Trickle-Down Theorems and Local-To-Global Analysis of Markov Chains

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Abstract

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This thesis covers multiple results related to high dimensional counting and sampling problems, as well as the broader theory of high dimensional expanders. Central to these results are "local-to-global" phenomena that allow the study of high dimensional distributions through multiple forms of localization. In particular, we prove new trickle-down theorems and apply them to problems in different fields. This includes proving rapid mixing of the natural Markov chain for sampling from graph colorings in a previously unsolved regime, and obtaining significantly improved bounds on the local spectral expansion of recent constructions of sparse high dimensional expanders, which are of particular interest in coding theory and complexity theory. We also use a local-to-global perspective to provide evidence that the natural down-up walk on the space of NBC bases of a matroid may not mix rapidly. Finally, we apply a local-to-global technique to take a step towards characterizing the coefficients of homogeneous completely log-concave polynomials, which also implies fast mixing of the natural random walk for sampling from the high dimensional distributions associated with such polynomials.

Contents

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Chapter 1

Introduction

A (weighted) graph is γ-spectral expander if the second largest eigenvalue of its normalized adjacency matrix is at most γ. The notion of spectral expansion is closely related to strong connectivity properties of graphs and has enabled advances in many areas of mathematics and computer science including number theory, group theory, coding theory, approximation algorithms, pseudo-randomness, embeddability of metric spaces, and probabilistically checkable proofs. See, e.g., [INW94; LLR95; SS96; HLW06; Din07; Lub12; ABS15] for some references.

High dimensional expansion is a generalization of spectral expansion in graphs to higher dimensional objects called simplicial complexes. The study of high dimensional expanders has revolutionized several areas of mathematics and theoretical computer science (e.g. see [GK23] for a survey). This includes analysis of Markov chains (e.g. [Ana+19; AL20; ALO20; CLV21; Ali+21; ALO22; Ana+22; WZZ24]), locally testable codes (e.g. [Dik+20; Din+22; DLZ23]), agreement tests (e.g. [DK17; DD19], quantum LDPC codes (e.g. [EKZ24; KT21]), and elsewhere (e.g. [AGT19; Din+21b; Din+21a; Baf+21]).

One of the main properties of high dimensional expanders is the existence of "local-to-global phenomena" that allow "global" properties of a high dimensional simplicial complex to be investigated through analysis of "local" spectral properties. To provide a more clear description of the local-to-global phenomena, we need to define simplicial complexes and high dimensional expanders more formally.

A simplicial complex X on a finite ground set $[n] = \{0, \ldots, n\}$ is a downwards closed collection of subsets of $[n]$, i.e., if $\tau \in X$ and $\sigma \subset \tau$, then $\sigma \in X$. The elements of X are called *faces*, and the maximal faces are called *facets*. We say that a face τ has dimension k if $|\tau| = k + 1$ and write $\dim(\tau) = k$. We say that X is a pure d-dimensional complex if every facet has dimension d. In this dissertation, we only work with pure simplicial complexes. Moreover, the co-dimension of a face $\tau \in X$ is defined as $\operatorname{codim}(\tau) = d - \dim(\tau)$. For any $0 \le i \le d$, define $X(i) = \{\tau \in X : \dim(\tau) = i\}$. We equip X with a probability distribution π supported on all facets of X, and denote this pair by (X, π).

For a face $\tau \in X$, the *link* of τ is a simplicial complex defined as

$$X_\tau = \{\sigma \setminus \tau : \sigma \in X, \sigma \supset \tau\}.$$

For a face $\tau \in X$, the distribution π induces a conditional distribution π_τ on facets of X_τ

where for each facet $\sigma \in X_\tau$,

$$\pi_\tau(\sigma) = \frac{\pi(\sigma \cup \tau)}{\sum_{\text{facet } \sigma' \in X_\tau} \pi(\sigma' \cup \tau)}.$$

Taking the link of a face τ of a simplicial complex can be seen as form of localization. Intuitively, the larger the dimension of τ is, the more "local" and low dimensional the link of τ will be. Note that the link of any face of co-dimension 2 is a 1-dimensional simplicial complex, which is simply a weighted graph. Such links could be considered as the most local nontrivial links.

While the links of a simplicial complex (X, π) provide localized perspectives into a complex, the links of faces of co-dimension more than 2 are still high dimensional. Thus, one can consider an additional level of localization by looking the "underlying graph" of these links. For any face τ of co-dimension at least 2, the *1-skeleton* of the link (X_τ, π_τ) is a weighted graph whose vertices are the 0-dimensional faces of X_τ, its edges are the 1-dimensional faces of X_τ, and the edge weights are given by $\mathbb{P}_{\sigma\sim\pi_\tau}[\{x, y\} \subseteq \sigma]$ for each edge $\{x, y\}$. Note that when τ is of co-dimension 2, the complex (X_τ, π_τ) is itself a weighted graph. We say that a complex X is *totally connected* if the 1-skeleton of the link of any face τ of co-dimension at least 2 is connected.

The 1-skeletons of the links of a simplicial complex (X, π) encode many important properties of the complex. As a result, one can try to investigate interesting global properties of high dimensional simplicial complex through studying the 1-skeletons of its links, following a local-to-global analysis. One of the most useful properties of the 1-skeletons is their spectral expansion.

Definition 1 (Local Spectral High Dimensional Expander [DK17; KO18; Dik+18])**.** *We say that the link of a face τ of co-dimension at least 2 of a d-dimensional (weighted) complex (X, π) is a λ-local spectral expander if the 1-skeleton of (X_τ, π_τ) is a λ-spectral expander. We say that (X, π) is a $(\gamma_2, \gamma_3, \ldots, \gamma_{d+1})$-local spectral expander if the link of any face τ of co-dimension at least 2 is a $\gamma_{\text{codim}(\tau)}$-spectral expander. When the complex (X, π) is clear in the context, for an integer $2 \le k \le d+1$, we write γ_k to denote the maximum 2nd largest eigenvalue of the simple random walk on the 1-skeleton of all links of faces of co-dimension k of the complex.*

The notion of local spectral expansion has facilitated the investigation of various complex problems. This includes the problem of sampling from high dimensional distributions, which is the main focus of this dissertation. High dimensional distributions capture the interactions of a large set of interacting nodes. Since the domain of these distributions could be exponentially large in the number of nodes, investigating different properties of these distributions in a computationally efficient ways is often a complex problem. A common approach to this problem is sampling from high dimensional distributions using Markov chain Monte Carlo methods. In this case, the main challenge is to design a Markov chain whose stationary distribution is the given high dimensional distribution and prove that the chain rapidly converges to its stationary distribution. A common approach to analyzing the mixing time of a Markov chain is through spectral analysis, and in particular by upper-bounding the second largest eigenvalue of its transition probability matrix, which we often denote by λ_2. Since the largest eigenvalue of the transition probability matrix is always 1, this is equivalent to showing that $1 - \lambda_2$, often referred to as the *spectral gap* of the random walk, is large. A formal statement of this well-known result can be found in the preliminaries in Section 1.2.2.

There are various generic methods such as the Metropolis-Hastings method [Met+53] that, given any distribution, readily yield Markov chains whose stationary distribution is the given distribution. The following natural Markov chain is a variant of the Metropolis-Hastings method for sampling from any given distribution π on subsets of size k of a set S. Starting at a set σ of size k, we transition to another set σ' of size k via the following two-step process:

1. Select a uniformly random element $x \in \sigma$ and remove x from σ.
2. Randomly select a set σ of size k containing $\sigma \setminus x$ with probability proportional to $\pi(\sigma')$.

This generic recipe generalizes various well-known classes of Markov chains including the Glauber dynamics for sampling configurations in graphical models and the bases exchange walk for sampling bases of matroids. While these chains are fairly simple and natural, in many cases, analyzing their mixing time is quite challenging. In recent years, the study of high dimensional expanders, and a framework closely related to local expansion commonly referred to as the "spectral independence" framework, have led to breakthroughs in this area (see e.g. [Ana+19; CGM19; ALO20; CLV20; CLV21; Che+21b; ALO22; Ali+21; Che+21a; Bla+22; Ana+22; Ana+22; Liu+24; WZZ24]).

Given a distribution π on subsets of size $d + 1$ of a set S, one can consider a weighted d-dimensional simplicial complex X whose facets are the subsets of size d of S, and whose weight is given by π. The above Markov chain can be seen as a "down-up" walk on the facets of the simplicial complex, where in each transition, starting from a facet σ, the first step of the two-step process samples a face $\tau \subset \sigma$ of dimension $d - 1$ and the second step samples a facet $\sigma' \supset \tau$.

This formulation allows us to investigate the mixing time of the Markov chain using local-to-global properties of high dimensional expanders. The following theorem is an example of local-to-global theorems that derive bounds on the mixing time of the down-up walk using local spectral expansions of the underlying simplicial complex.

Theorem 1.0.1 ([AL20]). *Let (X, π) be a $(\gamma_2, \ldots, \gamma_{d+1})$-local spectral expander. Then the second eigenvalue of the transition probabilty matrix of the down-up walk $P^\vee$ is bounded by*

$$\lambda_2(P^\vee) \leq 1 - \frac{1}{d} \prod_{j=2}^{d+1} (1 - \gamma_j).$$

Roughly speaking, the above bound implies that if $\gamma_i = \frac{O(1)}{i}$, the random walk converges to its stationary distribution π in polynomial time.

There are a variety of tools and techniques to prove that a given simplicial complex is a local spectral expander, including geometry of polynomials (e.g. see [Ana+19; Ali+21; CLV22]), correlation decay bounds (e.g. see [ALO20; Che+21b]), and more (e.g. see [Liu21; Bla+22]). Another approach to this problem that was introduced by Oppenheim is the "trickle-down method [Opp18]." Central to Oppenheim's trickle-down theorem is another form of local-to-global argument. Roughly speaking, this theorem states that if the links of the "most local non-trivial links", i.e. the links of faces of co-dimension 2, are $\frac{c}{d}$-local spectral expanders for some $0 \leq c < 1$, then for any $k > 2$ the links of faces of co-dimension i are $\frac{O(1)}{k}$-local spectral expanders.

Theorem 1.0.2 (Oppenheim's Trickle-Down Theorem [Opp18])**.** *Given a totally connected complex (X, π), if the links of faces of co-dimension 2 are $\frac{1-\delta}{d}$-spectral expanders for some $0 < \delta \leq 1$, then the complex is a $(\gamma_2, \ldots, \gamma_{d+1})$-local spectral expander, where $\gamma_k \leq \frac{1-\delta}{d-(k-2)(1-\delta)} \leq \frac{1-\delta}{d\delta}$ for all $2 \leq k \leq d+1$.*

The trickle-down theorem has found numerous applications in proving bounds on local spectral expansion of simplicial complexes. To invoke the theorem one needs to inspect the links of all faces of co-dimension 2 to find the worst 2nd eigenvalue. If we get lucky and this number is below $1/d$, then, the trickle-down theorem kicks in and automatically bounds the spectral expansion of all links of the complex. One of the major issues with this theorem is that the required bound is too small and often not satisfiable.

Several chapters of this dissertation focus on new trickle-down theorems that address the above issue and other shortcomings of Oppenheim's trickle-down theorem, along with the applications of these theorems in counting, sampling, and coding theory. The remaining chapters can also be broadly seen as applications of local-to-global perspectives to counting and sampling problems. The results presented in this dissertation are based on four research papers that I co-authored during my Ph.D.studies [ALO22; AO23a; ALO23; AO23b].

In Chapter 2, we prove a "matrix trickle-down theorem," which is a spectral local-to-global framework for inductively bounding the local spectral expansion of a simplicial complex. This framework generalizes Oppenheim's trickle-down theorem by strengthening the induction hypothesis, replacing scalar bounds on local spectral expansion with matrix bounds on the normalized adjacency matrices of 1-skeletons.

In Chapter 3, we apply this generalized local-to-global framework to efficient sampling of graph colorings, a long-standing open problem in the field. We use the matrix trickle-down theorem to show that the natural Glauber dynamics for sampling graph colorings mixes in polynomial time in a previously unsolved regime.

In Chapter 4, we prove a trickle-down theorem for partite complexes that significantly improves Oppenheim's trickle-down, and can be applied in a black-box manner to a large family of complexes to derive non-trivial bounds on their local spectral expansions. For an application, using our theorem as a black-box, we show that links of faces of co-dimension k in recent constructions of bounded degree high dimensional expanders have spectral expansion at most $O(1/k)$ fraction of the spectral expansion of the links of the worst faces of co-dimension 2, refuting a conjecture in the field.

In Chapter 4, we study counting and optimization problems on NBC bases of a generic matroid through a local-to-global lens. Given a matroid $M = (E, \mathcal{I})$, and a total ordering over the elements E, a broken circuit is a circuit where the smallest element is removed, and an NBC independent set is an independent set in $\mathcal{I}$ with no broken circuit. Sampling NBC bases of a generic matroid can be achieved by sampling a facet of the simplicial complex defined by NBC bases of these matroids using the down-up walk. In this chapter, we use a local-to-global perspective to give evidence that the natural down-up walk on the space of NBC bases of a matroid may not mix rapidly. We do this by showing that for a family of matroids, it is NP-hard to sample facets of certain links of the NBC complex.

In Chapter 6, we introduce an expressive subclass of non-negative almost submodular set functions, called strongly 2-coverage functions, which include coverage functions and (sums of) matroid rank functions, and prove that the homogenization of the generating polynomial of any such function is completely log-concave. A corollary of this result is that, given any such function f, the natural down-up random walk can be used to rapidly sample a subset of any given size of the ground set with probability proportional to $f(S)$. This result is obtained using tools from geometry of polynomials, diverging from the more linear algebraic and combinatorial approaches used within the context of theory of simplicial complexes in the previous chapters. However, a local-to-global paradigm, expressed in the language of polynomials, remains central to the proofs in this section.

1.1 Main Results

In this section, we present a summary of our main results. Throughout this section, given a symmetric matrix $M \in \mathbb{R}^{n\times n}$, we index its eigenvalues as $\lambda_n(M) \leq \cdots \leq \lambda_1(M)$.

1.1.1 A Matrix Trickle-Down Theorem

In almost all applications of high dimensional simplicial complexes, one first needs to to prove that the underlying complex is a local spectral expander and then exploit local-to-global theorems to prove global properties of the underlying complex X. Oppenheim's trickle-down method [Opp18] is a technique for proving a complex is a local spectral expander that is also based on a local-to-global perspective. This method inductively derives bounds on the spectral expansion of the 1-skeletons of all links using bounds on the spectral expansion of links of co-dimension 2, i.e. the most "local" links, as the base case for induction.

In one of our main technical contributions, we give a recursive local-to-global framework for bounding the local spectral expansion of a weighted simplicial complex X which significantly generalizes Oppenheim's result by strengthening the induction hypothesis. In particular, instead of inductively proving scalar bounds on local spectral expansion, we recursively obtain matrix bounds on the adjacency matrices of the 1-skeletons of links, which then yield upper bounds on local spectral expansion.

Before stating the main results, we need some additional definitions from [ALO22]. Given a set S, we write $v \in \mathbb{R}^S$ and $A \in \mathbb{R}^{S\times S}$ to respectively denote a vector and a matrix indexed by S. For symmetric matrices A, B, we write $A \preceq B$ if and only if $B - A$ is positive semidefinite (PSD). For a $n \times n$ symmetric matrix M with eigenvalues $\lambda_n \leq \cdots \leq \lambda_1$, define $\rho(A) = \max\{|\lambda_1|, |\lambda_n|\}$.

Given a (weighted) complex (X, π), let $X(\leq i) := X(-1) \cup \cdots \cup X(i)$. For each face τ and each integer $-1 \leq i \leq \operatorname{codim}(\tau) - 1$, we write $\pi_{\tau,i}$ for the induced distribution on $X_\tau(i)$ given by

$$\pi_{\tau,i}(\eta) = \frac{1}{\binom{\operatorname{codim}(\tau)}{i+1}} \Pr_{\sigma\sim\pi_d}[\sigma \supset \eta \mid \sigma \supset \tau]. \tag{1.1}$$

One should view this as a marginal distribution conditioned on τ. We will often view $\pi_{\tau,i}$ as a vector in $\mathbb{R}_{\geq 0}^{X_\tau(i)}$.

Given a weighted graph $G = (V, E, w)$, the simple random walk is the following stochastic process: Given $X_0 = v \in V$, for every $u \sim v$, we have $X_1 = u$ with probability $\frac{w_{\{u,v\}}}{d_w(v)}$. For any τ of co-dimension at least 2, let $P_{(X,\pi),\tau} \in \mathbb{R}^{X(0)\times X(0)}$ denote the transition probability matrix of the simple random walk on the 1-skeleton of (X_τ, π_τ) padded with zeros outside the $X_\tau(0) \times X_\tau(0)$ block, i.e.

$$P_{(X,\pi),\tau}(x,y) = \frac{\mathbb{P}_{\sigma\sim\pi_\tau}[\{x,y\} \subset \sigma]}{\sum_{z\in x_\tau(0)} \mathbb{P}_{\sigma\sim\pi_\tau}[\{x,z\} \subset \sigma]}$$

for $x, y \in X_\tau(0)$, and $P_\tau(x,y) = 0$ otherwise. When the weighted complex (X, π) is clear from context, we write P_τ to denote $P_{(X,\pi),\tau}$. Note that given a weighted graph, the transition probability matrix of the simple random walk on the graph has the same eigenvalues as the graph's normalized adjacency matrix. As a result, (X, π) is a $(\gamma_2, \gamma_3, \ldots, \gamma_{d+1})$-local spectral expander if and only if for every face τ of co-dimension at least 2, we have $\lambda_2(P_\tau) \leq \gamma_k$. We often refer to the simple random walks on the 1-skeleton of links of a simplicial complex as the *local walks* of the complex. Thus, the local-to-global framework allows us to study the spectral expansion of the global (high dimensional) down-up walk on the facets of a simplicial complex through the spectral expansion of the local walks associated to the complex.

For any τ of co-dimension at least 2, we define the diagonal matrix $\Pi_{(X,\pi),\tau} \in \mathbb{R}^{X(0)\times X(0)}$ as follows: $\Pi_{(X,\pi),\tau}(x,x) = \pi_{\tau,0}(x)$ for $x \in X_\tau(0)$, and $\Pi_{(X,\pi),\tau}(x,x) = 0$ otherwise. When (X, π) is clear from context, we write Π_τ to denote $\Pi_{(X,\pi),\tau}$. Note that $\Pi_\tau P_\tau$ is a symmetric matrix.

Theorem 1.1.1 (Matrix Trickle-Down Theorem [ALO22])**.** *Let (X, π) be a totally connected weighted complex. Suppose $\{M_\tau \in \mathbb{R}^{X(0)\times X(0)}\}_{\tau\in X(\leq d-2)}$ is a family of symmetric matrices satisfying the following:*

1. ***Base Case:*** *For every τ of co-dimension 2, we have the spectral inequality*

$$\Pi_\tau P_\tau - 2\pi_{\tau,0}\pi_{\tau,0}^\top \preceq M_\tau \preceq \frac{1}{5}\Pi_\tau.$$

2. ***Recursive Condition:*** *For every τ of co-dimension at least $k \geq 3$, at least one of the following holds: M_τ satisfies*

$$M_\tau \preceq \frac{k-1}{3k-1}\Pi_\tau \quad \textit{and} \quad \mathbb{E}_{x\sim\pi_\tau} M_{\tau\cup\{x\}} \preceq M_\tau - \frac{k-1}{k-2} M_\tau \Pi_\tau^{-1} M_\tau. \tag{1.2}$$

Or, $(X_\tau, \pi_{\tau,k-1})$ is a product of weighted simplicial complexes $(Y_1, \mu_1), \ldots, (Y_t, \mu_t)$ and for every $\eta \in X_\tau(k-1)$,

$$M_\tau = \bigoplus_{1\leq i\leq t: d_{Y_i}\geq 1} \frac{d_{Y_i}(d_{Y_i}+1)}{k(k-1)} M_{\tau\cup\eta_{-i}},$$

where $\eta_{-i} = \eta \setminus Y_i(0)$.

Then for every $\tau \in X(\leq d-2)$, we have the bound $\lambda_2(\Pi_\tau P_\tau) \leq \rho(\Pi_\tau^{-1} M_\tau)$.

1.1.2 Application of the Matrix Trickle-Down Theorem in Sampling Graph Colorings

As an application of our matrix trickle-down Theorem, we show that the natural Glauber dynamics mixes rapidly and generates a random proper edge-coloring of a graph with maximum degree Δ whenever the number of colors is at least $q \geq \left(\frac{10}{3} + \epsilon\right)\Delta$, where $\epsilon > 0$ is arbitrary and the maximum degree satisfies $\Delta \geq C$ for a constant $C = C(\epsilon)$ depending only on ϵ [ALO22]. For edge-colorings, this improves upon prior work [Vig99; Che+19] which show rapid mixing when $q \geq \left(\frac{11}{3} - \epsilon_0\right)\Delta$, where $\epsilon_0 \approx 10^{-5}$ is a small fixed constant.

Given an (undirected) graph $G = (V, E)$ with $n = |V|$ vertices and with maximum degree $\Delta \geq 1$, and q available colors, it is conjectured that the Glauber dynamics mixes in time $O(n \log n)$ for q as low as $\Delta + 2$ but despite significant attempts we are still very far from proving this conjecture. To this date, the best known result for general graphs is due Chen, Delcourt, Moitra, Perarnau and Postle [Che+19] who show that the Glauber dynamics mixes in polynomial time for $q \geq (11/6 - \epsilon)\Delta$ for some universal constant $\epsilon > 0$; this slightly improves on the classical works of Jerrum and Vigoda [Jer95; Vig99] which bounds the mixing time by a polynomial in n for $q \geq (11/6)\Delta$. The problem of sampling a random proper edge coloring of graphs can equivalently be seen as sampling a random proper vertex coloring of line graphs. Most of the recent analyses of the Glauber dynamics are focused on "locally sparse" graphs [HV03; Mol04; HV05; FV06; FV07; HVV07; Dye+13; Che+21b; Fen+22] where it was typically shown how to break the 11/6 barrier bound when the underlying graph has a large girth.

Unlike most recent trends which focus on sparse graphs with large girths, line graphs are very dense locally as they contain induced cliques of size $\Omega(\Delta)$. Note that for a graph with maximum degree Δ, the maximum degree of the line graph could be as large as 2Δ. Therefore, with the 11/6 barrier, one would need $q \geq 11/3\Delta$ to guarantee polynomial mixing for all edge coloring instances. In our main theorem we prove that this barrier can be broken for edge coloring of any graph with maximum degree Δ.

Theorem 1.1.2 ([ALO22])**.** *Let $G = (V, E)$ be a graph of maximum degree Δ. For any $0 < \epsilon \leq \frac{1}{10}$ such that $\frac{\ln^2 \Delta}{\Delta} \leq \frac{\epsilon^3}{15}$, and any collection of color lists $L = \{L(e)\}_{e \in E}$ satisfying $|L(e)| \geq \Delta(e) + (4/3 + 4\epsilon)\Delta$ where $\Delta(e)$ is the number of neighbors of e in the line graph of G, the spectral gap of the Glauber dyanmics for sampling proper L-edge-list-colorings on G is $\Omega(n^{-O(1/\epsilon)})$, so the mixing time is $O(n^{O(1/\epsilon)})$.*

Remark 1. Recently, [WZZ24] used the matrix trickle-down Theorem to prove that the Glauber dyanmics for sampling proper vertex colorings of line graphs with n vertices and maximum degree Δ mixes in time $O_\Delta(n \log n)$ when the number of available colors satisfies $q \geq (1 + o(1))\Delta$, thus almost resolving the aforementioned conjecture for line graphs.

1.1.3 An Improved Trickle-Down Theorem for Partite Complexes

The matrix trickle-down theorem provide a framework to derive bounds on local spectral expansion in regimes that Oppenheim's trickle-down theorem fails. However, unlike Oppenhiem's trickle-down theorem, the matrix trickle-down theorem cannot be applied in a black-box manner. We prove a generalization of the trickle-down theorem for *partite complexes* that can be applied in a black-box manner to bound local spectral expansion [AO23a]. A $(d+1)$-partite complex is a d-dimensional complex such that $X(0)$ can be (uniquely) partitioned into sets $T_0 \cup \cdots \cup T_d$ such that for every facet $\tau \in X(d)$, we have $|\tau \cap T_i| = 1$ for all $i \in [d]$. The type of any face $\tau \in X$ is defined as $\text{type}(\tau) = \{i \in [d] : |\tau \cap T_i| = 1\}$.

As opposed to Oppenheim's trickle-down theorem that requires *all* links of faces of co-dimension 2 to have expansion less than $1/d$, our improved trickle-down theorem for partite-complexes says that if "on average" the links of faces of co-dimension 2 have spectral expansion less than $1/d$, then the 1-skeleton of all links of co-dimension k are $O(1/k)$-spectral expanders. This theorem addresses several major pitfalls of Oppenheim's trickle-down theorem: theorem: First, the required bound on γ_2 is often not satisfiable. In particular, for many dense complexes in counting and sampling applications, the links of faces of co-dimension 2 are only $\Theta(1)$-spectral expanders, hence the trickle-down theorem does not apply. Yet, for many such complexes, one can show that $\gamma_k = O(1/k)$ for $k \geq \Omega(d)$ (see e.g., [ALO20; CLV21]). Second, even if $\gamma_2 \ll 1/d$, the trickle-down theorem only implies $\gamma_k \simeq \gamma_2$, i.e. γ_k does not increase too much as k increases. This is in contrast with the fact that, for many dense complexes, one can observe a steep decrease in spectral expansion as the co-dimension increases, i.e. $\gamma_k \lesssim \gamma_2/k$.

Before formally stating this result, we need the following definition.

Definition 2. *Given a $(d+1)$-partite complex (X, π) with parts $[d]$, for every $i \in [d]$, define*

$$\Delta_{(X,\pi)}(i) = |\{j \in [d] \setminus i : \exists \tau \ s.t. \ \text{type}(\tau) = [d] \setminus \{i,j\}, \lambda_2(P_\tau) > 0\}|,$$

i.e. $\Delta_{(X,\pi)}(i)$ is the number of parts $j \neq i$ for which there exists a face of type $[d] \setminus \{i,j\}$ whose link is not a 0-spectral expander. Moreover, define $\Delta_{(X,\pi)} = \max_{i \in [d]} \Delta_{(X,\pi)}(i)$. We drop the subscripts (X,π) when the complex is clear in the context.

Theorem 1.1.3 (Simplified Version of Improved Trickle-Down Theorem for Partite Complexes [AO23a])**.** *Let (X, π) be a $(d+1)$-partite (weighted) totally connected complex. For some $0 < \delta < 1$, assume that*

$$\gamma_2 \leq \frac{\delta^2}{10(1 + \ln \Delta)} \quad \text{and} \quad \gamma_2 \leq \frac{1-\delta}{\Delta + \ln \Delta}.$$

Then, the link of any face τ of co-dimension k of X has spectral expansion

$$\begin{cases} \frac{c(1-\delta)}{k\delta} & \text{if } k \geq \Delta, \\ \frac{c(1-\delta)\frac{k + \ln k}{\Delta + \ln \Delta}}{k\delta} & \text{if } k < \Delta, \end{cases}$$

for some constant $c \leq 2$ that depends on δ.

Note that, for $\Delta = d$, this theorem retrieves Oppenheim's trickle-down up to a lower order term in the condition on γ_2 and a constant in the bounds on local spectral expansions.

Applications to Sparse High Dimensional Expanders As an application of our result, we show that in recent constructions of bounded degree high dimensional expanders, the links of faces of co-dimension k have spectral expansion at most $O(1/k)$ times the maximum spectral expansion of a link of a face of co-dimension 2, refuting a conjecture in the field. One can generally divide the family of HDXs studied in recent works into two groups:

1. Dense Complexes. Here, we have a HDX with exponentially large number of facets, i.e. $|X(0)|^d$. One typically encounters these objects in studying Markov Chain Monte Carlo technique where we use a Markov Chain to sample from an exponentially large probability distribution. Perhaps the simplest such family is the complex of all independent sets of a matroid.

2. Sparse Complexes. Here we have a HDX where every vertex (of $X(0)$) only appear in constant number of facets, independent of $|X(0)|$. See, [LSV05b; KO18; OP22] for explicit constructions. These objects have been useful in constructing double samplers (e.g. [Din+21b]), agreement testers (e.g. [DK17; DD19]), and locally testable codes (e.g. [Din+22]).

For many dense complexes, one can observe a steep decrease in spectral expansion as the co-dimension increases, i.e., the complex is a $(\gamma_1, \ldots, \gamma_d)$-local spectral expander where $\gamma_k \lesssim \gamma_1/k$. Such a decrease has not been known for any sparse complex. This led some experts to conjecture that, perhaps, dense and sparse complexes exhibit a different pattern of local spectral expansion; in particular, unlike dense HDX, local spectral expansion does not decay for sparse complexes. As an application of our improved trickle-down theorem, we refute this conjecture by showing that the links of faces of co-dimension k in recent constructions of sparse high dimensional expanders have spectral expansion at most $O(1/k)$ fraction of the spectral expansion of the links of the worst faces of co-dimension 2. Here, we state our result for one such construction.

Kaufman and Oppenheim [KO18] obtained a simple construction of sparse $(d+1)$-partite complexes with $|X(0)| \geq p^s$ for any integer $s > d$ and prime power p such that every $x \in X(0)$ is in at most $p^{O(d^3)}$ many facets (hence the degree is independent of s). They argued that for any non-consecutive pair of parts $i, j \in [d]$, i.e., $j \neq i+1$ and $i \neq j+1 \pmod{d+1}$, we have $\epsilon_{\{i,j\}} = 0$ but $\epsilon_{\{i,i+1\}} \leq \frac{1}{\sqrt{p}}$ for any $i \in [d]$ ($i+1$ is taken modulo $d+1$). Consequently, $\Delta(i) = 2$ for any $i \in [d]$. Then, using Oppenheim's trickle-down theorem, they show that the complex is a $(\frac{1}{\sqrt{p}-(d-2)}, \ldots, \frac{1}{\sqrt{p}-d-2})$-local spectral expander for $p > (d-2)^2$. Simply plugging in these values into the above theorem, for $\delta = 1 - \frac{2}{\sqrt{p}}$ and $p \geq 193$ (independent of d) the assumptions of the theorem are satisfied. The resulting complex is $(\frac{2c}{\sqrt{p}\delta}, \ldots, \frac{2c}{d\sqrt{p}\delta})$-local spectral expander for some constant c. In other words, not only does the Kaufman-Oppenheim construction give an HDX for constant values of p independent of d, but also its local spectral expansion improves inverse linearly with the co-dimension.

1.1.4 On Sampling Non-Broken Bases of Matroids

We study counting and optimization problems on NBC bases of a generic matroid, which is closely related to counting and sampling of various combinatorial and geometric objects. One of our main results in this section presents evidence that, from a local-to-global perspective, the natural down-up walk on the space of NBC bases of a matroid may not mix rapidly [ALO23].

A matroid $M = (E, \mathcal{I})$ is consists of a finite ground set E and a collection $\mathcal{I}$ of subsets of E, called independent sets, satisfying:

Downward closure: If $S \subseteq T$ and $T \in \mathcal{I}$, then $S \in \mathcal{I}$.

Exchange axiom: If $S, T \in \mathcal{I}$ and $|T| > |S|$, then there exists an element $i \in T \setminus S$ such that $S \cup \{i\} \in \mathcal{I}$.

The rank of a set $S \subseteq E$ is the size of the largest independent set contained in S. All maximal independent sets of M, called the bases of M, have the same size r, which is called the rank of M.

Given a matroid $M = (E, \mathcal{I})$, set $C \subseteq E$ is a *circuit* iff $C \setminus \{e\} \in \mathcal{I}$ for any $e \in C$. Given a total ordering over the elements E, a *broken circuit* (with respect to a total ordering $\mathcal{O}$) is a set $C \setminus \{e\}$, where $C \subseteq E$ is a circuit and e is the smallest element of C with respect to $\mathcal{O}$. An independent set $S \subseteq E$ is a *non-broken* independent set (NBC independent set) if it contains no broken circuits. The number of NBC independent sets of size k in a graphic matroid is equal to the absolute value of the $(n-1)-k$-th coefficient of the chromatic polynomial of the underlying graph where n is the number of vertices. As a result, the problem of counting NCB independent sets of matroids is closely related to counting and sampling of various interesting combinatorial and geometric objects, such as acyclic orientations of a graph (e.g. [Sta73]), strongly connected orientations of a graph (e.g., [GL19]), regions defined by the intersection of hyperplanes (e.g. [Sta07]), and more (e.g. [BCT10]).

Given a matroid M with an arbitrary total ordering σ, one can run the down-up walk on the NBC bases of M. It is not hard to see that this chain is irreducible and converges to the uniform stationary distribution. Following the work of [Ana+19] it was conjectured that the down-up walk on the NBC bases of any matroid mixes rapidly [1].

Conjecture 1.1.4. *For any matroid M, and any total ordering $\mathcal{O}$ of the elements of M, the down-up walk on the NBC bases of a matroid mixes in polynomial time.*

The set of NBC independent sets of any matroid M (with respect to any ordering $\mathcal{O}$) form a simplicial complex that is known as the *broken circuit complex*. We denote this complex by $\mathrm{BC}(M, \mathcal{O})$. The down-up walk over this complex equipped with a uniform distribution over its facets is the same as the down-up walk over NBC bases.

To prove that the down-up walk mixes rapidly on the bases of any matroid, [Ana+19] proved that the independent set complex of any matroid M is a $(0, 0, \ldots, 0)$-local spectral expander. Building on this, a natural method to prove Conjecture 5.1.2 is to show that the broken circuit complex of any matroid M of rank r and for any total ordering is a $(\gamma_2, \ldots, \gamma_r)$-local spectral expander for $\gamma_i \leq \frac{O(1)}{i}$, and then apply Theorem 1.0.1.

Conjecture 1.1.5. *For any matroid M of rank r and any ordering $\mathcal{O}$ the broken circuit complex of M is a $(\gamma_2, \ldots, \gamma_r)$-local spectral expander for some $\gamma_i \leq \frac{O(1)}{i}$.*

[1] In fact, this conjecture was raised an an open problem in several recent workshops UC Santa Barbara workshop on New tools for Optimal Mixing of Markov Chains: Spectral Independence and Entropy Decay, and Simon's workshop on Geometry of Polynomials

One of our main results in this chapter is to disprove Conjecture 5.1.5 in a very strong form, namely for the class of (truncated) graphic matroids.

Theorem 1.1.6 ([ALO23]). *There exists an infinite sequence of (truncated) graphic matroids $M_1, M_2, \ldots$ with orderings $\mathcal{O}_1, \mathcal{O}_2, \ldots$, such that for every $n \geq 1$, M_n has poly(n) elements, and there exists a face τ of the broken circuit complex of $X = \mathrm{BC}(M_n, \mathcal{O})$ for which the down-up walk on the facets of the link X_τ has a spectral gap of at most $n^{-\Omega(n)}$.*

In fact, we even prove a stronger statement.

Theorem 1.1.7 ([ALO23]). *Given a matroid $M = (E, \mathcal{I})$ and a total ordering $\mathcal{O}$ and a set $S \subseteq E$, unless RP=NP, there is no FPRAS for counting the number of NBC bases of M that contain S.*

1.1.5 Complete Log Concavity of Coverage-Like Functions

A polynomial $p \in \mathbb{R}[z_1, \ldots, z_n]$ is *log-concave* over $\mathbb{R}^n_{\geq 0}$, if p is non-negative and $\log p$ is a concave function on $\mathbb{R}^n_{\geq 0}$. Note that the identically zero polynomials are log-concave. We say $p \in \mathbb{R}[z_1, \ldots, z_n]$ is *completely log-concave/Lorentzian* if for any $k \geq 0$, and any set of vectors $a_1, \ldots, a_k \in \mathbb{R}^n_{\geq 0}$, $D_{a_1} \ldots D_{a_k} p$ is non-negative and log-concave over $\mathbb{R}^n_{\geq 0}$, where for a vector $a \in \mathbb{R}^n$, $D_a = \sum_i a_i \partial_{z_i}$ is the directional derivative operator. Completely log-concave polynomials were introduced in [AOV18] and extended in [Ana+19; BH20].

We say a polynomial $p \in \mathbb{R}[z_1, \ldots, z_n]$ is *multiaffine* if every monomial of p is square-free, i.e., $\mathrm{supp}(p) \subseteq \{0,1\}^n$. A polynomial $p \in \mathbb{R}[z_1, \ldots, z_n]$ is d*-homogeneous* if $p(\alpha z) = \alpha^d p(z)$ for any $\alpha \in \mathbb{R}$. Given a non-negative function $f : 2^{[n]} \to \mathbb{R}_{\geq 0}$, we say f is a completely log-concave set function if the generating polynomial of f, i.e.,

$$p_f(z_1, \ldots, z_n) = \sum_{S \subseteq [n]} f(S) \cdot \prod_{i \in S} z_i$$

is completely log-concave. Note that p_f is a multiaffine polynomial.

Completely log-concave polynomials have nice properties that make them useful tools for design and analysis of algorithms and studying mathematical objects like matroids. One of these properties is that, as first shown in [Gur10], if the polynomial $\sum_{i=0}^n c_i y^{n-i} x^i$ is completely log-concave, then $c_0, \ldots, c_n$ is an ultra log-concave sequence. This property is used in [Ana+24] to prove Mason's ultra log-concavity conjecture for independent sets of matroids.

Another useful property is that, as first shown in [Ana+19] and later improved in [CGM19; Ana+21], given a d-homogeneous log-concave set function $f : 2^{[n]} \to \mathbb{R}_{\geq 0}$, a natural random walk can be used to rapidly sample a subset S of the ground set with probability proportional to $f(S)$. To put this in the language of simplicial complexes, consider a $(d-1)$-dimensional complex (X, π) whose facets are all subsets of size d of $[n]$, and the weight of each facet S is given by $\pi(S) = \frac{f(S)}{\sum_{S' \subset [n], |S'| = d} f(S')}$. As shown by [Ana+19], complete log-concavity of f implies that (X, π) is $(0, \ldots, 0)$-local spectral expander. As a result, Theorem 1.0.1 implies that the down-up random walk on the facets of the complex mixes rapidly.

An interesting aspect of homogeneous completely log-concave polynomials is that the family of sets that can serve as the support of these polynomials can be nicely characterized (see Theorem 6.1.1 for more details). A natural question is whether one can give a more fine characterization of the set of possible coefficients of homogeneous log-concave polynomials.

We take a step towards characterization of the coefficients of homogeneous log-concave polynomials. Given a polynomial $p \in \mathbb{R}[z_1, \dots, z_n]$, we define the homogenization of p as $\mathrm{Hom}(p, y) \coloneqq \sum_{i=0}^{n} y^{n+1-i} p_i$, where p_i is the i-homogeneous part of p, i.e. $p = p_0 + \cdots + p_d$ and p_i is a i-homogeneous polynomial. Note that if $\mathrm{Hom}(p_f, y)$ is completely log-concave, then all homogeneous parts of p_f are also log-concave. It can be shown that, given a non-negative set function $f \in 2^{[n]} \to \mathbb{R}_{\geq 0}$, if $\mathrm{Hom}(p_f, y)$ is completely log-concave, then f is almost log-submodular (see Corollary 6.1.3). Thus, to take a step toward finding a classification of coefficients of homogeneous completely log-concave polynomials, a natural question to ask is whether one can find a large subclass of non-negative almost log-submodular functions such that for every f in that subclass, $\mathrm{Hom}(p_f, y)$ is completely log-concave. An important subclass of non-negative log-submodular functions with numerous applications is the class of non-negative monotone submodular functions (see Fact 6.1.20). In this paper, we introduce an expressive subclass of non-negative monotone functions, called strongly 2-coverage functions, and prove that $\mathrm{Hom}(p_f, y)$ is completely log-concave. We show that the set of strongly 2-coverage functions includes several fundamental classes of non-negative monotone submodular functions including matroid rank functions, coverage functions, and, more generally, matroid rank sum functions, which are positive linear combination of rank functions and include a large subset of submodular functions that have been studied in the mechanism design literature [Cal+07; DRY11; Dug11; DV11]. As a consequence we prove the following theorem.

Theorem 1.1.8 ([AO23b]). *If $f : 2^{[n]} \to \mathbb{R}_{\geq 0}$ is a coverage function or a sum of matroid rank functions, then the sequence $f_0, f_1, \dots, f_n$ is ultra log-concave where $f_i = \sum_{S:|S|=i} f(S)$.*

Moreover, we introduce a strictly larger class of non-negative submodular functions called 2-coverage functions, which, for instance, also include the indicator function of independent sets of any matroid. We prove that if p_f is 2-coverage, all homogeneous parts of p_f are log-concave. As a consequence, given a 2-coverage function, one can use the results of [Ana+19; Ana+21; CGM19] to rapidly sample a subset S of $[n]$ with probability proportional to $f(S)$.

It is worth highlighting the connection of this result with the "local-to-global" paradigm. The results in this chapter are obtained using tools from geometry of polynomials, diverging from simplicial complexes. However, a local-to-global paradigm, expressed in the language of polynomials, remains central to the proofs. In particular, one can view taking partial derivatives of polynomials as a form of localization, similar to taking links of faces in simplicial complexes. From this perspective, Theorem 6.1.18 closely resembles Oppenheim's trickle-down theorem (Theorem 1.0.2), i.e. the first condition of Theorem 6.1.18 is similar to the condition of total connectivity in the trickle-down theorem, and the second condition resembles the assumed bound on spectral expansion of links of faces of co-dimension 2 (faces of size $d - 2$) in the trickle-down theorem.

1.2 Preliminaries

This section is based on the preliminaries sections of previously published papers [ALO22; AO23a]. First, we make some notational remarks. When it is clear from context, we write a to denote a singleton $\{a\}$. When we want to add a subscript b to an object denoted by x_a, we write $x_{a,b}$, i.e. $x_{a,b} \coloneqq (x_a)_b$. We use the same convention for superscripts. Throughout the dissertation, adding or removing $\emptyset$ as a subscript does not change the object. For an integer q, we denote $\{1, \dots, q\}$ by $[q]$. For a function $f : D \to R$ and a $D' \subseteq D$, we write $f|_{D'}$ to denote the restriction of f to domain D'.

1.2.1 Linear Algebra

For a $n \times n$ matrix A, we define $\|A\|_\infty = \max_{1 \le i \le n}\{\sum_{j=1}^n |A_{ij}|\}$. We write $\preceq$ to denote the Loewner order, i.e. for any symmetric matrices $A, B \in \mathbb{R}^{S \times S}$, we write $A \preceq B$ if $B - A$ is positive semidefinite.

Fact 1.2.1. *For any symmetric matrix $A \in \mathbb{R}^{S\times S}$ where $A_{i,j} \neq 0$ only for $i, j \in S' \subseteq S$, we have $A \preceq \|A\|_\infty I^{S'}$.*

Fact 1.2.2. *For matrices $A, B \in \mathbb{R}^{m\times n}$ and positive $\epsilon > 0$, we have the inequalities $AB^\top + BA^\top \preceq \epsilon AA^\top + \frac{1}{\epsilon} BB^\top$ and $(A + B)(A + B)^\top \preceq (1 + \epsilon)AA^\top + (1 + 1/\epsilon)BB^\top$.*

Proof. Since $\epsilon > 0$ we can write

$$0 \preceq \left(\sqrt{\epsilon}A - \frac{1}{\sqrt{\epsilon}}B\right)\left(\sqrt{\epsilon}A - \frac{1}{\sqrt{\epsilon}}B\right)^\top = \epsilon AA^\top + \frac{1}{\epsilon}BB^\top - AB^\top - BA^\top.$$

Rearranging yields the first inequality $AB^\top + BA^\top \preceq \epsilon AA^\top + \frac{1}{\epsilon}BB^\top$. Adding $AA^\top + BB^\top$ to both sides yields the second inequality. □

Lemma 1.2.3. *Let $A, B \in \mathbb{R}^{n\times n}$ be symmetric matrices such that $A\cdot(I-\alpha A) \preceq B\cdot(I-\alpha B)$ for a positive real number $\alpha > 0$. If $A, B \preceq \frac{1}{2\alpha} \cdot I$, then $A \preceq B$. Note that we crucially do not require $A, B \succeq 0$.*

Proof. It suffices to prove the claim when $\alpha = 1$, since the general claim then follows by replacing A, B with $\alpha A, \alpha B$, respectively.

First, observe the matrix map $M \mapsto M(I-M)$ is a bijection between $\left\{M \in \mathbb{R}^{n\times n} : M \preceq \frac{1}{2}I\right\}$ and $\left\{T \in \mathbb{R}^{n\times n} : T \preceq \frac{1}{4}I\right\}$ with inverse

$$T \mapsto \frac{1}{2}I - \left(\frac{1}{4}I - T\right)^{1/2}. \tag{1.3}$$

The way to see this is via the eigendecomposition: if $M = \sum_{i=1}^n \mu_i\varphi_i\varphi_i^\top$ for an orthonormal eigendecomposition $\{\varphi_i\}_{i=1}^n$ with corresponding eigenvalues $\{\mu_i\}_{i=1}^n$, then $M(I - M) = \sum_{i=1}^n \mu_i(1 - \mu_i)\varphi_i\varphi_i^\top$. Hence, to prove this claim, it suffices to show that the real function $x \mapsto x(1 - x)$ is a bijection between $(-\infty, 1/2]$ and $(-\infty, 1/4]$. To see this, observe that the quadratic $x(1-x)=\mu$ has roots $x = \frac{1}{2} \pm \left(\frac{1}{4}-\mu\right)^{1/2}$, and since we enforced that $\mu \le \frac{1}{4}$, we must choose x to be the smaller root, i.e. $\frac{1}{2} - \left(\frac{1}{4}-\mu\right)^{1/2} \le \frac{1}{2}$ gives the inverse function.

Knowing this explicit inverse function, we now return to the proof of the lemma. Since $A, B \preceq \frac{1}{2}I$, we may apply the inverse Eq. (1.3) to $A(I-A)$ (resp. $B(I-B)$) to recover A (resp. B). Hence, to prove the claim, it suffices to establish operator monotonicity of Eq. (1.3). A quick calculation reveals that this is equivalent to operator monotonicity of $M \mapsto \sqrt{M}$ for positive semidefinite $M \in \mathbb{R}^{n\times n}$, which is well-known and follows, for instance, by using the Löwner-Heinz Theorem [Löw34]. □

Given a weighted graph $G = (V, E, w)$, the simple random walk is the following stochastic process: Given $X_0 = v \in V$, for every $u \sim v$, we have $X_1 = u$ with probability $\frac{w_{\{u,v\}}}{d_w(v)}$ and we let P be the transition probability matrix of the walk. A graph $G = (V, E)$ paired with a weight function $w : E \to \mathbb{R}_{\ge 0}$ is ϵ-expander if $\lambda_2(P) \le \epsilon$, where $\lambda_2(P)$ is the second largest eigenvalue of P. For such a graph we write $d_w(x) = \sum_{y\sim x} w(\{x,y\})$ to denote the weighted degree of a vertex x and $\mathrm{vol}(S) = \sum_{x\in S} d_w(x)$ to denote the volume of a set $S \subseteq V$.

Lemma 1.2.4 (Cheeger's Inequality)**.** *For any graph $G = (V, E)$ with weights $w : E \to \mathbb{R}_{\ge 0}$ and any $S \subseteq V$,*

$$\frac{w(E(S,\overline{S}))}{\min\{\mathrm{vol}(S), \mathrm{vol}(\overline{S})\}} \le \sqrt{2(1-\lambda_2)}$$

where λ_2 is the second largest eigenvalue of the simple random walk on G

Fact 1.2.5 (Expander Mixing Lemma)**.** *Given a (weighted) graph $G = (V, E, w)$, for any set $S \subseteq V$,*

$$\left| w(E(S)) - \frac{\mathrm{vol}(S)^2}{\mathrm{vol}(V)} \right| \le \lambda_2 \mathrm{vol}(S),$$

where λ_2 is the second largest eigenvalue of the simple random walk on G.

Definition 3 (Conductance)**.** *Given a weighted d-regular graph $G = (V, E, w)$, with weights $w : E \to \mathbb{R}_{\ge 0}$, for $S \subseteq V$, the conductance of S is defined as*

$$\phi(S) = \frac{w(S,\overline{S})}{d|S|},$$

where $w(S,\overline{S})$ is the sum of the weights of edges in the cut $(S,\overline{S})$. Note that since G is regular, the weighted degree of every vertex is d. The conductance of G is defined as

$$\phi(G) = \min_{S:|S|\le|V|/2} \phi(S).$$

The following theorem is well-known and follows from the easy side of the Cheeger's inequality.

Theorem 1.2.6. *For any regular graph $G = (V, E)$ and any set $S \subseteq V$ and $|S| \le |V|/2$*

$$\frac{1-\lambda_2(P)}{2} \le \phi(G) \le \phi(S) \le \frac{|N(S)|}{|S|}$$

where $1-\lambda_2(P)$ is the spectral gap of the simple random walk on G.

1.2.2 Markov Chains

Let P be the transition probability matrix of a Markov chain on a finite state space Ω with stationary distribution π. We say P is irreducible if P is connected. We say P is reversible w.r.t. π if for all $x, y \in \Omega$, we have $\pi(x)P(x,y) = \pi(y)P(y,x)$. In this case, the matrix P becomes self-adjoint w.r.t. the natural inner product $\langle \cdot, \cdot \rangle_\pi$ induced by π on $\mathbb{R}^\Omega$ given by $\langle \phi, \psi \rangle_\pi = \mathbb{E}_\pi[\phi\psi]$. Throughout, we only work with irreducible reversible Markov chains.

We will be interested in the mixing of our Markov chains, which quantifies the rate of convergence to stationarity. Specifically, for an initial starting distribution μ on Ω and error parameter $\epsilon > 0$, define

$$t_{\mathrm{mix}}(P, \epsilon, \mu) \stackrel{\text{def}}{=} \min\{t \geq 0 : \left\|\mu P^t - \pi\right\|_{\mathrm{TV}} \leq \epsilon\},$$

where $\|\mu - \nu\|_{\mathrm{TV}} = \frac{1}{2}\sum_{x \in \Omega} |\mu(x) - \nu(x)|$ gives the total variation distance between two distributions μ, ν on Ω. We write $t_{\mathrm{mix}}(P, \epsilon) = \sup_\mu t_{\mathrm{mix}}(P, \epsilon, \mu)$, where the supremum is over all possible starting distributions μ. The mixing time of P is defined as $t_{\mathrm{mix}}(P) = t_{\mathrm{mix}}(P, 1/4)$.

It is well-known that the mixing time is controlled by various constants arising from classical functional inequalities. To introduce these, we write $\mathcal{E}_P(\phi, \psi) = \langle \phi, (I - P)\psi \rangle_\pi$ for the Dirichlet form of P, $\mathrm{Var}_\pi(\phi) = \mathbb{E}_\pi[\phi^2] - \mathbb{E}_\pi[\phi]^2$ for the variance of ϕ w.r.t. π, and $\mathrm{Ent}_\pi(\phi) = \mathbb{E}_\pi[\phi \log \phi] - \mathbb{E}_\pi[\phi] \log \mathbb{E}_\pi[\phi]$. With these in hand, we define

1. Spectral Gap: $\lambda(P) \stackrel{\text{def}}{=} \inf_{f \neq 0} \frac{\mathcal{E}(f,f)}{\mathrm{Var}_\pi(f)}$
2. Modified Log-Sobolev Constant: $\rho(P) \stackrel{\text{def}}{=} \inf_{f \geq 0} \frac{\mathcal{E}(f, \log f)}{\mathrm{Ent}_\pi(f)}$
3. Standard Log-Sobolev Constant: $\kappa(P) \stackrel{\text{def}}{=} \inf_{f \geq 0} \frac{\mathcal{E}(\sqrt{f}, \sqrt{f})}{\mathrm{Ent}_\pi(f)}$

Bounds on these constants have the following consequences for the mixing time.

Proposition 1.2.7. *For an irreducible reversible Markov chain P on a finite state space Ω with stationary distribution π, we have the following inequalities:*

$$t_{\mathrm{mix}}(P) \leq O\left(\frac{1}{\lambda(P)} \log \frac{1}{\pi_{\min}}\right) \qquad \text{([LPW17])}$$

$$t_{\mathrm{mix}}(P) \leq O\left(\frac{1}{\rho(P)} \log\log \frac{1}{\pi_{\min}}\right) \qquad \text{([BT03])}$$

$$t_{\mathrm{mix}}(P) \leq O\left(\frac{1}{\kappa(P)} \log\log \frac{1}{\pi_{\min}}\right) \qquad \text{([DS96])}$$

We note that bounds on the standard and modified log-Sobolev constants also yield sub-Gaussian concentration estimates [Goe04; Sam05; BLM16], but we will not need these in our applications.

1.2.3 Simplicial Complexes

Garland's Method We will need the following simple facts, which follow simply by applying the Law of Total Probability appropriately and using the definition of the local walks P and local distributions π. Nearly identical equations were first observed and found to be useful by Garland [Gar73] in the context of understanding cohomology of simplicial complexes. They also lie at the heart of understanding expansion phenomena, in particular local spectral expansion, in simplicial complexes [Opp18; KO18].

Lemma 1.2.8 ([Opp18])**.** *Given a weighted simplicial complex (X, π_d), we may decompose ΠP as*

$$\Pi P = \mathbb{E}_{x\sim\pi}\Pi_x P_x.$$

Proof. We check entry by entry, applying the Law of Total Probability along the way. Note that the matrices on both sides have zero diagonal and so it suffices to check equality for the (y, z)-entry, where $y \neq z$.

$$\begin{aligned}
\mathbb{E}_{x\sim\pi}(\Pi_x P_x)(y,z) &= \sum_{x\neq y,z} \pi(x)\Pi_x(y)P_x(y,z) \\
&= \sum_{x\neq y,z} \cdot \frac{1}{d+1}\Pr_{\sigma\sim\pi_d}[x\in\sigma]\cdot\pi_x(y)\cdot\frac{\pi_x(\{y,z\})}{2\pi_x(y)} && \text{(Definition)} \\
&= \sum_{x\neq y,z} \frac{1}{d+1}\Pr_{\sigma\sim\pi_d}[x\in\sigma]\cdot\frac{1}{d(d-1)}\Pr_{\sigma\sim\pi_d}[y,z\in\sigma \mid x\in\sigma] && \text{(Definition)} \\
&= \frac{1}{d(d+1)}\cdot\frac{1}{d-1}\sum_{x\neq y,z}\Pr_{\sigma\sim\pi_d}[x,y,z\in\sigma] && \text{(Rearranging)} \\
&= \frac{1}{d(d+1)}\Pr_{\sigma\sim\pi_d}[y,z\in\sigma] && \text{(Law of Total Probability)} \\
&= \pi(y)P(y,z) && \text{(Definition)} \\
&= (\Pi P)(y,z)
\end{aligned}$$

□

Lemma 1.2.9 ([Opp18])**.** *Given a weighted simplicial complex (X, π_d), we may decompose ΠP^2 as*

$$\Pi P^2 = \mathbb{E}_{x\sim\pi}\pi_x\pi_x^\top$$

Proof. The main observation is that the rows of P are precisely the vectors π_x, and that the rows of ΠP are precisely the vectors $\pi(x)\pi_x$. The claim immediately follows. □

Local-to-Global Theorems As alluded to earlier, one of the beautiful insights of [DK17; KO18] is that local spectral expansion implies quantitative bounds on the spectral gap of the down-up walk.

Theorem 1.2.10 ([AL20])**.** *Let (X, π) be a $(\gamma_2, \ldots, \gamma_{d+1})$-local spectral expander. Then the second eigenvalue of the transition probabilty matrix of the down-up walk $P^\vee$ is bounded by*

$$\lambda_2(P^\vee) \leq 1 - \frac{1}{d} \prod_{j=2}^{d} (1 - \gamma_j).$$

Analogous results have also been proved for the decay of entropy [CLV21; GM20; Ali+21]. We state this result here since our main technical result Theorem 2.1.3 is a general method to obtain local spectral expansion for any weighted simplicial complex.

The above theorem has been used successfully in several prior works [Ana+19; ALO20; CLV20; Che+21b; Fen+22; Ana+21] to establish polynomial time mixing for various dynamics. However, the exponent of these running times are typically large, depending sensitively on the constants in the γ_j. Recently, it was shown that for weighted simplicial complexes arising from spin systems on bounded-degree graphs, we can do significantly better. We will need the following to establish optimal mixing times in our applications. We state it specifically for colorings, as we do not analyze any other spin systems in this dissertation.

Theorem 1.2.11 ([CLV21; Bla+22])**.** *Let (X, π) be a weighted simplicial complex arising from the uniform distribution over proper list-colorings of a graph $G = (V, E)$ with $|V| = n$ and maximum degree Δ. If for some constant C independent of n, (X, π) is a $(\gamma_2, \ldots, \gamma_n)$-local spectral expander with $\gamma_k \leq \frac{C}{k}$ for all k, then the spectral gap, standard and modified log-Sobolev constants are all $\Omega_{C,\Delta}(1/n)$. In particular, the Glauber dynamics mixes in $O_{C,\Delta}(n \log n)$ steps.*

Chapter 2

A Matrix Trickle-Down Theorem

2.1 Introduction

In this section, we prove a matrix trickle-down theorem, generalizing Oppenheim's influential trickle-down theorem, as a new technique to prove that a high dimensional simplicial complex is a local spectral expander. The results of this section were previously published in [ALO22].

Previously, we presented a statement of Oppenheim's trickle-down theorem in Theorem 1.0.2. Here, we present a different statement of the theorem. Recall that given a weighted simplicial complex (X, τ), for any face τ of co-dimension at least 2, P_τ is the transition probability matrix of the simple walk on the 1-skeleton on X_τ. When τ is a singleton $\{x\}$, we simplify the notation and write P_x for $P_{\{x\}}$.

Theorem 2.1.1 (Trickle-Down Theorem [Opp18])**.** *Given a weighted simplicial complex (X, π_d), suppose the following holds:*

1. ***Connectivity:*** *$\lambda_2(P) < 1$, i.e. the local walk P is connected/irreducible.*
2. ***Spectral Bound for Links Above:*** *For some $0 \le \lambda \le 1/2$, we have the bound $\lambda_2(P_x) \le \lambda$ for all $i \in X(0)$.*

Then the local walk P actually satisfies the spectral bound $\lambda_2(P) \le \frac{\lambda}{1-\lambda}$.

Remark 2. The original statement in [Opp18] does not have the assumption $\lambda \le 1/2$, but the two are completely equivalent, since if $\lambda > 1/2$, then $\frac{\lambda}{1-\lambda} > 1$, making the statement vacuously true.

This result has already had many applications, such as the construction of bounded-degree high-dimensional expanders [KO18] and sampling algorithms for matroids [Ana+19; CGM19; Ana+21] and other combinatorial structures [AL20].

One can easily verify that Theorem 1.0.2 can be derived through an inductive application of the following theorem. In particular, if the links of faces of co-dimension 2 are $(1-\delta)/d$-spectral expanders for some $0 < \delta \le 1$, then by applying Theorem 2.1.1 in a totally black-box fashion, it follows that (X, π) is a $(\gamma_2, \ldots, \gamma_{d+1})$-local spectral expander where $\gamma_k \le \frac{1-\delta}{d-(k-2)(1-\delta)} \le \frac{1-\delta}{d\delta}$ for all $2 \le k \le d+1$. But if the links of faces of co-dimension 2 have expansion more than $1/d$, Oppenheim's trickle-down theorem fails to give a non-trivial bound on local spectral expansions. Unfortunately, in many applications, in the worst case, the links of faces of co-dimension 2 have expansion $\Theta(1)$.

2.1.1 Main Results

Our main technical contribution in this chapter is to provide a framework which significantly generalizes Theorem 2.1.1, and makes the trickle-down theorem applicable to wider classes of weighted simplicial complexes, including those arising from proper colorings. One of our main insights is to replace the hypothesis that $\lambda_2(P_x) \le \lambda$, which merely provides a uniform bound on all nontrivial eigenvalues of P_x, by a matrix bound "$P_x \preceq M_x$". The hope is that the matrix M_x itself can simultaneously be easily bounded, as well as provide information on where the "bad" eigen-spaces of P_x are. So, roughly speaking, although many of the top dimensional links P_τ's may have a constant second eigenvalue, by carefully choosing M_τ's one can "average out" these bad eigen-spaces to show that the link of a lower dimensional face has small eigenvalues.

Before formalizing this in the following theorem, we need to make an important notational remark. In the rest of this chapter, for any d-dimensional simplicial complex X, we denote the distribution on its facets by π_d. For every face of co-dimension at least 2, $\pi_{\tau,i}$ is the probability distribution induced by π_d on the faces of dimension i of the link of τ, as defined in Eq. (1.1). Since we regularly work with probability distributions induced on faces of dimension 0, in this chapter, we write π_τ to denote $\pi_{\tau,0}$.

Theorem 2.1.2 (Matrix Trickle-Down Theorem [ALO22]). *Given a weighted simplicial complex (X, π_d), suppose the following holds:*

1. ***Connectivity:*** *$\lambda_2(P) < 1$, i.e. the local walk P is connected/irreducible.*
2. ***Generalized Spectral Bound for Links Above:*** *There is a family of symmetric matrices $\{M_x\}_{x \in X(0)}$ such that*
$$\Pi_x P_x - \alpha \pi_x \pi_x^\top \preceq M_x \preceq \frac{1}{2\alpha + 1} \Pi_x$$
for all $i \in X(0)$.

Then the local walk P actually satisfies the spectral bound $\Pi P - \left(2 - \frac{1}{\alpha}\right) \pi\pi^\top \preceq M$, and in particular $\lambda_2(P) \le \rho(\Pi^{-1} M)$, where M is any symmetric matrix satisfying $M \preceq \frac{1}{2\alpha}\Pi$ and $\mathbb{E}_{x\sim\pi} M_x \preceq M - \alpha M \Pi^{-1} M$.

By recursive application of Theorem 2.1.2, we get the following theorem.

Theorem 2.1.3 (Inductive Matrix Trickle-Down Theorem [ALO22]). *Let (X, π_d) be a totally connected weighted simplicial complex. Suppose $\{M_\tau \in \mathbb{R}^{X(0)\times X(0)}\}_{\tau \in X(\le d-2)}$ is a family of symmetric matrices satisfying the following:*

1. ***Base Case:*** *For every τ of co-dimension 2, we have the spectral inequality*
$$\Pi_\tau P_\tau - 2\pi_\tau \pi_\tau^\top \preceq M_\tau \preceq \frac{1}{5}\Pi_\tau.$$
2. ***Recursive Condition:*** *For every τ of co-dimension at least $k \ge 3$, M_τ satisfies*
$$M_\tau \preceq \frac{k-1}{2k-1}\Pi_\tau \quad \text{and} \quad \mathbb{E}_{x\sim\pi_\tau} M_{\tau\cup\{x\}} \preceq M_\tau - \frac{k-1}{k-2} M_\tau \Pi_\tau^{-1} M_\tau.$$

Then $\lambda_2(P_\tau) \leq \rho(\Pi_\tau^{-1} M_\tau)$ for all $\tau \in X(\leq d-2)$, where ρ represents the spectral radius. In particular, (X, π) is a $(\gamma_2, \ldots, \gamma_{d+1})$-local spectral expander with $\gamma_k = \max_{\tau \in X(d-k)} \rho(\Pi_\tau^{-1} M_\tau)$.

Remark 3. Typically, it is not difficult to construct some family of matrices $\{M_\tau\}$ satisfying the assumptions of Theorem 2.1.3. The key challenge is choose M_τ's such that one can bound $\rho(\Pi_\tau^{-1} M_\tau) \leq O(1/\operatorname{codim}(\tau))$. One of our second key insights is that the matrices M_τ can be designed to have convenient sparsity patterns depending on (X, π_d) which allow for straightforward bounds on $\rho(\Pi_\tau^{-1} M_\tau)$. For instance, in our application to proper colorings, our matrices M_τ will have rows and columns corresponding to vertex-color pairs vc, and they will be supported on the "proper coloring constraints", namely pairs uc, vc' of vertex-color pairs where the vertices u, v are neighbors and the two colors c, c' are identical. We demonstrate the usefulness of this approach on sampling proper colorings in graphs below.

2.2 Proof of Oppenheim's Trickle-Down Theorem via Matrix Trickle-Down Theorem

To see that theorem Theorem 2.1.2 theorem generalizes Theorem 2.1.1, note that if $\lambda_2(P_x) \leq \lambda \leq 1/2$ for all $i \in X(0)$ then $M_x = \lambda \Pi_x$, $\alpha = 1 - \lambda$, and $M = \frac{\lambda}{1-\lambda}\Pi$ satisfies the assumptions of the above theorem; in particular,

$$M_x = \lambda \Pi_x \leq \frac{1}{2(1-\lambda)+1}\Pi_x = \frac{1}{2\alpha+1}\Pi_x \qquad (\lambda \leq 1/2)$$

$$\Pi_x P_x - (1-\lambda)\pi_x \pi_x^\top \preceq \lambda \Pi_x = M_x \qquad (\lambda_2(P_x) \leq \lambda \text{ and } \pi_x P_x = \pi_x)$$

$$M = \frac{\lambda}{1-\lambda}\Pi \preceq \frac{1}{2(1-\lambda)}\Pi = \frac{1}{2\alpha}\Pi \qquad (\lambda < 1/2)$$

$$\begin{aligned}\mathbb{E}_{x\sim\pi} M_x &= \lambda \mathbb{E}_{x\sim\pi}\Pi_x = \lambda \Pi \\ &= \frac{\lambda}{1-\lambda}\Pi - (1-\lambda)\left(\frac{\lambda}{1-\lambda}\right)^2 \Pi \\ &= M - \alpha M \Pi^{-1} M.\end{aligned}$$

So, assuming connectivity of P, we get $\lambda_2(P) \leq \rho(\Pi^{-1}M) = \frac{\lambda}{1-\lambda}$ from Theorem 2.1.2 as desired.

2.3 Proof of Matrix Trickle-Down Theorem

Let us now prove Theorem 2.1.2. To do this, we first need the following lemma.

Lemma 2.3.1. Let (X, π_d) be a weighted simplicial complex. Suppose for a symmetric matrix M and an $\alpha \geq 1/2$, the matrix inequalities $M, \Pi P - \left(2 - \frac{1}{\alpha}\right)\pi\pi^\top \preceq \frac{1}{2\alpha}\Pi$ hold and

$$\Pi P - \alpha \Pi P^2 \preceq M - \alpha M \Pi^{-1} M \tag{2.1}$$

Then we have the bound $\Pi P - \left(2 - \frac{1}{\alpha}\right)\pi\pi^\top \preceq M$.

Proof. Our goal is to apply Lemma 1.2.3 to suitably chosen A, B. Define $Q = P - \left(2 - \frac{1}{\alpha}\right) \mathbf{1}\pi^\top$. A quick calculation shows that $Q - \alpha Q^2 = P - \alpha P^2$, and so by multiplying both sides of Eq. (2.1) by $\Pi^{-1/2}$, we see that Eq. (2.1) is equivalent to

$$\Pi^{1/2}Q\Pi^{-1/2} - \alpha\Pi^{1/2}Q^2\Pi^{-1/2} \preceq \Pi^{-1/2}M\Pi^{-1/2} - \alpha\Pi^{-1/2}M\Pi^{-1}M\Pi^{-1/2}$$

Taking $A = \Pi^{1/2}Q\Pi^{-1/2}$ and $B = \Pi^{-1/2}M\Pi^{-1/2}$, we see by assumption that A, B are symmetric matrices satisfying $A, B \preceq \frac{1}{2\alpha}I$ and $A(I - \alpha A) \preceq B(I - \alpha B)$. It follows by Lemma 1.2.3 that $A \preceq B$, which is equivalent to $\Pi P - \left(2 - \frac{1}{\alpha}\right) \pi\pi^\top = \Pi Q \preceq M$ as desired. □

With this lemma in hand, let us now prove Theorem 2.1.2.

Proof of Theorem 2.1.2. The conclusion follows immediately from Lemma 2.3.1, and so it suffices to verify the conditions of the lemma. By assumption, we already have $M \preceq \frac{1}{2\alpha}\Pi$. Furthermore, $\Pi_x P_x - \alpha\pi_{x,0}\pi_{x,0}^\top \preceq M_x \preceq \frac{1}{2\alpha+1}\Pi_x$ implies that $\lambda_2(P_x) \leq \frac{1}{2\alpha+1}$. Since $\lambda_2(P) < 1$, by Theorem 2.1.1 (the original trickle-down Theorem), $\lambda_2(P) \leq \frac{1}{2\alpha}$. Combined with the inequality $2 - \frac{1}{\alpha} \geq 1 - \frac{1}{2\alpha}$, which holds since $\alpha \geq 1/2$, it follows that $\Pi P - \left(2 - \frac{1}{\alpha}\right) \pi_0\pi_0^\top \preceq \frac{1}{2\alpha}\Pi$.

All that remains is to verify Eq. (2.1). Observe that

$$\begin{aligned}
\Pi P &= \mathbb{E}_{x\sim\pi}\Pi_x P_x && \text{(Lemma 1.2.8)} \\
&\preceq \mathbb{E}_{x\sim\pi}\left[\alpha\pi_x\pi_x^\top + M_x\right] && \text{(Assumption)} \\
&= \alpha\Pi P^2 + \mathbb{E}_{x\sim\pi}M_x && \text{(Lemma 1.2.9)} \\
&\preceq \alpha\Pi P^2 + M - \alpha M\Pi^{-1}M && \text{(Assumption)}
\end{aligned}$$

Rearranging, we obtain that $\Pi P - \alpha\Pi P^2 \preceq M - \alpha M\Pi^{-1}M$ as desired. □

2.4 A Slight Extension of Matrix Trickle-Down Theorem

Next, we prove an extension of the matrix trickle-down method to take into account when simplicial complexes factor as products of smaller complexes. This will be useful in the context of proper colorings when the input graph is broken into several connect components by coloring some of the vertices.

Given two pure simplicial complexes X, Y of dimensions d_X, d_Y respectively with disjoint ground sets, we may form another pure simplicial complex Z of dimension-d_Z for $d_Z := d_X + d_Y + 1$ called the product $X \times Y$ of X, Y by taking the ground set of $X \times Y$ to be the disjoint union of the ground sets of X, Y, taking the facets of $X \times Y$ to be of the form $\tau \cup \sigma$, where τ, σ are facets of X, Y respectively, and then taking downwards closure. If π_{X,d_X}, π_{Y,d_Y} are distributions on the facets of X, Y respectively, we then form corresponding product distribution $\pi_{Z,d_Z} = \pi_{X,d_X} \times \pi_{Y,d_Y}$ on the facets of $X \times Y$ by taking $\pi_{Z,d_Z}(\tau \cup \sigma) = \pi_{X,d_X}(\tau) \cdot \pi_{Y,d_Y}(\sigma)$ for facets τ, σ of X, Y respectively.

Lemma 2.4.1. Given a weighted simplicial complex $(Z, \pi_{Z,d_Z}) = (X \times Y, \pi_{X,d_X} \times \pi_{Y,d_Y})$, *we may decompose* $P_{Z,\emptyset}$ *into* $P_{X,\emptyset}, P_{Y,\emptyset}$ *via the formula*

$$P_Z - \frac{d_Z+1}{d_Z}\mathbf{1}\pi_Z^\top = \begin{bmatrix} \frac{d_X}{d_Z}\left(P_X - \frac{d_X+1}{d_X}\mathbf{1}\pi_X^\top\right) & 0 \\ 0 & \frac{d_Y}{d_Z}\left(P_Y - \frac{d_Y+1}{d_Y}\mathbf{1}\pi_Y^\top\right) \end{bmatrix}$$

where we put a zero matrix for P_X *if* $d_X = 0$ *(and similarly for* P_Y*). In particular, if* M_X, M_Y *are symmetric matrices satisfying* $\Pi_X P_X - \frac{d_X+1}{d_X}\pi_X\pi_X^\top \preceq M_X$ *and* $\Pi_Y P_Y - \frac{d_Y+1}{d_Y}\pi_Y\pi_Y^\top \preceq M_Y$ *(we let* $M_X = 0$ *if* $d_X = 0$*, and similarly* $M_Y = 0$ *if* $d_Y = 0$*), then*

$$\Pi_Z P_Z - \frac{d_Z+1}{d_Z}\pi_Z\pi_Z^\top \preceq \begin{bmatrix} \frac{d_X(d_X+1)}{d_Z(d_Z+1)}M_X & 0 \\ 0 & \frac{d_Y(d_Y+1)}{d_Z(d_Z+1)}M_Y \end{bmatrix}$$

Proof. Let $x \in X(0)$ and $y \in Y(0)$. Then,

$$P_Z(x,y) \underset{\text{Eq. (5.1)}}{=} \frac{1}{d_Z}\Pr_{\sigma\sim\pi_{Z,d_Z}}[y \in \sigma \mid x \in \sigma] \underset{\text{By independence}}{=} \frac{1}{d_Z}\Pr_{\sigma\sim\pi_{Z,d_Z}}[y \in \sigma] \underset{\text{Eq. (1.1)}}{=} \frac{d_Z+1}{d_Z}\pi_Z(y).$$

Therefore $(P_Z - \frac{d_Z+1}{d_Z}\mathbf{1}\pi_Z^\top)(x,y) = 0$ for all $x \in X(0), y \in Y(0)$ (similarly $(P_Z - \frac{d_Z+1}{d_Z}\mathbf{1}\pi_Z^\top)(y,x) = 0$). For $x, x' \in X(0)$, if $x \neq x'$, by Eq. (5.1) we have

$$P_Z(x,x') = \frac{1}{d_Z}\Pr_{\sigma\sim\pi_{Z,d_Z}}[x' \in \sigma \mid x \in \sigma] = \frac{1}{d_Z}\Pr_{\sigma\sim\pi_{X,d_X}}[x' \in \sigma | x \in \sigma] = \frac{d_X}{d_Z}P_X(x,x').$$

If $x = x'$, then $P_Z(x,x) = \frac{d_X}{d_Z}P_X(x,x) = 0$ by definition of the local random walk. Note that, if $d_X = 0$, $P_X(x,x')$ is defined to be 0 by the statement of the theorem. Furthermore, for $x, x' \in X(0)$, by Eq. (1.1)

$$\mathbf{1}\pi_Z^\top(x,x') = \frac{1}{d_Z+1}\Pr_{\sigma\sim\pi_{Z,d_Z}}[x' \in \sigma] = \frac{1}{d_Z+1}\Pr_{\sigma\sim\pi_{X,d_X}}[x' \in \sigma] = \frac{d_X+1}{d_Z+1}\mathbf{1}\pi_Z^\top(x,x').$$

Putting these together, $(P_Z - \frac{d_Z+1}{d_Z}\pi_Z\pi_Z^\top)(x,x') = \left(\frac{d_X}{d_Z}(P_X - \frac{d_X+1}{d_X}\mathbf{1}\pi_X^\top)\right)(x,x')$ for all $x, x' \in X(0)$. One can see that a similar statement holds for the lower-right block. This finishes the proof of the first part. To get the second part, it is enough to note that when $d_X = 0$, on the $X(0) \times X(0)$ block of $P_Z - \frac{d_Z+1}{d_Z}\mathbf{1}\pi_Z^\top$ we have $-\frac{d_X+1}{d_Z}\mathbf{1}\pi_X^\top \preceq 0$. □

Theorem 2.4.2 ([ALO22]). *Let* (X, π_d) *be a totally connected weighted complex. Suppose* $\{M_\tau \in \mathbb{R}^{X(0)\times X(0)}\}_{\tau\in X(\le d-2)}$ *is a family of symmetric matrices satisfying the following:*

1. ***Base Case:*** *For every* τ *of co-dimension 2, we have the spectral inequality*

$$\Pi_\tau P_\tau - 2\pi_\tau\pi_\tau^\top \preceq M_\tau \preceq \frac{1}{5}\Pi_\tau.$$

2. ***Recursive Condition:*** *For every* τ *of co-dimension at least* $k \ge 3$*, at least one of the following holds:* M_τ *satisfies*

$$M_\tau \preceq \frac{k-1}{3k-1}\Pi_\tau \quad \text{and} \quad \mathbb{E}_{x\sim\pi_\tau} M_{\tau\cup\{x\}} \preceq M_\tau - \frac{k-1}{k-2}M_\tau\Pi_\tau^{-1}M_\tau. \tag{2.2}$$

Or, $(X_\tau, \pi_{\tau,k-1})$ is a product of weighted simplicial complexes $(Y_1, \mu_1), \ldots, (Y_t, \mu_t)$ and for every $\eta \in X_\tau(k-1)$,

$$M_\tau = \bigoplus_{1 \le i \le t : d_{Y_i} \ge 1} \frac{d_{Y_i}(d_{Y_i}+1)}{k(k-1)} M_{\tau \cup \eta_{-i}},$$

where $\eta_{-i} = \eta \setminus Y_i(0)$.

Then for every $\tau \in X(\le d-2)$, we have the bound $\lambda_2(\Pi_\tau P_\tau) \le \rho(\Pi_\tau^{-1} M_\tau)$.

Proof. Apply Theorem 2.1.2 and Lemma 2.4.1 inductively. □

Chapter 3

Application of Matrix Trickle-Down in Sampling Graph Colorings

3.1 Introduction

In this section, we apply the matrix trickle-down theorem to show that the natural Glauber dynamics for sampling graph colorings mixes in polynomial time, in a previously unsolved regime. The results of this section were previously published in [ALO22].

Given an (undirected) graph $G = (V, E)$ with $n = |V|$ vertices and with maximum degree $\Delta \geq 1$ can we generate a uniformly random proper coloring of vertices of G using q colors? For $q \leq \Delta$, there is no efficient algorithm (in the sense of an FPRAS) to approximately count proper q-colorings (at least when q is even) unless NP = RP, even for Δ-regular graphs which are triangle-free [GŠV15]. This fundamental question in the field of counting and sampling has puzzled researchers for decades. One can study a natural Markov Chain (MC), known as the Glauber dynamics, to generate a random proper coloring: Given a proper vertex coloring of G, choose a uniformly random vertex $v \in G$ and re-color v, namely choose a uniformly random color which is not present in any of the neighbors of v.

It is not hard to see that for $q \geq \Delta + 2$ this chain is irreducible and has unique stationary distribution which is uniform over all proper colorings of G. It is conjectured that the Glauber dynamics mixes in time $O(n \log n)$ for q as low as $\Delta + 2$ but despite significant attempts we are still very far from proving this conjecture.

To this date, the best known result for general graphs is due Chen, Delcourt, Moitra, Perarnau and Postle [Che+19] who show that the Glauber dynamics mixes in polynomial time for $q \geq (11/6 - \epsilon)\Delta$ for some universal constant $\epsilon > 0$; this slightly improves on the classical works of Jerrum and Vigoda [Jer95; Vig99] which bounds the mixing time by a polynomial in n for $q \geq (11/6)\Delta$.

Most of the recent analyses of the Glauber dynamics are focused on "locally sparse" graphs [HV03; Mol04; HV05; FV06; FV07; HVV07; Dye+13; Che+21b; Fen+22] where it was typically shown how to break the 11/6 barrier bound when the underlying graph has a large girth. We note that these assumptions are typically very strong as it can be seen that triangle graph free graphs can be colored with as little as $O(\Delta/\log \Delta)$ many colors [Joh96].

The results on locally sparse graphs typically exploit (strong) correlation decay properties:

Roughly speaking, they imply that if we color a vertex v with a color c, then the marginal probability of coloring a "far away" vertex u with a color c' does not change, or changes very mildly. Although it is conjectured that vertex coloring exhibits correlation decay, more formally known as "strong spatial mixing" property, for $q \geq \Delta + O(1)$, to this date, we are lacking techniques to establish such a statement (see e.g., [GMP05; Yin14; GKM15; Eft+19]). In this chapter, we study random proper edge coloring of graphs, which equivalently can be seen as a random proper vertex coloring of line graphs. Unlike most recent trends which focus on sparse graphs with large girths, line graphs are very dense locally as they contain induced cliques of size $\Omega(\Delta)$. To the best of our knowledge, prior to our result, the only result on edge coloring which goes significantly beyond the 11/6 barrier is recent work of Delcourt, Heinrich and Perarnau [DHP20] which shows that the Glauber dynamics mixes rapidly when the underlying graph is a tree and $q \geq \Delta + 1$. Note that for a graph with maximum degree Δ, the maximum degree of the line graph could be as large as 2Δ. Therefore, with the 11/6 barrier, one would need $q \geq 11/3\Delta$ to guarantee polynomial mixing for all edge coloring instances.

3.1.1 Main Results

In our main theorem we prove that the aforementioned barrier can be broken for edge coloring of any graph with maximum degree Δ.

Theorem 3.1.1 (Main [ALO22]). *Let $G = (V, E)$ be a graph of maximum degree Δ. For any $0 < \epsilon \leq \frac{1}{10}$ such that $\frac{\ln^2 \Delta}{\Delta} \leq \frac{\epsilon^3}{15}$, and any collection of color lists $L = \{L(e)\}_{e \in E}$ satisfying $|L(e)| \geq \Delta(e) + (4/3 + 4\epsilon)\Delta$ where $\Delta(e)$ is the number of neighbors of e in the line graph of G, the spectral gap of the Glauber dyanmics for sampling proper L-edge-list-colorings on G is $\Omega(n^{-O(1/\epsilon)})$, so the mixing time is $O(n^{O(1/\epsilon)})$. Furthermore, if $\Delta \leq O(1)$, the modified and standard log-Sobolev constants are $\Omega_{\epsilon,\Delta}(1/n)$.*

We remark that our general mixing time bound has no dependence on Δ or q. So, the algorithm runs in polynomial time even for graphs of unbounded degree.

In a second contribution we show that for any list vertex coloring instance where G is a tree with max degree Δ, and the size of the list of every vertex v is at least $\Delta(v) + \epsilon\Delta$ for $\epsilon = \Omega(\frac{\ln \Delta}{\sqrt{\Delta}})$ the Glauber dyanics mixes rapidly and generates a uniformly random vertex coloring of G.

Theorem 3.1.2. *Let $G = (V, E)$ be a tree of maximum degree Δ. For any $0 < \epsilon \leq 1$ such that $\frac{\ln^2 \Delta}{\Delta} \leq \frac{\epsilon^2}{100}$ and any collection of color lists $L = \{L(v)\}_{v \in V}$ satisfying $|L(v)| \geq \Delta(v) + \epsilon\Delta$, the spectral gap of the Glauber dynamics for sampling proper L-vertex-list-colorings on G is $\Omega(n^{-O(1/\epsilon)})$, therefore the mixing time is $O(n^{O(1/\epsilon)})$. Furthermore, if $\Delta \leq O(1)$, the modified and standard log-Sobolev constants are $\Omega_{\epsilon,\Delta}(1/n)$.*

The above theorem, although it is not as strong as [MSW04], shows that Glauber dynamics mixes rapidly even when we have a list coloring problem on a tree and furthermore it gives a possible avenue to exploit our techniques to prove that Glauber dynamics mixes rapidly on any graph when $q \geq (1 + \epsilon)\Delta$. We expect that upon further investigation our techniques can be coupled with the extensive literature on random proper colorings of graphs with large girth to break the $11/6 - \epsilon$ barrier.

To establish the above results, we view the Glauber dynamics as a high dimensional walk on a simplicial complex and we prove that this complex is a local spectral expander. As discussed in Section 1.2.3, by local-to-global theorems, local spectral expansion immediately implies a bound on the spectral gap of the Glauber dynamics and therefore on its mixing time. Given a graph G and a set of q colors, we build a simplicial complex called the *coloring complex* as follows: We let each facet in X be a set of vertex color pairs that corresponds to a proper coloring of the vertices of G. The *edge-coloring complex* of a graph G is the vertex-coloring complex of the line graph of G. With this definition in hand, the Glauber dynamics can be viewed as a walk on the facets of the coloring complex X. Unlike recent developments [ALO20; Che+21b; Fen+22; CLV20; CLV21; Fri+21] that exploit the correlation decay property to prove local spectral expansion, we use the matrix trickle-down theorem to bound the local spectral expansion of this complex. We expect our technique to find more applications in analysis of Markov chains as well as in other applications of simplicial complexes.

We remark that recently, building on this work, [WZZ24] used the matrix trickle-down theorem to prove that the Glauber dyanmics for sampling proper vertex colorings of line graphs with n vertices and maximum degree Δ mixes in time $O_\Delta(n \log n)$ when the number of available colors satisfies $q \geq (1 + o(1))\Delta$.

3.1.2 Related Prior Work

Sampling uniformly random proper colorings of bounded-degree graphs is a well-studied problem going back to the 1990s [Jer95; SS97; Vig99] with applications to statistical physics. As alluded to before, nearly all prior results showing rapid mixing of the Glauber dynamics for this problem used variants of the coupling method or the correlation decay property.

Another method of attack for graph coloring is based on deterministic algorithms where one typically exploits Weitz's elegant algorithmic framework [Wei04] based on the strong spatial mixing or the Barvinok's polynomial interpolation method [Sok01; GK12; LY13; LSS19; Ben+21].

Recently, it was observed [ALO20; Che+21b; Fen+22] that the perspective of high-dimensional expanders, and in particular local spectral expansion, yields powerful new methods to obtain rapid mixing for multi-state spin systems. These results crucially relied on the correlation decay property to bound the local spectral expansion of desired distribution. Specifically, for colorings, [Che+21b; Fen+22] showed that the correlation decay results of [GKM15] give strong local spectral expansion bounds for proper colorings on triangle-free graphs when $q > \alpha\Delta$, where $\alpha \approx 1.763 < 2$ is a constant. They concluded rapid mixing of the Glauber dynamics in this regime, a result that seems difficult to obtain using coupling arguments.

However, despite the power of this approach, the main difficulty is that obtaining correlation decay for proper colorings is extremely challenging. Along this line, our main technical theorem deviates from this recent trend as we directly bound the local spectral expansion of the underlying complex by induction instead of appealing to the correlation decay property.

3.2 Preliminaries

Notational Remark In this chapter, for any d-dimensional simplicial complex X, we denote the distribution on its facets by π_d. For every face of co-dimension at least 2, $\pi_{\tau,i}$ is the probability distribution induced by π_d on the faces of dimension i of the link of τ, as defined in Eq. (1.1). Since we regularly work with probability distributions induced on 0-dimensional faces, in this chapter we write π_τ to denote $\pi_{\tau,0}$.

Matrices and Vectors Given a set S, we write $v \in \mathbb{R}^S$ and $A \in \mathbb{R}^{S\times S}$ to respectively denote a vector and a matrix indexed by S. We see a probability distribution p over a set S as a vector $p \in \mathbb{R}^S_{\geq 0}$. For any matrix $A \in \mathbb{R}^{S\times S}$ and any $S' \subseteq S$, $A^{S'} \in \mathbb{R}^{S\times S}$ is defined to be $A^{S'}(x,y) = A(x,y)$ for $x, y \in S'$, and 0 on all other entries. We say a matrix A is diagonal if $A(x,y) = 0$ whenever $x \neq y$. We say matrix $A \in \mathbb{R}^{S\times S}$ is hollow if $A(x,x) = 0$ for all $x \in S$. For any matrix A define A^+ as $A^+(x,y) \coloneqq \max\{0, A(x,y)\}$ and $A^- \coloneqq A - A^+$. Furthermore, $\mathrm{diag}(A)$ is a diagonal matrix whose diagonal elements match those of A, i.e. $A(x,x) = \mathrm{diag}(A)(x,x)$. We define $\text{off-diag}(A) \coloneqq A - \mathrm{diag}(A)$.

Graphs Fix a graph $G = (V,E)$. We denote the degree of each vertex v by $\Delta_G(v)$ and the maximum degree of the graph by Δ_G. Let $n_G \coloneqq |V|$ and $m_G \coloneqq |E|$. Furthermore, let $G[U]$ be the induced subgraph of G on $U \subseteq V$. We may drop the subscripts when it is clear from context. For any $u, v \in V$, we write $u \sim v$ if $\{u,v\} \in E$. Furthermore, for any $e \in E$, we write $e \sim v$ whenever vertex v is an endpoint of e, and for an edge $f \neq e$ we write $e \sim f$ when e and f share an endpoint. For an edge $e = \{u,v\}$, define $\Delta(e)$ as the number of edges that share an endpoint with e, i.e. $\Delta(e) \coloneqq \Delta(u) + \Delta(v) - 2$. The line graph $L(G)$ of a graph G is defined as follows: every vertex in $L(G)$ corresponds to an edge in G and there is an edge between two vertices in $L(G)$ if their corresponding edges in G share an endpoint.

Coloring Complexes of Disconnected Graphs A natural example of product of simplicial complexes is the vertex-coloring complex of a disconnected graph. Say $G = (V,E)$ is a graph with n vertex and ℓ connected components $G[U_1], \ldots, G[U_\ell]$ and associated complexes $X_1, \ldots X_\ell$. If (X, π_{n-1}) is the complex associated with G and π_{n-1} is the uniform distribution over its facets, then we can write $(X, \pi_{n-1}) = (X_1, \mu_1) \times \cdots \times (X_l, \mu_\ell)$, where μ_i is the uniform distribution over facets of X_i. Suppose we have associated a matrix $A_\tau \in \mathbb{R}^{X(0)\times X(0)}$ to any non-empty face τ of co-dimension at least 2 and assume that for any $1 \leq i \leq \ell$, when τ_{-i} and σ_{-i} are two arbitrary colorings of all connected components except i, then $A_{\tau_{-i}} = A_{\sigma_{-i}}$. We associate a block-diagonal matrix

$$f_\times(X, \{A_\tau\}_{\emptyset \subsetneq \tau \in X(\leq n-3)}) \coloneqq \sum_{1 \leq i \leq \ell : |U_i| \neq 1} A_{\tau_{-i}}.$$

When X is the edge-coloring complex of a graph G, it is the vertex-coloring complex of the line graph of G, therefore $f_\times(X, \{A_\tau\}_{\emptyset \subsetneq \tau \in X(\leq n-3)})$ is given by the above definition.

3.3 Vertex Coloring

Fix an integer q and graph $G = (V, E)$ and a function L that maps each $v \in V$ to a subset of $[q]$. We call (G, L) a vertex-list-coloring instance. For any $u, v \in V$, we write $u \sim_c v$ when $u \sim v$ and $c \in L(u) \cap L(v)$. Furthermore, we define a β-extra color vertex-list-coloring instance as follows.

Definition 4. *We say a vertex-list-coloring instance* (G, L) *is a* β*-extra-color instance if for each* $v \in V$, $|L(v)| \geq \beta + \Delta_G(v)$.

We call an assignment $\sigma : V \to [q]$ a L-vertex-list-coloring of G if $\sigma(v) \in L(v)$ for all $v \in V$; we say σ is proper if $\sigma(u) \neq \sigma(v)$ whenever $u \sim v$. When it is clear from context we say σ is a proper coloring to mean it is a proper L-vertex-list-coloring. We say τ is proper partial coloring on $U \subset V$ when it is a proper $L|_U$-vertex-list-coloring for $G[U]$. We may view list-colorings σ as sets of vertex-color pairs (v, c), which we denote by vc for convenience. When the graph G and color lists L are clear from context, we write π_{n-1} for the uniform distribution over proper L-vertex-list-coloring of $G = (V, E)$. For a proper partial coloring on $U \subset V$, $v \in V \setminus U$ and $c \in L(v)$, define

$$p(vc|\tau) \coloneqq \mathbb{P}_{\sigma\sim\pi_{n-1}}(\sigma(v) = c|\forall u \in U : \sigma(u) = \tau(u)).$$

In order to analyze the Glauber dynamics for an vertex-list-coloring instance (G, L), we can build an $(n-1)$-dimensional simplicial complex such that the down-up walk on its facets is the same as the Glauber dynamics on (G, L). Our aim is to apply Theorem 2.4.2 to bound the second eigenvalue of the transition probability matrix of the local walks and then use **??** to bound the second eigenvalue of the transition probability matrix of the global down-up walk on the facets.

Definition 5 (Simplicial Complex of a Vertex-List-Coloring Instance). *Given a vertex-list-coloring instance* (G, L)*, let* $X(G, L)$ *be a pure* $(n-1)$*-dimensional simplicial complex specified by the following facets:* $\{(v, \sigma(v))\}_{v\in V}$ *is a facet if and only if* σ *is a proper* L*-vertex-list-coloring for* G*.*

When it is clear from context, we abbreviate $X(G, L)$ to X. Note that for all $0 \leq k \leq n$, any face τ of co-dimension k is a proper partial coloring on a subset of vertices U of size $n - k$, i.e., k vertices remain uncolored. Furthermore, $X_\tau(0)$ can be seen as the set of all vc such that $c \in L(v)$, $v \notin U$ and for any $u \sim v$, $uc \notin \tau$. We define $V_\tau \coloneqq \{v : \exists c, vc \in X_\tau(0)\}$. Let $G_\tau \coloneqq G[V_\tau]$ and $\Delta_\tau(.)$ be the degree function of G_τ and Δ_τ be its maximum degree. We define $L_\tau(v) \coloneqq \{c \in L(v) : vc \in X_\tau(0)\}$ to be the list of remaining colors available to v after coloring vertex u with c for each $uc \in \tau$. Let $l_\tau(v) \coloneqq |L_\tau(v)|$ and $l_\tau(u, v) \coloneqq |L_\tau(u) \cap L_\tau(v)|$. We write $v \sim_{\tau,c} u$ for vertices v, u when $v \sim u$ and $c \in L_\tau(v) \cap L_\tau(u)$. Furthermore, for $U \subseteq V \setminus V_\tau$, let $\tau|_U \coloneqq \{vc \in \tau : v \in U\}$.

3.3.1 Diagonal Matrix Bounds

To demonstrate the essence of our approach better, we start by restricting our attention to when the matrix bounds $\{M_\tau\}_{\tau\in X:\mathrm{codim}(\tau)\geq 2}$ in Theorem 2.4.2 are diagonal matrices. Using diagonal matrix bounds, we analyze the Glauber dynamics for $(1 + \epsilon)\Delta$-extra-color vertex-list-coloring instances.

Theorem 3.3.1. *Suppose (G, L) is a $(1+\epsilon)\Delta$-extra-color vertex-list-coloring instance for an $0 < \epsilon \leq 1$ such that $\frac{\ln(\Delta)+2}{\Delta} \leq \frac{\epsilon^2}{40}$, and let (X, π_{n-1}) be its associated weighted simplicial complex. For any $2 \leq k \leq n$ and $\tau \in X$ of co-dimension k we have $\lambda_2(P_\tau) \leq \frac{\frac{5}{2\epsilon}}{k-1}$.*

Combined with local-to-global theorems (Theorems 1.0.1 and 1.2.11), this yields a mixing time of $O(n \log n)$ for bounded-degree graphs, and $n^{O(1/\epsilon)}$ in general, in this setting where we have at least $(1+\epsilon)\Delta$ additional colors available to each vertex. Again, we emphasize that this mixing result in itself is not new; a simple coupling argument can already recover $O(n \log n)$ mixing for $(\Delta+1)$-extra-color vertex-list-coloring instances. However, we will see later on how our proof technique can be used to obtain new mixing results for sampling edge colorings which, to the best of our knowledge, cannot be recovered via simple coupling arguments.

To prove the above statement, first for any τ of co-dimension 2, we find a diagonal matrix F_τ such that $\Pi_\tau P_\tau \preceq 2\pi_\tau\pi_\tau^\top + \Pi_\tau F_\tau$. Then, for all τ of co-dimension at least 3, we apply Theorem 2.4.2 to $M_\tau = \frac{\Pi_\tau F_\tau}{k-1}$ to find a sufficient condition on the diagonal matrix F_τ based on matrices $F_{\tau'}$ for faces $\tau \subsetneq \tau'$, to get $\lambda_2(P_\tau) \leq \frac{\rho(F_\tau)}{k-1}$. To find F_τ for faces τ of co-dimension 2, we state a more general proposition that is also useful for approaches that use non-diagonal matrix bounds.

Proposition 3.3.2. *Given a vertex-list-coloring instance (G, L), consider the weighted complex (X, π_{n-1}). For any face τ of co-dimension 2 such that $G_\tau = (\{u, v\}, \{uv\})$ is connected we have,*

$$\Pi_\tau P_\tau - 2\pi_\tau\pi_\tau^\top \preceq \sqrt{\Pi_\tau}\widetilde{M}_\tau\sqrt{\Pi_\tau}, \tag{3.1}$$

where $\widetilde{M}_\tau$ is a block diagonal matrix with a block $\widetilde{M}_\tau^c$ for every color c such that

$$\widetilde{M}_\tau^c = \begin{pmatrix} \frac{1}{(l_\tau(u)-1)(l_\tau(v)-1)} & \frac{-1}{\sqrt{(l_\tau(u)-1)(l_\tau(v)-1)}} \\ \frac{-1}{\sqrt{(l_\tau(u)-1)(l_\tau(v)-1)}} & \frac{1}{(l_\tau(u)-1)(l_\tau(v)-1)} \end{pmatrix}$$

and all other entries are 0.

Proof. For clarity, we drop τ from all notation in the proof. Write $\widetilde{M} = \widetilde{M}_d + \widetilde{M}_o$, where $\widetilde{M}_d = \text{diag}(\widetilde{M})$ and $\widetilde{M}_o = \text{off-diag}(\widetilde{M})$. First observe that for $c \in L(u)$,

$$\pi(uc) = \begin{cases} \frac{l(v)-1}{2(l(u)l(v)-l(u,v))} & \text{if } c \in L(u) \cap L(v) \\ \frac{l(v)}{2(l(u)l(v)-l(u,v))} & \text{otherwise} \end{cases}$$

and a similar identity holds for any $c \in L(v)$. Also, observe that

$$\Pi P = \frac{J - J^u - J^v}{2(l(u)l(v) - l(u,v))} + \sqrt{\Pi}\widetilde{M}_o\sqrt{\Pi},$$

where J, J^u, J^v are the all-ones matrix, all-ones matrix on uc rows/columns and all-ones matrix on vc rows/columns, respectively. So, subtracting $\widetilde{M}_o$ from both sides of (3.1) and multiplying by $2(l(u)l(v) - l(u,v))$ it is enough to show

$$J - J^u - J^v - 4(l(u)l(v) - l(u,v))\pi\pi^\top \preceq 2(l(u)l(v) - l(u,v))\sqrt{\Pi}\widetilde{M}_d\sqrt{\Pi} =: N_d \tag{3.2}$$

Write $\ell = l(v)\mathbf{1}^u + l(u)\mathbf{1}^v$. Also, let $s \in \mathbb{R}^{l(u)+l(v)}$ where $s(xc) = 1$ if $c \in L(u) \cap L(v)$ and $s(xc) = 0$ otherwise for $x \in \{u, v\}$. Then, by Fact 1.2.2 we can write,

$$4(l(u)l(v) - l(u,v))\pi\pi^\top = \frac{(\ell - s)(\ell - s)^\top}{l(u)l(v) - l(u,v)} \underbrace{\succeq}_{Fact\ 1.2.2} \frac{\ell\ell^\top + ss^\top - \frac{1}{2}\ell\ell^\top - 2ss^\top}{l(u)l(v) - l(u,v)}$$
$$\succeq \frac{\ell\ell^\top}{2l(u)l(v)} - \frac{ss^\top}{l(u)l(v) - l(u,v)}$$

Plugging this into (3.2) it is enough to show that

$$J - J^u - J^v + \frac{ss^\top}{l(u)l(v) - l(u,v)} = \mathbf{1}^u \mathbf{1}^{v\top} + \mathbf{1}^v \mathbf{1}^{u\top} + \frac{ss^\top}{l(u)l(v) - l(u,v)} \preceq \frac{\ell\ell^\top}{2l(u)l(v)} + N_d \tag{3.3}$$

First, observe that by another application of Fact 1.2.2, $l^2(v)\mathbf{1}^u\mathbf{1}^{u\top} + l^2(u)\mathbf{1}^v\mathbf{1}^{v\top} \succeq l(u)l(v)(\mathbf{1}^u\mathbf{1}^{v\top} + \mathbf{1}^v\mathbf{1}^{u\top})$. So,

$$\frac{\ell\ell^\top}{2l(u)l(v)} = \frac{(l(v)\mathbf{1}^u + l(u)\mathbf{1}^v)(l(v)\mathbf{1}^u + l(u)\mathbf{1}^v)^\top}{2l(u)l(v)} \succeq \mathbf{1}^u\mathbf{1}^{v\top} + \mathbf{1}^v\mathbf{1}^{u\top}$$

Let $I^\cap \in \mathbb{R}^{(l(u)+l(v))\times(l(u)+l(v))}$ be the identity matrix only on entries xc, xc where $x \in \{u, v\}$ and $c \in L(u) \cap L(v)$. Finally, (3.3) simply follows from the fact that

$$\frac{ss^\top}{l(u)l(v) - l(u,v)} \preceq \frac{l(u,v)}{l(u)l(v) - l(u,v)} I^\cap \preceq N_d$$

where the first inequality uses that the only non-zero rows of $ss^\top$ correspond to a common color and the sum of the entries of any such row is exactly $l(u,v)$ and the last inequality uses that $\frac{l(u,v)}{l(u)l(v)-l(u,v)} \leq \frac{1}{\max\{l(u),l(v)\}-1}$ and that $N_d(uc, uc) = \frac{1}{l(u)-1}, N_d(vc, vc) = \frac{1}{l(v)-1}$ if $c \in L(u) \cap L(v)$ and it is zero otherwise. □

Note that in the above proposition, $\widetilde{M_\tau} \preceq (\frac{1}{\beta} + \frac{1}{\beta^2}) I^{X_\tau(0)}$, which gives us the diagonal matrix F_τ for any τ of co-dimension 2 such that G_τ is connected. Now, using Theorem 2.4.2, we derive a set of sufficient conditions on the family $\{F_\tau\}_{\tau\in X:\mathrm{codim}(\tau)\geq 2}$ to get $\lambda_2(P_\tau) \leq \frac{\rho(F_\tau)}{k-1}$ for all τ of co-dimension $2 \leq k \leq n$.

Proposition 3.3.3. Given a β-extra-color vertex-list-coloring instance (G, L), with corresponding weighted simplicial complex (X, π_{n-1}), suppose $\{F_\tau \in \mathbb{R}^{X(0)\times X(0)}\}_{\tau\in X:\mathrm{codim}(\tau)\geq 2}$ is a family of diagonal matrices supported on $X_\tau(0)\times X_\tau(0)$ such that $F_\tau = f_\times(X_\tau, \{F_{\tau\cup\sigma}\}_{\emptyset\subsetneq\sigma\in X_\tau(\leq \mathrm{codim}(\tau)-3)})$ if G_τ is disconnected and otherwise,

1. *For all τ of co-dimension 2: $F_\tau(vc, vc) = \frac{1}{\beta} + \frac{1}{\beta^2}$ for all $vc \in X_\tau(0)$.*

2. *For all τ of co-dimension $k \geq 3$: $F_\tau \preceq \frac{(k-1)^2}{3k-1} I^{X_\tau(0)}$ and for all $vc \in X_\tau(0)$*

$$\sum_{uc' \in X_{\tau\cup vc}(0)} p(uc'|\tau \cup vc) F_{\tau\cup uc'}(vc, vc) \leq (k-2)F_\tau(vc, vc) - F_\tau^2(vc, vc).$$

Then, for all $k \geq 2$ and τ of co-dimension k, $\lambda_2(P_\tau) \leq \frac{\rho(F_\tau)}{k-1}$.

Proof. We prove that the conditions of Theorem 2.4.2 hold for $M_\tau := \frac{\Pi_\tau F_\tau}{k-1}$ for any face τ of co-dimension at least 2. The desired condition holds for any τ of co-dimension 2 by Proposition 3.3.2. Now, let $k \geq 3$. First assume that G_τ is disconnected with connected components $G_\tau[U_1], \ldots, G_\tau[U_\ell]$ and associated complexes $Y_1, \ldots Y_\ell$. We can write $(X, \pi_{\tau,k-1}) = (Y_1, \mu_1) \times \cdots \times (Y_\ell, \mu_\ell)$, where μ_i is the uniform distribution over facets of Y_i. For an $\alpha \in X_\tau(k-1)$ let $\alpha_{-i} := \alpha \setminus \alpha|_{U_i}$. Therefore,

$$\sum_{1\leq i\leq \ell: d_{Y_i}\geq 1} \frac{d_{Y_i}(d_{Y_i}+1)}{(k-1)k} M_{\tau\cup\alpha_{-i}} \underset{\text{def of } M_{\tau\cup\alpha_{-i}}}{=} \sum_{1\leq i\leq \ell: d_{Y_i}\geq 1} \frac{d_{Y_i}(d_{Y_i}+1)}{(k-1)k} \frac{\Pi_{\tau\cup\alpha_{-i}}}{d_{Y_i}} F_{\tau\cup\alpha_{-i}}$$
$$= \sum_{1\leq i\leq \ell: d_{Y_i}\geq 1} (\Pi_\tau)^{X_{\tau\cup\alpha_{-i}}(0)} \frac{F_{\tau\cup\alpha_{-i}}}{k-1} \underset{\text{def of } F_\tau}{=} \frac{\Pi_\tau F_\tau}{k-1} = M_\tau$$

as desired.

Now, assume that G_τ is connected. Note that since each entry of F_τ is at most $\frac{(k-1)^2}{3k-1}$, we have $M_\tau \preceq \frac{k-1}{3k-1}\Pi_\tau$. Therefore, it only remains to show that $\mathbb{E}_{vc\sim\pi_\tau} M_{\tau\cup vc} \preceq M_\tau - \frac{k-1}{k-2} M_\tau \Pi_\tau^{-1} M_\tau$. This is equivalent to showing that

$$\Pi_\tau^{-1}\mathbb{E}_{uc'\sim\pi_\tau}\left[\Pi_{\tau\cup uc'}\frac{F_{\tau\cup uc'}}{k-2}\right] \preceq \frac{F_\tau}{k-1} - \frac{F_\tau^2}{(k-2)(k-1)}.$$

One can check that

$$\mathbb{E}_{uc'\sim\pi_\tau}\left[\Pi_\tau^{-1}\Pi_{\tau\cup uc'}\frac{F_{\tau\cup vc}}{k-2}\right](vc,vc) = \frac{\sum_{uc'\in X_{\tau\cup vc}(0)} p(uc'|\tau\cup vc) F_{\tau\cup uc'}(vc,vc)}{(k-1)(k-2)}.$$

Therefore, it is enough that

$$\frac{\sum_{uc'\in X_{\tau\cup vc}(0)} p(uc'|\tau\cup vc) F_{\tau\cup uc'}(vc,vc)}{(k-1)(k-2)} \leq \frac{F_\tau(vc,vc)}{k-1} - \frac{F_\tau^2(vc,vc)}{(k-1)(k-2)},$$

which holds by assumption. □

Now, to complete the proof of Theorem 3.3.1 it only remains to find $\{F_\tau\}_{\tau\in X(\leq n-2)}$ that satisfies the above conditions. The proof can be found in the Appendix A. Let us remark why we need the assumption $\beta > \Delta$ in this proof. Consider the worst case example, where G is a complete graph with $\Delta+1$ vertices. In that case, by symmetry, $F_\tau(vc,vc) = \frac{1}{\beta} + \frac{1}{\beta^2}$ for all faces of co-dimension 2, and every matrix F_τ is a multiple of identity on $X_\tau(0) \times X_\tau(0)$. So, the conditions on F_τ reduces to the following systems of inequalities:

$$(k-1)f(k-2) \leq (k-2)f(k-1) - f(k-1)^2 \quad \forall 3 \leq k \leq \Delta, f(1) = \frac{1}{\beta} + \frac{1}{\beta^2}.$$

It is not hard to see that such a system does not have a solution up to $k = \Delta + 1$ when $\beta \leq \Delta$.

3.3.2 Vertex-List-Coloring for Trees Using Non-Diagonal Matrix Bounds

By allowing the matrix bounds $\{M_\tau\}_{\tau \in X:\mathrm{codim}(\tau)\geq 2}$ in Theorem 2.4.2 to be non-diagonal matrices, one can hope to get a tighter result. In this section, for any constant $\epsilon > 0$ we analyze the Glauber dynamics for ϵ-extra-color vertex-List-Coloring instances when the graph is a tree.

Theorem 3.3.4. *Consider an arbitrary constant $\epsilon > 0$ and a $\epsilon\Delta$-extra-color vertex-list-coloring instance (G, L) such that G is a tree and $\frac{\ln^2(\Delta)}{\Delta} \leq \frac{\epsilon^2}{100}$. For the weighted simplicial complex (X, π_{n-1}), any $2 \leq k \leq n$ and $\tau \in X$ of co-dimension k we have $\lambda_2(P_\tau) \leq \frac{\frac{1}{20}+\frac{1}{\epsilon}}{k-1}$.*

For any $k \geq 2$ and τ of co-dimension at least k, assume that M_τ is of the form

$$M_\tau = \Pi_\tau \frac{F_\tau}{k-1} + \sqrt{\Pi_\tau}\frac{A_\tau}{k-1}\sqrt{\Pi_\tau},$$

for a diagonal matrix F_τ and a hollow matrix A_τ. The goal is to find F_τ and A_τ such that M_τ satisfies the conditions of Theorem 2.4.2. This is easily doable if $k = 2$ by Proposition 3.3.2. A natural approach is to define A_τ for any τ of co-dimension at least 3 such that

$$\sqrt{\Pi_\tau}\frac{A_\tau}{k-1}\sqrt{\Pi_\tau} = \mathbb{E}_{vc\sim\pi_\tau}\sqrt{\Pi_{\tau\cup vc}}\frac{A_{\tau\cup vc}}{k-2}\sqrt{\Pi_{\tau\cup vc}},$$

when G_τ is connected. Note that the following definition is not restricted to trees.

Definition 6 (Family of Matrices $\{A_\tau\}_{\tau\in X:\mathrm{codim}(\tau)\geq 2}$). *Given a vertex-list-coloring instance (G, L) and its associated weighted simplicial complex (X, π_{n-1}), define $\{A_\tau\}_{\tau\in X:\mathrm{codim}(\tau)\geq 2}$ as follows: let $A_\tau := f_\times(X, \{A_{\tau\cup\sigma}\}_{\emptyset\subsetneq\sigma\in X_\tau(\mathrm{codim}(\tau)-3)})$ if G_τ is disconnected and otherwise,*

1. *For any τ of co-dimension 2, say $G_\tau = (\{u, v\}, \{uv\})$; define $A_\tau \in \mathbb{R}^{X(0)\times X(0)}$ to be a hollow block diagonal matrix with a block for every color such that $A_\tau(uc, vc) = A_\tau(vc, uc) = \frac{-1}{\sqrt{(l_\tau(u)-1)(l_\tau(v)-1)}}$, for $c \in L_\tau(v)\cap L_\tau(u)$, and all other entries are 0.*

2. *For any τ of co-dimension $k \geq 3$, let*

$$A_\tau := \frac{k-1}{k-2}\sqrt{\Pi_\tau^{-1}}\left(\mathbb{E}_{vc\sim\pi_\tau}\sqrt{\Pi_{\tau\cup vc}}A_{\tau\cup vc}\sqrt{\Pi_{\tau\cup vc}}\right)\sqrt{\Pi_\tau^{-1}}. \tag{3.4}$$

Observe that A_τ is symmetric and hollow. Furthermore, its non-zero entries correspond to $v \sim_{\tau,c} u$, and when G_τ is connected,

$$A_\tau(vc, uc) = \frac{1}{k-2}\sum_{wc'\in X_\tau(0):w\neq v,u}\sqrt{p(wc'|vc\cup\tau)p(wc'|uc\cup\tau)}A_{\tau\cup wc'}(vc, uc).$$

We can bound entries of A_τ for any $\tau \in X(\leq n-3)$ as follows.

Proposition 3.3.5. *Consider a β-extra-color vertex-list-coloring instance (G, L), For any face τ of co-dimension at least 2 and $uc, vc \in X_\tau(0)$ such that $v \sim_c u$, $-\frac{1}{\beta} \leq A_\tau(vc, uc) \leq 0$.*

Proof. Let τ be a face of co-dimension $k \geq 2$. We prove by induction on k. It clearly holds for $k = 2$ by definition. For $k > 2$ and $vc, uc \in X_\tau(0)$, we have

$$\begin{aligned} A_\tau(vc, uc) &= \frac{1}{k-2} \sum_{w \in V_\tau : w \neq u,v} \sum_{c' \in L_\tau(w)} \sqrt{p(wc'|vc \cup \tau) p(wc'|uc \cup \tau)} A_{\tau \cup wc'}(vc, uc) \\ &\geq \frac{1}{k-2} \sum_{w \in V_\tau : w \neq u,v} \sum_{c' \in L_\tau(w)} \sqrt{p(wc'|vc \cup \tau) p(wc'|uc \cup \tau)} \cdot \frac{-1}{\beta} && \text{(by IH)} \\ &\geq \frac{1}{k-2} \cdot \frac{-1}{\beta} \sum_{w \in V_\tau : w \neq u,v} \left(\sum_{c' \in L_\tau(w)} p(wc'|vc \cup \tau) \right) \left(\sum_{c' \in L_\tau(w)} p(wc'|uc \cup \tau) \right) \\ & && \text{(by Cauchy-Schwarz)} \\ &\geq \frac{1}{k-2} \cdot \frac{-1}{\beta} \sum_{w \in V_\tau : w \neq u,v} 1 = -\frac{1}{\beta}. \end{aligned}$$

One can further see that $A_\tau(vc, uc) \leq 0$ follows from the induction hypothesis. □

Now, we apply Theorem 2.4.2 to derive sufficient conditions on the family $\{F_\tau\}_{\tau \in X, \mathrm{codim}(\tau) \geq 2}$ to get $\lambda_2(P_\tau) \leq \frac{\rho(F_\tau + A_\tau)}{k-1}$ for all τ of co-dimension $2 \leq k \leq n$.

Proposition 3.3.6. *Let (G, L) be a β-extra-color vertex-list-coloring instance such that G is a tree, and let (X, π_{n-1}) be its associated weighted complex. Take an arbitrary vertex and make G a rooted tree with root vertex r. For $v \neq r$, we write $a(v)$ to denote immediate ancestor of v, i.e., parent of v. Let $\{F_\tau \in \mathbb{R}^{X(0) \times X(0)}\}_{\tau \in X : \mathrm{codim}(\tau) \geq 2}$ be a family of diagonal matrices supported on $X_\tau(0) \times X_\tau(0)$ such that $F_\tau = f_\times(X_\tau, \{F_{\tau \cup \sigma}\}_{\emptyset \subsetneq \sigma \in X_\tau(\leq \mathrm{codim}(\tau) - 3)})$ if G_τ is disconnected and otherwise,*

1. *For all τ of co-dimension 2: F_τ is defined as $F_\tau(vc, vc) = \frac{1}{\beta^2}$ for $vc \in X_\tau(0)$.*

2. *For any τ of co-dimension $k \geq 3$: $F_\tau \preceq (\frac{(k-1)^2}{3k-1} - \frac{1}{\beta}) I^{X_\tau(0)}$, and for all $vc \in X_\tau(0)$,*

$$\sum_{uc' \in X_{\tau \cup vc}(0)} p(uc'|\tau, vc) F_{\tau \cup uc'}(vc, vc) \leq (k-2) F_\tau(vc, vc) - 2F_\tau^2(vc, vc) - \gamma_\tau(vc), \quad (3.5)$$

where $\gamma_\tau(vc) = \frac{4\Delta_\tau(v)}{\beta^2}$ if v is the root of the rooted tree G_τ, and $\gamma_\tau(vc) = \frac{4(\Delta_\tau(v) + \Delta_\tau(a(v)) - 1)}{\beta^2}$ otherwise.

Then, for all $k \geq 2$ and τ of co-dimension k, $\lambda_2(P_\tau) \leq \frac{\rho(F_\tau + A_\tau)}{k-1}$, where A_τ is defined in Definition 6.

Proof. We prove that the conditions of Theorem 2.4.2 hold for $M_\tau := \Pi_\tau \frac{F_\tau}{k-1} + \sqrt{\Pi_\tau} \frac{A_\tau}{k-1} \sqrt{\Pi_\tau}$ for $\tau \in X(\leq n-3)$. Note that the desired condition holds for any τ of co-dimension 2 by definition. Now, take $k \geq 2$. Assume G_τ is disconnected. Using the definition of A_τ and our assumption about F_τ, the proof of this case is similar to what we argued in Proposition 3.3.3. Now, assume that G_τ is connected. Note that by Proposition 3.3.5, the absolute value of every off-diagonal entry of A_τ is at most $\frac{1}{\beta}$ and that there are at most $(k-1)$ non-zero entries per row. Therefore, $\sqrt{\Pi_\tau}A_\tau\sqrt{\Pi_\tau} \preceq \frac{1}{\beta}\Pi_\tau$. Since each entry of F_τ is at most $\frac{(k-1)^2}{3k-1} - \frac{1}{\beta}$, we have $M_\tau \preceq \frac{k-1}{3k-1}\Pi_\tau$. Therefore, it only remains to show that $\mathbb{E}_{vc\sim\pi_\tau} M_{\tau\cup vc} \preceq M_\tau - \frac{k-1}{k-2} M_\tau \Pi_\tau^{-1} M_\tau$. This is equivalent to showing that

$$\sqrt{\Pi_\tau^{-1}}\mathbb{E}_{vc\sim\pi_\tau}\left[\Pi_{\tau\cup vc}\frac{F_{\tau\cup vc}}{k-2} + \sqrt{\Pi_{\tau\cup vc}}\frac{A_{\tau\cup vc}}{k-2}\sqrt{\Pi_{\tau\cup vc}}\right]\sqrt{\Pi_\tau^{-1}} \preceq \frac{F_\tau}{k-1} + \frac{A_\tau}{k-1} - \frac{(F_\tau + A_\tau)^2}{(k-2)(k-1)}. \tag{3.6}$$

We starting by proving an upper bound on A_τ^2. Define $A_{\tau,even}(vc, uc) = A_{\tau,even}(uc, vc) \coloneqq A(vc, uc)$ for all $vc, uc \in X_\tau(0)$ such that v is at an even distance from the root of G_τ and $a(u) = v$, and let other entries be 0. Define $A_{\tau,odd} \coloneqq A_{\tau,even} - A_{\tau,odd}$. By Fact 1.2.2,

$$A_\tau^2 \preceq 2A_{\tau,even}^2 + 2A_{\tau,odd}^2.$$

Furthermore, $A_{\tau,even}^2(vc, uc) = A_{\tau,even}^2(uc, vc) \neq 0$ only if $v = u$ and v is at an even distance from the root or when $a(v) = a(u)$ and u, v are at an odd distance from the root. A similar fact holds for $A_{\tau,odd}^2$. Therefore, if we let $\gamma_\tau(wc) = 0$ for $wc \notin X_\tau(0)$, we get

$$4(A_{\tau,even}^2 + A_{\tau,odd}^2) \preceq \mathrm{diag}(\gamma_\tau),$$

where we applied Fact 1.2.1, and used Proposition 3.3.5 to bound the absolute value of the entries of $A_{\tau,even}^2$ and $A_{\tau,odd}^2$. Therefore, by Fact 1.2.2, the RHS of Eq. (3.6) is bounded by

$$\frac{F_\tau}{k-1} + \frac{A_\tau}{k-1} - \frac{2F_\tau^2 + 2A_\tau^2}{(k-2)(k-1)} \succeq \frac{F_\tau}{k-1} + \frac{A_\tau}{k-1} - \frac{2F_\tau^2 + \mathrm{diag}(\gamma_\tau)}{(k-1)(k-2)}. \tag{3.7}$$

On the other hand, by the definiton of A_τ (see Eq. (3.4)), the LHS of Eq. (3.6) is equal to

$$\mathbb{E}_{uc'\sim\pi_\tau}\left[\Pi_\tau^{-1}\Pi_{\tau\cup uc'}\frac{F_{\tau\cup uc'}}{k-2}\right] + \frac{A_\tau}{k-1}.$$

Furthermore, for $vc \in X_\tau(0)$,

$$\mathbb{E}_{uc'\sim\pi_\tau}\left[\Pi_\tau^{-1}\Pi_{\tau\cup uc'}\frac{F_{\tau\cup uc'}}{k-2}\right](vc, vc) = \frac{\sum_{uc'\in X_{\tau\cup vc}(0)} p(uc'|\tau\cup vc)F_{\tau\cup uc'}(vc, vc)}{(k-1)(k-2)}$$

Combining this with Eq. (3.7), the desired inequality in Eq. (3.6) follows from the assumption (see Eq. (3.5)). □

Finally, with this in hand, we prove Theorem 3.3.4 similar to what we did for Theorem 3.3.1. The proof can be found in Appendix A.

3.4 Edge Coloring

Consider a graph $G = (V, E)$ and a function $L : E \to 2^{[q]}$. The pair (G, L) is called an edge-list-coloring instance. For a vertex v and an edge e, we write $e \sim_c v$ when $e \sim v$ and $c \in L(e)$. Furthermore, for any $e, f \in E$, we write $e \sim_c f$ when $e \sim f$ and $c \in L(e) \cap L(f)$. Furthermore, we define a β-extra-color edge-list-coloring instance as follows.

Definition 7. *We say an edge-list-coloring instance* (G, L) *is a* β*-extra-color instance if for each* $e \in E$, $|L(e)| \geq \beta + \Delta_G(e)$.

An assignment $\sigma : E \to [q]$ is a L-edge-list-coloring of G if $\sigma(e) \in L(e)$ for all $e \in E$. We say σ is proper if $\sigma(e) \neq \sigma(f)$ whenever $e \sim f$. When it is clear from context we say σ is a proper coloring to mean it is a proper L-edge-list-coloring. We say τ is proper partial coloring on $H \subset E$ when it is a proper $L|_H$-edge-list-coloring for (V, H). We may view a proper coloring as a set of edge-color pairs (e, c) which we denote by ec for simplicity of notation. We denote the uniform distribution over proper L-edge-list-colorings of G by π_{m-1} when (G, L) are clear from context. For a proper partial coloring on $H \subset E$ and $e \in E \setminus H$, define

$$p(ec|\tau) := \mathbb{P}_{\sigma \sim \pi_{m-1}}(\sigma(e) = c | \forall f \in H : \sigma(f) = \tau(f)).$$

To analyze the Glauber dynamics on an edge-list-coloring instance we associate a simplicial complex to it.

Definition 8 (Simplicial Complex of an Edge-List-Coloring Instance). *Given an edge-list-coloring instance* (G, L)*, let* $X(G, L)$ *be a pure* $(m-1)$*-dimensional simplicial complex specified by the following facets:* $\{(e, \sigma(e))\}_{e \in E}$ *is a facet if and only if* σ *is a proper* L*-edge-list-coloring for* G.

When it is clear from context, we abbreviate $X(G, L)$ to X. Note that for all $0 \leq k \leq m$, any face τ of co-dimension k is a partial coloring on a subset of edges H of size $m - k$ (k edges remain uncolored). Furthermore, $X_\tau(0)$ can be seen as the set of all ec such that $c \in L(e)$, $e \notin H$ and for any $f \sim e$, $fc \notin \tau$. Analogous to vertex-list-colorings, the Glauber dynamics on (G, L) is the down-up walk on the facets of (X, π_{m-1}). So, as we did before for vertex-list-colorings, our aim is to apply Theorem 2.4.2 to the simplicial complex to bound the second eigenvalue of the transition probability matrix of the local walks and then apply **??** to get a bound for the transition probability matrix of the down-up walk on the facets. The following is the main theorem of this section.

Theorem 3.4.1. *Let* (G, L) *be a* $(\frac{4}{3} + 4\epsilon)\Delta$*-extra-color edge-list-coloring instance for some* $0 < \epsilon \leq \frac{1}{10}$ *such that* $\frac{\ln^2(\Delta)}{\Delta} \leq \frac{\epsilon^3}{15}$*, and let* (X, π_{m-1}) *be its associated weighted simplicial complex. For any* $2 \leq k \leq m$ *and* $\tau \in X$ *of co-dimension* k *we have* $\lambda_2(P_\tau) \leq \frac{\epsilon + \frac{1}{\epsilon}}{k-1}$.

We remark that our analysis here is not tight and we expect that the factor 4/3 can be improved with a more careful analysis.

We proceed by introducing some notation and definitions. Given a face $\tau \in X$, let E_τ be the set of uncolored edges, i.e. $E_\tau := \{e : \exists c, ec \in X_\tau(0)\}$. Let $G_\tau = (V, E_\tau)$ and $\Delta_\tau(.)$ be the degree function of G_τ. Similarly, if $e = \{u, v\}$, define $\Delta_\tau(e)$ to be number of edges in G_τ that share an endpoint with e, i.e. $\Delta_\tau(e) = \Delta_\tau(u) + \Delta_\tau(v) - 2$. We define $L_\tau(e) := \{c \in L(e) : ec \in X_\tau(0)\}$. Let $l_\tau(e) := |L_\tau(e)|$ and $l_\tau(e, f) := |L_\tau(e) \cap L_\tau(f)|$. Furthermore, we write $e \sim_{\tau,c} v$ when $e \sim v$ and $c \in L_\tau(e)$. Similarly, define $e \sim_{\tau,c} f$ for edges e and f. Finally, for any matrix $B \in \mathbb{R}^{X(0) \times X(0)}$, define the restriction of B to $v \in V$ as $B^v(ec, fc) := B(ec, fc)$ for any $e, f \sim v$, and 0 on all other entries. Let $B^c \in \mathbb{R}^{X(0) \times X(0)}$ be defined as $B^c(ec, fc) := B(ec, fc)$ for all $e \sim_c f$, and 0 on all other entries.

Now, similar to our approach to vertex-coloring for trees, for any $k \geq 2$ and face τ of co-dimension k, assume that M_τ is of the form

$$M_\tau = \Pi_\tau \frac{F_\tau}{k-1} + \sqrt{\Pi_\tau}\frac{A_\tau}{k-1}\sqrt{\Pi_\tau}, \tag{3.8}$$

for a diagonal matrix F_τ and a hollow matrix A_τ. The goal is again to find F_τ and A_τ such that M_τ satisfies the conditions of Theorem 2.4.2. For $k = 2$, Proposition 3.3.2 gives us such matrices. For $k \geq 3$, as opposed to what we did for vertex-coloring of trees, we let $\sqrt{\Pi_\tau}\frac{A_\tau}{k-1}\sqrt{\Pi_\tau}$ deviate from $\mathbb{E}_{vc\sim\pi_\tau}\sqrt{\Pi_{\tau\cup vc}}\frac{A_{\tau\cup vc}}{k-2}\sqrt{\Pi_{\tau\cup vc}}$ in order to control the growth of F_τ.

Definition 9 (Family of Matrices $\{A_{\tau,\epsilon}\}_{\tau\in X,\mathrm{codim}(\tau)\geq 2}$). *Let (G, L) be a β-extra-color edge-list-coloring instance, and let(X, π_{m-1}) be its associated weighted complex. For $\epsilon > 0$, define $\{A_{\tau,\epsilon}\}_{\tau\in X,\mathrm{codim}(\tau)\geq 2}$ as follows: let $A_{\tau,\epsilon} := f_\times(X, \{A_{\tau\cup\sigma,\epsilon}\}_{\emptyset\subsetneq\sigma\in X_\tau(\mathrm{codim}(\tau)-3)})$ if the line graph of G_τ is disconnected and otherwise,*

1. *For any face τ of co-dimension 2, let $A_{\tau,\epsilon} \in \mathbb{R}^{X(0)\times X(0)}$ be a hollow block diagonal matrix with a block for every color such that*

$$A_{\tau,\epsilon}(ec, fc) = A_{\tau,\epsilon}(fc, ec) := -\frac{1}{\sqrt{(l_\tau(e)-1)(l_\tau(f)-1)}},$$

for $e, f \in E_\tau$ and any $c \in L_\tau(e) \cap L_\tau(f)$, and all other entries are 0.

2. *For any $k \geq 3$ and a face τ of co-dimension k, define $A_{\tau,\epsilon} := \bar{A}_{\tau,\epsilon} + \frac{\text{off-diag}(S_{\tau,\epsilon})}{k-2}$, where $\bar{A}_{\tau,\epsilon}$ and $S_{\tau,\epsilon}$ are defined as follows:*

$$\bar{A}_{\tau,\epsilon} := \frac{k-1}{k-2}\sqrt{\Pi_\tau^{-1}}\left(\mathbb{E}_{gc\sim\pi_\tau}\sqrt{\Pi_{\tau\cup gc}}A_{\tau\cup gc,\epsilon}\sqrt{\Pi_{\tau\cup gc}}\right)\sqrt{\Pi_\tau^{-1}}, \tag{3.9}$$

$$S^v_{\tau,\epsilon} := \begin{cases} 4(1+\epsilon)\left((\bar{A}^{+,v}_{\tau,\epsilon})^2 + (\bar{A}^{-,v}_{\tau,\epsilon})^2\right) & \text{if } \Delta_\tau(v) \leq \frac{\beta}{4(1+\epsilon)}, \\ 2(1+\epsilon)(\bar{A}^v_{\tau,\epsilon})^2 & \text{otherwise.} \end{cases} \tag{3.10}$$

and $S_{\tau,\epsilon} = \sum_v S^v_{\tau,\epsilon}$.

Observe that all three matrices $\bar{A}_{\tau,\epsilon}, S_{\tau,\epsilon}, A_{\tau,\epsilon}$ are symmetric and hollow. Furthermore, the non-zero entries of these matrices correspond to $e, f \sim_{\tau,c} v$, for some $v \in G$ and when the line graph of G_τ is connected,

$$\bar{A}_{\tau,\epsilon}(ec, fc) = \frac{1}{k-2}\sum_{gc'\in X_\tau(0):g\neq e,f}\sqrt{p(gc'|ec\cup\tau)p(gc'|fc\cup\tau)}A_{\tau\cup gc',\epsilon}(ec, fc). \tag{3.11}$$

When it is clear from context, we drop ϵ from the subscripts of matrices defined above .

Lemma 3.4.2. *For any $\tau \in X$, and $ec, fc \in X_\tau(0)$, if the line graph of G_τ is connected, then*

$$\operatorname*{avg}_{g\in E_\tau, g\neq e,f}\ \min_{c'\in L_\tau(g)} A_{\tau\cup gc',\epsilon}(ec, fc) \leq \bar{A}_{\tau,\epsilon}(ec, fc) \leq \operatorname*{avg}_{g\in E_\tau, g\neq e,f}\ \max_{c'\in L_\tau(g)} A_{\tau\cup gc',\epsilon}(ec, fc)$$

Proof. Let τ be a face of co-dimension $k \geq 2$. It clearly holds for $k = 2$ by definition. For $k > 2$ and $ec, fc \in X_\tau(0)$, we have

$$\begin{aligned}
\bar{A}_\tau(ec, fc) &= \frac{1}{k-2} \sum_{g \in E_\tau : g \neq e,f} \sum_{c' \in L_\tau(g)} \sqrt{p(gc'|ec \cup \tau) p(gc'|fc \cup \tau)} A_{\tau \cup gc'}(ec, fc) \\
&\leq \frac{1}{k-2} \sum_{g \in E_\tau : g \neq e,f} \left(\sum_{c' \in L_\tau(g)} \sqrt{p(gc'|ec \cup \tau) p(gc'|fc \cup \tau)} \right) \cdot \left(\max_{c' \in L_\tau(g)} A_{\tau \cup gc'}(ec, fc) \right) \\
&\leq \frac{1}{k-2} \sum_{g \in E_\tau : g \neq e,f} \left(\left(\sum_{c' \in L_\tau(e)} p(gc'|ec \cup \tau) \right) \left(\sum_{c' \in L_\tau(g)} p(gc'|fc \cup \tau) \right) \right) \left(\max_{c' \in L_\tau(g)} A_{\tau \cup gc'}(ec, fc) \right) \\
&\qquad \text{(by Cauchy-Schwarz)} \\
&\leq \frac{1}{k-2} \sum_{g \in E_\tau : g \neq e,f} \max_{c' \in L_\tau(g)} A_{\tau \cup gc'}(ec, fc) = \operatorname*{avg}_{g \in E_\tau, g \neq e,f} \max_{c' \in L_\tau(g)} A_{\tau \cup gc'}(ec, fc).
\end{aligned}$$

The other side of the inequality follows from a similar argument. □

In order to find diagonal matrices $\{F_\tau\}_{\tau \in X:\text{codim}(\tau) \geq 2}$ such that $\{M_\tau\}_{\tau \in X:\text{codim}(\tau) \geq 2}$ as defined by Eq. (3.8) satisfies the conditions of Theorem 2.4.2, we would need to prove some bounds on the entries of $\{A_{\tau,\epsilon}\}_{\tau \in X, \text{codim}(\tau) \geq 2}$ and $\{S_{\tau,\epsilon}\}_{\tau \in X, \text{codim}(\tau) \geq 2}$.

Proposition 3.4.3. *Suppose (G, L) is a β-extra-color edge-list-coloring instance where $\beta = (\frac{4}{3} + 4\epsilon)\Delta$ for an $0 < \epsilon \leq \frac{1}{10}$ such that $2\epsilon^{-2} \leq \Delta$. For any $\tau \in X$ of* codim$(\tau) \geq 2$*, the matrix A_τ defined in Definition 9 satisfies the following: for any vertex $v \in G$, color c and $e, f \sim_{\tau,c} v$,*

(i) if $\Delta_\tau(v) \leq \frac{\beta}{4(1+\epsilon)}$, then $-\frac{1}{\beta} \leq A_{\tau,\epsilon}(ec, fc) \leq 4(1+\epsilon)\frac{\Delta_\tau(v)-2}{\beta^2}$,

(ii) otherwise, if $\Delta_\tau(v) \geq \frac{\beta}{4(1+\epsilon)}$, then $|A_{\tau,\epsilon}(ec, fc)| \leq \frac{1}{1.5\beta - 2(1+2\epsilon)\Delta_\tau(v)}$.

Proof. Fix a vertex v. We prove the claim inductively for any pair of edges incident to v.

Case (i). Let τ be any face of co-dimension $k \geq 2$. We prove by induction on $\Delta_\tau(v) + k$. We start with the base case, that is when $\Delta_\tau(v) + k = 4$, i.e., $\Delta_\tau(v) = k = 2$. It is easy to see that, for any color c and $e, f \sim_{\tau,c} v$, we have $-\frac{1}{\beta} \leq A_\tau(ec, fc) \leq 0$, by definition. Now, we prove the claim for $k \geq 2$ and $2 \leq \Delta_\tau(v) \leq \frac{\beta}{4(1+\epsilon)}$ such that $k + \Delta_\tau(v) \geq 5$. If the line graph of G_τ is not connected, then the statement trivially holds. Otherwise, by Lemma 3.4.2, for any color c and $e, f \sim_{\tau,c} v$ we can write

$$\begin{aligned}
\bar{A}_\tau(ec, fc) &\leq \frac{\Delta_\tau(v) - 2}{k-2} \max_{gc' \in X_\tau(0): g \sim v, g \neq e,f} A_{\tau \cup gc'}(ec, fc) + \frac{k - \Delta_\tau(v)}{k-2} \max_{gc' \in X_\tau(0): g \not\sim v} A_{\tau \cup gc'}(ec, fc) \\
&\leq \frac{\Delta_\tau(v) - 2}{k-2} 4(1+\epsilon) \frac{\Delta_\tau(v) - 3}{\beta^2} + \frac{k - \Delta_\tau(v)}{k-2} 4(1+\epsilon) \frac{\Delta_\tau(v) - 2}{\beta^2} \\
&= \frac{4(1+\epsilon)(\Delta_\tau(v) - 2)(k-3)}{\beta^2 (k-2)} \leq \frac{1}{\beta} \qquad (3.12)
\end{aligned}$$

where the second to last inequality follows by IH and the last inequality follows by $\Delta_\tau(v) \leq \frac{\beta}{4(1+\epsilon)}$. Similarly,

$$\bar{A}_\tau(ec, fc) \geq -\frac{\Delta_\tau(v) - 2}{k-2} \min_{\substack{gc' \in X_\tau(0) \\ g \sim v, g \neq e, f}} A_{\tau \cup gc'}(ec, fc) - \frac{k - \Delta_\tau(v)}{k-2} \min_{\substack{gc' \in X_\tau(0) \\ g \not\sim v, g \neq e, f}} A_{\tau \cup gc'}(ec, fc) \geq -\frac{1}{\beta}. \tag{3.13}$$

Therefore, by (3.10)

$$S_\tau(ec, fc) = 4(1+\epsilon) \sum_{\substack{g \sim_{\tau,c} v \\ g \neq e, f}} \left[\bar{A}^+_\tau(ec, gc)\bar{A}^+_\tau(gc, fc) + \bar{A}^-_\tau(ec, gc)\bar{A}^-_\tau(gc, fc)\right] \leq 4(1+\epsilon)(\Delta_\tau(v) - 2)\frac{1}{\beta^2}. \tag{3.14}$$

where the last inequality follows by Eqs. (3.12) and (3.13) and that v has at most $\Delta_\tau(v) - 2$ edges that can be colored by c, other than e, f. So combining with (3.13), we get $A_\tau(ec, fc) = \bar{A}_\tau(ec, fc) + \frac{S_\tau(ec, fc)}{k-2} \geq -\frac{1}{\beta}$. Similarly, (3.12) and (3.14) gives

$$A_\tau(ec, fc) \leq \frac{4(1+\epsilon)(\Delta_\tau(v) - 2)(k-3)}{\beta^2(k-2)} + \frac{4(1+\epsilon)(\Delta_\tau(v) - 2)}{k-2}\frac{1}{\beta^2} = 4(1+\epsilon)\frac{\Delta_\tau(v) - 2}{\beta^2}.$$

Case (ii). For τ of co-dimension $k \geq 2$ we prove the claim by induction on $\Delta_\tau(v) + k$. The base case is when $\Delta_\tau(v) = k = \frac{\beta}{4(1+\epsilon)}$, which we already proved in case (i) (note that we always have $k \geq \Delta_\tau(v)$). Now, we prove the claim for $\Delta_\tau(v) > \frac{\beta}{4(1+\epsilon)}$ (and $k \geq \Delta_\tau(v)$). If the line graph of G_τ is disconnected then the statement trivially holds. Otherwise, for all colors c and $e, f \sim_{\tau,c} v$, we can write

$$\begin{aligned} |\bar{A}_\tau(ec, fc)| &= \frac{\Delta_\tau(v) - 2}{k-2} \max_{gc' \in X_\tau(0): g \sim v, g \neq e, f} |A_{\tau \cup gc'}(ec, fc)| + \frac{k - \Delta_\tau(v)}{k-2} \max_{gc' \in X_\tau(0): g \not\sim v} |A_{\tau \cup gc'}(ec, fc)| \\ &\leq \frac{\Delta_\tau(v) - 2}{k-2} \frac{1}{1.5\beta - 2(1+2\epsilon)(\Delta_\tau(v) - 1)} + \frac{k - \Delta_\tau(v)}{k-2} \frac{1}{1.5\beta - 2(1+2\epsilon)\Delta_\tau(v)} \quad (3.15) \\ &\leq \frac{1}{1.5\beta - 2(1+2\epsilon)\Delta_\tau(v)}. \quad (3.16) \end{aligned}$$

where the second to last inequality follows by the IH. Furthermore, by (3.10),

$$|S_\tau(ec, fc)| = 2(1+\epsilon) \left| \sum_{g \sim_{\tau,c} v, g \neq e, f} \bar{A}_\tau(ec, gc)\bar{A}_\tau(gc, fc) \right| \leq 2(1+\epsilon)\frac{\Delta_\tau(v) - 2}{(1.5\beta - 2(1+2\epsilon)\Delta_\tau(v))^2}.$$

where the inequality follows by (3.16) and that v has at most $\Delta_\tau(v) - 2$ edges other than e, f that can be colored by c. Recall $A_\tau = \bar{A}_\tau + \frac{S_\tau}{k-2}$. So, the above inequality with (3.15) gives

$$|A_\tau(ec, fc)| \leq \frac{\Delta_\tau(v) - 2}{k-2} \frac{1}{1.5\beta - 2(1+2\epsilon)(\Delta_\tau(v) - 1)} + \frac{k - \Delta_\tau(v)}{k-2} \frac{1}{1.5\beta - 2(1+2\epsilon)\Delta_\tau(v)}$$

$$+\frac{2(1+\epsilon)}{k-2}\cdot\frac{\Delta_\tau(v)-2}{(1.5\beta-2(1+2\epsilon)\Delta_\tau(v))^2}$$
$$\leq \frac{1}{1.5\beta-2(1+2\epsilon)\Delta_\tau(v)}.$$

where in the second inequality we used that $\epsilon \leq \frac{1}{10}$ and $\Delta \geq 2\epsilon^{-2}$, and that

$$1.5\beta - 2(1+2\epsilon)\Delta_\tau(v) \geq 1.5\left(\frac{4}{3}+4\epsilon\right)\Delta - 2(1+2\epsilon)\Delta = 6\epsilon\Delta - 4\epsilon\Delta \geq 0.$$

□

Corollary 3.4.4. *Given a β-extra-color edge-list-coloring instance (G, L) where $\beta = (\frac{4}{3}+4\epsilon)\Delta$ for an $0 < \epsilon \leq \frac{1}{10}$ such that $2\epsilon^{-2} \leq \Delta$,*

(i) For any $\tau \in X$ with $\mathrm{codim}(\tau) \geq 2$, $v \in G$, and $e, f \sim_{\tau,c} v$, $|A_{\tau,\epsilon}(ec, fc)| \leq \frac{1}{2\epsilon\Delta} \leq \frac{\epsilon}{4}$.

(ii) For any $\tau \in X$ with $\mathrm{codim}(\tau) \geq 3$, $v \in G$, and $e, f \sim_{\tau,c} v$, $S_{\tau,\epsilon}(ec, fc) \leq \frac{(1+\epsilon)(\Delta_\tau(v)-2)}{2\epsilon^2\Delta^2}$.

(iii) For any $\tau \in X$ with $\mathrm{codim}(\tau) \geq 3$, any color c, and $(e = \{u, v\}, c) \in X_\tau(0)$ c, $S_{\tau,\epsilon}(ec, ec) \leq \frac{(1+\epsilon)(\Delta_\tau(v)+\Delta_\tau(u)-2)}{2\epsilon^2\Delta^2}$.

Proof. First, we verify (i). Using Proposition 3.4.3, when $\Delta_\tau(v) \leq \frac{\beta}{4(1+\epsilon)}$ we have $|A_\tau(ec, fc)| \leq \frac{1}{\beta}$, and when $\Delta_\tau(v) > \frac{\beta}{4(1+\epsilon)}$ we have

$$|A_\tau(ec, fc)| \leq \frac{1}{1.5(4/3+4\epsilon)\Delta - 2(1+2\epsilon)\Delta} \leq \frac{1}{2\epsilon\Delta},$$

where we used $\Delta_\tau(v) \leq \Delta$. So, by Lemma 3.4.2, we get $|\bar{A}_{\tau,\epsilon}(ec, fc)| \leq \frac{1}{2\epsilon\Delta}$ for any τ of co-dimension at least 3. Now, we verify (ii). If $\Delta_\tau(v) \leq \frac{\beta}{4(1+\epsilon)}$ then by Eq. (3.10),

$$S_\tau(ec, fc) = 4(1+\epsilon)\sum_{g\sim_{\tau,c}v, g\neq e,f} \bar{A}_\tau^{+,v}(ec, gc)\bar{A}_\tau^{+,v}(gc, fc) + \bar{A}_\tau^{-,v}(ec, gc)\bar{A}_\tau^{-,v}(gc, fc)$$
$$\leq 4(1+\epsilon)\sum_{g\sim_{\tau,c}v, g\neq e,f} \max_{hc'\in X_\tau(0), h\neq e,g} |A_{\tau\cup hc'}(ec, gc)| \max_{hc'\in X_\tau(0), h\neq f,g} |A_{\tau\cup hc'}(gc, fc)|$$
(by Lemma 3.4.2)
$$\leq 4(1+\epsilon)(\Delta_\tau^c(v)-2)\frac{1}{\beta^2}.$$

Otherwise, if $\Delta_\tau(v) \geq \frac{\beta}{4(1+\epsilon)}$, with a similar use of Lemma 3.4.2,

$$S_\tau(ec, fc) = 2(1+\epsilon)\sum_{g\sim_{\tau,c}v, g\neq e,f} \bar{A}_\tau^v(ec, gc)\bar{A}_\tau^v(gc, fc) \leq 2(1+\epsilon)(\Delta_\tau^c(v)-2)\left(\frac{1}{2\epsilon\Delta}\right)^2,$$

where the first inequality uses part (i). Finally, (ii) follows from $\frac{4(1+\epsilon)}{\beta^2} \leq \frac{1+\epsilon}{2\epsilon^2\Delta^2}$. It remains to prove (iii). For a vertex u let $\alpha(u) = 4(1+\epsilon)$ if $\Delta_\tau(u) \leq \frac{\beta}{4(1+\epsilon)}$ and $\alpha_u = 2(1+\epsilon)$ otherwise. By an argument similar to (ii)

$$S_\tau(ec, ec) \leq \alpha(u)\sum_{f\sim_{\tau,c}u, f\neq e} \max_{gc'\in X_\tau(0), g\neq e,f} |A_{\tau\cup gc'}(ec, fc)|^2 + \alpha(v)\sum_{f\sim_{\tau,c}v, f\neq e} \max_{gc'\in X_\tau(0), g\neq e,f} |A_{\tau\cup gc'}(ec, fc)|^2$$

$$\leq (\Delta_\tau(u) + \Delta_\tau(v) - 2) \max\left\{\frac{4(1+\epsilon)}{\beta^2}, \frac{2(1+\epsilon)}{4\epsilon^2\Delta^2}\right\}$$
$$\leq \frac{(\Delta_\tau(u) + \Delta_\tau(v) - 2)(1+\epsilon)}{2\epsilon^2\Delta^2}$$

This completes the proof.

□

The following lemma is a crucial part of our proof as it will help us bound the term $M_\tau \Pi_\tau^{-1} M_\tau$ in Eq. (2.2) effectively.

Lemma 3.4.5. *Consider a graph $G = (V, E)$, and some weight function $w : E \to \mathbb{R}_{\geq 0}$. Let A be the weighted adjacency matrix of its line graph. Then*

$$A^2 \preceq 2\sum_{v \in V} (A^v)^2,$$

where $A^v(e, f) = A(e, f)$ if $e, f \sim v$ and 0 otherwise.

Proof. It is enough to show that for all $x \in \mathbb{R}^E$, $x^\top A^2 x \leq 2\sum_{v\in V} x^\top (A^v)^2 x$. We have

$$x^\top A^2 x = \|Ax\|_2^2 = \sum_{e\in E} (Ax(e))^2 = \sum_{e \in E} \langle A_e, x\rangle^2$$

where A_e is the row indexed by e. Now, let $e = \{u, v\} \in E$. We can write $\langle A_e, x\rangle = \langle (A^u)_e, x\rangle + \langle (A^v)_e, x\rangle$. Therefore, by an application of Fact 1.2.2

$$\sum_{e\in E} \langle A_e, x\rangle^2 \preceq 2 \sum_{e=\{u,v\}\in E} \langle (A^u)_e, x\rangle^2 + \langle (A^v)_e, x\rangle^2 = 2\sum_{v}\sum_{e\sim v} (A^v x(e))^2 = 2\sum_{v\in V} x^\top (A^v)^2 x.$$

□

Now, we apply Theorem 2.4.2 to derive sufficient conditions on the family $\{F_\tau\}_{\tau\in X, \text{codim}(\tau)\geq 2}$ to get $\lambda_2(P_\tau) \leq \frac{\rho(F_\tau + A_\tau)}{k-1}$ for all τ of co-dimension $2 \leq k \leq m$.

Proposition 3.4.6. *Let (G, L) be a $(\frac{4}{3} + 4\epsilon)\Delta$-extra-color edge-list-coloring instance such that $0 \leq \epsilon \leq \frac{1}{10}$ and $\Delta \geq 2\epsilon^{-2}$, and let (X, π_{m-1}) be its associated weighted simplicial complex. Suppose that $\{F_\tau \in \mathbb{R}^{X(0)\times X(0)}\}_{\tau\in X:\text{codim}(X)\geq 2}$ is a family of diagonal matrices supported on $X_\tau(0) \times X_\tau(0)$ such that $F_\tau = f_\times(X_\tau, \{F_{\tau\cup\sigma}\}_{\emptyset\subsetneq\sigma\in X_\tau(\leq\text{codim}(\tau)-3)})$ if the line graph of G_τ is connected and otherwise,*

1. *For all τ of co-dimension 2: F_τ is defined as $F_\tau(ec, ec) = \frac{1}{(\frac{4}{3}+4\epsilon)^2\Delta^2} = \frac{1}{\beta^2}$ for $ec \in X_\tau(0)$ and 0 on all other entries.*
2. *For all τ of co-dimension $k \geq 3$: $F_\tau \preceq (\frac{(k-1)^2}{3k-1} - \frac{1}{2\epsilon\Delta})I^{X_\tau(0)}$, and for any $ec \in X_\tau(0)$*

$$\sum_{gc' \in X_{\tau\cup ec}(0)} p(gc'|\tau \cup ec) F_{\tau\cup gc'}(ec, ec) \leq (k-2)F_\tau(ec, ec) - \left(\frac{2+\epsilon}{\epsilon}\right) F_\tau^2(ec, ec) - \gamma_\tau(ec), \tag{3.17}$$

where $\gamma_\tau(ec) = \frac{(1+\epsilon)\Delta_\tau(e)}{2\epsilon^2\Delta^2} + \frac{(1+\epsilon)^2(2+3\epsilon+\epsilon^2)}{\epsilon^5\Delta^2}$.

Then for all $k \geq 2$ and τ of co-dimension k, $\lambda_2(P_\tau) \leq \frac{\rho(F_\tau + A_\tau)}{k-1}$, where A_τ is defined in Definition 9.

Proof. We prove that the conditions of Theorem 2.4.2 hold for $\{M_\tau\}_{\tau \in X(\leq m-3)}$ defined as follows:

$$M_\tau := \Pi_\tau \frac{F_\tau}{k-1} + \sqrt{\Pi_\tau} \frac{A_\tau}{k-1} \sqrt{\Pi_\tau} \quad \forall \tau \in X, k = \operatorname{codim}(\tau) \geq 2$$

Note that the condition of the theorem holds for any τ of co-dimension 2 by definition. So, we prove the statement for τ of co-dimension at least 3. Assume the line graph of G_τ is disconnected. Using the definition of A_τ and our assumption about F_τ, the proof of this case is similar to what we argued in Proposition 3.3.3. Now, assume that the line graph of G_τ is connected. Note that by Corollary 3.4.4, the absolute value of every off-diagonal entry of A_τ is at most $\frac{1}{2\epsilon\Delta}$ and that there are at most $(k-1)$ non-zero entries per row. Therefore, $\sqrt{\Pi_\tau} \frac{A_\tau}{k-1} \sqrt{\Pi_\tau} \preceq \frac{1}{2\epsilon\Delta} \Pi_\tau$. Combined with the bound on entries of diagonal matrix F_τ, this implies that $M_\tau \preceq \frac{k-1}{3k-1} \Pi_\tau$. Therefore, it only remains to show that $\mathbb{E}_{gc \sim \pi_\tau} M_{\tau \cup gc} \preceq M_\tau - \frac{k-1}{k-2} M_\tau \Pi_\tau^{-1} M_\tau$. This is equivalent to showing that

$$\sqrt{\Pi_\tau^{-1}} \mathbb{E}_{gc \sim \pi_\tau} \left[\Pi_{\tau \cup gc} \frac{F_{\tau \cup gc}}{k-2} + \sqrt{\Pi_{\tau \cup gc}} \frac{A_{\tau \cup gc}}{k-2} \sqrt{\Pi_{\tau \cup gc}} \right] \sqrt{\Pi_\tau^{-1}} \preceq \frac{F_\tau}{k-1} + \frac{A_\tau}{k-1} - \frac{(F_\tau + A_\tau)^2}{(k-2)(k-1)}. \tag{3.18}$$

We proceed by first proving a lowerbound on the RHS. By two applications of Fact 1.2.2, we can write

$$\begin{aligned}
(F_\tau + A_\tau)^2 &\preceq \left(1 + \frac{2}{\epsilon}\right) F_\tau^2 + \left(1 + \frac{\epsilon}{2}\right) A_\tau^2 \\
&= \left(1 + \frac{2}{\epsilon}\right) F_\tau^2 + \left(1 + \frac{\epsilon}{2}\right) \left(\bar{A}_\tau + \frac{\text{off-diag}(S_\tau)}{k-2}\right)^2 \\
&\preceq \left(1 + \frac{2}{\epsilon}\right) F_\tau^2 + (1+\epsilon)\bar{A}_\tau^2 + \frac{(3 + \epsilon + 2/\epsilon)\, \text{off-diag}(S_\tau)^2}{(k-2)^2}.
\end{aligned} \tag{3.19}$$

We proceed by finding a diagonal matrix to upperbound $\bar{A}_\tau^2$. For any $c \in [q]$, $\bar{A}_\tau^c$ is the weighted adjacency matrix of a line graph. Therefore, by Lemma 3.4.5, $(\bar{A}_\tau^c)^2 \preceq 2\sum_{v \in V} (\bar{A}_\tau^{c,v})^2$. Since $\bar{A}_\tau^2 = \sum_{c \in [q]} (\bar{A}_\tau^c)^2$, we get that

$$\bar{A}_\tau^2 \preceq 2 \sum_{v \in V} (\bar{A}_\tau^v)^2 \preceq 4 \sum_{v \in V} ((\bar{A}_\tau^{+,v})^2 + (\bar{A}_\tau^{-,v})^2).$$

where in the second inequality we used Fact 1.2.2. Therefore, by definition of S_τ (see Eq. (3.10)),

$$(1+\epsilon)\bar{A}_\tau^2 \preceq S_\tau = (k-2)(A_\tau - \bar{A}_\tau) + \operatorname{diag}(S_\tau).$$

So, by (3.19), we can lowerbound the RHS of (3.18) as follows

$$\frac{F_\tau}{k-1} + \frac{A_\tau}{k-1} - \frac{(F_\tau + A_\tau)^2}{(k-2)(k-1)} \succeq \frac{F_\tau}{k-1} + \frac{\bar{A}_\tau}{k-1} - \frac{(1 + \frac{2}{\epsilon})F_\tau^2}{(k-1)(k-2)}$$

$$- \frac{\text{diag}(S_\tau)}{(k-1)(k-2)} - \frac{(3+\epsilon+\frac{2}{\epsilon})\,\text{off-diag}(S_\tau)^2}{(k-1)(k-2)^3}.$$

On the other hand, by definition of $\bar{A}_\tau$ (see (3.9)), the LHS of (3.18) is equal to

$$\mathbb{E}_{gc\sim\pi_\tau}\left[\Pi_\tau^{-1}\Pi_{\tau\cup gc}\frac{F_{\tau\cup gc}}{k-2}\right] + \frac{\bar{A}_\tau}{k-1},$$

and

$$\mathbb{E}_{gc\sim\pi_\tau}\left[\Pi_\tau^{-1}\Pi_{\tau\cup gc}\frac{F_{\tau\cup gc}}{k-2}\right](ec,ec) = \sum_{gc'\in X_{\tau\cup ec}(0)} p(gc'|\tau\cup ec)F_{\tau\cup gc'}(ec,ec).$$

Comparing this with the assumption (see Eq. (3.17)), and letting $\gamma_\tau(ec) = 0$ for all $ec \notin X_\tau(0)$, it is enough to show that

$$\text{diag}(\gamma_\tau) \succeq \text{diag}(S_\tau) + \frac{(3+\epsilon+\frac{2}{\epsilon})\,\text{off-diag}(S_\tau)^2}{(k-2)^2}.$$

First, notice,

$$\frac{\text{off-diag}(S_\tau)^2}{(k-2)^2} \preceq \frac{\|\text{off-diag}(S_\tau)\|_\infty^2\, I^{X_\tau(0)}}{(k-2)^2} \preceq \frac{(1+\epsilon)^2(\Delta-2)^2 4(\Delta-1)^2}{4\epsilon^4\Delta^4(k-2)^2} I^{X_\tau(0)} \preceq \frac{(1+\epsilon)^2}{\epsilon^4\Delta^2} I^{X_\tau(0)},$$

where the second inequality is by Fact 1.2.1, noting that by part (ii) of Corollary 3.4.4, every off-diagonal entry of S_τ is at most $\frac{(1+\epsilon)(\Delta-2)}{2\epsilon^2\Delta^2}$ and that there are at most $2(\Delta-1)$ non-zero entries per row. Finally, the statement follows from part (iii) of Corollary 3.4.4 which shows $S_\tau(ec,ec) \le \frac{(1+\epsilon)\Delta_\tau(e)}{2\epsilon^2\Delta^2}$ for any $ec \in X_\tau(0)$. □

With this in hand, we prove Theorem 3.4.1.

Proof of Theorem 3.4.1. For any $ec = \{u,v\}c \in X_\tau(0)$ define,

$$F_\tau(ec,ec) \coloneqq \begin{cases} 0 & \text{if } \Delta_\tau(e) = 0 \\ f_1(\Delta_\tau(g)) & \text{if } \Delta_\tau(e) = 1, g\sim e, \\ f_2(\Delta_\tau(e)) & \text{if } \Delta_\tau(e) \ge 2. \end{cases}$$

where $f_1(i) \coloneqq \frac{1}{(\frac{4}{3}+4\epsilon)^2\Delta^2} + \frac{(4\epsilon^{-5}+0.6\epsilon^{-2})\sum_{k=1}^{i-1}\frac{1}{k}}{\Delta^2}$ for any $i \ge 2$, and $f_2(i) \coloneqq \frac{5\epsilon^{-5}\ln(\Delta)+(4\epsilon^{-5}+\epsilon^{-2}i)\sum_{k=1}^{i-1}\frac{1}{k}}{\Delta^2}$ for $i \ge 2$. We prove that this satisfies the conditions of Proposition 3.4.6. Then, the statement follows from the fact that

$$\lambda_2(P_\tau) \le \frac{\rho(F_\tau + A_\tau)}{k-1} \le \frac{\epsilon+\frac{1}{\epsilon}}{k-1},$$

where the last inequality follows by (3.20) below and the fact that every entry of A_τ is at most $\frac{1}{2\epsilon\Delta}$ (by Corollary 3.4.4) and that every row of A_τ has at most $2(\Delta-1)$ non-zero entries.

The condition for links of co-dimension 2 holds by definition as $f_1(1) = \frac{1}{(4/3+4\epsilon)^2\Delta^2}$. Assume that $k \ge 3$ and τ is of co-dimension k. Similar to the proof of Appendix A.1, when the line graph of G_τ is disconnected, the condition holds. Now, assume that the line graph of G_τ is connected. It follows that, for all $1 \le i \le 2\Delta$,

$$f_2(i) \le \frac{9\epsilon^{-5}(\ln(\Delta)+1)}{\Delta^2} + \frac{2\epsilon^{-2}(\ln(\Delta)+2)}{\Delta} \underset{\frac{\ln^2(\Delta)}{\Delta} \le \frac{\epsilon^3}{15}}{\le} \frac{3\epsilon}{100} + \frac{2\epsilon}{15} \underset{\epsilon \le 0.1, k\ge 2}{\le} \frac{(k-1)^2}{3k-1} - \frac{1}{2\epsilon\Delta}. \tag{3.20}$$

A similar inequality holds for f_1 and $1 \le i \le 2\Delta$. It remains to check the condition Eq. (3.17) in Proposition 3.4.6. We need to show that for any $ec \in X_\tau(0)$,

$$\sum_{fc' \in X_{\tau\cup ec}(0)} p(fc'|\tau \cup ec) F_{\tau \cup fc'}(ec, ec) \preceq (k-2)F_\tau(ec, ec) - \left(1 + \frac{2}{\epsilon}\right) F_\tau^2(ec, ec) - \gamma_\tau(ec),$$

for

$$\gamma_\tau(ec) = \frac{(1+\epsilon)\Delta_\tau(e)}{2\epsilon^2\Delta^2} + \frac{(1+\epsilon)^2(2+3\epsilon+\epsilon^2)}{\epsilon^5\Delta^2} \underset{\epsilon \le 0.1}{\le} \frac{0.6\Delta_\tau(e)\epsilon^{-2}}{\Delta^2} + \frac{3\epsilon^{-5}}{\Delta^2}.$$

Case 1: $\Delta_\tau(e) = 1, g \sim_\tau e$. Since the line graph of G_τ is connected and τ is of co-dimension at least 3, $\Delta_\tau(g) \ge 2$. So, it is enough to show that,

$$\begin{aligned}\sum_{fc' \in X_\tau(0)} p(fc'|\tau \cup ec) F_{\tau \cup fc'}(ec, ec) &= (\Delta_\tau(g)-1) f_1(\Delta_\tau(g)-1) + (k - \Delta_\tau(g) - 1) f_1(\Delta_\tau(g)) \\ &\le (k-2) f_1(\Delta_\tau(g)) - \left(1+\frac{2}{\epsilon}\right) f_1^2(\Delta_\tau(g)) - \frac{0.6\epsilon^{-2} + 3\epsilon^{-5}}{\Delta^2}.\end{aligned} \tag{3.21}$$

Now, note that

$$\begin{aligned}&(k-2) f_1(\Delta_\tau(g)) - (\Delta_\tau(g)-1) f_1(\Delta_\tau(g)-1) - (k - \Delta_\tau(g) - 1) f_1(\Delta_\tau(g)) \\ &= (\Delta_\tau(g)-1)(f_1(\Delta_\tau(g)) - f_1(\Delta_\tau(u)-1)) = \frac{0.6\epsilon^{-2} + 4\epsilon^{-5}}{\Delta^2}.\end{aligned}$$

Furthermore,

$$\left(1+\frac{2}{\epsilon}\right) f_1^2(\Delta_\tau(g)) \underset{\epsilon \le 0.1}{\le} \frac{2.1}{\epsilon}\left(\frac{5\epsilon^{-5}\ln\Delta}{\Delta^2}\right)^2 \underset{\frac{\ln^2(\Delta)}{\Delta} \le \frac{\epsilon^3}{15}}{\le} \frac{\epsilon^{-5}}{\Delta^2}.$$

Putting these together, we get Eq. (3.21).

Case 2: $\Delta_\tau(e) \ge 2$. For convenience in writing the recursion, let $f_2(1) = \frac{5\epsilon^{-5}\ln(\Delta)}{\Delta^2}$. Following similar calculations, it is enough to show that

$$(\Delta_\tau(e)-1) f_2(\Delta_\tau(e)) - \Delta_\tau(e) f_2(\Delta_\tau(e)-1) \ge \left(1+\frac{2}{\epsilon}\right) f_2^2(\Delta_\tau(e)) + \frac{0.6\epsilon^{-2}\Delta_\tau(e) + 3\epsilon^{-5}}{\Delta^2}. \tag{3.22}$$

Note that in the LHS of the above equation we should write $f_1(\Delta_\tau(g))$ if $\Delta_\tau(e)-1=1$ and g is the only remaining neighbour (of e), but since $f_1(i) \le \frac{5\epsilon^{-5}\ln(\Delta)}{\Delta^2} = f_2(1)$ for all $1 \le i \le 2\Delta$ the above inequality is valid. Note that, by definition

$$(\Delta_\tau(e)-1)f_2(\Delta_\tau(e)) - \Delta_\tau(e)f_2(\Delta_\tau(e)-1) = \frac{\epsilon^{-2}\Delta_\tau(e)+4\epsilon^{-5}}{\Delta^2}.$$

Furthermore, $\frac{\ln^2(\Delta)}{\Delta} \ge \frac{\epsilon^3}{15}$ and $\epsilon \le 0.1$ imply that $\ln\Delta \ge 10$, and we can write

$$\begin{aligned}\left(1+\frac{2}{\epsilon}\right) f_2^2(\Delta_\tau(e)) &\underset{\epsilon\le 0.1}{\le} \frac{2.1(\ln\Delta+2)^2}{\epsilon\Delta^4}\cdot\left(\epsilon^{-2}\Delta_\tau(e)+9\epsilon^{-5}\right)^2 \\ &\underset{Fact\ 1.2.2}{\le} \frac{2.1(\ln\Delta+2)^2}{\epsilon\Delta^4}\cdot\left(1.2(\epsilon^{-2}\Delta_\tau(e))^2+6(9\epsilon^{-5})^2\right) \\ &\underset{\ln\Delta\ge 10,\frac{\ln^2(\Delta)}{\Delta}\le\frac{\epsilon^3}{15}}{\le} \frac{0.4\epsilon^{-2}\Delta_\tau(e)+\epsilon^{-5}}{\Delta^2}.\end{aligned}$$

This finishes the proof of Eq. (3.22). □

Chapter 4

An Improved Trickle-Down Theorem for Partite Complexes

4.1 Introduction

In this chapter, we use the matrix trickle-down framework to prove a strengthening of the trickle-down theorem for partite complexes. The results of this section were previously published in [AO23a].

Given a $(d+1)$-partite d-dimensional simplicial complex, we show that if "on average" the links of faces of co-dimension 2 are $\frac{1-\delta}{d}$-(one-sided) spectral expanders, then the link of any face of co-dimension k is an $O(\frac{1-\delta}{k\delta})$-(one-sided) spectral expander, for all $3 \leq k \leq d+1$. For an application, using our theorem as a black-box, we show that links of faces of co-dimension k in recent constructions of bounded degree (sparse) high dimensional expanders have spectral expansion at most $O(1/k)$ fraction of the spectral expansion of the links of the worst faces of co-dimension 2, a significant improvment over previously known bounds. This refutes a conjecture that, unlike dense high dimensional expanders, local spectral expansion does not decay for sparse complexes. A more detailed introduction to this application can be found in Section 1.1.3.

4.1.1 Main Results

We start by stating two special cases of our theorem. We need the following definition.

Definition 10. *Given a $(d+1)$-partite complex (X, π) with parts $[d]$, for every $i \in [d]$, define*

$$\Delta_{(X,\pi)}(i) = |\{j \in [d] \setminus i : \exists \tau \text{ s.t. } \operatorname{type}(\tau) = [d] \setminus \{i, j\}, \lambda_2(P_\tau) > 0\}|,$$

i.e. $\Delta_{(X,\pi)}(i)$ is the number of parts $j \neq i$ for which there exists a face of type $[d] \setminus \{i, j\}$ whose link is not a 0-spectral expander. Moreover, define $\Delta_{(X,\pi)} = \max_{i \in [d]} \Delta_{(X,\pi)}(i)$. We drop the subscripts (X, π) when the complex is clear in the context.

Theorem 4.1.1. *Let (X, π) be a $(d+1)$-partite (weighted) totally connected complex. For some $0 < \delta < 1$, assume that*

$$\gamma_2 \leq \frac{\delta^2}{10(1 + \ln \Delta)} \quad \textit{and} \quad \gamma_2 \leq \frac{1-\delta}{\Delta + \ln \Delta}.$$

Then, the link of any face τ of co-dimension k of X has spectral expansion

$$\begin{cases} \frac{c(1-\delta)}{k\delta} & \text{if } k \geq \Delta, \\ \frac{c(1-\delta)\frac{k+\ln k}{\Delta+\ln\Delta}}{k\delta} & \text{if } k < \Delta, \end{cases}$$

for some constant $c \leq 2$ that depends on δ.

Note that, for $\Delta = d$, this theorem retrieves Oppenheim's trickle-down theorem (Theorem 1.0.2) up to a lower order term in the condition on γ_2 and a constant in the bounds on local spectral expansions.

When $\Delta \ll d$, this theorem is a significant improvement over Theorem 1.0.2. Roughly speaking, this theorem says that, if the complex has many faces of co-dimension 2 whose links are 0-expanders, one needs to satisfy a much weaker condition on γ_2 to get $O(1/k)$-spectral expansion for faces of co-dimension k. In other words, the faces of co-dimension 2 that have perfect spectral expansion can compensate for faces of co-dimension 2 that have bad spectral expansion.

Next, we state the second special case of our theorem. For every integer n, let $H_n = \sum_{i=1}^{n} \frac{1}{i}$ be the n-th harmonic number. Moreover, for any $1 \leq i \leq n$ define $H_n(i) = \sum_{j=i}^{n} \frac{1}{j}$ and let $H_n(0) = H_n(1)$.

Theorem 4.1.2. *Let (X, π) be a $(d+1)$-partite (weighted) totally connected complex. For any distinct $i, j \in [d]$, let*

$$\epsilon_{\{i,j\}} = \max_{\tau:\text{type}(\tau)=[d]\setminus\{i,j\}} \lambda_2(P_\tau)$$

be the 2nd largest eigenvalue of the simple random walk matrices on (X_τ, π_τ) for all τ of type $[d] \setminus \{i, j\}$. For some $0 < \delta < 1$, assume that for every $i \in [d]$,

$$\begin{aligned} \epsilon_{\{i,j\}} \cdot H_d &\leq \frac{\delta^2}{10}, \forall j \neq i \quad \textit{and} \\ \sum_{\ell=1}^{d} \epsilon_{\{i,j_\ell\}} \cdot \frac{H_d(\ell-1)}{d} &\leq \frac{1-\delta}{d}, \end{aligned}$$

where $j_0 \ldots, j_d$ is an ordering of $[d] \setminus i$ such that $\epsilon_{\{i,j_0\}} \leq \cdots \leq \epsilon_{\{i,j_d\}}$. Then, X is $(\frac{c(1-\delta)}{\delta}, \ldots, \frac{c(1-\delta)}{d\delta})$-local spectral expander for some constant $c \leq 2$ that depends on δ.

We remark that $1 \leq \frac{\sum_{\ell=1}^{d} H_d(\ell-1)}{d} \leq 1 + \frac{\ln d}{d}$. So, roughly speaking, the latter condition can be seen as $\mathbb{E}_j[\epsilon_{\{i,j\}}] \leq \frac{1-\delta}{d}$ for every $i \in [d]$, where the expectation is weighted according to $\frac{H_d(.)}{d}$. This is an improvement over the stronger condition in **??**. Now, we state the main theorem.

Theorem 4.1.3 (Main). *Let (X, π) be a $(d+1)$-partite (weighted) totally connected complex. For any distinct $i, j \in [d]$, let $\epsilon_{\{i,j\}} = \max_{\tau:\text{type}(\tau)=[d]\setminus\{i,j\}} \lambda_2(P_\tau)$ be the 2nd largest eigenvalue of the simple random walk matrices on (X_τ, π_τ) for all τ of type $[d] \setminus \{i, j\}$. For some $0 < \delta < 1$, assume that for every $i \in [d]$,*

$$\epsilon_{\{i,j\}} \cdot H_{\Delta-1} \leq \frac{\delta^2}{10}, \forall j \neq i \quad \textit{and} \tag{4.1}$$

$$\sum_{\ell=1}^{\Delta(i)} \epsilon_{\{i,j_\ell\}} \cdot H_{\Delta(i)-1}(\ell-1) \quad \leq \quad 1-\delta, \tag{4.2}$$

where $j_0 \dots, j_d$ is an ordering of $[d] \setminus i$ such that $\epsilon_{\{i,j_0\}} \leq \cdots \leq \epsilon_{\{i,j_d\}}$. Then, the 1-skeleton of X is a $\frac{c(1-\delta)}{d\delta}$-expander for $c = \frac{2(1+\frac{\delta^2}{10})}{(1+\delta)}$.

Remark 4. If, for some $\delta > 0$, the conditions of the above theorem hold for a complex (X, π), then the conditions also hold for the same δ for all links (X_τ, π_τ) (of faces of co-dimension at least 2). Therefore, this theorem implies that X is $(\frac{c(1-\delta)}{\delta}, \dots, \frac{c(1-\delta)}{d\delta})$-local spectral expander for $c = \frac{2(1+\frac{\delta^2}{10})}{(1+\delta)}$. One can prove tighter bounds if they apply this theorem to any link (X_τ, π_τ) individually and possibly use better bounds on $\Delta_{(X_\tau,\pi_\tau)}(i)$.

Next, we show how Theorem 4.1.1 follows from our main theorem.

Proof of Theorem 4.1.1. Fix a face τ of co-dimension k. For brevity we abuse notation and write Δ_τ denote $\Delta_{(X_\tau,\pi_\tau)}$. If $k \geq \Delta$ the statement follows from the above remark. In particular, for any $i, j \in [d]$

$$\epsilon_{\{i,j\}} \cdot H_{\Delta_\tau - 1} \leq \gamma_2 \cdot H_{\Delta-1} \leq \gamma_2 \cdot (1 + \ln \Delta) \leq \frac{\delta^2}{10},$$
$$\sum_{\ell=1}^{\Delta_\tau(i)} \epsilon_{\{i,j_\ell\}} \cdot H_{\Delta_\tau(i)-1}(\ell-1) \leq \gamma_2(\Delta + \ln \Delta) \leq 1-\delta.$$

So, we can apply Theorem 4.1.3 to obtain that the link of τ has spectral expansion $\frac{c(1-\delta)}{k\delta}$ for some constant $c \leq 2$.

Otherwise, to bound the spectral expansion of (X_τ, π_τ), let $\delta_k = 1 - (1-\delta)\frac{k+\ln k}{\Delta+\ln\Delta} \geq \delta$. For $i, j \in [d]$

$$\epsilon_{\{i,j\}} \cdot H_{\Delta_\tau-1} \leq \gamma_2 \cdot H_{k-1} \leq \frac{\delta^2 \cdot H_{k-1}}{10(1+\ln\Delta)} \underset{\delta \leq \delta_k}{\leq} \frac{\delta_k^2}{10},$$

and

$$\sum_{\ell=1}^{\Delta_\tau(i)} \epsilon_{\{i,j_\ell\}} \cdot H_{\Delta_\tau(i)-1} \underset{\epsilon_{i,j_\ell} \leq \gamma_2, \Delta_\tau(i) \leq k}{\leq} \gamma_2(k + \ln k)$$
$$\leq \frac{(1-\delta)(k+\ln k)}{\Delta + \ln\Delta} = 1 - \delta_k.$$

Therefore, applying Theorem 4.1.3 to (X_τ, π_τ), we obtain that (X_τ, π_τ) is a $\frac{c(1-\delta_k)}{k\delta}$-expander for some constant $c \leq 2$. □

Applications to Graph Coloring Consider a graph $G = ([n], E)$ with degree function $\Delta : [n] \to \mathbb{Z}_{\geq 0}$ and maximum degree Δ, paired with a collection of color lists $\{L(i)\}_{i\in[n]}$ satisfying $L(i) \geq \Delta(i) + (1+\eta)\Delta$ for all $i \in [n]$ and for some $0 < \eta \leq 0.9$ such that

$\frac{1+\ln\Delta}{\Delta} \leq \frac{\eta^2}{40}$. We define the $(n+1)$-partite coloring complex $X(G,L)$ specified by the following facets: $\{i,\sigma(i)\}_{i\in[n]}$ is a facet if and only if σ is a proper L-coloring of G, i.e. $\sigma(i) \in L(i)$ for each $i \in [n]$ and $\sigma(i) \neq \sigma(j)$ if $\{i,j\} \in E$. It is not hard to see that if $\{i,j\} \notin E$, then $\epsilon_{\{i,j\}} = 0$. Moreover, if $\{i,j\} \in E$, then $\epsilon_{\{i,j\}} \leq \frac{1}{(1+\eta)\Delta} + \frac{1}{(1+\eta)^2\Delta^2}$ (see Proposition 3.3.2). Once can verify that if we apply the above theorem to the coloring complex $X(G,L)$ with $\delta = \frac{\eta}{2}$, we get that $X(G,L)$ is a $\left(\frac{4}{\eta}, \frac{4}{2\eta}, \ldots, \frac{4}{(|V|-1)\cdot\eta}\right)$-local spectral expander, and thus the Glauber dynamics for sampling a random proper coloring mixes in polynomial time. This retrieves Theorem 3.3.1 up to constants.

Applications to Sparse High Dimensional Expanders Kaufman and Oppenheim [KO18] obtained a simple construction of sparse $(d+1)$-partite complexes with $|X(0)| \geq p^s$ for any integer $s > d$ and prime power p such that every $x \in X(0)$ is in at most $p^{O(d^3)}$ many facets (hence the degree is independent of s). They argued that for any non-consecutive pair of parts $i,j \in [d]$, i.e., $j \neq i+1$ and $i \neq j+1 \pmod{d+1}$, we have $\epsilon_{\{i,j\}} = 0$ but $\epsilon_{\{i,i+1\}} \leq \frac{1}{\sqrt{p}}$ for any $i \in [d]$ ($i+1$ is taken modulo $d+1$). Consequently, $\Delta(i) = 2$ for any $i \in [d]$. Then, using **??**, they show that the complex is a $(\frac{1}{\sqrt{p}-(d-2)}, \ldots, \frac{1}{\sqrt{p}-d-2})$-local spectral expander for $p > (d-2)^2$. Simply plugging in these values into the above theorem, for $\delta = 1 - \frac{2}{\sqrt{p}}$ and $p \geq 193$ (independent of d) the assumptions of the theorem are satisfied. The resulting complex is $(\frac{2c}{\sqrt{p}\delta}, \ldots, \frac{2c}{d\sqrt{p}\delta})$-local spectral expander for $c \approx 1.15$. In other words, not only does the Kaufman-Opennheim construction give a HDX for constant values of p independent of d, but also its local spectral expansion improves inverse linearly with the co-dimension.

O'Donnell and Pratt [OP22] constructed $(d+1)$-partite (sparse) high-dimensional expanders, with unbounded dimension d, via root systems of simple Lie Algebras, namely families A_d for $d \geq 1$, B_d for $d \geq 2$, C_d for $d \geq 3$ and D_d for $d \geq 4$. For explicit descriptions of these root systems, see e.g. [Car89, Sec. 3.6]. O'Donnell and Pratt showed that, similar to the Kaufman-Oppenheim construction, the resulting d-dimensional complex X satisfies $|X(0)| \geq p^{\Theta(m)}$ whereas every vertex is only in $p^{\Theta(d^2)}$ many facets and for any $i,j \in [d]$, $\epsilon_{i,j} \leq \sqrt{2/p}$. Then, using **??** they concluded that the complex is a $(\frac{1}{\sqrt{p/2}-d+1}, \ldots, \frac{1}{\sqrt{p/2}-d+1})$-local spectral expander. Upon further inspection of the explicit set of roots, one can verify that $\Delta \leq 2$ for complexes based on A_d, B_d, C_d root systems and $\Delta \leq 3$ for the D_d root system. Plugging in these values in the above theorem and setting $\delta = 1-2\sqrt{2/p}$ for A_d, B_d, C_d complexes and $\delta = 1-3.5\sqrt{2/p}$ for the D_d complex, if $p \geq 376$ for A_d, B_d, C_d complexes and $p \geq 729$ for the D_d complex, we get that these complexes are $(\frac{c'}{\sqrt{p}\delta}, \ldots, \frac{c'}{d\sqrt{p}\delta})$-local spectral expander for some constant $c' > 1$.

The well known Ramanujan complexes, also known as LSV complexes, are generalizations of Ramanujan graphs that were introduced by Lubotsky, Samuels, and Vishne in [LSV05b] and explicitly constructed in [LSV05a]. As shown in [EK16], any d-dimenssional LSV complex X that is q-thick for some fixed prime power q and $d \geq 2$ has a bounded degree (the number of facets that contain each $x \in X(0)$ only deponents on q and d, and is constant in the size of the ground set n which can be arbitrarily large). Moreover, the link of every proper face of type S is a spherical building complex in which $\Delta(i) = |\{j \neq i : \epsilon_{\{i,j\}} > 0\}|$ is at most 2 for every $i \in [d] \setminus S$. Furthermore, the worst expansion among links co-dimension 2 is $\frac{c}{\sqrt{q}}$, for some constant c independent of q, d, n. So, there is a constant q_0 such that if $q \geq q_0$, 4.1.3 implies that the link of any (proper) face of X of co-dimension k is a $\frac{c'}{(k-1)\sqrt{q}}$-spectral expander for some constant $c' > 0$ independent of q, d, n. This improves over the bound $\frac{C(d)}{\sqrt{q}}$ proved in [EK16], where $C(d) \geq 2^d(d+1)!$.

4.1.2 Proof Overview

At a high-level, this proof builds on the matrix trickle-down framework discussed in Chapter 2. The Oppenheim's trickle-down theorem follows from an inductive argument that derives a bound on the second eigenvalue of the simple walk on 1-skeleton of each link (X_τ, π_τ) using the largest second eigenvalue of the simple walk on the 1-skeleton of links $(X_{\tau'}, \pi_{\tau'})$ for all faces $\tau' \supset \tau$ of size $|\tau| + 1$. The reason that one has to take the largest 2nd eigenvalue as opposed to the average in each inductive step is that the eigenspaces of these simple walks are very different. The matrix trickle-down framework overcomes this issue by substituting the scalar bounds on the second eigenvalues with matrices that upper bound the transition probability matrices of the simple walks on the 1-skeletons of links. However, as opposed to Oppenhiem's trickle-down theorem, the matrix trickle-down framework cannot be applied in a black-box manner to bound the spectral expansion of the 1-skeletons of all links only by bounding the spectral expansion of the 1-skeletons of links of faces of co-dimension 2. The main result of this chapter can be seen as applying the matrix trickle-down framework with a carefully chosen set of upper-bound matrices to prove an improved trickle-down theorem for partite complexes that can be applied in the same black-box fashion, just known an "average" second eigenvalue.

Our technical results in this chapter are twofold: First, we observe that for any two disjoint sets of parts $S, T \subseteq [d]$, if the links of all faces of co-dimension 2 whose types intersect with both S, T are 0-spectral expanders, then for any $\sigma \in X$ of type S and τ of type T we get

$$\mathbb{P}_{\eta\sim\pi}[\sigma \subset \eta | \tau \subset \eta] = \mathbb{P}_{\eta\sim\pi}[\sigma \subset \eta] \quad \text{and} \quad \mathbb{P}_{\eta\sim\pi}[\tau \subset \eta | \sigma \subset \eta] = \mathbb{P}_{\eta\sim\pi}[\tau \subset \eta],$$

namely, the conditional distributions on these types are independent (see Lemma 4.2.3 for details). This observation significantly simplifies invoking the Matrix trickle-down framework. Armed with this tool, we invoke the matrix trickle-down theorem using a carefully chosen family of (diagonal) matrices as the matrix bounds. These matrices are recursively defined based on an "average" of the spectral expansions of the links of all faces of co-dimension 2, See the proof of Theorem 4.1.3 for the construction of these matrices.

4.1.3 Preliminaries

Graphs Given a graph $G = (V, E)$, for any $v \in V$, let $\Delta_G(v)$ be the degree of v in G, and let Δ_G be the maximum degree of G. Moreover, given a subset $S \subseteq V$, $G[S]$ denotes the induced subgraph of G on the set of vertices S. For any $S \subseteq V$, define $G_S = G[V \setminus S]$. For simplicity of notation, when G is clear from context, we denote $\Delta_G(v)$ by $\Delta(v)$ for any $v \in V$, and for any $S \subseteq V$, we denote $\Delta_{G_S}(v)$ by $\Delta_S(v)$ for any $v \in V \setminus S$. Similarly, we denote the maximum degree of G and G_S by Δ and Δ_S respectively. Moreover, when G is clear from context, we write $u \sim v$ if u, v are adjacent vertices in G and $u \sim_S v$ if $u, v \in V \setminus S$ and $u \sim v$.

Simplicial Complexes We say that a simplicial complex X is *gallery connected* if for any face τ of co-dimension at least 2 and any pair of facets σ, σ' of X_τ there is a sequence of facets of X_τ, $\sigma = \sigma_0, \sigma_1, \ldots, \sigma_\ell = \sigma'$, such that for all $0 \le i < \ell$, $|\sigma_i \Delta \sigma_{i+1}| = 2$. It is shown in [Opp18, Prop 3.6] that if X is totally connected, then it is gallery connected.

Given a $(d+1)$-partite complex, we say that an $x \in X(0)$ is of type i and write $\text{type}(x) = i$ if $x \in T_i$. Similarly, the type of a face $\tau \in X$ is defined as $\text{type}(\tau) = \{i \in [d] : |\tau \cap T_i| = 1\}$.

Lemma 4.1.4. *Consider a totally connected $(d+1)$-partite complex X with parts indexed by $[d]$. For any $S \subseteq [d]$, The induced subgraph of the 1-skeleton of X on vertices of type S is connected.*

Proof. Take x, y of type $i, j \in S$ and facets η, η' such that $x \in \eta$, $y \in \eta'$. Total connectivity implies that there is a sequence $\eta = \eta_1, \ldots, \eta_t = \eta'$ of facets such that $\eta_i \cap \eta_{i+1} \neq \emptyset$ for all $1 \le i \le t-1$. Let $\sigma_1 \subseteq \eta_1, \ldots, \sigma_t \subseteq \eta_t$ be faces of type $\{i, j\}$. Note that this is possible because any facet in a partite complex has exactly one vertex from each of the $[d]$ parts. Finally, $\sigma_1, \ldots, \sigma_t$ gives a path between x, y. □

The following facts hold for weighted partite complexes:

Observation 4.1.5. *Consider a weighted $(d+1)$-partite complex (X, π). For any face τ of co-dimension $k \ge 1$, we have $k\pi_{\tau,0}(x) = \Pr_{\sigma \sim \pi_\tau}[x \in \sigma]$ for all $x \in X_\tau(0)$.*

Observation 4.1.6. *Consider a weighted $(d+1)$-partite complex (X, π) with parts indexed by $[d]$. For any face τ of co-dimension $k \ge 1$ and $i \in [d] \backslash \text{type}(\tau)$, $\sum_{x:\text{type}(x)=i} \Pr_{\sigma \sim \pi_\tau}[x \in \sigma] = 1$.*

The following definition is useful for proving the main theorem:

Definition 11. *For any $(d+1)$-partite complex (X, π) with parts indexed by $[d]$, define a graph $G_{(X,\pi)}$ on the set of vertices $[d]$, where any distinct $i, j \in [d]$ are adjacent in $G_{(X,\pi)}$ if there exists τ of type $[d] \setminus \{i, j\}$ such that the second eigenvalue of (X_τ, π_τ) is positive.*

Remark 5. For any $(d+1)$-partite complex (X, π) with parts indexed by $[d]$, for every $i \in [d]$, $\Delta(i)$ (see Definition 10) is the degree of i in graph $G_{(X,\pi)}$ and Δ is the maximum degree of $G_{(X,\pi)}$.

Note that if $\text{codim}(\tau) = k$, the link X_τ is a k-partite complex with parts indexed by $[d] \setminus S$. One can verify that given a face τ of type S, the set of edges of $G_{(X_\tau, \pi_\tau)}$ is a subset of the edges of $(G_{(X,\pi)})_S$, i.e., the induced subgraph of $G_{(X,\pi)}$ on $[d] \setminus S$. When (X, π) is clear from context, we write G for $G_{(X,\pi)}$ and G_S for $(G_{(X,\pi)})_S$.

Product of Weighted Complexes Given weighted complexes $(Y_1, \mu_1), \ldots, (Y_\ell, \mu_\ell)$ defined on disjoint ground sets and of dimensions $d_1, \ldots, d_\ell$ respectively, and weighted complexes (X, π) of dimension d, we write $(X, \pi) = (Y_1, \mu_1) \times \cdots \times (Y_\ell, \mu_\ell)$ if $X(d) = \{\cup_{i \in [\ell]} \tau_i : \tau_1 \in Y_1(d_1), \ldots, \tau_\ell \in Y_\ell(d_\ell)\}$ and $\pi(\cup_{i \in [\ell]} \tau_i) = \prod_{i \in [\ell]} \mu_i(\tau_i)$ for all $\tau_1 \in Y_1(d_1), \ldots, \tau_\ell \in Y_\ell(d_\ell)$. We denote the generating polynomial of (X, π) by $g_{(X,\pi)}$, i.e.

$$g_{(X,\pi)} = \sum_{\tau \in X(d)} \pi(\tau) \prod_{x \in \tau} z_x.$$

One can verify that $(X, \pi) = (X_1, \mu_1) \times \cdots \times (X_\ell, \mu_\ell)$ if and only if $g_{(X,\pi)} = g_{(X_1,\mu_1)} \times \cdots \times g_{(X_\ell,\mu_\ell)}$. Note that this is true because we assume that for any weighted simplicial complex, the given distribution on facets is non-zero on all facets.

4.2 Simplifying Matrix Trickle-Down's Conditions to Scalar Inequalities

In this section, given a $(d+1)$-partite complex (X, π), we apply the matrix trickle-down theorem to derive a set of conditions on a family of vectors $\{f_S \in \mathbb{R}^{[d]}\}_{S \subset [d], |S|<d}$ that will guarantee that $\lambda_2(P_\tau) \leq \frac{\max_{i\in[d]} f_S(i)}{k-1}$ for all $k \geq 2$ and τ of co-dimension k and type S. We prove the following theorem.

Theorem 4.2.1. Consider a totally connected $(d+1)$-partite complex (X, π) with parts indexed by $[d]$ and graph $G = G_{(X,\pi)}$. Suppose we are given a family of vectors $\{f_S \in \mathbb{R}^{[d]}\}_{S\subset[d],|S|<d}$ such that for all $S \subset [d]$ of size $(d+1)-k$, the support of f_S is a subset of $[d] \setminus S$, and the following holds:

- *If G_S is disconnected, then $f_S = \sum_{1\leq i\leq \ell:|I_i|\geq 2} f_{[d]\setminus I_i}$, where $I_1 \cup \cdots \cup I_\ell$ are the vertices of the connected components of G_S. Note that if all connected components are of size 1, then $f_S = 0$.*
- *Otherwise if G_S is connected, we have $\max_{i\in[d]} f_S(i) \leq \frac{(k-1)^2}{3k-1}$ and*
  1. *Base Case: If $k = 2$, then for every face τ of type S, $\lambda_2(P_\tau) \leq \max_{i\in[d]\setminus S} f_S(i)$.*
  2. *Recursive Condition: If $k \geq 3$, then*
  $$\sum_{j\in[d]\setminus(S\cup i)} f_{S\cup j}(i) \leq (k-2)f_S(i) - f_S^2(i),$$
  for all $i \in [d] \setminus S$.

Then, for all $k \geq 2$ and τ of co-dimension k and type S, $\lambda_2(P_\tau) \leq \frac{\max_{i\in[d]} f_S(i)}{k-1}$.

The main sets of conditions in the above theorem are the inequalities in Item 1 and Item 2. To get some intuition about these conditions, it is helpful to compare the above theorem with the standard trickle-down theorem (**??**). There, one shows that if $\lambda_2(P_{\tau\cup\{x\}}) \leq \lambda$ for all $x \in X_\tau(0)$, then $\lambda_2(P_\tau) \leq \alpha$, where α satisfies

$$\lambda \leq \alpha - \alpha^2(1 - \lambda). \tag{4.3}$$

Then, **??** follows by recursively applying these inequalities.

In the above theorem, instead of a single upper bound on $\lambda_2(P_\tau)$ for faces τ of co-dimension 2, one bounds the expansion of the links of all faces of co-dimension 2 of each type separately, allowing higher degrees of freedom. For any face τ of type S and co-dimension $k = |S|$, the function $\frac{f_S(.)}{k-1}$ will serve as the digonal entries of a matrix upper-bound P_τ. Then, the inequality $\frac{\sum_{j\in[d]\setminus(S\cup i)} f_{S\cup j}(i)}{k-2} \leq f_S(i) - \frac{f_S^2(i)}{k-2}$ is the natural analogue of Eq. (4.3) which requires f_S to be at least "the average" of $f_{S\cup j}$ for all $j \in [d] \setminus S$ plus a square error term.

Before proving the above theorem, we show that if G_S is disconnected with parts $G[I_1], \ldots, G[I_\ell]$ for some $S \subset [d]$ of size at most $d-1$, then for any τ of type S, $(X_\tau, \pi_{\tau,k-1})$ can be written as the product of its links of types $[d] \setminus I_i$ for all $1 \leq i \leq \ell$. This allows us to prove a better upper-bound on $\lambda_2(P_\tau)$ for such faces τ by simply "concatenating" upper-bounds on each connected component of G_S.

Lemma 4.2.2. Consider a 2-partite complex (X, π) with parts S, T. If (X, π) is 0-expander, then $(X, \pi) = (X_z, \pi_z) \times (X_y, \pi_y)$ for any $y \in S$ and $z \in T$.

Proof. Note that (X, π) is a weighted bipartite graph with parts S, T. Let $A \in \mathbb{R}^{X(0)\times X(0)}$ be the adjacency matrix of (X, π). Let $A_{S,T}(y, z) = A(y, z)$ for $y \in S$, $z \in T$ and 0 on other entries. Moreover, let $A_{T,S} = A - A_{S,T}$. Then, for any vector $v \in \mathbb{R}^{X(0)}$, we get $A = A_{S,T}v_T + A_{T,S}v_S$, where v_S, v_T are respectively supported on S, T and $v = v_S + v_T$. Thus, if $Av = \lambda v$, then $Av' = -\lambda v'$, for $v' = (-v_S + v_T)$. So if μ is an eigenvalue of A, then $-\mu$ is also an eigenvalue of A. Thus, if (X, π) is 0-expander, the rank of A is 2. This implies that there are vectors $w_S \in \mathbb{R}^S$ and $w_T \in \mathbb{R}^T$ such that $\pi(\{y, z\}) = A(y, z) = A(z, y) = w_S(y)w_T(z)$ for $y \in S$, $z \in T$. Without loss of generality, assume $\|w_S\|_1 = \|w_T\|_1 = 1$. Then, for any $y \in S$ and $z \in T$, we have $\pi_z(y) = \frac{\pi(\{y,z\})}{\sum_{x\in S}\pi(\{x,z\})} = w_S(y)$. Similarly $\pi_y(z) = w_T(z)$. Thus $\pi(\{y, z\}) = \pi_y(z)\pi_z(y)$. This finishes the proof. □

Lemma 4.2.3. Consider a totally connected $(d + 1)$-partite complex (X, π) with parts indexed by $[d]$ and its associated graph $G = G_{(X,\pi)}$. Let $I_1 \cup \cdots \cup I_\ell$ be a partition of $[d]$ such that for any $1 \le i \le \ell$ the induced graph $G[I_i]$ is a connected component or the union of several connected components of G. Then $(X, \pi) = (X_{\sigma_{-1}}, \pi_{\sigma_{-1}}) \times \cdots \times (X_{\sigma_{-\ell}}, \pi_{\sigma_{-\ell}})$, where σ_{-i} is an arbitrary face of type $[d] \setminus I_i$ for any $1 \le i \le \ell$.

Proof. We prove the statement by induction on d. For $d = 1$, the statement simply follows from Lemma 4.2.2. Now, assume that $d > 1$. If $|I_i| = 1$ for all $1 \le i \le \ell$, then $\ell \ge 3$. In this case, let $S = I_1 \cup I_2$. Otherwise, WLOG assume that $|I_1| \ge 2$ and let $S = I_1$. First, we show that $g_{(X,\pi)}$ can be written as $g_{(X,\pi)} = h \cdot h'$, where h is a polynomial in $\{z_y : \text{type}(y) \in I \setminus S\}$ and h' is a polynomial in terms of variables in $\{z_y : \text{type}(y) \in S\}$. By the induction hypothesis, for any $i \in S$, $x \in T_i$, and any face $\sigma \in X$ of type S such that $x \in \sigma$,

$$\partial_{z_x} g_{(X,\pi)} = f^x \cdot g^x, \tag{4.4}$$

where f^x is a polynomial in terms of variables in $\{z_y : \text{type}(y) \in S \setminus i\}$ and g^x is a polynomial in terms of variables in $\{z_y : \text{type}(y) \in I \setminus S\}$. Now, take arbitrary $i, j \in S$ such that $i \neq j$. Then, Eq. (4.4) implies that for any face $\{x, y\}$ of type $\{i, j\}$,

$$\partial_{z_x}\partial_{z_y} g_{(X,\pi)} = (\partial_{z_y} f^x)g^x = (\partial_{z_x} f^y)g^y.$$

It thus follows that g^x is a multiple of g^y. One can see this simply by substituting 1 for all variables in $\{z_y : \text{type}(y) \in S \setminus \{i, j\}\}$. Moreover, since g^x and g^y are generating polynomials of distributions, i.e. the coefficients sum up to 1, we get $g^x = g^y$. Therefore, we get that for any distinct x, y such that $\text{type}(x), \text{type}(y) \in S$ and $\{x, y\}$ is a face, $g^x = g^y$. Applying Lemma 4.1.4, we get $g^x = g^y$ for all $x, y \in \cup_{i\in S}T_i$. Thus, there exist a polynomial h in variables $\{z_y : \text{type}(y) \in I \setminus S\}$ such that we can rewrite Eq. (4.4) for any x with $\text{type}(x) \in S$ as

$$\partial_{z_x} g_{(X,\pi)} = f^x \cdot h,$$

where f^x is a polynomial in terms of variables in $\{z_y : \text{type}(y) \in S \setminus i\}$. Finally, since X is a partite complex,

$$|S|g_{(X,\pi)} = \sum_{i\in S}\sum_{x\in T_i} z_x \partial_{z_x} g_{(X,\pi)} = h \cdot \sum_{i\in S}\sum_{x\in T_i} z_x f^x = h \cdot h', \tag{4.5}$$

where $h' = \sum_{i\in S}\sum_{x\in T_i} z_x f^x$ is a polynomial in $\{z_y : \text{type}(y) \in S\}$. It remains to show that for any face σ of type S, we have $h = g_{(X_\sigma,\pi_\sigma)}$, and for any τ of type $[d]\setminus S$, we have $h' = g_{(X_\tau,\pi_\tau)}$. Fix arbitrary faces σ of type S and τ of type $[d]\setminus S$. Noting that $g_{(X,\pi)}$ is a multiple of $h\cdot h'$, and that h' is in variables associated to elements whose types are in S and h is in variables associated to elements whose types are in $[d]\setminus S$, we conclude that h' has a monomial that is a multiple of $\prod_{x\in\sigma} z_x$ and h has a monomial that is a multiple of $\prod_{x\in\tau} z_x$. First, take $(\prod_{x\in\sigma}\partial_{z_x})$ from both sides of *Eq. (4.5)*. We get that $g_{(X_\sigma,\pi_\sigma)}$ is a positive multiple of h. Similarly, taking $(\prod_{x\in\tau}\partial_{z_x})$ from both sides of *Eq. (4.5)*, we get that $g_{(X_\tau,\pi_\tau)}$ is a positive multiple of h'. Thus, noting that the coefficients of generating polynomials sum up to 1, we get $h = g_{(X_\sigma,\pi_\sigma)}$ and $h' = g_{(X_\tau,\pi_\tau)}$ as desired. Repeating the same argument inductively on the complex (X_σ,π_σ) proves the claim. □

Now we are ready to prove Theorem 4.2.1.

Proof of Theorem 4.2.1. We apply the matrix trickle-down theorem (Theorem 2.4.2). For every $S\subset[d]$ such that $|S|<d$, define a diagonal matrix $D_S\in\mathbb{R}^{X(0)\times X(0)}$ as $D_S(x,x) = f_S(\text{type}(x))$ for all $x\in X(0)$. We prove that the conditions of **??** hold for $M_\tau = \frac{\Pi_\tau D_S}{k-1}$ for an arbitrary face $\tau\in X$ of co-dimension at least $k\geq 2$ and type S.

First, we show that the condition $M_\tau \preceq \frac{k-1}{3k-1}\Pi_\tau$ holds. If G_S is connected, $\max_{i\in[d]} f_S(i) \leq \frac{(k-1)^2}{3k-1}$ holds by assumption. If G_S is disconnected, $\max_{i\in[d]} f_S(i) \leq \frac{(k-1)^2}{3k-1}$ follows from the assumptions that $f_S = \sum_{1\leq i\leq \ell:|I_i|\geq 2} f_{[d]\setminus I_i}$, where $I_1\cup\cdots\cup I_\ell$ are the vertices of connected components of G_S. Because the supports of vectors $f_{[d]\setminus I_i}$ are disjoint by assumption and $\frac{(k-1)^2}{3k-1}$ is an increasing function for $k\geq 2$, we get $D_\tau \preceq \frac{(k-1)^2}{3k-1}I$, and thus, $M_\tau \preceq \frac{k-1}{3k-1}\Pi_\tau$.

To prove the rest of the conditions hold, first assume that $k=2$. If G_S is two disconnected vertices, we get $f_S = 0$, and therefore, $D_S = 0$. Thus, we get $\Pi_\tau P_\tau - \pi_{\tau,0}\pi_{\tau,0}^\top \preceq 0 = \Pi_\tau D_S = M_\tau$, as desired. If G_S is connected, the base case assumption (1) implies that $\lambda_2(P_\tau)\leq D_S(x,x)$ for all $x\in X_\tau(0)$. Therefore, $\Pi_\tau P_\tau - \pi_{\tau,0}\pi_{\tau,0}^\top \preceq \Pi_\tau D_S = M_\tau$.

Now we prove the rest of the conditions for $k\geq 3$. First assume that G_S is disconnected and $G[I_1],\ldots,G[I_\ell]$ are its connected components for some partition $I_1\cup\cdots\cup I_\ell$ of $[d]\setminus S$. Fix any $\sigma\in X_\tau(k-1)$. By Lemma 4.2.3, $(X_\tau,\pi_\tau) = (X_{\tau\cup\sigma_{-1}},\pi_{\tau\cup\sigma_{-1}})\times\cdots\times(X_{\tau\cup\sigma_{-\ell}},\pi_{\tau\cup\sigma_{-\ell}})$ where for every $1\leq j\leq\ell$, σ_{-j} is ca subset of σ that has type $[d]\setminus(S\cup I_j)$. Therefore, we get $\Pr_{\eta\sim\pi_{\tau\cup\sigma_{-j}}}[x\in\eta] = \Pr_{\eta\sim\pi_\tau}[x\in\eta]$ for all $1\leq j\leq\ell$ and $x\in X_{\tau\cup\sigma_{-j}}(0)$. Combining this with Observation 4.1.5, we get $k_j\cdot\pi_{\tau\cup\sigma_{-j},0}(x) = k\cdot\pi_{\tau,0}(x)$, where $k_j = |I_j|$ for all $1\leq j\leq\ell$. Thus we can write

$$\begin{aligned}
\sum_{1\leq j\leq\ell:|I_j|\geq 2}\frac{(k_j-1)k_j}{(k-1)k}M_{\tau\cup\sigma_{-j}} &= \sum_{1\leq j\leq\ell:|I_j|\geq 2}\frac{(k_j-1)k_j}{(k-1)k}\frac{\Pi_{\tau\cup\sigma_{-j}}D_{[d]\setminus I_j}}{k_j-1}\\
&= \sum_{1\leq j\leq\ell:|I_j|\geq 2}\frac{k_j}{k(k-1)}\frac{k}{k_j}\Pi_\tau D_{[d]\setminus I_j}\\
&= \frac{\Pi_\tau}{k-1}\sum_{1\leq j\leq\ell:|I_j|\geq 2} D_{[d]\setminus I_j}\\
&= \frac{\Pi_\tau D_S}{k-1} = M_\tau,
\end{aligned}$$

where the first equality follows from the definition of $M_{\tau\cup\sigma_{-j}}$, and in the second to last equality, we used the fact that $\sum_{1\le j\le \ell:|I_j|\ge 2} f_{[d]\setminus I_j} = f_S$ and thus $\sum_{1\le j\le \ell:|I_j|\ge 2} D_{[d]\setminus I_j} = D_S$ by definition of D_S. Now, assume that G_S is connected. It is enough to show that $\mathbb{E}_{x\sim\pi_{\tau,0}} M_{\tau\cup x} \preceq M_\tau - M_\tau \Pi_\tau^{-1} M_\tau$. This is equivalent to showing that for any $x \in X_\tau(0)$

$$\mathbb{E}_{y\sim\pi_{\tau,0}}\left[\frac{(\Pi_\tau^{-1}\Pi_{\tau\cup y} D_{S\cup \mathrm{type}(y)})(x,x)}{k-2}\right] \le \frac{D_S(x,x)}{k-1} - \frac{D_S^2(x,x)}{(k-2)(k-1)} \tag{4.6}$$

One can check that for any $x \in X_\tau(0)$ of type i

$$\begin{aligned}
&\mathbb{E}_{y\sim\pi_{\tau,0}}\left[\frac{\Pi_\tau^{-1}\Pi_{\tau\cup y} D_{S\cup \mathrm{type}(y)}(x,x)}{k-2}\right] \\
&= \frac{\sum_{y\in X_{\tau\cup x}(0)} \Pr_{\sigma\sim\pi_{\tau\cup x}}[y\in\sigma] D_{S\cup\mathrm{type}(y)}(x,x)}{(k-1)(k-2)} \\
&= \sum_{j\in[d]\setminus S} \frac{f_{\tau\cup j}(i)}{(k-1)(k-2)} \sum_{\substack{y\in X_{\tau\cup x(0)}:\\ \mathrm{type}(y)=j}} \Pr_{\sigma\sim\pi_{\tau\cup x}}[y\in\sigma] \\
&= \frac{\sum_{j\in[d]\setminus S} f_{\tau\cup j}(i)}{(k-1)(k-2)},
\end{aligned}$$

where in the last equality, we used Observation 4.1.6. Thus, substituting $D_S(x,x) = f_S(\mathrm{type}(x))$ in the RHS of Eq. (4.6), it is enough to show that for any $i \in [d]\setminus S$

$$\frac{\sum_{j\in[d]\setminus S} f_{\tau\cup j}(i)}{(k-1)(k-2)} \le \frac{f_S(i)}{k-1} - \frac{f_S^2(i)}{(k-1)(k-2)},$$

which holds by Item 2. □

4.3 Proof of Main Theorem

We are ready to prove Theorem 4.1.3.

Proof of Theorem 4.1.3. We find a family of vectors $\{f_S \in \mathbb{R}^{[d]}\}_{S\subset[d]:|S|<d}$ that satisfy the conditions of theorem Theorem 4.2.1. Let $G = G_{(X,\pi)}$. Based on the conditions of Theorem 4.2.1, vectors $\{f_S \in \mathbb{R}^{[d]}\}_{S\subset[d]:|S|<d}$ can be defined as functions of $\{\epsilon_{\{i,j\}}\}_{i,j\in[d],i\neq j}$. Recall that edges of G capture pairs $\{i,j\}$ for which $\epsilon_{\{i,j\}} > 0$. Assign every edge $\{i,j\}$ of G with weight $\epsilon_{\{i,j\}}$. We restrict our attention to functions that are very local with respect to G, i.e. for every S and $i \in [d]\setminus S$, we assume $f_S(i)$ only depends on $\Delta_S(i)$ and the weights of edges adjacent to i in G_S if $\Delta(i) > 1$. It turns out that if $\Delta(i) = 1$, we would need to also take into account the degree of the unique neighbor of i. More formally, consider the following family of vectors $\{f_S \in \mathbb{R}^{[d]}\}_{S\subset[d]:|S|<d}$: for any $S \subset [d]$ such that $|S| < d$, let f_S be of the following form: for any $i \in S$, let $f_S(i) = 0$, and for any $i \in [d] \setminus S$ define

$$f_S(i) = \begin{cases} 0 & \text{if } \Delta_S(i) = 0, \\ \epsilon_{\{i,j\}} \cdot g_{i,j}(\Delta_S(j)) & \text{if } \Delta_S(i) = 1 \text{ and } i \sim_S j, \\ \sum_{j \sim_S i} \epsilon_{\{i,j\}} \cdot h_i\left(\Delta_S(i)\right) & \text{if } \Delta_S(i) \geq 2, \end{cases}$$

where for every $i \in [d]$ and $j \sim i$, the functions $g_{i,j}, h_i : \{1, \ldots, \Delta\} \to \mathbb{R}_{\geq 0}$ are defined later in a way that guarantees that $\{f_S\}_{S \subset [d]:|S|<d}$ satisfies the assumptions of Theorem 4.2.1 (see Eq. (4.9), Eq. (4.11)).

First, consider the case that G_S is disconnected. Note that for any $S, S' \subset [d]$ such that $|S|, |S'| < d$, if $\{j \in [d] : j \sim_S i\} = \{j \in [d] : j \sim_{S'} i\}$ for some $i \notin S, S'$, then $f_S(i) = f_{S'}(i)$. Let $I_1, \ldots, I_\ell$ be the vertices of connected components of G_S. Since the neighborhood of each vertex in any connected component of $G_{S'}$ is the same as its neighborhood in G_S, we get $f_S = \sum_{1 \leq i \leq \ell : |I_i| \geq 2} f_{[d] \setminus I_i}$.

Now, assume G_S is connected. Take an arbitrary $k \geq 2$ and $S \subset [d]$ of size $(d+1) - k$. First we verify the set of conditions given in Item 1 and Item 2. First, assume that $k = 2$. Let $[d] \setminus S = \{i, j\}$. By definition of $\epsilon_{\{i,j\}}$, for any τ of type S, $\lambda_2(P_\tau) \leq \epsilon_{\{i,j\}}$. Thus, if we define $g_{\ell,t}(1) = 1$ for all distinct $\ell, t \in [d]$, then we get $\lambda_2(P_\tau) \leq \epsilon_{\{i,j\}} = \epsilon_{\{i,j\}} g_{i,j}(1) = f_S(i) = f_S(j)$, as desired. Now, assume that $k \geq 3$. Fix an arbitrary $i \in [d] \setminus S$. Our goal is to define $g_{i,j}, h_i : \{1, \ldots, \Delta\} \to \mathbb{R}_{\geq 0}$ for all $j \sim i$ such that $g_{i,j}(1) = 1$ for all $j \sim i$ and the following inequality is satisfied:

$$\sum_{j \in [d] \setminus (S \cup i)} f_{S \cup j}(i) \leq (k-2) f_S(i) - f_S^2(i). \tag{4.7}$$

To keep the notation concise, relabel the elements such that i is relabeled to 0 and $\epsilon_{\{0,1\}} \geq \cdots \geq \epsilon_{\{0,d\}}$. Moreover, define $\epsilon_j = \epsilon_{\{0,j\}}$ for any $j \in [d] \setminus 0$.

Case 1: $\Delta_S(0) = 1$, and $j \sim_S 0$. Since G_S is connected and $(d+1) - |S| \geq 3$, we have $\Delta_S(j) \geq 2$. Define $t = \Delta_S(j)$. We have

$$\begin{aligned} &\sum_{\ell \in [d] \setminus (S \cup 0)} f_{S \cup \ell}(0) \\ &= f_{S \cup j}(0) + \sum_{\ell \in [d] \setminus (S \cup 0) : \ell \sim_S j} f_{S \cup \ell}(0) + \sum_{\ell \in [d] \setminus (S \cup 0) : \ell \not\sim_S j, \ell \neq j} f_{S \cup \ell}(0) \\ &= 0 + (t-1) \cdot \epsilon_j \cdot g_{0,j}(t-1) + (k-t-1) \cdot \epsilon_j \cdot g_{0,j}(t). \end{aligned}$$

On the other hand, $(k-2) f_S(0) - f_S(0)^2 = (k-2) \cdot \epsilon_j \cdot g_{0,j}(t) - \epsilon_j^2 \cdot g_{0,j}^2(t)$. So it is enough to satisfy

$$(t-1) \cdot \epsilon_j \cdot (g_{0,j}(t) - g_{0,j}(t-1)) \geq \epsilon_j^2 \cdot g_{0,j}^2(t). \tag{4.8}$$

Now, define $g_{0,j} : \{1, \ldots, \Delta\} \to \mathbb{R}_{\geq 0}$ as follows: recall that we defined $g_{0,j}(1) = 1$. For any $2 \leq \ell \leq \Delta$, let

$$g_{0,j}(\ell) = 1 + 1.3 \cdot \epsilon_j \cdot H_{\ell-1}. \tag{4.9}$$

Using assumption 4.1, $\epsilon_j H_{\Delta-1} \leq \frac{\delta^2}{10} \leq \frac{1}{10}$. Thus

$$\epsilon_j^2 \cdot g_{0,j}^2(t) \leq \epsilon_j^2 \left(1 + 1.3\epsilon_j \left(1 + H_{\Delta-1}\right)\right)^2 < 1.3\epsilon_j^2.$$

Substituting $g_{0,j}(t)$ according to Eq. (4.9) and using the above bound, one can verify that Eq. (4.8) holds.

Case 2: $\Delta_S(0) \geq 2$. For simplicity of notation, define $t = \Delta_S(0)$ and $\alpha = \sum_{j:j\sim_S 0} \epsilon_j$. Define $h_0(1) = \max_{j:j\sim 0} g_{0,j}(\Delta)$.

$$\begin{aligned}
&\sum_{j\in[d]\setminus(S\cup 0)} f_{S\cup j}(0) \\
&= \sum_{j\in[d]\setminus(S\cup 0):j\sim_S 0} f_{S\cup j}(0) + \sum_{j\in[d]\setminus(S\cup 0):j\not\sim_S 0} f_{S\cup j}(0) \\
&\leq \left(\sum_{j\in[d]\setminus(S\cup 0):j\sim_S 0} (\alpha - \epsilon_{\{0,j\}})\right) \cdot h_0(t-1) + (k-t-1)\cdot\alpha\cdot h_0(t) \\
&= (t-1)\cdot\alpha\cdot h_0(t-1) + (k-t-1)\cdot\alpha\cdot h_0(t).
\end{aligned}$$

Note that if $t \geq 3$, the first inequality is an equality by definition. If $t = 2$, the first inequality follows from the definition of $h_0(1)$. Thus, it is enough to satisfy

$$\begin{aligned}
\sum_{j\in[d]\setminus(S\cup 0)} f_{S\cup j}(0) &= (t-1)\cdot\alpha\cdot h_0(t-1) + (k-t-1)\cdot\alpha\cdot h_0(t) \\
&\leq (k-2)\cdot\alpha\cdot h_0(t) - \alpha^2\cdot h_0^2(t) \\
&= (k-2) f_S(0) - f_S^2(0).
\end{aligned}$$

Equivalently, it suffices to satisfy

$$(t-1)(h_0(t) - h_0(t-1)) \geq \alpha \cdot h_0^2(t). \tag{4.10}$$

Now, define $h_0 : \{1,\ldots,\Delta\} \to \mathbb{R}_{\geq 0}$ as follows: recall that we defined $h_0(1) = \max_{j:j\sim 0} g_{0,j}(\Delta)$. For any $2 \leq \ell \leq \Delta$, define

$$h_0(\ell) = \frac{h_0(1)}{1 - c\left(\sum_{j=1}^{\ell} \epsilon_j H_{\ell-1}(j-1)\right)}. \tag{4.11}$$

We need to prove Eq. (4.10) for a carefully chosen c. Let β be such that $h_0(t) = \frac{h_0(1)}{\beta}$. We get $h_0(t-1) = \frac{h_0(1)}{\beta + c(\sum_{j=1}^{t} \frac{\epsilon_j}{t-1})}$, and thus,

$$(t-1)(h_0(t) - h_0(t-1)) = \frac{h_0(1)\cdot c\sum_{j=1}^{t} \epsilon_j}{\beta\cdot(\beta + \frac{c\sum_{j=1}^{t}\epsilon_j}{t-1})}.$$

Note that $\alpha \cdot h_0^2(t) = \frac{\alpha\cdot h_0^2(1)}{\beta^2}$. Thus, to satisfy Eq. (4.10), it is enough to show that

$$\beta\cdot c\cdot\left(\sum_{j=1}^{t}\epsilon_j\right) \geq \alpha\cdot h_0(1)\cdot\left(\beta + \frac{c\sum_{j=1}^{t}\epsilon_j}{t-1}\right).$$

Note that

$$h_0(1) \leq \max_{j\sim i} g_{0,j}(\Delta) = 1 + 1.3\epsilon_1 H_{\Delta-1} \underset{\text{by } 4.1}{\leq} 1 + 1.3\frac{\delta^2}{10}. \tag{4.12}$$

Moreover, $\sum_{j=1}^{t} \epsilon_j \geq \sum_{j:j\sim_S 0} \epsilon_j = \alpha$. Thus, letting $c = 1 + c'\delta$ for some $c' > 0$ that we choose later, it is enough to show that

$$\beta \cdot (c' - 0.13\delta)\delta \geq (1 + 0.13\delta) \cdot \frac{(1 + c'\delta)\sum_{j=1}^{t} \epsilon_j}{t-1}.$$

Using $\frac{\sum_{j=1}^{t} \epsilon_j}{t-1} \leq 2\epsilon_1 \underset{4.1}{\leq} \frac{\delta^2}{5}$, it is enough to show that

$$\beta \cdot (c' - 0.13\delta) \geq (1 + 0.13\delta)(1 + c'\delta)\frac{\delta}{5}. \tag{4.13}$$

On the other hand,

$$\begin{aligned} \beta &\geq 1 - (1 + c'\delta)\left(\sum_{j=1}^{\Delta(0)} \epsilon_j H_{\Delta(0)-1}(j-1)\right) \\ &\underset{4.2}{\geq} 1 - (1 + c'\delta)(1-\delta) = \delta(1 - c' + c'\delta), \end{aligned} \tag{4.14}$$

Thus, to satisfy Eq. (4.13), it is enough to show that $(1-c'+c'\delta)(c'-0.13\delta) \geq (1+1.13\delta)(1+c'\delta)\frac{1}{5}$. Letting $c' = \frac{1}{2}$, this inequality holds for every $0 < \delta < 1$. This establishes *Eq. (4.7)*. So we verified conditions Item 1 and Item 2 are satisfied.

To show that all conditions of 4.2.1 are satisfied, it remains to show that $\max_{i\in[d]} f_S(i) \leq \frac{(k-1)^2}{3k-1}$. Note that $\sum_{j:j\sim i} \epsilon_{\{i,j\}} \leq \Delta_S \cdot \epsilon_1 \underset{4.1}{\leq} \Delta_S \cdot \frac{\delta^2}{10}$ for all $i \in [d] \setminus S$. Thus, we get $\max_{i\in[d]} f_S(i) \leq \Delta_S \cdot \frac{\delta^2}{10} \max_{i\in[d]\setminus S} \cdot h_i(\Delta_S(i))$. Moreover, using Eq. (4.12) and Eq. (4.14) with $c' = \frac{1}{2}$ (we can write this inequality for every i), we get

$$h_i(\Delta_S(i)) \leq h_i(\Delta(i)) \leq \frac{1 + \frac{\delta^2}{10}}{\delta(\frac{1}{2} + \frac{\delta}{2})}, \tag{4.15}$$

Thus, we can write

$$\max_{i\in[d]} f_S(i) \leq \Delta_S \cdot \frac{\delta^2}{10}\frac{1 + \frac{\delta^2}{10}}{\delta(\frac{1}{2} + \frac{\delta}{2})} \leq \frac{\Delta_S}{5} \leq \frac{k-1}{5} \leq \frac{(k-1)^2}{3k-1},$$

as desired. So we proved that $\{f_S\}_{S\subset[d]:|S|<d}$ satisfies the conditions of Theorem 4.2.1. Now, we are ready to bound $\lambda_2(P_\tau)$ for any face τ of co-dimension $k \geq 2$ and type S. First, we show that for every $i \in [d] \setminus S$, $\sum_{j:j\sim_S i} \epsilon_{\{i,j\}} \leq 1 - \delta$. Note that

$$\sum_{\ell=1}^{\Delta(i)} H_{\Delta(i)-1}(\ell - 1) = \sum_{\ell=2}^{\Delta(i)} \frac{\ell}{\ell - 1} = 2 + \sum_{\ell=3}^{\Delta(i)} \frac{\ell}{\ell - 1} \geq \Delta(i).$$

Thus, we can write

$$\sum_{j:j\sim_S i} \epsilon_{\{i,j\}} \leq \left(\sum_{\ell=1}^{\Delta(i)} \frac{H_{\Delta(i)-1}(\ell-1)}{\Delta(i)} \right) \left(\sum_{j\sim i} \epsilon_{\{i,j\}} \right)$$
$$\leq \sum_{\ell=1}^{\Delta(i)} H_{\Delta(i)-1}(\ell-1) \cdot \epsilon_{\{i,j_\ell\}} \leq 1-\delta, \tag{4.16}$$

where we assumed that $i_1, \ldots, j_d$ is an ordering of $[d] \setminus S$ such that $\epsilon_{j_1} \leq \cdots \leq \epsilon_{j_d}$. Using this inequality and Eq. (4.15), we get

$$\lambda_2(P_\tau) \leq \frac{\max_{i\in[d]\setminus S} f_S(i)}{k-1} \leq \frac{\max_{i\in[d]}(\sum_{j\sim_S i} \epsilon_{\{i,j\}}) \cdot h_i(\Delta_S(i))}{k-1}$$
$$\leq \frac{(1-\delta)\cdot \max_{i\in[d]} h_i(\Delta(i))}{k-1} \leq \frac{(1-\delta)\cdot \frac{2(1+\frac{\delta^2}{10}\delta)}{\delta(\delta+1)}}{k-1},$$

as desired. □

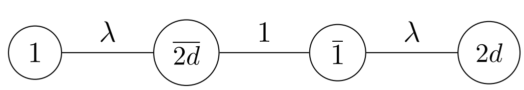
4.4 Obstructions to Trickle-Down

In this section, we exhibit some barriers to any class of trickle-down theorems that only look at 2nd eigenvalue of links of co-dimension 2.

Example 1. *We exhibit a family of partite complexes such that $\gamma_2 \leq \lambda$, but $\gamma_{2d+1} \geq \Omega(\lambda)$. Consider a 2d-partite complex (X, π) on the ground set $\cup_i^{2d} T_i$, where for all $i \in \{1, \ldots, d\}$, $T_i = \{i_{in}, i_{out}\}$ are the elements of type i. A set $\tau \in T_1 \times \cdots \times T_{2d}$ is a facet of X iff for all $i_{in}, j_{in} \in \tau$ we either have $i_{in}, j_{in} \leq d$ or $i_{in}, j_{in} \geq d+1$. For such a τ, we define $w(\tau) = \lambda^{\|\tau\|_1}$ where $\|\tau\|_1$ is the number of $i \in \tau$, for some $0 < \lambda < 1$ that we choose later (the weight of the complex is the probability distribution induced by this weight function, but for simplicity of calculations, we do not normalize the weights here).*

As a side note, facets of this complex correspond to the set of independent sets of $K_{d,d}$ the complete bipartite graph on the sets $\{1, \ldots, d\}, \{d+1, \ldots, 2d\}$. It is not hard to see that we have $\gamma_2 = \frac{\lambda}{1+\lambda}$. To see this, note that a worst link to take is $\tau = \{2_{out}, \ldots, (2d-1)_{out}\}$ with the the following 1-skeleton.

Notice that this graph is a $\frac{\lambda}{1+\lambda}$-spectral expander.



We claim that the 2nd eigenvalue of the 1-skeleton of the link of the empty set is at least $\Omega(\frac{\lambda}{1+\lambda})$. First note that this graph has 4d vertices. We partition its vertices into 4 sets of equal size, $A = \{1_{in}, \ldots, d_{in}\}, \bar{A} = \{1_{out}, \ldots, d_{out}\}$, $B = \{(d+1)_{in}, \ldots, (2d)_{in}\}$, and $\bar{B} = \{(d+1)_{out}, \ldots, (2d)_{out}\}$.

For simplicity of calculations, we can write the weight of every edge of the 1-skeleton as the sum of the weights of all facets that contain that edge. For every $i_{in} \in A, j_{out} \in \bar{A}$,

$$w_{i_{in},j_{out}} = \sum_{k=0}^{d-2} \binom{d-2}{k} \lambda^{k+1} = \lambda(1+\lambda)^{d-2}$$

Running a similar calculation for all possible pairs of elements, we obtain the following 1-skeleton of the empty set (after dividing all edge weights by $(1+\lambda)^{d-2}$).

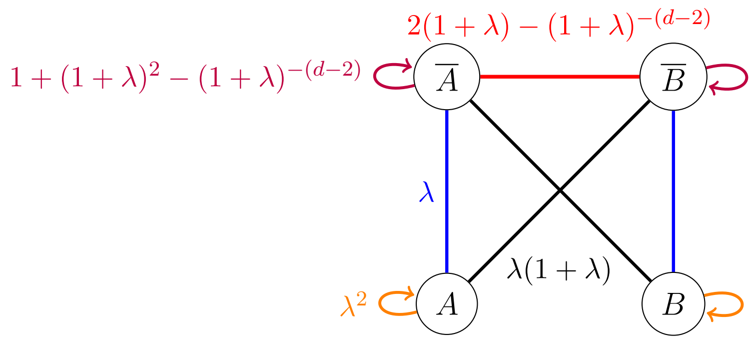


Figure 4.1: 1-skeleton of the link of $\emptyset$ for complex 1. Edges of the same color have the same weight.

We assume $1/d \ll \lambda \ll 1$ and $d \to \infty$ so we ignore $(1+\lambda)^{-(d-2)}$ low order term. It follows that $d_w(i_{in}) \approx 2d\lambda(1+\lambda)$ and $d_w(i_{out}) \approx 2d(2+3\lambda+\lambda^2)$ for i. It follows that,

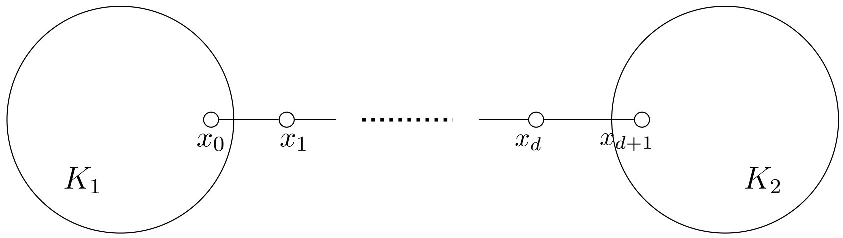
$$\begin{aligned}\left| w(E(A \cup B)) - \frac{\text{vol}(A \cup B)^2}{\text{vol}(V)} \right| &\geq \left| d^2\lambda^2 - \frac{(2d(2d\lambda(1+\lambda)))^2}{2d(2d(2+4\lambda+2\lambda^2))} \right| \\ &= \left| d^2\lambda^2 - 2d^2\lambda^2 \right|\end{aligned}$$

So, by Fact 1.2.5 for $S = A \cup B$ we have

$$\lambda_2 \geq \frac{d^2\lambda^2}{\text{vol}(A \cup B)} = \frac{d^2\lambda^2}{4d^2\lambda(1+\lambda)} \geq \frac{\lambda}{4(1+\lambda)}.$$

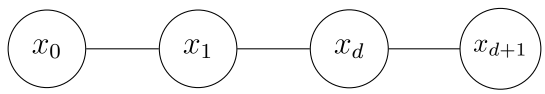
Note that if we apply Theorem 4.1.3 to the setting of the above example, we obtain $\Delta(i) = d$ for all $1 \leq i \leq 2d$. So, if $\lambda \cdot d \leq 1 - \delta$, (and $\lambda d \log d \leq \delta^2/10$), our theorem implies that the 2nd eigenvalue of the link of the emptyset is at most $\frac{c(1-\delta)}{2d \cdot \delta} \leq \frac{c\lambda}{\delta}$ which is consistent with the above calculations.

Example 2. *In this example we construct a totally connected (non-partite) $(d-1)$-dimensional weighted simplicial complex (X, π) such that links of co-dimension 2 are* 0.5*-spectral expanders, but the 1-skeleton of the link of the emptyset is only a* $1 - exp(-d)$*-spectral expander. See [Liu+23] for a random family of 3-dim complexes exhibiting a similar growth of eigenvalues. This in particular shows that local spectral expansion can increase significantly for (non-partite) complexes. Let $B(V, E)$ be the following* barbell *graph: Consider two disjoint cliques K_1, K_2 each with $2d$ vertices and connect them with a path of length $d+2$ namely $x_0 \in K_1, x_1, \ldots, x_d, x_{d+1} \in K_2$. Note that $x_1, \ldots, x_d$ do not belong to the cliques.*



Now we define a $d-1$-dimensional weighted complex on vertices V. The facets of X are precisely sets $S \subseteq V$ with $|S| = d$ such that the induced graph $B[S]$ is connected. For simplicity of calculations we define the weight of every facet to be 1, i.e., we don't normalize the weights to be a probability distribution.

We claim that any link X_τ of co-dimension 2 is $\Omega(1)$-spectral expander. Indeed the worst link is X_τ for $\tau = \{x_2, \ldots, x_{d-1}\}$. The 1-skeleton of X_τ is the following graph with second eigenvalue 0.5.



Now, we claim that the 1-skeleton of $X_\emptyset$, G_0, has 2nd eigenvalue $1 - \frac{1}{exp(d)}$. First notice G_0 has the same vertex set as B. Now, consider the set $S = K_1 \cup \{x_0, x_1, \ldots, x_{d/2}\}$. For simplicity of calculations, we let the weight of every edge $\{x, y\}$ of G_0 be sum of the weights of all facets that contains x, y. First, observe that the weight of every edge in K_1 is (at least) $\binom{2d-2}{d-2}$. On the other hand, the weight of any edge in the cut $(S, \overline{S})$ is at most $2\binom{2d}{d/2}$. Putting these together it follows that

$$\phi(S) = \frac{w(E(S, \overline{S}))}{\mathrm{vol}(S)} \geq \frac{2\frac{d}{2} \cdot \frac{5d}{2} \cdot \binom{2d}{d/4}}{\binom{2d}{2} \cdot \binom{2d-2}{d-2}} \approx 2^{-d/2},$$

for a large enough d. Therefore, by Cheeger's inequality, Lemma 1.2.4, the second eigenvalue of G_0 is at least $1 - 2^{-\Omega(d)}$ for d large enough.

Chapter 5

On Optimization and Counting of Non-Broken Bases of Matroids

5.1 Introduction

In this chapter, we study counting and optimization problems on NBC bases of a generic matroid from a local-to-global perspective. This chapter is based on results published in [ALO23].

Given a matroid $M = (E, \mathcal{I})$, and a total ordering over the elements E, a set $C \subseteq E$ is a *circuit* iff $C \setminus \{e\} \in \mathcal{I}$ for any $e \in C$. A *broken circuit* (with respect to a total ordering $\mathcal{O}$) is a set $C \setminus \{e\}$, where $C \subseteq E$ is a circuit and e is the smallest element of C with respect to $\mathcal{O}$. An independent set $S \subseteq E$ is a *non-broken* independent set (NBC independent set) if it contains no broken circuits. The set of NBC independent sets of any matroid M define a simplicial complex called the broken circuit complex which has been the subject of intense study in combinatorics.

A recent breakthrough in the sampling and counting field proved that the down-up walk on the bases of a matroid mixes rapidly to the (uniform) stationary distribution and can be used to count the number of bases of a matroid [Ana+19; CGM19], resolving the conjecture of Mihail and Vazirani from 1989 [MV89]. A central question that has puzzled researchers since then is sampling a non-broken (circuit) basis (NBC basis) of a matroid [BCT10].

The NBC independent sets are closely related to several interesting combinatorial objects. The number of NBC independent sets of size k in a graphic matroid is equal to the absolute value of the $(n-1)-k$-th coefficient of the chromatic polynomial of the underlying graph where n is the number of vertices. As a corollary the following facts hold:

Fact 5.1.1. *The following facts are well-known about the counts of NBC bases/independent sets of different family of matroids.*

- *The number of all NBC independent sets of a graphic matroid is equal to the the number of acyclic orientations of the graph [Sta73].*
- *The number of all NBC independent sets of a co-graphic matroid is equal to the number of strongly connected orientations of the graph (see e.g., [GL19]).*

- *The number of non-broken spanning trees of a graph is equal to the number of parking functions with respect to a unique source vertex [BCT10]*
- *The number of NBC independent of sets of linear matroid with vectors $v_1, \ldots, v_n$ is equal to the number of regions defined by the intersection of the orthogonal hyperplanes (see e.g., [Sta07]).*

We emphasize that although the set of NBC independent sets/bases of a matroid are functions of the underlying total order $\mathcal{O}$, the number NBC independent sets of rank k for any $0 \leq k \leq r$ are invariant under $\mathcal{O}$ [Sta07]. We remark that, to the best of our knowledge as of this date, none of the above counting problems are known to be computationally tractable.

Given a matroid M with an arbitrary total ordering $\mathcal{O}$, one can analogously run the down-up walk only on the NBC bases of M. It is not hard to see that this chain is irreducible and converges to the uniform stationary distribution. Following the work of [Ana+19] it was conjectured that the down-up walk on the NBC bases of any matroid mixes rapidly [1].

Conjecture 5.1.2. *For any matroid M, and any total ordering $\mathcal{O}$ of the elements of M, the down-up walk on the NBC bases of a matroid mixes in polynomial time.*

It turns out that the above conjecture, if true, would yield a generalization of the result of [Ana+19], because of the following fact.

Fact 5.1.3 ([Bry77]). *For any matroid M one can construct another matroid M' with an ordering $\mathcal{O}$ with only one extra element such that there is a bijection between bases of M and non-broken bases of M'.*

Furthermore, if the above conjecture is true, then since matroids are closed under truncation, one can also count the number of all NBC independent sets of M, thus resolving all of the open problems in Fact 5.1.1.

A promising reason to expect these problems to be tractable in the first place is the remarkable work of Adiprasito, Huh and Katz who proved the Rota's conjecture showing that the face numbers of a broken circuit complex (see below for definition) of any matroid forms a log-concave sequence [AHK18]. For comparison, it is well-known that the coefficients of the matching polynomial of any graph form a log-concave sequence and the classical algorithm of Jerrum-Sinclair [JS89] gives an efficient algorithm to count the number of matchings of any graph (although to date we still do not know an efficient algorithm to count the number of **perfect** matchings of general graphs).

5.1.1 Background

The set of NBC independent sets of any matroid M (with respect to any ordering $\mathcal{O}$) form a pure simplicial complex that is known as the *broken circuit complex*. We denote this complex by $\mathrm{BC}(M, \mathcal{O})$. The purity of this complex follows from the following fact.

[1] In fact, this conjecture was raised an an open problem in several recent workshops UC Santa Barbara workshop on New tools for Optimal Mixing of Markov Chains: Spectral Independence and Entropy Decay, and Simon's workshop on Geometry of Polynomials

Fact 5.1.4. *For every NBC independent set I, there exists an NBC base B such that $I \subseteq B$.*

The face numbers of the complex $BC(M, \mathcal{O})$ is the sequence $n_0, n_1, \dots, n_r$ where n_i is the number of NBC independent sets of rank i. As alluded to above this sequence is invariant over $\mathcal{O}$. The down-up walk over this complex equipped with a uniform distribution over its facets is the same as the down-up walk over NBC bases we explained before.

Recall that for any face τ of size $0 \le k \le d-2$, the local walk for τ is a Markov chain on the ground set of X_τ with transition probability matrix P_τ is defined as

$$P_\tau(x, y) = \frac{1}{d-k-1} \Pr_{\sigma \sim \pi_\tau}[y \in \sigma \mid \sigma \supset \tau \cup \{x\}]. \tag{5.1}$$

for distinct x, y in the ground set of X_τ.

Theorem 1.0.1 shows that the spectral expansion of the global down-up walk $P^\vee$ on a simplicial complex can be bounded through bounding the expansion of its local walks. To prove that the down-up walk mixes rapidly on the bases of any matroid, [Ana+19] proved that the independent set complex of any matroid M is a $(0, 0, \dots, 0)$-local spectral expander. Building on this, a natural method to prove Conjecture 5.1.2 is to show that the broken circuit complex of any matroid M of rank r and for any total ordering is a $(\gamma_2, \dots, \gamma_r)$-local spectral expander for $\gamma_i \le \frac{O(1)}{i}$.

Conjecture 5.1.5. *For any matroid M of rank r and any ordering $\mathcal{O}$ the broken circuit complex of M is a $(\gamma_2, \dots, \gamma_r)$-local spectral expander for some $\gamma_i \le \frac{O(1)}{i}$*

5.1.2 Main Results

Our main result is to disprove Conjecture 5.1.5 in a very strong form, namely for the class of (truncated) graphic matroids.

Theorem 5.1.6 ([ALO23]). *There exists an infinite sequence of (truncated) graphic matroids $M_1, M_2, \dots$ with orderings $\mathcal{O}_1, \mathcal{O}_2, \dots,$ such that for every $n \ge 1$, M_n has poly(n) elements, and there exists a face τ of the broken circuit complex of $X = \mathrm{BC}(M_n, \mathcal{O})$ for which the down-up walk on the facets of the link X_τ has a spectral gap of at most $n^{-\Omega(n)}$.*

In fact, we even prove a stronger statement

Theorem 5.1.7 ([ALO23]). *Given a matroid $M = (E, \mathcal{I})$ and a total ordering $\mathcal{O}$ and a set $S \subseteq E$, unless RP=NP, there is no FPRAS for counting the number of NBC bases of M that contain S.*

Although this theorem does not refute Conjecture 5.1.2, it shows that one probably need different techniques (or probably a different chain) to sample/count NBC bases of a matroid. Indeed, one may even need a different proof for the performance of down-up walk to sample ordinary bases of matroids.

To complement our main results we also prove that, unlike optimization on bases of a matroid, optimization is NP-hard on the NBC bases of matroids. Moreover, unless NP= RP, there is no FPRAS for computing the sum of the weights of all NBC bases of a matroid subject to an external field, while the same computation over the bases of a matroid has a FPRAS.

Theorem 5.1.8 ([ALO23]). *Given a matroid $M = (E, \mathcal{I})$ with $|E| = n$ elements, an arbitrary total ordering $\mathcal{O}$, and weights $w_1, \ldots, w_n$, it is NP-hard to find the maximum weight NBC basis of M, where the weight of a NBC basis B is $\sum_{i \in B} w_i$.*

Theorem 5.1.9 ([ALO23]). *Given a matroid $M = (E, \mathcal{I})$ with $|E| = n$ elements, a total ordering $\mathcal{O}$, and weights $\{1 \leq \lambda_e \leq O(n)\}_{e \in E}$, unless NP = RP, there is no FPRAS for computing the partition function of the λ-external field applied to uniform distribution of NBC independent sets, i.e., there is no FPRAS for computing :*

$$\sum_{B \text{ NBC Base}} \prod_{e \in B} \lambda_e.$$

It is well known that a 0/1-polytope (i.e. the convex hull of a subset $S \subseteq \{0,1\}^n$) has all vertices of equal hamming weight r and edges of ℓ_2 length $\sqrt{2}$ iff the polytope is a matroid base polytope of rank r [Gel+87]. Moreover, assuming the Mihail-Vazirani conjecture, there is efficient algorithm to sample a uniformly random vertex of a 0/1-polytope with constant sized edge length [MV89].

We show that, unlike matroids, the NBC Base polytope, i.e. the convex hull of the indicator vectors of all NBC bases of a matroid M, has edges of arbitrarily long length.

Theorem 5.1.10 ([ALO23]). *For any n, there exists a graphic matroid M with n elements and a total ordering $\mathcal{O}$ such that the convex hull of all NBC bases of M has edges of ℓ_2 length at least $\Omega(\sqrt{n})$.*

5.1.3 Preliminaries

Given a graph $G = (V, E)$, we denote the number of independent sets of size i of G by $i_k(G)$ For every set $S \subseteq V$, we define $N(S) \coloneqq \{v \notin S : \exists u \in S, \{u, v\} \in E\}$ as the set of neighbors of S in G.

A graphic matroid $M = (E, \mathcal{I})$ is a matroid defined on the edges of a graph $G = (V, E)$ and its independent sets are all subsets of edges that do not contain any cycle. It is easy to verify that circuits of M correspond to cycles of G.

Definition 12 (Matroid Truncation). *Let $M = (E, \mathcal{I})$ be a matroid of rank r. The truncation of M to rank $r' \leq r$ removes all independent sets of size strictly greater than r'. It is easy to see that the truncation of any matroid M to any $r' \leq r$ is also a matroid.*

Let M' be the truncation to rank r' of a graphic matroid of rank r defined on the edges of a graph G. The bases of M' correspond to forests with r' edges and the circuits of M' are the circuits of G along with all spanning forests of size $r' + 1$.

The following fact about polytopes follows from convexity.

Fact 5.1.11. *For any polytope $P \subseteq \mathbb{R}^d$ with vertices $v_1, \ldots, v_n \in \mathbb{R}^d$, $\{v_i, v_j\}$ is an edge of P iff there exists a weight function $w \in \mathbb{R}^d$ such that*

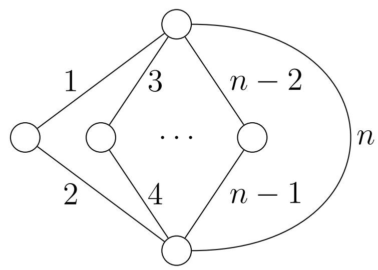
$$\langle w, v_i \rangle = \langle w, v_j \rangle > \langle w, v_k \rangle,$$

for any $k \neq i, j$.

5.2 Proof of Main Theorems

We start with proving Theorem 5.1.10.

Proof of Theorem 5.1.10. Let n be odd. Consider the following graphic matroid M (with n edges), with the ordering $\mathcal{O}$: $1 < 2 < \cdots < n$ defined by the edges of the following graph:



We show that for $B = \{n\} \cup \{2i-1 : 1 \leq i \leq \frac{n-1}{2}\}$ and $B' = \{1\} \cup \{2i : 1 \leq i \leq \frac{n-1}{2}\}$, $\{B, B'\}$ forms an edge in the NBC matroid base polytope denoted as P_M. We define a $w \in \mathbb{R}^n$ and then use Fact 5.1.11 to prove the statement. Let $w_n = \frac{n+1}{2}$, and for any $1 \leq i \leq \frac{n-1}{2}$, let $w_{2i} = 1$ and $w_{2i-1} = 0$. It is easy to check that the function $\langle w, \mathbf{1}_B\rangle = \langle w, \mathbf{1}_{B'}\rangle = \frac{n+1}{2}$ and $\langle w, \mathbf{1}_{B''}\rangle < \frac{n+1}{2}$ for all NBC basis $B'' \neq B, B'$. Therefore $\{B, B'\}$ forms and edge in P_M. The statements follows from the fact that $\|\mathbf{1}_B - \mathbf{1}_{B'}\|_2 = \sqrt{n}$. □

Next, we prove Theorem 5.1.8 via a reduction from the MAX-INDEP-SET problem: Given a graph $G = (V, E)$, a weight function $w : V \to \mathbb{R}_{\geq 0}$, and an integer k, decide whether G has an independent set of weight at least k or not.

Note that independent sets of G and independents sets of a BC complex/matroid are two different notions. To complete the proof we use the following well-known hardness result.

Theorem 5.2.1 ([Kar72]). MAX-INDEP-SET *is NP-complete.*

Proof of Theorem 5.1.8. We prove this by a reduction from MAX-INDEP-SET. Let $G = (V, E)$ be a graph, a vertex weight function $w : V \to \mathbb{R}_{\geq 0}$ and k an integer. Construct a new graph $G' = (V', E')$ from G by first copying G and then adding a new vertex z and edges $e_v = \{z, v\}$ for all $v \in V$. We define $w' : E' \to \mathbb{R}_{\geq 0}$ as $w'(e_v) = w(v)$ for every $v \in V$, and $w'(e) = 0$ for every $e \in E$. Moreover, consider the following total ordering $\mathcal{O}$ on E':

$$E < \{e_v : v \in V\},$$

where the ordering within each set is arbitrary. Let M be the graphic matroid defined by the edges of G', we will be look at bases/independent sets of $\mathrm{BC}(M, \mathcal{O})$.

Claim 5.2.2. *There exists an independent set of G of weight at least k iff there exists an NBC basis of M with weight at least k.*

We prove the claim in a straightforward manner. Suppose there is an independent set $I \subseteq V$ of G with $w(I) \geq k$ and consider the set $I' \subseteq E'$ defined by $I' = \{e_v : v \in I\}$. By definition, $w'(I') \geq k$. We argue that I' does not contain any broken circuit. Assume otherwise that there is a broken circuit $C \setminus \{e\} \subseteq I'$. Since C corresponds to a cycle in G' and

$C \setminus \{e\}$ is contained in I', it is not hard to see that $C \setminus \{e\} = \{e_v, e_{v'}\}$ for some $v, v' \in I$ and $e = \{v, v'\}$ is an edge in G. But this is a contradiction with the fact that I is an independent set of G. Hence I' is a NBC independent set. Since the broken circuit complex is pure (see Fact 5.1.4), there exists an NBC basis B containing I' which has weight $w'(B) \geq w'(I') \geq k$.

For the other direction, suppose we have a NBC basis $B' \subseteq E'$ of weight $w'(k) \geq k$, and define $I \subseteq V$ by $I = \{v : e_v \in B'\}$. Since all edges coming from E have zero weight, $w(I) = w'(B') \geq k$. To see that I is an independent set of G', note that if there is an edge $\{v, v'\}$ for some $v, v' \in I$, we have $e_v, e_{v'} \in B'$, then $\{e_v, e_{v'}\}$ forms a broken circuit according to the ordering $\mathcal{O}$. Therefore I is an independent set of G of weight at least k. □

It's important to note that the above proof works under the crucial assumption that the order $\mathcal{O}$ is chosen carefully based on the weights (and in some sense in the same order of the weights).

We can amplify the ideas in the previous construction to also argue Theorem 5.1.6. This is done by constructing a Broken Circuit complex for which the down-up walk of a carefully chosen link has inverse exponentially small spectral gap.

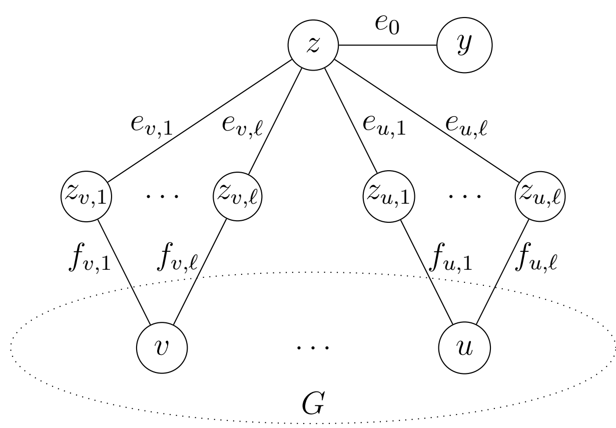


Figure 5.1: A schematic of the graph G' in the proofs of Theorem 5.1.6 and Theorem 5.1.7 where $G = K_{n,n}$ is the complete bipartite graph in the former and it is a hard instance of $\sharp$INDEP-SET-INC$(7, \frac{2}{19})$ in latter.

Proof of Theorem 5.1.6. Take the complete bipartite graph $G = K_{n,n} = (A, B, E = A \times B)$, with $|A| = |B| = n$. Also, let $V = A \cup B$. Let $\ell \geq 1$ be a parameter that we choose later, and construct a new graph

$$G' = (V' = V \cup \{y, z\} \cup \{z_{v,i} : v \in V, i \in [\ell]\}, E' = E \cup \{e_0\} \cup \{e_{v,i}, f_{v,i} : v \in V, i \in [\ell]\})$$

where $e_0 = \{y, z\}, e_{v,i} = \{z, z_{v,i}\}, f_{v,i} = \{z_{v,i}, v\}$ (see Fig. 5.1). For a sanity check, note that $|V| = 2n$ and $|V'| = 2\ell n + 2n + 2$.

Let $M = (E', \mathcal{I})$ be the graphic matroid defined by G' truncated to rank $2\ell n + n + 1$, i.e., the bases of M are forests of G' with exactly $2\ell n + n + 1$ edges. Now, consider the following total ordering $\mathcal{O}$ on E':

$$e_0 < E < \{e_{v,i} : v \in V, i \in [\ell]\} < \{f_{v,i} : v \in V, i \in [\ell]\},$$

where the ordering within each set is arbitrary. Moreover, let $X \coloneqq \mathrm{BC}(M, \mathcal{O})$, and define

$$\tau = \{e_{v,i} : v \in V, i \in [\ell]\}.$$

For simplicity of notation, let $F_A \coloneqq \{f_{v,i} : v \in A, i \in [\ell]\}$ and $F_B \coloneqq \{f_{v,i} : v \in B, i \in [\ell]\}$.

Claim 5.2.3. *For any facet S of X_τ, either $S \cap F_A = \emptyset$, or $S \cap F_B = \emptyset$,*

This follows from the fact that G is a complete bipartite graph and edges in E are smaller than $e_{v,i}$'s and $f_{u,j}$'s; so if $S \cap F_A, S \cap F_B \neq \emptyset$, then it has a broken circuit.

Therefore, the set of facets of X_τ can be partitioned into $2n+1$ sets $(\cup_{i=1}^n \mathcal{S}_{A,i}) \cup (\cup_{i=1}^n \mathcal{S}_{B,i}) \cup \mathcal{S}_0$, where $\mathcal{S}_{A,i}$ is the set of all facets S with $|S \cap F_A| = i$, $\mathcal{S}_{B,i}$ is the set of all facets S with $|S \cap F_B| = i$, and $\mathcal{S}_0$ is the set of all facets with $|S \cap (F_A \cup F_B)| = 0$. Let $\mathcal{S}_A \coloneqq \cup_{i=1}^n \mathcal{S}_{A,i}$ and similarly define $\mathcal{S}_B$. We show that $\frac{|N(\mathcal{S}_A)|}{|\mathcal{S}_A|} \leq n^{-\Omega(n)}$, where $N(\mathcal{S}_A)$ is the set of neighbors of $\mathcal{S}_A$ in the down-up walk $P_\tau^\vee$ on the facets of τ. WLOG we can assume that $|\mathcal{S}_A|$ is at most half of all facets. Applying Theorem 1.2.6, this would imply that $1 - \lambda_2(P_\tau^\vee) \leq n^{-\Omega(n)}$.

First, note that for every facet $S \in \mathcal{S}_A$ and $T \in \mathcal{S}_B \setminus \mathcal{S}_{B,1}$, we get $P^\vee(S, T) = 0$ since $|S \Delta T| > 2$. So, $N(\mathcal{S}_A) \subseteq \mathcal{S}_{B,1} \cup \mathcal{S}_0$. First, notice $|\mathcal{S}_0| \leq \binom{|E|}{n} \leq n^{2n}$. Furthermore, $|\mathcal{S}_{B,1}| \leq \binom{n}{1} \ell \binom{|E|}{n-1} \leq \ell n^{2n}$. This follows from the fact that any facet in $\mathcal{S}_{B,1}$ can be written as $\{f_{v,i_v}\} \cup \{e_0\} \cup K$ for some $v \in A$, $i_v \in [\ell]$, and subset $K \subseteq E$ of size $n-1$. Lastly, $|\mathcal{S}_A| \geq |\mathcal{S}_{A,n}| = \ell^n$. This is because every choice of $\{i_v\}_{v \in A}$ corresponds to a set in $\mathcal{S}_{A,n}$ whose sets are of the form $\{f_{v,i_v} : v \in V\} \cup \{e_0\}$. These sets all don't contain a broken circuit because the circuits introduced through truncation are exactly the forests with $2\ell n + n + 2$ edges. However, any proper superset of $\{f_{v,i_v} : v \in V\} \cup \{e_0\}$ must include e_0, so looking at the circuit introduced by the superset, the corresponding broken circuit will always remove e_0. Putting it all together,

$$1 - \lambda_2(P_\tau^\vee) \leq \frac{|N(\mathcal{S}_A)|}{|\mathcal{S}_A|} \leq \frac{n^{2n}(1+\ell)}{\ell^n} \underset{\text{assuming } \ell \geq n^3}{\leq} n^{-\Omega(n)}.$$

as desired.

□

We prove Theorem 5.1.9 and Theorem 5.1.7 by a reduction from $\sharp$INDEP-SET-INC$(7, \frac{2}{19})$, defined as the following.

Definition 13 ($\sharp$INDEP-SET-INC$(7, \frac{2}{19})$). *Given a 7-regular graph $G = (V, E)$ that satisfies $i_k(G) \leq i_{\lfloor \frac{2|V|}{19} \rfloor}(G)$ for any $k < \lfloor \frac{2|V|}{19} \rfloor$, where $i_k(G)$ are the independent sets of G of size k, count the number of independent sets of size $\lfloor \frac{2|V|}{19} \rfloor$.*

Theorem 5.2.4. *Unless* NP = RP*, there is no randomized algorithm with constant approximation ratio for* $\sharp$INDEP-SET-INC$(7, \frac{2}{19})$.

We leave the proof of this for the appendix (see Appendix B.1). Now, we are ready to prove Theorem 5.1.7. The high-level structure of the proof is similar to the proof of Theorem 5.1.6 where we apply a similar gadget to graphs on which it is hard to count independent sets (as opposed to the complete bipartite graph).

Proof of Theorem 5.1.7. For simplicity of notion, let $\alpha \coloneqq \frac{2}{19}$. We prove by a reduction from $\sharp$INDEP-SET-INC$(7, \frac{2}{19})$. Take any arbitrary 7-regular graph $G = (V, E)$ whose number of independent sets of size $\lfloor \alpha |V| \rfloor$ is at least the number of its independent sets of size k for any $k < \lfloor \alpha |V| \rfloor$. Let $n \coloneqq |V|$ and N be the number of independent sets of size $\lfloor \alpha n \rfloor$ of G. Also, define $\ell \geq 1$ to be a parameter that we choose later.

Now, construct a new graph

$$G' = (V' = V \cup \{y, z\} \cup \{z_{v,i} : v \in V, i \in [\ell]\}, E' = E \cup \{e_0\} \cup \{e_{v,i}, f_{v,i} : v \in V, i \in [\ell]\})$$

where $e_0 = \{y, z\}, e_{v,i} = \{z, z_{v,i}\}, f_{v,i} = \{z_{v,i}, v\}$ (see Fig. 5.1). Let $M = (E', \mathcal{I})$ be the graphic matroid defined by G truncated at rank $\ell n + \lfloor \alpha n \rfloor + 1$, i.e., the bases of M are forests of G' with exactly $\ell n + \lfloor \alpha n \rfloor + 1$ edges. Now, consider the following ordering $\mathcal{O}$ on E':

$$e_0 < E < \{e_{v,i} : v \in V, i \in [\ell]\} < \{f_{v,i} : v \in V, i \in [\ell]\},$$

where the ordering within each set is arbitrary. Moreover, let $X \coloneqq \mathrm{BC}(M, \mathcal{O})$, and define

$$\tau = \{e_{v,i} : v \in V, i \in [\ell]\}.$$

We claim that the number of facets of X_τ is at least $\ell^{\lfloor \alpha n \rfloor} N$ and at most $2\ell^{\lfloor \alpha n \rfloor} N$. So, a 1.5-approximation to the number facets of X_τ, i.e., the number NBC bases of M that contain τ, gives a 3-approximation to N, the number of independent sets of size $\lfloor \alpha n \rfloor$ of G.

We use the following crucial observation:

Claim 5.2.5. *For any facet S of X_τ, $\{v : \exists f_{v,i} \in S\}$ is an independent set of G and for any $f_{v,i}, f_{v,j} \in S$ we have $i = j$.*

Conversely, for any $S \subseteq \{f_{v,i} : v \in V, i \in [\ell]\}$, such that the set $\{v : \exists f_{v,i} \in S\}$ is an independent set of size $\lfloor \alpha n \rfloor$ of G, and $f_{v,i}, f_{v,j} \in S \implies i = j$, we have $S \cup \{e_0\}$ is a facet of X_τ.

The proof simply follows from the fact that edges of E are smaller than $e_{v,i}$'s, and $f'_{u,j}s$ in $\mathcal{O}$. By the second part of the claim, we can write

$$|X_\tau(\lfloor \alpha n \rfloor + 1)| = \ell^{\lfloor \alpha n \rfloor} N + |\{S \in X_\tau(\lfloor \alpha n \rfloor + 1) : S \cap E \neq \emptyset\}| \geq \ell^{\lfloor \alpha n \rfloor} N. \tag{5.2}$$

Define $i_k \coloneqq i_k(G)$ as the number of independent sets of size k of graph G. By the first part of the above claim we can write,

$$|\{S \in X_\tau(\lfloor \alpha n \rfloor + 1) : S \cap E \neq \emptyset\}| \leq \sum_{k=0}^{\lfloor \alpha n \rfloor - 1} \ell^k \cdot i_k \cdot \binom{|E|}{\lfloor \alpha n \rfloor - k} \leq \sum_{k=0}^{\lfloor \alpha n \rfloor - 1} \ell^k \cdot i_k \cdot |E|^{\lfloor \alpha n \rfloor - k} \tag{5.3}$$

$$\underset{\text{using } i_k \leq N}{\leq} N |E|^{\lfloor \alpha n \rfloor} \sum_{k=0}^{\lfloor \alpha n \rfloor - 1} (\ell / |E|)^k \tag{5.4}$$

$$\underset{\text{assuming } \ell \geq 2|E|}{\leq} N |E|^{\lfloor \alpha n \rfloor} (\ell / |E|)^{\lfloor \alpha n \rfloor} \leq N \ell^{\lfloor \alpha n \rfloor} \tag{5.5}$$

Putting these together with Eq. (5.2) concludes the proof. □

Proof of Theorem 5.1.9. For simplicity of notion, let $\alpha := \frac{2}{19}$. The proof is similar to the proof of Theorem 5.1.7 by a reduction from ♯INDEP-SET-INC$(7, \frac{2}{19})$. Take any arbitrary 7-regular graph $G = (V, E)$ with $n := |V|$ vertices whose number of independent sets of size $\lfloor \alpha |V| \rfloor$ is at least the number of its independent sets of size k for any $k < \lfloor \alpha |V| \rfloor$. Construct a new graph

$$G' = (V' = V \cup \{y, z\}, E' = E \cup \{e_0 = \{y, z\}\} \cup \{e_v = \{v, z\} : v \in V\})$$

Let $M = (E', \mathcal{I})$ be the graphic matroid given by G' truncated to rank $\lfloor \alpha n \rfloor + 1$ and consider the following ordering $\mathcal{O}$ on E'':

$$e_0 < E < \{e_v : v \in V\},$$

where as usual the ordering within each set is arbitrary. Define weights $\lambda : E' \to \mathbb{R}_{\geq 0}$ as follows:

$$\lambda_e = \begin{cases} \ell & \text{if } e = e_v \text{ for some } v \in V, \\ 1 & \text{o.w.} \end{cases},$$

for some ℓ that we choose later. We argue that

$$\lambda^{\lfloor \alpha n \rfloor} N \leq \sum_B \prod_{e \in B} \lambda_e \leq 2\lambda^{\lfloor \alpha n \rfloor} N.$$

where here (and henceforth) the sum is over B's that are NBC bases of M, and therefore a 1.5-approximation to the partition function, i.e., the quantity in the middle, is a 3-approximation to N. Similar to the previous theorem we have the following claim.

Claim 5.2.6. *For any NBC base B of M, we have $\{v : e_v \in B\}$ is an independent set of G. Conversely, for any independent set I of G of size $|I| = \lfloor \alpha n \rfloor$, $\{e_0\} \cup \{e_v : v \in I\}$ is a NBC base of M.*

So,

$$\begin{aligned} \sum_B \prod_{e \in B} \lambda_e &= \sum_{B : B \cap E \neq \emptyset} \prod_{e \in B} \lambda_e + \sum_{B : B \cap E = \emptyset} \prod_{e \in B} \lambda_e \\ &= \sum_{B : B \cap E \neq \emptyset} \prod_{e \in B} \lambda_e + \ell^{\lfloor \alpha n \rfloor} |\{S \subseteq V : S \text{ independent set of } G, |S| = \lfloor \alpha n \rfloor\}| \end{aligned} \tag{5.6}$$

Define i_k as the number of independent sets of size k of graph G. We have

$$\sum_{B : B \cap E \neq \emptyset} \prod_{e \in B} \lambda_e \leq \sum_{k=0}^{\lfloor \alpha n \rfloor - 1} \ell^k i_k \binom{|E|}{\lfloor \alpha n \rfloor - k} \underset{\substack{\text{using } i_k \leq N, \\ \text{assuming } \ell \geq 2|E|}}{\leq} \ell^{\lfloor \alpha n \rfloor} N$$

where the last inequality follows from the same calculations as in Eq. (5.3). □

Chapter 6

Complete Log Concavity of Coverage-Like Functions

6.1 Introduction

We introduce an expressive subclass of non-negative almost submodular set functions, called strongly 2-coverage functions which include coverage and (sums of) matroid rank functions, and prove that the homogenization of the generating polynomial of any such function is completely log-concave, taking a step towards characterizing the coefficients of (homogeneous) completely log-concave polynomials. This chapter is based on results published in [AO23b].

We previously stated the definition of completely log-concave polynomials as well as a number of other useful definitions in Section 1.1.5. Here, we provide some additional background. An interesting aspect of homogeneous completely log-concave polynomials is that the family of sets that can serve as the support of these polynomials can be nicely characterized. Given a polynomial $p = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} c_\alpha z^\alpha$, the Newton polytope of p is defined as

$$\mathrm{supp}(p) = \{\alpha \in \mathbb{Z}^n_{\geq 0} : c_\alpha \neq 0\}, \quad \mathrm{Newt}(p) = \mathrm{conv}(\mathrm{supp}(p)).$$

The following theorem gives a nice characterization of the supports of homogeneous completely log-concave polynomials.

Theorem 6.1.1 ([Ana+24; BH20]). *Given a homogeneous completely log-concave polynomial $p \in \mathbb{R}_{\geq 0}[z_1, \ldots, z_n]$, the Newton polytope of p is a generalized permutahedron, namely it is a polytope all of whose edges are parallel to $\mathbf{1}_i - \mathbf{1}_j$ for $1 \leq i < j \leq j$ and every integer point in this polytope also belongs to the supp(p). Conversely, for any generalized permutahedran, there is a homogeneous completely log-concave polynomial with support equal to all integer points in the polytope.*

An immediate consequence of the above theorem is that if p is homogeneous multiaffine and log-concave then $\mathrm{Newt}(p)$ is the base polytope of a matroid M with ground set of elements $[n]$ (see [Gel+87]). Having the above theorem that characterizes supports of homogeneous log-concave polynomials, a natural question is whether one can give a more fine characterization of the set of possible coefficients of homogeneous log-concave polynomials.

While such characterizations are not known, there are a number of results that give interesting necessary conditions for coefficients of log-concave polynomials. One such condition is implied by the following lemma.

Lemma 6.1.2 ([Ana+24]). *A polynomial* $h(y,z) = a+by+cz+dyz \in \mathbb{R}[y,z]$ *with non-negative coefficients is log-concave if and only if* $2bc \geq ad$.

Corollary 6.1.3 ([BH20]). *If* p_f *is log-concave for a non-negative set function* $f \in 2^{[n]} \to \mathbb{R}_{\geq 0}$, *then* f *is almost log-submodular, i.e. for any* $S \subset [n]$ *and* $i, j \in [n] \setminus S$,

$$2f(S \cup \{i\})f(S \cup \{i\}) \geq f(S)f(S \cup \{i,j\}).$$

Proof. To see this, note that the set of multiaffine log concave polynomials is closed under differentiation and specialization (see Proposition 6.1.16). In particular, the following polynomial is log-concave:

$$\begin{aligned} q(z_i, z_j) &= \left((\prod_{k \in S} \partial_{z_k}) p_f(z_1, \ldots, z_n) \right)_{|\{z_\ell = 0\}_{\ell \in [n] \setminus (S \cup \{i,j\})}} \\ &= f(S) + f(S \cup \{i\}) z_i + f(S \cup \{j\}) z_j + f(S \cup \{i,j\}) z_i z_j. \end{aligned}$$

Thus, by Lemma 6.1.2, f is almost submodular. □

Given a polynomial $p \in \mathbb{R}[z_1, \ldots, z_n]$, we define the homogenization of p as $\mathrm{Hom}(p, y) := \sum_{i=0}^{n} y^{n+1-i} p_i$, where p_i is the i-homogeneous part of p, i.e. $p = p_0 + \cdots + p_d$ and p_i is a i-homogeneous polynomial. Then, Corollary 6.1.3 implies that given a non-negative set function $f \in 2^{[n]} \to \mathbb{R}_{\geq 0}$, if $\mathrm{Hom}(p_f, y)$ is completely log-concave, then f is almost log-submodular. Thus, to take a step toward finding a classification of coefficients of homogeneous completely log-concave polynomials, a natural question to ask is whether one can find a large subclass of non-negative almost log-submodular functions such that for every f in that subclass, $\mathrm{Hom}(p_f, y)$ is completely log-concave. Note that if $\mathrm{Hom}(p_f, y)$ is completely log-concave, then all homogeneous parts of p_f are also log-concave. An important subclass of non-negative log-submodular functions with numerous applications is the class of non-negative monotone submodular functions (see Fact 6.1.20). Note that, however, even within this subclass, one can find functions that are not log-concave, and thus their homogenizations are not log-concave either (see Corollary 6.5.2 for an example).

In this paper, we introduce an expressive subclass of non-negative monotone functions, called strongly 2-coverage functions, and prove that $\mathrm{Hom}(p_f, y)$ is completely log-concave. We show that the set of strongly 2-coverage functions includes several fundamental classes of non-negative monotone submodular functions including matroid rank functions, coverage functions, and, more generally, matroid rank sum functions, which are positive linear combination of rank functions and include a large subset of submodular functions that have been studied in the mechanism design literature [Cal+07; DRY11; Dug11; DV11]. As a consequence we prove the following theorem.

Theorem 6.1.4 ([AO23b]). *If* $f : 2^{[n]} \to \mathbb{R}_{\geq 0}$ *is a coverage function or a sum of matroid rank functions, then the sequence* $f_0, f_1, \ldots, f_n$ *is ultra log-concave where* $f_i = \sum_{S:|S|=i} f(S)$.

Moreover, we introduce a strictly larger class of non-negative submodular functions called 2-coverage functions, which, for instance, also include the indicator function of independent sets of any matroid. We prove that if p_f is 2-coverage, all homogeneous parts of p_f are log-concave. As a consequence, given a 2-coverage function, one can use the results of [Ana+19; Ana+21; CGM19] to rapidly sample a subset S of $[n]$ with probability proportional to $f(S)$.

6.1.1 Main Results

Before presenting the results, we need a few more definitions.

A polynomial $p \in \mathbb{R}[z_1, \ldots, z_n]$ is *decomposable* if it can be written as a sum of two nonzero polynomials f and g such that f and g are supported on disjoint sets of variables. We say p is *indecomposable* otherwise. For a vector $\alpha \in \mathbb{Z}^n_{\geq 0}$ and a polynomial $p \in \mathbb{R}[z_1, \ldots, z_n]$, define $\partial^\alpha p \in \mathbb{R}[\{z_i\}_{i \notin \tau}]$ as

$$\partial^\alpha p := (\prod_{i=1}^{n} \partial_{z_i}^{\alpha_i})p.$$

When $p \in \mathbb{R}[z_1, \ldots, z_n]$ is a multiaffine polynomial, we might represent partial derivatives by subsets $\tau \subseteq [n]$, i.e. $\partial^\tau p := (\prod_{i \in \tau} \partial_{z_i})p$. In this case, we sometimes write p_τ to denote $\partial^\tau p$.

For a function $f : 2^{[n]} \to \mathbb{R}$ and an integer $0 \leq d \leq n$ define the d-homogeneous restriction of f, $f^{(d)} : 2^{[n]} \to \mathbb{R}$ as follows:

$$f^{(d)}(S) = \begin{cases} f(S) & \text{if } |S| = d, \\ 0 & \text{otherwise.} \end{cases}.$$

For any set function $f : 2^{[n]} \to \mathbb{R}_{\geq 0}$ and $\tau \subseteq [n]$, define f_τ as the (non-negative) function with generating polynomial $(p_f)_\tau$, where p_f is the generating polynomial of f. We say f is d-homogeneous (resp. indecomposable) if p_f is d-homogeneous (resp. indecomposable). Finally, we define $\deg(f)$ to be the degree of polynomial p_f.

We define the following classes of set functions.

Definition 14 (2-Coverage Set Functions). *A set function $f : 2^{[n]} \to \mathbb{R}_{\geq 0}$ is 2-coverage with respect to an integer $2 \leq d \leq \deg(f)$ if the following conditions holds.*

(i) For any $\tau \subseteq [n]$ with $|\tau| \leq d-2$, $(f^{(d)})_\tau$ is indecomposable.

(ii) For all $\tau \subseteq [n]$ with $|\tau| = d-2$, there exists $S \subseteq [n] \setminus \tau$[1], a coverage function $g : 2^S \to \mathbb{R}_{\geq 0}$, and a linear set function $\ell : 2^S \to \mathbb{R}_{\geq 0}$ (that are possibly dependent on τ), such that

- *$\ell(\{i\}) \leq g(\{i\})$ for all $i \in S$*
- *For any $T \subseteq [n] \setminus \tau$ of size $|T| = 2$,*

$$f_\tau(T) = \begin{cases} 0 & \textit{if } T \not\subseteq S, \\ g(T) - \frac{1}{2}\ell(T), & \textit{otherwise} \end{cases}$$

[1]Intuitively S corresponds to the non-loop elements of the "contracted" version of f.

We say $f : 2^{[n]} \to \mathbb{R}_{\geq 0}$ is 2-coverage if it is 2-coverage with respect to any $2 \leq d \leq \deg(f)$.

Definition 15 (Strongly 2-Coverage Set Functions). *A set function $f : 2^{[n]} \to \mathbb{R}_{\geq 0}$ is strongly 2-coverage if the following holds. For all $\tau \subseteq [n]$ such that $0 \leq |\tau| \leq n-2$, there exists a* coverage *function $g : 2^{[n]\setminus\tau} \to \mathbb{R}$, such that $(f_\tau)^{(1)} = g^{(1)} + f(\tau)$, and $(f_\tau)^{(2)} = g^{(2)} + f(\tau)$.*

The following propositions capture basic properties of these classes of set functions. A proof of these lemmas are included in section Section 6.2.

Proposition 6.1.5. *Let $f : 2^{[n]} \to \mathbb{R}_{\geq 0}$ be strongly 2-coverage, then f is monotone and submodular.*

Proposition 6.1.6. *Any strongly 2-coverage function $f : 2^{[n]} \to \mathbb{R}_{\geq 0}$ is* 2*-coverage.*

Proposition 6.1.7. *The set of strongly 2-coverage functions on $[n]$ is a convex cone, i.e., if $f_1, f_2 : 2^{[n]} \to \mathbb{R}$ are strongly 2-coverage set functions, then for any $\alpha \geq 0$, αf_1 and $f_1 + f_2$ are strongly 2-coverage functions.*

The following proposition provides some examples of strongly 2-coverage and 2-coverage functions. A proof of this proposition is included in section Section 6.2.

Proposition 6.1.8.

(i) For any matroid $M = ([n], I)$, its rank function $\mathrm{rk}_M : 2^{[n]} \to \mathbb{R}$ is a strongly 2-coverage set function.

(ii) For any matroid $M = ([n], I)$, the indicator function of its independent sets is 2*-coverage. Combined with Theorem 6.1.9, this gives another proof for the complete log-concavity of bases generating polynomial of a matroid that is proved in [AOV18].*

A consequence of Proposition 6.1.8 and Proposition 6.1.7 is that matroid rank sum functions are strongly 2-coverage. Therefore, by Fact 6.1.23 and Proposition 6.1.22, the joint entropy functions and coverage functions are strongly 2-coverage.

One of our main results is the following theorem. A proof of this theorem is included in Section 6.3.

Theorem 6.1.9. *If $f : 2^{[n]} \to \mathbb{R}_{\geq 0}$ is 2-coverage with respect to some $2 \leq d \leq \deg(f)$, then $f^{(d)}$ is completely log-concave.*

One of the consequences of Theorem 6.1.9 is that for any $f : 2^{[n]} \to \mathbb{R}_{\geq 0}$ that is 2-coverage with respect to some $2 \leq d \leq \deg(f)$, one can sample a set $S \subseteq [n]$ of size d proportionate to $f(S)$ in polynomial time.

Corollary 6.1.10. *Given a set function $f : 2^{[n]} \to \mathbb{R}_{\geq 0}$ that is 2-coverage with respect to some $2 \leq d \leq \deg(f)$, let μ_d be the distribution induced by $f^{(d)}$, i.e. $\mu_d(S) = \frac{f(S)}{\sum_{S \subset [n], |S|=d} f(S)}$, for any set S of size d. For any $\epsilon > 0$, starting from an arbitrary set S_0, the up-down walk P on sets of size d generates a sample from $\hat{\mu}_d$ such that $\|\hat{\mu}_d - \mu_d\|_{\mathrm{TV}} \leq \epsilon$ in time $O(d \log(d/\epsilon))$, i.e.,*

$$t_{mix}(P, S_0, \epsilon) \leq O(d \log(d/\epsilon)).$$

The following theorem is our other main result. This theorem is proved in Section 6.4.

Theorem 6.1.11. *Let $f : 2^{[n]} \to \mathbb{R}_{\geq \nvdash}$ be a strongly 2-coverage set function. Then, the polynomial $q_f(y, x_1, \dots, x_n) \coloneqq \sum_{i=0}^{n} y^{n+1-i} \sum_{S \subseteq [n], |S|=i} f(S) x^S$ is completely log-concave.*

The following corollary simply follows by an application of Proposition 6.1.19.

Corollary 6.1.12. *Let $f : 2^{[n]} \to \mathbb{R}_{\geq \nvdash}$ be a strongly 2-coverage set function. Let c_k be the k-th coefficient of $p_f(y, x) = \sum_{i=0}^{n} (\sum_{S \subseteq [n], |S|=i} f(S)) y^{n+1-i} x^i$. Then, for $1 < k < n$, we get*

$$\left(\frac{c_k}{\binom{n+1}{k}} \right)^2 \geq \left(\frac{c_{k-1}}{\binom{n+1}{k-1}} \right) \left(\frac{c_{k+1}}{\binom{n+1}{k+1}} \right)$$

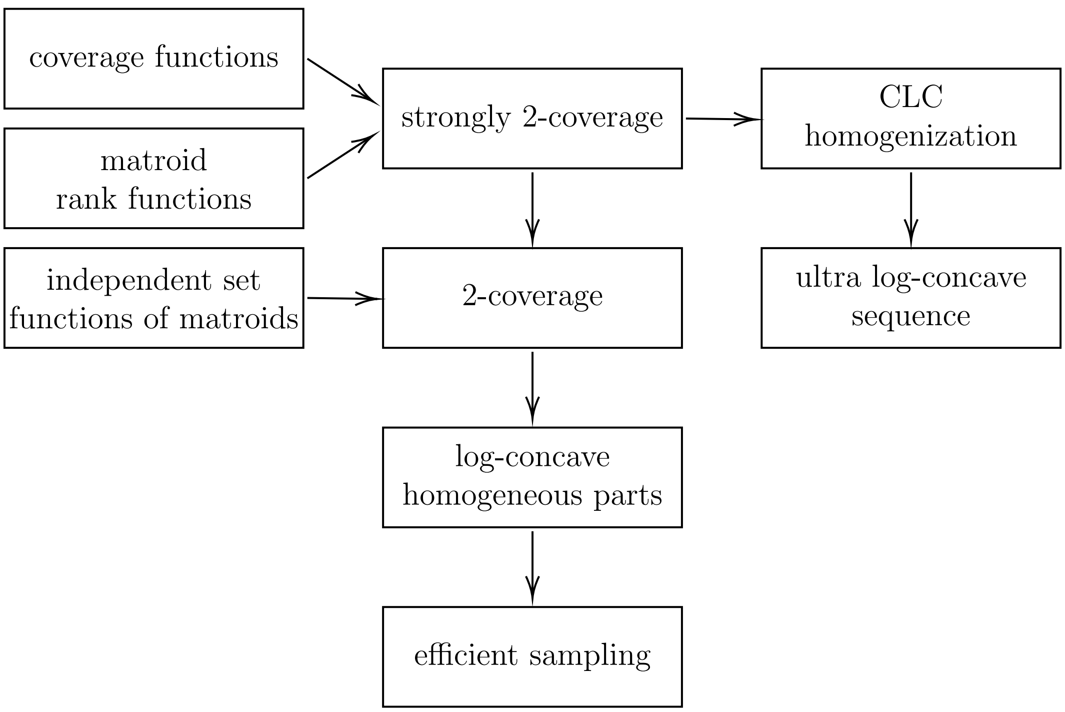


Figure 6.1: Summary of results

6.1.2 Preliminaries

Throughout the paper, we assume that for any set function $f^{[n]} \to \mathbb{R}_{\geq 0}$ that we work with, we have $f(\emptyset) = 0$.

Fact 6.1.13. *For any $x, y \in \mathbb{R}$, $-2xy \leq cx^2 + \frac{1}{c} y^2$.*

Linear Algebra We write J_n to denote the $n \times n$ all-ones matrix. For any integer $n > 0$ and indices i, j, define $E_{n,ij} \in \mathbb{R}^{n \times n}$ as $E_{n,ij}(i, j) = 1$ and let every other entry to be zero. We drop the subscript n when the dimension is clear from context.

Lemma 6.1.14. *For any diagonal matrix* $D \succeq 0$, $JD+DJ$ *has at most one positive eigenvalue.*

Proof. We can write

$$JD + DJ = (D + I)J(D + I) - DJD - J.$$

Now, the statement follows from the fact that $(D+I)J(D+I)$, J, and DJD are all rank-one PSD matrices. □

Lemma 6.1.15 ([Ana+19]). *Let* $A \in \mathbb{R}^{n\times n}$ *be a symmetric matrix. For any* $P \in \mathbb{R}^{m\times n}$, *if* A *has at most one positive eigenvalue,* $PAP^\top$ *has at most one positive eigenvalue.*

Log Concave Polynomials

Proposition 6.1.16 ([BH20; Ana+24]). *Given a polynomial* $p \in \mathbb{R}[z_1, \ldots, z_n]$

(i) If p *is (completely) log-concave, then* $p_{|z_i=a}$ *is also (completely) log-concave for any* $a \in \mathbb{R}_{\geq 0}$ *and* $1 \leq i \leq n$.

(ii) If p *is (completely) log-concave, any* $c, \lambda_1, \ldots, \lambda_n \in \mathbb{R}_{\geq 0}$, $cp(\lambda_1 z_1, \ldots, \lambda_n z_n)$ *is also (completely) log-concave.*

(iii) If p *is completely log-concave, then* $\partial_{z_i} p$ *is also completely log-concave for any* $1 \leq i \leq n$. *Moreover, if* p *is multiaffine and log-concave, then* $\partial_{z_i} p$ *is also log-concave for any* $1 \leq i \leq n$.

Proof. Item (i) and Item (ii) follow from definition. The first part of Item (iii) also follows from the definition of log-concavity. We prove the second part without loss of generality for $\partial_{z_1} p$. Let $q_t(z_1, \ldots, z_n) = t^{-1}(tz_1, \ldots, z_n)$. By Item (ii), q_t is log-concave for any $t \geq 0$. Note that since p is multiaffine, we can write $p = z_1 g + h$, where $g, h \in \mathbb{R}[z_2, \ldots, z_n]$, and hence $\partial_{z_1} p = g$. Assume that g is not log-concave. Therefore, for some $0 \leq \lambda \leq 1$ and $a, b \in \mathbb{R}_{\geq 0}$, $g(\lambda a + (1 - \lambda)b) > g(a)^\lambda + g(b)^{1-\lambda}$. Note that $q_t = z_1 g + t^{-1} h$. So for sufficiently large $t \in \mathbb{R}_{\geq 0}$, $q_t(\lambda a + (1 - \lambda)b) > \lambda q_t(a) + (1 - \lambda) q_t(b)$, which is a contradiction with the fact that q_t is log-concave. □

For d-homogeneous polynomials, the following lemma gives an equivalent condition to log-concavity that in many cases is much easier to verify.

Lemma 6.1.17 ([AOV18]). *Let* $p \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]$ *be a* d*-homogeneous polynomial for some integer* $d \geq 0$. *For* $a \in \mathbb{R}^n_{\geq 0}$ *such that* $p(a) \neq 0$, p *is log-concave at* a *if and only if,* $\nabla^2 p(a)$ *has at most one positive eigenvalue.*

The following theorem by [Ana+24], provides a very useful tool for proving a d-homogeneous polynomial is completely log-concave.

Theorem 6.1.18 ([Ana+24]). *Let* $p \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]$ *be a* d*-homogeneous polynomial. Then,* p *is completely log-concave if*

(i) For any $\alpha \in \mathbb{Z}^n_{\geq 0}$, *of size* $\|\alpha\|_1 \leq d - 2$, $\partial^\alpha p$ *is indecomposable.*

(ii) For any $\alpha \in \mathbb{Z}^n_{\geq 0}$, of size $\|\alpha\|_1 = d-2$, the quadratic polynomial $\partial^\alpha p$ is log-concave.

The following proposition is a slight modification of a statement by Gurvits [Gur10]. For completeness, we include a short proof for this slightly modified version.

Proposition 6.1.19. *If $p = \sum_{k=0}^n c_k y^{n-k+1} z^k \in \mathbb{R}[y,z]$ is completely log- concave, then the sequence $c_0, ..., c_n$ satisfies the following: for any $1 < k < n$,*

$$\left(\frac{c_k}{\binom{n+1}{k}}\right)^2 \geq \left(\frac{c_{k-1}}{\binom{n+1}{k-1}}\right)\left(\frac{c_{k+1}}{\binom{n+1}{k+1}}\right).$$

We call such sequences ultra log-concave.

Proof. Fix $1 < k < n$. Define $q(y,z) \coloneqq \partial_y^{n-k}\partial_z^{k-1} p$. Using the fact that $\partial_y^{n+1-m}\partial_z^m p = (n+1-m)!m!c_m = (n+1)!\frac{c_m}{\binom{n+1}{m}}$ for any $0 \leq m \leq n+1$, we compute $\nabla^2 q$ as follows

$$\nabla^2 q = \begin{bmatrix} \partial_y^2 q & \partial_y\partial_z q \\ \partial_y\partial_z q & \partial_z^2 q \end{bmatrix} = (n+1)! \begin{bmatrix} \frac{c_{k-1}}{\binom{n+1}{k-1}} & \frac{c_k}{\binom{n+1}{k}} \\ \frac{c_k}{\binom{n+1}{k-1}} & \frac{c_{k+1}}{\binom{n+1}{k+1}} \end{bmatrix}.$$

Since p is completely log-concave, by Proposition 6.1.16, q is log-concave. By Lemma 6.1.17, the 2×2 matrix $\nabla^2 q$ has at most one positive eigenvalue. Since all entries of $\nabla^2 q$ are positive, the matrix has exactly one positive eigenvalue (and one negative eigenvalue), therefore its determinant is non-positive. Therefore, we have

$$\left(\frac{c_k}{\binom{n+1}{k}}\right)^2 - \left(\frac{c_{k-1}}{\binom{n+1}{k-1}}\right)\left(\frac{c_{k+1}}{\binom{n+1}{k+1}}\right) \leq 0,$$

as desired. □

Note that, our definition of ultra log-concavity is slightly different from the more common definition which says that a sequence $c_0, \ldots, c_n$ is ultra log-concave if it satisfies $\left(\frac{c_k}{\binom{n}{k}}\right)^2 \geq \left(\frac{c_{k-1}}{\binom{n}{k-1}}\right)\left(\frac{c_{k+1}}{\binom{n}{k+1}}\right)$.

Submodular Functions A function $2^{[n]} \to \mathbb{R}$ is *submodular* if is has the diminishing return property, i.e.

$$\forall S \subseteq T \subseteq [n], i \in [n] : f(S \cup \{i\}) - f(S) \geq f(T \cup \{i\}) - f(T).$$

Such a function is *monotone* if $f(A) \leq f(B)$ for all $A \subseteq B$.

Fact 6.1.20. *Any non-negative monotone submodular function $f : 2^{[n]} \to \mathbb{R}$ is log-submodular, i.e. $\log f$ is submodular.*

Proof. Fix set $S \subseteq T \subseteq [n]$. It is enough to show that

$$\log \frac{f(S \cup \{i\})}{f(S)} = \log f(S \cup \{i\}) - \log f(S) \geq \log f(T \cup \{i\}) - \log f(T) = \log \frac{f(T \cup \{i\})}{f(T)}.$$

But one can easily verify that the diminishing return property combined with the fact that f is non-negative and monotone implies $\frac{f(S\cup\{i\})}{f(S)} \geq \frac{f(T\cup\{i\})}{f(T)}$. Since log is an increasing function, this finishes the proof. □

A fundamental class of non-negative monotone submodular functions are coverage functions.

Definition 16 (Coverage Functions). *Given a finite universe U and sets $A_1, \ldots, A_n \subseteq U$ and a measure w on U, we define $f : 2^{[n]} \to \mathbb{R}_{\geq 0}$ to be*

$$\forall T \subseteq [n] : f(T) = w\left(\bigcup_{i \in T} A_i\right).$$

Linear set functions are a subclass of coverage functions.

Definition 17 (Linear Set Functions). *$f : 2^{[n]} \to \mathbb{R}_{\geq 0}$ is a linear function if*

$$\forall T \subseteq [n] : f(T) = \sum_{i \in T} f(\{i\}).$$

We will use the following proposition which characterizes coverage set functions.

Proposition 6.1.21 ([CH12]). *A function $f : 2^{[n]} \to \mathbb{R}_{\geq 0}$ is a coverage function if and only if there exists a non-negative real number x_T for any $T \subseteq [n]$, such that for any $S \subseteq [n]$ we can write $f(S) = \sum_{T: T \cap S \neq \emptyset} x_T$.*

Many interesting classes of nonnegative monotone submodular functions are special cases of coverage functions. For example, given a set of random variables $\Omega = \{Y_1, \ldots, Y_n\}$, the joint entropy function defines a coverage function on $2^{[n]}$.

Proposition 6.1.22. *Given random variables $\Omega = \{Y_1, \ldots, Y_n\}$, let $H : 2^{\Omega} \to \mathbb{R}_{\geq 0}$ be the joint entropy function of these variables. Define $f : 2^{[n]} \to \mathbb{R}_{\geq 0}$ as $f(S) = H(Y_S)$ for any $S \subseteq [n]$, where $Y_S := \{Y_i | i \in S\}$. Then, f is a coverage function.*

A proof of this proposition can be found in Appendix C.1

Coverage functions are themselves special cases of matroid rank functions, as any coverage function can be written as a weighted sum of rank 1 matroids each corresponding to the coverage of an individual element.

Fact 6.1.23. *Any coverage function can be written as positive sum of matroid rank functions.*

6.2 Basic Properties of 2-Coverage and Strongly 2-Coverage Functions

Proof of Proposition 6.1.5. Fix $A \subseteq [n]$. We show that for any $B \supseteq A$, $f(A) \leq f(B)$ and for all $i \in [n]$, $f(A \cup \{i\}) - f(A) \geq f(B \cup \{i\}) - f(B)$. We prove by induction on $|B \setminus A|$. The claims trivially hold for $|B \setminus A| = 0$. Now, let $|B \setminus A| = \ell$ for some $1 \leq \ell \leq n - |A|$. Take some $j \in B \setminus A$ and let $C := B \setminus \{j\}$. By induction hypothesis, $f(A) \leq f(C)$. Furthermore, by definition there a coverage function $g : 2^{[n]\setminus C} \to \mathbb{R}_{\geq 0}$ such that $f(B) = f_C(\{j\}) = g(\{j\}) + f(C)$. Therefore, $f(A) \leq f(C) \leq f(B)$. To prove the submodularity condition, note that by induction hypothesis, we have $f(A \cup \{i\}) - f(A) \geq f(C \cup \{i\}) - f(C)$ for all $i \in [n]$. Furthermore, by submodularity of g, we have

$$f(C\cup\{i\})-f(C) = g(\{i\}) \geq g(\{j,i\})-g(\{j\}) = f(C\cup\{i,j\})-f(C\cup\{j\}) = f(B\cup\{i\})-f(B).$$

Therefore, $f(A \cup \{i\}) - f(A) \geq f(B \cup \{i\}) - f(B)$ as desired. □

Proof of Proposition 6.1.6. It is enough to show that given for any $2 \leq d \leq \deg(f)$, and any $\tau \subseteq [n]$ with $|\tau| \leq d-2$, $(f^{(d)})_\tau$ is indecomposable. If $(f^{(d)})_\tau = 0$ this claim follows trivially. Assume $(f^{(d)})_\tau \neq 0$ and let $p := p_{f^{(d)}}$ be the generating polynomial of $f^{(d)}$. For the sake of contradiction, assume that $p_\tau = h + g$ such that h, g are both non-zero. Let h and g be respectively supported on disjoint set of variables S_h and S_g. By Proposition 6.1.5, we know that f is monotone. Therefore, for any $x_i \in S_h$, $x_j \in S_g$, we must have

$$f(\tau \cup \{i\}) \leq f(\tau \cup \{i,j\}) = f_\tau(\{i,j\}) = 0.$$

This implies that $f_\tau(\{i\}) = f(\tau \cup \{i\}) = 0$. Similarly, $f_\tau(\{j\}) = 0$. Therefore, f_τ is 0 on all sets of size 1. By monotonicity of f, we get $f_\tau = 0$ which contradicts our assumption. □

Proof of Proposition 6.1.7. Fix a $\tau \in [n]$ such that $0 \leq |\tau| \leq n-2$. Let $g_1, g_2 : 2^{[n]\setminus\tau} \to \mathbb{R}_{\geq 0}$ be coverage functions such that $((f_1)_\tau)^{(2)} = (g_1)^{(2)}$ and $((f_2)_\tau)^{(2)} = (g_2)^{(2)}$. The statement follows from the fact that if g_1, g_2 are coverage functions, then αg and g_1+g_2 are also coverage functions. □

Proof of Proposition 6.1.8. We prove Item (i). Let $f := \mathrm{rk}_M$. For any $\tau \subseteq [n]$ with $|\tau| \leq n-2$, $f_\tau(S) = \mathrm{rk}_M(\tau)+\mathrm{rk}_{M/\tau}(S)$ for all sets $S \subseteq [n]\setminus\tau$ of size 1 or 2. Note that $f(\tau) = \mathrm{rk}_M(\tau)$. To satisfy the condition of Definition 15, it is enough to show that there is a coverage function g that takes the same values as $\mathrm{rk}_{M/\tau}(S)$ on sets of size 1 and 2. Using matroid partition property, M/τ can be partitioned into sets $S_0, \dots, S_k$ such that the following holds.

$$\mathrm{rk}_{M/\tau}(\{x,y\}) = \begin{cases} 0 & \text{if } x, y \in S_0 \\ 1 & \text{if } x \in S_0, y \in S_i \quad \text{for some } 1 \leq i \leq k \\ 1 & \text{if } x, y \in S_i \quad \text{for some } 1 \leq i \leq k \\ 2 & \text{if } x \in S_i, y \in S_j \quad \text{for some } 1 \leq i < j \leq k \end{cases}$$

Define g as follows. For any $i \in \{1, \dots, k\}$ and $x \in S_i$, let $A_x := \{i\}$, and for any $x \in S_0$, let $A_x = \emptyset$. For any $T \in [n] \setminus \tau$, define $g(T) := |\cup_{x\in T} A_x|$. One can easily check that $(f_\tau)^{(1)} = g^{(1)} + f(\tau)$, and $(f_\tau)^{(2)} = g^{(2)} + f(\tau)$.

Now, we prove Item (ii). Let $M = ([n], I)$ be a matroid of rank r and let $f : 2^{[n]} \to \mathbb{R}$ be an indicator function of independent sets, i.e. $f(S) = \mathbf{1}_I(S)$. We want to show that f is 2-coverage. The indecomposability holds because of the exchange property of matroids. We verify the second condition. Fix $2 \le d \le r$ and $\tau \subseteq [n]$ with $|\tau| = d-2$. Let S be the set of non-loop elements of M/τ. Using the matroid partition property, S can be partitioned into sets $S_1, \dots, S_k$ such that

$$(f_\tau)^{(2)}(\{x,y\}) = \begin{cases} 0 & x, y \in S_i \quad \text{for some } 1 \le i \le k \\ 1 & x \in S_i, y \in S_j \quad \text{for some } 1 \le i < j \le k \end{cases}$$

For any $1 \le i \le k$ and $x \in S_i$, let $A_x := \{i\}$. Define the coverage function $g : 2^S \to [n]$ as $g(T) = |\cup_{x \in T} A_x|$ for every $T \subseteq S$. Furthermore, consider the linear set function $\ell : 2^S \to [n]$ given by $\ell(\{i\}) = 1$ for all $i \in S$. It is easy to check that $f_\tau(T) = g(T) - \frac{\ell}{2}(T)$, for any $T \subseteq S$ of size 2. □

6.3 Complete Log-Concavity of Homogeneous Parts

In this section, we prove that if a set function f is 2-coverage with respect to some $2 \le d \le \deg(f)$, then $f^{(d)}$ is log-concave.

Lemma 6.3.1. *Let $g : 2^{[n]} \to \mathbb{R}_{\ge 0}$ be a coverage function and D be a diagonal matrix with $D(i,i) = g(\{i\})$ for all $i \in [n]$. Then, $R := (DJ + JD) - \nabla^2 p_{g^{(2)}} \succeq D$, where $p_{g^{(2)}}$ is the generating polynomial of $g^{(2)}$.*

Proof. By Proposition 6.1.21, there exists non-negative $\{x_T\}_{T \subseteq [m]}$ such that for any $i \neq j \in [n]$

$$\begin{aligned} g(\{i,j\}) &= \sum_{T : T \cap \{i,j\} \neq \emptyset} x_T = \sum_{T : i \in T} x_T + \sum_{T : j \in T} x_T - \sum_{T : i,j \in T} x_T \\ &= g(\{i\}) + g(\{j\}) - \sum_{T : i,j \in T} x_T. \end{aligned}$$

Note that for all $i \neq j \in [n]$, $(DJ + JD)(i,j) = g(\{i\}) + g(\{j\})$. Therefore, we can write R as

$$R(i,j) = \begin{cases} 2g(\{i\}) & \text{if } i = j \\ \sum_{T : i,j \in T} x_T & \text{otherwise.} \end{cases}$$

Furthermore, for any $T \subseteq [m]$ define matrix B_T as $B_T(i,j) = 1$ if $\{i,j\} \subseteq T$ and 0 otherwise. We can rewrite R as follows

$$R = \sum_{T \subseteq [m]} x_T B_T + D.$$

For any T, $B_T \succeq 0$. Therefore, since $x_T \ge 0$, we have $\sum_{T \subseteq [m]} x_T B_T \succeq 0$. This implies that $R \succeq D$, as desired. □

Proof of Theorem 6.1.9. Let $p := p_{f^{(d)}}$ be the generating polynomial of $f^{(d)}$. We use Theorem 6.1.18 to prove the theorem. Note that the indecomposability condition holds by definition. Now, we use Lemma 6.1.17 to prove that the second condition of Theorem 6.1.18 holds. Fix $\tau = \{i_1, \ldots, i_{d-2}\}$. We want to show that p_τ is log-concave. If p_τ is identically zero, the condition trivially holds. So, assume that $p_\tau \neq 0$. By Lemma 6.1.17, it is enough to show that $\nabla^2(p_\tau)$ has at most one positive eigenvalue. By definition, there is a set $S \subseteq [n] \setminus \tau$, a coverage function $g : 2^{[S]} \to \mathbb{R}_{\geq 0}$, and a linear set function $\ell : 2^S \to \mathbb{R}_{\geq 0}$ such that for any $T \subseteq [n] \setminus \tau$ of size 2, $f_\tau(T) = g(T) - \frac{\ell(T)}{2}$ if $T \subseteq S$, and $f_\tau(T) = 0$ otherwise. Therefore, it is enough to show that $Q = \nabla^2 p_{g^{(2)}} - \nabla^2 p_{\frac{\ell(2)}{2}}$ has at most one positive eigenvalue, where Q is the principle minor of $\nabla^2(p_\tau)$ obtained by restricting it to the rows and columns indexed by S. We can write, $Q = \nabla^2 p_{g^{(2)}} - (\frac{JC+CJ}{2} - C)$ where C is a diagonal matrix with $C(i,i) = \ell(\{i\})$. Furthermore, let D be a diagonal matrix with $D(i,i) = g(\{i\})$ for all $i \in S$. Note that by Lemma 6.3.1, we can write $p_{g^{(2)}} + D \preceq DJ + JD$. So

$$\begin{aligned}\nabla^2(p_\tau) = \nabla^2 p_{g^{(2)}} + C - \frac{JC+CJ}{2} &\preceq \nabla^2 p_{g^{(2)}} + D - \frac{JC+CJ}{2} \\ &\preceq DJ + JD - \frac{JC+CJ}{2} = (D - \frac{C}{2})J + J(D - \frac{C}{2}).\end{aligned}$$

Thus, by Lemma 6.1.14, $\nabla^2(p_\tau)$ has at most one positive eigenvalue. □

6.4 Complete Log-Concavity of the Homogenization of the Generating Polynomial of Strongly 2-Coverage Functions

Proof of Theorem 6.1.11. We use Theorem 6.1.18. First, we show that q_f is indecomposable. We want to show that for any $0 \leq k \leq n-1$, $0 \leq \ell \leq n-1-k$ and any $\tau = \{i_1 \ldots, i_\ell\}$, $\partial^\tau \partial_y^k q_f(y,x)$ is indecomposable (assuming it is non-zero). Note that, for some polynomial $g(y,x)$, we can write $\partial^\tau \partial_y^k q_f(y,x) = yg(y,x) + \partial^\tau p_{f^{(n-k+1)}}(x)$, where $p_{f^{(n-k+1)}}$ is the generating polynomial of $f^{(n-k+1)}$. Now, by Proposition 6.1.6, $\partial^\tau p_{f^{(n-k+1)}}$ is indecomposable. Furthermore, $yg(y,x)$ is indecomposable since y appears in all of its monomials. It is enough to show that there exists a variable that appears in monomials of both $yg(y,x)$ and $\partial^\tau p_{f^{(n-k+1)}}$. If $\partial^\tau p_{f^{(n-k+1)}}$ is identically zero, we are done. Otherwise, since f_τ is monotone and submodular by Proposition 6.1.5, there exists j such that $f_\tau(\{j\}) > 0$. Therefore, $x_j y^{n-\ell-k}$ is a monomial in $yg(y,x)$. By monotonicity of f_τ, for any set $S \subseteq [n] \setminus \tau$ of size $n-k-\ell+1$ such that $j \in S$, x^S is a monomial in $\partial^\tau p_{f^{(n-k+1)}}$. This finishes the proof of indecomposability. Now, we prove that the second condition of Theorem 6.1.18 holds. It is enough to show that for any $0 \leq k \leq n-1$ and $\tau = \{i_1 \ldots, i_k\}$, $\partial^\tau \partial_y^{n-1-k} q_f(y,x)$ is log-concave. Let $p = \frac{\partial^\tau \partial_y^{n-1-k} q_f(y,x)}{(n-1-k)!}$. We can write

$$p = \frac{(n-k+1)(n-k)}{2} f(\tau) y^2 + (n-k) \sum_{i \in [n] \setminus \tau} f(\tau \cup \{i\}) y x_i + \sum_{\{i,j\} \subseteq [n] \setminus \tau} f(\tau \cup \{i,j\}) x_i x_j.$$

Without loss of generality, assume that p is supported on variables $\{x_1, \ldots, x_m\} \cup \{y\}$ for $m := n-k$. We compute the Hessian matrix. In order to write the Hessian as a matrix indexed by $\{0,\ldots,m\}$, we assume that y corresponds to 0 and for all $1 \le i \le m$, x_i corresponds to integer i. With this indexing, we can write the Hessian matrix as

$$H = \begin{bmatrix} (m+1)mf(\tau) & mf_\tau(\{1\}) & \ldots & mf_\tau(\{m\}) \\ mf_\tau(\{1\}) & 0 & \ldots & f_\tau(\{1,m\}) \\ \vdots & \vdots & \ldots & \vdots \\ mf_\tau(\{m\}) & f_\tau(\{m,1\}) & \ldots & 0 \end{bmatrix}.$$

By Lemma 6.1.17, it is enough to show that H has at most one positive eigenvalue. By Lemma 6.1.15, this is equivalent to showing that the following matrix has at most one positive eigenvalue.

$$G = \begin{bmatrix} \frac{m+1}{m}f(\tau) & g(\{1\})+f(\tau) & \ldots & g(\{m\})+f(\tau) \\ g(\{1\})+f(\tau) & 0 & \ldots & g(\{1,m\})+f(\tau) \\ \vdots & \vdots & \ldots & \vdots \\ g(\{m\})+f(\tau) & g(\{m,1\})+f(\tau) & \ldots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & g(\{1\}) & \ldots & g(\{m\}) \\ g(\{1\}) & 0 & \ldots & g(\{1,m\}) \\ \vdots & \vdots & \ldots & \vdots \\ g(\{m\}) & g(\{m,1\}) & \ldots & 0 \end{bmatrix} + \begin{bmatrix} \frac{m+1}{m}f(\tau) & f(\tau) & \ldots & f(\tau) \\ f(\tau) & 0 & \ldots & f(\tau) \\ \vdots & \vdots & \ldots & \vdots \\ f(\tau) & f(\tau) & \ldots & 0 \end{bmatrix}.$$

where the coverage function g satisfies $f(S) = f(\tau) + g(S \setminus \tau)$ for sets S such that $|\tau|+1 \le |S| \le |\tau|+2$ and $\tau \subseteq S$. In the above line, let H, K denote the first and second matrix respectively. Let $D \in R^{(m+1)\times(m+1)}$ be a diagonal matrix such that $D(0,0) = \frac{m+1}{2m}f(\tau)$ and $D(i,i) = \frac{m-1}{2m}f(\tau)$ for $1 \le i \le m$. Let $D' \in R^{(m+1)\times(m+1)}$ be a diagonal matrix with $D'(0,0) = 0$ and $D'(i,i) = g(\{i\})$ for $1 \le i \le m$. Note that by Lemma 6.1.14, $(D+D')J+J(D+D')$ has exactly one positive eigenvalue. Therefore, to show that $H+K$ has at most one positive eigenvalue, it is enough to prove that $(D+D')J+J(D+D')-H-K \succeq 0$. Note that $D'J + JD' - H \succeq 0$. To see this, first note that the first row and column of $D'J + JD' - H$ are zero. Furthermore, one can check that after eliminating the first row and column of K, the remaining matrix is equal to the Hessian of the generating polynomial of $g^{(2)}$. Thus, the claim follows from Lemma 6.3.1. Therefore, it is enough to show that $DJ + JD - K \succeq 0$. Again, it is easy to check that the first row and column of $DJ + JD - K$ are zero. After removing the first row and column of $DJ + JD - K$, the remaining matrix is equal to $f(\tau)(I_m - \frac{1}{m}J_m)$. Now, note that $J_m \preceq mI_m$, as m is the the largest eigenvalue of J_m. Therefore, $DJ + JD - K \succeq 0$, as desired.

□

6.5 Negative Results

We showed that for an expressive class of nonnegative monotone submodular functions, the generating polynomial of d-homogenous restriction of the function is completely log-concave for any $d \ge 1$. We show that this claim does not hold for all nonnegative monotone submodular functions.

Proposition 6.5.1. *There exist integers $n \geq d \geq 1$ and a non-negative monotone submodular function $f : 2^{[n]} \to \mathbb{R}_{\geq 0}$ such that $f^{(d)}$ is not log-concave.*

Proof. Let [12] be the ground set. Furthermore, define $w_0 = \cdots = w_5 = 1$, $w_6 = \cdots = w_9 = 2$, and $w_{10} = w_{11} = 0$. Now, take the following set function $f : 2^{[12]} \to \mathbb{R}$ to be $f(S) = \min\{\sum_{i \in S} w_i, 2\}$. Note that this function is a budget additive function and is non-negative, monotone and submodular. But, one can verify that the polynomial $p(x) = \sum_{\{i,j\} \subseteq [12]} f(\{i,j\}) x_i x_j$ is not log-concave as $\nabla^2 p(x)$ has two positive eigenvalues. □

Corollary 6.5.2. *There exists monotone submodular function $f : 2^{[n]} \to \mathbb{R}_{\geq 0}$ such that f is not log-concave.*

Proof. This follows by the fact that if a function f is log-concave, all of its homogeneous parts are also log-concave. □

Gelfand, Goresky, MacPherson, and Serganova proved that the support of any homogeneous multiaffine log-concave polynomial correspond to bases of a matroid [Gel+87]. But there is not much known about the coefficient of these polynomials. A natural question to ask is that if the coefficients come from a monotone submodular function that is non-negative on non-empty sets. Another natural question to consider is whether the coefficients of these polynomials come from 2-coverage functions. The following proposition provides a counter-example to both of these statements.

Proposition 6.5.3. *There exist integers $n \geq d \geq 1$ and a d-homogeneous multiaffine log-concave $p \in R_{\geq 0}[x_1, \ldots, x_n]$ such that for any $f : 2^{[n]} \to \mathbb{R}_{\geq 0}$, if the generating polynomial of $f^{(d)}$ is equal to p, f is neither a monotone submodular function that is non-negative on non-empty sets nor a 2-coverage function.*

Proof. Let $p(x_1, x_2, x_3) := 3x_1x_2 + x_1x_3 + x_2x_3$. We have

$$\nabla^2 p = \begin{bmatrix} 0 & 3 & 1 \\ 3 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

One can easily check that $\nabla^2 p$ has exactly 1 positive eigenvalue. So, using Lemma 6.1.17, p is log-concave. Take an arbitrary $f : 2^{[3]} \to \mathbb{R}_{\geq 0}$ such that the generating polynomial of $f^{(2)}$ is equal to p. To show the first part of the statement, we assume that f is a monotone function that is non-negative on non-empty sets, and show that f is not submodular. By monotonicity, we have $f(\{1,2,3\}) \geq f(\{1,2\})$. Therefore

$$f(\{1,2,3\}) - f(\{1,3\}) \geq f(\{1,2\}) - f(\{1,3\}) = 2.$$

Moreover, $f(\{2,3\}) - f(\{3\}) \leq f(\{2,3\}) \leq 1$. Therefore, f is not submodular.

To show the second part of the statement, assume for contradiction that f is a 2-coverage function. Since f is non-zero on all sets of size 2, we must have $S = \{1,2,3\}$ (for $\tau = \emptyset$). So, there exists a coverage function $g : 2^{[3]} \to \mathbb{R}_{\geq 0}$ and a linear set function $\ell : 2^{[3]} \to \mathbb{R}_{\geq 0}$

such that $f^{(2)} = (g - \frac{\ell}{2})^{(2)}$ and that $\ell(\{i\}) \leq g(\{i\})$ for $i \in \{1, 2, 3\}$. We show that g is not submodular, so it cannot be a coverage function. Therefore

$$g(\{1,2\}) - g(\{1,3\}) = f(\{1,2\}) - f(\{1,3\}) + \frac{\ell(\{2\}) - \ell(\{3\})}{2} = 2 + \frac{\ell(\{2\}) - \ell(\{3\})}{2}.$$

This implies that $g(\{1,2,3\}) - g(\{1,3\}) \geq 2 + \frac{\ell(\{2\})-\ell(\{3\})}{2}$. Moreover,

$$\begin{aligned} g(\{2,3\}) - g(\{3\}) &= f(\{2,3\}) + \frac{\ell(\{2\}) + \ell(\{3\})}{2} - g(\{3\}) \\ &= 1 - g(\{3\}) + \frac{\ell(\{2\}) + \ell(\{3\})}{2} \underset{\ell(\{3\} \leq g(\{3\})}{\leq} 1 + \frac{\ell(\{2\}) - \ell(\{3\})}{2}. \end{aligned}$$

Combining these, we get $g(\{1,2,3\}) - g(\{1,3\}) \geq g(\{2,3\}) - g(\{3\})$, which is a contradiction with submodularity of g. □

Bibliography

[ABS15] Sanjeev Arora, Boaz Barak, and David Steurer. "Subexponential Algorithms for Unique Games and Related Problems". In: *J. ACM* 62.5 (Nov. 2015). ISSN: 0004-5411. DOI: 10.1145/2775105 (cit. on p. 4).

[AGT19] Vedat Levi Alev, Fernando Granha Jeronimo, and Madhur Tulsiani. "Approximating Constraint Satisfaction Problems on High-Dimensional Expanders". In: *2019 IEEE 60th Annual Symposium on Foundations of Computer Science (FOCS)*. Los Alamitos, CA, USA: IEEE Computer Society, Nov. 2019, pp. 180–201. DOI: 10.1109/FOCS.2019.00021 (cit. on p. 4).

[AHK18] Karim Adiprasito, June Huh, and Eric Katz. "Hodge theory for combinatorial geometries". In: *Annals of Mathematics* 188.2 (2018), pp. 381–452. DOI: https://doi.org/10.4007/annals.2018.188.2.1 (cit. on p. 65).

[AL20] Vedat Levi Alev and Lap Chi Lau. "Improved analysis of higher order random walks and applications". In: *Proceedings of the 52nd Annual ACM SIGACT Symposium on Theory of Computing*. STOC 2020. Chicago, IL, USA: Association for Computing Machinery, 2020, pp. 1198–1211. ISBN: 9781450369794. DOI: 10.1145/3357713.3384317 (cit. on pp. 4, 6, 20, 21).

[Ali+21] Yeganeh Alimohammadi, Nima Anari, Kirankumar Shiragur, and Thuy-Duong Vuong. "Fractionally log-concave and sector-stable polynomials: counting planar matchings and more". In: *Proceedings of the 53rd Annual ACM SIGACT Symposium on Theory of Computing*. STOC 2021. Virtual, Italy: Association for Computing Machinery, 2021, pp. 433–446. ISBN: 9781450380539. DOI: 10.1145/3406325.3451123 (cit. on pp. 4, 6, 20).

[ALO20] Nima Anari, Kuikui Liu, and Shayan Oveis Gharan. "Spectral Independence in High-Dimensional Expanders and Applications to the Hardcore Model". In: *SIAM Journal on Computing* 0.0 (2020), FOCS20-1-FOCS20–37. DOI: 10.1137/20M1367696. eprint: https://doi.org/10.1137/20M1367696 (cit. on pp. 4, 6, 11, 20, 29).

[ALO22] Dorna Abdolazimi, Kuikui Liu, and Shayan Oveis Gharan. "A Matrix Trickle-Down Theorem on Simplicial Complexes and Applications to Sampling Colorings". In: *2021 IEEE 62nd Annual Symposium on Foundations of Computer Science (FOCS)*. Los Alamitos, CA, USA: IEEE Computer Society, 2022, pp. 161–172. DOI: 10.1109/FOCS52979.2021.00024 (cit. on pp. 4, 6–10, 16, 21, 22, 25, 27, 28).

[ALO23] Dorna Abdolazimi, Kasper Lindberg, and Shayan Oveis Gharan. "On Optimization and Counting of Non-Broken Bases of Matroids". In: *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (APPROX/RANDOM 2023)*. Ed. by Nicole Megow and Adam Smith. Vol. 275. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2023, 40:1–40:14. ISBN: 978-3-95977-296-9. DOI: 10.4230/LIPIcs.APPROX/RANDOM.2023.40 (cit. on pp. 7, 13, 14, 64, 66, 67).

[Ana+19] Nima Anari, Kuikui Liu, Shayan Oveis Gharan, and Cynthia Vinzant. "Log-concave polynomials II: high-dimensional walks and an FPRAS for counting bases of a matroid". In: *Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing*. STOC 2019. Phoenix, AZ, USA: Association for Computing Machinery, 2019, pp. 1–12. ISBN: 9781450367059. DOI: 10.1145/3313276.3316385 (cit. on pp. 4, 6, 13–15, 20, 21, 64–66, 75, 78).

[Ana+21] Nima Anari, Kuikui Liu, Shayan Oveis Gharan, Cynthia Vinzant, and Thuy-Duong Vuong. "Log-concave polynomials IV: approximate exchange, tight mixing times, and near-optimal sampling of forests". In: *Proceedings of the 53rd Annual ACM SIGACT Symposium on Theory of Computing*. STOC 2021. Virtual, Italy: Association for Computing Machinery, 2021, pp. 408–420. ISBN: 9781450380539. DOI: 10.1145/3406325.3451091 (cit. on pp. 14, 15, 20, 21, 75).

[Ana+22] Nima Anari, Vishesh Jain, Frederic Koehler, Huy Tuan Pham, and Thuy-Duong Vuong. "Entropic independence: optimal mixing of down-up random walks". In: *Proceedings of the 54th Annual ACM SIGACT Symposium on Theory of Computing*. STOC 2022. Rome, Italy: Association for Computing Machinery, 2022, pp. 1418–1430. ISBN: 9781450392648. DOI: 10.1145/3519935.3520048 (cit. on pp. 4, 6).

[Ana+24] Nima Anari, Kuikui Liu, Shayan Oveis Gharan, and Cynthia Vinzant. "Log-concave polynomials III: Mason's ultra-log-concavity conjecture for independent sets of matroids". In: *Proceedings of the American Mathematical Society* 152.05 (2024), pp. 1969–1981 (cit. on pp. 14, 73, 74, 78).

[AO23a] Dorna Abdolazimi and Shayan Oveis Gharan. "An Improved Trickle Down Theorem for Partite Complexes". In: *Proceedings of the Conference on Proceedings of the 38th Computational Complexity Conference*. CCC '23. Warwick, United Kingdom: Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2023. ISBN: 9783959772822. DOI: 10.4230/LIPIcs.CCC.2023.10 (cit. on pp. 7, 11, 16, 48).

[AO23b] Dorna Abdolazimi and Shayan Oveis Gharan. "Complete Log Concavity of Coverage-Like Functions". In: *arXiv preprint arXiv:2303.03741* (2023) (cit. on pp. 7, 15, 73, 74).

[AOV18] Nima Anari, Shayan Oveis Gharan, and Cynthia Vinzant. "Log-Concave Polynomials, Entropy, and a Deterministic Approximation Algorithm for Counting Bases of Matroids". In: *2018 IEEE 59th Annual Symposium on Foundations of*

Computer Science (FOCS). 2018, pp. 35–46. DOI: 10.1109/FOCS.2018.00013 (cit. on pp. 14, 76, 78).

[Baf+21] Mitali Bafna, Max Hopkins, Tali Kaufman, and Shachar Lovett. "High Dimensional Expanders: Eigenstripping, Pseudorandomness, and Unique Games". In: *Proceedings of the 2022 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*. 2021, pp. 1069–1128. DOI: 10.1137/1.9781611977073.47. eprint: https://epubs.siam.org/doi/pdf/10.1137/1.9781611977073.47 (cit. on p. 4).

[BCT10] Brian Benson, Deeparnab Chakrabarty, and Prasad Tetali. "G-parking functions, acyclic orientations and spanning trees". In: *Discrete Math.* 310.8 (Apr. 2010), pp. 1340–1353. ISSN: 0012-365X. DOI: 10.1016/j.disc.2010.01.002 (cit. on pp. 13, 64, 65).

[Ben+21] Ferenc Bencs, Ewan Davies, Viresh Patel, and Guus Regts. "On zero-free regions for the anti-ferromagnetic Potts model on bounded-degree graphs". In: *Annales de l'Institut Henri Poincaré D* 8.3 (2021), pp. 459–489 (cit. on p. 29).

[BH20] Petter Branden and June Huh. "Lorentzian polynomials". English (US). In: *Annals of Mathematics* 192.3 (Nov. 2020). Publisher Copyright: © 2020. Department of Mathematics, Princeton University, pp. 821–891. ISSN: 0003-486X. DOI: 10.4007/annals.2020.192.3.4 (cit. on pp. 14, 73, 74, 78).

[Bla+22] Antonio Blanca, Pietro Caputo, Zongchen Chen, Daniel Parisi, Daniel Štefankovič, and Eric Vigoda. "On Mixing of Markov Chains: Coupling, Spectral Independence, and Entropy Factorization". In: *Proceedings of the 2022 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*. 2022, pp. 3670–3692. DOI: 10.1137/1.9781611977073.145. eprint: https://epubs.siam.org/doi/pdf/10.1137/1.9781611977073.145 (cit. on pp. 6, 20).

[BLM16] Stéphane Boucheron, Gábor Lugosi, and Pascal Massart. *Concentration Inequalities: A Nonasymptotic Theory of Independence*. 1st ed. Oxford University Press, 2016 (cit. on p. 18).

[Bry77] Tom Brylawski. "The broken-circuit complex". In: *Trans. Amer. Math. Soc.* 234.2 (1977), pp. 417–433 (cit. on p. 65).

[BT03] Sergey Bobkov and Prasad Tetali. "Modified log-sobolev inequalities, mixing and hypercontractivity". In: *Proceedings of the Thirty-Fifth Annual ACM Symposium on Theory of Computing*. STOC '03. San Diego, CA, USA: Association for Computing Machinery, 2003, pp. 287–296. ISBN: 1581136749. DOI: 10.1145/780542.780586 (cit. on p. 18).

[Cal+07] Gruia Calinescu, Chandra Chekuri, Martin Pál, and Jan Vondrák. "Maximizing a Submodular Set Function Subject to a Matroid Constraint (Extended Abstract)". In: *Proceedings of the 12th International Conference on Integer Programming and Combinatorial Optimization*. IPCO '07. Ithaca, NY, USA: Springer-Verlag, 2007, pp. 182–196. ISBN: 9783540727910. DOI: 10.1007/978-3-540-72792-7_15 (cit. on pp. 15, 74).

[Car89] Roger Carter. *Simple groups of Lie type*. John Wiley & Sons, 1989 (cit. on p. 51).

[CGM19] Mary Cryan, Heng Guo, and Giorgos Mousa. “Modified log-Sobolev Inequalities for Strongly Log-Concave Distributions”. In: *2019 IEEE 60th Annual Symposium on Foundations of Computer Science (FOCS)*. Los Alamitos, CA, USA: IEEE Computer Society, Nov. 2019, pp. 1358–1370. DOI: 10.1109/FOCS.2019.00083 (cit. on pp. 6, 14, 15, 21, 64, 75).

[CH12] Deeparnab Chakrabarty and Zhiyi Huang. “Testing Coverage Functions”. In: *Automata, Languages, and Programming*. Ed. by Artur Czumaj, Kurt Mehlhorn, Andrew Pitts, and Roger Wattenhofer. Berlin, Heidelberg: Springer Berlin Heidelberg, 2012, pp. 170–181. ISBN: 978-3-642-31594-7 (cit. on p. 80).

[Che+19] Sitan Chen, Michelle Delcourt, Ankur Moitra, Guillem Perarnau, and Luke Postle. “Improved Bounds for Randomly Sampling Colorings via Linear Programming”. In: *Proceedings of the 2019 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*. 2019, pp. 2216–2234. DOI: 10.1137/1.9781611975482.134. eprint: https://epubs.siam.org/doi/pdf/10.1137/1.9781611975482.134 (cit. on pp. 10, 27).

[Che+21a] Xiaoyu Chen, Weiming Feng, Yitong Yin, and Xinyuan Zhang. “Rapid Mixing of Glauber Dynamics via Spectral Independence for All Degrees”. In: *SIAM Journal on Computing* (2021), FOCS21-224-FOCS21–298. DOI: 10.1137/22M1474734. eprint: https://doi.org/10.1137/22M1474734 (cit. on p. 6).

[Che+21b] Zongchen Chen, Andreas Galanis, Daniel Štefankovič, and Eric Vigoda. “Rapid Mixing for Colorings via Spectral Independence”. In: *Proceedings of the 2021 ACM-SIAM Symposium on Discrete Algorithms (SODA)*. 2021, pp. 1548–1557. DOI: 10.1137/1.9781611976465.94. eprint: https://epubs.siam.org/doi/pdf/10.1137/1.9781611976465.94 (cit. on pp. 6, 10, 20, 27, 29).

[CLV20] Zongchen Chen, Kuikui Liu, and Eric Vigoda. “Rapid Mixing of Glauber Dynamics up to Uniqueness via Contraction”. In: *FOCS*. IEEE, 2020, pp. 1307–1318 (cit. on pp. 6, 20, 29).

[CLV21] Zongchen Chen, Kuikui Liu, and Eric Vigoda. “Optimal mixing of Glauber dynamics: entropy factorization via high-dimensional expansion”. In: STOC 2021. Virtual, Italy: Association for Computing Machinery, 2021, pp. 1537–1550. ISBN: 9781450380539. DOI: 10.1145/3406325.3451035 (cit. on pp. 4, 6, 11, 20, 29).

[CLV22] Zongchen Chen, Kuikui Liu, and Eric Vigoda. “Spectral Independence via Stability and Applications to Holant-Type Problems”. In: *2021 IEEE 62nd Annual Symposium on Foundations of Computer Science (FOCS)*. 2022, pp. 149–160. DOI: 10.1109/FOCS52979.2021.00023 (cit. on p. 6).

[DD19] Yotam Dikstein and Irit Dinur. “Agreement Testing Theorems on Layered Set Systems”. In: *2019 IEEE 60th Annual Symposium on Foundations of Computer Science (FOCS)*. Los Alamitos, CA, USA: IEEE Computer Society, Nov. 2019, pp. 1495–1524. DOI: 10.1109/FOCS.2019.00088 (cit. on pp. 4, 12).

[DHP20] Michelle Delcourt, Marc Heinrich, and Guillem Perarnau. "The Glauber dynamics for edge-colorings of trees". In: *Random Structures & Algorithms* 57.4 (2020), pp. 1050–1076. DOI: https://doi.org/10.1002/rsa.20960. eprint: https://onlinelibrary.wiley.com/doi/pdf/10.1002/rsa.20960 (cit. on p. 28).

[Dik+18] Yotam Dikstein, Irit Dinur, Yuval Filmus, and Prahladh Harsha. "Boolean Function Analysis on High-Dimensional Expanders". In: Leibniz International Proceedings in Informatics (LIPIcs) 116 (2018), 38:1–38:20. ISSN: 1868-8969 (cit. on p. 5).

[Dik+20] Yotam Dikstein, Irit Dinur, Prahladh Harsha, and Noga Ron-Zewi. "Locally testable codes via high-dimensional expanders". In: *arXiv preprint arXiv:2005.01045* (2020) (cit. on p. 4).

[Din+21a] Irit Dinur, Yuval Filmus, Prahladh Harsha, and Madhur Tulsiani. "Explicit SoS Lower Bounds from High-Dimensional Expanders". In: *12th Innovations in Theoretical Computer Science Conference (ITCS 2021)*. Ed. by James R. Lee. Vol. 185. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2021, 38:1–38:16. ISBN: 978-3-95977-177-1. DOI: 10.4230/LIPIcs.ITCS.2021.38 (cit. on p. 4).

[Din+21b] Irit Dinur, Prahladh Harsha, Tali Kaufman, Inbal Livni Navon, and Amnon Ta-Shma. "List-Decoding with Double Samplers". In: *SIAM Journal on Computing* 50.2 (2021), pp. 301–349. DOI: 10.1137/19M1276650. eprint: https://doi.org/10.1137/19M1276650 (cit. on pp. 4, 12).

[Din+22] Irit Dinur, Shai Evra, Ron Livne, Alexander Lubotzky, and Shahar Mozes. "Locally testable codes with constant rate, distance, and locality". In: *Proceedings of the 54th Annual ACM SIGACT Symposium on Theory of Computing*. STOC 2022. Rome, Italy: Association for Computing Machinery, 2022, pp. 357–374. ISBN: 9781450392648. DOI: 10.1145/3519935.3520024 (cit. on pp. 4, 12).

[Din07] Irit Dinur. "The PCP theorem by gap amplification". In: *J. ACM* 54.3 (June 2007), 12–es. ISSN: 0004-5411. DOI: 10.1145/1236457.1236459 (cit. on p. 4).

[DK17] Irit Dinur and Tali Kaufman. "High Dimensional Expanders Imply Agreement Expanders". In: *2017 IEEE 58th Annual Symposium on Foundations of Computer Science (FOCS)*. 2017, pp. 974–985. DOI: 10.1109/FOCS.2017.94 (cit. on pp. 4, 5, 12, 19).

[DLZ23] Irit Dinur, Siqi Liu, and Rachel Yun Zhang. "New codes on high dimensional expanders". In: *arXiv preprint arXiv:2308.15563* (2023) (cit. on p. 4).

[DP21] Ewan Davies and Will Perkins. "Approximately Counting Independent Sets of a Given Size in Bounded-Degree Graphs". In: *48th International Colloquium on Automata, Languages, and Programming (ICALP 2021)*. Ed. by Nikhil Bansal, Emanuela Merelli, and James Worrell. Vol. 198. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2021, 62:1–62:18. ISBN: 978-3-95977-195-5. DOI: 10.4230/LIPIcs.ICALP.2021.62 (cit. on p. 103).

[DRY11] Shaddin Dughmi, Tim Roughgarden, and Qiqi Yan. "From convex optimization to randomized mechanisms: toward optimal combinatorial auctions". In: *Proceedings of the Forty-Third Annual ACM Symposium on Theory of Computing*. STOC '11. San Jose, California, USA: Association for Computing Machinery, 2011, pp. 149–158. ISBN: 9781450306911. DOI: 10.1145/1993636.1993657 (cit. on pp. 15, 74).

[DS96] P. Diaconis and L. Saloff-Coste. "Logarithmic Sobolev inequalities for finite Markov chains". In: *The Annals of Applied Probability* 6.3 (1996), pp. 695–750. DOI: 10.1214/aoap/1034968224 (cit. on p. 18).

[Dug11] Shaddin Dughmi. "A truthful randomized mechanism for combinatorial public projects via convex optimization". In: *Proceedings of the 12th ACM Conference on Electronic Commerce*. EC '11. San Jose, California, USA: Association for Computing Machinery, 2011, pp. 263–272. ISBN: 9781450302616. DOI: 10.1145/1993574.1993614 (cit. on pp. 15, 74).

[DV11] Shaddin Dughmi and Jan Vondr´k. "Limitations of Randomized Mechanisms for Combinatorial Auctions". In: *2011 IEEE 52nd Annual Symposium on Foundations of Computer Science*. 2011, pp. 502–511. DOI: 10.1109/FOCS.2011.64 (cit. on pp. 15, 74).

[Dye+13] Martin Dyer, Alan Frieze, Thomas P. Hayes, and Eric Vigoda. "Randomly coloring constant degree graphs". In: *Random Structures & Algorithms* 43.2 (2013), pp. 181–200. DOI: https://doi.org/10.1002/rsa.20451. eprint: https://onlinelibrary.wiley.com/doi/pdf/10.1002/rsa.20451 (cit. on pp. 10, 27).

[Eft+19] Charilaos Efthymiou, Andreas Galanis, Thomas P. Hayes, Daniel Štefankovič, and Eric Vigoda. "Improved Strong Spatial Mixing for Colorings on Trees". In: *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (APPROX/RANDOM 2019)*. Ed. by Dimitris Achlioptas and László A. Végh. Vol. 145. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2019, 48:1–48:16. ISBN: 978-3-95977-125-2. DOI: 10.4230/LIPIcs.APPROX-RANDOM.2019.48 (cit. on p. 28).

[EK16] Shai Evra and Tali Kaufman. "Bounded degree cosystolic expanders of every dimension". In: *Proceedings of the Forty-Eighth Annual ACM Symposium on Theory of Computing*. STOC '16. Cambridge, MA, USA: Association for Computing Machinery, 2016, pp. 36–48. ISBN: 9781450341325. DOI: 10.1145/2897518.2897543 (cit. on pp. 51, 52).

[EKZ24] Shai Evra, Tali Kaufman, and Gilles Zémor. "Decodable Quantum LDPC Codes beyond the $\sqrt{n}$ Distance Barrier Using High-Dimensional Expanders". In: *SIAM Journal on Computing* 53.6 (2024), FOCS20-276-FOCS20–316. DOI: 10.1137/20M1383689. eprint: https://doi.org/10.1137/20M1383689 (cit. on p. 4).

[Fen+22] Weiming Feng, Heng Guo, Yitong Yin, and Chihao Zhang. "Rapid Mixing from Spectral Independence beyond the Boolean Domain". In: *ACM Trans. Algorithms* 18.3 (Oct. 2022). ISSN: 1549-6325. DOI: 10.1145/3531008 (cit. on pp. 10, 20, 27, 29).

[Fri+21] Tobias Friedrich, Andreas Göbel, Martin S. Krejca, and Marcus Pappik. "A spectral independence view on hardspheres via block dynamics". arXiv:2102.07443. 2021 (cit. on p. 29).

[FV06] Alan Frieze and Juan Vera. "On randomly colouring locally sparse graphs". English. In: *Discrete Mathematics & Theoretical Computer Science* Vol. 8, 1 (Jan. 2006). ISSN: 1365-8050. DOI: 10.46298/dmtcs.360 (cit. on pp. 10, 27).

[FV07] Alan Frieze and Eric Vigoda. "A survey on the use of Markov chains to randomly sample colourings". In: *Combinatorics, Complexity, and Chance: A Tribute to Dominic Welsh*. Oxford University Press, Jan. 2007. ISBN: 9780198571278. DOI: 10.1093/acprof:oso/9780198571278.003.0004. eprint: https://academic.oup.com/book/0/chapter/193380402/chapter-ag-pdf/44625038/book_25792_section_193380402.ag.pdf (cit. on pp. 10, 27).

[Gar73] H. Garland. "p-adic curvature and the cohomology of discrete subgroups of p-adic groups". In: *Annals of Mathematics* 97.3 (1973), pp. 375–423 (cit. on p. 19).

[Gel+87] I.M Gelfand, R.M Goresky, R.D MacPherson, and V.V Serganova. "Combinatorial geometries, convex polyhedra, and schubert cells". In: *Advances in Mathematics* 63.3 (1987), pp. 301–316. ISSN: 0001-8708. DOI: https://doi.org/10.1016/0001-8708(87)90059-4 (cit. on pp. 67, 73, 85).

[GK12] David Gamarnik and Dmitriy Katz. "Correlation decay and deterministic FPTAS for counting colorings of a graph". In: *J. of Discrete Algorithms* 12 (Apr. 2012), pp. 29–47. ISSN: 1570-8667. DOI: 10.1016/j.jda.2010.10.002 (cit. on p. 29).

[GK23] Roy Gotlib and Tali Kaufman. *No Where to Go But High: A Perspective on High Dimensional Expanders*. 2023. arXiv: 2304.10106 [math.CO] (cit. on p. 4).

[GKM15] David Gamarnik, Dmitriy Katz, and Sidhant Misra. "Strong spatial mixing of list coloring of graphs". In: *Random Struct. Algorithms* 46.4 (July 2015), pp. 599–613. ISSN: 1042-9832. DOI: 10.1002/rsa.20518 (cit. on pp. 28, 29).

[GL19] Emeric Gioan and Michel Las Vergnas. "The active bijection for graphs". In: *Advances in Applied Mathematics* 104 (2019), pp. 165–236. ISSN: 0196-8858. DOI: https://doi.org/10.1016/j.aam.2018.11.001 (cit. on pp. 13, 64).

[GM20] Heng Guo and Giorgos Mousa. "Local-to-Global Contraction in Simplicial Complexes". In: *arXiv preprint arXiv:2012.14317* (2020) (cit. on p. 20).

[GMP05] Leslie Ann Goldberg, Russell Martin, and Mike Paterson. "Strong Spatial Mixing with Fewer Colors for Lattice Graphs". In: *SIAM Journal on Computing* 35.2 (2005), pp. 486–517. DOI: 10.1137/S0097539704445470. eprint: https://doi.org/10.1137/S0097539704445470 (cit. on p. 28).

[Goe04] Sharad Goel. "Modified logarithmic Sobolev inequalities for some models of random walk". In: *Stochastic Processes and their Applications* 114.1 (2004), pp. 51–79. ISSN: 0304-4149. DOI: https://doi.org/10.1016/j.spa.2004.06.001 (cit. on p. 18).

[GŠV15] Andreas Galanis, Daniel Štefankovič, and Eric Vigoda. "Inapproximability for Antiferromagnetic Spin Systems in the Tree Nonuniqueness Region". In: *J. ACM* 62.6 (Dec. 2015). ISSN: 0004-5411. DOI: 10.1145/2785964 (cit. on p. 27).

[GŠV16] ANDREAS GALANIS, DANIEL ŠTEFANKOVIČ, and ERIC VIGODA. "Inapproximability of the Partition Function for the Antiferromagnetic Ising and Hard-Core Models". In: *Combinatorics, Probability and Computing* 25.4 (2016), pp. 500–559. DOI: 10.1017/S0963548315000401 (cit. on p. 103).

[Gur10] Leonid Gurvits. "On multivariate Newton-like inequalities". In: *Advances in Combinatorial Mathematics*. Ed. by Ilias S. Kotsireas and Eugene V. Zima. Berlin, Heidelberg: Springer Berlin Heidelberg, 2010, pp. 61–78. ISBN: 978-3-642-03562-3 (cit. on pp. 14, 79).

[HLW06] Shlomo Hoory, Nathan Linial, and Avi Wigderson. "Expander graphs and their applications". In: *Bulletin of the American Mathematical Society* 43.4 (2006), pp. 439–561 (cit. on p. 4).

[HV03] T.P. Hayes and E. Vigoda. "A non-Markovian coupling for randomly sampling colorings". In: *44th Annual IEEE Symposium on Foundations of Computer Science, 2003. Proceedings.* 2003, pp. 618–627. DOI: 10.1109/SFCS.2003.1238234 (cit. on pp. 10, 27).

[HV05] Tom Hayes and Eric Vigoda. "Coupling with the stationary distribution and improved sampling for colorings and independent sets". In: *Proceedings of the Sixteenth Annual ACM-SIAM Symposium on Discrete Algorithms*. SODA '05. Vancouver, British Columbia: Society for Industrial and Applied Mathematics, 2005, pp. 971–979. ISBN: 0898715857 (cit. on pp. 10, 27).

[HVV07] Thomas P. Hayes, Juan C. Vera, and Eric Vigoda. "Randomly coloring planar graphs with fewer colors than the maximum degree". In: *Proceedings of the Thirty-Ninth Annual ACM Symposium on Theory of Computing*. STOC '07. San Diego, California, USA: Association for Computing Machinery, 2007, pp. 450–458. ISBN: 9781595936318. DOI: 10.1145/1250790.1250857 (cit. on pp. 10, 27).

[INW94] Russell Impagliazzo, Noam Nisan, and Avi Wigderson. "Pseudorandomness for network algorithms". In: *Proceedings of the Twenty-Sixth Annual ACM Symposium on Theory of Computing*. STOC '94. Montreal, Quebec, Canada: Association for Computing Machinery, 1994, pp. 356–364. ISBN: 0897916638. DOI: 10.1145/195058.195190 (cit. on p. 4).

[Jer95] Mark Jerrum. "A very simple algorithm for estimating the number of k-colorings of a low-degree graph". In: *Random Structures & Algorithms* 7.2 (1995), pp. 157–165. DOI: https://doi.org/10.1002/rsa.3240070205. eprint: https://onlinelibrary.wiley.com/doi/pdf/10.1002/rsa.3240070205 (cit. on pp. 10, 27, 29).

[Joh96] A. Johansson. "Asymptotic choice number for triangle free graphs". Unpublished manuscript. 1996 (cit. on p. 27).

[JS89] Mark Jerrum and Alistair Sinclair. "Approximating the Permanent". In: *SIAM Journal on Computing* 18.6 (1989), pp. 1149–1178. DOI: 10.1137/0218077. eprint: https://doi.org/10.1137/0218077 (cit. on p. 65).

[Kar72] Richard M. Karp. "Reducibility among Combinatorial Problems". In: *Complexity of Computer Computations: Proceedings of a symposium on the Complexity of Computer Computations, held March 20–22, 1972, at the IBM Thomas J. Watson Research Center, Yorktown Heights, New York, and sponsored by the Office of Naval Research, Mathematics Program, IBM World Trade Corporation, and the IBM Research Mathematical Sciences Department*. Ed. by Raymond E. Miller, James W. Thatcher, and Jean D. Bohlinger. Boston, MA: Springer US, 1972, pp. 85–103. ISBN: 978-1-4684-2001-2. DOI: 10.1007/978-1-4684-2001-2_9 (cit. on p. 68).

[KO18] Tali Kaufman and Izhar Oppenheim. "Construction of new local spectral high dimensional expanders". In: *STOC*. Ed. by Ilias Diakonikolas, David Kempe, and Monika Henzinger. ACM, 2018, pp. 773–786. DOI: 10.1145/3188745.3188782 (cit. on pp. 5, 12, 19, 21, 51).

[KT21] Tali Kaufman and Ran J. Tessler. "New cosystolic expanders from tensors imply explicit Quantum LDPC codes with $\Omega(\sqrt{n} log^k n)$ distance". In: *Proceedings of the 53rd Annual ACM SIGACT Symposium on Theory of Computing*. STOC 2021. Virtual, Italy: Association for Computing Machinery, 2021, pp. 1317–1329. ISBN: 9781450380539. DOI: 10.1145/3406325.3451029 (cit. on p. 4).

[Liu+23] Siqi Liu, Sidhanth Mohanty, Tselil Schramm, and Elizabeth Yang. "Local and Global Expansion in Random Geometric Graphs". In: *Proceedings of the 55th Annual ACM Symposium on Theory of Computing*. STOC 2023. Orlando, FL, USA: Association for Computing Machinery, 2023, pp. 817–825. ISBN: 9781450399135. DOI: 10.1145/3564246.3585106 (cit. on p. 62).

[Liu+24] Kuikui Liu, Sidhanth Mohanty, Amit Rajaraman, and David X. Wu. "Fast Mixing in Sparse Random Ising Models*". In: *2024 IEEE 65th Annual Symposium on Foundations of Computer Science (FOCS)*. Los Alamitos, CA, USA: IEEE Computer Society, Oct. 2024, pp. 120–128. DOI: 10.1109/FOCS61266.2024.00018 (cit. on p. 6).

[Liu21] Kuikui Liu. "From Coupling to Spectral Independence and Blackbox Comparison with the Down-Up Walk". In: *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (APPROX/RANDOM 2021)*. Ed. by Mary Wootters and Laura Sanità. Vol. 207. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2021, 32:1–32:21. ISBN: 978-3-95977-207-5. DOI: 10.4230/LIPIcs.APPROX/RANDOM.2021.32 (cit. on p. 6).

[LLR95] Nathan Linial, Eran London, and Yuri Rabinovich. "The geometry of graphs and some of its algorithmic applications". In: *Combinatorica* 15 (1995), pp. 215–245. DOI: 10.1007/BF01200757 (cit. on p. 4).

[Löw34] Karl Löwner. "Über monotone Matrixfunktionen". In: *Mathematische Zeitschrift* 38.1 (1934), pp. 177–216. DOI: 10.1007/BF01170633 (cit. on p. 17).

[LPW17] David A. Levin, Yuval Peres, and Elizabeth L. Wilmer. *Markov Chains and Mixing Times*. 2nd ed. American Mathematical Society, 2017 (cit. on p. 18).

[LSS19] Jingcheng Liu, Alistair Sinclair, and Piyush Srivastava. "A Deterministic Algorithm for Counting Colorings with 2-Delta Colors". In: *2019 IEEE 60th Annual Symposium on Foundations of Computer Science (FOCS)*. 2019, pp. 1380–1404. DOI: 10.1109/FOCS.2019.00085 (cit. on p. 29).

[LSV05a] Alexander Lubotzky, Beth Samuels, and Uzi Vishne. "Explicit constructions of Ramanujan complexes of type Ãd". In: *European Journal of Combinatorics* 26.6 (2005). Combinatorics and Representation Theory, pp. 965–993. ISSN: 0195-6698. DOI: https://doi.org/10.1016/j.ejc.2004.06.007 (cit. on p. 51).

[LSV05b] Alexander Lubotzky, Beth Samuels, and Uzi Vishne. "Ramanujan complexes of type A d". In: *Israel journal of Mathematics* 149 (2005), pp. 267–299 (cit. on pp. 12, 51).

[Lub12] Alexander Lubotzky. "Expander graphs in pure and applied mathematics". In: *Bulletin of the American Mathematical Society* 49.1 (2012), pp. 113–162 (cit. on p. 4).

[LY13] Pinyan Lu and Yitong Yin. "Improved FPTAS for Multi-spin Systems". In: *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques*. Ed. by Prasad Raghavendra, Sofya Raskhodnikova, Klaus Jansen, and José D. P. Rolim. Berlin, Heidelberg: Springer Berlin Heidelberg, 2013, pp. 639–654. ISBN: 978-3-642-40328-6 (cit. on p. 29).

[Met+53] Nicholas Metropolis, Arianna W Rosenbluth, Marshall N Rosenbluth, Augusta H Teller, and Edward Teller. "Equation of state calculations by fast computing machines". In: *The journal of chemical physics* 21.6 (1953), pp. 1087–1092 (cit. on p. 6).

[Mol04] Michael Molloy. "The Glauber Dynamics on Colorings of a Graph with High Girth and Maximum Degree". In: *SIAM Journal on Computing* 33.3 (2004), pp. 721–737. DOI: 10.1137/S0097539702401786. eprint: https://doi.org/10.1137/S0097539702401786 (cit. on pp. 10, 27).

[MSW04] Fabio Martinelli, Alistair Sinclair, and Dror Weitz. "Fast mixing for independent sets, colorings and other models on trees". In: *Proceedings of the Fifteenth Annual ACM-SIAM Symposium on Discrete Algorithms*. SODA '04. New Orleans, Louisiana: Society for Industrial and Applied Mathematics, 2004, pp. 456–465. ISBN: 089871558X (cit. on p. 28).

[MV89] M. Mihail and U. Vazirani. "On the expansion of 0/1 polytopes". In: *Journal of Combinatorial Theory*. B (1989) (cit. on pp. 64, 67).

[OP22] Ryan O'Donnell and Kevin Pratt. "High-Dimensional Expanders from Chevalley Groups". In: *37th Computational Complexity Conference (CCC 2022)*. Ed. by Shachar Lovett. Vol. 234. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2022, 18:1–18:26. ISBN: 978-3-95977-241-9 (cit. on pp. 12, 51).

[Opp18] Izhar Oppenheim. "Local spectral expansion approach to high dimensional expanders part I: Descent of spectral gaps". In: *Discrete and Computational Geometry* 59.2 (2018), pp. 293–330 (cit. on pp. 7, 8, 19, 21, 53).

[Sam05] Marcus D. Sammer. "Aspects of Mass Transportation in Discrete Concentration Inequalities". PhD thesis. Georgia Institute of Technology, Apr. 2005 (cit. on p. 18).

[Sly10] Allan Sly. "Computational Transition at the Uniqueness Threshold". In: *Proceedings of the 2010 IEEE 51st Annual Symposium on Foundations of Computer Science*. FOCS '10. USA: IEEE Computer Society, 2010, pp. 287–296. ISBN: 9780769542447. DOI: 10.1109/FOCS.2010.34 (cit. on p. 103).

[Sok01] Alan D. Sokal. "Bounds on the Complex Zeros of (Di)Chromatic Polynomials and Potts-Model Partition Functions". In: *Comb. Probab. Comput.* 10.1 (Jan. 2001), pp. 41–77. ISSN: 0963-5483 (cit. on p. 29).

[SS14] Allan Sly and Nike Sun. "Counting in two-spin models on d-regular graphs". In: *The Annals of Probability* 42.6 (2014), pp. 2383–2416. DOI: 10.1214/13-AOP888 (cit. on p. 103).

[SS96] M. Sipser and D.A. Spielman. "Expander codes". In: *IEEE Transactions on Information Theory* 42.6 (1996), pp. 1710–1722. DOI: 10.1109/18.556667 (cit. on p. 4).

[SS97] Jesús Salas and Alan D. Sokal. "Absence of phase transition for antiferromagnetic Potts models via the Dobrushin uniqueness theorem". In: *Journal of Statistical Physics* 86.3 (1997), pp. 551–579 (cit. on p. 29).

[Sta07] Richard P. Stanley. "An Introduction to Hyperplane Arrangements". In: 2007 (cit. on pp. 13, 65).

[Sta73] Richard P. Stanley. "Acyclic orientations of graphs". In: *Discrete Mathematics* 5.2 (1973), pp. 171–178. ISSN: 0012-365X. DOI: https://doi.org/10.1016/0012-365X(73)90108-8 (cit. on pp. 13, 64).

[Vig99] E. Vigoda. "Improved bounds for sampling colorings". In: *40th Annual Symposium on Foundations of Computer Science (Cat. No.99CB37039)*. 1999, pp. 51–59. DOI: 10.1109/SFFCS.1999.814577 (cit. on pp. 10, 27, 29).

[Wei04] Dror Weitz. "Mixing in Time and Space for Discrete Spin Systems". AAI3147046. PhD thesis. 2004. ISBN: 0-496-05573-9 (cit. on p. 29).

[WZZ24] Yulin Wang, Chihao Zhang, and Zihan Zhang. "Sampling Proper Colorings on Line Graphs Using $(1+o(1))\Delta$ Colors". In: *Proceedings of the 56th Annual ACM Symposium on Theory of Computing*. STOC 2024. Vancouver, BC, Canada: Association for Computing Machinery, 2024, pp. 1688–1699. ISBN: 9798400703836. DOI: 10.1145/3618260.3649724 (cit. on pp. 4, 6, 10, 29).

[Yin14] Yitong Yin. "Spatial Mixing of Coloring Random Graphs". In: *Automata, Languages, and Programming*. Ed. by Javier Esparza, Pierre Fraigniaud, Thore Husfeldt, and Elias Koutsoupias. Berlin, Heidelberg: Springer Berlin Heidelberg, 2014, pp. 1075–1086 (cit. on p. 28).

Appendix A

Remaining Proofs from Section on Sampling Vertex Colorings

A.1 Proof of Theorem 3.3.1

Proof. For each τ of co-dimension at least 2, let $F_\tau \in \mathbb{R}^{X(0)\times X(0)}$ be a diagonal matrix supported on $X_\tau(0) \times X_\tau(0)$ defined as follows: for any $vc \in X_\tau(0)$,

$$F_\tau(vc, vc) := \begin{cases} 0 & \text{if } \Delta_\tau(v) = 0, \\ f_1(\Delta_\tau(u)) & \text{if } \Delta_\tau(v) = 1 \text{ and } u \sim_\tau v, \\ f_2(\Delta_\tau(v)) & \text{if } \Delta_\tau(v) \geq 2, \end{cases}$$

where $f_1(i) = \frac{1}{(1+\epsilon)\Delta} + \frac{1+2\sum_{j=1}^{i-1}\frac{1}{j}}{(1+\epsilon)^2\Delta^2}$ for $i \geq 1$ and $f_2(i) = \frac{i}{(1+\frac{\epsilon}{2})\Delta-(i-1)-\frac{4}{\epsilon}\sum_{j=1}^{i-1}\frac{1}{j}}$ for $i \geq 2$. We show that the conditions of Proposition 3.3.3 hold for $\{F_\tau\}_{\tau\in X:\mathrm{codim}(\tau)\geq 2}$. Then, the statement follows from the fact that $\rho(F_\tau) \leq \frac{5}{2\epsilon}$. This is true because $\frac{\ln(\Delta)+2}{\Delta} \leq \frac{\epsilon^2}{40}$ implies that for any $1 \leq i \leq \Delta$ the denominator of $f_2(i)$ is at least $\frac{2\epsilon}{5}\Delta$ and thus $f_1(i) \leq f_2(i) \leq \frac{5}{2\epsilon}$.

The condition for links of co-dimension 2 holds by definition. Assume τ is of co-dimension at least 3. When G_τ is disconnected one can check the condition holds because of the fact that the degrees of vertices of a connected component do not change by removing vertices from other connected components of the graph. Now, assume that G_τ is connected. Note that for every $v \in V_\tau$, we have

$$\frac{(k-1)^2}{3k-1} \geq \frac{\Delta_\tau(v)}{5} \underset{\frac{\ln(\Delta)+2}{\Delta}\leq\frac{\epsilon^2}{40}}{\geq} \frac{8}{\epsilon^2} \underset{f_2(\Delta_\tau(v))\leq\frac{5}{2\epsilon},\epsilon\leq 1}{\geq} f_2(\Delta_\tau(v)) \geq f_1(\Delta_\tau(v)).$$

Therefore, it is enough to show that $\sum_{uc'\in X_{\tau\cup vc}(0)} p(uc'|\tau \cup vc)F_{\tau\cup uc'}(vc, vc) \preceq (k-2)F_\tau(vc) - F_\tau^2(vc)$ for any $vc \in X_\tau(0)$.

Case 1: $\Delta_\tau(v) = 1$, and $u \sim_\tau v$. Since G_τ is connected and τ is of co-dimension at least 3, $\Delta_\tau(u) \geq 2$. We have

$$\sum_{wc'\in X_\tau(0)} p(wc'|\tau \cup vc)F_{\tau\cup wc'}(vc, vc) = (\Delta_\tau(u)-1)f_1(\Delta_\tau(u)-1) + (k-\Delta_\tau(u)-1)f_1(\Delta_\tau(u)).$$

On the other hand, $\frac{\ln(\Delta)+2}{\Delta} \le \frac{\epsilon^2}{40}$ and $\epsilon \le 1$ imply that $\frac{1+2\sum_{j=1}^{i-1}\frac{1}{j}}{(1+\epsilon)\Delta} \le \frac{1}{20(1+\epsilon)}$ for any $1 \le i \le \Delta$. Therefore,

$$F_\tau^2(vc,vc) = f_1^2(\Delta_\tau(u)) \le \left(\frac{1}{(1+\epsilon)\Delta} + \frac{1}{20(1+\epsilon)\Delta}\right)^2 \le \frac{2}{(1+\epsilon)^2\Delta^2}.$$

Therefore $(k-2)F_\tau(vc) - F_\tau^2(vc) \ge (k-2)f_1(\Delta_\tau(u)) - \frac{2}{(1+\epsilon)^2\Delta^2}$ and thus it is enough to show that

$$(\Delta_\tau(u)-1)\left(f_1\left(\Delta_\tau(u)\right) - f_1\left(\Delta_\tau(u)-1\right)\right) \ge \frac{2}{(1+\epsilon)^2\Delta^2}.$$

But this inequality holds with equality.

Case 2: $\Delta_\tau(v) \ge 2$**.** One can check that $\frac{\ln(\Delta)+1}{\Delta} \le \frac{\epsilon^2}{40}$ implies $\frac{1}{(1+\epsilon)\Delta} + \frac{2\sum_{j=1}^{\Delta-1}\frac{1}{j}+1}{(1+\epsilon)^2\Delta^2} \le \frac{1}{(1+\frac{\epsilon}{2})\Delta}$. For convenience define $f_2(1) = \frac{1}{1+\frac{\epsilon}{2}}$ and notice that $f_1(i) \le f_2(1)$ for any $1 \le i \le \Delta$. We want to show that

$$\begin{aligned}\sum_{uc' \in X_\tau(0)} p(uc'|\tau \cup vc)F_{\tau\cup uc'}(vc,vc) &= \Delta_\tau(v)f_2(\Delta_\tau(v)-1) + (k-\Delta_\tau(v)-1)f_2(\Delta_\tau(v)) \\ &\le (k-2)f_2(\Delta_\tau(v)) - f_2^2(\Delta_\tau(v)).\end{aligned}$$

Let $i \coloneqq \Delta_\tau(v)$. We have

$$(i-1)f_2(i) - if_2(i-1) = \frac{i(i-1)+\frac{4}{\epsilon}i}{\left((1+\frac{\epsilon}{2})\Delta - (i-1) - \frac{4}{\epsilon}\sum_{j=1}^{i-1}\frac{1}{j}\right)\left((1+\frac{\epsilon}{2})\Delta - (i-2) - \frac{4}{\epsilon}\sum_{j=1}^{i-2}\frac{1}{j}\right)}.$$

Therefore

$$(i-1)f_2(i) - if_2(i-1) - f_2^2(i) = \frac{(\frac{4}{\epsilon}-1)i\left((1+\frac{\epsilon}{2})\Delta - (i-1) - \frac{4}{\epsilon}\sum_{j=1}^{i-1}\frac{1}{j}\right) - i^2(1+\frac{4}{\epsilon}\frac{1}{i-1})}{\left((1+\frac{\epsilon}{2})\Delta - (i-1) - \frac{4}{\epsilon}\sum_{j=1}^{i-1}\frac{1}{j}\right)^2\left((1+\frac{\epsilon}{2})\Delta - (i-2) - \frac{4}{\epsilon}\sum_{j=1}^{i-2}\frac{1}{j}\right)}.$$

The denominator is positive for $1 \le i \le \Delta$ and for the numerator we have

$$(\frac{4}{\epsilon}-1)i\left((1+\frac{\epsilon}{2})\Delta - (i-1) - \frac{4}{\epsilon}\sum_{j=1}^{i-1}\frac{1}{j}\right) - i^2(1+\frac{4}{\epsilon}\frac{1}{i-1}) \underset{i\le\Delta}{\ge} (1-\frac{\epsilon}{2})i\Delta - \frac{16i}{\epsilon^2}(\ln(\Delta)+1) - \frac{4i^2}{\epsilon(i-1)}$$

Canceling out an i, and using $\frac{\ln(\Delta)+2}{\Delta} \le \frac{\epsilon^2}{40}$ and $\epsilon \le 1$ the RHS is non-negative. □

A.2 Proof of Theorem 3.3.4

Proof. As before we make the tree rooted at an arbitrary vertex. For any $k \ge 2$, any τ of co-dimension k, let $F_\tau \in \mathbb{R}^{X(0)\times X(0)}$ be a diagonal matrix supported on $X_\tau(0) \times X_\tau(0)$ defined as follows: for any $vc \in X_\tau(0)$,

$$F_\tau(vc,vc) \coloneqq \begin{cases} 0 & \text{if } \Delta_\tau(v) = 0, \\ f_1(\Delta_\tau(u)) & \text{if } \Delta_\tau(v) = 1,\ u \sim v, \text{ and } v \text{ is the root of a component of } G_\tau, \\ f_2(\Delta_\tau(v)) & \text{if } \Delta_\tau(v) \ge 2 \text{ and } v \text{ is the root of a component of } G_\tau, \\ f_3(\Delta_\tau(v), \Delta_\tau(a(v))) & \text{if } \Delta_\tau(v) \ge 1 \text{ and } a(v) \in V_\tau, \end{cases}$$

where $a(v)$ is the immediate ancestor of v in G, and $f_1(i) \coloneqq \frac{5(\sum_{j=1}^{i-1}\frac{1}{j})+1}{\epsilon^2\Delta^2}$ for $i \geq 1$, $f_2(i) \coloneqq \frac{5(\ln(\Delta)+1+i\sum_{j=1}^{i-1}\frac{1}{j})}{\epsilon^2\Delta^2}$ for $i \geq 2$, and $f_3(i,j) \coloneqq \frac{5(\ln(\Delta)+1+j+i\sum_{k=1}^{i-1}\frac{1}{k}}{\epsilon^2\Delta^2}$ for $i, j \geq 1$. We prove that the conditions of Proposition 3.3.6 hold for $\{F_\tau\}_{\tau\in X:\mathrm{codim}(\tau)\geq 2}$. Then, the statement follows from the fact that

$$\lambda_2(P_\tau) \leq \frac{\rho(F_\tau + A_\tau)}{k-1} \leq \frac{\rho(F_\tau)+\frac{1}{\epsilon}}{k-1} + \frac{\frac{1}{20}+\frac{1}{\epsilon}}{k-1},$$

where the second to last inequality follows by (A.1) below and the fact that every entry of A_τ is at most $\frac{1}{\epsilon\Delta}$ and that every row of A_τ has at most Δ non-zero entries. The last inequality uses that since $\frac{\ln^2(\Delta)}{\Delta} \leq \frac{\epsilon^2}{100}$ and $\epsilon \leq 1$, for all $1 \leq i \leq \Delta$

$$f_1(i), f_2(i), f_3(i) \leq \frac{5(\ln(\Delta)+2)}{\epsilon^2\Delta} \leq \frac{1}{20}. \tag{A.1}$$

The condition for links of co-dimension 2 holds by definition. Assume that τ is of co-dimension $k \geq 3$. Similar to the proof of Theorem 3.3.1, when G_τ is disconnected, the condition holds. Now, assume G_τ is connected. By Eq. (A.1), for all $vc \in X_\tau(0)$ $F_\tau(vc,vc) \leq \frac{1}{20} \leq \frac{(k-1)^2}{3k-1} - \frac{1}{\beta}$. Therefore, it is enough to show that for any $vc \in X_\tau(0)$,

$$\sum_{wc' \in X_{\tau\cup vc}(0)} p(wc'|\tau \cup vc) F_{\tau\cup wc'}(vc,vc) \preceq (k-2)F_\tau(vc,vc) - 2F_\tau^2(vc,vc) - \gamma_\tau(vc),$$

for $\gamma_\tau(vc)$ defined in Proposition 3.3.6.

Case 1: $\Delta_\tau(v) = 1$ and v is the root of a component of G_τ. Let u be the only neighbor of v. Since G_τ is connected and τ is of co-dimension at least 3, $\Delta_\tau(u) \geq 2$. We need to show that

$$\sum_{wc' \in X_{\tau\cup vc}(0)} p(wc'|\tau\cup vc)F_{\tau\cup wc'}(vc,vc) = (\Delta_\tau(u)-1)f_1(\Delta_\tau(u)-1) + (k-\Delta_\tau(u)-1)f_1(\Delta_\tau(u)) \tag{A.2}$$

$$\leq (k-2)f_1(\Delta_\tau(u)) - 2f_1^2(\Delta_\tau(u)) - \frac{4}{\epsilon^2\Delta^2}. \tag{A.3}$$

But,

$$\begin{aligned}&(k-2)f_1(\Delta_\tau(u)) - (\Delta_\tau(u)-1)f_1(\Delta_\tau(u)-1) - (k-\Delta_\tau(u)-1)f_1(\Delta_\tau(u))\\ &= (\Delta_\tau(u)-1)(f_1(\Delta_\tau(u)) - f_1(\Delta_\tau(u)-1)) = \frac{5}{\epsilon^2\Delta^2}.\end{aligned}$$

Furthermore, one can see that $\frac{\epsilon^2}{100}$ implies that $2f_1^2(\Delta_\tau(u)) \leq \frac{50(\ln(\Delta)+2)^2}{\epsilon^4\Delta^4} \leq \frac{1}{\epsilon^2\Delta^2}$. This completes the proof of Eq. (A.3).

Case 2: $\Delta_\tau(v) \geq 2$, and v is the root of a component of G_τ. Note that f_1 is bounded above by $\frac{5(\ln(\Delta)+1)}{\epsilon^2\Delta^2}$. For convenience in writing the recursion, let $f_2(1) = \frac{5(\ln(\Delta)+1)}{\epsilon^2\Delta^2}$. Following similar calculations, it is enough to show that

$$(\Delta_\tau(v)-1)f_2(\Delta_\tau(v)) - (\Delta_\tau(v))f_2(\Delta_\tau(v)-1) \geq 2f_2^2(\Delta_\tau(v) + \frac{4\Delta_\tau(v)}{\epsilon^2\Delta^2}. \tag{A.4}$$

But one can see that, by definition

$$(\Delta_\tau(v)-1)f_2(\Delta_\tau(v))-(\Delta_\tau(v))f_2(\Delta_\tau(v)-1)=\frac{5\Delta_\tau(v)}{\epsilon^2\Delta^2},$$

And,

$$2f_2^2(\Delta_\tau(v))\leq\frac{50(\Delta_\tau(v)+2)^2\ln^2(\Delta)}{\epsilon^4\Delta^4}\underset{\Delta\geq100}{\leq}\frac{55\Delta_\tau^2(v)\ln^2(\Delta)}{\epsilon^4\Delta^4}\underset{\frac{\ln^2(\Delta)}{\Delta}\leq\frac{\epsilon^2}{100}}{\leq}\frac{0.55\Delta_\tau^2(v)}{\epsilon^2\Delta^3}\leq\frac{\Delta_\tau(v)}{\epsilon^2\Delta^2}. \tag{A.5}$$

This finishes the proof of Eq. (A.4).

Case 3: $\Delta_\tau(v)\geq1$ and $a(v)\in V_\tau$. For convenience in writing the recursion, for $1\leq i\leq\Delta$, let $f_3(i,0)=\frac{5(\ln(\Delta)+1+i\sum_{k=1}^{i-1}\frac{1}{k})}{\epsilon^2\Delta^2}$ and note that $\max_{1\leq j\leq\Delta}f_1(j)\leq f_2(i)\leq f_3(i,0)$. Similar to the previous cases, after simplifying the recursion, one can see that it is enough that for $i=\Delta_\tau(v)$ and $j=\Delta_\tau(a(v))$

$$(i-1)f_3(i,j)-if_3(i-1,j)+(j-1)(f_3(i,j)-f_3(i,j-1))\geq2f_3^2(i,j)+\frac{4i}{\epsilon^2\Delta^2}+\frac{4(j-1)}{\epsilon^2\Delta^2}.$$

Now, note that

$$(i-1)f_3(i,j)-if_3(i-1,j)+(j-1)(f_3(i,j)-f_3(i,j-1))=\frac{5i}{\epsilon^2\Delta^2}+\frac{5(j-1)}{\epsilon^2\Delta^2}.$$

Furthermore,

$$2f_3^2(i)=2(f_2^2(i)+\frac{5(j-1)}{\epsilon^2\Delta^2})^2\leq2.5f_2^2(i)+\frac{250(j-1)^2}{\epsilon^4\Delta^4}\underset{\Delta\geq100\epsilon^{-2}}{\leq}\frac{i}{\epsilon^2\Delta^2}+\frac{j-1}{\epsilon^2\Delta^2},$$

where the last inequality uses the calculations in Eq. (A.5). This finishes the proof. □

Appendix B

Remaining Proof from Section on Non-Broken Bases of Matroids

B.1 Proof of Theorem 5.2.4

In this section we prove Theorem 5.2.4. We use a reduction from the problem of computing the partition function of the Hardcore model when the fugacity is above the critical threshold. Define $\sharp\mathrm{HC}(\Delta, \lambda)$ as follows: given a Δ-regular graph $G = (V, E)$, compute the partition function $Z_G(\lambda) = \sum_I \lambda^{|I|}$, where the sum is taken over the family of independent sets $I \subseteq V$ of G. The critical threshold is defined as $\lambda_c(\Delta) := \frac{(\Delta-1)^{\Delta-1}}{(\Delta-2)^{\Delta}}$.

Theorem B.1.1 ([Sly10; SS14; GŠV16]). *The following holds for any fixed $\epsilon > 0$, integer $\Delta \geq 3$ and $\lambda > \lambda_c(\Delta)$: unless NP=RP, for any $\lambda > \lambda_c(\Delta)$ there is no polynomial-time algorithm for for approximating $\sharp\mathrm{HC}(\Delta, \lambda)$ up to a $1 + \epsilon$ multiplicative factor.*

We give a polynomial-time algorithm that given a $e^{\pm\epsilon/2}$-approximation for $\sharp\mathrm{INDEP\text{-}SET\text{-}INC}(7, \frac{2}{19})$ (see Definition 13), approximates $\sharp\mathrm{HC}(7, \frac{2}{3})$ up to a $e^{\pm\epsilon}$-multiplicative error. Since $\frac{2}{3} > \lambda_c(7) = \frac{6^6}{5^7} \geq 0.6$, this finishes the proof of Theorem 5.2.4. Our reduction is a modification of Theorem 16 in [DP21].

Theorem B.1.2. *There exists a polynomial-time algorithm that for any given $\epsilon \leq 1$, satisfies the following properties:*

1. *Given an instance $G = (V, E)$ of $\sharp\mathrm{HC}(7, \frac{2}{3})$, the algorithm constructs an instance $G' = (V', E')$ of the problem $\sharp\mathrm{INDEP\text{-}SET\text{-}INC}(7, \frac{2}{19})$ with size polynomial in $|G|$.*
2. *Given a $e^{\pm\epsilon/2}$-multiplicative approximation to the number of independent sets of size $\lfloor \frac{2|V'|}{19} \rfloor$ of G', a $e^{\pm\epsilon}$-approximation of $Z_G(\frac{2}{3})$ can be computed in polynomial time.*

Proof. Given a 7-regular graph $G = (V, E)$, we define G' as the disjoint union of G with $r := \frac{c^2 n^2}{\epsilon}$ copies of the complete graph K_8, where $n = |V|$, for some $c > 1$ that we choose later. For simplicity of notation, let $N := |V'| = n + 8r$, $\alpha := \frac{2}{19}$, $\lambda := \frac{2}{3}$. It is enough to show that G' is an instance of $\sharp\mathrm{INDEP\text{-}SET\text{-}INC}(7, \frac{2}{19})$ and

$$e^{-\epsilon/2} \frac{i_{\lfloor \alpha N \rfloor}(G')}{\binom{r}{\lfloor \alpha N \rfloor} 8^{\lfloor \alpha N \rfloor}} \leq Z_G(\lambda) \leq e^{\epsilon/2} \frac{i_{\lfloor \alpha N \rfloor}(G')}{\binom{r}{\lfloor \alpha N \rfloor} 8^{\lfloor \alpha N \rfloor}}, \tag{B.1}$$

where as usual $i_k(G)$ is the number of independent sets of size k in G, and

$$\binom{r}{\lfloor \alpha N \rfloor} 8^{\lfloor \alpha N \rfloor} = i_{\lfloor \alpha N \rfloor}(rK_8).$$

Here, rK_8 is a shorthand for the graph which is a disjoint union of r copies of K_8. We first show that Eq. (B.1) holds. Note that

$$i_{\lfloor \alpha N \rfloor}(G') = \sum_{j=0}^{n} i_j(G) i_{\lfloor \alpha N \rfloor - j}(rK_8) = i_{\lfloor \alpha N \rfloor}(rK_8) \sum_{j=0}^{n} i_j(G) \frac{i_{\lfloor \alpha N \rfloor - j}(rK_8)}{i_{\lfloor \alpha N \rfloor}(rK_8)}.$$

Thus, to show Eq. (B.1), it is enough to prove that for every $1 \leq j \leq n$,

$$e^{-\epsilon/2} \cdot \frac{i_{\lfloor \alpha N \rfloor - j}(rK_8)}{i_{\lfloor \alpha N \rfloor}(rK_8)} \leq \lambda^j \leq e^{\epsilon/2} \cdot \frac{i_{\lfloor \alpha N \rfloor - j}(rK_8)}{i_{\lfloor \alpha N \rfloor}(rK_8)}. \tag{B.2}$$

We can write

$$\frac{i_{\lfloor \alpha N \rfloor - j}(rK_8)}{i_{\lfloor \alpha N \rfloor}(rK_8)} = \frac{\binom{r}{\lfloor \alpha N \rfloor - j} 8^{\lfloor \alpha N \rfloor - j}}{\binom{r}{\lfloor \alpha N \rfloor} 8^{\lfloor \alpha N \rfloor}} = \frac{1}{8^j} \prod_{i=0}^{j-1} \frac{\lfloor \alpha N \rfloor - i}{r - \lfloor \alpha N \rfloor + j - i}. \tag{B.3}$$

To prove the upper bound, first note that

$$\frac{\alpha N}{r - \alpha N + j} \underset{\alpha N \geq 8\alpha r}{\geq} \frac{8\alpha r}{r(1 - 8\alpha) + n} \underset{\substack{n = \sqrt{\epsilon r}/c \\ \alpha = 2/19}}{=} \frac{16}{3} \left(\frac{1}{1 + \frac{19\sqrt{\epsilon}}{3c\sqrt{r}}} \right). \tag{B.4}$$

This implies that $\frac{\alpha N}{r - \alpha N + j} \geq 1$. So, $\frac{\alpha N}{r - \alpha N + j} \leq \frac{\lfloor \alpha N \rfloor - i}{r - \lfloor \alpha N \rfloor + j - i}$ for every $i < r - \lfloor \alpha N \rfloor + j$. Thus,

$$\frac{1}{8^j} \cdot \prod_{i=0}^{j-1} \frac{\lfloor \alpha N \rfloor - i}{r - \lfloor \alpha N \rfloor + j - i} \geq \frac{1}{8^j} \cdot \left(\frac{\alpha N}{r - \alpha N + j} \right)^j$$
$$\underset{\substack{\text{Eq. (B.4)} \\ j \leq n = \sqrt{\epsilon r}/c}}{\geq} \frac{1}{8^j} \cdot (\tfrac{16}{3})^j \left(\frac{1}{1 + \frac{19\sqrt{\epsilon}}{3c\sqrt{r}}} \right)^{\sqrt{\epsilon r}/c} \geq (\tfrac{2}{3})^j e^{-\epsilon/2} = \lambda^j e^{-\epsilon/2},$$

for a large enough $c > 1$. Combining this with Eq. (B.3), we get the upper bound in Eq. (B.2).

To prove the lower bound, note that

$$\frac{1}{8^j} \cdot \prod_{i=0}^{j-1} \frac{\lfloor \alpha N \rfloor - i}{r - \lfloor \alpha N \rfloor + j - i} \underset{j - i \geq 0}{\leq} \frac{1}{8^j} \cdot \left(\frac{\lfloor \alpha N \rfloor}{r - \lfloor \alpha N \rfloor} \right)^j \underset{\lfloor \alpha N \rfloor = \lfloor \frac{16r}{19} + \frac{2\sqrt{\epsilon r}}{19c} \rfloor}{\leq} \frac{1}{8^j} \cdot \left(\frac{\frac{16r}{19}(1 + \frac{\sqrt{\epsilon}}{8c\sqrt{r}})}{\frac{3r}{19}(1 - \frac{2\sqrt{\epsilon}}{3c\sqrt{r}})} \right)^j$$
$$\leq (\tfrac{2}{3})^j e^{\epsilon/2} = \lambda^j \cdot e^{\epsilon/2},$$

for a large enough $c > 1$. Combining this with Eq. (B.3), the lower bound in Eq. (B.2), thus (B.1) follows.

It remains to show that G' is an instance of $\sharp$INDEP-SET-INC$(7, \frac{2}{19})$, i.e. $i_k(G') \leq i_{\lfloor \alpha N \rfloor}(G')$ for any $k < \lfloor \alpha N \rfloor$. For any $k < \lfloor \alpha N \rfloor$, and any independent set S in the original graph G, let $T_{S,k}$ be the set of all independent sets of size k of G' whose intersection with the vertices of G is S. It is enough to show that there exists a constant n_0 such that if $n \geq n_0$, then we have $|T_{S,k}| \leq |T'_{S,\lfloor \alpha N \rfloor}|$ for every independent set $S \subseteq V$ of G and $k < \lfloor \alpha N \rfloor$. We prove a stronger statement that there exists a constant n_0 such that if $n \geq n_0$, then for any fixed independent set $S \subseteq V$, $|T_{S,k}|$ is increasing as a function of k for all $k \leq \lfloor \alpha N \rfloor$. It is enough to show that $\frac{|T_{S,k}|}{|T_{S,k-1}|} \geq 1$ for any $|S| \leq k \leq \alpha N$. Note that $|T_{S,k}| = \binom{r}{k-|S|} 8^{k-|S|}$. So we have

$$\frac{|T_{S,k}|}{|T_{S,k-1}|} = \frac{\binom{r}{k-|S|} 8^{k-|S|}}{\binom{r}{k-1-|S|} 8^{k-1-|S|}} = 8 \cdot \frac{r - k + |S| + 1}{k - |S|} \geq 8 \cdot \frac{r-k}{k} \geq 8 \frac{\frac{3r}{19} - n}{\frac{16r}{19} + n},$$

where the last inequality comes from the fact that $k \leq \frac{2N}{19} = \frac{2}{19}(8r + n) \leq \frac{16r}{19} + n$. But since $r = \frac{c^2 n^2}{\epsilon}$, there is a constant n_0 such that for $n \geq n_0$, we have $\frac{\frac{3r}{19} - n}{\frac{16r}{19} + n} \geq \frac{1}{8}$. This shows that $\frac{|T_{S,k}|}{|T_{S,k-1}|} \geq 1$, which finishes the proof. $\square$

Appendix C

Entropy as a Coverage Function

C.1 Proof of Proposition 6.1.22

Proof. We use Proposition 6.1.21. For any $T \subseteq [n]$, define $x_T := I(Y_T|Y_{\bar{T}})$, where $Y_{\bar{T}} = [n] \setminus T$ and I is the multivariate mutual information. We prove inductively that $H(Y_S|Z) = \sum_{T:T\cap S\neq\emptyset} I(Y_T|Y_{\bar{T}}, Z)$ for an arbitrary set of random variables Z. This would imply that for any $S \subseteq [n]$, $f(S) = H(Y_S) = \sum_{T:T\cap S\neq\emptyset} x_T$, which would finish the proof. When $n = 1$, the statement trivially holds. Assuming the statement is true for $n = k - 1$, we prove it for for $n = k$. First, let $S = \{i\}$. We have

$$\sum_{T:T\cap S\neq\emptyset} I(Y_T \mid Y_{\bar{T}}, Z) = \sum_{T:i\in T} I(Y_T \mid Y_{\bar{T}}, Z)$$
$$= (\sum_{T:i\in T,T\neq\{i\}} I(Y_{T\setminus\{i\}}|Y_{\bar{T}}, Z) - I(Y_{T\setminus\{i\}}|Y_{\bar{T}\cup\{i\}}), Z) + H(Y_i|Y_{\bar{i}}, Z),$$

where we used the fact that for any set of random variables $X_1, \ldots, X_k$, $k \geq 2$, and any set of random variables Z,

$$I(X_1, \ldots, X_k|Z) := I(X_1, \ldots, X_{k-1}|Z) - I(X_1, \ldots, X_{k-1}|X_k, Z).$$

Using induction hypothesis

$$\sum_{T:i\in T,T\neq\{i\}} I(Y_{T\setminus\{i\}}|Y_{\bar{T}}, Z) - I(Y_{T\setminus\{i\}}|Y_{\bar{T}\cup\{i\}}, Z)$$
$$= \sum_{T\subseteq[n]\setminus\{i\}:T\cap([n]\setminus\{i\})\neq\emptyset} I(Y_T|Y_{[n]\setminus(T\cup\{i\})}, Z) - I(Y_T|Y_{[n]\setminus T}|Z)$$
$$= H(Y_{[n]\setminus\{i\}}|Z) - H(Y_{[n]\setminus\{i\}}|Y_i, Z)$$
$$= H(Y_{[n]\setminus\{i\}}|Z) - H(Y_{[n]}|Z) + H(Y_i|Z)$$
$$= -H(Y_i|Y_{\bar{i}}, Z) + H(Y_i|Z)$$

Therefore,

$$\sum_{T:T\cap S\neq\emptyset} I(Y_T \mid Y_{\bar{T}}, Z) = H(Y_i|Z).$$

Now, assume that this equation holds for any S such that $|S| < l$. We want to show that it holds for $|S| = l$. Choose $i \in S$. We have

$$\sum_{T:T\cap S\neq\emptyset} I(Y_T \mid Y_{\bar{T}}, Z) = \sum_{T:T\cap(S\setminus\{i\})\neq\emptyset} I(Y_T|Y_{\bar{T}}, Z) + \sum_{T\cap S=\{i\}} I(Y_T|Y_{\bar{T}}, Z).$$

Note that $\sum_{T:T\cap(S\setminus\{i\})\neq\emptyset} I(Y_T|Y_{\bar{T}}, Z) = H(Y_{S\setminus\{i\}}|Z)$ by the second induction hypothesis. Furthermore,

$$\sum_{T\cap S=\{i\}} I(Y_T|Y_{\bar{T}}, Z) = (\sum_{T\cap S=\{i\},T\neq\{i\}} I(Y_{T\setminus\{i\}}|Y_{\bar{T}}, Z) - I(Y_{T\setminus\{i\}}|Y_{\bar{T}\cup\{i\}}, Z)) + H(Y_i|Y_{\bar{i}}).$$

Similar to what we did before, using the first induction hypothesis we get

$$\begin{aligned}\sum_{T\cap S=\{i\},T\neq\{i\}} I(Y_{T\setminus\{i\}}|Y_{\bar{T}}, Z) - I(Y_{T\setminus\{i\}}|Y_{\bar{T}\cup\{i\}}, Z) &= H(Y_{[n]\setminus S} \mid Y_{S\setminus\{i\}}, Z) - H(Y_{[n]\setminus S}|Y_S, Z) \\ &= H(Y_{[n]\setminus S} \mid Y_{S\setminus\{i\}}, Z) \\ &\quad - H(Y_{([n]\setminus S)\cup\{i\}} \mid Y_{S\setminus\{i\}}, Z) + H(Y_i \mid Y_{S\setminus\{i\}}, Z) \\ &= -H(Y_i|Y_{\bar{i}}, Z) + H(Y_i \mid Y_{S\setminus\{i\}}, Z).\end{aligned}$$

Therefore,

$$\sum_{T\cap S=\{i\}} I(Y_T|Y_{\bar{T}}, Z) = H(Y_i \mid Y_{S\setminus\{i\}}, Z).$$

So

$$\sum_{T:T\cap S\neq\emptyset} I(Y_T \mid Y_{\bar{T}}, Z) = H(Y_i \mid Y_{S\setminus\{i\}}, Z) + H(Y_{S\setminus\{i\}}, Z) = H(Y_S|Z),$$

as desired. □