Quantitative Density Statements for Translation Surfaces

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Abstract

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The main results in this thesis are quantitative descriptions of the orbits of two dynamical systems on translation surfaces. First, we study the action of a discrete subgroup of $SL_2(\mathbb{R})$ on a closed square-tiled surface and quantify the density of the orbits by proving a Diophantine estimate. Second, we study the linear flow on a translation surface and identify a quantitative density condition on the flow that is equivalent to the boundedness of an associated geodesic in the moduli space of translation surfaces.

Acknowledgements

I am deeply grateful to the many people who made this journey successful. I came from outside the world of math, and while it did not take long to fall in love with the subject, my journey was certainly not without missteps. I was fortunate to have people around me who gave me the strength and fortitude to keep going.

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QUANTITATIVE DENSITY THEOREMS FOR TRANSLATION SURFACES

JOSH SOUTHERLAND

CONTENTS

1. INTRODUCTION

Translation surfaces arise in many settings, including the study of rational billiards, a model of a Boltzman gas, and electron transport on Fermi-surfaces [57]. Over time, researchers realized that these models are intimately connected to complex geometry and Teichmüller dynamics, and began studying the $SL_2(\mathbb{R})$-action on the space of translation surfaces. The recurring theme: orbit closures of the $SL_2(\mathbb{R})$-action govern the behavior of certain dynamical systems on individual translation surfaces.

The results in this thesis concern two different dynamical systems on translation surfaces. First, we study the action of the (abundant) set of derivatives of affine linear maps on square-tiled surfaces, which correspond to studying the stabilizer of the $SL_2(\mathbb{R})$-action. In this setting, the stabilizer is an arithmetic group, and we show that subgroups of these arithmetic groups exhibit Diophantine properties.

Second, we study the vertical (north) linear flow on a translation surface. As we will see, a translation surface has a well-defined “north", so there is unambigously a linear flow in the north direction. This flow is very closely tied to the geodesic flow in the moduli space, and we use this relationship to show that the vertical flow is sufficiently dense on the surface if and only if the geodesic in the moduli space remains bounded.

1.1. Definition of a translation surface.

A *translation surface* is a pair (X,ω) where X is a compact, connected Riemann surface without boundary and ω a non-zero holomorphic differential on X.

There is an equivalent definition of a translation surface that is more intuitive: a translation surface is an equivalence class of polygons or sets of polygons in the plane $\mathbb{C}$ such that each edge is identified by translation to a parallel edge on the opposite side of the polygon (or opposite side of a polygon in the set of polygons). The equivalence is given by a cut-and-paste procedure that preserves the positive imaginary direction relative to the ambient $\mathbb{C}$ [53], [57].

Note that by imposing the condition that sides are identified to opposite edges of the polygon we ensure that the positive imaginary direction is well-defined, as well as the positive real-direction.

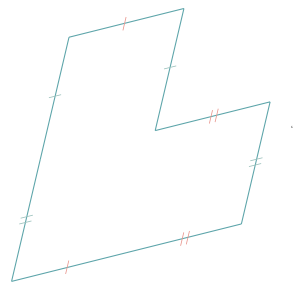

FIGURE 1. Translation surface

Translation surfaces are flat surfaces away from a finite set of singular points, and at the singular points are cone points whose angles are integer multiples of 2π. We can give one more alternative definition of a translation surface as a special type of compact flat surface with Euclidean cone points: let X be a closed, topological surface and let $\Sigma = \{p_1, p_2, \cdots, p_n\}$ be a finite collection of points on the surface such that

(1) $X \setminus \Sigma$ has a translation structure, i.e. it has an atlas of charts into $\mathbb{C}$ such that the transition maps are translations.
(2) In a neighborhood of each $p_i \in \Sigma$, there is a chart into $2(k_i+1)$ half-disks glued as pictured below. Here, we are allowing cone angles at the points in Σ, where the angle around the point p_i is $2\pi(k_i+1)$ for k_i a positive integer.

The half plane construction is pictured in Figure 2.

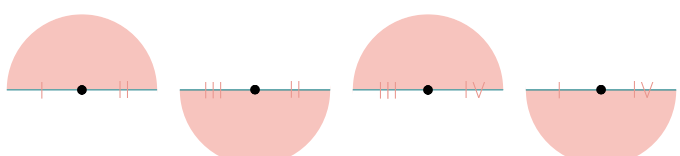

FIGURE 2. Half-plane construction, $k = 1$

We call the points p_i singularities. Since $X \setminus \Sigma$ has a complex structure, it is orientable. Additionally, the cone angles at the singularities are multiples of 2π, so the holonomy around these points is trivial. This means we can make a global choice of "north" on the surface away from Σ. By default, we typically choose the direction associated with the positive imaginary direction in the charts. This surface, equipped with the charts described and a direction indicating north, is a *translation surface*.

For the equivalence of these definitions, the reader is encouraged to visit [53]. The zeroes of the holomorphic differential in the first definition correspond to the singular points, and the order of the zeroes correspond to k_i, the measure of excess angle at the singularities. Moreover, given a compact, connected Riemann surface without a boundary and a non-zero holomorphic differential, we can find a set of charts with translations as transition functions by integrating against the form ω. In a neighborhood of the zeroes, charts given by integration will be precisely the charts described in (2) above.

1.2. **The $SL_2(\mathbb{R})$-action.** The group $SL_2(\mathbb{R})$ acts on the moduli space of translation surfaces, where the action of a matrix is just the usual linear action on the polygons. Since the linear action sends parallel lines to parallel lines, the action sends a translation surface to a (potentially different) translation surface. In fact, the natural action in this setting is $GL_2^+(\mathbb{R})$, but for our purposes, the action of $SL_2(\mathbb{R})$ is more relevant since we are only interested in volume preserving maps, and, as will become clear, recurrence in the moduli space. Note that if we opt to use the third definition of a translation surface given above, the action is by post-composition in charts.

The geometric origins of the action can be found in the work of Thurston [50] and Veech [55]. In Veech's influential paper, Veech answers a question posed by Thurston: do there exist translation surfaces in the moduli space which are periodic in the sense that the stabilizer of the $SL_2(\mathbb{R})$-action is a lattice in $SL_2(\mathbb{R})$? Veech answered in the affirmative, and such surfaces became known as lattice (or Veech) surfaces. This is discussed in the subsequent section.

Although irrelevant to our immediate goals, the following is structurally important for the moduli space. The number of singularities that can be on a surface is bounded by $2g-2$, where g is the genus of the surface. This is a consequence of Riemann–Roch. More precisely, $\sum_{i=1}^n k_i = 2g-2$, where k_i is as above. Hence, we can stratify the moduli space where each stratum is of the form $\mathcal{H}(k_1, k_2, \cdots, k_n)$, where $(k_1, k_2, \cdots, k_n)$ is a partition of $2g-2$, n is the number of singularities, and k_i captures a measure of the excess angle at the singularity.

For a more detailed introduction to translation surfaces, the reader is encouraged to consult [13], [53], [57].

1.3. **Two Dynamical Systems.**

1.3.1. *Action of the Veech Group.* Consider the $SL_2(\mathbb{R})$-action. The stabilizer of the action at a translation surface (X,ω) is called the *Veech group* of this surface, denoted $SL(X,\omega)$. For an example of a stabilizing element, consider the unit square with opposite sides identified by translation (a torus) and let $g = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

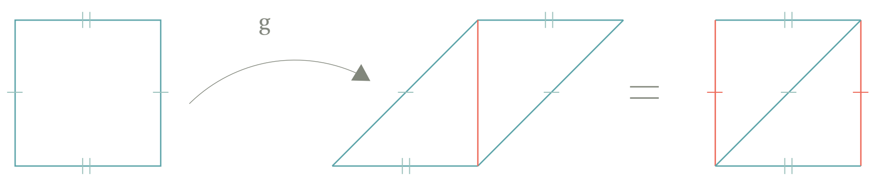

FIGURE 3. Stabilizing element of the Veech group

Using the cut-and-paste procedure pictured in Figure 3, we can reassemble the new polygon as the old, while respecting the “north" direction on the surface. This example shows us that the Veech group is not always trivial. Visually, the action of the matrix appears related to linear maps on the surface, and in fact, this is true. We can identify the Veech group with the collection of derivatives of affine linear maps on the surface [55].

As referenced above, in 1989, Veech discovered a class of translation surfaces that have “large" stabilizers, specifically, stabilizers that are lattices in $SL_2(\mathbb{R})$ [55]. Such lattices are necessarily non-cocompact, finite covolume, discrete subgroups of $SL_2(\mathbb{R})$ [30]. We call these surfaces *lattice surfaces.* Veech groups of lattice surfaces contain a hyperbolic element, which can be represented as a matrix with expanding and contracting eigenspaces. The corresponding linear action of this element, after several applications, sufficiently “mixes" the points on the surface. In fact, the map will be *ergodic* (with respect to the Lebesgue measure on the surface). For example, consider Arnold's cat map shown in Figure 4. One can imagine that after several iterations, the cat in Figure 4 will be quite blurred, illustrating that the points are moving around substantially.

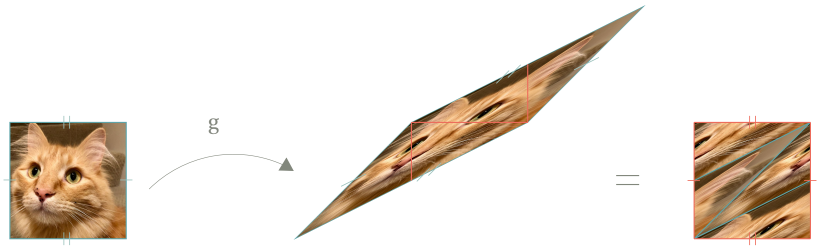

FIGURE 4. Arnold's Cat Map [23]

Since the Veech group of a lattice surface contains a hyperbolic element, the action of the Veech group is *ergodic*. Hence, we can ask questions about the *density* of the orbits. One way to do this is to frame the problem as a shrinking target problem. Fix a lattice surface S with Veech group Γ, and pick any $y \in S$. Let $B_g(y)$ denote the open ball of radius $\phi(||g||)$ (a decreasing function of the operator norm). Does almost every $x \in S$ have the property that $g \cdot x \in B_g(y)$ for infinitely many $g \in \Gamma$? How fast can ϕ decrease (the target shrink) before this no longer holds?

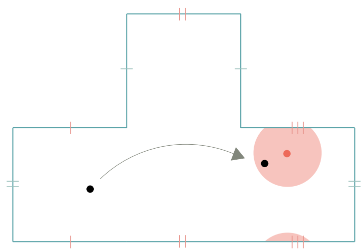

FIGURE 5. Hitting the target

In 2016, Finkelshtein studied a shrinking target problem on the square torus [21]. The torus is an example of a translation surface and $SL_2(\mathbb{Z})$ is its Veech group. In fact, $SL_2(\mathbb{Z})$ is a lattice subgroup of $SL_2(\mathbb{R})$, so the torus is an example of a lattice surface. Finkelshtein showed that the action of $SL_2(\mathbb{Z})$ on the torus exhibits certain Diophantine estimates. Finkelshtein's proof relies on a fundamental connection between the dynamics of the Veech group action and the Laplacian on the torus.

The action of the Veech group on the surface induces a group representation, the *Koopman representation*, $\pi : SL_2(\mathbb{Z}) \to U(L^2(\mathbb{T}^2))$, where $\pi(g)f(x) = f(g^{-1}x)$. Recall that the eigenfunctions of the Laplacian, $\Delta = -(\partial_x^2 + \partial_y^2)$, are solutions to $\Delta f =$

λf. We can compute eigenfunctions: $e^{2\pi i m x} e^{2\pi i n y}$, where $(m,n) \in \mathbb{Z}^2$. Let $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$, then

$$\pi(g) e^{2\pi i m x} e^{2\pi i n y} = e^{2\pi i (dm - cn) x} e^{2\pi i (an - bm) y}.$$

This is significant: the Koopman representation sends eigenspaces of the Laplacian to eigenspaces. In other words, *the action of the Veech group plays nicely with the spectral properties of the Laplacian.* In fact, we can say precisely how the eigenspaces are permuted by noting how (m,n) is permuted: by multiplying on the left by the inverse transpose of g.

We leverage a similar technique to study the action of the Veech group on a *square-tiled surface*, a translation surface that is finitely branched cover of the square torus. This problem is challenging for the following reason: the action of the Veech group on a translation surface does not, in general, respect the eigenspaces of the Laplacian.

We are able to bypass these difficulties by leveraging properties of the branched cover over the torus. Our main result shows that the action of a subgroup of a Veech group acting on a square-tiled surface exhibits similar Diophantine properties that are governed by the *critical exponent* of the subgroup. Recall the definition of critical exponent:

Definition 1.1 (Critical Exponent, δ_Γ)**.** Let Γ be a Fuchsian group. The critical exponent, δ_Γ, is

$$\delta_\Gamma := \limsup_{R\to\infty} \frac{\log(\#\{g \in \Gamma : d_{\mathbb{H}}(g.x_0, x_0) \le R\})}{R},$$

for any x_0, where $g.x_0$ denotes the action of g on x_0 by Möbius transformation. δ_Γ is independent of the basepoint x_0.

The critical exponent δ_Γ is the exponent required for convergence in the Poincaré series of the group Γ [5], [44], which is equivalent to the exponential growth rate of the number of points in the orbit of Γ acting on the upper half-plane [47] seen in Definition 1.1.

Patterson [44] showed that for a finitely generated Fuchsian group Γ, the critical exponent is precisely the Hausdorff dimension of the limit set, $\Lambda = \overline{\Gamma x} \cap S^1$, where S^1 is the circle at infinity. Sullivan [47] showed that in the general case of a Fuchsian group, the critical exponent is the Hausdorff dimension of the *radial* limit set, $\Lambda_r \subset \Lambda$ consisting of all points in the limit set such that there exists a sequence $\lambda_n x \to y$ remaining within a bounded distance of a geodesic ray ending at y.

These various interpretations are particularly relevant to our work since we obtain Theorem 1.1 indirectly through spectral estimates of the boundary representation of the subgroup Γ.

Theorem 1.1. Let (X,ω) be a square-tiled surface, and let Γ be a subgroup of the Veech group $SL(X,\omega)$. For any $y \in X$, for Lebesgue a.e. $x \in X$, the set

$$\{g \in \Gamma : |gx - y| < ||g||^{-\alpha}\}$$

is

(1) finite for every $\alpha > \delta_\Gamma$
(2) infinite for every $\alpha < \delta_\Gamma$

where δ_Γ is the critical exponent of the subgroup Γ, and $||\cdot||$ is the operator norm of g (as a linear transformation on $\mathbb{R}^2$).

In fact, Theorem 1.1 holds for parallelogram-tiled surfaces as well.

1.3.2. *Linear Flow on a Translation Surface.* There is another dynamical system commonly studied on translation surfaces: the linear flow Φ_t on the surface (Figure 6). This is the geodesic flow on the translation surface with the singular points removed. If a trajectory hits a singular point, we stop.

As alluded to above, one motivation for studying such a system is its relationship to billiard trajectories on polygons with angles that are rational multiples of π. A billiard trajectory is a straight trajectory that "bounces" off the edges of the polygon following

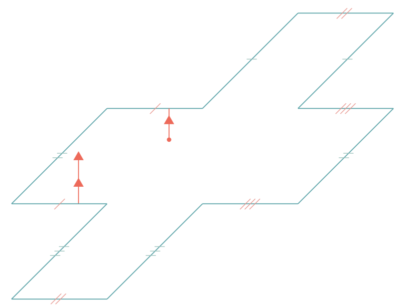

FIGURE 6. Linear flow segment on a translation surface

the rule that the angle of incidence is equal to the angle of reflection. Such polygons can be “unfolded" to “straighten" the billiard path. Since the angles are rational multiples of π, the number of directions a billiard path can follow is finite. After reflecting the polygon finitely many times, we arrive at only a single direction. The unfolded polygon is a translation surface, and the corresponding dynamical system is the linear flow [31], [53], [57].

Recall that a *translation surface* is a pair (X, ω) where X is a compact, connected Riemann surface without boundary and ω a non-zero holomorphic differential on X. If we fix the genus of the underlying Riemann surface, the moduli space Ω_g of pairs (X, ω) forms a vector bundle over $\mathcal{M}_g$, the moduli space of genus g Riemann surfaces, where the fiber over $X \in \mathcal{M}_g$ is the g-complex dimensional vector space $\Omega(X)$ of holomorphic 1-forms on X.

Throughout the remainder of this section, and in Section 4 below, we drop the notation (X, ω) for a translation surface and use only ω.

Recall that the moduli space of translation surfaces is equipped with an $SL_2(\mathbb{R})$-action, where the action is the usual linear action on the plane. Elements of the form $g_t = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}$ for any $t \in \mathbb{R}$ form a one-parameter subgroup which we will refer to as the (Teichmüller) *geodesic flow* (Figure 7).

We identify a quantitative density condition on the vertical flow of a translation surface ω that is equivalent to boundedness of the associated geodesic in the moduli

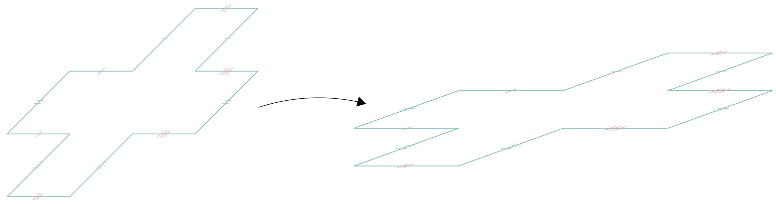

FIGURE 7. Translation surface ω and $g_t\omega$

space. The condition is inspired by papers of Beck and Chen, where they study billiard trajectories on similar objects [6], [7].

Theorem 1.2. Let $\omega \in \Omega_g$ be a translation surface. The linear flow on ω is superdense if and only if the associated Teichmüller geodesic $\{g_t\omega\}_{t>0}$ is bounded.

We prove this result in Section 4 by using the *diameter* of the translation surface to control the quantitative density of the vertical (northward) linear flow.

2. Shrinking Targets

In this section, we will give a technical description of a shrinking target problem and identify the main obstacle that we must overcome to solve one. Throughout this section, $(X, \mathscr{B}, \mu)$ is a probability space, and $T : X \to X$ is a measure-preserving transformation, unless otherwise indicated.

2.1. **Set-up.** First, recall Poincaré recurrence.

Theorem 2.1 (Poincaré recurrence)**.** Let $(X, \mathscr{B}, \mu)$ be a probability space, let $T : X \to X$ be a measure preserving transformation, and let $E \in \mathscr{B}$. Define a semigroup action on X by the group $\mathbb{N}$ as follows: $n \cdot x := T^n(x)$ for any $n \in \mathbb{N}$, where $T^0 = \mathrm{Id}$. Then for almost every point $x \in E$, the set

$$\{n \in \mathbb{N} : n \cdot x \in E\}$$

is infinite. (In other words, the set of points in E that return to E infinitely often has full measure in E.)

In most dynamical settings, and in particular, our setting, the measurable maps that we are studying are more than measure preserving: they are often ergodic, or at the very least, decomposable into ergodic components. In this setting, we can strengthen Poincaré Recurrence.

Recall that ergodicity implies that the time average eventually equals the space average:

Theorem 2.2 (Pointwise Ergodic Theorem)**.** Let $(X, \mathscr{B}, \mu)$ be a probability space and let $T : X \to X$ be ergodic. Then for any $f \in L^1(X, \mu)$, we have that

$$\frac{1}{N} \sum_{n=0}^{N} f(T^n(x)) \longrightarrow \int_X f \, d\mu$$

as $N \to \infty$, where the convergence is pointwise.

If $T : X \to X$ is ergodic, then for any measurable set $E \in \mathscr{B}$ almost every $x \in X$ will land in E infinitely often. In other words, $T^n(x) \in E$ for infinitely many $n \in \mathbb{N}$. And, in fact, we know how often the point returns. As $n \to \infty$, the ratio of x landing in E and x landing outside of E converges to the measure of the set E. To see this, let χ_E be the indicator function of the set E, and apply the pointwise ergodic theorem:

$$\frac{1}{N}\sum_{n=0}^{N}\chi_A(T^n(x)) \longrightarrow \int_X \chi_A d\mu = \mu(A)$$

as $N \to \infty$.

In short, ergodicity tells us that the almost every point in X lands in *any* measurable set infinitely often. However, it does not give us quantitative information about the density of the orbits. If we were interested in such information, we could ask the following: given a measurable set E, how quickly can we shrink the set E (shrink the set for each application of the transformation T) and still have almost every $x \in X$ land in the shrinking sequence of sets infinitely often? More concretely, assume X is a metric space, let $y \in X$, and let $B_{\phi(n)}(y)$ be a ball centered at y with radius $\phi(n)$, where $\phi : \mathbb{Z}_{\geq 0} \to \mathbb{R}_{>0}$ is a decreasing function. How quickly can we decrease the function ϕ so that $T^n(x) \in B_{\phi(n)}(y)$ infinitely often for almost every $x \in X$?

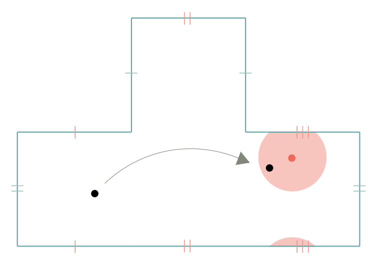

FIGURE 8. Hitting the target

Historically, the key to solving such shrinking target problems has been to use the Borel-Cantelli lemma and its partial converse.

Lemma 2.1 (Borel-Cantelli lemma and partial converse)**.** Let $(X, \mathscr{B}, \mu)$ be a probability space and let E_n be a sequence of measurable sets.

(1) (Borel-Cantelli lemma) If $\sum_n \mu(E_n) < \infty$, then the set of points $x \in X$ such that x occurs infinitely often has measure 0 ($\limsup_{n\to\infty} E_n$ has measure 0).

(2) Conversely, if the E_n are pairwise independent, and $\sum_n \mu(E_n) = \infty$, then the set $x \in X$ such that x occurs infinitely often has full measure ($\limsup_{n\to\infty} E_n$ has full measure).

To see how the lemma helps us solve a shrinking target problem, consider the following. If $T^n(x)$ lands in the target $B_{\phi(n)}(y)$, then $T^{-n}(B_{\phi(n)}(y))$ must contain x. See Figure 9.

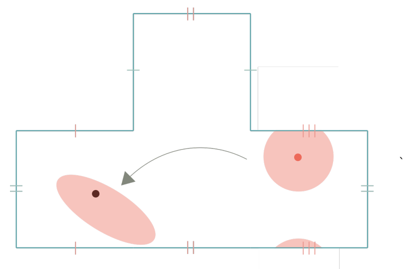

FIGURE 9. Will hit the target (Application of T^{-1} to target)

Now consider the following sum, assuming T is a measure-preserving:

$$\sum_{n=0}^{\infty} \mu(T^{-n}(B_{\phi(n)}(y))) = \sum_{n=0}^{\infty} \mu(B_{\phi(n)}(y)).$$

The first part of Lemma 2.1 tells us that if this sum converges, then the set of $x \in X$ such that $x \in T^{-n}(B_{\phi(n)}(y))$ infinitely often has measure zero. In other words, for almost every $x \in X$, there are at most *finitely many* $n \in \mathbb{N}$ such that $T^n(x) \in B_{\phi(n)}(y)$.

Similarly, if the sum above diverges, and we have that the sets $B_{\phi(n)}(y)$ are pairwise independent, then we can conclude that the set of points $x \in X$ such that $x \in T^{-n}(B_{\phi(n)}(y))$ infinitely often has full measure. In other words, for almost every $x \in X$, $T^n(x) \in B_{\phi(n)}(y)$ for *infinitely many* $n \in \mathbb{N}$.

By observing convergence or divergence of this sum, we can determine how fast $\phi(n)$ can decrease, or rather, how fast we can shrink the target. But, there is a catch. We often cannot say much regarding *pairwise independence* of the sets. In fact, this property is often absent, which is what makes a shrinking problem both interesting and challenging. We must look for a way to replace this hypothesis.

2.2. **A Brief History.** In 1966, Philipp used the Borel-Cantelli lemma in order to prove certain Diophantine estimates. He did this by formulating a quantitative version of the Borel-Cantelli lemma and used it to show that not only does the $2x$-map on the circle exhibit a shrinking target property, but so does the continued fraction map and the θ-adic expansion map [45].

Theorem 2.3 (Quantitative Borel-Cantelli lemma)**.** Let E_n be a sequence of measurable sets in an arbitrary probability space (X,μ). Denote $A(N,x)$ the number of integers $n \leq N$ such that $x \in E_n$. Define

$$\phi(N) = \sum_{n\leq N} \mu(E_n)$$

Suppose that there exists a convergent series $\sum C_k$ with $C_k \geq 0$ such that for all integers $n > m$ we have

$$\mu(E_n \cap E_m) \leq \mu(E_n)\mu(E_m) + \mu(E_n)C_{n-m}.$$

Then

$$A(N,x) = \phi(N) + O(\phi^{\frac{1}{2}}(N)\log^{\frac{3}{2}+\epsilon}(\phi(N)))$$

for any $\epsilon > 0$, for almost every $x \in X$.

Notice how this quantitative version gives an error estimate so that we can understand just how “far" from pairwise independence the sequence of measurable sets we can be.

In 1982, Sullivan used a similar idea to prove a logarithm law that describes the cusp excursions of generic geodesics on noncompact, finite volume hyperbolic spaces [48]. Sullivan constructed a quantitative Borel-Cantelli lemma by replacing the pairwise independence condition with a geometric condition imposed on shrinking sets in the cusps. In 1995, Hill and Velani coined the term "shrinking target" in their fundamental work on the subject [26]. Their work begins with an elegant description of the set-up (which we have expanded to include other formulations of shrinking target questions above), then they study the Hausdorff dimensions of the sets of points that hit a target (a Julia set) infinitely often for certain expanding rational maps on the Riemann sphere. Hill and Velani have also studied an analogous shrinking target problem corresponding to $\mathbb{Z}$-actions of affine linear (not necessarily measure preserving) maps on tori [27].

Kleinbock and Margulis used a Borel-Cantelli argument to prove a far-reaching result that generalizes Sullivan's logarithm law to noncompact, finite volume locally symmetric spaces [35]. They replace the pairwise independence condition with exponential decay of correlations of smooth functions on the space. Athreya and Margulis proved that unipotent flows satisfy an analogous logarithm law, using probabilistic methods, techniques from the geometry of numbers, and the exponential decay of correlations of smooth functions on the space [3], [4].

Following the literature on shrinking targets, we give the following definition.

Definition 2.1 (Borel-Cantelli [1], [20])**.** Let G be a group acting by measure-preserving transformations on a probability space $(X, \mathscr{B}, \mu)$ and let Γ be a subgroup. We say that a sequence of measurable sets $\{E_g\}_{g\in\Gamma}$ is *Borel-Cantelli, (BC),* if $\sum_{g\in\Gamma} \mu(E_g) = \infty$ and

$$\mu(\{x \in X : g x \in E_g \text{ infinitely often}\}) = 1.$$

Theorem 3.3 in Section 3 identifies conditions for our target sets to be BC. As with Finkelshtein's result [21], our result has the added benefit that we can deduce Diophantine properties of thin subgroups of the Veech group.

For the interested reader, Athreya has provided an expository article on the relationship between shrinking targets, logarithm laws, and Diophantine estimates. The article also speaks to how shrinking target properties can manifest in the various types of dynamical systems [1].

3. Application to Square-tiled Surfaces

The goal of this section is to prove Theorem 1.1. We begin by reviewing known properties of the Veech group of square-tiled surfaces, and then introduce a representation of the Veech group of the torus. The main contribution is in Section 3.3: we can use a geometric (covering) argument to yield spectral estimates for the representation of the Veech group of a square-tiled surface that can be used to run a Borel-Cantelli argument. In fact, we will show that the results extend to parallelogram-tiled surfaces.

3.1. Properties of the Veech group.

Let (X,ω) be a lattice surface, and $SL(X,\omega)$ its Veech group. $SL(X,\omega)$ is a non-cocompact lattice subgroup of $SL_2(\mathbb{R})$, which implies the following: the group contains a hyperbolic element, hence the action of the Veech group on (X,ω) is ergodic. In fact, we have more than this: see Section 5 Appendix A. For the interested reader, proofs of these statements can be found or constructed from the following sources: [30], [32].

We will use the following fact, one direction of which was originally proven by Veech [55]. The equivalence was proven by Gutkin and Judge [24], [25].

Recall that we say two subgroups Γ_1 and Γ_2 of $SL_2(\mathbb{R})$ are *commensurate* if $\Gamma_1 \cap \Gamma_2$ has finite index in both Γ_1 and Γ_2. We say that two subgroups Γ_1 and Γ_2 of $SL_2(\mathbb{R})$ are *commensurable* if Γ_1 is commensurate to a conjugate of Γ_2.

Theorem 3.1. The (X,ω) is a square-tiled surface if and only if $SL(X,\omega)$ is commensurate with $SL_2(\mathbb{Z})$. Similarly, (X,ω) is a parallelogram-tiled surface if and only if $SL(X,\omega)$ is commensurable with $SL_2(\mathbb{Z})$.

3.2. Spectral estimates and hyperbolic properties.

In this section, we will focus on properties of the square torus $\mathbb{T}^2$. The square torus the simplest example of a square-tiled surface, and the Veech group of the torus is $SL_2(\mathbb{Z})$.

We begin with a definition. Let $(X,\mathscr{B},\mu)$ be a probability space, let $U(L^2(X,\mu))$ be the space of unitary operators, and let Γ be a group acting by measure preserving transformations. Define the *Koopman representation* $\pi:\Gamma\to U(L^2(X,\mu))$ by $\pi(g)f(x) = f(g^{-1}x)$. Below, we will consider the Koopman representation of subgroups of the

Veech group acting on a square-tiled surface. Since constant functions are invariant, it will make sense to consider $\pi_0 : \Gamma \to U(L^2_0(X,\mu))$, where $L^2_0(X,\mu)$ is closed subspace of functions orthogonal to the constant functions.

We will reduce our considerations to *convex cocompact* subgroups of $SL_2(\mathbb{Z})$. Such groups have well-behaved hyperbolic properties: convex cocompact subgroups of $\mathrm{Isom}(\mathbb{H})$ are examples of *quasi-ruled hyperbolic spaces*. Such spaces admit a *visual boundary*, which is not unlike the boundary of hyperbolic space. We can use the hyperbolic properties of the group to bound the number of elements in subsets of the group. As we might expect, the asymptotic growth is exponential.

Lemma 3.1. [21] Let $\Gamma \subset \mathrm{Isom}(\mathbb{H})$ and fix a basepoint $z_0 \in \mathbb{H}$. Let δ_Γ denote the critical exponent of Γ. Define $S_{n,k} \subset \Gamma$ to be the set of elements such that $k < d_{\mathbb{H}}(gx, x) \leq n$, where gx denotes the action by Möbius transformation. Then, there exists a $k \in \mathbb{N}$ with $k < n$ such that

$$|S_{n,k}| = e^{\delta_\Gamma n + O(1)}.$$

Note that any convex cocompact subgroup of $SL_2(\mathbb{Z})$ has this property.

Definition 3.1. Let $\Gamma \subset SL_2(\mathbb{Z})$ be a convex cocompact subgroup and define the *shell* S_n to be the shell such that the asymptotics in Lemma 3.1 hold.

Lemma 3.2. With S_n as above, and letting $||\cdot||$ denote the operator norm of g thought of as a linear transformation on $\mathbb{R}^2$, we have

$$\max\{||g|| : g \in S_{2n}\} \leq e^n.$$

Proof. Fix an isometry between hyperbolic space and the Teichmüller space of tori [19]. The hyperbolic distance can be related to the dilatation of g (thought of as a linear map on $\mathbb{R}^2$). For elements in a convex cocompact subgroup, the dilatation is the square of the operator norm. ∎

Given a representation of a convex cocompact subgroup of $SL_2(\mathbb{Z})$, we have an induced representation on the visual boundary of the group. This action is not necessarily measure-preserving, but it does preserve a measure class (it is a *quasi-regular representation*). See, for example, §3 in [21] and Appendix A in [10]. For our purposes, defining the Koopman representation will be sufficient, but note that the proof of Theorem 3.2 uses an induced boundary representation.

Theorem 3.2. [21] Let $\mathbb{T}^2$ be a square torus, let $\Gamma \subset SL_2(\mathbb{Z})$ be a convex cocompact subgroup with critical exponent δ_Γ, and let π_0 denote the Koopman representation on $L^2_0(\mathbb{T}^2)$. Let μ_n be a uniform probability measure on $S_n \subset \Gamma$. Then

$$||\pi_0(\mu_n)|| \leq e^{-\frac{1}{2}\delta_\Gamma n + 2\log n + O(1)},$$

where $\pi_0(\mu_n) = \sum_{g \in \operatorname{supp} \mu_n} \mu_n(g)\pi_0(g)$.

This result was proven by Finkelshtein to solve a similar shrinking target problem on a torus. In our set-up, the spectral estimates of convex cocompact subgroups of $SL_2(\mathbb{Z})$ play the role of the pairwise independence assumption in the Borel-Cantelli lemma.

3.3. **A covering argument.**

Lemma 3.3. Let (X,ω) be a square-tiled surface and let $SL(X,\omega)$ be its Veech group. Then, there exist a finite index subgroup $\Gamma' \subset SL(X,\omega)$ and a branched cover $q : X \to \mathbb{T}^2$ such that the cover is equivariant with respect to the action of any subgroup Γ'.

This follows from Theorem 3.1, and the work in [25]. A similar statement holds for parallelogram-tiled surfaces, but we need to take compose a cover to a (non-square) torus with an affine map to a square torus.

Although irrelevant to our current goals, it is worth noting that we know precisely which square-tiled surfaces (and parallelogram-tiled surfaces) for which the full Veech group descends to an action on the square torus (or to an action conjugate to an action on the square torus). Recall that the *saddle connections* of a translation surface (Y,ν)

are straight line trajectories that start at a singular point and end at a singular point, passing through no singular points in-between. The *holonomy vectors* are the values we get when we integrate the saddle connections over the holomorphic one-form ν. The *period lattice* is the the lattice generated by the holonomy vectors.

For a square-tiled surface (X,ω) (tiled by unit squares), the holonomy vectors must be a subset of $\mathbb{Z}\oplus\mathbb{Z}[i]$. We say a square-tiled surface is *reduced* if the period lattice is $\mathbb{Z}\oplus\mathbb{Z}[i]$. Since the action of the Veech group preserves the period lattice, we see that the Veech group must be a subgroup of $SL_2(\mathbb{Z})$, and further, that for any $g\in SL(X,\omega)$, the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{g} & X \\ \downarrow q & & \downarrow q \\ \mathbb{T}^2 & \xrightarrow{g} & \mathbb{T}^2 \end{array}$$

where $q(x)=\int_p^x \omega \mod \mathbb{Z}\oplus\mathbb{Z}[i]$. (Note that this map respects the choice of north on the square-tiled surface.)

If the period lattice is not $\mathbb{Z}\oplus\mathbb{Z}[i]$, then there will be a parabolic element in the Veech group with non-integral entries. Such an element cannot descend with respect to the cover.

There is a similar picture for parallelogram-tiled surfaces, where the tiling parallelogram has sides $a,b\in\mathbb{C}$ and unit area. Let P denote the translation surface given by identifying opposite sides of the parallelogram. Then, we say that (X,ω) is a *reduced* parallelogram-tiled surface if the period lattice is $\mathbb{Z}[a]\oplus\mathbb{Z}[b]$. For a reduced parallelogram-tiled surface, the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{g} & X \\ \downarrow q & & \downarrow q \\ P & \xrightarrow{g} & P \\ \downarrow h & & \downarrow h \\ \mathbb{T}^2 & & \mathbb{T}^2 \end{array}$$

where $q(x) = \int_p^x \omega \mod \mathbb{Z}[a] \oplus \mathbb{Z}[b]$, $h \in SL_2(\mathbb{R})$, and as above, the choice of north on the translation surfaces is respected with the exception of the action of h.

We now turn our attention to functions on the space. The measure on each space is the usual Lebesgue measure, but we scale it so that it is a probability measure.

Lemma 3.4. Let (X, ω) be a square-tiled surface that can be partitioned into m squares $\{S_1, S_2, \cdots, S_m\}$, where each partition contains the interior of a square along with two sides. Then let $H = \{f \in L^2(X) : f = (h, h, \cdots, h)\}$

$$L^2(X, \nu) \cong H \oplus H^\perp$$

where the measure is the Lebesgue measure on the space.

Lemma 3.5. Let $\iota : L^2(\mathbb{T}^2) \to H$ be defined by $\iota(f) = (f, f, \cdots, f) \in H$ and note that $\operatorname{Im}(\iota) = H$. Then there exists a finite index subgroup $\Gamma' \subset SL(X, \omega)$ such that for every $g \in \Gamma'$, the following diagram commutes.

$$\begin{array}{ccc} H & \xrightarrow{\pi_H(g)} & H \\ \iota\uparrow & & \iota\uparrow \\ \mathrm{L}^2(\mathbb{T}^2) & \xrightarrow{\pi(g)} & \mathrm{L}^2(\mathbb{T}^2) \end{array}$$

where $\pi_H : \Gamma' \to U(H)$ is the Koopman representation of $SL(X, \omega)$ on H, and $\pi : \Gamma' \to U(L^2(\mathbb{T}^2))$ is the Koopman representation of Γ' on $L^2(\mathbb{T}^2)$.

Proof. This follows from the equivariance of the covering map, Lemma 3.3. In fact, the group representation π_H is well-defined because of Lemma 3.3. ∎

Corollary 3.1. With hypothesis as in Lemma 3.5, for every $g \in \Gamma'$,

$$||\pi_H(g)|| = ||\pi(g)||.$$

As a consequence, and by applying Theorem 3.2, we have the following.

Corollary 3.2. Let $H_0 \subset H$ such that $H_0 = \iota(L^2_0(\mathbb{T}^2))$, the subspace of H orthogonal to the constant functions. Let $\pi_{H_0} : \Gamma' \to U(H_0)$, which is well-defined since the space of constant functions is invariant under the representation π_H defined above. Let μ_n be the measure from Theorem 3.2. Then

$$||\pi_{H_0}(\mu_n)|| = ||\pi_0(\mu_n)|| \leq e^{-\frac{1}{2}\delta_\Gamma n + 2\log n + O(1)}.$$

The technique above provides a framework for lifting spectral estimates using a cover, so estimates on “primitive" translation surfaces, those that do not cover (with finite branching) other translation surfaces, can be lifted to surfaces covered by the primitive surface.

3.4. **Borel-Cantelli argument.** In this section, we show how to use the spectral estimates to run a Borel-Cantelli argument similar to [21], but with variations to accommodate a reduction to a finite index subgroup and the tiling of the surface.

Theorem 3.3. Let (X,ω) be a square-tiled (or parallelogram-tiled) surface with the Lebesgue measure μ and let $\Gamma \subset SL(X,\omega)$ be a convex cocompact subgroup with critical exponent δ_Γ. Let $\mathrm{Targ}_{\phi(||g||)}$ be a monotonic family of Lebesgue subsets of measure $\pi\phi(||g||)^2$, where $\phi : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$. Then for any $y \in X$, and almost every $x \in X$, the set

$$\{g \in \Gamma : gx \in \mathrm{Targ}_{\phi(||g||)}\}$$

is

(1) finite, if $\sum_{n=1}^{\infty} n^{2(\delta_\Gamma - 1)}\phi(n)^2 < \infty$.
(2) infinite, if $\sum_{n=1}^{\infty} (\log n)^4 n^{-(2\delta_\Gamma+1)}\phi(n)^{-2} < \infty$.

Proof. First, we show that the sum converges under the first condition, which by Borel-Cantelli (Lemma 2.1) implies that the set is finite. First, recall Lemma 3.2. For $g \in S_{2n}$, we have that $||g|| \leq e^n$.

$$\sum_{g\in\Gamma}\mu(g^{-1}\mathrm{Targ}_{\phi(||g||)}) = \sum_{n=1}^{\infty}\sum_{g\in S_{2n}}\pi\phi(||g||)^2$$

We focus on the tail of the sequence. For sufficiently large k, we have

$$\begin{aligned}\sum_{n=k}^{\infty}\sum_{g\in S_{2n}}\pi\phi(||g||)^2 &\le O(1)\sum_{n=k}^{\infty}e^{(2+\varepsilon)\delta_\Gamma n}\phi(||g||)^2,\text{ by Lemma 3.1.}\\ &\le O(1)\sum_{n=k}^{\infty}e^{(2+\varepsilon)\delta_\Gamma n}\phi(e^n)^2\end{aligned}$$

the tail converges if and only if $\sum_{n=k}^{\infty}n^{(2+\varepsilon)\delta_\Gamma-1}\phi(n)^2$ converges. Thus, by Borel-Cantelli, we can conclude that for almost every $x\in X$ the set $\{g\in\Gamma : gx\in\mathrm{Targ}_{\phi(||g||)}\}$ is finite.

The more difficult part of the proof, we show the set $\{g\in\Gamma : gx\in\mathrm{Targ}_{\phi(||g||)}\}$ is infinite when ϕ satisfies the condition in (2).

First, let

$$E_n = X\setminus\bigcup_{\{g\in\Gamma:||g||<e^n\}}g^{-1}\mathrm{Targ}_{\phi(||g||)})$$

and let $E=\limsup_{n\to\infty}E_N$. If we show that $\mu(E)=0$, then we will have shown that the set of $x\in X$ that hit the target only finitely many times has measure 0, hence the set of $x\in X$ hitting the target infinitely often has full measure, and the result follows.

We begin by reducing to a finite index subgroup $\Gamma'\subset\Gamma$ so that we can apply the results of Theorem 3.2. For finite index subgroups of convex cocompact subgroups we have that $\delta_{\Gamma'}=\delta_\Gamma$. Moreover, the finite index subgroup is convex cocompact (being finitely generated without parabolic elements).

Let $\chi_{\mathrm{Targ_n}}$ be the characteristic function of the target set $\mathrm{Targ}_{\phi(e^n)}$ and let χ_{E_n} be the characteristic of the set E_n. Let M be the number of squares tiling (X,ω) and let

$q : X \to \mathbb{T}^2$ be the covering map. Project the characteristic functions to $H_0 \subset L^2(X)$. To do this, we will define two bounded linear operators. First, define $A : L^2(X) \to H$ by

$$A(f) = A(f_1, f_2, \cdots, f_M) = (\tilde{f}, \tilde{f}, \cdots, \tilde{f})$$

where $\tilde{f}(x) = \frac{1}{M} \sum_{y \in q^{-1}(q(x))} f(y)$. Note that $\int_X A(f)\, d\mu = \int_X f\, d\mu$.
Now define the projection $P : L^2(X) \to H_0$ by

$$P(f) = A(f) - \int_X f\, d\mu.$$

P is a self-adjoint, idempotent operator: First, $\langle Pf, g\rangle = \langle f, Pg\rangle$:

$$\begin{aligned}
\langle Pf, g\rangle - \langle f, Pg\rangle &= \int_X \left(A(f) - \int_X f\, d\mu\right) g\, d\mu - \int_X \left(A(g) - \int_X g\, d\mu\right) f\, d\mu \\
&= \int_X A(f) g\, d\mu - \int_X f\, d\mu \int_X g\, d\mu - \int_X A(g) f\, d\mu + \int_X f\, d\mu \int_X g\, d\mu \\
&= \int_X A(f) g\, d\mu - \int_X A(g) f\, d\mu \\
&= 0,
\end{aligned}$$

where the last line follows from considering the integral over each square.
Second, $P^2(f) = P(f)$:

$$\begin{aligned}
P^2(f) &= P(A(f) - \int_X f\, d\mu) \\
&= P(A(f) - \int_X A(f)\, d\mu) \\
&= P(A(f)) - P(\int_X A(f)\, d\mu) \\
&= A(A(f)) - \int_X A(f)\, d\mu - 0 \\
&= A(f) - \int_X f\, d\mu \\
&= P(f).
\end{aligned}$$

We can project the characteristic sets:

$$\begin{aligned}
T_n &:= P(\chi_{Targ_n}) = A(\chi_{\mathrm{Targ}_n}) - \mu(\mathrm{Targ}_n) \\
B_n &:= P(\chi_{E_n}) = A(\chi_{E_n}) - \mu(E_n).
\end{aligned}$$

Now, observe the following

$$\begin{aligned}
||T_n||_2^2 &\leq (1-\mu(\mathrm{Targ}_n))\mu(\mathrm{Targ}_n) \leq \mu(\mathrm{Targ}_n) \\
||B_n||_2^2 &\leq (1-\mu(E_n))\mu(E_n) \leq \mu(E_n)
\end{aligned}$$

Moreover,

$$< T_n, B_n >= \mu(\mathrm{Targ}_n)\mu(E_n) - \mu(\mathrm{Targ}_n \cap E_n) = \mu(\mathrm{Targ}_n)\mu(E_n)$$

and for any $g \in S_{2n} \subset \Gamma'$ (where $||g|| < e^n$ by Lemma 3.2), we have

$$\langle \pi_{H_0}(g) T_n, B_n \rangle = \mu(\mathrm{Targ}_n)\mu(E_n),$$

hence,

$$\langle \pi_{H_0}(\mu_{2n}) T_n, B_n \rangle = \mu(\mathrm{Targ}_n)\mu(E_n).$$

By Cauchy-Schwarz, we can relate the measures to the operator norm of $||\pi_{H_0}(\mu_{2n})||$.

$$\langle \pi_{H_0}(\mu_{2n}) T_n, B_n \rangle \leq ||\pi_{H_0}(\mu_{2n})|| \mu(\mathrm{Targ}_n)^{\frac{1}{2}} \mu(E_n)^{\frac{1}{2}}$$

By applying the spectral estimate in Theorem 3.2 in combination with the previous two equations, we can deduce

$$\begin{aligned} \mu(E_n) &\leq ||\pi_{H_0}(\mu_{2n})|| \mu(\mathrm{Targ}_n)^{-1} \\ &= O(1) n^4 e^{-2\delta_{\Gamma'} n} \phi(e^n)^{-2}. \end{aligned}$$

And now we can deduce that in order for the following sum to converge,

$$\begin{aligned} \sum_{n=1}^{\infty} \mu(E_n) &\leq O(1) \sum_{n=1}^{\infty} n^4 e^{-2\delta_{\Gamma'}} \phi(e^n)^{-2} \\ &\leq O(1) \sum_{n=1}^{\infty} n^4 e^{-2\delta_{\Gamma}} \phi(e^n)^{-2} \end{aligned}$$

we need

$$\sum_{n=1}^{\infty} (\log n)^4 n^{-(2\delta_\Gamma + 1)} \phi(n)^{-2}$$

to converge, as desired.

∎

Theorem 1.1 essentially follows from Theorem 3.3. Setting $\phi(||g||) = ||g||^{-\alpha}$, we see that Theorem 1.1 holds for convex cocompact subgroups of the Veech group of square-tiled surfaces. To extend the result to all groups we employ the following lemma.

Lemma 3.6. [21] Let $\Gamma \subset SL_2(\mathbb{Z})$. For any $\varepsilon > 0$, there exists a convex cocompact subgroup $\Gamma' \subset \Gamma$ such that $\delta_{\Gamma'} \geq \delta_\Gamma - \varepsilon$.

Theorem 1.1. Let (X,ω) be a square-tiled surface, and let Γ be a subgroup of the Veech group $SL(X,\omega)$. For any $y \in X$, for Lebesgue a.e. $x \in X$, the set

$$\{g \in \Gamma : |gx - y| < ||g||^{-\alpha}\}$$

is

(1) finite for every $\alpha > \delta_\Gamma$
(2) infinite for every $\alpha < \delta_\Gamma$

where δ_Γ is the critical exponent of the subgroup Γ, and $||\cdot||$ is the operator norm of g (as a linear transformation on $\mathbb{R}^2$).

Proof. Apply Lemma 3.6 to find a convex cocompact subgroup with critical exponent within ϵ of the critical exponent of Γ. Run the same argument given in the proof of Theorem 3.3. ∎

4. Superdensity and Bounded Geodesics

There is a long history of interactions between the linear flow on individual translation surfaces and a dynamical system on the moduli space of translation surfaces, in particular, between the linear flow on a translation surface and geodesic flow on the moduli space. Masur proved what is now known as Masur's Criterion by building on of earlier work with Kerckhoff and Smillie [33], [40], [41]. Masur used it as a tool to give an upper bound on the Hausdorff dimension of quadratic differentials whose vertical linear flow is not uniquely ergodic.

Theorem (Masur's Criterion)**.** Let g_t denote the geodesic flow on the moduli space of translation surfaces and let ω be a translation surface. If $g_t \cdot \omega$ is non-divergent, that is, it returns to a compact set infinitely often, then the vertical straight line flow is uniquely ergodic.

We identify a quantitative density condition on the vertical flow of a translation surface ω that is equivalent to boundedness of the associated geodesic in moduli space. The condition is inspired by papers of Beck and Chen, where they study billiard trajectories on similar objects [6], [7].

Definition 4.1 (Superdensity)**.** Let ω be a translation surface. We say the linear flow Φ_t is *superdense* if there exists a constant $c > 0$ such that for every $T > 0$, the segment of the flow Φ_t for $t \in [0, cT]$ is within $\frac{1}{T}$ of every point on ω.

Beck and Chen show that a linear flow on a square-tiled surface is superdense if and only if the slope in the associated direction is a badly-approximable number.

We give the following generalization.

Theorem 1.2. Let $\omega \in \Omega_g$ be a translation surface. The linear flow on ω is superdense if and only if the associated Teichmüller geodesic $\{g_t \omega\}_{t>0}$ is bounded.

We prove this result by using the *diameter* of the translation surface to control the quantitative density of the vertical (northward) linear flow.

As a corollary, we have the following:

Corollary 4.1. If the linear flow on ω is superdense, it is uniquely ergodic. However, uniquely ergodic flows need not be superdense.

4.0.1. *Related results.* There have been a number of results that help explain the phenomenon described in Masur's criterion. For instance, Cheung and Masur constructed a half-translation surface (where we allow side identifications by translation and rotation by π) whose vertical flow is uniquely ergodic and the corresponding geodesic in the moduli space of Riemann surfaces diverges to infinity [15]. Not long after, Cheung and Eskin showed that if the geodesic diverges to infinity slowly enough, then the vertical linear flow is guaranteed to be uniquely ergodic [14].

Similar to our result, Chaika and Treviño found a closely related condition derived from the *flat geometry* of the surface that implies unique ergodicity of the vertical linear flow on a translation surface [12], [51]. Let $\delta(g_t\omega)$ be the systole on $g_t\omega$, by which we mean the shortest length of a non-contractible set of saddle connections. If $\int_0^\infty \delta^2(g_t\omega)\,dt$ diverges, then the vertical linear flow is uniquely ergodic. In short, the length of the shortest contractible set of saddle connections cannot get too short too quickly. The geodesic must stay sufficiently far from the boundary of the moduli space (in some compact set) for a sufficient amount of time.

Our result differs in that it identifies a condition on the forward-time geodesic in moduli space that is equivalent to a quantitative density condition on the linear flow. This is akin to results in homogeneous dynamics that seek to quantify the density of orbits. For example, the quantitative version of the Oppenheim conjecture seeks to give explicit quantitative information about the density of the orbits of unipotent flows [17], [18], [38]. Recently Lindenstrauss, Margulis, Mohammadi, and Shah gave effective bounds on time that the unipotent flow can spend near homogeneous subvarieties of an arithmetic quotient G/Γ [37]. This will become a tool for proving quantitative density statements about unipotent flows in this setting.

Various forms of quantitative density results have been explored for translation and half-translation surfaces. For instance, Forni showed that for almost all half-translation surfaces, the deviation from the ergodic average of the flow on the surface is governed by the Lyapunov exponents of the Kontsevich-Zorich cocycle [22]. The results tell us that we can use compact sets (rather, the amount of time the $g_t\omega$ spends in compact sets) to control the rate of convergence to the ergodic average of the linear flow on ω. Athreya and Forni used this idea to give the analogous result for the measure zero set of rational billiards [2]. In the case of a superdense flow, Theorem 4.4 in [2] gives us control over the second Lyapunov exponent and, in particular, shows that a superdense flow admits faster convergence to the ergodic average than any linear flow that is not superdense.

4.1. **Proof of theorem.**

Lemma 4.1. Let $d(\omega)$ be the diameter of the translation surface ω. Then d is continuous. Furthermore, if d is bounded on a set $K \subset \mathcal{H}(\kappa)$, then K is compact.

Proof. Let ω_n be a sequence of translation surfaces whose diameter is bounded by D. Following Masur and Smillie, we take a Delauney triangulation of each surface. Then the edge lengths in the triangulation are bounded by twice the diameter of the surface [41], hence $2D$. Thus, since each edge length is bounded, we can construct a limiting surface ω_∞. ■

Lemma 4.2. Let ω denote a translation surface and g_t the geodesic flow. If there exists a compact set K such that $g_t\omega \in K$ for all $t > 0$, then the vertical (north or south) linear flow on ω is superdense.

Proof. Let ω denote a translation surface such that $g_t\omega$ is a subset of a compact set K for all $t \in \mathbb{R}$. Then the diameter has a maximum on K. Call it D. We will use this maximum to find a constant c such that a vertical linear flow segment of length cT is within $\frac{1}{T}$ of every point on ω.

For any $T > 0$ and any $\varepsilon > 0$, let $N = DT$ and let Φ_s denote the vertical linear flow on ω. Consider the vertical straight line segment starting at any point on the surface $L_{\varepsilon N} := \{\Phi_s : s \in [0, \varepsilon N]\}$. Note that the length of $L_{\varepsilon N}$ is εN.

Apply $g_{\log(N)}$ to ω. See Figure 10.

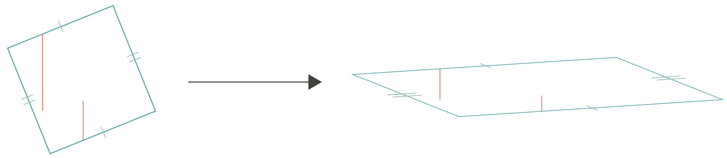

FIGURE 10. Apply $g_{\log(N)}$

Let $g_{\log(N)}L_{\varepsilon N}$ denote the image of $L_{\varepsilon N}$ under the action. Then $g_{\log(N)}L_{\varepsilon N}$ has length ε. Since $g_{\log(N)}\omega \in K$, the diameter of $g_{\log(N)}\omega$ is bounded by D. Let $U = \{x \in g_{\log(N)}\omega : \operatorname{dist}(x, g_{\log(N)}L_{\varepsilon N}) < D\}$ and notice that U covers $g_{\log(N)}\omega$. Apply $g_{-\log(N)}$ to U, and we have a set that covers ω. See Figure 11.

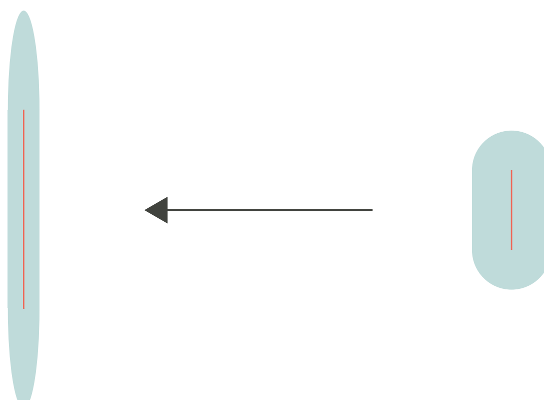

FIGURE 11. Apply $g_{-\log(N)}$ to U

Extend the vertical geodesic segment $L_{\varepsilon N}$ by DN in the positive and negative directions and reparametrize by $\tilde{s} = s + DN$. Let L_D be the reparametrized curve, so that $L_D = \{\Phi_{\tilde{s}} : \tilde{s} \in [0, \varepsilon N + 2DN]\}$. Notice that $[0, \varepsilon N + 2DN] = [0, (\varepsilon D + 2D^2)T]$.

Now, observe that for any $x \in \omega$, we have that $\operatorname{dist}(x, L_D) < \frac{D}{N} = \frac{1}{T}$. For any $T > 0$, the vertical segment $[0, (\varepsilon D + 2D^2)T]$ is within $\frac{1}{T}$ of every point on the surface. Since ϵ is arbitrary, we see that we can pick our constant c to be $2D^2$. The vertical linear flow is superdense as desired. ■

Lemma 4.3. If the vertical (north or south) linear flow on ω is superdense, then there exists a compact set K such that $g_t\omega \in K$ for all $t > 0$.

Proof. Let ω be such that the vertical linear flow Φ_s is superdense. Then, for any initial point on the surface, there exists a constant c such that for any $T > 0$, the vertical (north or south) segment $L_{cT} := \{\Phi_s : s \in [0, cT]\}$ of length cT is within $\frac{1}{T}$ of every point on ω. Let $U = \{x \in \omega : \text{dist}(x, L_{cT}) < \frac{1}{T}\}$ and note that U covers ω. We can think of U as a wrapper on ω that gives us a handle on certain lengths.

Let $\tilde{t} \in [1, \infty)$ and apply $g_{\log(\tilde{t})}$ to U. Notice that for every $\tilde{t}$, we have a cover of $g_{\log(\tilde{t})}\omega$. The diameter of $g_{\log(\tilde{t})}\omega$ is bounded by either $D = \sqrt{\left(\frac{cT}{\tilde{t}}\right)^2 + \left(\frac{2\tilde{t}}{T}\right)^2}$ or $D' = \frac{cT}{\tilde{t}} + \frac{2}{T\tilde{t}}$. See Figure 12.

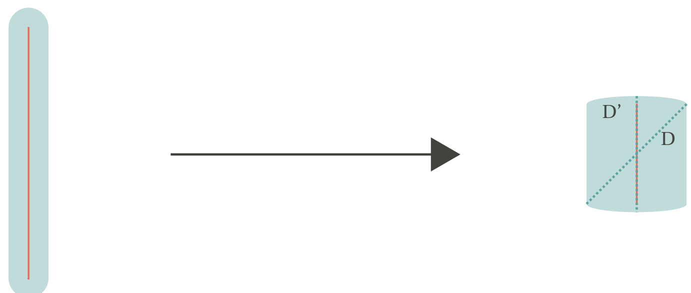

FIGURE 12. Apply $g_{\log(\tilde{t})}$ to U

If $D \geq D'$, we can pick $T = \frac{\sqrt{2}\tilde{t}}{\sqrt{c}}$ and note that the diameter is bounded by $4c$. If $D' > D$, we can find a $T > 0$ such that the diameter is bounded by $c + 2$.

Now, let $t = \log \tilde{t}$, and our argument shows that for any $t > 0$, the diameter is bounded by $\max\{4c, c + 2\}$. The forward time trajectory is contained in a compact set. ■

Theorem 1.2 follows immediately from Lemma 4.2 and Lemma 4.3.

5. APPENDIX A: SOME ERGODIC THEORY

Definition 5.1 (Ergodic)**.** Let $(X, \mathscr{B}, \mu)$ be a probability space. A measure preserving transformation T of $(X, \mathscr{B}, \mu)$ is called *ergodic* if all measurable sets $B \in \mathscr{B}$ with the property that $T^{-1}(B) = B$ satisfy $\mu(B) = 0$ or $\mu(B) = 1$.

Proposition 5.1 (Equivalent Characterization of Ergodicity)**.** Let $(X, \mathscr{B}, \mu)$ be a probability space and let $T : X \to X$ be a measure preserving transformation. Then T is ergodic if and only if whenever $f : X \to \mathbb{C}$ is a measurable function such that $f \circ T = f$ almost everywhere, then f is equal to a constant almost everywhere.

Proof. First assume T is ergodic. Now let f be a measurable function such that $f \circ T = f$ almost everywhere. We will assume f is real-valued, then apply the following results to the real and complex parts of the function seperately.

For any $k \in \mathbb{Z}$ and $n \in \mathbb{N}$, define sets $E_{k,n}$ as follows.

$$E_{k,n} := f^{-1}\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)\right)$$

Since $E_{k,n}$ is measurable, and $f \circ T(E_{k,n}) = f(E_{k,n})$ for almost every $x \in E_{k,n}$, so up to a set of measure 0, $T^{-1}(E_{k,n}) = E_{k,n}$. Let $E_{k,n}$ be the full measure set, and by our assumption of ergodicity, $\mu(E_{k,n}) = 0$ or $\mu(E_{k,n}) = 1$.

For each n notice that $\bigsqcup_{k \in \mathbb{Z}} E_{k,n} = X$, where the union is of disjoint sets. Then, there must be exactly one set $E_{k,n}$ such that $\mu(E_{k_n,n}) = 1$. Call this set $E_{k_n,n}$. Let $Y = \cap_{n=1}^{\infty} E_{k_n,n}$, and we have that $\mu(Y) = 1$ by continuity from above. Since the intersection is non-empty, containing a single point, we see that f must be constant almost everywhere.

We now show the converse. Assume that whenever $f \circ T = f$ almost everywhere, then f is equal to a constant almost everywhere. Let $B \in \mathscr{B}$ such that $T^{-1}(B) = B$. Recall that χ_B, the characteristic function on the set B, is a measurable function and notice that $T^{-1}(B^C) = B^C$. Then $\chi_B \circ T(B) = \chi_B \circ T(T^{-1}B) = \chi_B(B)$, and similarly, $\chi_B \circ T(B^C) = \chi_B \circ T(T^{-1}B^C) = \chi_B(B^C)$. Thus, $\chi_B \circ T = \chi_B$ for all $x \in B \cup B^C = X$.

Hence χ_B is constant almost everywhere and we can conclude $\chi_B(x) = 0$ or $\chi_B = 1$ almost everywhere. Thus, either B has full measure or B is a null set. ∎

Definition 5.2 (Ergodic Group Action)**.** Let G be a group acting on $(X, \mathscr{B}, \mu)$ a probability space, where each induced transformation is measurable. We say the group acts ergodically if for any G-invariant measurable set B, either $\mu(B) = 0$ or $\mu(B) = 1$.

Remark 5.1. For a set to be G-invariant, it must be invariant under the action of all elements $g \in G$. If a single element $g \in G$ acts ergodically, then any G-invariant set must be invariant under the action of g, and we can conclude the measure of such a set is 0 or 1.

Definition 5.3 (Strong and Weak Mixing)**.** Let $(X, \mathscr{B}, \mu)$ be a probability space.

a) A measure preserving transformation T of $(X, \mathscr{B}, \mu)$ is called *strong mixing* if for all measurable sets $A, B \in \mathscr{B}$, $\mu(T^{-n}A \cap B) \to \mu(A)\mu(B)$ as $n \to \infty$.
b) A measure preserving transformation T of $(X, \mathscr{B}, \mu)$ is called *weak mixing* if for all measurable sets $A, B \in \mathscr{B}$, $\frac{1}{N}\sum_{n=1}^{N-1} |\mu(T^{-n}A \cap B) - \mu(A)\mu(B)| \to 0$ as $N \to \infty$.

Remark 5.2. Strong-mixing is sometimes called *mixing*.

Remark 5.3. To help contextualize the mixing definitions above, recall the mean ergodic theorem, which tells us that $\frac{1}{N}\sum_{n=0}^{N-1} f \circ T^n \to \int_X f\, d\mu$ in L^2. Using properties of Hilbert spaces we can deduce $\frac{1}{N}\sum_{n=0}^{N-1}\langle f \circ T^n, g\rangle \to \int_X f\, d\mu \int_X g\, d\mu$ in L^2. Since characteristics are in $L^2(X, \mu)$, we can show that ergodicity implies that for all measurable sets $A, B \in \mathscr{B}$, $\frac{1}{N}\sum_{n=1}^{N-1} \mu(A \cap T^{-n}B) \to \mu(A)\mu(B)$ as $N \to \infty$. In fact, this is another equivalent characterization of ergodicity.

Proposition 5.2 (Equivalent Characterization of Strong Mixing)**.** Let $(X, \mathscr{B}, \mu)$ be a probability space and let $T : X \to X$ be a measure preserving transformation. Then T is strongly-mixing if and only if for all $f, g \in L^2(X, \mu)$, $\langle f \circ T^n, g\rangle \to \int_X f\, d\mu \int_X g\, d\mu$ as $n \to \infty$.

Proof. First assume T is strongly mixing. Let $\chi_A, \chi_B \in L^2(X, \mu)$ be characteristic functions on the measurable sets A and B. Then

$$\begin{aligned}
\langle \chi_A \circ T^n, \chi_B \rangle &= \int_X \chi_A \circ T^n(x)\, \chi_B(x)\, d\mu \\
&= \int_X \chi_{T^{-n}A}(x) \chi_B(x) d\mu \\
&= \int_X \chi_{T^{-n}A\cap B}(x) d\mu \\
&= \int_{T^{-n}A\cap B} d\mu \\
&= \mu(T^{-n}A\cap B) \\
&\to \mu(A)\mu(B), \text{ as } n\to\infty, \text{ by assumption} \\
&= \int_X \chi_A d\mu \int_X \chi_B d\mu.
\end{aligned}$$

The result follows for simple functions by the same argument. Then, by leveraging the density of simple function in $L^2(X,\mu)$, we attain desired result for all functions $f,g\in L^2(X,\mu)$.

For the converse, assume for all $f,g \in L^2(X,\mu)$, $\langle f\circ T^n, g\rangle \to \int_X f\,d\mu \int_X g\,d\mu$ as $n\to\infty$. Let A and B be any two measurable sets, and let $f=\chi_A$ and $g=\chi_B$. Then, reversing the argument above, we have

$$\begin{aligned}
\mu(T^{-n}A\cap B) &= \int_X \chi_{T^{-n}A\cap B}(x)\,d\mu \\
&= \int_X \chi_{T^{-n}A}(x)\chi_B(x)\,d\mu \\
&= \int_X \chi_A\circ T^n(x)\,\chi_B(x)\,d\mu \\
&= \langle \chi_A\circ T^n, \chi_B\rangle \\
&\to \int_X \chi_A\,d\mu \int_X \chi_B\,d\mu, \text{ as } n\to\infty, \text{ by assumption} \\
&= \mu(A)\mu(B).
\end{aligned}$$

■

With this characterization of mixing, we use the fact that the Veech group contains a hyperbolic element to prove the following statement. This proof technique was first seen by the author in [16].

Theorem 5.1. There is an element in the Veech group of a lattice surface whose induced transformation is strongly mixing.

Proof. Let (X,ω) be a lattice surface and let $SL(X,\omega)$ be its Veech group. Let μ be the Lebesque measure measure on X, and normalize so that $\mu(X)=1$. There exists a hyperbolic element $g\in SL(X,\omega)$. g has two distinct real eigenvalues, neither of which is of absolute value 1. Thus, in thinking about the action of g on $\mathbb{R}^2$, we have an eigenbasis with a stable and unstable foliation of the space. Locally, this action is the same.

Let $f\in L^2(X)$ and let h be a weak accumulation point of $f\circ T_g^n$ where T_g is the map induced by the action of the hyperbolic element. Pick a local chart on X and pick coordinates in this chart in the direction of the eigenspaces of g. This gives us a local foliation of a stable and unstable manifold. Here, we can apply Hopf's argument. Hopf's argument tells us that if we have any weak limit, then it must be constant a.e. Thus, we have a locally constant function. Since the measure (Lebesgue) has connected support, and X is a metric space, h is constant a.e in X.

Thus, we have shown the following: every weak accumulation point of $f \circ T_g^n$ must be constant almost everywhere. But, since $\mu(X) = 1$, we have that $h(x) = \int_X h\,d\mu$. Hence,

$$f \circ T_g^n \longrightarrow \int_X h\,d\mu, \text{ where the convergence is weak.}$$

Since T_g is measure-preserving, $\int f \circ T_g^n\,d\mu = \int_X f\,d\mu$. But then $\int f \circ T_g^n\,d\mu$ is constant for every n, thus $\int f \circ T_g^n\,d\mu = \int_X h\,d\mu$, and we can conclude that $\int_X f\,d\mu = \int_X h\,d\mu$. In other words,

$$f \circ T_g^n \longrightarrow \int_X f\,d\mu, \text{ where the convergence is weak.}$$

By compactness of X, if this is an accumulation point, it will be a limit point. Unraveling the definition of weak convergence, we can see that for any $\tilde{f} \in L^2$

$$\left\langle f \circ T_g^n, h \right\rangle \longrightarrow \left\langle \int_X f, h \right\rangle,$$

where the inner product is the usual inner produce on L^2, and we see that T_g is mixing.

$$\int_X (f \circ T_g^n)\,h\,d\mu \longrightarrow \int_X \left(\int_X f\,d\mu \right) h\,d\mu = \int_X f\,d\mu \int_X h\,d\mu$$

■

Corollary 5.1. The action of the Veech group on a lattice surface is ergodic.

Proof. A measurable transformation that is strongly mixing is ergodic, and since there exists an element in the Veech group acting ergodically, the action of the Veech group is ergodic. ■

There is also a notion of strong and weak mixing for a group action. For this, we will define the unitary operator corresponding to the group action.

Definition 5.4 (Unitary operator, U_g and U_g^0)**.** Let g be an ergodic measure preserving transformation on $(X, \mathscr{B}, \mu)$. Define $U_g : L^2(X) \to L^2(X)$ by $U_g(f)(x) = f(gx)$. Similarly, define $U_g^0 : L_0^2(X) \to L_0^2(X)$ by $U_g(f)(x) = f(gx)$.

Definition 5.5 (Strongly mixing group action)**.** The action of a locally compact group G on a probability space $(X,\mathscr{B},\mu)$ is *strongly mixing* if all the matrix coefficients

$$g \to \left\langle U_g^0 f, h \right\rangle,\ f, h \in L_0^2(X,\mu)$$

vanish at infinity, i.e. is in $C_0(G)$, the space of all continuous functions that vanish at infinity. Recall that on a compact space, this is the same as the set of all continuous functions.

Notably, the action of a $SL_2(\mathbb{Z})$ on the torus is not strongly mixing. Let H denote the subgroup generated by the parabolic element $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. The action of H on T^2 fixes the character

$$\chi : \mathbb{T}^2 \to \mathbb{T},\ \chi(x_1, x_2) = x_2$$

As $n \to \infty$, the character for the elements $\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ is constant. Notice that H is a non-compact, closed subgroup of $SL_2(\mathbb{Z})$, and by Remark 2.11 (iii) in [8], we see that the action induced $SL_2(\mathbb{Z})$ cannot be strongly mixing. All closed, non-compact subgroups of a strongly mixing group action inherit the property of strong mixing.

Definition 5.6 (Weakly mixing group action)**.** The action of a locally compact group G on a probability space $(X,\mathscr{B},\mu)$ is *weakly mixing* if the diagonal action on $X \times X$ is ergodic.

We can show that the action of $SL_2(\mathbb{Z})$ on the torus is weakly mixing by using a Fourier series argument and the definition above. See [8].

The following equivalent characterization (one of many) was proven by Bergelson and Rosenblatt [9]. We make note of this particular characterization since the Koopman representation is underlying the spectral estimates in Section 3.

Definition 5.7. Let G be a locally compact, second countable group whose action is weakly mixing on a probability space $(X, \mathscr{B}, \mu)$. Then $L_0^2(X)$ contains no nontrivial finite dimensional invariant subspaces of $(U_g)_{g \in G}$.

6. APPENDIX B: CRITICAL EXPONENT

In this appendix, we record several facts regarding Fuchsian groups and their corresponding critical exponents.

Lemma 6.1. [32] A Fuchsian group is finitely generated if and only if it is geometrically finite.

Lemma 6.2. [21] A Fuchsian group is convex-cocompact if and only if it is finitely generated and does not contain parabolic elements.

Lemma 6.3. Let Γ be a convex cocompact Fuchsian group, and assume that Γ' is a finite index subgroup. Then Γ' is also convex cocompact, and furthermore $\delta_\Gamma = \delta_{\Gamma'}$, where δ_Γ and $\delta_{\Gamma'}$ are the critical exponents of Γ and Γ', respectively.

Lemma 6.4 (Sullivan [47])**.** Let G be a Fuchsian group. Then

$$\delta_G = \sup\{\delta_H : H \subset G \text{ is finitely generated}\}.$$

Thus, if G be a Fuchsian group, then for any $\epsilon > 0$, there exists a geometrically finite Fuchsian group H such that $\delta_H > \delta_G - \epsilon$.

Lemma 6.5 (Stadlbauer [46])**.** Let G be an essentially free Kleinian group (geometrically finite Fuchsian groups are essentially free) and let $N \triangleleft G$ be a normal subgroup. Then $\delta_N = \delta_G$ if and only if G/N is amenable.

Lemma 6.6 (Susskind [49])**.** The intersection of two geometrically finite subgroups G, H of a discrete group in $\mathrm{Isom}(\mathbb{H}^n)$ are geometrically finite. Furthermore, the limit set $\Lambda(G\cap H) = \Lambda(G)\cap\Lambda(H)\cup P$, where P is a set of isolated points in the regular set of $G\cap H$.

Note that geometrically finite Fuchsian groups are finitely generated, and that for finitely generated Fuchsian groups (Fuchsian groups of the first kind), the limit set is the radial limit set. This means that $\delta_{G\cap H} = \dim_H(\Lambda(G\cap H))$.

Lemma 6.7 (Selberg's Lemma)**.** Let Γ be a finitely generated discrete group. Then Γ contains a finite-index subgroup without torsion.

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