures that the j th subclusters, which arise by applying (5.2.1) j times to the sets E_{k-j} are vertically $2 \cdot 2^{-2j} = \frac{1}{2}2^{-2(j-1)}$ -separated. The reason the height-bound can be improved, is because the j th subclusters are actually contained in smaller strips. See for instance, E_1 , where the first subclusters are contained in lines, and E_2 where the first subclusters are contained in the strips $\mathbb{R} \times [0, \frac{3}{16}] \cup [\frac{12}{16}, \frac{15}{16}]$.

(6) follows from the fact that vertically-closest connected components in E_k come from the connected components of E_1 which are $3 \cdot 2^{-2}$ separated. After being scaled by 2^{-2} in (5.2.1) another $(k-1)$ times the separation is reduced to a distance of $3 \cdot 2^{-2k}$ as claimed. This coincides with the precise formula in (5) and could be considered as a base case for induction on j for the interested reader.

(7) Throughout the proof of (7), we fix integers $1 \leq j < k$ and $k \geq 2$.

Claim 1: For all $x \in E_k$ there exists some $x' \in E_j$ with

$$\text{dist}(x, x') \leq 2^{-2j} \tag{5.2.3}$$

Proof of Claim 1. Note that the scaling in (5.2.1) ensures that for some ℓ , we know that every $x \in E_{\ell+1}$ is within a distance $3 \cdot 2^{-2(\ell+1)}$ of a point in E_ℓ . Iterating verifies the claim

by showing for $x \in E_k$ there exists $x' \in E_j$ such that

$$\text{dist}(x, x') \leq \sum_{\ell=j+1}^k 3 \cdot 2^{-2\ell} \leq 3 \sum_{\ell=j+1}^{\infty} 2^{-2\ell} = 4 \cdot 2^{-2(j+1)}.$$

Claim 2: There exists c independent of j such that for all $5 \cdot 2^{-2j} \leq r \leq 11 \cdot 2^{-2j}$ and all $x' \in E_j$,

$$\beta_{\mu_j; 2}^1(x', r)^2 \geq c$$

Proof of Claim 2. Let $J \subset E_j$ be the connected component containing x' . By (4)-(6) of this proposition, it follows that for any $r \geq 5 \cdot 2^{-2j} = \sqrt{(3 \cdot 2^{-2j})^2 + (4 \cdot 2^{-2j})^2}$, the ball $B_r(x')$ contains J and 3 other connected components of E_j . Consequently, there are two horizontal lines L^u and L^d such that $B_r(x') \cap (L^u \cup L^d)$ contains at least 4 connected components of E_j . Part (1) of this proposition ensures,

$$\min\{\mu_j(L^u \cap B_r(x')), \mu_j(L^d \cap B_r(x'))\} \geq 2 \cdot 2^{-2j}. \quad (5.2.4)$$

Moreover, part (6) ensures that the distance between L^u and L^d is $3 \cdot 2^{-2j}$, which combined with (5.2.4) forces that any line L satisfies,

$$\mu_j\left(\left\{y \in B_r(x') \mid \text{dist}(y, L) \geq 3 \cdot 2^{-2j-1}\right\}\right) \geq 2 \cdot 2^{-2j}. \quad (5.2.5)$$

Finally, (5.2.5) implies

$$\inf_L \int_{B_r(x')} \left(\frac{\text{dist}(y, L)}{r}\right)^2 \frac{d\mu_j(y)}{r} \geq \left(\frac{3 \cdot 2^{-2j}}{r}\right)^2 \left(\frac{2 \cdot 2^{-2j}}{r}\right) \geq \left(\frac{3 \cdot 2^{-2j}}{12 \cdot 2^{-2j}}\right)^2 \left(\frac{2 \cdot 2^{-2j}}{12 \cdot 2^{-2j}}\right) = c$$

which verifies Claim 2.

Claim 3: There exists c' such that for all $x \in E_k$ and all integers $1 \leq j < k$ and ρ such that

$$6 \cdot 2^{-2j} \leq \rho \leq 12 \cdot 2^{-2j},$$

$$\beta_{\mu_k;2}^1(x, \rho)^2 \geq c'. \quad (5.2.6)$$

Proof of Claim 3. Claim 1 ensures that for all $5 \cdot 2^{-2j} \leq r \leq 11 \cdot 2^{-2j}$ there exists $x' \in E_j$ such that $B_r(x') \subset B_\rho(x)$. As in Claim 2, fix lines L^d and L^u such that $B_r(x) \cap (L^u \cup L^d)$ contains at least 4 connected components of E_j . Choose a so that $L^d = \mathbb{R} \times \{a\}$ and $L^u = \{a + (0, 3 \cdot 2^{-2j})\} + \mathbb{R} \times \{0\}$. Moreover, suppose the left-most connected component of L^u has right-most endpoint with x -value equal to c_1 . Define $L_v = \{c_1 + 2^{-2j}\} \times \mathbb{R}$ and $L_h = a + 2^{-2j}$. By Proposition 5.2.1(5,6), the neighborhoods $N_v = B_{2^{-2j}}(L_v)$ and $N_h = B_{2^{-2j}}(L_h)$ are disjoint from E_ℓ for all $\ell \geq j$. See Figure 5.1.

Consequently, for any line L the neighborhood $B_{2^{-2j-1}}(L)$ can intersect at most 4 of the “quadrants” made by the neighborhoods of N_v and N_L . Making a generous estimate since the ball may cut-off part of one of the quadrants in Figure 5.1, we conclude

$$\mu_k \left(\{y \in B_r(x') \mid \text{dist}(y, L) \geq 2^{-2j-2}\} \right) \geq 2^{-2j-2} \quad (5.2.7)$$

where the measure-bound comes Proposition 5.2.1(1). Since $B_r(x') \subset B_\rho(x)$ and $1 \leq \frac{\rho}{r} \leq C < \infty$ Claim 3 follows from (5.2.7) analogously to how Claim 2 followed from (5.2.5).

Finally, we verify (7) because

$$\int_{6 \cdot 2^{-2j}}^1 \beta_{\mu_k}(x, \rho)^2 \frac{d\rho}{\rho} \geq \sum_{j=2}^k \int_{6 \cdot 2^{-2j}}^{11 \cdot 2^{-2j}} c' \frac{d\rho}{\rho} = c(k-2)$$

□

We construct Σ_0 from approximations to the 4-corner Cantor set by first defining

$$E(n) = (2^{-2n}, 0) + 2^{-2n} E_{2^{2n}} \quad \text{and} \quad \Sigma_0 = \bigcup_n E(n) \cup ([0, 1] \times \{0\}). \quad (5.2.8)$$

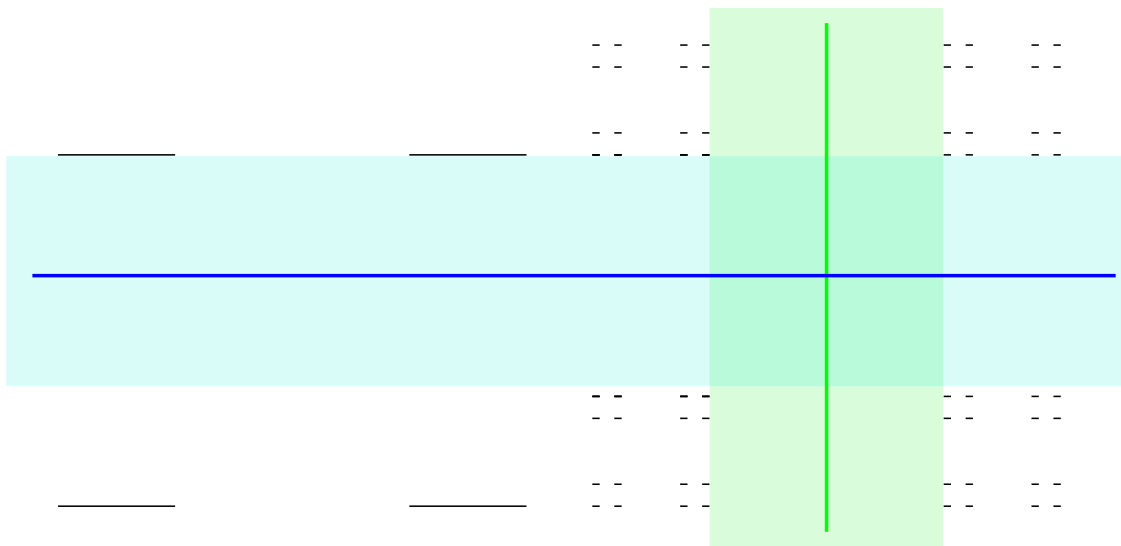


Figure 5.1: When $j = k - 2$, the picture displays a subclusters of equal length for E_j and E_k on the left and right respectively. In E_k , the line L_v and its neighborhood N_v are in green, whereas the line L_h and its neighborhood N_h are drawn where it would pass through both E_j and E_k

Proposition 5.2.2. Σ_0 has the following properties.

1. $0 < \mathcal{H}^1(\Sigma_0) < \infty$ and Σ_0 is countably 1-rectifiable.
2. If $j \geq 0$ is an integer and $Q \in \Delta^j$ is such that $Q \cap \Sigma_0 \neq \emptyset$, then

$$Q \cap \Sigma_0 = \begin{cases} \Sigma_0 \cap [0, \ell(Q))^2 & x_Q = (0, 0) \\ x_Q + 2^{-2j} E_k \text{ for some } k & x_Q \neq (0, 0) \text{ and } \pi_y(x_Q) \neq 0 \\ x_Q + 2^{-2j} E_k \cup [0, \ell(E_k)) \times \{0\} & x_Q \neq (0, 0) \text{ and } \pi_y(x_Q) = 0. \end{cases} \quad (5.2.9)$$

3. $C_{\Sigma_0}(0, \delta) = +\infty$ for all $\delta > 0$.

Proof. (1) Σ_0 has positive and finite mass due, Proposition 5.2.1(1) and the geometric scaling

in (5.2.8). It is also the countable union of countably 1-rectifiable sets by Proposition 5.2.1(1).

(2) The case when $x_Q = (0, 0)$ is clear. Suppose $x_Q \neq (0, 0)$. There exists unique a, b such that

$$x_Q = (a2^{-2j}, b2^{-2j}). \quad (5.2.10)$$

If $j = 0$, $Q \cap \Sigma_0 \neq \emptyset$, and $\Sigma_0 \subset [0, 1]^2$ forces $a = b = 0$. Therefore, $j \geq 1$. Since $h(E_{2^{2n}}) < \ell(E_{2^{2n}})$ and the $E(n)$ only use a translation in the positive horizontal direction of $E_{2^{2n}}$ and a homogeneous scaling, it follows that $\Sigma_0 \cap Q \neq \emptyset$ implies $0 \leq b < a$ so that $a \geq 1$. Since, $\ell(Q) = 2^{-2j}$ it follows that $a2^{-2j} \geq \ell(Q)$. Comparing the translation and scaling sizes in (5.2.1), $a \geq 2^{2j}\ell(Q)$ implies

$$\Sigma_0 \cap Q = \begin{cases} Q \cap E(n) & b \geq 1 \\ Q \cap (E(n) \cup [0, \ell(E(n))) \times \{b\}) & b = 0 \end{cases} \quad (5.2.11)$$

for some specific $n \leq j$. For simplicity of writing, assume we're in the first case. Then, $2^{2n}(Q \cap E(n) - (2^{-2n}, 0)) = (2^{2n}(Q - (2^{-2n}, 0))) \cap E_{2^{2n}}$ or equivalently

$$Q \cap E(n) = (2^{-2n}, 0) + 2^{-2n} \left(2^{2n} (Q - (2^{-2n}, 0)) \cap E_{2^{2n}} \right). \quad (5.2.12)$$

In light of (5.2.12), it follows that (5.2.2) implies the 2nd case of (5.2.9) since $2^{2n}(Q - (2^{-2n}, 0)) \in \Delta^{j-n}$ and $n \leq j$. Analogously the $b = 0$ case corresponds to the 3rd case of (5.2.9).

(3) Fix $\delta > 0$. Choose N so that $11 \cdot 2^{-2N} < \delta/2$, so that for all $n \geq N$, $E(n) \subset B_\delta(0)$. Then, with $\mu = \mathcal{H}^1 \llcorner \Sigma_0$ and $\mu_n = \mathcal{H}^1 \llcorner E(n)$, it follows from Proposition 5.2.1 (1,7), Proposition 2.5.1 (2), and the scaling in (5.2.8) that

$$C_{\Sigma_0}(0, \delta) \geq \sum_{n \geq N} \int_{E(n)} \int_0^{2^{-2n}} \beta_{\mu_n; 2}^1(x, r)^2 \frac{dr}{r} d\mu_n(x) \geq \sum_{n \geq N} c(2^{2n} - 2) \mathcal{H}^1(E(n)),$$

which diverges and completes the proof. \square

We wish to iterate Σ_0 densely along itself while being careful to maintain Ahlfors upper- and lower-regularity. This is attained by scaling, and being careful where we iterate.

Definition 5.2.3 (Tail points). We say a point y is a *tail point of E* if $0 < \mathcal{H}^1(E) < \infty$ and there exists a tetradic number r and $\delta > 0$ such that

$$y + r\Sigma_0 \cap B_\delta \subseteq E.$$

Note, if $y \in B_\delta(x)$ is a tail point of a set E , then $C_E(x, \delta) \equiv \infty$. See Claim 1 of Theorem 5.2.7.

Definition 5.2.4 (Iterative construction). Let Σ_0 be as above. Supposing that Σ_{i-1} has been defined, we define a (possibly empty) special collection of tetradic points,

$$D^i = \left\{ x \in 2^{-2i}\mathbb{Z}^2 \mid \left(x + [0, 2^{-2i})^2 \right) \cap \Sigma_{i-1} = x + [0, 2^{-2i}) \times \{0\} \right\}, \quad (5.2.13)$$

and define Σ_i by

$$\Sigma_i = \Sigma_{i-1} \cup \left\{ \cup_{x \in D^i} x + 2^{-8i}\Sigma_0 \right\}. \quad (5.2.14)$$

Define,

$$A_0 = \cup_{j \in \mathbb{N}} \Sigma_j. \quad (5.2.15)$$

Proposition 5.2.3. *The sets $\{\Sigma_j\}_{j=0}^\infty$ and $\{D^j\}_{j=1}^\infty$ as in Definition 5.2.4 have the following properties:*

- (1) $\Sigma_{j-1} \subset \Sigma_j$ for all $j \geq 1$,
- (2) Σ_j is contained in countably many horizontal line segments with tetradic heights.
- (3) D^j is non-empty infinitely often.
- (4) If I is a connected component of Σ_j then $\partial I \subset \ell(I)\mathbb{Z}^2$.

(5) Σ_j contains no connected component of length at least 2^{-2j} that contain no tail point.

Proof. Indeed, (1) follows from (5.2.14).

(2) Follows by induction. For Σ_0 it follows from Proposition 5.2.1 (1) combined with the scaling in (5.2.8). For general Σ_j induction holds due to the fact that each scaled copy of Σ_0 in (5.2.14) has a tail point on the dyadic lattice D^i which is coarser than the tetradic scaling factor of Σ_0 .

(3) follows from (2). (5) follows from (4) and the definition of D^j in (5.2.13).

(4) If I is a connected component of Σ_j then there exists $y \in D^i$ some $i \leq j$ such that I is a connected component of $y + 2^{-8i}\Sigma_0$. But then, $2^{8i}(I - y)$ is a connected component of Σ_0 . Since $y \in 2^{-2i}\mathbb{Z}^2$, Propositions 5.2.1(1) and 5.2.2(2) ensure $\partial(2^{8i}(I - y)) \in 2^{8i}\ell(I)\mathbb{Z}^2$ which verifies (4).

□

Definition 5.2.5 (Associated cubes). Any cluster (or subcluster) E has associated to it the dyadic cube $Q_E = x_E + [0, \ell(E))^2$. In particular, by Proposition 5.2.1 (5) it follows that if clusters E, E' are disjoint with $\ell(E) = \ell(E')$, then $Q_E, Q_{E'}$ are disjoint cubes. Moreover, for some cluster E , the root point of Q_E and the root point of E coincide.

Definition 5.2.6. We associate to the base set Σ_0 the following family of cubes

$$\mathcal{Q}_{\Sigma_0} = \{[0, 2^{-2i})^2 : i \geq 0\} \cup \{Q_E : E \text{ is a subcluster of } E(n) \subset \Sigma_0, n \geq 1\} \quad (5.2.16)$$

By similarity, for any $y \in D^i$ we associate to $y + 2^{-8i}\Sigma_0$ the family of cubes

$$\mathcal{Q}_y = \left(y + 2^{-8i}\mathcal{Q}_{\Sigma_0} \right) \cup \left(y + \{[0, 2^{-2k})^2 : i \leq k\} \right). \quad (5.2.17)$$

We will let

$$\mathcal{Q} = \bigcup_{i \geq 0} \bigcup_{y \in D^i} \mathcal{Q}_y \quad (5.2.18)$$

which we stratify by scale in the following sense

$$\mathcal{Q}^i = \{Q \in \mathcal{Q} \mid \ell(Q) = 2^{-2i}\} \quad (5.2.19)$$

and we enumerate the elements \mathcal{Q}^i so that

$$\mathcal{Q}^i = \{Q_j^i\}_{j=1}^{N(i)}. \quad (5.2.20)$$

Finally, for $Q \in \mathcal{Q}$ and any positive integer ℓ we let $\mathcal{C}_\ell(Q) = \{Q' \in \mathcal{Q} \mid \ell(Q') = 2^{-2\ell}\ell(Q)\}$, and call $\mathcal{C}_\ell(Q)$ the ℓ th *descendent cubes* of Q .

Lemma 5.2.4. *For all $i \geq 0$ and all cubes, $Q_j^i \in \mathcal{Q}^i$, $\Sigma_i \cap Q_j^i$ is similar to one of the following:*

1. $(2^{-2k}\Sigma_0 \cup [0, 1) \times \{0\}) \cap [0, 1)^2$ for some integer k .
2. $E \cap Q_E$ for some sub-cluster $E \subset E(n)$ for some integer $n \geq 1$

This follows immediately from the explicit definition of cubes.

Lemma 5.2.5. *$\mathcal{Q}^j \subset \Delta^j$ and for all $Q \in \Delta^j$, then either $Q \cap \Sigma_j = \emptyset$ or $Q \in \mathcal{Q}_j$.*

This follows from an induction argument similar to the proofs of Propositions 5.2.1 (1) and 5.2.2 (2). The key observation in the induction is that the scaling in (5.2.14) ensures that all tail points added in the j th stage have root points in tetradic lattices that are coarser than the length of the scaled copy of Σ_0 being added.

Corollary 5.2.6. *The cubes \mathcal{Q} have the following nice properties:*

1. Each collection \mathcal{Q}_i is a disjoint collection of cubes, and for any $Q \in \mathcal{Q}$ and any integer $\ell \geq 0$, $\mathcal{C}_\ell(Q)$ is a disjoint collection of subcubes of Q .

2. For all non-negative integers i and j ,

$$\Sigma_i \subseteq \cup_{Q \in \mathcal{Q}_j} Q \quad (5.2.21)$$

3. In particular, for any $Q_0 \in \mathcal{Q}_i$

$$\Sigma_i \cap Q_0 = \Sigma_i \cap \left(\cup_{Q \in \mathcal{C}_1(Q)} Q \right) \quad (5.2.22)$$

Theorem 5.2.7. *The set A_0 is 1-rectifiable, 1-Ahlfors regular, and satisfies the property that for all $x \in A_0$, for every $\delta > 0$, $C_{A_0}(x, \delta) = \infty$.*

Proof. Indeed, by Lemma 5.2.2 (1), Σ_0 is 1-rectifiable, and A_0 is a countable union of scaled translations of Σ_0 so A_0 is 1-rectifiable .

Next, we show that A_0 is 1-Ahlfors regular. Indeed, it suffices to show that there exists $0 < c \leq C < \infty$ independent of i such that for for any $j \geq 0$, $Q \in \Delta^j$, and $Q \cap A_0 \neq \emptyset$,

$$c\ell(Q) \leq \mathcal{H}^1(Q \cap A_0) \leq C\ell(Q). \quad (5.2.23)$$

We do this by showing similar bounds for $\frac{\mathcal{H}^1(Q \cap \Sigma_j)}{\ell(Q)}$ for cubes $Q \in \Delta^j$ that intersect Σ_j , and then proving that not too much additional mass is added to the cube Q .

Due to Lemma 5.2.5 the condition that $Q \in \Delta^j$ and $Q \cap A_j \neq \emptyset$ is equivalent to $Q \in \mathcal{Q}_j$. Since $Q \in \mathcal{Q}_j$ Lemma 5.2.4 characterizes what $Q \cap \Sigma_j$ looks like and we conclude

$$\ell(Q) \leq \mathcal{H}^1(Q \cap \Sigma_j) \leq 3\ell(Q), \quad (5.2.24)$$

by considering each of the three cases in Lemma 5.2.4. Indeed, each cube either contains its entire bottom portion, or contains a cluster E with $\ell(E) = \ell(Q)$. In either case this implies the lower bound in (5.2.24). On the other hand, we know that a rough upper-bound is to

assume that $Q \cap \Sigma_j$ contains a cluster with a line segment at the bottom, and contains Σ_0 scaled by 2^{-2k} , then by Proposition 5.2.1, the upper bound in (5.2.24) follows.

It remains to show that (5.2.24) implies (5.2.23). Due to Proposition 5.2.3 (1), the lower-bound in (5.2.23) is inherited directly from (5.2.24). The upper-bound follows with the additional observation that for $\ell \geq j$,

$$\mathcal{H}^1(Q \cap \Sigma_{\ell+1} \setminus \Sigma_\ell) \leq \#|D_{\ell+1}|2^{-8(\ell+1)}\mathcal{H}^1(\Sigma_0) \leq 2^{-4(\ell+1)}\mathcal{H}^1(\Sigma_0).$$

Summing over $\ell \geq j$ verifies (5.2.23). It is a standard argument to go from Ahlfors regularity in tetradic/dyadic cubes to in balls, see for instance the brief description in the proof of Proposition 5.2.1(3). Since the cubes in \mathcal{Q} are all the tetradic cubes with non-empty intersection with A_0 , we have regularity in tetradic cubes.

Finally, to see that $C_{A_0}(x, \delta) = \infty$ it suffices to show the following claim.

Claim 1- If $x \in A_0$ and $\delta > 0$, then there is a tail point in $A_0 \cap B_{\delta/2}(x)$.

Briefly assuming that Claim 1 holds, the fact that $C_{A_0}(x, \delta) = \infty$ for all $x \in A_0$ and $\delta > 0$ follows since if y is the tail point in $B_{\delta/2}(x)$ then, by Proposition 5.2.2 (3) and monotonicity of integrals of non-negative functions:

$$C_{A_0}(x, \delta) \geq C_{A_0}(y, \delta/2) \geq C_{\Sigma_0}(0, \epsilon_y) = \infty,$$

where $\epsilon_y > 0$ is some scale dependent on which D^i the tail point y is in.

To verify Claim 1, fix x and δ as in the claim. Adopting the convention that $\Sigma_{-1} = \emptyset$ fix i_0 such that $x \in \Sigma_{i_0} \setminus \Sigma_{i_0-1}$. Choose k to be the smallest natural number such that $\text{diam}(2^{-8k}\Sigma_0) \leq \delta/4$.

Case 1- $B_{\delta/4}(x) \cap \Sigma_k$ contains a tail. Since $\Sigma_k \subset A_0$ in this case the claim holds.

Case 2- Otherwise, choose $k_0 \geq k$ such that

$$\begin{cases} (\Sigma_{k_0-1} \setminus \Sigma_k) \cap B_{\delta/4}(x) = \emptyset \\ (\Sigma_{k_0} \setminus \Sigma_k) \cap B_{\delta/4}(x) \neq \emptyset, \end{cases}$$

that is k_0 is the first stage after k where something new is added to the ball $B_{\delta/4}(x)$. The way something new is added to the ball $B_{\delta/4}(x)$ in the k_0 th stage is if there exists y such that,

$$\{y + 2^{-8k_0}\Sigma_0\} \cap \{\Sigma_{k_0} \cap B_{\delta/4}(x)\} \neq \emptyset.$$

But then, y is a tail point of Σ_{k_0} and consequently of A_0 . By our choice of k , we conclude

$$|x - y| < \text{diam}(2^{-4k_0}\Sigma_0) + \delta/4 \leq \delta/2.$$

Hence the tail point y is indeed in $B_{\delta/2}(x)$. So, by Proposition 2.5.1(2)

$$C_{A_0}(x, \delta) \geq C_{A_0}(y, \delta/2) \geq cC_{\Sigma_0}(0, \delta') = \infty.$$

This completes the theorem. □

Chapter 6

APPENDICES

6.1 Appendix A

The purpose of this section is to justify Lemma 3.2.6. We use the language of porous sets. For a non-empty set $E \subset \mathbb{R}^n$, $x \in E$, and radius $0 < r$, we write

$$P(E, x, r) = \sup\{0, h : h > 0, B_h(y) \subset B_r(x) \setminus E \text{ for some } y \in B_r(x)\}. \quad (6.1.1)$$

For $\alpha > 0$, we say that E is α -porous if

$$\liminf_{r \rightarrow 0} \frac{P(E, x, r)}{r} > \alpha \quad (6.1.2)$$

for all $x \in E$.

We shall say that E is α -porous down to scale r_0 if

$$\frac{P(E, x, r)}{r} > \alpha \quad (6.1.3)$$

for all $x \in E$ and for all $r_0 \leq r$.

Remark 6.1.1. By definition, for $\Omega^\pm \in \mathcal{D}(n, \alpha, M_0)$, the boundary, $\partial\Omega^\pm$, is $\frac{1}{M_0}$ -porous. Similarly, $B_r(\partial\Omega^\pm)$ is $\frac{1}{2M_0}$ -porous down to scale $r_0 = 2rM_0$.

Lemma 6.1.2. Let $E \subset \mathbb{R}^n$ be a non-empty, bounded set, $E \subset [0, 1]^n$ with $0 \in E$. If E is

α -porous down to scale $r_0 \ll 1$, then there is a $k = k(\alpha)$, $k' = k'(n)$, and $N \leq \frac{-1}{k+k'} \log_2(r_0)$

$$\text{Vol}(E) \leq \left(1 - \frac{1}{2^{k+k'(n)}}\right)^N.$$

Moreover, there exists an $0 < \epsilon = \epsilon(\alpha, n)$ and a constant $c(n, \alpha)$ such that,

$$\mathcal{M}_{r_0}^{n-\epsilon}(E) \leq (1 - c)^N$$

Proof. Let $\{Q_j^i\}_j$ be the collection of dyadic sub-cubes, $Q_j^i \subset [0, 1]^n$ with $\ell(Q_j^i) = 2^{-i}$. Let $k \in \mathbb{N}$ be the smallest number such that $2^{-k} \leq \alpha$. Note that for any $y \in [0, 1]^n$ with $B_{\frac{\alpha}{2}}(y) \subset [0, 1]^n$, there exists a dyadic cube, $Q_j^{k+k'(n)} \subset B_{\frac{\alpha}{4}}(y)$, where j is the smallest integer such that $k'(n) \geq 2 + \frac{1}{2} \log_2(n)$. Let $\frac{1}{2}Q_j^i$ denote an axis-parallel cube with the same center as Q_j^i , but side length half that of Q_j^i .

Now, we apply the standard argument. Tile $[0, 1]^n$ by $Q_j^{k+k'(n)}$. By our porosity assumption, there exists a $Q_{j'}^{k+k'(n)}$ which does not intersect E . Thus,

$$\begin{aligned} \text{Vol}(E) &\leq \sum_{j \neq j'} \text{Vol}(Q_j^{k+k'(n)}) \\ &\leq (2^{(k+k'(n))n} - 1)2^{(-k-k'(n))n} \\ &\leq \left(1 - \frac{1}{2^{k+k'(n)}}\right) \end{aligned}$$

Now, within each $Q_j^{k+k'(n)}$ which intersects E , either E intersects $\frac{1}{2}Q_j^{k+k'(n)}$, or it doesn't. If

If $E \cap \frac{1}{2}Q_j^{k+k'(n)} = \emptyset$, then we tile $Q_j^{k+k'(n)}$ by cubes, $\{Q_\ell^{2(k+k'(n))}\}_\ell$ and overestimate,

$$\begin{aligned} \text{Vol}(E \cap Q_j^{k+k'(n)}) &\leq \sum_{\ell: Q_\ell^{2(k+k'(n))} \cap (E \cap Q_j^{k+k'(n)}) \neq \emptyset} \text{Vol}(Q_\ell^{2(k+k'(n))}) \\ &\leq (2^{2(k+k'(n))n} - 1) 2^{-2(k+k'(n))n} \text{Vol}(Q_j^{k+k'(n)}) \\ &\leq \left(1 - \frac{1}{2^{k+k'(n)}}\right) \text{Vol}(Q_j^{k+k'(n)}) \end{aligned}$$

If $E \cap \frac{1}{2}Q_j^{k+k'(n)} \neq \emptyset$, then there exists a ball, $B_{2^{-k-k'(n)-1}}(x) \subset Q_j^{k+k'(n)}$, centered on $x \in E$. By our porosity assumption and choice of $k'(n)$, we can still tile $Q_j^{k+k'(n)}$ by $Q_\ell^{2(k+k'(n))}$ and be guaranteed that at least one such sub-cube does not intersect $E \cap Q_j^{k+k'(n)}$. Thus, we overestimate in the same manner as above.

We can continue, inductively, only stopping at the first N such that $2^{-(N+1)(k+k'(n))} < r_0$. This gives the desired bound,

$$\text{Vol}(E) \leq \left(1 - \frac{1}{2^{k+k'(n)}}\right)^N.$$

Taking a bit more care, we can actually improve these estimates. Let $0 < \epsilon = \epsilon(\alpha, n)$ be such that,

$$\left(1 - \frac{1}{2^{k+k'(n)}}\right) < \left(2^{\epsilon(k+k'(n))} - \frac{1}{2^{(k+k'(n))(n-\epsilon)}}\right) < 1.$$

Then, we bound $\mathcal{M}_{r_0}^{n-\epsilon}(E)$ as follows,

$$\begin{aligned} \mathcal{M}_{r_0}^{n-\epsilon}(E) &= \inf\left\{\sum_i r^{n-\epsilon} : x_i \in E, r_0 \leq r, E \subset \cup_i B_r(x_i)\right\} \\ &\leq \sum_j \ell(Q^{N(k+k'(n))})^{n-\epsilon} \\ &\leq \left(2^{\epsilon(k+k'(n))} - \frac{1}{2^{(k+k'(n))(n-\epsilon)}}\right)^N \end{aligned}$$

□

As immediate corollaries, we have the following statements.

Corollary 6.1.3. If $E \subset \mathbb{R}^n$ is α -porous, then there exists an $0 < \epsilon = \epsilon(\alpha, n)$ such that $\overline{\dim_{\mathcal{M}}}(E) \leq n - \epsilon$.

Proof. Recall that $\overline{\dim_{\mathcal{M}}}(E) = \inf\{s : \mathcal{M}^{*,s}(E) = 0\}$ and that $\mathcal{M}^{*,s}(E) = \limsup_{r_0 \rightarrow 0} \mathcal{M}_{r_0}^{n-\epsilon}(E)$.

By taking $0 < \epsilon$ to be as in Lemma 6.1.2, we have that

$$\begin{aligned} \mathcal{M}_{r_0}^{n-\epsilon}(E) &\leq \left(2^{\epsilon(k+k'(n))} - \frac{1}{2^{(k+k'(n))(n-\epsilon)}}\right)^N \\ &\leq (1 - c)^N \end{aligned}$$

where $c = c(\alpha, n, \epsilon)$ and $N = N(\alpha, n, r_0)$, as in the previous lemma. Thus, letting $r_0 \rightarrow 0$, $N \rightarrow \infty$ and we have that $\mathcal{M}^{n-\epsilon}(E) = 0$. □

Recalling Remark 6.1.1, Corollary 6.1.3 gives Lemma 3.2.6.

Corollary 6.1.4. Let $\Sigma \subset \mathbb{R}^n$ be the mutual boundary of a pair of unbounded two-sided NTA domains with NTA constant $1 < M_0$. Then, there is an $0 < \epsilon = \epsilon(M_0, n)$ such that $\overline{\dim_{\mathcal{M}}}(E) \leq n - \epsilon$.

6.2 Appendix B

Now we investigate the quantities $\lambda(p, r, v)$. Recall that for $v \in \mathcal{A}(\Lambda, \alpha, M_0)$, $p \in B_1(0)$ and $0 < r \leq 1$ we define,

$$\lambda(p, r, v_\epsilon) = \frac{\int_{\partial B_r(p)} (v_\epsilon(y) - v_\epsilon(p)) \nabla v_\epsilon(y) \cdot (y - p) d\sigma(y)}{H(r, p, v_\epsilon)}$$

$$\lambda(p, r, v) = \lim_{\epsilon \rightarrow 0} \lambda(p, r, v_\epsilon)$$

Note that since v is merely Lipschitz, $\lambda(p, r, v)$ may not be defined directly for all admissible p and r . However, since ∇v is defined point-wise in $\mathbb{R}^n \setminus \partial\Omega^\pm$, $\lambda(p, r, v)$ can be defined directly for $p \in B_1(0)$, $0 < r \leq 1$ such that $\partial B_r(p) \cap \partial\Omega^\pm$ has zero surface measure. Thus, we will use the following formulation,

$$\lambda(p, r, v) = \frac{\int_{\partial B_r(p)} (v(y) - v(p)) \nabla v(y) \cdot (y - p) d\sigma(y)}{H(r, p, v)}$$

for all $p \in B_1(0)$, $0 < r \leq 1$ such that $\mathcal{H}^{n-1}(\partial B_r(p) \cap \partial\Omega^\pm) = 0$.

In this section is dedicated to proving the following lemma.

Lemma 6.2.1. Let $v_i \in \mathcal{A}(\Lambda, \alpha, M_0)$ and $x_i \in B_{1/4}(0)$ and $0 < r_i \leq 1/32$. Let $\rho_i \in [2r_i, 7r_i]$. If v_i is $(0, 2^{-i})$ -symmetric in $B_{8r_i}(x_i)$ and $\|\ln(h_i)\|_\alpha < 2^{-i}$, then we may extract a subsequence such that,

$$T_{x_i, 8r_i} v_i \rightarrow v_\infty \quad T_{x_i, 8r_i} y_i \rightarrow y \quad \frac{\rho_i}{r_i} \rightarrow \rho,$$

and

$$\lambda(y_i, \rho_i, v_i) = \lambda(T_{x_i, 8r_i} y_i, \frac{\rho_i}{r_i}, T_{x_i, 8r_i} v_i) \rightarrow \lambda(y, \rho, v_\infty).$$

We note that all but the last convergence result are already established by Lemma 3.5.1

and compactness, respectively. Furthermore, by the modes of convergence of Lemma 3.5.1, $H(y_j, \frac{\rho_j}{r_j}, T_{x_j, 8r_j} v_j) \rightarrow H(y, \rho, v_\infty)$. Therefore, we only consider the numerator,

$$\int_{\partial B_{\frac{\rho_j}{r_j}}(T_{x_i, 8r_i} y_i)} (T_{x_i, 8r_i} v_i(z) - T_{x_i, 8r_i} v_i(T_{x_i, 8r_i} y_i)) \nabla T_{x_i, 8r_i} v_i(z) \cdot (z - y) d\sigma(z).$$

We begin with an auxiliary lemma.

Lemma 6.2.2. Let $\{\Sigma_p\}$ be the collection of zero sets of homogeneous harmonic polynomials of degree $\leq D$. For every $0 < r_0$, there is a constant, $C(r_0, D)$, such that for all Σ_p with degree $\leq D$, for all $y \in B_1(0)$ and all $r \in [2, 7]$,

$$\begin{aligned} \mathcal{H}^{n-1}(\partial B_r(y) \cap B_{r_0}(\Sigma_p)) &\leq C(r_0, D) \\ C(r_0, D) &\rightarrow 0 \text{ as } r_0 \rightarrow 0 \end{aligned}$$

The proof is broken up into the following lemmata.

Lemma 6.2.3. There exists a constant, $0 < c < 1$, such that for all $y \in B_1(0)$ and $r \in [2, 7]$, if $x \in \partial B_r(y)$, and $\vec{\eta}_{y,r}(x)$ the unit outward normal to $\partial B_r(y)$ at x , then,

$$\frac{x}{|x|} \cdot \vec{\eta}_{y,r}(x) \geq 1 - c$$

Proof. Consider the function, $f : \overline{B_9(0)} \setminus B_1(0) \times \overline{B_1(0)} \cap \{(x, y) : |x - y| \geq 2\} \rightarrow \mathbb{R}$ given by

$$f(x, y) = |x| - \left\langle \frac{x}{|x|}, y \right\rangle.$$

Note that the domain of f is the intersection of two compact sets and is therefore compact. Furthermore, f is continuous. We argue that $f \neq 0$. Because $|x| \geq 1$ and $|y| \leq 1$, the only way that $|x| = \langle \frac{x}{|x|}, y \rangle$, is if $|x| = |y| = 1$ and $y = x$. However, $y = x$ is not in the domain in question. Since f is a continuous function on a compact domain which never vanishes there exists a constant $\tilde{c} > 0$ such that $f(x, y) \geq \tilde{c}$. Now we note that

$$\frac{x}{|x|} \cdot \vec{\eta}_{y,r}(x) = \left\langle \frac{x}{|x|}, \frac{x-y}{|x-y|} \right\rangle = \frac{1}{|x||x-y|} f(x, y) \geq \frac{1}{90} \tilde{c} = 1 - c. \quad \square$$

Lemma 6.2.4. Let $y \in B_1(0)$ and $r \in [2, 7]$. There exists a radius, $0 < \rho(n) \leq 1/8$, such that for any $x \in \partial B_r(y)$, there is a bi-Lipschitz map, $\phi : B_\rho(x) \rightarrow \mathbb{R}^n$ with the following properties.

1. $\phi(\partial B_r(y) \cap B_\rho(x)) \subset \mathbb{R}^{n-1} \times \{0\}$
2. For any closed cone over $\{0\}$, $C \subset \mathbb{R}^n$, there is a radius $0 < \rho_1(n) \leq \rho(n)$, such that

$$\phi(C \cap \partial B_r(y) \cap B_{\frac{1}{2}\rho}(x)) \times (-\rho_1, \rho_1) \subset \phi(C \cap B_\rho(x))$$

Proof. Let $\psi : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{S}^{n-1} \times \mathbb{R}$, be the following map,

$$\psi(x) = \left(\frac{x}{|x|}, |x| \right) \in \mathbb{R}^{n-1} \times \mathbb{R},$$

which changes from Cartesian coordinates to polar coordinates. Let $0 < \rho \leq 1/8$. Since $\text{dist}(B_\rho(x), \{0\}) \geq \frac{1}{2}$, ψ , is a diffeomorphism, and in particular, bi-Lipschitz on $B_\rho(x)$.

Furthermore, since $\partial B_r(y) \cap B_\rho(x)$ is a smooth, embedded sub-manifold, and $B_\rho(x)$ is $\frac{1}{2}$ -separated from the origin, $\text{dist}(\cdot, \{0\})$ is a smooth function on $\partial B_r(y) \cap B_\rho(x)$. Thus, $\psi(\partial B_r(y) \cap B_\rho(x))$ is a smooth graph over $U = \{\frac{z}{|z|} \in \mathbb{S}^{n-1} : z \in \partial B_r(y) \cap B_\rho(x)\}$. Let Γ be this smooth function so that we may write $\psi(\partial B_r(y) \cap B_\rho(x)) = \text{graph}_{U \subset \mathbb{S}^{n-1}}(\Gamma)$.

Let $\phi = (Id_\theta, Id - \Gamma) \circ \psi$. Recall that restricted to $B_\rho(x)$, ψ is a diffeomorphism, and so we must only argue that $\nabla(Id - \Gamma) \neq 0$. Recalling Lemma 6.2.3, we have that for all $v \in T_x \partial B_r(y)$ with $|v| = 1$, the directional derivative $\partial_v \Gamma = v \cdot \frac{x}{|x|} \leq c < 1$. Since there is no directional derivative such that $\partial_v \Gamma = 1$, $(Id_\theta, Id - \Gamma)$ is a diffeomorphism on $\psi(B_\rho(x))$. Since the composition of diffeomorphisms is a diffeomorphism, $\phi|_{B_\rho(x)}$ is a diffeomorphism, and hence bi-Lipschitz on $B_{\frac{7}{8}\rho}(x)$.

That ϕ satisfies the first property is immediate. To check the second, we note that the maps

in question are diffeomorphisms, and that $\overline{C \cap \partial B_r(y) \cap B_{\frac{1}{2}\rho}(x)}$ is compact. Hence,

$$\min_{z \in \overline{C \cap \partial B_r(y) \cap B_{\frac{1}{2}\rho}(x)}} \text{dist}(z, \phi(B_{\frac{3}{4}\rho}(x))^c) = \rho_1,$$

exists and is positive. □

Next, we note that by [8] Theorem 1.1 (vii), $\overline{\dim_{\mathcal{M}}(\Sigma_p)} \leq n - 1$. Now, we argue that for appropriate spheres, $\partial B_r(y) \cap \Sigma_p$ have zero $n - 1$ upper Minkowski content.

Remark 6.2.5. For any $E \subset \mathbb{R}^n$, we shall use the notation,

$$P(E, \epsilon) = \max\{k : \text{there are disjoint balls } B_\epsilon(x_i), i = 1, \dots, k, x_i \in E\}.$$

Note that if $\{B_\epsilon(x_i)\}$ are a maximal disjoint collection of balls with centers $x_i \in E$, then $B_\epsilon(E) \subset \bigcup_i^{P(E, \epsilon)} B_{3\epsilon}(x_i)$ and $\text{Vol}(\bigcup_i^{P(E, \epsilon)} B_\epsilon(x_i)) \leq \text{Vol}(B_\epsilon(E))$. Therefore,

$$P(E, \epsilon)\omega_n \epsilon^n \leq \text{Vol}(B_\epsilon(E)),$$

and therefore, if $\mathcal{M}^{*,n-1}(E) = 0$, then $\limsup_{\epsilon \rightarrow 0} \frac{P(E, \epsilon)\omega_n \epsilon^n}{\epsilon} = 0$, as well. See [23] Chapter 5 for further details.

Lemma 6.2.6. Let $\Sigma_p \subset \mathbb{R}^n$ be as above. For all $y \in B_1(0)$ and all $r \in [2, 7]$,

$$\mathcal{M}^{*,n-1}(\partial B_r(y) \cap \Sigma_p) = 0$$

Proof. We argue by contradiction. Suppose that there were a $y \in B_1(0)$ and an $r \in [2, 7]$, such that

$$\mathcal{M}^{*,n-1}(\partial B_r(y) \cap \Sigma_p) > 0.$$

Let $0 < \rho$ be the radius from Lemma 6.2.4. Letting $\{x_i\}$ be a maximal $\frac{1}{2}\rho$ -net in $\partial B_r(y)$,

we we may select a ball, $B_\rho(x_i)$, such that $\mathcal{M}^{*,n-1}(\partial B_r(y) \cap B_{\frac{1}{2}\rho}(x_i) \cap \Sigma_p) > 0$. If not, then, because there are only finitely many such balls, and upper Minkowski content is finitely additive, this would contradict the assumption that $\mathcal{M}^{*,n-1}(\partial B_r(y) \cap \Sigma_p) > 0$.

For ϕ the map guaranteed by Lemma 6.2.4, then, there exists some $0 < \rho_1$ such that,

$$\phi(\Sigma_p \cap \partial B_r(y) \cap B_{\frac{1}{2}\rho}(x)) \times (-\rho_1, \rho_1) \subset \phi(\Sigma_p \cap B_\rho(x)).$$

Since ϕ is bi-Lipschitz,

$$\mathcal{M}^{*,n-1} \left(\phi(\partial B_r(y) \cap B_{\frac{1}{2}\rho}(x) \cap \Sigma_p) \right) > 0.$$

Therefore,

$$\begin{aligned} \overline{\dim_{\mathcal{M}}} \left(B_{\frac{1}{2}\rho}(x) \cap \Sigma_p \right) &= \overline{\dim_{\mathcal{M}}} \left(\phi(B_{\frac{1}{2}\rho}(x) \cap \Sigma_p) \right) \\ &\geq \overline{\dim_{\mathcal{M}}} \left(\phi(\Sigma_p \cap \partial B_r(y) \cap B_{\frac{1}{2}\rho}(x)) \times (-\rho_1, \rho_1) \right) \\ &\geq n - 1 + 1 = n \end{aligned}$$

This contradicts [8] Theorem 1.1 (vii), which states that $\overline{\dim_{\mathcal{M}}}(\Sigma_p) \leq n - 1$. □

Lemma 6.2.7. Let $\Sigma_p \subset \mathbb{R}^n$ be the zero set of homogeneous harmonic polynomials of degree $\leq D$. Let $y \in B_1(0)$ and $r \in [2, 7]$. For every $0 < r_0 \ll 1$ small enough,

$$\partial B_r(y) \cap B_{r_0}(\Sigma_p) \subset \partial B_r(y) \cap B_{(1+C)r_0}(\partial B_r(y) \cap \Sigma_p)$$

where C only depends upon the constant c in Lemma 6.2.3.

Proof. Let $X \in \partial B_r(y) \cap B_{r_0}(\Sigma_p)$. Then, there is a point, $S \in \Sigma_p$ such that $X \in B_{r_0}(S)$. Since $cS \in \Sigma_p$ for all scalars $0 < c$, there exists a $cS \in \partial B_r(y) \cap \Sigma_p$.

We now argue that for all $0 < r_0$ sufficiently small, there is a constant, C , such $|S - cS| \leq Cr_0$

where C only depends upon the constant c in Lemma 6.2.3. If such a constant exists for all r_0 sufficiently small, then $X \in \partial B_r(y) \cap B_{(1+C)r_0}(cS)$, which is the desired result.

First, note that by convexity, $|S - cS| \leq |S - T|$ where $T = c'S \in T_x(\partial B_r(y)) + X$. Therefore, we reduce to estimating $|S - T|$. Next, note that if $\frac{X}{|X|} \cdot \vec{\eta}_{y,r}(X) \geq 1 - c$ as in Lemma 6.2.3, then,

$$\max_{v \in T_x(\partial B_r(y))} \left\{ v \cdot \frac{X}{|X|} \right\} \leq c < 1,$$

where c is the same constant as in Lemma 6.2.3.

Now, let $\theta(\rho) = \max\{\frac{X}{|X|} \cdot \frac{z}{|z|} : z \in B_\rho(\frac{X}{|X|})\}$. Note that by containment, $\theta(\rho) \rightarrow 0$ monotonically as $\rho \rightarrow 0$. This quantity gives an upper bound on the “visual radius” of $B_r(X)$ since $|X| > 1$. Therefore, there is a $\rho(c)$ such that for all $r_0 \leq \rho(c)$ and all $s \in B_{r_0}(x)$ and all $v \in T_x(\partial B_r(y))$ with $|v| = 1$,

$$\begin{aligned} \frac{s}{|s|} \cdot v &= \text{dist}_G\left(\frac{s}{|s|}, v\right) \\ &\leq \text{dist}_G\left(\frac{s}{|s|}, \frac{x}{|x|}\right) + \text{dist}_G\left(\frac{x}{|x|}, v\right) \\ &\leq \theta(\rho) + c \\ &\leq \frac{1-c}{2} + c < 1 \end{aligned}$$

Therefore, in the triangle $\Delta(S, X, T)$, we have that,

$$|S - X| \leq \rho,$$

and

$$\angle(\overline{XT}, \overline{ST}) \geq \cos^{-1}\left(\frac{1-c}{2} + c\right).$$

We use the Law of Sines,

$$\begin{aligned} \frac{|S - T|}{\sin(\angle(\overline{XT}, \overline{XS}))} &= \frac{|X - S|}{\sin(\angle(\overline{XT}, \overline{ST}))} \\ &\leq \frac{1}{\sqrt{1 - (\frac{1-c}{2} + c)^2}} \rho \end{aligned}$$

Since the sum of the angles of a triangle must sum to $\frac{\pi}{2}$, we have that for all $\rho \leq \rho(c)$,

$$\begin{aligned} |S - T| &\leq \frac{1}{\sqrt{1 - (\frac{1-c}{2} + c)^2}} \sin\left(\frac{\pi}{2} - \cos^{-1}\left(\frac{1-c}{2} + c\right)\right) \rho \\ &\leq C(c) \rho. \end{aligned}$$

□

Lemma 6.2.8. Let $\Sigma_p \subset \mathbb{R}^n$ be the zero set of homogeneous harmonic polynomials of degree $\leq D$. Let $y \in B_1(0)$ and $r \in [2, 7]$. For every $0 < r_0 \ll 1$ small enough, there is a $\epsilon(r) > 0$ such that if $|y - y'| < \epsilon$ and $|r - r'| < \epsilon$, then

$$\mathcal{H}^{n-1}(\partial B_{r'}(y') \cap B_{r_0}(\Sigma_p)) \leq C(n) \mathcal{H}^{n-1}(\partial B_r(y) \cap B_{r_0}(\Sigma_p)).$$

Proof. Note that $\partial B_r(y) \cap B_{r_0}(\Sigma_p)$ is relatively open in $\partial B_r(y)$. There is a finite collection of balls, $\{B_{2r_0}(x_i)\}_{i \in I}$, with centers, $x_i \in \partial B_r(y) \cap \Sigma_p$, such that the collection $\{B_{r_0}(x_i)\}_{i \in I}$ is pairwise disjoint, and for all $0 < r_0$ small enough,

$$\partial B_r(y) \cap B_{2r_0}(\Sigma_p) \subset \bigcup_{i \in I} B_{(1+C)2r_0}(x_i) \tag{6.2.1}$$

by Lemma 6.2.7.

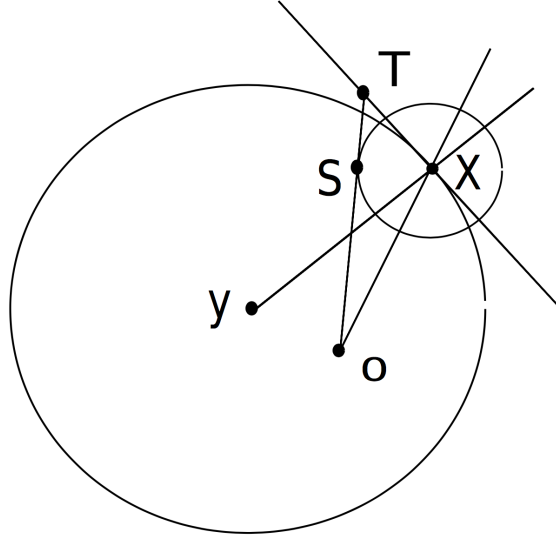


Figure 6.1: An illustration of the geometry in Lemma 6.2.7.

Furthermore, by taking $0 < r_0 \ll 1$ small enough, and $0 < \epsilon$ small enough with respect to r_0 , we may assume that

$$\mathcal{H}^{n-1}(\partial B_{r'}(y') \cap B_{(1+C)2r_0}(x_i)) \leq C((1+C)2r_0)^{n-1} \sum_{i \in I} (r_0)^{n-1} \leq C(n) \mathcal{H}^{n-1}(\partial B_r(y) \cap B_{r_0}(\Sigma_p)).$$

Note that if $|y - y'| < \epsilon$ and $|r - r'| < \epsilon$, then $\text{dist}_{\mathcal{H}}(\partial B_{r'}(y'), \partial B_r(y)) \leq 2\epsilon$. Therefore, $\text{dist}_{\mathcal{H}}(\Sigma_p \cap \partial B_{r'}(y'), \Sigma_p \cap \partial B_r(y)) \leq 2\epsilon$. Therefore, by taking $\epsilon \leq \frac{1}{2}r_0$, we have,

$$\partial B_{r'}(y') \cap B_{r_0}(\Sigma_p) \subset \partial B_r(y) \cap B_{2r_0}(\Sigma_p) \subset \bigcup_{i \in I} B_{(1+C)2r_0}(x_i).$$

Therefore,

$$\begin{aligned}
\mathcal{H}^{n-1}(\partial B_{r'}(y') \cap B_{r_0}(\Sigma_p)) &\leq \mathcal{H}^{n-1}(\partial B_{r'}(y') \cap \bigcup_{i \in I} B_{(1+C)2r_0}(x_i)) \\
&\leq C((1+C)2)^{n-1} \sum_{i \in I} r_0^{n-1} \\
&\leq C(n) \mathcal{H}^{n-1}(\partial B_r(y) \cap B_r(\Sigma_p)).
\end{aligned}$$

□

Lemma 6.2.9. For any $A \subset \mathbb{R}^n$, $y_0 \in \mathbb{R}^n$, and $0 < r$, if $0 < r_1 < r_2$,

$$\mathcal{H}^{n-1}(\partial B_r(y) \cap B_{r_1}(A)) \leq \mathcal{H}^{n-1}(\partial B_r(y) \cap B_{r_2}(A))$$

This is simply a result of containment and the monotonicity of measures.

6.2.1 Proof of Lemma 6.2.2

Proof. We argue by contradiction. Suppose that there were a sequence, Σ_{p_i} , of homogeneous harmonic polynomials of degree $\leq D$ with $H(1, 0, P_i) = 1$, of point, $y_i \in B_1(0)$, radii $r_i \in [2, 7]$ such that $\mathcal{H}^{n-1}(\partial B_{r_i}(y_i) \cap B_{2^{-i}}(\Sigma_{p_i})) \geq c > 0$. Let y_0, r_0 be the limit point and radius, respectively. Since the Σ_{p_i} are homogeneous harmonic polynomials of degree $\leq D$, for all $0 < r$ and all i ,

$$\begin{aligned}
N(r, 0, P_i) &= r \frac{\int_{B_r(0)} |\nabla P_i|^2 dx}{\int_{\partial B_r(0)} (P_i - P_i(0))^2 d\sigma} \\
&\leq D
\end{aligned}$$

Therefore, choose $r = 12$. By Equations (3.4.1), can lower bound $H(12, 0, P_i)$. Thus, we have

that,

$$\begin{aligned}
\int_{B_{12}(0)} |\nabla P_i|^2 dx &\leq \frac{D}{12} H(12, 0, P_i) \\
&\leq \frac{1}{12} D \left(\frac{12}{1}\right)^{(n-1)+2D} H(1, 0, P_i) \\
&\leq C(n, D)
\end{aligned}$$

Thus, the collection of homogeneous harmonic polynomials of degree $\leq D$ with $H(1, 0, P_i) = 1$ is bounded in $W^{1,2}(B_{12}(0))$. By standard elliptic results, then it is sequentially compact in $C^\infty(B_{11}(0))$. Therefore, may extract a subsequence Σ_{p_i} such that $P_i \rightarrow P$ in $C^\infty(B_{11}(0))$. Using an argument nearly identical to that of Lemma 3.2.4, $C^\infty(B_{11}(0))$ convergence implies that $\Sigma_{p_i} \cap B_{10}(0) \rightarrow \Sigma_p \cap B_{10}(0)$ locally in the Hausdorff metric on compact subsets.

That $\Sigma_{p_i} \cap B_{10}(0) \rightarrow \Sigma_p \cap B_{10}(0)$ locally in the Hausdorff metric on compact subsets, implies that for any fixed $0 < \tilde{r}$, there is an $i(\tilde{r})$ such that $B_{2^{-i}}(\Sigma_{p_i}) \cap B_{10}(0) \subset B_{\tilde{r}}(\Sigma_p) \cap B_{10}(0)$ for all $i \geq i(\tilde{r})$. Therefore,

$$\lim_{i \rightarrow \infty} \mathcal{H}^{n-1}(\partial B_{r_i}(y_i) \cap B_{2^{-i}}(\Sigma_{p_i})) \leq \lim_{i \rightarrow \infty} \mathcal{H}^{n-1}(\partial B_{r_i}(y_i) \cap B_{\tilde{r}}(\Sigma_p))$$

Additionally, for any fixed $0 < \tilde{r} \ll 1$, once $|y_i - y_0| < \epsilon(\tilde{r})$ and $|r_i - r_0| < \epsilon(\tilde{r})$ as in Lemma 6.2.8, we have that,

$$\lim_{i \rightarrow \infty} \mathcal{H}^{n-1}(\partial B_{r_i}(y_i) \cap B_{\tilde{r}}(\Sigma_{p_i})) \leq C' \mathcal{H}^{n-1}(\partial B_{r_0}(y_0) \cap B_{\tilde{r}}(\Sigma_p \cap \partial B_{r_0}(y_0))).$$

Now, let \tilde{r} be sufficiently small so that for any $x_i \in \partial B_{r_0}(y_0)$, we can bound $\mathcal{H}^{n-1}(B_{2\tilde{r}}(x_i)) \leq 10(2\tilde{r})^{n-1}$. Then, for $\{B_{\tilde{r}}(x_i)\}$ a maximal disjoint collection of balls with $x_i \in \Sigma_p \cap \partial B_{r_0}(y_0)$,

we have

$$\begin{aligned}
\mathcal{H}^{n-1}(\partial B_{r_0}(y_0) \cap B_{\tilde{r}}(\Sigma_p \cap \partial B_{r_0}(y_0))) &\leq \mathcal{H}^{n-1}(\partial B_{r_0}(y_0) \cap \bigcup_i B_{2\tilde{r}}(x_i)) \\
&\leq P(\Sigma_p \cap \partial B_{r_0}(y_0), 2\tilde{r}) \mathcal{H}^{n-1}(\partial B_{r_0}(y_0) \cap B_{2\tilde{r}}(x_i)) \\
&\leq 102^{n-1} P(\Sigma_p \cap \partial B_{r_0}(y_0), \tilde{r}) (\tilde{r})^{n-1}.
\end{aligned}$$

Now, by Remark 6.2.5 and Lemma 6.2.6 we have that

$$\limsup_{\tilde{r} \rightarrow 0} P(\Sigma_p \cap \partial B_{r_0}(y_0), \tilde{r}) (\tilde{r})^{n-1} = 0.$$

Thus, we can choose $0 < \tilde{r}$ sufficiently small so that

$$\begin{aligned}
\lim_{i \rightarrow \infty} \mathcal{H}^{n-1}(\partial B_{r_i}(y_i) \cap B_{2^{-i}}(\Sigma_{p_i})) &\leq \lim_{i \rightarrow \infty} \mathcal{H}^{n-1}(\partial B_{r_i}(y_i) \cap B_{\tilde{r}}(\Sigma_p)) \\
&\leq C' \mathcal{H}^{n-1}(\partial B_{r_0}(y_0) \cap B_{\tilde{r}}(\Sigma_p)) \\
&< c
\end{aligned}$$

This contradicts the assumption that $\mathcal{H}^{n-1}(\partial B_{r_i}(y_i) \cap B_{2^{-i}}(\Sigma_{p_i})) \geq c > 0$ for all i . Therefore, we have the desired result. \square

6.2.2 Proof of Lemma 6.2.1

Proof. We now argue that under the assumptions of Lemma 6.2.1, there exists a subsequence such that $\lim_{j \rightarrow \infty} \lambda(y'_j, \tilde{r}_j, T_{x_j, 8r_j} v_j) = \lambda(y_\infty, \tilde{r}, v_\infty)$. Note that Lemma 3.2.10 gives that $H(\tilde{r}_j, y'_j, T_{x_j, 8r_j} v_j) \rightarrow H(\tilde{r}, y_\infty, v_\infty)$. Therefore, we only consider the numerator.

Using Lemma 6.2.2, we argue as in Lemma 3.2.7. That is, for admissible locations and scales, $\mathcal{H}^{n-1}(B_r(\{v_\infty = 0\}) \cap \partial B_{r'}(y)) \rightarrow 0$ as $r \rightarrow 0$. Thus, for any $\theta > 0$ we can find an $r(\theta) > 0$

such that for all $0 < r < r(\theta)$,

$$\mathcal{H}^{n-1}(B_r(\{v_\infty = 0\}) \cap \partial B_{r'}(y)) \leq \theta.$$

Now, recall that $T_{x_i, 8r_i} v_i$ are harmonic away from $T_{x_i, 8r_i} \partial \Omega_i^\pm$, and that therefore $W^{1,2}$ convergence in $B_R(0) \setminus B_r(\{v_\infty = 0\})$ implies C^∞ convergence in $B_R(0) \setminus B_r(\partial \Omega_\infty^\pm)$. Recall that $T_{x_i, 8r_i} v_i$ are uniformly Lipschitz in $B_8(0)$ with Lipschitz constant $C(\alpha, M_0, \Gamma)$ by Corollary 3.4.5.

$$\begin{aligned} & \limsup_{i \rightarrow \infty} \int_{\partial B_{\frac{\rho_i}{r_i}}(T_{x_i, 8r_i} y_i)} (T_{x_i, 8r_i} v_i(z) - T_{x_i, 8r_i} v_i(y)) \nabla T_{x_i, 8r_i} v_i(z) \cdot (z - y) d\sigma(z) \\ & \leq \limsup_{i \rightarrow \infty} \int_{\partial B_{\frac{\rho_i}{r_i}}(T_{x_i, 8r_i} y_i) \cap B_{r(\theta)}(\partial \Omega_\infty^\pm)} (T_{x_i, 8r_i} v_i(z) - T_{x_i, 8r_i} v_i(y)) \nabla T_{x_i, 8r_i} v_i(z) \cdot (z - y) d\sigma(z) \\ & + \limsup_{i \rightarrow \infty} \int_{\partial B_{\frac{\rho_i}{r_i}}(T_{x_i, 8r_i} y_i) \setminus B_{r(\theta)}(\partial \Omega_\infty^\pm)} (T_{x_i, 8r_i} v_i(z) - T_{x_i, 8r_i} v_i(y)) \nabla T_{x_i, 8r_i} v_i(z) \cdot (z - y) d\sigma(z) \\ & \leq \limsup_{i \rightarrow \infty} \int_{\partial B_{\frac{\rho_i}{r_i}}(T_{x_i, 8r_i} y_i) \cap B_{r(\theta)}(\partial \Omega_\infty^\pm)} C(\alpha, M_0, \Gamma) d\sigma(z) \\ & + \limsup_{i \rightarrow \infty} \int_{\partial B_{\frac{\rho_i}{r_i}}(T_{x_i, 8r_i} y_i) \setminus B_{r(\theta)}(\partial \Omega_\infty^\pm)} (T_{x_i, 8r_i} v_i(z) - T_{x_i, 8r_i} v_i(y)) \nabla T_{x_i, 8r_i} v_i(z) \cdot (z - y) d\sigma(z) \\ & \leq C(\alpha, M_0, \Gamma) \theta + \int_{\partial B_{\bar{r}}(y_\infty) \setminus B_{r(\theta)}(\partial \Omega_\infty^\pm)} (v_\infty(z) - v_\infty(y)) \nabla v_\infty(z) \cdot (z - y) d\sigma(z) \end{aligned}$$

Furthermore, by the same reasoning,

$$\int_{\partial B_{\bar{r}}(y_\infty) \cap B_{r(\theta)}(\partial \Omega_\infty^\pm)} (v_\infty(z) - v_\infty(y)) \nabla v_\infty(z) \cdot (z - y) d\sigma(z) \geq -C(\alpha, M_0, \Gamma),$$

and so we have,

$$\begin{aligned} & \int_{\partial B_{\bar{r}}(y_\infty) \setminus B_{r(\theta)}(\partial\Omega_\infty^\pm)} (v_\infty(z) - v_\infty(y)) \nabla v_\infty(z) \cdot (z - y) d\sigma(z) \\ & \leq C(\alpha, M_0, \Gamma)\theta + \int_{\partial B_{\bar{r}}(y_\infty)} (v_\infty(z) - v_\infty(y)) \nabla v_\infty(z) \cdot (z - y) d\sigma(z), \end{aligned}$$

Therefore,

$$\begin{aligned} \limsup_{i \rightarrow \infty} \int_{\partial B_{\frac{\rho_i}{r_i}}(T_{x_i, 8r_i} y_i)} (T_{x_i, 8r_i} v_i(z) - T_{x_i, 8r_i} v_i(y)) \nabla T_{x_i, 8r_i} v_i(z) \cdot (z - y) d\sigma(z) \\ \leq 2C(M_0)\theta + \int_{\partial B_{\bar{r}}(y_\infty)} (v_\infty(z) - v_\infty(y)) \nabla v_\infty(z) \cdot (z - y) d\sigma(z) \end{aligned}$$

Letting $\theta \rightarrow 0$,

$$\begin{aligned} \limsup_{i \rightarrow \infty} \int_{\partial B_{\frac{\rho_i}{r_i}}(T_{x_i, 8r_i} y_i)} (T_{x_i, 8r_i} v_i(z) - T_{x_i, 8r_i} v_i(y)) \nabla T_{x_i, 8r_i} v_i(z) \cdot (z - y) d\sigma(z) \\ \leq \int_{\partial B_{\bar{r}}(y_\infty)} (v_\infty(z) - v_\infty(y)) \nabla v_\infty(z) \cdot (z - y) d\sigma(z). \end{aligned}$$

The same argument (using $-C(\alpha, M_0, \Gamma)$) may be used to lower bound,

$$\begin{aligned} \liminf_{i \rightarrow \infty} \int_{\partial B_{\frac{\rho_i}{r_i}}(T_{x_i, 8r_i} y_i)} (T_{x_i, 8r_i} v_i(z) - T_{x_i, 8r_i} v_i(y)) \nabla T_{x_i, 8r_i} v_i(z) \cdot (z - y) d\sigma(z) \\ \geq \int_{\partial B_{\bar{r}}(y_\infty)} (v_\infty(z) - v_\infty(y)) \nabla v_\infty(z) \cdot (z - y) d\sigma(z). \end{aligned}$$

All together then, we have that

$$\begin{aligned} \lim_{i \rightarrow \infty} \int_{\partial B_{\frac{\rho_i}{r_i}}(T_{x_i, 8r_i} y_i)} (T_{x_i, 8r_i} v_i(z) - T_{x_i, 8r_i} v_i(y)) \nabla T_{x_i, 8r_i} v_i(z) \cdot (z - y) d\sigma(z) \\ = \int_{\partial B_{\tilde{r}}(y_\infty)} (v_\infty(z) - v_\infty(y)) \nabla v_\infty(z) \cdot (z - y) d\sigma(z) \end{aligned}$$

Thus, $\lim_{i \rightarrow \infty} \lambda(y'_i, \tilde{r}_i, T_{x_j, 8r_j} v_j) = \lambda(y_\infty, \tilde{r}, v_\infty)$. □

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