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# Demographic Change, Social Security, and Lifetime Decision Making

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**Abstract**

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Professor Stephen Turnovsky

In the first chapter of my thesis, I build a two period OLG model with a realistic demographic structure, an endogenous retirement age, a state dependent utility from consumption specification, a unique and intuitive disutility of labor function, and a pay as you go (PAYG) social security system to investigate the impact of demographic change on lifetime decisions of consumption-saving and retirement. The demographic changes are in the form of mortality rates declining and compression of morbidity happening together. The demographic structure of the model is calibrated using the survival probability data of the United States in 2017. After numerically solving for the steady state, alternative social security mechanisms are offered and numerically evaluated to see whether they can improve the economic outcome in the face of demographic change. In addition to numerically solving the steady state of the model with this specification, I also characterize the consumption-saving behavior of both individuals and the cohorts along the balanced growth path in this chapter - before and after demographic change. In the second chapter of my thesis, I build on the study from the first chapter by removing the state dependent utility from consumption specification from the theoretical model to investigate the impact of including the state dependent utility from consumption function in the model. I repeat the numerical exercises in the first chapter by using the model without the state dependent utility from consumption function.

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## Chapter 1

# EFFECTS OF DEMOGRAPHIC CHANGES ON SAVING AND RETIREMENT AGE CHOICES IN THE PRESENCE OF STATE DEPENDENT UTILITY FROM CONSUMPTION AND AGE DEPENDENT DISUTILITY OF LABOR

### *1.1 Introduction*

Economic growth literature has been using "Overlapping Generations Models" (OLG) since the early seminal papers by Samuelson [66] and Diamond [32] to answer a wide array of questions. Later during 1980s<sup>1</sup>, economists increasingly incorporated demography as a factor in OLG models to explore even more complicated problems<sup>2</sup>.

In these early studies, rather unrealistic, ad-hoc, but effective assumptions, such as the constant mortality hazard assumption in [15], were used to introduce the demographic structure in OLG models. Since 2000s, researchers have been using more realistic theoretical survival functions to simulate demographic change. The parameters of these theoretical survival functions are commonly estimated using actual survival data, which is available free of charge today. After calibration, these studies conduct numerical simulations and offer policy suggestions whenever feasible.

However, when these more recent studies in the economic growth literature are investigated, it can be seen that most of them focus on fertility decline and / or mortality decline as sources of exogenous demographic change and don't consider morbidity at all. Morbidity is

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<sup>1</sup>Some of the seminal papers that included demography in OLG models during '80s were [15], [21], [74].

<sup>2</sup>Demographic structure of the society is inherently linked to its economic structure. Demography impacts long run economic growth rates, inequality of wealth and income distributions, the presence and choice of the society's social security net, intergenerational transfers, relative prices of factors of production and hence the income shares of the same factors, the effectiveness of monetary policy, the magnitude of the fiscal multiplier among many other economic variables.

the condition of being in a physically / mentally impaired situation. This, of course, impacts all imaginable lifetime decisions: to receive education or to join the labor force, to save or to consume, to keep on working or to retire among others.

Changing duration of lifetime spent in morbidity is a source of demographic change individuals face all over the world. As mentioned in the above paragraph, very few studies in the economic growth literature include changing periods of morbidity as a demographic factor. Also, to my knowledge, in those rare instances where morbidity is even considered as a source of demographic change, it only affects the disutility individuals suffer by participating in the labor market. To fill this gap in the literature, I build a two period OLG model with a realistic demographic structure, labor-leisure decision on the extensive margin, and a pay as you go (PAYG) social security system in this chapter. Two mechanisms are included in the model to formalize compression of morbidity: 1. a state dependent utility function which gives the agents a lower marginal utility from consumption when they are in the morbid state and 2. a unique disutility of labor specification which exponentially increases the disutility of labor as they spend more time in the morbid state.

After building the theoretical model with a generic survival function and analytically deriving the steady state conditions as much as possible, I discuss the pros and cons of alternative specific survival functions that can be used to introduce demographic change in the model. For reasons explained in the calibration section of this chapter, I choose the Boucekkine survival function ([18]) as the specific theoretical demographic structure in the model and calibrate its parameters by using the survival data of the United States in 2017. For comparison purposes, I repeat the same exercise with the Gompertz-Makeham survival function ([39] and [57]) as well.

Other important structural parameters of the model are calibrated using US life expectancy and retirement age data in 2017. After calibration, the time paths of individuals' and cohorts' consumption and asset holdings in the steady state of the model are calculated numerically and plotted.

Next, I numerically investigate the impact of different combinations of demographic

change (mortality decline by itself, compression of morbidity by itself, mortality decline and compression of morbidity happening together)<sup>3</sup> on individual decisions of retirement and saving. After obtaining the results of these numerical simulations, I introduce different social security specifications in the model to see whether they can improve economic outcomes.

In Appendix-D, at the end of the paper, I also numerically calculate, plot and compare the the time paths of individuals' and cohorts' consumption and asset holdings in the steady state of the model before and after demographic change (mortality decline and compression of morbidity happening together.)

Empirical evidence for demographic change is discussed in the next section.

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<sup>3</sup>It is important to note here that in this chapter, I assume the fertility rate exogenously declines to keep the population growth rate constant when mortality decline happens. Hence, I don't introduce fertility as an endogenous variable that can be controlled by the agents in the model.

## 1.2 Empirical Evidence

There have been three important demographic trends in most of the countries on earth since the last century. These are namely, the declining mortality rates, decreasing lengths of morbidity periods and declining fertility rates.

First, due to discoveries and advances in medical science, there have been great declines in mortality rates over the course of the 20th century all over the world. These declines resulted in greater life expectancies at all ages in both developed and developing countries. For instance according to the Human Mortality Database (HMD), [5], in 1933 in the United States life expectancy at birth (ie: life expectancy of the birth cohort) was 60.9 years for the whole population. In 1975, this number increased to 72.54 years and in 2017 to 78.88 years.

Similar changes have also been observed in the mortality rates of developing countries. For instance, in Ukraine, a developing country which systematically has had the lowest life expectancy among the 40 countries in the Human Mortality Database, there have been great improvements in the last century. In 1959 in Ukraine (1959 is the earliest year data is available for Ukraine in the HMD [5]), life expectancy at birth was 69.45 years for the whole population. In 1985, it increased to 69.94 years and in 2013 to 71.35 years.

The main reason for the decline in mortality rates over the course of the 20th century has been mankind's conquest of infectious diseases, such as tuberculosis, smallpox and diphtheria. A century ago, the great majority of deaths were attributed to infectious diseases ([25],[59]) which are nowadays easily curable. Today we are living in a second era, where the greater portion of illnesses and deaths are attributable to chronic diseases, such as diabetes, cirrhosis and osteo-arthritis. Prominent medical researches such as James Fries ([37]) predict that in the next century cures will be gradually found for these chronic diseases and there will be even further increases in life expectancy all over the world.

Second, the other important demographic factor that has been changing simultaneously with mortality rates is the number of years individuals spend in debilitating illness. In the medical literature, this period of an individual's life span is commonly referred to as the

”morbidity period” (for instance see [37], [38], [70]). Morbidity is ”a broad form of disability, frailty, impairment of functioning at activities of daily living, or other decreases in health-related quality of life” according to [70]. Note that impairment in this context can be either physical or mental. For instance, if an individual falls down the stairs and fractures their hip, they’ll be physically impaired until they get a hip replacement. But if an individual is ailed by the Alzheimer’s disease, they’ll be mentally impaired until the age they die.

During the last century, as medical discoveries and advances were taking place, our understanding of and possible responses to physical / mental impairment causing diseases improved as well. For instance, there have been frequent campaigns in the United States alone since 1960s to combat teenage smoking and childhood obesity, to reduce the number of individuals with hypertension and high cholesterol levels, and to promote exercise and healthy diets. At the same time, inventions, such as mobility scooters, have been created to improve the quality of life for impaired individuals. We can expect even more advanced responses to physical / mental impairment in the next century. For instance, artificial organs, such as 3D printed hearts, kidneys and livers, will no doubt be the cure for a huge number of people in the next century.

All of these advances mentioned in the previous paragraph (public campaigns to reduce impairment causing diseases, inventions to help or outright cure people with physical / mental impairment) have decreased the ”morbidity period” for individuals who have access to them. For instance according to the [3] (National Long Term Care Survey), which is a longitudinal survey designed to study changes in the health and functional status of older Americans (aged 65+), for 65+ men in the US, expectation of days with physical impairment declined by 17.5 % and expectation of days with cognitive impairment declined by 27.7 % from 1984 to 2004. The same rates of decline were 19 % and 36.1 % for 65+ women in the US during the same time period.

It is important to note here that the portion of lifetime spent in morbidity (the morbidity period) depends on two factors: the age at which physical / mental impairment begins (onset of debilitating illness) and the life expectancy of the individual (the terminal age). Due to

medical advances, both of these ages have been increasing since last century. Theoretically speaking there are three possible scenarios that can happen in the future with respect to these two ages. If the age at which morbidity begins increases faster than terminal age, individuals will spend a smaller portion of their lifetime in morbidity. This situation is termed as "compression of morbidity" and this is what happened in the US between 1984 and 2004 according to [3]. [38] posits this is what will happen in the next century for developed countries.

However, it is also possible that terminal age will increase faster than the age at which impairment begins. This situation is called "expansion of morbidity". Ernest Gruenberg famously referred to this possible scenario as "Failure of success" in [40]. [47] posits this is what will take place in developed countries in the future.

Finally, it is also possible to have an intermediate scenario in the future ([58]). In this scenario, these two ages would increase proportionally and so the "morbidity period" would remain the same as a percentage of individuals' lifetime.

The third important demographic change that took place in the same time period was the decline in fertility rates. Historical fertility rates are publicly available, among other sources, at [1] (Worldbank Open Data, Total Births per Woman). For instance, according to [1], in the United States in 1960 the fertility rate, defined as the number births per woman, was 3.65. The same rate was 1.80 in the US in 2016. For comparison, in Ukraine, in 1960, fertility rate rate was 2.24 and declined to 1.47 in 2016.

The decline in the fertility rates are frequently attributed to increasing labor force participation rate of women, increasing opportunity cost of having and raising children for families and advent of generic contraceptive drugs. There is a huge body of literature, which tries to explain the globally declining fertility rates with different mechanisms and methods of empirical analysis. For some important papers in this literature, see Gary Becker's seminal paper [14] and also [75], [62], [10].

### **1.3 Motivation**

In light of the demographic trends explained in the previous section, there are at least three pieces of motivation for writing this paper. First, in the existing literature regarding the economic effects of demographic change on individual decision making, there is no consensus about how consumption-saving and labor-leisure decisions are affected in the wake of increasing life expectancy. More specifically, there is a large number of papers in the literature that have different predictions about how saving behavior and retirement age respond to increasing life expectancy.

Intuitively, how individuals will respond to increasing life expectancy alone is not very clear. On one hand, they might increase their retirement age when their life expectancy increases to accumulate more assets for their post-retirement years. In this scenario they wouldn't be likely to increase their saving rate. They can leave it at the same level or perhaps even decrease it.

On the other hand, it might also be the case that individuals increase their saving rate during their working years to finance their consumption in their post-retirement years. In this scenario they wouldn't be likely to increase their retirement age. Again, they can leave it at the same level or perhaps even decrease it.

Finally, there could be an intermediate scenario with both a higher retirement age and a higher saving rate.

To complicate the matter further, in addition to increasing life expectancy, the proportion of lifetime spent in morbidity changes simultaneously. As mentioned earlier, there can be either expansion or compression of morbidity in the future. It is not clear how increasing life expectancy, coupled with changing periods of morbidity, will affect two important life time decisions, consumption-saving and retirement. The first motivation in writing this paper is to quantitatively investigate this question.

As part of this objective, I build a two period OLG model with a realistic demographic structure and impose different combinations of increasing life expectancy (due to declin-

ing mortality) and compression of morbidity on the model to disentangle and gauge their respective effects.

The second motivation is to investigate whether introducing a social security system can improve the situation of the agents in the model as they face demographic change. If so, what kind of social security specification would be best? Should the society adopt a pay as you go (PAYG) system? If yes, should there be a flexible or a mandatory retirement age in the PAYG system? Would, instead, introducing a fully funded (FF) system be more useful in certain demographic change scenarios? All of these questions are quantitatively explored in the numerical simulations section of this paper.

The third motivation for writing this paper is to explore the re-distributive effects created by demographic change and the choice of the social security system of the society. Since increasing life expectancy and compression of morbidity cause changes in individuals' saving and retirement decisions, they impact the share of the population living in retirement and change the budget constraints of different cohorts. These changes naturally influence the income and wealth distributions of the society.

From a policy perspective, in answer to the distributional effects created by demographic change, a social security system can be introduced (or perhaps an existing one can be modified) to improve the equitability of income and wealth distributions. However, these questions need to be quantitatively explored before any policy suggestion can be made. This is another reason for setting up a theoretical model, calibrating it and running simulations.

The rest of the paper is organized as follows. An extensive literature review is provided in the next section. Fifth section introduces the theoretical model and analytically derives solutions wherever possible. Sixth section explains the calibration of the model. Seventh section explains how numerical simulations are conducted and reports their results. The last section concludes by discussing the findings of the paper and possible extensions to the model.

## 1.4 Literature Review

### 1.4.1 Demographic structure and its use in OLG models

Uncertain finite lifespan seems to have been first introduced as a feature in economic models during 1960s to explore a large variety of questions<sup>4</sup>. I think these models can be seen as the first attempt to include a primitive demographic structure in a model environment in the economics literature.

Yaari [77] investigated how a random finite lifespan affects individuals' consumption behavior in a utility maximization problem under different specifications. In one of these specifications, he famously used "actuarial notes" to introduce life insurance and annuities in his model environment<sup>5</sup>. In another specification, he included a bequest motive in his model and explored the problem of portfolio allocation between regular notes and "actuarial notes".

Hakansson [41] followed the example set by Yaari [77]. He explored the consumption and investment paths in a model with an uncertain lifetime that has a known probability distribution, a utility function intended to represent the individual's bequest motive, and the ability to purchase lifetime insurance.

During 1970s, inclusion of an uncertain lifetime (which has a known probability distribution) became an increasingly popular feature in both economics and finance literatures. Richard Scott in [64] set up a continuous time model with an arbitrary but known distribution of lifetime to derive optimal consumption, portfolio and life insurance rules. One of the main conclusions of that paper was that investors had a "human capital" component of wealth, which is independent of their preferences and risky market opportunities available to them.

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<sup>4</sup>In the words of Menahem Yaari in [77], early prominent figures such as Alfred Marshal and Irving Fisher "... were aware of the uncertainty of survival but for one reason or another did not expound on how a consumer might react to this uncertainty."

<sup>5</sup>This assumption, as will be seen later, has been frequently used in recent models with more realistic demographic structures.

Barro and Friedman in [13] compared consumption decisions of individuals under the assumption of an uncertain lifetime and under the case where lifespan of an individual is announced at the time of birth. Their conclusion was that in a model with uncertain lifetime and no investment in human capital, individuals' utility is greater than the utility they would receive in the case of a certain lifespan.

Following in this tradition, Levhari and Mirman [53] built a  $T$  period overlapping generations (OLG) model in which both  $T$  and the return to savings are stochastic with known distributions. Their aim was to quantitatively investigate the impact of lifetime uncertainty on optimal consumption decisions. They concluded that changing the distribution of lifetime uncertainty creates two competing effects: 1. It decreases consumption of the agents in the model due to higher probability of having a longer life and 2. It increases consumption due to the desire of "sure consumption" in the present.

Eliakim Katz [46] built on the results of Levhari and Mirman [53] to reach the counter-intuitive conclusion that an increase in the probability of survival into the future can decrease the individuals' consumption and hence their welfare. Although he used a simple two period OLG model to reach this conclusion, he commented that his result can be extended to the case of  $T$  periods.

The influential paper by Olivier Blanchard [15] was the next milestone in this thread of the literature. This paper begins by noting that many issues in economics depend on the horizon of the agents. Blanchard builds an OLG model with an uncertain lifespan to investigate some of these issues. The instantaneous mortality hazard (ie: probability density of dying at any given age) is equal to a constant  $p$ , hence the life expectancy of the agents equals  $p^{-1}$  in his model. This means that, again, the uncertain lifespan of the individuals has a known probability distribution in this paper.

Blanchard pointed out that OLG models are heterogenous agent models by their very nature since agents at different ages have different levels and compositions of wealth. Also their marginal propensity to consume out of total wealth is not the same since their remaining lifespans are different. This makes aggregating individuals' decisions of consumption and

saving difficult if not impossible. The assumption mentioned in the previous paragraph, constant instantaneous mortality hazard of  $p$ , makes aggregation possible in this paper<sup>6</sup>.

As for its objective, the paper investigated: 1. The role of uncertain and finite lifespans in determining the steady state level of the real interest rate and 2. The role of fiscal policy (government spending, deficits and debt) in determining the real interest rate in the case of closed and open economies.

The crude but effective assumption of an age invariant instantaneous mortality hazard in Blanchard's paper [15] drew wide spread criticism in the following years, since this assumption means that all agents despite of their age have the same life expectancy. Hamid Faruquee published a series of papers in late 90s and early 2000s to offer possible solutions to this problem. For instance, his 2003 paper [34] introduced an age-specific mortality hazard function into the Yaari-Blanchard-Weil<sup>7</sup> framework to investigate the effects of public debt.

Faruquee, in his 2003 paper [34], used a hyperbolic tangent survival function and calibrated its parameters by using Gompertz's Laws<sup>8</sup>. It is important to note that he did not use empirically observed survival data to calibrate the parameters of the hyperbolic tangent survival function in this paper.

In another paper around the same time, [35], Faruquee introduced cohort specific birth rates and a time varying population growth rate, instead of using age specific mortality hazards, to investigate the question of what happens to optimal consumption and saving behavior under demographic change.

Researchers continued using OLG models with different survival function specifications to investigate an ever expanding array of issues in the next two decades. For example, Bloom and Canning in an 2007 paper [16] investigated the question of how improvements

<sup>6</sup>Blanchard remarks that Diamond [32] overcomes this problem by assuming a very simple population and age structure, which makes it possible to avoid the need for aggregation.

<sup>7</sup>The reader will already be familiar with the first two papers, [77] and [15], having read this literature review. The third paper,[74], builds an OLG model with infinitely lived agents to investigate the issues of Ricardian equivalence, efficiency of the competitive equilibrium and existence of bubbles.

<sup>8</sup>I elaborate on Gompertz's Laws in the calibration section of this paper.

in healthy life expectancy affects the age of retirement and the saving rate of agents. Their motivation was the observation that, theoretically, average retirement age should increase and the saving rate of agents should not change very much when healthy life expectancy increases. However, in reality, presence of retirement incentives in social security systems distort individuals' chosen retirement age in the face of increasing healthy life expectancy. They set up an OLG model with a constant mortality hazard  $\lambda$  (similar to Blanchard's 1985 paper [15]), endogenous retirement age, and a disutility of labor function that depends on age and life expectancy<sup>9</sup>. They used their model's steady state conditions to come up with structural equations for the endogenous retirement age and lifetime saving rate that can be estimated using empirical data. They concluded that the impact of increasing life expectancy on the chosen retirement age and the saving rate of agents depends on the social security specification.

The 2007 paper by D'Albis [26] is a theoretical study that uses an OLG model with a generic, non-specified instantaneous mortality hazard function to prove the existence of a population growth rate which maximizes long-run capital per capita. He examined the properties of this optimal population growth rate and then compared it with the respective cases implied by the frameworks of some of the seminal papers in the literature, including [32], [15], [21] and [74]. He concluded by explaining why such an optimal population growth rate does not exist in these seminal papers<sup>10</sup>.

Lau in his 2009 paper [48] built on the theoretical study in [26]. He used actual survival data for the United States in 2004 to calculate the relationship between population growth rate and capital accumulation in two theoretical cases: 1. agents in the model are not allowed to retire until they die and 2. agents in the model are made to retire at an exogenously

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<sup>9</sup>To my knowledge, this is one of the first papers, if not the first, that includes a theoretical mechanism for compression of morbidity in the economics literature. As life expectancy increases, disutility of labor declines across all cohorts in their theoretical model. I comment on the details of this specification in the calibration section of this chapter.

<sup>10</sup>In [32] and [15], there is a negative relationship between the population growth rate and capital accumulation.

specified mandatory retirement age of 65 years. In both these cases, numerical calculations indicated a negative relationship between the population growth rate and capital accumulation for the US in 2004. Lau concluded that this is likely to be the case for the other industrialized countries as well.

Ben Heijdra’s 2009 study [42] investigates the impact of two kinds of demographic change, 1. a decline in the fertility rate, which they call a ”baby bust” and 2. a longevity increase, in a small open economy where agents can choose their retirement age. Later they introduce a simple pay as you go (PAYG) social security system in their model and also consider the impact of two changes related to this system: 1. an increase in the payroll tax which finances the PAYG system and 2. a decline in the benefits paid to the retired agents. This paper uses the Gompertz-Makeham (GM) survival function<sup>11</sup> for its numerical calculations and estimates the parameters of the GM function by using survival data from the Netherlands.

Bloom and Canning’s 2014 paper [17] is another study with an OLG model, which uses the GM survival function for its theoretical demographic structure. The parameters of the GM survival function are calibrated using survival data from the US. Other features of their model include an endogenous retirement age, perfect capital markets, and a disutility of labor function that is proportional to the instantaneous mortality hazard at each age<sup>12</sup>. The paper compares the respective impacts of increasing healthy life expectancy and rising incomes on the optimal retirement age. Their numerical calculations signal that over the last century, rising incomes led to lower optimal retirement ages by dominating the impact of increasing healthy life expectancy.

Bruce and Turnovsky’s 2013 paper [20] has an OLG model and utilizes the De’Moivre specification, which is a 2 parameter survival function, for its theoretical survival function.

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<sup>11</sup>For a detailed discussion of the GM survival function, please see the calibration section of this chapter.

<sup>12</sup>Recall the disutility of labor specification in [16], where disutility of labor doesn’t only depend on age but also on life expectancy. In [17], life expectancy doesn’t affect the disutility of labor. Instead it increases exponentially with age following the instantaneous mortality hazard implied by the GM survival function. With this specification, when there is a decline in the mortality rates, this doesn’t only lead to a higher life expectancy but also lower disutility of labor for all ages. This is another way of introducing ”compression of morbidity” in a model environment.

This study explains that by changing the two parameters of the De'Moivre survival function, it's possible to obtain the demographic structures of the infinitely lived agent model, Samuelson model [66], and the Blanchard model [15]. Hence, the De'Moivre specification is said to offer a "unified demographic structure", which nests these other specifications as its limiting cases. There's an exogenously given mandatory retirement age in the model and the survival function is not calibrated using actual survival data. The study considers the impact of several different demographic changes on the saving rate of the agents in the model, and the economic growth rate.

Mierau and Turnovsky's 2014 paper [60] is another recent study that has an OLG model with a realistic demographic structure, no retirement and no disutility of labor. This paper uses the Boucekkine survival function (BCL) as its theoretical survival function and calibrates its parameters using the actual survival data of the US in 2006. One major contribution of this paper is the study of local transitional dynamics created by demographic changes, such as a mortality decline leading to higher life expectancy for the agents. The paper considers the impact of three exogenous changes on the steady state of the model: 1. An increase in productivity, 2. A decrease in the fertility rate, and 3. A decrease in the mortality rate.

[61] is a second study that was published by the same authors in 2014. Its theoretical structure is very similar to the paper that was mentioned in the above paragraph, again an OLG model with a realistic demographic structure, no retirement and no disutility of labor. This study again uses the BCL survival function and calibrates its parameters using US survival data from 2006. The main difference between these two studies is the presence of a single sector, endogenous production function<sup>13</sup> in the latter study. This paper investigates the local transitional dynamics around the balanced growth path and concludes that it is unstable. Also, using numerical simulations, it is shown that changes in the population growth rate has different economic consequences depending on the source of the change (whether it is due to a change in the fertility or the mortality rate).

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<sup>13</sup>This is the famous AK type production function in the economic growth literature. Interested readers can see the influential paper [65].

### 1.4.2 *Impact of demographic change on saving and retirement choices*

In this section, I list and briefly summarize some of the more influential papers that investigate the impact of demographic change on both consumption-saving and endogenous retirement decisions.

The first paper in this thread of the demographic change literature that I could find<sup>14</sup> is the 1991 study by Chang [23]. This paper follows Blanchard's influential study [15] in assuming that lifetime is a random variable that is exponentially distributed, with a constant instantaneous mortality hazard. A decline in this constant mortality hazard means an increase in expected lifetime. Agents respond to this change by altering their saving behavior and the endogenously chosen retirement age. The paper studies these decisions under the assumption of perfect and imperfect annuity markets. Their conclusions are as follows: 1. Under the assumption of uncertain lifetimes and perfect annuity markets, when there is an increase in life expectancy, the adjustment in the chosen retirement age depends on whether agents accumulate or decumulate their assets in old age. If they accumulate more assets in retirement, retirement age declines. Otherwise it increases. 2. With imperfect annuity markets, increasing life expectancy has a smaller impact on delaying the retirement age. Agents can actually decrease their chosen retirement age in response to increasing life expectancy if the annuity market is sufficiently far from perfect.

Dora Costa's 1988 paper [24] is a frequently cited empirical study from the '90s that explored the trends in the retirement and labor force participation rates since late 19th century in the US and the determinants of these trends. By studying several sources of empirical data, they find that as life expectancy increased and individuals began spending a larger proportion of their lifetime in a healthy condition in the US, the average retirement age has

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<sup>14</sup>An earlier seminal paper by Eytan Sheshinski, [68], investigates the problem of how the endogenously chosen retirement age adjusts in response to a change in the social security system. But there is a constant lifespan and the level of consumption is exogenously fixed in this early paper. Hence, demographic change and consumption-saving choice are not present. One nice feature of the model in this paper is the inelastic labor supply of individuals. My understanding is that this is one of the first papers, if not the first, that makes use of this assumption in a quantitative economic model.

fallen, number of years spent in retirement has gone up, and the fraction of older cohorts that participate in the labor market has been falling steadily. They propose that, in the face of demographic change, the following factors influenced the chosen retirement age of individuals: income, presence of a social security system, average duration of unemployment spells, sectoral shifts in the economy, health, and life style in retirement. They conclude by predicting that the fiscal problems facing the social security systems are likely to be compounded by the increasing longevity and morbidity improvements of the elderly population, since these changes have in the past caused the retirement rates to increase in the US.

The 2002 paper by Boucekkine et al. [18] is the next seminal paper in this thread of the literature. Agents in this paper have an uncertain lifespan, and optimally choose not only their consumption stream and retirement age but also the date they join the labor force (ie: the years they invest in schooling) to maximize their expected utility. The lifetime of the agents is given by a specification, which came to be known as the Boucekkine Survival function in the demographic change literature<sup>15</sup>. This survival function is perhaps the most important contribution of this paper. The other important contribution of the paper is that it characterizes the stability properties of the balanced growth path and computes the transition path following a demographic change. The paper studies how economic growth is affected by demographic change. Since fertility rate is an exogenously given constant and the idea of morbidity doesn't even exist in this study's model environment, demographic change is simply a change in the mortality parameters which leads to a different life expectancy. The paper's main conclusions are as follows: 1. Increasing life expectancy leads to longer schooling and later retirement, but has an ambiguous effect on economic growth. 2. Long term relationship between fertility and economic growth is hump shaped.

Lee and Goldstein's 2003 paper [51] is an empirical study which investigates whether certain periods in individuals' lives (time spent studying, working, in retirement, and before reproduction) adjust proportionally in response to increasing lifespans. The "proportional

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<sup>15</sup>Boucekkine Survival function (BCL) along with several other alternative survival function specifications are discussed in detail in the calibration section of this paper.

re-scaling hypothesis” is the idea that every life cycle stage and boundary simply adjusts in proportion to changing life expectancy<sup>16</sup>. From an economics view point, if human and physical capital stocks increase more rapidly compared to the size of the labor force (which is determined by the retirement age, labor force participation rate, fertility and mortality rates etc.), this causes wages to rise and the interest rate to fall. In that case, time spent in retirement is not likely to rise proportionately to longevity. By analyzing empirical data for the US, this study concludes that as life expectancy increased and morbidity declined, individuals spent more time in school, delayed having children, entered the labor force later and spent a bigger proportion of their lives in retirement.

Bloom and Canning’s 2007 paper [16], which was mentioned in the previous section, is the next influential study in the literature about the impact of demographic change on consumption-saving and retirement age choices. There are two kinds of demographic change in this paper: declining mortality rates leading to longer life expectancy and compression of morbidity (ie: individuals spending a smaller portion of their lives in a morbid condition). Their main finding is that in the face of demographic change, the adjustment of saving rates and the chosen retirement age depends on the social security system (in particular on the retirement incentives in the social security system) in place. Hence, the social security system provides an institutional barrier preventing the proportional rescaling hypothesis in practice.

Heijdra’s 2009 paper [42], Lau’s 2009 paper [48], and Bloom and Canning’s 2014 paper [17] are three other influential papers around this time period that have model environments which include uncertain lifetimes for their agents, an endogenous retirement age in addition to the usual consumption-saving choice. These three papers were summarized in detail in the previous section, so I won’t repeat their additional respective features and conclusions here.

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<sup>16</sup>The proportional rescaling hypothesis is studied under the subject of ”biological invariants” in biology. An example of such an invariant would be the relationship between the mean age of reproduction and life expectancy at birth. Biological invariants in nature are the result of evolutionary forces, but the increasing longevity of modern humans is not due to these forces. Instead, advances in science, changes in life style and social organization have increased the modern human’s life span. Hence, proportional rescaling is not very likely to apply to modern human life cycle.

Some of the other frequently cited studies in this literature that were not covered in the previous section are the 2009 paper by Zhang [78], 2010 paper by Kalemli-Ozcan and Weil [45], and the 2012 paper by Lau and et al. [50]. I will explain these papers briefly before closing this section.

The 2009 paper by Zhang [78] explores the question of how the retirement age, life cycle saving and human capital investment in children respond to rising longevity. The agents in their model survive to adulthood with certainty. But after the period of adulthood begins, agents can only survive to the age of retirement with a constant, exogenous probability of survival,  $p$ . They receive altruistic utility from the consumption of their children, invest in their childrens' education, and choose their consumption stream and age of retirement. Compression of morbidity is not present in this paper as a source of demographic change. Their findings can be summed up as follows: 1. Rising longevity causes the retirement age to increase in this model. 2. If initial life expectancy (before demographic change) is sufficiently low, rising longevity also causes investment in childrens' education and the saving rate to increase. Both of these changes lead to a higher economic growth rate. 3. Introduction of a mandatory retirement age reduces investment in childrens' education and the economic growth rate but raises the saving rate.

Kalemli-Ozcan and Weil's 2010 paper [45] sets up an OLG model with an uncertain lifespan to formally explain the rising retirement rates observed in the US over the course of the 20th century. They use two different specifications in their numerical exercises to calculate the survival probabilities of their agents: the exponential survival function and the Boucekkine survival function (BCL). They reach the conclusion that there are two effects competing with each other in determining the retirement age. They call these two effects the "horizon effect" and the "uncertainty effect". By the "horizon effect", they mean an increase in the retirement age due to increasing life expectancy. The "uncertainty effect" in the paper just means that declining mortality rates make it more likely for agents to enjoy their savings and leisure time in post-retirement years. Hence, this effect leads to later retirement. In the case of the US economy in the 20th century, they further reach the

conclusion that the uncertainty effect has been the dominant one.

The 2012 study by Lau et al. [50] investigates how a mortality change at an arbitrary age affects the optimal retirement age. This line of study is motivated by the empirical observation that during demographic transition, mortality decline first affects younger cohorts and then older cohorts. Their conclusion is that mortality reductions at younger ages cause an increase in agents' expected lifetime human wealth and this in turn may lead to earlier retirement, but mortality reductions at older ages delay retirement with certainty.

## 1.5 The Model

### 1.5.1 Households

Agents in the model are born the moment they join the labor force, which is assumed to be at 20 years old. They live through two different periods in the model: 1. The working age period, which corresponds to the years between 20 years of age and retirement, 2. The post retirement period.

Lifetime of the agents in the model is uncertain. The probability that an agent born at calendar date  $\nu$  will survive to a future calendar date  $t \geq \nu$  is given by:

$$S(t - \nu) = e^{-M(t-\nu)} \quad (1.1)$$

where  $M(t-\nu)$  is the cumulative mortality hazard function and represents the cumulative probability of dying before or at age  $t - \nu$ . Instantaneous mortality hazard at age  $t - \nu$  can be recovered from the survival function by using the following equation:

$$\mu(t - \nu) = -\frac{S'(t - \nu)}{S(t - \nu)} \quad (1.2)$$

Further, cumulative mortality hazard function can be obtained from the instantaneous mortality hazard by using the equation:

$$M(t - \nu) = \int_0^{t-\nu} \mu(\tau) d\tau \quad (1.3)$$

Probability of surviving to age 0 in the model is  $S(0) = e^{-M(0)} = e^{-0} = 1$ . Agents in the model can live up to a terminal age  $D$ . The terminal age implied by the survival function can be obtained by setting  $S(D) = e^{-M(D)} = e^{-\infty} = 0$ .

I assume that agents provide their labor inelastically until the age they choose to retire. Hence labor-leisure choice is only available on the extensive margin. Letting  $R$  denote the retirement age, a cohort  $\nu$  individual (ie: an agent who was born at date  $\nu$ ) has the following labor supply function at calendar date  $t$ :

$$L(\nu, t) = \begin{cases} 1 & \text{if } 0 \leq t - \nu < R \\ 0 & \text{if } R \leq t - \nu \leq D \end{cases} \quad (1.4)$$

Agents don't return back to the labor force after they choose to retire. Given their uncertain lifetime, a cohort  $\nu$  individual chooses their future consumption stream  $\{C(\nu, t)\}_{\nu}^{\nu+D}$  and retirement age  $R$  to maximize their expected lifetime utility:

$$E[\Lambda(\nu)] = \int_{\nu}^{\nu+D} [\gamma_0 I + (1 + \gamma_1 I)u(c(\nu, t))]e^{-\rho(t-\nu)-M(t-\nu)} dt - \int_{\nu}^{\nu+R} \phi(t-\nu)e^{-\rho(t-\nu)-M(t-\nu)} dt \quad (1.5)$$

subject to the flow budget constraint:

$$\frac{\partial A(\nu, t)}{\partial t} = \begin{cases} [r(t) + \mu(\tau - \nu)]A(\nu, t) + (1 - T)w(t) - c(\nu, t) & \text{if } t - \nu \leq R \\ [r(t) + \mu(t - \nu)]A(\nu, t) + bw(\nu + R) - c(\nu, t) & \text{if } t - \nu > R \end{cases} \quad (1.6)$$

Agents have no assets when they are born and can not die in debt or leave a bequest to their offspring, hence:  $A(\nu, \nu) = 0$  and  $A(\nu, \nu + D) = 0$ .

The term  $[\gamma_0 I + (1 + \gamma_1 I)u(c(\nu, t))]$ , in the objective function is intended to capture the detrimental effects of chronic diseases such as diabetes, arthritis, cancer etc. on the utility agents receive from consumption after a certain age  $X$ .  $\gamma_0 < 0$  is called the "level variable"

and  $\gamma_1 < 0$  the "state variable" in the literature ([36]).  $I$  stands for a binary variable that has the value of 1 if  $t - \nu \geq X$ , otherwise it is equal to 0.

With this formulation, at age  $X$ , agents begin suffering from chronic health conditions heavily enough that they lose a fixed amount level of utility. This is represented by  $\gamma_0$ . Also beginning from age  $X$ , utility from consumption no longer provides the same marginal utility. Due to chronic health problems agents' utility from consumption is multiplied by  $[1 + \gamma_1]$  where again  $\gamma_1 < 0$ .<sup>17</sup> As an example, if  $\gamma_1 = -0.1$ ,  $u(c(\nu, t))$  is multiplied by 0.9.

For the flow utility function  $u(c(\nu, t))$ , I choose the isoelastic utility function:

$$u(c(\nu, t)) = \frac{c(\nu, t)^{1-\frac{1}{\sigma}} - 1}{1 - \frac{1}{\sigma}} \quad (1.7)$$

where  $\sigma$  is the intertemporal elasticity of substitution.

The second part of the objective function,  $-\int_{\nu}^{\nu+Ret} \phi(t - \nu) e^{-\rho(t-\nu)-M(t-\nu)} dt$ , represents the disutility of labor suffered by the agents throughout their working life. The flow disutility of a cohort  $\nu$  individual at calendar date  $t$  is given by the following function:

$$\phi(t - \nu) = \delta_0 e^{\delta_1 \max\{t-\nu-X, 0\}} \quad (1.8)$$

This specification for the flow disutility function indicates that the disutility of participating in the labor market is equal to a constant  $\delta_0 > 0$  before age  $X$ . After age  $X$ , flow disutility increases at rate  $\delta_1 > 0$ .<sup>18</sup> Intuitively, this can be interpreted as the effects of chronic conditions accumulating over time and leading to greater disutility of working.

Following [77], I assume that there is a perfectly competitive insurance industry issuing actuarially fair lifetime annuities. Agents purchase these lifetime annuities for saving. The

<sup>17</sup>There are two theories in the literature regarding how marginal utility of consumption changes with deteriorating health. The literature indicates that in the case of United States marginal utility of consumption declines with deteriorating health ([36]). Since the model in this paper is calibrated by using the demographic structure of United States, I use negative values for  $\gamma_0$  and  $\gamma_1$ . Positive values for  $\gamma_0$  and  $\gamma_1$  would indicate that consumption goods are substitutes for good health. In that case, marginal utility of consumption would increase with deteriorating health.

<sup>18</sup>I explain some of the alternative specifications used in the literature for the disutility of labor function in the calibration section of this paper.

annuities pay an extra return above the risk free rate that is equal to the instantaneous mortality hazard of their holder, provided their holder doesn't die at that particular instant. Hence, a cohort  $\nu$  individual at time  $t$  receives a return equal to  $[r(t) + \mu(t - \nu)]$  if they don't die at time  $t$ . If the holder of the annuity happens to die at that instant, their savings are inherited by the insurance industry and distributed between the surviving members of the society<sup>19</sup>.

There is a "pay as you go" (PAYG) social security system in the benchmark model. During their working live, agents pay a social security tax,  $T$ , as their contribution to social security system. At each instant, their contributions are distributed as benefits to the retired agents in the society. Please note that the PAYG system has a replacement ratio,  $b < 1$ , between the benefits retirees receive and their wage at the time of retirement. As an example, if  $b = 0.4$ , a retired agent receives 40% of their wage at the time of retirement as their benefit until they die. I assume that the payroll tax,  $T$ , adjusts at each instant to keep the PAYG system solvent. Hence, the social security system can not run a deficit or surplus.

I provide a version of this model with a "Fully Funded" (FF) social security system, another version with no social security system at all and finally a version with mandatory retirement age and solve for their respective steady states in the numerical simulations sections of this paper. In the next section, I discuss the demographic structure of the population.

### 1.5.2 Population's Demographic Structure

I begin by assuming that the size of the population grows at the constant rate  $n$ . Letting  $P(\nu)$  denote the size of the population at time  $\nu$ , population size at a future date  $t \geq \nu$  is equal to:  $P(t) = P(\nu)e^{n(t-\nu)}$ .

Further I assume that the birth rate is constant at rate  $\beta$  throughout time. Size of the birth cohort,  $P(\nu, \nu)$  at any time  $\nu$  is then given by:  $P(\nu, \nu) = \beta P(\nu)$ .

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<sup>19</sup>It is also possible to borrow by selling lifetime annuities. If an individual happens to die when in debt, Yaari in [77] assumes that their outstanding debt is forgiven.

Next, following Mierau and Turnovsky's 2014 paper, [61], I define the average instantaneous mortality hazard across cohorts:

$$\bar{\mu} \equiv \int_{t-D}^t \mu(t-\nu) \frac{P(\nu, t)}{P(t)} d\nu \quad (1.9)$$

where

$$P(\nu, t) = P(\nu, \nu) e^{-M(t-\nu)} = \beta P(\nu) e^{-M(t-\nu)} \quad (1.10)$$

is the number of individuals left alive from time  $\nu$  birth cohort at calendar date  $t \geq \nu$ . The population growth rate  $n$  is just equal to the difference between the exogenously given constant birth rate and average mortality rate:  $n = \beta - \bar{\mu}$ .

This last equality indicates that if average mortality rate declines due to demographic change, population growth rate increases, unless of course there is a simultaneous decline in the exogenously given birth rate.

Size of the population at any time  $t$  can be obtained by aggregating over the surviving members of each cohort at that time:

$$P(t) = \int_{t-D}^t P(\nu, t) d\nu = \beta P(\nu) e^{-M(t-\nu)} d\nu \quad (1.11)$$

Combining  $P(t) = P(\nu) e^{n(t-\nu)}$  with equation (1.10) yields an equation which defines the "demographic steady state" that is discussed in the seminal work [55]:

$$\beta \int_{t-D}^t e^{-n(t-\nu)-M(t-\nu)} d\nu = 1 \quad (1.12)$$

This equation relates an estimated survival function to the birth rate,  $\beta$ , and the population growth rate,  $n$  of the society. I'll return back to it later in the calibration section to calculate the implied birth rate,  $\beta$ .

### 1.5.3 Firms

Since the focus of this paper is to investigate the impact of demographic change on individual decision making and aggregate macroeconomic variables, I use a simple specification for the production side. I assume firms produce the single good in the economy by using the neo-classical production function:

$$Y(t) = F(K(t), A(t)L(t)) \quad (1.13)$$

where  $K(t)$  is capital,  $L(t)$  is labor and  $A(t)$  is labor augmenting tech progress at calendar date  $t$ . I assume  $A(t)$  increases over time at exogenous rate  $\gamma_A$ :  $A(t) = A(0)e^{\gamma_A t}$ .

Dividing both sides of (13) with  $A(t)L(t)$  yields the production function in intensive form:

$$Y(t) \equiv f(k(t)) = F\left(\frac{K(t)}{A(t)L(t)}, 1\right) \quad (1.14)$$

Returns to capital and labor are given by:

$$r^* = f'(k^*) - \delta \quad (1.15)$$

$$w^*(t) = A(t)[f(k^*) - k^* f'(k^*)] = A(t)Z \quad (1.16)$$

where  $Z = [f(k^*) - k^* f'(k^*)]$  is a constant. Since  $k^*$  is constant in the steady state of the model, the equilibrium return to capital,  $r^*$ , is also time invariant. Equilibrium return to labor,  $w^*(t)$ , is not constant and grows at the rate of labor augmenting tech progress in the steady state.

#### 1.5.4 Solving for the steady state of the benchmark model

I begin by writing the current value Hamiltonian using the objective function in equation (1.5) and the budget constraint in equation (1.6):

$$H = e^{-\rho(t-\nu)-M(t-\nu)}[\gamma_0 I + [1 + \gamma_1 I]u(c(\nu, t))] + \lambda[(r(t) + \mu(t - \nu))A(\nu, t) + (1 - T)w(t)L(\nu, t) + bw(\nu + R)(1 - L(\nu, t)) - c(\nu, t)] \quad (1.17)$$

Taking the partial derivative of the Hamiltonian with respect to consumption yields:

$$\lambda = u'(c(\nu, t))e^{-\rho(t-\nu)-M(t-\nu)} \quad (1.18)$$

Similarly, taking the derivative of the Hamiltonian with respect to asset holdings I get:

$$\frac{\dot{\lambda}(\nu, t)}{\lambda(\nu, t)} = -[r(t) + \mu(t - \nu)] \quad (1.19)$$

Taking the time derivative of (17) and plugging the isoelastic utility function  $u(c(\nu, t)) = \frac{c(\nu, t)^{1-\frac{1}{\sigma}}-1}{1-\frac{1}{\sigma}}$  in (18), I obtain the growth rate of consumption at time  $t$  of a cohort  $\nu$  individual along the steady state:

$$\gamma_c(\nu, t) = \frac{\partial c(\nu, t)/\partial t}{c(\nu, t)} = \sigma[r(t) - \rho] \quad (1.20)$$

Next I take the partial derivative of the objective function in (5) with respect to the retirement age,  $R$ , and use the growth rate in equation (1.19) to obtain:

$$c(\nu, \nu)^{-1/\sigma} \left[ \int_{\nu}^{\nu+D} \frac{\partial c(\nu, \tau)}{\partial R} e^{-r^*(\tau-\nu)-M(\tau-\nu)} d\tau \right] + \gamma_1 c(\nu, \nu)^{-1/\sigma} \left[ \int_{\nu+X}^{\nu+D} \frac{\partial c(\nu, \tau)}{\partial R} e^{-r^*(\tau-\nu)-M(\tau-\nu)} d\tau \right] - [\phi(R^*)e^{-\rho R^*-M(R^*)}] = 0 \quad (1.21)$$

where  $R^*$  is the optimal retirement age chosen by the agent. Readers can refer to the appendix at the end of this paper for a detailed derivation of equation (1.20). Three

pieces of information are required to solve (1.20): consumption of a cohort  $\nu$  individual at the beginning of life,  $c(\nu, \nu)$ , integral of the discounted partial derivative of consumption of a cohort  $\nu$  individual at time  $t$  with respect to retirement age,  $\frac{\partial c(\nu, \tau)}{\partial R} e^{-r^*(t-\nu)-M(t-\nu)}$ , in the period  $[\nu, \nu + D]$ , and the same integral in the period  $[\nu + X, \nu + D]$  where  $X$  is the age at which chronic diseases begin decreasing the utility of consumption and increasing the disutility of labor.

First, to obtain  $c(\nu, \nu)$ , I plug  $c(\nu, \tau) = e^{\gamma c(t-\nu)} c(\nu, \nu)$  in the lifetime budget constraint (operative from date of birth,  $\nu$ ) and use equation (1.19). Restructuring the resulting equation yields:

$$c(\nu, \nu) = \frac{A(0)Z \left[ (1-T) \int_{\nu}^{\nu+D} e^{\gamma_A \tau - r^*(\tau-\nu) - M(\tau-\nu)} d\tau + b e^{\gamma_A(\nu+R)} \int_{\nu+R}^{\nu+D} e^{-r^*(\tau-\nu) - M(\tau-\nu)} d\tau \right]}{\left[ \int_{\nu}^{\nu+D} e^{(\sigma-1)r^*(\tau-\nu) - \sigma\rho(\tau-\nu) - M(\tau-\nu)} d\tau \right]} \quad (1.22)$$

Second, I obtain the integral of the discounted partial derivative,  $\frac{\partial c(\nu, \tau)}{\partial R} e^{-r^*(t-\nu)-M(t-\nu)}$ , in the period  $[\nu, \nu + D]$  by differentiating the lifetime budget constraint (operative from date of birth,  $\nu$ ) with respect to the retirement age.

$$\left[ \int_{\nu}^{\nu+D} \frac{\partial c(\nu, \tau)}{\partial R} e^{-r^*(\tau-\nu) - M(\tau-\nu)} d\tau \right] = \left[ (1-T-b)w(\nu+R)e^{-r^*R - M(R)} \right] + \left[ b\gamma_A w(\nu+R) \int_{\nu+R}^{\nu+D} e^{-r^*(\tau-\nu) - M(\tau-\nu)} d\tau \right] \quad (1.23)$$

Finally, to obtain the integral of the discounted partial derivative,  $\frac{\partial c(\nu, \tau)}{\partial R} e^{-r^*(t-\nu)-M(t-\nu)}$ , in the period  $[\nu + X, \nu + D]$ , I write the budget constraint of the agent in the period  $[\nu, \nu + X]$  and take its derivative with respect to the retirement age. After a significant amount of manipulation, I obtain equation (1.24). (Detailed derivations of all equations are provided

in the appendix.)

$$\begin{aligned}
\left[ \int_{\nu+X}^{\nu+D} \frac{\partial c(\nu, \tau)}{\partial R} e^{-r^*(\tau-\nu)-M(\tau-\nu)} d\tau \right] &= \left[ \int_{\nu}^{\nu+X} \frac{[A(0)Z e^{\sigma[r^*-\rho][\tau-\nu]}] [b\gamma_A e^{\gamma_A(\nu+R)}] [e^{-2r^*(\tau-\nu)-2M(\tau-\nu)}]}{\left[ \int_{\nu}^{\nu+D} e^{(\sigma-1)r^*(\tau-\nu)-\sigma\rho(\tau-\nu)-M(\tau-\nu)} d\tau \right]} d\tau \right] \\
+ [(1-T-b)w(\nu+R)e^{-r^*R-M(R)}] &+ \left[ b\gamma_A w(\nu+R) \int_{\nu+R}^{\nu+D} e^{-r^*(\tau-\nu)-M(\tau-\nu)} d\tau \right]
\end{aligned} \tag{1.24}$$

Inserting (1.22), (1.23), (1.24) into equation (1.21), I obtain an equation which determines the optimal retirement age. Intuitively the left hand side of this equation is the marginal benefit of not retiring at age  $R^*$ . It includes the marginal utility that can be received by consuming the after tax labor income that is earned by not retiring at age  $R^*$ , discounted to age 0. Right hand side of this equation is simply  $\phi(R^*)e^{-\rho R^*-M(R^*)}$ . That represents the marginal disutility suffered by the agent if they don't retire at age  $R^*$ , again discounted to age 0. Hence, at the optimal retirement age the marginal utility and disutility of not retiring, discounted to age 0, are equal.

It is important to note that (1.21) is an implicit equation which has variables that depend on  $k^*$  and  $R^*$  on both sides. So I need at least one more equation that depends on  $k^*$  and  $R^*$  to pin down the steady state values of  $k^*$  and  $R^*$ . I obtain that necessary equation by using the market clearing condition in steady state:

$$Y^* = f(K^*, A(t)L^*) = C^* + I^* \tag{1.25}$$

where  $Y^*, K^*, L^*$  are aggregate output, capital and labor supply in the steady state. Similarly,  $C^*$  and  $I^*$  are aggregate consumption and investment in the steady state. In steady state investment will be just sufficient to keep  $K^*$  constant. Hence it will have to account for population growth, labor augmenting tech progress and depreciation of capital:

$$I^* = (n + \gamma_A + \delta)K^* \tag{1.26}$$

plugging (26) into (25) and dividing both sides with  $A(t)L^*$  yields the market clearing condition in intensive form:

$$y^* = (k^*)^\alpha = c^* + i^* = c^* + (n + \gamma_A + \delta)k^* \quad (1.27)$$

The reader might be confused that in equations (1.25), (1.26), (1.27),  $R^*$  does not seem to show up. To clear that confusion, it must be noted that there is endogenous retirement in this model. This means that the optimal retirement age chosen by the agents  $R^*$  determines the aggregate labor supply  $L^*(t)$  in the steady state.  $L^*(t)$  in turn determines the steady state level of capital per effective worker, since  $k^* = \frac{K^*}{A(t)L^*(t)}$ . Hence, equations (1.25), (1.26), (1.27) represent the market clearing condition after  $k^*$  and  $R^*$  have both been calculated numerically together.

In the following pages, I derive a formula for  $c^*$  and it turns out to be a function of factor prices and the optimal retirement age. Since factor prices are functions of  $k^*$  and  $k^*$  is a function of  $R^*$  (due to endogenous retirement in the model),  $c^*$  is a function of  $k^*(R^*)$  and  $R^*$ . Hence I define  $c^* \equiv c^*(k^*(R^*), R^*)$ . Plugging this definition in (27) yields:

$$(k^*(R^*))^\alpha = c^*(k^*(R^*), R^*) + (n + \gamma_A + \delta)k^*(R^*) \quad (1.28)$$

Equations (1.28) and (1.21) are two implicit equations in  $k^*$  and  $R^*$ . It is unfortunately not possible to solve them analytically, but they can be solved using any modern numerical computing platform. I elaborate further on the numerical solution process in the solution methodology appendix at the end of the paper.

I explain the derivation of  $c^*(k^*(R^*), R^*)$  next. Begin by using the following equation for aggregate consumption in the steady state:

$$C^*(t) = \int_{t-D}^t P(\nu, t)C^*(\nu, t)d\nu \quad (1.29)$$

where  $P(\nu, t)$  is the number of people who survive from time  $\nu$  birth cohort at time  $t$ . Similarly,  $C(\nu, t)$  is the consumption of a time  $\nu$  birth cohort individual at time  $t$ . Hence,

equation (1.29) aggregates the consumption of surviving members of each cohort that is alive at time  $t$  to obtain aggregate consumption at that time.

Next, plug equation (1.10) in (1.29) to obtain:

$$C^*(t) = \int_{t-D}^t \beta P(t) e^{-n(t-\nu)-M(t-\nu)} C^*(\nu, t) d\nu \quad (1.30)$$

where  $C^*(\nu, t) = C^*(\nu, \nu) e^{\gamma_C(t-\nu)}$ . Further, by using equation (1.20) it can be shown that growth rate of consumption at time of birth is  $\gamma_A$  across birth cohorts in the steady state:  $C^*(\nu, \nu) = C^*(0, 0) e^{\gamma_A \nu}$ . (Details of the derivation are provided in the appendix.) Hence  $C^*(\nu, t) = C^*(0, 0) e^{[\gamma_A \nu + \gamma_C(t-\nu)]}$ . Plugging this expression in (30) and multiplying right hand side with  $e^{\gamma_A t}$  and  $e^{-\gamma_A t}$  results in:

$$C^*(t) = [e^{\gamma_A t} C^*(0, 0) \beta P(t)] \left[ \int_{t-D}^D e^{[-(n+\gamma_A-\gamma_C)(t-\nu)-M(t-\nu)]} d\nu \right] \quad (1.31)$$

$C^*(0, 0)$  in equation (1.31) can be obtained by assigning  $\nu = 0$  in equation (1.22). After plugging  $C^*(0, 0)$  in (1.31), I divide (1.31) with the effective labor supply  $A(t)L(t)$  to obtain consumption per unit of effective labor,  $c^* = Z M_1 M_2$ , where  $Z = [f(k^*) - k^* f'(k^*)]$ ,  $M_1$  and  $M_2$  are constants given by the following expressions:

$$M_1 = \frac{\left[ (1-T) \int_{\nu}^{\nu+D} e^{\gamma_A(\tau-\nu)-r^*(\tau-\nu)-M(\tau-\nu)} d\tau + b e^{\gamma_A R} \int_{\nu+R}^{\nu+D} e^{-r^*(\tau-\nu)-M(\tau-\nu)} d\tau \right]}{\left[ \int_{\nu}^{\nu+D} e^{(\sigma-1)r^*(\tau-\nu)-\sigma\rho(\tau-\nu)-M(\tau-\nu)} d\tau \right]} \quad (1.32)$$

$$M_2 = \frac{\left[ \int_{t-D}^t e^{[-(n+\gamma_A-\gamma_C)(t-\nu)-M(t-\nu)]} d\nu \right]}{\left[ \int_{t-R}^t e^{-n(t-\nu)-M(t-\nu)} d\nu \right]} \quad (1.33)$$

The integrals in equation (1.32) result from a single birth cohort's consumption and retirement decisions in their lifetime. Hence, these integrals are over the life span of a single birth cohort. They represent "longitudinal constraints". On the other hand integrals in equation (1.33) result from the consumption and retirement decisions of all the cohorts that are alive at time  $t$ . Hence, these integrals are over the births dates of all the cohorts that are alive at time  $t$ . They represent "cross-sectional constraints".

Integrals representing "longitudinal" and "cross-sectional" constraints such as the ones in this paper are common in papers that have OLG models with realistic demographic structures. For example, similar but simpler integrals can be found in [26] and [48]. I should also mention at this point that there is another method for deriving consumption per effective worker proposed in the literature. [26] explains that  $k^*$  can be obtained by solving a fixed point problem. Interested readers can investigate equation (19) in [26].

As explained earlier consumption per effective worker,  $c^*$ , turns out be a function of  $k^*$  and  $R^*$ . Once  $c^*$  is plugged in equation (1.27) (the market clearing equation in intensive form), (1.21) and (1.27) can be solved numerically to obtain values for  $k^*$  and  $R^*$ . Then these values for  $k^*$  and  $R^*$  can be plugged in  $c^*(k^*, R^*)$ . After that the flow budget constraint can be used to calculate the asset holdings of an individual throughout their lifetime.

The only term that is missing from the derivations explained in the preceding pages is the tax rate,  $T$ , which clears the PAYG social security system. The tax rate shows up in the equations (1.22) and (1.23), which are required to solve (1.21) and in equation (1.32), which is required to solve (1.27). Hence it is not possible to numerically solve (1.21) and (1.27) without a formula for the tax rate. I explain the derivation of the tax rate,  $T$ , next.

I begin by writing an expression for the total contributions paid by working age agents at a generic calendar date  $t$ :

$$\int_{t-R}^t Tw^*(t)P(\nu, t)d\nu \quad (1.34)$$

where  $P(\nu, t)$  is the number of people surviving from time  $\nu$  birth cohort at time  $t$ . Plugging equation (1.10) in (1.34), I obtain:

$$\int_{t-R}^t Tw^*(t)\beta P(t)e^{-n(t-\nu)-M(t-\nu)}d\nu \quad (1.35)$$

Total benefits received by the retired agents at time  $t$  is similarly given by:

$$\int_{t-D}^{t-R} bw^*(\nu + R)P(t)e^{-n(t-\nu)-M(t-\nu)}d\nu \quad (1.36)$$

where I make the assumption that retired agents receive a fraction of their wages at the time of their retirement as their pension benefits in their post retirement years. This fraction is the "replacement ratio" of pension benefits and denoted with the letter  $b$ .

Setting the total contributions paid by working age agents equal to the total benefits received by the retired agents yields the tax rate which clears the PAYG social security system:

$$T = \frac{\int_{t-D}^{t-R} b w^*(\nu + R) e^{-n(t-\nu)-M(t-\nu)} d\nu}{\int_{t-R}^t w^*(t) e^{-n(t-\nu)-M(t-\nu)} d\nu} \quad (1.37)$$

Equation (1.37) can be simplified by noting that the wages grow at rate  $\gamma_A$  in the steady state. This implies  $w^*(\nu + R) = w^*(t) e^{-\gamma_A(t-\nu-R)}$ . Using this equality in (37) yields:

$$T = \frac{b e^{\gamma_A R} \int_{t-D}^{t-R} e^{-(n+\gamma_A)(t-\nu)-M(t-\nu)} d\nu}{\int_{t-R}^t e^{-n(t-\nu)-M(t-\nu)} d\nu} \quad (1.38)$$

Please note once again that I assume the PAYG social security system does not run either a surplus or a deficit at any given moment. Hence the tax rate  $T$  adjusts in the face of exogenous changes, such as the demographic changes that are investigated in the numerical analysis section of this paper, to keep the PAYG system's budget in balance.

Before closing this section, I provide the expression for the payroll tax which clears a fully funded (FF) social security system<sup>20</sup>.

In the case of a fully funded system, the contributions paid by the working agents go to a government body or a bank / investment company / fund which invests the proceeds in securities<sup>21</sup>. When the agents retire, they draw down the accumulated stock of savings on their individual accounts.

<sup>20</sup>I investigate the effects of switching from a pay as you go (PAYG) system to a fully funded (FF) system in the numerical simulations section of this paper.

<sup>21</sup>In the United States, traditional IRAs (Individual Retirement Funds) can be administered by the state or federal government or private banks / investment companies. They are very popular examples of fully funded systems. One of the reasons for their popularity is that contributions to traditional IRAs are tax-deductible in the US.

From the above explanation, it is clear that the total contributions paid by an agent in the model throughout their working life accumulate to the number given by the following integral by the time of their retirement:

$$\int_{t-R}^t T w^*(\tau) e^{\int_x^R [r^* + \mu(u) - \gamma_A] dz} dx \quad (1.39)$$

where  $t$  is the calendar date of retirement,  $T$  is the rate of the payroll tax,  $\tau \in [t-R, t]$  is a generic calendar date,  $R$  is the endogenously chosen retirement age,  $\mu(u)$  is the instantaneous mortality hazard at age  $u \in [0, R]$ , and  $\gamma_A$  is the rate of labor augmenting tech progress in the production function.

In post retirement, an agent in the model receives the following sum of benefits:

$$\int_t^{t-R+D} b w^*(t) dt \quad (1.40)$$

where  $b$  is the replacement ratio of the fully funded system,  $w^*(t)$  is the wage received by the agent at the time of retirement and  $D$  is the maximum age the agents in the model can survive (ie: the terminal age).

Setting equations (1.39) and (1.40) equal yields the following expression for the payroll tax  $T$ :

$$T = \frac{b[D - R]}{\int_0^R e^{\int_x^R [r^* + \mu(u) - \gamma_A] dz} dx} \quad (1.41)$$

Please note that in the above derivation, I assumed that the body who is administering the fully funded system invests the accumulated contributions of the working agents in the lifetime annuity market. In practice, there are likely to be regulatory restrictions which might prevent this investment. It might be required by law that the proceeds of the fully funded system be invested in a risk free asset. In that case, equation (1.41) would be modified to the following equation:

$$T = \frac{b[D - R]}{\int_0^R e^{[r^* - \gamma_A]u} du} \quad (1.42)$$

## 1.6 Calibration

### 1.6.1 Demographic Structure and Variables

For the survival function I choose the specification suggested in the 2002 paper by Boucekkine et al., [18]:

$$S(t - \nu) = e^{-M(t-\nu)} = \frac{\mu_0 - e^{\mu_1(t-\nu)}}{\mu_0 - 1} \quad (1.43)$$

where  $\mu_0 > 1$  and  $\mu_1 > 0$ .<sup>22</sup> In this specification, the parameter  $\mu_0$  controls the mortality hazard of younger cohorts. Similarly the parameter  $\mu_1$  controls the mortality hazard of older cohorts.

The instantaneous mortality hazard at any age  $t - \nu$  implied by the Boucekkine (BCL) survival function can be obtained as follows:

$$\mu(t - \nu) = -\frac{S'(t - \nu)}{S(t - \nu)} = \frac{\mu_1 e^{\mu_1(t-\nu)}}{\mu_0 - e^{\mu_1(t-\nu)}} \quad (1.44)$$

Next the cumulative mortality hazard at any age  $t - \nu$  associated with the BCL survival function can be obtained from the instantaneous mortality hazard function by using the equation:

$$M(t - \nu) = \int_0^{t-\nu} \mu(\tau) d\tau = \ln(\mu_0 - 1) - \ln(\mu_0 - e^{\mu_1(t-\nu)}) \quad (1.45)$$

Agents in the model can live up to a terminal age  $D$ . Setting  $S(D) = 0$  yields:

$$D = \frac{\ln(\mu_0)}{\mu_1} \quad (1.46)$$

The unconditional life expectancy predicted by the BCL survival function is calculated

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<sup>22</sup>Boucekkine et al. [18] notes the theoretical possibility that  $0 < \mu_0 < 1$  and  $\mu_1 < 0$ . However, nonlinear least squares estimates of the BCL survival function explained in the next page indicate that for the United States in 2017,  $\mu_0 > 1$  and  $\mu_1 > 0$  happened to be the empirically valid case.

by using: <sup>23</sup>

$$\lambda = \int_{\nu}^{\nu+D} (t - \nu)(-\mu_1) \frac{e^{\mu_1(t-\nu)}}{1 - \mu_0} dt = \left[ -\frac{1}{\mu_1} + \frac{\mu_0 \ln(\mu_0)}{(1 - \mu_0)(-\mu_1)} \right] \quad (1.47)$$

To estimate  $\mu_0$  and  $\mu_1$ , I minimize the following objective function:

$$\sum_{t-\nu=0}^{t-\nu=90} [S_{Data}(t - \nu) - S_{BCL}(t - \nu)]^2 \quad (1.48)$$

In equation (1.47),  $S_{Data}$  stands for the actual conditional survival probabilities (ie: actual probabilities of surviving to a future age of  $20 + t - \nu$  conditional on having survived to age 20) in the United States in 2017. <sup>24</sup> Since agents in the model are born the moment they join the labor force, which I assume happens when they are 20 years old, actual age of 20 corresponds to age 0 in the model and actual age 110 corresponds to age 90 in the model. Hence the lower and upper limits of the summation in equation (1.47) are  $t - \nu = 0$  to  $t - \nu = 90$ .

$S_{BCL}$  in equation (1.47) stands for the estimated conditional survival probabilities of the agents from actual age 20 to actual age 110. <sup>25</sup> These actual ages again correspond to ages 0 and 90 in the model.

Solving the minimization problem, I obtain the following estimates for youth and old age mortality parameters:  $\mu_0 = 65.1154$ ,  $\mu_1 = 0.0548$ . In figure-1, I plot both the actual and the estimated conditional survival probabilities in the US in 2017. The estimated parameter values produce quite a nice fit to the actual conditional survival probabilities in the US in 2017, with the exception of cohorts that are 95 years and older. Around 95 years of age, estimated conditional survival probabilities begin to diverge from actual conditional survival

<sup>23</sup>In equation (1.43) assuming  $\mu_0 > 0$ , if the old age mortality parameter  $\mu_1 \rightarrow 0$ , life expectancy collapses to  $1/\mu_0$ . This is the constant mortality case explored in the 1985 seminal paper by Blanchard [15].

<sup>24</sup>Source of the data for the conditional survival probabilities is Human Mortality Database, [5]. The reason I choose to use survival data from 2017 is because that is the latest survival data available for the United States in HMD at the time I am writing this paper.

<sup>25</sup>The reason I am using 110 as the maximum possible age in the estimation of parameters  $\mu_0$  and  $\mu_1$  is because 110 is the oldest age available in 2017 survival data for the US in [5].

probabilities significantly. [61] note that this is not a serious problem, since only a small fraction of the US society survives beyond age 90 even with increasing life expectancy today.

26

Using equation (1.45), it is possible to obtain the terminal age predicted by the estimated BCL survival function:  $D = \frac{\ln(\mu_0)}{\mu_1} = \frac{\ln(65.1154)}{0.0548} = 76.207$  years in the model, ie:  $20 + 76.207 = 96.207$  years in reality. This number is significantly lower than the maximum of 110 years that can be found in Human Mortality Database [5] for the US in 2017. This is the second shortcoming of the BCL survival function. Due to its poor fit of the conditional survival probabilities of the older cohorts, BCL survival function predicts a lower terminal age than the empirically observed value.

Next, the unconditional life expectancy at birth predicted by the BCL survival function can be obtained by using equation (1.46):  $\lambda = \left[ -\frac{1}{\mu_1} + \frac{\mu_0 \ln(\mu_0)}{(1-\mu_0)(-\mu_1)} \right] = 59.149$  years in the model, ie:  $20 + 59.149 = 79.149$  years in reality. Although still lower than its empirically observed counterpart (82.01 years in the US in 2017)<sup>27</sup>, this estimate is interestingly much better in its accuracy than the value predicted by the BCL survival function for the terminal age.

Given the shortcomings of the BCL survival function explained in the paragraphs above, I will briefly discuss the alternative survival function specifications that can be chosen in its place. One of the well known alternatives to the BCL survival function in the demographic literature is the three parameter Gompertz-Makeham (GM) survival function:  $S(t - \nu) = e^{-[\mu_0(t-\nu) + (\frac{\mu_1}{\mu_2})(e^{\mu_2(t-\nu)} - 1)]}$ . This function's origins can be traced back to two seminal papers from 19th century, [39] and [57]. Compared to BCL survival function, it provides a much better fit to the survival probabilities of older cohorts. Recently this function has been utilized by Bloom and Canning [17] and by Heijdra [42].

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<sup>26</sup>According to US Census 2010 ([2]), 0.6% of the population survived beyond age 90 and only 0.1% of the population survived beyond age 95 in 2010 in the US.

<sup>27</sup>Source of the empirically observed life expectancy at birth in the US in 2017 is page 16 of the following Population Estimates and Projections Report prepared by US Census Bureau: <https://www.census.gov/content/dam/Census/library/publications/2020/demo/p25-1145.pdf>.

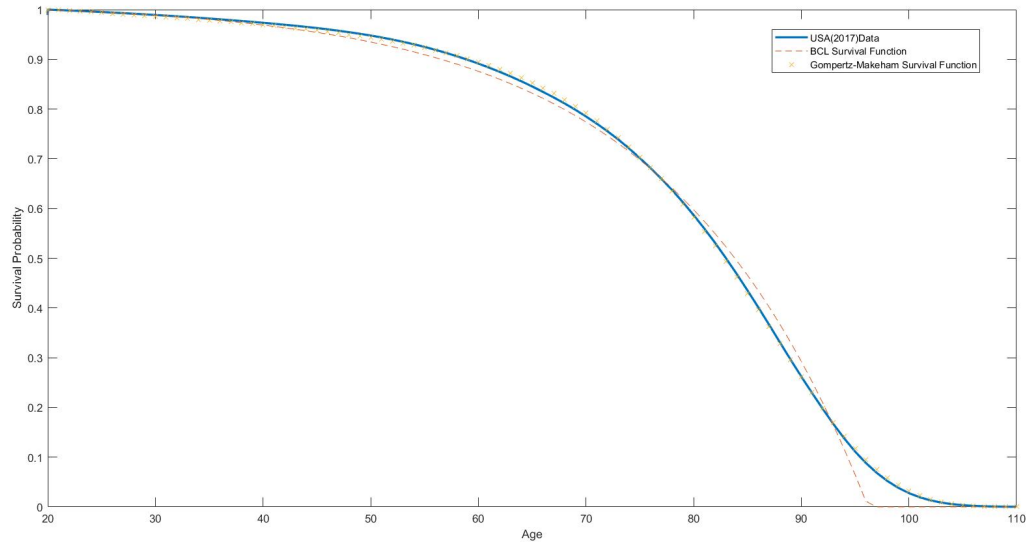


Figure 1.1: Actual and fitted survival probabilities with BCL and GM survival functions - US (2017)

The instantaneous mortality hazard implied by the Gompertz-Makeham survival function is simply  $\mu_{GM}(t - \nu) = \mu_0 + \mu_1 e^{\mu_2(t-\nu)}$ . In the demographic literature, the term  $\mu_0$  in the preceding equation is known as the "baseline mortality" and does not depend on age. The remaining part of the equation  $\mu_1 e^{\mu_2(t-\nu)}$  is known as the "senescent component" and depends on age. The main shortcoming of the GM specification in empirical work is possible correlation between "baseline mortality" and the "senescent component", which causes estimates of  $\mu_0, \mu_1, \mu_2$  to be less precise. In econometric terms, p-values of the estimates can not be trusted to identify whether estimates are statistically significant or not.<sup>28</sup> This makes the use of GM survival function in empirical work, such as the calibration and numerical analysis exercises in this paper, more difficult.

Another alternative to the BCL survival function is the de Moivre specification  $S(t - \nu) =$

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<sup>28</sup>In simpler terms, due to collinearity, even minor changes in the variables (baseline mortality and senescent component) lead to wildly different values and signs for estimated parameters  $\mu_0, \mu_1, \mu_2$  in empirical work.

$(1 - \frac{t-\nu}{\omega})^{\theta\omega-1}$  where  $\omega$  is the maximum attainable age and  $\theta$  is a parameter such that  $\theta\omega \geq 1$ . This specification is even older than the GM survival function, since it was first suggested in [31]. Its main advantage is its versatility. As explained in [20], it includes different survival functions such as the constant mortality hazard survival function used in Blanchard's 1985 paper [15] as limiting cases.<sup>29</sup> London in 1997, [54], describes the de Moivre specification as "the first continuous probability distribution to be suggested for use as a model of human survival." Despite its versatility, de Moivre survival function is a rough approximation to actual survival probabilities.

Third alternative survival function that can be used instead of BCL is the hyperbolic tangent function that was suggested by Faruquee in his 2003 paper, [34],  $S(t - \nu) = e^{-[c_0+c_1(\tanh(t,\nu,k))]}$  where  $\tanh$  is the hyperbolic tangent function and  $c_0, c_1, k$  are all parameters. In this paper, Faruquee investigates the effects of introducing this survival function in the Blanchard 1985 model, [15].

The advantage of using hyperbolic functions is that they have straightforward integrals and derivatives, which make calculating instantaneous and cumulative mortality hazards and implied life expectancies easier. As a disadvantage, in the words of Bruce and Turnovsky, [20], the hyperbolic tangent function approximates the GM survival function. Hence, from an empirical point it is more reasonable to use the GM function.

Faruquee in his 2003 paper, [34], doesn't calibrate his hyperbolic survival function with empirical survival probabilities either. Instead he borrows "Gomperty's Laws" from [15] to obtain constraints for the parameters of his hyperbolic survival function.<sup>30</sup>

Other than these three well known alternatives to the BCL specification, there are a multitude of even more complicated survival functions in the modern demographic literature

<sup>29</sup>Letting  $\omega \rightarrow \infty$ , one obtains  $S(t - \nu) = e^{-\mu(t-\nu)}$  where  $\mu$  is the constant mortality hazard. This is the survival function used in the 1985 paper by Blanchard [15]. Due to its mortality hazard being constant, Faruquee in [34] and Bruce and Turnovsky in [20] refer to this survival function as the "the perpetual youth" specification.

<sup>30</sup>Gomperty's Laws are ad hoc rules such as assuming death rate is roughly constant and near zero between ages 20 and 40 or cumulative mortality hazard is 16 % at age 80. Of course using ad hoc rules such as these is hard to justify today when empirical survival data is readily available for estimation.

such as the Siler 5 Component Competing Hazard model ([69])<sup>31</sup> or Heligman-Pollard 8 component model ([43])<sup>32</sup>. Due to the higher number of parameters in these models, they provide better fits to the empirically observed survival probabilities. But they are rarely used in economic models that include realistic demographic structures. Part of the reason is that BCL and GM survival function, both of which have just 2 parameters, are doing a well enough job capturing the same survival probabilities. Also having more parameters in a survival model, creates econometric methodology related problems such as obtaining unbiased estimates.

Weighing the advantages and disadvantages of all these alternatives for the survival function, I have decided to use the BCL specification in this paper. To investigate whether results would be sensitive to the survival model chosen, I also provide calibration results for a version of the model with the GM survival function in Table-1.2.

### 1.6.2 State dependent utility from consumption in the benchmark model

Given that there is a rapidly growing body of work which uses "health-state dependence" to explain a number of important questions in economics, it is unfortunate that state-dependent utility specifications are not frequently included in models of economic growth that investigate the effects of demographic change. One important contribution of the current work is the inclusion of "health-state dependence" by assuming a utility function of the form:

$$\gamma_0 I + (1 + \gamma_1 I)u(c(\nu, t)) \tag{1.49}$$

This is a standard formulation for "health-state dependence" and has been used in various

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<sup>31</sup>Competing hazard model ([69]) incorporates three additive risk components that compete with each other over different periods of life: one of them is dominant during the pre-maturity period, second one is a constant hazard component which is dominant during maturity and the third is a Gompertz hazard component which is dominant during senescence.

<sup>32</sup>Heligman-Pollard 8 component model ([43]) seems to have been used to investigate a wide range of problems recently. For instance, see [71], [67], [33].

works including Finkelstein (2013) [36], DeNardi (2006) [30], and Palumbo (1999)[63].

There are two states in this specification: healthy and unhealthy. In the above equation,  $I$  is an indicator function which attains a value of 0 if the individual is healthy and 1 if unhealthy. I assume there is a threshold age  $X$  at which agents deterministically transition from the healthy state to the unhealthy state.<sup>33</sup> This threshold age,  $X$ , is the same as the one that shows up in the disutility of labor specification that I explain in the next section.

In this specification,  $\gamma_0$  affects the level of utility received by the agent but doesn't change the shape of the utility function. On the other hand  $\gamma_1$  determines the shape of the utility function depending on the state of the individual. Hence, [36] refer to  $\gamma_0$  as the "level variable" and  $\gamma_1$  as the "state variable". To illustrate how  $\gamma_1$  changes the shape of the utility function, I plot three utility functions with different  $\gamma_1$  values in Figure-2 below using the CRRA type sub-utility function,  $u(c(\nu, t)) = \frac{c(\nu, t)^{1-\frac{1}{\sigma}} - 1}{1-\frac{1}{\sigma}}$ , in (48).

The uppermost utility function in Figure-2 corresponds to a good health state, ie:  $\gamma_1 = 0$ . The middle utility function corresponds to a bad health state where marginal utility of consumption decreases by 10% after threshold age  $X$  due to chronic health conditions, ie:  $\gamma_1 = -0.1$ . Finally, the bottom curve is the utility function in a bad health state where marginal utility of consumption decreases by 20% after age  $X$ , ie:  $\gamma_1 = -0.2$ .

As can be seen in Figure-2, when it is assumed that marginal utility of consumption decreases as health deteriorates, the utility curves fan out as consumption increases. In the literature this case is called "negative state dependence".<sup>34</sup> Intuitively in this case many consumption goods (such as travel, or sports equipments such as skis or footballs) are complements to good health. According to at least two empirical studies, [30] and [36], this

<sup>33</sup>The alternative to having a deterministic transition is to have a stochastic transition mechanism which has an increasing probability of transitioning from a healthy to an unhealthy state as the agent gets older. This stochastic transition mechanism can be introduced in a continuous time model by using an AR(1) process which can be approximated with a Markov transition matrix for use in discrete time models.

<sup>34</sup>If it is assumed that marginal utility of consumption increases as health deteriorates, the utility curves would fan in as consumption increases. In the literature this case is called "positive state dependence". Intuitively in this case many consumption goods (such as prepared meals or assistance with self care) are substitutes for good health. Interestingly, according to a recent study by Wang and Wang (2020) [73], this seems to be the valid case for China.

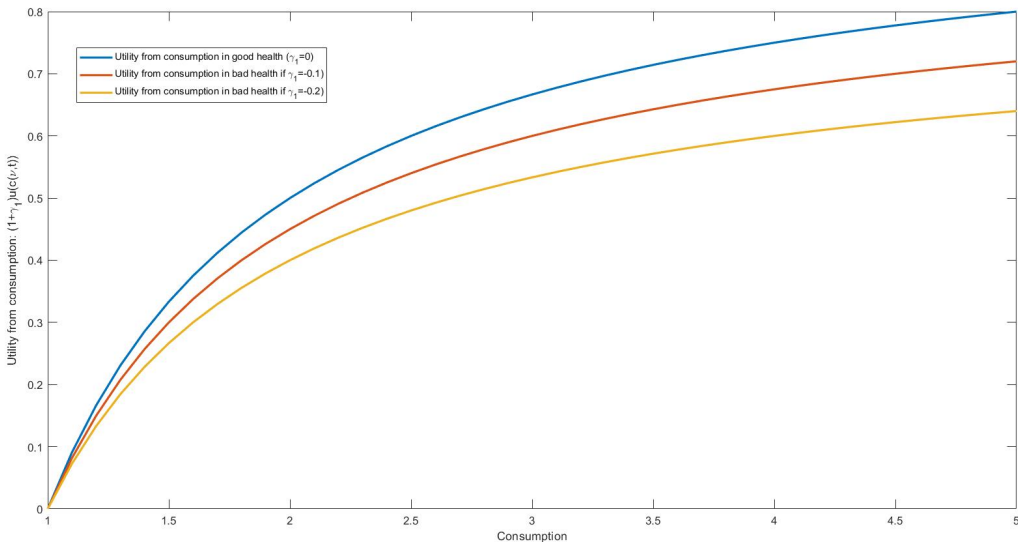


Figure 1.2: Utility function with  $\gamma_1 = 0, -0.1, -0.2$

is the relevant case for United States. Since the demographic structure of the current work is calibrated by using US survival data, I use the assumption of "negative state dependence" in this paper.

To gain some idea about empirical range of the parameters  $\gamma_0$  and  $\gamma_1$ , one can see the study in Finkelstein et al. (2013) [36]. That study investigates the impact of chronic diseases on the happiness of 11,514 Americans in their sample who are older than 50 years old by using the first seven waves of the HRS.<sup>35</sup> They define the sick state as an individual having the median number of chronic diseases (which is two diseases in their sample) or more.

Their estimates indicate that an increase of one chronic disease is associated with a statistically significant 1.1% decline in the probability the individual is happy. Marginal utility of those people in the sick state (2 or more diseases) is estimated to be 29% lower than the marginal utility of those in the healthy state (less than two diseases).

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<sup>35</sup>HRS stands for the Health&Retirement Study conducted by the University of Michigan, which is a longitudinal panel study that surveys a sample of approximately 20,000 Americans every two years.

I adopt the value of -0.022 for  $\gamma_0$  given above (two diseases correspond to a 2.2% decline in the probability the individual is happy) when numerically calibrating my model. This means that after the threshold age,  $X$ , agents in the model suffer a flat 2.2% reduction in the utility they receive from consumption.

As for the value of  $\gamma_1$  (the decline in the marginal utility of consumption), using as large a value as -0.29 yields negative values for the parameter  $\delta_0$  in the disutility of labor function in my model, which is covered in the next section. This violates the condition that both the marginal benefit of not retiring and the marginal cost of not retiring at the optimal retirement age,  $R^*$ , need to be positive (see left and right hand sides of equation (1.21) in the model). Intuitively, having a negative  $\delta_0$  implies that marginal cost of not retiring at the optimal retirement age  $R^*$  is negative. This would mean that agents retire earlier than age  $R^*$ , which would make  $R^*$  not the optimal choice.

I calibrate the parameters  $\gamma_1$  and  $\delta_0$  together by numerically trying many different pairs to make sure that the following constraints are not violated: 1. The positivity constraint explained in the above paragraph is not violated, ie:  $\delta_0$  must be positive, 2. As explained in the next section, chosen values for  $\gamma_1$  and  $\delta_0$  must ensure a retirement age of 67 and a life expectancy of 78.5 as calibration targets, 3. Negative state dependence must hold, ie:  $\gamma_1$  must be negative.

Through this trial and error approach, I find a value of -0.06 for  $\gamma_1$ , ie: a 6% decline in the marginal utility of consumption after age  $X$ . This number seems much smaller than the -0.29 reported in [36], but it must be noted that their estimated equations are highly stylized and do not consider individuals, who have less than two chronic conditions, sick. In practice, it would make much more sense for someone who is suffering from a single condition to have a lower reduction in their marginal utility of consumption than -29%. Of course, there are also individuals who are older than 50 years old and who don't have any chronic problems either. Hence, using the lower number of -6% can be interpreted as taking these facts into account.

Finally, it is important to note here that this simple specification for state dependent

utility is just a starting point for further research. It is included in the current work to quantitatively investigate the impact of having a "state dependent" utility from consumption specification on the retirement and saving decisions of the agents. I am planning to include a more realistic transition mechanism between the good health and the bad health state (which will necessarily include a stochastic element) in the next version of this paper. Another possible extension is to increase the number of possible states by introducing more than one bad health state. These worsening bad health states will be associated with further declining marginal utility of consumption.

### 1.6.3 Disutility of labor specification in the benchmark model

I assume disutility of labor at age  $t - \nu$  is given by:

$$\phi(t - \nu) = \delta_0 e^{\delta_1 \max\{t - \nu - X, 0\}} \quad (1.50)$$

where  $X$  is the threshold age at which health related problems begin increasing the disutility of labor exponentially. The parameter  $\delta_0$  is the disutility of labor before age  $X$  and  $\delta_1$  is the rate at which disutility increases exponentially after age  $X$ .

As explained in the previous section, I calibrate the parameter  $\delta_0$  together with  $\gamma_1$  using two constraints and two empirical observations. Constraints have already been explained ( $\delta_0$  must be positive and  $\gamma_1$  must be negative). The two empirical observations are the life expectancy at birth and the retirement age in the US in 2017. Life expectancy at birth in the US in 2017 was 78.5 years according to [5]. For the retirement age in 2017, there are a few alternatives that can be used. According to US Social Security Administration ([8]), the average retirement age in the US in 2017 was 62 years. However, beginning from 2017, to become entitled to full (unreduced) retirement benefits individuals had to work until 66 years and 2 months. <sup>36</sup> There is also financial bonus for delaying retirement in the US

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<sup>36</sup>This used to be 65 years before 2017 according to [8].

<sup>37</sup>. Given these incentives to retire later, a lot of Americans choose to retire later than 62 years. According to Survey of Household Economics and Decision Making ([7]), which is a survey conducted by the Federal Reserve, approximately 65% of Americans retired before or at age 62 and 91% of Americans retired before or at age 67. Hence, instead of the average retirement age, I use 67 years to calibrate  $\delta_0$ .

In the numerical calibration exercises, I calculate the value of  $\delta_0$  together with  $\gamma_1$  that would cause the agents to retire at age 67 if they have a life expectancy of 78.5 years using BCL and GM survival functions, without violating the constraints that  $\delta_0 > 0$  and  $\gamma_1 < 0$ .

Next, for the value of  $\delta_1$ , I use the increase in average health spending by age reported in "Kaiser Family Foundation Analysis of Medical Expenditure Panel Survey, 2016" ([4]). This number is 0.017 or 1.7%.

Using the BCL survival function to calibrate  $\delta_0$  and setting  $\delta_1 = 0.017$ , I plot the disutility of labor function in Figure-1.3 below:

Before closing this section, I discuss some of the alternative specifications that can be chosen for the disutility of labor function and the advantage of using the particular specification in equation (1.49).

One alternative in the literature to the specification I choose is the function  $\phi(t - \nu) = ke^{\frac{t-\nu}{\lambda}}$  where  $k$  is a constant which measures the intensity of the disutility of work<sup>38</sup>,  $\lambda$  is the life expectancy at time of birth, and  $t - \nu$  is the age of the agent. This function was first suggested by Bloom and Canning in their 2007 paper, [16], to formalize the idea of "compression of morbidity". In this specification, as mortality rates decline, life expectancy  $\lambda$  increases and the exponent of the function decreases. This causes the disutility of labor to decline for all cohorts alive at time  $t$ . Although analytically convenient, disutility of

<sup>37</sup>The full benefit age in the United States in 2017 was 66 years and 2 months old. An individual who did not begin drawing benefits until the latest possible age of 70 years was able to receive 8 percent higher benefits for each year they did not collect benefits.

<sup>38</sup>One interesting venue for research is to allow for the magnitude of disutility of labor to vary as the nature of employment changes. As an example, blue collar and white collar jobs intuitively have different values for their intensity of disutility.

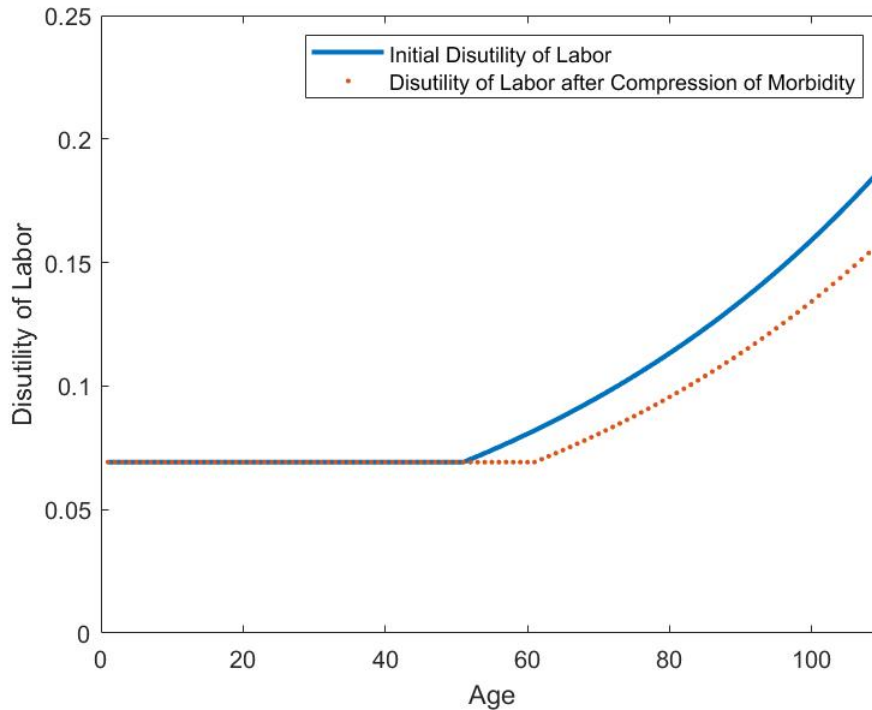


Figure 1.3: Disutility of Labor

labor declining for all cohorts simultaneously is hard to justify empirically. Intuitively, this is similar to claiming that a cure for a condition that increases disutility of labor (such as a new procedure for hip replacement surgery or a cure for Alzheimer’s disease) or a medical invention (such as a mobility scooter) benefits all of the cohorts alive at time  $t$  by reducing their disutility of labor simultaneously. It is difficult to imagine how younger cohorts benefit from these cures and inventions.

Another alternative specification in the literature is assuming that disutility of labor is proportional to the instantaneous mortality hazard. The 2014 paper by Bloom and Canning, [17], uses this assumption by employing the GM survival function. In that study, disutility of labor is formalized by the equation:  $d\mu(t) = d(\rho + ae^{bt})$  where  $d$  is a constant used to weigh the disutility of labor and  $\mu(t) = \rho + ae^{bt}$  is the instantaneous mortality hazard function

implied by the GM survival function.

The advantage of the specification I choose to use is in the way it formalizes compression of morbidity. Equation (1.49) implies that agents spend their lives in a morbid condition between age  $X$  and the terminal age  $D$ , since after age  $X$  disutility of labor begin increasing exponentially. With this specification compression of morbidity can be introduced as an increase in age  $X$ <sup>39</sup>. I illustrate this idea in Figure-1.3.

In Figure-1.3, I initially assume that morbidity begins affecting agents at age  $X = 50$  years. Before 50 years old, agents' disutility of labor is constant at  $\delta_0$ . After  $X = 50$  years, disutility begins increasing exponentially. Then medical advances happen which increase the threshold age  $X$  to 60 years, ie: compression of morbidity happens, individuals now spend a smaller portion of their lives in a morbid condition. In their improved situation, agents can enjoy the same disutility of labor until they are 60 years old. Only after 60 years, disutility begins increasing.

As can be seen in Figure-1.3, with the specification in equation (1.49), as compression of morbidity happens, only older cohorts' disutility of labor decreases. With this disutility of labor function, younger cohorts don't benefit from medical advances such as cures for ailments that affect older cohorts. This unique disutility of labor function is much easier to justify intuitively and is one of the contributions of the current work.

#### 1.6.4 *Structural Parameters and Equilibrium of the Benchmark Model*

The structural parameters are reported in Table-1 under BCL and GM survival functions. Implied economic variables are calculated by also using BCL and the GM survival functions and reported in Table-1.2.

I begin with the CRRA type sub-utility function by setting the elasticity of intertemporal substitution,  $\sigma = 0.5$ . The parameter  $\gamma_0$  is set to 0.022 and  $\gamma_1$  is calibrated to -0.06 in the

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<sup>39</sup>Theoretically it is also possible that the parameter  $\delta_1$  (rate at which disutility of labor increases after age  $X$ ) can decrease as the threshold age  $X$  increases due to medical advances. Proving this theory empirically is more challenging though than showing that age  $X$  has indeed been increasing in reality.

state dependent utility specification. I use  $\delta_1 = 0.017$  in the disutility of labor function and  $\delta_0$  is calibrated to 0.0911.

For the pure discount rate, I use  $\rho = 0.035$ <sup>40</sup>. The total discount rate of an individual from the average age cohort is  $\rho + \mu(\bar{u})=0.0414$ <sup>41</sup>.

Next, on the firms side, I assume output is produced using a Cobb-Douglas production function  $y(t) = k(t)^\alpha$  where  $y(t)$  is output per effective worker and  $k(t) = \left[ \frac{K(t)}{L(t)A(t)} \right]$  is capital per effective worker.  $A(t)$  is the level of labor augmenting tech progress at time  $t$  which increases at rate  $\gamma_A = 0.02$ , ie:  $A(t) = A(0)e^{\gamma_A t}$ . In the numerical exercises, I assume  $A(0) = 1$ . Income share of capital  $\alpha$  is set equal to 0.35 and the depreciation rate  $\delta$  to 0.05.

Demographic parameters used for numerically solving the model come from fitting the BCL survival function to the actual survival probabilities of the United States in 2017. That fitting exercise produces a terminal age of  $D = 96.207$  years and a life expectancy of  $\lambda = 79.149$  years. For the population growth rate, I use  $n = 0.01$ <sup>42</sup>.

I choose  $b=0.4$  as the replacement ratio between the benefit retired agents receive and the wage they earn at the precise moment of their retirement<sup>43</sup>. In other words,  $bw(\nu + R) = 0.4w(\nu + R^*)$  is the amount of benefit drawn by a retired agent of time  $\nu$  birth cohort.

Given these values for the structural parameters, I solve the model numerically by using the estimated BCL survival function to obtain the optimal retirement age,  $R^* = 43.9483$ , ie:  $20 + 43.9483 = 63.9483$  in actual life years. This number is slightly lower than the retirement

<sup>40</sup>Intertemporal elasticity of substitution and the discount rate were carefully chosen from a reasonable range of numbers in the literature so that the model produces a sort of hump shape for the asset holdings-age profile of agents in numerical simulations.

<sup>41</sup> $\mu(\bar{u})$  is calculated by pluggin  $\bar{u}$  in the instantenous mortality hazard associated with BCL survival function, equation (1.43). The average age  $\bar{u}$  is calculated by using the formula  $\left[ \frac{\beta\mu_0}{\mu_0-1} \right] \left[ \frac{1}{n^2} \right] \left[ e^{-nD}(-nD - 1) + 1 \right] - \left[ \frac{\beta}{\mu_0-1} \right] \left[ \frac{1}{(\mu_1-n)^2} \right] \left[ e^{(\mu_1-n)D}((\mu_1-n)D - 1) + 1 \right]$ .

<sup>42</sup>If I plug  $n = 0.01$  in the the "demographic steady state condition" (equation (1.12)), I obtain  $\beta = 0.0228$ , ie:2.28% as the birth rate. Compared with the actual birth rate of 1.766 (source:[1]), this number is too high. [61] has a similarly high estimate for the fertility rate and interpret this as the effect of the population growth rate taking into account immigration, which is not modeled in this paper.

<sup>43</sup>The empirical counterpart of the replacement ratio in the US is also 0.4. See the following document by the US Social Security Administration: <https://www.ssa.gov/policy/docs/ssb/v68n2/v68n2p1.html>.

<b>Table-1.1</b> Structural and Demographic Parameters of the Benchmark Model			
Parameter	Symbol	BCL Function	GM Function
Intertemporal Elasticity of Substitution	$\sigma$	0.5	0.5
Initial decline in utility from consumption	$\gamma_0$	0.022	0.022
Change in marginal utility of consumption	$\gamma_1$	-0.06	-0.07
Rate of increase in disutility of labor	$\delta_1$	0.017	0.017
Initial disutility of labor	$\delta_0$	0.0911	0.1224
Rate of tech progress	$\gamma_A$	0.02	0.02
Initial level of tech	A(0)	1	1
Income share of capital	$\alpha$	0.35	0.35
Depreciation Rate	$\delta$	0.05	0.05
Pure discount rate	$\rho$	0.035	0.035
Total discount rate	$\rho + \mu(\bar{u})$	0.0414	0.0580
Youth mortality	$\mu_0$	65.1154	0.0011
Old age mortality	$\mu_1$	0.0548	0.0001
2nd old age parameter in GM	$\mu_2$	NA	0.0980
Life expectancy	$\lambda$	79.149	82.1493
Terminal age	$D$	96.207	110
Population growth rate	n	0.01	0.01
Birth rate (implied)	$\beta$	0.0228	0.0219
Replacement ratio	b	0.4	0.4

Table 1.1: Structural and Demographic Parameters of the Benchmark Model

<b>Table-1.2</b> Equilibrium Values of Economic Variables in the Benchmark Model			
Variable	Symbol	BCL Function	GM Function
Retirement age	$R^*$	43.9483	43.8494
Capital per effective worker	$k^*$	4.8385	5.2982
Output per effective worker	$y^*$	1.7364	1.7924
Capital / output ratio	$k^*/y^*$	2.7865	2.9559
Return to capital	$r^*$	0.0756	0.0684
Wage rate at time 0	$w^*(0)$	1.1287	1.1651
Wage rate at time t	$w^*(t)$	$1.1287e^{0.02t}$	$1.1651e^{0.02t}$
Proportion of life spent working	$\Gamma$	0.7060	0.6780
Consumption per capita	$c^*$	1.3493	1.3686
Consumption of first birth cohort (at time of birth)	$c(0,0)$	1.0347	1.1294
Consumption of a generic birth cohort (at time of birth)	$c(\nu, \nu)$	$1.0347e^{0.02\nu}$	$1.1294e^{0.02\nu}$
Consumption of time $\nu$ birth cohort individual at time $t$	$c(\nu, t)$	$c(\nu, \nu)e^{0.0203(t-\nu)}$	$c(\nu, \nu)e^{0.0167(t-\nu)}$
Growth rate of consumption at birth	$\gamma_{c(\nu, \nu)}$	0.02	0.02
Growth rate of consumption across time	$\gamma_{c(\nu, t)}$	0.0203	0.0167
Payroll tax	T	0.0943	0.1043

Table 1.2: Equilibrium Values of Economic Variables in the Benchmark Model

age of 67 years I assumed in the calibration section. As explained earlier, there are financial incentives to retire later in the current structure of the US Social Security System. The absence of these incentive mechanisms in the current work can be part of the reason  $R^*$  turns out to be lower than 67 years.

Steady state capital per worker,  $k^*$ , is calculated as 4.838519 in the benchmark model. For comparison, Mierau and Turnovsky (2014) [61] has a capital per worker value in steady state equal to 5.6206 which is much higher than my result. This reflects the fact that there is a social security system and an endogenously chosen retirement age in my model, which enables the agents to draw retirement benefits. This reduces the incentive to save for future consumption in my model and produces the lower value of  $k^* = 4.8385$ .

Using  $k^*$ , it is possible to calculate output per worker in the steady state as  $y^* = (k^*)^\alpha = 1.7364$ . This yields a capital output ratio of  $k^*/y^* = 2.7865$ .

The return to capital in the steady state,  $r^*$ , implied by the model is 0.0756, ie: 7.56%. This rather high return to saving reflects the lower amount of capital per worker in the steady state of the model.

As per equation (1.16), wage rate is not constant in the model and grows at the rate of tech progress. Assuming  $A(0) = 0$ , wage rate at time  $t = 0$  is given by  $w(0) = (1 - \alpha)(k^*)^\alpha = 1.1287$  and at a generic time  $t$  in the steady state is equal to  $w^*(t) = 1.1287e^{0.02t}$ .

The payroll tax that clears the pay as you go (PAYG) social security system of the benchmark model is calculated as  $T = 0.0943$ , or 9.43%<sup>44</sup>.

Next, the proportion of lifetime spent working in the steady state is calculated as 0.7060, ie: 70.6% of the agents' adult lives after the age of 20. The empirical counterpart to this proportion is 0.7509<sup>45</sup>. Hence, the proportion predicted by the model is pretty close to the

<sup>44</sup>Even though the current work doesn't use the payroll tax as a calibration target, for comparison purposes, "Social Security and Medicare Withholding Rate" is 6.2% for the employees and another 6.2% for the employers, for a total of 12.4% in the US. Source: <https://www.irs.gov/taxtopics/tc751>

<sup>45</sup>The theoretical value of 70.6% is calculated numerically by using the ratio  $\frac{\int_0^R S(u)du}{\int_0^D S(u)du}$ . As explained in the calibration section, approximately 91% of the Americans retired before or at age 67 in 2017. Taking the retirement age of 67 and the observed life expectancy of 82.01 years in 2017, I obtain the value of 0.7509 as the empirical counterpart to the theoretical value.

empirical value.

Aggregate consumption per capita is calculated as 1.3493. Consumption of the very first birth cohort at the time of their birth in the model,  $c(0, 0)$  is equal to 1.0347.

Consumption of the birth cohorts at the time of their birth grows across cohorts at rate  $\gamma_A = 0.02$ . Hence,  $c(\nu, \nu) = c(0, 0)e^{\gamma_A \nu} = 1.0347e^{0.02\nu}$  where  $\nu \geq 0$ .

Similarly, consumption of an individual of time  $\nu$  birth cohort grows at rate  $\gamma_c = \sigma(r^* - \rho) = 0.0203$ , ie: at a rate of 2.03%. This means,  $c(\nu, t) = c(\nu, \nu)e^{0.0203(t-\nu)}$ . This equation can further be written using  $c(0, 0)$ :  $c(\nu, t) = c(0, 0)e^{0.02\nu}e^{0.0203(t-\nu)}$  where  $c(0, 0)$  has already been calculated as 1.0347.

Bottomline is, the consumption paths of individuals both across birth cohorts and across time are known. Hence, given their flow budget constraint, it is also possible to calculate the paths of their asset holdings, again, both across birth cohorts and across time<sup>46</sup>. I plot the steady state consumption and saving paths of the agents and the cohorts as a function of age in the next section.

## 1.7 Numerical Simulations

### 1.7.1 General Equilibrium Behavior of Individuals

Along the balanced growth path, the consumption of a time  $\nu$  birth cohort individual grows at rate  $\gamma_c = \sigma(r^* - \rho)$ . For illustration purposes, I pick an individual from time  $\nu=10$  birth cohort and plot their consumption from birth (age 20) to terminal age ( $20+D=96$  years) in Figure-1.4 on the next page.

Obtaining the asset holdings of any random individual in the population is straightforward as well. Begin with the flow budget constraint of a time  $\nu$  birth cohort individual, which can be found in equation (1.6), and integrate it forward from their time of birth  $\nu$  to any time  $t$  such that  $\nu \leq t \leq \nu + D$ . This yields the following period budget constraint for

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<sup>46</sup>In other words, both longitudinal and cross-sectional paths of the consumption and saving decisions of the agents have been calculated numerically at this point.

the chosen interval  $[\nu, t]$ :

$$\begin{aligned}
A(\nu, t) &+ \left[ \int_{\nu}^t (1 - T)w(\tau)L(\nu, \tau)e^{-r^*(\tau-\nu)}S(\tau - \nu)d\tau \right] e^{r^*(t-\nu)}[S(t - \nu)]^{-1} \\
&+ \left[ bw(\nu + R^*) \int_{\nu}^t (1 - L(\nu, \tau))e^{-r^*(\tau-\nu)}S(\tau - \nu)d\tau \right] e^{r^*(t-\nu)}[S(t - \nu)]^{-1} \quad (1.51) \\
&= \left[ \int_{\nu}^t c(\nu, \tau)e^{-r^*(\tau-\nu)}S(\tau - \nu)d\tau \right] e^{r^*(t-\nu)}[S(t - \nu)]^{-1}
\end{aligned}$$

where  $L(\nu, \tau)$  is the inelastic labor supply of the individual. Before retirement  $L(\nu, \tau)$  equals 1 and after retirement 0. Again for illustration purposes, I pick an individual from time  $\nu=10$  birth cohort and plot their asset holdings from birth (age 20) to terminal age ( $20+D=96$  years) in Figure-1.5 on the next page.

Interestingly, Figure-1.5 reveals that despite all the additional features introduced in this paper (state dependent utility from consumption, disutility of labor, endogenous retirement choice, presence of a social security system, uncertain lifetime, realistic demographic structure), the OLG model still can produce some sort of hump shape for the asset holdings-age profile of the agents. Unlike the typical hump shape in textbooks<sup>47</sup>, the distribution to the left of the peak has a different shape to the right of the peak in Figure-1.5.

The unique shape of the asset holdings-age profile of individuals in Figure-1.5 is mainly due to three factors: 1. The budget constraint of the agents before and after the chosen retirement age,  $R^*$ , 2. Growth rate of consumption across time,  $\gamma_c(\nu, t)$ , 3. Growth rate of wages,  $\gamma_A$ <sup>48</sup>.

In the working age period of their lives, there are two sources of income for the agents, labor income and return on whatever assets they save after financing consumption. By in-

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<sup>47</sup>For an example of the typical hump-shaped asset holdings-age profile, the reader can see the excellent textbook, Turnovsky(2000) [72].

<sup>48</sup>Assuming different values for structural parameters such as  $\rho, \sigma, A(0)$ , it is possible to obtain a different value for  $\gamma_c(\nu, t)$ . The shape of the asset-holdings age profile is very sensitive to the relationship between  $\gamma_c(\nu, t)$  and  $\gamma_A$ . Assuming a slightly lower consumption growth rate or a slightly higher wage growth rate than the figures reported in the calibration section of this paper, it is possible to make agents accumulate assets exponentially until terminal age. Similarly assuming a slightly higher consumption growth rate or a slightly lower wage growth rate, agents in the model can be made to accumulate debt exponentially.

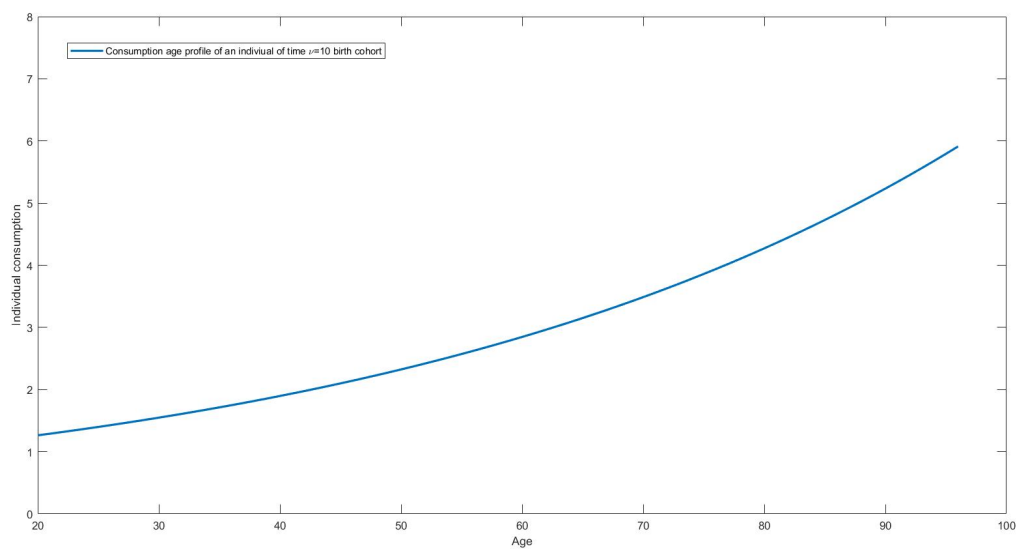


Figure 1.4: Consumption-age profile of an individual of time  $\nu = 10$  birth cohort

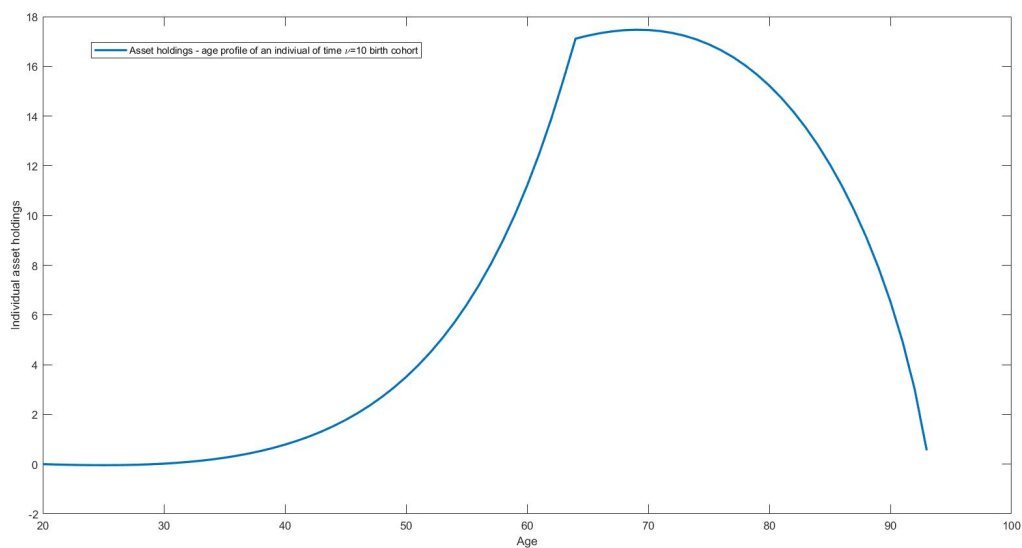


Figure 1.5: Asset holdings-age profile of an individual of time  $\nu = 10$  birth cohort

specting Figure-1.5, it can be seen that during this period agents are able to accumulate assets until their chosen retirement age,  $R^* = 64.9801$  years. Despite the fact that consumption grows across time more rapidly than their labor income ( $\gamma_c(\nu, t) > \gamma_A$ ), due to the relatively low levels of consumption especially in their early years, there is sufficient income left over to save for the future. As their assets accumulate, they get greater and greater returns on their savings and this enables them to accumulate assets at an increasing rate.

After retirement, agents lose their labor income but begin receiving a constant stream of pension benefits from the pay as you go (PAYG) social security system in addition to the return on their savings. The constant income from pension benefits and the returns from asset holdings remains sufficient to cover ever growing consumption expenditures and even accumulate further assets for approximately 6 years after retirement<sup>49</sup>.

After approximately 70 years, income of the retired agents is no longer sufficient to finance the increasing consumption expenditure. They begin selling their assets and their asset holding decline to 0 at the terminal age  $D = 96.207$ .

Before closing this section, it is important to note that the current model has an interesting prediction for the asset holdings of the agents in the final 3 years of their lives (ie: at ages 94, 95, and 96 years). At age 93 asset holdings of the agents approach 0. Since the consumption expenditure keeps growing despite declining asset holdings, the model predicts that the agents begin borrowing by using lifetime annuities at age 94. As explained earlier, when an agent borrows in the lifetime annuity market, they have to pay an extra interest, which is equal to their instantaneous mortality hazard, on top of the risk free rate, ie:  $\mu(t - \nu) + r^*$ . The problem with borrowing so late in life is that after age 93, there is a very rapid decline in survival probabilities. Hence, the cost of borrowing gets extremely high at ages 94, 95 and 96. By borrowing at age 94, the model predicts that the agent manages

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<sup>49</sup>From figure-1.5, it can be seen that even after retirement at age  $R^* = 63.9483$ , agents are able to accumulate assets until around age 70. Since labor income is lost in this period and replaced with a constant stream of pension benefits, asset accumulation happens at a decreasing rate during the 6 years between  $R^* = 63.9483$  and 70.

to get into a catastrophic borrowing spiral in the last 3 years of his life<sup>50</sup>. This would violate the boundary condition  $A(\nu, \nu + D) = 0$ .

From an empirical standpoint it is hard to justify the existence of a debt market that would let 94, 95 or 96 year old agents borrow excessive amounts of money to finance their growing consumption expenditure, given the very high instantaneous mortality hazards at those ages. This is one failure of the predictive power of the current model.

Fortunately, again from an empirical standpoint, there aren't many individuals of ages 94, 95 and 96 living in the United States in any given year. For example, in 2017 (the target year in the calibration exercises of this paper), proportion of people in the 95-99 year old bracket in the US was less than 0.2% in the population pyramid of the United States in 2017 (Source: [6]).

In the next section I plot and explain the steady state paths of consumption and asset holding of cohorts.

### 1.7.2 General Equilibrium Behavior of Cohorts

The number of individuals surviving from a generic time  $\nu$  birth cohort at time  $t$  was given in equation (1.10):  $P(\nu, t) = \beta P(\nu) S(t - \nu)$ , where  $\beta$  is the birth rate,  $P(\nu)$  is the size of the population at time  $\nu$  and  $S(t - \nu)$  is the estimated survival probability which is calculated using the BCL survival function in the benchmark model. Using the fact that population grows at a constant rate  $n$ , equation (1.10) can be written as:  $P(\nu, t) = \beta P(t) e^{-n(t-\nu)} S(t-\nu)$ .

Plugging in the values for the demographic parameters that were estimated in the calibration section<sup>51</sup>, I plot the evolution of time  $\nu=10$  birth cohort's size in Figure 1.6, which can be found in the following pages.

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<sup>50</sup>I assume that if an agent dies in debt, their debt is forgiven just like in [77].

<sup>51</sup>Using equation (1.12),  $\beta$  was obtained as 0.0228, or 2.28%.  $\mu_0$  and  $\mu_1$  were estimated as 65.1154 and 0.0548 respectively.  $n$  was assumed to be 0.01%. To calculate size of the population at time  $t$ , I assume the size of the initial population  $P(0) = 1$ , which means  $P(t) = e^{0.01t}$ .

Multiplying the size of the time  $\nu=10$  birth cohort with the level of consumption of a surviving individual from that cohort at each age ( $P(\nu = 10, t) * C(\nu = 10, t)$ ), I obtain the evolution of the cohort's consumption from age 20 to terminal age  $D=96$  and plot it in Figure-1.7, which is the middle figure on the next page.

Inspecting Figure-1.7, it can be seen that the model predicts a hump shaped consumption profile over the lifespan of a cohort. This is an intriguing result since the consumption of any random individual in the model grows exponentially at a constant rate,  $\gamma_C(\nu, t)$ .

Intuitively, there are two competing factors at work that produce the profile in Figure-1.7. On one hand, each surviving member in the cohort wants to enjoy an increasing consumption stream. On the other hand, as age increases, survival probabilities decline and hence there are less and less individuals surviving from a given cohort. In the case of time  $\nu=10$  birth cohort, the first factor (individuals wanting more consumption) dominates until about age 75, so the total consumption of the cohort increases. After age 75, individuals from the time  $\nu=10$  birth cohort die rapidly enough that the cohort's total consumption begins declining.

Next, I do the same exercise for the assets of a given cohort. The total asset holdings of time  $\nu$  birth cohort at time  $t$  is equal to  $P(\nu, t) * A(\nu, t)$ . To illustrate, I calculate the total assets of time  $\nu=10$  birth cohort individuals over their lifespan and plot it in Figure-1.8, which is at the bottom of the next page.

Figure-1.8 has a roughly similar shape to individual's asset holdings-age profile in Figure-1.5. There is an important difference between the two figures though. The peak of the cohort's total assets occurs at the age of retirement, whereas the individuals keep on accumulating assets for another 6 years after retirement. This means that after retirement, sufficiently many individuals from time  $\nu =10$  birth cohort die at every instant that the total assets of the cohort decreases.

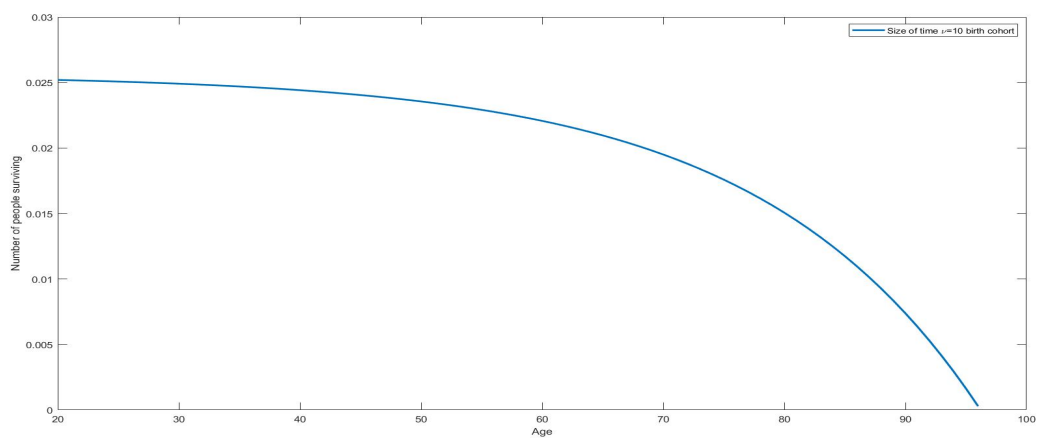


Figure 1.6: Evolution of the size of time  $\nu = 10$  birth cohort with age

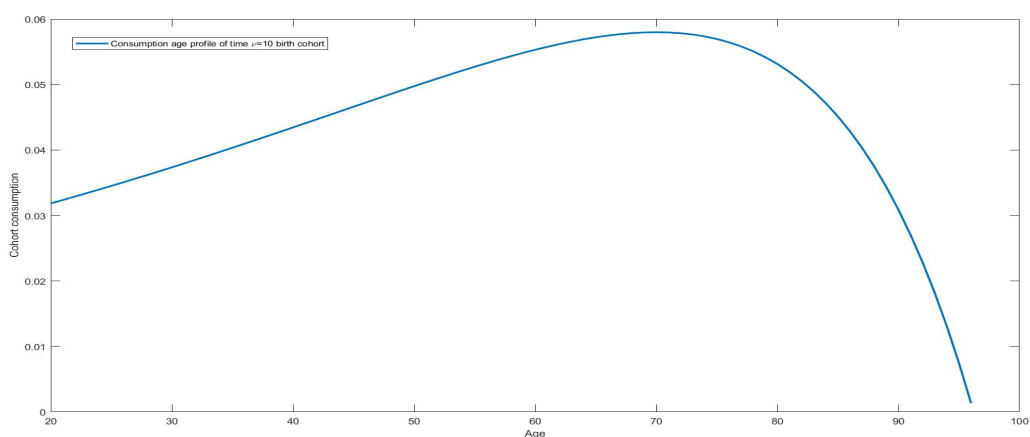


Figure 1.7: Consumption-age profile of time  $\nu = 10$  birth cohort

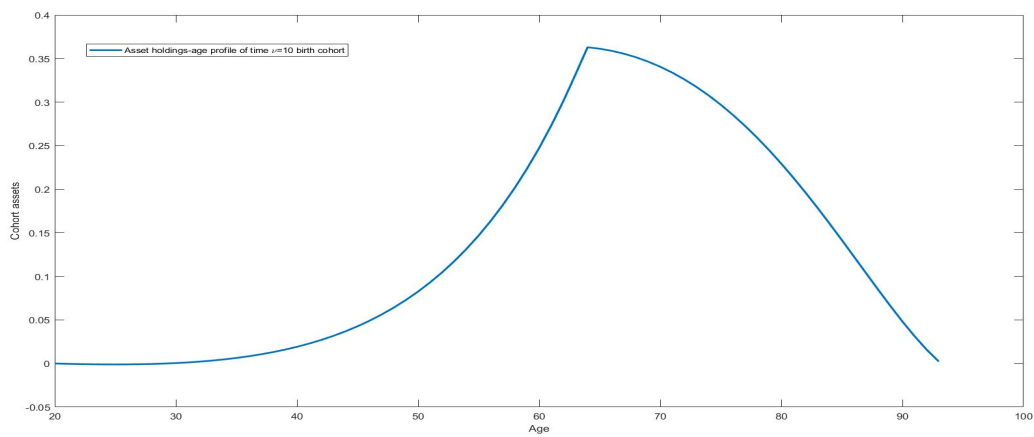


Figure 1.8: Asset holdings-age profile of time  $\nu = 10$  birth cohort

### 1.7.3 Demographic Change

In this section I investigate the effects of exogenous demographic change on the behavior of individuals and cohorts in the steady state of the model. In particular, I introduce two different kinds of demographic change: increase in life expectancy and compression of morbidity.

To disentangle the effects of these two different kinds of demographic change, I solve the model numerically using three different demographic change scenarios: 1. Life expectancy increases and compression of morbidity happens simultaneously, 2. Life expectancy increases but compression doesn't happen, ie: individuals still spend the same portion of their lives in either physical or mental morbidity, 3. Life expectancy doesn't change but compression of morbidity happens, ie: individuals can now spend a smaller portion of their lives in a morbid condition.

Before reporting the results of numerical simulations under these three different scenarios, I explain how I introduce increasing life expectancy and compression of morbidity into the model environment.

I begin with the methodology to introduce increasing life expectancy. There are two parameters in the BCL survival function,  $S(t - \nu) = e^{-M(t-\nu)} = \frac{\mu_0 - e^{\mu_1(t-\nu)}}{\mu_0 - 1}$ , which can be manipulated to change the conditional survival probabilities at every age,  $\mu_0$  the parameter controlling young age mortality and  $\mu_1$  the parameter controlling old age mortality. Decreasing either  $\mu_0$  or  $\mu_1$  causes the conditional survival probabilities to increase. This in turn leads to an increase in the conditional life expectancy at each age  $U$ , which is given by:

$$\lambda(U) = \int_U^D S(u) du \quad (1.52)$$

where  $U \in [20, D]$ . For the simulation exercises in this section, I assume the old age mortality parameter,  $\mu_1$  declines from 0.0548 to 0.0463<sup>52</sup>, which increases the life expectancy

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<sup>52</sup>The reason I choose to decrease the old age mortality parameter instead of the young age mortality parameter is because of the fact that developed countries have already enjoyed massive declines in young age mortality in the past century. This can be empirically observed in the evolution of figures such as child

at birth<sup>53</sup> (ie: at age 20 in real life) from 59.149 years in the benchmark calibration to 70 years.

It is also important to note that with the decline in  $\mu_1$ , the terminal age increases from 96 years to 110 years<sup>54</sup>. I plot the conditional survival probabilities in Figure-1.9 and the conditional life expectancies in Figure-1.10 on the next page for all ages between 20 and the terminal age before and after the young age mortality decline.

Next I explain how compression of morbidity is introduced in the model environment. By definition, compression of morbidity happens when individuals can spend a smaller portion of their lives in a morbid condition. This proportion is determined by two numbers in the current model environment: 1.  $X$ , the age at which chronic conditions begin affecting the individual seriously enough that their marginal utility of consumption declines and disutility of labor begins increasing exponentially, 2.  $\lambda$ , life expectation at birth.

Given  $X$  and  $\lambda$  (which is determined by the estimated BCL survival function in the benchmark calibration),  $\frac{\lambda-X}{\lambda}$  is the proportion of life agents in the model can expect to live in a morbid condition. In the benchmark calibration, this proportion is equal to 0.3683, or 36.83% of their total lifespan. After the decline in old age mortality, life expectancy goes up to  $20 + 70 = 90$  years. To keep the proportion of life spent in a morbid condition at 0.3683, the threshold age  $X$  must increase to 56.8530. Any threshold age greater than 56.8530 causes the proportion of life spent in morbidity to decline. Similarly, any threshold age lower than 56.8530 causes individuals to spend a bigger portion of their life in morbidity, ie: expansion of morbidity happens.

In the numerical simulations, I assume  $X$  goes up from 50 to 68.4 simultaneously as life

deaths at birth in the United States. Future increases in life expectancy in developed countries are likely to come from developing cures for chronic conditions that have a greater impact on the older cohorts of the population.

<sup>53</sup>See equation (1.43) for a straightforward formula without any integrals to calculate life expectancy at birth with pen and paper. Using  $U=20$  in equation (1.48) produces the same result as equation (1.43) but requires a computer to solve.

<sup>54</sup>Terminal age is calculated by using  $D = \frac{\ln(\mu_0)}{\mu_1}$ .

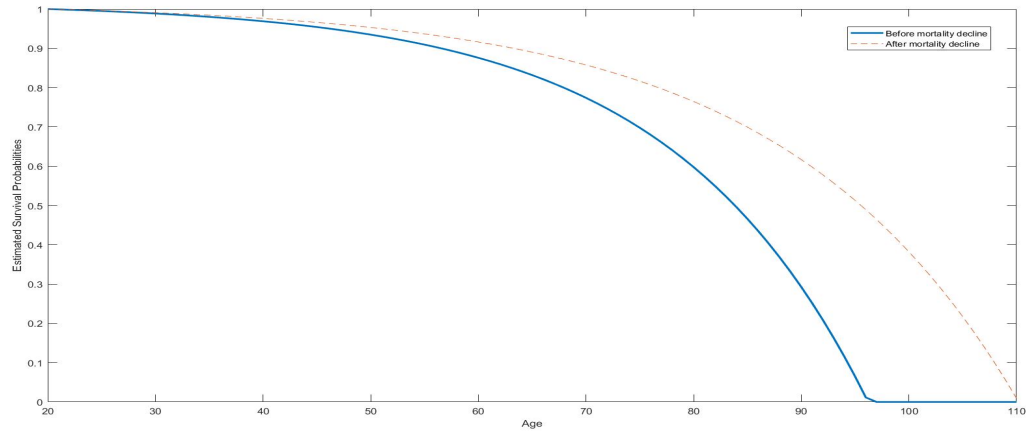


Figure 1.9: Conditional survival probabilities before and after old age mortality decline

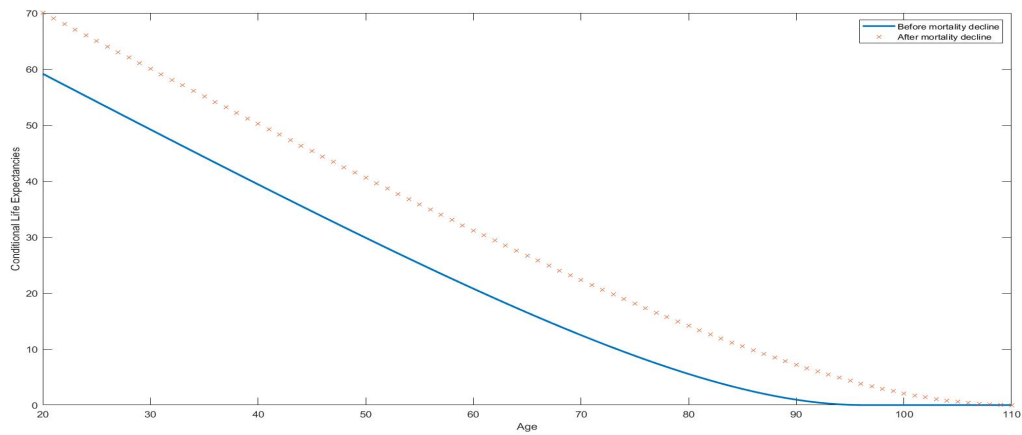


Figure 1.10: Conditional life expectancies before and after old age mortality decline

expectancy increases to  $20+70=90$  years<sup>55,56</sup>. This causes the proportion of life spent in morbidity to decline to 0.24, or 24%.

I report the results of the three demographic change scenarios in Table-1.3 on the next page. In the first scenario, I assume mortality decline and compression of morbidity happen together. Hence, life expectancy increases to 90 years simultaneously as the portion of life spent in morbidity declines to 24%. The main numerical results in this scenario are as follows:

1. optimal retirement age chosen by the agents,  $R^*$ , goes up to 69.0975 years from 63.9483 years in the baseline scenario (see Table-1.2 for results of the baseline calibration exercise),
2. Steady state capital per worker increases to 5.067 from 4.8385 in the baseline scenario,
3. Proportion of lifetime spent working after age 20 goes down from 67.04% in the baseline scenario to 70.60%.

Inspecting the numerical results in scenario-1, it is clear that in the presence of increasing life expectancy and compression of morbidity happening together, agents in the model respond by both increasing their retirement age and saving more to prepare for their post retirement years. Comparing the impact of this combination of demographic change on retirement and saving decisions, it can be seen that the main adjustment happens at the retirement age chosen.  $R^*$  increases sufficiently enough (by more than 5 years) to enable agents to increase their savings by a small margin. I will return back to this finding later in scenario- 3, in which only compression of morbidity happens. Despite the increasing retirement age, terminal age rising causes the proportion of adult life (ie: after age 20) spent

<sup>55</sup>Intuitively, this can be thought as two kinds of medical advance happening in tandem. While some medical researchers are coming up with cures for conditons that impact the lifespan of individuals, others focus their efforts on finding solutions for health problems that impact individuals' capability to enjoy consumption goods and participate in the labor market.

<sup>56</sup>Another possible reason for these simultaneous demographic changes is unintended spillovers of one kind of medical research to the other domain, ie: a cure or invention that was intended to reduce disutility can end up being the solution to a life threatening disease. As a real life example, in 2015 a team of Danish researchers, while trying to find a way to protect people against malaria in Africa, discovered that by attaching malaria proteins to cancer cells, cancer tumours can be destroyed. Malaria, if properly treated, is not lethal but causes major disutility to the infected. In this example, a cure intended to reduce disutility ended up being a possible cure for a life threatening disease. Source: <https://healthsciences.ku.dk/newsfaculty-news/2018/08/scientists-discover-new-method-of-diagnosing-cancer-with-malaria-protein>

working to decline to 67.04%.

The two demographic changes happening together puts a strain on the pay as you go (PAYG) social security system in the model. The payroll tax that keeps the social security system solvent has to increase from 9.43% to 9.93% (exactly half a percentage point increase)<sup>57</sup>.

Due to higher saving of the individuals, steady state capital per effective worker,  $k^*$  increases compared to baseline calibration results. This also leads to higher output and consumption per effective worker numbers,  $y^*$  and  $c^*$ . The higher  $k^*$  value decreases the return to saving,  $r^*$ , but increases the wage rate,  $w^*(t)$ .

Initial consumption levels of all cohort are higher under scenario-1. Growth rate of consumption over time slightly decreases compared to baseline. This is due to the fact that agents are now saving more to finance their post-retirement consumption. It is an interesting question whether at all points in time, consumption and asset holdings of individuals are higher compared to baseline<sup>58</sup>.

Next, I move on to scenario-2 where only life expectancy increases due to the decline in young age mortality but the proportion of lifetime spent in morbidity remains the same. As life expectancy increases to 90 years, to keep the portion of life spent in morbidity constant at 36.83% (which is the number under baseline calibration), I assume the threshold age  $X$  goes up to 56.8530 years.

Comparing the results between the first and second scenarios in Table-1.3 reveals crucial insights. I list my findings down below:

First, when only mortality decline happens, optimal retirement age,  $R^*$ , increases by a

<sup>57</sup>There are two competing forces at work here. One of them is individuals retiring later in this scenario. This increases the total contributions paid to the system. The second force is the increase in number of individuals in retirement because of higher life expectancy and terminal age. In this particular case, the second force dominates the first one since the payroll tax has to increase to make sure the social security system doesn't begin running a deficit.

<sup>58</sup>I compare the consumption and asset holdings of an individual of time  $\nu=10$  birth cohort and the  $\nu=10$  birth cohort itself before and after demographic change (mortality decline and compression happening together) with further numerical simulations in Appendix D at the end of this paper.

<b>Table-1.3</b> Equilibrium Values of Economic Variables in the 3 Demographic Change Scenarios				
Variable	Baseline	Mortality decline & compression	Only mortality decline	Only compression
$R^*$	43.9483	49.0975	48.5842	44.4378
$k^*$	4.8385	5.0760	5.1662	4.7552
$y^*$	1.7364	1.7658	1.7767	1.7259
$k^*/y^*$	2.7865	2.8747	2.9078	2.7553
$r^*$	0.0756	0.0718	0.0704	0.0770
$w^*(0)$	1.1287	1.1477	1.1548	1.1218
$w^*(t)$	$1.1287e^{0.02t}$	$1.1477e^{0.02t}$	$1.1548e^{0.02t}$	$1.1218e^{0.02t}$
$\Gamma$	0.7060	0.6704	0.6640	0.7129
$c^*$	1.3493	1.3597	1.3634	1.3455
$c(0,0)$	1.0347	1.080	1.0985	1.0191
$c(\nu, \nu)$	$1.0347e^{0.02\nu}$	$1.080e^{0.02\nu}$	$1.0985e^{0.02\nu}$	$1.0191e^{0.02\nu}$
$c(\nu, t)$	$c(\nu, \nu)e^{0.0203(t-\nu)}$	$c(\nu, \nu)e^{0.0184(t-\nu)}$	$c(\nu, \nu)e^{0.0177(t-\nu)}$	$c(\nu, \nu)e^{0.0210(t-\nu)}$
$\gamma_{c(\nu, \nu)}$	0.02	0.02	0.02	0.02
$\gamma_{c(\nu, t)}$	0.0203	0.0184	0.0177	0.0210
T	0.0943	0.0993	0.1020	0.0913

Table 1.3: Equilibrium Values of Economic Variables in the 3 Demographic Change Scenarios

smaller amount to 68.5842. This suggests that mortality decline and compression of morbidity both have the same effect on the retirement age. They both increase  $R^*$ , but numbers in Table-1.3 indicate the impact of mortality decline is bigger.

Second, when only mortality decline is operative,  $k^*$  increases by a bigger margin to 5.1662. This implies that mortality decline and compression of morbidity have opposite effects on saving. While mortality decline increases saving, compression of morbidity, by itself, lowers saving. Hence, these two kinds of demographic change also have opposite effects on output and consumption per effective worker,  $y^*$  and  $c^*$  in Table-1.3.

Third, percentage of adult life (life after age 20) spent working decreases in both scenarios 1 and 2. But the magnitude of the decrease is bigger when agents only face mortality decline. This suggests that mortality decline and compression of morbidity have opposite effects on this variable. While mortality decline decreases proportion of life spent working, compression of morbidity increases it.

Fourth, the payroll tax rate,  $T$ , which clears the PAYG social security system increases under both scenarios 1 and 2. But the magnitude of the increase in  $T$  is bigger under scenario-2. This is a signal that compression of morbidity, by itself, enables the payroll tax rate to decrease.

The values of the other implied economic variables in Table-1.3 are in line with the findings I listed above. For instance, because saving in scenario-1 (where mortality decline and compression happen together) is higher than in scenario-2 (where only mortality decline happens), return to saving  $r^*$  is higher and the wage rate  $w^*(t)$  is lower in scenario-1 compared to scenario-2 (where only mortality decline happens).

Consumption levels of all cohorts at time of birth are higher compared to the baseline when only mortality decline happens. Growth rate of consumption across time,  $\gamma_C(\nu, t)$  is smaller compared to the baseline. These are all because agents in the model have a higher saving rate compared to the baseline when they only face mortality decline.

Next, I investigate the case of compression of morbidity happening by itself (scenario-3) to support the conclusions I listed earlier. By inspecting the number in Table-1.3, I verify

that in this scenario, agents in the model increase their chosen retirement age and decrease their saving compared to the baseline calibration. Due to lower saving,  $c^*$  and  $y^*$  are lower,  $r^*$  is higher and  $w^*(t)$  is lower in this scenario compared to the baseline.

When there is only compression morbidity happening, consumption levels of all cohorts at time of birth turn out to be lower compared to the baseline. Predictably, growth rate of consumption over time is higher than the figure under the baseline. These changes reflect the fact agents save less when they only face compression of morbidity.

As predicted earlier, compression of morbidity increases the proportion of adult life spent working. This relieves some of the pressure on the PAYG social security system, since now agents in the model are working in a larger portion of their lives and spending a smaller portion in retirement. Hence, the payroll tax rate, which keeps the PAYG system in balance, now declines to 9.13%. This is a 0.3% decline compared to the baseline scenario.

#### *1.7.4 Social Security Alternatives*

In this section, I will discuss the consequences of adopting three different alternatives to the pay as you go (PAYG) social security system in the benchmark model. These alternatives are namely, 1. Not having a social security system at all, 2. Imposing a mandatory retirement age, 3. Adopting a fully funded system. I report the effects of choosing each one of these alternatives on the equilibrium values of economic variables in Table-1.4 on the next page.

I begin with the first alternative of abolishing the existing PAYG social security system without replacing it without any other system. Agents in the model respond to this change by greatly increasing their retirement age from 43.9483 to 55.0514 in the model. At the same time they significantly reduce their saving, which causes the capital per effective worker to decline from 4.8385 to 4.2392. Accordingly, output and consumption per effective worker also decrease.

Due to lower  $k^*$ , return to saving,  $r^*$  increases by a large margin (about 1.3%) and the wage rate declines. Consumption at time of birth is lower for all cohorts, but growth rate of consumption across time increases.

<b>Table-1.4</b> Equilibrium Values of Economic Variables with Different Social Security Alternatives				
Variable	PAYG	No Social Security	Mandatory Retirement	Fully Funded System
$R^*$	43.9483	55.0514	47	45.4969
$k^*$	4.8385	4.2392	4.4722	4.9844
$y^*$	1.7364	1.6579	1.6892	1.7545
$k^*/y^*$	2.7865	2.5570	2.6475	2.8408
$r^*$	0.0756	0.0869	0.0822	0.0732
$w^*(0)$	1.1287	1.0776	1.0980	1.1405
$w^*(t)$	$1.1287e^{0.02t}$	$1.0776e^{0.02t}$	$1.0980e^{0.02t}$	$1.1405e^{0.02t}$
$\Gamma$	0.7060	0.8518	0.7486	0.7278
$c^*$	1.3493	1.3187	1.3291	1.3558
$c(0,0)$	1.0347	0.9917	0.9712	1.1265
$c(\nu, \nu)$	$1.0347e^{0.02\nu}$	$0.9917e^{0.02\nu}$	$0.9712e^{0.02\nu}$	$1.1265e^{0.02\nu}$
$c(\nu, t)$	$c(\nu, \nu)e^{0.0203(t-\nu)}$	$c(\nu, \nu)e^{0.0260(t-\nu)}$	$c(\nu, \nu)e^{0.0236(t-\nu)}$	$c(\nu, \nu)e^{0.0191(t-\nu)}$
$\gamma_{c(\nu, \nu)}$	0.02	0.02	0.02	0.02
$\gamma_{c(\nu, t)}$	0.0203	0.0260	0.0236	0.0191
T	0.0943	0	0.0771	0.0633

Table 1.4: Equilibrium Values of Economic Variables with Different Social Security Alternatives

Next, I move on to the alternative of imposing a mandatory retirement age in the presence of the current PAYG social security system. I assume that the agents are made to retire at age 67, ie: 47 years in the model. This is an increase to the endogenously chosen retirement age of 43.9498 in the benchmark model with the PAYG social security system. Hence, even though the PAYG system is still available for their benefit, I am forcing the agents in the model to work longer<sup>59</sup>.

Agents respond to this policy by decreasing their saving rate. This is because of two reasons: 1. Since agents have to work longer, they don't need to save as much as before to pay for their post-retirement consumption, 2. The amount of time they spend after retirement is now lower. There isn't as urgent a need as before to save money for post-retirement consumption.

Just like in the first policy alternative considered, the lower saving rate leads to a lower capital, output and consumption per effective worker. Due to lower  $k^*$ , return to capital goes up and the wage rate declines. However, compared to the first policy alternative, the decline in saving is smaller magnitude. Hence, the changes in factor prices are less pronounced.

Due to lower saving for future consumption, agents in all birth cohorts enjoy a smaller consumption level at time of birth, but again consumption grows at a higher rate across time. Predictably, the higher mandatory retirement age relieves the stress on the PAYG system and enables the agents to enjoy a smaller payroll tax. Even though I am forcing the agents to work for approximately another 2 years (on top of the 43.9498 years they would willingly choose to work in the presence of the PAYG system), there is a significant decline in the payroll tax from 9.43% to 7.71%.

Increasing life expectancy will certainly put more pressure on the PAYG systems that are in existence today by creating a greater body of retired individuals. Theoretically, compression

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<sup>59</sup>The purpose of such a policy (imposing a mandatory retirement age which is greater than the one agents choose for themselves) would be to two fold: 1. It would increase the contributions paid to the system by forcing people to work longer, 2. It would decrease the amount of benefits paid to the retirees by reducing the number of retirees. This policy choice, which would inevitably be politically unpopular, would be especially useful in the presence of increasing life expectancy which was explored in the previous section.

of morbidity can help mitigate the situation by enabling the agents to retire later. But as we learn from the numerical simulation exercises in the previous section, there is no guarantee that compression of morbidity will be the dominant demographic change. Given these circumstances, increases in the mandatory retirement age can help make the PAYG social security system more sustainable (as we saw in the second policy alternative above) and ready to face future demographic challenges.

Finally, I investigate the effects of switching to a full funded system from the PAYG system. In this particular example, I assume that the contributions of the working age agents are invested in the risk free asset and not in the lifetime annuity market that was explained in the model section of the paper<sup>60</sup>.

Switching to the full funded system causes the agents in the model to increase both their chosen retirement age and their saving rate. Since  $k^*$  increases, both  $c^*$  and  $y^*$  rise as well compared to the baseline with the PAYG system.

Due to higher saving, return to saving  $r^*$  declines, the wage rate  $w^*(t)$  increases and all cohorts enjoy a higher amount of consumption at time of birth. But as expected growth rate of consumption across time declines if this policy alternative is adopted.

Finally, the switch to the fully funded system produces a decline in the rate of the payroll tax which keeps the social security system in equilibrium.

## 1.8 Conclusion

I set out to answer two questions when I began writing this chapter. First, how will the consumption-saving and the retirement age choices of individuals be affected by the two kinds of demographic change (namely mortality decline and compression of morbidity) that we are likely to experience in the future decades? Second, in terms of policy responses, can

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<sup>60</sup>I can relax this assumption easily by assuming that contributions in the fully funded system are invested in the lifetime annuity market. But from an empirical standpoint, it is easier to justify that the government body which runs the fully funded social security system chooses less risky assets. In the numerical simulation exercises I tried, this assumption slightly impacts the magnitude of the changes created by the switch to the fully funded system. But the direction of the changes are the same whether I assume the contributions are invested in the risk free asset or the lifetime annuity market.

anything be done with a social security system to improve the situation of the people in response to demographic change?

To answer these questions, I built a two period OLG model (working and retirement periods) with an endogenous retirement age choice, a realistic demographic structure, and two mechanisms to formalize the idea of morbidity: a state dependent utility function and a unique disutility of labor specification. After considering other alternatives, I decided to use the Boucekkine survival function to incorporate the demographic structure in the paper.

Next, I calibrated the parameters of the demographic structure of the model by using actual survival data for the United States in 2017. Similarly, I used the life expectancy and the retirement age observed in the US in 2017 as calibration targets (along with two theoretical constraints) to numerically calibrate two important parameters (initial disutility of labor and the decline in the marginal utility of consumption) in the model. Intertemporal elasticity of substitution and the discount rate were also carefully chosen so that the model produces a sort of hump shape for the asset holdings-age profile of agents in numerical simulations (ie: to prevent agents from dying with positive asset holdings or in debt.)

After calibration, I numerically calculated and plotted the paths of an example individual's consumption and asset holdings in the steady state of the theoretical model. The evolution of total consumption and asset holdings of an example cohort ( $\nu=10$  birth cohort) was likewise calculated and plotted.

With the sanity check provided by these numerical simulations, I introduced three combinations of demographic change (mortality decline and compression of morbidity happening together, mortality decline happening by itself, and compression of morbidity by itself) into the model environment to gauge and disentangle their individual effects. The main conclusions from that exercise are as follows: 1. Mortality decline and compression of morbidity both increase the retirement age chosen by the agents, 2. In the face of mortality decline, agents save more to prepare for their post-retirement years. But when faced with compression of morbidity, they choose to work longer and enjoy more consumption today by reducing their saving, 3. Either demographic change (mortality decline or compression of morbidity) pro-

duce opposite results on the initial level of consumption of individuals and the growth rate of consumption over time, 4. Mortality decline causes a drop in the proportion of adult life spent working. This puts a greater burden on pay as you go (PAYG) social security systems. Compression of morbidity leads to an increase in the proportion of adult life spent working. This removes part of the burden on the PAYG systems.

Finally, I numerically investigated the effects of introducing three different alternatives (having no social security system at all, including a mandatory retirement age into a PAYG system, or adopting a fully funded (FF) system) to the PAYG social security system in the model. Important findings from those simulations are as follows: 1. Abolishing the existing PAYG system without replacing it with some sort of social security net causes the agents to greatly increase their retirement age. Simultaneously agent in the model reduce their saving, because they don't have to spend as much time as before in post retirement, 2. Imposing a mandatory retirement age, which is greater than the endogenously chosen one by the agents, decreases the saving rate, since again, agents don't have to live as long as before in post-retirement, 3. Replacing the existing PAYG system with a fully funded system increases both the chosen retirement age and the saving of the agents.

In light of the findings listed above, my policy suggestions would be as follows: 1. If demographers forecast that decline in mortality rates (ie: increasing life expectancy) will be the dominant demographic trend in coming decades, there is going to be a growing pressure on existing PAYG systems. In this case, my policy suggestion would be either to introduce a mandatory retirement age within the existing PAYG system (which won't be politically popular), or offer greater financial incentives to individuals to make them retire on their own. According to numerical simulations, switching to a fully funded system seems like another alternative.

On the other hand, if demographers predict that compression of morbidity will be the dominant trend in the next century, lack of capital accumulation because of lower saving will be the main problem facing the society. In this case, my policy suggestion would be to switch to a fully funded system.

Before finishing, I'd like to mention the improvements that can be introduced in the present version of the paper. It is important to note that the current paper models morbidity in a highly stylized fashion. Agents in the model don't suffer sufficiently severe health problems (mental or physical) to influence their lifetime decisions before age 50 in the model environment. After age 50, state dependent utility specification in the paper assumes there is a constant decline in marginal utility from consumption. In reality, it is more likely that individuals will suffer from an expanding array of health related problems (which will compound each others' effects causing further declines in marginal utility of consumption) as they get older. Also, there is the possibility that individuals can still suffer from severe conditions before the age of 50. I am planning to address these issues by introducing multiple health states (which have more serious effects on the agents' utility from consumption and disutility of labor as the state of the agent gets worse) and an associated stochastic transition mechanism between those states in the next version of this paper.

Also, there is a huge literature on the re-distributive effects of social security systems. Investigating the effects of adopting different systems in the presence of demographic change is another interesting question that can be explored within the framework of the current paper. This requires the inclusion of global dynamics in the theoretical model. This is a bit challenging, since the dynamics will be between two steady states in the model that have different associated social security systems. This is something I have been working on.

## Chapter 2

# DEMOGRAPHIC CHANGE IN THE ABSENCE OF STATE DEPENDENT UTILITY FROM CONSUMPTION AND ITS EFFECTS ON SAVING AND RETIREMENT DECISIONS

### *2.1 Introduction*

One of the major contributions of the study in the previous chapter was the inclusion of a state dependent utility from consumption in an OLG model with demographic change. As mentioned in the literature review section of chapter-1, even those rare papers in the literature that include changing periods of morbidity (ie: compression of morbidity) as a source of demographic change only consider its effects on disutility of labor. Hence, one of the natural questions that follow is what kind of effects the inclusion of the state dependent utility from consumption creates in the context of the study in chapter-1. The purpose of the current chapter is to answer this question.

Intuitively, the state dependent utility from consumption and the age dependent disutility of labor functions can either reinforce or offset each other's effects on individual decision making. This distinction crucially depends on whether the utility received from consumption increases (positive state dependence) or decreases (negative state dependence) with worsening health states. In the case of positive state dependence, as individuals get older and transition to worsening health states, they begin receiving more utility from consumption, and would like to work longer despite the increasing disutility of labor they will suffer by participating in the labor market. In this scenario, the two channels created by the state dependent utility from consumption and the age dependent disutility of labor functions counteract each other's effect on the chosen retirement age and the saving of the agents. On the other hand, in the case of negative state dependence, worsening health states reduce the

utility from consumption and would cause the agents to retire earlier. In this scenario, the two channels reinforce each other's effects on the retirement age and saving of the agents.

In chapter-1, state dependent utility from consumption was introduced with a two state (healthy and unhealthy) utility function. Agents transitioned from the healthy to the unhealthy state deterministically at the exogenously given threshold age  $X$ . Further, following the empirical study in Finkelstein (2013) [36], negative state dependence was assumed. In particular, as agents transitioned to the unhealthy state, it was assumed that there would be a level reduction in the utility of consumption and marginal utility of consumption would be reduced by a constant factor. With this specification, as explained in the previous paragraph, I expect the two channels to reinforce each other's impact on saving and retirement age as agents get older. But at least three issues remain undiscussed at this point.

First, in terms of the magnitude of their impact, which channel is more influential, state dependent utility from consumption or the age dependent disutility of labor? Second, is the order of their impacts the same on saving and retirement? Perhaps, it might be the case that state dependent utility from consumption has a bigger impact on saving and age dependent disutility of labor has a bigger impact on the retirement age? Third, how do the respective effects of the two channels change in response to demographic change?

Given these unresolved questions, I decided to write the current chapter. In the following pages, I provide the version of the model which does not include the state dependent utility from consumption, and then I explain the calibration of this version of the model. After that, I redo the numerical exercises of chapter-1 with this new version of the model. First, I introduce demographic change into the model and report the results. Next, just like in chapter-1, I change the social security system in the model and investigate the impacts. Finally, I conclude by commenting on the results of these exercises.

## 2.2 Model

### 2.2.1 Households

The removal of state dependent utility from consumption in the model of Chapter-1 can be achieved by setting the parameters  $\gamma_0$  and  $\gamma_1$  equal to zero in equation (1.5). In this case, the objective function of the agent, who was born at time  $\nu$ , at calendar date  $t \geq \nu$  becomes:

$$E[\Lambda(\nu)] = \int_{\nu}^{\nu+D} u(c(\nu, t))e^{-\rho(t-\nu)-M(t-\nu)} dt - \int_{\nu}^{\nu+R} \phi(t-\nu)e^{-\rho(t-\nu)-M(t-\nu)} dt \quad (2.1)$$

where  $\phi(t-\nu)$  is the disutility of labor suffered by the agent at age  $t-\nu$ .

The agent maximizes this objective function subject to the same flow budget constraint as in Chapter-1:

$$\frac{\partial A(\nu, t)}{\partial t} = \begin{cases} [r(t) + \mu(\tau - \nu)]A(\nu, t) + (1 - T)w(t) - c(\nu, t) & \text{if } t - \nu \leq R \\ [r(t) + \mu(t - \nu)]A(\nu, t) + bw(\nu + R) - c(\nu, t) & \text{if } t - \nu > R \end{cases} \quad (2.2)$$

To refresh the reader's memory, it is assumed that agents invest or borrow by using the lifetime annuities issued by a perfectly competitive insurance industry. These annuities pay a return equal to  $[r(t) + \mu(\tau - \nu)]$ , where  $r(t)$  is the return on capital at time  $t$  and  $\mu(\tau - \nu)$  is the instantaneous mortality hazard that is given in equation (2.6). Also, there is a social security system to which agents pay the payroll tax  $T$  before the endogenously chosen retirement age,  $R$ . After retirement, agents receive benefits equal to a replacement ratio,  $b$ , times their wage at the moment of retirement.

I assume that agents are born without any assets,  $A(\nu, \nu) = 0$ . Also agents don't carry any accumulated assets or debt to the terminal age,  $D$ :  $A(\nu, \nu + D) = 0$ . This means that an agent, who survives to the terminal age,  $D$ , can not leave a bequest to their offspring or die in debt.

Since, there is still uncertain lifetime in the model, it is also crucial to state the following. At the moment of death, if an agent has any accumulated assets, following [77], I assume

that those assets are inherited by the insurance industry. Similarly, if an agent happens to have any accumulated debt at the moment of their death, it is assumed that their debt is forgiven by the insurance industry.

To make the results in this chapter comparable with the results of the previous chapter, I use the iso-elastic utility function for the flow utility from consumption:

$$u(c(\nu, t)) = \frac{c(\nu, t)^{1-\frac{1}{\sigma}} - 1}{1 - \frac{1}{\sigma}} \quad (2.3)$$

where  $\sigma$  is the intertemporal elasticity of substitution.

Similarly, the flow disutility is given by the following function:

$$\phi(t - \nu) = \delta_0 e^{\delta_1 \max\{t-\nu-X, 0\}} \quad (2.4)$$

The function in equation (2.4) was described earlier in chapter-1. Hence, I won't elaborate on it in detail in this chapter. However, in the calibration section of this chapter, I provide the values that are chosen in this version of the model for the parameters  $\delta_0$  and  $\delta_1$  and describe the process for choosing them.

The following generic functions are used to describe the demographic structure in the model: 1. The survival function,  $S(t - \nu)$ , which is the probability of surviving to age  $t - \nu$ , 2. The cumulative mortality hazard function,  $M(t - \nu)$ , which is the probability of dying before age  $t - \nu$ , 3. The instantaneous mortality hazard function,  $\mu(t - \nu)$ , which is the value of the probability density of dying at instant  $t - \nu$ .

These three functions are related to each other with the following relationships:

$$S(t - \nu) = e^{-M(t-\nu)} \quad (2.5)$$

$$\mu(t - \nu) = -\frac{S'(t - \nu)}{S(t - \nu)} \quad (2.6)$$

$$M(t - \nu) = \int_0^{t-\nu} \mu(\tau) d\tau \quad (2.7)$$

The terminal age  $D$  is defined by the equation  $S(D) = e^{-M(D)} = 0$ .

### 2.2.2 Equations that Determine the Steady State of the Model

As explained in the previous chapter, there are two endogenous variables that need to be numerically calculated to solve for the steady of the model,  $k^*$  and  $R^*$ . Hence, at least two equations are required to determine the steady state.

The first of these equations is the one that describes the condition for the retirement age. This was equation (1.21) in chapter-1. In the current version of the model, since there is no state dependent utility from consumption, this equation becomes:

$$c(\nu, \nu)^{-1/\sigma} \left[ \int_{\nu}^{\nu+D} \frac{\partial c(\nu, \tau)}{\partial R} e^{-r^*(\tau-\nu)-M(\tau-\nu)} d\tau \right] - [\phi(R^*)e^{-\rho R^*-M(R^*)}] = 0 \quad (2.8)$$

Left hand side of this equation is the discounted marginal utility from consumption that can be enjoyed by using the income from working at the time of the chosen retirement age,  $R^*$ . The right hand side is the discounted disutility of working at the instant of the retirement age. It is important to note that due to uncertainty in lifetime, discounting is done by using: 1. A pure discount factor:  $e^{-r^*(\tau-\nu)}$  and 2. The cumulative mortality hazard function:  $e^{-M(\tau-\nu)}$ . Both the left and right hand sides of equation (2.8) are discounted to the birth date of the agent.

Due to the absence of state dependent utility from consumption, equation (2.8) is easier to solve. Two things are required to solve equation (2.8): Consumption at time of birth,  $c(\nu, \nu)$  and integral of the discounted partial derivative of consumption with respect to the endogenous retirement age,  $\int_{\nu}^{\nu+D} \frac{\partial c(\nu, \tau)}{\partial R} e^{-r^*(\tau-\nu)-M(\tau-\nu)} d\tau$ . For the reader's convenience, I

Expressions for these two quantities were given in equations (1.22) and (1.23) in the

previous chapter<sup>1</sup>. I provide these equations down below for the reader's convenience:

$$c(\nu, \nu) = \frac{A(0)Z \left[ (1-T) \int_{\nu}^{\nu+D} e^{\gamma_A \tau - r^*(\tau-\nu) - M(\tau-\nu)} d\tau + b e^{\gamma_A(\nu+R)} \int_{\nu+R}^{\nu+D} e^{-r^*(\tau-\nu) - M(\tau-\nu)} d\tau \right]}{\left[ \int_{\nu}^{\nu+D} e^{(\sigma-1)r^*(\tau-\nu) - \sigma\rho(\tau-\nu) - M(\tau-\nu)} d\tau \right]} \quad (2.9)$$

$$\left[ \int_{\nu}^{\nu+D} \frac{\partial c(\nu, \tau)}{\partial R} e^{-r^*(\tau-\nu) - M(\tau-\nu)} d\tau \right] = \left[ (1-T-b)w(\nu+R)e^{-r^*R - M(R)} \right] + \left[ b\gamma_A w(\nu+R) \int_{\nu+R}^{\nu+D} e^{-r^*(\tau-\nu) - M(\tau-\nu)} d\tau \right] \quad (2.10)$$

The other equation required to solve for the steady state of the model is the market clearing condition, which dictates that output of the economy must be equal to consumption plus gross investment:  $Y^* = C^* + I^*$ . In intensive form, this equation becomes<sup>2</sup>:

$$y^* = (k^*)^\alpha = c^* + i^* = c^* + (n + \gamma_A + \delta)k^* \quad (2.11)$$

where  $\gamma_A$  is the rate of labor-augmenting tech progress in the production function and  $\delta$  is the depreciation rate of capital. Gross investment accounts for population growth, labor augmenting tech progress, and depreciation of the existing capital stock.

It is crucial to remind the reader at this point that due to endogenous retirement in the model, capital per effective worker is a function of the endogenous retirement age:  $k^*(R^*)$ . Hence, consumption per effective worker is given by:  $c^* \equiv c^*(k^*(R^*), R^*)$ .

The equation for  $c^*$  remains the same as in chapter-1:  $c^* = ZM_1M_2$  where  $Z = f(k^*) - k^*f'(k^*)$ <sup>3</sup>. As explained in the previous chapter,  $M_1$  includes "longitudinal constraints" that emerge as the consequence of an individual's decisions over their lifetime.  $M_2$  includes "cross-sectional constraints" that emerge as the result of the decisions of all the cohorts that are

<sup>1</sup>Detailed derivations of these equations can be found in Appendix-B.

<sup>2</sup>I assume a Cobb-Douglas production function for this version of the model.

<sup>3</sup>If the Cobb-Douglas production function is used,  $Z$  can be replaced with  $(1-\alpha)k^{*\alpha}$ .

alive at a given calendar date. Expressions for  $M_1$  and  $M_2$  can be found in equations (1.32) and (1.33).

To solve for the steady state of the model, I assume the same structure for the production side that was assumed in chapter-1. Finally, an equation for the social security tax is required to numerically solve for  $k^*$  and  $R^*$  by using equations (2.8) and (2.11). For comparability purposes, I assume a pay as you go social security system in this version of the model. The payroll tax, which adjusts over time to keep the social security system in balance, is given by equation (1.41).

### 2.3 Calibration

With negative state dependence, the state dependent utility from consumption and age dependent disutility of labor functions reinforce each other in their influence on the saving and retirement age choices of the individuals. Hence, the removal of the state dependent utility from consumption function from the model in chapter-1 reduces the incentive to retire earlier and to set aside money for post-retirement consumption. This means that the calibrated values for the parameters of the disutility of labor function in chapter-1 will no longer ensure the calibration targets are met. As an example, with an exogenous life expectancy of 78.5 years, it will no longer be possible to achieve a calibration target of 67 years for the retirement age.

The parameters of the disutility of labor function need to be re-calibrated to address this problem. These parameters are  $\delta_0$  and  $\delta_1$  in the function:  $\phi(t - \nu) = \delta_0 e^{\delta_1 \max\{t - \nu - X, 0\}}$ . As a reminder, this function implies that chronic health problems become serious enough after a threshold age  $X$  to exponentially increase the disutility of labor. The parameter  $\delta_0$  represents the disutility of labor before the threshold age  $X$  and  $\delta_1$  is the rate at which disutility of labor increases with age after  $X$ . In chapter-1,  $\delta_1$  was assumed to be 0.017<sup>4</sup> and  $\delta_0$  was jointly calibrated with  $\gamma_1$  (one of the parameters of the state dependent utility from

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<sup>4</sup>The increase in healthcare expenditure with age was used as a proxy for the rate of increase in disutility of labor with age in chapter-1.

consumption function) using other constraints explained in the calibration section of that chapter.

Since the state dependent utility from consumption function is no longer functional in this chapter, using a value as low as 0.017 for  $\delta_1$  would lead to a retirement age greater than 67 years<sup>5</sup>. But how does the endogenous decisions of the agents (ie: their saving and retirement age choices) respond to a change in  $\delta_1$ ? To answer that question, I solved the model numerically with several different values for  $\delta_1$  in the absence of the state dependent utility from consumption function using a calibration target of 67 years for the retirement age and an exogenous life expectancy of 78 years<sup>6</sup>. I report the results of this sensitivity analysis in Table-2.1. Calibrated values of the  $\delta_0$  parameter are also reported in the last column of this table.

Visually inspecting the results reported in Table-2.1, it is possible to reach two important conclusions: 1. As  $\delta_1$  increases, since disutility of labor increases at a faster rate, individuals retire at younger ages, 2. With higher  $\delta_1$  values, since agents retire earlier, they also save more during their working lives to prepare for their post-retirement years. In other words, higher  $\delta_1$  values mean lower capital per effective worker,  $k^*$ , values.

For the numerical exercises reported in the next section, I chose the value of 0.04 for the  $\delta_1$  parameter. The reason behind this choice is straightforward. The calibrated model in chapter-1, which included the state dependent utility (SDU) from consumption, produced a result of 43.9483 years for the retirement age,  $R^*$ <sup>7</sup> Using  $\delta_1=0.04$  in the current version of the model (ie: without the SDU function), yields a very similar result of 44.3907. This is one factor that would make the numerical exercises of this chapter comparable to the ones in chapter-1.

<sup>5</sup>Similarly, removing the state dependent utility from consumption function leads to lower saving choices before retirement

<sup>6</sup>I kept the values of all other structural and demographic parameters the same as in chapter-1 for the sensitivity analysis exercise.

<sup>7</sup>This corresponds to  $43.9483+20=63.9483$  real life years. This is quite close to the calibration target of 67 years.

<b>Table-2.1</b> Sensitivity Analysis with Changing Values of the $\delta_1$ Parameter				
$\delta_1$	$k^*$	$R^*$	Ratio spent working	$\delta_0$
0.035	7.0888	46.6790	0.7442	0.3202
0.04	6.5980	44.3907	0.7122	0.2801
0.045	6.4222	43.7482	0.7031	0.2490
0.05	6.3154	43.4142	0.6983	0.2241
0.055	6.2396	43.2004	0.6953	0.2037
0.06	6.1811	43.0504	0.6931	0.1868
0.065	6.1334	42.9349	0.6914	0.1724
0.07	6.0930	42.8401	0.6901	0.1601
0.075	6.0578	42.7580	0.6889	0.1494
0.08	6.0266	42.6836	0.6878	0.1401
0.085	5.9984	42.6138	0.6868	0.1318
0.09	5.9727	42.5465	0.6858	0.1245
0.095	5.9491	42.4799	0.6849	0.1180
0.1	5.9272	42.4131	0.6839	0.1121

Table 2.1: Sensitivity Analysis with Changing Values of the  $\delta_1$  Parameter

It is important to note that using  $\delta_1=0.04$  implies a calibrated value of 0.2801 for  $\delta_0$ . I assume the same values for the structural and demographic parameters of the model that were reported in Table-1.1 for the model of this chapter. This is, again, to keep the results of the numerical analysis exercises as comparable as possible between the two chapters.

The equilibrium values of the economic variables in the two versions of the model (with and without the SDU function) are reported in Table-2.2. Before closing this section, I comment on the comparison between these results.

First, due to the deliberate choice for the  $\delta_1$  parameter, the retirement ages are very close

to each other in the two versions of the model. Second, crucially, the version of the model without the state dependent utility (SDU) from consumption function produces higher saving rates, which is reflected in the higher capital per effective worker value. This result seems counter-intuitive, but it can be explained as follows. The removal of the SDU specification from the model causes agents to enjoy more utility from consumption after the threshold age,  $X$ . By itself, this effect would cause the agents to reduce their saving and enjoy more consumption. However, as will be explained down below, consumption expenditure keeps growing throughout the agents' lifespan. This means agents who would like to enjoy an every increasing amount of consumption in their retirement years need to save more during their working lives. This effect is compounded by the fact that agents would like to enjoy more consumption after age  $X$  without the SDU specification.

The higher  $k^*$  value produced by the version of the model without the SDU function creates a plethora of other results. First, both output and capital / output ratios are higher in the model without the SDU. Second, the return to capital,  $r^*$ , is suppressed and the wage rate,  $w^*(t)$ , is elevated due to higher capital accumulation. Next, consumption per capita is higher without the SDU function. Intuitively, this can be explained as follows. Inspecting the results in Table-2.2, consumption at time of birth is higher for all cohorts in the absence of the SDU function. But at the same time, the higher capital stock per effective worker value causes the growth rate of consumption to be lower throughout the agents' lifespan. Still, having begun from a higher level of consumption, agents enjoy a greater value of consumption per effective worker when the SDU function channel is not present in the model.

Finally, it should also be noted that removing the SDU function does not seem to affect the payroll tax,  $T$ , which keeps the pay as you go social security system in balance, very much. The numbers for the payroll tax are very close to each other under either specification of the model.

<b>Table-2.2</b> Equilibrium with and without State Dependent Utility (SDU) from Consumption			
Variable	Symbol	with SDU	without SDU
Retirement age	$R^*$	43.9483	44.3907
Capital per effective worker	$k^*$	4.8385	6.5980
Output per effective worker	$y^*$	1.7364	1.9355
Capital / output ratio	$k^*/y^*$	2.7865	3.4089
Return to capital	$r^*$	0.0756	0.0527
Wage rate at time 0	$w^*(0)$	1.1287	2.9406
Wage rate at time t	$w^*(t)$	$1.1287e^{0.02t}$	$2.9406e^{0.02t}$
Proportion of life spent working	$\Gamma$	0.7060	0.7122
Consumption per capita	$c^*$	1.3493	1.4107
Consumption of first birth cohort (at time of birth)	$c(0,0)$	1.0347	1.4684
Consumption of a generic birth cohort (at time of birth)	$c(\nu, \nu)$	$1.0347e^{0.02\nu}$	$1.4684e^{0.02\nu}$
Consumption of time $\nu$ birth cohort individual at time $t$	$c(\nu, t)$	$c(\nu, \nu)e^{0.0203(t-\nu)}$	$c(\nu, \nu)e^{0.0088(t-\nu)}$
Growth rate of consumption at birth	$\gamma_{c(\nu, \nu)}$	0.02	0.02
Growth rate of consumption across time	$\gamma_{c(\nu, t)}$	0.0203	0.0088
Payroll tax	T	0.0943	0.0916

Table 2.2: Equilibrium with and without State Dependent Utility from Consumption

## 2.4 Numerical Simulations

### 2.4.1 Demographic Change

In this section, I introduce the three demographic change scenarios, which were explored in the previous chapter, into the version of the model without the SDU function and report the results in Table-2.2. Next, I compare the new results with the results from chapter-1. Before beginning, I should remind the reader that the first demographic change scenario is both mortality decline and compression of morbidity happening together. The second scenario is mortality decline happening by itself and the third scenario is compression of morbidity happening on its own.

Inspecting the results in Table-2.3, I reach the following conclusions. First, mortality decline, by itself, increases the chosen retirement age,  $R^*$ , whereas compression of morbidity decreases  $R^*$  on its own. When happening together (ie: scenario 1), mortality decline is the more dominant factor.

Second, both mortality decline and compression of morbidity lower the saving of the agents. This leads to a lower level of capital per effective worker,  $k^*$ , under all scenarios. But compression of morbidity seems to be more effective in decreasing saving.

Next, because  $k^*$  is lower under all scenarios, return to capital,  $r^*$ , increases and the wage rate,  $w^*(t)$ , decreases under all combinations of demographic change. Similarly, both output per effective worker,  $y^*$ , and capital / output ratio,  $k^*/y^*$ , is lower under all scenarios.

As for consumption related economic variables, it is clear that both mortality decline and compression of morbidity lead to lower consumption levels at time of birth for all cohorts and the growth rate of consumption over time to increase. However, because the lower levels of consumption due to either kind of demographic change is the dominant factor, consumption per effective worker,  $c^*$ , ends up being lower under all scenarios.

Finally, the payroll tax,  $T$ , which clears the pay as you go social security system increases under both kinds of demographic change.

<b>Table-2.3</b> Equilibrium in the 3 Demographic Change Scenarios without SDU				
Variable	Benchmark	Mortality decline & compression	Only mortality decline	Only compression
$R^*$	44.3907	48.1808	47.2382	42.3673
$k^*$	6.5980	5.2530	5.8146	5.3633
$y^*$	1.9355	1.7871	1.8517	1.8001
$k^*/y^*$	3.4089	2.9394	3.1401	2.9794
$r^*$	0.0527	0.0691	0.0615	0.0675
$w^*(0)$	2.9406	1.1616	1.2036	1.1701
$w^*(t)$	$2.9406e^{0.02t}$	$1.1616e^{0.02t}$	$1.2036e^{0.02t}$	$1.1701e^{0.02t}$
$\Gamma$	0.7122	0.6590	0.6472	0.6832
$c^*$	1.4107	1.3668	1.3866	1.3711
$c(0,0)$	1.4684	1.1171	1.2523	1.1446
$c(\nu, \nu)$	$1.4684e^{0.02\nu}$	$1.1171e^{0.02\nu}$	$1.2523e^{0.02\nu}$	$1.1446e^{0.02\nu}$
$c(\nu, t)$	$c(\nu, \nu)e^{0.0088(t-\nu)}$	$c(\nu, \nu)e^{0.017(t-\nu)}$	$c(\nu, \nu)e^{0.0132(t-\nu)}$	$c(\nu, \nu)e^{0.0163(t-\nu)}$
$\gamma_{c(\nu, \nu)}$	0.02	0.02	0.02	0.02
$\gamma_{c(\nu, t)}$	0.0088	0.017	0.0132	0.0163
T	0.0916	0.1041	0.1091	0.1041

Table 2.3: Equilibrium in the 3 Demographic Change Scenarios without SDU

Before concluding this section, I compare the results reported in Table-2.3 with the results in Table-1.3 from chapter-1. My conclusions can be summarized as follows:

1. Both with and without the SDU function, the chosen retirement age,  $R^*$ , increases in response to mortality decline. But in response to compression of morbidity,  $R^*$  increases if SDU function is present. Without the SDU specification,  $R^*$  decreases in response to compression. In both cases (with or without SDU), mortality decline is more dominant.

2. Both with and without the SDU function, effective capital per worker,  $k^*$ , decreases in response to compression of morbidity (ie: lower saving). However,  $k^*$  increases in response to mortality decline if SDU function is operational in the model. If not,  $k^*$  decreases in response to mortality decline.

3. In the presence of SDU, due to the contrasting effects of mortality decline and compression of morbidity on  $k^*$ , the following variables move in the opposite directions under demographic change scenarios 2 and 3: growth rate of consumption across time,  $(\gamma_{c(\nu,\nu)})$ , consumption per effective worker,  $(c^*)$ , factor returns,  $r^*$  and  $w^*(t)$ , output per effective worker,  $y^*$  and capital output ratio,  $k^*/y^*$ , the payroll tax,  $T$ . Without the SDU specification, all these variables move in the same direction in response to mortality decline or compression of morbidity.

4. When SDU function is included in the model, consumption at time of birth for any cohort,  $\gamma_{c(\nu,\nu)}$ , increases under either kind of demographic change. But when the SDU specification is removed from the model,  $\gamma_{c(\nu,\nu)}$  decreases under either kind of demographic change.

Intuitively, there could be a number of reasons behind the contrasting effects of the two kinds of demographic change (mortality decline and compression of morbidity) on the equilibrium values of economic variables, when the SDU function is included or removed from the model. The one explanation, that I think makes most sense, is the asymmetric impact of compression of morbidity in the two specifications of the model. When the SDU function is included in the model, there are two channels through which compression of morbidity can operate: the state dependent utility from consumption and the age dependent disutility of labor. Removing the SDU function gets rid of one of these channels. This affects both the magnitude and the direction of the impact of either kind of demographic change on the equilibrium values in Table-1.4 and Table-2.3.

I introduce the different kinds of social security alternatives that were discussed in chapter-1 into the version of the model without the SDU function in the next section. The results are reported in Table-2.4.

<b>Table-2.4</b> Equilibrium without SDU under Different Social Security Alternatives				
Variable	PAYG	No Social Security	Mandatory Retirement	Fully Funded System
$R^*$	44.3907	49.4167	47	38.7229
$k^*$	6.5980	6.8456	4.5095	6.5053
$y^*$	1.9355	1.9606	1.6941	1.9259
$k^*/y^*$	3.4089	3.4916	2.6619	3.3778
$r^*$	0.0527	0.0502	0.0815	0.0536
$w^*(0)$	2.9406	1.2744	1.1012	1.2518
$w^*(t)$	$2.9406e^{0.02t}$	$1.2744e^{0.02t}$	$1.1012e^{0.02t}$	$1.2518e^{0.02t}$
$\Gamma$	0.7122	0.7812	0.7486	0.6295
$c^*$	1.4107	1.4130	1.3261	1.4055
$c(0,0)$	1.4684	1.6271	0.9790	1.3098
$c(\nu, \nu)$	$1.4684e^{0.02\nu}$	$1.6271e^{0.02\nu}$	$0.9790e^{0.02\nu}$	$1.3098e^{0.02\nu}$
$c(\nu, t)$	$c(\nu, \nu)e^{0.0088(t-\nu)}$	$c(\nu, \nu)e^{0.0076(t-\nu)}$	$c(\nu, \nu)e^{0.39(t-\nu)}$	$c(\nu, \nu)e^{0.0093(t-\nu)}$
$\gamma_{c(\nu, \nu)}$	0.02	0.02	0.02	0.02
$\gamma_{c(\nu, t)}$	0.0088	0.0076	0.3900	0.0093
T	0.0916	0	0.0711	0.1873

Table 2.4: Equilibrium without SDU under Different Social Security Alternatives

#### 2.4.2 Social Security Alternatives

Before reporting the results in this section, it is important to remind the reader that the benchmark model has the pay as you go (PAYG) social security system. The alternative social security specifications introduced into the model are: i. not having a social security net at all, ii. having a mandatory retirement age of 67 years (ie: 47 years in the model), and iii. replacing the PAYG system with a fully funded social security system.

After removing the SDU function from the model and introducing these three different alternatives to the PAYG system, I obtained the following results, which can be found in Table-2.4:

1. Removing the social security system and not replacing it with some other system causes the agents to increase their chosen retirement age,  $R^*$ . This is quite intuitive, since without a social security system agents prefer to work longer years. Next, replacing the PAYG system with a fully funded system causes  $R^*$  to significantly decrease.

2. First, if there is no social security net in place, this causes agents to accumulate more savings, hence  $k^*$  decreases. This again is very natural, since without a social security net agents try to finance their post retirement consumption by saving more in their working years. Second, if a mandatory retirement age of 67 years is introduced, this policy forces agents to work longer than they would under the PAYG system (which is  $44.3907+20=64.3907$  years). In response to this imposition, agents reduce their savings, since now they are made to spend about 2.6 years less time in retirement. Third, replacing the PAYG system with a fully funded system slightly reduces savings.

3. The return to capital,  $r^*$ , decreases under the no social security alternative and increases under the mandatory retirement policy and fully funded system, compared to the benchmark model with PAYG system. Wage rate,  $w^*(t)$  decreases under all alternatives.

4. Since the mandatory retirement age of 67 years is greater than the endogenously chosen value of 64.4907 years under the PAYG system, imposing the mandatory retirement system causes the payroll tax,  $T$ , to decline. But introducing the fully funded system causes the payroll tax to increase.

5. Both output per effective worker,  $y^*$ , and capital / output ratio,  $k^*/y^*$  increase under the no social security alternative, but decrease under the mandatory retirement age and fully funded system alternatives.

6. Consumption at time of birth for all cohorts,  $\gamma_{c(\nu,\nu)}$ , increases under the no social security alternative, but decrease under the mandatory retirement age and fully funded system alternatives. But the growth rate of consumption across time,  $\gamma_{c(\nu,t)}$ , decreases under

the no social security alternative, but increases under the mandatory retirement age and fully funded system alternatives. Both  $\gamma_{c(\nu,\nu)}$  and  $\gamma_{c(\nu,t)}$  have an impact on consumption per effective worker,  $c^*$ . However,  $\gamma_{c(\nu,\nu)}$  seems to be the dominant factor, since  $c^*$  increases under the no social security alternative, but decrease under the mandatory retirement age and fully funded system alternatives.

I compare these results with the results in Table-1.4 from chapter-1 before concluding this section. The most important difference between the two sets of results are as follows: First, when the SDU function is included in the model, replacing the PAYG system with a fully funded system causes both the retirement age,  $R^*$ , and capital per effective worker,  $k^*$ , to decrease. This is the opposite of what is observed in the version of the model without the SDU function, when the fully funded system is introduced. Second, if the social security net is totally removed, in the absence of the SDU function,  $k^*$  increases. But if the SDU function is operative in the model,  $k^*$  decreases. Other than these two important changes, the other social security alternatives produce the same impact on  $R^*$  and  $k^*$  whether the SDU function is operative in the model or not.

The contrasting changes on  $R^*$  and  $k^*$  with SDU and without SDU specifications of the model also influence the changes on the equilibrium values of the other economic variables, when the PAYG system is replaced with one of the alternatives. Closer inspection of the results reveals that the following variables move in the same direction if the SDU function is operative and either the mandatory retirement age is introduced or the PAYG system is replaced with the fully funded system: factor returns,  $r^*$  and  $w^*(t)$ , output per effective worker,  $y^*$ , and capital per effective worker,  $k^*/y^*$ , consumption per effective worker,  $c^*$ , growth rate of consumption across time,  $\gamma_{c(\nu,t)}$ , and consumption at time of birth for all cohorts,  $c(\nu, \nu)$ . In this particular case, if there is no social security net, these variable move in the opposite direction.

Interestingly, if the SDU function is present in the model, the variables mentioned above move in the same direction (which is opposite to the direction they would move if the SDU function is not operative), when the PAYG system is replaced with either no social security

or a mandatory retirement age is introduced. Also, in this case, if a full funded system is introduced, they move in the opposite direction.

Finally, whether the SDU function is included in the model or not, introducing the mandatory retirement age enable the payroll tax,  $T$ , to decline. But switching to the fully funded system causes the payroll tax to decline only if the SDU function is present in the model. If not, this change causes the tax to increase.

## **2.5 Conclusion**

In this chapter, I investigated the importance of the state dependent utility (SDU) from consumption function in the theoretical model of chapter-1. To that end, I removed the SDU function from the model and re-calibrated the parameters of the age dependent disutility of labor function to numerically calculate the equilibrium of this new version of the model.

Next, I repeated the demographic change exercises of the previous chapter without the SDU function. In particular, I compared the two sets of results (results from the model with the SDU function and the model without the SDU function) that emerge in response to different kinds of demographic change. Crucial results came to light from this exercise. First, it was revealed that compression of morbidity causes the retirement age,  $R^*$ , to increase if the SDU function is included in the model. If not,  $R^*$  decreases in response to compression of morbidity. Second, it was found that saving of the agents increases in response to mortality decline, if the SDU function is operative in the model. If not, their saving increases when mortality decline takes place. Also, the changes in the equilibrium values of the other economic variables in response to demographic change, both with and without the SDU function, have been discussed and compared in that section.

Finally, after removing the SDU function, I also repeated the exercise of changing the social security system in the model. Comparison of the two sets of numerical analysis results (results from the model with the SDU function and the model without the SDU function) again provided important insights. Most importantly, it was found that when the SDU function is present, introducing a fully funded system causes the retirement age,  $R^*$ , and

capital per effective worker,  $k^*$ , to decrease, but the opposite happens if the SDU function is removed from the model. Also, comparison of the results indicates that if the social security net is not available, in the absence of the SDU function,  $k^*$  increases. But if the SDU function is operative in the model,  $k^*$  decreases.

Given the contrasting results listed in the previous paragraphs, it is clear that including the state dependent function in the theoretical model of chapter-1 was no inconsequential task. It has on the contrary produced important effects for both the equilibrium values of the economic variables and the results of the numerical analysis exercises mentioned in both chapters 1 and 2. For this reason, I believe it would also be helpful to construct a model with a positive state dependent utility function from consumption and then repeat the same numerical exercises. Even more importantly perhaps, it will be very instructive if a model can be set up with more than two states and a stochastic transition mechanism between those two states. These are the extensions that I am planning to pursue in the near future.

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## Appendix A

### SOLUTION METHODOLOGY

I summarize the steps to calculate the steady state of the model in this appendix.

**Step-1:** Obtain the growth rate rate of consumption across time,  $\gamma_{C(\nu,t)}$ . Plug it in the objective function, equation (1.5) in the main text. Take the derivative with respect to the retirement age,  $R$ . This yields equation (1.21), which determines the optimal retirement age chosen by the agents in the model.

**Step-2:** Three things are required to solve equation (1.21):

1.  $c(\nu, \nu)$  , 2.  $\int_{\nu}^{\nu+D} \frac{\partial c(\nu, \tau)}{\partial R} e^{-r^*(\tau-\nu)-M(\tau-\nu)} d\tau$ , 3.  $\int_{\nu+X}^{\nu+D} \frac{\partial c(\nu, \tau)}{\partial R} e^{-r^*(\tau-\nu)-M(\tau-\nu)} d\tau$

To find  $c(\nu, \nu)$ , just plug  $\gamma_{c(\nu,t)}$  in the lifetime budget constraint and isolate  $c(\nu, \nu)$  on one side of the equation .

To get  $\int_{\nu}^{\nu+D} \frac{\partial c(\nu, \tau)}{\partial R} e^{-r^*(\tau-\nu)-M(\tau-\nu)} d\tau$ , take the derivative of the lifetime budget constraint with respect to the retirement age,  $R$ .

Finally, to retrieve  $\int_{\nu+X}^{\nu+D} \frac{\partial c(\nu, \tau)}{\partial R} e^{-r^*(\tau-\nu)-M(\tau-\nu)} d\tau$ , write the period budget constraint for the interval  $[\nu + X, \nu + D]$  and take its derivative with respect to the retirement age,  $R$ .

**Step-3:** Aggregate the individual consumption levels of agents across cohorts to obtain aggregate consumption in the steady state,  $C^*(t)$ , which is given by equation (1.31) in the

main text. Divide it with the effective labor supply  $A(t)L(t)$  to obtain aggregate per capita consumption in the steady state,  $c^*$ .

**Step-4:** Plug  $c^*$  in the market clearing condition in intensive form, which is equation (1.28) in the main text.

**Step-5:** Derive the equation for the payroll tax,  $T^*$ , which keeps the pay as you go (PAYG) social security system in balance in the steady state by setting the total contributions paid by the working agents equal to the total benefits received by the retired agents at a generic time  $t$ . This yields equation (1.38) in the main text.

**Step-6:** Using the calibrated values of the demographic and structural variables, numerically calculate the payroll tax rate,  $T^*$ , first.

**Step-7:** Plug  $T^*$  in equations (1.21) and (1.28). Equation (1.21) is an implicit equation in  $k^*$  and  $R^*$ . Equation (28) is the market clearing condition in intensive form, where  $c^*$  was plugged in Step-4. Solve equations (1.21) and (1.28) together numerically to obtain  $k^*$  and  $R^*$ . I used the *fsolve* function in MatLab to solve these equations together, but any modern numerical computing platform should be able to solve them easily.

**Step-8:** Plug  $k^*$  in factor return equations (1.15) and (1.16) to get the return to saving and wage rate in the steady state. Next plug  $k^*$  in equation (1.28) to find consumption and investment per effective worker in the steady state,  $c^*$  and  $i^*$ . Using the Cobb-Douglas production function, it is easy to get output per effective worker as  $y^* = (k^*)^\alpha$ .

I used the MatLab platform to calibrate the model, numerically solve for the steady state and then later on do the numerical simulation exercises in the paper.

## Appendix B

### DERIVATIONS

#### B.0.1 Derivation of Equation (1.20)

To derive the of growth rate of consumption across time,  $\gamma_{c(\nu,t)}$ , begin with the Hamiltonian associated with the agent's utility maximization problem:

$$H = e^{-\rho(t-\nu)-M(t-\nu)}[u(c(\nu,t)) - \phi(t-\nu)] + \lambda[(r(t) + \mu(t-\nu))A(\nu,t) + (1-T)w(t)L(\nu,t) + bw(\nu+R)(1-L(\nu,t)) - c(\nu,t)]$$

where  $u(c(\nu,t)) = \gamma_0 I + (1 + \gamma_1 I) \left[ \frac{c(\nu,t)^{1-\frac{1}{\sigma}} - 1}{1-\frac{1}{\sigma}} \right]$  and  $\phi(t-\nu) = \delta_0 e^{\delta_1 \max\{t-\nu-X,0\}}$ .

Take the derivative of the Hamiltonian with respect to consumption,  $c(\nu,t)$  to get:

$$\lambda = u'(c(\nu,t))e^{-\rho(t-\nu)-M(t-\nu)} \quad (\text{I})$$

Next, take the derivative of the Hamiltonian with respect to the state variable,  $A(\nu,t)$  and use the fact that  $-\frac{\partial H}{\partial A(\nu,t)} = \frac{\partial \lambda}{\partial t} = \dot{\lambda}(t,\nu)$  to get:

$$-\frac{\partial \dot{\lambda}(t,\nu)}{\partial \lambda(t,\nu)} = -[r(t) + \mu(t-\nu)] \quad (\text{II})$$

Taking the time derivative of (I) and combining it with (II) yields:

$$r(t) - \rho = -\frac{\partial c(\nu,t)}{\partial t} \frac{u''(c(\nu,t))}{u'(c(\nu,t))}$$

where  $u'(c(\nu, t)) = c^{-\frac{1}{\sigma}}[1 + \gamma_1 I]$  and  $u''(c(\nu, t)) = -\frac{1}{\sigma}[c^{-\frac{1}{\sigma}-1}][1 + \gamma_1 I]$  can be plugged in to get:

$$\gamma_c(\nu, t) = \frac{\partial c(\nu, t)/\partial t}{c(\nu, t)} = \sigma[r(t) - \rho]$$

### B.0.2 Derivations of Equations (1.21), (1.22), (1.23), (1.24)

To derive the equation which determines the retirement condition, equation (1.21) in the main text, begin by writing the objective function of the agents in the model:

$$\int_{\nu}^{\nu+D} [\gamma_0 I + (1 + \gamma_1 I)u(c(\nu, t))]e^{-\rho(t-\nu)-M(t-\nu)} dt - \int_{\nu}^{\nu+Ret} \phi(t - \nu)e^{-\rho(t-\nu)-M(t-\nu)} dt$$

Plug  $u(c(\nu, t)) = \gamma_0 I + (1 + \gamma_1 I) \left[ \frac{c(\nu, t)^{1-\frac{1}{\sigma}} - 1}{1-\frac{1}{\sigma}} \right]$  in the above objective function and take its derivative with respect to the retirement age.

When taking the derivative of the first part of the objective function (utility from consumption), use the equation  $c(\nu, t) = c(\nu, \nu)e^{\gamma_c(\nu, t)} = c(\nu, \nu)e^{\sigma[r^* - \rho]}$ . The derivative of this first part equals:

$$\left[ \int_{\nu}^{\nu+D} \frac{\partial \gamma_0 I}{\partial R} e^{-\rho(\tau-\nu)-M(\tau-\nu)} d\tau \right] + \left[ \int_{\nu}^{\nu+D} \frac{\partial c(\nu, \tau)}{\partial R} (1 + \gamma_1 I) c(\nu, \nu)^{-\frac{1}{\sigma}} e^{-\rho(\tau-\nu)-M(\tau-\nu)} e^{-r^*(\tau-\nu)+\rho(\tau-\nu)} d\tau \right]$$

The terms in the first bracket equal 0. The terms in the second bracket can be written as:

$$\left[ \int_{\nu}^{\nu+D} \frac{\partial c(\nu, \tau)}{\partial R} c(\nu, \nu)^{-\frac{1}{\sigma}} e^{-r^*(\tau-\nu)-M(\tau-\nu)} d\tau \right] + \left[ \int_{\nu}^{\nu+D} \gamma_1 I \frac{\partial c(\nu, \tau)}{\partial R} c(\nu, \nu)^{-\frac{1}{\sigma}} e^{-r^*(\tau-\nu)-M(\tau-\nu)} d\tau \right]$$

Noting that the indicator function  $I$  is equal to 0 in the interval  $0 < \tau - \nu < X$ , this last equation reduces to:

$$c(\nu, \nu)^{-1/\sigma} \left[ \int_{\nu}^{\nu+D} \frac{\partial c(\nu, \tau)}{\partial R} e^{-r^*(\tau-\nu)-M(\tau-\nu)} d\tau \right] + \gamma_1 c(\nu, \nu)^{-1/\sigma} \left[ \int_{\nu+X}^{\nu+D} \frac{\partial c(\nu, \tau)}{\partial R} e^{-r^*(\tau-\nu)-M(\tau-\nu)} d\tau \right]$$

Again, this is the derivative of the first part of the objective function (utility from consumption) with respect to the retirement age. To take the derivative of the second part of the objective function (disutility from labor) with respect to the retirement age, use Leibniz's Integral Rule:

$$\begin{aligned} \frac{\partial}{\partial R} \left[ - \int_{\nu}^{\nu+R} \phi(\tau - \nu) e^{\rho(\tau-\nu)-M(\tau-\nu)} d\tau \right] &= - \phi(\nu + R - \nu) e^{-\rho(\nu+R-\nu)-M(\nu+R-\nu)} \\ &\quad - \left[ \int_{\nu}^{\nu+R} \frac{\partial \phi(\tau - \nu)}{\partial R} e^{-\rho(\nu+R-\nu)-M(\nu+R-\nu)} \right] \end{aligned}$$

The terms in the bracket on the right hand side of this equation equal 0. Hence, the derivative of the second part reduces to  $-\phi(R) e^{-\rho(R)-M(R)}$ .

Combining the derivative of the first part (utility from consumption) and the second part (disutility from labor) of the objective function with respect to retirement age, I obtain equation (1.21) in the main text:

$$\begin{aligned} c(\nu, \nu)^{-1/\sigma} \left[ \int_{\nu}^{\nu+D} \frac{\partial c(\nu, \tau)}{\partial R} e^{-r^*(\tau-\nu)-M(\tau-\nu)} d\tau \right] + \gamma_1 c(\nu, \nu)^{-1/\sigma} \left[ \int_{\nu+X}^{\nu+D} \frac{\partial c(\nu, \tau)}{\partial R} e^{-r^*(\tau-\nu)-M(\tau-\nu)} d\tau \right] \\ - [\phi(R^*) e^{-\rho R^*-M(R^*)}] = 0 \end{aligned}$$

To solve equation (1.21), three terms need to be derived:

$$1. c(\nu, \nu), \quad 2. \left[ \int_{\nu}^{\nu+D} \frac{\partial c(\nu, \tau)}{\partial R} e^{-r^*(\tau-\nu)-M(\tau-\nu)} d\tau \right], \quad 3. \left[ \int_{\nu+X}^{\nu+D} \frac{\partial c(\nu, \tau)}{\partial R} e^{-r^*(\tau-\nu)-M(\tau-\nu)} d\tau \right]$$

I explain the detailed derivation of  $c(\nu, \nu)$  first. Begin with the lifetime budget constraint operative from time  $\nu$  (date of birth):

$$\left[ \int_{\nu}^{\nu+D} c(\nu, \tau) e^{-r^*(\tau-\nu)-M(\tau-\nu)} d\tau \right] = \left[ (1-T) \int_{\nu}^{\nu+D} w(\tau) e^{-r^*(\tau-\nu)-M(\tau-\nu)} d\tau \right] + \left[ bw(\nu+R) \int_{\nu+R}^{\nu+D} e^{-r^*(\tau-\nu)-M(\tau-\nu)} d\tau \right]$$

where the boundary conditions  $A(\nu, \nu) = A(\nu, \nu + D) = 0$  have been used. Utilizing the growth rate of consumption across time,  $\gamma_{c(\nu,t)}$ , on the left hand side and the growth rate of wages in equilibrium,  $\gamma_A$ , on the right hand side of this budget constraint yields:

$$\left[ \int_{\nu}^{\nu+D} c(\nu, \nu) e^{\sigma[r^*-\rho][\tau-\nu]} e^{-r^*(\tau-\nu)-M(\tau-\nu)} d\tau \right] = \left[ (1-T)A(0)Z \int_{\nu}^{\nu+D} e^{\gamma_A \tau} e^{-r^*(\tau-\nu)-M(\tau-\nu)} d\tau \right] + \left[ bA(0)Z e^{\gamma_A(\nu+R)} \int_{\nu+R}^{\nu+D} e^{-r^*(\tau-\nu)-M(\tau-\nu)} d\tau \right]$$

where  $A(0)Z$  is the wage level at time 0. Isolating  $c(\nu, \nu)$  on the left hand side of this last equation gives equation (1.22) in the main text:

$$c(\nu, \nu) = \frac{A(0)Z \left[ (1-T) \int_{\nu}^{\nu+D} e^{\gamma_A \tau} e^{-r^*(\tau-\nu)-M(\tau-\nu)} d\tau + b e^{\gamma_A(\nu+R)} \int_{\nu+R}^{\nu+D} e^{-r^*(\tau-\nu)-M(\tau-\nu)} d\tau \right]}{\left[ \int_{\nu}^{\nu+D} e^{(\sigma-1)r^*(\tau-\nu)-\sigma\rho(\tau-\nu)-M(\tau-\nu)} d\tau \right]}$$

Next, I explain the derivation of  $\left[ \int_{\nu}^{\nu+D} \frac{\partial c(\nu, \tau)}{\partial R} e^{-r^*(\tau-\nu)-M(\tau-\nu)} d\tau \right]$ . Return back to the lifetime budget constraint operative from time of birth,  $\nu$ . Take its derivative with respect to the retirement age and use Leibniz's Integral rule to get equation (1.23):

$$\left[ \int_{\nu}^{\nu+D} \frac{\partial c(\nu, \tau)}{\partial R} e^{-r^*(\tau-\nu)-M(\tau-\nu)} d\tau \right] = \left[ (1-T-b)w(\nu+R) e^{-r^*R-M(R)} \right] + \left[ b\gamma_A w(\nu+R) \int_{\nu+R}^{\nu+D} e^{-r^*(\tau-\nu)-M(\tau-\nu)} d\tau \right]$$

To obtain  $\left[ \int_{\nu+X}^{\nu+D} \frac{\partial c(\nu, \tau)}{\partial R} e^{-r^*(\tau-\nu)-M(\tau-\nu)} d\tau \right]$ , write the period budget constraint operative from age  $X$ , the age at which marginal utility of consumption decreases and disutility of labor begins increasing:

$$\begin{aligned} \left[ \int_{\nu+X}^{\nu+D} c(\nu, \tau) e^{-r^*(\tau-\nu)-M(\tau-\nu)} d\tau \right] e^{r^*X+M(X)} &= \left[ \int_{\nu+X}^{\nu+R} (1-T)w(\tau) e^{-r^*(\tau-\nu)-M(\tau-\nu)} d\tau \right] e^{r^*X+M(X)} + \\ \left[ bw(\nu+R) \int_{\nu+R}^{\nu+D} e^{-r^*(\tau-\nu)-M(\tau-\nu)} d\tau \right] e^{r^*X+M(X)} &+ A(\nu, \nu+X) \end{aligned}$$

Taking the derivative of this period budget constraint with respect to retirement age yields:

$$\begin{aligned} \left[ \int_{\nu+X}^{\nu+D} \frac{\partial c(\nu, \tau)}{\partial R} e^{-r^*(\tau-\nu)-M(\tau-\nu)} d\tau \right] &= \left[ \frac{\partial A(\nu, \nu+X)}{\partial R} \right] \left[ \frac{1}{e^{r^*X+M(X)}} \right] + \\ \left[ (1-T-b)e^{-r^*R-M(R)} + b\gamma_A \int_{\nu+R}^{\nu+D} e^{-r^*(\tau-\nu)-M(\tau-\nu)} d\tau \right] &w(\nu+R) \end{aligned} \quad (\text{III})$$

The partial derivative  $\left[ \frac{\partial A(\nu, \nu+X)}{\partial R} \right]$  on the right hand side of this equation is problematic, but it can be obtained by using the period budget constraint in the interval  $[\nu, \nu+X]$ :

$$\begin{aligned} \left[ \int_{\nu}^{\nu+X} c(\nu, \tau) e^{-r^*(\tau-\nu)-M(\tau-\nu)} d\tau \right] e^{r^*X+M(X)} &= \left[ \int_{\nu}^{\nu+X} (1-T)w(\tau) e^{-r^*(\tau-\nu)-M(\tau-\nu)} d\tau \right] e^{r^*X+M(X)} + \\ \left[ bw(\nu+R) \int_{\nu}^{\nu+X} e^{-r^*(\tau-\nu)-M(\tau-\nu)} d\tau \right] e^{r^*X+M(X)} &- A(\nu, \nu+X) \end{aligned}$$

Pension benefits in the interval  $[\nu, \nu+X]$  equal 0 (the second term in the brackets on the right hand side), since the agent is not retired in this period. Taking the derivative of the remaining terms with respect to the retirement age yields an equation for  $\frac{\partial A(\nu, \nu+X)}{\partial R}$ :

$$-\frac{\partial A(\nu, \nu+X)}{\partial R} = \left[ \int_{\nu}^{\nu+X} \frac{\partial c(\nu, \tau)}{\partial R} e^{-r^*(\tau-\nu)-M(\tau-\nu)} d\tau \right] [e^{r^*X+M(X)}] \quad (\text{IV})$$

To solve this equation further, I use the fact that  $c(\nu, \tau) = c(\nu, \nu) e^{\gamma_{c(\nu, t)}(\tau-\nu)}$  and hence  $\frac{\partial c(\nu, t)}{\partial R} = \frac{\partial c(\nu, \nu)}{\partial R} e^{\gamma_{c(\nu, t)}(\tau-\nu)}$ . An expression for  $c(\nu, \nu)$  was derived earlier and can be found in

equation (1.22). So I proceed by taking the derivative of equation (1.22) with respect to the retirement age:

$$\frac{\partial c(\nu, \nu)}{\partial R} = \frac{[ZA(0)b\gamma_A e^{\gamma_A(\nu+R)}][-e^{-r^*(\tau-\nu)-M(\tau-\nu)}]}{\left[ \int_{\nu}^{\nu+D} e^{(\sigma-1)r^*(\tau-\nu)} e^{-\sigma\rho(\tau-\nu)} e^{-M(\tau-\nu)} d\tau \right]}$$

Use this last equation in  $\frac{\partial c(\nu, t)}{\partial R} = \frac{\partial c(\nu, \nu)}{\partial R} e^{\gamma_{c(\nu, t)}(\tau-\nu)}$  to get  $\frac{\partial c(\nu, t)}{\partial R}$ . Then plug the resulting expression in equation (IV). This gives us an equation for  $\frac{\partial A(\nu, \nu+X)}{\partial R}$ :

$$\frac{\partial A(\nu, \nu+X)}{\partial R} = \left[ \int_{\nu}^{\nu+X} \frac{[A(0)Z e^{\sigma[r^*-\rho][\tau-\nu]}] [b\gamma_A e^{\gamma_A(\nu+R)}] [e^{-2r^*(\tau-\nu)-2M(\tau-\nu)}]}{\left[ \int_{\nu}^{\nu+D} e^{(\sigma-1)r^*(\tau-\nu)-\sigma\rho(\tau-\nu)-M(\tau-\nu)} d\tau \right]} d\tau \right] [e^{r^*X+M(X)}] \quad (\text{V})$$

Plugging (V) in equation (III) results in equation (1.24) in the main text:

$$\left[ \int_{\nu+X}^{\nu+D} \frac{\partial c(\nu, \tau)}{\partial R} e^{-r^*(\tau-\nu)-M(\tau-\nu)} d\tau \right] = \left[ \int_{\nu}^{\nu+X} \frac{[A(0)Z e^{\sigma[r^*-\rho][\tau-\nu]}] [b\gamma_A e^{\gamma_A(\nu+R)}] [e^{-2r^*(\tau-\nu)-2M(\tau-\nu)}]}{\left[ \int_{\nu}^{\nu+D} e^{(\sigma-1)r^*(\tau-\nu)-\sigma\rho(\tau-\nu)-M(\tau-\nu)} d\tau \right]} d\tau \right] \\ + [(1-T-b)w(\nu+R)e^{-r^*R-M(R)}] + \left[ b\gamma_A w(\nu+R) \int_{\nu+R}^{\nu+D} e^{-r^*(\tau-\nu)-M(\tau-\nu)} d\tau \right]$$

### B.0.3 Growth rate of consumption at time of birth across cohorts

In the main text, I mention that the consumption levels at time of birth grow across cohorts at the constant rate of tech progress,  $\gamma_A$ . In this section, I show the derivation of this fact. Begin with equation (1.22):

$$c(\nu, \nu) = \frac{A(0)Z \left[ (1-T) \int_{\nu}^{\nu+D} e^{\gamma_A \tau - r^*(\tau-\nu) - M(\tau-\nu)} d\tau + b e^{\gamma_A(\nu+R)} \int_{\nu+R}^{\nu+D} e^{-r^*(\tau-\nu) - M(\tau-\nu)} d\tau \right]}{\left[ \int_{\nu}^{\nu+D} e^{(\sigma-1)r^*(\tau-\nu) - \sigma\rho(\tau-\nu) - M(\tau-\nu)} d\tau \right]}$$

The denominator of this equation is a constant. Hence, the numerator will determine the growth rate of consumption at time of birth,  $\gamma_{c\nu, \nu}$ . Multiply the first term between the

brackets in the numerator with  $e^{\gamma_A \nu}$  and  $e^{-\gamma_A \nu}$  to get:

$$c(\nu, \nu) = \frac{A(0)Z \left[ (1 - T)e^{\gamma_A \nu} \int_{\nu}^{\nu+D} e^{\gamma_A(\tau-\nu) - r^*(\tau-\nu) - M(\tau-\nu)} d\tau + be^{\gamma_A(\nu)} e^R \int_{\nu+R}^{\nu+D} e^{-r^*(\tau-\nu) - M(\tau-\nu)} d\tau \right]}{\left[ \int_{\nu}^{\nu+D} e^{(\sigma-1)r^*(\tau-\nu) - \sigma\rho(\tau-\nu) - M(\tau-\nu)} d\tau \right]}$$

In the numerator of this equation, two  $e^{\gamma_A \nu}$  terms respectively multiply two constants that are additive between the brackets. Given that  $A(0)Z$  is also constant in the numerator, the growth rate of  $c(\nu, \nu)$  across birth cohorts is just equal to the growth rate of  $e^{\gamma_A \nu}$ , which is  $\gamma_A$ . In other words, consumption at time of birth grows at rate  $\gamma_A$  across birth cohorts.

## Appendix C

### THEORETICAL DEMOGRAPHIC STRUCTURE

In this appendix, I explain the derivations of some of the key equations describing the theoretical demographic structure of the model. As mentioned in the Calibration Section, I use the Boucekkine Survival function (BCL) in this paper. For the reader's benefit, I begin with some basic definitions related to the BCL survival function.

The survival function itself is given by:

$$S(t - \nu) = e^{-M(t-\nu)} = \frac{\mu_0 - e^{\mu_1(t-\nu)}}{\mu_0 - 1}$$

where where  $\mu_0 > 1$  is the youth mortality parameter and  $\mu_1 > 0$  is the old age mortality parameter. This equation gives the probability of surviving to age  $t - \nu$ .

The instantenous mortality hazard at age  $t - \nu$  is given by:

$$\mu(t - \nu) = -\frac{S'(t - \nu)}{S(t - \nu)} = \frac{\mu_1 e^{\mu_1(t-\nu)}}{\mu_0 - e^{\mu_1(t-\nu)}}$$

Integrating the instantenous mortality hazard function from age 0 to age  $t - \nu$  yields the cumulative mortality hazard at age  $t - \nu$ :

$$M(t - \nu) = \int_0^{t-\nu} \mu(\tau) d\tau = \ln(\mu_0 - 1) - \ln(\mu_0 - e^{\mu_1(t-\nu)})$$

The cumulative mortality hazard function represents the cumulative probability of dying before or at age  $t - \nu$ .

Using these three functions (survival, instantenous mortality and cumulative mortality

functions), the key equations of the demographic structure in the model are derived as follows.

First, the generic condition for the demographic steady state is given in equation (1.12) in the main text:

$$\beta \int_{t-D}^t e^{-n(t-\nu)-M(t-\nu)} d\nu = 1$$

Plugging the BCL survival function in the above equation yields the specific condition:

$$\beta \int_{t-D}^t e^{-n(t-\nu)} \left[ \frac{\mu_0 - e^{\mu_1(t-\nu)}}{\mu_0 - 1} \right] d\nu = \left[ \frac{\beta}{\mu_0 - 1} \right] \left[ \mu_0 \frac{1 - e^{-nD}}{n} + \frac{1 - e^{(\mu_1 - n)D}}{\mu_1 - n} \right] = 1$$

Next the number of agents from time  $\nu$  birth cohort surviving at time  $t$  was given in equation (1.10):

$$P(\nu, t) = P(\nu, \nu) e^{-M(t-\nu)} = \beta P(\nu) e^{-M(t-\nu)}$$

In the case of the BCL survival function, this equation becomes:

$$P(\nu, t) = \beta P(\nu) \left[ \frac{\mu_0 - e^{\mu_1(t-\nu)}}{\mu_0 - 1} \right] \quad (\text{VI})$$

This last equation will be useful in calculating the average age of the agents in the model, that is given by:

$$\bar{u} = \frac{\left[ \int_{t-D}^t \tau P(t - \tau, t) d\tau \right]}{P(t)} \quad (\text{VII})$$

where  $\tau$  is the age of the agent. Plugging equation (VI) in (VII) and noting the fact that population grows at constant rate  $n$ , the average age of the agents in the case of the BCL

survival function is obtained as:

$$\bar{u} = \left[ \int_{t-D}^t \tau \beta e^{-n(t-\nu)} \left[ \frac{\mu_0 - e^{\mu_1(t-\nu)}}{\mu_0 - 1} \right] d\tau \right] = \left[ \frac{\beta \mu_0}{\mu_0 - 1} \right] \left[ \frac{1}{n^2} \right] [e^{-nD}(-nD - 1) + 1] \\ - \left[ \frac{\beta}{\mu_0 - 1} \right] \left[ \frac{1}{(\mu_1 - n)^2} \right] [e^{(\mu_1 - n)D}((\mu_1 - n)D - 1) + 1]$$

The average instantaneous mortality hazard is given in equation (1.9) in the main text:

$$\bar{\mu} = \int_{t-D}^t \mu(t - \nu) \frac{P(\nu, t)}{P(t)} d\nu$$

Plugging (VI) in this last equation and once again using the fact that population grows at constant rate  $n$  yields the average instantaneous mortality hazard in the case of the BCL function:

$$\bar{\mu} = \int_{t-D}^t \beta e^{-n(t-\nu)} \left[ \frac{\mu_0 - e^{\mu_1(t-\nu)}}{\mu_0 - 1} \right] d\nu$$

The terminal age  $D$  in the model is simply obtained by using the fact that  $S(D) = 0$ . In the case of the BCL survival function this yields  $\mu_0 = e^{\mu_1 D}$ , which means:  $D = \frac{\ln(\mu_0)}{\mu_1}$ .

Finally, life expectancy at any age,  $u$ , can be calculated by using the generic formula:

$$\lambda(u) = \int_{\nu+u}^{\nu+D} (t - \nu) S(t - \nu) dt$$

where  $\nu$  is the date of birth. Using the BCL survival function in the above equation and setting  $u = 0$  yields the life expectancy at time of birth in the theoretical model of this paper:

$$\lambda = \left[ -\frac{1}{\mu_1} + \frac{\mu_0 \ln(\mu_0)}{(1 - \mu_0)(-\mu_1)} \right]$$

## Appendix D

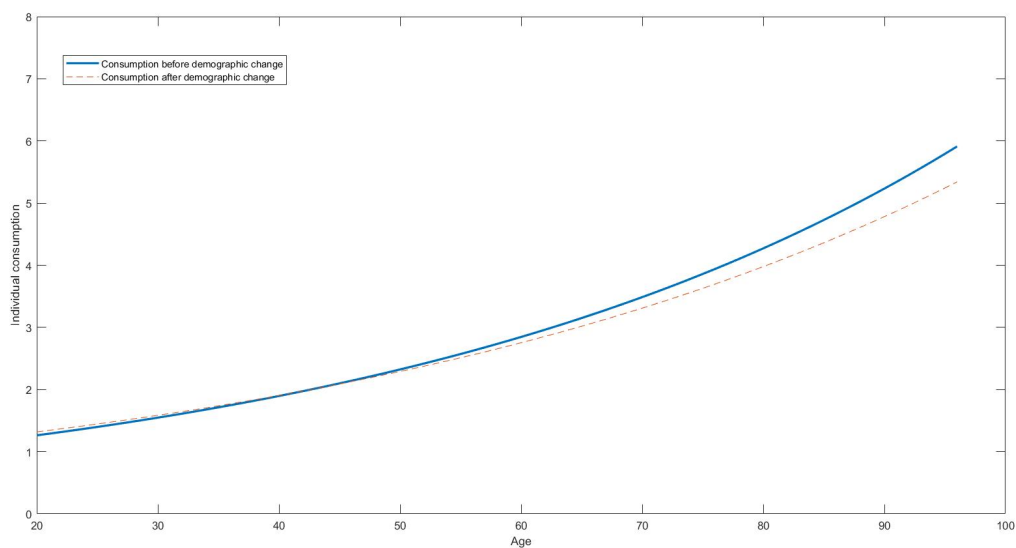
**GENERAL EQUILIBRIUM BEHAVIOR BEFORE AND AFTER  
DEMOGRAPHIC CHANGE**

Figure D.1: Consumption-age profile of an individual of time  $\nu = 10$  birth cohort before and after demographic change (mortality decline and compression happening together)

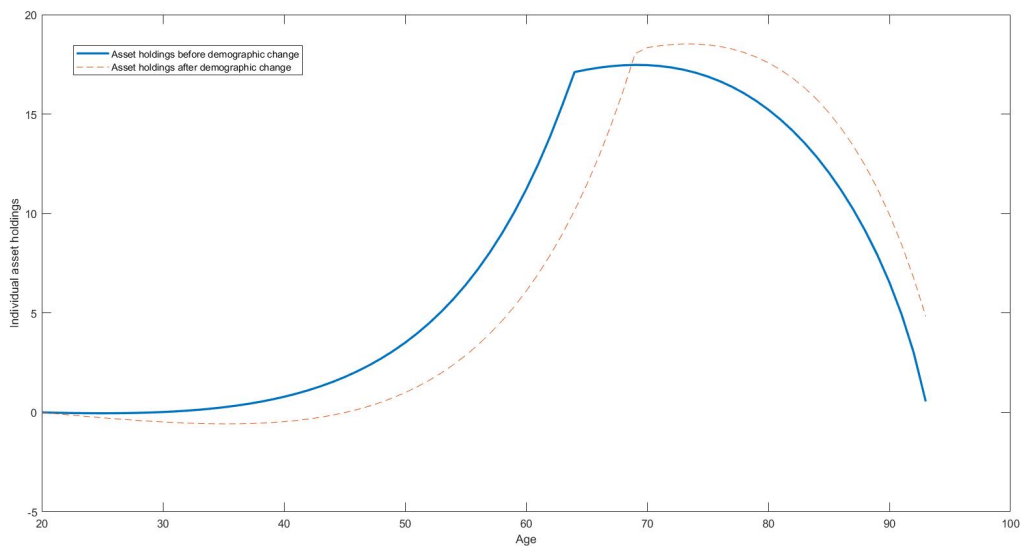


Figure D.2: Asset holdings-age profile of an individual of time  $\nu = 10$  birth cohort before and after demographic change (mortality decline and compression happening together)

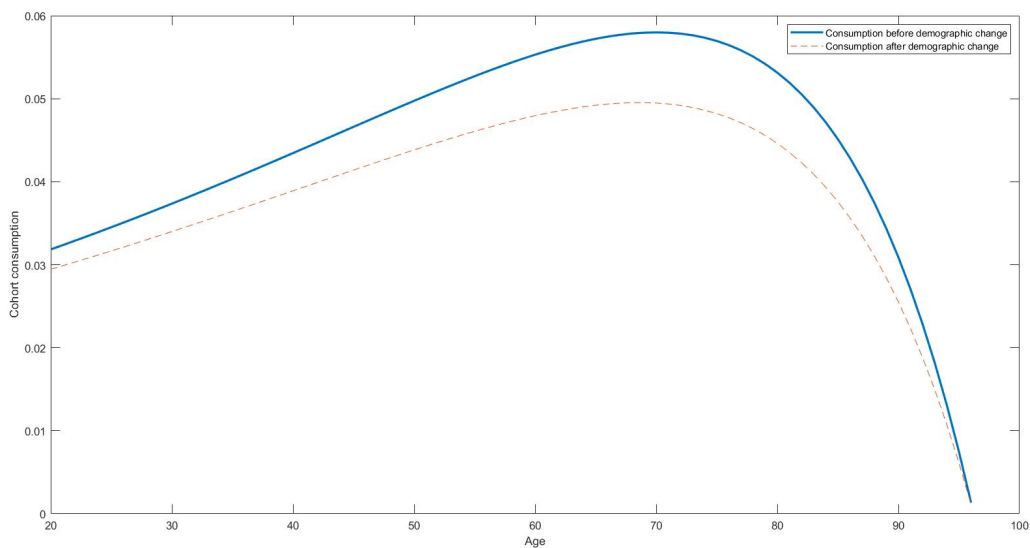


Figure D.3: Consumption-age profile of time  $\nu = 10$  birth cohort before and after demographic change (mortality decline and compression happening together)

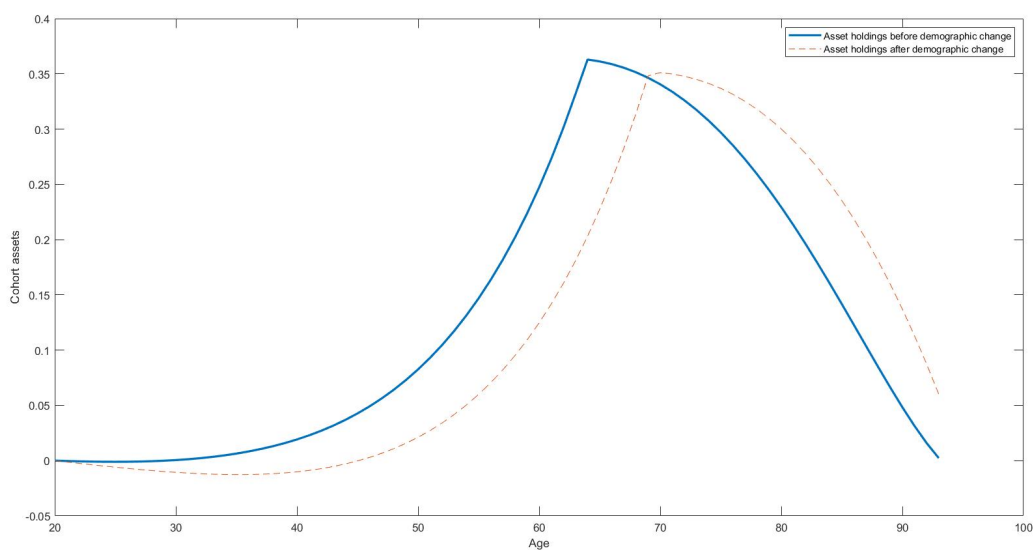


Figure D.4: Asset holdings-age profile of time  $\nu = 10$  birth cohort before and after demographic change (mortality decline and compression happening together)