Schubert Objects

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Abstract

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Schubert polynomials arose from questions involving enumerative and algebraic geometry, representation theory, and algebraic topology. They have been studied from a variety of perspectives, each with its own combinatorial object [1, 4, 5, 17, 15, 16, 28, 50, 66]. In this dissertation, the combinatorial objects which index the monomials in a Schubert polynomial are called *Schubert objects*. There are many such objects and one of the main goals of this dissertation is to illuminate the bijections between them. In addition to exploring the bijections between Schubert objects, we explore different methods of constructing them. The construction methods are all developed using trees of Schubert objects and taking the collection of leaves at the end of the tree. We introduce a new method to compute the decomposition of Schubert polynomials into key polynomials. We also define a new operator, called split, which provides an alternative approach to creating a tree of rc-graphs. A new Schubert object is explored, called an inversion filling. We discuss a special case of inversion fillings, the Grassmannian permutation case, which gives rise to a left divided difference operator on semistandard Young tableaux. In addition, we describe the previously known construction of skyline fillings and their connection to other Schubert objects.

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In research, most cannot progress alone and often must consult others in their journey. I am no exception. I would like to express my gratitude to those whom have helped mathematically along the way, especially Brant Jones, Kurt Luoto, and Sarah Mason. Along the same lines, there are those who have kept me sane throughout the years in graduate school. So to Juliet Anderson, Dan Finkel, Dave Rosoff, Food Thursday-ers, my entire family, and, above all, my husband Jim, I say, "Thanks for my sanity!"

INTRODUCTION

In three dimensions, how many lines intersect four given lines? This question is just one example of the types of problems Hermann Schubert was interested in when he wrote *Kalkül der abzählenden Geometrie (Calculus of Enumerative Geometry)* in 1879 [58]. Specifically, Schubert investigated the intersections of linear subspaces of a real, finite dimensional vector space. Hilbert expanded the popularity of the subject by including the rigourous foundation of Schubert's enumerative calculus as his 15th problem [25].

Intersection theory relates enumerative geometry to cohomology via the Chow ring. Solving problems in enumerative geometry thus reduces to finding solutions with the cohomology ring of the Grassmannian, or, more generally, of the flag manifold[19].

In the early 1970s, Bernstein, Gelfand, and Gelfand, and Demazure published papers which gave polynomial representatives for the cohomology class of a Schubert variety[3, 12]. Lascoux and Schützenberger extended this work through a series of papers[34, 37, 38, 39, 35, 33, 36, 41, 43]. They laid the framework for the study of enumerative geometry in combinatorics through polynomials. These polynomials, called Schubert polynomials, form an integer basis for the polynomial ring in infinitely many variables. Thus, when two Schubert polynomials are multiplied, their product can be expanded in the basis of Schubert polynomials. The coefficients of this expansion amazingly give the answers to the enumerative questions Schubert originally asked. In particular, they answer the question posed above. Assuming genericity of the lines, there are exactly two lines that intersect four given lines in three dimensional real space.

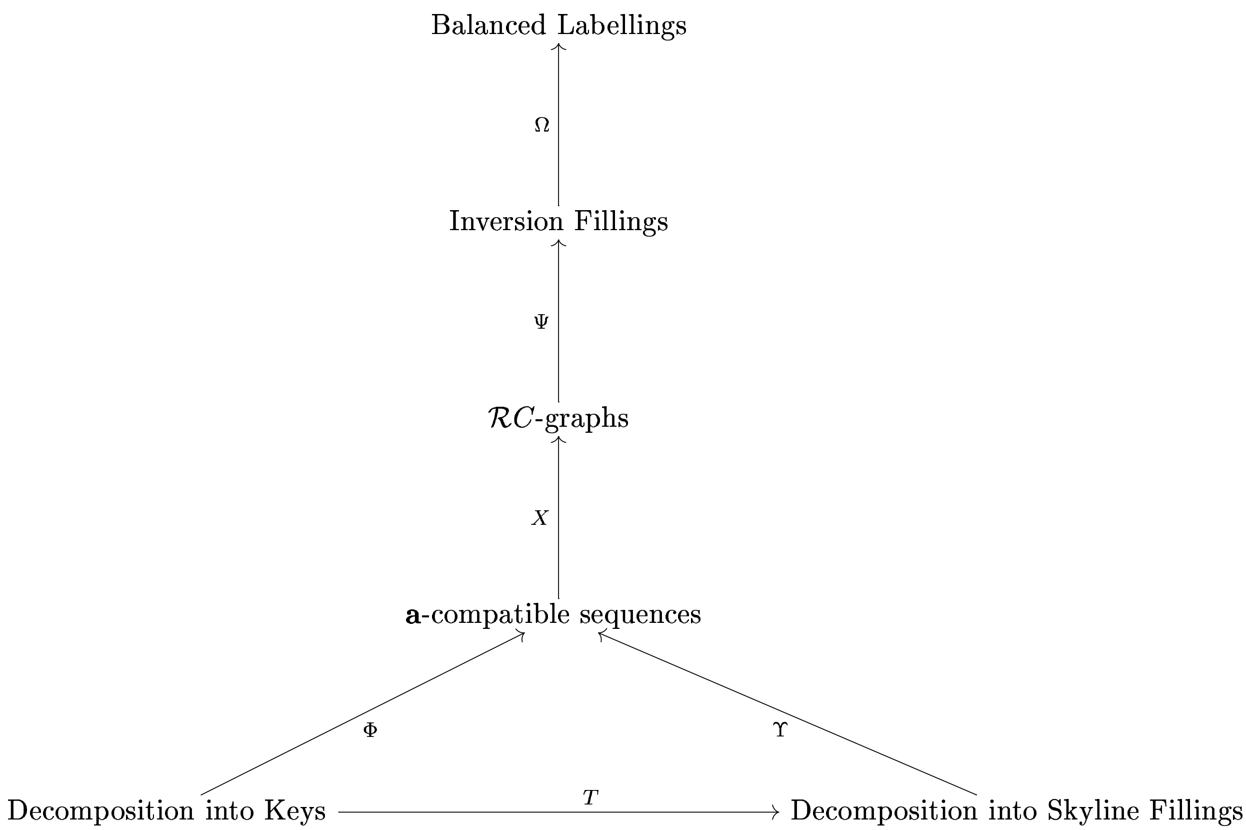
In this dissertation, we deal only with Schubert polynomials in the type A case. Schubert polynomials can be defined more generally for any classical Lie group[4, 21, 15]. In the type A case, let G be the general linear group, $GL(n)$ and B be the subset of G consisting of upper triangular matrices. Then G/B is isomorphic to the set consisting of points which are complete nested sequences of linear subspace of $\mathbb{C}^n$, called the flag manifold. Varieties in G/B decompose into irreducible components, called Schubert varieties, X_w, indexed by permutations in S_n. Borel showed the cohomology ring of the flag manifold, $H^*(Fl_n)$, is isomorphic to $\mathbb{Z}[x_1, \ldots, x_n]/\langle e_i(x_1, \ldots, x_n) \,|\, i = 1, \ldots, n\rangle$ as rings[7]. The image of $[X_w]$ is the Schubert polynomial, $\mathfrak{S}_{w_0 w}$ [3, 34]. For more explanation, see [47].

One can view Schur polynomials as a special case of Schubert polynomials. Much of the work developing Schubert objects is inspired the known results in the case of Schur polynomials and their "Schubert object", namely, semistandard Young tableaux. Many of the known results on Schur polynomials have been carried over to Schubert polynomials, like the Pieri rule and labeled chains in a certain poset. On the other hand, some results have not been successfully extended, like the Littlewood-Richardson rule.

Schubert calculus and Schubert polynomials, in particular, are of interest to areas outside mathematics as well. Physicists use Schur polynomials when studying quantum and theoretical physics [11, 26], computer scientists find the incidences of points and lines via Schubert calculus is used in computer graphics [65], and economists use Grassmannians in their research as well to compute economic equilibria [8, 64]. This subject is becoming increasingly more popular among mathematicians and non-mathematicians alike [6, 7, 19, 20, 22, 23, 31, 47, 48, 52].

Schubert polynomials arose from questions involving enumerative and algebraic geometry, representation theory, and algebraic topology. They have been studied from a variety of perspectives, each with its own combinatorial object [1, 4, 5, 17, 15, 16, 28, 50, 66]. In this dissertation, the combinatorial objects which index the monomials in a Schubert polynomial are called *Schubert objects*. There are many such objects and one of the main goals of this dissertation is to illuminate the bijections between them. In Figure 1, the bijections between these Schubert objects are labeled by capital letters.

Bijections have become increasingly important in mathematics today and lie at the heart of combinatorics. They are largely responsible for the recent introduction of combinatorics as a pure mathematical field. Posets, Möbius functions, graph theory, and many more combinatorial subjects are used together with bijections to prove hard results in other branches of pure and applied mathematics.



Below is a table which lists the sections in which each bijection appears.

Bijection	Section
Ω	4.2
Ψ	4.2
X	3.1
Φ	1.4
Υ	5.2
T	5.1

Figure 1: Chain of Schubert Bijections

In addition to exploring the bijections between Schubert objects, we explore different methods of constructing them. The construction methods are all developed using trees of Schubert objects and taking the collection of leaves at the end of the tree. Chapter 2 introduces a new method to compute the decomposition of Schubert polynomials into key polynomials. The slinky rule is a construction based on the set of all heaps of a permutation. We give several lemmas which restrict the set of heaps of a permutation which are useful. In addition, we re-prove the Remmel-Whitney rule and give a similar variant of the Littlewood-Richardson rule.

Chapter 3 describes mitosis, previously known, and the characterization of properties of this method of constructing rc-graphs. Chapter 3 also demonstrates a new operator, called split, which provides an alternative approach to creating a tree of rc-graphs. It defines a kind of "left" operation on rc-graphs, which gives a left operator on Schubert polynomials.

In Chapter 4, a new Schubert object is explored, called an inversion filling. Inversion fillings are related to other known Schubert objects, and their connections are presented here. We discuss a special case of inversion fillings, the Grassmannian permutation case, which gives rise to a left divided difference operator on semistandard Young tableaux. This is a construction on tableaux, and does not have an interpretation in terms of an operation on polynomials.

Finally, in Chapter 5, we describe the previously known construction of skyline fillings and their connection to other Schubert objects. In particular, a connection between compatible sequences and skyline fillings is proven. The dissertation concludes with open questions which arise in the course of the writing. Figure 2 gives some of the operators which appear, most of which are new.

Operator	Maps from	Maps to	Section
Slinky Rule	Heaps for w	P-tableaux for w	2.1
Mitosis$_i$	$\mathcal{RC}$-graphs for w	$\mathcal{RC}$-graphs for ws_i	3.2
Split$_i$	$\mathcal{RC}$-graphs for w	$\mathcal{RC}$-graphs for s_iw	3.4
Construction Algorithm	Inversion Fillings of w	Inversion Fillings of s_iw	4.3
Grassmannian μ_i	Contretableaux for w	Contretableaux for s_iw	4.4

Figure 2: Algorithms on Schubert Objects

Chapter 1

GENERAL BACKGROUND

In this chapter, we introduce our notation and several important theorems from the literature. Denote the set of permutations of n letters by S_n. Write a permutation in S_n in one line notation. That is, take $w = w_1 w_2 \dots w_n$ to be the permutation which takes i to w_i.

1.1 Reduced Words for a Permutation

Denote the permutation which transposes i and $i+1$ by s_i. These adjacent transpositions are known to generate S_n and satisfy the following Coxeter relations:

$$\begin{aligned} s_i s_i &= 1 \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} \\ s_i s_j &= s_j s_i \text{ for all } |i-j| > 1. \end{aligned} \tag{1.1}$$

Every permutation can be written as a product of adjacent transpositions. If $w = s_{a_1} \cdots s_{a_p}$ and no shorter product of adjacent transpositions equals w, then $s_{a_1} \cdots s_{a_p}$ is said to be *reduced*. The *length* of w, denoted $\ell(w)$, is equal to p. We have that

$$Red(w) := \{(a_1, a_2, \dots, a_p) : w = s_{a_1} s_{a_2} \cdots s_{a_p} \text{ and } p = \ell(w)\}$$

is the set of reduced words for w. There exists a unique element of longest length in S_n. Call this permutation the *longest element of* S_n. The longest element of S_n is the permutation $w_0^{(n)} = n(n-1)\dots 21$ of length $\binom{n}{2}$. Write w_0 if n is understood from the context. For a word $\mathbf{a} = (a_1, a_2, \dots, a_p)$, say $reverse(\mathbf{a}) = (a_p, a_{p-1}, \dots, a_1)$. Notice $\mathbf{a} \in Red(w)$ if and only if $reverse(\mathbf{a}) \in Red(w^{-1})$.

Algorithm 1.1.1. (Heap Algorithm) Let $\mathbf{a} = (a_1, \dots, a_p)$ be any word. Form the *heap* of

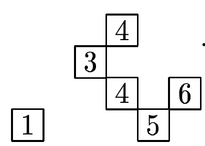
a, *heap*(**a**), as follows:

1. Drop a box filled with a_1 in column a_1 until it lands in row 1.

2. Assume boxes have been dropped in columns $a_1, \ldots, a_{j-1}$. Drop a box in column a_j such that this box has minimum height greater than all boxes in columns $a_j - 1$ and $a_j + 1$ already dropped. If there are no boxes in columns $a_j - 1$ and $a_j + 1$, then the box lands in row 1.

Note that if $|i-j| > 1$, dropping a box in column i and then dropping a box in column j produces the same result as dropping a box in column j and then dropping a box in column i. Thus, if $\mathbf{a}$ is related to $\mathbf{a}'$ by doing a sequence of $(i, j) \mapsto (j, i)$ moves for $|i - j| > 1$, then $\mathbf{a}$ and $\mathbf{a}'$ are said to be in the same commutativity class and have the same heap.

Example 1.1.2. The following is *heap*$(5, 1, 4, 3, 6, 4)$:



One can think of *heap*(**a**) as a partial order on the elements in **a**. Each box in column k is thought of as a distinct element, which is denoted by placing subscripts on the fillings. For a discussion of heaps, see [14, 63].

Definition 1.1.3. A *linear extension* of heap, H, is a word **b** which is a permutation of the elements of H such that k_r appears before $(k+1)_s$ in **b** if and only if k_r is below $(k+1)_s$ in H.

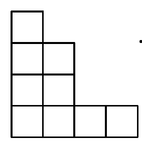
Notice that **a** is a linear extension of *heap*(**a**), but it is not necessarily the only linear extension. Each heap represents exactly one commutativity class. That is, the set of linear extensions of a heap H are are related by a sequence of $(i, j) \mapsto (j, i)$ moves for $|i - j| > 1$. Denote the set of heaps of a permutation w by $\mathcal{H}(w)$. The set of reduced words of a permutation is the same as the set of linear extensions of the heaps for that permutation.

Let $\mathbf{a} = (a_1, a_2, \ldots, a_r) \in Red(w)$ and suppose there exists a subexpression $\alpha = (a_{i_1}, \ldots, a_{i_p})$ of $\mathbf{a}$ such that $\alpha \in Red(v)$ for some v. Then, we say $v \leq w$ in *Bruhat order*.

For more information on Bruhat order, see [1]. If there exists $\mathbf{a} = (a_1, a_2, \ldots, a_r) \in Red(w)$ and $0 \leq q \leq r$ such that $(a_1, a_2, \ldots, a_q) \in Red(v)$, then say $v \leq_r w$ in *right weak order*. If there exists $\mathbf{b} = (b_1, b_2, \ldots, b_r) \in Red(w)$ and $0 \leq m \leq r$ such that $(b_{m+1}, \ldots, b_r) \in Red(v)$, then say $v \leq_\ell w$ in *left weak order*.

1.2 Semi-Standard Young Tableaux and Contretableaux

A partition is a weakly decreasing sequence of nonnegative integers, $\lambda_1 \geq \lambda_2 \geq \ldots$, finitely many of which are positive. To each partition λ, associate an array of boxes with λ_1 boxes on the first row, λ_2 boxes on the second row, etc., the rows numbered from bottom to top. This array is known as the *Ferrers diagram* for λ. For example, let $\lambda = (4, 2, 2, 1)$. The Ferrers diagram associated to λ is



.

The conjugate of λ is the partition $\overline{\lambda} = \overline{\lambda}_1 \geq \overline{\lambda}_2 \geq \ldots$ such that $\overline{\lambda}_i$ is equal to the number of boxes in the i^{th} column of the Ferrers diagram for λ. For $\lambda = (4, 2, 2, 1)$, we have that $\overline{\lambda} = (4, 3, 1, 1)$. Given λ, a filling T of the boxes of the Ferrers diagram for λ with positive integers is called a *tableau* of shape λ, denoted $\lambda = sh(T)$. The conjugate of a tableau T of shape λ is the tableau $\overline{T}$ of shape $\overline{\lambda}$ whose ij^{th} entry is the the ji^{th} entry of T.

Definition 1.2.1. If the columns of a tableau T increase strictly from bottom to top and the rows increase weakly from left to right, T is called a *semi-standard Young tableau.*

If the columns of a tableau U decrease strictly from bottom to top and the rows decrease weakly from left to right, U is called a *contretableau* or a *reverse semi-standard Young tableau.* The term contretableau comes from a slightly different definition in [42], but the definitions are equivalent.

Define $\mathcal{SSYT}$ (respectively, $\mathcal{SSYT}_\lambda$) to be the set of all semi-standard Young tableaux (respectively, the set of all semi-standard Young tableaux with shape λ). Denote $\mathcal{CT}$ (respectively, $\mathcal{CT}_\lambda$) to be the set of all contretableaux (respectively, the set of all contretableaux of shape λ).

Example 1.2.2. Below are a semi-standard Young tableau (left) and a contretableau (right). Both are examples of tableaux of shape $\lambda = (4, 2, 2, 1)$.

$$S = \begin{array}{llll} 4 & & & \\ 3 & 3 & & \\ 2 & 2 & & \\ 1 & 1 & 3 & 3 \end{array} \qquad U = \begin{array}{llll} 1 & & & \\ 2 & 1 & & \\ 3 & 2 & & \\ 4 & 3 & 3 & 3 \end{array}$$

The conjugates of S and U are

$$\overline{S} = \begin{array}{llll} 3 & & & \\ 3 & & & \\ 1 & 2 & 3 & \\ 1 & 2 & 3 & 4 \end{array} \qquad \overline{U} = \begin{array}{llll} 3 & & & \\ 3 & & & \\ 3 & 2 & 1 & \\ 4 & 3 & 2 & 1 \end{array} .$$

Call a tableau T *standard* if the entries in T are in bijection with the numbers 1 through $|\lambda| = \lambda_1 + \lambda_2 + \ldots$, where $\lambda = sh(T)$. The *column word* of a tableau T, denoted $col(T)$, is the word obtained by reading the tableau down each column, taking the columns in order from left to right. In Example 1.2.2,

$$col(S) = (4, 3, 2, 1, 3, 2, 1, 3, 3) \text{ and } col(U) = (1, 2, 3, 4, 1, 2, 3, 3, 3).$$

Knuth [27] defined the following equivalence relation on words: If $d < e < f$, then $\mathbf{x}dfe\mathbf{y} \sim \mathbf{x}fde\mathbf{y}$, $\mathbf{x}edd\mathbf{y} \sim \mathbf{x}ded\mathbf{y}$, $\mathbf{x}eed\mathbf{y} \sim \mathbf{x}ede\mathbf{y}$, and $\mathbf{x}edf\mathbf{y} \sim \mathbf{x}efd\mathbf{y}$, where $\mathbf{x}$ and $\mathbf{y}$ are the peripheral subwords. Let $\sim$ be the transitive closure of these relations. These equivalence classes are called *plactic classes* or *Knuth equivalence classes* in the literature.

Lemma 1.2.3. *[42] Given a word* $\mathbf{a}$,

1. *there is exactly one word* $\mathbf{b}$ *in the equivalence class of* $\mathbf{a}$ *such that* $\mathbf{b}$ *is the column word of a semi-standard Young tableau. Call this tableau* $\tau(\mathbf{a})$.

2. *there is exactly one word* $\mathbf{c}$ *in the equivalence class of* $\mathbf{a}$, *such that* $reverse(\mathbf{c})$ *is the column word of a contretableau. Call this contretableau* $T(\mathbf{a})$.

We refer the reader to [42] for the proof of Lemma 1.2.3. The following are the algorithms to compute $\tau(\mathbf{a})$ and $T(\mathbf{a})$ using the famous Robinson-Schenstead-Knuth correspondence

[27, 55, 57] and a variant of the RSK correspondence.

Algorithm 1.2.4. (RSK Correspondence) To form $\tau(\mathbf{a})$, start by inserting a_1 into the empty tableau and hence obtain the tableau of shape $\mu = (1)$ with entry a_1. Now assume $a_1, a_2, \ldots, a_{m-1}$ have already been inserted, yielding a tableau $\tau(a_1, \ldots, a_{m-1})$. Insert a_m into the tableau as follows:

1. Let $e_1 := a_m$.

2. Given e_i, scan the i^{th} row of $\tau(a_1, \ldots, a_{m-1})$ from left to right and find the first entry (say r) strictly *larger* than e_i. If such an entry does not exist, e_i is placed at the end of the i^{th} row and the process terminates.

3. Define $e_{i+1} := r$. Replace r with e_i in the tableau.

4. Repeat Steps 2 and 3 until Step 2 says to terminate. The result will be the tableau $\tau(a_1, a_2, ..., a_{m-1}, a_m)$.

Lemma 1.2.5. *[20] For a word* $\mathbf{a}$*,* $\tau(\mathbf{a})$ *is a semi-standard Young tableau such that* $\mathbf{a} \sim col(\tau(\mathbf{a}))$.

Algorithm 1.2.6. (Complementary RSK Correspondence) To form $T(\mathbf{a})$, start by inserting a_p into the empty tableau and hence obtain the tableau of shape $\mu = (1)$ with entry a_p. Now assume $a_{m+1}, a_{m+2}, \ldots, a_p$ have already been inserted, yielding a tableau, $T(a_{m+1}, \ldots, a_p)$. Insert a_m into the tableau as follows:

1. Let $e_1 := a_m$.

2. Given e_i, scan the i^{th} row of $T(a_{m+1}, \ldots, a_p)$ from left to right and find the first entry (say s) strictly *smaller* than e_i. If such an entry does not exist, e_i is placed at the end of the i^{th} row and the process terminates.

3. Define $e_{i+1} := s$. Replace s with e_i in the tableau.

4. Repeat Steps 2 and 3 until Step 2 says to terminate. The result will be the tableau $T(a_m, \ldots, a_p)$.

Lemma 1.2.7. *For a word* $\mathbf{a}$*,* $T(\mathbf{a})$ *is a contretableau such that* $\mathbf{a} \sim reverse(col(T(\mathbf{a})))$*.*

Proof. The steps in Algorithm 1.2.6 are the complements of the steps in Algorithm 1.2.4. Thus, the fact that $\tau(\mathbf{a})$ is a semi-standard Young tableau implies $T(\mathbf{a})$ is a contretableau.

In Algorithm 1.2.6, $\mathbf{a}$ is inserted from right to left into $T(\mathbf{a})$ and the rules of the RSK Correspondence are complemented. Define another equivalence relation: If $k > j > i$, then $\mathbf{x}kij\mathbf{y} \equiv \mathbf{x}ikj\mathbf{y}$, $\mathbf{x}jkk\mathbf{y} \equiv \mathbf{x}kjk\mathbf{y}$, $\mathbf{x}jjk\mathbf{y} \equiv \mathbf{x}jkj\mathbf{y}$, and $\mathbf{x}jki\mathbf{y} \equiv \mathbf{x}jik\mathbf{y}$, where $\mathbf{x}$ and $\mathbf{y}$ are the peripheral subwords. Let $\equiv$ be the transitive closure of these relations. By inspection, we have $\mathbf{a} \equiv \mathbf{b}$ if and only if $reverse(\mathbf{a}) \sim reverse(\mathbf{b})$. By Lemma 1.2.5, we have that $reverse(\mathbf{a}) \equiv col(T(\mathbf{a}))$. Thus, $\mathbf{a} \sim reverse(col(T(\mathbf{a})))$.

□

More can be said about the relationship between $\tau(\mathbf{a})$ and $T(\mathbf{a})$. The following theorem is the motivation behind Schensted's involvement in RSK.

Theorem 1.2.8. *[57] Suppose* $sh(\tau(\mathbf{a})) = \lambda$*. Then* $\lambda_1 + \ldots + \lambda_i$ *is the maximum sum of the lengths of* i *weakly increasing disjoint subsequences in* $\mathbf{a}$ *from left to right.*

Corollary 1.2.9. *We have that* $sh(T(\mathbf{a})) = sh(\tau(\mathbf{a}))$*.*

Proof. When forming $T(\mathbf{a})$, we have simply complemented the rules and inserted $\mathbf{a}$ from right to left. Thus, if $sh(T(\mathbf{a})) = \mu$, by Theorem 1.2.8, then $\mu_1 + \ldots + \mu_i$ is the maximum sum of the lengths of i weakly *decreasing* disjoint subsequences of a from right to left, which are exactly the weakly increasing subsequences of $\mathbf{a}$ from left to right. By Theorem 1.2.8, if $sh(\tau(\mathbf{a})) = \lambda$, then $\lambda_1 + \ldots + \lambda_i = \mu_1 + \ldots + \mu_i$ for every $i \geq 1$, and so $\lambda = \mu$. □

Example 1.2.10. Let $\mathbf{a} = (5, 3, 6, 4, 3, 1)$, then $\tau(\mathbf{a})$ and $T(\mathbf{a})$ are computed as follows:

$$
\begin{array}{llll}
\tau(5) = \begin{array}{|c|}\hline 5 \\ \hline \end{array} & & T(1) = \begin{array}{|c|}\hline 1 \\ \hline \end{array} & \\
\tau(5,3) = \begin{array}{l} 5 \\ 3 \end{array} & & T(3,1) = \begin{array}{l} 1 \\ 3 \end{array} & \\
\tau(5,3,6) = \begin{array}{ll} 5 & \\ 3 & 6 \end{array} & & T(4,3,1) = \begin{array}{l} 1 \\ 3 \\ 4 \end{array} & \\
\tau(5,3,6,4) = \begin{array}{ll} 5 & 6 \\ 3 & 4 \end{array} & & T(6,4,3,1) = \begin{array}{l} 1 \\ 3 \\ 4 \\ 6 \end{array} & . \\
\tau(5,3,6,4,3) = \begin{array}{ll} 5 & \\ 4 & 6 \\ 3 & 3 \end{array} & & T(3,6,4,3,1) = \begin{array}{ll} 1 & \\ 3 & \\ 4 & \\ 6 & 3 \end{array} & \\
\tau(\mathbf{a}) = \tau(5,3,6,4,3,1) = \begin{array}{ll} 5 & \\ 4 & \\ 3 & 6 \\ 1 & 3 \end{array} & & T(\mathbf{a}) = T(5,3,6,4,3,1) = \begin{array}{ll} 1 & \\ 3 & \\ 4 & 3 \\ 6 & 5 \end{array} &
\end{array}
$$

The following is one more variant on the association of tableaux to words. Given a reduced word $\mathbf{a} = (a_1, \ldots, a_p)$, associate a pair of semi-standard Young tableaux, $(P(\mathbf{a}), Q(\mathbf{a}))$. Do so by the following algorithm.

Algorithm 1.2.11. [13] (Edelman-Greene Insertion) To form $P(\mathbf{a})$, start by inserting a_1 into the empty tableau and hence obtaining a tableau of shape $\mu = (1)$ with entry a_1. Now assume $a_1, a_2, \ldots, a_{m-1}$ have already been inserted, yielding a tableau, $P(a_1, \ldots, a_{m-1})$. Insert a_m into the tableau as follows:

1. Let $e_1 := a_m$.

2. Given e_i, scan the i^{th} row of $P(a_1, \ldots, a_{m-1})$ from left to right and find the first entry (say r) larger than e_i. If such an entry does not exist, e_i is placed at the end of the i^{th} row and the process terminates.

3. If $r = e_i + 1$ and the entry just to the left of r is equal to e_i, then set $e_{i+1} := e_i + 1$. Otherwise, set $e_{i+1} := r$ and replace r with e_i in the tableau.

4. Repeat Steps 2 and 3 until Step 2 says to terminate. The result will be the tableau $P(a_1, a_2, \ldots, a_{m-1}, a_m)$.

At each insertion step, one box is added to the Ferrers diagram, $sh(P)$. Build up the recording tableau, $Q(\mathbf{a})$, by placing an m in the box that is added after inserting a_m.

Definition 1.2.12. Let $P(w) := \{P(\mathbf{a}) \,|\, \mathbf{a} \in \mathit{Red}(w)\}$.

Example 1.2.13. Let $\mathbf{a} = (5, 3, 6, 4, 3, 1)$, then $P(\mathbf{a})$ and $Q(\mathbf{a})$ are computed as follows:

$$P(5) = \begin{array}{|c|}\hline 5 \\ \hline\end{array} \qquad Q(5) = \begin{array}{|c|}\hline 1 \\ \hline\end{array}$$

$$P(5,3) = \begin{array}{|c|}\hline 5 \\ \hline 3 \\ \hline\end{array} \qquad Q(5,3) = \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array}$$

$$P(5,3,6) = \begin{array}{cc} 5 & \\ 3 & 6 \end{array} \qquad Q(5,3,6) = \begin{array}{cc} 2 & \\ 1 & 3 \end{array}$$

$$P(5,3,6,4) = \begin{array}{|c|c|}\hline 5 & 6 \\ \hline 3 & 4 \\ \hline\end{array} \qquad Q(5,3,6,4) = \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 & 3 \\ \hline\end{array}$$

$$P(5,3,6,4,3) = \begin{array}{cc} 5 & \\ 4 & 6 \\ 3 & 4 \end{array} \qquad Q(5,3,6,4,3) = \begin{array}{cc} 5 & \\ 2 & 4 \\ 1 & 3 \end{array}$$

$$P(5,3,6,4,3,1) = \begin{array}{cc} 5 & \\ 4 & \\ 3 & 6 \\ 1 & 4 \end{array} \qquad Q(5,3,6,4,3,1) = \begin{array}{cc} 6 & \\ 5 & \\ 2 & 4 \\ 1 & 3 \end{array}$$

Theorem 1.2.14. *[13] The Edelman-Greene Insertion Algorithm also has the following properties:*

1. *The map from reduced words to pairs of tableaux, given by mapping* $\mathbf{a} \mapsto (P(\mathbf{a}), Q(\mathbf{a}))$, *is injective.*

2. *If $P \in P(w)$, then for every standard Young tableau Q of shape $sh(P)$, there exists $\mathbf{a}' \in Red(w)$ such that $(P(\mathbf{a}'), Q(\mathbf{a}')) = (P, Q)$.*

3. *The rows and columns of $P(\mathbf{a})$ are strictly increasing. Therefore, $P(\mathbf{a})$ is a semi-standard Young tableau.*

4. *The word $\mathbf{a}$ has a descent in position j (meaning $a_j > a_{j+1}$) if and only if j is southeast of $j+1$ in $Q(\mathbf{a})$.*

5. *We have that $T \in P(w)$ if and only if $\overline{T} \in P(w^{-1})$.*

Corollary 1.2.15. *[13] For $P \in P(w)$, the number of $\mathbf{a} \in Red(w)$ such that $P(\mathbf{a}) = P$ is the same as the number of standard tableau of shape $sh(P)$. Therefore,*

$$|Red(w)| = \sum_{P \in P(w)} f^{sh(P)},$$

where f^λ is the number of standard Young tableau of shape λ.

Frame, Robinson, and Thrall [18] first computed $f^\lambda = \frac{n!}{\prod h_{ij}}$, where h_{ij} is the hook length for every box (i, j) in the Ferrers diagram for λ and $n = |\lambda| = \lambda_1 + \lambda_2 + \ldots$.

1.3 Schubert Polynomials and Compatible Sequences

Let $f \in \mathbb{Z}[x_1, x_2, \ldots]$. Following Bernstein-Gelfand-Gelfand, Demazure, and Lascoux-Schützenberger[3, 12, 34], define

$$\partial_i(f) = \frac{f - s_i \cdot f}{x_i - x_{i+1}},$$

where s_i acts on f by reversing the roles of x_i and x_{i+1} in f. Say ∂_i is the *divided difference operator*. Note $f - s_i \cdot f$ vanishes if $x_i = x_{i+1}$, so $f - s_i \cdot f$ is divisible by $x_i - x_{i+1}$. Thus, $\partial_i(f)$ is a polynomial in $\mathbb{Z}[x_1, x_2, \ldots]$. From the relations in (1.1) and the facts above, one can show ∂_i satisfies the following nilCoxeter relations:

$$\begin{aligned} \partial_i\partial_i &= 0 \\ \partial_i\partial_{i+1}\partial_i &= \partial_{i+1}\partial_i\partial_{i+1} \\ \partial_i\partial_j &= \partial_j\partial_i \text{ for all } |i-j|>1. \end{aligned}$$

Given a reduced word, $(a_1, a_2, \ldots, a_p) \in \mathit{Red}(w)$, for some permutation w, define $\partial_w = \partial_{a_1}\partial_{a_2}\cdots\partial_{a_p}$. We have that ∂_w is well defined, because compositions of ∂_i obey similar commutation relations as the generators of S_n when the word the composition corresponds to is a reduced word.

Let w be a permutation in S_n. Lascoux and Schützenberger[34] define the *Schubert polynomial* indexed by w to be

$$\mathfrak{S}_w := \partial_{w^{-1}w_0}(x_1^{n-1}x_2^{n-2}\ldots x_{n-1}). \tag{1.2}$$

Definition 1.3.1. Suppose $\mathbf{a} = (a_1, a_2, \ldots, a_p)$. Define an $\mathbf{a}$-*compatible sequence* to be a sequence $\mathbf{i} = (i_1, i_2, \ldots, i_p)$ such that

- $1 \leq i_1 \leq i_2 \ldots \leq i_p$,
- $i_j = i_{j+1}$ implies $a_j > a_{j+1}$, and
- $i_j \leq a_j$ for every j.

For a word $\mathbf{a}$, $C(\mathbf{a})$ is the set of all sequences $\mathbf{i}$ such that $\mathbf{i}$ is an $\mathbf{a}$-compatible sequence. For a permutation w, $C(w)$ is the set of all pairs $(\mathbf{a}, \mathbf{i})$ such that $\mathbf{a} \in \mathit{Red}(w)$ and $\mathbf{i} \in C(\mathbf{a})$.

Theorem 1.3.2. *[5, 16] For a permutation w, the Schubert polynomial indexed by w has the following expansion into monomials:*

$$\mathfrak{S}_w = \sum_{(\mathbf{a},\mathbf{i})\in C(w)} x_{i_1}x_{i_2}\ldots x_{i_{\ell(w)}}.$$

For more background on Schubert polynomials, see [47, 48].

1.4 Key Polynomials and Keys

Demazure [12] defined

$$\pi_i(f) = \partial_i(x_i f),$$

known as the Demazure operator. One can check that π_i satisfies the following relations:

$$\begin{aligned} \pi_i\pi_i &= \pi_i \\ \pi_i\pi_{i+1}\pi_i &= \pi_{i+1}\pi_i\pi_{i+1} \\ \pi_i\pi_j &= \pi_j\pi_i \text{ for all } |i-j|>1. \end{aligned}$$

Given a reduced word for some permutation w, $(a_1, a_2, \ldots, a_p) \in Red(w)$, define $\pi_w = \pi_{a_1}\pi_{a_2}\cdots\pi_{a_p}$. Because similar commutation relations hold on both the π_i and s_i, π_w is well defined on reduced words.

A *composition* is a sequence of nonnegative integers, finitely many of which are positive. Given a composition α, let $\mu(\alpha)$ be the unique permutation of smallest length which rearranges α into weakly decreasing order, which we call the partition $\lambda(\alpha)$. Following [42], define the *key polynomial* associated to α to be

$$\kappa_\alpha = \pi_{\mu(\alpha)}(x^{\lambda(\alpha)}). \tag{1.3}$$

Example 1.4.1. Let $\alpha = (2,1,3)$. Then $\mu(\alpha) = s_2 \cdot s_1$ and $\lambda(\alpha) = (3,2,1)$. Thus,

$$\begin{aligned} \kappa_{(2,1,3)} &= \pi_2 \cdot \pi_1(x_1^3x_2^2x_3) \\ &= \pi_2(x_1^3x_2^2x_3 + x_1^2x_2^3x_3) \\ &= x_1^3x_2^2x_3 + x_1^2x_2^3x_3 + x_1^2x_2^2x_3^2 + x_1^3x_2x_3^2 + x_1^2x_2x_3^3. \end{aligned}$$

Like Schubert polynomials, key polynomials also have a combinatorial interpretation.

Definition 1.4.2. A contretableau is a *key* if the set of entries in the $(k+1)^{st}$ column is a subset of the set of entries in the k^{th} column.

The contretableau in Example 1.2.2 is a key. The *content* of a tableau T is a composition

$\alpha = (\alpha_1, \alpha_2, ...)$ such that there are α_i boxes of T which are filled with an i. There is a bijection between keys and compositions. To each key T, associate the composition $\alpha = content(T)$. We have that α will always be a permutation of $sh(T)$. The inverse map associates a composition α with the contretableau with an i in column k if $1 \leq k \leq \alpha_i$. Call this $key(\alpha)$. The composition associated to the key in Example 1.2.2 is $\alpha = (2, 2, 4, 1)$.

Definition 1.4.3. To keys, impose the partial order that $K \trianglelefteq L$ if $sh(K) = sh(L)$ and K is entrywise weakly less than L.

This induces a partial order on compositions as well. By abuse of notation, say $\alpha \trianglelefteq \beta$ if $key(\alpha) \trianglelefteq key(\beta)$. It happens that $\alpha \trianglelefteq \beta$ if $\sum_{i=1}^{k} \alpha_i \geq \sum_{i=1}^{k} \beta_i$ for every $k \geq 1$ and α and β are rearrangements of the same partition. The partial order $\trianglelefteq$ is a suborder of what is known as *dominance order* on compositions.

Let $\mathbf{a} = (a_1, a_2, \ldots, a_p)$ be a word and $\mathbf{a} = a^{(1)} \cdot a^{(2)} \cdots a^{(k)}$ be the decomposition of $\mathbf{a}$ into maximal strictly decreasing subwords. Call $a^{(1)} \cdot a^{(2)} \cdots a^{(k)}$ the *column word* of $\mathbf{a}$, denoted $col(\mathbf{a})$. Say $colform(\mathbf{a})$ is the composition $\alpha = (|a^{(1)}|, \ldots, |a^{(k)}|)$. Denote the leftmost column of a, $a^{(1)}$, by $lcol(a)$ and the rightmost column of $\mathbf{a}$, $a^{(k)}$, by $rcol(\mathbf{a})$. We say a word is *column-frank* if $colform(\mathbf{a})$ is a rearrangement of $\overline{sh(T(\mathbf{a}))}$, the conjugate shape of $T(\mathbf{a})$, $T(\mathbf{a})$ as in Lemma 1.2.3.

Definition 1.4.4. Define the *right key* of a reduced word $\mathbf{a}$ to be the key $K_+(\mathbf{a})$ whose set of columns is the same as $\{rcol(t) \,|\, t \sim \mathbf{a}$ and t is column-frank$\}$ and $sh(K_+(\mathbf{a})) = sh(T(\mathbf{a}))$.

Define the *left key* of a reduced word $\mathbf{a}$ to be the key $K_-(\mathbf{a})$ whose set of columns is the same as $\{lcol(t) \,|\, t \sim \mathbf{a}$ and t is column-frank$\}$ and $sh(K_-(\mathbf{a})) = sh(T(\mathbf{a}))$.

It is not immediately obvious, but in fact, K_+ and K_- are well defined. For a proof, see the Appendix in [53]. By abuse of notation, for a semi-standard young tableau T,

$$K_+(T) = K_+(col(T)) \text{ and } K_-(T) = K_-(col(T)).$$

Also, by abuse of notation, for a contretableau U,

$$K_+(U) = K_+(reverse(col(U)) \text{ and } K_-(U) = K_-(reverse(col(U)).$$

Example 1.4.5. Suppose $\mathbf{a} = (5,3,6,4,3,1)$. Then $\mathbf{a}$ is $\sim$ equivalent to $(5,6,3,4,3,1)$, $(5,6,4,3,3,1)$, $(5,4,6,3,3,1)$, $(5,6,4,3,1,3)$, $(5,4,6,3,1,3)$, $(5,4,3,6,3,1)$, $(5,4,3,6,1,3)$, and $(5,4,3,1,6,3)$. Of the elements in the same $\sim$ equivalence class as $\mathbf{a}$, only $(5,3,6,4,3,1)$ and $(5,4,3,1,6,3)$ are column-frank. See Example 1.2.10 for the shape of $T(\mathbf{a})$. From the definitions, we have that

$$K_+(\mathbf{a}) = \begin{array}{|c|c} \hline 1 & \multicolumn{1}{|c}{} \\ \hline 3 & \multicolumn{1}{|c}{} \\ \hline 4 & \multicolumn{1}{c|}{3} \\ \hline 6 & \multicolumn{1}{c|}{6} \\ \hline \end{array}$$

and

$$K_-(\mathbf{a}) = \begin{array}{|c|c} \hline 1 & \multicolumn{1}{|c}{} \\ \hline 3 & \multicolumn{1}{|c}{} \\ \hline 4 & \multicolumn{1}{c|}{3} \\ \hline 5 & \multicolumn{1}{c|}{5} \\ \hline \end{array} \quad .$$

Theorem 1.4.6. *[39, 42] The key polynomial has the following expansion into monomials:*

$$\kappa_\alpha = \sum_{\substack{T \in \mathcal{SSYT} \\ K_+(T) \trianglelefteq key(\alpha)}} x^T,$$

where $x^T = \prod_{i \in T} x_i$.

An equivalence relation $\stackrel{\cdot}{\sim}$, similar to $\sim$, is constructed in [13, 39]. Define an equivalence relation on words. If $d < e < f$ and $i \geq 1$, then $\mathbf{x}dfe\mathbf{y} \stackrel{\cdot}{\sim} \mathbf{x}fde\mathbf{y}$, $\mathbf{x}edf\mathbf{y} \stackrel{\cdot}{\sim} \mathbf{x}efd\mathbf{y}$, and $\mathbf{x}i(i+1)i\mathbf{y} \stackrel{\cdot}{\sim} \mathbf{x}(i+1)i(i+1)\mathbf{y}$, where $\mathbf{x}$ and $\mathbf{y}$ are the peripheral subwords. Let $\stackrel{\cdot}{\sim}$ be the closure of these relations. These are called *nilplactic* or *Coxeter-Knuth equivalence classes* in the literature. In [53], it is shown that in each equivalence class of $\stackrel{\cdot}{\sim}$, there is exactly one word which is the column word of a semi-standard Young tableau. Furthermore, this semi-standard Young tableau is $P(\mathbf{a})$. A word is *nilcolumn-frank* if $colform(\mathbf{a})$ is a rearrangement of $\overline{sh(P(\mathbf{a}))}$, the conjugate shape of $P(\mathbf{a})$.

Definition 1.4.7. Define the *right nilkey* of a reduced word $\mathbf{a}$ to be the key $K_+^0(\mathbf{a})$ whose set of columns is the same as $\{rcol(t) \,|\, t \stackrel{\cdot}{\sim} \mathbf{a}$ and t is nilcolumn-frank$\}$ and $sh(K_+^0(\mathbf{a})) = sh(P(\mathbf{a}))$.

Define the *left nilkey* of a reduced word $\mathbf{a}$ to be the key $K_-^0(\mathbf{a})$ whose set of columns is the same as $\{rcol(t) \,|\, t \stackrel{\cdot}{\sim} \mathbf{a}$ and t is nilcolumn-frank$\}$ and $sh(K_-^0(\mathbf{a})) = sh(P(\mathbf{a}))$.

Again, K_-^0 and K_+^0 are well defined. See the Appendix in [53] for a proof. By abuse of notation, for a semi-standard young tableau T,

$$K_+^0(T) = K_+^0(col(T)) \text{ and } K_-^0(T) = K_-^0(col(T)).$$

Also, by abuse of notation, for a contretableau U,

$$K_+^0(U) = K_+^0(reverse(col(U))) \text{ and } K_-^0(U) = K_-^0(reverse(col(U))).$$

Example 1.4.8. Suppose $\mathbf{a} = (5,3,6,4,3,1)$. Then $\mathbf{a}$ is $\stackrel{\cdot}{\sim}$ equivalent to $(5,6,3,4,3,1)$, $(5,6,4,3,4,1)$, $(5,4,6,3,4,1)$, $(5,6,4,3,1,4)$, $(5,4,6,3,1,4)$, $(5,4,3,6,4,1)$, $(5,4,3,6,1,4)$, and $(5,4,3,1,6,4)$. Of the elements in the same $\stackrel{\cdot}{\sim}$ equivalence class as $\mathbf{a}$, only $(5,3,6,4,3,1)$ and $(5,4,3,1,6,4)$ are nilcolumn-frank. See Example 1.2.13 for the shape of $P(\mathbf{a})$. From the definitions, we have that

$$K_+^0(\mathbf{a}) = \begin{array}{|c|c} \hline 1 & \multicolumn{1}{c}{} \\ \hline 3 & \multicolumn{1}{c}{} \\ \hline 4 & \multicolumn{1}{c|}{4} \\ \hline 6 & \multicolumn{1}{c|}{6} \\ \hline \end{array}$$

and

$$K_-^0(\mathbf{a}) = \begin{array}{|c|c} \hline 1 & \multicolumn{1}{c}{} \\ \hline 3 & \multicolumn{1}{c}{} \\ \hline 4 & \multicolumn{1}{c|}{3} \\ \hline 5 & \multicolumn{1}{c|}{5} \\ \hline \end{array} \quad .$$

Theorem 1.4.9. *[39] The Schubert polynomial has the following expansion into key polynomials:*

$$\mathfrak{S}_w = \sum_{P \in P(w^{-1})} \kappa_{content(K_-^0(P))}.$$

We also have the following extension, due to Reiner and Shimozono. We view this as bijection Φ in Figure 1.

Theorem 1.4.10. *[53] Key polynomials have the following expansions:*

1.

$$\kappa_\alpha = \sum_{reverse(\mathbf{a}) \stackrel{\cdot}{\sim} col(P)} \sum_{\mathbf{i} \in C(\mathbf{a})} x_{\mathbf{i}},$$

where P is any semi-standard Young tableau satisfying content$(K^0_-(P)) = \alpha$.

2.

$$\kappa_\alpha = \sum_{reverse(\mathbf{a}) \sim col(T)} \sum_{\mathbf{i} \in C(\mathbf{a})} x_{\mathbf{i}},$$

where T is any semi-standard Young tableau satisfying content$(K_-(T)) = \alpha$.

3. *In particular,*

$$\kappa_\alpha = \sum_{reverse(\mathbf{a}) \sim key(\alpha)} \sum_{\mathbf{i} \in C(\mathbf{a})} x_{\mathbf{i}}.$$

1.5 Symmetric Functions

Definition 1.5.1. A *symmetric polynomial* in n variables is a function $f(x_1, x_2, \ldots, x_n)$ such that for every permutation $\sigma \in S_n$,

$$f(x_1, x_2, \ldots, x_n) = f(x_{\sigma 1}, x_{\sigma 2}, \ldots, x_{\sigma n}).$$

A *symmetric function* in infinitely many variables is a function $f(x_1, x_2, \ldots)$ such that for every n and for every permutation $\sigma \in S_n$,

$$f(x_1, \ldots, x_n, x_{n+1}, \ldots) = f(x_{\sigma 1}, \ldots, x_{\sigma n}, x_{n+1}, \ldots).$$

Definition 1.5.2. The *Schur polynomial* in k variables indexed by the partition λ is the function

$$s_\lambda(x_1, x_2, \ldots, x_k) = \sum_{T \in \mathcal{SSYT}^k_\lambda} x^T,$$

where $\mathcal{SSYT}^k_\lambda$ the subset of tableaux in $\mathcal{SSYT}_\lambda$ whose largest entry is at most k.

The *Schur function* indexed by the partition λ is the function

$$s_\lambda(x_1, x_2, \ldots) = \sum_{T \in \mathcal{SSYT}_\lambda} x^T.$$

For a thorough treatment of Schur functions, see [20, 62]. It is well known that Schur functions form a basis for the ring of symmetric functions with integer coefficients. Thus, for partitions λ, μ,

$$s_\lambda(x_1, x_2, \ldots)s_\mu(x_1, x_2, \ldots) = \sum c^\nu_{\lambda\mu} s_\nu(x_1, x_2, \ldots).$$

The coefficients $c^\nu_{\lambda\mu}$ are called *Littlewood-Richardson coefficients* and are known to be non-negative. There are many combinatorial rules for finding these coefficients. We will describe one such rule.

Given λ, μ, define $W(\lambda, \mu)$ to be the pair of tableaux (L_λ, L_μ), where $sh(L_\lambda) = \lambda$, $sh(L_\mu) = \mu$, L_λ is filled with the integers $1, \ldots, |\lambda|$, and L_μ is filled with the integers $|\lambda|+1, \ldots, |\lambda|+|\mu|$, starting from the bottom row, right to left. We call $W(\lambda, \mu)$ the *reverse numbering* of (λ, μ).

Example 1.5.3. Suppose $\lambda = (2, 2, 1), \mu = (2)$. The reverse numbering of (λ, μ) is

$$W(\lambda, \mu) = \left(L_\lambda = \begin{array}{|c|c} \hline 5 & \multicolumn{1}{|c}{} \\ \hline 4 & \multicolumn{1}{c|}{3} \\ \hline 2 & \multicolumn{1}{c|}{1} \\ \hline \end{array} \qquad L_\mu = \begin{array}{|c|c|} \hline 6 & 7 \\ \hline \end{array} \right).$$

Definition 1.5.4. Given tableaux (U, V), a standard Young tableau T is in the set $\mathfrak{W}(U, V)$ if and only if all of the following hold:

1. for $k < \ell$ in the same column in U or V, then ℓ is northwest of k in T,

2. for $k < \ell$ in the same row in U or V, then ℓ is southeast of k in T, and

3. $sh(T)$ is a partition of $|sh(U)| + |sh(V)|$.

Theorem 1.5.5. *[54] (The Remmel-Whitney Rule)For partitions* (λ, μ),

$$s_\lambda(x_1, x_2, \ldots)s_\mu(x_1, x_2, \ldots) = \sum_{T \in \mathfrak{W}(W(\lambda,\mu))} s_{sh(T)}(x_1, x_2, \ldots).$$

Schur functions arise in this dissertation because of their connection to Schubert polynomials. In fact, a Schur polynomial is a special case of a Schubert polynomial.

Definition 1.5.6. A permutation w in S_n is *Grassmannian* if there exists a k such that $w_i < w_{i+1}$ for all $i \neq k$. The set of such permutations is denoted $G(n)$. Given $1 \leq k < n$, the set of $w \in G(n)$ with $w_i < w_{i+1}$ for all $i \neq k$ is denoted $G(n;k)$.

There is a bijection from partitions whose Ferrers diagram fits inside a $k \times (n-k)$ box to $G(n;k)$. Given $\lambda = \lambda_1 \geq \ldots \geq \lambda_k \geq 0$, with $\lambda_1 \leq n-k$, associate the permutation $w \in G(n;k)$ such that $w_i = i + \lambda_{k+1-i}$ for $1 \leq i \leq k$ and the remaining integers are written in increasing order. The inverse map sends $w \in G(n;k)$ to the partition λ whose $(k-i+1)^{st}$ part is $|\{j > i : w_j < w_i\}|$ for $1 \leq i \leq k$. The following theorem has many proofs, one of which is found as Proposition 2.6.8 in [48].

Theorem 1.5.7. *[48] Suppose λ is the partition associated to the permutation w in $G(n;k)$. Then*

$$\mathfrak{S}_w(x_1, \ldots, x_n) = s_\lambda(x_1, x_2, \ldots, x_k).$$

There is another set of symmetric functions which has a connection to Schubert polynomials, called Stanley symmetric functions. Given a set $S \subset \{1, \ldots, n-1\}$ and a positive integer n, define

$$Q_{S,n}(x_1, x_2, \ldots) = \sum x_{i_1} \cdots x_{i_n}, \tag{1.4}$$

where the sum ranges over all sequences $1 \leq i_1 \leq \ldots \leq i_n$ such that $1 \leq i_j < i_{j+1}$ if $j \in S$. The set of ascents of a word $\mathbf{a}$ is the set $E(\mathbf{a}) := \{j \,|\, a_j < a_{j+1}\}$. Define the *Stanley Symmetric Function* indexed by w to be

$$F_w(x_1, x_2, \ldots) = \sum_{\mathbf{a} \in Red(w)} Q_{E(a),\ell(w)}(x_1, x_2, \ldots). \tag{1.5}$$

Stanley first introduced these functions in [60] to count the number of reduced decompositions of a permutation. Surprisingly, Stanley symmetric functions are in fact symmetric. Stemming from the work of Edelman-Greene[13], Stanley found that

$$F_w(x_1, x_2, \ldots) = \sum_{T \in P(w)} s_{sh(T)}(x_1, x_2, \ldots).$$

Let 1^n denote the identity permutation in S_n. For permutations u and v in S_m and S_l, respectively, we say $u \times v$ is the permutation $u_1 u_2 \dots u_m (v_1 + m)(v_2 + m) \dots (v_l + m)$ in S_{m+l}.

The relations used to define **a**-compatible sequences in Definition (1.3.1) are very similar to the relations used to define Equations (1.4) and (1.5). In fact, it was shown in [47] that

$$lim_{n \to \infty} \mathfrak{S}_{1^n \times w} = F_{w^{-1}}.$$

Similarly, we have that for permutations u and v in S_m and S_l, respectively,

$$lim_{n \to \infty} \mathfrak{S}_{1^n \times u} \mathfrak{S}_{1^{n+m} \times v} = lim_{n \to \infty} \mathfrak{S}_{1^n \times u \times v} = F_{(u \times v)^{-1}}.$$

Thus,

$$lim_{k \to \infty} s_\lambda(x_1, \dots, x_k) s_\mu(x_1, \dots, x_k) = s_\lambda(x_1, \dots) s_\mu(x_1, \dots) = F_{(u \times v)^{-1}},$$

where u and v are the permutations associated with λ and μ in $G(n; k)$. On the other hand, we also have

$$F_{(u \times v)^{-1}} = \sum_{T \in P((u \times v)^{-1})} s_{sh(T)}(x_1, x_2, \dots). \tag{1.6}$$

Therefore, we obtain another version of the Littlewood-Richardson Rule.

Corollary 1.5.8. *For partitions λ, μ associated with permutations u and v, we have*

$$s_\lambda(x_1, \dots) s_\mu(x_1, \dots) = \sum_{T \in P((u \times v)^{-1})} s_{sh(T)}(x_1, x_2, \dots). \tag{1.7}$$

Chapter 2

EDELMAN-GREENE TABLEAUX AND THE SLINKY RULE

The main theorem in this chapter, Theorem 2.1.5, says that all Edelman-Greene P-tableaux for w can be obtained by using the slinky rule. Section 2.2 discusses several consequences of and corollaries to the slinky rule.

2.1 The Slinky Rule

The slinky rule is defined on heaps. Recall the definition of a heap from Algorithm 1.1.1. We think of $heap(\mathbf{a})$ as a partial order on the elements of $\mathbf{a}$. Each element of $\mathbf{a}$ is thought of as distinct. So, for all boxes k in H, label the k's from bottom to top in strictly increasing order.

Algorithm 2.1.1. (Slinky Tree Algorithm) Given a heap H in $\mathcal{H}(w)$, form the *slinky tree* for H according to the following rules:

1. Start with the empty tableau as the root of the tree.

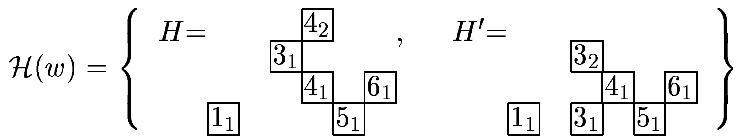
2. At the $(k+1)^{st}$ step, assume the $(k+1)^{st}$ column of H contains $(k+1)_1, \ldots, (k+1)_t$. Add each $(k+1)_s$ to the leaves of the tree created on the k^{th} step such that k_r **is northwest of** $(k+1)_s$ **in the tableau if and only if** k_r **is below** $(k+1)_s$ **in** H**.**

3. When all of H has been added to the tree, suppress the subscripts of the entries to form tableaux.

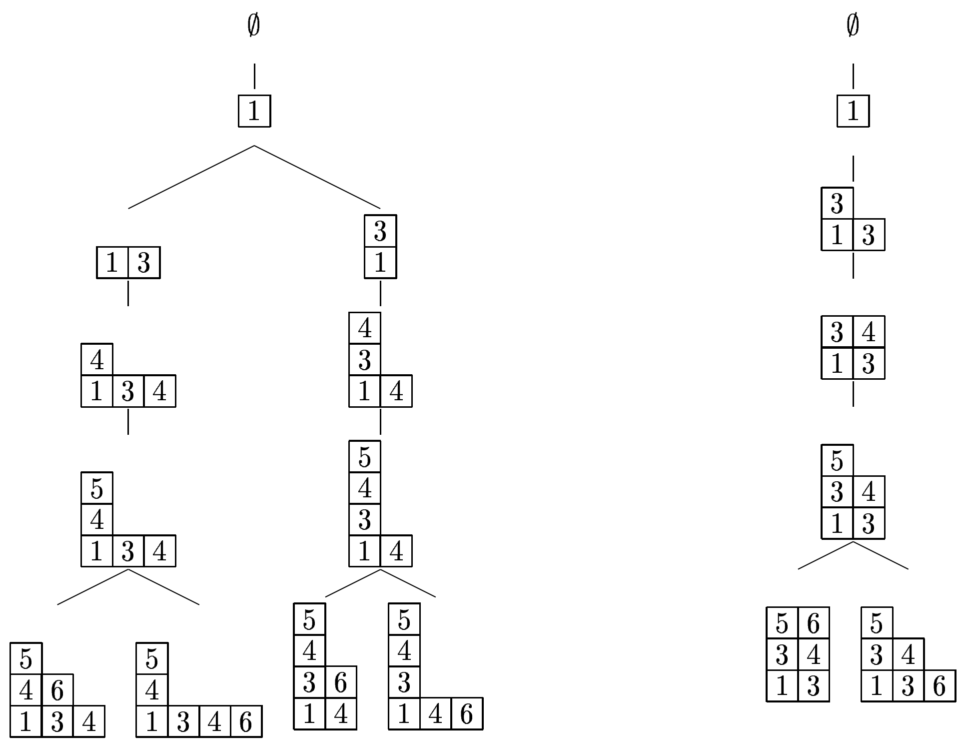
The collection of leaves on the last step is refered to as the set $S(H)$. This collection does not include the leaves of branches created along the way which are childless, refered to as *barren*. We can avoid some barren branches by replacing Step 2 with Step $2'$.

2′. At the $(k+1)^{st}$ step, assume the $(k+1)^{st}$ column of H contains $(k+1)_1, \ldots, (k+1)_t$. Add each $(k+1)_s$ to the leaves of the tree created on the k^{th} step such that k_r is northwest of $(k+1)_s$ in the tableau if and only if k_r is below $(k+1)_s$ in H. Furthermore, for $1 \leq i < t$, $(k+1)_i$ appears stictly northwest of $(k+1)_{i+1}$.

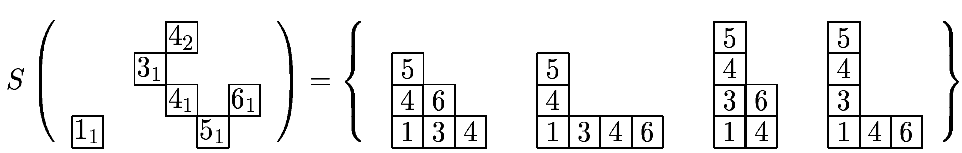
Example 2.1.2. The permutation $w = 2164375$ has two heaps,



The slinky tree for H is on the left and the slinky tree for H' is on the right.



So, we have



and

$$S\left(\begin{array}{cccccc} & & \boxed{3_2} & & & \\ & & & \boxed{4_1} & & \boxed{6_1} \\ \boxed{1_1} & & \boxed{3_1} & & \boxed{5_1} & \end{array}\right) = \left\{ \begin{array}{|c|c|} \hline 5 & 6 \\ \hline 3 & 4 \\ \hline 1 & 3 \\ \hline \end{array} \quad \begin{array}{lll} \boxed{5} & & \\ \boxed{3} & \boxed{4} & \\ \boxed{1} & \boxed{3} & \boxed{6} \end{array} \right\}.$$

For an example with barren branches, see Example 2.2.19.

Lemma 2.1.3. *Given T a tableau with strictly increasing rows and columns, the following are equivalent for every pair, $\{k_r, (k+1)_s\}$ in T:*

1. *k_r is northwest of $(k+1)_s$ in T.*

2. *k_r appears before $(k+1)_s$ in $col(T)$.*

3. *k_r appears below $(k+1)_s$ in $heap(col(T))$.*

Proof. By the definition of the column word of a tableau, k_r comes before $(k+1)_s$ in $col(T)$ if and only if k_r is in a column strictly west of $(k+1)_s$ in T or north of $(k+1)_s$ in the same column. Notice that, if k_r is in a column strictly west of $(k+1)_s$ in T, then k_r must be northwest of $(k+1)_s$ (weakly north and strictly west). Otherwise the entry in the same column as k_r and same row as $(k+1)_s$ would be less than or equal to k_r or greater than or equal to $(k+1)_s$. Either of these would violate the condition that the tableau must be strictly increasing in rows and columns. Also, observe that k_r can not be north of $(k+1)_s$ in the same column because then the columns would not be increasing. Thus, k_r comes before $(k+1)_s$ in $col(T)$ if and only if k_r is northwest of $(k+1)_s$ in T.

From the definition of $heap(col(T))$, k_r appears before $(k+1)_s$ in $col(T)$ if and only if k_r appears below $(k+1)_s$ in $heap(col(T))$. □

The following corollary says that any row and column strict tableau T is a leaf of the slinky tree for $heap(col(T))$.

Corollary 2.1.4. *Given T a tableau with strictly increasing rows and columns,*

$$T \in S(heap(col(T))).$$

Proof. This corollary follows from Step (2) of Algorithm 2.1.1 and Lemma 2.1.3. □

Theorem 2.1.5. *(The Slinky Rule) For every permutation w,*

$$\bigcup_{\mathcal{H}(w)} S(H) = P(w).$$

Proof. Suppose that $H \in \mathcal{H}(w)$ and $T \in S(H)$. Then k_r is northwest of $(k+1)_s$ in T if and only if k_r is below $(k+1)_s$ in H by Algorithm 2.1.1. By Lemma 2.1.3, this implies $col(T)$ is a linear extension of H, and hence $col(T) \in Red(w)$. We have $P(col(T)) = T$, so, $T \in P(w)$. Therefore, for every permutation w,

$$\bigcup_{H \in \mathcal{H}(w)} S(H) \subseteq P(w).$$

Suppose $T \in P(w)$. Then $col(T) \in Red(w)$ and $heap(col(T)) \in \mathcal{H}(w)$. By Corollary 2.1.4, $T \in S(heap(col(T))$. So for every permutation w,

$$P(w) \subseteq \bigcup_{H \in \mathcal{H}(w)} S(H).$$

□

Example 2.1.6. By Equation 1.6, Example 2.1.2, and Theorem 2.1.5, for $w = 2164375$,

$$F_{w^{-1}} = s_{4,2} + s_{4,1,1} + s_{3,3} + 2s_{3,2,1} + s_{3,1,1,1}.$$

2.2 Productive Heaps and Simply Productive Permutations

In this section, we describe ways to make the slinky rule more efficient. We say a heap H is *productive* if $P(H) \neq \emptyset$. That is, H is productive if the branches of the slinky tree for H are not all barren. From Theorem 2.1.5, it is clear that for $w \neq id$, there exists some $H \in \mathcal{H}(w)$ such that H is productive.

Lemma 2.2.1. *Suppose $w \neq id$ and H has the lexicographically largest content in $\mathcal{H}(w)$. Then H is productive.*

Proof. Given $\mathbf{a} \in Red(w)$, the column word of $P(\mathbf{a})$, $col(P(\mathbf{a}))$, is a linear extension of $heap(col(P(\mathbf{a})))$ by Lemma 2.1.3. Furthermore, all linear extensions of $heap(col(P(\mathbf{a})))$ are also in $Red(w)$. By Corollary 2.1.4, $P(\mathbf{a}) \in S(heap(col(P(\mathbf{a}))))$. At each step in Algorithm 1.2.11, the only change to the content of a word is possibly to increase elements of the word by one. Thus, the content of $P(\mathbf{a})$ is weakly lexicographically larger than the content of the word $\mathbf{a}$.

Suppose H has the lexicographically largest content in $\mathcal{H}(w)$. There is a unique such H[14]. Given a linear extension $\mathbf{b}$ of H, we observed that $col(P(\mathbf{b})) \in Red(w)$, so

$$heap(col(P(\mathbf{b})) \in \mathcal{H}(w).$$

The content of $\mathbf{b}$ is the same as the content of H, and the content of $P(\mathbf{b})$ is the same as the content of $heap(col(P(\mathbf{b}))$, which must be have the lexicographically largest content. Thus, if H has the lexicographically largest content, the content of $\mathbf{b}$ is the same as the content of $P(\mathbf{b})$. Therefore,

$$heap(col(P(\mathbf{b})) = heap(\mathbf{b}) = H.$$

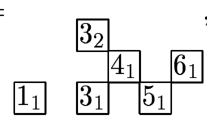
This implies $P(\mathbf{b}) \in S(H)$, and so, H is productive. □

Corollary 2.2.2. *If $heap(\mathbf{b})$ has the lexicographically largest content among $\mathcal{H}(w)$, then $P(\mathbf{b}) \in S(heap(\mathbf{b}))$.*

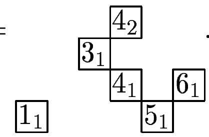
Observe Corollary 2.2.2 does not always hold. That is, given $\mathbf{a} \in Red(w)$, it is not necessarily true that $heap(\mathbf{a}) = heap(col(P(\mathbf{a})))$.

Example 2.2.3. Let $\mathbf{a} = (5, 3, 6, 4, 3, 1)$. Then

$heap(\mathbf{a}) =$



, $\quad P(\mathbf{a}) = \begin{array}{|c|c|} \hline 5 \\ \cline{1-1} 4 \\ \hline 3 & 6 \\ \hline 1 & 4 \\ \hline \end{array}$, and

$heap(col(P(\mathbf{a}))) =$  .

The proof of Lemma 2.2.1 does not produce the heap H which has the lexicographically largest content in $\mathcal{H}(w)$. Algorithms 2.2.4 and 2.2.5 describe how to find this heap.

Algorithm 2.2.4. (Lexicographically Largest Heap Algorithm) Given a permutation w, we describe how to find the lexicographically largest heap.

1. Define $c(w) := (c_1, c_2, \ldots)$ such that $c_i = |\{j > i : w_j < w_i\}|$. This is known as the *code* of w.

2. Let $\mathbf{a} = (a_1, \ldots, a_{\ell(w)})$ be the word such that $a_m = i + \sum_{j=1}^{i} c_j - m$ for $\displaystyle\sum_{j=1}^{i-1} c_j < m \leq \sum_{j=1}^{i} c_j$. Then $\mathbf{a} \in Red(w)$.

3. Then $H = heap(\mathbf{a})$ has the lexicographically largest content in $\mathcal{H}(w)$.

The fact that Algorithm 2.2.4 produces the heap with the lexicographically largest content follows from Edelman's work on lattice paths and the heap with the lexicographically smallest content [14].

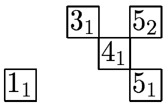
Algorithm 2.2.5. (Lexicographically Largest P-Tableau Algorithm) To find one tableau T in $S(H)$, do the following:

1. Form an array of boxes with c_i boxes in column i.

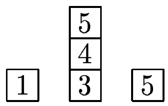
2. Fill in the box in column i at height j with $i + j - 1$.

3. Push all the boxes as far left as possible until they run into the wall next to the first column. The result is a tableau T in $S(H)$.

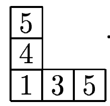
Example 2.2.6. For $w = 216354$, $c(w) = (1,0,3,0,1,0)$. From Step 2 of Algorithm 2.2.4, we have $\mathbf{a} = (1,5,4,3,5)$. So the heap with the lexicographically largest content in $\mathcal{H}(w)$ is


.

To find one T in $S(H)$, follow Algorithm 2.2.5 to get



and $T =$


.

Lemma 2.2.7. *Suppose H is a productive heap and column i of H contains n_i boxes. Then for each $r \geq 1$,*

$$\sum_{i=1}^{r-1} n_i \geq \binom{n_r}{2}.$$

Proof. For every $T \in S(H)$, there are n_r entries of T equal to r. Each row and column of T is strictly increasing. Thus, when T is restricted to the entries 1 through r, we again have a semi-standard Young tableau $T(r)$ of shape $\mu = \mu_1 \geq \ldots \geq \mu_\ell > 0$. Each entry r must be in its own row and column, by row and column strictness. Thus, this semi-standard Young tableau has at least n_r columns and rows. This implies μ has at least n_{r-1} descents (i.e. $\mu_i > \mu_{i-1}$), $\ell \geq n_r$, and $\mu_1 \geq n_r$. These together imply $\mu_i \geq n_r - i + 1$. Thus, for $1 \leq i \leq n_r$,

$$\sum_{i=1}^{r} n_i = |\mu| = \sum_{i=1}^{\ell} \mu_i \geq \sum_{i=1}^{n_r} n_r - i + 1 = \binom{n_r+1}{2} = n_r + \binom{n_r}{2}.$$

□

Lemma 2.2.8. *Let H be a heap. Suppose there is a sequence, $k_{r_0}, (k+1)_{r_1} \ldots, (k+\ell)_{r_\ell}$ such that for every $1 \leq i \leq \ell$, $(k+i-1)_{r_{i-1}}$ lies below (respectively, above) $(k+i)_{r_i}$ in H. Then for every $T \in S(H)$, $(k+\ell)_{r_\ell}$ is in at least the $(\ell+1)^{st}$ column (respectively, the $(\ell+1)^{st}$ row).*

Proof. Let H be a heap and $T \in S(H)$. Suppose k_{r_0} is below $(k+1)_{r_1}$ in H. Also suppose that for every $T \in S(H)$, k_{r_0} is in at least the i^{th} column. Then $(k+1)_{r_1}$ must be northwest of k_{r_0} in T. By the proof of Lemma 2.1.3, $(k+1)_{r_1}$ must be strictly north of k_{r_0} in T. Then

$(k+1)_{r_1}$ must be in at least the $(i+1)^{st}$ column.

We have that k_{r_0} must be in at least the first column and first row. Thus, if there is a sequence, $k_{r_0}, \ldots, (k+\ell)_{r_\ell}$ such that for every $1 \leq i \leq \ell$, $(k+i-1)_{r_{i-1}}$ lies below $(k+i)_{r_i}$ in H, then by induction, we know $(k+\ell-1)_{r_{\ell-1}}$ must be in at least the ℓ^{th} column in T. By the observation above, $(k+\ell)_{r_\ell}$ is in at least the $(\ell+1)^{st}$ column. A similar proof holds for the second case. $\square$

Corollary 2.2.9. *Suppose a heap H is productive. Also, suppose there are sequences, $k_{r_0}, (k-1)_{r_1} \ldots, (k-\ell)_{r_\ell}$ and $k_{s_0} = k_{r_0}, (k-1)_{s_1} \ldots, (k-m)_{s_m}$ such that for every $1 \leq i \leq \ell$, $(k-i+1)_{r_{i-1}}$ lies below $(k-i)_{r_i}$ in H and for every $1 \leq j \leq m$, $(k-j+1)_{s_{j-1}}$ lies below $(k-j)_{s_j}$ in H. Then*

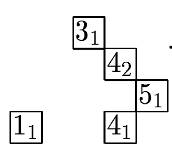
$$\sum_{i=1}^{k-1} n_i \geq \ell m - 1.$$

Proof. Lemma 2.2.8 says k_{r_0} must be in at least the ℓ^{th} row and m^{th} column. For $T \in S(H)$, the rectangular subtableau with northeast corner k_{r_0} contains entries at most $k-1$ except k_{r_0}. Therefore,

$$\sum_{i=1}^{k-1} n_i \geq \ell m - 1.$$

$\square$

Example 2.2.10. Suppose $H \in LH(w)$ and

$H =$  .

Observe $\boxed{5_1}$ is at the bottom of a height decreasing sequence of 3 boxes, thus must be in at least the third row of a tableau $T \in S(H)$. Also, $\boxed{5_1}$ is at the top of a height increasing sequence of 2 boxes, so must be in at least the second column of a tableau $T \in S(H)$. This requires that every tableau in $S(H)$ has at least 6 boxes, but $\ell(w) = 5 < 6$. So, $S(H) = \emptyset$ and thus, H is not productive.

Lemma 2.2.11. *Given a heap H, suppose $u_i(k_r)$ is the number of boxes in column i of H*

which are unrelated to k_r, when H is viewed as a partial order on the boxes. Also, suppose there are sequences, $k_{r_0}, (k-1)_{r_1} \ldots, (k-\ell)_{r_\ell}$ and $k_{s_0} = k_{r_0}, (k-1)_{s_1} \ldots, (k-m)_{s_m}$ such that for every $1 \leq i \leq \ell$, $(k-i+1)_{r_{i-1}}$ lies below $(k-i)_{r_i}$ in H and for every $1 \leq j \leq m$, $(k-j+1)_{s_{j-1}}$ lies below $(k-j)_{s_j}$ in H. Then

$$\sum_{i=1}^{k-1} u_i(k_{r_0}) \geq (\ell-1)(m-1).$$

Proof. Lemma 2.2.8 says k_{r_0} must be in at least the ℓ^{th} row and m^{th} column. For $T \in S(H)$, the rectangular subtableau with northeast corner k_{r_0} contains entries at most $k-1$ except k_{r_0}. Furthermore, every entry in the rectangular subtableau in at most the $(\ell-1)^{st}$ row and $(m-1)^{st}$ column is strictly south and strictly west of k_{r_0}. The conditions in Algorithm 2.1.1 guarantee that, if k_{r_0} is related to any entry in H, then it is either northwest or southeast of k_{r_0}. Thus, every entry in this subtableau is both unrelated to k_{r_0} and strictly less than k. Therefore,

$$\sum_{i=1}^{k-1} u_i(k_{r_0}) \geq (\ell-1)(m-1).$$

□

Definition 2.2.12. We say a permutation $w \neq 1$ is *simply productive* if there is only one heap of w which is productive. We know there is always at least one productive heap since $P(w) \neq \emptyset$.

Lemma 2.2.13. *Suppose H is a heap and H' is the heap obtained by reflecting H about a horizontal axis. Then H is productive if and only if H' is productive.*

Proof. Suppose a heap $H \in \mathcal{H}(w)$ is productive, so there exists some semi-standard Young tableau T in $S(H)$. Then there exists a reduced word, $\mathbf{a} \in Red(w)$ such that $H = heap(\mathbf{a})$. We have $reverse(\mathbf{a}) \in Red(w^{-1})$, so $H' = heap(reverse(\mathbf{a})) \in \mathcal{H}(w^{-1})$. Furthermore, the relations among the boxes in H are complementary to the relations of the boxes in H'. Thus, reflecting H about a horizontal axis and letting gravity act on the boxes (that is, a box in column k falls so that it does not pass any boxes in columns $k-1$ or $k+1$) yields the heap H'. The rules for finding $S(H)$ are simply reversed. Thus $\overline{T}$ is in $S(H')$. Recall, the rows of a P-tableau are also strictly increasing, so $\overline{T}$ is also a semi-standard Young tableau. $\square$

Corollary 2.2.14. *A permutation w is simply productive if and only if w^{-1} is simply productive.*

Corollary 2.2.15. *We have that $T \in P(w)$ if and only if $\overline{T} \in P(w^{-1})$.*

Note Corollary 2.2.15 is the same as Property 5 in Theorem 1.2.14, but has an alternative proof than was given in [13].

Let $w = w_1 w_2 \ldots w_n$ be written in one line notation. We say that w *contains* the pattern $\sigma_1 \ldots \sigma_k$ if there exist $i_1 < i_2 < \ldots < i_k$ such that $w_{i_1} w_{i_2} \ldots w_{i_k}$ is in the same relative order as $\sigma_1 \ldots \sigma_k$. If w does not contain the pattern $\sigma_1 \ldots \sigma_k$, w is said to *avoid* $\sigma_1 \ldots \sigma_k$.

Observation 2.2.16. If w is 321-avoiding, then it is simply produtive.

Since 321-avoiding permutations have only one heap [5], there can only be one productive heap for each 321-avoiding permutation.

Observation 2.2.17. If w is 2143-avoiding, then it is simply produtive.

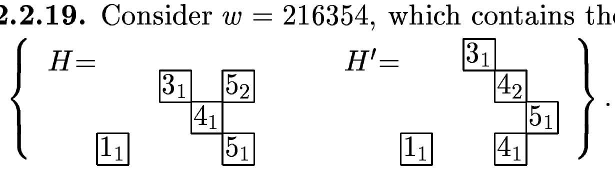
We see this because the Schubert polynomial indexed by a 2143-avoiding permutation is a key polynomial itself [13, 60]. The 2143-avoiding permutations are called *vexillary* in the literature. From Theorem 1.4.9, we have that a vexillary permutation can have only one P-tableau, and so can have only one productive heap.

An open question remains. Which permutations are simply productive? Through experimentation through Σ_8, we venture to state the following conjecture.

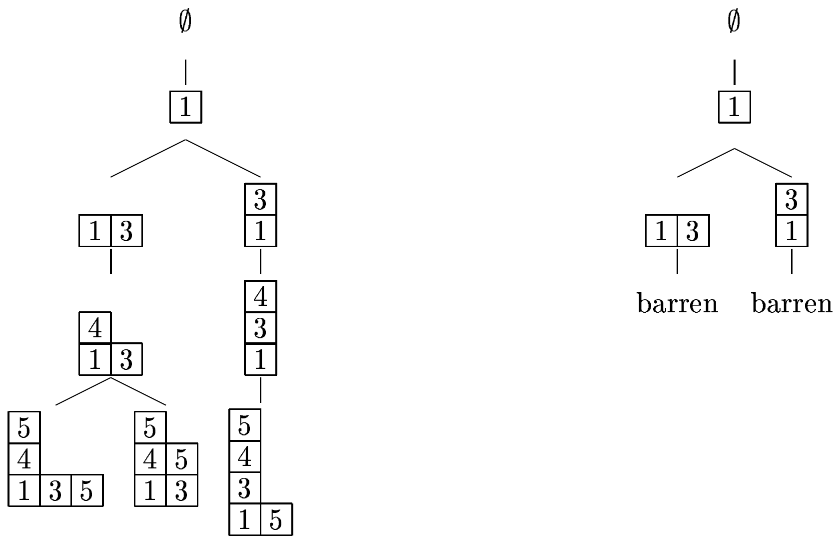
Conjecture 2.2.18. *If a permutation w avoids the permutation 21543, then w is simply productive.*

This conjecture is not sufficient as Example 2.2.19 shows.

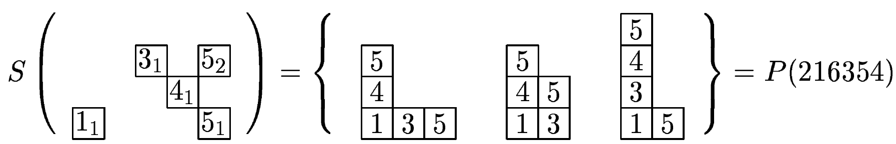
Example 2.2.19. Consider $w = 216354$, which contains the pattern 21543. We have

$\mathcal{H}(w) =$ 

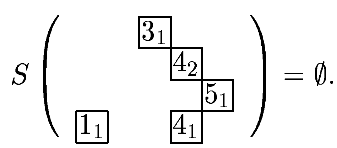
The slinky tree for H is on the left and for H' is on the right.



And so, we have that



and



Thus, w is a simply productive permutation.

2.3 The Remmel-Whitney Rule as a Special Case

The slinky rule bears resemblence to the Remmel-Whitney rule, Theorem 1.5.5. In fact, the Remmel-Whitney rule is a corollary of the slinky rule, as this section will show. We will be using much of the background in Section 1.5 here.

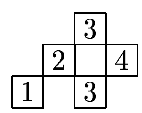
Consider partitions λ and μ associated to permutations u and v in $G(n;k)$. Recall

Equations (1.6) and (1.7):

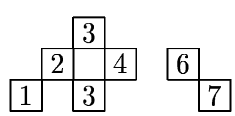
$$F_{(u\times v)^{-1}} = s_\lambda s_\mu = \sum_{T \in P(u\times v)^{-1}} s_{sh(T)}.$$

It remains to find the set $P((u \times v)^{-1})$. In fact, $P \in P((u \times v)^{-1})$ if and only if $\overline{P} \in P(u \times v)$ by Lemma 1.2.14, so it is sufficient to describe $P(u \times v)$. Observe $u \times v$ is 321-avoiding and thus, $u \times v$ only has one heap. In addition, we recognize the heap of $u \times v$ as the disjoint union of the heaps for u and $1^m \times v$. Given any Grassmanian permutation ω associated to the partition λ in $G(n; k)$, the code of ω is $c(\omega) = (\lambda_k, \lambda_{k-1}, \ldots, \lambda_1, 0, 0, \ldots)$. Its heap is the diagram computed in Algorithm 2.2.4. This is the diagram with λ_i boxes proceeding diagonally down to the right from the $(k - i + 1)^{st}$ column to the $(k - i + \lambda_i)^{th}$ column, the i^{th} diagonal on top of the $(i + 1)^{st}$ diagonal.

Example 2.3.1. For $\lambda = (2, 2, 1)$ with $k = 3$, the permutation associated to λ in $G(5; 3)$ is $u = 24513$ and the unique heap of u is below.



Example 2.3.2. Suppose $u = 24513$ in $G(5; 3)$ and $v = 312$ in $G(3; 1)$. Then $u \times v = 24513867$ in Σ_8.



By rotating each connected component of the unique heap of $u \times v$ by 135 degrees counterclockwise and letting the boxes fall, the result is a pair of fillings U and V of $\overline{\lambda}$ and $\overline{\mu}$, respectively. The filling, U, of $\overline{\lambda}$ has a k in the lower left corner, the filling, V, of $\overline{\mu}$ has an $m + k$ in the lower left corner, every row decreases strictly by 1 across from left to right, and each column increases by 1 from bottom to top. Call the resulting tableaux $(U, V) = tab(u \times v)$. Note, this action is only well defined on 321-avoiding permutations.

Example 2.3.3. Below is the component-wise rotated heap of $u \times v$, $tab(u \times v)$.

4	3			7
3	2	1		6

Recall the operator $\mathfrak{W}$ on tableaux defined in Section 1.5. It was used to describe the Remmel-Whitney rule. We will show its relation to $P(u \times v)$.

Lemma 2.3.4. *For permutations u and v in $G(n)$,*

$$P(u \times v) = \mathfrak{W}(tab(u \times v)).$$

Proof. Consider a column of $tab(u \times v) = (U, V)$. If $k < \ell$ are in the same column of U or V, then k is below ℓ, because the columns are strictly increasing from bottom to top. This also implies that ℓ is below k in the heap of $u \times v$. Therefore, for every tableaux $T \in P(u \times v)$, ℓ must be northwest of k in T. Likewise, if $k < \ell$ are in the same row of U or V, then k is to the right of ℓ, because the rows are strictly decreasing from left to right. This implies that ℓ is above k in the heap of $u \times v$. Therefore, ℓ appears southeast of k in T, for every $T \in P(u \times v)$. From these considerations and the definition of $\mathfrak{W}$, we have that $T \in \mathfrak{W}(tab(u \times v))$. These are the only restrictions on the tableaux in $P(u \times v)$ and are exactly the same restrictions as on $\mathfrak{W}(tab(u \times v))$. Thus, every tableau in $\mathfrak{W}(tab(u \times v))$ is also in $P(u \times v)$. Therefore,

$$P(u \times v) = \mathfrak{W}(tab(u \times v)).$$

□

We now give the main result in this section, as well as the original inspiration behind the slinky rule. The Remmel-Whitney rule then follows as a corollary, providing an alternative proof.

Lemma 2.3.5. *Given partitions λ and μ with associated Grassmannian permutations u and v, there exists a shape preserving bijection from $\mathfrak{W}(tab(u \times v))$ to $\mathfrak{W}(W(\overline{\lambda}, \overline{\mu}))$.*

Proof. We have that $tab(u \times v) = (U, V)$ and $W(\overline{\lambda}, \overline{\mu}) = (Y, Z)$ are both pairs of tableaux such that U and Y have shape $\overline{\lambda}$, and V and Z have shape $\overline{\mu}$. Furthermore, the rows of each tableaux are strictly decreasing from left to right and the columns stictly increase from top to bottom.

Let F, G, H be tableaux such that the rows of each tableaux are strictly decreasing from left to right and the columns stictly increase from top to bottom. Furthermore, suppose $sh(F) = sh(G)$, every entry of H is strictly greater than each entry of F and each entry of G, and suppose F and G are exactly the same except on row i of F and G. We wish to show that there is a shape preserving bijection from $\mathfrak{W}(F, H)$ to $\mathfrak{W}(G, H)$.

Suppose $T \in \mathfrak{W}(F, H)$. Given an entry f_{ij} in position (i, j) in F, replace f_{ij} in T with g_{ij} from G. Call this new tableau T_G. The only way that T_G is not in $\mathfrak{W}(G, H)$ is if T_G fails to be a semistandard tableau. Suppose there is a row in T_G that is not strictly increasing. Then the row has the form

$$\begin{array}{|c|c|c|c|c|}\hline \alpha & g_1 & \cdot & g_r & \beta \\ \hline \end{array},$$

where $\cdot$ indicates more entries from row i of G. The entries $g_1 \ldots g_r$ are in strictly increasing order because they have the same relative order as row i of F. Suppose $\alpha > g_1$. Then α is in the same column as some entry k in row i of G. Otherwise, α is below row i of G, therefore, $\alpha < g_1$ (every path that goes down and right in G is decreasing). If $k > \alpha$, then k lies west of g_1 in G. Therefore, k lies northwest of α and southeast of g_1, so α and g_1 cannot be in the same row. If $\alpha > g_1 > k$, then α is northwest of k and g_1 is southeast of k, so again, k must separate g_1 and α and this is not the case. If $\alpha > k > g_1$, then $\alpha > \ell > f_1$, where ℓ is the entry in F in the same position as k and f_1 is the entry in F in the same position as g_1. This would contradict that T is a semistandard Young tableau, so this cannot happen. If $g_1 = k$, that is, if α is in the same column as g_1, then α was also bigger than f_1(the entry in F in the same position as g_1), in which case T wouldn't be semistandard. The proof proceeds in the exact same manner for the case when $\beta < g_r$, and for the case when the columns are not strictly increasing. Therefore, $T_G \in$ and by symmetry, there exists a shape preserving bijection from $\mathfrak{W}(F, H)$ to $\mathfrak{W}(G, H)$. Likewise, there exists a shape preserving bijection from $\mathfrak{W}(H, F)$ to $\mathfrak{W}(H, F)$ when H has content strictly less than the content of both F and G.

There exists a sequence of tableaux which change content one row at a time from U to Y and V to Z such that each tableaux in the sequence has rows which are strictly decreasing from left to right and columns which stictly increase from top to bottom. Therefore, there is a shape preserving bijection from $\mathfrak{W}(tab(u \times v))$ to $\mathfrak{W}(W(\overline{\lambda}, \overline{\mu}))$. □

Corollary 2.3.6. *Given partitions λ and μ, we have that*

$$s_\lambda s_\mu = \sum_{U \in \mathfrak{W}(W(\lambda,\mu))} s_{sh(U)}.$$

Proof. Given partitions λ, μ, ν, we have that $c^{\nu}_{\lambda,\mu} = c^{\overline{\nu}}_{\overline{\lambda},\overline{\mu}}$, found in Manivel's book on Schubert polynomials [48]. Furthermore, if λ and μ correspond to permutations u and v, then

$$\begin{aligned} s_{\overline{\lambda}} s_{\overline{\mu}} &= \sum_{\overline{T} \in P((u \times v)^{-1})} s_{sh(T)} \\ &= \sum_{T \in P(u \times v)} s_{sh(T)} \\ &= \sum_{U \in \mathfrak{W}(W(\overline{\lambda},\overline{\mu}))} s_{sh(U)}. \end{aligned}$$

By symmetry of conjugation, the corollary is proved. □

Chapter 3
POPTOTIC CLASSES

Miller and Knutson introduced a recursive algorithm on $\mathcal{RC}$-graphs which acts much like the divided difference operator on monomials in Schubert polynomials called mitosis [28]. In this chapter, we describe $\mathcal{RC}$-graphs and the mitosis algorithm, as well as describe the class of algorithms with nice properties, called poptotic classes. The main theorem in this chapter is Theorem 3.3.8, which characterizes poptotic classes. Section 3.4 introduces an operator, split, similar to mitosis, which acts like a left divided difference operator on Schubert polynomials.

3.1 Introduction to $\mathcal{RC}$-graphs

$\mathcal{RC}$-graphs are a set of combinatorial objects associated with a permutation $w \in S_n$. In a sense, they record the planar history of a permutation and so are a more visual object to consider than **a**-compatible sequences. Most of the introductory material presented in this section appears in [1].

Suppose $\mathbf{a} = (a_1, \ldots, a_p) \in \mathit{Red}(w)$ is a reduced word for w. Recall from Section 1.3 that $\mathbf{i} = (i_1, \ldots, i_p)$ is an **a**-*compatible sequence* if

- $1 \leq i_1 \leq i_2 \ldots \leq i_p$,
- $i_j = i_{j+1}$ implies $a_j > a_{j+1}$, and
- $i_j \leq a_j$ for every j.

Let $\mathcal{C}(\mathbf{a})$ denote the set of all **a**-compatible sequences.

Definition 3.1.1. Given $\mathbf{a} \in \mathit{Red}(w)$ and $\mathbf{i} \in \mathcal{C}(\mathbf{a})$, the *rc-graph* of $(\mathbf{a}, \mathbf{i})$ is

$$D(\mathbf{a}, \mathbf{i}) := \{(i_k, a_k - i_k + 1) \,|\, 1 \leq k \leq p\}.$$

Let $\mathcal{RC}(w) := \{D(\mathbf{a}, \mathbf{i}) \,|\, \mathbf{a} \in Red(w), \mathbf{i} \in \mathcal{C}(\mathbf{a})\}$.

Example 3.1.2. Let $\mathbf{a} = (3, 1, 4, 5, 4)$ and $\boldsymbol{i} = (1, 1, 2, 3, 4)$. Then $D(\mathbf{a}, \boldsymbol{i}) \in \mathcal{RC}(214653)$ is

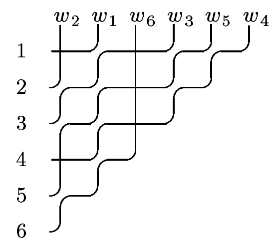
$$\begin{array}{cccccc} & 1 & 2 & 3 & 4 & 5 \\ 1 & + & \cdot & + & \cdot & \cdot \\ 2 & \cdot & \cdot & + & \cdot & \\ 3 & \cdot & \cdot & + & & \\ 4 & + & \cdot & & & \\ 5 & \cdot & & & & \end{array} \tag{3.1}$$

where a $+$ represents an occupied position and a $\cdot$ represents an unoccupied position in the graph.

Given a subset D of $\{(k, b) \in \mathbb{P} \times \mathbb{P} \,|\, k + b \leq n\}$, form a string diagram as follows: Draw n strings going up and to the right such that the i^{th} string begins at $(i, 1)$. When the string reaches (k, b), it crosses the other string arriving at (k, b) if $(k, b) \in D$ or avoids it otherwise. Such a string diagram is called a *pipe dream*. If the i^{th} string ends at position $(1, w_i)$, then $w(D) = w_1 w_2 \ldots w_n$. If no two strings in the line diagram for D cross more than once, then $D \in \mathcal{RC}(w(D))$.

For any subset D of $\{(k, b) \in \mathbb{P} \times \mathbb{P} \,|\, k+b \leq n\}$, an expression for $w(D)$ is easily recovered. Define a total reading order on the elements of D, $(k, a) < (\ell, b)$ if $k < \ell$ or $k = \ell$ and $a > b$. Then enumerate the elements of D with respect to this order $(k_1, b_1) < (k_2, b_2) < \ldots < (k_r, b_r)$. Let $a_j = b_j + k_j - 1$ and $i_j = k_j$. Then $\mathbf{a} = (a_1, a_2, \ldots, a_r)$ is an expression for $w(D)$ and $\mathbf{i} = (i_1, i_2, \ldots, i_r)$ is an $\mathbf{a}$-compatible sequence.

Example 3.1.3. Let $\mathbf{a} = (3, 1, 4, 5, 4)$ and $\boldsymbol{i} = (1, 1, 2, 3, 4)$. Below is the rc-graph $D(\mathbf{a}, \boldsymbol{i}) \in \mathcal{RC}(214653)$ with the strings filled in.



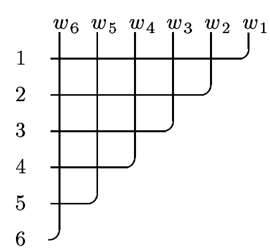
(3.2)

To recover the reduced expression from the diagram, read column numbers in the rows right to left, top to bottom, adding the row number minus 1 to the column number. Following this procedure, the reduced expression corresponding to this rc-graph is $\mathbf{a} = (3+0, 1+0, 3+1, 3+2, 1+3) = (3, 1, 4, 5, 4)$, as expected.

Note that $\ell(w_0) = \binom{n}{2} = |\{(k, b) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} : k + b \leq n\}|$. Therefore, there is a unique rc-graph for w_0, namely $D_0 = \{(k, b) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} : k + b \leq n\}$.

Example 3.1.4. Below is the rc-graph of $w_0 = 654321$ in S_6.

$D_0 =$



(3.3)

Given an rc-graph D, define the monomial

$$x^D = \prod_{(i,j) \in D} x_i.$$

Following Theorem 1.3.2, Fomin-Kirillov and Bergeron-Billey observed the corollary below. The bijection from $\mathbf{a}$-compatible sequences to rc-graphs is denoted X in Figure 1.

Corollary 3.1.5. *[1, 15] For a permutation $w \in S_n$,*

$$\mathfrak{S}_w(x) = \sum_{D \in \mathcal{RC}(w)} x^D.$$

Lemma 3.1.6. *[1] Given an rc-graph D, the transpose of D is also an rc-graph, $D^t \in \mathcal{RC}(w^{-1})$. Therefore, the map $\rho : \mathcal{RC}(w) \to \mathcal{RC}(w^{-1})$ such that $\rho(D) = D^t$ is an involution.*

Definition 3.1.7. Given $w \in S_n$, define

$$D_{bot}(w) := \{(i,c) \,|\, 1 \leq c \leq m_i\},$$

where $m_i = \#\{j \,|\, j > i \text{ and } w_j < w_i\}$. Define

$$D_{top}(w) := \{(c,j) \,|\, 1 \leq c \leq n_i\},$$

where $n_i = \#\{i \,|\, i < j \text{ and } w_i > w_j\}$.

It can be shown that $D_{bot}(w)$ and $D_{top}(w)$ correspond to the lexicographically largest and smallest compatible sequences for w, respectively, and are both in $\mathcal{RC}(w)$.

Suppose $D \in \mathcal{RC}(w)$ and R is a $2 \times (m+1)$ rectangle in $\mathbb{P} \times \mathbb{P}$ for $m \geq 1$. Also, suppose $R \cap D$ contains all of R except the southwest, northwest, and southeast corners of R. That is, $R \cap D$ looks like:

$$\begin{array}{cccccc} & j-m & & & & j \\ i & \cdot & + & + & + & + \\ i+1 & \cdot & + & + & + & \cdot \end{array} \tag{3.4}$$

Define the *chute*, C_{ij}, of D to be the set $C_{ij}(D) := \{(i+1, j-m)\} \cup D - \{(i,j)\}$, where the $2 \times (m+1)$ rectangle with northeast corner (i,j) is as above.

$$\begin{array}{cccccc} & j-m & & & & j \\ i & \cdot & + & + & + & + \\ i+1 & \cdot & + & + & + & \cdot \end{array} \mapsto \begin{array}{cccccc} & j-m & & & & j \\ i & \cdot & + & + & + & \cdot \\ i+1 & + & + & + & + & \cdot \end{array}$$

Such a rectangle R is refered to as a *chutable* rectangle. Also, define $\mathcal{CH}(D)$ to be the set of all $C_{i_0 j_0} \cdots C_{i_k j_k}(D)$ such that $(i_0, j_0), \ldots, (i_k, j_k)$ is a sequence of chutable rectangles.

Example 3.1.8. Let $D \in \mathcal{RC}(216534)$.

$$D = \begin{array}{cccccc} & 1 & 2 & 3 & 4 & 5 \\ 1 & + & \cdot & \cdot & \cdot & \cdot \\ 2 & \cdot & + & + & + & \\ 3 & \cdot & + & + & & \\ 4 & \cdot & + & & & \\ 5 & \cdot & & & & \end{array} \qquad C_{2,4}(D) = \begin{array}{cccccc} & 1 & 2 & 3 & 4 & 5 \\ 1 & + & \cdot & \cdot & \cdot & \cdot \\ 2 & \cdot & + & + & \cdot & \\ 3 & + & + & + & & \\ 4 & \cdot & + & & & \\ 5 & \cdot & & & & \end{array} \tag{3.5}$$

Lemma 3.1.9. *[1] Suppose $D \in \mathcal{RC}(w)$ and R is a chutable rectangle in D with northeast corner (i, j). Then $C_{ij}(D) \in \mathcal{RC}(w)$.*

Similarly, suppose $D \in \mathcal{RC}(w)$ and S is an $(m+1) \times 2$ rectangle in $\mathbb{P} \times \mathbb{P}$ for $m \geq 1$. Also, suppose $S \cap D$ contains all of S except the northeast, northwest, and southeast corners of S. That is, $S \cap D$ looks like:

$$\begin{array}{ccc} & j & j{+}1 \\ i{-}m & \cdot & \cdot \\ & + & + \\ & + & + \\ & + & + \\ & + & + \\ i & + & \cdot \end{array} \tag{3.6}$$

Define the *ladder*, $\mathcal{L}_{ij}$ of D to be the set $\mathcal{L}_{ij}(D) = \{(i-m, j+1)\} \cup D - \{(i, j)\}$, where the $(m+1) \times 2$ rectangle with southwest corner (i, j) is as above. Such a rectangle is refered to as a *climbable* rectangle. Also, define $\mathcal{L}(D)$ to be the set of all rc-graphs, $L_{i_0 j_0} \cdots L_{i_k j_k}(D)$ such that $(i_0, j_0), \ldots, (i_k, j_k)$ is a sequence of climbable rectangles.

Example 3.1.10. Let $D \in \mathcal{RC}(216534)$.

$$D = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ 1 & + & \cdot & \cdot & \cdot & \cdot \\ 2 & \cdot & + & + & + & \\ 3 & \cdot & + & + & & \\ 4 & \cdot & + & & & \\ 5 & \cdot & & & & \end{array} \qquad L_{4,2}(D) = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ 1 & + & \cdot & + & \cdot & \cdot \\ 2 & \cdot & + & + & + & \\ 3 & \cdot & + & + & & \\ 4 & \cdot & \cdot & & & \\ 5 & \cdot & & & & \end{array} \tag{3.7}$$

Lemma 3.1.11. *[1] Suppose $D \in \mathcal{RC}(w)$ and S is a climbable rectangle in D with southwest corner (i, j). Then $L_{ij}(D) \in \mathcal{RC}(w)$.*

Theorem 3.1.12. *[1] Suppose $w \in S_n$. Then the following all hold:*

1. *For every $D \in \mathcal{RC}(w)$, $\rho(C_{ij}(D)) = L_{ji}(\rho(D))$, where $\rho(D) = D^t$.*
2. *$\mathcal{RC}(w) = \mathcal{CH}(D_{top}(w))$.*
3. *$\mathcal{RC}(w) = \mathcal{L}(D_{bot}(w))$.*

Definition 3.1.13. A chute move C_{ij} is said to be *simple* if (i, j) is the northwest corner of a 2×2 chutable rectangle. Likewise, a ladder move is said to be *simple* if (i, j) is the southeast corner of a 2×2 climbable rectangle.

Observation 3.1.14. If w avoids the pattern 321, then w admits only simple chute and ladder moves.

3.2 Mitosis

Given a set D contained in $\{(k, b) \in \mathbb{P} \times \mathbb{P} \,|\, k + b \leq n\}$ and $1 \leq i < n$, define

$$start_i(D) := min\{j \in \mathbb{P} \,|\, (i, j) \notin D\}$$

and

$$\mathcal{J}_i(D) := \{1 \leq p < start_i(D) \,|\, (i + 1, p) \notin D\}.$$

Algorithm 3.2.1. [28, 50] (Mitosis Algorithm) For $p \in \mathcal{J}_i(D)$, form the *offspring* D^i_p as follows:

1. Delete (i,p) from D.

2. For every $j \in \mathcal{J}_i(D)$ such that $j < p$, move the cross at (i,j) down to $(i+1,j)$. That is, delete (i,j) from D and add $(i+1,j)$. The result is D^i_p.

Definition 3.2.2. For an rc-graph D,

$$mitosis_i(D) := \{D^i_p : p \in \mathcal{J}_i(D)\}.$$

For a set $\mathcal{P}$ of rc-graphs,

$$mitosis_i(\mathcal{P}) := \bigcup_{D \in \mathcal{P}} mitosis_i(D).$$

The offspring D^i_p can be seen as the subset obtained by deleting $(i, start_{i+1}(D))$ from D and doing consecutive chute moves on rows i and $i+1$ until ending with C_{ip}. One can define a similar rule using ladder moves. Section 3.4 explores this kind of rule.

The following theorem is the motivation behind using mitosis. The theorem shows that mitosis behaves similar to a divided difference operator on Schubert polynomials.

Theorem 3.2.3. *[28, 50] For $w \in S_n$,*

$$mitosis_i(\mathcal{RC}(w)) = \begin{cases} \mathcal{RC}(ws_i) & \text{if } \ell(w) > \ell(ws_i) \\ \emptyset & \text{if } \ell(w) < \ell(ws_i) \end{cases}. \tag{3.8}$$

Therefore, for $\mathbf{a} = (a_1, a_2, \ldots, a_p) \in Red(w^{-1}w_0)$,

$$\mathcal{RC}(w) = mitosis_{a_1} mitosis_{a_2} \cdots mitosis_{a_p}(D_0) = mitosis_{\mathbf{a}}(D_0).$$

Example 3.2.4. Suppose $w = 52134$, $w^{-1}w_0 = 15423$ and $\mathbf{a} = (4,3,4,2,3) \in Red(15423)$. The mitosis tree in Figure 3.1 has root D_0, and the children of D on level j (from the bottom) are precisely $mitosis_{a_j}(D)$. Thus, $\mathcal{RC}(52134) = \left\{\begin{matrix} + & + & + & + \\ + & \cdot & \cdot & \\ \cdot & \cdot & & \\ \cdot & & & \end{matrix}\right\}$ and $\mathfrak{S}_{52134} = x_1^4 x_2$.

3.3 Poptotic Classes

Example 3.2.4 illustrates an important point: some $\mathbf{a} \in Red(w^{-1}w_0)$ may give rise to barren nodes. This leads to a good question, which mitosis trees have barren nodes?

Definition 3.3.1. An rc-graph D is *i-barren* if $mitosis_i(D) = \emptyset$, i.e. D is childless in the mitosis tree for i.

The i in i-barren is omitted if it is understood from the context. Observe that D is i-barren if and only if $start_i(D) \leq start_{i+1}(D)$.

Definition 3.3.2. A reduced word $\mathbf{a} \in Red(w^{-1}w_0)$ is *poptotic* if the mitosis tree for $\mathbf{a}$ has no barren nodes. Otherwise, $\mathbf{a}$ is *apoptotic.*

Lemma 3.3.3. *If* $\mathbf{a}, \mathbf{b} \in Red(u)$ *and* $heap(\mathbf{a}) = heap(\mathbf{b})$, *then* $\mathbf{a}$ *is poptotic if and only if* $\mathbf{b}$ *is poptotic.*

Proof. Observe $mitosis_i(D)$ affects only rows i and $i+1$ of D and leaves all other rows untouched. Suppose $|i-j| > 1$. Then $mitosis_i$ does not affect the j^{th} and $(j+1)^{st}$ rows of D, or $mitosis_j(D)$ (and the same conversely). This implies $start_j(D) = start_j(E)$ and $start_{j+1}(D) = start_{j+1}(E)$ for every $E \in mitosis_i(D)$. By symmetry, $start_i(D) = start_i(F)$ and $start_{i+1}(D) = start_{i+1}(F)$ for every $F \in mitosis_j(D)$.

The i-barrenness of D depends only on the relationship of $start_i(D)$ and $start_{i+1}(D)$ as observed above. Thus, if $mitosis_i mitosis_j(D)$ has a barren node, then either $start_j(D) \leq start_{j+1}(D)$ or $start_i(F) \leq start_{i+1}(F)$ for some $F \in mitosis_j(D)$. This implies either $start_j(E) \leq start_{j+1}(E)$ for some $E \in mitosis_i(D)$ or $start_i(D) \leq start_{i+1}(D)$ and so $mitosis_j mitosis_i(D)$ has a barren node. Therefore, we have that $mitosis_i mitosis_j(D)$ has a barren node if and only if $mitosis_j mitosis_i(D)$ has a barren node.

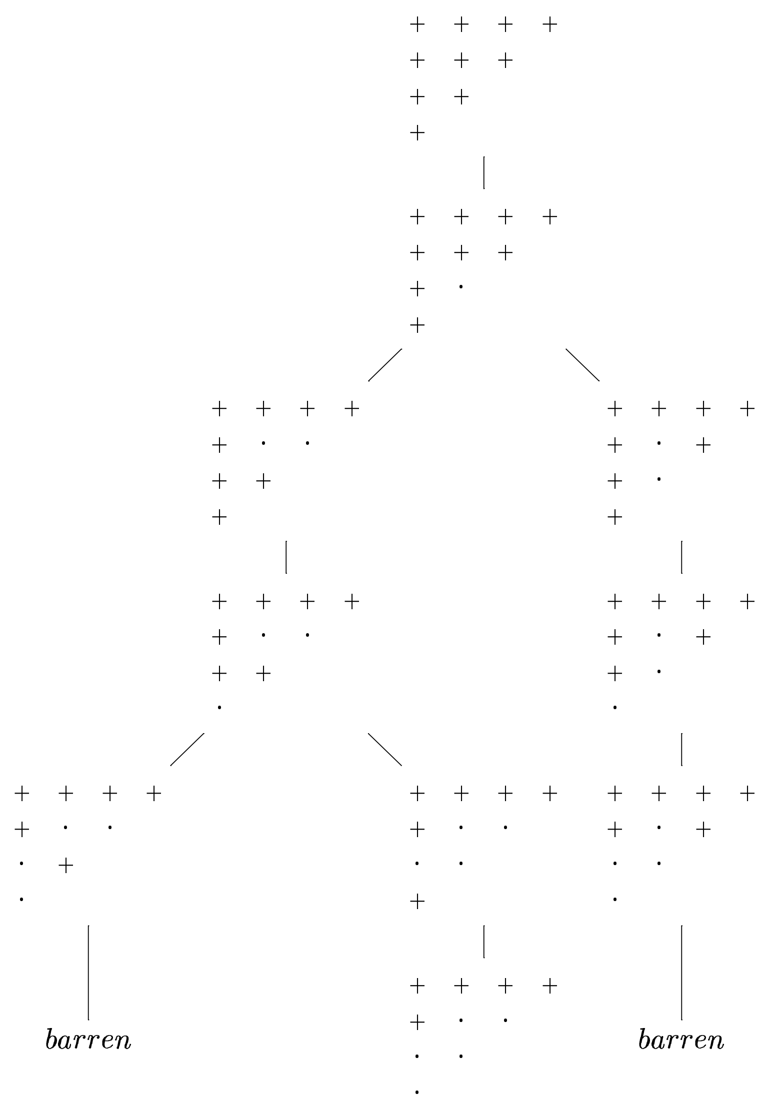


Figure 3.1: The mitosis tree for $\mathbf{a} = (4, 3, 4, 2, 3)$

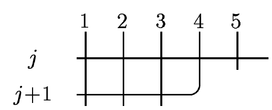
Recall from Section 1.1, for $\mathbf{a}, \mathbf{b} \in Red(u)$ such that $heap(\mathbf{a}) = heap(\mathbf{b})$, $\mathbf{a}$ and $\mathbf{b}$ differ only by a sequence of $(i,j) \leftrightarrow (j,i)$ switches, where $|i-j| > 2$. Thus, $mitosis_{\mathbf{a}}(D)$ has a barren node if and only if $mitosis_{\mathbf{b}}(D)$ has a barren node and so the lemma is proved. ☐

Lemma 3.3.4. *Suppose $w \in S_n$.*

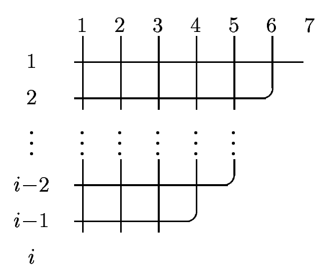
1. *If $w_j < w_{j+1}$, then $start_j(D) \leq start_{j+1}(D)$ for every $D \in \mathcal{RC}(w)$.*

2. *If $w_1 > w_2 > \ldots > w_i$, then $start_1(D) > start_2(D) > \ldots > start_i(D)$ for every $D \in \mathcal{RC}(w)$, and $start_h(D) = w_h$ for $1 \leq h \leq i$.*

Proof. To prove the first statement, observe that for any rc-graph D, if $start_j(D) > start_{j+1}(D)$, then string $j+1$ crosses string j at position $(j, start_{j+1}(D))$. This implies $w_j > w_{j+1}$. Thus, if $w_j < w_{j+1}$, then $start_j(D) \leq start_{j+1}(D)$ for every $D \in \mathcal{RC}(w)$.



To prove the second statement, suppose $D \in \mathcal{RC}(w)$. Observe, the claim is trivially true for $i = 1$. Proceed by induction on i. Suppose $w_1 > w_2 > \ldots > w_i$, then $start_1(D) > start_2(D) > \ldots > start_{i-1}(D)$ and $start_h(D) = w_h$ for $1 \leq h \leq i-1$ by the inductive hypothesis. Since $w_{i-1} > w_i$, strings $i-1$ and i must cross in D. By considering the string diagram for D, the only way for string i to cross string $i-1$ is if they do so before $start_{i-1}(D)$. It must be that $start_i(D) < start_{i-1}(D)$. Furthermore, string i continues straight up in the string diagram in column $start_i(D)$, so, $start_i(D) = w_i$.



☐

Miller remarks without proof in [50] that, if $\mathbf{a}$ is smallest in lexicographic order in $Red(w^{-1}w_0)$, then $\mathbf{a}$ is poptotic. We include a proof for completeness.

Lemma 3.3.5. *Suppose* $\mathbf{a} \in Red(w^{-1}w_0)$ *such that* $\mathbf{a}$ *is smallest in lexicographic order in* $Red(w^{-1}w_0)$*. If* $heap(\mathbf{b}) = heap(\mathbf{a})$*, then* $\mathbf{b}$ *is poptotic.*

Proof. To prove the lemma, first we must prove that $\mathbf{a}$ is poptotic. Lemma 3.3.3 proves the rest of the statement.

Let w be a permutation whose first ascent is i. Then ws_i has first ascent either at $i-1$ or strictly after i and ws_i covers w in right weak order. Suppose $E \in \mathcal{RC}(ws_i)$. We wish to show $start_i(E) > start_{i+1}(E)$, thereby proving E is not i-barren. If ws_i has first ascent strictly after i, then by Lemma 3.3.4,

$$start_1(E) > start_2(E) > \ldots > start_i(E) > start_{i+1}(E).$$

On the other hand, suppose ws_i has first ascent at $i-1$, then by Lemma 3.3.4,

$$start_1(E) > start_2(E) > \ldots > start_{i-1}(E).$$

Let $v = ws_i$. Then,

$$start_{i-1}(E) = v_{i-1} = w_{i-1} > w_i = v_{i+1}$$

and

$$v_i = w_{i+1} > w_i = v_{i+1}\,.$$

String $i+1$ must cross string $i-1$ in a column in E to the left of $start_{i-1}(E)$ because $start_{i-1} = v_{i-1}$. In order to do this, it must be that $start_{i-1}(E) > start_{i+1}(E)$. By assumption, $v_{i-1} < v_i$, so again by Lemma 3.3.4, $start_i(E) \geq start_{i-1}(E)$. Thus, $start_i(E) \geq start_{i-1}(E) > start_{i+1}(E)$. Therefore, E is not i-barren and so, $mitosis_i(\mathcal{RC}(ws_i))$ does not have any barren nodes.

One finds the smallest lexicographic reduced word for $w^{-1}w_0$ by finding the saturated path in right weak order $w \rightarrow ws_{a_1} \rightarrow ws_{a_1}s_{a_2} \rightarrow \ldots \rightarrow w_0$ such that a_j is the first ascent of $ws_{a_1} \cdots s_{a_{j-1}}$. Let $\mathbf{a} = (a_1, a_2, \ldots, a_{\binom{n}{2}-\ell(w)})$. Therefore, if $\mathbf{a}$ is smallest in lexicographic order in $Red(w^{-1}w_0)$, then **a** is poptotic.

□

Lemma 3.3.5 gives a sufficient criteria for a reduced word to be poptotic. Is it also necessary? This is equivalent to asking which reduced words **a** have mitosis trees which only have rc-graphs in which $start_i(D) > start_{i+1}(D)$? Before answering this question, we need more information about the other heaps.

Lemma 3.3.6. *Suppose $H \in \mathcal{H}(w)$ and $H \neq heap(\mathbf{a})$, where $\mathbf{a}$ is the lexicographically smallest reduced word for w. Then there exists a linear extension $\mathbf{b} = (b_1, b_2, \ldots, b_p)$ of H and $k \leq p-2$ such that $b_k = b_{k+2} = j+1$ and $b_{k+1} = j$ for some j.*

Proof. For $w \in S_n$, suppose $n = 2$. Then the statement is true, because there is only one heap for each permutation. Suppose the statement is true for $w \in S_n$, for all $n < m$, and let $u \in S_m$. If $u_m = m$, then u can be embedded in S_{m-1} and the claim is true by induction, so supppose $u_t = m$ for $t < m$. Then there exists a linear extension σ of H such that $\sigma = \sigma_1 \cdot (m-1, m-2, \ldots, t) \cdot \sigma_2$ and σ_1 can be embedded in S_{m-1}, where $\cdot$ denotes concatonation of reduced words. Also, assume σ_1 has maximal length with regard to these restrictions.

If σ_2 is not the empty word, then let j be the first element of σ_2. If $j > t$, then move j past t by using commuting moves until j is as far left as possible. It will have been stopped by a $j-1$. Thus, there is σ' in the same commutativity class as σ such that $\sigma' = \sigma_1 \cdot (m-1, m-2, \ldots, j, j-1, j, j-2, \ldots, t) \cdot \sigma_2'$ and the lemma is proved in this case.

Suppose $j < t$. If $j = t-1$, then it $u_t \neq m$ because m moves past the t position. So, we can assume that $j < t-1$. Move j past $m-1, m-2, \ldots, t$ to form $\tau = \sigma_1 \cdot j \cdot (m - 1, m-2, \ldots, t) \cdot \tau_2$ where τ_2 consists of all of σ_2 except the first element, j. Then $\sigma_1 \cdot j$ can be embedded into S_{m-1}, which contradicts the choice of σ_1.

Suppose σ_2 is not the empty word and $heap(\sigma_1)$ is the heap of the lexicographically smallest word for v where $\sigma_1 \in Red(v)$. Then $heap(\sigma) = heap(\mathbf{a})$ where $\mathbf{a}$ is the lexicographically smallest reduced word for w. Thus, if σ_2 is the empty word, then $heap(\sigma_1)$ is not the heap of the lexicographically smallest word for v where $\sigma_1 \in Red(v)$. The lemma holds for $heap(\sigma_1)$ by induction, and hence, for H. □

Define the relation $\preceq$ on $\mathcal{H}(w)$ by $H \prec G$ if there exist linear extensions $\mathbf{h}$ and $\mathbf{g}$ of H, and G such that $\mathbf{g}$ can be obtained from $\mathbf{h}$ by doing one $(j, j+1, j) \to (j+1, j, j+1)$ switch. Let $\preceq$ be the transitive closure of this relation.

Corollary 3.3.7. *Suppose $w \in S_n$. Let $H = heap(\mathbf{a}) \in \mathcal{H}(w)$, where $\mathbf{a}$ is the lexicographically smallest reduced word for w. Then $\preceq$ defines a ranked partial order on $\mathcal{H}(w)$. The rank of G is given by $\sum_{i \in G} i - \sum_{j \in H} j$. Consequently, H is the unique heap with the lexicographically smallest content for w.*

Proof. From Lemma 3.3.6, every heap $G \in \mathcal{H}(w)$ which is not H can be obtained from another heap F by doing one increasing braid move such that $\sum_{i \in G} i - \sum_{j \in F} j = 1$. Furthermore, if $H \preceq G$, then either $H = G$ or G can be obtained from H by doing a series of $(j, j+1, j) \mapsto (j+1, j, j+1)$ moves, implying the content of G is strictly greater than the content of H. So, $\preceq$ is a valid partial order. The only heap which can not be obtained from an increasing braid move is H. This implies H has the lexicographically smallest content in $\mathcal{H}(w)$. □

Theorem 3.3.8. *Suppose $\mathbf{a} \in Red(w^{-1}w_0)$. Then $\mathbf{a}$ is poptotic if and only if $heap(\mathbf{a})$ has the lexicographically smallest content in $\mathcal{H}(w^{-1}w_0)$.*

Proof. From Lemma 3.3.5, one has that if $heap(\mathbf{a})$ has the lexicographically smallest content, then $\mathbf{a}$ is poptotic.

To prove the converse, let $H = heap(\mathbf{a})$, and suppose H does not have the lexicographically smallest content in $\mathcal{H}(w)$. By Lemma 3.3.6, there exists a linear extension $\mathbf{b}$ of H and $k \leq \ell(w) - 2$ such that $b_k = j + 1 = b_{k+2}$ and $b_{k+1} = j$ for some $j < n$. Let $\mathcal{P} = mitosis_{i_{k+3}} \cdots mitosis_{i_{\ell(w)}}(D_0)$, and consider

$$mitosis_{i_k} mitosis_{i_{k+1}} mitosis_{i_{k+2}}(\mathcal{P}) = mitosis_{j+1} mitosis_j mitosis_{j+1}(\mathcal{P}).$$

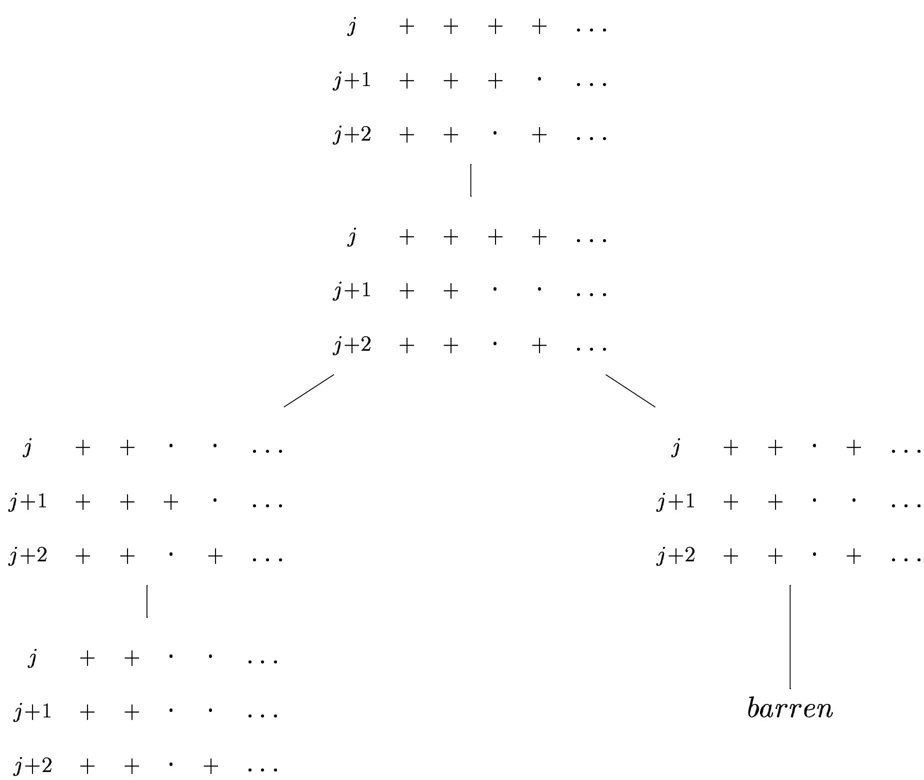
For $D \in mitosis_{j+1}(\mathcal{P})$, we have that $start_{j+2}(D) \geq start_{j+1}(D)$. Similarly, if $mitosis_j(D) \neq \emptyset$, then $start_j(D) > start_{j+1}(D)$. If $p = start_{j+1}(D)$, then $p \in \mathcal{J}_j(D)$ and $p = min \mathcal{J}_i(D)$.

Let $D' = D^j_p$. Then

$$start_{j+1}(D') = p = start_{j+1}(D) \leq start_{j+2}(D) = start_{j+2}(D')$$

which implies $mitosis_{j+1}(D') = \emptyset$. Thus, **b** is apoptotic. From Lemma 3.3.3, every linear extension of $heap(\mathbf{b})$ is also apoptotic. Therefore, **a** is apoptotic. $\square$

Pictorally, the following is an example of part of a mitosis tree for $(j+1, j, j+1)$ for some partial rc-graph D.



3.4 *A Left Divided Difference Operator on RC-Graphs*

The divided difference operator, ∂_i, defined in Section 1.3, has been studied and generalized extensively in order to find more information about Schubert polynomials [3, 12, 17, 15, 28, 34, 47]. Recall,

$$\partial_i(\mathfrak{S}_w) = \begin{cases} \mathfrak{S}_{ws_i} & \text{if } \ell(w) > \ell(ws_i) \\ 0 & \text{if } \ell(w) < \ell(ws_i) \end{cases}.$$

That is, ∂_i acts on a Schubert polynomial indexed by w by multiplying w on the right by s_i. In this section, we define a new operator, *split*, similar to mitosis on rc-graphs such that $split_i$ acts on a Schubert polynomial indexed by w by multiplying w on the left by s_i. A different left divided difference operator was defined on the equivariant cohomology of flag varieties recently by Tymoczko[66]. The two operators act in the same way on Schubert polynomials. Furthermore, we describe in Section 4.4 how the split operator acts on another Schubert object defined in Chapter 4.

Given a set D contained in $\{(k, b) \in \mathbb{P} \times \mathbb{P} \,|\, k + b \leq n\}$ and $1 \leq j < n$, define

$$top_j(D) := min\{i \,|\, (i, j) \notin D\},$$

and

$$\mathcal{T}_j(D) := \{1 \leq i < top_j(D) \,|\, (i, j + 1) \notin D\}.$$

Algorithm 3.4.1. (Split Algorithm) For $q \in \mathcal{T}_j(D)$, form the *offspring* ${}^j_qD$ as follows:

1. Delete (q, j) from D.

2. For every $i < q$ such that $i \in \mathcal{T}_j(D)$, move the cross at (i, j) right to $(i, j + 1)$. That is, delete (i, j) from D and add $(i, j + 1)$. The result is ${}^j_qD$.

Definition 3.4.2. For an rc-graph D,

$$split_j(D) := \{{}^j_qD : q \in \mathcal{T}_j(D)\}.$$

For a set $\mathcal{P}$ of rc-graphs,

$$split_j(\mathcal{P}) := \bigcup_{D \in \mathcal{P}} split_j(D).$$

The offspring ${}^{j}_{q}D$ can be seen as the subset obtained by deleting $(top_{j+1}(D), j)$ from D and doing consecutive ladder moves on columns j and $j+1$ until ending with L_{qj}. In fact, split is just the action of conjugating mitosis with the transpose operator.

Lemma 3.4.3. *For an rc-graph D,*

$$split_j(D) = \rho(mitosis_j(\rho(D))).$$

Proof. The lemma follows directly from the definitions. □

Theorem 3.4.4. *For $w \in S_n$,*

$$split_j(\mathcal{RC}(w)) = \begin{cases} \mathcal{RC}(s_j w) & \textit{if } \ell(w) > \ell(s_j w) \\ \emptyset & \textit{if } \ell(w) < \ell(s_j w) \end{cases}.$$

Therefore, for $\mathbf{a} = (a_1, a_2, \ldots, a_p) \in Red(ww_0)$,

$$\mathcal{RC}(w) = split_{a_1} split_{a_2} \cdots split_{a_p}(D_0) = split_{\mathbf{a}}(D_0).$$

Proof. Define $\rho(\mathcal{P}) := \{\rho(D) \,|\, D \in \mathcal{P}\}$. By Lemma 3.1.6, we have that $\rho(\mathcal{RC}(w)) = \mathcal{RC}(w^{-1})$. So by Lemma 3.4.3,

$$\begin{aligned} split_j(\mathcal{RC}(w)) &= \rho(mitosis_j(\mathcal{RC}(w^{-1}))) \\ &= \begin{cases} \rho(\mathcal{RC}(w^{-1}s_j)) & \text{if } \ell(w^{-1}) > \ell(w^{-1}s_j) \\ \rho(\emptyset) & \text{if } \ell(w^{-1}) < \ell(w^{-1}s_j) \end{cases} \\ &= \begin{cases} \mathcal{RC}(s_j w) & \text{if } \ell(w) > \ell(s_j w) \\ \emptyset & \text{if } \ell(w) < \ell(s_j w) \end{cases}. \end{aligned}$$

□

The advantage of using split is that split simply decreases one exponent in x^D by one. That is, for $D \in \mathcal{RC}(w)$,

$$\sum_{E \in split_j(D)} x^E = \sum_{q \in \mathcal{T}_j(D)} \frac{x^D}{x_q}. \tag{3.9}$$

Chapter 4

A NEW SCHUBERT OBJECT: INVERSION FILLINGS

There are many combinatorial objects which index the monomials in Schubert polynomials. We refer to these as *Schubert objects.* A couple Schubert objects have already been mentioned, namely, **a**-compatible sequences and rc-graphs. In this chapter, we introduce two more known such combinatorial objects, balanced tableaux and balanced labellings and one new set of such object, called *inversion fillings.*

Inversion fillings serve as a link between rc-graphs and balanced labellings. Lemma 4.2.6 gives a bijection from inversion fillings to balanced labellings and Theorem 4.2.9 provides a bijection from rc-graphs to inversion fillings, which completes the connection. Section 4.3 provides an algorithm to construct inversion fillings. Furthermore, Section 4.4 supplies a connection between contretableaux and rc-graphs for Grassmannian permutations via inversion fillings and the split operator from Section 3.4.

4.1 Balanced Tableaux and Balanced Labellings

Given a finite set $S \subset \mathbb{P} \times \mathbb{P}$ and $(i, j) \in S$, define the *left hook*, L_{ij}, of (i, j) in S to be the set

$$L_{ij} := \{(k, b) \in S \,|\, k = i, b \leq j\} \cup \{(k, b) \in S \,|\, i \leq k \text{ and } b = j\}$$

and the *right hook*, R_{ij}, of (i, j) in S to be the set

$$R_{ij} := \{(k, b) \in S \,|\, k = i, b \geq j\} \cup \{(k, b) \in S \,|\, i \leq k \text{ and } b = j\}.$$

Definition 4.1.1. A *balanced tableau* T of shifted staircase shape $\delta_n = \{n-1, n-2, ..., 2, 1\}$ is a filling of the boxes in the set $\{(i, j) \in \mathbb{P} \times \mathbb{P} \,|\, i < j \leq n\}$ with positive entries t_{ij} in the i^{th} row and j^{th} column such that, for each $i < j$, the number of entries in L_{ij} (including t_{ij} itself) less than or equal to t_{ij} is $j - i$. A balanced tableau T with entries t_{ij} is *standard* if

$\{t_{ij} \in T\} = \{1, \ldots, \binom{n}{2}\}$.

Example 4.1.2. Following are several examples of standard balanced tableaux of shape $\delta_4 = (3,2,1)$:

1	2	3
	4	5
		6

1	2	3
	6	5
		4

1	4	3
	6	5
		2

1	5	3
	6	4
		2

2	5	3
	6	4
		1

6	2	4
	1	3
		5

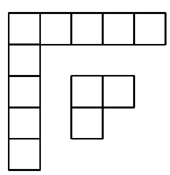
.

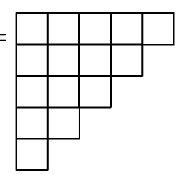
Theorem 4.1.3. *[13] There is a bijection from the set of standard balanced tableaux of shape δ_n to the set of maximal chains from the identity to w_0 in the right weak order.*

Fomin, Greene, Reiner, and Shimozono define an extension of balanced tableaux using the diagram of a permutation in [17].

Definition 4.1.4. The *diagram* of a permutation w is the set of boxes $D(w)$ in positions (i, w_j) such that $i < j$ and $w_i > w_j$.

Example 4.1.5. Following are the diagrams of $w = 625431$ and $w_0 = 654321$:

$D(625431) =$ 

$D(654321) =$ 

Notice that $D(w_0) = \{(i,j) \in \mathbb{P} \times \mathbb{P} \,|\, i + j \leq n\}$.

Definition 4.1.6. Suppose B is a filling of the boxes in $D(w)$ for a permutation w with entries b_{ij}. For $(i,j) \in D(w)$, rearrange the entries in the boxes in R_{ij} so that the entries increase from right to left across the row and increase top to bottom down the column of R_{ij}. If b_{ij} remains in position (i,j) in the rearrangement of R_{ij} for every $(i,j) \in B$, then B is *balanced*. If B is balanced and if no column contains two equal entries, then B is a *balanced labelling*. Denote the set of all balanced labellings for the diagram of a permutation w by $BL(w)$.

Example 4.1.7. Below is a balanced labelling for $w = 625431$. The $\boxed{\cdot}$'s are added to distinguish the rows and columns.

$D(625431) =$

2	2	1	1	1	·
3	·	·	·	·	·
6	·	3	5	·	·
7	·	2	·	·	·
8	·	·	·	·	·

Remark 4.1.8. [17] Let B be a filling of $D(w_0)$. Let B' be the filling of δ_n such that $b'_{ij} := b_{i,n-j+1}$. Then B is a balanced labelling of w_0 if and only if B' is a balanced tableaux for $w_0 \in S_n$.

Theorem 4.1.9. *[17] Let w be a permutation. Then*

$$\mathfrak{S}_w = \sum_{T \in BFL(w)} x^T,$$

where $BFL(w)$ denotes the set of balanced labellings of $D(w)$ such that $t_{ij} \leq i$ for every $(i,j) \in D(w)$.

Recall the Stanley symmetric functions from Section 1.5. Theorem 4.1.10 can be seen as an interpretation of Stanley symmetric functions in terms of balanced labellings.

Theorem 4.1.10. *[17] Given a permutation w,*

$$F_{w^{-1}} = \sum_{F \in BL(w)} x^F .$$

4.2 Inversion Fillings

In this section, we define *inversion fillings*. Subsequently, the relationships of inversion fillings to balanced tableaux and to balanced labellings are described.

Definition 4.2.1. Let $w \in S_n$ be a permutation. Define the *inversion set* of w to be

$$I(w) := \{(i,j) \in \mathbb{P} \times \mathbb{P} \,|\, i < j \text{ and } w_i > w_j\}.$$

Also, define

$$
\begin{aligned}
I(w)_{(k,-)} &:= \{j \mid k < j \text{ and } w_k > w_j\} \\
&= \{j \mid (k,j) \in I(w)\} \\
I(w)_{(-,k)} &:= \{i \mid i < k \text{ and } w_i > w_k\} \\
&= \{i \mid (i,k) \in I(w)\}.
\end{aligned}
$$

Notice,

$$
I(w) = \bigcup_{k \geq 1} I(w)_{(k,-)} = \bigcup_{k > 1} I(w)_{(-,k)}.
$$

The inversion set for a permutation w is unique to w. That is, if $w \neq u$, then $I(w) \neq I(u)$. Thus, w can be recovered from $I(w)$. Given any set S in $\mathbb{P} \times \mathbb{P}$, we also make the following definitions which will become useful. Define

$$
\begin{aligned}
S_{(k,-)} &:= \{j \mid k < j \text{ and } (k,j) \in S\} \\
S_{(-,k)} &:= \{i \mid i < k \text{ and } (i,k) \in S\} \qquad (4.1) \\
p_S(k) &:= |S_{(k,-)}| - |S_{(-,k)}| + k.
\end{aligned}
$$

The following elementary lemma will come in handy in a few proofs in this chapter. It is known in the literature as folklore.

Lemma 4.2.2. *Suppose $S \subset \{(i,j) \in \mathbb{P} \times \mathbb{P} : i < j\}$ is a finite set. Then S is the inversion set of some permutation w if and only if S has the following properties:*

1. *If (i,j) is in S, then for every $i < k < j$, either (i,k) is in S or (k,j) is in S.*

2. *If (i,k) and (k,j) are in S, then $(i,j) \in S$.*

In fact, $w(k) = p_S(k)$.

Proof. If w is a permutation, then certainly $I(w)$ has the two given properties.

For the proof in the other direction, suppose the finite set S has the two given properties. Observe that $|S_{(-,k)}| \leq k-1$ for every $k \geq 1$ which implies $p_S(k) \geq 1$ for all $k \geq 1$. Thus, p_S maps $\mathbb{P}$ to $\mathbb{P}$. To prove the lemma, first it is shown that $(i,j) \in S$ if and only if $p_S(i) > p_S(j)$ and $i < j$. It is shown at the same time that p_S is injective. Then, it is shown that p_S is surjective onto $\mathbb{P}$. For some n, $p_S(j) = j$ for all $j > n$. This will conclude the proof by setting $w_j = p_S(j)$, so $w \in S_n$.

Suppose $i < j$ and $(i,j) \in S$. Then the following all hold by the assumptions on S:

$$\text{If } i < k < j, \text{ then } S \cap \{(i,k),(k,j)\} \text{ is nonempty.} \tag{4.2}$$

$$\text{If } a < i < j \text{ and } (a,i) \in S, \text{ then } (a,j) \in S. \tag{4.3}$$

$$\text{If } i < j < z \text{ and } (j,z) \in S, \text{ then } (i,z) \in S. \tag{4.4}$$

From (4.3), it follows that $S_{(-,i)} \subset S_{(-,j)}$, and hence, $|S_{(-,j)} \setminus S_{(-,i)}| = |S_{(-,j)}| - |S_{(-,i)}|$. Likewise, (4.4) implies $S_{(j,-)} \subset S_{(i,-)}$, and thus, $|S_{(i,-)} \setminus S_{(j,-)}| = |S_{(i,-)}| - |S_{(j,-)}|$. From (4.2),

$$\left|\left(S_{(-,j)} \cup S_{(i,-)}\right) \cap \{k \,|\, i \leq k \leq j\}\right| = j - i + 1,$$

and so,

$$\begin{aligned}
&p_S(i) - i - p_S(j) + j \\
&= \left|S_{(-,j)}\right| - \left|S_{(-,i)}\right| + \left|S_{(i,-)}\right| - \left|S_{(j,-)}\right| \\
&= \left|S_{(-,j)} \setminus S_{(-,i)}\right| + \left|S_{(i,-)} \setminus S_{(j,-)}\right| \\
&\geq \left|\left(S_{(-,j)} \cup S_{(i,-)}\right) \cap \{k \,|\, i \leq k \leq j\}\right| \\
&= j - i + 1.
\end{aligned}$$

This implies $p_S(i) - p_S(j) > 0$, and thus, if $(i,j) \in S$, then $p_S(i) > p_S(j)$.

Now suppose $(i,j) \notin S$ and $i<j$. Then the following all hold by the assumptions on S:

$$\text{If } i<k<j, \text{ then } |S \cap \{(i,k),(k,j)\}| \leq 1. \tag{4.5}$$

$$\text{If } a<i<j \text{ and } (a,j) \in S, \text{ then } (a,i) \in S. \tag{4.6}$$

$$\text{If } i<j<z \text{ and } (i,z) \in S, \text{ then } (j,z) \in S. \tag{4.7}$$

From (4.6), it follows that $\left(S_{(-,j)} \cap \{a|a \leq i\}\right) \subset S_{(-,i)}$ and so

$$\left|S_{(-,i)} \setminus \left(S_{(-,j)} \cap \{a \,|\, a \leq i\}\right)\right| = \left|S_{(-,i)}\right| - \left|S_{(-,j)} \cap \{a \,|\, a \leq i\}\right| \geq 0.$$

Likewise, (4.7) says $\left(S_{(i,-)} \cap \{z \,|\, z \geq j\}\right) \subset S_{(j,-)}$, and thus,

$$\left|S_{(j,-)} \setminus \left(S_{(i,-)} \cap \{z \,|\, z \geq j\}\right)\right| = \left|S_{(j,-)}\right| - \left|S_{(i,-)} \cap \{z \,|\, z \geq j\}\right| \geq 0.$$

By (4.5),

$$\left|S_{(i,-)} \cap \{k \,|\, i<k<j\}\right| + \left|S_{(-,j)} \cap \{k \,|\, i<k<j\}\right| \leq j-i-1,$$

and so,

$$\begin{aligned}
&p_S(j) - j - p_S(i) + i \\
&= \left|S_{(j,-)}\right| - \left|S_{(i,-)}\right| + \left|S_{(-,i)}\right| - \left|S_{(-,j)}\right| \\
&= \left|S_{(j,-)}\right| - \left|S_{(i,-)} \cap \{z \,|\, z \geq j\}\right| - \left|S_{(i,-)} \cap \{k \,|\, i<k<j\}\right| \\
&+ \left|S_{(-,i)}\right| - \left|S_{(-,j)} \cap \{a \,|\, a \leq i\}\right| - \left|S_{(-,j)} \cap \{k \,|\, i<k<j\}\right| \\
&\geq - \left|S_{(i,-)} \cap \{k \,|\, i<k<j\}\right| - \left|S_{(-,j)} \cap \{k \,|\, i<k<j\}\right| \\
&\geq 1 - j + i\,.
\end{aligned}$$

This implies $p_S(j) - p_S(i) > 0$, and so we have, if $(i,j) \notin S$, then $p_S(i) < p_S(j)$. Thus, p_S is an injection from $\mathbb{P}$ into $\mathbb{P}$, and $(i,j) \in S$ if and only if $i<j$ and $p_S(i) > p_S(j)$.

Given k, it remains to show there exists j such that $p_S(j) = k$. We have that S is finite, sothere exists some element a such that for every $\ell \geq a$, $S_{(\ell,-)} \cup S_{(-,\ell)}$ is empty. This implies $p_S(\ell) = \ell$. But all $p_S(j)$'s are distinct and $(j,\ell) \notin S$. So, $\{p_S(j)|1 \leq j < a\} \subseteq \{1, ..., a-1\}$.

The two sets have the same cardinality because p is injective, so the sets are equal. Thus, p is surjective from $\mathbb{P}$ onto $\mathbb{P}$ and S is the inversion set for w where $w_j = p_S(j)$. $\square$

Given a set $S \subset \mathbb{P} \times \mathbb{P}$, represent S pictorially via the set of boxes in positions (i, j) for all $(i, j) \in S$. Fill the boxes with positive integers, refered to as a *filling* of S.

Definition 4.2.3. An *inversion filling* for S is a filling of S such that for every entry m_{ij} in position $(i, j) \in S$, m_{ij} lies between m_{ik} and m_{kj} for all $i < k < j$. If $(i, j) \notin S$, then m_{ij} is defined to be 0 and not drawn in the filling. An *inversion filling for w* is an inversion filling of the inversion set $I(w)$. The set of all inversion fillings for w is denoted by $IF(w)$.

An inversion filling for S is *semistandard* if the entries in each column are distinct. Denote this set by $SSIF(S)$. An inversion filling is *standard* if the filling is semistandard and the entries are exactly the integers from 1 to $|S|$. Denote this set by $SIF(S)$. Again,

$$SSIF(w) := SSIF(I(w)) \text{ and } SIF(w) := SIF(I(w)).$$

The inversion filling in Example 4.2.4 is a semistandard inversion filling. Inversion fillings are a different generalization of balanced tableaux as Lemma 4.2.5 points out.

Example 4.2.4. When drawing an inversion filling, it is helpful to know the column and row numbers. For this reason, the unfilled boxes (i, j) with $1 \leq i < j$ are represented by a $\boxed{\cdot}$. The following is an example of an inversion filling for $w = 625431$:

$$\begin{array}{|c|c|c|c|c|} \hline 2 & 1 & 1 & 1 & 2 \\ \hline & \cdot & \cdot & \cdot & 3 \\ \cline{2-5} & & 5 & 3 & 6 \\ \cline{3-5} & & & 2 & 7 \\ \cline{4-5} & & & & 8 \\ \cline{5-5} \end{array}.$$

Lemma 4.2.5. *For $w_0 \in S_n$, a filling T of $I(w_0) = \{(i, j) \in \mathbb{P} \times \mathbb{P} \,|\, i < j \leq n\}$ is a balanced tableau of shape δ_n if and only if T is an inversion filling for w_0.*

Proof. The lemma follows directly from Definitions 4.1.1 and 4.2.3. $\square$

Lemma 4.2.6. *There is a content preserving bijection, Ω, from the set of inversion fillings for w to the set of balanced labellings of w.*

Proof. Let M be an inversion filling for a permutation w. Rearrange the columns of M in the order of w. That is, given box $(i,j) \in I(w)$ with entry m_{ij} in M, make a filling of $D(w)$ by placing m_{ij} in box (i, w_j). Call this new filling M'.

By the definition of an inversion filling, for every $i < k < j$, m'_{iw_j} lies between m'_{iw_k} and m'_{kw_j}. From [17, Lemma 2.5], a filling of $D(w)$ is a balanced labelling if and only if, for every triple (i,k,j), the diagram restricted to rows i,k,j and columns w_i, w_k, w_j is balanced. So, M' is a balanced labelling.

Given a balanced labelling F of $D(w)$, again from [17, Lemma 2.5], for $i < k < j$, the same restricted subdiagram must be balanced. It is a quick verification to show that this implies f_{iw_j} lies between f_{iw_k} and f_{kw_j}. Thus, rearranging the columns of F in the order of w^{-1} gives an inversion filling. □

From Theorem 4.1.10 and Lemma 4.2.6, Corollary 4.2.7 is immediate.

Corollary 4.2.7. *Given a permutation w,*

$$F_{w^{-1}} = \sum_{F \in SSIF(w)} x^F,$$

where x^F is the monomial $\prod_{i \in F} x_i$.

Definition 4.2.8. Let $FIF(S)$ be the subset of $SSIF(S)$ such that, for each entry m_{ij}, we have $m_{ij} \leq i$. Call these inversion fillings *flag inversion fillings.*

Theorem 4.2.9. *There is a bijection Ψ from $\mathcal{RC}(w)$ to $FIF(w)$, such that $x^D = x^{\Psi(D)}$ for every rc-graph D.*

The proof of Theorem 4.2.9 is delayed until Section 4.3. It is placed here to emphasize the connections between inversion fillings and known Schubert objects.
Recall, Corollary 3.1.5,

$$\mathfrak{S}_w(x) = \sum_{R \in RC(w)} x^R.$$

As a corollary to Theorem 4.2.9 and Corollary 3.1.5, the flag semistandard inversion fillings also index the monomials in the Schubert polynomial indexed by w.

Corollary 4.2.10. *For a permutation $w \in S_n$,*

$$\mathfrak{S}_w = \sum_{F \in FIF(w)} x^F,$$

where x^F is the monomial $\prod_{i \in F} x_i$.

4.3 The Construction Algorithm and the Proof of Theorem 4.2.9

In this section, we construct inversion fillings such that each intermediate step produces all inversion fillings for permutations w' between the identity element and w in left weak order of fixed content. The construction is then used in the proof of Theorem 4.2.9.

Algorithm 4.3.1. (Construction Algorithm) Given a sequence of nonnegative integers $(b_1, b_2, \ldots, b_a)$, form the *construction tree* with content $(1^{b_1}, 2^{b_2}, \ldots, a^{b_a})$ according to the following rules:

1. Start with the empty inversion set, $I(id)$.

2. After all $a, a-1, \ldots, \ell+1$ have been inserted into the diagram, insert all ℓ's in the diagram one at a time, b_ℓ in total. Set $m_{ij} = \ell$ only if the following all hold:

   (a) $i < j$.

   (b) $p_S(i) = p_S(j) - 1$, where $p_S(k)$ is as in Equation (4.1) and S is the set of boxes already filled.

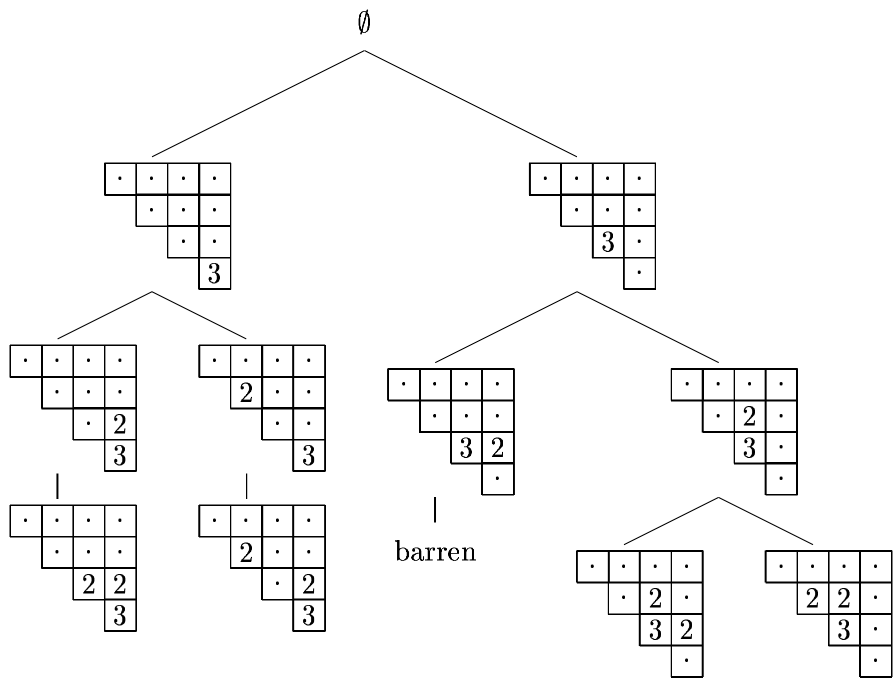
   (c) if $m_{st} = \ell$, then $t \neq j$ and $p_S(i) \geq p_S(s)$.

   To make sure the construction algorithm yields flag semi-standard fillings, the construction tree must be restricted by

   (d) $\ell \leq i$.

The collection of leaves on the last step is $CL(b_1, b_2, \ldots, b_a)$.

Example 4.3.2. Below is the construction tree for content $(1^0, 2^2, 3^1)$ of sets $S \subset \{(i,j) \in \mathbb{P} \times \mathbb{P} \,|\, i < j \leq 5\}$ which are flag semi-standard. Again, say a node is *barren* if it bears no children.



For a filling M of boxes in $\{(i,j) \in \mathbb{P}\times\mathbb{P} : i < j\}$, let $S(M)$ be the set of boxes (i,j) which are filled. We will use this convention for the next few proofs. Also, recall Equation (4.1) for the definitions of $S(M)_{(i,-)}$, $S(M)_{(-,j)}$, and $p_S(k)$.

Lemma 4.3.3. *At each step of the construction tree, the set of filled boxes in a node N of the tree is the inversion set for some permutation $w(N)$, where $w(N)$ is the permutation such that $S(N) = I(w(N))$.*

Proof. If M is the empty filling, then $S(M) = \emptyset$ is the inversion set for the identity permutation. Thus, M is a filling of $I(id)$. Proceed by induction on the size of $S(M)$.

Assume the lemma is true if $|S(M)| = \alpha - 1$, and suppose $|S(N)| = \alpha$ for some filling, N, which is a node in the contruction tree. It must be that N is a child of a filling M in the tree, such that $S(M) \cup \{(i,j)\} = S(N)$ and $p_S(i) = p_S(j) - 1$. Then, by the inductive hypothesis, p_S is a permutation and $S(M) = I(p_S)$.

It remains to show $I(p) \cup \{(i,j)\}$ is the inversion set for some permutation w when $|I(p)_{(i,-)}| - |I(p)_{(-,i)}| + i + 1 = |I(p)_{(j,-)}| - |I(p)_{(-,j)}| + j$. From the proof of Lemma 4.2.2,

$p_i + 1 = p_j$. Thus, multiplying p on the left by the adjacent transposition $s_{p_i} = (p_i, p_i + 1)$ changes $I(p)$ only by adding (i, j) to it, and so, $I(p) \cup \{(i, j)\} = I(s_{p_i} p)$. Therefore, $S(N) = S(M) \cup \{(i, j)\} = I(p) \cup \{(i, j)\}$ is the inversion set for $s_{p_i} p$. □

Corollary 4.3.4. *In the construction tree, N is a child of M only if $w(N) = s_r w(M)$ for some $r \geq 1$, where $w(N)$ is the permutation such that $S(N) = I(w(N))$.*

Lemma 4.3.5. *At each step of the construction tree, a node M is a semistandard inversion filling for $w(M)$.*

Proof. If M is the empty filling, then M is the unique inversion filling for the identity permutation. Proceed by induction on the size of $S(M)$.

Assume the lemma is true if $|S(M)| = \alpha - 1$, and suppose $|S(N)| = \alpha$ for some filling, N, which is a node in the contruction tree. It must be that N came from a filling M by adding an entry ℓ in position (i, j), such that $S(M) \cup \{(i, j)\} = S(N)$ and $p_S(i) = p_S(j) - 1$. Furthermore, $n_{ij} = \ell$ must be less than or equal to every entry in M, and strictly less than every entry in column j of M. Then, by the inductive hypothesis, $M \in SSIF(v)$ for some permutation v.

If N is not a semistandard inversion filling of $S(N)$, then, because $M \in SSIF(v)$, there must exist $r \geq 1$ such that at least one of the following hold:

(i) $r < i < j$ and m_{rj} does not lie between ℓ and m_{ri}.

(ii) $i < r < j$ and ℓ does not lie between m_{ir} and m_{rj}.

(iii) $i < j < r$ and m_{ir} does not lie between ℓ and m_{jr}.

If Statement (i) holds, then it must be that $m_{ri} \geq m_{rj}$, because m_{ij} is taken to be 0 when (i, j) is unfilled and $M \in SSIF(v)$. By construction, $m_{rj} \geq \ell$ which contradicts Statement (i). Similarly, Statement (iii) cannot hold.

The proof of the impossibility of (ii) needs one more observation. By the inductive hypothesis, $M \in SSIF(v)$ and $(i, j) \notin I(v)$. Thus, at most one of (i, r) and (r, j) is filled for each $i < r < j$. We also have that $|I(v)_{(i,-)}| - |I(v)_{(-,i)}| + i + 1 = |I(v)_{(j,-)}| - |I(v)_{(-,j)}| + j$, by the construction algorithm. From the proof of Lemma 4.2.2,

$$p_S(j) - p_S(i) \geq j - i - \left|I(v)_{(-,j)} \cap \{k \,|\, i < k < j\}\right| - \left|I(v)_{(i,-)} \cap \{k \,|\, i < k < j\}\right| \geq 1.$$

By assumption, $p_S(j) - p_S(i) = 1$. So,

$$\left|I(v)_{(-,j)} \cap \{k \,|\, i < k < j\}\right| + \left|I(v)_{(i,-)} \cap \{k \,|\, i < k < j\}\right| = j - i - 1\,.$$

This implies that, for every $i < r < j$, exactly one of $(i,r),(r,j)$ is filled in M. Thus, by filling (i,j) with ℓ and the fact that ℓ is weakly less than every entry in M proves (ii) cannot hold. □

Theorem 4.3.6. *Given a sequence $(b_1, b_2, \ldots, b_a)$ of nonnegative integers, the leaves of the construction tree, $CL(b_1, b_2, \ldots, b_a)$, is the set of all semistandard inversion fillings for every permutation with content $(1^{b_1}, 2^{b_2}, \ldots, a^{b_a})$.*

Proof. The claim is true when $\sum b_i = 0$, by Step 1 of Algorithm 4.3.1. Now, assume the claim is true when $\sum b_i = \alpha - 1$. Suppose w is a permutation such that $\ell(w) = \alpha$ and $G \in SSIF(w)$. Thus, if $content(G) = (b_1, b_2, \ldots, b_a)$, then $\sum b_i = \alpha$. We will show that for some $s < t$, the filling G minus one box (s,t) is a semistandard inversion filling for a permutation v. Afterwards, we will show that G can be constructed from this filling with the constuction algorithm.

Let $\ell := min\{g_{st} \,|\, (s,t) \in G\}$ and let $T := \{(i,j) \in I(w) \,|\, g_{ij} = \ell\}$. Within T, find (i,j) such that w_i is maximal and then, such that w_j is maximal. Call this pair (i_0, j_0). We wish to show that $J = I(w) \setminus \{(i_0, j_0)\}$ is an inversion set for some permutation v.

Suppose there exists $i_0 < k < j_0$ such that (i_0, k) and (k, j_0) are in J. Then either $g_{i_0 k}$ or $g_{k j_0}$ equals ℓ, because $g_{i_0 j_0}$ is between $g_{i_0 k}$ and $g_{k j_0}$. Since $(k, j_0) \in I(w)$, we have that $w_{i_0} > w_k > w_{j_0}$. If $g_{i_0 k} = \ell$, then the maximality of w_{j_0} is contradicted because $w_k > w_{j_0}$. Hence, the choice of (i_0, j_0) is contradicted. In addition, $g_{k j_0} \neq \ell$, by the definition of semistandard. So, one of (i_0, k) and (k, j_0) is not in J.

Suppose there exists $h < i_0 < j_0$ such that $(h, j_0) \in J$, but $(h, i_0) \notin J$. Then it would have to happen that $g_{i_0j_0} > g_{hj_0}$, which contradicts the minimality of $g_{i_0j_0}$. Suppose there exists $r > j_0 > i_0$ such that $(i_0, r) \in J$ but $(j_0, r) \notin J$. Then $g_{i_0j_0} \geq g_{i_0r}$ and, by the minimality of $g_{i_0j_0}$, $g_{i_0j_0} = g_{i_0r}$. But $r > j_0$ and $(j_0, r) \notin J$, so $w_r > w_{j_0}$. This contradicts the maximality of w_{j_0}, and again contradicts the choice of (i_0, j_0).

Thus, by Lemma 4.2.2, $J = I(v)$ for $v = s_{w_{j_0}}w$ and $\ell(v) = \alpha - 1$. Let F be the filling obtained from G by removing the filling in box (i_0, j_0) and all other entries remain the same. Then F is a semistandard inversion filling for v as just shown. Thus, F can be constructed by the algorithm by the inductive hypothesis. Also, inserting ℓ into the $(i_0, j_0)^{th}$ entry of F satisfies the conditions of the construction algorithm. Thus, G can be constructed, and the proof is complete by induction. □

Corollary 4.3.7. *Given the sequence of t 1's,* $(1, 1, \ldots, 1)$*, we have that* $CL(1, 1, \ldots, 1)$ *is the (infinite) set of all standard inversion fillings for all permutations of length t. Therefore, the number of fillings in* $CL(1, 1, \ldots, 1)$ *contained in* $[n] \times [n]$ *is equal to the number of reduced words of length t in* S_n*.*

From Theorem 4.1.3 and the fact that any standard inversion filling for $w \in S_n$ can be extended to a standard inversion filling for $w_0 \in S_n$ as stated in [13], we have:

Corollary 4.3.8. *There is a bijection from the set of maximal chains from the identity to w in left weak order to the set of standard inversion fillings for w. The maximal chains from the identity to w in left weak order index the reduced words for w, therefore there is a bijection* β *from* $Red(w)$ *to* $SIF(w)$*.*

Define an equivalence relation on $SSIF(S)$ such that $M \sim M'$ if for every $i < k < j$, $\{m_{ik}, m_{ij}, m_{kj}\}$ is in the same relative order as $\{m'_{ik}, m'_{ij}, m'_{kj}\}$.

Corollary 4.3.9. *Given* $\mathbf{a}, \mathbf{b} \in Red(w)$*, we have that* $heap(\mathbf{a}) = heap(\mathbf{b})$ *if and only if* $\beta(\mathbf{a}) \sim \beta(\mathbf{b})$ *in* $SIF(w)$*, where* β *is the bijection in Corollary 4.3.8.*

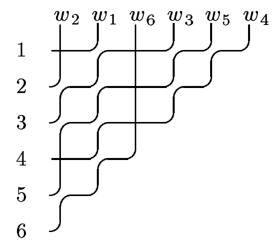
Finally, the necessary background for the proof of Theorem 4.2.9 has been stated.

Proof of Theorem 4.2.9. Given an rc-graph D for a permutation w, label the strings of D by the row in which the strings originate as in Example 3.1.3 (as opposed to the column they end up in). Each crossing of strings $s < t$ in D corresponds bijectively with an inversion $(s,t) \in I(w)$, and the crossing forces string s to proceed horizontally and string t to proceed vertically. Furthermore, the crossing occurs in row ℓ, for some $\ell \leq s$, implying this is the only place where string t appears in row ℓ. Define $\Psi(D)$ to be the filling M of $I(w)$ such that $\ell = m_{st}$, i.e. the row in which strings $s < t$ cross in D. Proceeding along any string s, the rows in which the crossings occur weakly decrease. If strings s and t cross and $s < k < t$, then either string t had to cross string k prior to crossing string s, or string s had to cross string k prior to crossing string t. Both of these could not have happened before strings s and t crossed. Likewise, both could not happen after s and t crossed. Thus, m_{st} lies between m_{sk} and m_{kt}. Furthermore, each string only moves right or up, so $m_{st} \leq s$ for each string s and all $s \leq t \leq n$. Thus, $\Psi(D) \in FIF(w)$.

It remains to find $\Psi^{-1} : FIF(w) \to \mathcal{RC}(w)$. The proof proceeds by induction on $\ell(w)$. For $\ell(w) = 0$ or 1, the claim is clear. Now, suppose the claim is true for $\ell(w) = \alpha - 1$. Given $G \in FIF(w)$ such that $\ell(w) = \alpha$, find $(i_0, j_0) \in G$ such that $g_{i_0 j_0} > 0$ is minimal, w_{i_0} is maximal, and then, j_0 such that w_{j_0} is maximal. Then, by the same considerations as in the proof of the Theorem 4.3.6, $F = \{g_{ij} > 0 \,|\, (i,j) \neq (i_0, j_0)\}$ is a flag inversion filling for a permutation v, with $\ell(v) = \ell(w) - 1$. This corresponds to an rc-graph E for v by the inductive hypothesis. Furthermore, $w = s_{v_{i_0}} v$. It must be that $v^{-1}(i_0) < v^{-1}(j_0)$ and so string i_0 lies just to the left of string j_0 in the one line notation v. In row ℓ of E, j_0 crosses no strings, because G is a flag inversion filling and so, all entries in column j_0 are distinct. This implies there is no ℓ in column j_0 of F and hence, j_0 does not cross any strings in row ℓ of E. No more crossings occur in any row higher than row ℓ in E, so strings i_0 and j_0 leave row ℓ next to each other. Thus, strings i_0 and j_0 lie next to each other in row ℓ. Therefore, adding a cross in row ℓ for strings i_0 and j_0 is an rc-graph for w and whose image in Ψ is G. □

Remark 4.3.10. The proof of Theorem 4.2.9 implies that for $F \in FIF(w)$, the filling of box (i,j) denotes the row on which string i crosses string j in the corresponding rc-graph, $\Psi^{-1}(F)$. Thus, the sequence in row i of F gives the positions of all the horizontal crossings of string i in $\Psi^{-1}(F)$. Likewise, column j gives the positions of all the vertical crossings of string j in $\Psi^{-1}(F)$.

Example 4.3.11. Let $\mathbf{a} = (3, 1, 4, 5, 4)$ and $\boldsymbol{i} = (1, 1, 2, 3, 4)$. Below is the rc-graph $D(\mathbf{a}, \boldsymbol{i}) \in \mathcal{RC}(214653)$ with the strings filled in.



(4.8)

The corresponding inversion filling, $F = \Psi(D(\mathbf{a}, \mathbf{i}))$ is below.

1	·	·	·	·
	·	·	·	·
		·	·	1
			4	3
				2

4.4 Grassmannian Inversion Fillings

Recall from Definition 1.5.6 that a permutation w in S_n is *Grassmannian* if there exists a k such that $w_i < w_{i+1}$ for all $i \neq k$. The set of such permutations is denoted $G(n)$. Given $1 \leq k < n$, the set of $w \in G(n)$ with $w_i < w_{i+1}$ for all $i \neq k$ is denoted $G(n; k)$.

In this section, we examine special properties of inversion fillings for Grassmannian permutations and use the split operator from Section 3.4 to obtain a left divided difference operator on contretableaux.

Observation 4.4.1. Suppose $w \in G(n; k)$. Then for every $1 \leq i < \ell < j \leq n$, at most one of (i, ℓ) and (ℓ, j) is in $I(w)$.

Proof. If $(i, \ell) \in I(w)$, then it must be that $i \leq k$ and $\ell > k$. For $j > \ell > k$, we have that $w_\ell < w_j$ and so $(\ell, j) \notin I(w)$. Likewise, if $(\ell, j) \in I(w)$, then it must be that $\ell \leq k$ and $j > k$. For $i < \ell \leq k$, we have that $w_i < w_\ell$. So $(i, \ell) \notin I(w)$. $\square$

Observation 4.4.2. Suppose $w \in G(n;k)$ and recall Definition 4.2.1. If $i < \ell \leq k$, then $I(w)_{(i,-)} \subset I(w)_{(\ell,-)}$. Likewise, if $k < \ell < j$, then $I(w)_{(-,j)} \subset I(w)_{(-,\ell)}$. Hence, $I(w)_{(-,j)} = \{i \,|\, a_j \leq i \leq k\}$ for some a_j and $a_j \leq a_{j+1}$, for every $j > k$, and $I(w)$ has partition shape.

Proof. If $(i,j) \in I(w)$ and $i < \ell \leq k$, then $w_i < w_\ell$, $w_i > w_j$, and $j > k$. Hence, $w_\ell > w_i > w_j$ and $\ell < j$, so $(\ell, j) \in I(w)$. Likewise, if $(i,j) \in I(w)$ and $j > \ell > k$, then $w_j < w_i$ and $w_j > w_\ell$, so $w_i > w_\ell$. Furthermore, $i \leq k$, so $(i,\ell) \in I(w)$. ☐

Recall, $\mathcal{CT}_\lambda$ is the set of all contretableaux of shape λ. Denote the subset of $\mathcal{CT}_\lambda$ with entries at most k by $\mathcal{CT}_\lambda^k$.

Lemma 4.4.3. *Suppose $w \in S_n$ is a Grassmannian permutation with descent k. Then there is a monomial preserving bijection γ from $FIF(w)$ to $\mathcal{CT}_\lambda^k$, where $\lambda_{k+1-i} = w_i - i$ for $1 \leq i \leq k$.*

Proof. Given a flag inversion filling F for w, suppose (i,ℓ) and (i,j) are in $I(w)$, with $i < \ell < j$. Then (ℓ, j) is not in $I(w)$ by Observation 4.4.1. Thus, it must be that $f_{ij} \leq f_{i\ell}$ since $f_{\ell j}$ is taken to be 0. Suppose (i,j) and (ℓ,j) are in $I(w)$ with $i < \ell < j$. Again, by Observation 4.4.1, (i,ℓ) is not in $I(w)$. So, it must be that $f_{ij} \leq f_{\ell j}$. We have that F is a flag inversion filling, specifically a semistandard inversion filling, where each column has only distinct entries, so $f_{ij} < f_{\ell j}$. Furthermore, from Observation 4.4.2, $I(w)$ has partition shape λ as in the lemma with largest row k. So, F is a contretableau when regarded as a filling of λ by ignoring the $\boxdot$'s. Call this tableau $\gamma(F)$. The only other restriction is that $f_{i\ell} \leq i$ for every $(i,\ell) \in I(w)$ because F is a *flag* inversion filling. Thus, F has maximum entry $f_{k,k+1}$, which can be at most k. So, $\gamma(F) \in \mathcal{CT}_\lambda^k$ and γ is an injection.

The only restrictions on F to be in $FIF(w)$ were that $f_{ij} \leq f_{i\ell}$ for $i \leq k < \ell < j$, $f_{ij} < f_{\ell j}$ for $i < \ell \leq k < j$, and $f_{ij} \leq i$. All these restrictions also hold on contretableaux, so the map is also surjective and hence a bijection. ☐

Example 4.4.4. Consider the permutation $w = 1357246 \in G(7;4)$. Below is an inversion filling F for w. Ignoring the $\boxdot$'s, F can be viewed as the contretableau on the right.

·	·	·	·	·	·
	·	·	1	·	·
		·	3	2	·
			4	3	3
				·	·
					·

1		
3	2	
4	3	3

It has been a question to find an operator δ_i on contretableaux such that $\delta_i(T)$ is set of contretableaux of same shape as T minus one outer corner box. The ordinary divided difference operators cannot hope to do this. In particular, $\partial_i(\mathfrak{S}_w)$ is $\mathfrak{S}_{ws_i}$ or 0. If w is Grassmannian with descent k, then ws_i is not Grassmannian with descent k for every i, other than in the case where $w = s_i$.

In Section 3.4, we introduced an operator on rc-graphs for w which produces all rc-graphs for s_iw, namely the split operator. Contrary to right multiplication by s_i, if w is Grassmannian with descent k and if $\ell(w) > \ell(s_iw)$, then s_iw is again a Grassmannian permutation with descent k. In light of Theorem 4.2.9 and Lemma 4.4.3, we describe how the split operator works on inversion fillings for Grassmannian permutations. In particular, we have the following diagram of maps for every permutation $w \in S_n$:

$$FIF(w) \xrightarrow{\Psi^{-1}} \mathcal{R}C(w) \xrightarrow{split_i} \mathcal{R}C(s_iw) \xrightarrow{\Psi} FIF(s_iw)$$

Definition 4.4.5. Suppose $w \in S_n$ and $F \in FIF(w)$. Then for $1 \leq i < n$,

$$\delta_i(F) := \Psi(split_i(\Psi^{-1}(F))).$$

For a collection of flag inversion fillings $\mathcal{F}$ for w,

$$\delta_i(\mathcal{F}) := \bigcup_{F \in \mathcal{F}} \delta_i(F).$$

From Theorems 3.4.4 and 4.2.9, the following corollary is immediate.

Corollary 4.4.6. *For a permutation $w \in S_n$,*

$$\delta_j(FIF(w)) = \begin{cases} FIF(s_j w) & \text{if } \ell(w) > \ell(s_j w) \\ \emptyset & \text{if } \ell(w) < \ell(s_j w) \end{cases} .$$

Observation 4.4.7. If $w \in G(n)$, then only columns $w^{-1}(i)$ and $w^{-1}(i+1)$ of F are affected by δ_i.

Observation 4.4.8. Suppose $w \in G(n;k)$. Then $\ell(s_i w) < \ell(w)$ if and only if there exist $h \leq k < j$ such that $w_h = i+1$ and $w_j = i$. Furthermore, if $\ell(s_i w) < \ell(w)$, then $I(s_i w) = I(w) \setminus \{(h,j)\}$.

Now we give an algorithm on contretableaux to produce contretableaux with one less box. It will be shown subsequently that the following algorithm acts like δ_i on $G(n;k)$ for some i.

Algorithm 4.4.9. (Grassmannian μ Algorithm) Suppose $w \in G(n;k)$ and $F \in FIF(w)$. Given $j > k$, let

$$a_j := \begin{cases} min\{i \mid (i,j) \in I(w)\} & \text{if } I(w)_{(-,j)} \neq \emptyset \\ 0 & \text{else} \end{cases} .$$

Define

$$t_j(F) = max\left(\{i \geq a_j \mid f_{ij} = i - a_j + 1 \text{ and } f_{ij} \neq f_{i,j+1}\} \cup \{0\}\right).$$

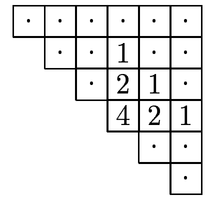
Suppose $1 \leq q \leq t_j(F) - a_j + 1$. Form ${}^j_q F$ as follows:

1. Delete box $(q + a_j - 1, j)$ from F.

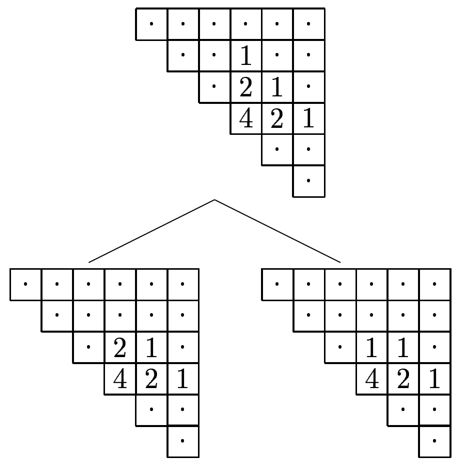
2. For every $a_j \leq i < q + a_j - 1$, move box (i,j) to position $(i+1,j)$. The result is ${}^j_q F$.

Then define

$$\mu_j(F) := \bigcup_{q=1}^{t_j - a_j + 1} {}^j_q F.$$

Example 4.4.10. Suppose $w = 1357246 \in G(7;4)$ and $F =$  .

Below, the leaves of the tree make up $\mu_5(F)$, $j = 5$, $a_j = 2$, and $t_j = 3$.



Lemma 4.4.11. *Suppose $w \in G(n;k)$. Then $\mu_j(FIF(w)) \subset FIF(s_i w)$ for $w_j = i$.*

Proof. From Algorithm 4.4.9 and Observation 4.4.8, we have that $G \in \mu_j(F)$ is a filling of $I(w) \setminus \{(h,j)\} = I(s_i w)$ when $w^{-1}(i) = j$ and $w^{-1}(h) = i+1$. By construction, G remains a flag inversion filling. Thus, $\mu_j(FIF(w)) \subset FIF(s_i w)$. $\square$

Theorem 4.4.12. *For a permutation $w \in G(n;k)$ and $F \in FIF(w)$,*

$$\mu_j(F) = \delta_i(F) \tag{4.9}$$

for $w_j = i$.

Proof. Let $w^{-1}(i) = j$ and $w^{-1}(i+1) = h$. Suppose $G \in \mu_j(F)$. Then $\Psi^{-1}(G)$ is an rc-graph for $s_i w$ by Lemma 4.4.11. Thus, there exists a unique rc-graph $D \in \mathcal{RC}(w)$ such that $\Psi^{-1}(G) = E \in split_i(D)$, and D agrees with E outside of columns i and $i+1$. That is, $E = {}^{i}_{q}D$ for some $q \in \mathcal{T}_i(D)$. So, $\Psi(D)$ and $\Psi(E)$ are flag inversion fillings which agree outside of columns j and h. (It is important to note here that $(s_i w)^{-1}(i) = w^{-1}(i+1) = h$, $(s_i w)^{-1}(i+1) = w^{-1}(i) = j$, and $(s_i w)^{-1}(m) = w^{-1}(m)$ for every $m \geq 1$ not equal to i or

$i+1$.) Furthermore, by Corollary 3.1.14, $\mathcal{T}_i(D) = \{q \mid 1 \le q \le t_j - a_j + 1\}$. So $\mu_{w^{-1}(i)}(F)$ and $\delta_i(F)$ have the same cardinality.

From Observation 4.4.8, $\ell(s_i w) < \ell(w)$ if and only if $h \le k < j$. If $\ell(s_i w) < \ell(w)$, then $I(w)_{(-,h)} = I(s_i w)_{(-,h)} = \emptyset$. All the vertical crosses for string j remain vertical crosses for string j except one is deleted, namely the vertical cross in row q. So, only column j of F changes in $\delta_i(F)$ by removing the q entry. Furthermore, every inversion filling in $\delta_i(F)$ can be viewed as a contretableau, so it is clear that G is the contretableau obtained by deleting q from column j of F and letting the boxes above fall down by one unit. Thus, $G \in \delta_i(F)$, implying $\mu_{w^{-1}(i)}(F) \subset \delta_i(F)$. Hence, $\mu_{w^{-1}(i)}(F) = \delta_i(F)$.

Now suppose $\ell(s_i w) > \ell(w)$. Then $\delta_i(F) = \emptyset$ and $j < h$. So either $I(w)_{(-,j)} = \emptyset$ or $I(w)_{(-,j)} = I(w)_{(-,h)}$ and $h = j+1$. Suppose $I(w)_{(-,j)} = I(w)_{(-,h)}$ and $h = j+1$. If $f_{ij} = i - a_j + 1$, then it also must be that $f_{i(j+1)} = i - a_j + 1$ by simple proof by induction. Thus, $t_j = 0$. In either case, $\mu_{w^{-1}(i)}(F) = \emptyset$. So, $\mu_{w^{-1}(i)}(F) = \delta_i(F)$. □

Corollary 4.4.13. *For a permutation $w \in G(n;k)$,*

$$\mu_j(FIF(w)) = \begin{cases} FIF(s_{w_j} w) & \text{if } \ell(w) > \ell(s_{w_j} w) \\ \emptyset & \text{if } \ell(w) < \ell(s_{w_j} w) \end{cases}.$$

From Theorem 4.4.12, the set of all contretableaux of a given shape with maximal entry k can be constructed. In particular, note that there exists a unique flag inversion filling for the permutation in $G(n;k)$ of longest length, $w = n-k+1, n-k+2, \ldots, n, 1, 2 \ldots, n-k$.

Example 4.4.14. Below is the unique flag inversion filling for $w = 456123$ in $G(6;3)$.

·	·	1	1	1
	·	2	2	2
		3	3	3
			·	·
				·

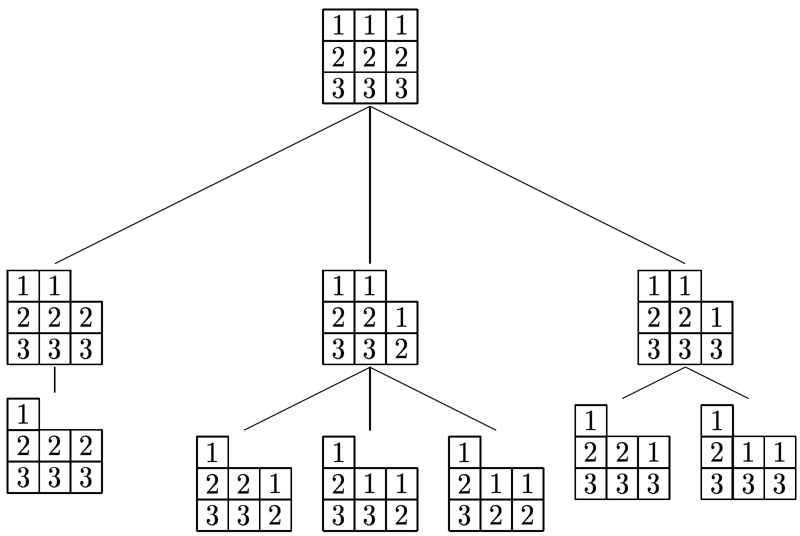
Corollary 4.4.15. *Let F_0 denote the unique flag inversion filling for the permutation of longest length in $G(n;k)$, and let $w \in G(n;k)$. Then*

$$FIF(w) = (\mu_n)^{n-\overline{\lambda_{n-k}}} \cdots (\mu_{k+1})^{n-\overline{\lambda_1}}(F_0),$$

where $(\mu_i)^\ell$ *is the composition of the* μ_i *operator* ℓ *times.*

Example 4.4.16. Suppose $w = 256134 \in G(6;3)$. Then w corresponds to the partition $\lambda = (3,3,1)$.

Below is the tree whose leaves on the bottom level comprise $FIF(w)$. The $\boxed{\cdot}$'s are left out to simplify the tree.



Chapter 5

SKYLINE FILLINGS

In this chapter, we introduce one more known Schubert object, semistandard skyline fillings. In Section 5.2, we give a bijection between the semistandard skyline fillings and $\mathbf{a}$-compatible sequences.

5.1 *Introduction to Standard Bases*

Standard bases were originally introduced by Lascoux and Schützenberger to attempt to link the work of Lakshmibai, Musili, and Seshadri with that of Demazure [12, 32, 42]. Standard bases are defined, like Schubert and key polynomials, in terms of the divided difference operator. Demazure defines

$$\theta_i(f) = x_{i+1}\partial_i(f). \tag{5.1}$$

Given a reduced word for a permutation w, $\mathbf{a} = (a_1, a_2, \ldots, a_p) \in Red(w)$, define

$$\theta_w = \theta_{a_1}\theta_{a_2}\cdots\theta_{a_p}.$$

We have that θ_w is well defined for reduced words because θ_i obeys similar commutation relations to s_i. One can check that

$$\begin{aligned} \theta_i\theta_i &= -\theta_i \\ \theta_i\theta_{i+1}\theta_i &= \theta_{i+1}\theta_i\theta_{i+1} \\ \theta_i\theta_j &= \theta_j\theta_i \text{ for all } |i-j| > 1. \end{aligned}$$

Let λ be a partition and w a permutation. Then define the *standard basis* indexed by λ and w to be

$$\mathfrak{U}(\lambda, w) = \theta_w(x^\lambda).$$

Recall, for a composition β, $\mu(\beta) = \mu$ is the permutation of shortest length such that $(\beta_{\mu_1}, \beta_{\mu_2}, \ldots) = \lambda(\beta)$ is a partition. Surprisingly, we have the following connection to key polynomials.

Theorem 5.1.1. *[42] Given a permutation w and a partition λ,*

$$\mathfrak{U}(\lambda, w) = \sum_{\substack{T \in \mathcal{SSYT}_\lambda \\ K_+(T) = key(w \cdot \lambda)}} x^T$$

where $w \cdot \lambda = (\lambda_{w_1}, \lambda_{w_2}, \ldots)$ is a composition.

We view the following corollary as the bijection T in Figure 1.

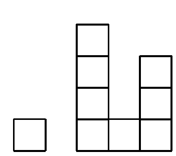
Corollary 5.1.2. *[42] For any composition α,*

$$\kappa_\alpha = \sum_{\beta \trianglelefteq \alpha} \mathfrak{U}(\lambda(\beta), (\mu(\beta))^{-1}),$$

where $\trianglelefteq$ is the partial order on compositions in Definition 1.4.3 and $\lambda(\beta)$ and $\mu(\beta)$ are as in Section 1.4.

We also define a combinatorial rule for standard bases, introduced recently by Mason[49]. To state the rule, we must define another combinatorial object. To a composition α, associate a diagram called a *skyline* with α_i boxes in column i. Fill the boxes of the skyline with positive integers to make a *skyline filling.* Skyline fillings were originally introduced in the context of nonsymmetric MacDonald polynomials [24]. Much of the remainder of this section can be found in [49].

Example 5.1.3. Let $\alpha = (1, 0, 4, 1, 3)$. The skyline associated to α is



The following is a skyline filling of shape $\alpha = (1, 0, 4, 1, 3)$:

		4		
		8		1
		1		5
3		2	2	1

An extra row, called the *basement*, is added to a skyline filling by adding an i underneath column i. Such a filling is called a *skyline augmented filling.*

Example 5.1.4. Below is a skyline augmented filling G of shape $\alpha = (1, 0, 4, 1, 3)$.

		4		
		4		8
		1		5
3		2	4	1
1	2	3	4	5

A *descent* in a skyline augmented filling F occurs when a box in F has a strictly greater entry than the box below it. If no box exists below, there is no descent. Thus, in Example 5.1.4, there is 1 descent in column 1, none in column 2, 1 in column 3, none in column 4, and 2 in column 5. The content of a skyline augmented filling G, $content(G)$, is the composition $\beta = (\beta_1, \beta_2, \ldots)$ such that there are exactly β_i entries of G equal to i not in the basement of G. The content of the skyline augmented filling Example 5.1.4 is $content(G) = (2, 1, 1, 3, 1, 0, 0, 1)$. Recall the definition of a contretableau from Definition 1.2.1.

Algorithm 5.1.5. [49] (Sky Algorithm) To each contretableau, T, associate a unique skyline augmented filling, $sky(T)$, as follows:

1. Conjugate T, so the rows are strictly decreasing and the columns are weakly decreasing from bottom to top. Call this $\overline{T}$. Start with the skyline aumented filling with no boxes except those in the basement. Call this filling F.

2. Beginning with row 1 of $\overline{T}$, going from left to right, insert each entry of $\overline{T}$ into row 1 of F by placing each entry in the first position (from left to right) that does not create a descent.

3. Suppose row 1 through $i-1$ of $\overline{T}$ have already been inserted into F. Insert row i from left to right into F by placing each entry in the first position in row i of F (from left to right) that both does not create descent and maintains that F has a composition shape. Thus, no box is created in row i without a box beneath to support it.

4. The process terminates when all entries of $\overline{T}$ have been placed in F. Define

$$sky(T) := F.$$

This association is illustrated by Figure 5.1. Observe that $sh(sky(T))$ is a rearrangement of $sh(T)$. From the construction, if T is a key, then $content(T) = sh(sky(T))$.

Definition 5.1.6. A skyline augmented filling which is the image of a contretableau under the sky algorithm is a *semi-skyline augmented filling.* The set of all semi-skyline fillings is denoted $\mathcal{SSAF}$. Denote the subset of $\mathcal{SSAF}$ of shape α by $\mathcal{SSAF}_\alpha$.

One can recover column i of T from $sky(T)$ as $sort(row_i(sky(T)))$, where $sort$ rearranges a word into decreasing order and row_i extracts the i^{th} row of the skyline filling. Thus, sky is an injective function and hence a bijection from the set of all contretableau $\mathcal{CT}$ to $\mathcal{SSAF}$ which preserves content.

Theorem 5.1.7. *[49] Given a partition λ and permutation w,*

$$\mathfrak{U}(\lambda, w) = \sum_{F \in \mathcal{SSAF}_{w\cdot\lambda}} x^F,$$

where $x^F = x^{content(F)}$.

From Theorems 5.1.1 and 5.1.7, we have the following corollaries.

Corollary 5.1.8. *[49] Given a contretableau T, $content(K_+(T)) = sh(sky(T))$.*

Corollary 5.1.9. *[49] For a permutation $w \in S_n$,*

$$\mathfrak{S}_w = \sum_{P\in P(w)} \sum_{\beta \trianglelefteq \alpha_P} \sum_{T \in SSAF_\beta} x^T,$$

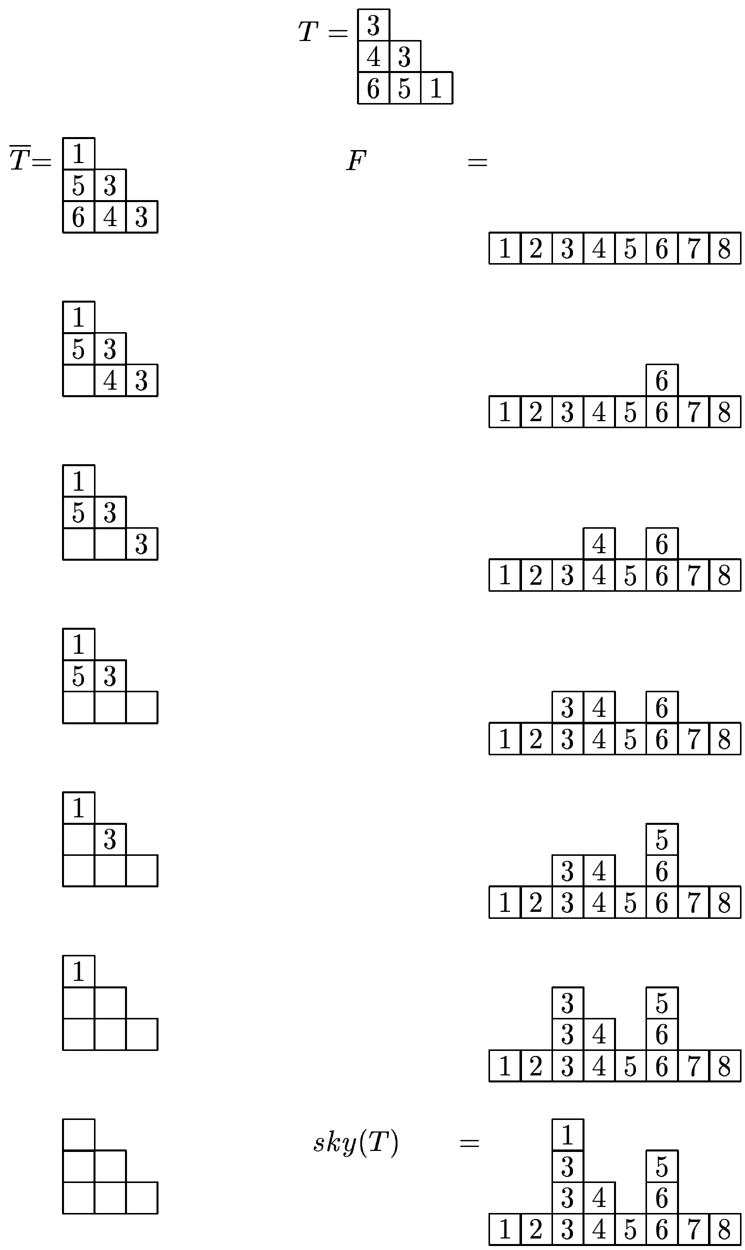


Figure 5.1: Sky algorithm on a contretableau T.

where $\alpha_P = content(K_+(P))$.

The reader should note that $\mathcal{SSAF}$ was defined in [49] in a much different manner. The definition in this paper is completely equivalent, as proved in [49].

5.2 Mapping a-Compatible Sequences to Skyline Fillings

Corollary 5.1.9 illustrates that semistandard augmented fillings are another Schubert object. How are they related to known Schubert objects? In the remainder of this chapter, we give a connection between **a**-compatible sequences and semistandard augmented fillings. The remainder of this chapter is dedicated to proving the following theorem:

Theorem 5.2.1. *There is a bijection,* Υ*, from* $\cup_{\alpha\in I}\{F \in SSAF_\beta | \beta \trianglelefteq \alpha\}$ *to* $\mathcal{C}(w)$ *where* $\mathfrak{S}_w = \sum_{\alpha\in I} \kappa_\alpha$.

Let $(\mathbf{a}, \mathbf{i}) \in \mathcal{C}(w)$. We will associate a pair of tableaux with $(\mathbf{a}, \mathbf{i})$, similar to the pair $(P(\mathbf{a}), Q(\mathbf{a}))$, constructed by the Edelman-Greene Insertion algorithm, Algorithm 1.2.11. For a reduced word $\mathbf{a} = (a_1, \dots, a_p)$, associate a pair of contretableaux, $(R(\mathbf{a}), S(\mathbf{a}))$. Do so by the following algorithm.

Algorithm 5.2.2. (Complementary Edelman-Greene Insertion) To form $R(\mathbf{a})$, start by inserting a_p into the empty contretableau and hence, obtaining a contretableau of shape (1) with entry a_p. Now assume $a_p, a_{p-1}, \dots, a_{q+1}$ have already been inserted and $R(a_{q+1}, \dots, a_p)$ is a tableau. Insert a_q into the tableau as follows:

1. Let $e_1 = a_q$.

2. Given e_i, scan the i^{th} row of $R(a_{q+1}, \dots, a_{p-1}, a_p)$ from left to right and find the first entry (say s) strictly smaller than e_i. If such an entry does not exist, e_i is placed at the end of the i^{th} row and the process terminates.

3. If $s = e_i - 1$ and the entry just to the left of s is e_i, then set $e_{i+1} = e_i - 1$. Otherwise, set $e_{i+1} = s$ and replace s with e_i in the tableau.

4. Repeat Steps (2) and (3) until Step (2) says to terminate. The result will be the tableau $R(a_q, a_{q+1}, \ldots, a_{p-1}, a_p)$.

At each insertion step, one box is added to the Ferrers diagram, $sh(R)$. Build up the recording contretableau, $S(\mathbf{a})$, by placing a q in the box that is added after inserting a_q.

Definition 5.2.3. Let $R(w) := \{R(\mathbf{a}) : \mathbf{a} \in Red(w)\}$.

Example 5.2.4. Suppose $\mathbf{a} = (5, 1, 4, 3, 6, 4)$ and $\alpha_{\mathbf{q}} = (a_q, \ldots, a_6)$ for $1 \leq q \leq 6$.

q	$\alpha_{\mathbf{q}}$	$R(\alpha_{\mathbf{q}})$	$S(\alpha_{\mathbf{q}})$
6	(4)	$\begin{array}{l} 4 \end{array}$	$\begin{array}{l} 6 \end{array}$
5	(6,4)	$\begin{array}{l} 4 \\ 6 \end{array}$	$\begin{array}{l} 5 \\ 6 \end{array}$
4	(3,6,4)	$\begin{array}{ll} 4 & \\ 6 & 3 \end{array}$	$\begin{array}{ll} 5 & \\ 6 & 4 \end{array}$
3	(4,3,6,4)	$\begin{array}{ll} 4 & 3 \\ 6 & 4 \end{array}$	$\begin{array}{ll} 5 & 3 \\ 6 & 4 \end{array}$
2	(1,4,3,6,4)	$\begin{array}{lll} 4 & 3 & \\ 6 & 4 & 1 \end{array}$	$\begin{array}{lll} 5 & 3 & \\ 6 & 4 & 2 \end{array}$
1	(5,1,4,3,6,4)	$\begin{array}{lll} 3 & & \\ 4 & 3 & \\ 6 & 5 & 1 \end{array}$	$\begin{array}{lll} 1 & & \\ 5 & 3 & \\ 6 & 4 & 2 \end{array}$

Recall from Section 1.4 the equivalence relation $\dot\sim$ on words. The rules of Edelman-Greene insertion have been complemented, so, $col(R(\mathbf{a})) \,\dot\sim\, reverse(\mathbf{a})$ and

$$reverse(col(R(\mathbf{a}))) \,\dot\sim\, \mathbf{a} \,\dot\sim\, col(P(\mathbf{a})).$$

The proof is similar to the proof of Lemma 1.2.7. Notice $col(R(\mathbf{a}))$ is the unique word in the equivalence class of $reverse(\mathbf{a})$ under $\dot\sim$ that is the column word of a contretableau. Under the abuse of notation as before, we have that $K_-^0(P(\mathbf{a})) = K_-^0(R(\mathbf{a}))$. Thus, from Theorem 1.2.14, we have the following corollary:

Corollary 5.2.5. *[13] The Reverse Edelman-Greene Insertion Algorithm also has the following properties:*

1. *The map from reduced words to pairs of tableaux, given by mapping* $\mathbf{a} \mapsto (R(\mathbf{a}), S(\mathbf{a}))$, *is injective.*

2. *If* $R \in R(w)$, *then, for every standard contretableau* S *of shape* $sh(R)$, *there exists* $\mathbf{a} \in Red(w)$ *such that* $(R(\mathbf{a}), S(\mathbf{a})) = (R, S)$.

3. *The rows, from left to right, and columns, from bottom to top, of* $R(\mathbf{a})$ *are strictly decreasing. Thus,* $R(\mathbf{a})$ *and* $S(\mathbf{a})$ *are contretableaux.*

4. *The word* $\mathbf{a}$ *has a descent in position* j *(meaning* $a_j > a_{j+1}$*) if and only if* $j+1$ *is southeast of* j *in* $S(\mathbf{a})$.

5. *If* $k > j$ *and if* j *is southeast of* k *in* $S(\mathbf{a})$, *then there must be* $j < \ell < k$ *such that* $a_\ell < a_{\ell+1}$.

6. *We have that* $T \in R(w)$ *if and only if* $\overline{T} \in R(w^{-1})$.

Definition 5.2.6. Given a reduced word $\mathbf{a}$ and a positive, weakly increasing sequence $\mathbf{i}$ of same length as $\mathbf{a}$, define $S(\mathbf{a}, \mathbf{i})$ to be the tableau obtained from $S(\mathbf{a})$ by replacing j in $S(\mathbf{a})$ with i_j.

Lemma 5.2.7. *If* $(\mathbf{a}, \mathbf{i}) \in \mathcal{C}(w)$, *then* $\overline{S(\mathbf{a}, \mathbf{i})}$ *is a contretableau.*

Proof. We have that $\mathbf{i}$ is a weakly increasing sequence and that $S(\mathbf{a})$ is a contretableau. Replacing j with i_j in $S(\mathbf{a}, \mathbf{i})$ guarantees that the rows and columns of $S(\mathbf{a}, \mathbf{i})$ are weakly decreasing. By Corollary 5.2.5, if $k > j$, then j is southeast of k in $S(\mathbf{a})$ if and only if there is an ascent in $\mathbf{a}$ between k and j. This implies that $i_k > i_j$. In particular, if j and k are in the same row, then j is southeast of k, and $i_k > i_j$. Hence, the rows of $S(\mathbf{a}, \mathbf{i})$ are strictly decreasing and $\overline{S(\mathbf{a}, \mathbf{i})}$ is a contretableau. □

Lemma 5.2.8. *If* $(\mathbf{a},\mathbf{i}),(\mathbf{a}',\mathbf{i}')\in\mathcal{C}(w)$ *such that* $R(\mathbf{a})=R(\mathbf{a}')$, *then* $S(\mathbf{a},\mathbf{i})=S(\mathbf{a}',\mathbf{i}')$ *if and only if* $(\mathbf{a},\mathbf{i})=(\mathbf{a}',\mathbf{i}')$.

Proof. Suppose $S(\mathbf{a},\mathbf{i})=S(\mathbf{a}',\mathbf{i}')$. Then $\mathbf{i}=\mathbf{i}'$, because

$$content(\mathbf{i})=content(S(\mathbf{a},\mathbf{i}))=content(S(\mathbf{a}',\mathbf{i}'))=content(\mathbf{i}'),$$

and the content of a weakly increasing sequence is unique. Let k be the largest entry in $S(\mathbf{a})$ that does not agree with the placement in $S(\mathbf{a}')$. That is, k occurs in a different position in $S(\mathbf{a})$ and $S(\mathbf{a}')$ and every entry larger than k agrees in $S(\mathbf{a})$ and $S(\mathbf{a}')$. Say the k in $S(\mathbf{a})$ is in position (r,s) (i.e. the r^{th} column and the s^{th} row), the k in $S(\mathbf{a}')$ is in position (c,d). Without loss of generality, assume (r,s) is southeast of (c,d). Say j is in position (r,s) in $S(\mathbf{a}')$. Then $j<k$ by choice of k, and, because $S(\mathbf{a},\mathbf{i})=S(\mathbf{a}',\mathbf{i}')$, then it must be that $i'_k=i'_j$. So, j is southeast of k in $S(\mathbf{a}')$, and, by Corollary 5.2.5, it must be that there is an ascent of $\mathbf{a}'$ between a'_j and a'_k. So, $i'_k>i'_j$ and $i'_k\neq i'_j$. Thus, no such k exists, so, $\mathbf{a}=\mathbf{a}'$ by Corollary 5.2.5, implying $(\mathbf{a},\mathbf{i})=(\mathbf{a}',\mathbf{i}')$. The other direction is clear.

□

Lemma 5.2.9. *Suppose* $\mathbf{a}\in Red(w)$ *and* $\mathbf{i},\tilde{\mathbf{i}}\in\mathcal{C}(\mathbf{a})$ *such that* $\tilde{\mathbf{i}}$ *is lexicographically largest in* $\mathcal{C}(\mathbf{a})$. *Then* $K_+\left(\overline{S(\mathbf{a},\mathbf{i})}\right)\trianglelefteq K_+\left(\overline{S(\mathbf{a},\tilde{\mathbf{i}})}\right)$, *where* $\trianglelefteq$ *denotes entrywise comparison.*

Proof. First, by Corollary 5.1.8, $sh(sky(\overline{S(\mathbf{a},\mathbf{i})})=content(K_+(\overline{S(\mathbf{a},\mathbf{i})}))$. Suppose $\mathbf{p}\in\mathcal{C}(\mathbf{a})$ is the lexicographically largest compatible sequence such that $K_+(\overline{S(\mathbf{a},\mathbf{p})})=K_+(\overline{S(\mathbf{a},\mathbf{i})})$. One can find such an $\mathbf{p}$ by the following construction. Suppose $\mathbf{a}$ has ascents in positions $r_1<r_2<\ldots<r_m$. For $r_{j-1}<s\leq r_j$, set $p_s=i_{r_j}$. Then, $\overline{S(\mathbf{a},\mathbf{p})}$ is a key. Furthermore, we have that $K_+(\overline{S(\mathbf{a},\mathbf{i})})\trianglelefteq K_+(\overline{S(\mathbf{a},\mathbf{p})})$, because $K_+(\overline{S(\mathbf{a},\mathbf{i})})=K_+(\overline{S(\mathbf{a},\mathbf{p})})$.

We also have that $\overline{S(\mathbf{a},\mathbf{p})}=K_+(\overline{S(\mathbf{a},\mathbf{p})})$ and $\overline{S(\mathbf{a},\tilde{\mathbf{i}})}=K_+(\overline{S(\mathbf{a},\tilde{\mathbf{i}})})$, because $\overline{S(\mathbf{a},\mathbf{p})}$ and $\overline{S(\mathbf{a},\tilde{\mathbf{i}})}$ are keys. So, in the remainder of the proof, we are merely checking that $S(\mathbf{a},\mathbf{p})\trianglelefteq S(\mathbf{a},\tilde{\mathbf{i}})$.

Suppose there exists some entry in $S(\mathbf{a},\mathbf{p})$ that is greater than the entry in the same position in $S(\mathbf{a},\tilde{\mathbf{i}})$. Suppose this is the position with entry j in $S(\mathbf{a})$. Then $p_j>\tilde{i}_j$. Let k be the largest such j. Then $p_k>\tilde{i}_k$, but $p_{k+1}\leq\tilde{i}_{k+1}$. Define $\mathbf{q}$ to be another weakly increasing sequence such that $q_\ell = \tilde{i}_\ell$ for $\ell \neq k$ and $q_k = \tilde{i}_k + 1$. We have that

$$\tilde{i}_{k+1} - \tilde{i}_k > p_{k+1} - p_k$$

implies

$$\tilde{i}_{k+1} - (\tilde{i}_k + 1) \geq p_{k+1} - p_k.$$

So, if $a_k < a_{k+1}$, then

$$q_{k+1} - q_k = \tilde{i}_{k+1} - (\tilde{i}_k + 1) \geq p_{k+1} - p_k \geq 1.$$

Furthermore, $a_k \geq p_k \geq \tilde{i}_k + 1$, so $\mathbf{q}$ is again an $\mathbf{a}$-compatible sequence. But $\mathbf{q}$ is lexicographically larger than $\tilde{\mathbf{i}}$, which contradicts our choice of $\tilde{\mathbf{i}}$. Thus $S(\mathbf{a}, \mathbf{p}) \trianglelefteq S(\mathbf{a}, \tilde{\mathbf{i}})$. □

Lemma 5.2.10. *For* $(\mathbf{a}, \mathbf{i}) \in \mathcal{C}(w)$, $K_+\left(\overline{S(\mathbf{a}, \mathbf{i})}\right) \trianglelefteq K_-^0\left(P(\mathbf{a})\right)$, *where* $\trianglelefteq$ *is entrywise comparison.*

Proof. The proof proceeds as follows. Let $\tilde{\mathbf{i}}$ be the $\mathbf{a}$-compatible sequence that is lexicographically largest in $\mathcal{C}(\mathbf{a})$ (the same $\mathbf{a}$ as in the start of the lemma). By Lemma 5.2.9, $K_+(\overline{S(\mathbf{a}, \mathbf{i})}) \trianglelefteq K_+(\overline{S(\mathbf{a}, \tilde{\mathbf{i}})})$. Then let $(\mathbf{a}', \mathbf{i}')$ be the pair such that $P(\mathbf{a}') = P(\mathbf{a})$ and $\mathbf{i}'$ is the lexicographically largest such compatible sequence. Then we will show $K_+(\overline{S(\mathbf{a}, \tilde{\mathbf{i}})}) \trianglelefteq K_+(\overline{S(\mathbf{a}', \mathbf{i}')})$. Notice that the lexicographically largest monomial in $\kappa_{content(K_-^0(P(\mathbf{a})))}$ is equal to both $x^{content(K_-^0(P(\mathbf{a})))}$ and $x_{i'_1} x_{i'_2} \dots x_{i'_{\ell(\mathbf{a})}} = x^{content(\mathbf{i}')}$, by Theorem 1.4.10. We have that $\overline{S(\mathbf{a}', \mathbf{i}')} = K_+(\overline{S(\mathbf{a}', \mathbf{i}')})$, and so,

$$content(\mathbf{i}') = content(\overline{S(\mathbf{a}', \mathbf{i}')}) = content(K_+(\overline{S(\mathbf{a}', \mathbf{i}')})).$$

The content of keys is unique, therefore this implies that

$$\overline{S(\mathbf{a}', \mathbf{i}')} = K_+(\overline{S(\mathbf{a}', \mathbf{i}')}) = K_-^0(P(\mathbf{a})).$$

We have that $\overline{S(\mathbf{a}, \tilde{\mathbf{i}})}$ and $\overline{S(\mathbf{a}', \mathbf{i}')}$ are both keys themselves, so the proof reduces to showing $\overline{S(\mathbf{a}, \tilde{\mathbf{i}})} \trianglelefteq \overline{S(\mathbf{a}', \mathbf{i}')}$. Suppose this is not the case. Then

$$\tilde{\mathbf{i}} = content(\overline{S(\mathbf{a}, \tilde{\mathbf{i}})} \ntrianglelefteq content(\overline{S(\mathbf{a}', \mathbf{i}')} = \mathbf{i}'$$

in dominance order. Thus, $\mathbf{i}'$ is not lexicographically larger than $\tilde{\mathbf{i}}$, which contradicts the choice of $\mathbf{i}'$, and completes the proof. $\square$

Proof of Theorem 5.2.1. From Lemmas 5.2.8 and 5.2.10, *sky* is an injection from $\mathcal{C}(w)$ (along with conjugation of $S(\mathbf{a}, \mathbf{i})$) into $\cup_{\alpha \in I}\{F \in SSAF_\beta | \beta \trianglelefteq \alpha\}$. By Theorems 1.3.2 and 1.4.9, these two sets have the same cardinality, so, *sky* is a bijection. $\square$

Corollary 5.2.11. *Suppose $P \in P(w)$ and $T \in \mathcal{CT}_{\overline{\lambda}}$, where $\lambda = sh(P)$. Then $\overline{T} = S(\mathbf{a}, \mathbf{i})$ for some $(\mathbf{a}, \mathbf{i}) \in \mathcal{C}(w)$ if and only if $T \trianglelefteq K_+(P)$.*

OPEN QUESTIONS

The subject of Schubert polynomials and Schubert objects leaves many open questions. We present several here which are relevant to this dissertation, but this list is, by no means, complete.

It should be noted that there are other Schubert objects which are not mentioned here. Some, including crystal graphs, were not mentioned because their connections to rc-graphs were already shown in [44]. On the other hand, there are a few that raise open questions. First, Bergeron and Sottile and later, Lenart and Sottile, give a new Schubert object, labeled chains in k-Bruhat order, which give a formula similar to Equation (3.9) [2, 45]. What is the bijection between the chains made by the split operator and the Bergeron-Sottile labeled chains?

There is another object whose connection to current Schubert objects is unknown, namely Kohnert diagrams [30]. A direct proof or a bijection between known Schubert objects would be a wonderful contribution.

Several questions arise in the thesis pertaining to the new Schubert object or construction methods. In Chapter 2, it was conjectured that 21543-avoiding permutations are simply productive. Can we characterize simply productive permutations? How many productive heaps does a given permutation have? Is there an upper bound on the number of productive heaps (other than the trivial answer)? How efficient is this method of finding P-tableaux?

The left divided difference operators on rc-graphs and inversion fillings from Chapters 3 and 4 make an interesting question. Does there exist an operator on polynomials that acts as a left divided difference operator on Schubert polynomials?

Lastly, we give the most famous and studied open problem in the area of Schubert polynomials. As mention in Chapter 1, Schubert polynomials form an integral basis for $\mathbb{Z}[x_1, x_2, \ldots]$. Thus, when multiplying two Schubert polynomials, the product can be expressed as a sum of Schubert polynomials,

$$\mathfrak{S}_u \mathfrak{S}_v = \sum_{w \in S_n} c_{uv}^w \mathfrak{S}_w.$$

From the isomorphism from the cohomology ring of the flag manifold to $\mathbb{Z}[x_1, x_2, \ldots]$, we know that these coefficients, c_{uv}^w, are all nonnegative. In the Grassmannian case, there is a construction on the Schubert objects (semistandard Young tableaux) which proves this nonnegativity, namely the Littlewood-Richardson rule. There are other constructions, such as the Remmel-Whitney rule, the slinky rule on Grassmannians as this dissertation has shown, and jeu de taquin (see [62] for a description). In the case of the flag manifold, it is wide open to find a proof of the nonnegativity based on a construction of some Schubert object. It should be noted that the Littlewood-Richardson rule has been extended in certain cases, such as Monk's rule[51], the Pieri rule[34, 40, 59], Kogan's Schur polynomial times a Schubert polynomial[29], and Coskun's Mondrian tableaux[9, 10]. However, none encompasses every case. One possibility is to try jeu de taquin on the set $S(\mathbf{a}, \mathbf{i})$ for a permutation.

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