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Green's Law and the Riemann Problem in Layered Media

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Abstract

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The propagation of long waves onto a continental shelf is of great interest in tsunami modeling, where understanding the amplification of waves during shoaling is of significant importance. When the linearized shallow water equations are solved with the continental shelf modeled as a sharp discontinuity, the ratio of the amplitudes is given by the transmission coefficient that can be obtained from the solution of a Riemann problem. On the other hand, when the slope is very broad relative to the wavelength of the incoming wave, then amplification is governed by Green's Law, which predicts a larger amplification than the transmission coefficient, and a much smaller reflection than given by the reflection coefficient of a sharp interface.

Exploring the relation between these results offers a perspective that allows us to view the solution to the shallow water equations as combinations of infinitely many transmitted and reflected waves. The thesis focuses on an asymptotic approximation to this solution, general enough for continuous non-homogeneous media, and presents some interesting results and scenarios along the way. The same phenomena and similar results exist for other physical settings described by wave equations, including those of linear acoustic waves and electromagnetic waves.

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DEDICATION

To my sister, Nidhi

and

all the students that have been part of the AMATH Masters Program

Chapter 1

INTRODUCTION

1.1 Background and motivation

The question of how much waves get amplified as they come onto the shore from the deep ocean is an important one in tsunami modeling. Asking this question in the context of multilayer shallow water equations let us stumble onto the problem which is the focus of this thesis. Multilayer equations are suitable models for long waves when the ocean is stratified due to differences in density and temperature. This often results in internal waves between layers of water that can be of much larger amplitudes than the surface waves. We initially thought of exploring the effect of internal waves on the amplification of the top layer of water in multi-layer shallow water equations. It quickly became apparent that the amplification even in the single layer case is not understood that well.

For the single layer shallow water equations, there are two seemingly contradictory results for the amplitude of the wave transmitted onto the shore, one predicted by Green's law and the other obtained from the Riemann problem. A major focus of the thesis is to show the connection between the two results. Parts of this thesis have been adapted from [3], which is joint work with Randall J. LeVeque and David I. Ketcheson.

1.2 Green's Law and the Riemann problem

If the wavelength of an ocean wave is significantly greater than the ocean depth, then the one-dimensional shallow water equations can be used to model the propagation of a plane wave onto a planar topography of the sort illustrated in fig. 1.1. If we also assume that the amplitude of the wave is very small relative to the water depth everywhere, then the linearized shallow water equations can be used as a model.

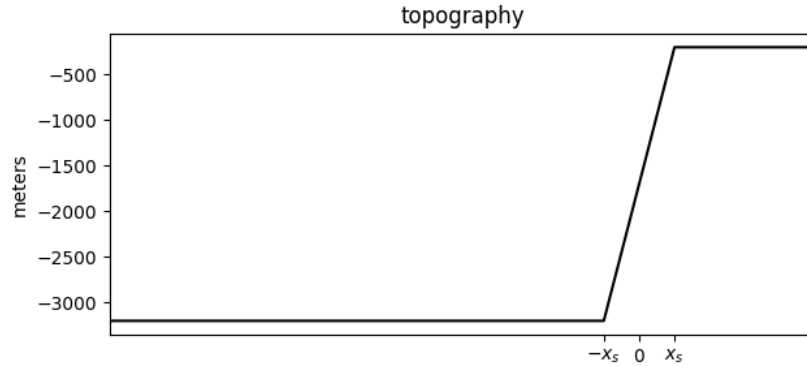


Figure 1.1: Topography, with linear continental slope for $-x_s \leq x \leq x_s$.

The linearized shallow water equations in one dimension can be written in conservation form as

$$\begin{aligned} \eta_t(x, t) + (h(x)u(x, t))_x &= 0, \\ u_t + g\eta_x(x, t) &= 0, \end{aligned} \tag{1.1}$$

where $h(x)$ is the undisturbed fluid depth that we are linearizing about, $\eta(x, t)$ is the surface elevation (with $\eta = 0$ corresponding to the undisturbed sea level), $u(x, t)$ is the depth-averaged horizontal velocity, and $g = 9.81 \text{ m/s}^2$ is the gravitational constant.

The coefficient matrix of the linear hyperbolic system (1.1) has eigenvalues and corresponding eigenvectors given by

$$\begin{aligned} \lambda_1 &= -\sqrt{gh(x)}, & r_1 &= \begin{bmatrix} 1 \\ -\sqrt{g/h(x)} \end{bmatrix}, \\ \lambda_2 &= \sqrt{gh(x)}, & r_2 &= \begin{bmatrix} 1 \\ \sqrt{g/h(x)} \end{bmatrix}. \end{aligned} \tag{1.2}$$

Green's Law predicts that the amplitude of a wave moving up a slowly varying slope will be magnified by a factor of

$$C_G = (h_\ell/h_r)^{1/4}. \tag{1.3}$$

This is valid if the wavelength of the wave is significantly smaller than the width of the slope.

We generally consider piecewise linear topography of the form

$$h(x) = \begin{cases} h_\ell, & x < x_\ell, \\ h_\ell + \left(\frac{h_r - h_\ell}{x_r - x_\ell}\right)(x - x_\ell), & x_\ell \leq x \leq x_r, \\ h_r, & x > x_r. \end{cases} \quad (1.4)$$

as shown in fig. 1.1. Here h_ℓ denotes the ocean depth and h_r the depth of the continental shelf. The region between x_ℓ and x_r is known as the continental slope. For the examples, we use $h_\ell = 3200$ m and $h_r = 200$ m, which are realistic values and were chosen to have a ratio of 16, so that $C_G = 2$.

To illustrate the shoaling behavior of waves governed by the shallow water equations, in figs. 1.2 and 1.3 we show the solution to these equations when the initial conditions consist of a square pulse wave approaching a continental slope with three different slopes. Although a square pulse is not at all realistic, for the linear equations the behavior is qualitatively the same whether we use a square pulse, a smooth hump such as a Gaussian, or a continuous wave train. In order to explain the behavior of the solution as the slope is made more steep we will study the case of a single jump discontinuity propagating toward the shelf and the square pulse is then natural to consider as a pair of propagating discontinuities.

Figure 1.2 shows a case in which a square pulse moves up a slope that is broad relative to the width of the pulse, a case where Green's Law should approximately apply. The initial surface displacement is shown on the left, and consists of a pulse with amplitude $A = 1$ m, and with fluid velocity $u = \sqrt{g/h_\ell}$ in the pulse, to give a purely right-going wave (based on the eigenvectors). The plot on the right shows the solution at a later time when the wave has been amplified due to shoaling by roughly the expected factor of $C_g = 2$. The pulse also becomes narrower as the wave speed decreases. There is a very small amplitude reflected wave in this case. There is also a bit of a negative tail

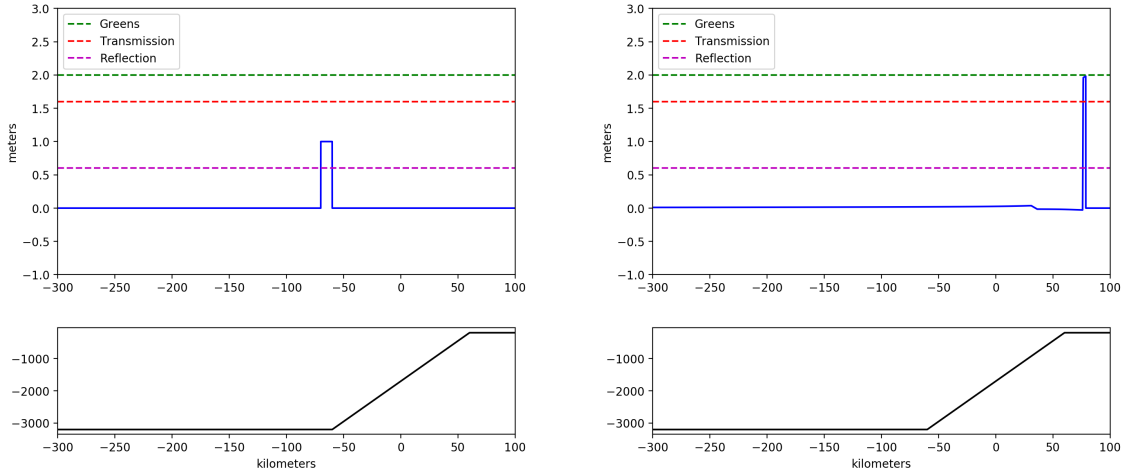


Figure 1.2: Left: The initial data consists of a square pulse of height 1 m propagating to the right toward a continental slope. Right: Solution at a later time, when the pulse has become narrower and taller on the shelf, with very little reflected energy. The amplitude is close to that predicted by Green’s Law. The bottom plots show the topography, with a different vertical scale.

following the transmitted pulse that we will say more about later.

Figure 1.3 shows solutions to the equations with the same initial conditions as in fig. 1.2 but on steeper slopes. In the left, the slope is replaced by a jump discontinuity at $x = 0$. In this case Green’s Law does not hold. Instead, the incident pulse is partially transmitted as a narrower pulse with height C_T and partially reflected as a left-moving wave with height C_R . Using the eigenvectors to the left and right of $x = 0$ and continuity of the solution, we can derive the standard transmission and reflection coefficients (see section 2.1):

$$C_T = \frac{2\sqrt{h_\ell}}{\sqrt{h_\ell} + \sqrt{h_r}}, \quad C_R = \frac{\sqrt{h_\ell} - \sqrt{h_r}}{\sqrt{h_\ell} + \sqrt{h_r}} = C_T - 1. \quad (1.5)$$

With our choice of h_ℓ and h_r we have $C_T = 1.6$ and $C_R = 0.6$.

Finally, the right side of Figure 1.3 shows a case where the width of the continental slope is comparable to the width of the incoming pulse. In this case the solution is

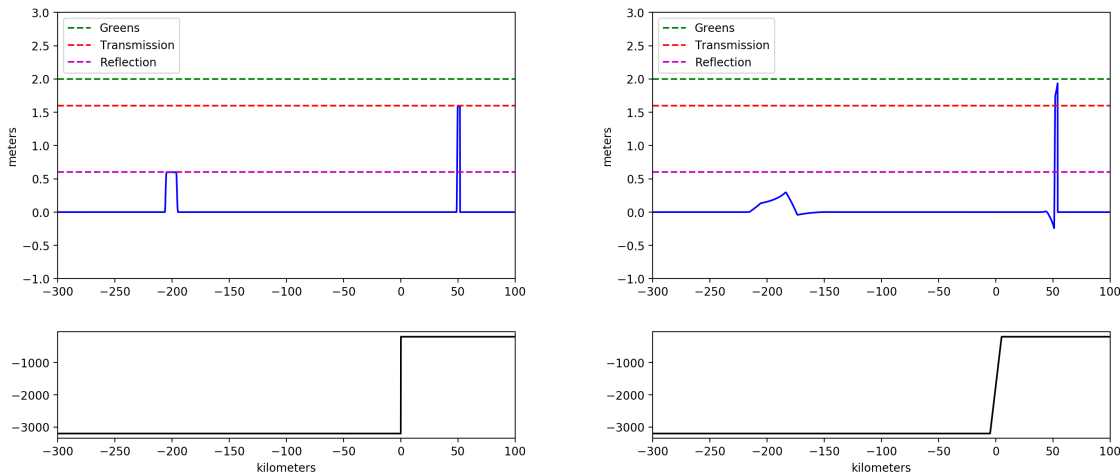


Figure 1.3: Left: Solution when the smooth slope of fig. 1.2 has been replaced with a sharp discontinuity, consisting of a right-going transmitted pulse on the shelf and a left-going reflected pulse in the ocean. Right: Solution with a step slope, showing more reflected energy than in the case shown in fig. 1.2.

more complicated, consisting of a transmitted wave on the shelf that has a peak value close to that predicted by Green’s Law but decays behind the peak and has a negative undershoot, along with a reflected wave with complicated structure. As the slope is made more broad, the amplitude of the reflected wave will decrease and eventually go to zero in the Green’s Law limit. Our goal is to fully describe waves in this transitional regime of step slopes and the manner in which this reflection disappears.

1.3 Overview

In this chapter, we described the curiosities around this problem and how we came across it. Chapter 2 explores Green’s law and the Riemann problem in more detail. We show the derivation Green’s law for linearized shallow water equations and explain how the transmission and reflection coefficients arise out of the Riemann problem.

Chapter 3 looks to establish a connection between the two results using a layered

media approach. This approach reveals to us that the peak value of the solution is governed by Green's law and the asymptotic value by the transmission coefficient.

The layered media offers us a new perspective on the solution to the shallow-water equations. Utilizing the idea of multiple reflections and transmissions, we are able to obtain a first-order analytic approximation to the solution which is usually a good approximation. Chapter 4 sets up the technique for obtaining these first-order approximation which can be extended to higher orders.

Chapter 5 explores a parallel method to that in Chapter 4 for approximations to the reflected and transmitted wave. This method uses the theory of characteristics and can be thought of as a differential form of the method in Chapter 4. The technique in this chapter parametrizes time and is easier to generalize for higher-order approximations.

While Chapter 4 and Chapter 5 leave us with integral expressions for reflected and transmitted waves, the asymptotic contributions of these waves are much easier to obtain. Chapter 6 details the general method to find all the terms and uses them to write two series expansions for the transmission coefficient in the Riemann problem. This chapter also touches briefly on when the higher order terms dominate.

Most of the thesis is about the amplification of the height of the wave that gets transmitted onto the shore. This begs questions about the mass and energy transmitted onto the shore and the width of the waves. Ketcheson and LeVeque had some answers which are showcased in Chapter 7.

The methods and results surrounding this problem can be extended to other physical phenomena governed by the wave equation. Among them, the linear acoustic equations are quite interesting and different from the shallow water equations since there are two parameters, the impedance and the wave speed which can vary independently depending on the media. Chapter 8 takes a brief look at them.

Lastly, Chapter 9 gives a final summary of the work done in the thesis.

Chapter 2

THE RIEMANN PROBLEM AND GREEN'S LAW

The results shown in figs. 1.2 and 1.3 seem contradictory. Before we attempt to connect the results, it seems like a good idea to take a closer look at the two results and the settings where they are used. This chapter shows the derivation of C_T , the transmission coefficient for the linearized shallow water equations similar to what has been done for acoustics in [7]. It also showcases a derivation of Green's law by following a similar procedure to [8].

2.1 *The Riemann problem*

We are faced with a question of what happens to a wave as it comes onto a continental shelf. A numerical analyst might feel motivated to use the Riemann problem to model the shelf. The Riemann problem illustrated in Figure 2.1 is an initial value problem with discontinuous data. Here, the discontinuity is in the bathymetry, with the left side being the deep ocean and the right side being the continental shelf.

Because of the discontinuity, the purely right going wave incident on the discontinuity splits into a left-going wave and a right-going wave. Expressing the shallow water equations as a hyperbolic system allows us to get expressions for the transmitted and reflected waves. The shallow-water equations in (1.1) are described in their conservation form. It is possible to write it in its non-conservative form by introducing the momentum $\mu = h(x)u$.

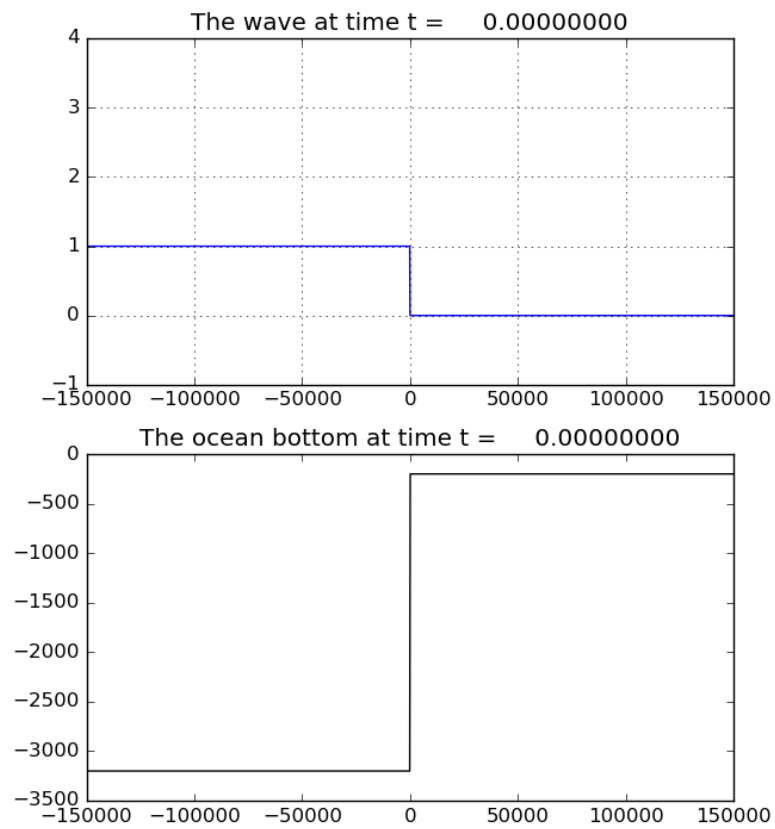


Figure 2.1: The Riemann problem : If the bathymetry was constant, a purely right-going wave would propagate unchanged onto the right. The sharp jump in the bathymetry creates a transmitted wave and reflected wave.

$$\eta_t + \mu_x = 0 \quad (2.1)$$

$$\mu_t + gh(x)\eta_x = 0, \quad (2.2)$$

This is a system of equations where each state (η, μ) can be represented as q . This allows us to write (2.1) as

$$q_t + Aq_x = 0 \quad (2.3)$$

where

$$A = \begin{bmatrix} 0 & 1 \\ gh(x) & 0 \end{bmatrix} \quad (2.4)$$

The eigenvalues of A are given by $\lambda_1 = -\sqrt{gh(x)}$ and $\lambda_2 = \sqrt{gh(x)}$ representing wave speeds for a left-going and a right-going wave.

The eigenvectors of A are

$$r^1 = \begin{bmatrix} 1 \\ -\sqrt{gh(x)} \end{bmatrix}, \quad r^2 = \begin{bmatrix} 1 \\ \sqrt{gh(x)} \end{bmatrix}, \quad (2.5)$$

Thus, the solution to the Riemann problem can be written as a left-going wave \mathcal{W}_1 and a right-going wave \mathcal{W}_2 , where \mathcal{W}_1 and \mathcal{W}_2 are given by

$$\mathcal{W}_1 = \alpha_1 \begin{bmatrix} 1 \\ -\sqrt{gh(x)} \end{bmatrix}, \quad \mathcal{W}_2 = \alpha_2 \begin{bmatrix} 1 \\ \sqrt{gh(x)} \end{bmatrix}, \quad (2.6)$$

From [7], we know that α_1 and α_2 can be obtained by solving

$$\begin{bmatrix} 1 & 1 \\ -\sqrt{gh_\ell} & \sqrt{gh_r} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = q_r - q_l \quad (2.7)$$

But the jump across q_r and q_l should be our initial purely right-going wave. Let $\Delta\eta$ represent the ocean surface elevation in the right going wave. Δu , the perturbation in

velocity should be $\sqrt{gh_\ell}\Delta\eta$ (since it is a purely right-going wave). Then,

$$q_r - q_l = \begin{bmatrix} \Delta\eta \\ \Delta u \end{bmatrix} = \Delta\eta \begin{bmatrix} 1 \\ \sqrt{gh_\ell} \end{bmatrix} \quad (2.8)$$

$$\begin{bmatrix} 1 & 1 \\ -\sqrt{gh_\ell} & \sqrt{gh_r} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \Delta\eta \begin{bmatrix} 1 \\ \sqrt{gh_\ell} \end{bmatrix} \quad (2.9)$$

$$\alpha_1 = \Delta\eta \frac{\sqrt{h_r} - \sqrt{h_\ell}}{\sqrt{h_\ell} + \sqrt{h_r}} \quad \alpha_2 = \Delta\eta \frac{2\sqrt{h_\ell}}{\sqrt{h_\ell} + \sqrt{h_r}} \quad (2.10)$$

The transmitted wave is given by

$$\mathcal{W}_2 = \alpha_2 \begin{bmatrix} 1 \\ \sqrt{gh(x)} \end{bmatrix} = \Delta\eta \frac{2\sqrt{h_\ell}}{\sqrt{h_\ell} + \sqrt{h_r}} \begin{bmatrix} 1 \\ \sqrt{gh(x)} \end{bmatrix} \quad (2.11)$$

The ratio of the amplitude of surface elevation of the transmitted wave to that of the incident wave ($\Delta\eta$) is given by

$$C_T = \frac{2\sqrt{h_\ell}}{\sqrt{h_\ell} + \sqrt{h_r}} \quad (2.12)$$

Thus, the eigenvalues and eigenvectors of the hyperbolic system reveals to us, the amplification factor in the Riemann problem. Is it a good estimate for the amplitude of transmitted wave as it goes onto the continental shelf? It should be, if it is reasonable to model the continental shelf as a sharp discontinuity.

2.2 Green's Law

In reality, continental slopes are spread out over dozens of kilometres and are very different from sharp discontinuities. There is a whole field called shoaling devoted to the amplification of waves as they move into shallower water and onto shore. George Green was one of the first people to get an expression now known as Green's Law for

the amplification of the wave in his 1838 essay [5]. The technique he used to find this expression became known as the Louville-Green method and formed the basis of WKB asymptotics. This section rederives Green's law for linearized shallow water equations in a similar manner to [8]

$$\begin{aligned}\eta_t(x, t) + (h(x)u(x, t))_x &= 0, \\ u_t + g\eta_x(x, t) &= 0,\end{aligned}\tag{2.13}$$

Starting with the conservation form of the linearized shallow water equations, we first differentiate each equation with respect to t and x respectively.

$$\begin{aligned}\eta_{tt} + hu_{xt} + h_x u_t &= 0, \\ u_{tx} + g\eta_{xx} &= 0,\end{aligned}\tag{2.14}$$

This allows us to combine the two equations.

$$\eta_{tt} - gh\eta_{xx} - gh_x\eta_x = 0,\tag{2.15}$$

Now, that we have a pde for the amplitude of the wave, we can start with an ansatz.

$$\eta(x, t) = A(x)e^{i(\omega t - \phi(x))}\tag{2.16}$$

where $A(x)$ is the amplitude of the wave. ω and $\phi(x)$ have physical interpretations of wave frequency and wave function. The wave number k is related to the wave function by

$$\phi_x(x) = k$$

$$\begin{aligned}\eta_x &= A_x e^{i(\omega t - \phi(x))} - ikA e^{i(\omega t - \phi(x))} \\ \eta_{xx} &= A_{xx} e^{i(\omega t - \phi(x))} - ikA_x e^{i(\omega t - \phi(x))} - ik_x A e^{i(\omega t - \phi(x))} - ikA_x e^{i(\omega t - \phi(x))} \\ &\quad - k^2 A e^{i(\omega t - \phi(x))}\end{aligned}\tag{2.17}$$

Plugging this in (2.15), we get

$$\begin{aligned}
& -w^2 A e^{i(wt-\phi(x))} - gh A_{xx} e^{i(wt-\phi(x))} + igh k_x A e^{i(wt-\phi(x))} + 2igh k A_x e^{i(wt-\phi(x))} \\
& + gh k^2 A e^{i(wt-\phi(x))} - gh_x A_x e^{i(wt-\phi(x))} + ikgh_x A e^{i(wt-\phi(x))} = 0
\end{aligned} \tag{2.18}$$

The technique used here looks at the most dominant terms in Equation (2.15) to get an expression for the spatial variation of the peak amplitude. Ignoring the terms with a derivative since they are relatively small, we get the following dispersion relation.

$$-w^2 + h(x)gk^2 = 0 \tag{2.19}$$

Equating the terms which contain a single derivative, we get

$$ihgk_x A e^{i(wt-\phi(x))} + 2ihgk A_x e^{i(wt-\phi(x))} + ikgh_x A e^{i(wt-\phi(x))} = 0 \tag{2.20}$$

$$hk_x A + 2hk A_x + kh_x A = 0$$

$$2\frac{A_x}{A} + \frac{k_x}{k} + \frac{h_x}{h} = 0 \tag{2.21}$$

To eliminate the wave number dependence, we differentiate (2.19).

$$-2ww_x + h_x gk^2 + 2hgk k_x = 0 \tag{2.22}$$

Since our ansatz is assumed to have only a single frequency, $w_x = 0$.

$$h_x gk^2 + 2hgk k_x = 0$$

$$\frac{k_x}{k} = -\frac{h_x}{2h} \tag{2.23}$$

Using this in (2.21), we get

$$2\frac{A_x}{A} - \frac{h_x}{2h} + \frac{h_x}{h} = 0$$

$$\frac{A_x}{A} = -\frac{h_x}{4h}$$

$$\log(A(x))' = -\frac{1}{4} \log(h(x))'$$

$$A_r = \left(\frac{h_\ell}{h_r}\right)^{\frac{1}{4}} A_i \quad (2.24)$$

Here, A_i represents the maximum amplitude of the incident wave. This asymptotic derivation eliminates most terms leaving only the dominant terms. So, A_r represents the maximum amplitude of the wave transmitted onto the continental shelf.

2.3 The discrepancy and a historical connection

Why does Green's law give us one expression for the amplification of the wave while we get a different one from the Riemann problem? The critical difference between the two problems lies in the shape of the bathymetry. The Riemann problem looks at a sharp jump while Green's law was derived for a smoothly varying slope. In fact, for Green's law to hold well, the wave length of the wave should be small compared to the width of the continental slope. So, the ratio of wavelength to the width of the slope in the setting of Green's law should be much less than 1. In the Riemann problem, the width of the continental slope is zero, making the ratio infinity.

At first glance, it may look like these results contradict each other. However, we will see soon in Chapter 3 that there is a close connection between them. The ratio of the wavelength to the slope width is also important and Section 3.4 highlights two different regimes where either result can dominate.

These problems also share an interesting historical connection. Green's theorem in mathematics is named after George Green who introduced a form of the divergence theorem in his famous 1828 essay. But the first proof of Green's theorem came a while later in the doctoral dissertation of Bernard Riemann!

Chapter 3

INTERPRETATION AS LAYERED MEDIA

To explore the connection between the amplitude predicted by Green's law and that by the Riemann problem, we need to look at the connection between the bathymetry in the two cases. A smooth slope can be viewed as the limiting case of a sequence of small jump discontinuities defining a layered medium. In fact, numerical simulations like those built on finite volume methods often utilize a piecewise constant approximation to represent a continuous function. So, it is natural to first consider a layered medium with only a few intermediate layers. The single jump discontinuity between depths h_ℓ and h_r that we used to define the transmission and reflection coefficients can be partitioned into $N + 1$ discontinuities separating N intermediate layers. Figure 3.1 shows examples with $N = 0, 1,$ and 2 . The figures show, in the $x-t$ plane, what happens to a single jump discontinuity in η of amplitude A that encounters the layered medium.

3.1 An intermediate layer

With no intermediate layer ($N = 0$), the wave splits into transmitted and reflected waves only once, with amplitudes $C_T A$ and $C_R A$ respectively. With $N = 1$ interior layer like in Figure 3.2, the solution is already much more complicated, with internal reflections in the layer that result in an infinite sequence of waves eventually departing to both the right and to the left, as shown in fig. 3.1(b). The good news is that we do not need to know all the waves to get an idea of the phenomenon since the amplitudes of later waves decay rapidly with time since each reflection coefficient is less than one.

Suppose the intermediate layer has depth h_m with $h_r < h_m < h_\ell$. A single intermediate layer implies two interfaces, one between h_ℓ and h_m and another between h_m

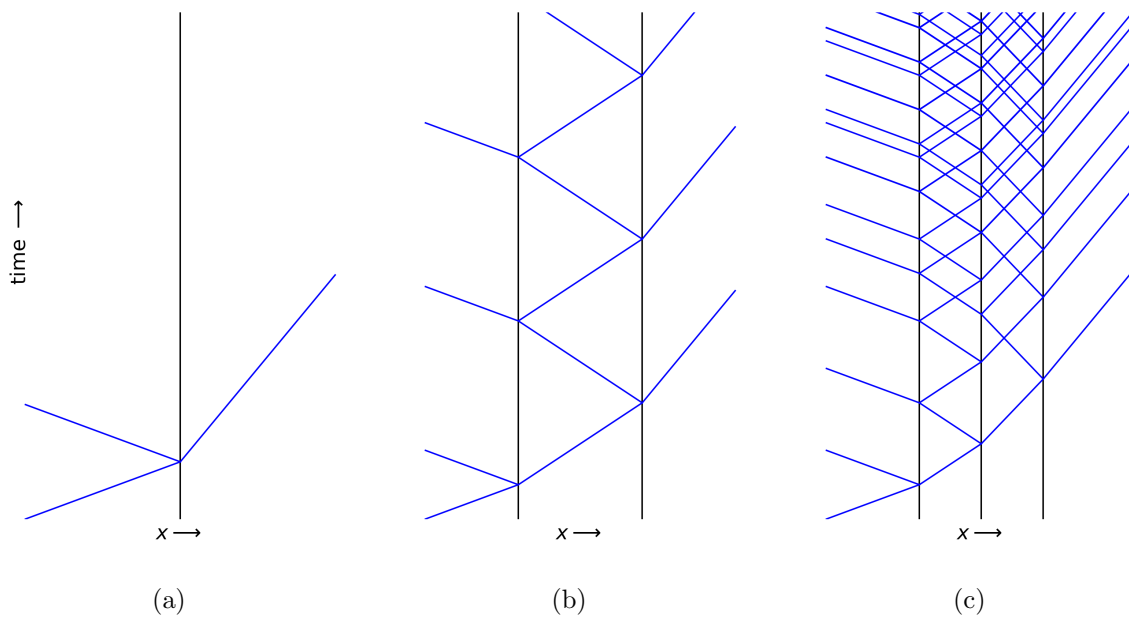


Figure 3.1: Plots in the $x-t$ plane showing an initial right-going wave interacting with a layered medium. (a) With a single interface, where only one transmitted and reflected wave are generated. (b) With two interfaces and a single intermediate layer. (c) With three interfaces and two intermediate layers.

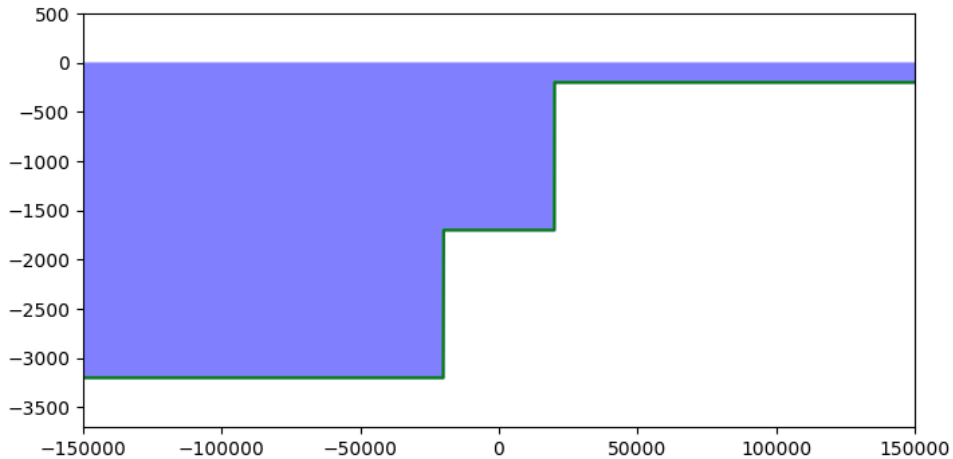


Figure 3.2: Adding an intermediate layer breaks the sharp jump into two smaller jumps. See Figure 7.2 for the transmitted and reflected waves that emerge in this setting.

and h_r . We define the following transmission coefficients:

$$C_T^{\ell m} = \frac{2\sqrt{h_\ell}}{\sqrt{h_\ell} + \sqrt{h_m}}, \quad C_T^{mr} = \frac{2\sqrt{h_m}}{\sqrt{h_m} + \sqrt{h_r}}, \quad C_T^{m\ell} = \frac{2\sqrt{h_m}}{\sqrt{h_\ell} + \sqrt{h_m}}, \quad (3.1)$$

where the superscript mr for example indicates transmission of a right-going wave from the middle layer to the right layer. Similarly, we can define the reflection coefficients for internal reflections in the intermediate layer as

$$C_R^{mr} = \frac{\sqrt{h_m} - \sqrt{h_r}}{\sqrt{h_m} + \sqrt{h_r}}, \quad C_R^{m\ell} = \frac{\sqrt{h_\ell} - \sqrt{h_r}}{\sqrt{h_m} + \sqrt{h_\ell}} \quad (3.2)$$

When a wave hits the first interface, it breaks into a transmitted and reflected wave. The transmitted wave hits the second interface and splits into a new pair of transmitted and reflected waves. The wave transmitted here is the first wave to reach the right side. The reflected wave gets partially reflected again at the first interface and get partially transmitted onto the right side as a wave with a much lower amplitude. The first wave to be transmitted onto the right side would have no internal reflections. If A_i is the

amplitude of the incident wave, the transmitted wave would have the amplitude $A_{t_1 t_2}$ where t_1, t_2 represent transmission through the first and second interface respectively.

$$A_{t_1 t_2} = C_T^{\ell m} C_T^{mr} A_i = \frac{2\sqrt{h_\ell}}{\sqrt{h_m} + \sqrt{h_\ell}} \frac{2\sqrt{h_m}}{\sqrt{h_m} + \sqrt{h_r}} A_i \quad (3.3)$$

The second wave would be reflected internally once between both interfaces before getting transmitted onto the right side. This wave would have the following amplitude

$$A_{t_1 r_2 r_1 t_2} = \frac{2\sqrt{h_\ell}}{\sqrt{h_m} + \sqrt{h_\ell}} \frac{\sqrt{h_r} - \sqrt{h_m}}{\sqrt{h_r} + \sqrt{h_m}} \frac{\sqrt{h_\ell} - \sqrt{h_m}}{\sqrt{h_\ell} + \sqrt{h_m}} \frac{2\sqrt{h_m}}{\sqrt{h_m} + \sqrt{h_r}} A_i \quad (3.4)$$

A wave that is reflected n times internally at both interfaces would have the following amplitude.

$$A_{t_1 (r_2 r_1)^n t_2} = \frac{2\sqrt{h_\ell}}{\sqrt{h_m} + \sqrt{h_\ell}} \left(\frac{\sqrt{h_r} - \sqrt{h_m}}{\sqrt{h_r} + \sqrt{h_m}} \frac{\sqrt{h_\ell} - \sqrt{h_m}}{\sqrt{h_\ell} + \sqrt{h_m}} \right)^n \frac{2\sqrt{h_m}}{\sqrt{h_m} + \sqrt{h_r}} A_i \quad (3.5)$$

Adding all the transmitted waves, we get a geometric series.

$$\begin{aligned} \sum_{n=0}^{\infty} A_{t_1 (r_2 r_1)^n t_2} &= \sum_{n=0}^{\infty} \frac{2\sqrt{h_\ell}}{\sqrt{h_m} + \sqrt{h_\ell}} \left(\frac{\sqrt{h_r} - \sqrt{h_m}}{\sqrt{h_r} + \sqrt{h_m}} \frac{\sqrt{h_\ell} - \sqrt{h_m}}{\sqrt{h_\ell} + \sqrt{h_m}} \right)^n \frac{2\sqrt{h_m}}{\sqrt{h_m} + \sqrt{h_r}} A_i \\ &= \frac{2\sqrt{h_\ell}}{\sqrt{h_r} + \sqrt{h_\ell}} A_i \end{aligned}$$

This is as if the middle layer didn't exist. This means that asymptotically for the two layer case, the amplitude of the transmitted wave should be given by the transmission coefficient from the Riemann problem. However, these waves do not arrive on the right side at the same time. There is a lag associated with these waves because of the time spent being internally reflected. The wave has to traverse the width of the middle layer twice for every pair of internal reflections. There is a fixed time lag between each departing wave of $2W_L\sqrt{gh_m}$, where W_L is the width of the layer.

Similarly, the sum of all left-going waves that depart into the deep ocean after $2n + 1$ internal reflections (for $n = 0, 1, 2, \dots$) is given by

$$\sum_{n=0}^{\infty} C_T^{\ell m} C_R^{mr} (C_R^{ml} C_R^{mr})^n C_T^{ml} A = C_R A, \quad (3.6)$$

where C_R is the reflection coefficient in the case of no layer.

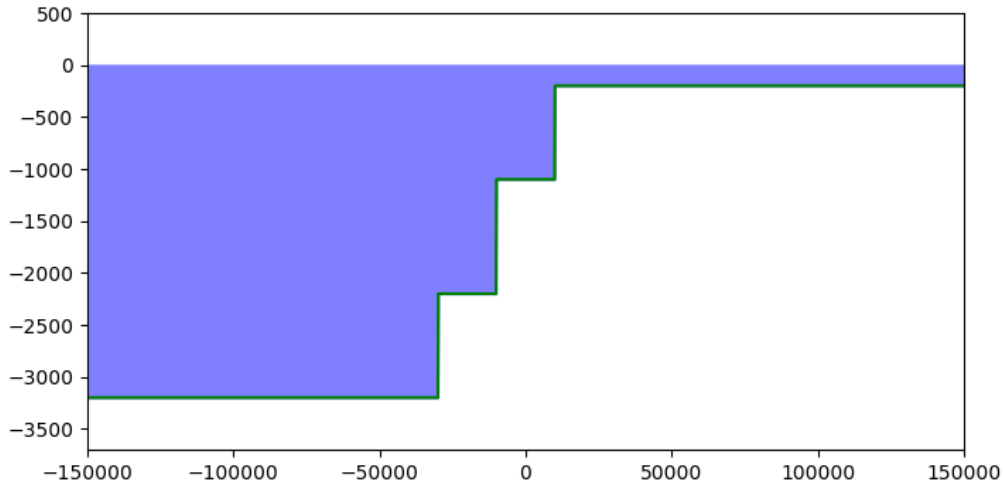


Figure 3.3: Two intermediate layers.

3.2 Two intermediate layers

It seems appropriate to see if this behavior persists with a two layer approximation as shown in Figure 3.3. Suppose the two layers have depths h_{m_1} and h_{m_2} with $h_r < h_{m_2} < h_{m_1} < h_\ell$. Since there are two layers, there are three interfaces. If x is the number of overall reflections in the first layer, y the number of reflections of the third interface and z the subset of y that reflect off the first interface, the amplitude of the transmitted wave can be represented by $A_{(r_2 r_1)^x t_2 r_3^y r_2^z (t_2 r_1 t_2)^{y-z}}$.

Adding all the amplitudes of all the possible transmitted waves numerically, we obtain the same transmission coefficient as that given by the Riemann problem. This gives more confidence to the theory that the asymptotic amplitude is as if the middle layers were not present. Section 7.1 in Chapter 7 explores how this is true regardless of the number of layers and for any bathymetry.

3.3 The peak value of the first transmitted wave

The amplitude of the first transmitted wave in the case of a single intermediate layer is given by (3.3). Since the reflection coefficient is always less than or equal to one, the first transmitted wave has the largest amplitude. In the case of two intermediate layer, the amplitude of the first transmitted wave is given by

$$A_{t_1 t_2 t_3} = \frac{2\sqrt{h_\ell}}{\sqrt{h_{m_1}} + \sqrt{h_\ell}} \frac{2\sqrt{h_{m_1}}}{\sqrt{h_{m_1}} + \sqrt{h_{m_2}}} \frac{2\sqrt{h_{m_2}}}{\sqrt{h_{m_2}} + \sqrt{h_r}} A_i \quad (3.7)$$

layers-a

$A_{t_1 t_2 t_3}$ is larger than $A_{t_1 t_2}$ and it turns out that increasing the number of layers increases the peak amplitude. Our aim is to find if the peak amplitude converges as the number of layers become infinite. To do this, we discretize the jump between the heights h_ℓ and h_r into smaller and smaller jumps and hope to find a limit.

The transmission coefficient between two layers of depth h and $h+\Delta h$ is given by

$$\frac{2\sqrt{h}}{\sqrt{h} + \sqrt{h + \Delta h}} = \frac{2}{1 + \sqrt{1 + \frac{\Delta h}{h}}}$$

If we choose h_i such that $h_i = h_\ell + i\Delta h$ and $h_{n+1} = h_r$, our transmission ratio at the $(i+1)$ th interface is given by

$$t_i = \frac{2}{1 + \sqrt{1 + \frac{\Delta h}{h_\ell + i\Delta h}}}$$

The peak amplitude is given by the limit of the product of these transmission coefficients.

$$\text{Limit} = \lim_{n \rightarrow \infty} \prod_{i=0}^n t_i = \lim_{n \rightarrow \infty} \prod_{i=0}^n \left(\frac{2}{1 + \sqrt{1 + \frac{\Delta h}{h_\ell + i\Delta h}}} \right)$$

Representing our limit by L ,

$$L = \lim_{n \rightarrow \infty} \prod_{i=0}^n \left(\frac{2}{1 + \sqrt{1 + \frac{\Delta h}{h_\ell + i\Delta h}}} \right)$$

$$\begin{aligned}\log(L) &= \lim_{n \rightarrow \infty} \sum_{i=0}^n \log \left(\frac{2}{1 + \sqrt{1 + \frac{\Delta h}{h_\ell + i \Delta h}}} \right) \\ &= \lim_{n \rightarrow \infty} \sum_{i=0}^n \log \left(\frac{2}{1 + \sqrt{1 + \frac{\Delta h}{h_i}}} \right)\end{aligned}$$

Using the Taylor expansion of the square root,

$$\sqrt{1 + \frac{\Delta h}{h_i}} = 1 + \frac{\Delta h}{2h_i} - \frac{\Delta h^2}{8h_i^2} + \dots$$

$$\log(L) = \lim_{n \rightarrow \infty} \sum_{i=0}^n \log \left(\frac{2}{2 + \frac{\Delta h}{2h_i} - \frac{\Delta h^2}{8h_i^2} + \dots} \right) \quad (3.8)$$

$$= \lim_{n \rightarrow \infty} \sum_{i=0}^n \log \left(\frac{1}{1 + \frac{\Delta h}{4h_i} - \frac{\Delta h^2}{16h_i^2} + \dots} \right) \quad (3.9)$$

$$= \lim_{n \rightarrow \infty} \sum_{i=0}^n -\log \left(1 + \frac{\Delta h}{4h_i} - \frac{\Delta h^2}{16h_i^2} + \dots \right) \quad (3.10)$$

Using the Taylor expansion of log,

$$\log(L) = - \lim_{n \rightarrow \infty} \sum_{i=0}^n \left(\frac{\Delta h}{4h_i} - \frac{\Delta h^2}{16h_i^2} + \dots \right) \quad (3.11)$$

But,

$$\Delta h = \frac{h_r - h_\ell}{n}$$

$$\log(L) = - \lim_{n \rightarrow \infty} \frac{h_r - h_\ell}{n} \sum_{i=0}^n \frac{1}{4h_i} + \lim_{n \rightarrow \infty} \left(\frac{h_r - h_\ell}{n} \sum_{i=0}^n \frac{(h_r - h_\ell)^2}{16nh_i^2} + \dots \right)$$

$$\log(L) = - \int_{h_\ell}^{h_r} \frac{1}{4x} dx = -\frac{1}{4} \log \left(\frac{h_r}{h_\ell} \right)$$

Thus,

$$L = \left(\frac{h_\ell}{h_r} \right)^{\frac{1}{4}} \quad (3.12)$$

This is exactly Green's Law. Figure 3.4 shows the result of a right-going wave going onto a continental shelf with a linear slope. The transmitted wave has a peak amplitude given by Green's Law at the leading wavefront and also an asymptotic amplitude

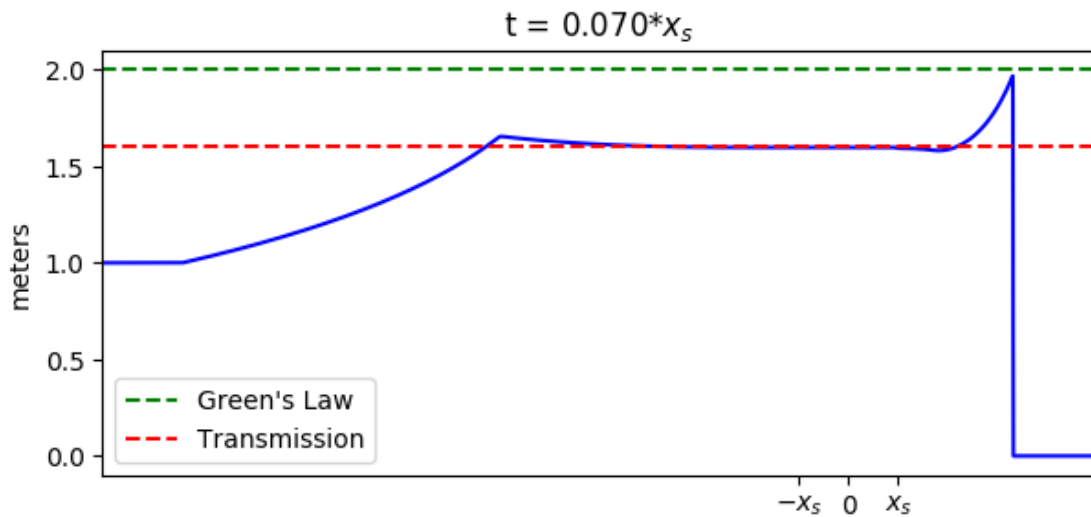


Figure 3.4: A hydraulic jump going up a continental shelf results in a transmitted wave with a peak amplitude given by Green’s law and which decays to an asymptotic amplitude given by the transmission coefficient from the Riemann problem.

given by the transmission coefficient from the Riemann problem. This is a fascinating structure and Figure 3.5 shows the time evolution that leads to it.

After seeing this result, one may be tempted to wonder if the technique used here is a form of the Louiville-Green method. The reason we think so, is that both techniques use ‘asymptotics’ to arrive at the same result. However, the word ‘asymptotics’ has different meanings in the two cases. The asymptotics in the derivation of Green’s law involve finding the largest terms in the partial differential equation and can be best described as a kind of dominant balance. The asymptotics done in this section uses a large number of layers to obtain the continuous limit of a discontinuous approximation of the bathymetry.

Furthermore, Green’s method finds the largest amplitude in the transmitted wave while the method done in this section finds the amplitude of the first wave transmitted onto the right side without any reflections. In our problem of a wave equation without

any dispersive or dissipative terms, the amplitude of the first transmitted wave just happens to be the maximum amplitude. Also, Green's method requires a smooth initial condition like a Gaussian hump but our method is done with the single bore as the initial condition, which is a discontinuity.

3.4 The case of a single bore

In section 3.3, we used a right-going hydraulic jump to demonstrate how the peak amplitude is given by C_G and the asymptotic amplitude is given by C_T . Such a hydraulic jump is referred to as a single bore. The bore is quite relevant to our problem because unlike figs. 1.2 and 1.3, it shows the complete connection between Green's Law and the Riemann problem. Since the linearized shallow water equations are being used here, any kind of incident wave can be expressed as a linear combination of bores of various amplitudes if it is piecewise constant. As data we take

$$\eta(x, 0) = \begin{cases} A, & x < -x_s, \\ 0, & x \geq -x_s \end{cases}, \quad u(x, 0) = \begin{cases} \sqrt{g/h_\ell}, & x < -x_s, \\ 0, & x \geq -x_s \end{cases}. \quad (3.13)$$

This can be viewed as a right-going hydraulic jump (or bore) moving across the ocean, at the instant when it first encounters the continental slope, as shown in the $t = 0$ plot in fig. 3.5. Figure 3.5 shows the form of this solution, as computed using Clawpack [1] on a very fine grid.

As the bore moves up the continental slope to some point $x > -x_s$, its amplitude increases according to Green's Law based on the depth $h(x)$ at the point the bore has reached and h_ℓ . Upon reaching the shelf at $x = x_s$, it has reached an amplitude $C_G A = (h_r/h_\ell)^{1/4} A$, and is followed by a decrease in amplitude down to the value $C_T A$ given by the transmission coefficient (1.5). As it propagates up the shelf, it also gives rise to a left-going wave that has the appearance of a smooth profile that rises to just above $C_T A$ before settling down to the value $C_T A$. This reflected wave thus has overall amplitude $C_T A - A = C_R A$.

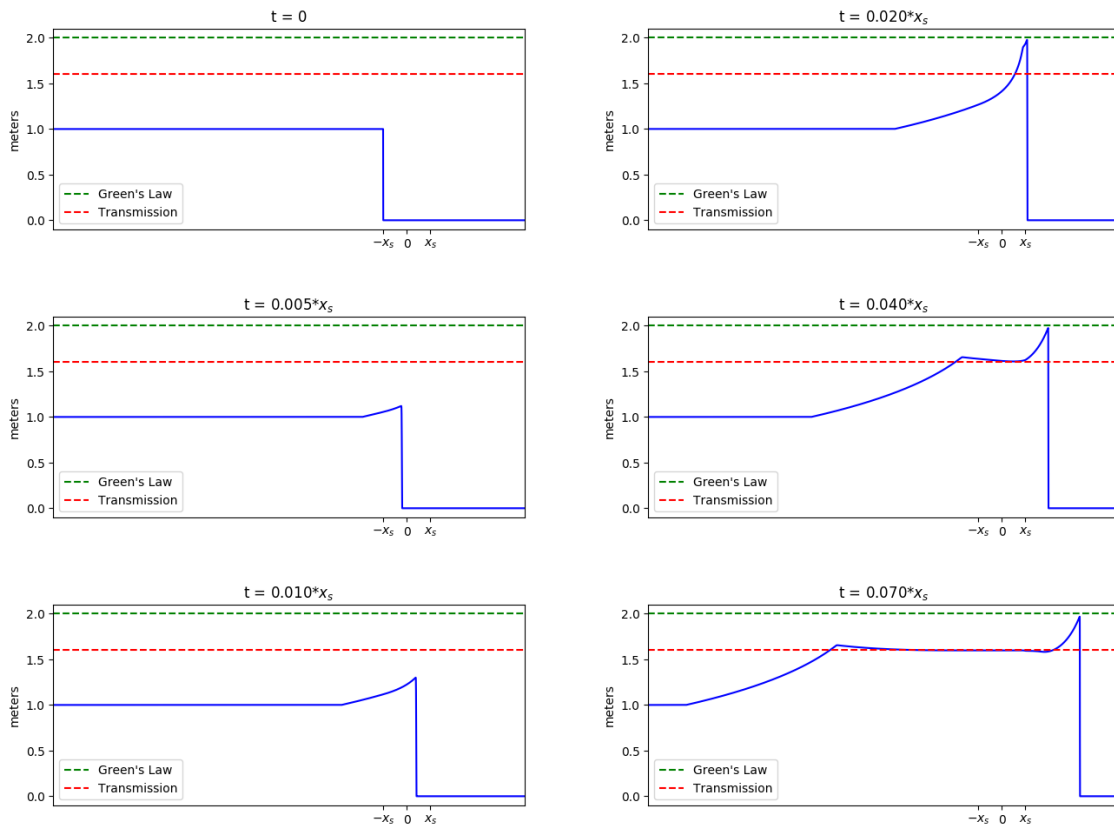


Figure 3.5: Left column: Initial data $\eta_{x_s}(x, 0)$ at $t = 0$, and evolution to $t = 0.01x_s$, where x_s is the half-width of the continental slope. Right column: Further evolution to $t = 0.07x_s$ seconds (when x_s is in meters). At later times the left going and right-going waves propagate outward over constant topography with no further change in shape.

Once we have observed the form of this solution, it is easy to understand the results shown in figs. 1.2 and 1.3 and the transition between them. The square pulse initial data considered in section 1.2 can be viewed as a jump discontinuity from $\eta = 0$ up to $\eta = A$ followed by another jump discontinuity from $\eta = A$ back down to $\eta = 0$. The solution for each of these initial conditions has the form discussed above, and the full solution for the pulse is hence (by linearity) the sum of the solutions for each jump separately. Figure 3.6 shows each of these solutions separately on the top, and the sum of the two on the bottom. These are shown for two different choices of the width w of the initial pulse. As the width decreases, the two waves cancel out almost everywhere except near the initial peak location, where the amplitude jumps up to $C_G A$. Hence in the limit of a narrow pulse relative to the width x_s of the continental slope ($w/x_s \rightarrow 0$), we recover the Green's Law limit of a pulse with amplitude $C_G A$. For larger values of w/x_s there is less cancellation of the left-going waves and a larger apparent reflected wave.

3.5 Numerical dissipation around the peak value

The peak amplitude of the transmitted wave over a continuous bathymetry should be the same as C_G . Numerically, this looks right at first glance. However, when we zoom into the plot, we see that the peak value comes very close but doesn't exactly touch the Green's value. Running the simulations again with a finer grid reveals a peak value closer to C_G than the one on the coarse grid as shown in fig. 3.7. This suggests that numerical dissipation is at play here and convergence tests show a first order convergence to C_G . Numerical dissipation becomes more pronounced in Section 6.4 where we explore cases with $h_\ell \gg h_r$.

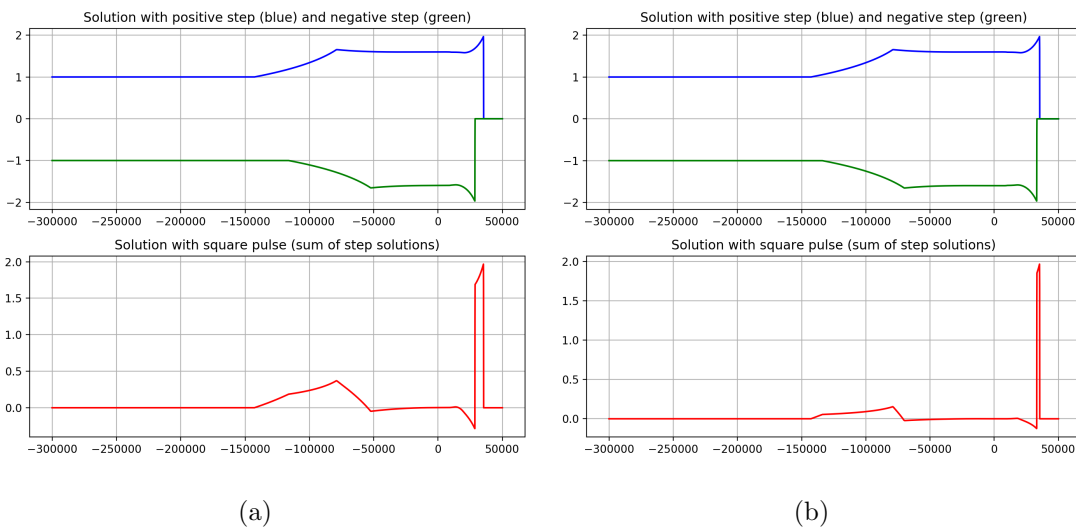


Figure 3.6: The solution to the shoaling problem with square pulse initial data can be found as the linear combination of the single bore solution and a shifted and negated version of the single bore solution. (a) A wider pulse has a larger shift between the two. (b) A narrower pulse has a smaller shift between the two, more cancellation of the reflected waves, and a transmitted pulse that more closely resembles the Green's Law prediction.

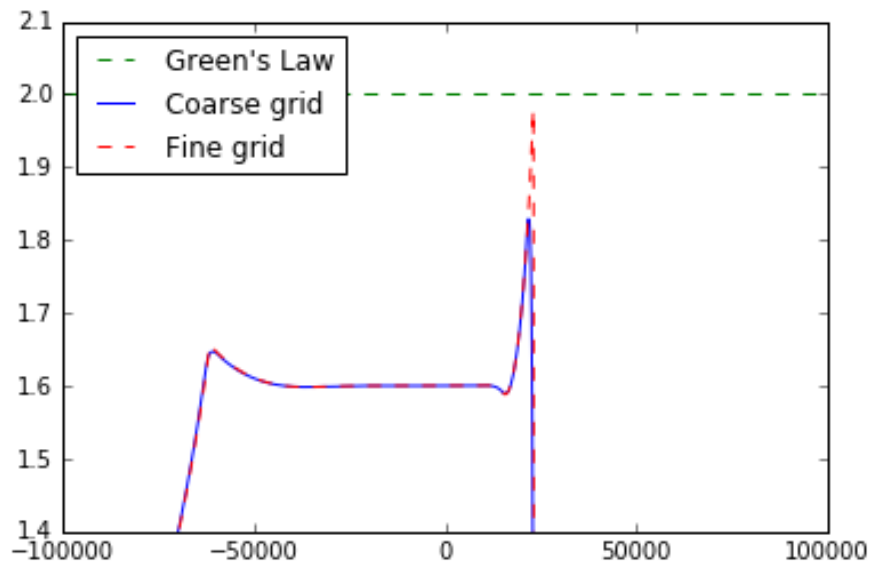


Figure 3.7: The transmitted wave has a very sharp peak because of the slow velocity on the shelf. Numerical dissipation appears and the need to use finer grid might arise.

Chapter 4

THE FIRST-ORDER REFLECTED WAVE AND THE SECOND-ORDER TRANSMITTED WAVE

In the last chapter, we saw that the transmitted wave has a peak amplitude C_G which decays down to an asymptotic amplitude C_T . This chapter builds on that to generate ‘first-order’ expressions for the reflected and transmitted waves beyond the peak and asymptotic values.

4.1 *The reflected wave*

According to Green’s law, there should be no reflection ¹. However, we know that for a sharp jump, the reflection coefficient of there wave is given by C_R

$$C_R = \frac{\sqrt{h_\ell} - \sqrt{h_r}}{\sqrt{h_\ell} + \sqrt{h_r}}$$

Suppose a wave is reflected between two layers of depth h and $h+\Delta h$, its reflection coefficient is given by

$$\frac{\sqrt{h} - \sqrt{h + \Delta h}}{\sqrt{h} + \sqrt{h + \Delta h}} \approx -\frac{1}{4} \frac{\Delta h}{h}$$

As Δh goes to zero, the reflection coefficient becomes zero which means that there would be no reflected wave. However, when Δh is just extremely small, there is a reflected wave with a very tiny amplitude. This wave by itself is insignificant but when added to other miniscule reflected waves creates a noticeable difference. These other waves are waves that get reflected once at other locations. If a wave was reflected three

¹The reflected wave would have an amplitude close to zero. But, we will see in Chapter 7 that the mass of the reflected water is not insignificant.

times at the interfaces between h_1 and $h_1 + \Delta h$, and h_2 and $h_2 + \Delta h$ and h_3 and $h_3 + \Delta h$, its reflection coefficient would be

$$\frac{1}{4^3} \frac{\Delta h^3}{h_1 h_2 h_3} \quad (4.1)$$

This particular wave is at a much lower scale than a wave that is reflected only once. For now, we assume that the combination of the once-reflected waves is much larger than the combination of thrice or five times reflected waves. This is not always true and Chapter 5 explores those cases.

The wave can be reflected at any interface in between h_ℓ and h_r . Suppose the wave starts from h_ℓ , gets transmitted completely until h_n where it gets reflected and finally, it gets transmitted completely back to h_ℓ . We already know the amplitude of a completely transmitted wave between two points in our scenario. Thus, we can write the amplitude of the reflected wave as

$$A_{r_n} = \left(\frac{h_\ell}{h_n}\right)^{\frac{1}{4}} \left(-\frac{1}{4} \frac{\Delta h}{h_n}\right) \left(\frac{h_n}{h_\ell}\right)^{\frac{1}{4}} A_i = -\frac{1}{4} \frac{\Delta h}{h_n} A_i \quad (4.2)$$

where A_i is the amplitude of the incident wave. The incident wave gets amplified as it goes over the continental slope and decays as it moves towards the left after being reflected. The amplification and decay cancel each other out as seen in the equation above.

Adding all the reflected waves up, we have

$$\begin{aligned} \sum_n A_{r_n} &= \lim_{N \rightarrow \infty} \sum_{n=1}^N -\frac{1}{4} \frac{\Delta h}{h_n} A_i \\ &= \lim_{N \rightarrow \infty} \frac{h_r - h_\ell}{N} \sum_{n=1}^N -\frac{A_i}{4h_n} \\ &= -\int_{h_\ell}^{h_r} \frac{A_i}{4x} dx \\ &= \frac{1}{4} \log \left(\frac{h_\ell}{h_r}\right) A_i \end{aligned} \quad (4.3)$$

The reflected wave comes back to the left side and adds on to the incident wave.

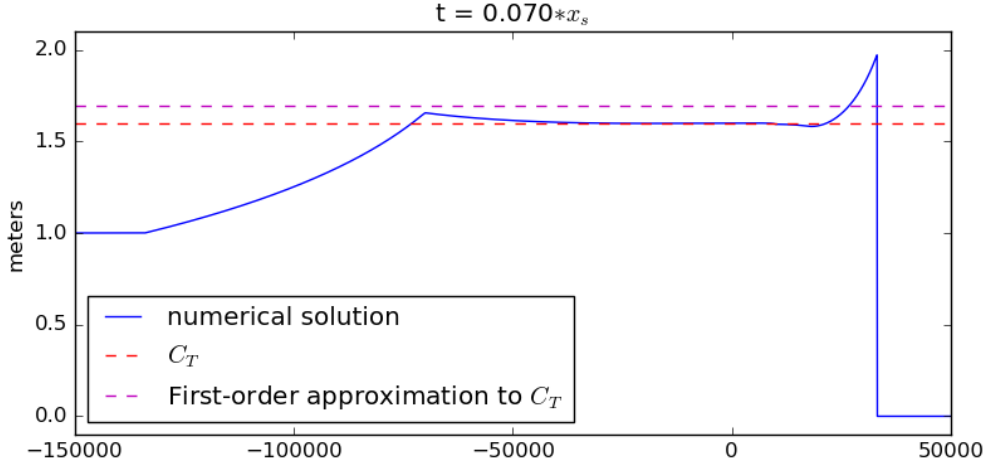


Figure 4.1: Adding up all the waves reflected once in (4.4) gives a ‘first-order’ approximation to C_T

Thus the asymptotic amplitude of the reflected wave is given by

$$\begin{aligned} A_r &= \left(1 + \frac{1}{4} \log \left(\frac{h_\ell}{h_r}\right)\right) A_i \\ &= (1 + \log(C_G)) A_i \end{aligned} \quad (4.4)$$

This ends up being a good ‘first-order’ approximation for the asymptotic amplitude C_T as seen in fig. 4.1. ‘First-order’ is in quotes because here it refers to the contributions from the once-reflected waves. This is not always the largest contribution and section 6.4 takes a look at when that is not the case.

Knowing the leading reflected wave amplitude (zero) and the asymptotic value, finding the curve corresponding to the reflection wave is not too difficult. A wave that is reflected at x_i comes out earlier than a wave that is reflected at x_j if $x_i < x_j$. The total amplitude of all waves reflected at or before x_i would be given by

$$A_{r_i} = \left(1 + \frac{1}{4} \log \left(\frac{h_\ell}{h_i}\right)\right) A_i \quad (4.5)$$

We also need to know far apart these waves are. Waves that get reflected near the left interface x_l would be closer to the leading edge of the reflected wave than the wave that gets reflected near the right interface x_r . The farther waves spend less time within the varying bathymetry. The time lag Δt is given by

$$\Delta t = 2 \int_{x_l}^{x_i} dt = 2 \int_{x_l}^{x_i} \frac{dx}{v(x)} = 2 \int_{x_l}^{x_i} \frac{dx}{\sqrt{gh(x)}}$$

The distance lag Δx or how far from the leading edge, each wave is, is given by

$$\Delta x = \sqrt{gh_\ell} \Delta t = 2\sqrt{h_\ell} \int_{x_l}^{x_i} \frac{dx}{\sqrt{h(x)}}$$

Since we have a linear slope,

$$h(x) = h_\ell + s(x - x_l)$$

where s is the slope given by

$$s = \frac{h_\ell - h_r}{x_l - x_r}$$

$$\Delta t = \frac{4(\sqrt{h_i} - \sqrt{h_\ell})}{s\sqrt{g}}, \Delta x = \frac{4\sqrt{h_\ell}(\sqrt{h_i} - \sqrt{h_\ell})}{s}$$

The expression for the reflected wave is a first order approximation because it only contains information about the waves that are reflected once internally. The expression is exact at the leading edge of the reflected wave and tapers from the actual solution as you go farther from the leading edge.

The next set of reflected waves that go to the left are the waves that have been reflected three times internally. These are waves with negative amplitude since $\Delta h < 0$ in (4.1). So, although the first order approximation is higher than the actual solution, the next set of waves reduce the approximation below the actual solution but closer to it. Eventually, adding the contributions from all the sets of waves, the approximation should converge to the numerical solution.

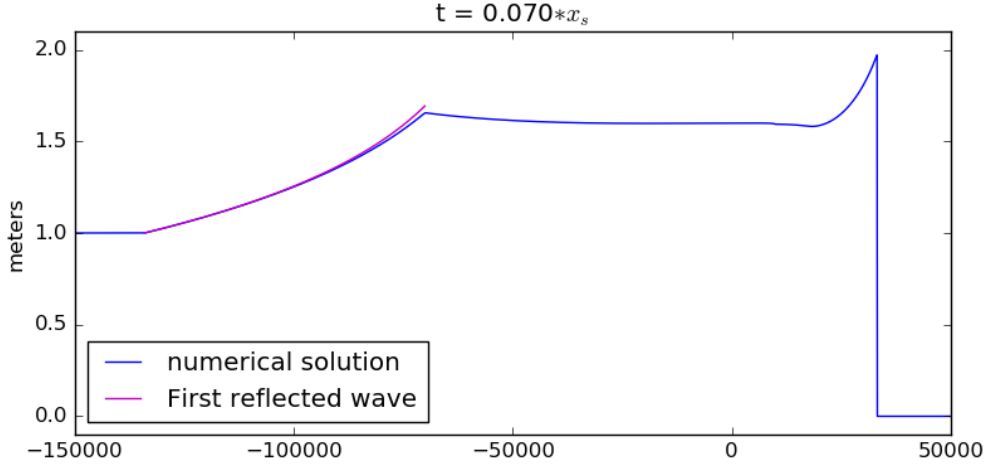


Figure 4.2: The magenta line represents (4.5) which is ‘first-order’ approximation to the reflected wave, exact at the left edge and while tapering up at the right edge as the thrice-reflected waves become prominent.

4.2 The transmitted wave

The very first wave that reaches the right side is the one that is completely transmitted. The next set of waves are those that have been reflected twice internally.

Suppose the wave is reflected first at h_a and then at h_b before coming out. The amplitude of such a wave is given by

$$\left(\frac{h_\ell}{h_a}\right)^{\frac{1}{4}} C_R(h_a) \left(\frac{h_a}{h_b}\right)^{\frac{1}{4}} C_R(h_b) \left(\frac{h_b}{h_r}\right)^{\frac{1}{4}} = C_R(h_a) C_R(h_b) \left(\frac{h_\ell}{h_r}\right)^{\frac{1}{4}}$$

So, $x_b < x_a$. Let the depth difference between h_r and h_ℓ be divided into N differences of magnitude Δh_1 . Similarly, let the depth difference between h_a and h_ℓ be divided into N differences of magnitude Δh_2 . Thus, the amplitude of the wave is

$$-\frac{\Delta h_1}{4h_a} \frac{\Delta h_2}{4h_b} \left(\frac{h_\ell}{h_r}\right)^{\frac{1}{4}} = -\frac{\Delta h_1 \Delta h_2}{4h_a h_b} \left(\frac{h_\ell}{h_r}\right)^{\frac{1}{4}}$$

For a particular h_a , there are N possible h_b values. Adding all these waves, we have

$$-\left(\frac{h_\ell}{h_r}\right)^{\frac{1}{4}} \frac{\Delta h_1}{16h_a} \sum_N \frac{\Delta h_2}{h_b} = -\left(\frac{h_\ell}{h_r}\right)^{\frac{1}{4}} \frac{\Delta h_1}{16h_a} \frac{h_a - h_\ell}{N} \sum_N \frac{1}{h_b}$$

As N goes to infinity, this becomes

$$-\left(\frac{h_\ell}{h_r}\right)^{\frac{1}{4}} \frac{\Delta h_1}{16h_a} \int_{h_\ell}^{h_a} \frac{1}{h} dh = -\left(\frac{h_\ell}{h_r}\right)^{\frac{1}{4}} \frac{\Delta h_1}{16h_a} \log\left(\frac{h_a}{h_\ell}\right)$$

There are also N possible values of h_a . Adding up those waves,

$$-\left(\frac{h_\ell}{h_r}\right)^{\frac{1}{4}} \sum_N \frac{\Delta h_1}{16h_a} \log\left(\frac{h_a}{h_\ell}\right) = -\left(\frac{h_\ell}{h_r}\right)^{\frac{1}{4}} \frac{h_r - h_\ell}{N} \sum_N \frac{1}{16h_a} \log\left(\frac{h_a}{h_\ell}\right)$$

As N goes to infinity, this becomes

$$-\left(\frac{h_\ell}{h_r}\right)^{\frac{1}{4}} \frac{1}{16} \int_{h_\ell}^{h_r} \frac{1}{h_a} \log\left(\frac{h_a}{h_\ell}\right) dh = -\left(\frac{h_\ell}{h_r}\right)^{\frac{1}{4}} \frac{1}{32} \left(\log\left(\frac{h_r}{h_\ell}\right)\right)^2$$

The first transmitted wave has an amplitude given by $\left(\frac{h_\ell}{h_r}\right)^{\frac{1}{4}}$. Adding the amplitudes of the twice reflected waves to it, we get the following second order approximation to the asymptotic transmitted amplitude.

$$A_T = \left(\frac{h_\ell}{h_r}\right)^{\frac{1}{4}} \left(1 - \frac{1}{32} \left(\log\left(\frac{h_r}{h_\ell}\right)\right)^2\right) A_i \quad (4.6)$$

$$A_T = C_G \left(1 - \frac{\log(C_G)^2}{2}\right) A_i \quad (4.7)$$

To go beyond the asymptotic amplitude and find the transmitted wave, we need the time lag or the distance lag of the wave getting reflected between h_a and h_b . The time lag Δt is given by the extra time spent being reflected internally

$$\Delta t = 2 \int_{x_b}^{x_a} dt = 2 \int_{x_b}^{x_a} \frac{dx}{v(x)} = 2 \int_{x_l}^{x_i} \frac{dx}{\sqrt{gh(x)}}$$

If we assume a linear slope, our time and distance lags are given by

$$\Delta t = \frac{4(\sqrt{h_a} - \sqrt{h_b})}{s\sqrt{g}}, \quad \Delta x = \frac{4\sqrt{h_r}(\sqrt{h_a} - \sqrt{h_b})}{s}$$

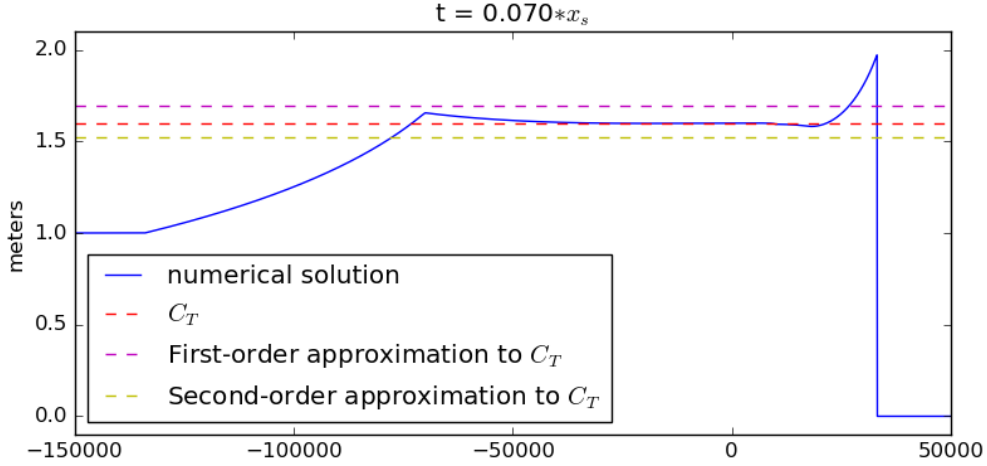


Figure 4.3: Like fig. 4.1, Equation (4.7) gives a ‘second-order’ approximation to C_T . The ‘first-order approximation is obtained by adding up the reflected waves while the ‘second-order’ approximation uses the transmitted waves that have been reflected twice internally.

To find the transmitted wave, we need to find the conditions on the values of h_a and h_b such that they lag behind by a particular distance d .

$$4 \frac{\sqrt{h_r}(\sqrt{h_a} - \sqrt{h_b})}{s} = d$$

$$\sqrt{h_a} - \sqrt{h_b} = \frac{sd}{4\sqrt{h_r}} = k$$

Given the first reflection happens at h_a , we have the following condition on h_b so that the waves that emerge lag only by a maximum of length d .

$$\sqrt{h_a} - k = \sqrt{h_b}$$

$$h_b = (\sqrt{h_a} - k)^2$$

Adding all the waves that are reflected first at h_a and then at or before h_b , we have

$$-\frac{\Delta h_1}{16h_a} \int_{(\sqrt{h_a}-k)^2}^{h_a} \frac{1}{h} dh = -\frac{\Delta h_1}{16h_a} \log \left(\frac{h_a}{(\sqrt{h_a} - k)^2} \right)$$

Now, h_a can be anywhere between $(\sqrt{h_\ell} + k)^2$ and h_r . But, between h_ℓ and $(\sqrt{h_\ell} + k)^2$, all the waves reflected twice come out with a distance lag less than or equal to d . So, we need to add a correction term by adding up all the waves coming from that.

$$\begin{aligned} & -\frac{1}{16} \int_{(\sqrt{h_\ell}+k)^2}^{h_r} \frac{1}{h_a} \log \left(\frac{h_a}{(\sqrt{h_a} - k)^2} \right) dh - \frac{1}{16} \int_{h_\ell}^{(\sqrt{h_\ell}+k)^2} \frac{1}{h_a} \log \left(\frac{h_a}{h_\ell} \right) dh \\ & = -\frac{1}{16} \int_{(\sqrt{h_\ell}+k)^2}^{h_r} \frac{1}{h_a} \log \left(\frac{h_a}{(\sqrt{h_a} - k)^2} \right) dh - \frac{1}{32} \left(\log \left(\frac{(\sqrt{h_\ell} + k)^2}{h_\ell} \right) \right)^2 \end{aligned}$$

And thus, we have the transmitted wave.

$$A_T(k) = \left(\frac{h_\ell}{h_r} \right)^{\frac{1}{4}} \left(1 - \frac{1}{16} \int_{(\sqrt{h_\ell}+k)^2}^{h_r} \frac{1}{h_a} \log \left(\frac{h_a}{(\sqrt{h_a} - k)^2} \right) dh - \frac{1}{32} \left(\log \left(\frac{(\sqrt{h_\ell} + k)^2}{h_\ell} \right) \right)^2 \right) \quad (4.8)$$

This is a second order approximation to the transmitted wave since it has the first transmitted wave and the twice reflected waves. It is exact at the peak value and tapers below the actual solution as seen in fig. 4.4.

4.3 Higher-order approximations

The transmitted wave in (4.8) is obtained by using only the waves that have been reflected twice. Incorporating the effects of the waves that have been reflected thrice or more would result in better approximations for the transmitted and reflected waves. The techniques utilized in this chapter uncovered sequences for the amplitude of waves and corresponding ones for the time lag. This can be extended to obtain the higher-order effects but the derivations become quite tedious. Chapter 5 explores an alternate method that parameterizes the amplitude in time and makes it easier to extend to higher-order waves.

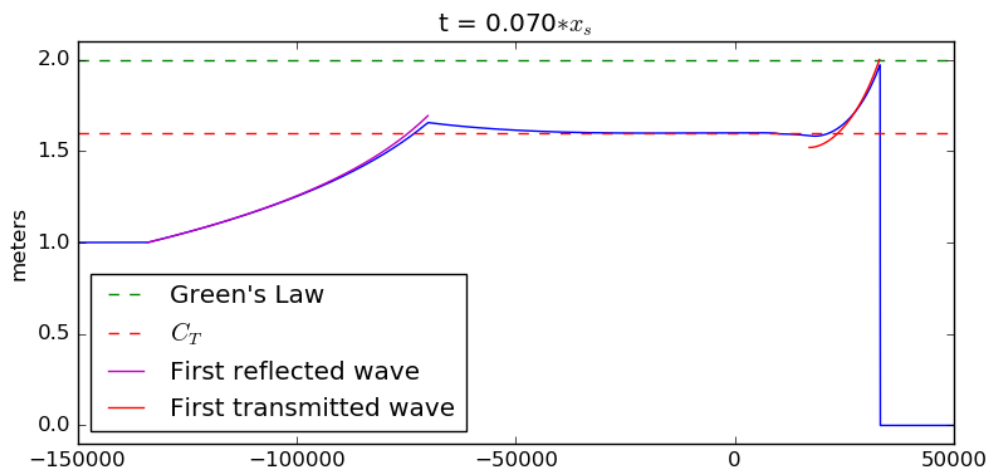


Figure 4.4: (4.8) provides a ‘second-order’ approximation to the transmitted wave. This does not match as well as our approximation for the reflected wave because of the dissipation around the peak value and due to the shorter lag of the four times-reflected waves on the right side of the shelf.

Chapter 5

A DIFFERENTIAL FORM

In Chapter 3 and Chapter 4, a layered media approach allowed us to see the connection between Green’s law and the transmission coefficient. It also allowed us to approximate the solution by considering the effects of the once-reflected and twice-reflected waves as asymptotic sums. This technique gets more complicated for thrice-reflected wave. So instead, we consider the continuous limit and derive ODEs for the evolution of the solution based on the differential form of the asymptotic sums.

In section 5.2 we discuss the first transmitted wave and show that its amplitude is always $C_G A$ for any variation in $h(x)$ other than a single discontinuity. In sections 5.3 and 5.4 we discuss waves that reflect 1 or 2 times respectively and derive expression which agree with the approximations to the transmitted and reflected waves in Chapter 4.

5.1 Characteristics and infinitesimal jumps

Consider a continental slope, now extending from $x_\ell = 0$ to $x_r > 0$, over which the depth $h(x)$ varies from h_ℓ to h_r . At each point x the characteristic (gravity wave) speed is $\sqrt{gh(x)}$, and we can define characteristic curves by solving the ODE

$$X'(t) = \sqrt{gh(X(t))}, \quad X(0) = 0. \quad (5.1)$$

This defines one characteristic curve that can be viewed as the path taken by a jump discontinuity moving from left to right that hits $x = 0$ at time $t = 0$ and then propagates up the slope to the continental shelf, the constant depth region that starts at x_r (with $h(x) \equiv h_r$ for $x > x_r$). This is what we call the “first transmitted wave” that suffers no reflection but grows with amplitude according to Green’s law, as will be derived in the

next section. This wave reaches x_r at some time t_r defined by solving $X(t_r) = x_r$. For the special case of a linear slope $h(x) = h_\ell + sx$ with $s = (h_r - h_\ell)/x_r < 0$, the ODE (5.1) can be solved explicitly, yielding

$$X(t) = \frac{1}{s} \left(\frac{1}{g} \left(\sqrt{gh_\ell} - \frac{1}{2}gst \right)^2 - h_\ell \right) \quad (5.2)$$

and solving for t_r gives

$$t_r = \frac{2}{gs} \left(\sqrt{gh_r} - \sqrt{gh_\ell} \right). \quad (5.3)$$

The family of all right-going characteristics can be obtained by shifting the curve $X(t)$ up or down in the $x-t$ plane, since the characteristic speed varies only with x . These curves satisfy $x = X(t - \tau)$ for some arbitrary fixed value of τ .

There is another family of left-going characteristics corresponding to the speed $-\sqrt{gh}$ of left-going gravity waves. Because this is simply the negative of the right-going speed at each point, these characteristics are obtained by reflecting the right-going characteristics about some line of constant t , and satisfy $x = X(\tau - t)$ for some arbitrary τ . Figure 5.1 shows some of these characteristic curves.

At any jump discontinuity in $h(x)$, a finite jump in η propagating along a right-going characteristic is partially reflected, and the reflection then propagates on a left-going characteristic. Where $h(x)$ is smoothly varying there is continuous reflection occurring and its magnitude can be approximated by locally approximating $h(x)$ by a step function with a small jump Δh over a small distance Δx . Such approximations are used below to derive ODEs for the variation in the reflected wave from a smooth slope, where Taylor expansion can be used to calculate that a small jump in h representing the variation over some distance Δx leads to a reflection coefficient

$$C_R(x; \Delta x) \approx \frac{\sqrt{h(x + \Delta x)} - \sqrt{h(x)}}{\sqrt{h(x + \Delta x)} + \sqrt{h(x)}} \approx -\frac{1}{4} \frac{h'(x)}{h(x)} \Delta x. \quad (5.4)$$

Part of the energy is also transmitted, with

$$C_T(x; \Delta x) = C_R(x; \Delta x) + 1 \approx 1 - \frac{1}{4} \frac{h'(x)}{h(x)} \Delta x. \quad (5.5)$$

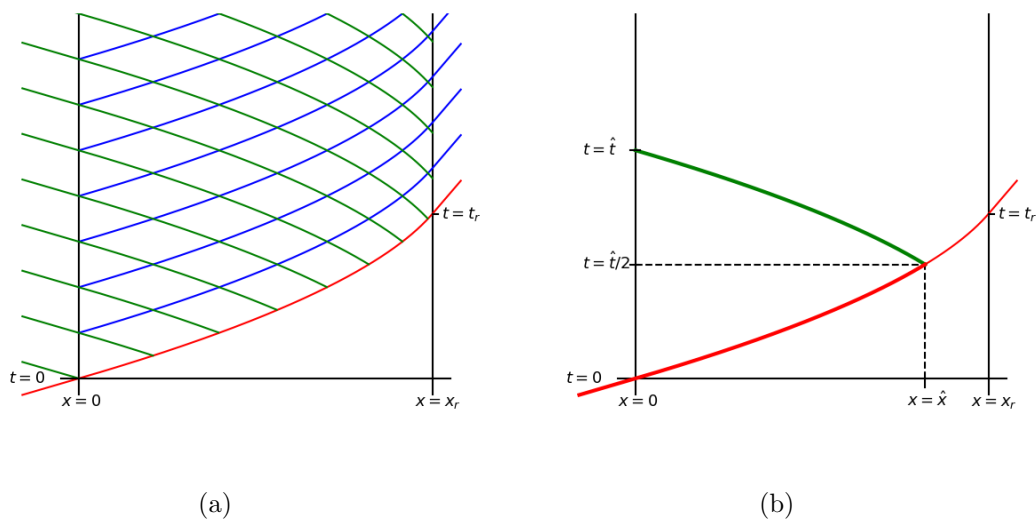


Figure 5.1: (a) Several right-going (blue) and left-going (green) characteristics in the $x-t$ plane. The red characteristic is the curve $x = X(t)$ along which the “first transmitted wave” propagates as discussed in section 5.2, reaching $x = x_r$ at time $t = t_r$. (b) Notation for section 5.3. The portion of the wave reflected at time $\hat{t}/2$ follows the green characteristic and exits at time \hat{t} .

5.2 The first transmitted wave

We now consider the first transmitted wave that suffers no internal reflection. Suppose $h(x)$ varies smoothly from h_ℓ to h_r as x varies from $x_\ell = 0$ to x_r and that an initial right-going wave with amplitude $\Delta\eta = A$ grows as it moves along the characteristic $x = X(t)$. Let $A(x)$ denote its amplitude when it has reached $x \leq x_r$. Based on the transmission coefficient $C_T(x; \Delta x)$ of (5.5), we can approximate

$$A(x + \Delta x) \approx \left(1 - \frac{1}{4} \frac{h'(x)}{h(x)} \Delta x\right) A(x) \quad (5.6)$$

and hence

$$(A(x + \Delta x) - A(x))/\Delta x \approx -\frac{1}{4} \frac{h'(x)}{h(x)} A(x). \quad (5.7)$$

Letting $\Delta x \rightarrow 0$ and dividing by $A(x)$ gives

$$A'(x)/A(x) = -\frac{1}{4} h'(x)/h(x). \quad (5.8)$$

Integrating this from $x = 0$ to the location of interest gives Green's Law:

$$A(x) = (h(x_\ell)/h(x))^{1/4} A. \quad (5.9)$$

In particular, at $x = x_r$ we recover the amplification factor C_G of (1.3). Note that this shows that the jump across the first transmitted wave is always $C_G A$ for any smoothly varying $h(x)$, and depends only on the values h_ℓ and h_r , not the shape of the variation.

We note that this same approach to modeling the evolution of $A(x)$ can also be applied to other wave equations; for example Madsen and Sorensen [8] apply to this a dispersive Boussinesq equation, in which case the factor $1/4$ appearing in (5.8) is replaced by a factor that varies with depth.

5.3 The once-reflected waves

Now consider waves that result from a single reflection. If the depth $h(x)$ varies continuously then the reflection at any particular x is infinitesimally small, but again we can

discretize locally and consider the waves reflecting over a small discrete layer. Instead of parameterizing by x , as we did to compute the transmitted wave, it is now more convenient to parameterize by \hat{t} , the time at which the reflected wave leaves the domain, as indicated in fig. 5.1(b). Note that the wave leaving at this time must have reflected at time $\hat{t}/2$ since the wave speeds (and hence travel times) are the same in each direction. We will derive an ODE for how the total cumulative reflected wave, denoted by $R_1(\hat{t})$, varies for $0 \leq t \leq 2t_r$, the time when the last once-reflected wave leaves the domain. The change from $R_1(\hat{t})$ to $R_1(\hat{t} + \Delta\hat{t})$ can be interpreted as the magnitude of the reflection from an interface with a jump in h from $h(\hat{x})$ to $h(\hat{x} + \Delta\hat{x})$. Here $\hat{x} = X(\hat{t}/2)$ and $\Delta\hat{x} \approx \sqrt{gh(\hat{x})}(\Delta\hat{t}/2)$. The factor $\Delta\hat{t}/2$ appears instead of $\Delta\hat{t}$ because over time $\Delta\hat{t}$ the wave then has time to travel back and forth through this layer of width $\Delta\hat{x}$, as now required.

The reflection coefficient for this hypothetical thin layer is thus

$$C_R(\hat{t}; \Delta\hat{t}) \approx -\frac{1}{8} \frac{h'(\hat{x})}{h(\hat{x})} \sqrt{gh(\hat{x})} \Delta\hat{t}. \quad (5.10)$$

The wave that is reflecting at this point is the initial wave of amplitude A that has been amplified due to Green's Law as it propagated up to \hat{x} , and hence it has amplitude $(h_\ell/h(\hat{x}))^{1/4}A$ at the time of reflection. This is multiplied by $C_R(\hat{t}; \Delta\hat{t})$ to obtain the magnitude of the reflected wave, which then propagates leftward for time $\Delta\hat{t}/2$ before reaching $x = 0$. During this leftward propagation it decays, it decays by a factor $(h_\ell/h(\hat{x}))^{-1/4}$, the inverse of the amplification factor during its rightward propagation. These factors thus cancel out like in (4.2), and it is only the reflection coefficient that comes into the expression for the growth of R_1 :

$$R_1(\hat{t} + \Delta\hat{t}) - R_1(\hat{t}) \approx C_R(\hat{t}; \Delta\hat{t})A. \quad (5.11)$$

Using (5.10) and the expression thus leads to the ODE

$$R_1'(\hat{t}) = -\frac{1}{8} \frac{h'(\hat{x})}{h(\hat{x})} \sqrt{gh(\hat{x})}A, \quad R_1(0) = 0. \quad (5.12)$$

Since $\hat{x} = X(\hat{t}/2)$, this can be solved simply by integration in \hat{t} . However, in computing the integral it is then convenient to make the change of variables $\hat{x} = X(\hat{t}/2)$ so that $d\hat{x} = \frac{1}{2}X'(\hat{t})d\hat{t} = \frac{1}{2}\sqrt{gh(X(\hat{t}))}d\hat{t}$ and integrating up to t then yields the following integral in \hat{x} :

$$\begin{aligned} R_1(t) &= -\frac{A}{4} \int_0^{X(t/2)} \frac{h'(\hat{x})}{h(\hat{x})} d\hat{x} \\ &= \frac{A}{4} \log(h_\ell/h(X(t/2))). \end{aligned} \quad (5.13)$$

In particular, at the time $t = 2t_r$ when the last singly-reflected wave arrives at $x = 0$, we have

$$R_1(2t_r) = \log((h_\ell/h_r)^{1/4}) A = \log(C_G)A \quad (5.14)$$

where C_G is the Green's Law amplification factor. The result here ends up being the same as derived in (4.4). The factor $\log(C_G)$ is larger than the reflection coefficient C_R from (1.5) (when $h_\ell > h_r$), e.g., for the case $h_\ell = 3200, h_r = 200$, we find $C_R = 0.6$ while $\log(C_G) = \log(2) \approx 0.69$. The waves arising from multiple internal reflections, which we ignored in calculating R_1 , reduce the amplitude so that asymptotically as $t \rightarrow \infty$ the reflected wave from a smooth slope matches the amplitude predicted by C_R for a discontinuous slope.

We have calculated the variation in $R_1(t)$ with time, the magnitude of the singly-reflected waves as they cross $x = 0$. The corresponding spatial variation as the reflection continues to move away from the slope is easy to determine from this, since the wave velocity $-\sqrt{gh_\ell}$ is constant for $x < 0$. We can thus approximate

$$\eta(x, t) \approx A + R_1(t + x/\sqrt{gh_\ell}) \quad (5.15)$$

for $-t\sqrt{gh_\ell} < x < 0$ and $t > 0$. In fig. 5.2 we plot this for the case of the linear slope, and compare this curve to the numerical solution from fig. 3.5. Of course the numerical solution also contains higher order reflections and was computed on a very fine grid, so this is a good approximation to the true solution. We see that the approximation to $\eta(x, t)$ from (5.15), obtained with only singly-reflected waves, agrees very well initially

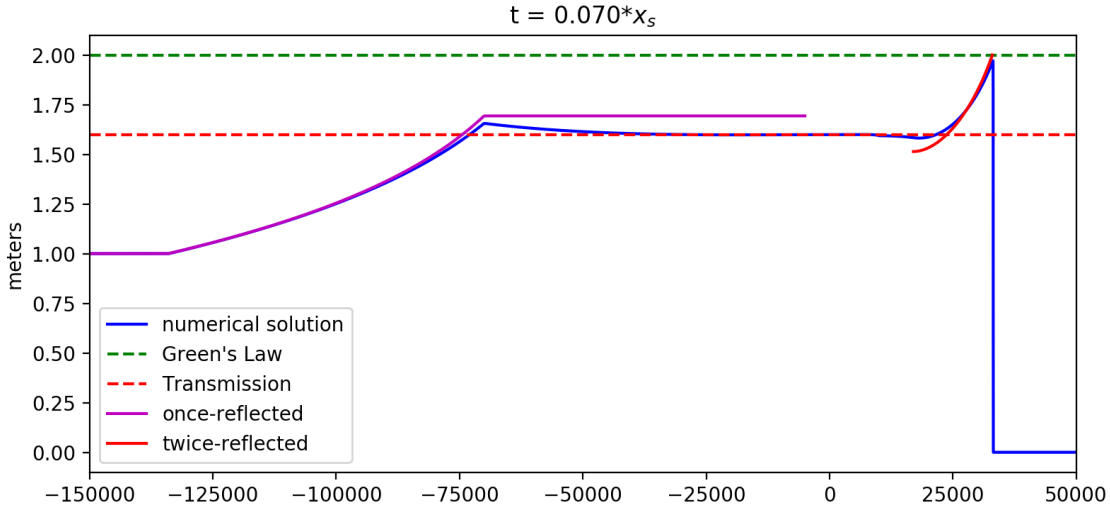


Figure 5.2: The numerical solution from fig. 3.5 is shown together with the curves obtained by considering only once-reflected or twice-reflected waves, as derived in sections 5.3 and 5.4. Since these expressions ignore waves with more reflections, they match the early portion of the outgoing waves best.

and starts to deviate for larger x since at later times the multiply-reflected waves begin to play more of a role. These eventually reduce the amplitude back to $C_R A$, the value predicted by the reflection coefficient for a discontinuous continental slope.

5.4 The twice-reflected waves

Now consider waves that reflect twice internally and exit as right-going waves at x_r . Again it is best to parameterize by the exit time. The first transmitted wave reaches x_r at time t_r and the last twice-reflected wave arrives at time $3t_r$, so consider time $t_r + \hat{t}$ for $0 < \hat{t} \leq 2t_r$. A wave that exits at this time must follow the characteristic curve $x = X(t)$ from time 0 up to some time t_1 , reflect at $x_1 = X(t_1)$ and then follow the left-going characteristic $x = X(2t_1 - t)$ until time $t_2 = t_1 + \hat{t}/2$ where it reflects a second time at $x_2 = X(2t_1 - t_2) = X(t_1 - \hat{t}/2)$ and then follows the right-going characteristic

$x = X(t - \hat{t})$ to its exit, reaching x_r at time $t_r + \hat{t}$. Note that the wave spends time $\hat{t}/2$ going backwards from x_1 to x_2 and another $\hat{t}/2$ going forwards to reach x_1 again, so that it is delayed by total time \hat{t} . In order for the second reflection point x_2 to lie in the region where $h(x)$ is varying, the first reflection time t_1 must satisfy $\hat{t}/2 \leq t_1 \leq t_r$. To determine the twice-transmitted waves departing at $t_r + \hat{t}$ we must integrate all contributions in this range. A second integration in \hat{t} will be required to determine the solution at x_r at time $t > t_r$, since we will first derive an equation for the variation in this as \hat{t} varies.

Recall from section 5.2 that the initial wave amplitude $A = \Delta\eta$ increases by $(h_\ell/h(x_1))^{1/4}$ as it propagates rightward from 0 to x_1 . Using the same formulas show that any reflected wave likewise decreases by $(h(x_2)/h(x_1))^{1/4}$ as it propagates leftward from x_1 to the second reflector x_2 , but then the reflected portion of this wave increases again by $(h(x_1)/h(x_2))^{1/4}$ as it propagates back to x_1 . These two factors cancel out, and so after the final propagation from x_1 to x_r the net growth of the wave due to all transmissions is simply $(h_\ell/h_r)^{1/4}$, the Green's Law factor, regardless of the location of the reflectors. In fact this is also true for waves with more internal reflections. However, each reflection also requires scaling the amplitude by an appropriate reflection coefficient. It is the product of these factors that causes rapid decay in the amplitude of multiply-reflected waves.

For the twice-reflected waves that reach x_r at time $t_r + \hat{t}$, we must multiply the transmission amplitude $(h_\ell/h_r)^{1/4}A$ by two reflection coefficients,

$$C_R(x_1; \Delta x_1) \approx -\frac{1}{4} \frac{h'(x_1)}{h(x_1)} \Delta x_1 \quad (5.16)$$

for the first reflection and

$$C_R(x_2; \Delta x_2) \approx +\frac{1}{4} \frac{h'(x_2)}{h(x_2)} \Delta x_2 \quad (5.17)$$

for the second. Note that the second factor has a positive sign since at this point a left-going wave reflects, and switching the states causes a negation of the reflection coefficient, as seen from (1.5). We must define suitable distances Δx_1 and Δx_2 in order

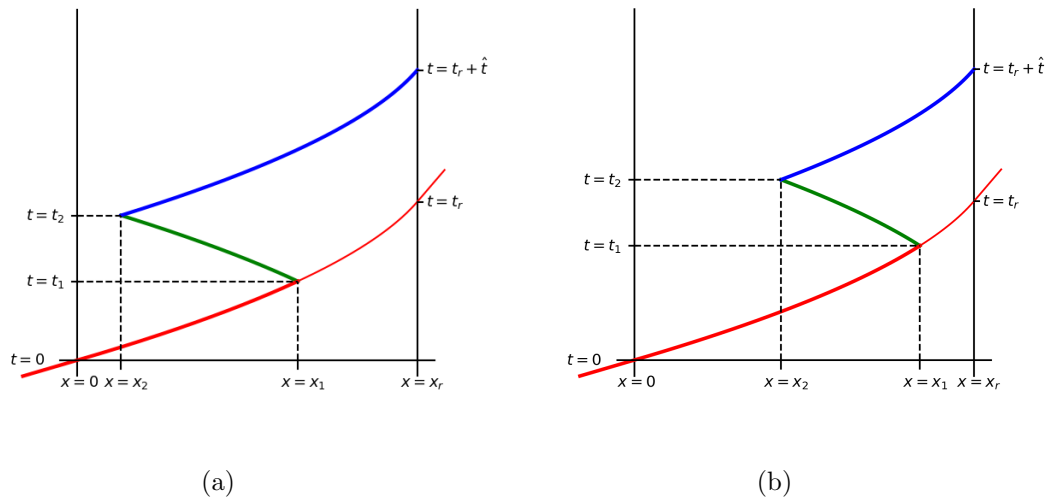


Figure 5.3: Notation for section 5.4. The portion of the wave first reflected at time t_1 follows the green characteristic and is then reflected a second time at $t = t_2$, exiting at time $t_r + \hat{t}$. Figures (a) and (b) show the same choice of $\hat{t} \in [0, 2t_r]$, but two different choices of first reflection time $t_1 \in [\hat{t}/2, t_r]$. In each case, $x_1 = X(t_1)$, $t_2 = t_1 + \hat{t}/2$, and $x_2 = X(t_1 - \hat{t}/2)$.

to fully define these reflection coefficients. These are related to discretizations of t_1 and \hat{t} that are used to define the two integrals that must be performed.

As noted above, we must integrate the reflections over t_1 from $\hat{t}/2$ to t_r in order to capture all possible locations of the first reflector. Discretize this time period into increments Δt_1 . Then this induces a spatial discretization with

$$\Delta x_1 = X(t_1 + \Delta t_1) - X(t_1) \approx X'(t_1)\Delta t_1 = \sqrt{gh(x_1)} \Delta t_1. \quad (5.18)$$

We also need to discretize the output times $t_r + \hat{t}$ using some $\Delta \hat{t}$. If we fix $x_1 = X(t_1)$ and let $\delta T_2(t_r + \hat{t}; t_1)$ represent the incremental effect of twice-reflected waves arriving at time $t_r + \hat{t}$ from reflections at time t_1 , then we want to write

$$\delta T_2(t_r + \hat{t} + \Delta \hat{t}; t_1) - \delta T_2(t_r + \hat{t}; t_1) \approx C_R(x_1; \Delta x_1)C_R(x_2; \Delta x_2)(h_\ell/h_r)^{1/4} A. \quad (5.19)$$

Since $x_2 = X(t_1 - \hat{t}/2)$, in this context it makes sense to set

$$\Delta x_2 \approx |X(t_1 - (\hat{t} + \Delta \hat{t})/2) - X(t_1 - \hat{t}/2)| \approx X'(t_2)(\Delta \hat{t}/2) = \frac{1}{2}\sqrt{gh(x_2)} \Delta \hat{t}. \quad (5.20)$$

Using these values with the expressions (5.16) and (5.17) then gives

$$\begin{aligned} \delta T_2(t_r + \hat{t} + \Delta \hat{t}; t_1) - \delta T_2(t_r + \hat{t}; t_1) & \\ & \approx -\frac{1}{16} \left(\frac{h'(x_1)}{h(x_1)} \right) \sqrt{gh(x_1)} \Delta t_1 \left(\frac{h'(x_2)}{h(x_2)} \right) \sqrt{gh(x_2)} (\Delta \hat{t}/2) (h_\ell/h_r)^{1/4} A \\ & \approx -\frac{g}{32} (h_\ell/h_r)^{1/4} A \left(\frac{h'(x_1)h'(x_2)}{(h(x_1)h(x_2))^{1/2}} \right) \Delta t_1 \Delta \hat{t}. \end{aligned} \quad (5.21)$$

Recall that in these expressions we use the shorthand $x_1 = X(t_1)$ and $x_2 = X(t_1 - \hat{t}/2)$.

Integrating over the time t_1 of the first reflection then gives

$$\begin{aligned} T_2(t_r + \hat{t} + \Delta \hat{t}) - T_2(t_r + \hat{t}) & \\ & \approx -\frac{g}{32} (h_\ell/h_r)^{1/4} A \int_{\hat{t}/2}^{t_r} \left(\frac{h'(X(t_1))h'(X(t_1 - \hat{t}/2))}{(h(X(t_1))h(X(t_1 - \hat{t}/2)))^{1/2}} \right) dt_1 \Delta \hat{t} \end{aligned} \quad (5.22)$$

and finally integrating in \hat{t} from 0 to time $t-t_r$ gives an expression for the second-reflected waves valid for all times $t_r < t < 3t_r$:

$$T_2(t) = -\frac{g}{32}(h_\ell/h_r)^{1/4}A \int_0^{t-t_r} \int_{\hat{t}/2}^{t_r} \left(\frac{h'(X(t_1))h'(X(t_1 - \hat{t}/2))}{[h(X(t_1))h(X(t_1 - \hat{t}/2))]^{1/2}} \right) dt_1 d\hat{t}. \quad (5.23)$$

Finally, this gives an approximation to the solution $\eta(x_r, t)$ over this time range as

$$\eta(x_r, t) \approx (h_\ell/h_r)^{1/4}A + T_2(t), \quad (5.24)$$

using only the initially transmitted wave and the twice-reflected waves. Since the wave speed $\sqrt{gh_r}$ is constant for $x > x_r$, this expression at x_r can be converted into an approximate wave form in space for the wave that propagates onto the shelf. Figure 5.2 shows the resulting curve along with the numerical solution from fig. 3.5.

5.5 Waves that are reflected thrice or more

In this chapter, the effects of successive waves were expressed in their differential form. Using the characteristics, we were also able to parameterize the amplitude of the waves in time. This allowed us to obtain a differential equation which could be integrated to reveal the reflected or transmitted wave like in (5.13) and (5.23).

This technique can be extended to find the effects from the waves that have been reflected thrice or more internally. We do not need to derive the differential equation each time. We can go straight from the time parameterization and the characteristics to the integral that shows the effects of the multiply-reflected waves. This is shown in good detail in [3].

Chapter 6

HIGHER ORDER TERMS AND THE COMPLETE ASYMPTOTIC EXPANSION

In the last chapter, we explored the contributions from waves that were reflected once and reflected twice. This allowed us to derive a first-order approximation to the transmitted wave and a second-order approximation to the reflected wave.

This chapter looks at the contribution of the waves that are reflected more than twice and showcases the method to derive the complete asymptotic expansion.

6.1 *The thrice-reflected wave*

In Section 4.1, we found the asymptotic sum of the first-reflected waves by adding up the amplitudes at various locations of h_n . In Section 4.2, we did the same thing for the twice-reflected waves by adding up all the possible values of h_b first and then adding up all the possible values of h_a . If we choose h_a to lie between h_ℓ and h_r , we have a physical constraint $h_a > h_b$ that forces h_b to lie between h_a and h_ℓ .

This constraint becomes very important when considering the asymptotic sum of the thrice reflected waves. For waves that are reflected three times, if h_1, h_2 and h_3 represent the heights at the first, second and third points of reflection, there would be the following constraints.

$$\begin{aligned} h_1 &\in (h_2, h_r) \\ h_3 &\in (h_2, h_r) \end{aligned} \tag{6.1}$$

Here, we can see that the range of h_1 and h_3 is determined by h_2 . The amplitude of a thrice-reflected wave would be given by

$$A_{R3i} = -\frac{\Delta h_1 \Delta h_2 \Delta h_3}{4^3 h_1 h_2 h_3} A_i \tag{6.2}$$

h_1 and h_3 lie between h_2 and h_r . So, it is prudent to add either of them first. For the sake of consistency with future expressions, we start with h_1 .

$$\begin{aligned}\sum_{h_1} A_{R3i} &= - \sum_{h_1} \frac{\Delta h_1 \Delta h_2 \Delta h_3}{4^3 h_1 h_2 h_3} A_i \\ &= - \frac{\Delta h_2 \Delta h_3}{4^3 h_2 h_3} \int_{h_2}^{h_r} \frac{dh_1}{h_1} A_i\end{aligned}\tag{6.3}$$

Then, h_3

$$\begin{aligned}\sum_{h_3} \sum_{h_1} A_{R3i} &= - \sum_{h_3} \sum_{h_1} \frac{\Delta h_1 \Delta h_2 \Delta h_3}{4^3 h_1 h_2 h_3} A_i \\ &= - \frac{\Delta h_2}{4^3 h_2} \int_{h_2}^{h_r} \frac{1}{h_3} \int_{h_2}^{h_r} \frac{dh_1 dh_3}{h_1} A_i\end{aligned}\tag{6.4}$$

Lastly, since the constraints have been imposed on h_1 and h_3 , we can add up all the possible values of h_2 between h_ℓ and h_r .

$$\begin{aligned}\sum_{h_2} \sum_{h_3} \sum_{h_1} A_{R3i} &= - \sum_{h_2} \sum_{h_3} \sum_{h_1} \frac{\Delta h_1 \Delta h_2 \Delta h_3}{4^3 h_1 h_2 h_3} A_i \\ &= - \frac{1}{4^3} \int_{h_\ell}^{h_r} \frac{1}{h_2} \int_{h_2}^{h_r} \frac{1}{h_3} \int_{h_2}^{h_r} \frac{dh_1 dh_3 dh_2}{h_1} A_i\end{aligned}\tag{6.5}$$

Thus, the total contribution of the thrice-reflected waves can be expressed in terms of a transmission coefficient R_3 .

$$\begin{aligned}R_3 &= - \frac{1}{4^3} \int_{h_\ell}^{h_r} \frac{1}{h_2} \int_{h_2}^{h_r} \frac{1}{h_3} \int_{h_2}^{h_r} \frac{dh_1 dh_3 dh_2}{h_1} A_i \\ &= - \frac{1}{4^3} \int_{h_\ell}^{h_r} \int_{h_2}^{h_r} \int_{h_2}^{h_r} \frac{dh_1 dh_3 dh_2}{h_2 h_3 h_1} \\ &= \frac{1}{3} \frac{1}{4^3} \log \left(\left(\frac{h_\ell}{h_r} \right)^3 \right) = \frac{\log(C_G)^3}{3}\end{aligned}\tag{6.6}$$

6.2 The complete asymptotic expansion

For waves that are reflected four times, there would be the following constraints.

$$\begin{aligned} h_1 &\in (h_2, h_r) \\ h_3 &\in (h_2, h_r) \\ h_4 &\in (h_\ell, h_3) \end{aligned} \tag{6.7}$$

$$T_4 = -\frac{1}{4^4} \left(\frac{h_\ell}{h_r}\right)^{\frac{1}{4}} \int_{h_\ell}^{h_r} \int_{h_2}^{h_r} \int_{h_\ell}^{h_3} \int_{h_2}^{h_r} \frac{dh_1 dh_4 dh_3 dh_2}{h_2 h_3 h_4 h_1} = \frac{5 \log(C_G)^4}{24} \left(\frac{h_\ell}{h_r}\right)^{\frac{1}{4}} \tag{6.8}$$

By now, we can see that when we add the constraint $h_5 \in (h_4, h_r)$ for the waves reflected five times, we need to add the reflections from possible locations of h_5 before those from h_4 and after h_1 .

$$R_5 = -\frac{1}{4^5} \int_{h_\ell}^{h_r} \int_{h_2}^{h_r} \int_{h_\ell}^{h_3} \int_{h_4}^{h_r} \int_{h_2}^{h_r} \frac{dh_1 dh_5 dh_4 dh_3 dh_2}{h_2 h_3 h_4 h_5 h_1} = \frac{2 \log(C_G)^5}{15} \tag{6.9}$$

All the reflected waves should add up to the transmission coefficient. Figure 6.1 shows even the third order approximation converging quickly to C_T .

$$\begin{aligned} C_T &= 1 + R_1 + R_3 + R_5 + \dots \\ &= 1 + \log(C_G) - \frac{\log(C_G)^3}{3} + \frac{2 \log(C_G)^5}{15} + \dots \end{aligned} \tag{6.10}$$

Adding up all the transmitted waves gives us an alternate asymptotic expansion and another visual illustration in Figure 6.2.

$$\begin{aligned} C_T &= C_G + T_2 + T_4 + T_6 + \dots \\ &= C_G \left(1 - \frac{\log(C_G)^2}{2} + \frac{5 \log(C_G)^4}{24} + \dots \right) \end{aligned} \tag{6.11}$$

Since the integral is in h , because of the constraints, there was thought to be a need for monotonicity of h . But since h can be parametrized by x which is monotonic and

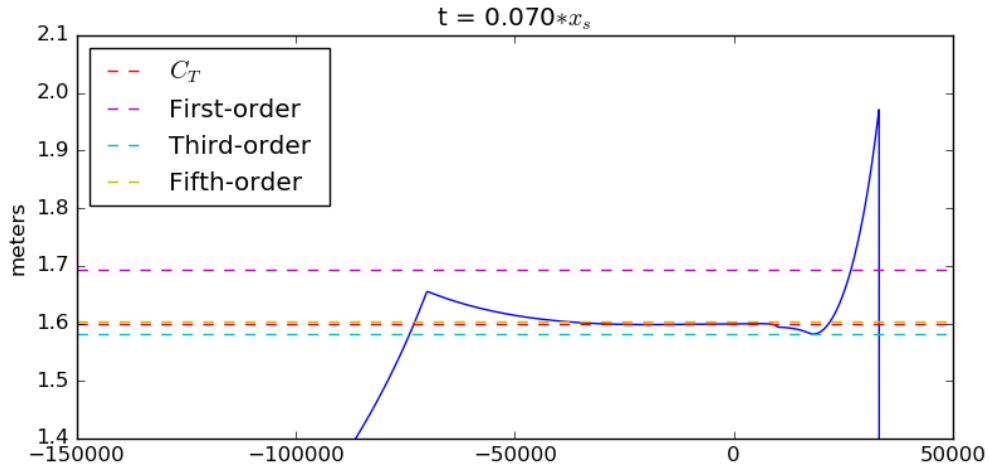


Figure 6.1: We see the asymptotic contributions from the reflected waves converge quickly to C_R , so that the state behind the reflected wave converges to $C_T = 1 + C_R$.

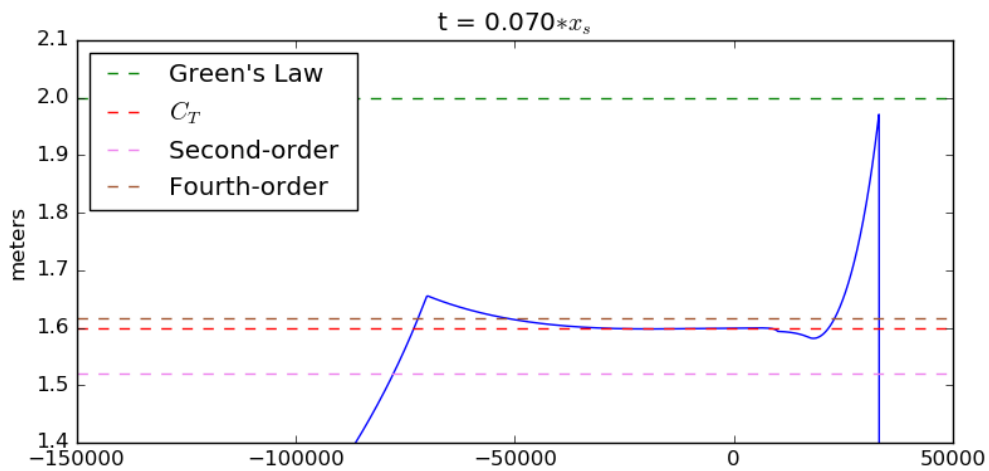


Figure 6.2: We see the asymptotic contributions from the transmitted waves converge quickly to C_T . Note that C_G is the zeroth order approximation.

since the values of the integrals only depend on the end points, we obtain the same asymptotic results regardless of the shape of the bathymetry.

To find the complete asymptotic effect of the n th reflected waves, we would take the corresponding integral to it in the form similar to (6.8) or (6.9) and use mathematica or some symbolic calculator to solve it. There is a simpler way to calculate the terms.

6.3 A generating function for the coefficients

David Ketcheson pointed out that it is possible to express the transmitted and reflected waves as

$$R_n = i^{n+1} a_n \frac{1}{4^n} (\log(h_\ell/h_r))^n \quad (6.12)$$

$$T_n = C_G i^n a_n \frac{1}{4^n} (\log(h_\ell/h_r))^n, \quad (6.13)$$

and attempt to find a_n . We can combine these sums by defining

$$C_n = a_n \left(\frac{i}{4}\right)^n (\log(h_\ell/h_r))^n,$$

Then each coefficient R_n^∞ is the real part of C_n^∞ for n odd, while the T_n^∞ are the values C_n^∞ for n even. Thus

$$\sum_{n=0}^{\infty} C_n^\infty = \frac{C_T}{C_G} + iC_R. \quad (6.14)$$

Let $z = \sqrt{h_\ell/h_r}$; then (6.14) reads

$$\begin{aligned} \sum_0^\infty a_n \left(\frac{i}{4} \log z^2\right)^n &= \frac{2z}{\sqrt{z}(z+1)} + i \frac{z-1}{z+1} \\ \sum_0^\infty a_n \left(\frac{i}{2} \log z\right)^n &= \frac{2\sqrt{z} + i(z-1)}{z+1} \\ \sum_0^\infty a_n \left(\frac{i}{2} \log z\right)^n &= i \frac{(\sqrt{z}-i)^2}{z+1}. \end{aligned}$$

Substituting $w = \log z$, we obtain

$$\sum_0^\infty a_n \left(\frac{i}{2}\right)^n w^n = i \frac{(e^{w/2} - i)^2}{e^w + 1} =: g(w).$$

The sum on the left is clearly a Taylor series for $g(w)$. Thus the values a_n can be obtained from derivatives of g :

$$a_n = (-2i)^n g^{(n)}(0)$$

The first several values of a_n are

$$1, 1, 1/2, 1/3, 5/24, 2/15, 61/720, 17/315, 277/8064, 62/2835.$$

The approximation converges fairly rapidly. These values are the same as those obtained from the integrals (6.8) and (6.9). For $h_\ell = 3200$ and $h_r = 200$, taking the first 30 terms in the sum (6.14) gives the following approximations:

$$C_R \approx 0.600000000010316$$

$$C_T \approx 1.600000000046760.$$

6.4 Can the higher-order terms dominate ?

The motivation for an asymptotic expansion was the belief that waves that suffer more reflections diminish in magnitude. While this is true, that does not hold for the cumulative effects of the waves. Even though the waves that get reflected more have a smaller amplitude, there are a lot more of them as seen in Figure 3.1. This leads to the question of when it is possible to ignore the higher order terms.

It is possible to show that the ratio between the magnitudes of two successive terms in (6.10) and (6.11) has a lower bound given by $\frac{\log(C_G)^2}{3}$. So, for the higher order terms to have a bigger impact than the lower order terms, we need the ratio to be bigger than one.

$$\begin{aligned} \frac{\log(C_G)^2}{3} &> 1 \\ C_G &> e^{\sqrt{3}} \\ h_\ell &> e^{4\sqrt{3}} h_r \end{aligned} \tag{6.15}$$

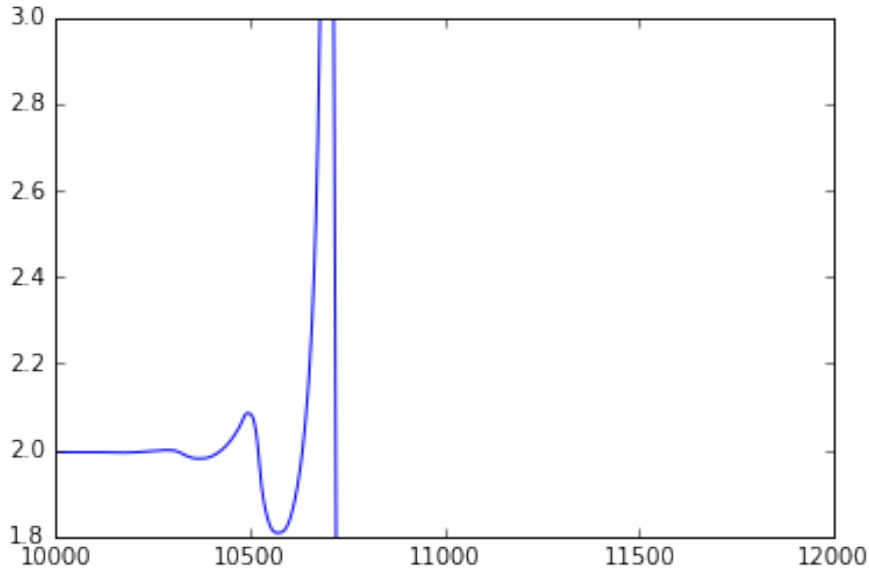


Figure 6.3: When $h_\ell > e^{4\sqrt{3}}h_r$, a first-order approximation is no longer valid. The higher terms dominate resulting in wild oscillations, especially in the transmitted wave.

Even for an h_r of 11 metres, h_ℓ becomes approximately 11227 metres which is deeper than Challenger Deep in the Mariana Trench and thus, is an unphysical depth on Earth. However it is still an interesting mathematical problem.

When the higher-order terms dominate, the transmitted and reflected waves have a lot of oscillations. The oscillations are more prominent in the transmitted wave than the reflected wave. This is at least partly because the leading waves in the reflected wave have been reflected close to h_ℓ . The amplitudes of these leading reflected waves are of the form $\left(\frac{\Delta h}{h_\ell}\right)^n$. Since $h_\ell \gg h_r$, the contributions of the leading reflected waves are negligible. That is not the case for the transmitted wave where the leading waves could have been reflected anywhere. So, their amplitudes are not negligible and cause oscillations right behind the peak value as seen in Figure 6.3.

Despite the oscillations, the transmitted and reflected waves still converge asymptotically to C_T and C_R because a_n is independent of the values of h_ℓ and h_r and (6.14) has

to hold. Furthermore, as h_ℓ gets larger and larger compared to h_r , C_G grows unbounded while C_T converges to 2. So, the peak value grows unbounded while the asymptotic value stays at 2. This would result in quite a lot of numerical dissipation.

Chapter 7

THE MASS CARRIED BY THE WAVES

Most of this thesis has been about the change in the amplitude of the wave as it goes onto the shore. However, the shallow water equations are a conservation law for the mass of the water column. So, this chapter looks at the mass carried by the transmitted wave and the reflected wave. This setting also allows us to show that the asymptotic amplitude of the transmitted or reflected wave is given by C_T regardless of the number of layers.

7.1 The asymptotic amplitude

The linearized shallow water equations can be written in the following form.

$$\begin{aligned}\eta_t + \mu_x &= 0 \\ \mu_t + gh(x)\eta_x &= 0,\end{aligned}\tag{7.1}$$

where $\mu = h(x)u$, represents the momentum of the wave. This can be written as $q_t + A(x)q_x = 0$ where the eigenvectors of A are

$$r^1 = \begin{bmatrix} 1 \\ -c \end{bmatrix}, \quad r^2 = \begin{bmatrix} 1 \\ c \end{bmatrix},\tag{7.2}$$

and corresponding eigenvalues $\lambda^{1,2} = \mp c$, where $c(x) = \sqrt{gh(x)}$. Thus left-going waves forces the elevation in wave height and the momentum to satisfy

$$\mu = -c\eta\tag{7.3}$$

while right-going waves satisfy

$$\mu = c\eta.\tag{7.4}$$

Consider bathymetry given by

$$h(x) = \begin{cases} h_\ell & x < -\delta \\ \tilde{h}(x) & |x| \leq \delta \\ h_r & x > \delta \end{cases} \quad (7.5a)$$

and a purely right-going incident wave coming from the left, with support on a compact interval that is contained in $(-\infty, -\delta)$. We are interested in the ‘net’ transmission and reflection of this wave due to the variation in h near $x = 0$.

In the limit $\delta \rightarrow 0$, $h(x)$ is a step function and the amplitude (in η) of the transmitted and reflected waves is described by the transmission and reflection coefficients:

$$C_T = \frac{2\sqrt{h_\ell}}{\sqrt{h_\ell} + \sqrt{h_r}} \quad (7.6)$$

$$C_R = 1 - C_T. \quad (7.7)$$

We have the following initial data

$$q(x, t) = \begin{cases} q_l & x < -\delta \\ q_r & x > -\delta \end{cases} \quad (7.8)$$

Here, q is the state of the system Equation (2.1) given by

$$q(x, t) = \begin{bmatrix} \eta(x, t) \\ \mu(x, t) \end{bmatrix} \quad (7.9)$$

In the case of the Riemann problem as $\delta \rightarrow 0$, C_T and C_R describe the middle state q_m that arises. Figure 7.1 gives a look at this.

With two layers, there are more waves going to each side, but after some time, the middle state separates the left going and right going waves. We can see this in Figure 7.2. Is the middle state the same as before ?

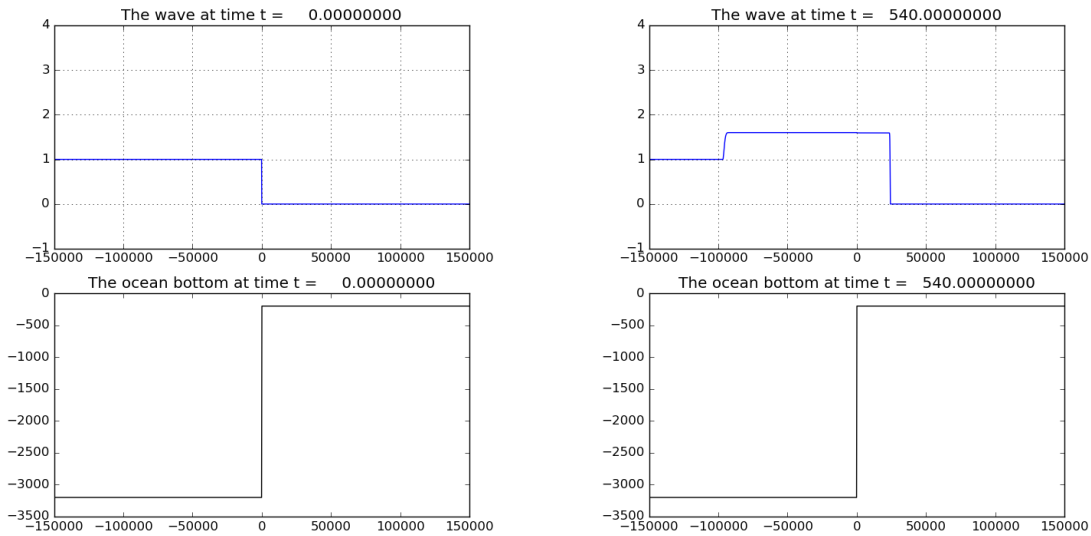


Figure 7.1: The two initial states q_l and q_r in a Riemann problem get separated by a middle state q_m determined by C_T

After some time, the solution to the right (of $x = 0$) consists only of right-going waves. Let q_{r_1} and q_{r_2} be the states on either side of the wave. Since the wave is purely right-going, the jump across the states should be along the right eigenvector.

$$q_{r_1} - q_{r_2} = c_1 r_2 \quad (7.10)$$

Since q_r is the initial state of the right side, it will remain the rightmost state on the right side. The middle state will be the leftmost state on the right side (of $x = 0$). q_r and q_m are separated by multiple right going waves. So, we can relate them by

$$q_r - q_m = c_1 r_2 + c_2 r_2 + \dots + c_n r_2 = c r_2 \quad (7.11)$$

where c is a constant

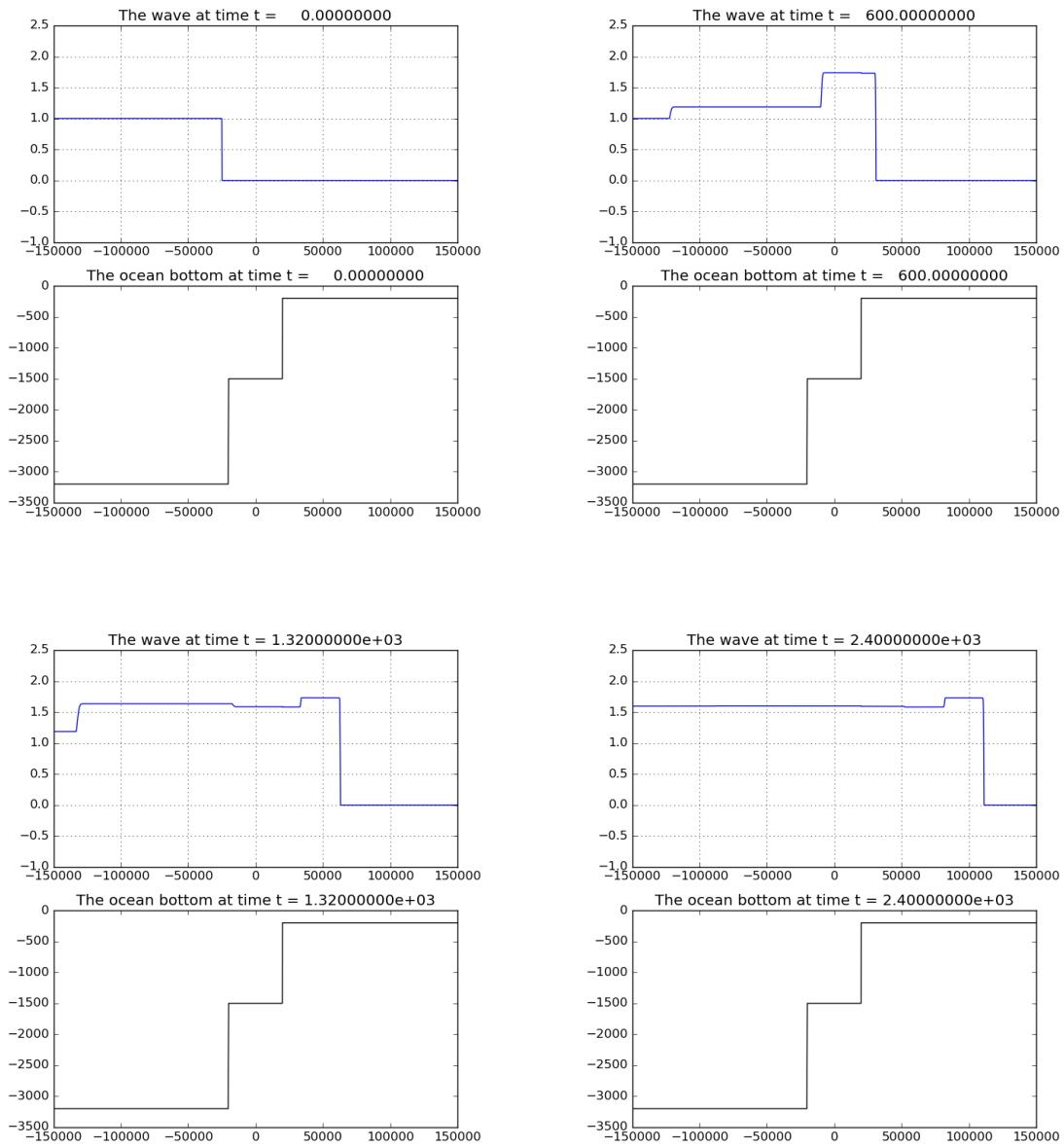


Figure 7.2: Multiple right-going and left-going waves emerge but the middle state becomes visible after some time.

Similarly, making the same argument about the left side of $x = 0$, we can see

$$q_r - q_m = dr_1 \quad (7.12)$$

where d is a constant

Looking at (7.11) and (7.12) together, we see that q_m has to be the middle state from the Riemann problem. This argument holds regardless of the number of layers. Thus, the elevation in the middle state will be amplified by a factor of C_T as compared to the incident amplitude.

7.2 The mass carried by the waves

The shallow-water equations can also be written in their conservation form:

$$\begin{aligned} \eta_t + (h(x)u)_x &= 0 \\ u_t + g\eta_x &= 0, \end{aligned} \quad (7.13)$$

from which we see that $\int \eta dx$ and $\int u dx$ are constant in time at least until the waves hit the boundaries. This means that the mass and the total velocity are conserved quantities

Let us define

$$\eta^0 = \int_{-\infty}^{\infty} \eta(x, 0) dx \quad (7.14)$$

$$\eta^R = \lim_{t \rightarrow \infty} \int_{-\infty}^0 \eta(x, t) dx \quad (7.15)$$

$$\eta^T = \lim_{t \rightarrow \infty} \int_0^{\infty} \eta(x, t) dx, \quad (7.16)$$

and similar quantities for u and μ . Here η^0 is the mass of the initial wave, while η^R (η^T) is the total mass of all the reflected (transmitted) waves. We want to find the relation between η^0 and η^T or η^R .

We can also define

$$u^0 = \int_{-\infty}^{\infty} u(x, 0) dx \quad (7.17)$$

$$u^R = \lim_{t \rightarrow \infty} \int_{-\infty}^0 u(x, t) dx \quad (7.18)$$

$$u^T = \lim_{t \rightarrow \infty} \int_0^{\infty} u(x, t) dx, \quad (7.19)$$

Since u and η are conserved quantities, by the first result we have that

$$\eta^0 = \eta^T + \eta^R \quad (7.20)$$

$$u^0 = u^T + u^R. \quad (7.21)$$

After a short time, the solution in the interval $(-\infty, 0)$ is purely left-going and the solution in the interval $(0, \infty)$ is purely right-going. Then, we have

$$\mu^T = c_r \eta^T \quad (7.22)$$

$$\mu^R = -c_l \eta^R \quad (7.23)$$

Since $\mu^T = h_r u^T$ and $\mu^R = h_\ell u^R$, this means that

$$u^T = \frac{c_r}{h_r} \eta^T$$

$$u^R = -\frac{c_l}{h_\ell} \eta^R.$$

Combining these equations we find

$$\eta^T = \frac{2\sqrt{h_r}}{\sqrt{h_\ell} + \sqrt{h_r}} \eta^0 \quad (7.24)$$

$$\eta^R = \frac{\sqrt{h_\ell} - \sqrt{h_r}}{\sqrt{h_\ell} + \sqrt{h_r}} \eta^0. \quad (7.25)$$

It is worth pointing out that the coefficient for transmission in (7.24) differs from the coefficient for the Riemann problem by a factor of $c_r/c_l = \sqrt{h_r/h_\ell}$. This is because the coefficient derived here relates masses, while the usual transmission coefficient relates amplitudes.

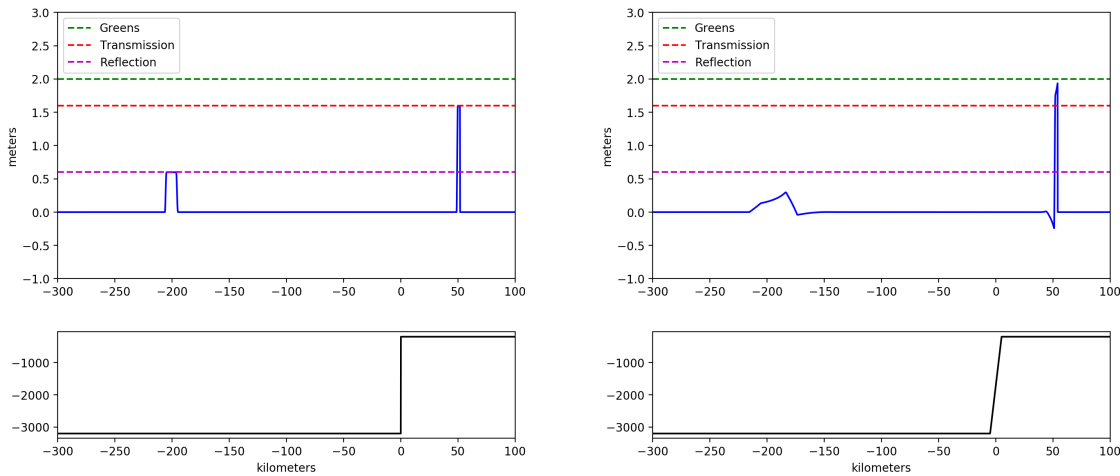


Figure 7.3: As the bathymetry gets less steep, the amplitude of the reflected wave decreases. The width of the wave increases so that the the same mass is reflected in both cases.

Thus, the amplification of the transmitted wave is different from the ratio of the transmitted mass to the initial mass. The transmitted mass also has no dependence on the bathymetry. This is only possible if the width of the transmitted and reflected pulses changes depending on the setting. Thus, in the setting of a sharp jump, the reflected wave has an amplitude of C_R and the same width as the incident pulse, while in the setting of Greens law, with a wide continental slope, the reflected wave becomes much wider than the incident pulse and carries the same mass with much smaller amplitude. The width of the transmitted wave gets modified by the reciprocal factor c_l/c_r (relative to the incident wave).

Chapter 8

LINEAR ACOUSTICS

Although we have mostly discussed wave propagation across layered media in the context of shallow water equations, everything in the present work can be applied to a range of other physical settings, including those of linear acoustic waves and electromagnetic waves, which are described by similar equations. For instance, for linear acoustics, we have

$$\begin{aligned} p_t + K(x)u_x &= 0 \\ u_t + \frac{1}{\rho(x)}p_x &= 0, \end{aligned} \tag{8.1}$$

where p is pressure, u is velocity, K is the bulk modulus, and ρ is the density. System (8.1) is equivalent to (1.1) if we identify (p, u) with (η, hu) and let $K(x) = 1$, $\rho(x) = 1/gh(x)$.

However, things are slightly different for the linear acoustics equation. The linear acoustics system (8.1) has eigenvalues and corresponding eigenvectors given by

$$\begin{aligned} \lambda_1 &= -\sqrt{K/\rho}, & r_1 &= \begin{bmatrix} -\sqrt{K\rho} \\ 1 \end{bmatrix}, \\ \lambda_2 &= \sqrt{K/\rho}, & r_2 &= \begin{bmatrix} \sqrt{K\rho} \\ 1 \end{bmatrix}. \end{aligned} \tag{8.2}$$

Usually, we define the sound speed $c = \sqrt{K/\rho}$ and the impedance $Z = \sqrt{K\rho}$. Unlike the shallow water equations (1.1) where there is only one free parameter $h(x)$, acoustics has two independent parameters K and ρ (or, equivalently, c and Z), which leads to interesting phenomena.

8.1 The transmission coefficient and the first transmitted wave

The transmission and reflection coefficients for the Riemann problem for linear acoustics can be easily derived [7] and is given by

$$C_T = \frac{2Z_r}{Z_\ell + Z_r}, \quad C_R = \frac{Z_r - Z_\ell}{Z_\ell + Z_r}. \quad (8.3)$$

For the Riemann problem, the incident wave is transmitted completely with no change to its amplitude if the impedance is constant even if the sound speed varies.

The amplitude of the first transmitted wave might not be the same as the peak amplitude predicted by Green's law due to the interplay of c and Z . Like we did in Section 3.3, we can show that the amplitude of the first transmitted wave denoted here by C_α is given by

$$C_\alpha = \left(\frac{Z_r}{Z_\ell} \right)^{\frac{1}{2}} = \left(\frac{K_r \rho_r}{K_\ell \rho_\ell} \right)^{\frac{1}{4}} \quad (8.4)$$

8.2 A Green's law for linear acoustics

The values in (8.3) and (8.4) seem insufficient in describing the behavior of the pressure wave across varying media. In this section, we do some calculations similar to [8] to get an equivalent of Green's law for acoustics.

Differentiating (8.1) with respect to t and x , we get

$$\begin{aligned} p_{tt} + K(x)u_{xt} &= 0 \\ u_{tx} + \frac{1}{\rho(x)}p_{xx} + \left(\frac{1}{\rho(x)} \right)_x p_x &= 0, \end{aligned} \quad (8.5)$$

This allows us to plug the first equation in the second to obtain

$$p_{tt} - K(x)\frac{1}{\rho(x)}p_{xx} + K(x)\left(\frac{1}{\rho(x)} \right)_x p_x = 0 \quad (8.6)$$

We start with an assumption that the pressure wave is of the form

$$p(x, t) = A(x)e^{i(\omega t - \gamma x)} \quad (8.7)$$

where $A(x)$ is the amplitude of the wave and w and γ have physical interpretations of wave frequency and wave number.

Plugging this in (8.6), we get

$$\begin{aligned}
& -w^2 A e^{i(wt-\gamma x)} \\
& -K(x) \frac{1}{\rho(x)} (A_{xx} e^{i(wt-\gamma x)} - i\gamma A_x e^{i(wt-\gamma x)} - i\gamma A_x e^{i(wt-\gamma x)} - \gamma^2 A e^{i(wt-\gamma x)} - i\gamma_x A e^{i(wt-\gamma x)}) \\
& + K(x) \left(\frac{1}{\rho(x)} \right)_x (A_x e^{i(wt-\gamma x)} - i\gamma A e^{i(wt-\gamma x)}) = 0
\end{aligned} \tag{8.8}$$

Ignoring the terms with a derivative since they are relatively small, we get the following dispersion relation.

$$-w^2 + K(x) \frac{1}{\rho(x)} \gamma^2 = 0 \tag{8.9}$$

Equating the terms which contain a single derivative, we get

$$\begin{aligned}
& -K(x) \frac{1}{\rho(x)} (-2i\gamma A_x e^{i(wt-\gamma x)} - i\gamma_x A e^{i(wt-\gamma x)}) + K(x) \left(\frac{1}{\rho(x)} \right)_x (-i\gamma A e^{i(wt-\gamma x)}) = 0 \\
& 2 \frac{1}{\rho(x)} \gamma A_x + \frac{1}{\rho(x)} \gamma_x A + \left(\frac{1}{\rho(x)} \right)_x \gamma A = 0 \\
& 2 \frac{A_x}{A} + \frac{\gamma_x}{\gamma} + \frac{\left(\frac{1}{\rho(x)} \right)_x}{\left(\frac{1}{\rho(x)} \right)} = 0
\end{aligned} \tag{8.10}$$

To eliminate the wave number dependence, we differentiate (8.9).

$$-2ww_x + \left(\frac{K(x)}{\rho(x)} \right)_x \gamma^2 + \frac{K(x)}{\rho(x)} 2\gamma\gamma_x = 0 \tag{8.11}$$

Since the frequency in time is constant for our ansatz, we get

$$\begin{aligned}
& \left(\frac{K(x)}{\rho(x)} \right)_x \gamma^2 + \frac{K(x)}{\rho(x)} 2\gamma\gamma_x = 0 \\
& \frac{\gamma_x}{\gamma} = - \frac{\left(\frac{K(x)}{\rho(x)} \right)_x}{2 \left(\frac{K(x)}{\rho(x)} \right)}
\end{aligned} \tag{8.12}$$

Using this in (8.10), we get

$$2\frac{A_x}{A} - \frac{\left(\frac{K(x)}{\rho(x)}\right)_x}{2\left(\frac{K(x)}{\rho(x)}\right)} + \frac{\left(\frac{1}{\rho(x)}\right)_x}{\left(\frac{1}{\rho(x)}\right)} = 0$$

$$\frac{A_x}{A} - \frac{\left(\frac{K(x)}{\rho(x)}\right)_x}{4\left(\frac{K(x)}{\rho(x)}\right)} + \frac{\left(\frac{1}{\rho(x)}\right)_x}{2\left(\frac{1}{\rho(x)}\right)} = 0 \quad (8.13)$$

$$\log(A(x))' - \frac{1}{4} \log\left(\frac{K(x)}{\rho(x)}\right)' + \frac{1}{2} \log\left(\frac{1}{\rho(x)}\right)' = 0$$

$$\left(\log(A(x)) - \frac{1}{4} \log\left(\frac{K(x)}{\rho(x)}\right) + \frac{1}{2} \log\left(\frac{1}{\rho(x)}\right)\right)' = 0$$

$$\log\left(A(x) \left(\frac{K(x)}{\rho(x)}\right)^{-\frac{1}{4}} \left(\frac{1}{\rho(x)}\right)^{\frac{1}{2}}\right)' = 0$$

$$A(x) \left(\frac{K(x)}{\rho(x)}\right)^{-\frac{1}{4}} \left(\frac{1}{\rho(x)}\right)^{\frac{1}{2}} = \text{constant}$$

$$A(x)(K(x)\rho(x))^{-\frac{1}{4}} = \text{constant} \quad (8.14)$$

$$C_G = \frac{A_r}{A_l} = \left(\frac{K_r \rho_r}{K_\ell \rho_\ell}\right)^{\frac{1}{4}} = \left(\frac{Z_r}{Z_\ell}\right)^{\frac{1}{2}} \quad (8.15)$$

Thus, we see that even for acoustics $C_\alpha = C_G$ which only depends on the impedance.

Chapter 9

CONCLUSIONS

We started with a question that seemed simple enough : What happens to a wave as it goes from deep ocean to shallower waters ? In the search for an answer, we found that there were two results that seemed to contradict each other, one given by the transmission coefficient from the Riemann problem and the other from Green's law. Furthermore, the slope of the bathymetry determined which result dominated.

To understand the connection between the two results, we explored the case of a single bore. The single bore forms a good basis for all kinds of waves because any wave can be approximated by a piecewise constant function, and hence viewed as a linear combination of multiple bores. Since we are dealing with the linearized shallow water equations, the solution is simply a linear combination of the solution to the multiple bores. In particular, we looked at a layered media interpretation of the continental slope. This allowed us to show that the amplitude of the leading wave transmitted onto the shore is given by Green's law while the asymptotic amplitude is given by the transmission coefficient.

By thinking of the continuous slope as infinitesimally tiny layers stacked next to each other, it was possible to obtain the leading amplitude and the asymptotic amplitude by adding all the transmitted waves or by looking at just the first transmitted wave. This idea of waves that get reflected and transmitted many times can be extended to find a good approximation to the reflected wave and the transmitted wave. This is a 'first-order' or 'second-order' approximation because we consider waves that have been reflected only once or twice. The thesis then details the general method of finding higher order approximations to the transmitted and reflected waves.

As we add up the waves that have been reflected internally once, twice and so on, their asymptotic effect should converge to that of the transmission coefficient C_T . This reveals two asymptotic expansions for C_T . It is possible to obtain all the terms in the asymptotic expressions by integrating over all the possible reflections but this is computationally expensive. A generating function is derived which massively simplifies the whole process.

It is possible that the ‘first-order’ approximation is not a good one. The ‘higher-order’ terms can dominate when the value of C_G gets too large. This leads to curious oscillations which are more prominent in the transmitted wave than the reflected wave and we have ideas why that is.

The phenomenon explored here and the results derived here can be generalized for other wave equations in heterogeneous media. So, we take a look at the linear acoustics equations and find a Green’s law for acoustics. Here, we see that although there are two parameters, the impedance and the sound speed, that can change in a heterogeneous media, the amplification is only governed by the impedance. Changes in the sound speed cause a pulse to narrow or widen and thus the variation of the sound speed does become important in the results for the mass and energy transmitted.

This problem initially started with a desire to understand how internal waves in multilayer shallow water equations impact the amplification of the wave. We found out that even the single layer case was not understood that well. Two results that seemed to contradict each other had a connection. This connection allowed us to get an expressions for the transmitted and reflected waves by creatively adding up the waves reflected at various locations. We essentially obtained a reduced-order solution for shallow water equations without actually (conventionally) solving the partial differential equation. That is quite a lot to uncover since starting from just two results that were different.

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