

THE MARTIN BOUNDARY IN NON-LIPSCHITZ DOMAINS

RICHARD F. BASS AND KRZYSZTOF BURDZY

University of Washington

ABSTRACT. The Martin boundary with respect to the Laplacian and with respect to uniformly elliptic operators in divergence form can be identified with the Euclidean boundary in C^γ domains, where $\gamma(x) = bx \log \log(1/x) / \log \log \log(1/x)$, b small. A counterexample shows that this result is very nearly sharp.

1. Introduction.

In the past few years a number of results that were previously known for Lipschitz domains have been shown to hold in much wider classes of domains. Among these, for example, are the mutual absolute continuity of harmonic measure and surface measure (in some L^p_1 domains) [JK2], the boundary Harnack principle (in all Hölder domains) [BB2, BBB], the parabolic boundary Harnack principle (in some L^p domains) [BB3], intrinsic ultracontractivity (in uniformly Hölder domains) [Bñ], and upper bounds for the heat kernel with Neumann boundary conditions (in all Hölder domains) [BH].

In this paper we consider the problem of the identification of the Martin boundary. Here the situation is quite a bit more delicate. We prove

Theorem 1.1. *Both the Martin boundary and the minimal Martin boundary are equal to the Euclidean boundary in bounded C^γ domains for*

$$(1.1) \quad \gamma(x) = bx \frac{\log \log(1/x)}{\log \log \log(1/x)}$$

provided b is sufficiently small.

For definitions of Martin boundary, minimal Martin boundary, and C^γ domains, see Section 2. Roughly, however, a C^γ domain is one where the boundary can be represented locally as the graph of a function whose modulus of continuity is no worse than γ . Much better results can be obtained in \mathbb{R}^2 by complex analytic techniques, so throughout this paper we restrict attention to \mathbb{R}^d , $d \geq 3$. Our results are also valid for the Martin boundary with respect to uniformly elliptic operators in divergence form with bounded and measurable coefficients.

Our results are very nearly sharp. We also prove

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Theorem 1.2. *For each $b > 0$ there exists a bounded C^γ domain in \mathbb{R}^3 with*

$$(1.2) \quad \gamma(x) = bx \log \log(1/x)$$

for which the Martin boundary is different from the Euclidean boundary.

The Martin boundary was introduced in [M] in order to give a representation of positive harmonic functions in a domain akin to the Poisson integral formula for a ball. The identification of the Martin boundary with the Euclidean boundary for Lipschitz domains was first given in [HW]. Alternate proofs have been given by [JK1] and [BB1]. In [JK3] Jerison and Kenig showed that the Martin boundary equals the Euclidean boundary in nontangentially accessible domains. Although nontangentially accessible domains share many of the scaling and potential-theoretic properties of Lipschitz domains, their boundaries can be quite wild. Ancona has a number of papers on Martin boundary; see [A] and the references therein.

In Section 2 we prove Theorem 1.1: we prove the equality of the Martin boundary with the Euclidean boundary in bounded C^γ domains, $\gamma(x)$ as in (1.1) and we prove that the Martin kernel associated with each Euclidean point is a minimal harmonic function. In Section 3 we give the necessary lemmas that allow us to extend the results of Section 2 to divergence form operators. And in Section 4 we prove Theorem 1.2.

Let us give a heuristic explanation for why the critical modulus of continuity for our problem is close to $x \log \log(1/x)$. We start with a domain D_1 which is the upper halfspace from which two hemispheres were removed, i.e.,

$$D_1 = \{x^1 > 0\} - (\{|x - (0, 1, 0, \dots, 0)| \leq 1\} \cup \{|x - (0, -1, 0, \dots, 0)| \leq 1\}).$$

Martin [M] proved that the Euclidean point 0 corresponds in D_1 to infinitely many Martin boundary points.

Now modify D_1 to obtain a domain D which has the same general shape as D_1 but lies above the graph of a function with modulus of continuity $x \log \log(1/x)$. The part of ∂D whose distance from 0 is between 2^{-k-1} and 2^{-k} consists mostly of two almost flat annuli; the distance a_k between them satisfies $a_k \log \log(1/a_k) \approx 2^{-k}$ from which we obtain $a_k \approx 2^{-k} / \log k$. The ratio of the length to width of the canal between these two annuli is $\approx 2^{-k} / (2^{-k} / \log k) = \log k$. The chances that Brownian motion conditioned to go to 0 will go from one side of the canal to the other are of order $\exp(-\log k) = 1/k$. The series $\sum 1/k$ is divergent, so by the Borel-Cantelli lemma the process will change the direction of approach infinitely often and there is only one Martin boundary point corresponding to the approach of Brownian motion to 0. A slight perturbation of the domain will make the series summable and the process will change its direction of approach only finitely many times. In this case, we will obtain, by symmetry, at least 2 distinct Martin boundary points, corresponding to Brownian motion approaching 0 from two sides.

The letter c with subscripts will denote constants with values in $(0, \infty)$ which may change from proposition to proposition but do not change within a proposition and its proof. For details on the Martin boundary see Doob [Do]. For more information on the relationship between Brownian motion and harmonic functions, see [Do] and [PS].

2. Martin boundary.

We begin with some notation. Suppose $d \geq 3$. The Euclidean closure of $A \subset \mathbb{R}^d$ will be denoted \bar{A} . Write $x = (x^1, \dots, x^d) = (\tilde{x}, x^d)$, where $\tilde{x} = (x^1, \dots, x^{d-1})$. Suppose $\gamma : [0, \infty) \rightarrow [0, \infty)$. We let C^γ be the set of all functions $f : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ such that $|f(x) - f(y)| \leq \sup_{0 < r \leq |x-y|} \gamma(r)$ for all $x, y \in \mathbb{R}^{d-1}$. We say D is a C^γ domain if for all $x \in D$ there exist a neighborhood U_x of x , an orthonormal coordinate system CS_x , and a C^γ function Γ_x such that

$$D \cap U_x = \{y : y = (\tilde{y}, y^d) \text{ in } CS_x, y^d > \Gamma_x(\tilde{y})\} \cap U_x.$$

Define

$$L_2(x) = \log \log(1/x) \quad \varphi(x) = \frac{\log \log(1/x)}{\log \log \log(1/x)}$$

if $x \in (0, \exp(-e^e))$, e otherwise.

We take $b > 0$, the value of b to be chosen later. Suppose $\Gamma : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ is in $C^{bx\varphi(x)}$, and for now suppose $D = \{x \in \mathbb{R}^d : x^d > \Gamma(\tilde{x})\}$. Let $d(x) = \text{dist}(x, \partial D)$, $\delta(x) = x^d - \Gamma(\tilde{x})$, and $B(x, r) = \{y \in \mathbb{R}^d : |y - x| < r\}$.

Let

$$(2.1) \Delta(x, a, r) = \{y \in D : \Gamma(\tilde{y}) < y^d < \Gamma(\tilde{y}) + a, |\tilde{y} - \tilde{x}| < r\},$$

$$\partial^u \Delta(x, a, r) = \{y \in \partial \Delta(x, a, r) : y^d = \Gamma(\tilde{y}) + a\}, \quad (\text{“u” for upper})$$

and

$$\partial^s \Delta(x, a, r) = \{y \in \partial \Delta(x, a, r) : |\tilde{y} - \tilde{x}| = r\} \quad (\text{“s” for side}).$$

Let (X_t, P^x) be standard Brownian motion in \mathbb{R}^d . For any Borel set A , let

$$T(A) = \inf\{t > 0 : X_t \in A\}.$$

Before plunging into the proof of Theorem 1.1, let us give a brief overview. To show that no Euclidean point corresponds to more than one Martin boundary point, it suffices to fix $x, x_0 \in D$ and to show $g(x, y)/g(x_0, y)$ is uniformly continuous for y near the boundary, where g is the Green function for the domain D . We do this by showing uniform continuity for $g(x, y)/h(y)$ and $g(x_0, y)/h(y)$, where $h(y)$ (given in (2.5)) is the probability of exiting a certain “box” $\Delta(0, \psi(a), \psi(a))$ from its upper side. Write $u(y)$ for $g(x, y)$. Consider two boxes $W_{k+1} \subset W_k$. By the usual Harnack inequality, the values of u along the upper side of W_k are all comparable, and similarly for h . By the techniques of [BB2] (see Lemma 2.4), the probability that the h -path transform of Brownian motion starting in W_{k+1} will exit W_k through its upper boundary can be bounded below. We conclude that the oscillation of u/h on W_{k+1} must be less than ρ_k times the oscillation of u/h on W_k for suitable ρ_k . Taking a decreasing sequence W_k and showing $\prod \rho_k = 0$ gives the uniform continuity of u/h .

The following lemma is Lemma 2.2 of [BB2].

Lemma 2.1. *There exists $c_1 \in (0, 1)$ such that if $k > 0$, $a > 0$, $r \geq ak$, and $y \in \Delta(x, a, r)$ with $\tilde{y} = \tilde{x}$, then*

$$P^y(X_{T(\partial \Delta(x, a, r))} \in \partial^s \Delta(x, a, r)) \leq c_1^k.$$

Lemma 2.2. (cf. Lemma 2.3 of [BB2]) *For each $b \in (0, 1)$ there exists $a_0 = a_0(b)$ and c_1 (independent of b) such that if $a < a_0$ and $\delta(y) = a$, then*

- (a) *if $r \geq 4a$, $P^y(X_{T(\partial\Delta(y, 2a, r))}) \in \partial^u \Delta(y, 2a, r) \geq \exp(-c_1 b \varphi(a))$;*
 (b) *if $r \geq 2^{k+1}a$, $P^y(X_{T(\partial\Delta(y, 2^k a, r))}) \in \partial^u \Delta(y, 2^k a, r) \geq \exp(-c_1 b \sum_{i=1}^k \varphi(2^{i-1}a))$.*

Proof. Since $\Gamma \in C^{bx\varphi(x)}$, we observe that $d(y) \geq c_2 a / b \varphi(a)$. Let $y_i = (\tilde{y}, y^d + i c_2 a / 2b \varphi(a))$, $B_i = B(y_i, c_2 a / b \varphi(a))$, $i = 0, 1, \dots, n$, where $n = [4b\varphi(a)/c_2] + 3$. Since D is the region above Γ , $d(y_i)$ is increasing, and each $B_i \subset D$.

Let $B = \bigcup_{i=1}^n B_i$. Let h be the harmonic function on B with boundary values 1 on $\partial B \cap \partial B_n$, and 0 on the remainder of ∂B . By standard properties of Brownian motion, $h(y_n) \geq c_3$. By Harnack's inequality in B_i , $h(y_i) \geq c_4 h(y_{i+1})$. So by an induction argument,

$$h(y) = h(y_0) \geq c_3 (c_4)^n = c_3 \exp(-n \log(c_4^{-1})) \geq \exp(-c_1 b \varphi(a)),$$

using $a < a_0$. By our choice of n , we see that the desired probability in (a) is bounded below by $P^y(X_{T(\partial B)} \in \partial B_n) = h(y)$.

If we let $S_i = \inf\{t : X_t \in \partial^u \Delta(y, 2^i a, 2 \cdot 2^i a)\}$, then (b) follows from (a) by the strong Markov property applied at the times S_i and an induction argument. \square

Lemma 2.3. *Suppose v is nonnegative and harmonic in $\Delta(y, 2^{k+1}a, 2^{k+2}a)$, $\delta(y) = a$, and $z = (\tilde{y}, y^d + 2^k a)$. Then*

$$v(y) \geq \exp(-c_1 b k \varphi(a)) v(z), \quad v(z) \geq \exp(-c_1 b k \varphi(a)) v(y).$$

Proof. Let $y_1 = (\tilde{y}, y^d + c_2 n a / 2b \varphi(a))$, where c_2 and n are as in the proof of Lemma 2.2. Constructing the chain of balls B_i connecting y and y_1 as in that proof, we see that

$$v(y) \geq \exp(-c_1 b \varphi(a)) v(z), \quad v(z) \geq \exp(-c_1 b \varphi(a)) v(y)$$

by Harnack's inequality. This proves the corollary in the case $k = 1$.

An induction argument together with the fact that

$$\exp(-c_1 b \sum_{i=1}^k \varphi(2^i a)) \geq \exp(-c_2 b k \varphi(a))$$

for $a < a_0$ gives the general case. \square

Lemma 2.4. *For each $b \in (0, 1)$ there exist $a_0 = a_0(b)$ and K_1 and c_1 (independent of b) such that if $a \in (0, a_0)$, $R \geq K_1 b a \varphi(a)$,*

$$(2.2) \quad \begin{aligned} H_1 &= \{X_{T(\partial\Delta(0, 2a, 2R))} \in \partial^s \Delta(0, 2a, 2R)\} & \text{and} \\ H_2 &= \{X_{T(\partial\Delta(0, 2a, 2R))} \in \partial^u \Delta(0, 2a, 2R)\}, \end{aligned}$$

then

$$P^x(H_1) \leq c_1 P^x(H_2) \quad \text{for all } x \in \partial\Delta(0, a, R).$$

Proof. We follow the proof of Theorem 2.4 of [BB2] very closely. Write $\beta_k = 2^{-k}$, $r_k = 2R - (R/8) \sum_{i=0}^k (1+i)^{-2}$. Note $r_0 = 15R/8$ and $\inf_k r_k > R$. Take a_0 small enough so that $b\varphi(a_0) \geq 1$.

Let

$$J_k = \{y \in D : y^d \in [\Gamma(\tilde{y}) + a\beta_{k+1}, \Gamma(\tilde{y}) + a\beta_k], |\tilde{y}| \leq r_k\}, \quad k = 0, 1, \dots$$

By Lemma 2.2(b), if $z \in J_0$, then $P^z(H_2) \geq \exp(-c_3 b\varphi(a))$, c_3 independent of a . On the other hand, if $z \in J_0$, then by Lemma 2.1, $P^z(H_1) \leq \exp(-c_4 K_1 b\varphi(a))$. So by taking K_1 large enough (independent of a_0), $P^z(H_1) \leq P^z(H_2)$ for $z \in J_0$.

Let $d_m = \sup_{z \in J_m} P^z(H_1)/P^z(H_2)$. From what we have just observed, $d_0 \leq 1$. Since $\Delta(0, a, R) \subset \bigcup_{k=0}^{\infty} J_k$, to prove the theorem it suffices to prove $\sup_m d_m < \infty$.

Fix m and suppose $z \in J_{m+1}$. For the remainder of the proof, write

$$\Delta_m = \Delta(z, a\beta_{m+1}, r_m - r_{m+1}), \quad U_m = T(\partial\Delta_m).$$

By the strong Markov property,

$$(2.3) \quad P^z(H_1) \leq E^z(P^{X_{U_m}}(H_1); X_{U_m} \in \partial^u \Delta_m) + P^z(X_{U_m} \in \partial^s \Delta_m)$$

and

$$(2.4) \quad P^z(H_2) \geq E^z(P^{X_{U_m}}(H_2); X_{U_m} \in \partial^u \Delta_m).$$

Since $\partial^u \Delta_m \subseteq J_m$ when $z \in J_{m+1}$, using (2.4) we see that the first term on the right hand side of (2.3) is bounded by

$$d_m E^z(P^{X_{U_m}}(H_2); X_{U_m} \in \partial^u \Delta_m) \leq d_m P^z(H_2).$$

By Lemma 2.1 the second term on the right of (2.3) is bounded above by

$$\exp(-c_4/a(m+2)^2 \beta_{m+1}) \leq \exp(-c_4 b K_1 \varphi(a)/m^2 \beta_{m+1}).$$

On the other hand, by Lemma 2.2(b), $P^z(H_2)$ is bounded below by

$$c_5 \exp(-c_3 b \sum_{i=-m-2}^0 \varphi(2^i a)) \geq c_5 \exp(-c_6 b m^3 \varphi(a)).$$

Take K_1 larger if necessary so that $2c_6 m^3 \leq c_4 K_1/m^2 \beta_{m+1}$ for all $m \geq 1$. We can choose c_7 such that for all m

$$c_5 \exp(-c_6 b m^3 \varphi(a)) \geq c_7^{-1} m^2 \exp(-c_4 b K_1 \varphi(a)/m^2 \beta_{m+1}).$$

So for $a < a_0$

$$\begin{aligned} P^z(H_1) &\leq d_m P^z(H_2) + \exp(-c_4 b K_1 \varphi(a)/m^2 \beta_{m+1}) \\ &\leq d_m P^z(H_2) + c_7 m^{-2} c_5 \exp(-c_6 b m^3 \varphi(a)) \\ &\leq d_m P^z(H_2) + c_7 m^{-2} P^z(H_2). \end{aligned}$$

Hence

$$d_{m+1} \leq d_m + c_7 m^{-2},$$

or $\sup_m d_m \leq d_0 + c_7 \sum_{m=1}^{\infty} m^{-2} \stackrel{df}{=} c_1 < \infty$, as required. \square

Define

$$\psi(a) = 2K_1 b a (\varphi(a))^5.$$

Lemma 2.5. *For each $b \in (0, 1)$ there exists $a_0 = a_0(b)$ and c_1 (independent of b) such that if $a < a_0$, v is nonnegative and harmonic in $\Delta(0, \psi(a), \psi(a))$, $|\tilde{x} - \tilde{y}| \leq 16K_1ba\varphi(a)$, and $\delta(x) = \delta(y) = a$, then*

$$v(y) \geq \exp(-c_1bL_2(a))v(x).$$

Proof. Let $w = (\tilde{x}, x^d + a(\varphi(a))^4)$, $z = (\tilde{y}, y^d + a(\varphi(a))^4)$. Since $\Gamma \in C^{bx\varphi(x)}$, we have

$$d(w) \geq \delta(w)/c_2b\varphi(\delta(w)) \geq \delta(w)/c_2b\varphi(a) > c_3a(\varphi(a))^3/b$$

if a_0 is small enough, and similarly for $d(z)$. Since $b < 1$, note also that $|z - w| \leq c_4a(\varphi(a))^2$. Provided a_0 is small enough, we can use Harnack's inequality in the ball $B(w, 2c_4a(\varphi(a))^2)$ to get $v(z) \geq c_5v(w)$.

By Lemma 2.3 we have

$$v(y) \geq \exp(-c_6b\varphi(a) \log(\varphi(a))^4)v(z), \quad v(w) \geq \exp(-c_6b\varphi(a) \log(\varphi(a))^4)v(x).$$

To complete the proof note that $\varphi(a) \log(\varphi(a))^4 \leq c_7L_2(a)$ for $a < a_0$ if a_0 is sufficiently small. \square

Now define

$$(2.5) \quad h(x) = P^x(X_{T(\partial\Delta(0, \psi(a), \psi(a)))}) \in \partial^u \Delta(0, \psi(a), \psi(a)).$$

Of course, h is positive and harmonic in $\Delta(0, \psi(a), \psi(a))$. Let (X_t, P_h^x) be the h -path transform of Brownian motion in $\Delta(0, \psi(a), \psi(a))$, i.e., Brownian motion conditioned by h ; see [Do] for details.

Lemma 2.6. *For each $b \in (0, 1)$ there exist $a_0 = a_0(b)$ and c_1, c_2 (independent of b) such that if $a < a_0$, $K_1ba\varphi(a) \leq R \leq 2K_1ba\varphi(a)$, H_2 is defined as in (2.2), and $x \in \Delta(0, a, R)$, then*

$$P_h^x(H_2) \geq c_1 \exp(-c_2bL_2(a)).$$

Proof. Note h has boundary values 0 on $\partial^s \Delta(0, \psi(a), \psi(a))$, hence $\partial h / \partial x^d = 0$ there. Since $h \leq 1$ and h has boundary values 1 on $\partial^u \Delta(0, \psi(a), \psi(a))$, then $\partial h / \partial x^d \geq 0$ on the upper boundary. Finally, since $h \geq 0$ and $h = 0$ on ∂D , then $\partial h / \partial x^d \geq 0$ on $\partial D \cap \partial \Delta(0, \psi(a), \psi(a))$. By the maximum principle, then, $\partial h / \partial x^d \geq 0$ in $\Delta(0, \psi(a), \psi(a))$.

Write Δ for $\Delta(0, 2a, 2R)$. Since h is harmonic,

$$\begin{aligned} h(x) &= E^x(h(X_{T(\partial\Delta)}); X_{T(\partial\Delta)} \in \partial^u \Delta) + E^x(h(X_{T(\partial\Delta)}); X_{T(\partial\Delta)} \in \partial^s \Delta) \\ &\leq E^x(h(X_{T(\partial\Delta)}); X_{T(\partial\Delta)} \in \partial^u \Delta) + (\sup_{\partial^s \Delta} h) P^x(X_{T(\partial\Delta)} \in \partial^s \Delta). \end{aligned}$$

By Lemma 2.4, $P^x(X_{T(\partial\Delta)} \in \partial^s \Delta) \leq c_3 P^x(X_{T(\partial\Delta)} \in \partial^u \Delta)$. Since $\partial h / \partial x^d \geq 0$, h is nondecreasing as a function of x^d , so $\sup_{\partial^s \Delta} h \leq \sup_{\partial^u \Delta} h$. By Lemma 2.5, $\sup_{\partial^u \Delta} h \leq \exp(c_4bL_2(a)) \inf_{\partial^u \Delta} h$. Therefore

$$\begin{aligned} h(x) &\leq E^x(h(X_{T(\partial\Delta)}); X_{T(\partial\Delta)} \in \partial^u \Delta) \\ &\quad + c_3 \exp(c_4bL_2(a)) (\inf_{\partial^u \Delta} h) P^x(X_{T(\partial\Delta)} \in \partial^u \Delta) \\ &\leq (1 + c_3 \exp(c_4bL_2(a))) E^x(h(X_{T(\partial\Delta)}); X_{T(\partial\Delta)} \in \partial^u \Delta). \end{aligned}$$

Hence

$$P_h^x(H_2) = \frac{E^x(h(X_{T(\partial\Delta)}); X_{T(\partial\Delta)} \in \partial^u\Delta)}{h(x)} \geq c_5 \exp(-c_4 b L_2(a)). \quad \square$$

Remark 2.7. Since we know the boundary Harnack principle holds for Hölder domains ([BBB]), it holds for the domain D that we have been considering, as well as any $C^{bx\varphi(x)}$ domain. Also, by the argument of [PS], Section 3.3, the domain D is regular with respect to the Dirichlet problem, and so is any $C^{bx\varphi(x)}$ domain.

We are now ready for our key step.

Lemma 2.8. *There exists $b_0 \in (0, 1)$ with the property that for each $b \in (0, b_0)$ there exists $a_0 = a_0(b)$ such that if $a < a_0$, u is nonnegative and harmonic in the domain $\Delta(0, \psi(a), \psi(a))$ and vanishes continuously on $\partial D \cap \partial\Delta(0, \psi(a), \psi(a))$, and h is as in (2.5), then u/h is uniformly continuous in $\Delta(0, a/2, a/2)$.*

Proof. Let $r_k = 2^{-k(\log k)^2}$, $x \in \partial\Delta(0, a/2, a/2)$,

$$W_k = \Delta(x, r_k, K_1 b r_k \varphi(r_k)), \quad V_k = \Delta(x, 2r_k, 2K_1 b r_k \varphi(r_k)).$$

By the boundary Harnack principle, $\sup_{W_k}(u/h) < \infty$. Let

$$O_k = \underset{W_k}{Osc}(u/h) = \sup_{W_k}(u/h) - \inf_{W_k}(u/h).$$

Now for k sufficiently large,

$$(2.6) \quad W_k \subset \Delta(0, a, a), \quad r_k \geq 2K_1 r_{k+1} \log(r_{k+1}^{-1})^5, \quad \text{and } V_{k+1} \subset W_k.$$

Observe u/h is P_h^x -harmonic in $\Delta(0, \psi(a), \psi(a))$.

Let $U = A(u/h) + B$, where A and B are chosen so that $\sup_{W_k} U = 1$ and $\inf_{W_k} U = 0$. Then U is P_h^x -harmonic in W_k . Since $V_{k+1} \subset W_k$, then U is bounded by 0 and 1 in V_{k+1} . By looking at $1 - U$ if necessary, we may suppose that $\sup_{\partial^u V_{k+1}} U \geq 1/2$. Note also that hU is nonnegative and harmonic (with respect to P^x) in W_k .

Suppose $y \in W_{k+1}$. Then

$$(2.7) \quad \begin{aligned} U(y) &\geq E_h^y(U(X_{T(\partial V_{k+1})}); X_{T(\partial V_{k+1})} \in \partial^u V_{k+1}) \\ &\geq \left(\inf_{\partial^u V_{k+1}} U \right) P_h^y(X_{T(\partial V_{k+1})} \in \partial^u V_{k+1}). \end{aligned}$$

By Lemma 2.6

$$(2.8) \quad P_h^y(X_{T(\partial V_{k+1})} \in \partial^u V_{k+1}) \geq c_1 \exp(-c_2 b L_2(r_{k+1})).$$

For some $x_1 \in \partial^u V_{k+1}$ we have that $U(x_1) \geq 1/4$. Since both h and hU are nonnegative and harmonic in W_k , then by Lemma 2.5

$$\begin{aligned} h(z) &\leq c_3 \exp(c_4 b L_2(r_{k+1})) h(x_1) \quad \text{and} \\ (hU)(z) &\geq c_3^{-1} \exp(-c_4 b L_2(r_{k+1})) (hU)(x_1) \end{aligned}$$

for $z \in \partial^u V_{k+1}$. Hence

$$(2.9) \quad \begin{aligned} U(z) &\geq c_3^{-2} \exp(-2c_4 b L_2(r_{k+1})) U(x_1) \\ &\geq (c_3^{-2}/4) \exp(-2c_4 b L_2(r_{k+1})). \end{aligned}$$

Recall that $U \leq 1$ on W_{k+1} . Substituting (2.8) and (2.9) in (2.7) gives

$$\frac{Osc U}{W_{k+1}} \leq \rho_{k+1} \frac{Osc U}{W_k},$$

where

$$\rho_k = 1 - c_5 \exp(-c_6 b L_2(r_k)).$$

Recalling the definition of U , we then get $O_{k+1} \leq \rho_{k+1} O_k$.

Choose $b_0 < 1 \wedge c_6^{-1}$. If $b < b_0$, then $\sum_{k=k_0}^{\infty} (1 - \rho_k) = \infty$ for any k_0 , hence $\prod_{k=k_0}^{\infty} \rho_k = 0$. The uniform continuity follows easily: if $b < b_0$, choose a_0 small as in Lemmas 2.2, 2.4–2.6. Then choose k_0 large as in (2.6). Given ϵ choose $k_1 > k_0$ so that $\prod_{k=k_0}^{k_1} \rho_k < \epsilon$. For any two points y_1, y_2 in $\Delta(x, r_{k_1+1}, K_1 r_{k_1+1} \varphi(r_{k_1+1}))$,

$$\begin{aligned} |(u/h)(y_1) - (u/h)(y_2)| &\leq \frac{Osc(u/h)}{W_{k_1+1}} \leq \left(\prod_{k=k_0+1}^{k_1+1} \rho_k \right) \frac{Osc(u/h)}{W_{k_0}} \\ &\leq \epsilon \sup_{\partial \Delta(0, a, a)} (u/h). \quad \square \end{aligned}$$

We now turn to the Martin boundary. Suppose $D \subset \mathbb{R}^d$, fix a base point $x_0 \in D$ and define the Martin kernel

$$(2.10) \quad K(y, x) = G_D(y, x) / G_D(y, x_0)$$

for $x, y \in D, y \neq x_0$, where G_D is the Green function for the domain D . There is a unique (up to a homeomorphism) compactification D^M of D with the property that K has a continuous extension to $(D^M - \{x_0\}) \times D$ and $K(z_1, \cdot) \equiv K(z_2, \cdot)$ if and only if $z_1 = z_2 \in D^M$ ([Do] 1 XII 3). The set $\partial^M D \stackrel{df}{=} D^M - D$ is called the Martin boundary of D .

A positive harmonic function h is minimal if whenever g is a positive harmonic function in D with $g \leq h$, then $g \equiv ch$ for some constant c . The set of all points $z \in \partial^M D$ such that $K(z, \cdot)$ is a minimal harmonic function is called the minimal Martin boundary and denoted $\partial_1^M D$.

Theorem 1.1 can then be expressed as follows. If D is a bounded C^γ domain with γ given by (1.1), b sufficiently small, then the function K has a continuous extension to $(\bar{D} - \{x_0\}) \times D$, we have $K(z_1, \cdot) \equiv K(z_2, \cdot)$ if and only if $z_1 = z_2$ and, moreover, $\partial_1^M D = \partial^M D$.

Suppose from now on that D is a $C^{bx\varphi(x)}$ domain, b less than the b_0 of Lemma 2.8.

Proof of Theorem 1.1. Pick $x_0, x \in D$. For $z \in \partial D$ pick a neighborhood U and a coordinate system CS so that

$$U \cap D = \{y : y^d > \Gamma(\tilde{y}) \text{ if } y = (\tilde{y}, y^d) \text{ in } CS\} \cap U.$$

Without loss of generality we may suppose that our coordinate system was chosen so that $z = 0$. Take a_0 small enough so that $\Delta(0, \psi(a), \psi(a)) \subset U \cap D$ and that $x_0, x \notin \partial\Delta(0, \psi(a), \psi(a))$ if $a < a_0$. Define h as in (2.5) and suppose u and v are positive and harmonic in $\Delta(0, \psi(a), \psi(a))$, vanishing continuously on $\partial D \cap \partial\Delta(0, \psi(a), \psi(a))$. By Remark 2.7 h also vanishes continuously on $\partial D \cap \partial\Delta(0, \psi(a), \psi(a))$. By Lemma 2.8 u/h and v/h are uniformly continuous in $\Delta(0, a/2, a/2)$.

If $x_1 \in \Delta(0, a, a)$, then $u(x_1)/h(x_1) \in (0, \infty)$, clearly, and by the boundary Harnack principle, there exists a constant c_1 such that

$$c_1^{-1}(u/h)(x_1) \leq (u/h)(y) \leq c_1(u/h)(x_1), \quad y \in \Delta(0, a, a).$$

The same holds for v/h . Therefore we see that $u/v = (u/h)/(v/h)$ is uniformly continuous in $\Delta(0, a/2, a/2)$.

We now choose $u(y) = G_D(x, y)$ and $v(y) = G_D(x_0, y)$. Since D is regular, u, v vanish continuously on ∂D . So for a fixed x , the function $K(y, x) = u(y)/v(y)$ has a continuous extension to $\bar{D} - \{x_0\}$. For each $z \in \bar{D} - \{x_0\}$, the function $K(z, \cdot)$ is positive and harmonic in $D - \{z\}$. This, the Harnack inequality, and the fact that $K(z, x_0) = 1$ for every z may be used to show that the family of functions $\{K(z, \cdot), z \notin B(x, r)\}$ is uniformly continuous on $B(x, r/2)$ if $B(x, r) \subset D$. This implies easily that K is jointly continuous on $(\bar{D} - \{x_0\}) \times D$, except on the diagonal.

We next need to show that the Martin kernels corresponding to two distinct points $w, z \in \partial D$ are distinct. The proof of this follows exactly the proof of Theorem 4.4 of [BB1], using the boundary Harnack principle of [BB2] or [BBB] in place of Theorem 1.1 of [BB1].

Finally, we need to show that each $K(z, \cdot)$ is a minimal harmonic function. To see this, note that the proof of Theorem 4.5 of [BB1] may be followed virtually word for word. This completes the proof of Theorem 1.1. \square

Remark 2.9.. The proof of Theorem 4.4 of [BB1] depends only on the boundary Harnack principle and not Lemma 2.8. So a Martin boundary point cannot correspond to more than one Euclidean boundary point in any domain for which the boundary Harnack principle holds, e.g., Hölder domains. On the other hand, the first example of Section 5 of [BB2] is that of a domain above the graph of a continuous function where the boundary Harnack principle does not hold; one can show that this domain has a Martin boundary point which corresponds to more than one Euclidean boundary point.

The proof of Theorem 1.1 shows, in particular, the following.

Theorem 2.10. *Suppose D is as in Theorem 1.1. If A compact is contained in V open and u and v are positive and harmonic in D and vanish continuously on $\partial D \cap V$, then u/v is uniformly continuous in $A \cap D$.*

Remark 2.11.. Define

$$\omega(x, A) = P^x(X_{T(\partial\Delta)} \in A), \quad x \in D, A \subset \partial D,$$

the harmonic measure of the set A with respect to the domain D . By Theorem 1 XII 10 of [Do],

$$\omega(x, A) = \int_A K(y, x) \omega(x_0, dy) \quad \text{for all } A \subset \partial D.$$

Consequently $\omega(x, A)/\omega(x_0, A) \rightarrow K(z, x)$ whenever A decreases to $\{z\}$. This is an extension to $C^{bx\varphi(x)}$ domains of one of the main results of [HW].

3. Divergence form operators.

In this section we consider the Martin boundary with respect to L -harmonic functions, where L is a uniformly elliptic operator in divergence form. We let L be the operator defined by

$$Lf(x) = \sum_{i,j=1}^d \frac{\partial}{\partial x^i} (a_{ij}(x) \frac{\partial f}{\partial x^j})(x),$$

where we assume the a_{ij} are bounded and uniformly elliptic:

(3.1) there exists c_L such that

$$c_L^{-1} \sum_{i=1}^d (y^i)^2 \leq \sum_{i,j=1}^d a_{ij}(x) y^i y^j \leq c_L \sum_{i=1}^d (y^i)^2, \quad (y^1, \dots, y^d) \in \mathbb{R}^d.$$

We also assume the a_{ij} are smooth, but the estimates we obtain will depend on the a_{ij} 's only through the bound c_L and not on the smoothness of the a_{ij} 's.

Let (X_t, P^x) be the strong Markov process corresponding to L . Let

$$(3.2) \quad F_a = \{x \in D : \delta(x) < a\}.$$

All potential-theoretic objects, e.g., the Green function, are defined relative to L in this section. See [BB2], Section 4, for a summary of estimates related to divergence form operators.

Lemma 3.1. *The analog of Lemma 2.1 holds.*

Proof. By Lemma 2.1 of [BB2] there exist c_1, c_2 such that $|B(x, 2a) \cap F_a^c|/|B(x, 2a)| \geq c_1$ if $a < c_2$, where $|\cdot|$ denotes Lebesgue measure. By the remarks and results of Section 4 of [BB2], $P^x(T(F_a^c) < T(\partial B(x, 2a))) \geq c_3$, with c_3 independent of x and a . Using this estimate we then follow the proof of Lemma 2.2 of [BB2]. \square

Using Moser's Harnack principle (see [BB2], Section 4) instead of the usual Harnack's inequality, Lemmas 2.3–2.5 go through as before. The main change required is in the proof of Lemma 2.6.

If $G_D(\cdot, \cdot)$ denotes the Green function for the domain D with respect to L , let $g(x) = G_{\mathbb{R}^d}(0, x)$, the Green function in \mathbb{R}^d with pole at 0.

Lemma 3.2. *There exists β depending only on the constant c_L such that*

$$P_g^x(T(\{0\}) < T(\partial B(0, 1))) \geq 1/2$$

whenever $x \in B(0, \beta)$.

Proof. By [FS] there exists c_1 depending only on c_L such that

$$(3.3) \quad c_1 |x|^{2-d} \leq g(x) \leq c_1^{-1} |x|^{2-d}.$$

For $a > 0$ let $A(a) = \{y : g(y) = a\}$. By (3.3), if a_1 is sufficiently large, $A(a_1) \subset B(0, 1)$. Again using (3.3), we may take β small (depending only on c_L and a_1) so that $B(0, \beta) \subset \{y : g(y) \geq 2a_1\}$.

Suppose $x \in B(0, \beta)$. If $a_2 > 2a_1 \vee 2g(x)$ is large enough so that $A(a_2) \subset B(0, \beta)$, then

$$\begin{aligned} P_g^x(T(A(a_2)) < T(A(a_1))) &\geq \frac{E^x(g(X_{T(A(a_2))}); T(A(a_2)) < T(A(a_1)))}{g(x)} \\ &\geq \frac{a_2}{g(x)} P^x(g(X_t) \text{ hits } a_2 \text{ before hitting } a_1) \\ &= \frac{a_2}{g(x)} \frac{g(x) - a_1}{a_2 - a_1} \\ &\geq 1 - a_1/g(x) \geq 1/2, \end{aligned}$$

using the fact that $g(X_t)$ is a martingale up until time $T(A(a_2))$. Now let $a_2 \rightarrow \infty$. \square

Suppose now that D is the region above a $C^{b,x\varphi(x)}$ function, $a < a_0$, H_2 and h defined as in (2.2) and (2.5).

Lemma 3.3. *If $x \in \partial\Delta(0, a, a)$ and $K_1ba\varphi(a) \leq R \leq 2K_1ba\varphi(a)$, then*

$$P_h^x(H_2) \geq c_1 \exp(-c_2bL_2(a)).$$

Proof. Suppose $v \in D$ with $\text{dist}(v, \Delta(0, 2a, 2R)) \in (6R, 12R)$. Write Δ_1 for $\Delta(0, 2a, 2R)$,

$$\Delta_2 = \Delta(v, \delta(v) + a, 2R) - \Delta(v, \delta(v) - a, 2R), \quad \partial^u \Delta_2 = \partial^u \Delta(v, \delta(v) + a, 2R).$$

If $G(x, y) = G_{\mathbb{R}^d}(x, y)$, note that there exists c_3 depending only on c_L such that

$$c_3^{-1} \leq G(z, w)/G(z, v) \leq c_3 \quad \text{and} \quad c_3^{-1} \leq G(v, z)/G(v, x) \leq c_3$$

for $z \in \partial\Delta(0, 2a, 2R)$, $w \in \partial\Delta(v, 2a, 2R)$ and $x \in \partial\Delta(0, a, R)$.

Now suppose $z \in \partial^u \Delta_1$, $w \in \partial^u \Delta_2$. Let $c_4 = \beta d(z)/2$, where β is as in Lemma 3.2. Arguing as in Lemmas 2.2, 2.3, and 2.5, $P^w(T(B(z, c_4)) < T(\partial D)) \geq c_5 \exp(-c_6bL_2(a))$. By (3.3) $G(z, x)/G(z, w) \geq c_7$ for $x \in \partial B(z, c_4)$. Hence $P_{G(z, \cdot)}^w(T(B(z, c_4)) < T(\partial D)) \geq c_8 \exp(-c_6bL_2(a))$.

If $x \in \partial B(z, c_4)$, then $P_{G(z, \cdot)}^x(T(\{z\}) < T(\partial D)) > c_9$; this follows from Lemma 3.2 and a scaling argument. So by the strong Markov property for the process $(X_t, P_{G(z, \cdot)}^w)$, we see that

$$(3.4) \quad P_{G(z, \cdot)}^w(T(\{z\}) < T(\partial D)) > c_{10} \exp(-c_6bL_2(a)).$$

We next turn to the proof of Theorem 3.2 of [BB2], using $\partial^u \Delta$ instead of $\partial^g \Delta$. Rather than repeating the rather long proof, we sketch the changes necessary. By (3.3), c_3, c_4, c_6 , and c_9 of that proof may be taken to be independent of a . By (3.4) c_5 of that proof is greater than or equal to $c_{10} \exp(-c_6bL_2(a))$. So from the conclusion of that theorem, we get

$$P_{G_D(v, \cdot)}^x(H_2) \geq c_{11} \exp(-c_6bL_2(a))$$

or

$$(3.5) \quad E^x(G_D(X_{T(\partial\Delta_1)}, v); H_2) \geq c_{11} \exp(-c_6 L_2(a)) G_D(x, v).$$

Now let $D_n = \{v \in D : \text{dist}(v, \partial\Delta(0, 2a, 2R)) < 9R\} - J_n$, where $J_n = \{x : d(x) < 1/n\}$. Note $\bar{D}_n \subset D$. By standard results on balayage and réduites, if h_n is L -harmonic in D_n , then $h_n(x) = \int G_D(x, v) \mu_n(dv)$ for some measure μ_n on ∂D_n . See [Do], p. 160, for the proofs in the case of the Laplacian; since the Green function corresponding to the operator L is symmetric, the proofs go through in the present case. If in addition h_n has boundary values 0 on ∂J_n , we have that μ_n is supported on $\partial D_n - \partial J_n$. Integrating (3.5) with respect to μ_n , we get

$$(3.6) \quad E^x(h_n(X_{T(\partial\Delta_1)}); H_2) \geq c_{11} \exp(-c_6 b L_2(a)) h_n(x)$$

for such h_n provided n is large enough so that $x \in D_n$.

Finally let $h_n(x) = E^x(h(X_{T(\partial D_n)}); X_{T(\partial D_n)} \notin \partial J_n)$. Then h_n is L -harmonic on D_n and 0 on ∂J_n . Letting $n \rightarrow \infty$ and using the fact that h is bounded by 1, we get that h_n converges to h in $\Delta(0, 3a, 3R)$ by dominated convergence. Then letting $n \rightarrow \infty$ in (3.6) and using dominated convergence again, we obtain our result. \square

Theorem 3.4. *Theorem 1.1 holds for the Martin boundary with respect to the operator L .*

Proof. We replace the use of Lemma 2.6 by Lemma 3.3, we use Moser's Harnack inequality instead of the usual one, and we use the estimates of Section 4 of [BB2] in the proof of Theorem 4.1 of [BB1]. With these changes the proofs in Section 2 go through. \square

4. Counterexample.

For each $b_1 > 0$ we will construct a bounded C^γ domain D with

$$\gamma(x) = b_1 x \log \log(1/x)$$

and such that the Euclidean and Martin compactifications of D are different. Our example is 3-dimensional. It is easy to see that $D \times (0, 1)^{d-3}$ provides a d -dimensional example of the same kind, for $d \geq 4$. The construction will take up the rest of the section and will be divided into steps.

In this section, P^x , P_h^x , etc. will refer to the standard and conditioned Brownian motion.

Step 1. First we will define D and argue that it is a C^γ domain. Recall that $L_2(x)$ denotes $\log \log(1/x)$. Let

$$f(x^1, x^3) = a|x^1|/L_2(|x^1|) + bx^3/L_2(x^3)$$

for $x^1 \in \mathbb{R}$, $x^3 > 0$. We will use the convention $1/L_2(0) = 0$. The values of a and b will be specified later. At this point it will suffice to say that a and b should be thought of as large numbers satisfying $0 < b < a < \infty$. Let

$$\begin{aligned} D &= \{x \in \mathbb{R}^3 : |x^2| < f(x^1, x^3), 0 < x^3 \leq 1/100 - |x^1|/10\} \\ &\cup \{x \in \mathbb{R}^3 : |x^2| < 1, 1/100 - |x^1|/10 < x^3 < 1 - |x^1|/10, x^3 > 0\}, \\ \partial^u D &= \{x \in \partial D : x^3 = 1 - |x^1|/10\}. \end{aligned}$$

It is evident that for every closed ball $B(0, r)$ with r positive and small, $D - B(0, r)$ is a Lipschitz domain. We have to show that it is a C^γ domain in a neighborhood of 0. Our estimates in this and subsequent steps hold only locally near 0 but this does not cause any loss of generality.

Let g be the inverse function of $t \rightarrow t/L_2(t)$, i.e., $t = g(s)$ if $s = t/L_2(t)$, for all $t \in [0, 1/e)$. It is elementary to check that

$$(4.1) \quad g(c_1 t) \leq 2c_1 t L_2(t)$$

for small t .

The equation $x^2 = f(x^1, x^3)$ describes the surface $\{x \in \partial D : x^1 > 0, x^2 > 0, x^3 > 0\}$ near 0. We may write the equation in an equivalent way as

$$bx^3/L_2(x^3) = x^2 - ax^1/L_2(x^1)$$

or

$$x^3 = g((1/b)(x^2 - ax^1/L_2(x^1))).$$

Hence, for small $r > 0$, $\partial D \cap B(0, r)$ is given by

$$x^3 = g((1/b)(|x^2| - a|x^1|/L_2(|x^1|))^+) \stackrel{df}{=} g_1(x^1, x^2).$$

The function $(x^1, x^2) \rightarrow (|x^2| - a|x^1|/L_2(|x^1|))^+$ is Lipschitz, so (4.1) implies that

$$|x^3 - z^3| = |g_1(x^1, x^2) - g_1(z^1, z^2)| \leq (c_2/b)|x^1 - z^1| \cdot L_2(|x^1 - z^1|)$$

for small $|x|$ and $|z|$. We conclude that D is a C^γ domain with $\gamma(x) = b_1 x L_2(x)$, provided we take $b > c_2/b_1$.

Step 2. This step is devoted to some preliminary estimates of the Green function in D which are very crude.

Fix a base point $x_0 = (0, 0, 3/4)$ and let $h(x) = G_D(x, x_0)$, the Green function with pole at x_0 . We will compare h to a harmonic function h_1 which is positive in D with boundary values 1 on $\partial^u D$ and 0 elsewhere.

First we will show that $h_1(x^1, x^2, x^3)$ is increasing in x^1 for $x^1 > 0$. Let $D_1 \stackrel{df}{=} \{x \in D : x^1 > 0\}$. Observe that $\partial h_1 / \partial x^1 \geq 0$ on $\partial^u D \cap \partial D_1$ because $h_1 \leq 1$ in D and $h_1 = 1$ on $\partial^u D$. For similar reasons, $\partial h_1 / \partial x^1 \geq 0$ on the remaining part of $\partial_1 D \cap \partial D$. For $x \in \partial D_1$ with $x^1 = 0$ we have $\partial h_1 / \partial x^1 = 0$, by symmetry. Since $\partial h_1 / \partial x^1$ is harmonic in D , we obtain $\partial h_1 / \partial x^1 \geq 0$ in D and, therefore, $h_1(x^1, x^2, x^3)$ is increasing in x^1 for $x^1 > 0$.

A similar argument shows that $h_1(x^1, x^2, x^3)$ is decreasing in x^2 for $x^2 > 0$; one has to consider the boundary values of $\partial h_1 / \partial x^2$ in $\{x \in D : x^2 > 0\}$. Finally, note that $h_1(x^1, x^2, x^3)$ is increasing in x^3 . In order to see this, examine the boundary values of $\partial h_1 / \partial x^3$ in D .

For an integer $k > 0$, let

$$(4.2) \quad F_k = \{x \in D : |x^1| = 2^{-k}, x^3 \leq 2^{-k}\} \cup \{x \in D : |x^1| \leq 2^{-k}, x^3 = 2^{-k}\},$$

$$V = \{x \in D : |x^1| \in (3/8, 5/8), x^3 < 5/8\} \cup \{x \in D : |x^1| < 5/8, x^3 \in (3/8, 5/8)\}.$$

The functions h and h_1 are positive and harmonic in V and vanish on $\partial D \cap \partial V$. Fix some $y_0 \in F_1$. The boundary Harnack principle (see Remark 2.7) implies that

$$\frac{h(x)}{h_1(x)} \geq c_3 \frac{h(y_0)}{h_1(y_0)}$$

and

$$\frac{h_1(x)}{h(x)} \geq c_3 \frac{h_1(y_0)}{h(y_0)}$$

for all $x \in F_1$. Let

$$c_4 = c_3 \min(h(y_0)/h_1(y_0), h_1(y_0)/h(y_0)).$$

For $|x| < 1/2$, $x \in D$,

$$h(x) = E^x(h(X(T(F_1)))) \geq c_4 E^x(h_1(X(T(F_1)))) = c_4 h_1(x).$$

For similar reasons, $h_1(x) \geq c_4 h(x)$. These two inequalities and the monotonicity properties of h_1 mentioned above imply that

$$h(y^1, y^2, y^3) \leq c_5 h(x^1, 0, x^3),$$

for $c_5 = c_4^{-2} < \infty$ and all $x, y \in D$, $|x| < 1/2$, $|y| < 1/2$, $|y^1| \leq x^1$, $y^3 \leq x^3$.

Step 3. We proceed to obtain much more accurate estimates of the function h .

We start with a lower bound for $h(x)$ for $x \in D$ with $2^{-k} > x^1 > 2^{-k}/40$, $x^2 = 0$, $2^{-k} > x^3 > 2^{-k}/40$. Let $z_k = (2^{-k}, 0, 2^{-k})$. The points x and z_k may be connected by a chain of balls B_1, B_2, \dots, B_{n+m} with the following properties. Each ball B_j has radius $r = a(2^{-k}/40)/2L_2(2^{-k}/40)$. The centers y_j of B_j are chosen so that $y_j^2 = 0$ for all j . We let $y_1 = x$, $y_j^3 = y_1^3$ for $j = 1, \dots, n$, $y_j^1 = y_{j-1}^1 + r/2$ for $j = 2, \dots, n-1$, $y_n^1 \leq y_{n-1}^1 + r/2$, $y_n^1 = 2^{-k}$. Note that

$$n \leq 2^{-k}/(r/2) + 1.$$

Let $y_j^1 = y_n^1 = 2^{-k}$ for $j \geq n+1$, $y_j^3 = y_{j-1}^3 + r/2$ for $j = n+1, \dots, n+m-1$, $y_{n+m}^3 \leq y_{n+m-1}^3 + r/2$, $y_{n+m} = z_k$.

We have $m \leq 2^{-k}/(r/2)$ and the total number $n+m$ of balls in the chain does not exceed

$$2 \cdot 2^{-k}/(r/2) + 1 \leq (8 \cdot 40/a)L_2(2^{-k}/40) + 1 \leq c_6 \log k$$

for large k . We may take $c_6 = 16 \cdot 40/a$.

All the balls B_j are contained in $D - \{x_0\}$ so we may apply the Harnack principle in each of them. We obtain

$$\begin{aligned} h(x) &\geq h(z_k) c_7^{n+m} \\ &\geq h(z_k) \exp(-c_8(n+m)) \\ &\geq h(z_k) \exp(-c_8 c_6 \log k) \\ &= h(z_k) k^{-c_8 c_6} \\ &= h(z_k) k^{-c_9/a}. \end{aligned} \tag{4.3}$$

Next we will find an upper bound on $h(x)$ for $x \in D$ with $x^1 = 0$, $x^3 \in [2^{-k}/4, 2^{-k}/2]$. Recall that $z_k = (2^{-k}, 0, 2^{-k})$. Suppose that $b/a \in (2^{-j-1}, 2^{-j}]$, where $j > 0$. If $y \in D$ with $|(y^1, y^3) - (x^1, x^3)| \in [2^{-k-n}, 2^{-k-n+1}]$, $2 \leq n \leq j$, then let

$$A(y) = \{z \in \mathbb{R}^3 : |z^1 - y^1| < a2^{-k-n}/L_2(2^{-k-n}), |z^2| < 8a2^{-k-n}/L_2(2^{-k-n}), \\ |z^3 - y^3| < a2^{-k-n}/L_2(2^{-k-n})\},$$

and let $K(y)$ be the ball with radius $a2^{-k-n}/2L_2(2^{-k-n})$ and center

$$(y^1, y^2 + 7a2^{-k-n}/L_2(2^{-k-n}), y^3).$$

We will show that $K(y) \subset A(y) - D$. Note that for k large

$$y^3 \leq 2^{-k}/2 + 2^{-k-n+1} \leq 2^{-k}/2 + 2^{-k-2+1} = 2^{-k}, \\ 2^{-j}y^3 \leq 2^{-k-j} < 2^{-k-n+1}, \\ L_2(y^3) \geq L_2(2^{-k}) \geq L_2(2^{-k-n+1})/2.$$

These inequalities and the fact that $b/a \leq 2^{-j}$ imply for k large and y, n as above,

$$by^3/L_2(y^3) \leq a2^{-j}y^3/L_2(y^3) \leq a2^{-k-n+1}/L_2(y^3) \leq 2a2^{-k-n+1}/L_2(2^{-k-n+1}).$$

It follows that

$$f(y^1, y^3) \leq a|y^1|/L_2(|y^1|) + by^3/L_2(y^3) \\ \leq a2^{-k-n+1}/L_2(2^{-k-n+1}) + 2a2^{-k-n+1}/L_2(2^{-k-n+1}) \\ = 3a2^{-k-n}/L_2(2^{-k-n}).$$

Now it is easy to check that $K(y) \subset A(y) - D$. This and Brownian scaling imply that

$$P^y[T(D^c) > T(A^c(y))] < P^y([(K(y)) > T(A^c(y))] = c_{10} < 1.$$

Let

$$T_0 = \inf\{t > 0 : |x - X_t| \geq 2^{-k-j}\}, \\ T_n = \inf\{t > T_{n-1} : X_t \notin A(X(T_{n-1}))\}, \quad n \geq 1.$$

A repeated application of the strong Markov property at the stopping times T_n gives

$$P^x(T(D^c) > T_n) \leq c_{10}^n.$$

Recall from Step 2 that $h(z) \leq c_5 h(z_k)$ for all $z \in F_k$ (see (4.2)). Note that at least $2^{-k-n}/(a2^{-k-n}/L_2(2^{-k-n})) - 1$ stopping times T_m must occur between the hitting times of

$$\{y \in D : |(y^1, y^3) - (x^1, x^3)| = 2^{-k-n}\}$$

and

$$\{y \in D : |(y^1, y^3) - (x^1, x^3)| = 2^{-k-n+1}\},$$

assuming the process starts from x and does not hit D^c . Hence, P^x -a.s.,

$$\{T(D^c) > T(F_k)\} \subset \{T(D^c) > T_m\},$$

where

$$\begin{aligned} m &\geq \sum_{n=2}^j [2^{-k-n}/(a2^{-k-n}/L_2(2^{-k-n})) - 1] \\ &\geq (j-1)[(1/2a) \log k - 1] \\ &\geq [\log(a/b)/\log 2 - 2](1/4a) \log k \end{aligned}$$

for large k . We obtain

$$\begin{aligned} h(x) &= E^x(h(X(T_{F_k}))1_{\{T(D^c) > T(F_k)\}}) \\ &\leq c_5 h(z_k) P^x(T(D^c) > T(F_k)) \\ &\leq c_5 h(z_k) P^x(T(D^c) > T_m) \\ &\leq c_5 h(z_k) c_{10}^m \\ &\leq c_5 h(z_k) k^{-c_{11}[\log(a/b)/\log 2 - 2]/4a}. \end{aligned}$$

Now choose a so that $0 < b < a$ and

$$c_{11}[\log(a/b)/\log 2 - 2]/4a \geq c_9/a + 2.$$

Then

$$(4.4) \quad h(x) \leq k^{-c_9/a-2} c_5 h(z_k)$$

for large k and x such that $x^1 = 0$, $x^3 \in [2^{-k}/4, 2^{-k}/2]$.

Step 4. Let

$$\begin{aligned} H &= \{x \in D : x^3 > 10|x^1|\}, \\ J &= \{x \in \mathbb{R}^3 : x^3 = 10x^1\}. \end{aligned}$$

If $x \in \mathbb{R}^3$, let $\mathcal{S}(x)$ denote the point obtained by reflecting across the plane J . We will show that $\tilde{H} \stackrel{df}{=} \mathcal{S}(H) \subset D$ for some choice of a and b .

Suppose that $\lambda > 0$ and let $N = \{x \in H : x^3 = \lambda\}$. Note that $x^2 \leq f(\lambda/10, \lambda)$ for $x \in N$. In order to show that $\tilde{H} \subset D$, it will suffice to prove that $\tilde{N} \stackrel{df}{=} \mathcal{S}(N) \subset D$ for every $\lambda > 0$. The set \tilde{N} lies in a plane whose projection on the (x^1, x^3) -plane is a straight line L with the slope $\tan(2 \arctan(1/10))$. We will show that $f(x^1, x^3)$ increases when $(x^1, x^3) \in L$, $x^1 > x^3/10$, and x^1 increases. For $x^1 > 0$, $x^3 > 0$,

$$\begin{aligned} \frac{\partial}{\partial x^1} f(x^1, x^3) &= a \left(\frac{1}{\log(1/x^1)(L_2(x^1))^2} + \frac{1}{L_2(x^1)} \right), \\ \frac{\partial}{\partial x^3} f(x^1, x^3) &= b \left(\frac{1}{\log(1/x^3)(L_2(x^3))^2} + \frac{1}{L_2(x^3)} \right). \end{aligned}$$

For small x^1, x^3 such that $x^1 > x^3/10$ we have

$$\frac{\partial}{\partial x^1} f / \frac{\partial}{\partial x^3} f > (a/2b) \log(1/x^3) / \log(10/x^3) > a/4b.$$

Thus, if we assume that a/b is large then f is increasing along L for $x^1 > x^3/10$ which completes the proof of the fact that $\tilde{H} \subset D$.

Note that we may assume that a/b is large without contradicting the assumptions on a and b imposed in Steps 1 and 3.

Step 5. Let

$$\begin{aligned} M_k &= \{x \in D : x^1 = 0, x^3 \in [2^{-k}/4, 2^{-k}/2]\}, \\ \tilde{J} &= \{x \in \mathbb{R}^3 : x^3 = -10x^1\}, \\ K &= J \cup \tilde{J}. \end{aligned}$$

Let Q_k be the union of the images of M_k by the reflections with respect to the planes J and \tilde{J} .

Consider an exit system of Brownian motion X from K (see [Bu] or [S] for the definition and properties of exit systems). Suppose that the process starts from $x \in J \cap D$. Then there will occur some excursions from K .

If the process X hits M_k before hitting D^c , then the part A of the last excursion from K before hitting M_k is contained in H . Suppose without loss of generality that that excursion starts from J . The symmetric image of A with respect to J is contained in $\tilde{H} \subset D$, by the previous step. Since the Brownian excursion law from a plane is symmetric, the process has at least the same chance of hitting Q_k before hitting D^c , as the chance of hitting M_k before hitting D^c (assuming it starts from $x \in J \cap D$).

Step 6. It is straightforward to check that for $x \in Q_k$, $|x|$ small, we have $x^1 > 2^{-k}/40$, $x^3 > 2^{-k}/40$, and $|x^2| < f(x^1, x^3)/2$. Hence (4.3) and the Harnack principle yield

$$h(x) \geq c_{12}h(z_k)k^{-c_9/a}$$

for $x \in Q_k$ and large k . This, the previous step and (4.4) imply for $x \in J \cap D$,

$$\begin{aligned} (4.5) \quad \frac{1}{h(x)} \int h(y)P^x(X(T(M_k)) \in dy, T(M_k) < T(D^c)) & \\ & \leq \frac{\int h(y)P^x(X(T(M_k)) \in dy, T(M_k) < T(D^c))}{\int h(y)P^x(X(T(Q_k)) \in dy, T(Q_k) < T(D^c))} \\ & \leq \frac{k^{-c_9/a-2}c_5h(z_k)P^x(T(M_k) < T(D^c))}{k^{-c_9/a}c_{12}h(z_k)P^x(T(Q_k) < T(D^c))} \\ & \leq c_{13}k^{-2}. \end{aligned}$$

Step 7. Now suppose that the Euclidean and Martin compactifications of D are identical. We will show that this assumption leads to a contradiction. The reader may consult [Do] 1 XII 12 for the definition of the minimal fine topology.

Let g be the unique (up to a multiplicative constant) minimal harmonic function in D corresponding to $0 \in \partial D$. Suppose that $\{Y_t, t \geq 0\}$ is a g -process in D starting from x_0 . The lifetime R of Y is finite by the results of [BB3]. The time-reversed process $Z_t \stackrel{df}{=} Y_{R-t}$ is an h -process in D starting from $0 \in \partial D$ ([Do] 3 III 2). Let

$$v(x) = \int h(y)P^x(X(T(M_k)) \in dy, T(M_k) < T(D^c)).$$

Then, according to (4.5),

$$\frac{v(x)}{G_D(x, x_0)} \leq c_{13}k^{-2},$$

for $x \in J \cap D$. By Theorem 1 XII 14 of [Do], the minimal fine limit of $v(x)/G_D(x, x_0)$ exists at $0 \in \partial D$ and is less than or equal to $c_{13}k^{-2}$. The chance that the process Z ever hits M_k is less than or equal to

$$E[P_h^{Z_t}(T(M_k) < \infty)] = E[v(Z_t)/G_D(Z_t, x_0)]$$

for every $t > 0$, by the Markov property, where E denotes expectation for the Z process starting at 0. The probabilistic interpretation of the minimal fine topology implies that as $t \rightarrow 0$, the integrand in the last expectation converges to a limit which does not exceed $c_{13}k^{-2}$. It follows that the probability of hitting M_k by Z is not greater than $c_{13}k^{-2}$. Since $\sum c_{13}k^{-2} < \infty$, only finitely many of the sets M_k are hit by Z . Hence Y approaches 0 from one side of the plane $\{x : x^1 = 0\}$. Both sides are equally probable, by symmetry, so the event that Y stays eventually in $\{x : x^1 > 0\}$ has probability 1/2. But this event belongs to the tail σ -field of Y and, therefore, it must have probability 0 or 1 ([Do] 2 X 11 (c1)). This is a contradiction which shows that the Euclidean and Martin compactifications of D are different.

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