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On the Control of Consensus Networks: Theory and Applications

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Abstract

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Signed networks allow the study of positive and negative interactions between agents. In this thesis, three papers are presented that address controllability of networked dynamics. First, controllability of signed consensus networks is approached from a symmetry perspective, for both linear and nonlinear consensus protocols. It is shown that the graph-theoretic property of signed networks known as structural balance renders the consensus protocol uncontrollable when coupled with a certain type of symmetry. Stabilizability and output controllability of signed linear consensus is also examined, as well as a data-driven approach to finding bipartite consensus stemming from structural balance for signed nonlinear consensus. Second, an algorithm is constructed that allows one to grow a network while preserving controllability, and some generalizations of this algorithm are presented. Submodular optimization is used to analyze a second algorithm that adds nodes to a network to maximize the network connectivity.

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I need to thank all of my current roommates in the Red Barn; your names are left out for reasons outlined in Chapter 1, but know that you all have a special place in my heart. I extend that to all of the people I have lived with since moving out and beginning my academic career.

¹I would thank you with all of my heart, but I'm pretty sure my physical heart doesn't appreciate the crude tonnes of caffeine I consume as a result

To all the fellow grad students I shared my time at the University of Washington with, thank you for your companionship. There are about 50 names here, so forgive me for not listing all of you.

I guess here is an appropriate place to thank the bartenders at the Big Time Brewery and the Toronado; thank you for providing me with a local watering hole, and a place in which several ideas for papers were born.

Lastly, but certainly not leastly, I would like to thank my friends Armin Mortazavi and Michael Grudich. You guys are the brothers I never had.

DEDICATION

To my fellow members of the human race; continue being excellent
and stop being not excellent. *Ad astra!*

Chapter 1

INTRODUCTION

1.1 *Why Networked Systems?*

Over the past year, I have lived with some very interesting people. One of my roommates, denoted ‘Bueller’¹, is a graduate from the music program at BYU. He now works in a marketing firm. Having lived in a few cities in the US, he has established a rather impressive social circle. Another of my roommates, whom I will call ‘Mandie’, is a SoCal-born-and-raised chemist. She moved in about 6 months after I did. Bueller, on the other hand, has lived in the house for years, despite obviously being able to afford much nicer housing.

One typical evening², we were discussing Mandie’s romantic life. A few beers in, and Bueller starts talking about this friend of his named Ryan Paul³. Mandie gets this look on her face, a look of amused indiscretion. Bueller picks up on this social cue, getting excited that Mandie has possibly been with this man in an intimate setting.

It turns out that Bueller lived with this guy for a while a few years ago. Mandie dated him in high school. Fifteen years later, Mandie and Bueller decide to rent rooms in the same house, only to find out after six months of cohabitation that they had this common person in their lives who was quite dear to both of them. This is indicative of the phenomenon known as the “six degrees of separation”, which has been explored both in film [1] and in academic settings [2]. This idea is simple. It hypothesizes that if one were to draw a *network* of people, with connections indicating acquaintance or friendship, then everyone in the network would be at most 6 friendships away from each other.

¹These names are manufactured to protect the privacy of my roommates, although they both told me they don’t care

²After a long day of proving theorems, publishing papers and establishing tenure

³Honestly, I have no idea what the guy’s name actually was, only that it was incredibly generic.

Although this hypothesis may not hold in general, the underlying concept is very relevant to the modern age. With the rise of the Internet and social media, very explicit social networks form and elucidate complex interactions between people on a scale not seen before.

Apart from the study of social dynamics, networks form as an intrinsic, underlying structure in many fields of science and engineering. In general, agents in a system may have some interaction that occurs between pairs of agents in that system. Theoretical applications of network science can be found in a broad variety of fields. Tensor networks appear in the study of quantum many body systems, which can be used to characterize exotic structures like matrix product states, and projected entangled pair states [3]. Social networks provide a structure for the study of opinion propagation across populations [4].

In engineering, interacting systems are often modelled via networks. Infrastructure networks provide a perspective on how cities operate [5], allowing for disaster mitigation planning [6]. Energy is the cornerstone of our civilization, and distribution of energy through power networks forms a critical support system [7].

Of particular interest are groups, or swarms, of vehicles. These can be ground vehicles, or unmanned aerial vehicles (UAVs). In order to facilitate grouped behaviour such as formation flight, agents in a swarm form a communication network.

It is clear that the realm of network science is full of diverse applications, and interesting research problems. In the next section, we highlight two broad questions regarding network science in engineering, and how this thesis contributes to solving them.

1.2 Realms of Networked Dynamics: Control and Design

When considering networked dynamical systems in engineering, there are two main concerns that broadly apply in a wide variety of settings. First, *how can one exert control on a networked dynamical system to drive the network to behave in a desired way?* This sort of control can be applied in various ways to different systems. For example, a power system may be controlled by time-dependent consumption and supply of electricity. A formation of UAVs may exchange information that allows the formation to autonomously decide their

heading. A user can direct the formation by injecting a signal to one UAV which then propagates to the rest of the formation, or the UAVs can take in sensor data which drives them to choose a new heading.

The second concern with networked dynamical systems is *how does the connection structure of the network affect how it behaves, and how can one a priori choose a connection structure that facilitates control or a desired behaviour?* Clearly, a lack of connections in a network can be detrimental. Adding on to a previous example, if a formation of UAVs contains two separate groups of UAVs that don't communicate with each other, then information from one group cannot propagate to the other group.

In this thesis, we summarize three papers (listed in § 1.4) that address these questions. First, we address the question of *control*. In particular, we extend results by Rahmani *et al.* [8] and Chapman & Mesbahi [9] that discuss *symmetry* conditions for the controllability of a certain class of networks. In summary, we show that a property of signed networks called *structural balance* when combined with a certain symmetry of the network, produces uncontrollability. We also consider the equivalent problem for nonlinear network dynamics, extending work by Auguilar and Gharesifard [10].

The question of *design* is also considered. We provide an algorithm that grows a large network from a small one that preserves controllability. We use a property known as *submodularity* to analyze a second algorithm that adds nodes to a network to increase the connectivity of that network.

Before beginning the technical content of this thesis, we summarize some necessary mathematics.

1.3 Mathematical Background

In this section, we highlight the basic definitions and notational conventions.

We denote \mathbb{R}_+ and \mathbb{R}_{++} as the sets of nonnegative and positive real numbers, respectively. A column vector with n elements is referred to as $v \in \mathbb{R}^n$ where v_i or $[v]_i$ both represent the i th element in v . We denote a matrix with p rows and q columns as $M \in \mathbb{R}^{p \times q}$. $[M]_{ij}$

denotes the element in the i th row and j th column of M . A square matrix $N \in \mathbb{R}^{n \times n}$ is called *symmetric* if $N^T = N$. For $w \in \mathbb{R}^n$ the vector $\text{diag}(w)$ is an $n \times n$ matrix with w on its diagonal entries. The vector e_i is the column vector with $[e_i]_i = 1$, and $[e_i]_{j \neq i} = 0$. The ones vector is denoted as $\mathbf{1}$, and the identity matrix is denoted $I_n := \text{diag}(\mathbf{1}_n)$. We denote the Moore-Penrose pseudoinverse of a matrix A as A^\dagger .

$|S|$ denotes the *cardinality* of a set S . The *annihilator* of a set S is defined as

$$S^\perp = \{v^* \in \mathbb{R}^n : \langle v, v^* \rangle = 0 \text{ for all } v \in S\},$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in the Euclidean space. The column space of a matrix M is denoted by $\mathcal{R}(M)$. We call $\mathcal{R}(P)$ *A-invariant* if there exists C such that $AP = PC$. We say A is similar to B if there is an invertible matrix R such that $R^{-1}AR = B$. Two similar matrices share the same eigenvalues, namely $\lambda \in \mathbb{C}$ satisfying $Av = \lambda v$. The *leading principal submatrix* of order k of X is the square submatrix of X formed by deleting the last $n - k$ rows and columns, for example:

$$X = \left[\begin{array}{cc|cc} 1 & 3 & 4 & 2 \\ 5 & 1 & 8 & 2 \\ \hline 4 & 3 & 7 & 1 \\ 0 & 8 & 5 & 2 \end{array} \right] \leftrightarrow A_k = \begin{bmatrix} 1 & 3 \\ 5 & 1 \end{bmatrix}.$$

1.3.1 Consensus Dynamics in Signed Networks

This notation is standard, and can be found in Mesbahi & Egerstedt [11].

Multi-agent systems with n agents are characterized by a *graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$. The notation $(\mathcal{V}, \mathcal{E}, W)$ refers to the following sets: $\mathcal{V} = \{v_1, v_2, \dots, v_n\}$ is the set of nodes, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denotes the set of edges, and $W \in \mathbb{R}_+^{n \times n}$ consists of weights assigned to edges. Nodes correspond to agents in the network, and an edge ij corresponds to a connection between agents i and j . The weight W_{ij} denotes the strength of that connection.

We call $A \in \mathbb{R}_+^{n \times n}$ the *adjacency matrix* where $A_{ij} = W_{ij} \neq 0$. The *degree matrix* $D \in \mathbb{R}^{n \times n}$ is a square diagonal matrix where $D_{ii} = \sum_{j \in \mathcal{N}(i)} A_{ij}$. The *graph Laplacian* is then

defined as $L = D - A$.

Then the *consensus dynamics* is defined as

$$\dot{x} = -Lx$$

with $x \in \mathbb{R}^n$ the state vector. A *path* of length r is given by a sequence of distinct nodes $v_{i_0}, v_{i_1}, \dots, v_{i_r}$ such that for $k = 0, 1, \dots, r-1$, we have that $i_k i_{k+1} \in \mathcal{E}$. When $v_0 = v_r$, the path is instead called a *cycle*.

A *signed graph* \mathcal{G}_s is a graph that admits negative weights, $W_{ij} \in \mathbb{R}$. We define the *signed graph Laplacian* as $L_s = D_s - A_s$, where the *signed adjacency matrix* is given by $[A_s]_{ij} = \pm A_{ij}$, and the *signed degree matrix* is given by $[D_s]_{ii} = \sum_{j \in \mathcal{N}(i)} |A_{ij}| = \sum_{j \in \mathcal{N}(i)} |W_{ij}|$. The generalization of consensus dynamics to signed graphs was introduced in [12], and the corresponding dynamical system is given by $\dot{x} = -L_s x$, i.e.

$$\dot{x}_i = - \sum_{j \in \mathcal{N}(i)} |W_{ij}| (x_i - \text{sgn}(W_{ij})x_j),$$

where sgn represents the sign function. A *gauge transformation* is a change of orthant order via a matrix $G_t \in \{\text{diag}(\sigma) : \sigma = [\sigma_1, \dots, \sigma_n], \sigma_i = \pm 1\}$. We know $G_t = G_t^T = G_t^{-1}$. Let the *gauge-transformed Laplacian* be given by $L_{G_t} = G_t L_s G_t = D - G_t A_s G_t$ where

$$(L_{G_t})_{ij} = \begin{cases} \sum_{k \in \mathcal{N}(i)} |A_{ik}| & j = i \\ -\sigma_i \sigma_j A_{ij} & j \neq i. \end{cases}$$

1.3.2 Automorphisms, Interlacing and Equitable Partitions

An *automorphism* of the graph \mathcal{G} is a permutation $\phi : \mathcal{V} \rightarrow \mathcal{V}$ of its nodes such that $\phi(i)\phi(j) \in \mathcal{E}$ if and only if $ij \in \mathcal{E}$. The permutation ϕ induces a mapping $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $[\varphi(x)]_i = x_{\phi(i)}$. Let the permutation matrix J be such that $[J]_{ij} = 1$ if $\phi(i) = j$ and zero otherwise. Therefore, the permutation matrix J is simply the Jacobian matrix of φ , in that $J = D\varphi$. Thus, ϕ represents the automorphism of \mathcal{G} if and only if $JA(\mathcal{G}) = A(\mathcal{G})J$ (see [11]).

In a similar way that we defined the functions ϕ and φ for the graph automorphism, consider the function $\mathbf{g} : \mathcal{V} \rightarrow \mathcal{V}$ encoding the action of the gauge on the nodes⁴. This induces a function $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $[g(x)]_i = \sigma_i x_i$. The gauge transformation is then the Jacobian of this function, in that $G_t = Dg$.

Suppose $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{m \times m}$ are both symmetric and $m \leq n$. Then the eigenvalues of B *interlace* the eigenvalues of A if for $i = 1, 2, \dots, m$, $\lambda_{n-m+i}(A) \leq \lambda_i(B) \leq \lambda_i(A)$ where $\lambda_1(A) \geq \lambda_2(A) \geq \dots \geq \lambda_n(A)$ are the eigenvalues of A in a non-increasing order [13].

The *cell* C is a subset of the graph nodes \mathcal{V} . A *nontrivial cell* is a cell with more than one node. A *partition* is a grouping of \mathcal{V} into different cells. An *r-partition* π of \mathcal{V} with cells $\{C_i\}_{i=1}^r$ is *equitable* if each node in C_j has the same number of neighbors in C_i , for all i, j . We call π a *nontrivial equitable partition* (NEP) if it contains at least one nontrivial cell. Let b_{ij} be the number of neighbors in C_j of a node in C_i . The *quotient* of \mathcal{G} over π , denoted by \mathcal{G}/π , is the directed graph with the cells of an equitable *r-partition* π as its nodes and b_{ij} edges directed from C_i to C_j . The adjacency matrix of the quotient is specified by $[A(\mathcal{G}/\pi)]_{ij} = b_{ij}$. A *characteristic vector* $p_i \in \mathbb{R}^n$ of a nontrivial cell C_i has 1's in components associated with C_i and 0's elsewhere. A *characteristic matrix* $P \in \mathbb{R}^{n \times r}$ of a partition π of \mathcal{V} is defined as $[p_i]_{i=1}^r$.

For example, for the equitable partition in figure 2.6 we get

$$A(\mathcal{G}/\pi) = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 0 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

⁴This definition is somewhat superfluous, and is included for completeness

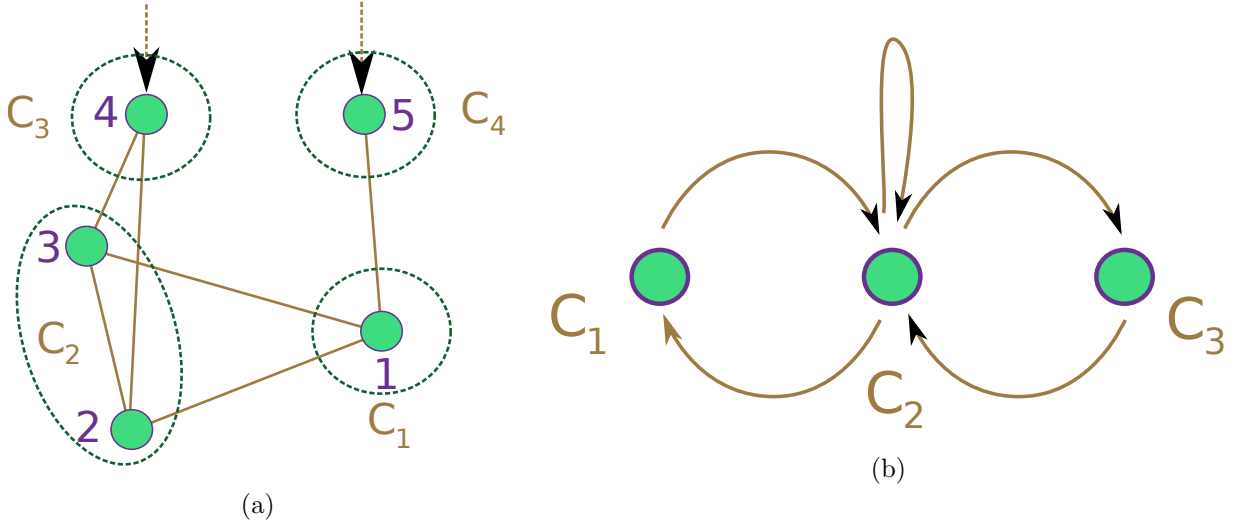


Figure 1.1: (a) Equitable partition of a sample graph, (b) quotient of the graph in (a)

Let $m \in \mathbb{Z}_+$, and $[m] := \{1, \dots, m\}$. We call a real-valued function $f : 2^{[m]} \rightarrow \mathbb{R}$ *nondecreasing* if for sets $J \subset K \subset [m]$, $f(J) \leq f(K)$. The function f is *submodular* if for subsets $J, K \subset [m]$, we have that $f(K) + f(J) \geq f(K \cup J) + f(K \cap J)$. Furthermore, f is *nonincreasing* if $-f$ is nondecreasing, and f is *supermodular* if $-f$ is submodular. f is *modular* if it is both supermodular and submodular.

A matrix H is *Hermitian* if $H = H^\dagger$, where ‘ \dagger ’ denotes the conjugate-transpose operation. Let Λ_E be a finite interval of \mathbb{R} , and thereby denote the set of $n \times n$ Hermitian matrices with eigenvalues contained in Λ_E as $H_n(\Lambda_E)$. We say that f is *operator monotone* on $H(\Lambda_E)$ if for all $n \geq 1$ and for all $A, B \in H_n(\Lambda_E)$, $A \succeq B \implies f(A) \succeq f(B)$. Lastly, let $A[K]$ denote the principal submatrix of A obtained by deleting the rows and columns of A corresponding to the elements in the set $[m] \setminus K$.

The function h is *even* if $h(-x) = h(x)$ and is *odd* if $h(-x) = -h(x)$. The function f is of class C^r if the derivatives $f, f', \dots, f^{(r)}$ exist and are continuous. The function $g \in C^\infty$, otherwise called *smooth*, has derivatives of all order. Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a vector field and let $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a smooth mapping. Then, F is φ -*invariant* if $(D\varphi(x))F(x) = F(\varphi(x))$

for all $x \in \mathbb{R}^n$ with $D\varphi(x)$ the Jacobian matrix of φ at x . Given a mapping $\gamma : \mathcal{M} \rightarrow \mathcal{M}$, the fixed point set of γ is denoted by $\text{Fix}(\gamma) = \{x \in \mathcal{M} | \gamma(x) = x\}$.

1.3.3 Nonlinear Dynamical Systems

We follow the same conventions as in [10]. Consider the controlled dynamical system

$$\dot{x} = F(x, u) \tag{1.1}$$

where $F : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a smooth mapping. The *accessible set* $\mathcal{A}(x_0, T)$ of the system (2.1) from x_0 at time T is the set of all end-points $\theta(T)$ where $\theta : [0, T] \rightarrow \mathbb{R}^n$ is a trajectory of (2.1). The accessible set of (2.1) from x_0 up to T is defined as $\mathcal{A}(x_0, \leq T) := \cup_{0 \leq \tau \leq T} \mathcal{A}(x_0, \tau)$. Then, the nonlinear dynamical system (2.1) is said to be *accessible* from the initial point x_0 if for every $T > 0$ the set $\mathcal{A}(x_0, \leq T)$ contains a non-empty interior.

1.3.4 Koopman Operator

Consider the dynamical system in (2.1) without control. The *Koopman operator* \mathcal{K} acts on functions of the state space (called *observables*) ψ by the action $\mathcal{K}\psi = \psi \circ F$. The function $\varphi(x)$ is an *eigenfunction* of \mathcal{K} corresponding to the *eigenvalue* μ if $\mathcal{K}\varphi(x) = e^{\mu t}\varphi(x)$. For an observable function g in the span of Koopman eigenfunctions, one can write $g(x) = \sum_{k=1}^{\infty} v_k \varphi_k(x)$, where the v_k 's are the *Koopman modes*. For the case of full-state observable $g(x) = x$ the states can be reconstructed as, $x(t) = g(x(t)) = \sum_{k=1}^{\infty} e^{\mu_k t} \varphi_k(x_0) v_k$, where we refer to $\{\mu_k, \varphi_k, v_k\}$ as the *Koopman triple*.

The Koopman operator is linear, but can be infinite-dimensional. To make this operator computationally usable, a numerical approximation may be obtained by Extended Dynamic Mode Decomposition (EDMD), resulting in a finite-dimensional approximation K [14].

1.4 Statement of Contributions and Included Works

The content of this thesis comes from three publications. S. Alemzadeh has given explicit permission that joint works with him may be included in this thesis. Explicit authorship

statements are listed after each paper.

§1.3 was taken from all three of the papers below. Chapter 2 is comprised of the following two papers:

Paper 1: Reprinted with permission: S. Alemzadeh, **M. Hudoba de Badyn** & M. Mesbahi. “Controllability and Stabilizability Analysis of Signed Consensus Networks”, *to be published in the Proc. IEEE Conference on Control Technology and Applications*, Kohala Coast, USA, 2017. © IEEE

This paper dicusses the graph-theoretic property called *structural balance* of signed networks. In particular, we extend the symmetry arguments from Rahmani *et al.* to derive uncontrollability conditions for signed networks [8] . We show that structural balance is the necessary property of consensus networks to induce uncontrollability when an input symmetry is present. An examination of stabilizability and output controllability for influenced consensus is included. S. Alemzadeh and M. Hudoba de Badyn both worked together to derive the single-input-single-output case, and shared the task of writing the paper. S. Alemzadeh worked on the multi-input-multi-output case, and M. Hudoba de Badyn performed the stabilizability and output controllability analysis. All work performed was the result of discussions with M. Mesbahi.

Paper 2: **M. Hudoba de Badyn**, S. Alemzadeh & M. Mesbahi. “Controllability and Data-Driven Identification of Bipartite Consensus on Nonlinear Signed Networks”, preprint.

This paper extends the controllability analysis from the preceding paper to nonlinear processes on networks, following the work by Aguilar and Gharesifard [10]. We show that for three classes of nonlinear network flows, structural balance and an input symmetry imply the network flows are not accessible from the origin. A data-driven method utilizing the Koopman operator was used to show that taking data from the system allows one to obtain information about the bipartition of agents in a structurally balanced system. S. Alemzadeh and M. Hudoba de Badyn worked together to derive the accessibility conditions, the data-driven method of examing bipartite consensus, and shared the task of writing the paper. All work performed was the result of discussions with M. Mesbahi.

Chapter 3 is comprised of the following paper:

Paper 3: Reprinted with permission: **M. Hudoba de Badyn** & M. Mesbahi, “Growing Controllable Networks via Whiskering and Submodular Optimization” in *Proc. 55th IEEE Conference on Decision and Control*, Las Vegas, USA, 2016, pp. 867-872. © IEEE

This paper considers the problem of designing networks that are controllable. We discuss an algorithm that iteratively attaches a cluster of nodes to each node in the network, preserving controllability at each iteration. Submodular optimization is then utilized to add nodes to a graph to maximize the second-smallest eigenvalue of the graph Laplacian, reducing the problem of adding nodes to a well-known algorithm by Ghosh and Boyd [15]. M. Hudoba de Badyn performed the work in this paper, following discussions with M. Mesbahi.

Chapter 2

CONTROL OF NETWORKS

2.1 Introduction

In this chapter, we discuss three results pertaining to the control of consensus networks. First, we identify a graph-theoretic property of signed networks called *structural balance*, which we then show that when combined with symmetry, produces uncontrollability of leader-follower consensus networks. We extend this result to *influenced consensus* using the signed-graph extension of signed fractional automorphisms.

Next, we provide the equivalent controllability result for signed nonlinear consensus protocols. In particular, we show for certain classes of nonlinear protocols that structural balance and a symmetry in the graph about a leader node causes the dynamics to be not accessible from the origin.

Lastly, we discuss a data-driven approach using the Koopman operator. In particular, we show that using data collected from the nonlinear network dynamics on a structurally balanced graph, one can infer a bipartite grouping of agents in the network into two groups that have positive interactions among each other, but only negative interactions exist between the two groups

This chapter is organized as follows. In § 3.2, we provide a literature review. We discuss the controllability of linear signed networks in § 2.3, and nonlinear signed networks in § 2.4. The data-driven identification of bipartite consensus is done in § 2.4.3. The chapter is concluded in § 2.5.

2.2 Literature Review

Networked dynamical systems have been studied in many different contexts over the past several decades. For example, the *Hegselmann-Krause dynamics* are used to study opinion sharing in the context of social networks [4]. These dynamics are of interest for two reasons. First, the dynamics themselves are difficult to analyze, and have thus been the source of some key theoretical developments in consensus protocols [16, 17, 18]. Second, the dynamics exhibit an emergent phenomenon known as *clustering*, whereby the agents in the network self-assemble into distinct groups of a single shared opinion.

A number of networks in both natural [19] and synthetic [20] systems exhibit *flocking* behaviour, where simple interactions create large-scale emergent and coherent structures in the network. For example, flocking is observed in fish [21, 22], birds [23], both Jackass and Adélie penguins [24, 25], parrots [26] and bats [27]. Early work by Reynolds in computer graphics defined flocking to be behaviour of a multi-agent network that follows three rules [28]:

1. **Flock Centering:** Agents in the flock try to stay close to their neighbours
2. **Collision Avoidance:** Agents try to avoid collisions with their neighbours
3. **Velocity Matching:** Agents try to match velocities with their neighbours

The *Vicsek model* is one such model of self-driving particles that average their headings with their neighbours, creating a large-scale flock [29]. Convergence of this model on a torus was proven by Tahbaz-Salehi and Jadbabaie [30] using stability results from Moreau [31].

Many other examples networked dynamics are similar to, or use flocking-like protocols directly. Olfati-Saber discussed applications of flocking to synthetic systems, such as robotics [32]. Stone & Veloso discussed machine learning applications to autonomous robotics [33], and Fabrizio *et al.* discussed autonomous formation flight [34]. Acín *et al.* discussed entanglement percolation in quantum networks [35]. Yang & Yagar studied traffic

control [36], and Yeung *et al.* looked at gene networks [37]. Several networked systems include both cooperative and antagonistic interactions, such as certain classes of social dynamics [38].

Consensus algorithms have been used in many scientific and engineering applications, including multi-agent systems [39, 40], robotics [41] and Kalman filtering [42]. A large amount of research has been dedicated to looking at the control of consensus [11]. The work by Rahmani *et al.* showed that certain symmetries of networks characterized by automorphisms of the topology of the network cause uncontrollability [8]. This was generalized by Chapman and Mesbahi who showed signed fractional automorphisms generate necessary and sufficient conditions for uncontrollability and unstabilizability of linear systems [9]. Further work examined methods of generating network topologies that are controllable for consensus, such as in [43] and [44].

Consensus algorithms on networks with antagonistic interactions were first considered by Altafini [45, 12]. The network property of *structural balance*, first considered in the study of social networks ([38, 46, 47]) was identified in Altafini's work as the property inducing bipartite consensus in which the agents converge to two disjoint clusters instead of a uniform consensus. Graph-theoretic properties of signed Laplacian dynamics were studied by Pan *et al.* [48]. Further research by Pan *et al.* has looked at identifying the bipartite structure of structurally balanced graphs using data from signed Laplacian dynamics and dynamic mode decomposition [49], adding to the works done by Harary and Kabell [50] and Facchetti *et al.* [51]. Recent contributions by Clark *et al.* have studied the leader selection problem in signed consensus [52].

The generalization to nonlinear consensus algorithms has been studied in numerous settings. Behaviour of nonlinear consensus protocols was considered by Srivastava *et al.* [53]. The extension of these consensus protocols to signed networks was studied by Altafini [45]. Moreover, the generalization of symmetry arguments for controllability was examined by Aguilar and Gharesifard [10].

There has been a recent interest in applying data-driven methods to control of networks,

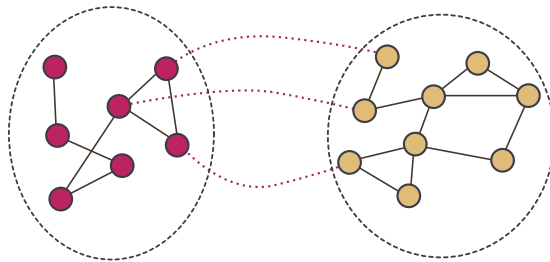


Figure 2.1: Example of network with antagonistic interactions between two groups of agents.

for example the Koopman operator. The Koopman operator is a dynamical framework in which one considers the propagation of observables of the state, rather than the state itself. The Koopman operator is linear, even for a non-linear system, but the trade-off is that the vector space of observables is generally infinite dimensional [54]. This formalism lends itself well to a data-driven approach, allowing one to approximate the Koopman operator by collecting data [14]. Research by Pan *et al.* has looked at identifying the bipartite structure of signed networks using data-driven methods [49], furthering work done by Facchetti *et al.* [51], and Harary and Kabell [50].

2.3 Signed Linear Networks

The consensus dynamics have been studied under the assumption that the underlying interactions have positive edge weights. Many networks in practice exhibit antagonistic interactions, for example social networks [38]. Such antagonistic interactions can be considered as negative edge weights in a graph model.

2.3.1 Signed Linear Consensus

The contributions of this section are as follows. We conduct a controllability analysis of signed Laplacian consensus using symmetry arguments developed by Rahmani *et al.* [8], for the individual SISO and MIMO cases of consensus dynamics with leader nodes. In particular, we identify the property of structural balance that when combined with symmetry causes

uncontrollability of signed consensus dynamics. The key feature of structurally balanced graphs that allows this analysis is that they admit a *gauge transformation* that allows the permutation matrix corresponding to the graph symmetry to be extended to a signed permutation matrix, which we show leads to uncontrollability when corresponding to a symmetry about input nodes. We then use tools developed by Chapman and Mesbahi in [9] to derive controllability and stabilizability conditions for influenced signed consensus dynamics.

2.3.2 Problem Statement

The analysis of this section consists of two main parts, which examine two different notions of influencing control on networks. In the first part, we examine the notion of uncontrollability due to symmetry and interlacing which was initially derived in [8] for signed consensus networks. The notion of control in this case is taking over the state of one or several nodes in the network, and using their edges to inject signals into the system. In the second part, we consider the case where the nodes are controlled by injecting a single-integrator signal to some nodes of the graph.

Signed consensus networks are of interest because the negative weights induce a phenomenon known as *clustering*, where agents will not converge to an agreement subspace, but rather converge to equal and opposite equilibria. The condition that causes clustering was identified by [45] as *structural balance*. We show that this topological feature surprisingly is exactly the condition that produces uncontrollability of signed networks.

Lemma 1. (See [45]) *The following statements are equivalent:*

1. *The signed graph \mathcal{G} is structurally balanced;*
2. *There exists a gauge G_t such that $G_t A_s G_t$ has only positive entries (i.e. the graph becomes unsigned);*
3. *All cycles in \mathcal{G} are positive;*

4. *The signed Laplacian L_s has a zero eigenvalue;*
5. *There exists a bipartition of \mathcal{V} such that the edge weights on the edges within the same set are positive, and the edges connecting the two sets are negative.*

One may recall that the unsigned consensus dynamics have an agreement subspace spanned by the eigenvector $\mathbf{1}$ corresponding to the zero eigenvalue. For signed consensus, the zero eigenvalue explicitly corresponds to disagreement. Then one may think that structural balance is therefore not a desirable quality for signed consensus, but it turns out that if the graph is not structurally balanced, the consensus dynamics become trivial in some sense, and converge to zero.

Theorem 1. *(See [45]) If L_s is structurally unbalanced in a signed graph \mathcal{G}_s , then*

$$\lim_{t \rightarrow \infty} x(t) = 0$$

where $x \in \mathbb{R}^n$ is the state vector with the corresponding dynamics $\dot{x} = -L_s x$. Otherwise, if L_s is structurally balanced, then $\lim_{t \rightarrow \infty} x(t) = (1/n) (\mathbf{1}^T G_t x(0)) G_t \mathbf{1}$.

Previous work by Rahmani *et al.* in [8] showed that symmetry with respect to a single input and interlacing for multiple input are sufficient for uncontrollability, and this was generalized by Chapman and Mesbahi in [9] to show that fractional symmetry with respect to the inputs is sufficient and necessary for uncontrollability.

In this section, we aim to show that structural balance is the property that combined with symmetry and interlacing leads to uncontrollability. There are examples of signed networks that are input symmetric but controllable, and we will discuss these in Section 2.3.5. Before proceeding to our main results, we summarize the various dynamics considered in this section.

There are several variants of consensus dynamics with respect to how inputs are injected into nodes. Given a connected signed graph \mathcal{G}_s , we can select one node and use that specific node to inject our input signal u . This corresponds to partitioning the Laplacian as follows

(see, for example, [11]):

$$L_s = \left[\begin{array}{c|c} A_s^f & B_s^f \\ \hline B_s^{fT} & A_s^i \end{array} \right],$$

where f and i denote the *floating* and *input* parts of the network respectively. Then, the dynamical system for signed consensus networks is

$$\dot{x} = -A_s^f x - B_s^f u, \quad (2.1)$$

where for a single-input-single-output (SISO) system we have $A_s^f \in \mathbb{R}^{(n-1) \times (n-1)}$, $B_s^f \in \mathbb{R}^{(n-1) \times 1}$, and A_s^i is a scalar. Similarly, one can define the same dynamics as (2.1) for the multiple-input-multiple-output (MIMO) system. In that case, $A_s^f \in \mathbb{R}^{n_f \times n_f}$ and $B_s^f \in \mathbb{R}^{n_f \times n_i}$ where n_f and n_i are the numbers of the nodes in floating and input subgraphs respectively. The *floating signed graph* is denoted by \mathcal{G}_s^f .

Lemma 2. [*Popov-Belevitch-Hautus (PBH) Test*] *The system described in (2.1) is controllable if and only if none of the eigenvectors of A_s^f are simultaneously orthogonal to all columns of B_s^f .*

We use lemma 2 further to discuss controllability of the SISO case in Section 2.3.3, and the MIMO case in Section 2.3.3.

The second variant of controlling consensus networks is to simply inject signals into nodes, without taking over the state of the node. We can therefore define the *influenced signed consensus dynamics* with q inputs and p outputs as

$$\dot{x} = -L_s x + B(I)u, \quad y = C(O)x, \quad (2.2)$$

where $I \subseteq N$ is the set of input nodes, $O \subseteq N$ the set of output nodes, and $B(I), C(O)$ are the matrices

$$B(I) = \begin{bmatrix} | & & | \\ e_{i_1} & \cdots & e_{i_q} \\ | & & | \end{bmatrix}, \quad C(O) = \begin{bmatrix} | & & | \\ f_{j_1} & \cdots & f_{j_p} \\ | & & | \end{bmatrix}^T,$$

in which e_i is the indicator vector for nodes $i \in I$, and f_j for nodes $j \in O$. The control signal vector is $u \in \mathbb{R}^q$. Both (output) controllability and (output) stabilizability of these dynamics is considered in Section 2.3.4.

2.3.3 Signed Graph Controllability

The SISO Case

A standard result by [8] shows that for a SISO consensus network, a symmetry about the input node is sufficient for uncontrollability. We extend this result for the signed consensus networks considered by [45] and [49]. In particular, we show that structural balance and input symmetry for the unsigned graph is sufficient for uncontrollability.

Remark 1. *This result shows signed Laplacian is in some sense more robust to symmetries about the input nodes. In particular, there are examples of unsigned graphs that are symmetric about an input and therefore uncontrollable, but whose signed counterparts exhibit controllability despite the symmetry. Therefore, there is not enough conditions to claim the uncontrollability of the system.*

Now, we state and prove a lemma which helps us provide the main result of this section.

Lemma 3. *Assume the unsigned graph \mathcal{G} enjoys input symmetry and the signed network \mathcal{G}_s is structurally balanced. Then, the following statements hold*

1. *There exists J' such that $J'A_s^f = A_s^f J'$*
2. *For the same J' as part 1, $J'^T B_s^f = B_s^f$*
3. *If v is the eigenvector corresponding to the eigenvalue λ of A_s^f . Then, $J'v$ (and hence $v - J'v$) would also be the eigenvectors corresponding to λ*

Proof. From the second statement of lemma 12 there exists G_t such that

$$G_t L_s G_t = L, \quad (2.3)$$

and hence

$$\begin{aligned} A_s^f &= G' A^f G' \\ B_s^f &= \sigma_n G' B^f. \end{aligned}$$

It follows

$$\begin{aligned} G_t \left[\begin{array}{c|c} A_s^f & B_s^f \\ \hline B_s^{fT} & A_s^i \end{array} \right] G_t &= \left[\begin{array}{c|c} G' A_s^f G' & G' B_s^f \sigma_n \\ \hline \sigma_n B_s^{fT} G' & \sigma_n A_s^i \sigma_n \end{array} \right] \\ &= \left[\begin{array}{c|c} A^f & B^f \\ \hline B^{fT} & A^i \end{array} \right], \end{aligned}$$

where $G' = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_{n-1})$. Without loss of generality, we can assume $\sigma_n = 1$ since (2.8) also holds for $-G$. This gives

$$B^f = G' B_s^f.$$

From [8] we know that if \mathcal{G} has the input symmetry structure, then there exists the permutation matrix J such that

$$J A^f = A^f J.$$

Let $J' = G' J G'$,

$$\begin{aligned} J' A_s^f &= G' J G' G' A^f G' = G' J A^f G' \\ &= G' A^f J G' = G' A^f G' G' J G' = A_s^f J', \end{aligned}$$

which proves part (1).

We also have

$$\begin{aligned} J'^T B_s^f &= J'^T G' B^f = G' J'^T G' G' B^f = G' J'^T B^f \\ &= -G' J'^T A^f \mathbf{1} = -G' A^f J'^T \mathbf{1} = B_s^f, \end{aligned}$$

which gives the result of part (2).

For the last part of the proof by definition we know $A_s^f v = \lambda v$. Then

$$A_s^f J' v = J' A_s^f v = \lambda J' v,$$

implying that $J' v$ is also an eigenvector for the same eigenvalue. Hence, assuming orthonormal eigenvectors for A_s^f , then $v - J' v$ would also be an eigenvector corresponding to λ . \square

The matrix J' introduced for a signed floating graph in lemma 2 is in some sense correspondent to the permutation matrix J in the unsigned case. In fact, the only difference between J and J' are the elements with the same rows or columns as the negative elements in G' . For example, for the graph of Figure 2.2, with node 5 as the input node, the matrices J , J' , and G' are found to be

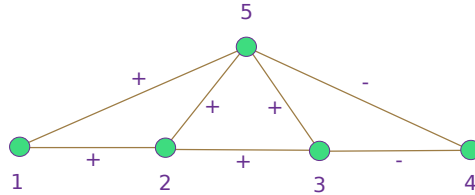


Figure 2.2: A network topology with input symmetry

$$J = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad G' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad J' = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, the first two parts of lemma 3 demonstrate the correlation between the signed and unsigned consensus networks using the gauge transformation G_t .

Now we have all the tools we need to provide the main theorem of this part.

Theorem 2. *The signed network system \mathcal{G}_s is uncontrollable if it is input symmetric and structurally balanced*

Proof. We use the results of lemma 3 and the PBH test to show the uncontrollability of \mathcal{G}_s .

Let (λ, v) be a pair of eigenvalue and eigenvector for A_s^f so that $A_s^f v = \lambda v$. Then, from part 3 of lemma 3 we know that $v - J'v$ is also and eigenvector for A_s^f . Then, from part 2 of lemma 3 we have

$$(v - J'v)^T B_s^f = v^T B_s^f - v^T J'^T B_s^f = v^T B_s^f - v^T B_s^f = 0,$$

which implies that the system is not controllable according to the PBH test (Lemma 2). \square

Remark 2. *A signed symmetry implies the existence an unsigned symmetry of \mathcal{G} . The converse is true when \mathcal{G} is structurally balanced.*

The MIMO Case

In this section, we examine how the notion of structural balance is interposed in the controllability analysis of multiple input signed networks. The results in this section are extensions to [8]. To this end, we leverage the machinery of interlacing and equitable partitions on graphs.

First, we restate and modify two fundamental lemmas from [13] to the signed case and then provide the analysis which leads to sufficient conditions on the uncontrollability of the system.

Definition 1. *Let G_t be the gauge transformation as in (2.8). Then, P' is the signed characteristic matrix defined as $P' = G_t P$.*

Lemma 4. *Let π be a partition of the structurally balanced signed graph \mathcal{G}_s , with adjacency matrix A_s and signed characteristic matrix P' . Then, π is equitable if and only if the column space of P' is A_s -invariant.*

Proof. (Necessity) assume π is equitable. From lemma 9.3.1 in [13] if π is equitable then $AP = P\hat{A}$ with $\hat{A} = A(\mathcal{G}_s/\pi)$. Then, it follows from the second statement in lemma 12

$$P\hat{A} = AP = G_t A_s G_t P \quad \Rightarrow \quad A_s P' = P' \hat{A}.$$

(Sufficiency) From lemma 9.3.2 in [13], π is equitable if there exists B such that $AP = PB$. Then, if such B exists, we get

$$PB = AP = G_t A_s G_t P \quad \Rightarrow \quad A_s P' = P' B.$$

□

Lemma 4 shows how the gauge transformation is injected into the analysis of signed networks. Indeed, based on the knowledge of the second statement in lemma 12, we use the structural balance to turn the signed system into the extensively developed un-signed consensus dynamics.

The next lemmas are provided for the sake of completeness.

Lemma 5. (See [8]) *Given a symmetric matrix $A \in \mathbb{R}^{n \times n}$, let S be a subspace of \mathbb{R}^n . Then, S^\perp is A -invariant if and only if S is A -invariant.*

Lemma 6. (See [13] Theorem 9.5.1) *Let $\Phi \in \mathbb{R}^{n \times n}$ be a real symmetric matrix, and let $R \in \mathbb{R}^{n \times m}$ be such that $R^T R = I_m$. Set $\Theta = R^T \Phi R$ and let $\nu_1, \nu_2, \dots, \nu_m$ be an orthogonal set of eigenvectors for Θ such that $\Theta \nu_i = \lambda_i(\Theta) \nu_i$, where $\lambda_i(\Theta) \in \mathbb{R}$ is an eigenvalue of Θ . Then, $\Phi R = R\Theta$ if the interlacing between Φ and Θ is tight.*

Remark 3. *As discussed in [8], we can now find an orthogonal decomposition of \mathbb{R}^n using the signed characteristic matrix P' as*

$$\mathbb{R}^n = \mathcal{R}(P') \oplus \mathcal{R}(Q'),$$

where $\mathcal{R}(Q') = \mathcal{R}(P')^\perp$. Then, an orthonormal basis for \mathbb{R}^n can be formed as

$$T = [\bar{P}' \mid \bar{Q}'], \tag{2.4}$$

where \bar{P}' and \bar{Q}' represent normalized P' and Q' respectively and satisfy $\bar{P}'^T \bar{Q}' = 0$ and $\bar{Q}'^T \bar{Q}' = I_{n-r}$.

Lemma 7. *Given a connected signed graph \mathcal{G}_s , the system (2.1) is uncontrollable if and only if L_s and A_s^f share at least one common eigenvalue.*

Lemma 7 is a derivation from lemma 7.9 in [8]. Since this is a general result depending on the Laplacian and its leading principal submatrix (floating graph), and the PBH test, the same holds for the signed case.

From this point, the goal is to show that for some specific graph partition and the structural balance of the network, L_s and A_s^f share similar eigenvalues which leads to uncontrollability. The next two lemmas assert that L_s and A_s^f are similar to some block diagonal matrices.

Lemma 8. *Suppose a structurally balanced signed graph \mathcal{G}_s has an NEP π with \bar{P}' and \bar{Q}' as in (2.9). Then, the signed Laplacian L_s is similar to the block diagonal matrix*

$$\bar{L}_s = \begin{bmatrix} L_{P'} & \mathbf{0} \\ \mathbf{0} & L_{Q'} \end{bmatrix},$$

where $L_{P'} = \bar{P}'^T L_s \bar{P}'$ and $L_{Q'} = \bar{Q}'^T L_s \bar{Q}'$.

Lemma 9. *Let \mathcal{G}_s^f be a signed floating graph, and A_s^f be defined as in (2.1) and \bar{P}' and \bar{Q}' be as in (2.9). If there exists an NEP π_f in \mathcal{G}_s^f and a π in the original structurally balanced signed graph \mathcal{G}_s such that all the nontrivial cells in π_f are also cells in π , then A_s^f is similar to the block diagonal matrix*

$$\bar{A}_s^f = \begin{bmatrix} A_{P'}^f & \mathbf{0} \\ \mathbf{0} & A_{Q'}^f \end{bmatrix},$$

with $A_{P'}^f = \bar{P}'^T A_s^f \bar{P}'$ and $A_{Q'}^f = \bar{Q}'^T A_s^f \bar{Q}'$.

The proofs are similar to lemmas 7.11, 7.12, and 7.14 in [8] and is skipped for succinctness. One just needs to consider the role of structural balance and the fact that the signed

characteristic matrix P' needs to be replaced for P due to the insertion of gauge transformation.

We are now well-equipped to address the main result of the section. In the following theorem, we see that in certain circumstances, the two block diagonal matrices share identical blocks and thus common eigenvalues. This leads to the uncontrollability of the system by lemma 7.

Theorem 3. *Given a connected structurally balanced signed graph \mathcal{G}_s with the floating graph \mathcal{G}_s^f , the system (2.1) is uncontrollable if there exist NEPs on \mathcal{G}_s and \mathcal{G}_s^f , π and π_f , such that π_f contains all nontrivial cells of π .*

The main scheme of the proof is similar to theorem 7.15 in [8]. We repeat the proof to show how the new orthogonal basis formed by \bar{P}' and \bar{Q}' work in the new setup of signed networks.

Proof. As a result of structural balance, let \bar{P}' and \bar{Q}' be defined as in (2.9). Following the convention in [8], let $\pi \cap \pi_f = \{C_1, C_2, \dots, C_{r_1}\}$ with $|C_i| \geq 2$, $i = 1, 2, \dots, r_1$. Let the nontrivial cells contain the first n_1 nodes. Since π_f contains all nontrivial cells of π , it follows

$$P' = \begin{bmatrix} P'_1 & \mathbf{0} \\ \mathbf{0} & I_{n-n_1} \end{bmatrix}_{n \times r} \quad \text{and} \quad P'_f = \begin{bmatrix} P'_1 & \mathbf{0} \\ \mathbf{0} & I_{n_f-n_1} \end{bmatrix}_{n_f \times r_f},$$

where $P'_1 \in \mathbb{R}^{n_1 \times r_1}$ contains the nontrivial part of the signed characteristic matrices. Let \bar{P}' and \bar{P}'_f be the normalization of P' and P'_f and define \bar{Q}' and \bar{Q}'_f as in (2.9). Then

$$\bar{Q}' = \begin{bmatrix} Q'_1 \\ \mathbf{0} \end{bmatrix}_{n \times (n_1-r_1)}, \quad \bar{Q}'_f = \begin{bmatrix} Q'_1 \\ \mathbf{0} \end{bmatrix}_{n_f \times (n_1-r_1)},$$

where $Q'_1 \in \mathbb{R}^{n_1 \times (n_1-r_1)}$ satisfies $Q'_1 P_1 = 0$. It follows that $\bar{Q}'_f = R^T \bar{Q}'$ with $R = [I_{n_f}, \mathbf{0}]^T$. Then, by lemmas 8 and 9, we get

$$\mathcal{L}_{Q'} = \bar{Q}'^T L_s \bar{Q}' = \bar{Q}'_f{}^T R^T L_s R \bar{Q}'_f = \bar{Q}'_f{}^T A_s^f \bar{Q}'_f = \mathcal{A}_{Q'}^f$$

This implies that L_s and A_s^f share a block matrix and thus have at least one equal eigenvalue. Therefore, by lemma 7 the system is uncontrollable. \square

Theorem 6 gives sufficient conditions for uncontrollability of the signed network system. However, this by no means is a necessary condition. e.g. a system can be uncontrollable and structurally unbalanced simultaneously.

2.3.4 Stabilizability and Output Controllability

Recent developments in controllability have extended the idea of using fractional symmetry to characterize controllability of linear systems.

Theorem 4 (Symmetry Controllability Test [9]). *Consider the general linear dynamics*

$$\dot{x} = Ax + Bu, \quad y = Cx,$$

with $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $C \in \mathbb{R}^{p \times n}$. For A diagonalizable and C full row rank, consider the conditions on a square matrix P :

- a. $P \neq I, AP = PA$ and $PB = B$
- b. $CP = ZC$ for some $Z \neq I$
- c. $\frac{1}{2}(P + P^T) \preceq I$ and $PA + (PA)^T \preceq A + A^T$

Then, there exists $P \in \mathbb{C}^{n \times n}$ such that

1. (a) $\iff (A, B)$ uncontrollable
2. (a) $\&$ (b) $\iff (A, B, C)$ is output uncontrollable
3. (a) $\&$ (c) $\iff (A, B)$ is unstabilizable
4. (a) $\&$ (b) $\&$ (c) $\iff (A, B, C)$ is output unstabilizable

Using Theorem 2 in [9], we can now characterize the controllability, output controllability, stabilizability and output stabilizability of the *influenced signed consensus dynamics*

$$\dot{x} = -L_s x + B(I)u, \quad y = C(O)x,$$

where $I \subseteq N$ is the set of input nodes, $O \subseteq N$ the set of output nodes, and $B(I), C(O)$ are the matrices

$$B(I) = \begin{bmatrix} | & & | \\ e_{i_1} & \cdots & e_{i_q} \\ | & & | \end{bmatrix}, \quad C(O) = \begin{bmatrix} | & & | \\ f_{j_1} & \cdots & f_{j_p} \\ | & & | \end{bmatrix}^T,$$

in which e_i is the indicator vector for nodes $i \in I$, and f_j for nodes $j \in O$.

Our main result is the following theorem, which identifies structural balance as the key feature for uncontrollability of the dynamics (2.2), on top of symmetry of the underlying unsigned graph. The crux of the argument is that the gauge transformation allows a signed symmetric automorphism to satisfy the criteria in Theorem 2 of [9], allowing a variant of the result in Corollary 5 of [9].

Theorem 5. *Let J be a non-trivial signed fractional automorphism of L . Suppose further that L_s is structurally balanced with gauge G_t . Consider dynamics (2.2), and the following conditions on $J_s := G_t J G_t$ with $B_s := G_t B(I)$ and $C_s := C(O) G_t$*

- a. $J_s B_s = B_s$
- b. $C_s(R) J_s C_s(\mathcal{V} \setminus R)^T = 0$, $C_s(R) J_s C_s(R)^T = Z \neq I$.
- c. $J_s v_i = v_i$ for all $v_i \sim \lambda_i(L_s) > 0$.

Then,

- 1. (a) $\iff (-L_s, B)$ uncontrollable
- 2. (a) & (b) $\iff (-L_s, B, C)$ is output uncontrollable

3. (a) \mathcal{E} (c) $\iff (-L_s, B, C)$ is output unstabilizable

4. (a) \mathcal{E} (b) \mathcal{E} (c) $\iff (-L_s, B, C)$ is output unstabilizable

The results hold when $B \rightarrow B_s$ or $C \rightarrow C_s$.

The proof of Theorem 8 is similar to the proof of Corollary 5 in [9], with several technical differences which we discuss here. In particular, our result only requires a fractional automorphism J of the underlying unsigned graph; the matrix J_s does not need to be a signed fractional automorphism, which in this sense generalizes Corollary 5 of [9].

The following lemmas establish the equivalence of controllability under a gauge transformation of the B and C matrices for structurally balanced L_s , and some useful identities that will elucidate the role of the gauge transformation in Theorem 8.

Lemma 10. *Let $(L_s, B(I))$ be the pair in the dynamics (2.2), and let $B_s(I) = G_t B(I)$ for any gauge transformation B (regardless of whether L_s is structurally balanced). Then, $(-L_s, B(I))$ is controllable if and only if $(-L_s, B_s(S))$ is controllable. Furthermore, letting $C_s(O) = C(O)G_t$, we have that $(-L_s, B(I), C(O))$ is output controllable if and only if $(-L, B_s(I), C_s(O))$ is output controllable.*

Lemma 11. *Consider the dynamics in (2.2). Suppose L_s is structurally balanced with gauge G_t .*

1. *Suppose that there is an automorphism J such that $JL_s = L_s J$. Then, $J_s L_s = L_s J_s$, where $J_s = G_t J G_t$.*
2. *Suppose J is input symmetric. Then, $J_s B_s = B_s$.*
3. *Suppose that there exists $Z \neq I$ such that $ZC(O) = C(O)J$. Then, $ZC_s(O) = C(O)J_s$.*

Proof. By Lemma 10, $J_s B_s = B_s$ is equivalent to Theorem 4(a).

Suppose that there exists $Z \neq I$ such that $ZC(O) = C(O)J$ to establish condition Theorem 4(b). Note that $C(O)C(O)^T = I$ and $C(O)C(N \setminus O)^T = 0$, and $[C(O)^T, C(N \setminus O)^T]$ is unitary. We can compute

$$\begin{aligned}
0 &= C(O)J - ZC(N \setminus O) \\
&= (C(O)J - ZC(N \setminus O))[C(O)^T, C(N \setminus O)^T] \\
&= [C(O)JC(O)^T - ZC(R)C(R)^T, \\
&\quad C(O)JC(N \setminus O)^T - ZC(R)C(N \setminus O)^T] \\
&= [C(O)JC(O)^T - Z, C(O)JC(N \setminus O)^T],
\end{aligned}$$

yielding $C(R)JC(N \setminus R)^T = 0$, $C(R)JC(R)^T = Z \neq I$. By Lemma 11(3), this is equivalent if we interchange $J \rightarrow J_s$ and $C \rightarrow C_s$.

Now, lets assume 8(a), and assume that

$$\begin{aligned}
& -J_s L_s - (L_s J_s)^T \succeq -L_s - L_s^T \\
\implies & \frac{1}{2} (J_s L_s + (J_s L_s)^T) \preceq L_s,
\end{aligned}$$

in order to establish Theorem 4(c). Since $(J_s + J_s^T)/2 \preceq I$, we can see that

$$\frac{\lambda_i(L_s)}{2} v_i^T (J_s + J_s^T) v_i \geq v_i^T L_s v_i = \lambda_i(L_s),$$

with equality if $J_s v_i = v_i$, and hence $\lambda_i(L_s) > 0$, which holds for the stable modes of $-L_s$. \square

We now prove Lemma 10.

Proof. Note that the column space of the controllability matrix of $(-L_s, B)$

$$\mathcal{C}(B(I)) := \begin{bmatrix} B & -L_s B & \cdots & (-L_s)^{n-1} B \end{bmatrix}$$

is spanned by columns of the form $(-L_s)^m e_i$ for $0 \leq m \leq n-1$. The action of a gauge G_t on $B(I)$ is to multiply each column of $B(I)$ by ± 1 , and so the column space of the controllability matrix of $(-L_s, B_s)$

$$\mathcal{C}(B(I)) := \begin{bmatrix} G_t B & \cdots & (-L_s)^{n-1} G_t B \end{bmatrix}$$

is spanned by columns of the form $\sigma_i(-L_s)^m e_i$ for $0 \leq m \leq n-1$, and $\sigma_i = \pm 1$. Clearly,

$$\text{span}\{(-L_s)^m e_i\} = \text{span}\{\sigma_i(-L_s)^m e_i\}$$

and so $\text{rank}[\mathcal{C}(B(I))] = \text{rank}[\mathcal{C}(B_s(I))]$. The same argument applies for output controllability, considering the output controllability matrix

$$\mathcal{C}(B(I), C(O)) := \begin{bmatrix} CB & \cdots & C(-L_s)^{n-1}B \end{bmatrix}.$$

□

We now prove Lemma 11.

Proof. 1) $J_s L_s = G_t J G_t G_t L G_t = G_t J L G_t = G_t L J G_t = G_t L G_t G_t J G_t = L_s J_s$. 2) We know $JB = B$. Hence, $J_s B_s = G_s J G_s G_s B = G_s J B = G_s B = B_s$. 3) $ZC(O)G_t = C(O)JG_t = C(O)G_t G_t J G_t = C_s(O)J_s$. □

Using the equivalences established in these two lemmas, the proof of Theorem 8 follows as the proof of Corollary 5 in [9], but using J_s instead of the signed fractional automorphism P .

2.3.5 Examples

In this section, we show examples pertaining to the discussions in this section. In particular, we show the difference between consensus under signed and unsigned consensus.

Bipartite Consensus

Signed Laplacians considered in this section exhibit clustering when structurally balanced. Consider the signed network in Figure 2.4a, and its unsigned counterpart. Note that the signed network is structurally balanced, since the product of the edge weights over all three cycles is positive. The consensus dynamics (with no control input) beginning at

$x_0 = [1, 2, 3, 4]^T$ for these two networks are shown in Figure 2.3. As one can see, the signed consensus converges to two clusters rather than a single equilibrium point.

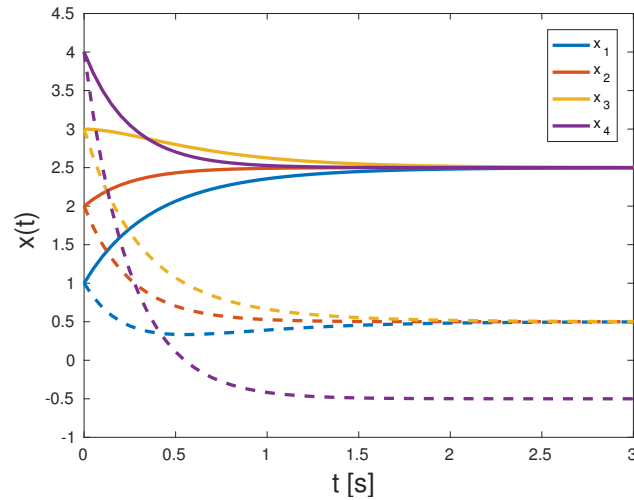


Figure 2.3: Solid lines: unsigned consensus. Dashed lines: signed consensus on structurally balanced graph



Figure 2.4: (a) Uncontrollable and structurally balanced graph with an input symmetry about node 4. (b) Controllable and structurally unbalanced graph with an input symmetry about node 4.

MIMO Perturbed Laplacian

In this example, we consider the MIMO case with two input signals injected onto nodes 4 and 5. The partition is equitable and $\pi = \{C_1, C_2, C_3, C_4\}$ and $\pi_f = \{C_1, C_2\}$. Moreover

$$A = \begin{bmatrix} 0 & -1 & -1 & 0 & -1 \\ -1 & 0 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad P' = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A(\mathcal{G}/\pi) = \begin{bmatrix} 0 & 2 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

The system in figure 2.5a is structurally balanced and π_f contains the nontrivial cell C_2 in π and thus the system is uncontrollable. However, changing one negative sign to positive as depicted in figure 2.5 makes the system structurally unbalanced and the system becomes controllable.

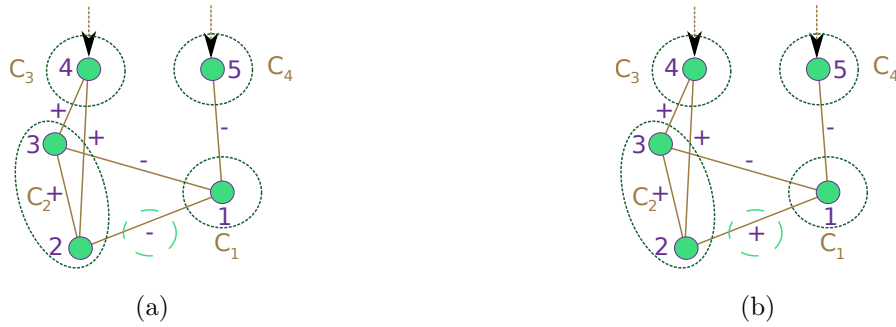


Figure 2.5: (a) Uncontrollable and structurally balanced graph with π_f containing nontrivial cells in π . (b) The same system with one change in the signs. No structural balance and controllable.

Influenced Consensus

Previously, we discussed that symmetry alone is not sufficient for uncontrollability. In this section we show an example of a symmetric signed network that is controllable.

Consider the influenced consensus networks in Figure 2.4. As one can see, the network in Figure 2.4a is structurally balanced, but the one in Figure 2.4b is not. The Laplacians for these two networks are, respectively,

$$L_1 = \begin{bmatrix} 2 & -1 & 0 & 1 \\ -1 & 3 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 2 & -1 & 0 & 1 \\ -1 & 3 & -1 & 1 \\ 0 & -1 & 2 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix}.$$

The controllability matrices for these two networks are, respectively:

$$\mathcal{C}_1 = \begin{bmatrix} 0 & 1 & 4 & 16 \\ 0 & 1 & 4 & 16 \\ 0 & 1 & 4 & 16 \\ 1 & 3 & 12 & 48 \end{bmatrix}, \quad \mathcal{C}_2 = \begin{bmatrix} 0 & 1 & 4 & 14 \\ 0 & 1 & 6 & 32 \\ 0 & 1 & 6 & 30 \\ 1 & 3 & 12 & 52 \end{bmatrix}.$$

As one can see, the network in Figure 2.4a is uncontrollable since \mathcal{C}_1 is rank-deficient, but the network in Figure 2.4b is controllable, and hence one can conclude that unsigned symmetry is not sufficient for uncontrollability.

2.4 Signed Nonlinear Consensus

The contributions of this section are twofold. First, we show that the property of structural balance, when combined with symmetries in the underlying graph, as well as certain symmetries of the nonlinear dynamics, causes uncontrollability in the context of the accessibility problem. In particular, we consider the same network flows studied in [53, 10, 45]; however we extend the controllability analysis to signed dynamics.

Secondly, we extend the bipartite identification problem considered by Pan *et al.* in [49] to the case of signed nonlinear consensus networks. In particular, we use a Koopman operator-theoretic approach alongside Extended Dynamic Mode Decomposition (EDMD) to extract a ‘Koopman mode’ whose sign structure reveals the bipartite structure.

This section is organized as follows. The problems considered are outlined in §2.4.1. In §2.4.2, we examine the controllability problem. Relevant examples are shown in §2.3.5.

2.4.1 Problem Statement

As previously stated, we tackle two problems regarding nonlinear signed consensus networks. In the first section, we extend the result in [10] to the signed networks. We examine the necessary conditions of uncontrollability in nonlinear signed network systems due to input and dynamics symmetry. In particular, we will show how the additional topological property of *structural balance* in signed networks plays a key role in driving uncontrollability.

Recall that the following lemma from [45] elucidates some properties of structural balance:

Lemma 12. (See [45]) *The following statements are equivalent:*

1. *The signed graph \mathcal{G} is structurally balanced;*
2. *There exists a gauge transformation G_t such that $G_t A_s G_t$ has only positive entries;*
3. *For all cycles in \mathcal{G} , the product of the edge weights on the cycle are positive;*
4. *The signed Laplacian L_s has a zero eigenvalue;*
5. *There exists a bipartition of \mathcal{V} such that the edge weights on the edges within the same set are positive, and the edges connecting the two sets are negative.*

In the second section of this paper, we show that a particular *Koopman mode* from the EDMD approximation of the Koopman operator contains the sign structure corresponding to the bipartition in Lemma 12(5). This extends the work by Pan *et al.* who considered the equivalent problem for linear signed consensus [49].

2.4.2 Nonlinear Controllability of Signed Networks

In this section, we extend previous work [10] to analyze the controllability of nonlinear consensus protocols to the case where these protocols run on a signed network. We consider three types of nonlinear consensus protocols, following the nomenclature in [45, 53, 10].

- **Absolute Nonlinear Flow**

$$\dot{x}_i = - \sum_{j \in N_i} [f(x_i) - \text{sgn}(a_{ij})f(x_j)] \quad (2.5)$$

- **Relative Nonlinear Flow**

$$\dot{x}_i = - \sum_{j \in N_i} f(x_i - \text{sgn}(a_{ij})x_j) \quad (2.6)$$

- **Disagreement Nonlinear Flow**

$$\dot{x}_i = -f \left(\sum_{j \in N_i} x_i - \text{sgn}(a_{ij})x_j \right) \quad (2.7)$$

To make the paper self-contained, we provide two main theorems from [10] which we use later to demonstrate uncontrollability.

Theorem 6. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a flow on \mathcal{G} . Assume φ is a non-identity symmetry on F . Then, for any leader l , the leader-follower network flow on \mathcal{G} induced by l is not accessible from the origin in \mathbb{R}^{n-1} .*

Theorem 7. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and let $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the dynamics in any of (2.5)-(2.7). Also, assume φ be an automorphism of \mathcal{G} . Then F is φ -invariant.*

Remark 4. *The more general case of (2.5)-(2.7) holds when each node has its own smooth nonlinear function $f_i : \mathbb{R} \rightarrow \mathbb{R}$. However, as shown in [10], $f_i = f_{\phi(i)}$ is a necessary and sufficient condition for all the subsequent controllability analysis to work. Since we assume the general function f for all nodes, this condition is automatically satisfied.*

The behavior of the dynamics (2.5)-(2.7) clearly depends on the choice of $f : \mathbb{R} \rightarrow \mathbb{R}$. In [45], several classes of functions were considered. First, the class of *translated positive, infinite sector nonlinearities* \mathcal{S} is defined as

$$\mathcal{S} := \left\{ f : [f(x) - f(x^*)](x - x^*) > 0 \text{ for } x \neq x^*, f(0) = 0, \int_{x^*}^x f(t)dt \rightarrow \infty \text{ as } |x| \rightarrow \infty \right\}. \quad (2.8)$$

See [47] for properties of this class of functions. A subset $\mathcal{S}_0 \subset \mathcal{S}$ of these functions that will be used later is the *untranslated ($x^* = 0$) positive, infinite sector nonlinearities* given by

$$\mathcal{S}_0 := \left\{ f : f(x^*)x > 0 \text{ for } x \neq 0, f(0) = 0, \int_0^x f(t)dt \rightarrow \infty \text{ as } |x| \rightarrow \infty \right\}. \quad (2.9)$$

The reason these classes of functions are interesting is that when combined with the dynamics introduced in (2.5)-(2.7), *clustering* occurs in a structurally balanced graph. This is summarized in the following theorem.

Theorem 8. (Theorems 3 & 4 in [45]) *Consider a graph \mathcal{G} . Assume either the dynamics (2.5) with $f \in \mathcal{S}$ or the dynamics (2.6) with $f \in \mathcal{S}_0$ running on \mathcal{G} . Then,*

$$\lim_{t \rightarrow \infty} x(t) = \frac{1}{n} (\mathbf{1}^T G_t x(0)) G_t \mathbf{1}$$

if and only if \mathcal{G} is structurally balanced (with gauge transformation G_t).

According to this theorem, for certain classes of functions, the dynamics will converge to two different clusters. These clusters are exactly those corresponding to the bipartite consensus condition in Lemma 12(5).

In the following subsections, we elaborate on the controllability of the dynamics (2.5)-(2.7) and show that a notion of symmetry about the input node, as well as structural balance, lead to uncontrollability. For each case we consider even (symmetric) and odd (anti-symmetric) functions f . From [45] we know if the underlying signed graph \mathcal{G} is structurally balanced,

then there exists a gauge transformation G_t that acts as a similarity transformation on the adjacency matrix of \mathcal{G} in that $G_t A_s(\mathcal{G}) G_t = A$ where A is the adjacency matrix of unsigned \mathcal{G} . We will show that G_t defines a useful coordinate transformation that allows an immediate application of the uncontrollability test derived by Aguilar and Ghahesifard [10].

Before that, we need the following definition to extend the notion of a graph symmetry to signed graphs.

Definition 2. Let φ be a non-identity automorphism on graph \mathcal{G} . Suppose that this graph is structurally balanced, with gauge transformation G_t induced by the function $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by $[g(x)]_i = \sigma_i x_i$. Then, we define signed automorphism operator as $\varphi' = g \circ \varphi \circ g$. Moreover, assume that J is the matrix representation of the permutation operator φ , in that $J = D\varphi$. Then, the analogous matrix $J' = G_t J G_t$ is the matrix representation of the signed permutation operator φ' , in that $J' = D(g \circ \varphi \circ g)$.

Definition 2 implies that unlike the unsigned case, the signed automorphism contains sign alterations of edge weights while permuting the nodes. Hence, if $\varphi(x_i) = x_r$, then $\varphi'(x_i) = \sigma_i \sigma_r x_r$. For example, consider the graph in Figure 2.6.

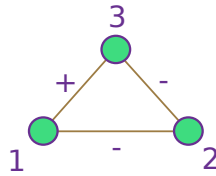


Figure 2.6: Example of a signed automorphism

The corresponding gauge transformation and automorphisms are defined as

$$G_t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad J' = G_t J G_t = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

One can note that while the structures of J and J' are similar, the signs of the elements are different. This also implies

$$\varphi'(x_1, x_2, x_3) = (-x_2, -x_1, x_3) \Rightarrow \begin{cases} \varphi'(x_1) = \sigma_1\sigma_2x_2 \\ \varphi'(x_2) = \sigma_1\sigma_2x_1 \\ \varphi'(x_3) = \sigma_3\sigma_3x_3 \end{cases}$$

Recall that we use $\phi(i)$ as the action of the automorphism on the index of a node rather than the more obscure notation $\varphi(v_i)$. For example, in Figure 2.6 we have that $\phi(1) = 2$ and $\phi(2) = 1$.

Absolute Nonlinear Flow

The original definition of this kind of flow allows the function f to vary across the nodes [53]. In our problem setup, however, all such functions are assumed to be equal, and thus the dynamics resemble linear consensus in that we can write $\dot{x} = -L_s f(x)$, where $f(x)$ is the function f applied entry-wise to the vector x .

In the following theorem, we will show that for absolute nonlinear flow with odd functions f , structural balance directly generalizes the uncontrollability conditions in [10]. For the case of even functions, we need to impose additional topological structure on the edge weights of the underlying graph.

Theorem 9. *Consider a structurally balanced graph \mathcal{G} with gauge transformation G_t and absolute nonlinear flow dynamics (2.5). Further suppose \mathcal{G} has a non-trivial signed automorphism φ' . Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a smooth odd function (for example, odd $f \in \mathcal{S}_0$ with f smooth). Then, for any vertex $j \in \text{Fix}(\varphi')$ chosen as the leader, the leader-follower network is not accessible from the origin in \mathbb{R}^{n-1} . Moreover, the same results holds for smooth even functions f if φ' preserves edge signs, in that $\text{sgn}(a_{ij}) = \text{sgn}(a_{\phi(i)\phi(j)})$.*

Proof. Following Equation (2.1), let F denote the network flow and assume the dynamics in (2.5). We will first note that for smooth odd functions f , the dynamics of the system

can become unsigned by a convenient coordinate transformation. Let $z = G_t x$, or $z_i = \sigma_i x_i$. Then, following [12] we get the equivalent dynamics

$$\begin{aligned}\dot{z}_i &= -\sigma_i \sum_{j \in \mathcal{N}_i} f(\sigma_i z_i) - \text{sgn}(a_{ij}) f(\sigma_j z_j) \\ &= -\sigma_i \sum_{j \in \mathcal{N}_i} \sigma_i f(z_i) - \text{sgn}(a_{ij}) \sigma_j f(z_j) \\ &= - \sum_{j \in \mathcal{N}_i} f(z_i) - f(z_j),\end{aligned}$$

which is defined as the unsigned absolute nonlinear flow. Then, from Theorems 6 and 7 we conclude the system is not accessible from the origin.

Now, suppose f is an even function. From (2.5) we have

$$\begin{aligned}F_i(\varphi'(x)) &= - \sum_{l \in N_r} f(\sigma_i \sigma_r x_r) - \text{sgn}(a_{ij}) f(\sigma_j \sigma_l x_l) \\ &= - \sum_{l \in N_r} f(x_r) - \text{sgn}(a_{ij}) f(x_l),\end{aligned}$$

where $r = \phi(i)$ and $l = \phi(j)$ and the property $f(\sigma_i \sigma_j x) = f(x)$ of even functions is used.

On the other hand,

$$F_{\phi(i)}(x) = F_r(x) = - \sum_{l \in N_r} f(x_r) - \text{sgn}(a_{rl}) f(x_l).$$

Hence, F is φ' -invariant if $\text{sgn}(a_{ij}) = \text{sgn}(a_{rl})$. \square

The condition on even functions in Theorem 9 can be interpreted as the sign symmetry of the graph, in that if the link between two nodes contains a negative weight, it will remain negative even after the signed automorphism. For instance, sign symmetry does not hold in the graph of Figure 2.6 since $a_{13} \neq a_{23}$. This condition is required in addition to the topological property of structural balance.

Relative Nonlinear Flow

The main result of this section shows that unlike the absolute nonlinear flow, structural balance on top of the signed variant of the conditions in Theorem 6 imply that the network

flow is not accessible from the origin for both even and odd functions, without any additional constraints on edge weight permutations.

Theorem 10. *Consider a structurally balanced graph \mathcal{G} with gauge transformation G_t and relative nonlinear flow dynamics. Further suppose \mathcal{G} has a non-trivial automorphism φ' . Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a smooth odd or even function (for example, odd $f \in \mathcal{S}_0$ with f smooth). Then, for any vertex $j \in \text{Fix}(\varphi')$ chosen as the leader, the leader-follower network is not accessible from the origin in \mathbb{R}^{n-1} .*

Proof. Let the same notations as in proof of Theorem 9 hold. We will show that both cases of odd and even functions f lead to φ' -invariance of the flow F and therefore the inaccessibility from the origin.

Let f be an odd function. Changing the coordinates by $z = G_t x$ yields

$$\begin{aligned} \dot{z}_i &= -\sigma_i \sum_{j \in \mathcal{N}_i} f(\sigma_i z_i - \text{sgn}(a_{ij}) \sigma_j z_j) \\ &= -\sigma_i \sum_{j \in \mathcal{N}_i} f(\sigma_i (z_i - \sigma_i \text{sgn}(a_{ij}) \sigma_j z_j)) \\ &= - \sum_{j \in \mathcal{N}_i} f(z_i - z_j), \end{aligned}$$

which is the unsigned relative nonlinear flow. Hence, F is φ' -invariant and inaccessibility from the origin follows from Theorems 6 and 7.

For even function f , from (2.6)

$$\begin{aligned} F_i(\varphi'(x)) &= - \sum_{l \in \mathcal{N}_r} f(\sigma_i \sigma_r x_r - \sigma_j \sigma_l \text{sgn}(a_{ij}) x_l) \\ &= - \sum_{l \in \mathcal{N}_r} f(\sigma_i \sigma_r (x_r - \sigma_r \sigma_l \sigma_i \sigma_j \text{sgn}(a_{ij}) x_l)) \\ &= - \sum_{l \in \mathcal{N}_r} f(x_r - \sigma_r \sigma_l x_l) \\ &= - \sum_{l \in \mathcal{N}_r} f(x_r - \text{sgn}(a_{rl}) x_l) \\ &= F_r(x), \end{aligned}$$

where we have used the fact that $\sigma_i \sigma_j \text{sgn}(a_{ij}) > 0$ hence $\sigma_i \sigma_j = \text{sgn}(a_{ij})$ for all i and $j \in \mathcal{N}_i$. \square

Disagreement Nonlinear Flow

Applying the same assumptions for this type of flow, we will prove similar results without proof for the sake of brevity.

Theorem 11. *Consider a structurally balanced graph \mathcal{G} with gauge transformation G_t and disagreement nonlinear flow dynamics. Further suppose \mathcal{G} has a non-trivial automorphism φ' . Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a smooth odd or even function (for example, odd $f \in \mathcal{S}_0$ with f smooth). Then, for any vertex $j \in \text{Fix}(\varphi')$ chosen as the leader, the leader-follower network is not accessible from the origin in \mathbb{R}^{n-1} .*

Proof. The proof is almost identical to that of Theorem 10 with the same change of coordinates for the odd functions, and similar sign analysis for the even case. \square

Remark 5. *The analysis of this section demonstrates that for all of the three nonlinear flows (2.5)-(2.7) structural balance in addition to φ' -invariance leads to system uncontrollability for even and odd functions f . The only exception is when the absolute nonlinear flow f is even. In this case, an edge-sign symmetry condition is also required.*

2.4.3 Identification of Bipartite Structure

In this subsection, we consider a numerical method to identify the bipartite consensus structure of nonlinear dynamics on a structurally balanced graph. We do this by exploiting Theorem 4, and numerically approximating the Koopman mode corresponding to the zero eigenvalue. The goal is to take data from an unknown dynamical system and numerically identify the two groups of agents between which only exist antagonistic interactions. i.e. the bipartition of nodes in Lemma 12(5).

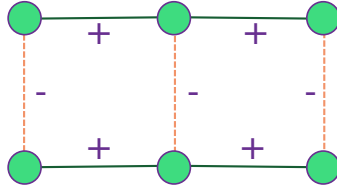


Figure 2.7: Underlying structurally balanced signed graph for EDMD example. Dashed edges indicate negative edges.

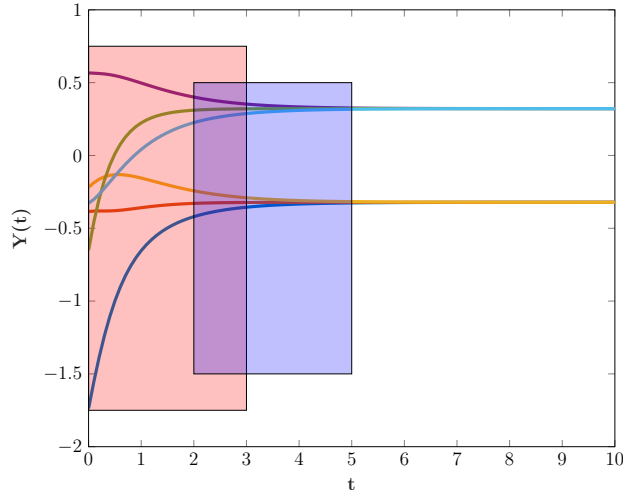


Figure 2.8: Evolution of Dynamics 2.6 with $f(\cdot) = \sin(\cdot)$ on the graph in Figure 2.7. Left (red) shaded region corresponds to \bar{v}_1^2 and the right (blue) shaded region corresponds to \bar{v}_1^3 .

Consider the dynamics (2.6) with $f(\cdot) = \sin(\cdot)$. The underlying structurally balanced graph is pictured in Figure 2.7. We use EDMD to approximate the first Koopman mode corresponding to the zero eigenvalue. Due to space constraints, we refer the reader to [14] for a detailed explanation of the EDMD algorithm; a summary of the relevant computations is shown in Algorithm 1. In particular, we use a dictionary of functions of the form $\psi_k = \prod_{i=1}^6 H(x_i, j_i)$ where $H(x_i, j_i)$ is the j_i th order Hermite polynomial of x_i . We use all such possible ψ_k for Hermite polynomials up to order 2. Simple combinatorics indicates that there are 729 such functions ψ_k for a 6-node graph; hence $N_k = 729$ in Algorithm 1. The dynamics are

Algorithm 1 Extended Dynamic Mode Decomposition

- 1: **procedure** EDMD
 - 2: *Initialize*
 - 3: **Data:** $\{[X_i, Y_i]\}_{i=1}^{n-1} = \{[X_i, X_{i+1}]\}_{i=1}^{n-1}$
 - 4: **Dictionary:** $\Psi_i = \psi_i(x_1, \dots, x_n)$, $1 \leq i \leq N_K$
 - 5: **Matrix:** $B \in \{0, 1\}^{n \times N_K}$ s.t. $\Psi_j = x_i \Leftrightarrow B_{ij} = 1$
 - 6: *Compute*
 - 7: $G = 1/(n-1) \sum_{m=1}^{n-1} \Psi(X_m)^* \Psi(X_m)$
 - 8: $A = 1/(n-1) \sum_{m=1}^{n-1} \Psi(X_m)^* \Psi(Y_m)$
 - 9: $K = G^\dagger A$
 - 10: *Decompose K Into*
 - 11: **Eigenvalues:** μ_i
 - 12: **Right Eigenvectors:** ξ_i
 - 13: **Left Eigenvectors:** w_i
 - 14: *Compute*
 - 15: **Koopman Modes:** $v_i = (w_i^* B)^T$, $1 \leq i \leq N_K$
-

shown in Figure 2.8 for an initial condition of $x_0 = (-1.73, -0.38, -0.21, 0.56, -0.65, -0.32)$. The EDMD procedure in Algorithm 1 was applied to three sets of data with a time-step of 0.1, as depicted in Figure 2.8: the full data (from $0 \leq t \leq 10$), the data in the red shaded region ($0 \leq t \leq 3$) and the data in the blue shaded region ($2 \leq t \leq 5$). The computed Koopman modes corresponding to $\bar{\lambda}_1 \approx 1$ for these regions respectively are

$$\bar{v}_1^1 = \begin{bmatrix} 0.018 \\ 0.026 \\ 0.016 \\ -0.020 \\ -0.020 \\ -0.011 \end{bmatrix}, \bar{v}_1^2 = \begin{bmatrix} 0.019 \\ 0.021 \\ 0.015 \\ -0.018 \\ -0.020 \\ -0.007 \end{bmatrix}, \bar{v}_1^3 = \begin{bmatrix} 0.016 \\ 0.023 \\ 0.015 \\ -0.018 \\ -0.016 \\ -0.012 \end{bmatrix},$$

which all contain sign structure corresponding to the bipartition depicted in Figure 2.7. Despite not using all available data, the EDMD procedure was able to extract the bipartition well before the dynamics converged (after which the bipartite structure is obvious from the data). This begs the question *how early can we detect the bipartite structure of the underlying dynamics?* We will address this question in future works.

2.5 Conclusion

In this section, we characterized the controllability and stabilizability of signed consensus networks, in the linear and nonlinear cases. We showed that the topological notion of structural balance is the significant condition that appears when making statements on uncontrollability. In particular, structural balance induces a gauge transformation that permits the extension of the classical controllability and stabilizability analysis of consensus networks to be applied to the signed Laplacian case. For the nonlinear case, we showed that structural balance, combined with a leader-node symmetry and φ -invariance of the flow dynamics, results in uncontrollability.

We elucidated the role of structural balance both in the SISO and MIMO cases of the

leader-follower consensus dynamics, and then extended this to the output controllability and stabilizability of the influenced consensus dynamics.

We then looked at the task of identifying the bipartite consensus of certain classes of nonlinear network flows. We showed that the sign structure of the first Koopman mode corresponds to this bipartite consensus, and then used Extended Dynamic Mode Decomposition to numerically approximate this sign structure.

Potential future works would be to extend the analysis to classes of nonlinear consensus networks, and to study applications of signed Laplacians to controller design. Future work will also determine bounds on the amount of data required to have an accurate approximation of the Koopman mode with respect to sign structure for nonlinear network flows.

Chapter 3

DESIGN OF NETWORKS

3.1 Introduction

In this chapter, we explore methods that allow the engineer to construct networks useful for consensus. We begin by describing an algorithm called *whiskering*, which allows one to grow a network that remains controllable. A few extensions of this algorithm are presented.

We follow up by using submodular optimization to analyze a second algorithm that allows one to add nodes to a network in order to maximize the connectivity of the network. This algorithm is shown to be equivalent to an edge-addition algorithm developed by Ghosh & Boyd [15].

This chapter is outlined as follows: we perform a literature review in § 3.2, and in § 3.3 we summarize relevant tools from submodular optimization. In §3.4, we discuss the graph whiskering process and its generalizations, and prove that they preserve controllability. Optimization problems involving these processes are then formulated and discussed. We implement algorithms to solve the optimization problems in §3.5. The chapter is concluded in §3.6, where future extensions of the work are discussed.

3.2 Literature Review

A great amount of effort has recently been focused on understanding how the connection structure, or topology, of a network affects the behaviour or performance of a dynamical process on that network [11]. To that end, a natural question is how one can systematically construct a network topology such that a certain performance metric defined over that behaviour is satisfied. A well-known method for constructing networks is of preferential attachment, where new nodes are attached to pre-existing nodes with a probability proportional

to the degree of those nodes [55]. The advantage of this method is that it produces networks with power-law degree distributions that resemble networks found in nature [56]. Another method for growing networks uses Kolmogorov-Sinai entropy as a heuristic parameter for evolving networks [57].

An area of recent focus is the study of controlling distributed systems [11, 10, 58, 8, 9, 59]. Some research pertains to how one may systematically construct a network that has favorable characteristics for consensus. Ghosh and Boyd developed an algorithm to select connections between agents in a network to maximize the connectivity of the network [15]. Chapman and Mesbahi showed how to construct large networks from graph products of atomic networks and examined their controllability properties [60, 43]. Yazicioglu and Egerstedt, and Abbas and Egerstedt worked on constructing networks for leader-follower selection [61, 62]. Liu *et al.* have discussed constructing graphs for scalable semi-supervised learning [63]. When designing a network graph, there are several methods to achieve various performance characteristics. For example, there has been recent work in using submodular optimization for picking input vectors [64, 65]. An excellent summary of submodular optimization applications to the control of networked systems is given in [66]. Measures by which one may gauge network performance and specify network topologies have been suggested in [67, 68].

Whiskering is a process for growing graphs where at each iteration, a vertex and edge is connected to every node in the graph. In this chapter, we discuss using this process, and generalizations of this process, to construct large graphs that are controllable. A process similar in spirit to whiskering are the ‘fractal’ networks studied in the control-theoretic setting by Li *et al.* in [69]

The contributions of this chapter are as follows. We extend the use of submodular optimization in network science to problems involving adding *nodes* to the network. We present a graph-growth method that preserves controllability of consensus on the graph, and provide relevant bounds on the network performance. Lastly, we develop a graph-growth algorithm, and formulate convex optimization problems, which we then solve for specific test cases.

3.3 Mathematical Preliminaries

This section consists of the relevant constructs we use later on for stating and proving our main results.

Recent works in submodular optimization have examined matrix functions (of say, A), such as the trace or the trace of powers of matrices, in the context of submodularity over a set $K \subseteq [n]$ on the principal submatrices $A[K]$ [70, 71]. We summarize the main results of these works, as well as a more specific result about submodularity of functions over principle sub-matrices of M -matrices.

Theorem 12 ([70, 71]). *Let f be a real continuous function on an interval Λ_E of \mathbb{R} . Furthermore, let f' be operator monotone on the interior of $H(\Lambda_E)$. Then, for all $A \in H_n(\Lambda_E)$, the map from $2^{[n]} \rightarrow \mathbb{R}$ given by $K \rightarrow \text{Tr } f(A[K])$ is supermodular.*

Theorem 13 ([70]). *Let A be an M -matrix of size $n \times n$. Then, for all subsets $J, K \subset [n]$ we have that for $0 \leq p \leq 1$:*

$$\text{Tr } A[K]^p + \text{Tr } A[J]^p \geq \text{Tr } A[K \cup J]^p + \text{Tr } A[K \cap J]^p$$

and for $1 \leq p \leq 2$ and for $p < 0$,

$$\text{Tr } A[K]^p + \text{Tr } A[J]^p \leq \text{Tr } A[K \cup J]^p + \text{Tr } A[K \cap J]^p.$$

In particular, the map $K \rightarrow \text{Tr } A[K]^{-1}$ is supermodular if A is an M -matrix.

3.4 Graph Growing

In this section, we introduce a method of growing graphs that preserves controllability, by adding a leaf to every node of the graph. We then generalize this process to adding more complicated structures.

Graph whiskering is a process for adding nodes to a graph, originally studied for the purpose of looking at monomial ideals [72]. For each whiskering iteration, a single unique

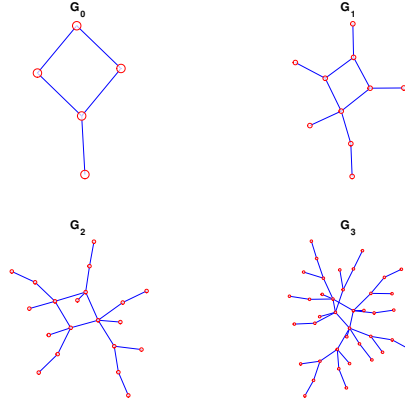


Figure 3.1: Whiskering a graph by adding a leaf to every node

node is connected to an already existing node in the graph, for all nodes in the graph. This corresponds to concatenating the Laplacian in the following form:

$$L \longrightarrow \begin{bmatrix} L + I & -I \\ -I & I \end{bmatrix} = L', \quad (3.1)$$

where we denote the operation as $L' = \mathcal{W}_1(L)$. Figure 3.1 shows an example of whiskering a graph three times. This process is of great interest because of several properties that make it useful for control theoretic analysis, namely it preserves controllability and provides guarantees on the performance of control exerted on the resulting network. The former property is captured in the following theorem.

Theorem 14. *Let $L' = \mathcal{W}_1(L)$. The pairs $(L', [b^T, b^T]^T)$ and $(L', [b^T, \mathbf{0}^T]^T)$ are controllable if and only if the pair (L, b) is controllable.*

Proof. We prove the contrapositive using the Popov-Belevitch-Hautus (PBH) test [11]. Suppose that (L, b) is uncontrollable. Then, there exists w such that $w^T b = 0$ and $w^T L = \lambda L$. We show by construction that there exists a left eigenvector of L' that is orthogonal to the columns of $[b^T, b^T]^T$ and of $[b^T, \mathbf{0}^T]$. We claim that $[w^T, \alpha^T]^T$ is an eigenvector of L' with

eigenvalue Λ , where

$$\alpha = \frac{1}{1 - \Lambda} w, \quad \Lambda = \frac{1}{2} \left(\sqrt{\lambda^2 + 4} + \lambda + 2 \right).$$

The Laplacian is symmetric, and so its left eigenvectors are transposed right eigenvectors.

Therefore, a computation yields

$$\begin{aligned} \begin{bmatrix} L + I & -I \\ -I & I \end{bmatrix} \begin{bmatrix} w \\ \alpha \end{bmatrix} &= \begin{bmatrix} (L + I)w - I\alpha \\ I\alpha - Iw \end{bmatrix} \\ &= \begin{bmatrix} (\lambda + 1)w - \alpha \\ \alpha - w \end{bmatrix} = \begin{bmatrix} \Lambda w \\ \Lambda \alpha \end{bmatrix}. \end{aligned}$$

This is orthogonal to the columns of $[b^T, b^T]^T$ and $[b^T, \mathbf{0}^T]^T$.

For the reverse direction, assume that $(L', b = [b_1^T, b_2^T]^T)$ is uncontrollable. Then by the PBH test, we have an eigenvector of L' orthogonal to the columns of b :

$$\begin{aligned} L' = \lambda \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} &= \begin{bmatrix} Lw_1 + (w_1 - w_2) \\ w_2 - w_1 \end{bmatrix} \\ [w_1^T, w_2^T] \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} &= \begin{bmatrix} w_1^T b_1 \\ w_2^T b_2 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}. \end{aligned} \tag{3.2}$$

It follows that

$$w_2 = \frac{1}{1 - \lambda} w_1, \quad Lw_1 = \left(\frac{1 - (1 - \lambda)^2}{1 - \lambda} \right) w_1,$$

and so w_1 is an eigenvector of L with eigenvalue $(1 - (1 - \lambda)^2)(1 - \lambda)^{-1}$. It is clear that from Equation (3.2), $w_1^T b_1 = \mathbf{0}$, and so (L, b_1) is uncontrollable. This result also holds when $b_2 = \mathbf{0}$, and the theorem follows. \square

Theorem 14 therefore provides a useful way of ensuring that a large graph is controllable. Since rank controllability tests for very large graphs are often inaccurate due to machine precision, one can start with a small graph on which a controllability test is easily performed and iterate this process several times until a sufficiently large network is obtained.

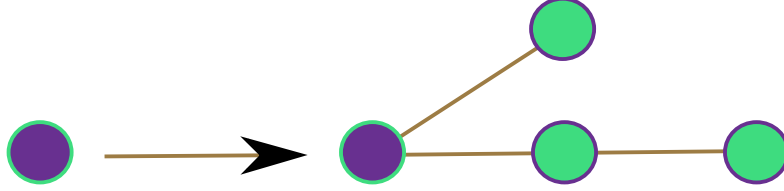


Figure 3.2: Adding a leaf and a path of length 2 to every node

A natural question to ask is what other types of growth processes preserve controllability? For example, one can attach a leaf and a path of length 2 to every node (as shown in Figure 3.2). This corresponds to concatenating the Laplacian as follows:

$$L \longrightarrow \begin{bmatrix} L + 2I & -I & -I & \mathbf{0} \\ -I & I & \mathbf{0} & \mathbf{0} \\ -I & \mathbf{0} & 2I & -I \\ \mathbf{0} & \mathbf{0} & -I & I \end{bmatrix} = L'. \quad (3.3)$$

We denote $L' = \mathscr{W}_2(L)$. It turns out that this growth process also preserves controllability.

Theorem 15. *Let $L' = \mathscr{W}_2(L)$. Then, it follows that the pairs $(L', [b^T, b_i^T, b_i^T, b_i^T]^T)$ (where $b_i \in \{b, \mathbf{0}\}$) are controllable if and only if the pair (L, b) is controllable.*

Proof. We again prove the contrapositive using the PBH test. Suppose that (L, b) is uncontrollable. Then, there exists w such that $w^T b = 0$ and $w^T L = \lambda L$. We show by construction that there exists a left eigenvector of L' that is orthogonal to the columns of $[b^T, b^T]^T$ and of $[b^T, \mathbf{0}^T]$. It can be verified in a similar computation as in the proof of Theorem 14 that $[w^T, \alpha^T, \beta^T, \gamma^T]$ is a left eigenvector of L' with eigenvalue Λ , where

$$\gamma = \frac{1}{1 - \Lambda} \beta, \quad \beta = \left(\frac{1}{1 - \Lambda} - \Lambda - 2 \right)^{-1} w, \quad \alpha = \frac{1}{1 - \Lambda} w,$$

and where Λ satisfies the equation $\lambda + 2 = \frac{\Lambda(\Lambda^3 - 4\Lambda + 2)}{\Lambda^3 - 2\Lambda + 1}$. Note that α and β are simply scalings of w , and therefore γ is also a scaling of w . Therefore, α, β and γ are all orthogonal to the columns of b : $\alpha^T b = \beta^T b = \gamma^T b = w^T b = \mathbf{0}$, and so it is clear that

$[w^T, \alpha^T, \beta^T, \gamma^T][b^T, b_i^T, b_i^T, b_i^T]^T = \mathbf{0}$ for $b_i \in \{b, \mathbf{0}\}$. It follows that $(L', [b^T, b_i^T, b_i^T, b_i^T]^T)$ is uncontrollable.

For the reverse direction, assume that the pair $(L', b = [b_1^T, b_2^T, b_3^T, b_4^T]^T)$ is uncontrollable. Then by the PBH test, there exists an eigenvector $w = [w_1^T, w_2^T, w_3^T, w_4^T]^T$ of L' orthogonal to the columns of b with eigenvalue λ such that

$$\begin{bmatrix} L + 2I & -I & -I & \mathbf{0} \\ -I & I & \mathbf{0} & \mathbf{0} \\ -I & \mathbf{0} & 2I & -I \\ \mathbf{0} & \mathbf{0} & -I & I \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \lambda \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}.$$

It follows from a simple computation that w_1 is an eigenvector of L with eigenvalue

$$\Lambda = \frac{\lambda^4 - 6\lambda^3 + 14\lambda^2 - 14\lambda + 4}{(\lambda - 1)(\lambda^2 - 3\lambda + 1)}.$$

Since w is orthogonal to the columns of b , it follows that $w_1^T b_1 = \mathbf{0}$, and the theorem follows. \square

As Theorem 15 suggests, there are many ways to grow graphs such that they remain controllable. In this case, we have shown that adding a *cluster* of nodes, namely a leaf and a path of length 2, to each node preserves controllability. It is natural to examine what types of node clusters in general each node can be replaced with to preserve controllability. The following theorem places some conditions on these clusters.

Theorem 16 (General Graph Growth). *Let L be an $n \times n$ graph Laplacian, and let L_δ be $n \times n$, C be $r \times r$ and B be $n \times r$ (where $r = kn$ for $k \in \mathbb{Z}_+$) such that the matrix*

$$L' = \begin{bmatrix} L + L_\delta & B \\ B^T & C \end{bmatrix}$$

is a graph Laplacian. Since we are interested in adding the same cluster to each node, we can write $L_\delta = sI$, where s is the number of edges added to the node when attaching it to the cluster. We have the following results.

1) Let $b_n, w_1 \in \mathbb{R}^n$ and $b_r, \beta \in \mathbb{R}^r$. Suppose $(L', b = [b_1, b_r])$ is uncontrollable. Then, there exists an eigenvector $W_1 \neq 0$ such that $L'W_1 = \Lambda w$, $W_1^T b = \mathbf{0}$ where, say $W_1 = [w_1^T, \beta^T]^T$. Then (L, b_1) is uncontrollable when $(\Lambda I - C)$ is invertible and when w_1 is an eigenvector of $B(\Lambda I - C)^{-1}B^T$.

2) Suppose (L, b) is uncontrollable. Then, there exists $w \neq 0$ such that $Lw = \lambda w$ with $w^T b = 0$. We thus have that $(L', [b, \mathbf{0}])$ is uncontrollable if there exists $\Lambda \geq 0$ such that $(\Lambda I - C)^{-1}$ is invertible, and w is an eigenvector of $B(\Lambda I - C)^{-1}B^T$ such that $B(\Lambda I - C)^{-1}B^T w = f(\Lambda)$, where Λ satisfies $\Lambda - \lambda - s = f(\Lambda)$.

Proof. We prove the two results separately.

1) Assuming the notation in Theorem 16-(1), suppose that $(L', b = [b_1, b_r])$ is uncontrollable. Then, by the PBH test, there exists an eigenvector $W_1 = [w_1^T, \beta^T]^T \neq 0$ such that $L'W_1 = \Lambda w$, $W_1^T b = \mathbf{0}$ yielding:

$$L'W_1 = \begin{bmatrix} (L + sI)w_1 + B\beta \\ B^T w_1 + C\beta \end{bmatrix} = \Lambda \begin{bmatrix} w_1 \\ \beta \end{bmatrix}. \quad (3.4)$$

Therefore, if $(\Lambda I - C)$ is invertible, the lower entry of the vector in Equation (3.4) gives $\beta = (\Lambda I - C)^{-1}B^T w_1$ and the first entry of the vector in Equation (3.4) gives

$$Lw_1 = [(\Lambda - s)I - B(\Lambda I - C)^{-1}B^T] w_1.$$

This equation admits w_1 as an eigenvector of L if the action of $B(\Lambda I - C)^{-1}B^T$ on w_1 is to scale w_1 by a fixed amount. In other words, if w_1 is an eigenvector of $B(\Lambda I - C)^{-1}B^T$, then it is an eigenvector of L , and since $w_1^T b_1 = \mathbf{0}$, the result follows.

2) Assuming the notation in Theorem 16-(2), suppose that (L, b) is uncontrollable. Then, there exists $w \neq 0$ such that $Lw = \lambda w$ with $w^T b = 0$. We seek an admissible solution for the equation

$$\begin{bmatrix} L + L_\delta & B \\ B^T & C \end{bmatrix} \begin{bmatrix} w \\ \beta \end{bmatrix} = \Lambda \begin{bmatrix} w \\ \beta \end{bmatrix} \quad (3.5)$$

in terms of the eigenvalue Λ of L' , and the lower part of the eigenvector, β . If $(\Lambda I - C)$ is invertible, we can write $\beta = (\Lambda I - C)^{-1}B^T w$. From the upper entry of the vector in Equation (3.5), we get the relation

$$(\Lambda - \lambda - s)Iw = B(\Lambda I - C)^{-1}B^T w.$$

Then, if w is an eigenvector of $B(\Lambda I - C)^{-1}B^T$, say $B(\Lambda I - C)^{-1}B^T w = f(\Lambda)w$, then we get an equation for Λ :

$$\Lambda - \lambda - s = f(\Lambda).$$

We add the stipulation that Λ must be *admissible*: $\Lambda \geq 0$ for it to be a Laplacian eigenvalue. Finally, it is clear that since $w^T b = 0$, we have that $[w^T, \beta^T][b^T, \mathbf{0}^T]^T = \mathbf{0}$. \square

In the next section, we discuss using a similar graph growing approach to optimize graph performance. We will also discuss bounds obtained using the submodularity theorems in § 3.3 on the performance of graphs generated using the whiskering method.

3.4.1 Optimization Algorithms: Adding Leaves

In this section, we discuss optimization problems that are related to growing graphs. In particular, we consider efficient addition of node clusters to a specific set of nodes in the graph.

Consider a connected graph \mathcal{G} and its Laplacian matrix $L_{\mathcal{G}}$. The second-smallest eigenvalue $\lambda_2(L_{\mathcal{G}})$ is a measure of how interconnected the graph is. It is also an inverse measure of how long it takes for agents connected with graph \mathcal{G} to achieve consensus by convergence to the agreement subspace. A well-known algorithm by Ghosh and Boyd [15] adds edges between unconnected nodes in \mathcal{G} to maximize $\lambda_2(L_{\mathcal{G}})$.

The algorithm considers a set of candidate edges between unconnected nodes in \mathcal{G} , and selects the k candidate edges that maximize $\lambda_2(L_{\mathcal{G}})$. For a set of m candidate edges $l = \{i, j\}$, let a_l be the vector with all-zero entries except $(a_l)_i = 1$ and $(a_l)_j = -1$ when $\{i, j\}$ is a candidate edge. The selection of k candidate edges from this set can be encoded with a

$\{0, 1\}^m$ -vector x , where $x_l = 1$ if the algorithm selects candidate edge l , and zero otherwise. The optimization problem is then written in terms of the individual Laplacians $a_l a_l^T$ for each edge l as follows:

$$\begin{aligned} & \text{maximize} && \lambda_2 \left(L_{\mathcal{G}} + \sum_{l=1}^m x_l a_l a_l^T \right) \\ & \text{subject to} && \mathbf{1}^T x = k \\ & && x \in \{0, 1\}^m. \end{aligned}$$

The standard relaxation of this problem into a semidefinite program (SDP) is of the form

$$\begin{aligned} & \text{maximize} && s \\ & \text{subject to} && s(I - \mathbf{1}\mathbf{1}^T/n) \preceq L(x) \\ & && \mathbf{1}^T x = k \\ & && 0 \leq x \leq 1 \\ & && L(x) = L_{\mathcal{G}} + \sum_{l=1}^m x_l a_l a_l^T. \end{aligned}$$

We present a modification of this algorithm whereby one wants to add *nodes* to the graph \mathcal{G} in such a way that the graph grows in order to maximize $\lambda_2(\mathcal{G})$. Suppose \mathcal{G} has n nodes. We want to choose one of these n nodes to attach leaves to in order to ‘grow’ the graph to maximize $\lambda_2(\mathcal{G})$. Recall that $L_{\mathcal{G}}[I]$ is the principal submatrix of $L_{\mathcal{G}}$ obtained by deleting the rows and columns of $L_{\mathcal{G}}$ corresponding to the elements in the set $I \setminus [m]$. Let L_{tot} denote the graph that has every node whiskered, as in Equation (3.1). We can write this problem as

$$\begin{aligned} & \text{maximize} && \lambda_2(L_{\mathcal{G}'}) \\ & \text{subject to} && L_{\mathcal{G}'} \in \{L_{\text{tot}}[[n] \cup \{i\}], i \in \{n+1, \dots, 2n\}\} \\ & && L_{\text{tot}} = \begin{bmatrix} L_{\mathcal{G}} + I & -I \\ -I & I \end{bmatrix}. \end{aligned} \tag{3.6}$$

This can be solved via exhaustive search over all possible whiskerings; however this becomes computationally intractable for large n . We can relax this problem to a modified Ghosh-Boyd Max- $\lambda_2(\mathcal{G})$ SDP as follows. Let e_i denote the i th standard basis vector in \mathbb{R}^n . Then, we introduce a single node into the system and create a set of n candidate edges

potentially connecting the new node to any pre-existing node in the graph. The individual Laplacian for each candidate edge is $a_i a_i^T$, where $a_i \in \mathbb{R}^{n+1}$ is of the form $a_i = [e_i, -1]^T$. The SDP relaxation is then

$$\begin{aligned}
& \text{maximize} && s \\
& \text{subject to} && s \left(I_{n+1} - \frac{(\mathbf{1}\mathbf{1}^T)_{n+1}}{n+1} \right) \preceq L(x) \\
& && \mathbf{1}^T x = 1 \\
& && 0 \leq x \leq 1 \\
& && L(x) = L'_G + \sum_{l=1}^n x_l a_l a_l^T \\
& && L'_G = \begin{bmatrix} L_G & \mathbf{0}_n \\ \mathbf{0}_n^T & 0 \end{bmatrix}, \mathbf{0}_n \in \mathbb{R}^n.
\end{aligned} \tag{3.7}$$

Note that $L[K]$ is positive-definite. We can also relate the inverse of this matrix to the *controllability Gramian* [67, 73] P , which is a measure of the steady-state covariance of the agent states. The matrix P is the positive-definite solution of the Lyapunov equation

$$-PL[2:n] - L[2:n]P^T = -I,$$

and is given by $P = \frac{1}{2}L[2:n]^{-1}$. Certain submodular functions of P (with respect to edge-addition) have been studied in [65].

The trace of P can be interpreted as an average amount of energy expended to move the agent states around the controllable subspace, and therefore it is of interest to be able to bound the value of $\text{Tr } P$ on the results of our algorithms. We do this using the supermodularity properties of M -matrices from Theorem 13.

Theorem 17. *Let L'_1 denote the whiskering process in Equation (3.1) and let L'_2 denote the whiskering process in Equation (3.3), where L_1 and L_2 are $n \times n$, and so L'_1 is $2n \times 2n$ and L'_2 is $4n \times 4n$. Let the controllability Gramians P'_1 of L'_1 and P'_2 of L'_2 be the respective solutions*

to

$$\begin{aligned} -P'_1 L'_1[2 : 2n] - L'_1[2 : 2n] P_1'^T &= -I \\ -P'_2 L'_2[2 : 4n] - L'_2[2 : 4n] P_2'^T &= -I. \end{aligned}$$

Then,

$$\text{Tr } P'_1 \geq n + C_1 \text{ and } \text{Tr } P'_2 \geq 4n + C_2,$$

where C_1, C_2 are constants depending on L_1, L_2 respectively.

Proof. We observe that the solution to

$$-P' L'[2 : 2n] - L'[2 : 2n] P'^T = -I$$

is given by $P' = \frac{1}{2} L'[2 : 2n]^{-1}$. From Theorem 13, using the fact that L' is an M -matrix, we have that

$$\begin{aligned} \text{Tr}(P_1) &= \text{Tr}(L'_1[2 : 2n]^{-1}) \\ &\geq \text{Tr}(L'_1[2 : n]^{-1}) + \text{Tr}(L'_1[n+1 : 2n]^{-1}) \\ &= \text{Tr}([L_1[2 : n] + I]^{-1}) + \text{Tr}(I) = C_1 + n, \end{aligned}$$

where $C_1 = \text{Tr}([L_1[2 : n] + I]^{-1})$ depends only on L_1 . The second result for L_2 follows from an identical calculation, noting that

$$\text{Tr} \begin{bmatrix} I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2I & -I \\ \mathbf{0} & -I & I \end{bmatrix}^{-1} = 2 \text{Tr}(I) + \text{Tr}(2I) = 4n.$$

□

We can use this result to bound the trace of the controllability Gramian when adding a single node to the system.

Theorem 18. *Consider the task of attaching a single node to the system with $n \times n$ Laplacian L to maximize λ_2 , as denoted in Problem (3.6). Let L' be the subsequent Laplacian, and so $P = \frac{1}{2}L'[2 : n + 1]^{-1}$. Then, $\text{Tr } P \geq C + 1$, where C is a constant depending only on L .*

Proof. Using Theorem 13 we can compute

$$\begin{aligned} \text{Tr}(P) &= \text{Tr}(L'[2 : n + 1]^{-1}) \\ &\geq \text{Tr}(L'[2 : n]^{-1}) + \text{Tr}(L'[n + 1 : n + 1]^{-1}) \\ &= \text{Tr}([L[2 : n] + e_i e_i^T]^{-1}) + 1 \geq C + 1, \end{aligned}$$

where i is the index of the attachment node chosen, and $C = \min_i(\text{Tr}([L[2 : n] + e_i e_i^T]^{-1}))$ is a constant depending only on L . \square

In the next section, we will consider adding a cluster, and provide a similar result on the performance $\text{Tr}(P)$.

3.4.2 Optimization Algorithms: Adding Clusters

In the previous section, we considered the problem of optimally adding leaves to some nodes to optimize the algebraic connectivity of the graph. We will now consider the problem of adding a cluster of a node and a length-2 path, as depicted in Figure 3.2.

Let $\mathbf{0}_n \in \mathbb{R}^n$ and define $a_{3 \rightarrow 4} = [\mathbf{0}_n^T, 0, 1, -1]^T$, $a_{i,1} = [e_i^T, -1, 0, 0]^T$ and $a_{i,2} = [e_i^T, 0, -1, 0]^T$. Then, choosing an attachment node to maximize λ_2 can be written as

$$\begin{aligned} &\text{maximize} && \lambda_2(L_{\mathcal{G}'}) \\ &\text{subject to} && L_{\mathcal{G}'} \in \{L_{\text{tot}}[[n] \cup \{i, i + n, i + 2n\}], \\ &&& i \in \{n + 1, \dots, 2n\}\} \\ &&& L_{\text{tot}} = \begin{bmatrix} L + 2I & -I & -I & \mathbf{0} \\ -I & I & \mathbf{0} & \mathbf{0} \\ -I & \mathbf{0} & 2I & -I \\ \mathbf{0} & \mathbf{0} & -I & I \end{bmatrix}. \end{aligned} \tag{3.8}$$

We can write the SDP relaxation as:

$$\begin{aligned}
& \text{maximize} && s \\
& \text{subject to} && s \left(I_{n+3} - \frac{(\mathbf{1}\mathbf{1}^T)_{n+3}}{n+3} \right) \preceq L(x) \\
& && \mathbf{1}^T x = 1 \\
& && 0 \leq x \leq 1 \\
& && L(x) = L'_G + a_{3 \rightarrow 4} a_{3 \rightarrow 4}^T \\
& && \quad + \sum_{l=1}^n x_l (a_{i,1} a_{i,1}^T + a_{i,2} a_{i,2}^T) \\
& && L'_G = \begin{bmatrix} L_G & \mathbf{0}_{n \times 3} \\ \mathbf{0}_{3 \times n} & \mathbf{0}_{3 \times 3} \end{bmatrix}.
\end{aligned} \tag{3.9}$$

Lastly, we provide a performance bound on the Gramian analogous to Theorem 18.

Theorem 19. *Consider the task of attaching a single node to the system with $n \times n$ Laplacian L to maximize λ_2 , as denoted in Problem (3.8). Let L' be the subsequent Laplacian, and so $P = \frac{1}{2} L'[2 : n+3]^{-1}$. Then, $\text{Tr} P \geq C + 4$, where C is a constant depending only on L .*

Proof.

$$\begin{aligned}
\text{Tr}(P) &= \text{Tr}(L'[2 : n+3]^{-1}) \\
&\geq \text{Tr}(L'[2 : n]^{-1}) + \text{Tr}(L'[n+1 : n+3]^{-1}) \\
&= \text{Tr}([L[2 : n] + 2e_i e_i^T]^{-1}) + 4 \geq C + 4,
\end{aligned}$$

where i is the index of the attachment node chosen, and $C = \min_i (\text{Tr}([L[2 : n] + 2e_i e_i^T]^{-1}))$ is a constant depending only on L . \square

3.5 Algorithm Implementation

In this section, we show examples of the optimization problems discussed in the previous section.

The optimization problems (3.6) and (3.8) were implemented using `cvx` [74, 75]. An additional relaxation method used to solve problems (3.7) and (3.8) discussed in [15], known

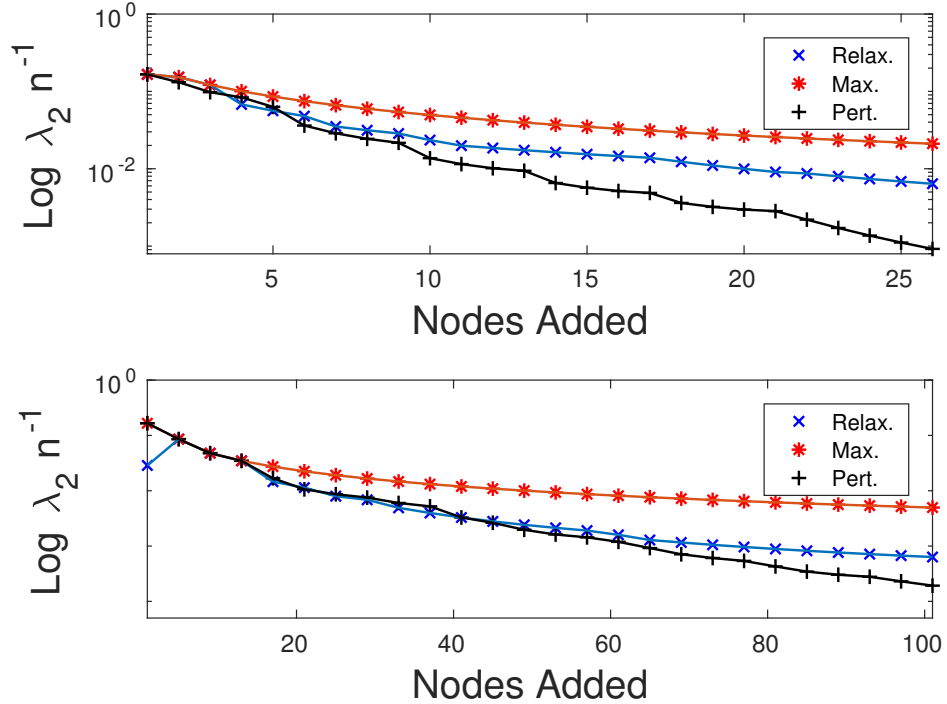


Figure 3.3: Max- λ_2 algorithm results for adding leaves (top) and path clusters (bottom), using the convex relaxation (\times), exhaustive search ($*$) and perturbation heuristic ($+$).

as the *perturbation heuristic*, was also implemented for the purpose of comparison. At each iteration of the heuristic algorithm, the node cluster is attached to one node chosen by selecting the node with the largest value of $(v_i - v_{n+1})^2$, where v satisfies $L'v = \lambda_2 v$. Here, v_{n+1} is the entry of v corresponding to the node in the cluster attaching the cluster to node i .

If there is more than one node attaching the cluster to the node in the graph, then without loss of generality, denote these (say, l) nodes as $n + 1, \dots, n + l$. Then, the perturbation heuristic is to find the node $i \in [n]$ maximizing $\sum_{j=1}^l (v_i - v_{n+j})^2$ at each iteration.

The results of running these algorithms for 25 iterations are shown in Figures 3.3, 3.4, and 3.5. The seed graphs, and final graphs after 25 iterations for each of the three techniques (exhaustive search, convex relaxation and perturbation heuristic) are shown in Figure 3.4

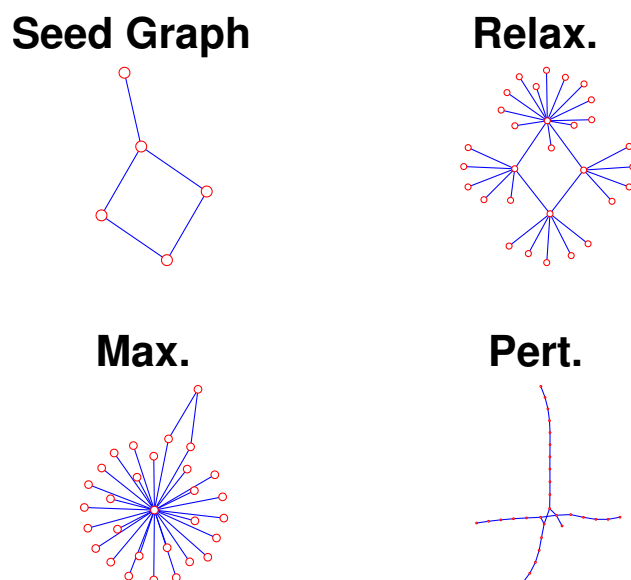


Figure 3.4: Seed graph, and final graph after 25 iterations of the leaf-adding problem using the SDP relaxation 3.7, exhaustive search over problem 3.6 and the perturbation heuristic.

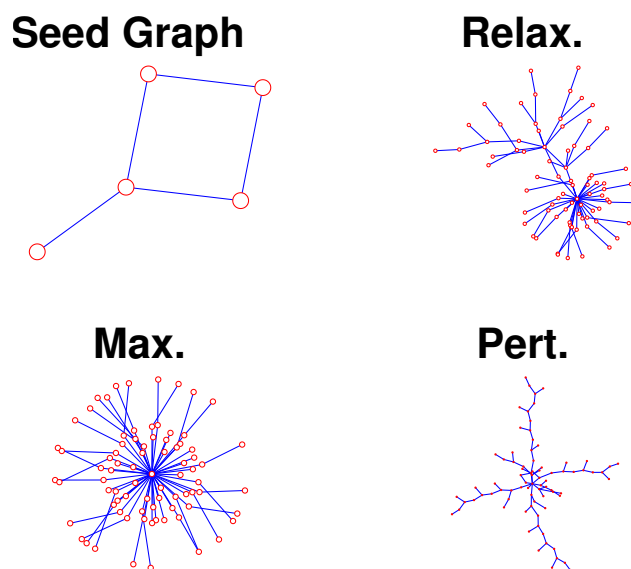


Figure 3.5: Seed graph, and final graph after 25 iterations of the path-cluster-adding problem using the SDP relaxation 3.9, exhaustive search over problem 3.8 and the perturbation heuristic.

for adding a single leaf, and in Figure 3.5 for adding the path cluster. For both cases, the convex relaxations (problems (3.7) and (3.9)) perform reasonably well and pick out slightly suboptimal solutions, as seen in Figure 3.3.

3.6 Conclusions and Future Works

In this chapter, we explored methods of constructing graphs by iterating a procedure that preserves controllability. In Theorem 14, we showed that adding a leaf to every node preserves controllability, and in Theorem 15 we showed that adding a cluster of two paths (of length 1 and 2) to each node also preserves controllability. We provided bounds on the performance of the resulting graph using submodularity. We obtained general conditions for preserving controllability under iterative graph growing in Theorem 16.

An interesting area of further work is to classify all graph clusters that one can attach to all nodes in the network that preserve controllability; in other words describe the matrix L' appearing in Theorem 16 in terms of graph objects. It would also be worth exploring the combination of the node-addition algorithms presented in the chapter with edge-adding algorithms, for example maximizing λ_2 [15] and adding edges to make cycles for more robust consensus [76].

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