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# On special Lagrangian equations

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**Abstract**

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In this paper we study the special Lagrangian equation and related equations. Special Lagrangian equation originates in the special Lagrangian geometry by Harvey-Lawson [HL1].

In subcritical phases, we construct singular solutions in dimension three and higher. We also convert our counterexamples to the ones for the minimal surface system equation.

In critical and supercritical phases, we derive a priori Hessian estimates in general higher dimensions ( $n \geq 4$ ). Our unified approach leads to sharper estimates for previously known three dimensional and convex solution cases.



## TABLE OF CONTENTS

	Page
Chapter 1: Introduction . . . . .	1
1.1 Calibration and special Lagrangian geometry . . . . .	1
1.2 Special Lagrangian equation . . . . .	4
1.3 Some previously known results . . . . .	8
1.4 Summary of main results . . . . .	10
1.5 Notes and examples . . . . .	13
Chapter 2: Hessian estimates for special Lagrangian equations with critical and supercritical phases in general dimensions . . . . .	15
2.1 Introduction . . . . .	15
2.2 Preliminary inequalities . . . . .	17
2.3 Proof of Theorem 1.1 . . . . .	26
Chapter 3: Singular solutions to special Lagrangian equations with subcrit- ical phases and minimal surface systems . . . . .	32
3.1 Introduction . . . . .	32
3.2 Cauchy-Kowalevskaya with critical phase $\Theta = \frac{\pi}{2}$ . . . . .	33
3.3 Rotate to subcritical phases $ \Theta  < \frac{\pi}{2}$ : proof of Theorem 1.2 . . . . .	39
3.4 Rotate to smooth solutions: proof of Theorem 1.3 . . . . .	47
3.5 Minimal surface system: proof of Theorem 1.4 . . . . .	50
Bibliography . . . . .	52

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And to all my family and friends, for every step they have helped me in my life.

# DEDICATION

to my grandfather



## Chapter 1

### INTRODUCTION

#### **Notation**

We use these differential forms

$$\begin{aligned} dz &= dz_1 \wedge \cdots \wedge dz_n \\ \alpha_\theta &\equiv \operatorname{Re} e^{i\theta} dz \\ \alpha &= \alpha_0, \quad \beta = \operatorname{Im} dz \end{aligned}$$

We use  $J$  to denote the complex structure on a complex manifold.

We use  $\mathcal{S}$  to denote the space of symmetric  $n \times n$  matrices.

We use  $\sigma_k$  to denote the elementary symmetric functions of eigenvalues of a matrix  $M$ .

$$\sigma_k(M) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \det M_{i_1 \dots i_k, i_1 \dots i_k} = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}$$

$M_{i_1 \dots i_k, i_1 \dots i_k}$  denote principal  $k \times k$  minors of  $M$

$\lambda_i$  denotes the  $i$ -th eigenvalue of  $M$ .

#### **1.1 Calibration and special Lagrangian geometry**

We follow Harvey-Lawson [HL1].

##### *1.1.1 Calibration*

**Definition.** Let  $X$  be a Riemannian manifold. A closed  $k$ -form  $\varphi$  that satisfies

$$\varphi|_\xi \leq \operatorname{vol}_\xi$$

for all oriented tangent  $k$ -planes  $\xi$  on  $X$ , is called a calibration.

An oriented  $k$ -submanifold (with boundary)  $M$  of  $X$  that satisfies

$$\varphi|_M = \text{vol}_M$$

is called a  $\varphi$ -submanifold of  $X$ .

**Example.** 1.  $\mathbb{R}^2$  with standard Euclidean metric,  $\varphi = dx$ . The  $\varphi$ -submanifolds are straight lines  $y \equiv C$ .

2.  $\mathbb{C}^n$  with metric  $g = |dz_1|^2 + \cdots + |dz_n|^2$ ,  $\omega = \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + \cdots + dz_n \wedge d\bar{z}_n)$ .

We verify  $\omega\left(\frac{\partial}{\partial e}, \frac{\partial}{\partial f}\right) = g\left(\frac{\partial}{\partial e}, J\frac{\partial}{\partial f}\right) \leq 1$ , with equality if and only if  $\frac{\partial}{\partial e} = J\frac{\partial}{\partial f}$ , where  $\frac{\partial}{\partial e}$  and  $\frac{\partial}{\partial f}$  form an orthonormal basis for any two-plane.

Thus  $\omega$ -submanifolds are precisely the 1-dimensional complex submanifolds of  $\mathbb{C}^n$ .

Wirtinger's inequality:  $\omega^k(\xi) \leq k!|\xi|$ , for any  $2k$ -plane  $\xi$ , with equality holds if and only if  $\xi$  is a complex  $k$ -plane.

Using this we see  $\omega^k/k!$ -submanifolds are the  $k$ -dimensional complex submanifolds of  $\mathbb{C}^n$ .

3. More generally, let  $(X, \omega)$  be a Kähler manifold. Then its complex submanifolds are calibrated by  $\omega^k/k!$ .

Any compact  $\varphi$ -submanifold of  $X$  is volume minimizing in its homology class. More generally, let  $M$  and  $M'$  be compact submanifolds (with boundary) of  $X$ , that satisfy:  $M$  is a  $\varphi$ -submanifold,  $\partial M = \partial M'$  and  $[M - M'] = 0$  in  $H_k(X; \mathbb{R})$ . Then we have

$$\text{vol}(M) = \int_M \text{vol}_M = \int_M \varphi = \int_{M'} \varphi \leq \int_{M'} \text{vol}_{M'} = \text{vol}(M').$$

**Example.** Let  $X = S^1 \times \mathbb{R}^1$  be the standard cylinder,  $\varphi = \text{vol}_{S^1}$ . Round circles are the  $\varphi$ -submanifolds, they minimize the length among all closed curves around the cylinder.

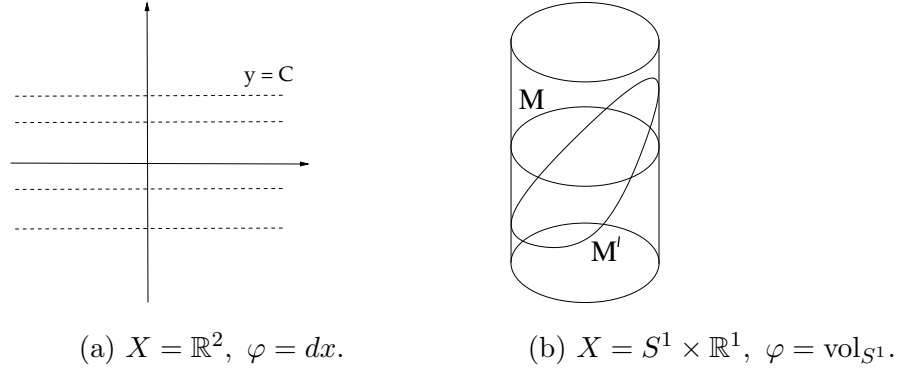


Figure 1.1: Calibrations

### 1.1.2 Special Lagrangian geometry

**Definition.** An oriented real  $n$ -plane  $\xi$  in  $\mathbb{C}^n$  is called Lagrangian if  $Je \perp \xi$ ,  $\forall e \in \xi$ .

**Definition.** An oriented  $n$ -plane  $\xi$  in  $\mathbb{C}^n$  is called special Lagrangian if

1.  $\xi$  is Lagrangian
2.  $\xi = A\xi_0$ , where  $A \in \mathbb{U}_n$  has  $\det A = 1$ ,  $\xi_0 = \mathbb{R}^n$ .

**Theorem** ([HL1, III Theorem 1.10]). For the form  $\alpha \equiv \text{Re } dz$ , we have  $\alpha(\xi) \leq |\xi|$  for all  $n$ -planes  $\xi$ , with equality if and only if  $\xi$  is special Lagrangian.

This implies the form  $\alpha \equiv \text{Re } dz$  is a calibration. It is called the special Lagrangian calibration on  $\mathbb{C}^n$ .

**Definition.** An  $n$ -dimensional oriented submanifold  $M$  of  $\mathbb{C}^n$  is called a (special) Lagrangian submanifold of  $\mathbb{C}^n$  if the tangent plane to  $M$  at each point is (special) Lagrangian.

A graph  $(x, U(x))$  is Lagrangian if it can be written as a gradient graph  $(x, Du(x))$  locally, for some potential function  $u$ .

The forms  $\alpha_\theta$  and  $\alpha = \alpha_0$  are equivalent under  $\mathbb{U}_n$  transformations. Thus we will also call  $\alpha_\theta$ -submanifolds *special Lagrangian with respect to the calibration  $\alpha_\theta$* .

**Theorem** ([HL1, III Theorem 2.3]). *Suppose  $u \in C^2(\Omega)$  with  $\Omega^{open} \subset \mathbb{R}^n$ . Let  $M$  denote the gradient graph  $(x, Du(x))$  in  $\mathbb{C}^n = \mathbb{R}^n \oplus i\mathbb{R}^n$ . Then  $M$  (with the correct orientation) is special Lagrangian if and only if*

$$\sum_{k=0}^{[(n-1)/2]} (-1)^k \sigma_{2k+1}(D^2u) = 0$$

or equivalently,

$$\text{Im} \det_{\mathbb{C}}(I + i D^2u) = 0.$$

The special Lagrangian submanifolds are just the Lagrangian manifolds which are minimal.

**Theorem** ([HL1, III Proposition 2.17]). *A connected submanifold  $M \subset \mathbb{R}^{2n} \cong \mathbb{C}^n$  is both Lagrangian and stationary if and only if  $M$  is special Lagrangian with respect to one of the calibrations  $\alpha_\theta \equiv \text{Re } e^{i\theta} dz$ .*

## 1.2 Special Lagrangian equation

We study the special Lagrangian equation

$$F(D^2u) = \sum_{i=1}^n \arctan \lambda_i = \Theta \tag{1.1}$$

where  $\lambda_i$  are the eigenvalues of the Hessian matrix  $D^2u$ . The constant  $|\Theta| < \frac{n\pi}{2}$  is called *phase* of the equation.

Take a product

$$\prod_i (1 + \sqrt{-1}\lambda_i) = \sum_{0 \leq 2k \leq n} (-1)^k \sigma_{2k} + \sqrt{-1} \sum_{1 \leq 2k+1 \leq n} (-1)^k \sigma_{2k+1}.$$

The argument of this complex number is  $\sum_{i=1}^n \arctan \lambda_i = \Theta$ . Thus we have the following algebraic form of (1.1):

$$\cos \Theta \sum_{1 \leq 2k+1 \leq n} (-1)^k \sigma_{2k+1} - \sin \Theta \sum_{0 \leq 2k \leq n} (-1)^k \sigma_{2k} = 0. \tag{1.2}$$

Equivalently, the gradient graph  $(x, Du(x))$  is special Lagrangian with respect to the calibration  $\alpha_\Theta \equiv \text{Re } e^{i\Theta} dz$ .

Notice that (1.1) only represents one branch of (1.2).

**Example.** *In two dimensions.*

When  $\Theta = 0$ , (1.1) is equivalent to the Laplacian equation

$$\Delta u = \lambda_1 + \lambda_2 = 0.$$

When  $\Theta = \pi/2$ , (1.1) is equivalent to the Monge-Ampère equation

$$\det(D^2u) = \lambda_1 \cdot \lambda_2 = 1.$$

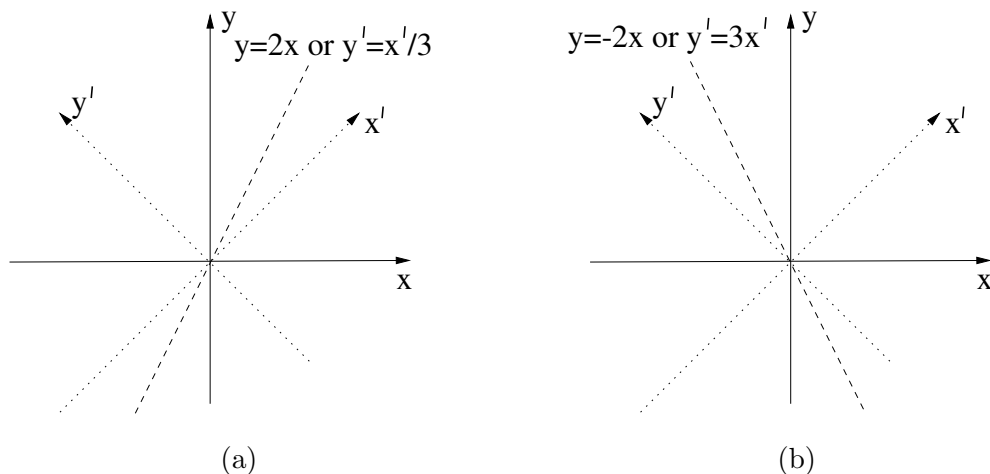
$\mathbb{U}^n$  transformations preserve the special Lagrangian condition, but will change its phase. We apply a  $\pi/4$  rotation in dimension two to illustrate this.

**Example.** Let  $u = x_1^2 - x_2^2$ , it is a harmonic function, we have  $\Theta = 0$ . Look at its gradient graph  $(y_1, y_2) = Du = (2x_1, -2x_2)$ . In complex coordinate  $z = (z_1, z_2) = (x_1 + iy_1, x_2 + iy_2)$ , we apply the  $\pi/4$  rotation  $z' = e^{-i\pi/4}z$ . The new gradient graph  $(y'_1, y'_2) = (\frac{1}{3}x'_1, 3x'_2)$  has a potential function  $u' = \frac{1}{6}x'_1{}^2 + \frac{3}{2}x'_2{}^2$  satisfying the Monge-Ampère equation, we have new phase  $\Theta = \pi/2$ .

Let  $u = x_1 \cdot x_2$ , again it is a harmonic function, we have  $\Theta = 0$ . This time however, the gradient graph  $(y_1, y_2) = (x_2, x_1)$ . After the rotation  $x' = \frac{1}{\sqrt{2}}(y+x)$ ,  $y' = \frac{1}{\sqrt{2}}(y-x)$ , becomes  $x'_1 = x'_2$ ,  $y'_1 = -y'_2$ , which is no longer a graph over  $x'$ -space. Geometrically it is a "vertical" plane with phase  $\pi/2$ . In terms of the Hessian of potential functions, the eigenvalues become 0 and  $\infty$ .

**Example.** *In dimension three, when  $\Theta = \pi/2$ , (1.1) is equivalent to*

$$1 - \sigma_2 = 0.$$

Figure 1.2:  $\pi/4$  rotation

The special Lagrangian equation (1.1) is a fully non-linear equation. The linearization operator at any solution  $u$  is a second order elliptic operator. When the Hessian  $D^2u$  is diagonalized at point  $p$ , we have

$$(F_{u_{ij}}) \sim \begin{pmatrix} \frac{1}{1+\lambda_1^2} & & \\ & \ddots & \\ & & \frac{1}{1+\lambda_n^2} \end{pmatrix}$$

Weak solutions to (1.1) can be defined without using divergence structure.

**Definition.** A continuous function  $u$  in  $\Omega \subset \mathbb{R}^n$  is a viscosity subsolution (supersolution) to (1.1) if

$$F(D^2\varphi(x_0)) \geq \Theta \quad (\leq \Theta)$$

for any  $x_0 \in \Omega$ ,  $\varphi \in C^2(\Omega)$  such that  $u - \varphi$  has a local maximum (minimum) at  $x_0$ .

We say  $u$  is a viscosity solution to (1.1) if it is both a viscosity subsolution and a viscosity supersolution.

Existence and uniqueness of the viscosity solution to the Dirichlet problem for special Lagrangian equation follows from the more general strictly elliptic equation,

using Perron's method and comparison principle ([CC, chapter 5]). An alternative approach can be found in Harvey-Lawson [HL2] [HL3].

A priori estimates are used to establish the regularity of  $C^0$  viscosity solutions.

For a general fully nonlinear second order elliptic equation, standard elliptic regularity theory applies to  $C^{2,\alpha}$  solutions.

**Theorem** ([CC Proposition 9.1]). *Let  $0 < \alpha < 1$  and  $u \in C^{2,\alpha}(\Omega)$  be a solution of*

$$F(D^2u, x) = f(x) \quad \text{in } \Omega.$$

*Assume that  $F \in C^\infty(\mathcal{S} \times \Omega)$  and  $f \in C^\infty(\Omega)$ . Then  $u \in C^\infty(\Omega)$ .*

To reach  $C^{2,\alpha}$  estimate, we need a Hessian estimate to bound  $\|D^2u\|_{L^\infty}$  using  $\|u\|_{L^\infty}$ . Then for *concave* functionals we can apply Evans-Krylov-Safonov theory. ([CC section 6.1] [GT section 17.4])

**Theorem** (Evans-Krylov). *Let  $F \in C^\infty(\mathcal{S})$  be concave and let*

$$u \in C^2(B_1) \quad \text{satisfy } F(D^2u) = 0 \quad \text{in } B_1.$$

*Then  $u \in C^{2,\alpha}(\bar{B}_{1/2})$  and*

$$\|u\|_{C^{2,\alpha}(\bar{B}_{1/2})} \leq C \|u\|_{C^{1,1}(\bar{B}_{3/4})},$$

*where  $0 < \alpha < 1$  and  $C$  are universal constants.*

The functional  $F(D^2u) = \sum_{i=1}^n \arctan \lambda_i$  is not concave or convex. But we have

**Lemma** ([Y3 Lemma 2.1][CNS Lemma 8.1]). *The level set in eigenvalue space  $\Gamma = \{(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n \mid \sum \arctan \lambda_i = \Theta\}$  is a convex hypersurface when  $|\Theta| \geq \frac{\pi}{2}(n-2)$ .*

When  $\Theta = \frac{\pi}{2}(n-2)$ , it is called *critical* phase. When  $\Theta > \frac{\pi}{2}(n-2)$ , it is called *supercritical* phase.

Thus in critical and supercritical phase cases, we can replace  $F$  by a concave functional resulting an equation equivalent to (1.1), Evans-Krylov-Safonov theory will still apply.

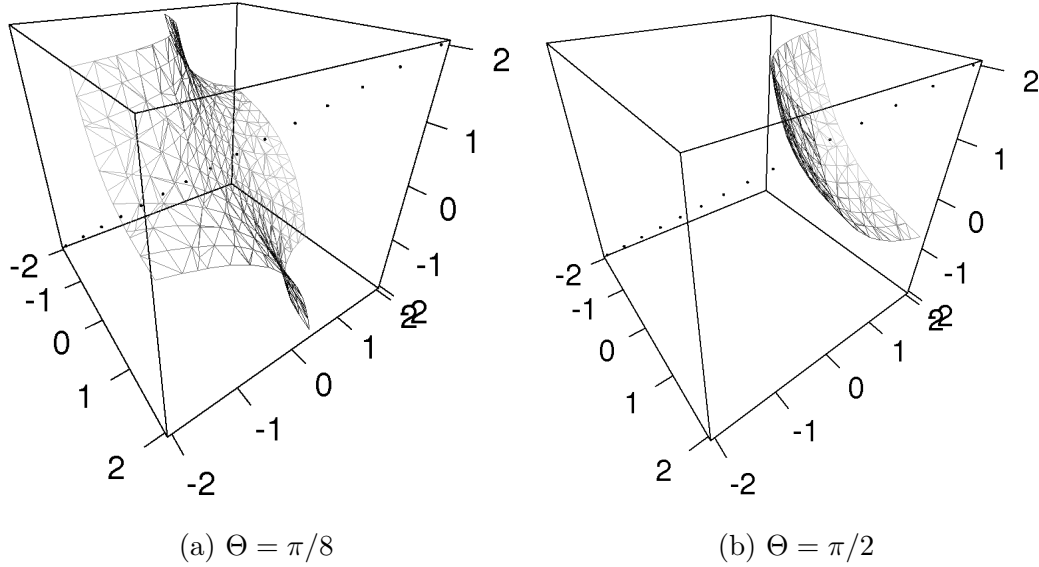


Figure 1.3: Level set of different phases in dimension three

### 1.3 Some previously known results

*Liouville type results and Bernstein type results*

**Theorem** ([Y2, Theorem 1.1]). *Suppose  $\mathcal{M} = (x, Du)$  is a minimal surface in  $\mathbb{R}^n \times \mathbb{R}^n$  and  $u$  is a smooth convex function in  $\mathbb{R}^n$ . Then  $\mathcal{M}$  is a plane.*

**Theorem** ([Y3, Theorem 1.1]). *Let  $u$  be a smooth solution in  $\mathbb{R}^n$  to (1.1) with  $|\Theta| > \frac{\pi}{2}(n-2)$ , then  $u$  is quadratic.*

*A priori estimates*

**Theorem** (Warren-Yuan [WY3, Theorem 1.1]). *Let  $u$  be a smooth solution to (1.1) with  $\Theta \in (-\pi, \pi)$  and  $n = 2$  on  $B_R(0) \subset \mathbb{R}^2$ . Then the following both hold*

$$|D^2u(0)| \leq C(2) \exp \left[ C(2) \max_{B_R(0)} |Du|^2 / R^2 \right],$$

and

$$|D^2u(0)| \leq C(2) \exp \left[ C(2) \frac{1}{|\sin \Theta|^{3/2}} \max_{B_R(0)} |Du|/R \right].$$

**Theorem** (Warren-Yuan [WY4, Theorem 1.1]). *Let  $u$  be a smooth solution to (1.1) with  $|\Theta| \geq \pi/2$  and  $n = 3$  on  $B_R(0) \subset \mathbb{R}^3$ . Then we have*

$$|D^2u(0)| \leq C(3) \exp \left[ C(3) \left( \cot \frac{|\Theta| - \pi/2}{3} \right)^2 \max_{B_R(0)} |Du|^7/R^7 \right].$$

and also

$$|D^2u(0)| \leq C(3) \exp \left\{ C(3) \exp \left[ C(3) \max_{B_R(0)} |Du|^3/R^3 \right] \right\}.$$

**Theorem** (Warren-Yuan [WY4, Theorem 1.2]). *Let  $u$  be a smooth solution to (1.1) with  $|\Theta| \geq (n-2)\frac{\pi}{2}$  and  $n \geq 2$  on  $B_{3R}(0) \subset \mathbb{R}^n$ . Then we have*

$$\max_{B_R(0)} |Du| \leq C(n) \left[ \operatorname{osc}_{B_{3R}(0)} \frac{u}{R} + 1 \right]. \quad (1.3)$$

**Theorem** (Chen-Warren-Yuan [CWY, Theorem 1.1]). *Let  $u$  be a smooth convex solution to (1.1) with  $\Theta \in (-\frac{n\pi}{2}, \frac{n\pi}{2})$  and  $n \geq 2$  on a ball  $B_R(0) \subset \mathbb{R}^n$ . Then we have*

$$|D^2u(0)| \leq C(n) \exp \left\{ C(n) \left[ \operatorname{osc}_{B_R(0)} \frac{u}{R^2} \right]^{3n-2} \right\},$$

where  $C(n)$  is a uniform dimensional constant.

*Counterexample in subcritical phases*

**Theorem** (Nadirashvili-Vlăduț [NV]). *For any  $\Theta \in (-\pi/2, \pi/2)$  and  $n = 3$ , there exists a small ball  $B \subset \mathbb{R}^3$  and an analytic function  $\phi$  on  $\partial B$ , for which the unique Harvey-Lawson solution  $u_\Theta$  of the Dirichlet problem (1.1) with boundary data  $\phi$ , satisfies:*

1.  $u_\Theta \in C^{1,1/3}$ ;
2.  $u_\Theta \notin C^{1,\delta}$  for all  $\delta > 1/3$ .

## 1.4 Summary of main results

### 1.4.1 Critical and supercritical phases

Main result in [WdY2] is stated as the following theorem.

**Theorem 1.1.** *Let  $u$  be a smooth solution to (1.1) with  $|\Theta| \geq (n-2)\pi/2$  and  $n \geq 3$  on  $B_R(0) \subset \mathbb{R}^n$ . Then we have*

$$|D^2u(0)| \leq C(n) \exp \left[ C(n) \max_{B_R(0)} |Du|^{2n-2} / R^{2n-2} \right];$$

and when  $|\Theta| = (n-2)\pi/2$ , we also have

$$|D^2u(0)| \leq C(n) \exp \left[ C(n) \max_{B_R(0)} |Du|^{2n-4} / R^{2n-4} \right].$$

Using the gradient estimate (1.3) we can bound  $D^2u(0)$  in terms of the solution  $u$  in  $B_R(0)$ .

One application of the above estimates is the regularity (analyticity) of the  $C^0$  viscosity solutions to (1.1) with  $|\Theta| \geq (n-2)\pi/2$ . In particular, the solutions of the Dirichlet problem with continuous boundary data to (1.1) with convex condition  $|\Theta| \geq (n-2)\pi/2$  enjoy interior regularity. In contrast, the Hessian estimates and the interior regularity for solutions to (1.1) with  $|\Theta| = \left[\frac{n-1}{2}\right]\pi$  in [CNS] by Caffarelli-Nirenberg-Spruck were derived under the  $C^4$  smoothness assumption on the boundary data.

Another quick consequence is a Liouville type result for global solutions to (1.1) with quadratic growth and  $|\Theta| = (n-2)\pi/2$ , (cf. [Y2] [Y3]).

**Corollary.** *Let  $u$  be a smooth solution in  $\mathbb{R}^n$  to (1.1) with  $|\Theta| = (n-2)\pi/2$  and  $|u(x)| < C|x|^2$  for some constant  $C$ , then  $u$  is a quadratic.*

### 1.4.2 Subcritical phases

In the *subcritical* phases  $|\Theta| < (n-2)\pi/2$  and  $n \geq 3$ , singular (viscosity) solutions to (1.1) are constructed by Nadirashvili-Vlăduț [NV] and [WdY1] showing that the

critical and supercritical phase condition in Theorem 1.1 is necessary. We state the results from [WdY1] in the following theorem.

**Theorem 1.2.** *There exist  $C^{1,1/(2m-1)}$  ( $m = 2, 3, 4, \dots$ ) viscosity solutions  $u^m$  to (1.1) with  $n = 3$  and each  $\Theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ , such that  $u^m \in C^{1,1/(2m-1)}(B_1) \cap C^\infty(B_1 \setminus \{0\})$  for  $B_1 \subset \mathbb{R}^3$  but  $u^m \notin C^{1,\delta}$  for any  $\delta > 1/(2m-1)$ .*

Rotating forth and back, we obtain our second (“smooth”) result.

**Theorem 1.3.** *There exists a family of smooth solutions  $u^\varepsilon$  to (1.1) in  $B_1 \subset \mathbb{R}^3$  with  $n = 3$  and each fixed  $\Theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$  such that*

$$\|Du^\varepsilon\|_{L^\infty(B_1)} \leq C \quad \text{but} \quad |D^2u^\varepsilon(0)| \rightarrow \infty \quad \text{as } \varepsilon \rightarrow 0.$$

For each  $u^\varepsilon$  with small  $\varepsilon$  fixed in Theorem 1.3, the Hessian  $|D^2u^\varepsilon(0)|$  (in the max eigenvalue norm) is strictly larger than its nearby values in the three dimensional domain of the solution to a now uniformly elliptic equation (1.1). (It can be seen by Property S.4 in Section 3.2 and tracing the eigenvalue dependency in Section 3.4.) This violates the maximum principle. This is in contrast to the two dimensional fully nonlinear uniformly elliptic equations, where it is classically known that the Hessian of any solution enjoys the maximum principle (cf. [GT, p. 301]).

To the solutions in the above two theorems, by adding quadratics of extra variables in higher dimensions  $n \geq 4$ , we immediately get the corresponding counterexamples for (1.1) with all subcritical phases  $|\Theta| < (n-2)\pi/2$ .

Note that here one cannot produce those a priori estimate breaking family of smooth solutions by the usual way, that is to solve the Dirichlet problem with smooth approximate boundary data of the boundary value of a singular solution, as Theorem 1.2 shows the *non-solvability* of smooth solution to the Dirichlet problem to (1.1) of subcritical phase even with smooth boundary data. The Dirichlet problem to the saddle branch of (1.1) with  $n = 3$  and  $\Theta = 0$  was “invited” by Caffarelli, Nirenberg, and Spruck in [CNS].

### 1.4.3 Minimal surface system

The minimal surface system for  $k$ -vector valued functions of  $n$ -variables is

$$\Delta_g U = \sum_{i,j=1}^n \frac{1}{\sqrt{g}} \partial_{x_i} (\sqrt{g} g^{ij} \partial_{x_j} U) = 0, \quad (1.4)$$

where the induced metric

$$g = I + (DU)^T DU.$$

When  $u$  is a solution to (1.1), the gradient graph  $(x, Du(x))$  is a minimal submanifold of  $\mathbb{R}^n \times \mathbb{R}^n$ , thus  $U = Du$  satisfies (1.4) with  $k = n$ .

We convert our counterexamples in subcritical phases of (1.1) to the ones for minimal surface system (1.4).

**Theorem 1.4.** *There exist a family of weak solutions  $U^m$  to (1.4) in  $B_1 \subset \mathbb{R}^3$  with  $n = 3$ ,  $k = 3$ , and  $m = 2, 3, 4, \dots$  such that*

$$U^m \in W^{1,p}(B_1) \quad \text{for any } p < \frac{2m+1}{2m-2} \quad \text{but } U^m \notin W^{1, \frac{2m+1}{2m-2}}(B_1).$$

*Furthermore, there exists a family of smooth solutions  $U^\varepsilon$  to (1.4) in  $B_1 \subset \mathbb{R}^3$  with  $n = 3$  and  $k = 3$  such that*

$$\|U^\varepsilon\|_{L^\infty(B_1)} \leq C \quad \text{but } |DU^\varepsilon(0)| \rightarrow \infty \quad \text{as } \varepsilon \rightarrow 0.$$

The vector valued functions  $U^m$  are taken as  $Du^m$  with  $u^m$  from Theorem 1.2, thus the first part of the theorem gives a negative answer to Nadirashvili's question whether there is an  $\varepsilon$  improvement of  $W^{2,1}$  solutions to special Lagrangian equation (1.1) in general. We are grateful for this question. In terms of minimal surface system (1.4), the question would be whether there is an  $\varepsilon$  improvement of  $W^{1,1}$  solutions.

When  $k = 1$ , (1.4) is the minimal surface equation. The gradient estimate in terms of the height of the minimal surfaces is the classic result by Bombieri-De Giorgi-Miranda [BDM], from which follows the regularity of weak or viscosity solutions.

For smooth solutions to (1.4) with  $n = 2$ , Gregori [G] extended Heinz's Jacobian estimate to get a gradient bound in terms of the heights of the two dimensional minimal surfaces with any codimension.

For smooth solutions to general minimal surface system (1.4) with certain constraints on the gradients themselves, a gradient estimate was obtained by Wang [W], using an integral method developed for codimension one minimal graphs.

Nonetheless, there do exist singular  $W^{1,2^-}$  weak solutions (in fact Lagrangian) to (1.4) with  $n = 2$ ; see Osserman [O]. Now gradient estimates for (1.4) with  $k = 2$  and  $n \geq 3$  still remain mysterious and challenging.

### 1.5 Notes and examples

In the 1950's, Heinz [H] derived a Hessian bound for the two dimensional Monge-Ampère type equation including (1.1) with  $n = 2$ ; see also Pogorelov [P1] for Hessian estimates for these equations including (1.1) with  $|\Theta| > \pi/2$  and  $n = 2$ . In the 1970's Pogorelov [P2] constructed his famous counterexamples, namely irregular solutions to three dimensional Monge-Ampère equations

$$\det(D^2u) = \sigma_3(D^2u) = 1; \tag{1.5}$$

those irregular solutions also serve as counterexamples for cubic and higher order symmetric  $\sigma_k$  equations (cf. [U2]). For solutions with certain *strict* convexity constraints to Monge-Ampère equations and  $\sigma_k$  equation ( $k \geq 2$ ), Hessian estimates are proved by Pogorelov [P2] and Chou-Wang [CW] respectively using the Pogorelov technique. Trudinger [T2] and Urbas [U3][U4], also Bao-Chen [BC] obtained (pointwise) Hessian estimates in terms of certain integrals of the Hessian, for  $\sigma_k$  equations and special Lagrangian equation (1.1) with  $n = 3$ ,  $\Theta = \pi$  respectively. Pointwise Hessian estimates for strictly convex solutions to quotient equations  $\sigma_n/\sigma_k$  were derived in terms of certain integrals of the Hessian by Bao-Chen-Guan-Ji [BCGJ].

We are also curious to know whether there exist other  $C^{1,\alpha}$  (no better) singu-

lar solutions to (1.1) with, in particular, irrational exponents  $\alpha$  between those odd reciprocals  $1/(2m-1)$ . Meanwhile, we guess that all for  $\alpha > \frac{1}{3}$ ,  $C^{1,\alpha}$  solutions to special Lagrangian equation (1.1) with  $n = 3$  should be regular (analytic). This regularity for  $C^{1,1}$  solutions to (1.1) in dimension three was shown in [Y1]. Earlier on, Urbas [U1, Theorem 1.1] proved the regularity for better than Pogorelov solutions, namely all  $C^{1,\alpha}$  for  $\alpha > 1 - \frac{2}{n}$  (convex) solutions to the (dual) Monge-Ampère equation  $\ln \det D^2 u = \ln \lambda_1 + \dots + \ln \lambda_n = c$  are  $C^{3,\beta}$  and eventually analytic. Finally recall that the singularities of Pogorelov-like singular solutions to Monge-Ampère equation extend (in fact, must, by Caffarelli [C]) to the boundary of the domain of the solutions; while the singularity of singular solutions so far constructed to special Lagrangian equation is in the interior of the domain.

## Chapter 2

**HESSIAN ESTIMATES FOR SPECIAL LAGRANGIAN  
EQUATIONS WITH CRITICAL AND SUPERCRITICAL  
PHASES IN GENERAL DIMENSIONS**

**2.1 Introduction**

Our strategies for the Hessian estimates go as follows. We bound the subharmonic function  $b = \ln \sqrt{1 + \lambda_{\max}^2}$  by its integral on the minimal surface using Michael-Simon's mean value inequality [MS]. Applying certain Sobolev inequalities, we estimate the integral of  $b$  by the integral of its gradient. The decisive choice  $b$  satisfies a Jacobi inequality: its Laplacian bounds its gradient; in turn, the integral of the gradient  $b$  is bounded by a weighted volume of the minimal Lagrangian graph. By a conformality identity, the weighted volume element is in fact the trace of the linearized operator of the special Lagrangian equation in algebraic form, which is a linear combination of the elementary symmetric functions of the Hessian. Taking advantage of the divergence structure of those functions, we bound the weighted volume in terms of the height of special Lagrangian graph, or the gradient of the solution.

However, there are two major difficulties in the execution for general dimension. The first one is to justify the nonlinear Jacobi inequality in the integral sense for the Lipschitz only function  $b$ , which was only achieved in dimension three by involved arguments [WY2]. The second one is to find, in the critical phase case, a relative isoperimetric inequality or equivalent Sobolev inequality for functions without compact support, which was circumvented only in dimension three thanks to the linear dependence on the Hessian for the linearized operator of now equivalent equation  $\sigma_2 = 1$  [WY2].

We overcome the first one by observing that the Jacobi inequality and its equivalent linear formulation hold in the viscosity sense, consequently in the potential sense. By Hervé-Hervé [HH, Theorem 1] (see also Watson [Wn, p. 246]), the linear inequality holds in the integral sense, in turn, so does the needed Jacobi inequality. Conceptually it is natural this way. For details, see the proof of Proposition 2.1.

To deal with the second difficulty, we instead apply the Sobolev inequality for functions with compact supports, but use a “twist-multiplication” trick to contain the terms involving derivatives of the cut-off functions (Step 4 in Section 2.3). This trick enables us to have a unified approach (for both the critical and supercritical cases) in all dimensions  $n \geq 3$ . Even in the known three dimensional [WY2,4] and convex cases [CWY], the simpler unified argument leads to sharper Hessian estimates.

Our unified arguments does not work for (1.1) with  $\Theta = 0$  and  $n = 2$ , as the Jacobi inequality fails (only) for harmonic functions. Elementary methods in [WY3] led to the sharp Hessian estimates in dimension two. (The sharp Hessian estimates in terms of the linear exponential dependence on the gradients, can be seen by the corresponding solutions to the Monge-Ampère equation or (1.1) with  $\Theta = \pi/2$  and  $n = 2$ , converted from Finn’s minimal surface [F, p. 355] via Heinz transformation [J, p. 133].)

As one can see that, not only our Hessian-slope estimates for “gradient” minimal graphs are analogous to the gradient-slope estimates for the codimension one minimal graphs, but also our arguments resemble the original integral proof by Bombieri-De Giorgi-Miranda [BDM] and the simplified one by Trudinger [T1] for the latter classical result. When one tries to adapt the later Korevaar pointwise technique [K], certain extra structure or assumption has to be used, as in [WY1]. Otherwise, an adaptation of the technique alone would lead to Hessian estimates for the Monge-Ampère equations, to which the Jacobi inequality is available. But this is inconsistent with Pogorelov’s singular solutions [P2].

**Notation.** First  $\partial_i = \frac{\partial}{\partial x_i}$ ,  $\partial_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}$ ,  $u_i = \partial_i u = D_i u$ ,  $u_{ji} = \partial_{ij} u$  etc., but

$\lambda_1, \dots, \lambda_n$  and  $b_k = \left( \ln \sqrt{1 + \lambda_1^2} + \dots + \ln \sqrt{1 + \lambda_k^2} \right) / k$  do not represent the partial derivatives. Also

$$\sigma_k(\lambda_1, \dots, \lambda_n) = \sum_{1 \leq i_1 < \dots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}.$$

Further,  $h_{ijk}$  will denote (the second fundamental form)

$$h_{ijk} = \frac{1}{\sqrt{1 + \lambda_i^2}} \frac{1}{\sqrt{1 + \lambda_j^2}} \frac{1}{\sqrt{1 + \lambda_k^2}} u_{ijk}.$$

when  $D^2u$  is diagonalized. Finally  $C(n)$  will denote various constants depending only on dimension  $n$ .

## 2.2 Preliminary inequalities

Taking the gradient of both sides of the special Lagrangian equation (1.1), we have

$$\sum_{i,j=1}^n g^{ij} \partial_{ij} (x, Du(x)) = 0, \quad (2.1)$$

where  $(g^{ij})$  is the inverse of the induced metric  $g = (g_{ij}) = I + D^2u D^2u$  on the surface  $(x, Du(x)) \subset \mathbb{R}^n \times \mathbb{R}^n$ . Simple geometric manipulation of (2.1) yields the usual form of the minimal surface equation

$$\Delta_g (x, Du(x)) = 0,$$

where the Laplace-Beltrami operator of the metric  $g$  is given by

$$\Delta_g = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^n \partial_i \left( \sqrt{\det g} g^{ij} \partial_j \right).$$

Because we are using harmonic coordinates  $\Delta_g x = 0$ , we see that  $\Delta_g$  also equals the linearized operator of the special Lagrangian equation (1.1) at  $u$ ,

$$\Delta_g = \sum_{i,j=1}^n g^{ij} \partial_{ij}.$$

The volume form, gradient and inner product with respect to the metric  $g$  are

$$\begin{aligned} dv_g &= \sqrt{\det g} \, dx, \\ \nabla_g v &= \left( \sum_{k=1}^n g^{1k} v_k, \dots, \sum_{k=1}^n g^{nk} v_k \right), \\ \langle \nabla_g v, \nabla_g w \rangle_g &= \sum_{i,j=1}^n g^{ij} v_i w_j, \quad \text{in particular} \quad |\nabla_g v|^2 = \langle \nabla_g v, \nabla_g v \rangle_g. \end{aligned}$$

We begin with some algebraic and trigonometric inequalities needed in this paper.

**Lemma 2.1.** *Suppose the ordered real numbers  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$  satisfy (1.1) with  $\Theta \geq (n-2)\pi/2$  and  $n \geq 2$ . Then we have*

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{n-1} > 0 \quad \text{and} \quad \lambda_{n-1} \geq |\lambda_n|, \quad (2.2)$$

$$\lambda_1 + (n-1)\lambda_n \geq 0, \quad (2.3)$$

$$\sigma_k(\lambda_1, \dots, \lambda_n) \geq 0 \quad \text{for all } 1 \leq k \leq n-1. \quad (2.4)$$

*Proof.* Set  $\theta_i = \arctan \lambda_i$ . Property (2.2) follows from the inequalities

$$\theta_{n-1} + \theta_n \geq (n-2)\pi/2 - (\theta_1 + \dots + \theta_{n-2}) \geq 0.$$

We only need to check property (2.3) when  $\lambda_n < 0$  or  $\theta_n < 0$ . We know

$$\frac{\pi}{2} > \frac{\pi}{2} + \theta_n \geq \left(\frac{\pi}{2} - \theta_1\right) + \dots + \left(\frac{\pi}{2} - \theta_{n-1}\right) > 0.$$

It follows that

$$\begin{aligned} -\frac{1}{\lambda_n} &= \tan\left(\frac{\pi}{2} + \theta_n\right) \\ &\geq \tan\left(\frac{\pi}{2} - \theta_1\right) + \dots + \tan\left(\frac{\pi}{2} - \theta_{n-1}\right) = \frac{1}{\lambda_1} + \dots + \frac{1}{\lambda_{n-1}} \\ &\geq (n-1) \frac{1}{\lambda_1}. \end{aligned} \quad (2.5)$$

Then we get (2.3).

Next we prove property (2.4) with  $k = n - 1$ . We only need to deal with the case  $\lambda_n < 0$ . From (2.5), we have

$$0 \geq \frac{1}{\lambda_1} + \cdots + \frac{1}{\lambda_{n-1}} + \frac{1}{\lambda_n} = \frac{\sigma_{n-1}(\lambda_1, \dots, \lambda_n)}{(\lambda_1 \cdots \lambda_{n-1}) \lambda_n}.$$

Using  $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} > 0 > \lambda_n$ , we get  $\sigma_{n-1}(\lambda_1, \dots, \lambda_n) \geq 0$ .

Finally we prove the whole property (2.4) inductively. Property (2.4) with  $n = 2$  is obvious (or by the above). Assume property (2.4) with  $n = m$  is true, that is

$$\sigma_j(\lambda_1, \dots, \lambda_m) \geq 0 \quad \text{for } 1 \leq j \leq m - 1,$$

provided  $\arctan \lambda_1 + \cdots + \arctan \lambda_m \geq (m - 2) \pi/2$ .

Let us prove (2.4) with  $n = m + 1$  for

$$\arctan \lambda_1 + \cdots + \arctan \lambda_{m+1} \geq (m - 1) \pi/2. \tag{2.6}$$

By the proved property (2.4) with  $k = n - 1 = m$ , we get  $\sigma_m(\lambda_1, \dots, \lambda_{m+1}) \geq 0$ . We only need to verify the other  $\sigma$  inequalities when the smallest number is negative, say  $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m > 0 > \lambda_{m+1}$ . (By (2.2), only the smallest  $\lambda_{m+1}$  can be negative.) We have

$$\sigma_{m-1}(\lambda_1, \dots, \lambda_{m+1}) = \sigma_{m-1}(\lambda_2, \dots, \lambda_{m+1}) + \lambda_1 \sigma_{m-2}(\lambda_2, \dots, \lambda_{m+1}).$$

From (2.6), we infer

$$\arctan \lambda_2 + \cdots + \arctan \lambda_{m+1} \geq (m - 2) \pi/2.$$

By the induction assumption, we should have

$$\sigma_{m-1}(\lambda_2, \dots, \lambda_{m+1}) \geq 0 \quad \text{and} \quad \sigma_{m-2}(\lambda_2, \dots, \lambda_{m+1}) \geq 0.$$

Thus we obtain  $\sigma_{m-1}(\lambda_1, \dots, \lambda_{m+1}) \geq 0$ . Similarly we prove  $\sigma_i(\lambda_1, \dots, \lambda_{m+1}) \geq 0$  for  $1 \leq i \leq m - 2$ . Therefore property (2.4) holds for all  $n \geq 2$ . This completes the proof of Lemma 2.1.  $\square$

**Lemma 2.2.** *Let  $u$  be a smooth solution to (1.1). Suppose that the Hessian  $D^2u$  is diagonalized and the eigenvalue  $\lambda_\gamma$  is distinct from all other eigenvalues of  $D^2u$  at point  $p$ . Then we have at  $p$*

$$\left| \nabla_g \ln \sqrt{1 + \lambda_\gamma^2} \right|^2 = \sum_{k=1}^n \lambda_\gamma^2 h_{\gamma\gamma k}^2 \quad (2.7)$$

and

$$\begin{aligned} \Delta_g \ln \sqrt{1 + \lambda_\gamma^2} &= \quad (2.8) \\ (1 + \lambda_\gamma^2) h_{\gamma\gamma\gamma}^2 &+ \sum_{k \neq \gamma} \left( \frac{2\lambda_\gamma}{\lambda_\gamma - \lambda_k} + \frac{2\lambda_\gamma^2 \lambda_k}{\lambda_\gamma - \lambda_k} \right) h_{kk\gamma}^2 \\ &+ \sum_{k \neq \gamma} \left[ 1 + \frac{2\lambda_\gamma}{\lambda_\gamma - \lambda_k} + \frac{\lambda_\gamma^2 (\lambda_\gamma + \lambda_k)}{\lambda_\gamma - \lambda_k} \right] h_{\gamma\gamma k}^2 \\ &+ \sum_{\substack{k > j \\ k, j \neq \gamma}} 2\lambda_\gamma \left[ \frac{1 + \lambda_k^2}{\lambda_\gamma - \lambda_k} + \frac{1 + \lambda_j^2}{\lambda_\gamma - \lambda_j} + (\lambda_j + \lambda_k) \right] h_{kj\gamma}^2. \end{aligned}$$

*Proof.* The calculation was done in Lemma 2.1 of [WY2]. □

**Lemma 2.3.** *Let  $u$  be a smooth solution to (1.1) with  $\Theta \geq (n - 2)\frac{\pi}{2}$ . Suppose that the ordered eigenvalues  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$  of the Hessian  $D^2u$  satisfy  $\lambda_1 = \dots = \lambda_m > \lambda_{m+1}$  at point  $p$ . Then the function  $b_m = \frac{1}{m} \sum_{i=1}^m \ln \sqrt{1 + \lambda_i^2}$  is smooth near  $p$  and satisfies at  $p$*

$$\Delta_g b_m \geq \left( 1 - \frac{4}{\sqrt{4n+1} + 1} \right) |\nabla_g b_m|^2. \quad (2.9)$$

*Proof.* Step 1. The function  $b_m$  is symmetric in  $\lambda_1, \dots, \lambda_m$ . Thus for  $m < n$ ,  $b_m$  is smooth when  $\lambda_m > \lambda_{m+1}$ , in particular near  $p$ , at which  $\lambda_1 = \dots = \lambda_m > \lambda_{m+1}$ . For  $m = n$ ,  $b_n$  is certainly smooth everywhere.

We again assume that Hessian  $D^2u$  is diagonalized at point  $p$ . Let us also first assume the first  $m$  eigenvalues  $\lambda_1, \dots, \lambda_m$  are distinct. Using (2.8) in Lemma 2.2, we calculate  $\Delta_g b_m$ ; after grouping those terms  $h_{\heartsuit\heartsuit\heartsuit}$ ,  $h_{\heartsuit\heartsuit\clubsuit}$  and  $h_{\heartsuit\clubsuit\heartsuit}$  in the summation,

we obtain

$$\begin{aligned}
m \Delta_g b_m &= \sum_{\gamma=1}^m \Delta_g \ln \sqrt{1 + \lambda_\gamma^2} \stackrel{p}{=} \\
&\sum_{k \leq m} (1 + \lambda_k^2) h_{kkk}^2 + \left( \sum_{i < k \leq m} + \sum_{k < i \leq m} \right) (3 + \lambda_i^2 + 2\lambda_i \lambda_k) h_{iik}^2 + \sum_{k \leq m < i} \frac{2\lambda_k (1 + \lambda_k \lambda_i)}{\lambda_k - \lambda_i} h_{iik}^2 \\
&+ \sum_{i \leq m < k} \frac{3\lambda_i - \lambda_k + \lambda_i^2 (\lambda_i + \lambda_k)}{\lambda_i - \lambda_k} h_{iik}^2 + \\
&\left\{ \begin{array}{l} 2 \sum_{i < j < k \leq m} (3 + \lambda_i \lambda_j + \lambda_j \lambda_k + \lambda_k \lambda_i) h_{ijk}^2 + \\ 2 \sum_{i < j \leq m < k} \left( 1 + \lambda_i \lambda_j + \lambda_j \lambda_k + \lambda_k \lambda_i + \lambda_i \frac{1 + \lambda_k^2}{\lambda_i - \lambda_k} + \lambda_j \frac{1 + \lambda_k^2}{\lambda_j - \lambda_k} \right) h_{ijk}^2 + \\ 2 \sum_{i \leq m < j < k} \lambda_i \left[ \lambda_j + \lambda_k + \frac{1 + \lambda_j^2}{\lambda_i - \lambda_j} + \frac{1 + \lambda_k^2}{\lambda_j - \lambda_k} \right] h_{ijk}^2 \end{array} \right. .
\end{aligned}$$

Now as a function of the matrices (then composed with smooth matrix function  $D^2u$  of  $x$ ),  $b_m$  is  $C^2$  at  $D^2u(p)$  with eigenvalues satisfying  $\lambda = \lambda_1 = \dots = \lambda_m > \lambda_{m+1}$ . Note that  $D^2u(p)$  can be approximated by matrices with distinct eigenvalues. Therefore the above expression for  $\Delta_g b_m$  at  $p$  still holds and simplifies to

$$\begin{aligned}
m \Delta_g b_m &\stackrel{p}{=} \\
&\sum_{k \leq m} (1 + \lambda^2) h_{kkk}^2 + \left( \sum_{i < k \leq m} + \sum_{k < i \leq m} \right) (3 + 3\lambda^2) h_{iik}^2 + \sum_{k \leq m < i} \frac{2\lambda (1 + \lambda \lambda_i)}{\lambda - \lambda_i} h_{iik}^2 + \\
&\sum_{i \leq m < k} \frac{3\lambda - \lambda_k + \lambda^2 (\lambda + \lambda_k)}{\lambda - \lambda_k} h_{iik}^2 + \\
&\left\{ \begin{array}{l} 2 \sum_{i < j < k \leq m} (3 + 3\lambda^2) h_{ijk}^2 + \\ 2 \sum_{i < j \leq m < k} \left[ 1 + \frac{2\lambda}{\lambda - \lambda_k} + \frac{\lambda^2 (\lambda + \lambda_k)}{\lambda - \lambda_k} \right] h_{ijk}^2 + \\ 2 \sum_{i \leq m < j < k} \lambda \left( \lambda_j + \lambda_k + \frac{1 + \lambda_j^2}{\lambda - \lambda_j} + \frac{1 + \lambda_k^2}{\lambda - \lambda_k} \right) h_{ijk}^2 \end{array} \right. \\
&\geq \sum_{k \leq m} \lambda^2 h_{kkk}^2 + \left( \sum_{i < k \leq m} + \sum_{k < i \leq m} \right) 3\lambda^2 h_{iik}^2 + \sum_{k \leq m < i} \frac{2\lambda^2 \lambda_i}{\lambda - \lambda_i} h_{iik}^2 + \\
&\sum_{i \leq m < k} \frac{\lambda^2 (\lambda + \lambda_k)}{\lambda - \lambda_k} h_{iik}^2,
\end{aligned}$$

where we used (2.2) of Lemma 2.1 in the inequality.

Similarly by (2.7) in Lemma 2.2 and the  $C^1$  continuity of  $b_m$  as a function of

matrices at  $D^2u(p)$ , we obtain

$$|\nabla_g b_m|^2 \stackrel{p}{=} \frac{1}{m^2} \sum_{1 \leq k \leq n} \lambda^2 \left( \sum_{i \leq m} h_{iik} \right)^2 \leq \frac{\lambda^2}{m} \sum_{1 \leq k \leq n} \left( \sum_{i \leq m} h_{iik}^2 \right).$$

From the above two inequalities, it follows that

$$m (\Delta_g b_m - \varepsilon |\nabla_g b_m|^2) \geq \lambda^2 \left[ \sum_{k \leq m} (1 - \varepsilon) h_{kkk}^2 + (\sum_{i < k \leq m} + \sum_{k < i \leq m}) (3 - \varepsilon) h_{iik}^2 + \sum_{k \leq m < i} \frac{2\lambda_i}{\lambda - \lambda_i} h_{iik}^2 \right] + \quad (2.10)$$

$$\lambda^2 \left[ \sum_{i \leq m < k} \left( \frac{\lambda + \lambda_k}{\lambda - \lambda_k} - \varepsilon \right) h_{iik}^2 \right] \quad (2.11)$$

with  $\varepsilon$  to be fixed.

Step 2. We show (2.10) and (2.11) in the above inequality are nonnegative for  $\varepsilon = 1 - 4/(\sqrt{4n+1} + 1)$ . For each fixed  $k$  in (2.10) and (2.11), set  $t_i = h_{iik}$ . By the minimal surface equation (2.1), we have

$$t_1 + \cdots + t_n = 0. \quad (2.12)$$

Step 2.1. For each fixed  $k \leq m$ , we prove the  $[ ]_k$  term in (2.10) is nonnegative. In the case with all  $\lambda_i \geq 0$ , the nonnegativity is straightforward. In the remaining worst case  $\lambda_{n-1} > 0 > \lambda_n$ . Without loss of generality, we assume  $k = 1$  for simple notation. Then we proceed as follows:

$$\begin{aligned} [ ]_1 &= \left\{ (1 - \varepsilon) t_1^2 + \sum_{i=2}^m (3 - \varepsilon) t_i^2 + \sum_{i=m+1}^{n-1} \frac{2\lambda_i}{\lambda - \lambda_i} t_i^2 \right\} + \frac{2\lambda_n}{\lambda - \lambda_n} t_n^2 \\ &= \left\{ (1 - \varepsilon) t_1^2 + \sum_{i=2}^m (3 - \varepsilon) t_i^2 + \sum_{i=m+1}^{n-1} \frac{2\lambda_i}{\lambda - \lambda_i} t_i^2 \right\} + \frac{2\lambda_n}{\lambda - \lambda_n} \left( \sum_{i=1}^{n-1} t_i \right)^2 \\ &\geq \left\{ (1 - \varepsilon) t_1^2 + \sum_{i=2}^m (3 - \varepsilon) t_i^2 + \sum_{i=m+1}^{n-1} \frac{2\lambda_i}{\lambda - \lambda_i} t_i^2 \right\} \\ &\quad \left[ 1 + \frac{2\lambda_n}{\lambda - \lambda_n} \left( \frac{1}{1 - \varepsilon} + \sum_{i=2}^m \frac{1}{3 - \varepsilon} + \sum_{i=m+1}^{n-1} \frac{\lambda - \lambda_i}{2\lambda_i} \right) \right], \end{aligned}$$

where we used (2.12) and a Cauchy-Schwarz inequality to reach the above inequality.

We now show the second factor  $[\ ]$  in the last term is also nonnegative:

$$\begin{aligned}
& \left[ 1 + \frac{2\lambda_n}{\lambda - \lambda_n} \left( \frac{1}{1 - \varepsilon} + \sum_{i=2}^m \frac{1}{3 - \varepsilon} + \sum_{i=m+1}^{n-1} \frac{\lambda - \lambda_i}{2\lambda_i} \right) \right] \\
&= \frac{2\lambda_n}{\lambda - \lambda_n} \left( \frac{\lambda - \lambda_n}{2\lambda_n} + \frac{1}{1 - \varepsilon} + \frac{m-1}{3 - \varepsilon} + \frac{\lambda - \lambda_{m+1}}{2\lambda_{m+1}} + \cdots + \frac{\lambda - \lambda_{n-1}}{2\lambda_{n-1}} \right) \\
&= \frac{2\lambda_n}{\lambda - \lambda_n} \left[ \frac{1}{1 - \varepsilon} + \frac{m-1}{3 - \varepsilon} + \frac{\lambda}{2} \left( \frac{1}{\lambda_1} + \cdots + \frac{1}{\lambda_1} \right) - \frac{n}{2} \right] \\
&= \frac{2\lambda_n}{\lambda - \lambda_n} \left[ \frac{1}{1 - \varepsilon} + \frac{m-1}{3 - \varepsilon} + \frac{\lambda \sigma_{n-1}}{2 \sigma n} - \frac{n}{2} \right] \\
&\geq \frac{2\lambda_n}{\lambda - \lambda_n} \left( \frac{1}{1 - \varepsilon} + \frac{m-1}{3 - \varepsilon} - \frac{n}{2} \right) \\
&\geq 0,
\end{aligned}$$

where we used  $\lambda_1 = \cdots = \lambda_m$ , (2.4), and  $\frac{1}{1-\varepsilon} + \frac{m-1}{3-\varepsilon} - \frac{n}{2} \leq 0$  under the assumption

$$\varepsilon \leq 2 - \frac{m}{n} - \sqrt{\left(1 - \frac{m}{n}\right)^2 + \frac{4}{n}}.$$

Therefore  $[\ ]_1 \geq 0$ .

Step 2.2. For each  $k$  between  $m$  and  $n$ , we have  $\lambda_k > 0$ , the  $[\ ]_k$  term in (2.11) satisfies

$$\begin{aligned}
[\ ]_k &= \sum_{i \leq m} \left( \frac{\lambda + \lambda_k}{\lambda - \lambda_k} - \varepsilon \right) t_i^2 \\
&\geq \sum_{i \leq m} (1 - \varepsilon) t_i^2 \geq 0,
\end{aligned}$$

as long as  $\varepsilon \leq 1$ .

For  $k = n$ , the  $[\ ]_n$  term in (2.11) becomes

$$\begin{aligned}
[\ ]_n &= \sum_{i \leq m} \left( \frac{\lambda + \lambda_n}{\lambda - \lambda_n} - \varepsilon \right) t_i^2 \\
&\geq \sum_{i \leq m} \left( \frac{n-2}{n} - \varepsilon \right) t_i^2 \geq 0,
\end{aligned}$$

where we used (2.3) and we also assumed  $\varepsilon \leq \frac{n-2}{n}$ .

Note that for  $n-1 \geq m \geq 1$

$$1 - \frac{4}{\sqrt{4n+1}+1} \leq 2 - \frac{m}{n} - \sqrt{\left(1 - \frac{m}{n}\right)^2 + \frac{4}{n}} \leq \frac{n-2}{n},$$

therefore we have proved (2.9) with  $n-1 \geq m \geq 1$ . When  $m = n$ , we have  $\lambda_1 = \dots = \lambda_n > 0$ . Then from (2.10) we see in a much easier way that (2.9) holds.

The proof of Lemma 2.3 is complete.  $\square$

**Proposition 2.1.** *Let  $u$  be a smooth solution to the special Lagrangian equation (1.1) with  $n \geq 2$  and  $\Theta \geq (n-2)\pi/2$  on  $B_R(0) \subset \mathbb{R}^n$ . Set*

$$b = \ln \sqrt{1 + \lambda_{\max}^2},$$

where  $\lambda_{\max}$  is the largest eigenvalue of Hessian  $D^2u$ , namely,  $\lambda_{\max} = \lambda_1 \geq \dots \geq \lambda_n$ .

Then  $b$  satisfies the integral Jacobi inequality

$$\int_{B_R} -\langle \nabla_g \varphi, \nabla_g b \rangle_g dv_g \geq \varepsilon(n) \int_{B_R} \varphi |\nabla_g b|^2 dv_g \quad (2.13)$$

for all non-negative  $\varphi \in C_0^\infty(B_R)$ , where  $\varepsilon(n) = 1 - 4/(\sqrt{4n+1}+1)$ .

*Proof.* If  $b(x) = b_1(x)$  is smooth everywhere, then the pointwise Jacobi inequality (2.9) in Lemma 2.3 with  $m = 1$  already implies the integral Jacobi inequality (2.13). In general, we know that  $\lambda_{\max}$  is only a Lipschitz function of the entries of the Hessian  $D^2u$ . By the assumption,  $D^2u(x)$  is smooth in  $x$ , thus  $b = b_1 = \ln \sqrt{1 + \lambda_{\max}^2}$  is Lipschitz in terms of  $x$ .

Set  $\varepsilon = \varepsilon(n)$ . We first show that

$$\Delta_g b \geq \varepsilon |\nabla_g b|^2 \quad \text{in the viscosity sense.}$$

Given any quadratic polynomial  $Q$  touching  $b$  from above at  $p$ . If  $p$  is a smooth point of  $b$ , by (2.9) with  $m = 1$ , we get

$$\Delta_g Q \geq \varepsilon |\nabla_g Q|^2 \quad \text{at } p.$$

Otherwise, eigenvalue  $\lambda_1$  is not distinct at  $p$ . Suppose  $\lambda_1 = \cdots = \lambda_k > \lambda_{k+1}$  at  $p$ . Then  $Q$  also touches the smooth  $b_k = \left( \ln \sqrt{1 + \lambda_1^2} + \cdots + \ln \sqrt{1 + \lambda_k^2} \right) / k$  from above at  $p$ , because

$$b(x) \geq b_k(x) \quad \text{and} \quad b(p) = b_k(p).$$

By pointwise Jacobi inequality (2.9) with  $m = k$ , we still have

$$\Delta_g Q \geq \varepsilon |\nabla_g Q|^2 \quad \text{at } p.$$

Next we switch to  $a = e^{-\varepsilon b}$  and  $a_k = e^{-\varepsilon b_k}$ , the above argument leads to

$$\Delta_g a \leq 0 \quad \text{in the viscosity sense.}$$

Relying on the definition of viscosity supersolutions, we see  $a$  is  $\Delta_g$ -superharmonic in the potential sense, namely,  $a \geq h$  in any regular domain  $\Omega$  for  $\Delta_g$ -harmonic function  $h$  with the boundary value  $a$  on  $\partial\Omega$  :

$$\begin{cases} \Delta_g h = 0 & \text{in } \Omega \\ h = a & \text{on } \partial\Omega \end{cases}.$$

By [HH, Theorem 1] (see also [Wn, p. 246]), we obtain

$$\Delta_g a \leq 0 \quad \text{in the distribution sense.}$$

Note  $a$  is Lipschitz because  $b$  is. We move to the integral Jacobi inequality as follows.

Take the test function  $\varphi e^{\varepsilon b}$  for and nonnegative  $\varphi \in C_0^\infty$ , we get

$$\begin{aligned} 0 &\geq \int_{B_R} \varphi e^{\varepsilon b} \Delta_g a \, dv_g = \int_{B_R} -\langle \nabla_g (\varphi e^{\varepsilon b}), \nabla_g a \rangle_g \, dv_g \\ &= \int_{B_R} \langle e^{\varepsilon b} (\nabla_g \varphi + \varepsilon \varphi \nabla_g b), \varepsilon e^{-\varepsilon b} \nabla_g b \rangle_g \, dv_g \\ &= \int_{B_R} \left( \varepsilon \langle \nabla_g \varphi, \nabla_g b \rangle_g + \varepsilon^2 \varphi |\nabla_g b|_g^2 \right) \, dv_g. \end{aligned}$$

Thus we arrive at the integral Jacobi inequality (2.13). □

### 2.3 Proof of Theorem 1.1

We assume that  $R = 2n + 1$  and  $u$  is a solution on  $B_{2n+1} \subset \mathbb{R}^n$  for simplicity of notation. By scaling  $v(x) = u\left(\frac{R}{2n+1}x\right) / \left(\frac{R}{2n+1}\right)^2$ , we still get the estimate in Theorem 1.1. We consider the case  $\Theta \geq (n-2)\pi/2$ . The negative phase case  $\Theta \leq -(n-2)\pi/2$  follows by symmetry.

Step 1. By the integral Jacobi inequality (2.13) in Proposition 2.1,  $b$  is subharmonic in the integral sense. Then  $b^{\frac{n}{n-2}}$  is also subharmonic in the integral sense on the minimal surface  $\mathfrak{M} = (x, Du)$ :

$$\begin{aligned} & \int - \left\langle \nabla_g \varphi, \nabla_g b^{\frac{n}{n-2}} \right\rangle_g dv_g \\ &= \int - \left\langle \nabla_g \left( \frac{n}{n-2} b^{\frac{n}{n-2}} \varphi \right) - \frac{2n}{(n-2)^2} b^{\frac{4-n}{n-2}} \varphi \nabla_g b, \nabla_g b \right\rangle_g dv_g \\ &\geq \int \left( \frac{n}{n-2} \varepsilon(n) \varphi b^2 |\nabla_g b|^2 + \frac{2n}{(n-2)^2} b^{\frac{4-n}{n-2}} \varphi |\nabla_g b|^2 \right) dv_g \geq 0 \end{aligned}$$

for all non-negative  $\varphi \in C_0^\infty$ , where we approximate  $b$  by smooth functions if necessary.

Applying Michael-Simon's mean value inequality [MS, Theorem 3.4] to the Lipschitz subharmonic function  $b^{\frac{n}{n-2}}$ , we obtain

$$b(0) \leq C(n) \left( \int_{\mathfrak{B}_1 \cap \mathfrak{M}} b^{\frac{n}{n-2}} dv_g \right)^{\frac{n-2}{n}} \leq C(n) \left( \int_{B_1} b^{\frac{n}{n-2}} dv_g \right)^{\frac{n-2}{n}},$$

where  $\mathfrak{B}_r$  is the ball with radius  $r$  and center at  $(0, Du(0))$  in  $\mathbb{R}^n \times \mathbb{R}^n$ , and  $B_r$  is the ball with radius  $r$  and center at 0 in  $\mathbb{R}^n$ . Choose a cut-off function  $\varphi \in C_0^\infty(B_2)$  such that  $\varphi \geq 0$ ,  $\varphi = 1$  on  $B_1$ , and  $|D\varphi| \leq 1.1$ ; we then have

$$\left( \int_{B_1} b^{\frac{n}{n-2}} dv_g \right)^{\frac{n-2}{n}} \leq \left( \int_{B_2} \varphi^{\frac{2n}{n-2}} b^{\frac{n}{n-2}} dv_g \right)^{\frac{n-2}{n}} = \left( \int_{B_2} (\varphi b^{1/2})^{\frac{2n}{n-2}} dv_g \right)^{\frac{n-2}{n}}.$$

Applying the Sobolev inequality on the minimal surface  $\mathfrak{M}$  [MS, Theorem 2.1] or [A, Theorem 7.3] to  $\varphi b^{1/2}$ , which we may assume to be  $C^1$  by approximation, we obtain

$$\left( \int_{B_2} (\varphi b^{1/2})^{\frac{2n}{n-2}} dv_g \right)^{\frac{n-2}{n}} \leq C(n) \int_{B_2} |\nabla_g (\varphi b^{1/2})|^2 dv_g.$$

Decomposing the integrand as follows

$$\begin{aligned} |\nabla_g (\varphi b^{1/2})|^2 &= \left| \frac{1}{2b^{1/2}} \varphi \nabla_g b + b^{1/2} \nabla_g \varphi \right|^2 \leq \frac{1}{2b} \varphi^2 |\nabla_g b|^2 + 2b |\nabla_g \varphi|^2 \\ &\leq \frac{1}{\ln(4/3)} \varphi^2 |\nabla_g b|^2 + 2b |\nabla_g \varphi|^2, \end{aligned}$$

where we used

$$b \geq \ln \sqrt{1 + \tan^2 \left( \frac{\pi}{2} - \frac{\pi}{n} \right)} \geq \ln \sqrt{4/3},$$

we get

$$\begin{aligned} b(0) &\leq C(n) \int_{B_2} |\nabla_g (\varphi b^{1/2})|^2 dv_g \\ &\leq C(n) \left( \int_{B_2} \varphi^2 |\nabla_g b|^2 dv_g + \int_{B_2} b |\nabla_g \varphi|^2 dv_g \right). \end{aligned}$$

Step 2. By (2.13) in Proposition 2.1,  $b$  satisfies the Jacobi inequality in the integral sense:

$$\frac{1}{\varepsilon(n)} \Delta_g b \geq |\nabla_g b|^2.$$

Multiplying both sides by the above non-negative cut-off function  $\varphi \in C_0^\infty(B_2)$ , then integrating, we obtain

$$\begin{aligned} \int_{B_2} \varphi^2 |\nabla_g b|^2 dv_g &\leq \frac{1}{\varepsilon(n)} \int_{B_2} \varphi^2 \Delta_g b dv_g \\ &= \frac{-1}{\varepsilon(n)} \int_{B_2} \langle 2\varphi \nabla_g \varphi, \nabla_g b \rangle dv_g \\ &\leq \frac{1}{2} \int_{B_2} \varphi^2 |\nabla_g b|^2 dv_g + \frac{2}{\varepsilon(n)^2} \int_{B_2} |\nabla_g \varphi|^2 dv_g. \end{aligned}$$

It follows that

$$\int_{B_2} \varphi^2 |\nabla_g b|^2 dv_g \leq \frac{4}{\varepsilon(n)^2} \int_{B_2} |\nabla_g \varphi|^2 dv_g.$$

So far we have reached

$$\begin{aligned} b(0) &\leq C(n) \left( \int_{B_2} |\nabla_g \varphi|^2 dv_g + \int_{B_2} b |\nabla_g \varphi|^2 dv_g \right) \\ &\leq C(n) \int_{B_2} b |\nabla_g \varphi|^2 dv_g \\ &\leq C(n) \int_{B_2} b \sum_{i=1}^n \frac{1}{1 + \lambda_i^2} \sqrt{\det g} dx, \end{aligned} \tag{2.14}$$

where in the second inequality, we again used  $b \geq \ln \sqrt{4/3}$ .

Step 3. Differentiating the complex identity

$$\begin{aligned} \ln V + \sqrt{-1} \sum_{i=1}^n \arctan \lambda_i &= \ln \prod_{i=1}^n (1 + \sqrt{-1} \lambda_i) \\ &= \ln \left[ \sum_{0 \leq 2k \leq n} (-1)^k \sigma_{2k} + \sqrt{-1} \sum_{1 \leq 2k+1 \leq n} (-1)^k \sigma_{2k+1} \right]. \end{aligned}$$

we obtain the (conformality) identity

$$\left( \frac{1}{1 + \lambda_1^2}, \dots, \frac{1}{1 + \lambda_n^2} \right) V = \left( \frac{\partial \Sigma}{\partial \lambda_1}, \dots, \frac{\partial \Sigma}{\partial \lambda_n} \right)$$

with  $V = \sqrt{\det g}$  and

$$\begin{aligned} \Sigma &= \cos \Theta \sum_{1 \leq 2k+1 \leq n} (-1)^k \sigma_{2k+1} - \sin \Theta \sum_{0 \leq 2k \leq n} (-1)^k \sigma_{2k} \\ &= \sigma_{n-1} - \sigma_{n-3} + \dots, \text{ in particular when } |\Theta| = (n-2) \frac{\pi}{2}. \end{aligned}$$

Taking trace, we then get

$$\begin{aligned} &\sum_{i=1}^n \frac{1}{1 + \lambda_i^2} V = \sum_{i=1}^n \frac{\partial \Sigma}{\partial \lambda_i} \\ &= \cos \Theta \sum_{1 \leq 2k+1 \leq n} (-1)^k (n-2k) \sigma_{2k} - \sin \Theta \sum_{0 \leq 2k \leq n} (-1)^k (n-2k+1) \sigma_{2k-1} \\ &= c_0 + c_1 \sigma_1 + \dots + c_{n-1} \sigma_{n-1}, \end{aligned} \tag{2.15}$$

where the coefficient  $c_i$  depends only on  $i, n$ , and  $\Theta$ . At the critical phase  $|\Theta| = (n-2)\pi/2$ , the leading term in (2.15) is  $\sigma_{n-2}$

$$\sum_{i=1}^n \frac{1}{1 + \lambda_i^2} V = 2\sigma_{n-2} - 4\sigma_{n-4} + \dots \tag{2.16}$$

In turn, (2.14) becomes

$$b(0) \leq C(n) \int_{B_2} b(c_0 + c_1 \sigma_1 + \dots + c_{n-1} \sigma_{n-1}) dx. \tag{2.17}$$

Step 4. Next we estimate the integrals  $\int b\sigma_k dx$  for  $1 \leq k \leq n-1$  inductively, using the divergence structure of  $\sigma_k(D^2u)$  :

$$\begin{aligned} k\sigma_k(D^2u) &= \sum_{i,j=1}^n \frac{\partial\sigma_k}{\partial u_{ij}} \frac{\partial^2 u}{\partial x_i \partial x_j} = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left( \frac{\partial\sigma_k}{\partial u_{ij}} \frac{\partial u}{\partial x_j} \right) \\ &= \operatorname{div} (L_{\sigma_k} Du), \end{aligned}$$

where  $L_{\sigma_k}$  denotes the matrix  $\left( \frac{\partial\sigma_k}{\partial u_{ij}} \right)$ . Let  $\psi$  be a smooth cut-off function on  $B_{\rho+1}$  such that  $\psi = 1$  on  $B_\rho$ ,  $0 \leq \psi \leq 1$ , and  $|D\psi| \leq 1.1$ . Noticing that  $\sigma_k > 0$  by (2.4) in Lemma 2.1 and  $b > 0$ , we have

$$\begin{aligned} \int_{B_\rho} b\sigma_k dx &\leq \int_{B_{\rho+1}} \psi b\sigma_k dx = \int_{B_{\rho+1}} \psi b \frac{1}{k} \operatorname{div} (L_{\sigma_k} Du) dx \\ &= \frac{1}{k} \int_{B_{\rho+1}} -\langle bD\psi + \psi Db, L_{\sigma_k} Du \rangle dx \\ &\leq C(n) \|Du\|_{L^\infty(B_{\rho+1})} \left[ \int_{B_{\rho+1}} b\sigma_{k-1} dx + \int_{B_{\rho+1}} [|\nabla_g b|^2 + \operatorname{tr}(g^{ij})] \sqrt{\det g} dx \right]. \end{aligned} \quad (2.18)$$

The last inequality was derived as follows. As all the above integrands are invariant under orthogonal transformations, at any point  $p \in B_{\rho+1}$ , we assume  $D^2u(p)$  is diagonalized. Then  $L_{\sigma_k}$  is also diagonal with positive entries  $\partial_{\lambda_i}\sigma_k$ . The positivity can be seen by applying Lemma 2.1 to all  $\lambda_1, \dots, \lambda_n$  but  $\lambda_i$ , whose corresponding phase is no less than  $(n-3)\pi/2$ . Thus  $0 < \partial_{\lambda_i}\sigma_k < (n-k+1)\sigma_{k-1}$ . Now we have

$$\begin{aligned} |\langle bD\psi + \psi Db, L_{\sigma_k} Du \rangle| &\stackrel{p}{\leq} \sum_{i=1}^n (b|D_i\psi| + \psi|D_ib|) \partial_{\lambda_i}\sigma_k |D_iu| \\ &\stackrel{p}{\leq} C(n) |Du(p)| \left( b\sigma_{k-1} + \sum_{i=1}^n |D_ib| \partial_{\lambda_i}\sigma_k \right). \end{aligned}$$

Recall  $k \leq n-1$ , then  $\partial_{\lambda_i}\sigma_k$  only consists of multiples of at most  $(n-2)$  eigenvalues without  $\lambda_i$ . ‘‘Twist’’ multiplying the two  $g^{\heartsuit\heartsuit}$  terms involving the missed  $\lambda_i$  and the

other eigenvalue, we obtain

$$\begin{aligned} |D_i b| \partial_{\lambda_i} \sigma_k &\stackrel{p}{\leq} |D_i b| \partial_{\lambda_i} \sigma_k (|\lambda_1|, \dots, |\lambda_n|) \\ &\stackrel{p}{\leq} C(n) \sum_{\alpha \neq i} \left( \frac{|D_i b|^2}{1 + \lambda_i^2} + \frac{1}{1 + \lambda_\alpha^2} \right) \sqrt{(1 + \lambda_1^2) \cdots (1 + \lambda_n^2)}. \end{aligned}$$

Summing up, we get

$$\begin{aligned} \sum_{i=1}^n |D_i b| \partial_{\lambda_i} \sigma_k &\stackrel{p}{\leq} C(n) \sum_{i=1}^n (g^{ii} |D_i b|^2 + g^{ii}) \sqrt{\det g} \\ &\stackrel{p}{=} C(n) [|\nabla_g b|^2 + \text{tr}(g^{ij})] \sqrt{\det g}. \end{aligned}$$

The inequality (2.18) has been established. To simplify the last integral in (2.18), we repeat the integral Jacobi argument in Step 2 to get

$$\int_{B_{\rho+1}} |\nabla_g b|^2 \sqrt{\det g} \, dx \leq C(n) \int_{B_{\rho+2}} \text{tr}(g^{ij}) \sqrt{\det g} \, dx.$$

Hence (2.18) becomes the following inductive inequality

$$\int_{B_\rho} b \sigma_k \, dx \leq C(n) \|Du\|_{L^\infty(B_{\rho+1})} \left[ \int_{B_{\rho+1}} b \sigma_{k-1} \, dx + \int_{B_{\rho+2}} \text{tr}(g^{ij}) \sqrt{\det g} \, dx \right]. \quad (2.19)$$

Step 4.1. We iterate (2.19) to derive

$$\begin{aligned} &\int_{B_2} b \sigma_k \, dx \\ &\leq C(n) \left\{ \begin{aligned} &\|Du\|_{L^\infty(B_{2+k})}^k \int_{B_{2+k}} b \, dx + \\ &\left[ \|Du\|_{L^\infty(B_{2+k})}^k + \cdots + \|Du\|_{L^\infty(B_{2+k})} \right] \int_{B_{2+k+1}} \text{tr}(g^{ij}) \sqrt{\det g} \, dx \end{aligned} \right\} \\ &\leq C(n) \left\{ \begin{aligned} &\|Du\|_{L^\infty(B_{2+k})}^{k+1} + \\ &\left[ \|Du\|_{L^\infty(B_{2+k})}^k + \|Du\|_{L^\infty(B_{2+k})} \right] \int_{B_{2+k+1}} \text{tr}(g^{ij}) \sqrt{\det g} \, dx \end{aligned} \right\}, \end{aligned}$$

where for the last inequality, we used Young's inequality and

$$\int_{B_{2+k}} b \, dx \leq C(n) \|Du\|_{L^\infty(B_{2+k})},$$

which follows from

$$b = \ln \sqrt{1 + \lambda_{\max}^2} < \lambda_{\max} < \lambda_1 + \lambda_2 + \cdots + \lambda_n = \Delta u$$

by (2.2) in Lemma 2.1. Putting all the estimates for  $b\sigma_k$ s in (2.17) together, we get

$$b(0) \leq C(n) \left\{ \begin{aligned} & \|Du\|_{L^\infty(B_{n+1})}^n + \|Du\|_{L^\infty(B_{n+1})} + \\ & \left[ \|Du\|_{L^\infty(B_{n+1})}^{n-1} + \|Du\|_{L^\infty(B_{n+1})} \right] \int_{B_{n+2}} \text{tr}(g^{ij}) \sqrt{\det g} \, dx \end{aligned} \right\}. \quad (2.20)$$

Step 4.2. We bound the last integral in the above inequality. Relying on the trace conformality identity (2.15), we derive

$$\begin{aligned} \int_{B_{n+2}} \text{tr}(g^{ij}) \sqrt{\det g} \, dx &= \int_{B_{n+2}} (c_0 + c_1\sigma_1 + \cdots + c_{n-1}\sigma_{n-1}) \, dx \\ &\leq C(n) \left[ \|Du\|_{L^\infty(B_{2n+1})}^{n-1} + 1 \right], \end{aligned} \quad (2.21)$$

where for the last inequality, we repeated the iteration integral estimates for (2.19) in Step 4.1 with  $b = 1$  (now much simpler)

$$\int_{B_\rho} \sigma_k \, dx \leq C(n) \|Du\|_{L^\infty(B_{\rho+1})} \int_{B_{\rho+1}} \sigma_{k-1} \, dx.$$

Finally from the above estimates (2.21) and (2.20), we conclude that

$$b(0) \leq C(n) \left[ \|Du\|_{L^\infty(B_{2n+1})}^{2n-2} + \|Du\|_{L^\infty(B_{2n+1})} \right]$$

and after exponentiating

$$|D^2u(0)| \leq C(n) \exp \left[ C(n) \|Du\|_{L^\infty(B_{2n+1})}^{2n-2} \right].$$

Note at the critical phase  $\Theta = (n-2)\pi/2$ , because of (2.16), the leading term in (2.17) and (2.21) is  $\sigma_{n-2}$ . The iteration integral estimates in Step 4.1 and 4.2 start from  $\sigma_{n-2}$ . Thus we really obtain

$$|D^2u(0)| \leq C(n) \exp \left[ C(n) \|Du\|_{L^\infty(B_{2n})}^{2n-4} \right].$$

The proof of Theorem 1.1 is complete.

## Chapter 3

**SINGULAR SOLUTIONS TO SPECIAL LAGRANGIAN  
EQUATIONS WITH SUBCRITICAL PHASES AND  
MINIMAL SURFACE SYSTEMS**

**3.1 Introduction**

Our construction goes as follows. In the first stage, we solve the special Lagrangian equation (1.1) with the critical phase by Cauchy-Kowalevskaya. The approximate solutions or initial data for the relatively “easier” corresponding quadratic equation

$$\sigma_2(D^2u) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = 1 \tag{3.1}$$

or

$$\Delta u = \det D^2u. \tag{3.2}$$

are built up via a *systematic* procedure, which allows us to have the approximation at arbitrarily high order (Property S.1 and S.2), and eventually those highly (“oddly”  $C^{1,1/(2m-1)}$ ) singular solutions in Theorem 1.2 and Theorem 1.4.

In the second stage, we take an “inversion”  $\frac{\pi}{2}$  rotation of the solutions from the first stage to obtain those singular solutions with phase 0 (Proposition 3.1). The singular solutions with other subcritical phases are achieved via a preliminary “horizontal” rotation before the “inversion”  $\frac{\pi}{2}$  rotation (Step 1 of Section 3.4). Some remarks are in order. Those  $\mathbb{U}(n)$  rotations are “obvious” to produce for the  $\mathbb{U}(n)$  invariant special Lagrangian equation (1.1). But it is by no means easy to justify that the special Lagrangian submanifold is still a graph in the rotated new coordinate system, thus a valid equation (1.1) to work on. (Earlier development of those  $\mathbb{U}(n)$  rotations for (1.1) can be found in [Y2] [Y3].) Here our *elementary analytic* justification for the

“inversion”  $\frac{\pi}{2}$  rotation (Proposition 4.1) avoids a hard and deep topological formula of [EL], which was employed in [NV]. Lastly we point out that the Legendre transformation (usually used for convex functions), is just the “inversion”  $\frac{\pi}{2}$  rotation followed by a conjugation for converting “gradient” graph  $(x, Du(x))$  to the one  $(Du^*(y), y)$  (now with saddle potentials  $u$  and  $u^*$ ).

In the third stage, we kick in a little bit extra to the preliminary “horizontal” rotations of Stage 2, then after the same “inversion”  $\frac{\pi}{2}$  rotation, we make up a corresponding little bit “backward” rotation to finally generate the desired family of smooth solutions in Theorem 1.3, which break a priori Hessian estimates for special Lagrangian equation (1.1) with subcritical phase.

### 3.2 Cauchy-Kowalevskaya with critical phase $\Theta = \frac{\pi}{2}$

As a preparation for the constructions in the next three sections, we solve the following special Lagrangian equation with critical phase in dimension three by Cauchy-Kowalevskaya. The quadratic nature of the equation at the critical phase is easier to work with than the cubic nature of the equations otherwise.

Our approximate solution  $P(x)$  to the equation

$$\begin{cases} \sigma_2(D^2u) = \frac{1}{2} \left[ (\Delta u)^2 - |D^2u|^2 \right] = 1 \quad \text{or} \quad \sum_{i=1}^3 \arctan \lambda_i = \frac{\pi}{2} \\ u_3(x_1, x_2, 0) = P_3(x_1, x_2, 0) \\ u(x_1, x_2, 0) = P(x_1, x_2, 0) \end{cases} \quad (3.3)$$

is a polynomial of degree  $2m$

$$P = \frac{1}{2} (x_1^2 + x_2^2) + \operatorname{Re} Z^m x_3 + \frac{m^2}{4} \rho^{2m-2} x_3^2 + \nu \sum_{j=0}^m a_j x_3^{2m-2j} \rho^{2j},$$

where  $Z = x_1 + \sqrt{-1}x_2 = \rho \exp(\sqrt{-1}\theta)$ , coefficients  $\nu$  and  $a_j$ s are to be determined later. We construct this  $P$  satisfying the following four properties, so does  $u$  then, for  $|x| = r \leq r_m$  with positive  $r_m$  depending only on  $m$ .

Property S.1.  $\sigma_2(D^2P) - 1 = [r^{3m-3}]$ , here  $[r^k]$  represents an analytic function starting from order  $k$ . Then the solution  $u$  coincides with  $P$  up to order  $3m-2$  ( $\geq 2m$  for  $m \geq 2, 3, 4, \dots$ ).

Property S.2. The three eigenvalues of  $D^2P$ , then also  $D^2u$  satisfy

$$\begin{aligned}\lambda_1 &= 1 + [r^{m-1}] \\ \lambda_2 &= 1 + [r^{m-1}] \\ -\delta_2(m) r^{2m-2} &\leq \lambda_3 \leq -\delta_1(m) r^{2m-2}\end{aligned}$$

Property S.3. The “gradient” graph

$$(x, Du) = \begin{pmatrix} x, x_1 + O(\rho)[r^{m-1}] + [r^{2m}], x_2 + O(\rho)[r^{m-1}] + [r^{2m}], \\ \operatorname{Re} Z^m + \frac{m^2}{2}\rho^{2m-2}x_3 - 2m\nu x_3^{2m-1} + \nu\rho^2[r^{2m-3}] + [r^{2m}] \end{pmatrix}.$$

Property S.4. The gradient  $Du$  satisfies

$$\delta_3(m) r^{2m-1} \leq |Du(x)| \leq \delta_4(m) r.$$

We first find the equation near a quadratic solution. Let

$$u = \frac{1}{2} (\mu_1 x_1^2 + \mu_2 x_2^2 + \mu_3 x_3^2) + w(x).$$

Then

$$\begin{aligned}\sigma_2(D^2u) - 1 &= \frac{1}{2} [(\Delta u)^2 - |D^2u|^2] - 1 \\ &= \frac{1}{2} \left[ (\mu_1 + \mu_2 + \mu_3 + \Delta w)^2 - \sum_{i=1}^3 (\mu_i + w_{ii})^2 - 2w_{12}^2 - 2w_{23}^2 - 2w_{13}^2 \right] - 1 \\ &= \mu_1(\Delta w - w_{11}) + \mu_2(\Delta w - w_{22}) + \mu_3(\Delta w - w_{33}) + \frac{1}{2} [(\Delta w)^2 - |D^2w|^2] \\ &\quad + \mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1 - 1.\end{aligned}$$

Set  $\mu_1 = \mu_2 = 1$  and  $\mu_3 = 0$ , we get

$$\begin{aligned}\sigma_2(D^2u) - 1 &= w_{11} + w_{22} + 2w_{33} + \frac{1}{2} [(\Delta w)^2 - |D^2w|^2] \\ &= \tilde{\Delta}w + \frac{1}{2} [(\Delta w)^2 - |D^2w|^2],\end{aligned}$$

where  $\tilde{\Delta} = \partial_{11} + \partial_{22} + 2\partial_{33}$ . To make the right hand side of the above equation vanish at high orders, we choose  $w = h + Q + H$ , where

$$\begin{aligned} h &= \operatorname{Re} Z^m x_3, \text{ an ad hoc "harmonic" function;} \\ Q &= \frac{m^2}{4} \rho^{2m-2} x_3^2, \text{ to match } \sigma_2(D^2 h); \\ H &= \nu \left( -x_3^{2m} + \sum_{j=1}^m a_j x_3^{2m-2j} \rho^{2j} \right), \text{ to make eigenvalue } \lambda_3 \text{ negative.} \end{aligned}$$

Then

$$\sigma_2(D^2 u) - 1 = \underbrace{\tilde{\Delta} h}_0 + \tilde{\Delta} Q + \tilde{\Delta} H + \frac{1}{2} [(\Delta h)^2 - |D^2 h|^2] + [r^{3m-3}].$$

A simple calculation leads to

$$D^2 h = \begin{bmatrix} \operatorname{Re} [m(m-1) Z^{m-2}] x_3 & -\operatorname{Im} [m(m-1) Z^{m-2}] x_3 & \operatorname{Re} m Z^{m-1} \\ & -\operatorname{Re} [m(m-1) Z^{m-2}] x_3 & -\operatorname{Im} m Z^{m-1} \\ & & 0 \end{bmatrix}.$$

It follows that

$$\sigma_2(D^2 h) = -[m(m-1)\rho^{m-2}]^2 x_3^2 - m^2 \rho^{2m-2}.$$

Thus

$$\tilde{\Delta} Q + \sigma_2(D^2 h) = [m(m-1)\rho^{m-2}]^2 x_3^2 + m^2 \rho^{2m-2} + \sigma_2(D^2 h) = 0.$$

Finally we fix the "harmonic"  $H$  satisfying  $\tilde{\Delta} H = 0$  with

$$\begin{aligned} a_0 &= -1 \\ a_j &= -\frac{2 \cdot (2m-2j+2)(2m-2j+1)}{(2j)^2} a_{j-1} \\ &= (-1)^{j+1} \frac{2^j 2m(2m-1) \cdots (2m-2j+1)}{2^2 4^2 \cdots (2j)^2} \text{ for } j \geq 1, \end{aligned}$$

and  $\nu$  is still pending. Therefore,  $P = \frac{1}{2}(x_1^2 + x_2^2) + h + Q + H$ , satisfies

$$\sigma_2(D^2 P) - 1 = [r^{3m-3}].$$

Now the analytic solution  $u$  to (3.3) with initial data  $P$  follows from Cauchy-Kowalevskaya. As in [NV], considering the linear equation for difference  $u - P$ , the Cauchy-Kowalevskaya procedure implies that the solution  $u$  coincides with  $P$  up to order  $3m - 2$  ( $\geq 2m$  for  $m \geq 2$ ). Thus Property S.1 is verified.

We move to Property S.2. We have

$$D^2u = \begin{bmatrix} 1 + [r^{m-1}] & [r^{m-1}] & \operatorname{Re} mZ^{m-1} + [r^{2m-2}] \\ & 1 + [r^{m-1}] & -\operatorname{Im} mZ^{m-1} + [r^{2m-2}] \\ & & \frac{m^2}{2}\rho^{2m-2} + H_{33} + [r^{2m-1}] \end{bmatrix}. \quad (3.4)$$

Because the eigenvalues are Lipschitz functions of the matrix entries, we get

$$\begin{aligned} \lambda_1 &= 1 + [r^{m-1}] \\ \lambda_2 &= 1 + [r^{m-1}]. \end{aligned}$$

By the quadratic Taylor expansion of the isolated eigenvalue  $\lambda_3$  in terms of the matrix entries near  $D^2u(0)$ , we obtain

$$\begin{aligned} \lambda_3 &= u_{33} - u_{13}^2 - u_{23}^2 + [r^{3m-3}] \\ &= \frac{m^2}{2}\rho^{2m-2} + \nu \sum_{j=0}^m (2m-2j)(2m-2j-1) a_j x_3^{2m-2j} \rho^{2j} \\ &\quad - m^2 \rho^{2m-2} + [r^{2m-1}] \quad \text{for } m \geq 2 \\ &= \nu [-2m(2m-1)x_3^{2m-2} + \tilde{a}_2 x_3^{2m-4} \rho^2 + \cdots + \tilde{a}_{m-1} \rho^{2m-2}] - \frac{m^2}{2} \rho^{2m-2} \\ &\quad + [r^{2m-1}] \\ &= H_{33} - \frac{m^2}{2} \rho^{2m-2} + [r^{2m-1}]. \end{aligned}$$

The ‘‘harmonic’’ function  $H_{33}$  cannot have a definite sign near the origin, but with the help of  $-\frac{m^2}{2}\rho^{2m-2}$  and small  $\nu$ , we make  $\lambda_3$  negative. Let  $\eta$  be a small positive constant to be chosen shortly.

Case 1:  $\eta|x_3| \geq \rho$ . We have

$$[-2m(2m-1)x_3^{2m-2} + \tilde{a}_2 x_3^{2m-4} \rho^2 + \cdots] = -[2m(2m-1) + O(1)\eta^2] x_3^{2m-2}.$$

Note  $r/\sqrt{1+\eta^2} \leq |x_3| \leq r$ , then

$$-\left\{ \begin{array}{c} \nu [2m(2m-1) + O(1)\eta^2] \\ + \frac{m^2}{2} + o(1) \end{array} \right\} r^{2m-2} \leq \lambda_3 \leq -\nu \left[ \begin{array}{c} \frac{2m(2m-1) + O(1)\eta^2}{(\sqrt{1+\eta^2})^{2m-2}} \\ + o(1) \end{array} \right] r^{2m-2}.$$

Case 2:  $\eta|x_3| < \rho$ . Note  $r\eta/\sqrt{1+\eta^2} \leq \rho \leq r$ , we have

$$[-2m(2m-1)x_3^{2m-2} + \tilde{a}_2 x_3^{2m-4} \rho^2 + \dots] = \frac{O(1)}{\eta^{2m-2}} \rho^{2m-2},$$

then

$$\lambda_3 = - \left[ \frac{m^2}{2} - \frac{\nu O(1)}{\eta^{2m-2}} \right] \rho^{2m-2} + [r^{2m-1}]$$

and

$$-\left[ \begin{array}{c} \frac{m^2}{2} - \frac{\nu O(1)}{\eta^{2m-2}} \\ + o(1) \end{array} \right] r^{2m-2} \leq \lambda_3 \leq \left\{ \begin{array}{c} - \left[ \frac{m^2}{2} - \frac{\nu O(1)}{\eta^{2m-2}} \right] \frac{\eta^{2m-2}}{(\sqrt{1+\eta^2})^{2m-2}} \\ + o(1) \end{array} \right\} r^{2m-2}.$$

We first choose  $\eta = \eta(m) > 0$  small, next  $\nu = \nu(\eta, m) > 0$  smaller, then there exist  $\delta_1 = \delta(\eta, m) > 0$  and  $\delta_2 = \delta_2(m) > 0$  such that

$$-\delta_2 r^{2m-2} \leq \lambda_3 \leq -\delta_1 r^{2m-2}$$

for  $r \leq r_m$ . Here  $r_m$  is within the valid radius for the Cauchy-Kowalevskaya solution  $u$ .

Property S.3 follows from  $u = P + [r^{3m-2}]$ .

Finally we prove Property S.4. The upper bound is straightforward. For the lower bound, from Property S.3, we have

$$\begin{aligned} |Du(x)|^2 &= (x_1 + [r^m])^2 + (x_2 + [r^m])^2 \\ &\quad + (\operatorname{Re} Z^m + \frac{m^2}{2} \rho^{2m-2} x_3 - 2m\nu x_3^{2m-1} + \nu \rho^2 [r^{2m-2}] + [r^{2m}])^2. \end{aligned}$$

Case 1:  $x_3^2 \geq \rho$ . From  $r^2 = \rho^2 + x_3^2 \leq (x_3^2 + 1)x_3^2$ , we know

$$|x_3| \geq r.$$

Note that the other terms than  $-2m\nu x_3^{2m-1}$  in  $u_3(x)$  have the following asymptotic behavior near the origin

$$\begin{aligned} |ReZ^m| &\leq \rho^m = x_3^{2m}, \\ \left| \frac{m^2}{2} \rho^{2m-2} x_3 \right| &\leq \frac{m^2}{2} |x_3|^{4m-3}, \\ \nu \rho^2 [r^{2m-2}] &= O(x_3^{2m+2}), \\ [r^{2m}] &= O(x_3^{2m}). \end{aligned}$$

It follows that

$$\begin{aligned} |Du(x)|^2 &\geq |u_3(x)|^2 = [-2m\nu x_3^{2m-1} + O(x_3^{2m})]^2 \\ &\geq \delta_3(m) x_3^{2(2m-1)} \geq \delta_3(m) r^{2(2m-1)} \end{aligned}$$

for  $|x| \leq r_m$  with positive  $r_m$  and  $\delta_3(m)$  to be fixed shortly.

Case 2:  $x_3^2 < \rho$ . From  $r^2 = \rho^2 + x_3^2 \leq (\rho + 1)\rho$ , we know

$$\rho > r^2.$$

Then

$$\begin{aligned} |Du(x)|^2 &\geq u_1^2(x) + u_2^2(x) = \rho^2 + 2x_1[r^m] + 2x_2[r^m] + 2[r^m]^2 \\ &= \rho^2 + O(\rho^{\frac{m+1}{2}}) \\ &\geq \frac{1}{2}\rho^2 \geq \frac{1}{2}r^4 \geq r^{2(2m-1)} \end{aligned}$$

for  $\rho \leq r \leq r_m$  with the positive  $r_m$  to be fixed next.

Now we choose positive  $\delta_3(m)$  small and the small positive  $r_m$  within the valid radius for Cauchy-Kowalevskaya solution  $u$  and Property S.2, Property S.4 is then completely justified.

Since  $u(r_mx)/r_m^2$  is still a solution to  $\sigma_2(D^2u) = 1$  in  $B_1 \subset \mathbb{R}^3$ . We may assume the above constructed solution is already defined in  $B_1 \subset \mathbb{R}^3$ . Note that  $D[u(r_mx)/r_m^2] = Du(r_mx)/r_m$  and  $D^2[u(r_mx)/r_m^2] = D^2u(r_mx)$ , we see that Property S.2 and Property S.4 are still valid in  $B_1$  with  $\delta_1, \delta_2, \delta_3$  replaced by  $r_m^{2m-2}\delta_1, r_m^{2m-2}\delta_2, r_m^{2m-2}\delta_2$  respectively, and  $\delta_4$  unchanged.

### 3.3 Rotate to subcritical phases $|\Theta| < \frac{\pi}{2}$ : proof of Theorem 1.2

In this section, we carry out the construction of the singular solutions in Theorem 1.2 by “horizontally” and  $\pi/2$  rotating the Cauchy-Kowalevskaya solutions from Section 3.2. The latter rotation, Proposition 3.1 is pivotal.

Step 1. Let  $\alpha \in [0, \pi/4)$ . We will take  $\alpha = \Theta/2$  for  $\Theta \in [0, \pi/2)$  in Step 3 of this section. We make a  $\mathbb{U}(3)$  rotation in  $\mathbb{C}^3$  :  $\tilde{z}' = e^{\alpha\sqrt{-1}}z'$  and  $\tilde{z}_3 = z_3$  with  $\tilde{z} = (\tilde{z}', \tilde{z}_3) = (\tilde{x}', \tilde{x}_3) + \sqrt{-1}(\tilde{y}', \tilde{y}_3)$  and  $z = (z', z_3) = (x', x_3) + \sqrt{-1}(y', y_3)$ . Because  $\mathbb{U}(3)$  rotations preserve the length and complex structure,  $\mathfrak{M} = (x, Dv(x))$  for  $x \in B_1$  is still a special Lagrangian submanifold in the new coordinate system with parametrization

$$\begin{cases} \tilde{x} = (x_1 \cos \alpha + u_1(x) \sin \alpha, x_2 \cos \alpha + u_2(x) \sin \alpha, x_3) \\ \tilde{y} = (-x_1 \sin \alpha + u_1(x) \cos \alpha, -x_2 \sin \alpha + u_2(x) \cos \alpha, u_3(x)) \end{cases}. \quad (3.5)$$

We show that  $\mathfrak{M}$  is also a “gradient” graph over  $\tilde{x}$  space. From Property S.2, we know that  $u(x', x_3)$  is a convex function in terms of  $x'$  for  $|x| \leq 1$ , or if necessary  $|x| \leq r_m$  with  $r_m$  depending only on  $m$ . From (3.4) we also assume  $|D'u_3(x)| = |(u_{13}, u_{23})(x)| \leq 1/2$  for  $|x| \leq r_m$ . Then we have

$$\begin{aligned} & \delta_5(m) |x - x^*|^2 \geq |\tilde{x}(x) - \tilde{x}(x^*)|^2 \quad (3.6) \\ & = \left| (x' - x'^*) \cos \alpha + \begin{bmatrix} D'u(x', x_3) - D'u(x', x_3^*) \\ + D'u(x', x_3^*) - D'u(x'^*, x_3^*) \end{bmatrix} \sin \alpha \right|^2 + |x_3 - x_3^*|^2 \\ & \geq \left[ \frac{1}{2} \left| (x' - x'^*) \cos \alpha + \left( \frac{D'u(x', x_3^*) - D'u(x'^*, x_3^*)}{2} \right) \sin \alpha \right|^2 \right. \\ & \quad \left. - |(D'u(x', x_3) - D'u(x', x_3^*)) \sin \alpha|^2 + |x_3 - x_3^*|^2 \right] \\ & \geq \left[ \frac{\cos^2 \alpha}{2} |x' - x'^*|^2 + \cos \alpha \sin \alpha \underbrace{\langle x' - x'^*, D'u(x', x_3^*) - D'u(x'^*, x_3^*) \rangle}_{\geq 0} \right. \\ & \quad \left. - \underbrace{\sin^2 \alpha}_{\leq 1} 2 \|D'u_3\|_{L^\infty(B_{r_m})} |x_3 - x_3^*|^2 + |x_3 - x_3^*|^2 \right] \\ & \geq \frac{\cos^2 \alpha}{2} |x' - x'^*|^2 + (1 - \sin^2 \alpha) |x_3 - x_3^*|^2 \\ & \geq \frac{1}{4} |x - x^*|^2. \quad (3.7) \end{aligned}$$

It follows that  $\mathfrak{M}$  is a special Lagrangian graph  $(\tilde{x}, D\tilde{u}(\tilde{x}))$  over a domain containing a ball of radius  $1/\sqrt{2}$  in  $\tilde{x}$  space. The Hessian of the potential function  $\tilde{u}$  satisfies

$$\begin{aligned}
D^2\tilde{u} &= \frac{\partial\tilde{y}}{\partial\tilde{x}} = \frac{\partial\tilde{y}}{\partial x} \left( \frac{\partial\tilde{x}}{\partial x} \right)^{-1} \\
&= \begin{bmatrix} -\sin\alpha + u_{11}\cos\alpha & u_{12}\cos\alpha & u_{13}\cos\alpha \\ u_{12}\cos\alpha & -\sin\alpha + u_{22}\cos\alpha & u_{23}\cos\alpha \\ u_{13} & u_{23} & u_{33} \end{bmatrix} \\
&\quad \begin{bmatrix} \cos\alpha + u_{11}\sin\alpha & u_{12}\sin\alpha & u_{13}\sin\alpha \\ u_{12}\sin\alpha & \cos\alpha + u_{22}\sin\alpha & u_{23}\sin\alpha \\ 0 & 0 & 1 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} \tan\left(\frac{\pi}{4} - \alpha\right) & & \\ & \tan\left(\frac{\pi}{4} - \alpha\right) & \\ & & 0 \end{bmatrix} + [r^{m-1}] \tag{3.8}
\end{aligned}$$

and

$$\det D^2\tilde{u} = \tan\left(\frac{\pi}{4} - \alpha\right) \left[ -\frac{m^2}{2}\rho^{2m-2} + \tan\left(\frac{\pi}{4} - \alpha\right) H_{33} \right] - [r^{2m-1}], \tag{3.9}$$

where the above abused notation  $[r^{m-1}]$  also represents a matrix whose entries are all analytic functions starting from order  $m-1$ , and (3.8) (3.9) follow from a simple calculation and the asymptotic behavior of  $D^2u$ , (3.4). We verify the following properties for  $D^2\tilde{u}$ . There exists a positive number  $\tilde{r}_{m,\alpha}$  depending only on  $m$  and  $\alpha \in [0, \pi/4)$  such that for  $|\tilde{x}| \leq \tilde{r}_{m,\alpha}$  we have:

Property T.1. The determinant  $\det D^2\tilde{u}(\tilde{x})$  is negative for small  $\tilde{x} \neq 0$ , indeed

$$\det D^2\tilde{u}(\tilde{x}) \approx -\tan\left(\frac{\pi}{4} - \alpha\right) |\tilde{x}|^{2m-2};$$

Property T.2. The upper left  $2 \times 2$  principle minor of the Hessian  $D^2\tilde{u}$ ,

$$2 \tan\left(\frac{\pi}{4} - \alpha\right) I \geq (D^2\tilde{u})' \geq \frac{\tan\left(\frac{\pi}{4} - \alpha\right)}{2} I;$$

Property T.3. The three eigenvalues  $\tilde{\lambda}_i$  of the Hessian  $D^2\tilde{u}$  satisfy

$$\begin{cases} \tilde{\theta}_1 = \arctan \tilde{\lambda}_1 = \left(\frac{\pi}{4} - \alpha\right) [1 + O(|\tilde{x}|^{m-1})] \\ \tilde{\theta}_2 = \arctan \tilde{\lambda}_2 = \left(\frac{\pi}{4} - \alpha\right) [1 + O(|\tilde{x}|^{m-1})] \\ \tilde{\theta}_3 = \arctan \tilde{\lambda}_3 \approx -\frac{1}{\tan(\frac{\pi}{4}-\alpha)} |\tilde{x}|^{2m-2} [1 + O(|\tilde{x}|^{m-1})] \end{cases} ;$$

where “ $\approx$ ” means two quantities are comparable up to a multiple of constant depending only on  $m$  and  $\alpha$ . Relying on (3.9), repeating the arguments for the estimate of  $\lambda_3$  in Section 3.2, using (3.7) and (3.5), we obtain Property T.1. Property T.2 follows from (3.8). From (3.7) (3.8) and the Lipschitz continuity of eigenvalues in terms of matrix entries, we derive the estimates for the first two eigenvalues in Property T.3. In turn, noticing  $\tilde{\lambda}_3 = \det D^2\tilde{u} / (\tilde{\lambda}_1\tilde{\lambda}_2)$ , relying on both (3.6) and (3.7) we get two sided estimates of the last eigenvalue.

Step 2. We proceed with the following proposition.

**Proposition 3.1.** *Let  $\mathcal{L} = (x, Df)$  be a Lagrangian surface in  $\mathbb{C}^3 = \mathbb{R}^3 \times \mathbb{R}^3$  with the smooth potential  $f$  over  $B_\rho \subset \mathbb{R}^3$ , satisfying:*

$$\begin{aligned} Df(0) &= 0, \\ \det D^2f(x) &< 0 \text{ for } x \neq 0, \\ \left\{ \begin{array}{l} \kappa^{-1}I \geq \begin{bmatrix} f_{11}(x) & f_{12}(x) \\ f_{21}(x) & f_{22}(x) \end{bmatrix} \geq \kappa I \\ |D'f_3(x)| = |(f_{13}, f_{23})(x)| \leq \frac{1}{2}, \text{ say} \end{array} \right\} \text{ for } x \in B_\rho, \kappa > 0. \end{aligned} \quad (3.10)$$

Then  $\mathcal{L}$  can be re-represented as a graph  $(\tilde{x}, \tilde{y}) = (\tilde{x}, D\tilde{f}(\tilde{x}))$  over the open set  $\Omega = Df(B_{\frac{1}{2}\kappa^2\rho})$  with  $\tilde{x} + \sqrt{-1}\tilde{y} = e^{-\frac{\pi}{2}\sqrt{-1}}(x + \sqrt{-1}y)$  and  $\tilde{f} \in C^1(\Omega) \cap C^\infty(\Omega \setminus \{0\})$ .

*Proof of Proposition 3.1.* Note that the  $\mathbb{U}(3)$  rotation by  $\pi/2$  is  $(\tilde{x}, \tilde{y}) = (y, -x)$ . This proposition really says that the map  $Df$  has a (unique) continuous inverse  $\Phi = -D\tilde{f}$ .

Step 2.1. We first prove  $Df$  is one-to-one on  $B_{\kappa^2\rho}$ . Consider a coordinate change

given by  $t = \Psi(x) = (f_1(x), f_2(x), x_3)$ . Then the Jacobian of  $\Psi$  is

$$\det D_x \Psi(x) = \det \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ 0 & 0 & 1 \end{bmatrix} (x) = \det \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} (x) > 0. \quad (3.11)$$

Hence  $\Psi$  is a local diffeomorphism on  $B_\rho$ . Note that  $\Psi$  is actually a distance expansion map. We have for all  $x, x^\#$  in  $B_\rho$

$$\begin{aligned} |\Psi(x) - \Psi(x^\#)|^2 &= \left| \begin{array}{c} D'f(x', x_3) - D'f(x^{\#'}, x_3) \\ + D'f(x^{\#'}, x_3) - D'f(x^{\#'}, x_3^\#) \end{array} \right|^2 + |x_3 - x_3^\#|^2 \\ &\geq \left[ \begin{array}{c} \frac{1}{2} |D'f(x', x_3) - D'f(x^{\#'}, x_3)|^2 - |D'f(x^{\#'}, x_3) - D'f(x^{\#'}, x_3^\#)|^2 \\ + |x_3 - x_3^\#|^2 \end{array} \right] \\ &= \left[ \begin{array}{c} \frac{1}{2} |D'f(x', x_3) - D'f(x^{\#'}, x_3) - \kappa(x' - x^{\#'}) + \kappa(x' - x^{\#'})|^2 \\ - |D'f(x^{\#'}, x_3) - D'f(x^{\#'}, x_3^\#)|^2 + |x_3 - x_3^\#|^2 \end{array} \right] \\ &\geq \left[ \begin{array}{c} \underbrace{\langle D'f(x', x_3) - D'f(x^{\#'}, x_3) - \kappa(x' - x^{\#'}), \kappa(x' - x^{\#'}) \rangle}_{\geq 0} \\ + \frac{1}{2} |\kappa(x' - x^{\#'})|^2 - 2 \|D'f\|_{L^\infty(B_\rho)}^2 |x_3 - x_3^\#|^2 + |x_3 - x_3^\#|^2 \end{array} \right] \\ &\geq \frac{\kappa^2}{2} |x' - x^{\#'}|^2 + \left(1 - 2 \|D'f\|_{L^\infty(B_\rho)}^2\right) |x_3 - x_3^\#|^2 \\ &\geq \frac{\kappa^2}{2} |x - x^\#|^2, \end{aligned} \quad (3.12)$$

where we used (3.10). Thus  $\Psi$  is a “global” diffeomorphism on  $B_\rho$ .

We claim that the  $\Psi$ -image of  $B_\rho$ ,  $\Psi(B_\rho) \supset B_{\frac{\kappa}{\sqrt{2}}\rho}^t$ . Otherwise, let  $t^\#$  be a boundary point of  $\Psi(B_\rho)$  in  $\mathring{B}_{\frac{\kappa}{\sqrt{2}}\rho}^t$ . We know there exist a sequence of points  $x_i \in B_\rho$  such that  $\Psi(x_i)$  goes to  $t^\#$  and  $x_i$  goes to  $x^\# \in \bar{B}_\rho$  as  $i$  goes to infinity. If  $x^\# \in \mathring{B}_\rho$ , then  $\Psi(x^\#) = t^\#$  by the continuity of  $\Psi$ . But this is impossible because  $\Psi(x^\#)$  is an interior point of  $\Psi(B_\rho)$  under the diffeomorphism of  $\Psi$ . If  $x^\# \in \partial B_\rho$ , from (3.12), we have

$$|t^\#| = \lim_{i \rightarrow \infty} |\Psi(x_i) - 0| \geq \lim_{i \rightarrow \infty} \frac{\kappa}{\sqrt{2}} |x_i - 0| = \frac{\kappa}{\sqrt{2}} |x^\#| = \frac{\kappa}{\sqrt{2}} \rho.$$

This contradicts  $t^\# \in \overset{\circ}{B}_{\frac{\kappa}{\sqrt{2}}\rho}^t$ .

From (3.10) (only the upper bound), then

$$|\Psi(x) - \Psi(x^\#)| \leq \|DD'f\|_{L^\infty(B_\rho)} |x - x^\#| \leq \kappa^{-1}|x - x^\#|,$$

it follows that  $\Psi^{-1}$  is also a distance expansion map with a factor  $\kappa$ . Apply the arguments above we get  $\Psi^{-1}(B_{\frac{1}{2}\kappa^2\rho}^t) \supset B_{\frac{1}{2}\kappa^2\rho}$  or  $B_{\frac{1}{2}\kappa\rho}^t \supset \Psi(B_{\frac{1}{2}\kappa^2\rho})$ .

Now for the injectivity of  $Df$  on  $B_{\frac{1}{2}\kappa^2\rho}$ , it suffices to show that

$$y(t) = Df \circ \Psi^{-1}(t) = (t_1, t_2, f_3(x(t)))$$

is one-to-one in  $B_{\frac{1}{2}\kappa\rho}^t$ . Suppose that  $y(t) = y(t^\#)$ , then

$$t_1^\# = t_1, \quad t_2^\# = t_2, \quad y_3(t^\#) = y_3(t).$$

Note that

$$\begin{aligned} \frac{\partial y_3(t_1, t_2, \xi)}{\partial t_3} &= \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ * & * & \frac{\partial y_3(t_1, t_2, \xi)}{\partial t_3} \end{bmatrix} \\ &= \det D_t(Df \circ \Psi^{-1}) = \det(D^2f)|_{\Psi^{-1}(t_1, t_2, \xi)} \cdot \det D_t\Psi^{-1}|_{(t_1, t_2, \xi)} < 0 \end{aligned}$$

for  $(t_1, t_2, \xi) \neq 0$ , where we used (3.11) and  $\det D^2f(x) < 0$  for  $x \neq 0$ . It follows that the function  $y_3(t_1, t_2, \xi)$  is strictly decreasing in  $\xi$ . Now  $y_3(t_1, t_2, t_3^\#) = y_3(t_1, t_2, t_3)$  implies  $t_3^\# = t_3$ . This shows that  $y = Df \circ \Psi^{-1}$  is one-to-one.

So far we have obtained the inverse function  $\Phi = (Df)^{-1}$  on  $\Omega = Df\left(B_{\frac{1}{2}\kappa^2\rho}\right)$ .

Step 2.2. We prove  $Df$  is an open map from  $B_\rho$  to  $\mathbb{R}^3$ . Since the Jacobian  $\det D^2f(x) \neq 0$  for  $x \neq 0$ ,  $Df$  is already a local diffeomorphism for  $x \neq 0$ . It suffices to show that the image of an open neighborhood of 0 in  $B_\rho$ , under  $Df$ , contains an open neighborhood of 0 in  $\mathbb{R}^3$ . Since  $\Psi$  is a diffeomorphism, we only need to show this property for  $Df \circ \Psi^{-1}$ . Indeed we only need to consider the image of the ball  $B_{2\eta}^t$  of radius  $2\eta$  centered at  $t = 0$  for a small  $\eta > 0$ . According to Step 2.1,

$y(t_1, t_2, \cdot)$  is strictly decreasing in the third variable. So  $2h_- = y_3(0, 0, \eta) < 0$  and  $2h_+ = y_3(0, 0, -\eta) > 0$ . By continuity of  $y = Df \circ \Psi^{-1}$ , there exists  $\eta' \in (0, \eta)$  such that  $y_3(t_1, t_2, \eta) < h_- < 0$  and  $y_3(t_1, t_2, -\eta) > h_+ > 0$  for  $|(t_1, t_2)| \leq \eta'$ . Then by intermediate value theorem (for function  $y_3(t_1, t_2, \cdot)$ ), the open set

$$\begin{aligned} & \{|(y_1, y_2)| < \eta'\} \times \{h_- < y_3 < h_+\} \subset Df \circ \Psi^{-1}(\{|(t_1, t_2)| \leq \eta'\} \times \{|t_3| \leq \eta\}) \\ & \subset Df \circ \Psi^{-1}(B_{2\eta}^t). \end{aligned}$$

Thus  $Df$  is an open map.

Step 2.3. Now  $\Omega = Df(B_{\kappa^2\rho})$  is an open neighborhood of  $y = 0$ , and  $\Phi$  is continuous on  $\Omega$  by the openness of  $Df$ . Lastly we find a potential for the Lagrangian submanifold  $\mathcal{L}$  now represented as  $(\tilde{x}, -\Phi(\tilde{x}))$ . Let

$$\tilde{f}(\tilde{x}) = \int_0^{\tilde{x}} -\Phi^1(s)ds_1 - \Phi^2(s)ds_2 - \Phi^3(s)ds_3.$$

Because  $D_{\tilde{x}}(-\Phi(\tilde{x})) = -(D^2f)^{-1}$  is symmetric when  $\tilde{x} \neq 0$  and  $\Phi$  is bounded,  $\tilde{f}(\tilde{x})$  is well-defined on  $\Omega$ . Further we know  $\tilde{f} \in C^1(\Omega) \cap C^\infty(\Omega \setminus \{0\})$ .

The proof for Proposition 3.1 is complete.  $\square$

*Remark.* For the purpose of Theorem 1.2, we can replace (3.10) by a weaker condition (3.11)  $\det [D^2f]' > 0$ . Consequently we have no estimate on the size of the existing neighborhood supporting the solution  $\tilde{u}$  in this section, then  $u^m$  for each single  $m$  and  $\Theta$ . The stronger assumption (3.10) is designed for Theorem 1.3 where we need a uniform control with respect to  $\varepsilon$  on the valid radius for the solutions  $\tilde{u}^\varepsilon$ , then  $u^\varepsilon$ . Lastly there is another argument for the openness of the particular map  $D\tilde{u} = Df$ , relying on the uniqueness of the pre-image of  $y = 0$  (which can be also derived from the distance expansion property at the origin (3.16)), instead of using the strict monotonicity property in Step 2.2.

Step 3. Equipped with Property T.1, T.2, and (3.8), we apply Proposition 4.1 to our function  $\tilde{u}(\tilde{x})$  with  $\tilde{x}$  replaced by  $\tilde{\tilde{x}}$ ,  $x$  replaced by  $\tilde{x}$ ,  $\rho = \tilde{r}_{m,\alpha}$ , and

$\kappa = \tan(\frac{\pi}{4} - \alpha)/2$ . Then we get a new  $C^1$  function  $\tilde{u}(\tilde{x})$ , defined on an open neighborhood of  $\tilde{x} = 0$ . By Property T.3, the three eigen-angles  $\tilde{\theta}_i = \arctan \tilde{\lambda}_i$  of  $D^2\tilde{u}(\tilde{x})$  away from the origin satisfy

$$\begin{cases} \tilde{\theta}_1 = \tilde{\theta}_1 - \frac{\pi}{2} = -\frac{\pi}{4} - \alpha + o(1) \\ \tilde{\theta}_2 = \tilde{\theta}_2 - \frac{\pi}{2} = -\frac{\pi}{4} - \alpha + o(1) \\ \tilde{\theta}_3 = \tilde{\theta}_3 - \frac{\pi}{2} + \pi \\ \quad = \frac{\pi}{2} - \frac{\delta_{m,\alpha}(x)}{\tan(\frac{\pi}{4} - \alpha)} |D\tilde{u}(\tilde{x})|^{2m-2} \left[ 1 + O\left(|D\tilde{u}(\tilde{x})|^{m-1}\right) \right] \end{cases}, \quad (3.13)$$

where the positive number  $\delta_{m,\alpha}(x)$  is bounded from both below and above uniformly with respect to  $\tilde{x}$ , and

$$\sum_{i=1}^3 \tilde{\theta}_i = -2\alpha.$$

We verify  $\tilde{u}$  is still a viscosity solution to (1.1) with  $\Theta = -2\alpha$  across the origin. For any quadratic  $Q$  touching  $\tilde{u}$  at the origin from below, we have under the ‘‘diagonalized’’ coordinate system for  $D^2\tilde{u}(0)$

$$D^2Q \leq \begin{bmatrix} \tan(-\frac{\pi}{4} - \alpha) & & \\ & \tan(-\frac{\pi}{4} - \alpha) & \\ & & \infty \end{bmatrix}.$$

It follows that the eigenvalues  $\lambda_i^*$  of  $D^2Q$  must satisfy

$$\arctan \lambda_1^* \leq -\frac{\pi}{4} - \alpha, \quad \arctan \lambda_2^* \leq -\frac{\pi}{4} - \alpha, \quad \text{and} \quad \arctan \lambda_3^* < \frac{\pi}{2}.$$

Then the quadratic satisfies

$$\sum_{i=1}^3 \arctan \lambda_i^* < -2\alpha.$$

Observe that we can never arrange any quadratic touching  $\tilde{u}$  from above at the origin. Then there is nothing to check. When those testing quadratics touch the smooth  $\tilde{u}$  away from the origin, the verification according to the definition of viscosity solutions is straightforward. Thus  $\tilde{u}$  is a viscosity solution to (1.1) with  $\Theta = -2\alpha$  in a neighborhood of the origin.

Step 4. Lastly we verify that the solution  $\tilde{u}$  is in fact  $C^{1,1/(2m-1)}$  but not  $C^{1,\delta}$  for any  $\delta > 1/(2m-1)$  in a neighborhood of the origin. The latter is easy. From Property S.3 and (3.7), we see that

$$(0, 0, \tilde{x}_3, D\tilde{u}(0, 0, \tilde{x}_3)) = (0, 0, \tilde{x}_3, [\tilde{x}_3^{2m}], [\tilde{x}_3^{2m}], -2m\varepsilon\tilde{x}_3^{2m-1} + [\tilde{x}_3^{2m}]).$$

It follows that

$$\frac{|\tilde{x}_3 - 0|}{|D\tilde{u}(0, 0, \tilde{x}_3) - D\tilde{u}(0)|^\delta} = \frac{|\tilde{x}_3|}{(2m\varepsilon + [\tilde{x}_3]^\delta) |\tilde{x}_3|^{(2m-1)\delta}} \rightarrow \infty$$

as  $\tilde{x}_3 \rightarrow 0$  for any  $\delta > \frac{1}{2m-1}$ . This shows that  $\tilde{u}$  is not  $C^{1,\delta}$  for any  $\delta > 1/(2m-1)$ .

Next we prove that  $\tilde{u}$  is  $C^{1,1/(2m-1)}$  by the argument in [NV]. Observe that for  $i = 1, 2, 3$

$$\begin{aligned} & \left[ \frac{|\tilde{u}_i(\tilde{x}) - \tilde{u}_i(\tilde{x}^*)|}{|\tilde{x} - \tilde{x}^*|^{1/(2m-1)}} \right]^{2m-1} \\ &= \left[ \frac{|\tilde{u}_i(\tilde{x}) - \tilde{u}_i(\tilde{x}^*)|}{|\tilde{u}_i^{2m-1}(\tilde{x}) - \tilde{u}_i^{2m-1}(\tilde{x}^*)|^{1/(2m-1)}} \right]^{2m-1} \frac{|\tilde{u}_i^{2m-1}(\tilde{x}) - \tilde{u}_i^{2m-1}(\tilde{x}^*)|}{|\tilde{x} - \tilde{x}^*|} \\ &\leq C(m) (2m-1) \sup_{\tilde{x}} |\tilde{u}_i(\tilde{x})|^{2m-2} |D\tilde{u}_i(\tilde{x})| \\ &\leq C(m) \sup_{\tilde{x}} |D\tilde{u}(\tilde{x})|^{2m-2} |D^2\tilde{u}(\tilde{x})| \\ &\leq C(m) \sup_{\tilde{x}} |D\tilde{u}(\tilde{x})|^{2m-2} \frac{1}{|D\tilde{u}(\tilde{x})|^{2m-2}} \\ &\leq C(m), \end{aligned}$$

where we used the fact that the scalar function  $t^{1/(2m-1)}$  is  $C^{1/(2m-1)}(\mathbb{R}^1)$  for the first inequality, and (3.13) for the third inequality.

Finally by scaling  $u^m(x) = \tilde{u}(\tau x)/\tau^2$  with valid radius  $\tau$  implicitly depending on  $m$  and the  $\tilde{r}_{m,\alpha}$  in Step 1 (We need to make this dependence explicit and then to have a uniform control with respect to  $\varepsilon$  on the valid radius for the solutions  $u^\varepsilon$  in Section 3.4. Our guaranteed valid radius goes to zero as  $m$  goes to infinity.), the desired solutions in Theorem 1.2 with each fixed  $\Theta \in (-\frac{\pi}{2}, 0]$  are achieved. By symmetry,  $-u^m$  are the sought solutions with phase  $\Theta \in [0, \frac{\pi}{2})$ .

### 3.4 Rotate to smooth solutions: proof of Theorem 1.3

In this section, we create the desired family of solutions by another corresponding families of  $\mathbb{U}(3)$  rotations in  $\mathbb{C}^3$  on top those two in Section 3.3. For any fixed  $\Theta \in [0, \frac{\pi}{2})$ , let  $4\gamma = \frac{\pi}{2} - \Theta > 0$ . We start the construction by taking small positive numbers  $\varepsilon \in (0, \gamma)$  and solution  $u$  with fixed  $m$ .

Step 1. We take the  $\mathbb{U}(3)$  rotation in Step 1 of Section 3.3 with  $\alpha = \frac{\Theta}{2} - \frac{3\varepsilon}{2}$ . The valid radius of the rotation and the estimates of the Hessian  $D^2\tilde{u}$  are still valid. To prepare the final rotations in the last Step of this section, we require the following estimates of  $D^2\tilde{u}$  with eigenvalues  $\tilde{\lambda}_i$  by shrinking the radius for  $\tilde{x}$  or  $|x| \leq r_\Theta$  :

$$\begin{cases} \tilde{\theta}_1^\varepsilon = \arctan \tilde{\lambda}_1^\varepsilon = \frac{\pi}{4} - \frac{\Theta}{2} + \frac{3\varepsilon}{2} + o(1) \geq \gamma \\ \tilde{\theta}_2^\varepsilon = \arctan \tilde{\lambda}_2^\varepsilon = \frac{\pi}{4} - \frac{\Theta}{2} + \frac{3\varepsilon}{2} + o(1) \geq \gamma \\ \tilde{\theta}_3^\varepsilon = \arctan \tilde{\lambda}_3^\varepsilon = -\frac{\delta_{m,\alpha}(\tilde{x})}{\tan(\frac{\pi}{4} - \frac{\Theta}{2} + \frac{3\varepsilon}{2})} |\tilde{x}|^{2m-2} [1 + O(|\tilde{x}|^{m-1})] \end{cases}, \quad (3.14)$$

where again the positive  $\delta_{m,\alpha}(\tilde{x})$  is bounded from both below and above uniformly with respect to  $\tilde{x}$  and  $\varepsilon$ , further the above estimates and  $r_\Theta$  are both uniform with respect to  $\varepsilon$ .

Step 2. Exactly as in Step 3 of Section 3.3, we apply Proposition 3.1 with  $\rho = r_\Theta$  and  $\kappa = \tan(\frac{\pi}{4} - \alpha)/2$  to  $(\tilde{x}, D\tilde{u})$  to get the potential  $\tilde{u}^\varepsilon$  with  $(\tilde{x}, D\tilde{u}^\varepsilon)$  for  $\tilde{x} \in D\tilde{u}(B_{\frac{1}{2}\kappa^2 r_\Theta})$ . It follows from (3.14) and (3.13) that

$$\begin{cases} \tilde{\theta}_1 = -\frac{\pi}{4} - \frac{\Theta}{2} + \frac{3\varepsilon}{2} + o(1) \geq \gamma - \frac{\pi}{2} \\ \tilde{\theta}_2 = -\frac{\pi}{4} - \frac{\Theta}{2} + \frac{3\varepsilon}{2} + o(1) \geq \gamma - \frac{\pi}{2} \\ \tilde{\theta}_3 = \frac{\pi}{2} - |o(1)| \end{cases} \quad (3.15)$$

for  $|\tilde{x}| = |D\tilde{u}(\tilde{x})| \leq \tilde{r}_\Theta$ . We need to show this  $\tilde{r}_\Theta$  and the above  $o(1)$  terms are still uniform with respect to  $\varepsilon$ , and also the  $|o(1)|$  term for  $\tilde{\theta}_3$  never vanishes when the input  $\tilde{x}$  does not vanish (actually this  $|o(1)|$  can be made explicit enough by (3.16)). All these can be seen from the following inequalities

$$\delta_6(m) |\tilde{x}| \geq |\tilde{x}(\tilde{x})| = |D\tilde{u}(\tilde{x})| \geq \delta_7(m) |\tilde{x}|^{2m-1}. \quad (3.16)$$

Indeed we see the first inequality by recalling (3.5)

$$\begin{aligned} D\tilde{u}(\tilde{x}(x)) &= \tilde{y}(x) = (\cos \alpha D'u(x) - \sin \alpha x', u_3(x)), \\ (D'u, u_3)(x', x_3) &= Du(x) \in C^1, \end{aligned}$$

and (3.6). We have to work a little harder for the second inequality. Because of (3.4) and  $\alpha \in (-\frac{3}{2}\gamma, \frac{\pi}{4} - 4\gamma)$ , the following convexity for function

$$u'_{x_3}(x') = \cos \alpha u(x', x_3) - \frac{\sin \alpha}{2} |x'|^2$$

is available

$$\cos \alpha [D^2u(x)]' - \sin \alpha \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \geq \frac{\cos \alpha - \sin \alpha}{2} I > 0$$

for  $|(x', x_3)| \leq r_\Theta$ , where we shrink  $r_\Theta$  if necessary. Then we get

$$\begin{aligned} |\tilde{y}(x)|^2 &= |(Du'_{x_3}(x'), u_3(x))|^2 = |(Du'_{x_3}(x') - bx' + bx', u_3(x))|^2 \\ &\geq |bx'|^2 + 2 \underbrace{\langle Du'_{x_3}(x') - bx', bx' \rangle}_{\geq 0} + |u_3(x)|^2 \\ &\geq b^2 |x'|^2 + |u_3(x)|^2, \end{aligned}$$

where we set  $b = (\cos \alpha - \sin \alpha)/2$  for simplicity of notation. In order to bound  $|x'|^2$  from below, we use Property S.3 to obtain

$$\begin{aligned} |D'u(x', x_3)|^2 &\leq 2|D'u(x', x_3) - D'u(0, x_3)|^2 + 2|D'u(0, x_3)|^2 \\ &\leq C_m |x'|^2 + |[r^{2m}]|^2, \quad \text{or} \\ |x'|^2 &\geq \frac{1}{C_m} |D'u(x)|^2 - |[r^{2m}]|^2. \end{aligned}$$

Hence

$$|\tilde{y}(x)|^2 \geq \frac{b^2}{C_m} |Du(x)|^2 - |[r^{2m}]|^2,$$

where we assumed the positive  $b^2/C_m \leq 1$  without loss of generality. By virtue of Property S.4, we get

$$\begin{aligned} |\tilde{y}(x)|^2 &\geq \frac{b^2}{C_m} |r^{2m-1}|^2 - |[r^{2m}]|^2 \\ &\geq \frac{(\delta_7(m))^2}{\delta_5(m)} (|x|^{2m-1})^2 \end{aligned}$$

for  $|x| \leq r_\Theta$  and small positive  $\delta_7(m, \alpha)$ , where again we shrink  $r_\Theta$  if necessary. By (3.6) we arrive at the second inequality of (3.16).

Step 3. We make a final family of  $\mathbb{U}(3)$  rotations in  $\mathbb{C}^3 : \tilde{\tilde{z}} = e^{\varepsilon\sqrt{-1}}\tilde{\tilde{z}}$ . Again because  $\mathbb{U}(3)$  rotation preserves the length and complex structure,  $\mathfrak{M} = (\tilde{\tilde{x}}, D\tilde{\tilde{u}}^\varepsilon)$  for  $|\tilde{\tilde{r}}| \leq \tilde{\tilde{r}}_\Theta$  still a smooth special Lagrangian submanifold with parameterization

$$\begin{cases} \tilde{\tilde{x}} = \tilde{x} \cos \varepsilon + D\tilde{u}^\varepsilon(\tilde{x}) \sin \varepsilon \\ \tilde{\tilde{y}} = -\tilde{x} \sin \varepsilon + D\tilde{u}^\varepsilon(\tilde{x}) \cos \varepsilon \end{cases} .$$

We show that  $\mathfrak{M}$  is still a ‘‘gradient’’ graph over  $\tilde{\tilde{x}}$  space. From (3.15) we know that the function  $\tilde{u}^\varepsilon(\tilde{x}) + \frac{1}{2} \tan(\frac{\pi}{2} - \gamma) |\tilde{x}|^2$  is convex. We then have

$$\begin{aligned} & \left| \tilde{\tilde{x}}(\tilde{x}) - \tilde{\tilde{x}}(\tilde{x}^*) \right|^2 = |(\tilde{x} - \tilde{x}^*) \cos \varepsilon + (D\tilde{u}^\varepsilon(\tilde{x}) - D\tilde{u}^\varepsilon(\tilde{x}^*)) \sin \varepsilon|^2 \\ &= \left| \begin{array}{l} (\tilde{x} - \tilde{x}^*) [\cos \varepsilon - \tan(\frac{\pi}{2} - \gamma) \sin \varepsilon] + \\ + [(D\tilde{u}^\varepsilon(\tilde{x}) - D\tilde{u}^\varepsilon(\tilde{x}^*)) + (\tilde{x} - \tilde{x}^*) \tan(\frac{\pi}{2} - \gamma)] \sin \varepsilon \end{array} \right|^2 \\ &\geq \left\{ +2 \left[ \begin{array}{l} \cos \varepsilon - \\ \tan(\frac{\pi}{2} - \gamma) \sin \varepsilon \end{array} \right] \sin \varepsilon \underbrace{\left\langle \tilde{x} - \tilde{x}^*, \left[ \begin{array}{l} (D\tilde{u}^\varepsilon(\tilde{x}) - D\tilde{u}^\varepsilon(\tilde{x}^*)) + \\ (\tilde{x} - \tilde{x}^*) \tan(\frac{\pi}{2} - \gamma) \end{array} \right] \right\rangle}_{\geq 0} \right\} \\ &= |\tilde{x} - \tilde{x}^*|^2 \cos^2 \varepsilon \left( 1 - \frac{\tan \varepsilon}{\tan \gamma} \right)^2 \geq \frac{1}{4} |\tilde{x} - \tilde{x}^*|^2 \end{aligned}$$

provided we take  $\varepsilon \in (0, \gamma)$  even smaller. It follows that the smooth  $\mathfrak{M}$  is a special Lagrangian graph  $(\tilde{\tilde{x}}, D\tilde{\tilde{u}}^\varepsilon(\tilde{\tilde{x}}))$  over a domain containing a ball of radius  $\frac{1}{2}\tilde{\tilde{r}}_\Theta$  in  $\tilde{\tilde{x}}$

space. The eigenvalues  $\tilde{\tilde{\lambda}}_i^\varepsilon$  of the Hessian  $D^2\tilde{\tilde{u}}^\varepsilon$  satisfy

$$\begin{cases} \tilde{\tilde{\theta}}_1^\varepsilon = \arctan \tilde{\tilde{\lambda}}_1^\varepsilon = -\frac{\pi}{4} - \frac{\Theta}{2} + \frac{3\varepsilon}{2} - \varepsilon + o(1) \\ \tilde{\tilde{\theta}}_2^\varepsilon = \arctan \tilde{\tilde{\lambda}}_2^\varepsilon = -\frac{\pi}{4} - \frac{\Theta}{2} + \frac{3\varepsilon}{2} - \varepsilon + o(1) \\ \tilde{\tilde{\theta}}_3^\varepsilon = \arctan \tilde{\tilde{\lambda}}_3^\varepsilon = \frac{\pi}{2} - \varepsilon - |o(1)| \end{cases} . \quad (3.17)$$

It follows that  $\tilde{\tilde{u}}^\varepsilon$  is smooth and satisfies

$$\arctan \tilde{\tilde{\lambda}}_1^\varepsilon + \arctan \tilde{\tilde{\lambda}}_2^\varepsilon + \arctan \tilde{\tilde{\lambda}}_3^\varepsilon = -\Theta \quad \text{in } B_{\frac{1}{2}\tilde{r}_\Theta}.$$

Finally set

$$u^\varepsilon(x) = -\frac{\tilde{\tilde{u}}^\varepsilon\left(\frac{1}{2}\tilde{r}_\Theta x\right)}{\left(\frac{1}{2}\tilde{r}_\Theta\right)^2}.$$

Observe that the gradients of the potential functions, or the heights of the special Lagrangian graphs are kept uniformly bounded with respect to  $\varepsilon$  under the above three families of  $\mathbb{U}(3)$  rotations. Combined with (3.17), we obtain the desired family of smooth solutions to (1.1) with  $n = 3$  and fixed  $\Theta \in [0, \pi/2)$ . By symmetry,  $-u^\varepsilon$  are the other family of solutions to (1.1) with  $n = 3$  and fixed  $\Theta \in (-\pi/2, 0]$ .

### 3.5 Minimal surface system: proof of Theorem 1.4

In this section, we prove Theorem 1.4. Take the singular solutions  $u^m$  from Theorem 1.2 with  $\Theta = 0$  and  $m = 2, 3, 4, \dots$ . Let

$$U^m = Du^m.$$

From Property S.2 and Proposition 3.1, we see that

$$|DU^m(y)| = |D^2u^m(y)| \approx \frac{1}{|Du^m(y)|^{2m-2}}.$$

Here “ $\approx$ ” means two quantities are equivalent up to a multiple of constant depending only on the dimension and  $m$ . Then we have

$$\begin{aligned} \int_{B_1} |DU^m(y)|^p dy &\approx \int_{B_1} \frac{1}{|Du^m(y)|^{(2m-2)p}} dy \\ &= \int_{Du^m(B_1)} \frac{1}{|x|^{(2m-2)p}} \left| \det [D^2u^m(y)]^{-1} \right| dx \\ &\approx \int_{Du^m(B_1)} \frac{1}{|x|^{(2m-2)p}} |x|^{2m-2} dx_1 dx_2 dx_3. \end{aligned}$$

It follows that

$$U^m \in W^{1,p}(B_1) \quad \text{for any } p < \frac{2m+1}{2m-2} \quad \text{but } U^m \notin W^{1, \frac{2m+1}{2m-2}}(B_1).$$

We next show that  $U^m$  satisfies (1.4) in the integral sense, namely

$$\int_{B_1} \sum_{i,j=1}^3 \sqrt{g} g^{ij} \langle \partial_{x_i} U, \partial_{x_j} \Phi \rangle dx = 0$$

for all  $\Phi \in C_0^\infty(B_1, \mathbb{R}^3)$ . This is because the integrand is 0 everywhere except at the origin and we have the following bound on the integrand near the origin. Diagonalizing  $D^2u^m$ , we see that

$$\begin{aligned} \left| \sum_{i,j=1}^3 \sqrt{g} g^{ij} \langle \partial_{x_i} U, \partial_{x_j} \Phi \rangle \right| &= \left| \sum_{i=1}^3 \sqrt{(1+\lambda_1^2) \cdots (1+\lambda_3^2)} \frac{\lambda_i}{1+\lambda_i^2} \partial_i \Phi^i \right| \\ &\leq C(3, m) |D^2u^m| |D\Phi| = C(3, m) |DU^m| |D\Phi| \in L^1, \end{aligned}$$

where we used again the fact (3.13) that two of the eigenvalues of  $D^2u^m$  are bounded. The first part of the Theorem 1.4 is proved.

The second part of Theorem 1.4 is straightforward if we take  $U^\varepsilon = Du^\varepsilon$  with smooth solutions  $u^\varepsilon$  in Theorem 1.3 for any fixed  $\Theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ .

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