# A Tensor-Triangulated Approach to Derived Categories of Non-Noetherian Rings

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**Abstract**

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We investigate the subcategories and Bousfield lattices of derived categories of general commutative rings, extending previous work done under a Noetherian hypothesis. Maps between rings $R \to S$ induce adjoint functors between unbounded derived categories $D(R) \rightleftarrows D(S)$, and we explore the induced relationships between thick and localizing subcategories, and Bousfield lattices. Several specific non-Noetherian rings are studied in depth. We also contextualize these results within the human dimension in which they occurred.

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I am grateful to John Palmieri for mentoring me, and everyone at the University of Washington for teaching me.

# DEDICATION

to HH and WH

# NOTATION INDEX

# Chapter 1

# INTRODUCTION AND BACKGROUND

We approach the unbounded derived category $D(T)$ of a (graded or ungraded) commutative ring $T$, a classical algebraic object, from the perspective of algebraic topology and tensor-triangulated category theory. We think of $D(T)$ as a monogenic stable homotopy category [HPS97]. It has a symmetric monoidal tensor product, $- \otimes_T^L -$, which we will denote as the smash product $- \wedge -$. The unit of this smash product, the module $T$ concentrated in degree zero, is the sphere object – it is a small, weak generator. We have arbitrary coproducts, given by degree-wise direct sums, and Brown representability holds for cohomology functors.

A triangulated subcategory of $D(T)$ is called *thick* if it is closed under retracts. An object is called *finite* if it is in the thick subcategory generated by the sphere object. A triangulated subcategory is called *localizing* if it is closed under arbitrary coproducts. We are interested in characterizing the localizing subcategories, and the thick subcategories of finite objects, in $D(T)$. When the ring $T$ is Noetherian and ungraded, this has been done [Nee92]: these are classified by subsets, and specialization-closed subsets, respectively, of the prime spectrum $\mathsf{Spec}\, T$. Furthermore, Thomason [Tho97] showed the thick subcategories of finite objects of an arbitrary (ungraded) commutative ring can be classified by certain subsets of $\mathsf{Spec}\, T$. More broadly, much work has been done towards understanding thick subcategories of finite objects in general triangulated and tensor-triangulated categories. See, e.g. [Bal05, BIK11].

However, localizing subcategories seem harder to pin down. One approach is to study Bousfield classes, since every Bousfield class is a localizing subcategory. The *Bousfield class* of an object $X$ is $\{W \mid W \wedge X = 0\}$, the acyclics of the homology theory corresponding to $X$. When there is a set of Bousfield classes, these form a complete lattice called the *Bousfield lattice* $\mathsf{BL}_{D(T)}$ of $D(T)$. We have some understanding of the Bousfield lattice of the stable homotopy category of spectra [Bou79b, HP99], of the derived category of the graded non-Noetherian ring $\Lambda_k$ [DP08], and of a general tensor-triangulated category [IK11]. The Bousfield lattice of the derived category $D(T)$ of a Noetherian ring $T$ is more or less completely understood – every localizing subcategory is a Bousfield class, and the Bousfield lattice is in bijection with subsets of $\mathsf{Spec}\, T$.

In Chapter 2 we look at the relationship between different derived categories, coming from different rings. A ring map $f : R \to S$ between commutative rings induces adjoint functors $f_\bullet : D(R) \rightleftarrows D(S) : i_\bullet$. By placing different hypotheses on the rings $R$ and $S$, and the map $f$, we deduce a range of results about the relationship between subcategories and Bousfield lattices of $D(R)$ and $D(S)$. For example, in Section 2.4 we suppose $f$ is surjective and $S$ is Noetherian. Proposition 2.3.2 shows that in this case there is a quotient lattice of $\mathsf{BL}_{D(R)}$ isomorphic to $\mathsf{BL}_{D(S)}$. Separately, we construct various objects in $D(R)$ and are able to compute their support and Bousfield classes (e.g. Props. 2.4.1 and 2.4.17, Cor. 2.4.14). The new results in this chapter range from formal structural results to specific computations.

In Chapter 4, we apply some of these results to several specific non-Noetherian graded rings: $\Lambda_{\mathbb{Z}_{(p)}}$, $\Lambda_{\mathbb{F}_p}$, and $\Lambda_{\mathbb{Q}}$. Specifically, we fix a prime $p$ and integers $n_i > 1$, let $k$ be an arbitrary countable field, and set

$$\Lambda_{\mathbb{Z}_{(p)}} := \frac{\mathbb{Z}_{(p)}[x_1, x_2, ...]}{(x_1^{n_1}, x_2^{n_2}, ...)}, \quad \Lambda_{\mathbb{F}_p} := \frac{\mathbb{F}_p[x_1, x_2, ...]}{(x_1^{n_1}, x_2^{n_2}, ...)}, \quad \Lambda_{\mathbb{Q}} := \frac{\mathbb{Q}[x_1, x_2, ...]}{(x_1^{n_1}, x_2^{n_2}, ...)}, \quad \text{and} \quad \Lambda_k := \frac{k[x_1, x_2, ...]}{(x_1^{n_1}, x_2^{n_2}, ...)},$$

with $\deg(x_i) = 2^i$. There is a surjection $g : \Lambda_{\mathbb{Z}_{(p)}} \to \Lambda_{\mathbb{F}_p}$ and an injection $\Lambda_{\mathbb{Z}_{(p)}} \hookrightarrow \Lambda_{\mathbb{Q}}$, and we use the functors $D(\Lambda_{\mathbb{Z}_{(p)}}) \rightleftarrows D(\Lambda_{\mathbb{F}_p})$ and $D(\Lambda_{\mathbb{Z}_{(p)}}) \rightleftarrows D(\Lambda_{\mathbb{Q}})$ to derive various similarities and differences between $D(\Lambda_{\mathbb{Z}_{(p)}})$, $D(\Lambda_{\mathbb{F}_p})$, and $D(\Lambda_{\mathbb{Q}})$ (e.g. Prop. 4.1.2, Cor. 4.2.8, and Thm. 4.2.11). In Theorem 4.2.5 we show that $g$ induces a splitting of Bousfield lattices:

$$\mathsf{BL}_{D(\Lambda_{\mathbb{Z}_{(p)}})} \cong \mathsf{BL}_{D(\Lambda_{\mathbb{F}_p})} \times \mathsf{BL}_{\mathsf{loc}(i_\bullet \Lambda_{\mathbb{Q}})}.$$

Chapter 6 constructs a collection of objects in $D(\Lambda_k)$ with periodic homology. Furthermore, we construct an object $\mathsf{Tel}$ whose homology in each degree is the graded dual $I(\Lambda_k) = \mathrm{Hom}_k^*(\Lambda_k, k)$. In Theorem 6.1.4, we use this $\mathsf{Tel}$ to show that there are objects in $D(\Lambda_k)$ that are not Bousfield equivalent to any module. Specifically, every $I(\Lambda_k)$-acyclic that is not $\mathsf{Tel}$-acyclic has this property. This answers an open question posed in [DP08].

**Further summaries of results can be found at the beginning of each chapter. Page v contains an index of notation.**

Each of these rigorous chapters concludes with a section addressing the experiential context of results in that chapter. This is a sort of meta-data, included along with the results, to allow for improved interfacing with the human dimensions of mathematics. Specifically, we discuss:

1. where and when the ideas arose,
2. what the key insights and central organizing principles are,
3. conceptual metaphors and mental images we use to reason about the ideas, and
4. the process of development of the ideas and results.

## *1.1 Metamathematics*

**Definition 1.1.1.** *Metamathematics* is the study of the human context of research mathematics, including the sociology of the math community, the psychology and cognitive science of mathematics, and implicitly the history and philosophy of math.

**Definition 1.1.2.** *Intersubjectivity* is the sharing of subjective states by two or more individuals [Sch06].

In addition to the conventional, rigorous chapters of mathematical results – 2, 4, and 6 – we include several chapters detailing work we have done towards building an understanding of real, lived mathematics. These chapters are less result-oriented and more exploratory, but nevertheless comprise original contributions. They constitute results in metamathematics, rather than mathematics proper, and have been written to be accessible to a non-mathematical audience.

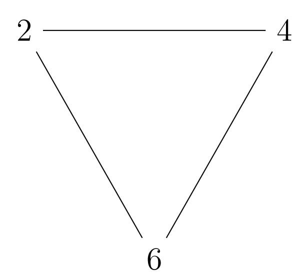
Chapter 3 discusses the role of contemplation in mathematics. Here, contemplation means reflection on the lived experience of doing math. We conjecture that such contemplation matters, and that mathematical research experiences are not simply subjective, but intersubjective. To establish evidence towards the validity of these claims, we discuss the Flavors and Seasons project.

In Chapter 5 we discuss a theory of interdisciplinary collaboration, and describe several collaborations at the frontier of contemporary mathematics and contemporary art. These works present new contributions to both math and art.

Chapter 7 builds on work of other mathematicians and philosophers reflecting on the culture of mathematics and community of math researchers. We present several specific examples of how the development of mathematics has been shaped by the sociological characteristics of the community of practitioners, and conversely, how the community has been shaped by the nature of mathematics. We discuss how alternative characteristics of mathematics itself suggest new or different ways of doing mathematics, and how changing the way mathematicians do mathematics might alter the development of mathematical content.

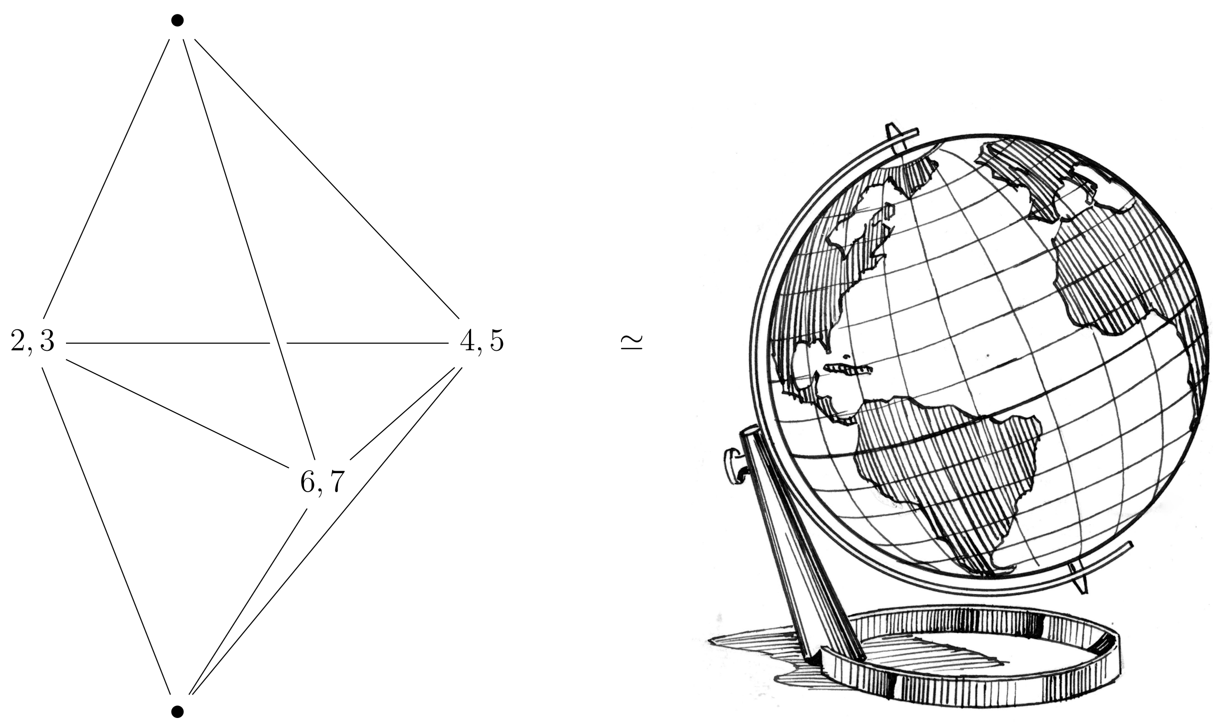
Chapters 3, 5, and 7 are entirely independent of Chapters 2, 4, and 6, but there is a subtle complementarity between the pairs 2 and 3, 4 and 5, and 6 and 7.

Structurally, the organization of this dissertation uses the metaphor of suspension of a 1-simplex. One may simply consider the rigorous mathematical results of Chapters 2, 4, and 6 – this is a one-dimensional mathematical object, a 1-simplex.




Incorporating the metamathematical chapters, and considering the dissertation as a whole, corresponds to taking the suspension of this 1-simplex. This two-dimensional object is topologically equivalent to a sphere or globe, with three distinguished points[1]. Introducing this additional dimension corresponds to engaging the human dimension of the mathematical experience.

[1] corresponding to Thailand, Siberia, and Seattle, as the astute reader will observe.




The remainder of this chapter is devoted to establishing mathematical definitions and background, and contains no original work. First we discuss axiomatic stable homotopy categories generally, unbounded derived categories specifically, subcategory classification, and the Bousfield lattice. Then we outline what is known about subcategory classification and Bousfield lattices our main categories of interest: a Noetherian stable homotopy category; the categories $D(\Lambda_k), D(\Lambda_{\mathbb{F}_p})$, and $D(\Lambda_{\mathbb{Q}})$; and the category of spectra.

## *1.2 Tensor-triangulated and stable homotopy categories*

All the categories we are interested in are monogenic stable homotopy categories, a special class of tensor-triangulated categories that we will now describe. Here we give some preliminary definitions and properties, mostly following [HPS97]. Let $\mathcal{D}$ be a triangulated category [Ver96, Nee01, Wei94]. Given $X$ and $Y$ in $\mathcal{D}$, let $[X, Y]$ denote the set of degree zero morphisms from $X$ to $Y$, and $[X, Y]_*$ the set of all morphisms. We will use $\Sigma$ to denote the suspension functor on a triangulated category. Let $\mathfrak{Ab}$ and $\mathfrak{Ab}_*$ denote the categories of abelian groups and graded abelian groups.

**Definition 1.2.1.** Let $\mathcal{D}$ be a triangulated category. A covariant additive functor $F : \mathcal{D} \to \mathfrak{Ab}$ is called *exact* if for every cofiber sequence

$$X \to Y \to Z \to \Sigma X$$

in $\mathcal{D}$, the following sequence is exact:

$$F(X) \to F(Y) \to F(Z).$$

Similarly, for contravariant functors. An additive functor between two triangulated categories is called exact if it commutes with suspension and sends cofiber sequences to cofiber sequences.

**Definition 1.2.2.** A *homology functor* is a covariant, exact functor $H : \mathcal{D} \to \mathfrak{Ab}$, such that the canonical map $\coprod H(X_\alpha) \to H(\coprod X_\alpha)$ is equivalence; in other words, $H$ sends coproducts to coproducts.

**Definition 1.2.3.** A *cohomology functor* is a contravariant, exact functor $H : \mathcal{D} \to \mathfrak{Ab}$, such that the canonical map $H(\coprod X_\alpha) \to \prod H(X_\alpha)$ is equivalence; in other words, $H$ sends coproducts to products.

**Definition 1.2.4.** We say that a cohomology functor $H : \mathcal{D} \to \mathfrak{Ab}$ is *representable in* $\mathcal{D}$ if there exists an object $Y$ in $\mathcal{D}$ and a natural isomorphism of functors from $H$ to $[-, Y]$. In other words, for every object $X$ in $\mathcal{D}$ we have a functorial isomorphism $H(X) \cong [X, Y]$. In this case, we say that $H$ is *represented by* $Y$.

Brown's Representability Theorem, giving conditions for when a cohomology functor on the homotopy category of spaces is representable, is an incredibly powerful tool in algebraic topology. One part of the definition of a stable homotopy category, as we will see, is that all cohomology functors are representable.

**Definition 1.2.5.** A *closed symmetric monoidal category* is a category $\mathcal{C}$ with

1. a *sphere object* $S^0$

2. a functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$, denoted $(X, Y) \mapsto X \wedge Y$ and called the *smash product*, that is associative, commutative, naturally functorial in both variables, and has $S^0$ as a unit (so $S^0 \wedge X \cong X \cong X \wedge S^0$)

3. for every $Y$ and $Z$ in $\mathcal{C}$, a *function object* $F(Y, Z)$ that is covariantly functorial in $Z$, contravariantly functorial in $Y$, and represents the functor $[- \wedge Y, Z]$. Thus we have a natural isomorphism $[X \wedge Y, Z] \cong [X, F(Y, Z)]$, functorial in each variable.

One consequence of this definition is that the smash product necessarily commutes with arbitrary coproducts. We are interested in triangulated categories with a smash product that behaves well with triangles; the following definition makes this precise.

**Definition 1.2.6.** A *tensor-triangulated category* is a closed symmetric monoidal category that is triangulated, such that

1. the smash product commutes with suspension – there is a natural equivalence $\Sigma X \wedge Y \to \Sigma(X \wedge Y)$, and the following diagram commutes:

$$\begin{array}{ccc} S^r \wedge S^s & \xrightarrow{\cong} & S^{r+s} \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle (-1)^{r+s}} \\ S^s \wedge S^r & \xrightarrow[\cong]{} & S^{r+s}, \end{array}$$

   where $S^r = \Sigma^r S^0$ and $c$ is the commutativity map.

2. the smash product is exact, i.e. the functors $X \wedge -$ and $- \wedge X$ are exact for all $X$.

3. the functors $F(X, -)$ and $F(-, Y)$ are exact (the latter only up to a sign).

When these conditions are satisfied, we say that the monoidal structure is *compatible with the triangulation.*

In fact, the sphere object is especially nice in a monogenic stable homotopy category. It satisfies the following properties.

**Definition 1.2.7.** An object $X$ in some category $\mathcal{D}$ is *small* if the natural map $\coprod [X, Y_\alpha] \to [X, \coprod Y_\alpha]$ is an equivalence for all coproducts that exist in $\mathcal{D}$.

**Definition 1.2.8.** An object $X$ in a triangulated category $\mathcal{D}$ is a *graded weak generator* if $[X, Y]_* = 0$ implies $Y = 0$.

Finally, we're ready to define a monogenic stable homotopy category.

**Definition 1.2.9.** A *monogenic stable homotopy category* is a tensor-triangulated category $\mathcal{C}$ such that

1. the sphere object $S^0$ is a small, graded weak generator,

2. all coproducts of objects of $\mathcal{C}$ exist, and

3. every cohomology functor on $\mathcal{C}$ is representable.

In [HPS97] a slightly more general definition is given, omitting the term *monogenic*, essentially weakening the requirement that $S^0$ be both small and a weak generator. However, the two main cases we're interested in – the category of spectra and the derived category of a ring – are both monogenic. *Henceforth, all stable homotopy categories will be assumed to be monogenic, as defined above.* Because $S^0$ is a graded weak generator, the functor $[S^0, -]_* : \mathcal{C} \to \mathfrak{Ab}_*$ plays a key role in computations, and is denoted $\pi_*(-)$.

**Definition 1.2.10.** Let $\mathcal{C}$ be a stable homotopy category. For each $X$ in $\mathcal{C}$, we call $[S^0, X]_* = \pi_*(X)$ the *homotopy groups of* $X$.

Note that the homotopy groups of the sphere object, $\pi_*(S^0)$, inherit a ring structure from the map $S^0 \wedge S^0 \to S^0$.

**Example 1.2.11.** Since this definition is motivated by topology, it's no surprise that the category of spectra (*the* stable homotopy category) is a stable homotopy category. So is the category of $p$-local spectra, which we will denote $\mathcal{S}$. These properties of $\mathcal{S}$ were demonstrated clearly in [Ada74]. In particular, Adams constructs the smash product operation on spectra, and shows that it yields a monoidal structure that is compatible with the triangulation. The sphere object is the sphere spectrum $S^0$, and coproducts are wedges. The ring $\pi_*(S^0)$ is quite complex.

**Example 1.2.12.** The unbounded derived category $D(R)$ of chain complexes of ungraded modules over an ungraded commutative ring $R$ also satisfies all these conditions. That $D(R)$ is triangulated was well-known; in fact, it was the derived category that motivated Verdier to create the notion of a triangulated category [Ver96]. In this case, the smash product is the total tensor product , with function objects $\mathbb{R}\mathrm{Hom}(-,-)$, and it takes some work to see that this is compatible with the triangulation [HPS97, Sect. 9.3]. The sphere object is the ring $R$ itself, or rather the image in $D(R)$ of the chain complex consisting of a single copy of $R$ concentrated in degree zero, and zero modules in every other degree. The ring $\pi_*(S^0)$ is again just $R$. Coproducts are direct sums constructed degree-wise.

In the case of $D(R)$, the homotopy groups are $[S^0, X]_* = [R, X]_* \cong H_*(X)$, just the ordinary homology of $X$ as a chain complex. This shows that $R$ is a graded weak generator; if $H_*(X) = 0$ then $X$ is exact, and thus equivalent to zero in $D(R)$. Because we are taking a tensor-triangulated approach, when discussing homology groups of a complex $X$ we will use the notation $\pi_*(X)$ rather than $H_*(X)$.

In fact, [HPS97, Sect. 9.3] explains that the unbounded derived category of graded modules over a graded commutative ring is also a stable homotopy category. In Chapters 4 and 6, and parts of Chapter 2, we will want to consider the derived category of graded modules over a graded ring. In the next section we discuss grading issues, and computations in the unbounded setting.

**Example 1.2.13.** A third example of a monogenic stable homotopy category, which we will only briefly mention, is denoted $\mathcal{C}((kG)^*)$. Here $G$ is a finite $p$-group, $k$ is a field, and $(kG)^*$, the dual of the group algebra, is a commutative Hopf algebra. The objects of $\mathcal{C}((kG)^*)$ are cochain complexes of injective $\mathcal{C}((kG)^*)$-comodules, and morphisms are cochain homotopy classes of maps. In this case, the unit of the smash product is an injective resolution of $k$. Any study of general stable homotopy categories may yield interesting consequences in this particular category.

## 1.3 Grading issues, and computations in $D(R)$

### 1.3.1 Grading issues

We will often want to consider the derived category of chain complexes of graded modules (and graded maps) over a graded ring; this is also a stable homotopy category [HPS97, 9.3]. Specifically, in Chapter 4 we work within the categories $D(\Lambda_{\mathbb{Z}_{(p)}})$, $D(\Lambda_{\mathbb{F}_p})$, and $D(\Lambda_{\mathbb{Q}})$, defined above, and in Chapter 6 we work in the category $D(\Lambda_k)$. These chapters draw on [DP08], which assumes a grading on rings and modules.

The results of Sections 2.1 through 2.3, however, apply in both the graded and ungraded setting, and we make no notational distinction.

In Sections 2.4 and 2.5, we only consider the ungraded setting: here $D(R)$ means the derived category of chain complexes of ungraded modules over an ungraded ring $R$. The (only) reason for this restriction is that we wish to apply the work of Neeman [Nee92] and Thomason [Tho97], and to date these results have not been extended to the graded setting. There is evidence that this extension will be completed in the near future, and perhaps already is. Indeed, in the recent preprint [DS11], localizing subcategories of the derived category of graded modules over a graded Noetherian ring are classified via subsets of the homogeneous spectrum, and this restricts to a classification of thick subcategories of finite objects [DS11, Thm.5.7]. In the preprint [DS12], Thomason's classification of thick subcategories of finite objects is extended to arbitrary graded commutative rings. Since we have not had the time to check all the details in these preprints, we have taken a cautious approach, and will not use them. We do note, however, that an innocent application of Thomason's classification theorem to the derived category $D(\Lambda_{\mathbb{Z}_{(p)}})$ of the graded ring $\Lambda_{\mathbb{Z}_{(p)}}$ is consistent with our results in Chapter 4 (see Proposition 4.1.2).

There are warnings and reminders at relevant places throughout, to remind the reader whether we are considering the graded or ungraded context, or both.

### *1.3.2 Computing in the unbounded derived category*

We will be working in the unbounded derived category $D(R)$ of a ring $R$, and will be computing smash products. In the general unbounded case, projective or flat resolutions are not useful. Instead, we will use cellular towers, from [HPS97, 2.3]. These exist in any stable homotopy category, so in particular will work in both the graded and ungraded setting. In the ungraded case, these are the same as the cell modules of [KM95, III] (if we consider an ungraded ring as a differential graded algebra concentrated at zero), which are themselves specific examples of $K$-flat resolutions [Spa88].

Thus let $D(R)$ be either the derived category of graded modules over a graded commutative ring $R$, or the derived category of (ungraded) modules over a commutative ring $R$. Let $K(R)$ denote the homotopy category of chain complexes.

**Definition 1.3.1.** A *cellular tower* in $K(R)$ is the sequential colimit of a sequence $X^0 \to X^1 \to \cdots$ of complexes $X^i$ in $K(R)$, such that $X^0 = 0$ and the cofiber of each map $X^k \to X^{k+1}$ is a coproduct of shifted sphere objects $\Sigma^j R$.

Proposition 2.3.1 in [HPS97] shows that every object in $D(R)$ is equivalent to a cellular tower.

Recall that the smash product $A \wedge - : D(R) \to D(R)$ is the derived functor of $A \otimes - := Tot^{\oplus}(S \otimes_R -) : K(R) \to K(R)$. We define $- \wedge B$ similarly, and of course usually think of $- \wedge -$ as a bifunctor.

**Definition 1.3.2.** A complex $Z$ in $K(R)$ is *K-flat* if for every acyclic complex $A$, $A \otimes Z$ is acyclic.

**Lemma 1.3.3.** *Every cellular tower is K-flat.*

**Proof.** This is a straightforward generalization of [KM95, Lemma 4.1], which is formal. If $A$ is acyclic, then clearly $A \otimes (\coprod \Sigma^i R)$ is as well. Since $A \otimes -$ is exact, we can induct up the sequence and pass to colimits. □

Then the Generalized Existence Theorem [Wei94, 10.5.9] (see also [Spa88, 6.5(a)]) implies that we can use cellular towers to compute the smash product. Specifically, to compute $Y \wedge Z$ in $D(R)$, we construct a cellular tower $X$ for $Y$, with a quasi-isomorphism $X \to Y$, and then $Y \wedge Z \cong X \otimes Z$.

## *1.4 Subcategory classification*

The natural subcollections to study, when considering a stable homotopy category, are those that are closed under the various operations that are possible within such a category. Let $\mathcal{C}$ be a stable homotopy category.

**Definition 1.4.1.** A full subcategory $\mathcal{D}$ of $\mathcal{C}$ is *triangulated* if it is closed under the formation of triangles; in other words if $X \to Y \to Z$ is an exact triangle in $\mathcal{C}$ and two of $X$, $Y$, and $Z$ are in $\mathcal{D}$, then so is the third.

**Definition 1.4.2.** A full subcategory $\mathcal{D}$ of $\mathcal{C}$ is *thick* if it is triangulated and closed under retracts; i.e. if $X \amalg Y$ is in $\mathcal{D}$, then $X$ and $Y$ are in $\mathcal{D}$.

**Definition 1.4.3.** A full subcategory $\mathcal{D}$ of $\mathcal{C}$ is *localizing* if it is thick and closed under the formation of arbitrary coproducts; i.e. $\coprod_\alpha X_\alpha$ is in $\mathcal{D}$ for any collection of $X_\alpha$ in $\mathcal{D}$.

The Eilenberg swindle [HPS97, Sect. 1.4] shows that any subcategory closed under triangles and coproducts is necessarily closed under retracts.

**Definition 1.4.4.** Given some collection $\mathsf{A}$ of objects in $\mathcal{C}$, the *thick subcategory generated by* $\mathsf{A}$, denoted $\mathsf{th}(\mathsf{A})$, is the intersection of all the thick subcategories containing $\mathsf{A}$. Likewise, we can define the *localizing subcategory generated by* $\mathsf{A}$, denoted $\mathsf{loc}(\mathsf{A})$. If $X$ and $Y$ are objects in $\mathcal{C}$ and $X$ is in $\mathsf{loc}(Y)$, we say that $X$ *can be built from* $Y$. An object is said to be *finite* if it is in the thick subcategory generated by the sphere object $S^0$. The collection of finite objects is sometimes denoted $\mathcal{F}$.

It turns out that in monogenic stable homotopy categories, an object is finite if and only if it is small.

Classifications of thick or localizing subcategories are very useful in practice, because it is often the case that the properties we are interested in are preserved under the formation of triangles, retracts, or coproducts. For example, consider the property $P$ of having homotopy groups of finite type. Since a cofiber sequence in $\mathcal{C}$ yields a long exact sequence of homotopy groups, we see that property $P$ is preserved under the formation of triangles and retracts. If $X$ in $\mathcal{C}$ happens to have homotopy groups of finite type, then for all $Y$ in $\mathsf{th}(X)$, we can conclude that $Y$ has homotopy groups of finite type as well.

Below, we will outline what is known about subcategory classifications of our main categories of interest. But first we discuss the Bousfield lattice.

## *1.5 Bousfield lattices*

**Definition 1.5.1.** Given an object $E$ in a stable homotopy category, define the *Bousfield class* of $E$ to be the collection

$$\langle E\rangle := \{X \mid E \wedge X = 0\}.$$

See [Bou79a, Bou79b, Rav84, HP99]. Because $S^0$ is a weak generator, $E \wedge X = 0$ if and only if $E_*(X) := \pi_*(E \wedge X) = 0$. Thus $\langle E\rangle$ is the collection of *$E$-acyclics* – the objects that are invisible to the homology functor $E_*$. It's not hard to see that every Bousfield class is a localizing subcategory. We say that two objects $E$ and $F$ are *Bousfield equivalent* if $\langle E\rangle = \langle F\rangle$, and this gives an equivalence relation. It is possible for some confusion to arise, since $\langle E\rangle$ might refer to the Bousfield equivalence class of $E$ (so, for example, $E$ is of course in $\langle E\rangle$), or $\langle E\rangle$ might refer to the localizing subcategory of $E$-acyclics (in which case, we may have $E\wedge E \neq 0$). To avoid confusion, if we intend to refer to the collection of $X$ such that $E \wedge X = 0$, we will use the term $E$-acyclics.

**Definition 1.5.2.** There is a partial ordering on Bousfield classes, given by reverse inclusion. Thus we say that $\langle E\rangle \leq \langle F\rangle$ when $X \wedge F = 0$ implies $X \wedge E = 0$.

The class of the sphere object $\langle S^0\rangle$ is the maximum class in this ordering, because $X \wedge S^0 = 0$ exactly when $X = 0$, which implies $X \wedge E = 0$ for all $X$. Also, $\langle 0\rangle$ is the minimum.

We can define an operation on Bousfield classes,

$$\langle X\rangle \vee \langle Y\rangle := \langle X \amalg Y\rangle,$$

this is in fact the join. Arbitrary joins exist, and are given by $\bigvee_\alpha \langle X_\alpha\rangle = \langle \coprod_\alpha X_\alpha\rangle$.

In general it is not known whether the collection of Bousfield classes form a set, rather than a proper class. The first result in this direction was by Ohkawa [Ohk89], who showed that in the category of spectra $\mathcal{S}$, the collection of Bousfield classes is a set. Recently, Iyengar and Krause showed that in a well-generated triangulated category, the collection of Bousfield classes forms a set [IK11, Thm. 3.1]. The derived category of a ring is a well-generated category.

A set of Bousfield classes allows us to define a meet operation, where $\langle X\rangle \curlywedge \langle Y\rangle$ is the join of (the *set* of) all the lower bounds of $\langle X\rangle$ and $\langle Y\rangle$. Thus in these two examples, the Bousfield classes form a poset. Because it has finite meets and arbitrary joins, it is a complete lattice.

**Definition 1.5.3.** When there is a set of Bousfield classes, the collection is called the *Bousfield lattice*, and denoted $\mathsf{BL}$.

Another straightforward operation on Bousfield classes is given by $\langle X\rangle \wedge \langle Y\rangle := \langle X \wedge Y\rangle$. This is a lower bound, but in general not the meet.

The smash product distributes over arbitrary joins, but in general the meet operation does not. However, there is a nice sub-poset within $\mathsf{BL}$ in which it does.

**Definition 1.5.4.** Let $\mathsf{DL}$ be the collection of Bousfield classes $\langle E\rangle$ such that $\langle E\rangle = \langle E\rangle \wedge \langle E\rangle$.

**Proposition 1.5.5.** *In* $\mathsf{DL}$*, the meet of* $\langle X\rangle$ *and* $\langle Y\rangle$ *is* $\langle X\rangle \wedge \langle Y\rangle$*. Thus* $\mathsf{DL}$ *is a frame, i.e. a complete lattice in which the meet distributes over arbitrary joins. The inclusion* $i : \mathsf{DL} \hookrightarrow \mathsf{B}$ *preserves arbitrary joins but does not preserve meets.*

We say a Bousfield class $\langle X\rangle$ is *complemented* if there exists a class $\langle X^c\rangle$ such that $\langle X\rangle \vee \langle X^c\rangle = \langle S^0\rangle$ and $\langle X\rangle \wedge \langle X^c\rangle = \langle 0\rangle$. Because we know there is a set of Bousfield classes, we can define a complementation operator $a(-)$ on Bousfield classes, by

$$a\langle X\rangle = \bigvee_{\langle X\rangle \wedge \langle Y\rangle = \langle 0\rangle} \langle Y\rangle.$$

It follows from the definition that $a^2\langle X\rangle = \langle X\rangle$ for all $X$, and $a(-)$ is order-reversing, so

$$\langle X\rangle \leq \langle Y\rangle \text{ if and only if } a\langle Y\rangle \leq a\langle X\rangle.$$

It is not hard to show that if $\langle X \rangle$ is complemented by $\langle X^c \rangle$, then $\langle X^c \rangle = a\langle X \rangle$.

**Definition 1.5.6.** Let $\mathsf{BA}$ denote the collection of all complemented Bousfield classes.

**Lemma 1.5.7.** *Suppose that $\langle X \rangle$ and $\langle Y \rangle$ are in $\mathsf{BA}$, and $\langle E \rangle$ is an arbitrary Bousfield class.*

1. $\langle E \rangle \leq \langle X \rangle$ *if and only if* $\langle E \rangle = \langle E \rangle \wedge \langle X \rangle$.

2. $\langle X \rangle \curlywedge \langle Y \rangle = \langle X \rangle \wedge \langle Y \rangle$ .

3. $\mathsf{BA} \subseteq \mathsf{DL}$.

4. $\mathsf{BA}$ *is a Boolean algebra; i.e. a distributive lattice in which every element has a complement.*

## *1.6 Examples*

### *1.6.1 Noetherian Stable Homotopy Categories*

**Definition 1.6.1.** A stable homotopy category is *Noetherian* if $\pi_*(S^0)$ is a Noetherian ring.

Amnon Neeman, in [Nee92], gives a complete classification of localizing subcategories, and thick subcategories of finite objects, for the derived category $D(R)$ of chain complexes of ungraded modules over an ungraded Noetherian ring $R$. In the derived category, the finite objects are those that are equivalent to a bounded below complex of projectives.

Benson, Carlson, and Rickard, in [BCR97], give a classification of thick subcategories in $\mathcal{C}((kG)^*)$ and their methods bear some similarity to Neeman's. These two categories are both examples of Noetherian stable homotopy categories. In [HPS97], these two examples are generalized, and a classification is given for general Noetherian stable homotopy categories, which we will now describe briefly.

**Definition 1.6.2.** Let $\mathcal{C}$ be a Noetherian stable homotopy category, with $R = \pi_*(S^0)$. Fix a prime ideal $\mathfrak{p} \leq R$.

1. Write $\mathfrak{p} = (y_1, y_2, ..., y_k)$. Each $y_i$ is a self-map of the sphere. Let $S/y_i$ be the cofiber of the map $S^0 \xrightarrow{y_i} S^0$, and define $S/\!\!/\mathfrak{p} = S/y_1 \wedge S/y_2 \wedge \cdots S/y_k$. These are called *Koszul objects* in [BIK11] (in [HPS97] the authors use the notation $S/\mathfrak{p}$). It turns out that different choices of generators $y_i$ generate the same thick subcategory $\mathsf{th}(S/\!\!/\mathfrak{p})$, and this is good enough for our purposes.

2. Define $K(\mathfrak{p}) = S^0_{\mathfrak{p}} \wedge S/\mathfrak{p} = (S/\mathfrak{p})_{\mathfrak{p}}$ to be the localization of $S/\mathfrak{p}$ at $\mathfrak{p}$. (For a general ring $T$ and prime ideal $\mathfrak{q} \leq T$, let $T_{\mathfrak{q}}$ denote localization of $T$ at $\mathfrak{q}$, as usual.)

3. In the case of the derived category of a Noetherian ring $R$, for each prime ideal $\mathfrak{p}$ of $R$ we have a residue field $k_{\mathfrak{p}}$. Let $\overline{k_{\mathfrak{p}}}$ denote $k_{\mathfrak{p}}$ as a chain complex in $D(R)$, concentrated in degree zero. Then [HPS97, 9.1] shows that for each prime ideal $\mathfrak{p}$ of $R$, $\mathsf{loc}(K(\mathfrak{p})) = \mathsf{loc}(\overline{k_{\mathfrak{p}}})$, so $\langle K(\mathfrak{p})\rangle = \langle \overline{k_{\mathfrak{p}}}\rangle$.

**Theorem 1.6.3.** *The $\langle K(\mathfrak{p})\rangle$ satisfy the following.*

1. $\langle K(\mathfrak{p})\rangle \wedge \langle K(\mathfrak{p})\rangle = \langle K(\mathfrak{p})\rangle$ *for all* $\mathfrak{p}$.

2. $\langle K(\mathfrak{p})\rangle \wedge \langle K(\mathfrak{q})\rangle = 0$ *when* $\mathfrak{p} \neq \mathfrak{q}$.

3. $\langle S^0\rangle = \coprod_{\mathfrak{p}\in \mathsf{Spec}\, R} \langle K(\mathfrak{p})\rangle$.

In order to classify the subcategories of $\mathcal{C}$ in a succinct way, we require the following hypothesis: for each $\mathfrak{p} \in \mathsf{Spec}\, R$, the Bousfield class $\langle K(\mathfrak{p}) \rangle$ is minimal among nontrivial Bousfield classes. This hypothesis is satisfied by both $\mathcal{C}((kG)^*)$ and $D(R)$, for Noetherian $R$.

**Theorem 1.6.4.** *Suppose that each $\langle K(\mathfrak{p}) \rangle$ is minimal. Then every localizing subcategory is a Bousfield class, and the Bousfield classes form a lattice. The Bousfield lattice is in one-to-one correspondence with the subsets of $\mathsf{Spec}\, R$. The lattice of thick subcategories of finite objects is in one-to-one correspondence with the subsets of $\mathsf{Spec}\, R$ that are closed under specialization.*

Recall that a subset $T \subseteq \mathsf{Spec}\, R$ is *closed under specialization* if $\mathfrak{p} \in T$ and $\mathfrak{p} \leq \mathfrak{q}$ implies that $\mathfrak{q} \in T$. This is equivalent to $T$ being a union of Zariski closed sets.

The bijection is given in terms of supports.

**Definition 1.6.5.** Given an object $X$ in $\mathcal{C}$, the *support of* $X$ is

$$\mathrm{supp}(X) = \{\mathfrak{p} \mid K(\mathfrak{p})_*(X) \neq 0\} = \left\{\mathfrak{p} \mid \overline{k_{\mathfrak{p}}} \wedge X \neq 0\right\}.$$

If $\mathcal{D}$ is a subcategory of $\mathcal{C}$, define $\mathrm{supp}(\mathcal{D}) = \bigcup_{X \in \mathcal{D}} \mathrm{supp}(X)$.

Let $\mathcal{A}$ be a localizing subcategory of $\mathcal{C}$, and let $T$ be a subset of $\mathsf{Spec}\, R$. The first correspondence in the theorem is given by the following:

$$\{\text{localizing subcategories of } \mathcal{C}\} \longleftrightarrow \{\text{subsets of } \mathsf{Spec}\, R\},$$

$$\mathcal{A} \longmapsto \mathrm{supp}(\mathcal{A}) = \{\mathfrak{p} \mid S /\!\!/ \mathfrak{p} \in \mathcal{A}\} \subseteq \mathsf{Spec}\, R,$$

with inverse

$$T \longmapsto \mathsf{loc}\,(K(\mathfrak{p}) \mid \mathfrak{p} \in T) = \langle W \rangle,$$

where $W = \coprod_{\mathfrak{q} \notin T} K(\mathfrak{q})$.

When we restrict to finite objects, the correspondence becomes

$$\{\text{thick subcategories of } \mathcal{F}\} \longleftrightarrow \{\text{subsets of } \mathsf{Spec}\, R \text{ closed under specialization}\},$$

$$\mathcal{A} \longmapsto \text{supp}(\mathcal{A}) = \{\mathfrak{p} \mid S/\!\!/\mathfrak{p} \in \mathcal{A}\} \subseteq \mathsf{Spec}\, R,$$

with inverse

$$T \longmapsto \mathsf{th}\,(S/\!\!/\mathfrak{p} \mid \mathfrak{p} \in T) = \{W \text{ in } \mathcal{F} \mid \ \text{supp}(W) \subseteq T\}\,.$$

Both these correspondences are order-preserving bijections of posets.

The previous theorem implies that the Bousfield lattice of a Noetherian stable homotopy category is a Boolean algebra on the classes $\langle K(\mathfrak{p})\rangle$. This implies that every class is complemented, and

$$\mathsf{BA} = \mathsf{DL} = \mathsf{BL}.$$

We conclude with two other strong results found in [HPS97].

**Theorem 1.6.6.** *Let $\mathcal{C}$ be a Noetherian stable homotopy category in which each $\langle K(\mathfrak{p})\rangle$ is minimal. The telescope conjecture holds (i.e., every smashing localization is a finite localization). Also, the objects $K(\mathfrak{p})$ detect nilpotence.*

### *1.6.2 Generalization to Non-Noetherian $D(R)$*

In general, the techniques used to prove the above strong results for Noetherian stable homotopy categories will not carry over to the non-Noetherian case. However, there is one example of a result that generalizes nicely. Thomason [Tho97] gave a classification for thick subcategories of finite objects in $D(R)$ for an arbitrary ungraded commutative ring $R$, which reduces to the above result in the case where $R$ is Noetherian. To date, Thomason's result only applies in the ungraded setting.

**Notation 1.6.7.** Given an ideal $I$ in a ring $R$, let $V(I)$ denote the closure of $I$ in $\mathsf{Spec}\, R$. That is, $V(I) = \{\mathfrak{p} \in \mathsf{Spec}\, R \ : \ I \subseteq \mathfrak{p}\}$.

**Theorem 1.6.8.** *[Tho97] Let $R$ be any ungraded commutative ring. There is a one-to-one correspondence between thick subcategories of finite objects in $D(R)$ and subsets of $\mathsf{Spec}\, R$ of the form $\bigcup_\alpha V(I_\alpha)$, where each $I_\alpha$ is finitely generated.*

Such subsets are called *Thomason-closed*. Just as above, a thick subcategory $\mathcal{A}$ of $\mathcal{F}$ corresponds to $\mathrm{supp}(\mathcal{A}) \subseteq \mathsf{Spec}\, R$, and a subset $T \subseteq \mathsf{Spec}\, R$ corresponds to $\{W \text{ in } \mathcal{F} \mid \mathrm{supp}(W) \subseteq T\}$.

Note that in the case where $R$ is Noetherian, every subset of the form $\bigcup_\alpha V(I_\alpha)$ is closed under specialization, because ideals in a Noetherian ring are finitely generated.

### *1.6.3 The categories $D(\Lambda_{\mathbb{Z}_{(p)}}), D(\Lambda_{\mathbb{F}_p}), D(\Lambda_{\mathbb{Q}})$, and $D(\Lambda_k)$.*

**Definition 1.6.9.** Fix a prime $p$ and integers $n_i > 1$, $i \geq 1$. Let $k$ be an arbitrary countable field. Define the following rings.

$$\Lambda_{\mathbb{Z}_{(p)}} := \frac{\mathbb{Z}_{(p)}[x_1, x_2, ...]}{(x_1^{n_1}, x_2^{n_2}, ...)}, \quad \Lambda_{\mathbb{F}_p} := \frac{\mathbb{F}_p[x_1, x_2, ...]}{(x_1^{n_1}, x_2^{n_2}, ...)}, \quad \Lambda_{\mathbb{Q}} := \frac{\mathbb{Q}[x_1, x_2, ...]}{(x_1^{n_1}, x_2^{n_2}, ...)}, \quad \text{and} \quad \Lambda_k := \frac{k[x_1, x_2, ...]}{(x_1^{n_1}, x_2^{n_2}, ...)}.$$

Grade the $x_i$ so that these rings are graded-connected and finitely-generated in each degree. For example, one can take $\deg(x_i) = 2^i$. Dwyer and Palmieri studied the derived category of the slightly more general ring $\Lambda_k$ in [DP08] (although they called it $\Lambda$), without specifying a countable field. Of course, all of the results about $D(\Lambda_k)$ in [DP08] apply to $D(\Lambda_{\mathbb{F}_p})$ and $D(\Lambda_{\mathbb{Q}})$. In Chapter 4 we will exhibit differences between $D(\Lambda_{\mathbb{F}_p})$ and $D(\Lambda_{\mathbb{Q}})$, and thus must make the distinction. However, the proofs in Chapter 6 work for $D(\Lambda_k)$, and so there we work in that larger context.

The motivation for choosing these rings is that they are non-Noetherian, locally finite, graded connected, graded commutative, have few prime ideals. Furthermore, all elements of positive degree are nilpotent. The same is true of the homotopy groups of the $p$-sphere spectrum $\pi_*(S^0)$ in $\mathcal{S}$.

In this section, we outline some of the results in [DP08].

**Theorem 1.6.10.** *[DP08, Cor. B] The Bousfield lattice of $D(\Lambda_k)$ has cardinality $2^{2^{\aleph_0}}$.*

This shows that the Bousfield lattice of $D(\Lambda_k)$ is quite different than that of a Noetherian ring. With a Noetherian ring, the Bousfield lattice is limited by $\mathsf{Spec}\, R$. For example, consider the Noetherian rings

$$\Lambda_k[m] := k[x_1, x_2, ..., x_m]/\left(x_i^{n_i} \text{ for all } i \leq m\right).$$

The Bousfield lattice of each $D(\Lambda_k[m])$ has only two classes: $\langle 0 \rangle$ and $\langle \Lambda_k[m] \rangle$.

**Theorem 1.6.11.** *[DP08, Thm. 6.1] In $D(\Lambda_k)$, there are objects of arbitrarily high smash-nilpotence height. That is, for any $n \geq 1$ there is an object $X_n$ in $D(\Lambda_k)$ such that the $n$-fold smash product of $X_n$ with itself is nonzero, while the $(n+1)$-fold smash product is zero.*

Let $I(\Lambda_k) = \mathrm{Hom}_k^*(\Lambda_k, k)$ be the graded vector space dual of $\Lambda_k$. In Section 7 of [DP08], it is shown that every nonzero Bousfield class $\langle X \rangle$ in $\mathsf{BL}_{D(\Lambda_k)}$ satisfies $\langle I(\Lambda_k) \rangle \leq \langle X \rangle$. One consequence is that the Boolean algebra $\mathsf{BA}$ is trivial in $\mathsf{BL}_{D(\Lambda_k)}$. In Proposition 4.2.2 we show that this is not the case in $\mathsf{BL}_{D(\Lambda_{\mathbb{Z}_{(p)}})}$.

Question 5.8 in [DP08] asks if every object in the derived category of a ring is Bousfield equivalent to a module. In Chapter 6 we show that in $D(\Lambda_k)$ this is not the case (see Theorem 6.1.4).

### *1.6.4 The stable homotopy category of spectra*

One of the most significant results, in terms of both elegance and utility, in stable homotopy theory in the last several decades is the classification of the thick subcategories of finite objects in the category of $p$-local spectra [HS98].

These subcategories are determined by the Morava $K$-theories $K(n)$. For each $n \geq 1$, $K(n)$ is a ring spectrum (actually a *field spectrum* - every module object over $K(n)$ is equivalent to a wedge of suspensions of $K(n)$), and has coefficient ring $\pi_*(K(n)) \cong \mathbb{F}_p[v_n, v_n^{-1}]$ with $|v_n| = 2(p^n - 1)$. The $K(n)$ are constructed from the

Brown-Peterson spectrum $BP$. We define $K(0) = H\mathbb{Q}$ and $K(\infty) = H\mathbb{F}_p$, Eilenberg-Maclane spectra. Set $\mathcal{C}_0 = \mathcal{F}$, and for $n \geq 1$ define

$$\mathcal{C}_n := \{X \text{ in } \mathcal{F} \;:\; K(n-1)_*(X) = 0\} = \langle K(n-1)\rangle \cap \mathcal{F}.$$

**Theorem 1.6.12.** *(Thick Subcategory Theorem) [HS98] A subcategory $\mathcal{D}$ of $\mathcal{F}$ is thick if and only if $\mathcal{D} = \mathcal{C}_n$ for some $n$. These subcategories form a nested strictly decreasing filtration of $\mathcal{F}$:*

$$\cdots \subsetneq \mathcal{C}_{n+1} \subsetneq \mathcal{C}_n \subsetneq \mathcal{C}_{n-1} \subsetneq \cdots \subsetneq \mathcal{C}_1 \subsetneq \mathcal{C}_0.$$

A spectrum $X$ in $\mathcal{C}_n - \mathcal{C}_{n+1}$ is said to be of type $n$, and we write $\text{type}(X) = n$. Mitchell [Mit85] showed that this filtration is strictly decreasing. Hopkins and Smith use their thick subcategory theorem to prove

**Theorem 1.6.13.** *(Class-invariance theorem) [HS98] Let $X$ and $Y$ be finite spectra. Then $\langle X\rangle \leq \langle Y\rangle$ if and only if $\text{type}(X) \geq \text{type}(Y)$.*

For each $n \geq 0$, let $F(n)$ denote an arbitrary finite spectrum of type $n$. Thus there is a well-defined class $\langle F(n)\rangle$, and $\langle F(n)\rangle \leq \langle F(m)\rangle$ precisely when $n \geq m$. Every finite spectrum $X$ of type $n$ has $\langle X\rangle = \langle F(n)\rangle$. This gives us a complete understanding of the Bousfield classes of finite spectra.

Bousfield introduced the notion of Bousfield classes on $\mathcal{S}$ in [Bou79a] and [Bou79b]. Further work was done in [Rav84] and [HP99]. Bousfield shows that every ring spectrum and every finite spectrum is in $\mathsf{DL}$. The Brown-Comenetz dual $I$ of the sphere is not in $\mathsf{DL}$, since $I \wedge I = 0$. Every finite spectrum is in $\mathsf{BA}$, but the inclusion $\mathsf{BA} \subset \mathsf{DL}$ is proper since for example $\langle H\mathbb{Z}\rangle$ is a ring spectrum not in $\mathsf{BA}$. Thus in contrast with the Noetherian case, here we have

$$\mathsf{BA} \subsetneq \mathsf{DL} \subsetneq \mathsf{BL}.$$

We've briefly outlined significant structural similarities between derived categories of rings, and the category of spectra. But we've also illustrated significant differences, particularly between derived categories of Noetherian rings, and spectra. The derived category of general commutative rings, or specifically non-Noetherian rings, presents a fascinating middle ground.

# Chapter 2

## RING MAPS AND THE BOUSFIELD LATTICE

In this chapter, in order to better understand the Bousfield lattice and localizing subcategories in the derived category of an arbitrary commutative ring, we use ring maps to relate the derived categories of different rings. First, as discussed in Section 1.3, we must be careful to distinguish the graded and ungraded cases.

**WARNING:** The results in Sections 2.1, 2.2, and 2.3 hold for either derived categories of graded modules over graded rings, or for derived categories of ungraded modules over ungraded rings. We will use the same notation for either of these cases. Thus $\mathsf{Mod}\text{-}R$ denotes either the category of right $R$-modules, or the category of graded right $R$-modules. However, in Sections 2.4 and 2.5 we will restrict to the ungraded case.

A commutative ring map $f : R \to S$ induces a functor on module categories $f_* : \mathsf{Mod}\text{-}R \to \mathsf{Mod}\text{-}S$, where $f_*(M) = M \otimes_R S$. This induces a functor $Ch(R) \to Ch(S)$ on complexes. Let $f_\bullet : D(R) \to D(S)$ be the derived functor $f_\bullet = Lf_*$, and let $i_\bullet = Li : D(S) \to D(R)$ be its right adjoint, induced by the forgetful functor $i : \mathsf{Mod}\text{-}S \to \mathsf{Mod}\text{-}R$. Placing various hypotheses on the map $f$ gives a range of results. When $R$ or $S$ is non-Noetherian, most of these results are new. Some results that are not new have been included because it is difficult to find them in the literature, especially in the graded case; furthermore, we wish to illustrate the language and methods of the tensor-triangulated approach towards derived categories. For completeness, the case of a map between two Noetherian rings is explored in Section 2.5, although some of these results follow in a straightforward way from the classifications of [Nee92, HPS97].

The first three sections establish general properties of $f_\bullet$ and $i_\bullet$, and focus on Bousfield lattices. For example, there are two interesting well-known sublattices $\mathsf{DL}$ and $\mathsf{BA}$ within the Bousfield lattice $\mathsf{BL}$, and Propositions 2.1.9 and 2.1.14 show that $f_\bullet$ and $i_\bullet$ define order-preserving operations between the Bousfield lattices of $D(R)$ and $D(S)$, such that the map $f_\bullet$ sends $\mathsf{DL}_{D(R)}$ to $\mathsf{DL}_{D(S)}$ and $\mathsf{BA}_{D(R)}$ to $\mathsf{BA}_{D(S)}$.

Furthermore, if we assume $f_\bullet i_\bullet \langle X \rangle = \langle X \rangle$ for all $X$ (which occurs, for example, when $f$ is surjective and $S$ is Noetherian), then Proposition 2.3.6 shows that $i_\bullet$ injects $\mathsf{DL}_{D(S)}$ into $\mathsf{DL}_{D(R)}$ and $\mathsf{BA}_{D(S)}$ into $\mathsf{BA}_{D(R)}$, and $f_\bullet$ surjects $\mathsf{DL}_{D(R)}$ onto $\mathsf{DL}_{D(S)}$ and $\mathsf{BA}_{D(R)}$ onto $\mathsf{BA}_{D(S)}$.

We also define a quotient lattice $\mathsf{BL}_{D(R)}/J$ of the Bousfield lattice of $D(R)$, and show (Prop. 2.3.2) that when $f_\bullet i_\bullet \langle X \rangle = \langle X \rangle$ for all $X$, $f_\bullet$ induces an isomorphism

$$\overline{f_\bullet} : \mathsf{BL}_{D(R)}/J \xrightarrow{\cong} \mathsf{BL}_{D(S)},$$

with inverse $i_\bullet$. This is a complete splitting in the case of a surjection of Noetherian rings. In Chapter 4, we will apply Prop. 2.3.2 to a specific map on non-Noetherian rings, and use it to deduce a complete splitting of Bousfield lattices (Theorem 4.2.5).

Sections 2.4 and 2.5 invoke the classifications of Neeman and Thomason, and for this reason (only) we restrict to derived categories of ungraded modules over ungraded rings. Section 2.4 assumes $f : R \to S$ is surjective and $S$ is Noetherian, but $R$ may be non-Noetherian.

The first half of Section 2.4 is devoted to computing the effect of $f_\bullet$ and $i_\bullet$ on specific objects in $D(R)$ and $D(S)$. Proposition 2.4.1 shows that $f_\bullet(\overline{k_{f^{-1}\mathfrak{p}}})$ and $\overline{k_\mathfrak{p}}$ generate the same localizing subcategory. Given a prime $\mathfrak{p} \in \mathsf{Spec}\, S$ and a choice of pre-images of generators for $\mathfrak{p}$, we construct a new, unstudied finite object $R/\!\!/\tilde{\mathfrak{p}}$ in $D(R)$. Proposition 2.4.17 shows that for every choice of $R/\!\!/\tilde{\mathfrak{p}}$, $\mathrm{supp}(R/\!\!/\tilde{\mathfrak{p}}) = V(\tilde{\mathfrak{p}})$, where $V(-)$ is the closure in $\mathsf{Spec}\, R$ and $\mathrm{supp}(-)$ denotes the support (see Section 1.6). Lemma 2.4.4 shows $f_\bullet(R/\!\!/\tilde{\mathfrak{p}}) = S/\!\!/\mathfrak{p}$. We also have (Cor. 2.4.14, Lemma 2.4.15, Lemma 2.4.16) that

$$\operatorname{supp}(R/\!\!/\tilde{\mathfrak{p}}) \cap f^{-1}(\mathsf{Spec}\, S) = V(f^{-1}\mathfrak{p}) = f^{-1}(V(\mathfrak{p})) = \operatorname{supp}(i_\bullet(S/\!\!/\mathfrak{p})).$$

Furthermore, we construct objects $K(\tilde{\mathfrak{p}}) := R/\!\!/\tilde{\mathfrak{p}} \wedge R_{f^{-1}\mathfrak{p}}$ in $D(R)$. Lemma 2.4.5 shows that in the quotient lattice $\mathsf{BL}_{D(R)}/J$ the class $\langle K(\tilde{\mathfrak{p}})\rangle$ is well-defined, and equal to $\langle i_\bullet K(\mathfrak{p})\rangle$. Proposition 2.4.7 shows that these classes play the same role in $\mathsf{BL}_{D(R)}/J$ that the $\langle K(\mathfrak{p})\rangle$ play in $\mathsf{BL}_{D(S)}$. For example, in $\mathsf{BL}_{D(R)}/J$ we have $\langle R\rangle = \coprod_{p\in \mathsf{Spec}\, S}\langle K(\tilde{\mathfrak{p}})\rangle$, and each $\langle K(\tilde{\mathfrak{p}})\rangle$ is a minimal nonzero Bousfield class.

Our hope is that, when $R$ is non-Noetherian, with further study the $R/\!\!/\tilde{\mathfrak{p}}$ can help to understand the finite objects of $D(R)$, and the classes $\langle K(\tilde{\mathfrak{p}})\rangle$, or at least the $\langle i_\bullet K(\mathfrak{p})\rangle$, might serve as useful tools for understanding the full Bousfield lattice $\mathsf{BL}_{D(R)}$.

The remainder of Section 2.4, and most of Section 2.5 (where we assume $f : R \to S$ is a surjection of Noetherian rings), investigate subcategories. We show that $f_\bullet$ and $i_\bullet$ give well-defined operations on thick and localizing subcategories that, via the notion of support and the map $f^{-1} : \mathsf{Spec}\, S \to \mathsf{Spec}\, R$, respect the classification theorems. In Section 2.5 these results are as elegant as one might hope, but perhaps not a surprise to someone familiar with Neeman's classification.

These subcategory results in Section 2.4 are slightly weaker, but new and perhaps more interesting. For example, assuming $f : R \to S$ is a surjection and $S$ Noetherian, Proposition 2.4.20 shows that

$$f^{-1}(\operatorname{supp}(f_\bullet X)) \subseteq \operatorname{supp}(X) \cap f^{-1}(\mathsf{Spec}\, S), \text{ and}$$

$$f^{-1}(\operatorname{supp}(f_\bullet \mathsf{B})) \subseteq \operatorname{supp}(\mathsf{B}) \cap f^{-1}(\mathsf{Spec}\, S),$$

where $X$ is an arbitrary object in $D(R)$ and $\mathsf{B}$ is a thick subcategory of finite objects in $D(R)$. Equality holds when $X = i_\bullet Y$ for some $Y$ in $D(S)$. Proposition 2.4.13 shows that if $i_\bullet S$ is finite and $\mathsf{A}$ is a thick subcategory of finite objects in $D(S)$, then $\operatorname{supp}(i_\bullet \mathsf{A}) = f^{-1}(\operatorname{supp}(\mathsf{A}))$. As one might hope or expect, when $R$ is also Noetherian, the above inclusions are equalities (Props. 2.5.1 and 2.5.9), and the statements hold for $\mathsf{A}$ and $\mathsf{B}$ localizing subcategories as well (Props. 2.5.1 and 2.5.5).

The chapter is organized to follow a gradual strengthening of hypotheses on the rings $R$ and $S$ and the map $f : R \to S$. We have chosen this organization so as to clearly indicate which results rely on which hypotheses, and help build intuition about the differences between the Noetherian and general case. Again, some of the results are standard in algebraic geometry, but difficult to find in the literature, especially in the graded case; for these we have chosen to include (new?) proofs, that use the language and methods of tensor-triangulated category theory.

## 2.1 *General* $f : R \to S$

In this section, let $f : R \to S$ be any ring homomorphism, and $f_* : \mathsf{Mod}\text{-}R \to \mathsf{Mod}\text{-}S$ as above.

**Definition 2.1.1.** Let $f_\bullet$ be the left derived functor $f_\bullet = Lf_* = L(- \otimes_R S) : D(R) \to D(S)$. Let $i_\bullet = Li : D(S) \to D(R)$ be the derived functor of the forgetful functor $i : \mathsf{Mod}\text{-}S \to \mathsf{Mod}\text{-}R$.

Then [HPS97, 9.3.1] shows that $f_\bullet$ is a stable morphism - it is exact, has $f_\bullet(R) = S$, and $f_\bullet(X \wedge Y) = f_\bullet X \wedge f_\bullet Y$. It is a left adjoint, and it commutes with coproducts. The right adjoint of $f_\bullet$ is $i_\bullet$, which is exact and commutes with coproducts and products. The functor $i_\bullet$ is injective in the sense that $i_\bullet(X) = 0$ implies $X = 0$, simply because an acyclic complex of $S$-modules is acyclic whether we think of it as a complex of $R$-modules or $S$-modules. The adjointness means

$$\mathrm{Hom}^*_{D(S)}(f_\bullet X, Y) \cong \mathrm{Hom}^*_{D(R)}(X, i_\bullet Y).$$

The following lemma will be used frequently. Recall the discussion of cell modules in Section 1.3.

**Lemma 2.1.2.** *For all objects $A$ in $D(R)$ and $B$ in $D(S)$, we have*

$$i_\bullet(f_\bullet A \wedge B) = A \wedge i_\bullet B.$$

**Proof.** This is the projection formula, proven in [Wei94, 10.8.5] for the bounded derived category. It relies on the fact that $f_*$ sends projectives to projectives, and in the bounded derived category every object is equivalent to a complex of projectives. In the unbounded derived category, every object is equivalent to a cellular tower (see Section 1.3). Because $f_\bullet$ sends $R$ to $S$, it sends cellular towers in $D(R)$ to cellular towers in $D(S)$, and this suffices to extend the proof. □

**Corollary 2.1.3.** *For all objects $A$ in $D(R)$ and $B$ in $D(S)$,*

$$f_\bullet A \wedge B = 0 \text{ if and only if } A \wedge i_\bullet B = 0.$$

**Remark 2.1.4.** Take $z \in R = [R, R]_*$, and consider the morphism $R \xrightarrow{z} R$ in $D(R)$. Applying $f_\bullet$ to this, we get

$$\begin{aligned}\left(f_\bullet(R) \xrightarrow{f_\bullet(z)} f_\bullet(R)\right) &= \left(R \otimes_R S \xrightarrow{z\otimes 1} R \otimes_R S\right) \\ &= \left(R \otimes_R S \xrightarrow{1\otimes f(z)} R \otimes_R S\right) = \left(S \xrightarrow{f(z)} S\right).\end{aligned}$$

From this we conclude the following.

**Lemma 2.1.5.** *The functor $f_\bullet$ takes finite objects to finite objects.*

### *2.1.1 Ideals and prime ideals*

We introduce two important classes of objects. For any finitely generated ideal $\mathfrak{r} = (z_1, ..., z_n)$ in a ring $T$, let $T/\!\!/\mathfrak{r}$ denote the wedge $T/z_1 \wedge T/z_2 \wedge \cdots \wedge T/z_n$, where

$T/z_i$ is the cofiber of the map $T \xrightarrow{z_i} T$. These are often called *Koszul objects*, as in Definition 1.6.2. They depend on the choice of generators, but are well-defined at the level of thick subcategories.

**Lemma 2.1.6.** *Given two finitely generated ideals $\mathfrak{r}$, $\mathfrak{t}$ in a ring $T$, if $\mathfrak{r} \subseteq \mathfrak{t}$ then $T/\!\!/\mathfrak{t} \in \mathsf{th}(T/\!\!/\mathfrak{r})$. Therefore different choices of generators of an ideal $\mathfrak{r}$ will generate the same thick subcategory $\mathsf{th}(T/\!\!/\mathfrak{r})$.*

**Proof.** This is basically Lemma 6.0.9 in [HPS97]. The proof there requires the ideals be finitely generated, but not prime. □

Now, for a prime ideal $\mathfrak{p}$ in a ring $T$, let $T_\mathfrak{p}$ be the localization at $\mathfrak{p}$. Then let $k_\mathfrak{p} = T_\mathfrak{p}/\mathfrak{p}T_\mathfrak{p}$ be the residue field of $\mathfrak{p}$; let $\overline{k_\mathfrak{p}}$ be this field thought of in $D(T)$. Then [HPS97, 3.7.2] shows that $\overline{k_\mathfrak{p}}$ is a skew field object in $D(T)$.

For every prime ideal $\mathfrak{p}$, the object $\overline{k_\mathfrak{p}} \wedge \overline{k_\mathfrak{p}}$ is nonzero, because it has homology $\mathrm{Ext}^*_T(k_\mathfrak{p}, k_\mathfrak{p}) \neq 0$.

**Lemma 2.1.7.** *Let $\mathfrak{q}$, $\mathfrak{p}$ be prime ideals of a ring $T$. If $\mathfrak{p} \neq \mathfrak{q}$, then $\overline{k_\mathfrak{p}} \wedge \overline{k_\mathfrak{q}} = 0$.*

**Proof.** Without loss of generality, take $r \in \mathfrak{p} \backslash \mathfrak{q}$. Then since $k_\mathfrak{q}$ is $\mathfrak{q}$-local, and $r \notin \mathfrak{q}$, the map $k_\mathfrak{q} \xrightarrow{r} k_\mathfrak{q}$ is an isomorphism, and induces an equivalence in $D(T)$. On the other hand, $k_\mathfrak{p}$ is $\mathfrak{p}$-torsion, so $k_\mathfrak{p} \xrightarrow{r} k_\mathfrak{p}$ is nilpotent (some power of it is zero). Since

$$\overline{k_\mathfrak{p}} \wedge \overline{k_\mathfrak{q}} \xrightarrow{1 \wedge r} \overline{k_\mathfrak{p}} \wedge \overline{k_\mathfrak{q}} \quad \text{is an equivalence, and}$$

$$\overline{k_\mathfrak{p}} \wedge \overline{k_\mathfrak{q}} \xrightarrow{r \wedge 1} \overline{k_\mathfrak{p}} \wedge \overline{k_\mathfrak{q}} \quad \text{is nilpotent,}$$

we must have $\overline{k_\mathfrak{p}} \wedge \overline{k_\mathfrak{q}} = 0$. □

### *2.1.2 Bousfield lattice*

Here we show that the functors $f_\bullet$ and $i_\bullet$ induce maps between the Bousfield lattices of $D(R)$ and $D(S)$. If we consider a Bousfield class $\langle X \rangle$ as the localizing subcategory of $X$-acyclics, then we can map this to $f_\bullet(\langle X\rangle)$ as a subcollection in $D(S)$. However, in general $f_\bullet(\langle X\rangle)$ will not be triangulated, because $f_\bullet \circ i_\bullet \neq 1_{D(S)}$. Instead we make the following definitions.

**Definition 2.1.8.** Define an operation $f_\bullet : \mathsf{BL}_{D(R)} \to \mathsf{BL}_{D(S)}$ by $\langle X\rangle \mapsto \langle f_\bullet X\rangle$. Also, define an operation $i_\bullet : \mathsf{BL}_{D(S)} \to \mathsf{BL}_{D(R)}$ by $\langle X\rangle \mapsto \langle i_\bullet X\rangle$. For the rest of this document, $f_\bullet\langle X\rangle$ and $i_\bullet\langle X\rangle$ will mean $\langle f_\bullet X\rangle$ and $\langle i_\bullet X\rangle$.

**Proposition 2.1.9.** *Both $f_\bullet$ and $i_\bullet$, as defined above, are well-defined, order-preserving operations on Bousfield lattices, and both preserve arbitrary joins.*

**Proof.** First we show that $\langle Y\rangle \leq \langle X\rangle$ implies $\langle i_\bullet Y\rangle \leq \langle i_\bullet X\rangle$. Suppose $\langle Y\rangle \leq \langle X\rangle$ and $W \wedge i_\bullet X = 0$. Then Corollary 2.1.3 implies $f_\bullet W \wedge X = 0$. Thus $f_\bullet W \wedge Y = 0$, and $W \wedge i_\bullet Y = 0$.

This implies that if $\langle Y\rangle = \langle X\rangle$, then $\langle i_\bullet Y\rangle = \langle i_\bullet X\rangle$, so $i_\bullet$ is well-defined and order-preserving.

Now suppose $\langle Y\rangle \leq \langle X\rangle$ and $f_\bullet X \wedge W = 0$. Then from Corollary 2.1.3, $X \wedge i_\bullet W = 0$, so $Y \wedge i_\bullet W = 0$, which implies $f_\bullet Y \wedge W = 0$. Therefore $f_\bullet$ is order-preserving and well-defined.

It's clear that $f_\bullet$ and $i_\bullet$ preserve arbitrary joins. $\square$

As an aside, note that this implies $\langle i_\bullet X\rangle \leq \langle i_\bullet S\rangle$ for all $X$ in $D(S)$.

**Definition 2.1.10.** Let $J$ be the image of $Ker f_\bullet$ in $\mathsf{BL}_{D(R)}$, in other words $J = \{\langle X\rangle \mid f_\bullet\langle X\rangle = \langle 0\rangle\}$. Also define

$$\langle M\rangle := \bigvee_{\langle Y\rangle \in J} \langle Y\rangle.$$

**Proposition 2.1.11.** *The subposet $J$ is a complete principal ideal in $\mathsf{BL}_{D(R)}$, and $f_\bullet$ induces a poset map*

$$\overline{f_\bullet} : \mathsf{BL}_{D(R)}/J \to \mathsf{BL}_{D(S)}.$$

**Proof.** Suppose $\langle Y\rangle \leq \langle X\rangle$ and $\langle f_\bullet X\rangle = \langle 0\rangle$. Then $\langle f_\bullet Y\rangle \leq \langle f_\bullet X\rangle$, so $\langle f_\bullet Y\rangle = \langle 0\rangle$ and $J$ is a lattice ideal. It is complete because it is closed under arbitrary joins. From the definition of $\langle M\rangle$, we see that $J = \{\langle X\rangle \mid \langle X\rangle \leq \langle M\rangle\}$ is principal.

This is not enough to guarantee an induced map on the quotient lattice (see [HP99, 3.11]). For this, we also need to know that if $\langle X\rangle \equiv \langle Y\rangle (\text{mod } J)$, then $f_\bullet\langle X\rangle = f_\bullet\langle Y\rangle$. As in [HP99, Sect. 3], $\langle X\rangle$ and $\langle Y\rangle$ are equivalent if and only if $\langle X\rangle \vee \langle M\rangle = \langle Y\rangle \vee \langle M\rangle$. But then since $\langle f_\bullet M\rangle = \langle 0\rangle$,

$$\langle f_\bullet X\rangle = \langle f_\bullet X\rangle \vee \langle f_\bullet M\rangle = f_\bullet(\langle X\rangle \vee \langle M\rangle) = \langle f_\bullet Y\rangle \vee \langle f_\bullet M\rangle = \langle f_\bullet Y\rangle.$$

□

**Notation 2.1.12.** For brevity, we will denote $\mathsf{BL}_{D(R)}/J$ by $\mathsf{BL}/J$.

Note that in $\mathsf{BL}/J$ we have $\overline{f_\bullet}\langle X\rangle = \langle 0\rangle$ if and only if $\langle X\rangle = \langle 0\rangle$.

**Lemma 2.1.13.** *In $\mathsf{BL}_{D(R)}$ we have $i_\bullet \circ f_\bullet\langle X\rangle \leq \langle X\rangle$, and in $\mathsf{BL}/J$ we have $i_\bullet \circ \overline{f_\bullet}\langle X\rangle = \langle X\rangle$, for all $\langle X\rangle$. Thus in $\mathsf{BL}/J$, if $\overline{f_\bullet}\langle X\rangle = \overline{f_\bullet}\langle Y\rangle$, then $\langle X\rangle = \langle Y\rangle$.*

**Proof.** An object $W$ has $W \wedge i_\bullet f_\bullet Y = 0$ iff $f_\bullet W \wedge f_\bullet Y = 0$ iff $f_\bullet(W \wedge Y) = 0$.

Therefore $W \wedge Y = 0$ implies $W \wedge (i_\bullet f_\bullet Y) = 0$. And in $\mathsf{BL}/J$ the converse holds as well.

If $\langle \overline{f_\bullet} X\rangle = \langle \overline{f_\bullet} Y\rangle$, then $\langle X\rangle = \langle i_\bullet \overline{f_\bullet} X\rangle = \langle i_\bullet \overline{f_\bullet} Y\rangle = \langle Y\rangle$. □

### *2.1.3* BA *and* DL

Here we begin an investigation of the effects of $f_\bullet$ and $i_\bullet$ on the sublattices $\mathsf{BA}$ and $\mathsf{DL}$ of $\mathsf{BL}$. See also Section 2.3.2, and [Bou79a, HP99, DP08, IK11] for more results on $\mathsf{BA}$ and $\mathsf{DL}$.

**Proposition 2.1.14.** *$f_\bullet$ maps $\mathsf{DL}_{D(R)}$ into $\mathsf{DL}_{D(S)}$, and $\mathsf{BA}_{D(R)}$ into $\mathsf{BA}_{D(S)}$. If $\langle X\rangle$ in $\mathsf{BA}_{D(R)}$ has complement $\langle X^c\rangle$, then $\langle f_\bullet X\rangle$ has complement $\langle f_\bullet(X^c)\rangle$.*

**Proof.** If $\langle Y\rangle = \langle Y \wedge Y\rangle$, then $\langle f_\bullet Y\rangle = \langle f_\bullet Y \wedge f_\bullet Y\rangle$.

If $\langle X\rangle$ has $\langle X\rangle \vee \langle X^c\rangle = \langle R\rangle$ and $\langle X\rangle \wedge \langle X^c\rangle = \langle 0\rangle$, then $\langle f_\bullet X\rangle \vee \langle f_\bullet X^c\rangle = \langle f_\bullet R\rangle = \langle S\rangle$ and $\langle f_\bullet X\rangle \wedge \langle f_\bullet X^c\rangle = \langle 0\rangle$, so $\langle (f_\bullet X)^c\rangle = \langle f_\bullet X^c\rangle$. □

## 2.2 Surjective $f : R \to S$

In this section, let $f : R \to S$ be a surjective ring map.

**Lemma 2.2.1.** *The map $f^{-1} : \mathsf{Spec}\, S \to \mathsf{Spec}\, R$ is injective.*

**Proof.** $\mathfrak{p} = f(f^{-1}(\mathfrak{p})) = f(f^{-1}(\mathfrak{q})) = \mathfrak{q}$. □

In fact, as a map of topological spaces, $f^{-1}$ is a homeomorphism onto its image [Har77, ex.2.18].

**Lemma 2.2.2.** *Let $\mathfrak{p} \subseteq S$ be a prime ideal. Then $f_\bullet(R_{f^{-1}\mathfrak{p}}) = R_{f^{-1}\mathfrak{p}} \otimes_R S = S_\mathfrak{p}$ as objects in $D(S)$.*

**Proof.** First note that $R_{f^{-1}\mathfrak{p}}$ is flat over $R$, so

$$f_\bullet(R_{f^{-1}\mathfrak{p}}) = L(- \otimes_R S)(R_{f^{-1}\mathfrak{p}}) = R_{f^{-1}\mathfrak{p}} \otimes_R S.$$

We have an $S$-module map $\phi : R_{f^{-1}\mathfrak{p}} \otimes_R S \longrightarrow S_\mathfrak{p}$ given by

$$\phi\left(\frac{a}{b} \otimes c\right) = \frac{f(a)c}{f(b)}.$$

If $\frac{f(a)c}{f(b)} = 0$ then $f(a)c = 0$, so $\frac{a}{b} \otimes c = \frac{1}{b} \otimes f(a)c = 0$.

Given $\frac{y}{z} \in S_\mathfrak{p}$, with $z \in S \backslash \mathfrak{p}$, since $f$ is surjective we have $b \in R \backslash f^{-1}\mathfrak{p}$ such that $f(b) = z$. Then

$$\phi\left(\frac{1}{b} \otimes y\right) = \frac{y}{f(b)} = \frac{y}{z}.$$

□

The next proposition plays an important role in later computations, and will be strengthened in Section 2.4.

**Proposition 2.2.3.** *Let $\mathfrak{p} \subseteq S$ be a prime ideal. Then $f_\bullet(\overline{k_{f^{-1}\mathfrak{p}}})$ is in the localizing subcategory generated by $\overline{k_\mathfrak{p}}$.*

**Proof.** Let $\mathfrak{q} = f^{-1}\mathfrak{p}$. Because $f_\bullet = L(- \otimes_R S)$, $f_\bullet(\overline{k_\mathfrak{q}})$ is $\overline{k_\mathfrak{q}} \otimes^L_R S$ considered as a complex of $S$-modules. First we will show that, as complexes of $S$-modules, $\overline{k_\mathfrak{q}} \otimes^L_R S \cong \overline{k_\mathfrak{q}} \otimes^L_R S_\mathfrak{p}$. Then we will compute $\overline{k_\mathfrak{q}} \otimes^L_R S_\mathfrak{p}$.

In the local ring $R_\mathfrak{q}$, projectives are free. Since $k_\mathfrak{q}$ is $\mathfrak{q}$-local, it has a free $R_\mathfrak{q}$-resolution in $D(R_\mathfrak{q})$. Since $R_\mathfrak{q}$ is flat over $R$, this resolution is also a flat $R$-resolution for $\overline{k_\mathfrak{q}}$ in $D(R)$. Thus to show $\overline{k_\mathfrak{q}} \otimes^L_R S \cong \overline{k_\mathfrak{q}} \otimes^L_R S_\mathfrak{p}$ it suffices to show

$$R_\mathfrak{q} \otimes_R S \cong R_\mathfrak{q} \otimes_R S_\mathfrak{p} \text{ as } S\text{-modules.}$$

This follows from Lemma 2.2.2 and the fact that $S_\mathfrak{p}$ is $\mathfrak{q}$-local as an $R$-module, so its $\mathfrak{q}$-localization $R_\mathfrak{q} \otimes_R S_\mathfrak{p} \cong S_\mathfrak{p}$.

To compute $\overline{k_\mathfrak{q}} \otimes^L_R S_\mathfrak{p}$, note that the surjection $R \to S$ induces a surjection $R_\mathfrak{q} \to S_\mathfrak{p}$. As above, we can take a free $R_\mathfrak{q}$-resolution of $S_\mathfrak{p}$ in $D(R_\mathfrak{q})$, and think of this as a flat $R$-resolution of $S_\mathfrak{p}$ in $D(R)$.

$$\cdots \longrightarrow \bigoplus_{J_2} R_\mathfrak{q} \longrightarrow \bigoplus_{J_1} R_\mathfrak{q} \longrightarrow R_\mathfrak{q} \longrightarrow 0$$

Then since $k_\mathfrak{q}$ is $\mathfrak{q}$-local, $\overline{k_\mathfrak{q}} \otimes^L_R S_\mathfrak{p}$ is represented by

$$\cdots \longrightarrow \bigoplus_{J_2} k_\mathfrak{q} \longrightarrow \bigoplus_{J_1} k_\mathfrak{q} \longrightarrow k_\mathfrak{q} \longrightarrow 0.$$

If $f : R \to S$ has kernel $I$, then $S \cong R/I$, and $S_\mathfrak{p} \cong R_\mathfrak{q}/I$. Then $\mathfrak{q} = f^{-1}\mathfrak{p} = \mathfrak{p} + I$. Therefore $I \cdot k_\mathfrak{q} = I \cdot (R_\mathfrak{q}/\mathfrak{q}R_\mathfrak{q}) = 0$ and $k_\mathfrak{q}$ is also an $S$-module. In fact, as $S$-modules, $k_\mathfrak{q} \cong k_\mathfrak{p}$, since

$$k_\mathfrak{q} = \frac{R_\mathfrak{q}}{(\mathfrak{p}+I)R_\mathfrak{q}} \cong \frac{(R_\mathfrak{q}/I)}{\mathfrak{p}(R_\mathfrak{q}/I)} \cong \frac{S_\mathfrak{p}}{\mathfrak{p}S_\mathfrak{p}} = k_\mathfrak{p}.$$

Therefore $f_\bullet(\overline{k_{f^{-1}\mathfrak{p}}})$ is represented in $D(S)$ by

$$\cdots \longrightarrow \bigoplus_{J_2} k_\mathfrak{p} \longrightarrow \bigoplus_{J_1} k_\mathfrak{p} \longrightarrow k_\mathfrak{p} \longrightarrow 0.$$

This shows that $f_\bullet(\overline{k_{f^{-1}\mathfrak{p}}})$ is in the localizing subcategory generated by $\overline{k_\mathfrak{p}}$. □

## 2.3 Maps $f : R \to S$ satisfying $f_\bullet i_\bullet \langle X\rangle = \langle X\rangle$ for all $X$

In several situations that we are interested in, the map $f : R \to S$ satisfies $f_\bullet i_\bullet \langle X\rangle = \langle X\rangle$ for all $X$. For example, this happens when $f$ is surjective and $S$ is Noetherian (see Section 2.4). It also happens with the specific non-Noetherian projection $f : \Lambda_{\mathbb{Z}_{(p)}} \to \Lambda_{\mathbb{F}_p}$ that we examine in Chapter 4.

Note that $f_\bullet i_\bullet \langle X\rangle = \langle X\rangle$ means $i_\bullet$ is injective in the sense that $i_\bullet\langle X\rangle = i_\bullet\langle Y\rangle$ implies $\langle X\rangle = \langle Y\rangle$.

**Lemma 2.3.1.** *The following are equivalent:*

1. $f_\bullet i_\bullet \langle X\rangle = \langle X\rangle$ *for all* $\langle X\rangle$

2. $i_\bullet W \wedge i_\bullet X = 0$ *if and only if* $i_\bullet(W \wedge X) = 0$

3. $i_\bullet\langle Y \wedge X\rangle = \langle i_\bullet Y\rangle \wedge \langle i_\bullet X\rangle$.

**Proof.** For (1) ⇔ (2), note that $W \wedge f_\bullet i_\bullet X = 0$ iff $i_\bullet W \wedge i_\bullet X = 0$, and $W \wedge X = 0$ iff $i_\bullet(W \wedge X) = 0$.

For (1) ⇔ (3), note that $W \wedge i_\bullet(Y \wedge X) = 0$ iff $f_\bullet W \wedge (Y \wedge X) = 0$ iff $(f_\bullet W \wedge Y) \wedge X = 0$, and $W \wedge i_\bullet X \wedge i_\bullet Y = 0$ iff $(f_\bullet W \wedge f_\bullet i_\bullet X) \wedge Y = 0$ iff $(f_\bullet W \wedge Y) \wedge (f_\bullet i_\bullet X) = 0$. □

**Proposition 2.3.2.** *If* $f_\bullet i_\bullet \langle X\rangle = \langle X\rangle$ *for all* $\langle X\rangle$*, then* $f_\bullet : \mathsf{BL}_{D(R)} \to \mathsf{BL}_{D(S)}$ *is onto, and we have an isomorphism of posets*

$$\overline{f_\bullet} : \mathsf{BL}_{D(R)}/J \xrightarrow{\cong} \mathsf{BL}_{D(S)},$$

*where* $i_\bullet$ *is the inverse.*

**Proof.** It's clear that $f_\bullet$ and $\overline{f_\bullet}$ are onto. We showed in Lemma 2.1.13 that $\overline{f_\bullet}$ is injective. □

We will apply this proposition in Chapter 4, to get an interesting splitting of the Bousfield lattices of the derived categories of specific non-Noetherian rings.

### *2.3.1 Poset adjoints*

As a poset map, because $i_\bullet$ preserves joins on $\mathsf{BL}_{D(S)}$, it has a poset map right adjoint $r : \mathsf{BL}_{D(R)} \to \mathsf{BL}_{D(S)}$, see [HP99, 3.5]. We know

$$r\langle Y\rangle = \bigvee \{\langle X\rangle \mid i_\bullet\langle X\rangle \leq \langle Y\rangle\}, \text{ and}$$

$$i_\bullet\langle X\rangle \leq \langle Y\rangle \text{ if and only if } \langle X\rangle \leq r\langle Y\rangle.$$

**Proposition 2.3.3.** *If $f_\bullet i_\bullet\langle X\rangle = \langle X\rangle$ for all $\langle X\rangle$, then $f_\bullet\langle X\rangle = r\langle X\rangle$ for all $\langle X\rangle$, so*

$$\langle i_\bullet X\rangle \leq \langle Y\rangle \textit{ if and only if } \langle X\rangle \leq \langle f_\bullet Y\rangle.$$

**Proof.** Lemma 2.1.13 implies that $f_\bullet\langle X\rangle \leq r\langle X\rangle$ for all $\langle X\rangle$. For the other direction, it suffices to show that if $\langle i_\bullet X\rangle \leq \langle Y\rangle$, then $\langle X\rangle \leq \langle f_\bullet Y\rangle$.

If $\langle i_\bullet X\rangle \leq \langle Y\rangle$ and $W \wedge f_\bullet Y = 0$, then Corollary 2.1.3 implies $i_\bullet W \wedge Y = 0$, so $i_\bullet W \wedge i_\bullet X = 0$. It follows from Lemma 2.3.1 that $i_\bullet(W \wedge X) = 0$, so $W \wedge X = 0$. □

The $\mathsf{BL}$ operation $f_\bullet$ also preserves arbitrary joins, so has a poset map right adjoint. On the object level, we know that $f_\bullet$ is left adjoint to $i_\bullet$, and so it is natural to ask if $i_\bullet$ is the poset adjoint of $f_\bullet$.

**Proposition 2.3.4.** *Assume $f_\bullet i_\bullet\langle X\rangle = \langle X\rangle$ for all $X$. Then on the level of Bousfield classes, we have*

$$\langle f_\bullet X\rangle \leq \langle Y\rangle \Leftarrow \langle X\rangle \leq \langle i_\bullet Y\rangle,$$

*but the forward direction need not hold. In the quotient $\overline{f_\bullet} : \mathsf{BL}/J \to \mathsf{BL}$ we do indeed have*

$$\overline{f_\bullet}\langle X\rangle \leq \langle Y\rangle \Leftrightarrow \langle X\rangle \leq i_\bullet\langle Y\rangle,$$

*so $\overline{f_\bullet}$ and $i_\bullet$ are poset adjoints.*

**Proof.** First suppose $\langle X\rangle \leq \langle i_\bullet Y\rangle$ and $W \wedge Y = 0$. Then $i_\bullet(W \wedge Y) = 0$, which using Lemma 2.3.1 means $i_\bullet W \wedge i_\bullet Y = 0$, so $i_\bullet W \wedge X = 0$, and $W \wedge f_\bullet X = 0$.

On the other hand, suppose $\langle f_\bullet X\rangle \leq \langle Y\rangle$ and $W \wedge i_\bullet Y = 0$. Then $f_\bullet W \wedge Y = 0$, $f_\bullet W \wedge f_\bullet X = 0$, and $f_\bullet(W \wedge X) = 0$. At the $\mathsf{BL}$ level, this does not necessarily mean $W \wedge X = 0$. (Take, for example, $Y = 0$, $W = R$, and $X$ any object such that $f_\bullet X = 0$.) In the quotient, however, $\overline{f_\bullet}(W \wedge X) = 0$ does imply $W \wedge X = 0$. □

### *2.3.2* BA *and* DL

**Lemma 2.3.5.** *When $f_\bullet i_\bullet \langle X\rangle = \langle X\rangle$ for all $\langle X\rangle$, the map $i_\bullet$ sends $\mathsf{BA}_{D(S)}$ into $\mathsf{BA}_{D(R)}$. If $\langle X\rangle \in \mathsf{BA}_{D(S)}$ has complement $\langle X^c\rangle$, then $\langle i_\bullet X\rangle \in \mathsf{BA}_{D(R)}$ has complement $\langle i_\bullet(X^c)\rangle \vee \langle M\rangle$. In particular, $\langle i_\bullet S\rangle$ is complemented, with complement $\langle M\rangle$ (see Definition 2.1.10).*

**Proof.** We first show that $\langle i_\bullet S\rangle$ is complemented, with complement $\langle M\rangle$. Note that $\langle i_\bullet S\rangle \equiv \langle R\rangle (\mathrm{mod}\ J)$, because $\overline{f_\bullet}\langle i_\bullet S\rangle = \langle f_\bullet i_\bullet S\rangle = \langle S\rangle = \langle f_\bullet R\rangle = \overline{f_\bullet}\langle R\rangle$, and $\overline{f_\bullet}$ is injective on $\mathsf{BL}/J$. Therefore $\langle i_\bullet S\rangle \vee \langle M\rangle = \langle R\rangle \vee \langle M\rangle = \langle R\rangle$. On the other hand, $f_\bullet M = 0$, so $S \wedge f_\bullet M = 0$, so $i_\bullet S \wedge M = 0$.

We noted earlier that $\langle i_\bullet X\rangle \leq \langle i_\bullet S\rangle$ for all $X$. Therefore $i_\bullet X \wedge M = 0$ for all $X$.

Now suppose $\langle X\rangle \in \mathsf{BA}_{D(S)}$, so $\langle X\rangle \vee \langle X^c\rangle = \langle S\rangle$ and $\langle X\rangle \wedge \langle X^c\rangle = \langle 0\rangle$. This implies $\langle i_\bullet X\rangle \vee \langle i_\bullet X^c\rangle = \langle i_\bullet S\rangle$ and $\langle i_\bullet X\rangle \wedge \langle i_\bullet X^c\rangle = \langle 0\rangle$.

We calculate that

$$\langle i_\bullet X\rangle \vee (\langle i_\bullet X^c\rangle \vee \langle M\rangle) = \langle i_\bullet S\rangle \vee \langle M\rangle = \langle R\rangle \vee \langle M\rangle = \langle R\rangle,$$

because $\langle i_\bullet S\rangle \equiv \langle R\rangle (\mathrm{mod}\ J)$.

Also, we have

$$\langle i_\bullet X\rangle \wedge (\langle i_\bullet X^c\rangle \vee \langle M\rangle) = (\langle i_\bullet X\rangle \wedge \langle i_\bullet X^c\rangle) \vee (\langle i_\bullet X\rangle \wedge \langle M\rangle)$$

$$= \langle 0\rangle \vee (\langle i_\bullet X\rangle \wedge \langle M\rangle) = \langle 0\rangle.$$

This shows that the complement of $\langle i_\bullet X\rangle$ is $\langle i_\bullet X^c\rangle \vee \langle M\rangle$. □

**Proposition 2.3.6.** *Suppose $f_\bullet i_\bullet\langle X\rangle = \langle X\rangle$ for all $\langle X\rangle$. The following hold.*

1. *The map $f_\bullet$ sends $\mathsf{DL}_{D(R)}$ onto $\mathsf{DL}_{D(S)}$, and the map $i_\bullet$ injects $\mathsf{DL}_{D(S)}$ into $\mathsf{DL}_{D(R)}$.*

2. *The map $f_\bullet$ sends $\mathsf{BA}_{D(R)}$ onto $\mathsf{BA}_{D(S)}$, and $i_\bullet$ injects $\mathsf{BA}_{D(S)}$ into $\mathsf{BA}_{D(R)}$.*

3. *The map $\overline{f_\bullet}$ establishes a poset isomorphism between (the image of) $\mathsf{DL}$ in $\mathsf{BL}/J$ and $\mathsf{DL}$ in $D(S)$, with inverse $i_\bullet$.*

4. *The map $\overline{f_\bullet}$ establishes a poset isomorphism between (the image of) $\mathsf{BA}$ in $\mathsf{BL}/J$ and $\mathsf{BA}$ in $D(S)$, with inverse $i_\bullet$.*

**Proof.** Lemma 2.3.1 implies that if $\langle Y\rangle = \langle Y \wedge Y\rangle$, then $\langle i_\bullet Y\rangle = \langle i_\bullet Y \wedge i_\bullet Y\rangle$, so $i_\bullet$ sends $\mathsf{DL}$ to $\mathsf{DL}$. The rest follows from Propositions 2.1.14 and 2.3.2, Lemma 2.3.5, and the fact that $f_\bullet$ is surjective and $i_\bullet$ is injective. □

**Question 2.3.7.** *How do the results of this entire chapter change if we instead consider injective maps $f : R \to S$?*

## 2.4 Surjective $f : R \to S$ with $S$ Noetherian, $R$ not necessarily Noetherian; ungraded setting

**In this section, all rings and modules are ungraded.** See Section 1.3 for a discussion on grading issues.

As discussed in Sections 1.3 and 1.6, much is known about the Bousfield lattice of the derived category of a Noetherian ring $S$, in the ungraded case. Recall $\mathsf{BL}_{D(S)}$ is isomorphic to the Boolean algebra on the classes $\left\{\langle\overline{k_\mathfrak{p}}\rangle \mid \mathfrak{p} \in \mathsf{Spec}\, S\right\}$. Every localizing subcategory is a Bousfield class; every smashing localization is a finite localization; and there are classifications of localizing and thick subcategories, corresponding to subsets and specialization-closed subsets of $\mathsf{Spec}\, S$. Given $\mathfrak{p} \in \mathsf{Spec}\, S$, we have the finite Koszul object $S/\!\!/\mathfrak{p}$, and $K(\mathfrak{p}) := S_\mathfrak{p} \wedge S/\!\!/\mathfrak{p}$. Furthermore, $\mathsf{loc}(K(\mathfrak{p})) = \mathsf{loc}(\overline{k_\mathfrak{p}})$. See [HPS97, Ch.6] or [Nee92] for details.

Much less is known in the general case of a commutative (ungraded) ring. In this section let $f : R \to S$ be surjective, with $S$ Noetherian, but $R$ not necessarily Noetherian. We will attempt to use $f_\bullet$ and $i_\bullet$ to pull back structure to $D(R)$. We will construct various objects in $D(R)$ and examine their behavior under $f_\bullet$ and $i_\bullet$, and we will look at the action of $f_\bullet$ and $i_\bullet$ on the collection of thick subcategories of finite objects in $D(R)$.

First, we can strengthen Proposition 2.2.3.

**Proposition 2.4.1.** *Let $\mathfrak{p} \subseteq S$ be a prime ideal. Then $f_\bullet(\overline{k_{f^{-1}\mathfrak{p}}})$ and $\overline{k_\mathfrak{p}}$ generate the same localizing subcategory, and $\langle f_\bullet(\overline{k_{f^{-1}\mathfrak{p}}})\rangle = \langle\overline{k_\mathfrak{p}}\rangle$.*

**Proof.** In Proposition 2.2.3 we showed that $f_\bullet(\overline{k_{f^{-1}\mathfrak{p}}})$ is in $\mathsf{loc}(\overline{k_\mathfrak{p}})$, and the proof made it clear that $f_\bullet(\overline{k_{f^{-1}\mathfrak{p}}})$ is nonzero. When $S$ is Noetherian, $\mathsf{loc}(\overline{k_\mathfrak{p}})$ is a minimal nonzero localizing subcategory, so $0 \neq \mathsf{loc}(f_\bullet(\overline{k_{f^{-1}\mathfrak{p}}})) \subseteq \mathsf{loc}(\overline{k_\mathfrak{p}})$ implies equality. It is always the case that $\mathsf{loc}(X) = \mathsf{loc}(Y)$ implies $\langle X\rangle = \langle Y\rangle$. $\square$

### *2.4.1 Koszul and $K(\mathfrak{p})$ objects in $D(R)$*

Now we define an analog of Koszul and $K(\mathfrak{p})$ objects for $D(R)$, and investigate their properties. Recall that in this section we are assuming $f : R \to S$ is surjective and $S$ is Noetherian.

**Definition 2.4.2.** Let $\mathfrak{p} \subseteq S$ be a prime ideal. Since $S$ is Noetherian, choose generators $\mathfrak{p} = (z_1, ..., z_n)$. Since $f$ is surjective, we can choose $y_i \in R$ such that $f(y_i) = z_i$ for all $i$. Define $\tilde{\mathfrak{p}} = (y_1, ..., y_n) \subseteq R$. Define the object $R/\!\!/\tilde{\mathfrak{p}} = R/y_1 \wedge R/y_2 \wedge \cdots \wedge R/y_n$. Then define $K(\tilde{\mathfrak{p}}) = R/\!\!/\tilde{\mathfrak{p}} \wedge R_{f^{-1}\mathfrak{p}}$. Different choices of preimages yield different $R/\!\!/\tilde{\mathfrak{p}}$, as the following example illustrates.

In general, $\tilde{\mathfrak{p}}$ need not be a prime ideal of $R$. Note that for every possible choice of $y_i$'s, we will always have $\tilde{\mathfrak{p}} \subseteq f^{-1}(\mathfrak{p})$. Also, note that $f(\tilde{\mathfrak{p}}) = \mathfrak{p}$.

**Example 2.4.3.** Consider the projection $f : \mathbb{Z}_{(p)} \to \mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)} = \mathbb{F}_p$. Let $\mathfrak{p} = (0) \subseteq \mathbb{F}_p$. If we choose $0 \in f^{-1}(0)$, then $\tilde{\mathfrak{p}} = (0)$ and $R/\!\!/\tilde{\mathfrak{p}}$ is the complex

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z}_{(p)} \xrightarrow{0} \mathbb{Z}_{(p)} \longrightarrow 0 \longrightarrow \cdots ,$$

which is $\mathbb{Z}_{(p)} \oplus \Sigma\mathbb{Z}_{(p)}$.

If instead we choose $p \in f^{-1}(0)$, then $\tilde{\mathfrak{p}}' = (p)$ and $R/\!\!/\tilde{\mathfrak{p}}'$ is

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z}_{(p)} \xrightarrow{p} \mathbb{Z}_{(p)} \longrightarrow 0 \longrightarrow \cdots ,$$

which is $\mathbb{F}_p$.

Therefore $R/\!\!/\tilde{\mathfrak{p}}$ and $R/\!\!/\tilde{\mathfrak{p}}'$ are not equal, nor do they generate the same thick subcategory.

Since $f^{-1}\mathfrak{p} = (p)$, and $\mathbb{Z}_{(p)}$ is $(p)$-local, $K(\tilde{\mathfrak{p}}) = R/\!\!/\tilde{\mathfrak{p}}$ and $K(\tilde{\mathfrak{p}}') = R/\!\!/\tilde{\mathfrak{p}}'$. Thus in this example, $K(\tilde{\mathfrak{p}})$ and $K(\tilde{\mathfrak{p}}')$ are not equal, nor do they generate the same thick or localizing subcategories. Furthermore, $\langle K(\tilde{\mathfrak{p}})\rangle = \langle \mathbb{Z}_{(p)}\rangle$, and $\langle K(\tilde{\mathfrak{p}}')\rangle = \langle \mathbb{F}_p\rangle$, and these are not equal because, for example, $\mathbb{Q} \wedge \mathbb{F}_p = 0$.

However, in Lemma 2.4.5 we'll show that $\langle K(\tilde{\mathfrak{p}})\rangle$ is well-defined in $\mathsf{BL}/J$, independent of choice of $\tilde{\mathfrak{p}}$.

**Lemma 2.4.4.** *For all $\mathfrak{p} \in \mathsf{Spec}\, S$ and all choices of $\tilde{\mathfrak{p}}$, we have $f_\bullet(R/\!\!/\tilde{\mathfrak{p}}) = S/\!\!/\mathfrak{p}$ and $f_\bullet K(\tilde{\mathfrak{p}}) = K(\mathfrak{p})$.*

**Proof.** For each $i$, if we apply $f_\bullet$ to the cofiber sequence $R \xrightarrow{y_i} R \longrightarrow R/y_i$, we get (see Remark 2.1.4)

$$\left(f_\bullet R \xrightarrow{f_\bullet y_i} f_\bullet R \longrightarrow f_\bullet(R/y_i)\right)$$

$$= \left(S \xrightarrow{f(y_i)} S \longrightarrow f_\bullet(R/y_i)\right)$$
$$= \left(S \xrightarrow{z_i} S \longrightarrow f_\bullet(R/y_i)\right).$$

Therefore $f_\bullet(R/y_i) = S/z_i$. Then using Lemma 2.2.2 we have

$$f_\bullet K(\tilde{\mathfrak{p}}) = f_\bullet \left(R_{f^{-1}(\mathfrak{p})} \wedge R/y_1 \wedge \cdots \wedge R/y_n\right)$$

$$= f_\bullet R_{f^{-1}(\mathfrak{p})} \wedge f_\bullet(R/y_1) \wedge \cdots \wedge f_\bullet(R/y_n)$$

$$= S_\mathfrak{p} \wedge S/z_1 \wedge \cdots S/z_n = K(\mathfrak{p}).$$

□

**Lemma 2.4.5.** *For all* $\mathfrak{p} \in \mathsf{Spec}\, S$ *and all choices of* $\tilde{\mathfrak{p}}$*, in* $\mathsf{BL}_{D(R)}$ *we have* $\langle i_\bullet K(\mathfrak{p})\rangle \le \langle K(\tilde{\mathfrak{p}})\rangle$*, and in* $\mathsf{BL}/J$ *we have* $\langle i_\bullet K(\mathfrak{p})\rangle = \langle K(\tilde{\mathfrak{p}})\rangle$*. Therefore* $\langle K(\tilde{\mathfrak{p}})\rangle$ *is well-defined in* $\mathsf{BL}/J$*, independent of choice of* $\tilde{\mathfrak{p}}$*.*

**Proof.** Suppose $X \wedge K(\tilde{\mathfrak{p}}) = 0$. Then $f_\bullet(X \wedge K(\tilde{\mathfrak{p}})) = f_\bullet X \wedge K(\mathfrak{p}) = 0$, so $X \wedge i_\bullet K(\mathfrak{p}) = 0$.

In $\mathsf{BL}/J$, $\overline{f_\bullet}$ is injective, so we can follow the logic in the other way: if $X \wedge i_\bullet K(\mathfrak{p}) = 0$, then $f_\bullet X \wedge K(\mathfrak{p}) = 0$, so $\overline{f_\bullet}(X \wedge K(\tilde{\mathfrak{p}})) = 0$, so $X \wedge K(\tilde{\mathfrak{p}}) = 0$. □

**Proposition 2.4.6.** *For all* $X$ *in* $D(S)$*, we have* $f_\bullet i_\bullet \langle X\rangle = \langle X\rangle$*.*

**Proof.** Every $\langle X\rangle$ in $\mathsf{BL}_{D(S)}$ is the join of some $\langle K(\mathfrak{p})\rangle$'s. Since $i_\bullet$ and $f_\bullet$ both preserve joins, we can reduce to the case where $\langle X\rangle = \langle K(\mathfrak{p})\rangle$. Lemma 2.4.5 showed $\langle i_\bullet K(\mathfrak{p})\rangle = \langle K(\tilde{\mathfrak{p}})\rangle$ in $\mathsf{BL}/J$. Therefore $\overline{f_\bullet}\langle i_\bullet K(\mathfrak{p})\rangle = \overline{f_\bullet}\langle K(\tilde{\mathfrak{p}})\rangle$, so $\langle f_\bullet i_\bullet K(\mathfrak{p})\rangle = \langle K(\mathfrak{p})\rangle$. □

This proposition allows us to apply the results of Section 2.3. Proposition 2.3.2 gives a poset isomorphism $\overline{f_\bullet} : \mathsf{BL}/J \to \mathsf{BL}_{D(S)}$, with inverse $i_\bullet$. This immediately implies the following.

**Proposition 2.4.7.** *In* $\mathsf{BL}/J$ *the following hold.*

1. *If* $\mathfrak{p} \neq \mathfrak{p}'$*, then* $\langle K(\tilde{\mathfrak{p}})\rangle \wedge \langle K(\tilde{\mathfrak{p}}')\rangle = \langle 0\rangle$*.*

2. $\langle K(\tilde{\mathfrak{p}})\rangle \wedge \langle K(\tilde{\mathfrak{p}})\rangle = \langle K(\tilde{\mathfrak{p}})\rangle$*.*

3. $\langle K(\tilde{\mathfrak{p}})\rangle$ *is a minimal nonzero Bousfield class.*

4. $\langle R\rangle = \coprod_{p \in \mathsf{Spec}\, S} \langle K(\tilde{\mathfrak{p}})\rangle$*.*

**Proof.** All the proofs work by pushing into $\mathsf{DL}_{D(S)}$. For example, we'll prove (4). Suppose $X$ in $D(R)$ has $X \wedge K(\tilde{\mathfrak{p}}) = 0$ for all $\mathfrak{p} \in \mathsf{Spec}\, S$. We want to show that $X = 0$. But we know $f_\bullet X \wedge f_\bullet K(\tilde{\mathfrak{p}}) = 0$, which is to say $f_\bullet X \wedge K(\mathfrak{p}) = 0$ for all $\mathfrak{p} \in \mathsf{Spec}\, S$. Since $\langle S\rangle = \coprod_{\mathsf{Spec}\, S} \langle K(\mathfrak{p})\rangle$, this means $f_\bullet X = 0$. Since $f_\bullet$ is injective on $\mathsf{BL}/J$, we have $X = 0$. ☐

**Question 2.4.8.** *To what extent can these results be pulled back to* $\mathsf{BL}_{D(R)}$*?*

For example, (1) might hold in $\mathsf{BL}_{D(R)}$, but (4) most definitely does not. It would be especially interesting to prove or disprove (3) in $\mathsf{BL}_{D(R)}$. The corresponding results about $\langle K(\mathfrak{p})\rangle$ in $\mathsf{BL}_{D(S)}$ almost completely describe $\mathsf{BL}_{D(S)}$; any progress in answering the above question would contribute significantly to our understanding in the non-Noetherian case.

The following lemmas will be used later.

**Lemma 2.4.9.** *Let* $\tilde{\mathfrak{p}} = (y_1, ..., y_n)$ *be as in Definition 2.4.2. Then*

$$\langle R /\!\!/ \tilde{\mathfrak{p}}\rangle \leq \cdots \leq \langle R/(y_1, y_2)\rangle \leq \langle R/y_1\rangle.$$

**Proof.** This comes from repeated application of Lemma 2.1.6. For example, $(y_1) \subseteq (y_1, y_2)$ implies that $R/(y_1, y_2) \in \mathsf{th}(R/y_1)$, so $\langle R/(y_1, y_2)\rangle \leq \langle R/y_1\rangle$. ☐

**Lemma 2.4.10.** *If* $\mathfrak{p} \not\subseteq \mathfrak{p}'$ *are prime ideals of* $S$*, then*

$$R /\!\!/ \tilde{\mathfrak{p}} \wedge R_{f^{-1}(\mathfrak{p}')} = 0 \text{ and } R /\!\!/ \tilde{\mathfrak{p}} \wedge \overline{k_{f^{-1}(\mathfrak{p}')}} = 0.$$

**Proof.** Take $z_1 \in \mathfrak{p}\backslash\mathfrak{p}'$, and complete it with $z_i$ so that $(z_1, z_2, ..., z_n) = \mathfrak{p}$. For each $i$, choose $y_i \in f^{-1}(z_i)$, and define $\tilde{\mathfrak{p}} = (y_1, ..., y_n)$. Note that $y_1 \in f^{-1}(\mathfrak{p})$ but $y_1 \notin f^{-1}(\mathfrak{p}')$.

Now, in $D(R)$, the object $R/y_1$ is represented by the chain complex

$$\cdots \to 0 \to R \xrightarrow{y_1} R \to 0 \to \cdots .$$

Therefore $R/y_1 \wedge R_{f^{-1}(\mathfrak{p}')}$ is represented by the chain complex

$$\cdots \to 0 \to R_{f^{-1}(\mathfrak{p}')} \xrightarrow{y_1} R_{f^{-1}(\mathfrak{p}')} \to 0 \to \cdots .$$

Since $y_1 \in R\backslash f^{-1}(\mathfrak{p}')$, it is invertible in $R_{f^{-1}(\mathfrak{p}')}$. Therefore the above complex is acyclic, and $R/y_1 \wedge R_{f^{-1}(\mathfrak{p}')} = 0$.

The previous lemma tells us that $\langle R/\!\!/\tilde{\mathfrak{p}}\rangle \leq \langle R/y_1\rangle$, so $R/\!\!/\tilde{\mathfrak{p}} \wedge R_{f^{-1}(\mathfrak{p}')} = 0$.

Similarly, $R/y_1 \wedge \overline{k_{f^{-1}(\mathfrak{p}')}}$ is represented by the chain complex

$$\cdots \to 0 \to \frac{R_{f^{-1}(\mathfrak{p}')}}{f^{-1}(\mathfrak{p}')R_{f^{-1}(\mathfrak{p}')}} \xrightarrow{y_1} \frac{R_{f^{-1}(\mathfrak{p}')}}{f^{-1}(\mathfrak{p}')R_{f^{-1}(\mathfrak{p}')}} \to 0 \to \cdots .$$

The map $y_1$ is also invertible here, so this chain complex is acyclic, and we conclude $R/\!\!/\tilde{\mathfrak{p}} \wedge \overline{k_{f^{-1}(\mathfrak{p}')}} = 0$. □

### *2.4.2 Some support calculations*

Thomason [Tho97] gave a way of understanding thick subcategories of finite objects in $D(R)$, via subsets of $\mathsf{Spec}\, R$ and support. Here we first calculate the support of $R/\!\!/\tilde{\mathfrak{p}}$ and $i_\bullet(S/\!\!/\mathfrak{p})$, and in fact $\mathrm{supp}(i_\bullet X)$ for all $X$ in $D(S)$ .

Recall that, for the Noetherian ring $S$, the classification gives a one-to-one correspondence between thick subcategories of finite objects in $D(S)$ and specialization-closed subsets of $\mathsf{Spec}\, S$, via the notion of support. For an object $X$ in $D(S)$, $\mathrm{supp}(X) = \{\mathfrak{p} \in \mathsf{Spec}\, S \;:\; \overline{k_\mathfrak{p}} \wedge X \neq 0\}$. A specialization-closed subset $W \subseteq \mathsf{Spec}\, S$

corresponds to

$$\mathsf{th}\,(S/\!\!/\mathfrak{p}\ :\ \mathfrak{p} \in W) = \{X \text{ in } \mathcal{F}\ :\ \operatorname{supp}(X) \subseteq W\},$$

and a thick subcategory $\mathsf{A}$ corresponds to

$$\operatorname{supp}(\mathsf{A}) = \{\mathfrak{p} \in \mathsf{Spec}\, S\ :\ \text{ there exists } X \text{ in } \mathsf{A} \text{ with } \mathfrak{p} \in \operatorname{supp}(X)\}.$$

For a commutative ring $R$ that is not necessarily Noetherian, we replace specialization-closed subsets with *Thomason-closed* subsets of $\mathsf{Spec}\, R$, which are those of the form $\bigcup_\alpha V(I_\alpha)$, where each $I_\alpha$ is finitely generated.

**Notation 2.4.11.** Throughout Sections 2.4 and 2.5, fix

$$T := f^{-1}(\mathsf{Spec}\, S) \subseteq \mathsf{Spec}\, R, \text{ and } U := (\mathsf{Spec}\, R)\backslash T.$$

Given an ideal $I$ in a ring $R$, let $V(I)$ denote the closure of $I$ in $\mathsf{Spec}\, R$. That is

$$V(I) = \{\mathfrak{p} \in \mathsf{Spec}\, R\ :\ I \subseteq \mathfrak{p}\}.$$

Our computations rely on the following observation.

**Lemma 2.4.12.** *Given* $\mathfrak{q} \in \mathsf{Spec}\, R$*, we have*

$$\mathsf{loc}(f_\bullet \overline{k_\mathfrak{q}}) = \begin{cases} \mathsf{loc}(\overline{k_{f(\mathfrak{q})}}), & \textit{if } \mathfrak{q} \in T \\ 0, & \textit{if } \mathfrak{q} \in U \end{cases}$$

**Proof.** In Proposition 2.4.1 we showed that $\mathsf{loc}(f_\bullet(\overline{k_{f^{-1}\mathfrak{p}}})) = \mathsf{loc}(\overline{k_\mathfrak{p}})$. Now take $\mathfrak{q} \in U$. Then for all $\mathfrak{p} \in \mathsf{Spec}\, S$, $f^{-1}\mathfrak{p} \neq \mathfrak{q}$. By Lemma 2.1.7, $\overline{k_{f^{-1}\mathfrak{p}}} \wedge \overline{k_\mathfrak{q}} = 0$. Therefore

$$0 = f_\bullet(0) = f_\bullet(\overline{k_{f^{-1}\mathfrak{p}}} \wedge \overline{k_\mathfrak{q}}) = f_\bullet \overline{k_{f^{-1}\mathfrak{p}}} \wedge f_\bullet \overline{k_\mathfrak{q}} = \overline{k_\mathfrak{p}} \wedge f_\bullet \overline{k_\mathfrak{q}},$$

for all $\mathfrak{p} \in \mathsf{Spec}\, S$. Since $S$ is Noetherian, we have

$$\langle S \rangle = \coprod_{\mathfrak{p} \in \mathsf{Spec}\, S} \langle \overline{k_\mathfrak{p}} \rangle.$$

This implies that $f_\bullet \overline{k_\mathfrak{q}} = f_\bullet \overline{k_\mathfrak{q}} \wedge S = 0$. ☐

**Proposition 2.4.13.** *Let $f : R \to S$ be a surjective ring map with $S$ Noetherian. Let $X$ be an arbitrary object of $D(S)$. Then we have*

$$\operatorname{supp}(i_\bullet X) = f^{-1}(\operatorname{supp}(X)).$$

**Proof.** A prime $\mathfrak{q}$ is in $\operatorname{supp}(i_\bullet X)$ if and only if $\overline{k_\mathfrak{q}} \wedge i_\bullet X \neq 0$. The projection formula says $\overline{k_\mathfrak{q}} \wedge i_\bullet X = i_\bullet(f_\bullet \overline{k_\mathfrak{q}} \wedge X)$, and $i_\bullet$ is injective, so $\mathfrak{q}$ is in $\operatorname{supp}(i_\bullet X)$ if and only if $f_\bullet \overline{k_\mathfrak{q}} \wedge X \neq 0$.

If $f_\bullet \overline{k_\mathfrak{q}} \wedge X \neq 0$ then $f_\bullet \overline{k_\mathfrak{q}} \neq 0$, so Lemma 2.4.12 forces $\mathfrak{q} \in f^{-1}(\mathsf{Spec}\, S)$, say $\mathfrak{q} = f^{-1}\mathfrak{p}$. By Proposition 2.4.1, $\langle f_\bullet \overline{k_\mathfrak{q}} \rangle = \langle \overline{k_\mathfrak{p}} \rangle$, so $\overline{k_\mathfrak{p}} \wedge X \neq 0$. Thus $\mathfrak{p} \in \operatorname{supp} X$, and $\mathfrak{q} = f^{-1}\mathfrak{p} \in f^{-1}(\operatorname{supp} X)$.

If $\mathfrak{q} = f^{-1}\mathfrak{p} \in f^{-1}(\operatorname{supp} X)$ then again by Proposition 2.4.1, $\overline{k_\mathfrak{p}} \wedge X \neq 0$ implies $f_\bullet \overline{k_\mathfrak{q}} \wedge X \neq 0$. □

**Corollary 2.4.14.** *Take $\mathfrak{p} \in \mathsf{Spec}\, S$. Then*

$$\operatorname{supp}(i_\bullet(S /\!\!/ \mathfrak{p})) = f^{-1}(V(\mathfrak{p})).$$

**Proof.** Proposition 6.1.7(c) in [HPS97] shows that $\operatorname{supp}(S /\!\!/ \mathfrak{p}) = V(\mathfrak{p})$. □

**Lemma 2.4.15.** *Take $\mathfrak{p} \in \mathsf{Spec}\, S$. Then*

$$f^{-1}(V(\mathfrak{p})) = V(f^{-1}(\mathfrak{p})).$$

**Proof.** Given $\mathfrak{q} \in f^{-1}(V(\mathfrak{p}))$, we have $\mathfrak{q} = f^{-1}(\mathfrak{r})$ for some $\mathfrak{r}$ with $\mathfrak{p} \subseteq \mathfrak{r}$. Then $f^{-1}(\mathfrak{p}) \subseteq f^{-1}(\mathfrak{r}) = \mathfrak{q}$. Thus $\mathfrak{q} \in V(f^{-1}(\mathfrak{p}))$.

It's clear that $f^{-1}\mathfrak{p} \in f^{-1}(V(\mathfrak{p}))$. Since $V(\mathfrak{p})$ is a closed subset of $\mathsf{Spec}\, S$, and $f^{-1}$ is a homeomorphism onto its image in $\mathsf{Spec}\, R$ (see Lemma 2.2.1), then $f^{-1}(V(\mathfrak{p}))$ is a closed subset of $\mathsf{Spec}\, R$. This implies that the closure $V(f^{-1}(\mathfrak{p}))$ is contained in $f^{-1}(V(\mathfrak{p}))$. □

**Lemma 2.4.16.** *Take $\mathfrak{p} \in \mathsf{Spec}\, S$. Then for all choices of $\tilde{\mathfrak{p}}$ we have*

$$f^{-1}(V(\mathfrak{p})) = V(\tilde{\mathfrak{p}}) \cap T.$$

**Proof.** Take $\mathfrak{q} \in V(\tilde{\mathfrak{p}}) \cap T$, and take $\mathfrak{r} \in \mathsf{Spec}\, S$ so $f^{-1}(\mathfrak{r}) = \mathfrak{q}$. Then $\tilde{\mathfrak{p}} \subseteq \mathfrak{q}$, so $\mathfrak{p} = f(\tilde{\mathfrak{p}}) \subseteq f(\mathfrak{q}) = \mathfrak{r}$. Thus $\mathfrak{r} \in V(\mathfrak{p})$, so $\mathfrak{q} \in f^{-1}(V(\mathfrak{p}))$.

Since $f^{-1}(V(\mathfrak{p})) \subseteq f^{-1}(\mathsf{Spec}\, S)$, it remains to show that $f^{-1}(V(\mathfrak{p})) \subseteq V(\tilde{\mathfrak{p}})$. In light of Corollary 2.4.14, take $\mathfrak{q} \in \mathrm{supp}(i(S/\!\!/\mathfrak{p}))$, so $i(S/\!\!/\mathfrak{p}) \wedge \overline{k_{\mathfrak{q}}} \neq 0$. Suppose $\mathfrak{p} = (z_1, ..., z_n)$. We will show that every choice of $f^{-1}(z_i)$ is in $\mathfrak{q}$ for all $i$; this will imply that $\tilde{\mathfrak{p}} \subseteq \mathfrak{q}$, so $\mathfrak{q} \in V(\tilde{\mathfrak{p}})$.

Suppose, towards a contradiction, that there is some $f^{-1}(z_i)$ not contained in $\mathfrak{q}$. As a map of $R$-modules,

$$R_{\mathfrak{q}} \xrightarrow{f^{-1}(z_i)} R_{\mathfrak{q}}$$

is an isomorphism; as a map of objects in $D(R)$ it is an equivalence. Therefore applying $f_\bullet$,

$$f_\bullet R_{\mathfrak{q}} \xrightarrow{z_i} f_\bullet R_{\mathfrak{q}}$$

is an equivalence. Applying $- \wedge f_\bullet(R/\mathfrak{q}R)$ and noting that $f_\bullet R_{\mathfrak{q}} \wedge f_\bullet(R/\mathfrak{q}R) = f_\bullet \overline{k_{\mathfrak{q}}}$ (since $R_{\mathfrak{q}}$ is a flat $R$-module), we see that $f_\bullet \overline{k_{\mathfrak{q}}} \xrightarrow{z_i} f_\bullet \overline{k_{\mathfrak{q}}}$ is an equivalence. The cofiber $S/z_i \wedge f_\bullet \overline{k_{\mathfrak{q}}}$ is zero. Smashing with the other $S/z_j$, we see that $S/\!\!/\mathfrak{p} \wedge f_\bullet \overline{k_{\mathfrak{q}}} = 0$. But $i_\bullet(S/\!\!/\mathfrak{p} \wedge f_\bullet \overline{k_{\mathfrak{q}}}) = i(S/\!\!/\mathfrak{p}) \wedge \overline{k_{\mathfrak{q}}} \neq 0$ by hypothesis. □

In a Noetherian ring $S$, the Koszul objects $S/\!\!/\mathfrak{p}$ have $\mathrm{supp}(S/\!\!/\mathfrak{p}) = V(\mathfrak{p})$. The next proposition is an extension of this to our more general setting. Combined with the previous lemmas and corollary, this proposition gives a good picture of the difference between $i_\bullet(S/\!\!/\mathfrak{p})$ and the objects $R/\!\!/\tilde{\mathfrak{p}}$.

**Proposition 2.4.17.** *Take $\mathfrak{p} \in \mathsf{Spec}\, S$. For all choices of $R/\!\!/\tilde{\mathfrak{p}}$, we have*

$$\mathrm{supp}(R/\!\!/\tilde{\mathfrak{p}}) = V(\tilde{\mathfrak{p}}).$$

**Proof.** First we show the $\subseteq$ direction. Suppose $\mathfrak{q} \in \mathsf{Spec}\, R$ has $R/\!\!/\tilde{\mathfrak{p}} \wedge \overline{k_{\mathfrak{q}}} \neq 0$. We must show that $\tilde{\mathfrak{p}} \subseteq \mathfrak{q}$.

Suppose $\mathfrak{p} = (z_1, ..., z_n)$, and let $y_i \in f^{-1}(z_i)$, for $1 \leq i \leq n$, be choices of preimages, so that $\tilde{\mathfrak{p}} = (y_1, ..., y_n)$. Then for each $i$, Lemma 2.4.10 implies that $R/y_i \wedge \overline{k_{\mathfrak{q}}} \neq 0$.

If we consider the triangle

$$R \wedge \overline{k_{\mathfrak{q}}} \xrightarrow{y_i \wedge 1} R \wedge \overline{k_{\mathfrak{q}}} \longrightarrow R/y_i \wedge \overline{k_{\mathfrak{q}}},$$

we see that the map

$$\overline{k_{\mathfrak{q}}} \xrightarrow{y_i} \overline{k_{\mathfrak{q}}}$$

is not an equivalence. Thus as a map of $R$-modules, $y_i : k_{\mathfrak{q}} \to k_{\mathfrak{q}}$ is not an isomorphism. But $y_i \in R$, and $k_{\mathfrak{q}}$ is $R_{\mathfrak{q}}$-local, so everything in $R \backslash \mathfrak{q}$ is invertible. Therefore $y_i \in \mathfrak{q}$. Since this is true for all $i$, $\tilde{\mathfrak{p}} \subseteq \mathfrak{q}$.

Now we show the $\supseteq$ direction. Let $\mathfrak{q}$ be a prime ideal of $R$ such that $\tilde{\mathfrak{p}} \subseteq \mathfrak{q}$. We must show that $R/\!\!/\tilde{\mathfrak{p}} \wedge \overline{k_{\mathfrak{q}}} \neq 0$. As above, suppose $\tilde{\mathfrak{p}} = (y_1, ..., y_n)$. Since each $y_i \in \tilde{\mathfrak{p}} \subseteq \mathfrak{q}$, the map $\overline{k_{\mathfrak{q}}} \xrightarrow{y_i} \overline{k_{\mathfrak{q}}}$ is zero. Then the triangles

$$R \wedge \overline{k_{\mathfrak{q}}} \xrightarrow{y_i \wedge 1} R \wedge \overline{k_{\mathfrak{q}}} \longrightarrow R/y_i \wedge \overline{k_{\mathfrak{q}}}$$

$$\overline{k_{\mathfrak{q}}} \xrightarrow{0} \overline{k_{\mathfrak{q}}} \longrightarrow k_{\mathfrak{q}} \oplus \Sigma k_{\mathfrak{q}}$$

show that $R/y_i \wedge \overline{k_{\mathfrak{q}}}$ is quasi-isomorphic to $k_{\mathfrak{q}} \oplus \Sigma k_{\mathfrak{q}}$. Therefore $R/\!\!/\tilde{\mathfrak{p}} \wedge \overline{k_{\mathfrak{q}}}$ is quasi-isomorphic to a direct sum of $2^n$ copies of $k_{\mathfrak{p}}$, and is nonzero. $\square$

**Example 2.4.18.** With the same notation as the previous example (2.4.3), where $f : \mathbb{Z}_{(p)} \to \mathbb{F}_p$ and $\mathfrak{p} = (0)$, we have

$$\text{supp}(R/\!\!/\tilde{\mathfrak{p}}) = V(\tilde{\mathfrak{p}}) = V((0)) = \{(0), (p)\} = \mathsf{Spec}\, \mathbb{F}_p, \text{and}$$

$$\text{supp}(R/\!\!/\tilde{\mathfrak{p}}') = V((p)) = \{(p)\}.$$

These two sets intersect with $T = \{(p)\}$ to give $f^{-1}(V(\mathfrak{p})) = \{(p)\}$.

Furthermore,

$$i_\bullet(S/\!\!/\mathfrak{p}) = i_\bullet(\mathbb{F}_p/(0)) = \mathbb{F}_p \oplus \Sigma\mathbb{F}_p,$$

so

$$\mathsf{th}(i_\bullet(S/\!\!/\mathfrak{p})) = \mathsf{th}(\mathbb{F}_p) = \mathsf{th}(R/\!\!/\tilde{\mathfrak{p}}').$$

Therefore, as in Corollary 2.4.14,

$$\text{supp}(i_\bullet(S/\!\!/\mathfrak{p}) = \text{supp}(R/\!\!/\tilde{\mathfrak{p}}') = \{(p)\} = f^{-1}(V(\mathfrak{p})).$$

### *2.4.3 Thick subcategories*

Now we will look at how $f_\bullet$ and $i_\bullet$ relate the thick subcategories of $D(R)$ and $D(S)$, and how they work with $f^{-1} : \mathsf{Spec}\, S \to \mathsf{Spec}\, R$. The following diagram may be helpful.

$$\begin{array}{ccc}
\left\{\begin{array}{c}\text{nice subsets}\\ \text{of } \mathsf{Spec}\, R\end{array}\right\} & \xleftarrow{\quad f^{-1}\quad} & \left\{\begin{array}{c}\text{nice subsets}\\ \text{of } \mathsf{Spec}\, S\end{array}\right\} \\
{\scriptstyle supp(-)}\upharpoonleft\downharpoonright & & {\scriptstyle supp(-)}\upharpoonleft\downharpoonright \\
\left\{\begin{array}{c}\text{thick subcategories of}\\ \text{finite objects in } D(R)\end{array}\right\} & \underset{i_\bullet}{\overset{f_\bullet}{\rightleftarrows}} & \left\{\begin{array}{c}\text{thick subcategories of}\\ \text{finite objects in } D(S)\end{array}\right\}
\end{array}$$

In the next section, where we assume $R$ is Noetherian, we will improve on these results and extend them to localizing subcategory classification. With this in mind, we give the following definitions.

**Definition 2.4.19.** Given a thick subcategory $\mathsf{A}$ of finite objects in $D(R)$, define $f_\bullet\mathsf{A}$ to be the intersection of all thick subcategories of finite objects in $D(S)$ that contain $f_\bullet X$ for all $X$ in $\mathsf{A}$. Given localizing subcategories $\mathsf{C}$ of $D(R)$ and $\mathsf{D}$ of $D(S)$, define $f_\bullet\mathsf{C}$ and $i_\bullet\mathsf{D}$ similarly.

We know that $f_\bullet$ takes finite objects to finite objects, but in general $i_\bullet$ does not, so we do not have a nice operation of $i_\bullet$ on thick subcategories of finite objects of $D(S)$. For this, it suffices to have $i_\bullet S$ finite in $D(R)$, and we will add this hypotheses at the end of this subsection.

Note that for principal subcategories we have $f_\bullet(\mathsf{th}(X)) = \mathsf{th}(f_\bullet X)$ and $f_\bullet(\mathsf{loc}(X)) = \mathsf{loc}(f_\bullet X)$, and likewise for $i_\bullet$. Also, since the full subcategory of finite objects is essentially small in the derived category of any commutative ring, if $\mathsf{A}$ is a thick subcategory of finite objects, we can identify $f_\bullet\mathsf{A}$ with the thick subcategory generated by the set $\{f_\bullet X \ : \ X \in \mathsf{A}\}$.

**Proposition 2.4.20.** *Let $f : R \to S$ be a surjective ring map, with $S$ Noetherian. Recall that $T := f^{-1}(\mathsf{Spec}\, S)$. Let $\mathsf{B}$ be a thick subcategory of finite objects in $D(R)$, and let $X$ be an arbitrary object in $D(R)$. Then*

$$f^{-1}(\mathrm{supp}(f_\bullet X)) \subseteq \mathrm{supp}(X) \cap T, \ \textit{and}\ f^{-1}(\mathrm{supp}(f_\bullet \mathsf{B})) \subseteq \mathrm{supp}(\mathsf{B}) \cap T.$$

*Equality holds when $X = R/\!\!/\tilde{\mathfrak{p}}$, for all choices of $R/\!\!/\tilde{\mathfrak{p}}$, or when $X = i_\bullet Y$ for some $Y$ in $D(S)$.*

**Proof.** Given $\mathfrak{q} = f^{-1}(\mathfrak{r}) \in f^{-1}(\mathrm{supp}(f_\bullet X))$, $f_\bullet X \wedge \overline{k_{\mathfrak{r}}} \neq 0$ so Proposition 2.4.1 implies $f_\bullet X \wedge f_\bullet \overline{k_{\mathfrak{q}}} \neq 0$. Therefore $X \wedge \overline{k_{\mathfrak{q}}} \neq 0$, and $\mathfrak{q} \in \mathrm{supp}(X) \cap T$.

Because $f_\bullet\mathsf{B}$ is the thick subcategory generated by all the $f_\bullet X$, $X \in \mathsf{B}$, we have that

$$Y \wedge f_\bullet X = 0 \text{ for all } X \in \mathsf{B} \text{ if and only if } Y \wedge W = 0 \text{ for all } W \in f_\bullet\mathsf{B}.$$

Therefore, given $\mathfrak{q} =\in f^{-1}(\mathrm{supp}(f_\bullet\mathsf{B}))$, there is some $X \in \mathsf{B}$ with

$$\mathfrak{q} \in f^{-1}(\mathrm{supp}(f_\bullet X)) \subseteq \mathrm{supp}(X) \cap T \subseteq \mathrm{supp}(\mathsf{B}) \cap T.$$

When $X = R/\!\!/\tilde{\mathfrak{p}}$, we have

$$f^{-1}(\mathrm{supp}(f_\bullet(R/\!\!/\tilde{\mathfrak{p}}))) = f^{-1}(\mathrm{supp}(S/\!\!/\mathfrak{p})) = f^{-1}(V(\mathfrak{p})) = \mathrm{supp}(R/\!\!/\tilde{\mathfrak{p}}) \cap T.$$

Now suppose $X = i_\bullet Y$ for some $Y \in D(S)$, and take $\mathfrak{q} = f^{-1}(\mathfrak{r}) \in \mathrm{supp}(X) \cap T$. Then $i_\bullet Y \wedge \overline{k_\mathfrak{q}} \neq 0$, and the projection formula implies that $Y \wedge f_\bullet \overline{k_\mathfrak{q}} \neq 0$ so $Y \wedge \overline{k_\mathfrak{r}} \neq 0$. By the injectivity of $i_\bullet$, and Lemma 2.3.1, this implies $i_\bullet Y \wedge i_\bullet \overline{k_\mathfrak{r}} \neq 0$. Again, the projection formula gives $i_\bullet(f_\bullet X \wedge \overline{k_\mathfrak{r}}) \neq 0$, so $f_\bullet X \wedge \overline{k_\mathfrak{r}} \neq 0$, and $\mathfrak{q} = f^{-1}(\mathfrak{r}) \in f^{-1}(\mathrm{supp}(f_\bullet X))$. □

**Question 2.4.21.** *For what class of objects in D(R), and what class of thick subcategories of D(R), are the inclusions in Proposition 2.4.20 an equality?*

Given a specialization-closed subset $W \subseteq \mathsf{Spec}\, S$, we would like to have $f^{-1}(W) \subseteq \mathsf{Spec}\, R$ a Thomason-closed subset. In general this is not the case. For example, consider the projection $k[x_1, x_2, ...] \to k$, where $k$ is a field. Then $W = \{(0)\} = \mathsf{Spec}\, S$ has $f^{-1}(W) = \{(x_1, x_2, ...)\}$, and this isn't Thomason-closed. However, we have the following sufficient condition.

**Lemma 2.4.22.** *If $T = f^{-1}(\mathsf{Spec}\, S)$ is Thomason-closed, then $f^{-1}(W)$ is Thomason-closed for every specialization-closed subset $W \subseteq \mathsf{Spec}\, S$.*

**Proof.** Given a specialization-closed subset $W$ of $\mathsf{Spec}\, S$, it is not hard to see that

$$f^{-1}(W) = \bigcup_{\mathfrak{p} \in W} f^{-1}(V(\mathfrak{p})).$$

Combining this with Lemma 2.4.16 gives

$$f^{-1}(W) = \left( \bigcup_{\mathfrak{p} \in W} V(\tilde{\mathfrak{p}}) \right) \cap T.$$

Thus is suffices to show that the intersection of two Thomason-closed subsets is Thomason-closed. This follows from the fact that

$$V((y_1, y_2, ..., y_n)) \cap V((z_1, z_2, ..., z_m)) = V((y_1, y_2, ..., y_n, z_1, z_2, ..., z_m)).$$

□

**Lemma 2.4.23.** *If $f^{-1}(0) \subseteq R$ is finitely generated, then $T$ is Thomason-closed.*

**Proof.** Since $V(0) = \mathsf{Spec}\, S$, $T = f^{-1}(V(0)) = V(f^{-1}(0))$. In general, for an ideal $I = (y_\alpha \;:\; \alpha \in A)$,

$$\bigcap_A V(y_\alpha) = V(I).$$

This follows from a straightforward argument using definitions.

The intersection of two Thomason-closed subsets is Thomason-closed, and by induction a finite intersection of Thomason-closed subsets is Thomason-closed. Therefore if $I = f^{-1}(0)$ is finitely generated, then $T$ is Thomason-closed. □

If we assume that $i_\bullet S$ is a finite object in $D(R)$, then $i_\bullet$ sends finite objects to finite objects, and gives a well-defined operation on thick subcategories of finite objects in $D(S)$. Furthermore, the objects $i_\bullet(S/\!\!/\mathfrak{p})$ will be finite, for all $\mathfrak{p} \in \mathsf{Spec}\, S$.

**Lemma 2.4.24.** *If $i_\bullet S$ is finite, then $T = f^{-1}(\mathsf{Spec}\, S)$ is Thomason-closed.*

**Proof.** Corollary 2.4.14 gives

$$T = \bigcup_{\mathfrak{p} \in \mathsf{Spec}\, S} f^{-1}(V(\mathfrak{p})) = \operatorname{supp}\left(\mathsf{th}\left(i_\bullet(S/\!\!/\mathfrak{p}) \;:\; \mathfrak{p} \in \mathsf{Spec}\, S\right)\right).$$

Thus $T$ is the support of a thick subcategory of finite objects in $D(R)$, hence Thomason-closed by the classification theorem. □

In the case where $i_\bullet S$ is finite, then, Lemma 2.4.22 implies that $f^{-1}$ takes specialization-closed subsets to Thomason-closed subsets. This also follows from the next proposition.

**Proposition 2.4.25.** *Let $f : R \to S$ be a surjective ring map, with $S$ Noetherian. Suppose $i_\bullet S$ is finite, and let $\mathsf{A}$ be a thick subcategory of finite objects in $D(S)$. Then*

$$supp(i_\bullet \mathsf{A}) = f^{-1}(supp(\mathsf{A})).$$

**Proof.** Because $i_\bullet \mathsf{A}$ is the thick subcategory generated by all the $i_\bullet X$, $X \in \mathsf{A}$, we have that

$$Y \wedge i_\bullet X = 0 \text{ for all } X \in \mathsf{A} \text{ if and only if } Y \wedge W = 0 \text{ for all } W \in i_\bullet \mathsf{A}.$$

This implies that, for $\mathfrak{q} \in \mathsf{Spec}\, R$, $\overline{k_\mathfrak{q}} \wedge W \neq 0$ for some $W \in i_\bullet \mathsf{A}$ if and only if $\overline{k_\mathfrak{q}} \wedge i_\bullet X \neq 0$ for some $X \in \mathsf{A}$. Using this, the argument in Proposition 2.4.13 gives that $\mathrm{supp}(i_\bullet \mathsf{A}) = f^{-1}(\mathrm{supp}(\mathsf{A}))$. □

**Lemma 2.4.26.** *Assume $i_\bullet S$ is finite. For any finite object $X$ in $D(S)$, we have*

$$\mathsf{th}(f_\bullet i_\bullet X) = \mathsf{th}(X).$$

**Proof.** Because $X$ is finite and $S$ is Noetherian, $\mathsf{th}(X) = \mathsf{th}(S /\!/ \mathfrak{p} \ : \ \mathfrak{p} \in W)$ for some $W \subseteq \mathsf{Spec}\, S$. Specifically, $W = \mathrm{supp}(X)$. Since $f_\bullet$ and $i_\bullet$ are exact and commute with coproducts, we get

$$\mathsf{th}(f_\bullet i_\bullet X) = \mathsf{th}\left(f_\bullet i_\bullet (S /\!/ \mathfrak{p}) \ : \ \mathfrak{p} \in W\right).$$

Thus we can reduce to showing that $\mathsf{th}(f_\bullet i_\bullet (S /\!/ \mathfrak{p})) = \mathsf{th}(S /\!/ \mathfrak{p})$. Since $f_\bullet i_\bullet S$ is finite, these are both thick subcategories of finite objects, so if suffices to show that they have the same support.

Proposition 2.4.20 says that

$$f^{-1}(\mathrm{supp}(f_\bullet i_\bullet (S /\!/ \mathfrak{p}))) = \mathrm{supp}(i_\bullet (S /\!/ \mathfrak{p})) \cap T = f^{-1}(V(\mathfrak{p})).$$

Therefore $\mathrm{supp}(f_\bullet i_\bullet (S /\!/ \mathfrak{p})) = V(\mathfrak{p}) = \mathrm{supp}(S /\!/ \mathfrak{p})$. □

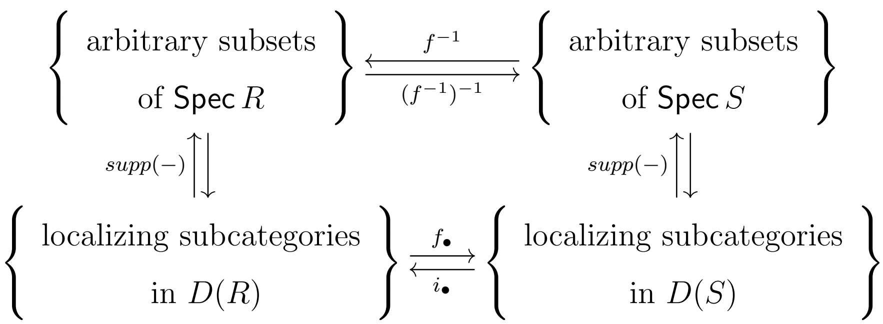
## 2.5 *Surjective $f : R \to S$, with $R$ and $S$ both Noetherian; ungraded setting*

All the results from the last section can be improved and generalized when $R$ is also Noetherian. **As in Section 2.4, we are working in the ungraded setting.** Some of the results in this section follow easily from the Noetherian classification theorems. In Subsection 2.5.3 we show that the map $\overline{f_\bullet}$ induces a splitting

$$\mathsf{BL}_{D(R)} \cong \mathsf{BL}_{D(S)} \times J.$$

### *2.5.1 Localizing subcategories*

The localizing subcategories of $D(R)$ and $D(S)$ are classified by arbitrary subsets of $\mathsf{Spec}\, R$ and $\mathsf{Spec}\, S$. More specifically, the lattice $\mathbf{2}^{\mathsf{Spec}\, R}$ of subsets of $\mathsf{Spec}\, R$ is isomorphic to the lattice of localizing subcategories of $D(R)$, denoted $\mathsf{Loc}(D(R))$. Under this isomorphism, $\{\mathfrak{p}\}$ corresponds to $\mathsf{loc}(\overline{k_\mathfrak{p}})$; in particular, $\mathrm{supp}(\overline{k_\mathfrak{p}}) = \{\mathfrak{p}\}$.




Throughout this section, fix $T := f^{-1}(\mathsf{Spec}\, S) \subseteq \mathsf{Spec}\, R$, and $U := \mathsf{Spec}\, R \backslash T$.

The map $f^{-1}$ induces a map $\mathbf{2}^{\mathsf{Spec}\, S} \to \mathbf{2}^{\mathsf{Spec}\, R}$, and $f_\bullet$ induces a map $\mathsf{Loc}(D(R)) \to \mathsf{Loc}(D(S))$. The following lemma states that, via the lattice isomorphisms described above, we have "$(f^{-1})^{-1} = f_\bullet$".

**Proposition 2.5.1.** *Let $f : R \to S$ be a surjective ring map between Noetherian rings. Let $\mathsf{A}$ be a localizing subcategory of $D(R)$. Then*

$$\mathrm{supp}(f_\bullet \mathsf{A}) = (f^{-1})^{-1}\left(\mathrm{supp}(\mathsf{A})\right) = f\left(\mathrm{supp}(\mathsf{A}) \cap T\right).$$

*Equivalently, $f^{-1}\left(\mathrm{supp}(f_\bullet \mathsf{A})\right) = \mathrm{supp}(\mathsf{A}) \cap T$. The same holds if we replace $\mathsf{A}$ with $X$, for any object $X$ in $D(R)$.*

**Proof.** Let $\mathsf{A}$ be a localizing subcategory of $D(R)$. Then $\mathsf{A} = \mathsf{loc}(\overline{k_\mathfrak{q}} \;:\; \mathfrak{q} \in W)$ for some $W \subseteq \mathsf{Spec}\, R$, and in this case $\mathrm{supp}(\mathsf{A}) = W$. Using Lemma 2.4.12, we compute that

$$f_\bullet \mathsf{A} = f_\bullet \mathsf{loc}\left(\overline{k_\mathfrak{q}} \;:\; \mathfrak{q} \in W\right) = \mathsf{loc}\left(f_\bullet \overline{k_\mathfrak{q}} \;:\; \mathfrak{q} \in W\right) = \mathsf{loc}\left(\overline{k_{f(\mathfrak{q})}} \;:\; \mathfrak{q} \in W \cap T\right).$$

In other words, $\mathrm{supp}(f_\bullet \mathsf{A}) = f(W \cap T) = f(\mathrm{supp}(\mathsf{A}) \cap T) = (f^{-1})^{-1}(\mathrm{supp}(\mathsf{A}))$.

In general, $\mathrm{supp}(X) = \mathrm{supp}(\mathsf{loc}(X))$ for all $X \in D(R)$. Because $f_\bullet(\mathsf{loc}(X)) = \mathsf{loc}(f_\bullet X)$, the result holds with $X$ in place of $\mathsf{A}$. □

**Corollary 2.5.2.** *For all $X$ in $D(R)$,*

$$\mathrm{supp}(i_\bullet f_\bullet X) = \mathrm{supp}(X) \cap T.$$

**Proof.** Using Propositions 2.4.13 and 2.5.1 we compute

$$\mathrm{supp}(i_\bullet f_\bullet X) = f^{-1}(\mathrm{supp}(f_\bullet X)) = f^{-1}\left(f(\mathrm{supp}(X) \cap T)\right) = \mathrm{supp}(X) \cap T.$$

□

**Remark 2.5.3.** In [IK11], the authors consider the general case of a tensor triangulated functor $F : \mathcal{T} \to \mathcal{U}$ between well generated tensor triangulated categories. They define support $\mathrm{Sp}(-)$ in terms of prime elements in the distributive lattice $\mathsf{DL}$, and show that if $F$ is well-defined on the level of Bousfield classes and preserves coproducts, then there is a continuous map $\mathrm{Sp}(F) : \mathrm{Sp}(\mathcal{U}) \to \mathrm{Sp}(\mathcal{T})$ such that

$$\mathrm{supp}_\mathcal{U}(FX) = \mathrm{Sp}(F)^{-1}(\mathrm{supp}_\mathcal{T}(X)) \text{ for all } X.$$

In the case where $R$ and $S$ are Noetherian, their notion of support agrees with ours [IK11, 6.10]. If we consider $i_\bullet : D(S) \to D(R)$, then Proposition 2.4.13 says that

$$\mathrm{Sp}(i_\bullet)^{-1} = f^{-1} : \mathsf{Spec}\, S \to \mathsf{Spec}\, R.$$

If we consider $f_\bullet : D(R) \to D(S)$, then Proposition 2.5.1 says that

$$\mathrm{Sp}(f_\bullet)^{-1} = (f^{-1})^{-1} : \mathsf{Spec}\, R \to \mathsf{Spec}\, S.$$

**Example 2.5.4.** Take an arbitrary subset $W \subseteq \mathsf{Spec}\, S$. The localizing subcategory of $D(R)$ corresponding to $f^{-1}(W)$ is

$$\mathsf{loc}\left(\overline{k_{f^{-1}\mathfrak{p}}} \ : \ \mathfrak{p} \in W\right).$$

Applying $f_\bullet$ to this gives $\mathsf{loc}\left(\overline{k_\mathfrak{p}} \ : \ \mathfrak{p} \in W\right)$ in $D(S)$, which corresponds to $W$ under the classification.

Now we look at the action of $i_\bullet$ on localizing subcategories.

**Proposition 2.5.5.** *Let $f : R \to S$ be a surjective ring map between Noetherian rings. Let $\mathsf{B}$ be a localizing subcategory of $D(S)$. Then*

$$\mathrm{supp}(i_\bullet \mathsf{B}) = f^{-1}\left(\mathrm{supp}(\mathsf{B})\right).$$

**Proof.** Since $R$ and $S$ are Noetherian, every localizing subcategory is principal. Using fact that $\mathrm{supp}(X) = \mathrm{supp}(\mathsf{loc}(X))$ for all $X$, the result follows from Proposition 2.4.13. □

This lemma implies that, via the lattice isomorphisms $\mathbf{2}^{\mathsf{Spec}\, R} \cong \mathsf{Loc}(D(R))$ and $\mathbf{2}^{\mathsf{Spec}\, S} \cong \mathsf{Loc}(D(S))$, we have "$f^{-1} = i_\bullet$". Because $(f^{-1})^{-1} \circ f^{-1}$ is the identity on $\mathsf{Spec}\, S$, the composition $f_\bullet \circ i_\bullet$ is the identity on $\mathsf{Loc}(D(S))$. We make this more precise in the next proposition.

**Proposition 2.5.6.** *Take $\mathfrak{p} \in \mathsf{Spec}\, S$. Then*

$$\mathsf{loc}(i_\bullet \overline{k_\mathfrak{p}}) = \mathsf{loc}(\overline{k_{f^{-1}\mathfrak{p}}}), \ \textit{so}\ \langle i_\bullet \overline{k_\mathfrak{p}} \rangle = \langle \overline{k_{f^{-1}\mathfrak{p}}} \rangle.$$

*Thus for all localizing subcategories $\mathsf{B}$ of $D(S)$, we have $f_\bullet i_\bullet \mathsf{B} = \mathsf{B}$.*

**Remark 2.5.7.** The last statement might seem to follow from the one-to-one correspondence between localizing subcategories and Bousfield classes, in light of the fact that $\langle f_\bullet i_\bullet X \rangle = \langle X \rangle$ for all $X$ in $D(S)$. It does, but some care is necessary; see Section 2.5.3.

**Proof.** For all $\mathfrak{q} \in U$, we have $i_\bullet\overline{k_\mathfrak{p}} \wedge \overline{k_\mathfrak{q}} = i_\bullet(\overline{k_\mathfrak{p}} \wedge f_\bullet\overline{k_\mathfrak{q}}) = i_\bullet(0) = 0$, by Lemma 2.4.12. For all $\mathfrak{q} \in T\backslash\{f^{-1}(\mathfrak{p})\}$, Lemma 2.1.7 implies $\overline{k_\mathfrak{p}} \wedge \overline{k_{f(\mathfrak{q})}} = 0$, so by Proposition 2.4.1 we have

$$i_\bullet\overline{k_\mathfrak{p}} \wedge \overline{k_\mathfrak{q}} = i_\bullet(\overline{k_\mathfrak{p}} \wedge f_\bullet\overline{k_\mathfrak{q}}) = i_\bullet(0) = 0.$$

Therefore

$$\mathsf{loc}(i_\bullet\overline{k_\mathfrak{p}}) \subseteq \left\langle \coprod_{\mathfrak{q}\neq f^{-1}(\mathfrak{p})} \overline{k_\mathfrak{q}} \right\rangle = \mathsf{loc}\left(\overline{k_{f^{-1}\mathfrak{p}}}\right).$$

On the other hand, $\overline{k_\mathfrak{p}} \wedge \overline{k_\mathfrak{p}} \neq 0$, Proposition 2.4.1, and the injectivity of $i_\bullet$ imply

$$i_\bullet\overline{k_\mathfrak{p}} \wedge \overline{k_{f^{-1}\mathfrak{p}}} = i_\bullet(\overline{k_\mathfrak{p}} \wedge f_\bullet\overline{k_{f^{-1}\mathfrak{p}}}) \neq 0.$$

This shows that $i_\bullet\overline{k_\mathfrak{p}} \neq 0$, so $\mathsf{loc}(i_\bullet\overline{k_\mathfrak{p}})$ is non-zero. Because $\mathsf{loc}(\overline{k_{f^{-1}\mathfrak{p}}})$ is a minimal non-zero localizing subcategory, we have equality.

The last statement follows from the classification of localizing subcategories and Proposition 2.4.1. □

### *2.5.2 Thick subcategories*

When $R$ is Noetherian we can strengthen the results of Section 2.4.3.

**Lemma 2.5.8.** *The function $f^{-1} : \mathsf{Spec}\, S \to \mathsf{Spec}\, R$ takes specialization-closed subsets to specialization-closed subsets.*

**Proof.** A set of prime ideals is specialization-closed if and only if it is the union of closed subsets. We know that $f^{-1}(\bigcup W_\alpha) = \bigcup f^{-1}(W_\alpha)$. Because $f$ is surjective, the map $f^{-1}$ is a homeomorphism onto its image, thus closed. □

We have the following improvement on Proposition 2.4.20.

**Proposition 2.5.9.** *Let $f : R \to S$ be a surjective ring map between Noetherian rings. Let $\mathsf{B}$ be a thick subcategory of finite objects in $D(R)$. Then*

$$f^{-1}(\mathrm{supp}(f_\bullet\mathsf{B})) = \mathrm{supp}(\mathsf{B}) \cap T.$$

**Proof.** Because $R$ and $S$ are Noetherian, all thick subcategories of finite objects are principal. The result follows from Proposition 2.5.1. □

**Lemma 2.5.10.** *Given $\mathfrak{p} \in \mathsf{Spec}\, S$, the objects $f_\bullet(R/\!\!/f^{-1}\mathfrak{p})$ and $S/\!\!/\mathfrak{p}$ generate the same thick subcategory.*

**Proof.** Since $f^{-1}\mathfrak{p}$ is finitely generated, $R/\!\!/f^{-1}\mathfrak{p}$ is defined, and is finite. Therefore it suffices to show that $f_\bullet(R/\!\!/f^{-1}\mathfrak{p})$ and $S/\!\!/\mathfrak{p}$ have the same support. We know $\mathrm{supp}(S/\!\!/\mathfrak{p}) = V(\mathfrak{p})$, and use Proposition 2.5.1 to calculate

$$\mathrm{supp}(f_\bullet(R/\!\!/f^{-1}\mathfrak{p})) = f(\mathrm{supp}(R/\!\!/f^{-1}\mathfrak{p}) \cap T)$$

$$= f(V(f^{-1}\mathfrak{p}) \cap T) = f(f^{-1}(V(\mathfrak{p}))) = V(\mathfrak{p}).$$

□

### *2.5.3 Bousfield lattices*

With the Noetherian rings $R$ and $S$, there is a correspondence between the lattice of localizing subcategories and the Bousfield lattice:

$$\mathsf{loc}\left(\overline{k_\mathfrak{q}} \;:\; \mathfrak{q} \in W\right) \longleftrightarrow \left\langle \bigvee_{\mathfrak{q} \in W^c} \overline{k_\mathfrak{q}} \right\rangle.$$

Some caution is required, however, because we have defined different operations $i_\bullet$ on Bousfield classes and on localizing subcategories, and these do not always agree.

For example, as a Bousfield class, we have $i_\bullet : \langle 0 \rangle \mapsto \langle i_\bullet 0 \rangle = \langle 0 \rangle$. But

$$\langle 0 \rangle = \mathsf{loc}(S) = \mathsf{loc}\left(\coprod_{\mathfrak{q} \in \mathsf{Spec}\, S} \overline{k_\mathfrak{q}}\right),$$

and, in light of Proposition 2.5.6, as a localizing subcategory we have

$$i_\bullet : \mathsf{loc}\left(\coprod_{\mathfrak{q} \in \mathsf{Spec}\, S} \overline{k_\mathfrak{q}}\right) \mapsto \mathsf{loc}\left(\coprod_{\mathfrak{q} \in \mathsf{Spec}\, S} i_\bullet \overline{k_\mathfrak{q}}\right) = \mathsf{loc}\left(\coprod_{\mathfrak{p} \in T} \overline{k_\mathfrak{p}}\right) = \left\langle \bigvee_{\mathfrak{p} \in U} \overline{k_\mathfrak{p}} \right\rangle.$$

This is not surprising, because as an operation on Bousfield classes or on localizing subcategories, $i_\bullet$ is not surjective. The difference is always precisely $\langle \bigvee_{\mathfrak{p}\in U} \overline{k_\mathfrak{p}} \rangle = \mathsf{loc}\left(\coprod_{\mathfrak{p}\in T} \overline{k_\mathfrak{p}}\right)$.

On the other hand, because the operations induced by $f_\bullet$ are both surjective (Propositions 2.5.6 and 2.4.6), the two definitions always agree.

When $R$ is Noetherian, the lattice bijection $\mathsf{BL}_{D(R)}/J \xrightarrow{\sim} \mathsf{BL}_{D(S)}$ is actually a splitting of the Bousfield lattice $\mathsf{BL}_{D(R)}$.

Lemma 2.4.12 implies that

$$J = \left\{ \langle X \rangle \mid \langle X \rangle \leq \left\langle \coprod_{\mathfrak{q}\in U} \overline{k_\mathfrak{q}} \right\rangle \right\}.$$

Because $\mathsf{BL}_{D(R)}$ is just a Boolean algebra on the classes $\langle \overline{k_\mathfrak{q}} \rangle$ for $\mathfrak{q} \in \mathsf{Spec}\, R$, and we have the partition $\mathsf{Spec}\, R = T \amalg U$, we get that

$$\mathsf{BL}_{D(R)} \cong \mathsf{BL}/J \times J \cong \mathsf{BL}_{D(S)} \times J.$$

**Remark 2.5.11.** In Proposition 6.12 of [IK11], the authors show that any smashing localization functor gives a splitting of the Bousfield lattice. If we take $L : D(R) \to D(R)$ to be finite localization at $\mathsf{th}(R/f^{-1}\mathfrak{p} \ : \ \mathfrak{p} \in \mathsf{Spec}\, S)$, then the support of this thick subcategory is $T$, which is specialization-closed. The $L$-acyclics are

$$\mathsf{loc}\left(\coprod_{\mathfrak{q}\in T} \overline{k_\mathfrak{q}}\right) = \left\langle \coprod_{\mathfrak{q}\in U} \overline{k_\mathfrak{q}} \right\rangle,$$

and $L$ is smashing since finite. The splitting from [IK11, 6.12] is exactly the same as the one above induced by $f_\bullet$, described above.

## 2.6 Experiential context

Most of these results were developed in spring and summer of 2011, mainly on the beach in Thailand. I took a leave of absence and was staying in a beach hut on Tonsai beach. This beach is accessible only by boat, with only a few restaurants for food and minimal distraction. I diligently worked 40 hours a week on research.

The main insight was that each ring had a derived category with rich structure, and ring maps would induce functors between derived categories, so I should look at how these interacted with the rich structure. It seemed like such a simple idea that surely someone had already thought of it. I had been rereading papers by Neeman [Nee92] and Hovey-Palmieri-Strickland [HPS97], as well as Weibel's book on homological algebra [Wei94], so my mind was primed. I had a sense that this was a decent question.

Sometimes I think of rings as fruit, and derived categories as pies baked from the fruit, but this only gets me so far. If there's a primary mental image, it's tied to the commutative diagrams based on the classification theorems, such as the one just before Proposition 2.4.20. Each corner of that diagram is a nebulous blob with substructure, organized by inclusion. Spec usually appears as a thin rectangular Venn diagram. Noetherian rings feel like carefully mapped out, friendly, brightly-lit objects, but non-Noetherian rings have betraying corners of infinite darkness. Within the categories are the finite objects, and these subcategories feel very tangible and easy to work with. They feel constructible and contained, clustered around the sphere object. Each category has a small family of interesting explicit objects: $S$, $S/\!\!/\mathfrak{q}$, $\overline{k_{\mathfrak{q}}}$, $K(\mathfrak{q})$, $\overline{k_{f^{-1}\mathfrak{p}}}$, $R/\!\!/\tilde{\mathfrak{p}}$, $K(\tilde{\mathfrak{p}})$, etc. These offer finite possibilities for calculations, and pushing them up and down and left and right through these diagrams began as a confusing process but has gradually come to feel unmysterious.

As I was immersed in the topic, at the beach with notebooks, questions continually presented themselves. I felt that I was making good progress at building results, lemma after lemma, and keeping track of new questions as they popped up. Most of the proofs were, and are, light and straightforward. Some are well-known. However, in fall 2011, as I tried to write up the results more rigorously, I kept finding holes and mistakes, which required significant adjustments. Many statements had to be qualified, or weakened. Only at the end did I connect these results with recent work of Iyengar and Krause [IK11] and start working out explicit examples.

The lack of interaction with other mathematicians and the long hours of research necessitated a careful self-awareness of my process. It was essential I stay focused and productive, on my own, and maintain the clarity and perspective to analyze my own results, questions, and progress. Without this self-reflection, the experience would not have been productive or meaningful.

# Chapter 3

# CONTEMPLATION IN MATHEMATICS

Here we look at the role that contemplation plays, or might play, in the mathematical research experience. By *contemplation* we mean reflection on the lived experience of doing math. For example, one might contemplate the feeling of aesthetic pleasure induced by certain concepts and proofs. One might reflect on the emotional and cognitive context of a particular research work session. Or one might contemplate the norms and tacit knowledge of the math research community.

It is evident that such contemplation adds depth to the mathematical experience, and contributes to the enjoyment and meaning we find in doing math. Reflecting on the human dimensions of math might give us greater appreciation of the content, or might help us find our place in a community of peers. But is there a more direct way in which contemplation of the research experience might inform the research process itself, in a way that spurs on the development of new and good math?

Our answer is in the positive, and in this section we present a case study as supporting evidence.

First, however, we address some common qualms about discussing the research process. (Here we are basing our assessment on personal communication with researchers, and on our personal understanding of the norms and values of the research community.) Aren't all such discussions inherently personal and subjective? What am I going to learn from listening to you talk about your process? And do these conversations ever *go* anywhere? Without well-defined terms, how can we even know we are talking about the same things, let alone build relationships between concepts and establish any sort of understanding?

Now certainly it is true that we must make a strong distinction between talking *in* math, and talking *about* math. Math concepts are precisely those that are well-defined, stable, and universal – in a word, objective. The above criticisms seem to operate within a dichotomy of pure objectivity and pure subjectivity. Since meta-discussions are not purely objective, they are viewed as a waste of time – something for philosophers and math educators to worry about but not research mathematicians.

This identification "math = rigor" grew out of history. Starting in the mid-19th century, following the discovery of non-Euclidean geometries, and attempts in analysis to reconcile the discrete with the continuous, there was a drive towards increasing rigor and establishing firm foundations in mathematics. Although the Foundations movement faltered, the increasingly abstract nature of mathematics has supported the push towards Formalism and the belief that "math = rigor" [Bye07, Ch.7].

However, this is an oversimplification. Instead, we see a continuous spectrum, ranging from the purely subjective to the purely objective. Seemingly objective mathematical concepts, e.g. the derivative, have a subjective component. Bill Thurston discusses this example [Thu94], saying that the derivative might have a dozen definitions, each one functioning in a different context, providing a different intuition, couched in different visual/spacial/kinesthetic/structural frames. What a derivative is changes depending on who you are, when you are looking, and which direction you are facing. At the same time, as we will argue below, even the most personal and seemingly subjective math experiences have an objective component. Abandoning an insistence on pure rigor, we can use somewhat well-defined terms to discuss somewhat stable internal phenomena.

At this point, we make two claims, and will attempt to provide evidence for their validity.

**Claim 3.0.1.** Contemplation of the research experience matters. Awareness of, and reflection on, the research process will result in more and better mathematical results.

We claim that the more you understand how you do math, the more you will understand how you do your best math, and the more effective and efficient you will be at doing so. This may seem obvious, but anecdotally we have seen that it is not a common view.

**Claim 3.0.2.** Mathematical research experiences have an intersubjective reality. You and I share common experience, and we should think about this and talk about it together.

Following White [Whi47], we place the locus of mathematical reality in human culture. Math is a human practice, couched in a cultural context. Different practitioners have assimilated common modes of thought and process through years of apprenticeship and imitation. We believe that, when you or I sit down to work through the details of a proof, for example, our thoughts follow similar (but not identical!) paths. Working through a proof also induces or encourages a certain common state of mind, a certain common quality of awareness. And there is also some commonality in the possible emotions associated with working through a proof. In this sense, we would say that the experience of "working through a proof" demonstrates an *intersubjective* reality [VS99]. It is part of a more universal body of mathematical wisdom and experience, transcending the individual, that is often neglected.

To date, the most thorough investigation of intersubjectivity of the mathematical experience comes from Jacques Hadamard, an accomplished number theorist and differential geometer, who was intrigued specifically by the process of mathematical discovery. Hadamard built on the ideas and analysis of Poincaré [Poi13] to suggest a common psychological process of discovery: immersive preparation, stuckness and dissonance, insight, and analytical follow-up. In his book [Had54], he establishes evidence, proposes reasons, and discusses exceptions to his theory. He relies on his personal experience, in-depth discussions with colleagues, and responses from a general survey, "An Inquiry into the Working Methods of Mathematicians," conducted in

1902. Although at the time the field of psychology was in its infancy, Hadamard's perceptions and analyses, as a mathematician attempting to understand his own mind, are valid and compelling.

## 3.1 Case study: The Flavors and Seasons Project

The author has been intentionally contemplating his mathematical experience for years, and using such reflections to do better math. Inspired by Hadamard's work, The Flavors and Seasons project grew out of a desire to standardize these reflections in a format that would be accessible to other mathematicians.

A *flavor* is a shorter math research experience, one that lasts for a few hours or days. A *season* is a longer research experience, lasting for weeks or months. The project is an attempt to document the flavors and seasons of research mathematics, to build a database of mathematical experiences. This is done using a list of standard questions, that are answered as precisely and concretely as possible. Since most flavors repeat themselves, and the seasons last for some time, the emphasis during documentation is on those aspects of the experience that seem to repeat or endure.

### *3.1.1 Questions*

The questions for each flavor are:

1. What brings it about mathematically?
2. Emotional/logistical context?
3. What thoughts are there?
4. Quality of awareness?
5. What emotions?
6. What ideas does it resolve to, after how much time?
7. How frequent is this flavor?
8. Good and bad ways to change/follow it up?

The questions for each season are:

1. What mathematical activities? What level of rigor?
2. What relevant interactions with other mathematicians?
3. How does it feel, what is the mood?
4. What state of mind? stable vs. chaotic? focused vs. dispersed?
5. What type of self-reflection during the experience, and did it help?
6. An everyday metaphor for the experience?
7. An example of a good day and a bad day?
8. What did you do when you were stuck?
9. When and why did it end?

Each flavor and season report is written up and posted on a website[1]. The project is ongoing, but at time of writing, there are eighteen flavors and five seasons posted. We give some examples below. Reports will often include hyperlinked references to related writings of other mathematicians. Some are attracting comments from readers.

The purpose of publicizing such reflections is to provide empirical evidence for Claims 3.0.1 and 3.0.2. The answers to flavor question #8 and season question #5 often reveal concrete ways in which reflecting on process allows for improved math-doing. And by focusing on the recurring and more-universal aspects of these experiences, the hope is that other mathematicians will find some overlap with their own process, and thereby affirm the intersubjective reality of our shared experience.

[1] flavorsandseasons.wordpress.com

### *3.1.2 Current flavors and seasons*

The current list of flavors is:

Breakthrough I: Getting stuck.

Breakthrough II: Sticking with stuckness.

Breakthrough III: Rest.

Breakthrough IV: Insight.

Discussing with a colleague.

Dull mind.

Getting my hands dirty, to clear up confusion.

Intentioned immersion.

Math anxiety - “I should be doing more math.”

Mathache.

Preparing a (research) talk.

Proving myself wrong, via counterexample.

Stunned and sublime.

Translating a proof from one context to another.

Trying to explain math to a stranger.

Unintentioned immersion.

Using my math powers for evil.

Working through a proof.

The current list of seasons is:

Building a theory / Imagining what could be.

Crescendo.

Doing computations.

Entering a field.

Pulling together and writing up.

### *3.1.3 Examples*

We conclude with two representative examples, verbatim from the project website.

Flavor: Proving myself wrong, via counterexample.

1. *What's going on mathematically?*

After working towards proving something for a while, I find a counterexample. This involves an insight, followed by a verification.

2. *What is the emotional and logistical context?*

These counterexamples usually show up suddenly. The most dramatic and surprising cases are after working towards a particular result for weeks, because my expectation is that I've been getting closer and closer to a complete proof. So counterexamples hit when I'm hopeful, maybe even overly idealistic.

3. *What thoughts are there?*

The initial insight is a surprising "Aha" moment, accentuated by the fact that most counterexamples have a simplicity and necessity that seems to stab directly into the essence of the problem. This is immediately followed by some concerned analysis of the situation - does this mean I just wasted three weeks? is there a way to fix it? But before a complete reassessment, there's a careful verification, to prove that the counterexample is a counterexample, i.e. to prove myself wrong.

4. *What quality of awareness?*

It's like the rug has been pulled out from underneath me. There's a shock and surprise, grounded in certainty, that then trickles outward along logic pathways and finds a deserted city. Or worse, the city I thought I knew is now filled with people that speak a language I don't understand. On a deep level, it's an unsettled, shifting, almost paranoid wandering in this strange new city, searching for any familiar faces. But on the shallow level, there's a sharp certainty and cleanness, as my proved counterexample resonates within itself.

These are the times when I'm most aware of the non-logical, heuristic, mysterious "intuition" I have built up about the math I do. I had a mathematical worldview in which Proposition X was true - this sense of the way things work guided me, helped me make sense of it all. But now that I have found a counterexample, it's not just the statement of Proposition X, but the whole worldview, that needs to be adjusted.

5. *What emotions?*

Of course, I usually feel disappointed and frustrated, depending on the severity of the situation. At worst, it can devolve into fatigue and meaninglessness. (I'm fortunate that the most time I've thrown away on a false proposition is 2.5 weeks; I'm sure it gets much worse than that.) There's also an undeniable sense of finality, that comes with proving any result - "at least now I know for sure." It is a very strange feeling, to prove yourself wrong. This certainty is a feeling I almost only get from math, and for some reason I feel it more strongly when I've been proven wrong than when I've been proven right.

I'll usually take a break from the problem for a bit, and then I feel some revulsion towards it. Maybe it's a feeling of being betrayed, but I don't want anything to do with the question. This goes away soon, though.

6. *What does it resolve to, after how much time?*

A good mathematician would say that in every counterexample there's new ideas to follow up. Maybe I just need to tweak my hypotheses; maybe the counterexample is pointing towards the essence of what's going on; maybe the fact that Proposition X fails is a "good" thing, that e.g. allows for more interesting behavior. I can usually start to pick up the pieces after a few hours.

7. *How frequent is this flavor?*

Oh, I'm such a bad research mathematician, this happens way too much.

8. *What are good/bad ways to change or follow it up?*

Bad: take it personally and get discouraged. Good: take a deep breath and get to work picking up the pieces. Mathematical intuition isn't built overnight, and without surprises math would be boring.

Season: Pulling together and writing up.

1. *What mathematical activities? What level of rigor?*

After a period of creative research (of days, weeks, months,...), it's necessary to consolidate and pull together results. This involves very carefully retracing steps, chronologically, and lining up ideas and proofs. Initially, results are scattered throughout my notes; or proofs haven't been written down; or some statements are wrong, or outdated, or improved upon.

Because the original path to the result is almost always not the most direct, everything must be restructured. The goal here is to present ideas and proofs in the conventional form, an explanation to a particular audience. So there is a pure logic component, of lining up arguments correctly, but also a conversational component, as I decide how much detail to include, how much exposition, how much rigor.

This task is relatively easy and straightforward. It can feel administrative at times, for example when compiling a list of references. Virtually all math is type-set in Tex, so writing up involves hours and hours of typing Tex code, which is not very intellectually gripping.

I try to keep a running list of random ideas or questions that pop into my head as I'm writing something up. But I won't pursue these until I've finished, since switching back and forth seems to make the writing up process less efficient.

2. *What relevant interactions with other mathematicians?*

This is maybe the most independent extended math experience I know. I might need to check some work with someone else, but presumably at this stage I've already solidified the results. It's helpful to ask for tips on Tex syntax. I might have someone check that I've included the right amount of justification and exposition for my target audience. When submitting a paper, there is a well-established process of refereeing, which involves recursive feedback and reworking, and this can drag out past any self-contained “writing up” experience.

3. *How does it feel, what is the mood?*

Pulling things together can be affirming, and satisfying. It feels good to solidify knowledge. Writing up can be relaxing, or mildly frustrating. It’s unnerving when I find a mistake I made a long time ago, and have to fix it.

4. *What state of mind? stable vs. chaotic? focused vs. dispersed?*

Pulling together feels easy, methodical, and uniquely compartmentalized. I only need to worry about one proof or handful of ideas at a time, and can safely ignore the periphery. Of course, I try to stay open to the occasional random new idea or question, but I intentionally stop my mind from wandering too much from the concrete task at hand. Writing up results is conversational and performative - my mind traces through the ideas as though I were explaining them out loud, in real time.

This kind of math is also relatively easy to turn on and off. Sometimes while doing it my mind wanders from math, and I get lost in a daydream. It’s maybe the closest that math comes to being a “day job.”

5. *What type of self-reflection during the experience, and did it help?*

As mentioned above, I try to keep a balance between capturing any possibly-valuable peripheral thoughts, and not getting too distracted from finishing the write-up. So I allow myself to use the restructuring and reviewing as an opportunity to gain perspective, but this perspective only comes if I keep some distance and don’t get wrapped up in following new leads. Maintaining this balance requires some self-reflection. In fact, it seems that the better I’m attuned to this balance, the closer I can get to simultaneously maximizing perspective and efficiency.

6. *An everyday metaphor for the experience?*

Pulling together and writing up is just like washing dishes. The goal is to sanitize all the mess of discovery, and to dry off any trace of the restructuring. We present a stack of dry, clean, glistening ideas, full of order and necessity, untouched by humans. These spotless ideas are complete in themselves, but sit ready to be used and rearranged as vessels and tools for someone else's new mess.

The dish-washing process is narrative (to me), relaxing, and mechanical. I can let my mind wander, to some extent. There are definitely more efficient and less efficient ways to do it.

7. *An example of a good day and a bad day?*

A good day ends with a few new pages of nice, clean Texed math. On a bad day, I find a gap or hole, and can't fix it.

8. *What did you do when you were stuck?*

Getting stuck might mean finding a gap in some argument; this needs to be fixed. Or it might mean that I lost or can't find some proof, so I have to reprove it. Or it might be that I don't know the Tex syntax for the symbol I want, which means I have to hunt through Tex documentation.

9. *When and why did it end?*

It ends when the results are typed up and clean.

#### *3.1.4 Discussion and future directions*

The above examples may seem too personal and subjective to be meaningful. We believe they have some intersubjective validity. Or they may seem to articulate such common and apparent shared mathematical experiences that they are content-less. However, the most apparent and simple self-observations are usually the most universal and pervasive. As a practitioner contemplating one's own experience, it is difficult to observe oneself from the outside, and very difficult to articulate tacit knowledge.

An analogy can be made by looking at the art community, where the value of articulating and discussing process is apparent. An art critique, among a group of painters for example, consists of an analysis of the aesthetics, themes, and ideas contained in a piece, and a discussion of the "how" behind the work. How art is made, how art has been made, and how art could be made in the future, is inextricably woven into the artistic discourse. Many, or most, artists, are constantly contemplating their process, and experimenting with different modes of working. Clearly, the experience of doing art is valued as an important dimension of art, worth thinking and talking about, just as much as the objects of art.

It is perhaps easier, and even essential, to separate the objects of mathematics and the doing of mathematics. But there is no reason to devalue the experience of doing mathematics, and the contemplation and discussion of that experience.

A more explicit and thorough understanding of mathematical practice among the math community could lead to only a more explicit and thorough training in best practices. To be the most valid and helpful, such an analysis must come from the community of researchers, not from math educators or social scientists. The consequence would be better mathematicians doing better mathematics.

# Chapter 4

## NON-NOETHERIAN DERIVED CATEGORIES

In this chapter, we look at some specific non-Noetherian rings and their derived categories. We apply many of the results of Chapter 2, to investigate thick and localizing subcategories, and Bousfield lattices.

As in Definition 1.6.9, fix a prime $p$ and integers $n_i > 1$ for $i \geq 1$, and set

$$\Lambda_{\mathbb{Z}_{(p)}} := \frac{\mathbb{Z}_{(p)}[x_1, x_2, ...]}{(x_1^{n_1}, x_2^{n_2}, ...)}, \quad \Lambda_{\mathbb{F}_p} := \frac{\mathbb{F}_p[x_1, x_2, ...]}{(x_1^{n_1}, x_2^{n_2}, ...)} \quad \text{and} \quad \Lambda_{\mathbb{Q}} := \frac{\mathbb{Q}[x_1, x_2, ...]}{(x_1^{n_1}, x_2^{n_2}, ...)}.$$

We grade the $x_i$ so that $\Lambda_{\mathbb{Z}_{(p)}}$, $\Lambda_{\mathbb{F}_p}$, and $\Lambda_{\mathbb{Q}}$ are graded-connected and finitely-generated in each module degree, for example by setting $\deg(x_i) = 2^i$. **The rings $\Lambda_{\mathbb{Z}_{(p)}}$, $\Lambda_{\mathbb{F}_p}$, and $\Lambda_{\mathbb{Q}}$ are graded, and we consider the derived categories $D(\Lambda_{\mathbb{Z}_{(p)}})$, $D(\Lambda_{\mathbb{F}_p})$, and $D(\Lambda_{\mathbb{Q}})$ of chain complexes of graded modules.**

The motivation behind considering these particular rings comes from stable homotopy theory. In the stable homotopy category, the automorphisms of the smash product unit are the stable homotopy groups of spheres. This ring is surely one of the most complex and fascinating objects in topology. It is common to localize the category of spectra at a prime $p$ or at $\mathbb{Q}$, and instead study the $p$-local or $\mathbb{Q}$-local category of spectra. An arithmetic square result then allow us to approach the stable homotopy category using pullbacks from the localized categories.

Let $\mathcal{S}$ be the $p$-local category of spectra, and $\pi_*(S^0) = [S^0, S^0]_*$ the automorphisms of the smash unit, the $p$-local sphere. Then $\pi_*(S^0)$ is a graded-connected $\mathbb{Z}_{(p)}$-module, is non-Noetherian, and is finitely-generated in each degree. Furthermore, the prime spectrum of $\pi_*(S^0)$ has order two, and every element of positive degree is nilpotent. The two quotient domains are $\mathbb{Z}_{(p)}$ and $\mathbb{F}_p$, in degree zero.

In this sense, $\pi_*(S^0)$ is structurally similar to the above ring $\Lambda_{\mathbb{Z}_{(p)}}$. The categories $\mathcal{S}$ and $D(\Lambda_{\mathbb{Z}_{(p)}})$ are both stable homotopy categories (Section 1.2), and the automorphism group of the smash unit in $D(\Lambda_{\mathbb{Z}_{(p)}})$ is precisely $\Lambda_{\mathbb{Z}_{(p)}}$.

Motivated by this analogy, [DP08] studied the categories $D(\Lambda_{\mathbb{F}_p})$ and $D(\Lambda_{\mathbb{Q}})$ (or rather, the slightly more general category $D(\Lambda_k)$; see Section 1.6.9). In this chapter we'll extend many of their results to the category $D(\Lambda_{\mathbb{Z}_{(p)}})$. As we'll see, the main differences between $D(\Lambda_{\mathbb{Z}_{(p)}})$ and $D(\Lambda_{\mathbb{F}_p})$ arise from the nontrivial prime ideal $(p) \subsetneq \mathbb{Z}_{(p)}$, and from the fact that $\Lambda_{\mathbb{Z}_{(p)}}$ is not self-dual.

Our main result, Theorem 4.2.5 states that the projection $\Lambda_{\mathbb{Z}_{(p)}} \to \Lambda_{\mathbb{F}_p}$ and the injection $\Lambda_{\mathbb{Z}_{(p)}} \hookrightarrow \Lambda_{\mathbb{Q}}$ induce a splitting of the Bousfield lattices,

$$\mathsf{BL}_{D(\Lambda_{\mathbb{Z}_{(p)}})} \cong \mathsf{BL}_{D(\Lambda_{\mathbb{F}_p})} \times \mathsf{BL}_{\mathsf{loc}(i_\bullet \Lambda_{\mathbb{Q}})},$$

and this restricts to splittings of the sublattices $\mathsf{BA}$ and $\mathsf{DL}$.

We also examine the following question.

**Question 4.0.1.** *In the above splitting, is* $\mathsf{BL}_{\mathsf{loc}(i_\bullet \Lambda_{\mathbb{Q}})} \cong \mathsf{BL}_{D(\Lambda_{\mathbb{Q}})}$*?*

Proposition 4.1.2 shows that there is at least one nontrivial thick subcategory of finite objects in $D(\Lambda_{\mathbb{Z}_{(p)}})$, generated by $i_\bullet \Lambda_{\mathbb{F}_p}$. This implies (Corollary 4.2.3) that unlike $\mathsf{BL}_{D(\Lambda_{\mathbb{F}_p})}$, the Bousfield lattice of $D(\Lambda_{\mathbb{Z}_{(p)}})$ does not have a minimum nonzero class. On the other hand, Corollary 4.2.8 states that $\mathsf{BL}_{D(\Lambda_{\mathbb{Z}_{(p)}})}$ has the same cardinality as $\mathsf{BL}_{D(\Lambda_{\mathbb{F}_p})}$, namely $2^{2^{\aleph_0}}$. Theorem 4.2.11 is a nilpotence theorem for $D(\Lambda_{\mathbb{Z}_{(p)}})$.

## 4.1 $D(\Lambda_{\mathbb{Z}_{(p)}})$ and $D(\Lambda_{\mathbb{F}_p})$: thick subcategories

Let $f : \Lambda_{\mathbb{Z}_{(p)}} \to \Lambda_{\mathbb{F}_p}$ be the projection induced by $\mathbb{Z}_{(p)} \to \mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)} = \mathbb{F}_p$. This induces adjoint maps $f_\bullet$ and $i_\bullet$ between $D(\Lambda_{\mathbb{Z}_{(p)}})$ and $D(\Lambda_{\mathbb{F}_p})$, as discussed in Chapter 2. There is also a ring map $g : \mathbb{Z}_{(p)} \to \mathbb{F}_p$, and we looked at $g_\bullet : D(\mathbb{Z}_{(p)}) \to D(\mathbb{F}_p)$ in Examples 2.4.3 and 2.4.18. We have the following commutative diagram.

$$\begin{array}{ccc} \Lambda_{\mathbb{Z}_{(p)}} & \xrightarrow{f} & \Lambda_{\mathbb{F}_p} \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle j} \\ \mathbb{Z}_{(p)} & \xrightarrow[g]{} & \mathbb{F}_p \end{array}$$

**Proposition 4.1.1.** *The maps $f, g, h,$ and $j$ induce functors on derived categories such that the following diagram commutes.*

$$\begin{array}{ccc} D(\Lambda_{\mathbb{Z}_{(p)}}) & \xrightarrow{f_\bullet} & D(\Lambda_{\mathbb{F}_p}) \\ {\scriptstyle h_\bullet}\downarrow & & \downarrow{\scriptstyle j_\bullet} \\ D(\mathbb{Z}_{(p)}) & \xrightarrow[g_\bullet]{} & D(\mathbb{F}_p) \end{array}$$

**Proof.** As ring maps, $j \circ f = g \circ h$, so it suffices to show that $j_\bullet \circ f_\bullet = (j \circ f)_\bullet$, in the general case of composed ring maps $R \xrightarrow{f} S \xrightarrow{j} T$. These induce maps

$$\begin{array}{ccc} \textsf{Mod-}R & \xrightarrow{-\otimes_R S} & \textsf{Mod-}S \\ & {\scriptstyle -\otimes_R T}\searrow & \downarrow{\scriptstyle -\otimes_S T} \\ & & \textsf{Mod-}T \end{array} \quad ,$$

and this diagram commutes because for all $M$,

$$M \otimes_R T \cong (M \otimes_R S) \otimes_S T.$$

This implies that $(j \circ f)_* = j_* \circ f_*$ at the chain complex level, and so also at the homotopy category level.

In unbounded derived categories, every object is equivalent to a cellular tower (see Section 1.3). Because $f_\bullet$ sends $R$ to $S$, it sends cellular towers to cellular towers. The General Existence Theorem [Wei94, 10.8.2] then implies there is a natural isomorphism

$$(j \circ f)_\bullet = L((j \circ f)_*) \cong L(j_*) \circ L(f_*) = j_\bullet \circ f_\bullet.$$

□

In a slight abuse of notation, we let $i_\bullet$ denote all the forgetful functors, since the context will be clear.

**Proposition 4.1.2.** *There is at least one nontrivial thick subcategory of finite objects in $D(\Lambda_{\mathbb{Z}_{(p)}})$, generated by $i_\bullet\Lambda_{\mathbb{F}_p}$.*

**Proof.** It's not hard to see that in $D(\Lambda_{\mathbb{Z}_{(p)}})$ there is a quasi-isomorphism between $i_\bullet\Lambda_{\mathbb{F}_p}$ and the chain complex

$$\left(\cdots \to 0 \to \Lambda_{\mathbb{Z}_{(p)}} \xrightarrow{p} \Lambda_{\mathbb{Z}_{(p)}} \to 0 \to \cdots\right).$$

Let $\mathfrak{q} = (x_1, x_2, ...) \in \operatorname{\mathsf{Spec}} \Lambda_{\mathbb{Z}_{(p)}}$. First we will show that $i_\bullet\Lambda_{\mathbb{F}_p} \wedge \overline{k_\mathfrak{q}} = 0$.

To compute $i_\bullet\Lambda_{\mathbb{F}_p} \wedge \overline{k_\mathfrak{q}}$, we can replace $i_\bullet\Lambda_{\mathbb{F}_p}$ with the projective resolution above, and then tensor with $k_\mathfrak{q}$. Thus $i_\bullet\Lambda_{\mathbb{F}_p} \wedge \overline{k_\mathfrak{q}}$ is represented by

$$\left(\cdots \to 0 \to k_\mathfrak{q} \xrightarrow{p} k_\mathfrak{q} \to 0 \to \cdots\right).$$

Of course, in $k_\mathfrak{q}$ everything in $\Lambda_{\mathbb{Z}_{(p)}} \backslash (x_1, x_2, ...)$ is invertible, so $p$ is invertible and this chain complex is exact. This shows that $i_\bullet\Lambda_{\mathbb{F}_p} \wedge \overline{k_\mathfrak{q}} = 0$.

It follows that everything in $\mathsf{th}(i_\bullet\Lambda_{\mathbb{F}_p})$ smashes with $\overline{k_\mathfrak{q}}$ to zero, so $\Lambda_{\mathbb{Z}_{(p)}} \notin \mathsf{th}(i_\bullet\Lambda_{\mathbb{F}_p})$. Therefore $\mathsf{th}(i_\bullet\Lambda_{\mathbb{F}_p}) \subsetneq \mathsf{th}(\Lambda_{\mathbb{Z}_{(p)}})$. Clearly $\mathsf{th}(i_\bullet\Lambda_{\mathbb{F}_p}) \neq \mathsf{th}(0)$. □

**Remark 4.1.3.** In our graded context, Thomason's classification of thick subcategories does not apply. But it's perhaps worth noting that if we were to neglect the grading on $\Lambda_{\mathbb{Z}_{(p)}}$ and $\Lambda_{\mathbb{F}_p}$, we could conclude that $\mathsf{th}(i_\bullet\Lambda_{\mathbb{F}_p})$ is the unique nontrivial thick subcategory of finite objects in $D(\Lambda_{\mathbb{Z}_{(p)}})$. For there is one nontrivial Thomason-closed subset of $\operatorname{\mathsf{Spec}} \Lambda_{\mathbb{Z}_{(p)}}$, namely $\{(p, x_1, x_2, ...)\}$. This corresponds to a unique nontrivial thick subcategory $\mathcal{D} = \{W \in \mathcal{F} \ : \ \mathrm{supp}(W) \subseteq \{(p, x_1, x_2, ...)\}\}$. Since $i_\bullet\Lambda_{\mathbb{F}_p} \neq 0$, the above calculation shows that $i_\bullet\Lambda_{\mathbb{F}_p}$ must be supported in $\{(p, x_1, x_2, ...)\}$, and therefore be in $\mathcal{D}$. Thus, again, $0 \neq \mathsf{th}(i_\bullet\Lambda_{\mathbb{F}_p}) \subseteq \mathcal{D}$, so $\mathsf{th}(i_\bullet\Lambda_{\mathbb{F}_p}) = \mathcal{D}$.

### *4.2* $D(\Lambda_{\mathbb{Z}_{(p)}})$, $D(\Lambda_{\mathbb{F}_p})$, *and* $D(\Lambda_{\mathbb{Q}})$: *Bousfield lattices*

**Proposition 4.2.1.** *Let $f : \Lambda_{\mathbb{Z}_{(p)}} \to \Lambda_{\mathbb{F}_p}$ be the obvious projection. For all $X$ in $D(\Lambda_{\mathbb{F}_p})$, we have $f_\bullet i_\bullet X \cong X \oplus \Sigma X$. Therefore $\mathsf{th}(f_\bullet i_\bullet X) = \mathsf{th}(X)$ and $\langle f_\bullet i_\bullet X\rangle = \langle X\rangle$.*

**Proof.** First, using $f_\bullet \Lambda_{\mathbb{Z}_{(p)}} = \Lambda_{\mathbb{F}_p}$, and Remark 2.1.4, we see that $f_\bullet i_\bullet \Lambda_{\mathbb{F}_p}$ in $D(\Lambda_{\mathbb{F}_p})$ is

$$\left(\cdots \to 0 \to \Lambda_{\mathbb{F}_p} \xrightarrow{0} \Lambda_{\mathbb{F}_p} \to 0 \to \cdots\right),$$

which is just $\Lambda_{\mathbb{F}_p} \oplus \Sigma\Lambda_{\mathbb{F}_p}$.

Next, let $W \in D(\Lambda_{\mathbb{F}_p})$ be a finite object. Then $W$ can be represented by a bounded-below complex of projective $\Lambda_{\mathbb{F}_p}$-modules. Since $\Lambda_{\mathbb{F}_p}$ is local, projectives are free, and we can represent $W$ by some free resolution

$$\cdots \to \coprod_{I_3} \Lambda_{\mathbb{F}_p} \xrightarrow{d_2} \coprod_{I_2} \Lambda_{\mathbb{F}_p} \xrightarrow{d_1} \coprod_{I_1} \Lambda_{\mathbb{F}_p} \to 0.$$

Each differential is a direct sum of maps $\Lambda_{\mathbb{F}_p} \to \Lambda_{\mathbb{F}_p}$, which we can think of as elements of $\Lambda_{\mathbb{F}_p}$.

Applying $i_\bullet$, we get

$$\cdots \to \coprod_{I_3} i_\bullet\Lambda_{\mathbb{F}_p} \xrightarrow{d_2} \coprod_{I_2} i_\bullet\Lambda_{\mathbb{F}_p} \xrightarrow{d_1} \coprod_{I_1} i_\bullet\Lambda_{\mathbb{F}_p} \to 0,$$

which is

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \coprod_{I_2} \Lambda_{\mathbb{Z}_{(p)}} & \xrightarrow{\overline{d_1}} & \coprod_{I_1} \Lambda_{\mathbb{Z}_{(p)}} & \longrightarrow & 0 & \longrightarrow & 0 & \\
 & \oplus & \searrow^{\oplus p} & \oplus & \searrow^{\oplus p} & \oplus & & & \\
\cdots \longrightarrow & \coprod_{I_3} \Lambda_{\mathbb{Z}_{(p)}} & \xrightarrow[\overline{d_2}]{} & \coprod_{I_2} \Lambda_{\mathbb{Z}_{(p)}} & \xrightarrow[\overline{d_1}]{} & \coprod_{I_1} \Lambda_{\mathbb{Z}_{(p)}} & \longrightarrow & 0. &
\end{array}$$

Here $\overline{d_i}$ is a direct sum of maps $\Lambda_{\mathbb{Z}_{(p)}} \to \Lambda_{\mathbb{Z}_{(p)}}$ that correspond to preimages via $f : \Lambda_{\mathbb{Z}_{(p)}} \to \Lambda_{\mathbb{F}_p}$ of the elements of $\Lambda_{\mathbb{F}_p}$ comprising $d_i$. Since this is a bounded-below complex of projective $\Lambda_{\mathbb{Z}_{(p)}}$-modules, we can compute $f_\bullet(i_\bullet W)$ by applying $-\otimes_{\Lambda_{\mathbb{Z}_{(p)}}} \Lambda_{\mathbb{F}_p}$ directly. Then again using Remark 2.1.4, we see that this gives $f_\bullet i_\bullet W = W \oplus \Sigma W$.

Now take $X \in D(\Lambda_{\mathbb{F}_p})$ to be arbitrary. Then [HPS97, Thm.4.2.4] implies that every object in $D(\Lambda_{\mathbb{F}_p})$ is the filtered colimit of finite objects. Suppose $X = \mathsf{colim}\, W_i$, for finite $W_i$. Then since $f_\bullet$ and $i_\bullet$ commute with coproducts and are exact, they commute with filtered colimits [Mar83, Prop.7(a), Appendix 1.2], and

$$f_\bullet i_\bullet X = f_\bullet i_\bullet(\mathsf{colim}\, W_i) = \mathsf{colim}(f_\bullet i_\bullet W_i) = \mathsf{colim}(W_i \oplus \Sigma W_i)$$

$$= \mathsf{colim}\, W_i \oplus \Sigma(\mathsf{colim}\, W_i) = X \oplus \Sigma X.$$

□

The previous result is significant, since it allows us to apply all the results of Section 2.3 to $D(\Lambda_{\mathbb{Z}_{(p)}}) \rightleftarrows D(\Lambda_{\mathbb{F}_p})$. In particular, it implies that $f_\bullet$ induces a lattice isomorphism

$$\mathsf{BL}_{D(\Lambda_{\mathbb{Z}_{(p)}})}/J \xrightarrow{\cong} \mathsf{BL}_{D(\Lambda_{\mathbb{F}_p})},$$

where $J = \{\langle X\rangle \mid f_\bullet\langle X\rangle = \langle 0\rangle\}$. Our next goal is to show that this is actually a splitting.

Given a self-map, the cofiber and the sequential colimit form a complemented pair of Bousfield classes [HPS97, 3.6.9]. The cofiber of $\Lambda_{\mathbb{Z}_{(p)}} \xrightarrow{p} \Lambda_{\mathbb{Z}_{(p)}}$ is $i_\bullet \Lambda_{\mathbb{F}_p}$. The sequential colimit $p^{-1}\Lambda_{\mathbb{Z}_{(p)}}$ is a module concentrated in degree zero, with zeroth homology

$$\mathsf{colim}(\Lambda_{\mathbb{Z}_{(p)}} \xrightarrow{p} \Lambda_{\mathbb{Z}_{(p)}} \to \cdots) = \frac{\mathbb{Q}[x_1, x_2, ...]}{(x_1^{n_1}, x_2^{n_2}, ...)} =: \Lambda_{\mathbb{Q}}.$$

The map $g : \Lambda_{\mathbb{Z}_{(p)}} \to \Lambda_{\mathbb{Q}}$, coming from $\mathbb{Z}_{(p)} \hookrightarrow \mathbb{Q}$, induces functors $g_\bullet$ and $i_\bullet$ between $D(\Lambda_{\mathbb{Z}_{(p)}})$ and $D(\Lambda_{\mathbb{Q}})$, and we can identify the sequential colimit $p^{-1}\Lambda_{\mathbb{Z}_{(p)}}$ in $D(\Lambda_{\mathbb{Z}_{(p)}})$ with $i_\bullet \Lambda_{\mathbb{Q}}$. Therefore in the Bousfield lattice of $D(\Lambda_{\mathbb{Z}_{(p)}})$ we have

$$\langle i_\bullet \Lambda_{\mathbb{F}_p}\rangle \vee \langle i_\bullet \Lambda_{\mathbb{Q}}\rangle = \langle \Lambda_{\mathbb{Z}_{(p)}}\rangle \text{ and } \langle i_\bullet \Lambda_{\mathbb{F}_p}\rangle \wedge \langle i_\bullet \Lambda_{\mathbb{Q}}\rangle = \langle 0\rangle.$$

**Proposition 4.2.2.** *The sublattice* $\mathsf{BA}$ *is non-trivial in the Bousfield lattice of* $D(\Lambda_{\mathbb{Z}_{(p)}})$*, whereas* $\mathsf{BA}$ *is trivial in the Bousfield lattice of* $D(\Lambda_{\mathbb{F}_p})$*.*

**Proof.** We've just demonstrated non-trivial complemented classes in $D(\Lambda_{\mathbb{Z}_{(p)}})$. Corollary 7.4 in [DP08] shows that $\mathsf{BA}_{D(\Lambda_{\mathbb{F}_p})}$ is trivial. $\square$

One of the most interesting properties of the Bousfield lattice of $D(\Lambda_{\mathbb{F}_p})$ is that the class $\langle I(\Lambda_{\mathbb{F}_p})\rangle$ is a minimum among nonzero classes, where $I(\Lambda_{\mathbb{F}_p}) = \mathrm{Hom}^*_{\mathbb{F}_p}(\Lambda_{\mathbb{F}_p}, \mathbb{F}_p)$ is the graded vector-space dual to $\Lambda_{\mathbb{F}_p}$, concentrated in chain degree zero (see Section 1.6.3). It's not hard to see that if every nonzero $\langle X\rangle$ has $\langle X\rangle \geq \langle I(\Lambda_{\mathbb{F}_p})\rangle$, then there are no nontrivial complemented Bousfield classes. The contrapositive of this implies the following.

**Corollary 4.2.3.** *Unlike in $\mathsf{BL}_{D(\Lambda_{\mathbb{F}_p})}$, in the Bousfield lattice of $D(\Lambda_{\mathbb{Z}_{(p)}})$ there is no Bousfield class that is a minimum among nonzero classes.*

We know that $\mathsf{th}(i_\bullet\Lambda_{\mathbb{F}_p})$ is nontrivial thick subcategory of finite objects in $D(\Lambda_{\mathbb{Z}_{(p)}})$. Let $L : D(\Lambda_{\mathbb{Z}_{(p)}}) \to D(\Lambda_{\mathbb{Z}_{(p)}})$ be finite localization at $\mathsf{th}(i_\bullet\Lambda_{\mathbb{F}_p})$ (see e.g. [HPS97, Ch.3] for a discussion of Bousfield localization). Recall that we say an object $X$ is *$L$-acyclic* when $LX = 0$.

**Proposition 4.2.4.** *The localization functor $L$ has the following acyclics and locals.*

$$L\text{-}acyclics = \mathsf{loc}(i_\bullet\Lambda_{\mathbb{F}_p}) = i_\bullet\Lambda_{\mathbb{Q}}\text{-}acyclics = L\Lambda_{\mathbb{Z}_{(p)}}\text{-}acyclics,$$
$$L\text{-}locals = \mathsf{loc}(i_\bullet\Lambda_{\mathbb{Q}}) = i_\bullet\Lambda_{\mathbb{F}_p}\text{-}acyclics.$$

**Proof.** All finite localizations are smashing localizations, which means $LX = L\Lambda_{\mathbb{Z}_{(p)}} \wedge X$. Thus the $L$-acyclics are precisely the $L\Lambda_{\mathbb{Z}_{(p)}}$-acyclics. Finite localization at $\mathsf{th}(i_\bullet\Lambda_{\mathbb{F}_p})$ means also that the $L$-acyclics are $\mathsf{loc}(i_\bullet\Lambda_{\mathbb{F}_p})$.

Next we show that the $L$-acyclics are the same as the $i_\bullet\Lambda_{\mathbb{Q}}$-acyclics. Suppose $X$ is $L$-acyclic. Then $X \in \mathsf{loc}(i_\bullet\Lambda_{\mathbb{F}_p})$. Since $i_\bullet\Lambda_{\mathbb{F}_p} \wedge i_\bullet\Lambda_{\mathbb{Q}} = 0$, this implies that $X \wedge i_\bullet\Lambda_{\mathbb{Q}} = 0$. Suppose instead that $X$ has $X \wedge i_\bullet\Lambda_{\mathbb{Q}} = 0$. Then

$$\langle X \wedge i_\bullet\Lambda_{\mathbb{F}_p}\rangle \vee \langle X \wedge i_\bullet\Lambda_{\mathbb{Q}}\rangle = \langle X \wedge \Lambda_{\mathbb{Z}_{(p)}}\rangle = \langle X\rangle,$$

so $\langle X \wedge i_\bullet\Lambda_{\mathbb{F}_p}\rangle = \langle X\rangle$ and $\langle X\rangle \leq \langle i_\bullet\Lambda_{\mathbb{F}_p}\rangle$. Since $i_\bullet\Lambda_{\mathbb{F}_p} \in \mathsf{loc}(i_\bullet\Lambda_{\mathbb{F}_p})$ is $L$-acyclic, we have $L\Lambda_{\mathbb{Z}_{(p)}} \wedge i_\bullet\Lambda_{\mathbb{F}_p} = 0$, so $L\Lambda_{\mathbb{Z}_{(p)}} \wedge X = LX = 0$, and $X$ is $L$-acyclic.

In general, a localization functor $L : \mathcal{C} \to \mathcal{C}$ determines a colocalization functor $C : \mathcal{C} \to \mathcal{C}$, such that for each $X$ we have an exact triangle $CX \to X \to LX$ [HPS97, 3.1]. With a smashing localization, the classes $\langle LS^0 \rangle$ and $\langle CS^0 \rangle$ are complemented, where $S^0$ is the sphere object in $\mathcal{C}$. Furthermore, the $L$-locals are precisely the $CS^0$-acyclics. It's not hard to show that complements are unique. Thus in the present context, since $L : D(\Lambda_{\mathbb{Z}_{(p)}}) \to D(\Lambda_{\mathbb{Z}_{(p)}})$ is smashing and $\langle L\Lambda_{\mathbb{Z}_{(p)}} \rangle = \langle i_\bullet \Lambda_{\mathbb{Q}} \rangle$ is complemented by $\langle i_\bullet \Lambda_{\mathbb{F}_p} \rangle$, we see that the $L$-locals are precisely the $i_\bullet \Lambda_{\mathbb{F}_p}$-acyclics.

It remains to show that the $L$-locals are precisely $\mathsf{loc}(i_\bullet \Lambda_{\mathbb{Q}})$. Since $L$ is smashing, $L$ commutes with coproducts, and the $L$-locals form a localizing subcategory. Thus $i_\bullet \Lambda_{\mathbb{Q}} \wedge i_\bullet \Lambda_{\mathbb{F}_p} = 0$ implies that $\mathsf{loc}(i_\bullet \Lambda_{\mathbb{Q}})$ is contained in the $L$-locals.

Now assume that $X$ is $L$-local. We will show that $X \in \mathsf{loc}(i_\bullet \Lambda_{\mathbb{Q}})$. Recall that we have an inclusion $g : \Lambda_{\mathbb{Z}_{(p)}} \to \Lambda_{\mathbb{Q}}$. First we compute $g_\bullet i_\bullet \Lambda_{\mathbb{Q}}$, using the $K$-flat resolutions in [Spa88] (see also Section 1.3), and noting that $K$-flat replacement, and the following two quoted results, apply in the graded setting as well as the ungraded setting. In $D(\Lambda_{\mathbb{Z}_{(p)}})$, $i_\bullet \Lambda_{\mathbb{Q}}$ is $\mathsf{colim}(\Lambda_{\mathbb{Z}_{(p)}} \xrightarrow{p} \Lambda_{\mathbb{Z}_{(p)}} \to \cdots)$. Since $\Lambda_{\mathbb{Z}_{(p)}}$ is flat as a $\Lambda_{\mathbb{Z}_{(p)}}$-module, [Spa88, Prop. 5.2] implies that it is $K$-flat. Then [Spa88, Prop. 5.4(c)] shows that filtered colimits of $K$-flat complexes are $K$-flat. Furthermore, since $f_\bullet$ commutes with coproducts, it commutes with direct limits. So to compute $g_\bullet(i_\bullet \Lambda_{\mathbb{Q}})$, we use this colimit description, and get

$$\begin{aligned} g_\bullet i_\bullet \Lambda_{\mathbb{Q}} &= g_\bullet \left( \mathsf{colim}(\Lambda_{\mathbb{Z}_{(p)}} \xrightarrow{p} \Lambda_{\mathbb{Z}_{(p)}} \to \cdots) \right) \\ &= \mathsf{colim}(g_\bullet \Lambda_{\mathbb{Z}_{(p)}} \xrightarrow{p} g_\bullet \Lambda_{\mathbb{Z}_{(p)}} \to \cdots) \\ &= \mathsf{colim}(\Lambda_{\mathbb{Q}} \xrightarrow{p} \Lambda_{\mathbb{Q}} \to \cdots) \\ &= \Lambda_{\mathbb{Q}}. \end{aligned}$$

The inclusion $\mathbb{Z}_{(p)} \hookrightarrow \mathbb{Q}$ induces a morphism $\Lambda_{\mathbb{Z}_{(p)}} \to i_\bullet \Lambda_{\mathbb{Q}}$ in $D(\Lambda_{\mathbb{Z}_{(p)}})$. Let $F$ be the fiber of this map. Then we have an exact triangle

$$F \wedge i_\bullet \Lambda_{\mathbb{Q}} \to i_\bullet \Lambda_{\mathbb{Q}} \to i_\bullet \Lambda_{\mathbb{Q}} \wedge i_\bullet \Lambda_{\mathbb{Q}}.$$

The projection formula, Lemma 2.1.2, implies that $i_\bullet\Lambda_{\mathbb{Q}} \wedge i_\bullet\Lambda_{\mathbb{Q}} = i_\bullet(g_\bullet i_\bullet\Lambda_{\mathbb{Q}} \wedge \Lambda_{\mathbb{Q}}) = i_\bullet g_\bullet i_\bullet\Lambda_{\mathbb{Q}}$, and we just showed that this is $i_\bullet\Lambda_{\mathbb{Q}}$. Therefore $F \wedge i_\bullet\Lambda_{\mathbb{Q}} = 0$ in $D(\Lambda_{\mathbb{Z}_{(p)}})$. From the complementation relationship, this implies $\langle F \wedge i_\bullet\Lambda_{\mathbb{F}_p}\rangle = \langle F\rangle$, so $\langle F\rangle \leq \langle i_\bullet\Lambda_{\mathbb{F}_p}\rangle$.

Now if $X$ is $L$-local, then $X \wedge i_\bullet\Lambda_{\mathbb{F}_p} = 0$, so $X \wedge F = 0$. From the triangle $F \to \Lambda_{\mathbb{Z}_{(p)}} \to i_\bullet\Lambda_{\mathbb{Q}}$, this implies that the map $X = X \wedge \Lambda_{\mathbb{Z}_{(p)}} \to X \wedge i_\bullet\Lambda_{\mathbb{Q}}$ is an equivalence. Since $X \in \mathsf{loc}(\Lambda_{\mathbb{Z}_{(p)}}) = D(\Lambda_{\mathbb{Z}_{(p)}})$, we have

$$X = X \wedge i_\bullet\Lambda_{\mathbb{Q}} \in \mathsf{loc}(\Lambda_{\mathbb{Z}_{(p)}} \wedge i_\bullet\Lambda_{\mathbb{Q}}) = \mathsf{loc}(i_\bullet\Lambda_{\mathbb{Q}}).$$

□

The next theorem is the main result of this section, and will allow us to extend many results about $D(\Lambda_{\mathbb{F}_p})$ to $D(\Lambda_{\mathbb{Z}_{(p)}})$. Recall that a lattice is a poset with joins and meets. Given lattices $\mathsf{A}$ and $\mathsf{B}$, the product lattice is defined as the set product $\mathsf{A} \times \mathsf{B}$, with $(a, b) \leq (c, d)$ precisely when $a \leq c$ and $b \leq d$, and joins and meets are defined termwise. A sublattice is simply a sub-poset that is a lattice under the same meet and join operations. A morphism of lattices is a set map that preserves joins and meets (and hence is order-preserving).

Up to this point, we have looked at Bousfield lattices of categories of the form $D(R)$. However, we can apply the Bousfield equivalence relation to any localizing subcategories as well.

Recall that for a fixed prime $p$ and integers $n_i > 1$, we set $\deg(x_i) = 2^i$ and define

$$\Lambda_{\mathbb{Z}_{(p)}} := \frac{\mathbb{Z}_{(p)}[x_1, x_2, ...]}{(x_1^{n_1}, x_2^{n_2}, ...)}, \quad \Lambda_{\mathbb{F}_p} := \frac{\mathbb{F}_p[x_1, x_2, ...]}{(x_1^{n_1}, x_2^{n_2}, ...)} \quad \text{and} \quad \Lambda_{\mathbb{Q}} := \frac{\mathbb{Q}[x_1, x_2, ...]}{(x_1^{n_1}, x_2^{n_2}, ...)}.$$

**Theorem 4.2.5.** *The functor $f_\bullet : D(\Lambda_{\mathbb{Z}_{(p)}}) \to D(\Lambda_{\mathbb{F}_p})$ induces a lattice isomorphism*

$$\mathsf{BL}_{D(\Lambda_{\mathbb{Z}_{(p)}})} \cong \mathsf{BL}_{D(\Lambda_{\mathbb{F}_p})} \times \mathsf{BL}_{\mathsf{loc}(i_\bullet\Lambda_{\mathbb{Q}})}, \ \textit{where}$$

$$\langle X\rangle \mapsto (f_\bullet\langle X\rangle, \langle X \wedge i_\bullet\Lambda_{\mathbb{Q}}\rangle).$$

*The inverse is given by*

$$(\langle Y\rangle, \langle Z\rangle) \mapsto i_\bullet\langle Y\rangle \vee \langle Z\rangle.$$

**Proof.** First we will establish an isomorphism

$$\mathsf{BL}_{D(\Lambda_{\mathbb{Z}_{(p)}})} \xrightarrow{\sim} \mathsf{BL}_{\mathsf{loc}(i_\bullet\Lambda_{\mathbb{F}_p})} \times \mathsf{BL}_{\mathsf{loc}(i_\bullet\Lambda_{\mathbb{Q}})}, \text{ where}$$

$$\langle X\rangle \mapsto \left(\langle X \wedge i_\bullet\Lambda_{\mathbb{F}_p}\rangle, \langle X \wedge i_\bullet\Lambda_{\mathbb{Q}}\rangle\right).$$

Note that $X \in \mathsf{loc}(\Lambda_{\mathbb{Z}_{(p)}})$ implies $X \wedge i_\bullet\Lambda_{\mathbb{F}_p} \in \mathsf{loc}(\Lambda_{\mathbb{Z}_{(p)}} \wedge i_\bullet\Lambda_{\mathbb{F}_p}) = \mathsf{loc}(i_\bullet\Lambda_{\mathbb{F}_p})$, and likewise for $X \wedge i_\bullet\Lambda_{\mathbb{Q}}$. This is a morphism of lattices, because smashes distribute across joins.

Next, note that $W \in \mathsf{loc}(i_\bullet\Lambda_{\mathbb{Q}})$ if and only if $\langle W\rangle \leq \langle i_\bullet\Lambda_{\mathbb{Q}}\rangle$. The forward direction is always true. For the reverse direction use Proposition 4.2.4: $i_\bullet\Lambda_{\mathbb{Q}} \wedge i_\bullet\Lambda_{\mathbb{F}_p} = 0$, so $W \wedge i_\bullet\Lambda_{\mathbb{F}_p} = 0$ and $W$ is $L$-local. A symmetric argument shows that $W \in \mathsf{loc}(i_\bullet\Lambda_{\mathbb{F}_p})$ if and only if $\langle W\rangle \leq \langle i_\bullet\Lambda_{\mathbb{F}_p}\rangle$.

We can check that an inverse to the above lattice morphism is given by $(\langle X\rangle, \langle Y\rangle) \mapsto \langle X\rangle \vee \langle Y\rangle$. On the one hand, we have

$$\langle X \wedge i_\bullet\Lambda_{\mathbb{F}_p}\rangle \vee \langle X \wedge i_\bullet\Lambda_{\mathbb{Q}}\rangle = \langle X\rangle \wedge \left(\langle i_\bullet\Lambda_{\mathbb{F}_p}\rangle \vee \langle i_\bullet\Lambda_{\mathbb{Q}}\rangle\right) = \langle X\rangle \wedge \langle \Lambda_{\mathbb{Z}_{(p)}}\rangle = \langle X\rangle.$$

On the other hand, Proposition 4.2.4 shows that, for $X \in \mathsf{loc}(i_\bullet\Lambda_{\mathbb{F}_p})$ and $Y \in \mathsf{loc}(i_\bullet\Lambda_{\mathbb{Q}})$,

$$\left(\langle X \vee Y\rangle \wedge \langle i_\bullet\Lambda_{\mathbb{F}_p}\rangle, \langle X \vee Y\rangle \wedge \langle i_\bullet\Lambda_{\mathbb{Q}}\rangle\right) = (\langle X\rangle, \langle Y\rangle),$$

because $X$ is $L$-acyclic and $Y$ is $L$-local.

As in Chapter 2, define $J$ to be the image of $Kerf_\bullet$ in $\mathsf{BL}_{D(\Lambda_{\mathbb{Z}_{(p)}})}$. In Proposition 2.1.11, we show that $J = \{\langle X\rangle \mid \langle X\rangle \leq \langle M\rangle\}$, where $\langle M\rangle$ is the join of all elements of $J$. Because of Proposition 4.2.1, we can apply Lemma 2.3.5, which says that the complement of $\langle i_\bullet\Lambda_{\mathbb{F}_p}\rangle$ is $\langle M\rangle$. But we know the complement of $\langle i_\bullet\Lambda_{\mathbb{F}_p}\rangle$ is $\langle i_\bullet\Lambda_{\mathbb{Q}}\rangle$. Thus we conclude that

$$J = \{\langle X\rangle \mid \langle X\rangle \leq \langle i_\bullet\Lambda_{\mathbb{Q}}\rangle\} = \{\langle X\rangle \mid X \in \mathsf{loc}(i_\bullet\Lambda_{\mathbb{Q}})\} = \mathsf{BL}_{\mathsf{loc}(i_\bullet\Lambda_{\mathbb{Q}})}.$$

Combining the above lattice isomorphism with Proposition 2.3.2, $f_\bullet$ induces a lattice isomorphism with inverse $i_\bullet$:

$$\mathsf{BL}_{\mathsf{loc}(i_\bullet\Lambda_{\mathbb{F}_p})} = \mathsf{BL}_{D(\Lambda_{\mathbb{Z}_{(p)}})}/J \xrightarrow{f_\bullet} \mathsf{BL}_{D(\Lambda_{\mathbb{F}_p})}.$$

We've shown there are isomorphisms

$$\mathsf{BL}_{D(\Lambda_{\mathbb{Z}_{(p)}})} \xrightarrow{\sim} \mathsf{BL}_{\mathsf{loc}(i_\bullet\Lambda_{\mathbb{F}_p})} \times \mathsf{BL}_{\mathsf{loc}(i_\bullet\Lambda_{\mathbb{Q}})} \xrightarrow{(f_\bullet,1)} \mathsf{BL}_{D(\Lambda_{\mathbb{F}_p})} \times \mathsf{BL}_{\mathsf{loc}(i_\bullet\Lambda_{\mathbb{Q}})}.$$

The theorem is proved, once we compute that

$$f_\bullet\langle X \wedge i_\bullet\Lambda_{\mathbb{F}_p}\rangle = \langle f_\bullet X\rangle \wedge \langle f_\bullet i_\bullet\Lambda_{\mathbb{F}_p}\rangle = \langle f_\bullet X\rangle \wedge \langle \Lambda_{\mathbb{F}_p}\rangle = \langle f_\bullet X\rangle.$$

□

**Corollary 4.2.6.** *The isomorphism in Theorem 4.2.5 induces a splitting of the distributive lattices and Boolean algebras*

$$\mathsf{DL}_{D(\Lambda_{\mathbb{Z}_{(p)}})} \cong \mathsf{DL}_{D(\Lambda_{\mathbb{F}_p})} \times \mathsf{DL}_{\mathsf{loc}(i_\bullet\Lambda_{\mathbb{Q}})},$$

$$\mathsf{BA}_{D(\Lambda_{\mathbb{Z}_{(p)}})} \cong \mathsf{BA}_{D(\Lambda_{\mathbb{F}_p})} \times \mathsf{BA}_{\mathsf{loc}(i_\bullet\Lambda_{\mathbb{Q}})}.$$

**Proof.** This is straightforward from the definitions. □

**Question 4.2.7.** *In the above splitting, is* $\mathsf{BL}_{\mathsf{loc}(i_\bullet\Lambda_{\mathbb{Q}})} \cong \mathsf{BL}_{D(\Lambda_{\mathbb{Q}})}$*?*

Although this would be an elegant result, it is by no means clear. The isomorphism $\mathsf{BL}_{\mathsf{loc}(i_\bullet\Lambda_{\mathbb{F}_p})} \cong \mathsf{BL}_{D(\Lambda_{\mathbb{F}_p})}$ relied on Proposition 2.3.2, which uses the surjection $\Lambda_{\mathbb{Z}_{(p)}} \to \Lambda_{\mathbb{F}_p}$. We wouldn't expect this result to hold for the injection $\Lambda_{\mathbb{Z}_{(p)}} \hookrightarrow \Lambda_{\mathbb{Q}}$. Furthermore, the natural candidates for an isomorphism between $\mathsf{BL}_{\mathsf{loc}(i_\bullet\Lambda_{\mathbb{Q}})}$ and $\mathsf{BL}_{D(\Lambda_{\mathbb{Q}})}$ do not seem too promising. First, such an isomorphism would need to send the maximum $\langle i_\bullet\Lambda_{\mathbb{Q}}\rangle$ of $\mathsf{BL}_{\mathsf{loc}(i_\bullet\Lambda_{\mathbb{Q}})}$ to the maximum $\langle\Lambda_{\mathbb{Q}}\rangle$ of $\mathsf{BL}_{D(\Lambda_{\mathbb{Q}})}$. The map $f_\bullet : \mathsf{BL}_{\mathsf{loc}(i_\bullet\Lambda_{\mathbb{Q}})} \to \mathsf{BL}_{D(\Lambda_{\mathbb{Q}})}$ doesn't do this, since $\langle f_\bullet i_\bullet\Lambda_{\mathbb{Q}}\rangle \neq \langle\Lambda_{\mathbb{Q}}\rangle$. Second, the collection $\{i_\bullet X \;:\; X \in \mathsf{loc}(\Lambda_{\mathbb{Q}})\} \subset \mathsf{loc}(i_\bullet\Lambda_{\mathbb{Q}})$ is not triangulated, and is significantly smaller than $\mathsf{loc}(i_\bullet \Lambda_{\mathbb{Q}})$. Thus one wouldn't expect the map $i_\bullet : \mathsf{BL}_{D(\Lambda_{\mathbb{Q}})} \to \mathsf{BL}_{\mathsf{loc}(i_\bullet \Lambda_{\mathbb{Q}})}$ to be onto.

The next result shows that the Bousfield lattice of $D(\Lambda_{\mathbb{Z}_{(p)}})$ and $D(\Lambda_{\mathbb{F}_p})$ have the same cardinality.

**Corollary 4.2.8.** *The Bousfield lattice of $D(\Lambda_{\mathbb{Z}_{(p)}})$ has cardinality $2^{2^{\aleph_0}}$.*

**Proof.** Corollary B in [DP08] states that the Bousfield lattice of $D(\Lambda_{\mathbb{F}_p})$ has cardinality $2^{2^{\aleph_0}}$, so $BL_{D(\Lambda_{\mathbb{Z}_{(p)}})}$ is at least as large. However, $\Lambda_{\mathbb{Z}_{(p)}}$ is countable, so [DP01, Thm. 1.2] implies that $\mathsf{BL}_{D(\Lambda_{\mathbb{Z}_{(p)}})}$ has cardinality at most $2^{2^{\aleph_0}}$. □

In the remainder of this chapter, we will illustrate one significant difference and one significant similarity between $D(\Lambda_{\mathbb{F}_p})$ and $D(\Lambda_{\mathbb{Z}_{(p)}})$.

An important result in [DP08] was Theorem 7.1, showing that $\Lambda_{\mathbb{F}_p}$ is local with respect to every nonzero homology theory in $D(\Lambda_{\mathbb{F}_p})$. More specifically, it states

**Theorem 4.2.9.** *[DP08, 7.1] Fix objects $E$ and $X$ in $D(\Lambda_{\mathbb{F}_p})$. If $E \neq 0$ and $X \wedge E = 0$, then $\mathbb{R}Hom_{\Lambda_{\mathbb{F}_p}}(X, \Lambda_{\mathbb{F}_p}) = 0$.*

This theorem fails in $D(\Lambda_{\mathbb{Z}_{(p)}})$.

**Counterexample 4.2.10.** Let $E = \mathbb{Q}$. Since $\mathbb{Q}$ is $p$-local, we can make it a $\Lambda_{\mathbb{Z}_{(p)}}$-module where each $x_i$ acts trivially. Let $X$ be the chain complex

$$0 \longrightarrow \Lambda_{\mathbb{Z}_{(p)}} \xrightarrow{p} \Lambda_{\mathbb{Z}_{(p)}} \longrightarrow 0,$$

concentrated in degrees 0 and $-1$. We claim that $X \wedge \mathbb{Q} = 0$, but $\mathbb{R}\mathrm{Hom}_{\Lambda_{\mathbb{Z}_{(p)}}}(X, \Lambda_{\mathbb{Z}_{(p)}}) \neq 0$.

First, since $X$ is a chain complex of projectives and $\mathbb{Q}$ is a module, $X \wedge \mathbb{Q}$ is equivalent to the chain complex

$$0 \longrightarrow \Lambda_{\mathbb{Z}_{(p)}} \wedge \mathbb{Q} \xrightarrow{p} \Lambda_{\mathbb{Z}_{(p)}} \wedge \mathbb{Q} \longrightarrow 0,$$

$$\text{or} \quad 0 \longrightarrow \mathbb{Q} \xrightarrow{p} \mathbb{Q} \longrightarrow 0.$$

Since $p : \mathbb{Q} \to \mathbb{Q}$ is a $\mathbb{Z}_{(p)}$-isomorphism and the $x_i$ are acting trivially, it is a $\Lambda_{\mathbb{Z}_{(p)}}$-isomorphism, so $X \wedge \mathbb{Q} = 0$ in $D(\Lambda_{\mathbb{Z}_{(p)}})$.

Second, we show that $\mathbb{R}\mathrm{Hom}_{\Lambda_{\mathbb{Z}_{(p)}}}(X, \Lambda_{\mathbb{Z}_{(p)}}) \neq 0$. Consider the degree-zero chain map $f : X \to \Lambda_{\mathbb{Z}_{(p)}}$ given by

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Lambda_{\mathbb{Z}_{(p)}} & \xrightarrow{p} & \Lambda_{\mathbb{Z}_{(p)}} & \longrightarrow & 0 \\ & & {\scriptstyle 1}\downarrow & \swarrow_{s} & & & \\ 0 & \longrightarrow & \Lambda_{\mathbb{Z}_{(p)}} & \longrightarrow & 0. & & \end{array}$$

In order for $f$ to be null-homotopic, we would need a map $s : \Lambda_{\mathbb{Z}_{(p)}} \to \Lambda_{\mathbb{Z}_{(p)}}$ such that $1 = sp$. This would require $s(p) = 1$, which is impossible since $p$ is not inverted in $\mathbb{Z}_{(p)}$. Therefore $f$ is not null-homotopic. Now, $X$ and $\Lambda_{\mathbb{Z}_{(p)}}$ are both cellular complexes in the sense of [Wei94, Ex.10.4.5], and we have the equivalence of categories

$$K_{cell}(\Lambda_{\mathbb{Z}_{(p)}}) \cong D(\Lambda_{\mathbb{Z}_{(p)}}).$$

Therefore

$$\mathrm{Hom}^0_{D(\Lambda_{\mathbb{Z}_{(p)}})}(X, \Lambda_{\mathbb{Z}_{(p)}}) = \mathrm{Hom}^0_{K_{cell}(\Lambda_{\mathbb{Z}_{(p)}})}(X, \Lambda_{\mathbb{Z}_{(p)}}) \neq 0,$$

so $\mathbb{R}\mathrm{Hom}_{\Lambda_{\mathbb{Z}_{(p)}}}(X, \Lambda_{\mathbb{Z}_{(p)}}) \neq 0$.

We conclude with a nilpotence theorem for $D(\Lambda_{\mathbb{Z}_{(p)}})$, similar to Theorem 8.2 of [DP08]. In that theorem, the authors show that in $D(\Lambda_{\mathbb{F}_p})$, the field object $\mathbb{F}_p$ detects nilpotence. See [DP08] or [HPS97, Ch.5] for the relevant definitions. In $D(\Lambda_{\mathbb{Z}_{(p)}})$ we have the following.

**Theorem 4.2.11.** *The object $\mathbb{Z}_{(p)}$ detects nilpotence in $D(\Lambda_{\mathbb{Z}_{(p)}})$, although it is not a field object.*

**Proof.** The proof in [DP08], which is based on the proof of the nilpotence theorem in the category of spectra [HS98], works with a few alterations.

First, we must adapt Lemma 4.3(a) in [DP08], which says that $\Lambda_{\mathbb{F}_p}(S) \in \mathsf{th}(\Lambda_{\mathbb{F}_p}(T))$ if and only if $T$ is cofinite in $S$ (see Definition 6.0.1). In the proof of this lemma, we need only replace each $\mathbb{F}_p$ with $\mathbb{Z}_{(p)}$ and the phrase "(in)finite-dimensional vector space" with"(in)finitely-generated $\mathbb{Z}_{(p)}$-module." For example, if $R \subseteq \mathbb{N}$ is nonempty, then $\mathrm{Ext}^*_{\Lambda_{\mathbb{Z}_{(p)}}(R)}(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)})$ is a non-finitely-generated $\mathbb{Z}_{(p)}$-module. We conclude that in $D(\Lambda_{\mathbb{Z}_{(p)}})$, $\Lambda_{\mathbb{Z}_{(p)}}(S) \in \mathsf{th}(\Lambda_{\mathbb{Z}_{(p)}}(T))$ if and only if $T$ is cofinite in $S$.

Second, note that $\mathbb{Z}_{(p)} = \mathsf{lim}_n \Lambda_{\mathbb{Z}_{(p)}}(n, n+1, ...)$. The rest of the proof of [DP08, Thm. 8.2] goes through, and shows $\mathbb{Z}_{(p)}$ detects nilpotence.

However, $\mathbb{Z}_{(p)}$ is not a field object. If it were, it would detect thick subcategories [HPS97, Cor. 5.2.3], which would imply every thick subcategory of finite objects in $D(\Lambda_{\mathbb{Z}_{(p)}})$ is trivial. We showed in Proposition 4.1.2 that this not the case. $\square$

### 4.3 Experiential context

The idea to investigate $D(\Lambda_{\mathbb{Z}_{(p)}})$ came during the summer of 2009, while I was riding the Trans-Siberian railway between the east coast of Russia and Irkutsk. I had been reading [Mar83, HPS97, HS98, DP08], and so was thinking about $\mathcal{S}$ and $D(\Lambda_{\mathbb{F}_p})$ (or rather $D(\Lambda_k)$) from an axiomatic perspective. It seemed natural to consider $\Lambda_{\mathbb{Z}_{(p)}}$, which, as discussed in the introduction of this chapter, is sort of in between $\Lambda_{\mathbb{F}_p}$ and $\mathcal{S}$.

The rest of the work was done in summer 2011, mostly in South-East Asia. It was a natural idea, to go through [DP08] carefully, seeing what changed when $\Lambda_{\mathbb{F}_p}$ was replaced by $\Lambda_{\mathbb{Z}_{(p)}}$. After I had developed the results and machinery of Chapter 2, I simply applied it to the map $\Lambda_{\mathbb{Z}_{(p)}} \to \Lambda_{\mathbb{F}_p}$, which is quite nice. This was not as much an insight as an application or example.

My mental images for this chapter are similar to those used in Chapter 2. $\mathsf{Spec}\,\Lambda_{\mathbb{F}_p}$

is the simplest of sets, and $\mathsf{Spec}\,\Lambda_{\mathbb{Z}_{(p)}}$ is just one element more complicated. It took several months in summer 2011 to develop comfort and intuition about $\mathbb{Z}_{(p)}$. I see the difference between $\Lambda_{\mathbb{Z}_{(p)}}$ and $\Lambda_{\mathbb{F}_p}$ as conceptually the same as the difference between $\mathbb{Z}_{(p)}$ and $\mathbb{F}_p$, namely that there is one non-trivial prime ideal, and that $\mathbb{Z}_{(p)}$ is not self-dual. I do not yet have a satisfying intuition of product lattices, which is preventing me from grasping the significance of Theorem 4.2.5 and asking new questions.

In Siberia I was traveling with my collaborator and hometown friend, artist Elizabeth McTernan. As I contemplated the space between topology and algebra, she contemplated the space between neighbors and coasts, for a piece we later performed at the south shore of Lake Baikal. And together we discussed the space between mathematics and art.

# Chapter 5

# RECENT WORK AT THE MATH-ART FRONTIER

The main results of Chapters 2, 4, and 6 could be said to reside at the interface of homological algebra and algebraic topology. Inspired by algebraic topology, we have asked questions about the subcategories and Bousfield lattice of classical homological algebraic objects. Any answers we find, in turn, inform our thinking about the general behavior of subcategories and the Bousfield lattice, in triangulated category theory and in algebraic topology.

In this chapter, we discuss recent work at the interface of mathematics and art. Specifically, we will describe and analyze two collaborations with artists, the video piece *Cuculetsu* with visual artist Shannon Wallace, and the performance piece *Imagining Negative-Dimensional Space*, with Berlin-based artist Elizabeth McTernan. The former was recently screened at a five-day workshop on mathematics and the arts, in December 2011, at the Banff International Research Station, Canada. The latter will debut at the Bridges Math-Art Conference in Baltimore, MD, in July 2012.

In analogy with our results at the algebra-topology interface, these artworks translate between the domains of mathematics and art, using the different modes of thought to ask and answer interesting questions. By performing these pieces as works of art, we are able to exhibit aspects of mathematics, and the doing of mathematics, that are poorly represented in conventional mathematical discourse. At the same time, using the content of mathematics – the ideas, the theorems, and the culture – as material, these pieces present novel artistic statements.

Before we elaborate on these two specific pieces of art, we will discuss a general theory of math-art collaboration.

### *5.1 Math-art collaboration theory*

There is a long history of interaction between mathematics and the arts. Indeed, one could argue that since its conception, math has been inspired by, and has in turn inspired, art. However, these math-art works do not always involve the most contemporary mathematics and the most contemporary art. In a review of current math-art work [HS10], it is still common to see Escher drawings and fractals. While pleasant to behold, the math behind these works is rarely deep or contemporary.

In recent years, there has been an increase in popular theater (e.g. *Proof*), film (e.g. *A Beautiful Mind*, *Good Will Hunting*), and television (e.g. *Numb3rs*) about mathematicians. The tendency is towards more nuanced and researched representations of the mathematical lifestyle, but by and large these works still center on mathematical legends and clichés. Furthermore, these depictions rarely contain significant mathematics, but instead focus only on the personal stories of the mathematicians themselves.

The above examples can mostly be characterized as *art about math*. We can think of no examples of such works contributing to mathematics itself. Not only do they lack contemporary mathematics, these common one-sided examples hardly qualify as collaborations.

Recently, the artist Aaron Bocanegra and the mathematician William Kronholm have produced some genuine math-art collaborations. Furthermore, they have developed a theory of math-art collaboration [BK12], which we describe here.

**Definition 5.1.1.** *Interdisciplinary collaboration* is the practice of multiple individuals from multiple disciplines engaging in creative acts which mutually benefit and enrich each discipline.

**Axiom 5.1.2.** It is possible, meaningful, and desirable for artists and mathematicians to engage in interdisciplinary collaboration.

**Axiom 5.1.3.** Collaboration between mathematicians and artists should:

- Actively engage mathematicians and artists in a project,
- Contribute to the field of art,
- Contribute to the field of mathematics,
- Inspire new directions in art,
- Inspire new directions in mathematics.

Thus in order to be considered a true math-art collaboration, at the frontier of both math and art, a piece must involve truly contemporary mathematics and art, and add something new on both sides of the math-art interface. The work described in the following sections should be considered in light of these axioms.

## *5.2 Cuculetsu*

Cuculetsu tries to simultaneously capture the sustained absorption and isolation of deep mathematical thought, and the whimsy and creativity that blossoms therein. It is a 6'16" video, available on YouTube[1]. Shot in a single, 49-minute take, the video was compressed by a factor of eight and put to music. During the video, the author writes in black at a floor-to-ceiling whiteboard, working steadily on a thread of ideas. At the same time, the visual artist Shannon Wallace is filling the whiteboard with colored illustrations - some abstract, some fantastic, some tragic. The two of us do not speak or interact per se, except to move around each other and erase each others' work. One is able to watch the interaction between the mathematical and artistic modes of thought in realtime.

[1]http://youtu.be/bqFJwbXHoww

Mathematically, we see many commutative diagrams presented and used. Such diagrams have a definite visually aesthetic quality, as has been recognized by many mathematicians and non-mathematicians (see [Hof02] for a discussion of commutative diagrams in fine art). In this way, the viewer is able to appreciate, in some sense, the aesthetics of the mathematical content.

However, the video exhibits so much more than just this superficial and static visual element. As a performance, artificially accelerated, it becomes a dance between the mathematician and the illustrator. The creative act of the mathematician is more gradual and subtle, but in the final minute of the video the viewer witnesses a mathematical "Aha" moment. By the end, the board has been wiped clean, leaving no trace of the toil or discovery.

The coherent narrative and soundtrack belie the fact that the piece was improvised and unscripted. Our starting premise was, "Let's see what happens," with no further discussion. We stopped the video recorder at 49 minutes, because it felt right. This artistic process places the piece in the genre of performance art and *Happening*. As we discuss in the next section, Happenings hold a prominent place in 20th century art.

Furthermore, by exhibiting the video with a soundtrack, uploaded to YouTube, we are engaging one of the most contemporary genres of pop art. The YouTube home video - a mostly-boring, usually personal, everyday experience that is captured on video, roughly edited, put to music, and uploaded - is revolutionizing what society considers performance.

The reaction from mathematicians who have viewed the video has been positive, with many saying that the video captures, in a small way, what it is like to do math. Besides the actual result found during the shooting, the video portrays a mode of working, call it *loosely-performative whiteboard research work*, that has consequently been fruitful for the author. By demonstrating and documenting this method of working, we have introduced the possibility that other mathematicians will adopt it and find it fruitful.

### *5.3 Imagining Negative-Dimensional Space*

The goal of this performance piece is to induce the experience of contemplating negative-dimensional space. *Imagining Negative-Dimensional Space* takes the form of a 75-minute participatory workshop, accessible to a layperson. It is a collaboration with the Berlin-based artist Elizabeth McTernan, who has exhibited work throughout Europe and the US, as well as in Japan and Vietnam.

By a "negative-dimensional space," we simply mean an object in the category of spectra. The workshop is not a lecture about stable homotopy theory, but is a performance art piece of the performance/lecture genre.

In a conference paper co-authored with McTernan [WM12], we outline the art-historical context for this piece.

> For the past century, intersecting with advances in Modernism, formalism, and material exploration, there has been a fundamental shift in the material paradigm of art. First the invention of photography forced us to reconsider the meaning of image-making itself, shifting the emphasis towards the act of making and the non-pictorial function of material, a.k.a. Modernism. Then, with advent of more streamlined mass production in the 1950s and 1960s, the importance of materials themselves came under question, and Western art was philosophically launched into Phenomenology, post-structuralism, and a (then) new notion of idea-as-material, a.k.a. Conceptual art.
>
> Thus, we now have a long-established operating system wherein artists approach works with an immaterial theoretical structure, and any material qualities that do emerge do so out of formal necessity, but otherwise are

> not essential to the "existence" of the piece. Formality (that is, that which pertains to Form) is not a prerequisite for art. Substance is defined as something altogether different.
>
> Another movement that developed out of Conceptual art was the work of Happenings in the 1960s, which over subsequent generations has transmuted into various forms of performance art. Much of performance art is not explicitly theatrical with clear-cut subject-object terms, but is socially "relational." Relational art is concisely defined as "a set of artistic practices which take as their theoretical and practical point of departure the whole of human relations and their social context, rather than an independent and private space" [Bou98].
>
> Therefore, the act of performance can manifest in many more ambiguous forms than that which is tableau-oriented, such as public intervention, public interruption, invisible actions, symposiums, lectures, workshops - all producing intersubjective encounters, disbanding conventional producer-audience roles, toppling objectness. But while these actions exist in ambiguous forms, let it be clear that the conceptual intentionality is highly precise. The purpose is not to behold a discrete object, but rather to re-examine the subject, reconsider ourselves.

Here, mathematics directly informs contemporary art, by presenting a new type of object-less art. At the core is negative-dimensional space, which does not really exist as such. An elaborate mathematical explanation might give one the taste of experiencing spectra, but in the performance we refrain from such an explanation. Instead, the participant is left with imagining shadows of an object that does not exist in any real sense.

To facilitate this imagining, during the performance we guide participants through a range of thought experiments. These mental exercises demonstrate and underscore the limitations of conventional understanding of space and dimensions. In particular, we describe various ways of constructing $(n+1)$-spheres or $(n-1)$-spheres from $n$-spheres, and attempt to iterate these outside the range $0 \leq n \leq 2$.

Through guided movement and repetitions of various types, we invoke the analogy with positive numbers and the number line. Just as we can replace the metaphor "numbers = piles" with "numbers = points on a line" [LN00], can we break free of our understanding of dimension and build a new metaphor? Such breakthroughs in imagery appear suggestively throughout the workshop.

In our attempt to communicate a rigorous understanding with non-rigorous means, we also embrace the current trend in performance art known as ritual art. The workshop is designed to encourage kinesthetic intelligence, and the non-rational thought that blossoms in trance-like states. We conduct the workshop, at times, as a ceremony of esoteric knowledge. In addition to helping induce an understanding of negative-dimensional space, the use of ritual functions as a critique of the role of mathematics in society, following [Enz99].

Written descriptions or other documentation of performance art pieces are necessarily deficient and problematic. However, we see *Imagining Negative-Dimensional Space* as a true math-art collaboration, in the sense of Bocanegra and Kronholm.

# Chapter 6

# A TELESCOPE CONSTRUCTION IN $D(\Lambda)$

Recall from Definition 1.6.9, that for a countable field $k$ and integers $n_i > 1, i \geq 1$, we define

$$\Lambda = \Lambda_k = \frac{k[x_1, x_2, x_3, ...]}{(x_1^{n_1}, x_2^{n_2}, x_3^{n_3}, ...)}.$$

For notational simplicity, in this chapter we will write $\Lambda$ for $\Lambda_k$. **Let $D(\Lambda)$ be the unbounded derived category of chain complexes of graded modules over $\Lambda$.** We grade the $x_i$ so that $\Lambda$ is graded-connected and locally finite (i.e. finite-dimensional in each degree). In previous chapters, we have done this by setting $\deg(x_i) = 2^i$ specifically. However, in this chapter, different sections require us to grade $\Lambda$ differently, as will become clear. The category $D(\Lambda)$ was studied in [DP08]. As discussed in Section 1.6.9, $D(\Lambda)$ is a generalization of the categories $D(\Lambda_{\mathbb{F}_p})$ and $D(\Lambda_{\mathbb{Q}})$ of Chapter 4. The following objects were defined and developed in [DP08].

**Definition 6.0.1.** 1. Let $S \subseteq \mathbb{N}$ be any subset of the natural numbers. Define

$$\Lambda(S) = \frac{k[x_i \ : \ i \in S]}{(x_i^{n_i} \ : \ i \in S)}.$$

We give this a $\Lambda$-action by having $x_j$, $j \notin S$, act trivially.

2. Let $\mathbb{R}\mathrm{Hom}_k(-,-)$ denote $\mathbb{R}\mathrm{Hom}$ in the derived category of graded $k$-modules. For any $X$ in $D(\Lambda)$, define $I(X) = \mathbb{R}\mathrm{Hom}_k(X, k)$. Note that for a graded $\Lambda$-module $N$, since $k$ is self-injective, $I(N)$ is (represented by) $\mathrm{Hom}_k^*(N, k)$, the graded $k$-vector space dual. Since $\Lambda$ is commutative, this has a right $\Lambda$-module structure defined by $(f.\sigma)(x) = (\sigma.f)(x) = f(x.\sigma)$ for $f \in I(\mathbb{N})$, $\sigma \in \Lambda$, and $x \in N$. In the same way, for any object $X$ in $D(\Lambda)$, we can think of $I(X)$ as an object of $D(\Lambda)$.

3. For $S \subseteq \mathbb{N}$, set $I(S) = I(\Lambda(S)) \cong \mathrm{Hom}_k^*(\Lambda(S), k)$.

Using this notation, then, we have $I(\mathbb{N}) = I(\Lambda(\mathbb{N})) = I(\Lambda) \cong \mathrm{Hom}_k^*(\Lambda, k)$, the graded dual of $\Lambda$. As a graded $\Lambda$-module, $I(\mathbb{N})$ is concentrated in non-positive degrees, and $I(I(\mathbb{N})) \cong \Lambda$ because $\Lambda$ is finite-dimensional in each degree. Of course, we will repeatedly consider $\Lambda$-modules as chain complexes, concentrated at chain degree zero. Note that, as we are taking the tensor-triangulated approach (see Section 1.2), we will denote the homology group $H_n(X)$ of any object $X$ as $\pi_n(X)$.

**Notation 6.0.2.** Given a graded $\Lambda$-module $M = \oplus_{i \in \mathbb{Z}} M_i$, let $M[s]$ denote the shifted $\Lambda$-module with $(M[s])_i = M_{i-s}$.

In Section 6.1, we construct an object $\mathsf{Tel}$ and an isomorphism $\tau : \pi_n(\mathsf{Tel}) \xrightarrow{\sim} I(\mathbb{N})[r]$ for all $n \in \mathbb{Z}$, and a fixed $r$. In the case that each $n_i = 2$, we show that $r = 2$; in the general case $r$ depends on the choices of $n_i$. Using this object, we prove the following, our main result of this chapter.

**Theorem 6.1.4**. In $D(\Lambda)$ there are objects that are not Bousfield equivalent to any module. Specifically, every $I(\mathbb{N})$-acyclic object that is not $\mathsf{Tel}$-acyclic cannot be Bousfield equivalent to a module.

Note that the proof is not constructive. In particular, we do not know if the object $\mathsf{Tel}$ is Bousfield equivalent to a module; it is itself $\mathsf{Tel}$-acyclic and $I(\mathbb{N})$-acyclic, so the theorem doesn't apply to it.

This answers a question posed by Dwyer and Palmieri in [DP08]. In the derived category of an ungraded Noetherian ring, an object $X$ is Bousfield equivalent to the module (see [Nee92] or Section 1.6)

$$\bigoplus_{\mathfrak{p} \in \mathrm{supp}(X)} k_{\mathfrak{p}}.$$

Question 5.8 in [DP08] asks if for an arbitrary commutative ring $R$, every object in $D(R)$ is Bousfield equivalent to a module. Theorem 6.1.4 shows this is not the case, and thus illustrates another significant difference between the Noetherian and non-Noetherian settings.

Section 6.2 generalizes Section 6.1 slightly. For a given $S \subseteq \mathbb{N}$, we build an object, also denoted $\mathsf{Tel}$, and an isomorphism $\tau : \pi_n(\mathsf{Tel}) \xrightarrow{\sim} I(S)[r]$ for all $n \in \mathbb{Z}$, and a fixed $r$. The constant $r$ depends in a nontrivial way on the values of $n_i$ in the definition of $\Lambda$, and we have not worked out a general formula for this dependence. Thus from Section 6.2 on, we will neglect this shift in module degrees, writing only $\pi_n(\mathsf{Tel}) \cong I(S)$ and the like.

In Section 6.3, we generalize the construction further. Given an integer $p \geq 1$ and any choice of subsets $S_i \subseteq \mathbb{N}$, $1 \leq i \leq p$, we construct an object $\mathsf{Tel}$ so that $\pi_n(\mathsf{Tel})$ is periodic, of period $p$, with values $I(S_i)$. We also show how to alter the construction to selectively kill off these periodic homology groups. Thus, for example, given $S \subseteq \mathbb{N}$, we can construct an object whose $n$th homology is $I(S)$ if $p$ divides $n$, and zero otherwise.

Although we use $\mathsf{Tel}$ to refer to all these different telescope constructions, and repeatedly use $C$ to refer to certain chain complexes, the context of each section will be clear and we hope no confusion will arise.

In Section 6.4, we dualize each of these constructions, using Brown-Comenetz duality, to make objects $\mathsf{Mic}$ that have homology $\Lambda$, or $\Lambda(S)$, or periodic homology of $\Lambda(S_i)$'s and zeros. One interesting asymmetry is that, for the $\mathsf{Tel}$ with $\pi_n(\mathsf{Tel}) \cong I(\mathbb{N})$ for all $n$, $\langle \mathsf{Tel} \rangle \neq \langle I(\mathbb{N}) \rangle$ (Lemma 6.1.2), but for the $\mathsf{Mic}$ with $\pi_n(\mathsf{Mic}) \cong \Lambda$ for all $n$, we have $\langle \mathsf{Mic} \rangle = \langle \Lambda \rangle$ (Proposition 6.4.1).

## 6.1 *An object* Tel *with* $\pi_n(\mathsf{Tel}) \cong I(\mathbb{N})$ *for all* $n$

In this section, we show that there are objects in $D(\Lambda)$ that are not Bousfield equivalent to a module, by constructing an object Tel with $\pi_n(\mathsf{Tel}) \cong I(\mathbb{N})$ for all $n$.

### 6.1.1 ***CASE 1****: In the definition of* $\Lambda$*, fix* $n_i = 2$ *for all* $i$*.*

Let $\deg(x_i) = 2^i$, and let $C$ in $D(\Lambda)$ be represented by the chain complex

$$0 \longrightarrow \Lambda \xrightarrow[x_1]{} \Lambda[-2] \xrightarrow[x_1x_2]{} \Lambda[-8] \xrightarrow[x_2x_3]{} \Lambda[-20] \xrightarrow[x_3x_4]{} \Lambda[-44] \xrightarrow[x_4x_5]{} \Lambda[-92] \xrightarrow[x_5x_6]{} ...,$$

where we are grading homologically, and $C$ is concentrated in non-positive degrees. The module shifts are necessary to guarantee the maps are degree zero; one can check that in degree $-n$ we have $\Lambda[4-3\cdot 2^n]$. For notational simplicity, we will often neglect to include these shifts, instead writing the following.

$$0 \longrightarrow \Lambda \xrightarrow[x_1]{} \Lambda \xrightarrow[x_1x_2]{} \Lambda \xrightarrow[x_2x_3]{} \Lambda \xrightarrow[x_3x_4]{} \Lambda \xrightarrow[x_4x_5]{} \Lambda \xrightarrow[x_5x_6]{} ...,$$

where we are grading homologically, and $C$ is concentrated in non-positive degrees.

Define $f : C \to \Sigma^2 C$ to be the following chain map.

$$\begin{array}{cccccccccccccccc}
 & & & & 0 & \xrightarrow{0} & \Lambda & \xrightarrow{x_1} & \Lambda[-2] & \xrightarrow{x_1x_2} & \Lambda[-8] & \xrightarrow{x_2x_3} & \Lambda[-20] & \xrightarrow{x_3x_4} & \Lambda[-44] & \longrightarrow \cdots \\
 & & & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle x_3} & & \downarrow{\scriptstyle x_1x_4} & & \downarrow{\scriptstyle x_2x_5} & & \downarrow{\scriptstyle x_3x_6} & & \downarrow{\scriptstyle x_4x_7} & \\
0 & \xrightarrow[0]{} & \Lambda & \xrightarrow[x_1]{} & \Lambda[-2] & \xrightarrow[x_1x_2]{} & \Lambda[-8] & \xrightarrow[x_2x_3]{} & \Lambda[-20] & \xrightarrow[x_3x_4]{} & \Lambda[-44] & \xrightarrow[x_4x_5]{} & \Lambda[-92] & \xrightarrow[x_5x_6]{} & \Lambda[-188] & \longrightarrow \cdots \\
 & & (2) & & (1) & & (0) & & (-1) & & (-2) & & (-3) & & (-4) &
\end{array}$$

Or simply

$$\begin{array}{cccccccccccccccccc}
 & & & & 0 & \xrightarrow{0} & \Lambda & \xrightarrow{x_1} & \Lambda & \xrightarrow{x_1x_2} & \Lambda & \xrightarrow{x_2x_3} & \Lambda & \xrightarrow{x_3x_4} & \Lambda & \xrightarrow{x_4x_5} & \Lambda & \longrightarrow \cdots . \\
 & & & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle x_3} & & \downarrow{\scriptstyle x_1x_4} & & \downarrow{\scriptstyle x_2x_5} & & \downarrow{\scriptstyle x_3x_6} & & \downarrow{\scriptstyle x_4x_7} & & \downarrow{\scriptstyle x_5x_8} & \\
0 & \xrightarrow[0]{} & \Lambda & \xrightarrow[x_1]{} & \Lambda & \xrightarrow[x_1x_2]{} & \Lambda & \xrightarrow[x_2x_3]{} & \Lambda & \xrightarrow[x_3x_4]{} & \Lambda & \xrightarrow[x_4x_5]{} & \Lambda & \xrightarrow[x_5x_6]{} & \Lambda & \xrightarrow[x_6x_7]{} & \Lambda & \longrightarrow \cdots \\
 & & (2) & & (1) & & (0) & & (-1) & & (-2) & & (-3) & & (-4) & & (-5) &
\end{array}$$

One can check that with the module grading $\deg(x_i) = 2^i$, this is in fact a chain map.

Now define $\mathsf{Tel}$ to be the sequential colimit

$$\mathsf{Tel} = \mathsf{colim}\left( C \xrightarrow{f} \Sigma^2 C \xrightarrow{\Sigma^2 f} \Sigma^4 C \longrightarrow \cdots \right).$$

This is a minimal weak colimit [Mar83, 3.1], so it is unique up to equivalence, and satisfies

$$\pi_n(\mathsf{Tel}) \cong \mathsf{colim}\left[\pi_n(C) \longrightarrow \pi_n(\Sigma^2 C) \longrightarrow \cdots\right].$$

**Proposition 6.1.1.** *For all $n \in \mathbb{Z}$, there is a graded $\Lambda$-module isomorphism*

$$\tau : \pi_n(\mathsf{Tel}) \xrightarrow{\sim} I(\mathbb{N})[r],$$

*for a fixed $r$. When all $n_i = 2$ in the definition of $\Lambda$, then $r = 2$.*

We defer the proof to the next subsection, in order to focus on the main result of this chapter, Theorem 6.1.4. We will prove two lemmas about the Bousfield class of this $\mathsf{Tel}$, and use them to prove the theorem.

**Lemma 6.1.2.** $\langle \mathsf{Tel} \rangle \neq \langle I(\mathbb{N}) \rangle$.

**Proof.** Let $K$ be the cofiber of $f : C \to \Sigma^2 C$. We know that $K$ is not zero, because Proposition 6.1.1 implies that $f$ is not an equivalence. The following are known about $C$, $K$, and $\mathsf{Tel}$ [HPS97, Prop.3.6.9]:

$$\langle C \rangle = \langle K \rangle \vee \langle \mathsf{Tel} \rangle \text{ and } \langle 0 \rangle = \langle K \rangle \wedge \langle \mathsf{Tel} \rangle.$$

Furthermore, [DP08, Cor. 7.3] shows that $\langle I(\mathbb{N}) \rangle \leq \langle X \rangle$ for all nonzero $X$ in $D(\Lambda)$.

Suppose, towards a contradiction, that $\langle \mathsf{Tel} \rangle = \langle I(\mathbb{N}) \rangle$. Then $\langle \mathsf{Tel} \rangle \leq \langle K \rangle$, so $\langle C \rangle = \langle K \rangle \vee \langle \mathsf{Tel} \rangle = \langle K \rangle$. This implies $\langle 0 \rangle = \langle C \rangle \wedge \langle \mathsf{Tel} \rangle$, so $C \wedge \mathsf{Tel} = 0$. This would force $C \wedge I(\mathbb{N}) = 0$.

But we will now show that $C \wedge I(\mathbb{N}) \neq 0$. As remarked in [DP08, Lemma 3.4], tensor-hom adjointness at the module level yields the following for all $X$ and $Y$ in $D(\Lambda)$.

$$\mathbb{R}\mathrm{Hom}_\Lambda(X, \mathbb{R}\mathrm{Hom}_k(Y, k)) = \mathbb{R}\mathrm{Hom}_k(X \wedge Y, k).$$

In particular, setting $Y = \Lambda$ gives $I(X) = \mathbb{R}\mathrm{Hom}_k(X, k) \cong \mathbb{R}\mathrm{Hom}_\Lambda(X, I(\mathbb{N}))$ for all $X$ in $D(\Lambda)$. Using this, we compute

$$\begin{aligned} I(C \wedge I(\mathbb{N})) &\cong \mathbb{R}\mathrm{Hom}_\Lambda(C \wedge I(\mathbb{N}), I(\mathbb{N})) \\ &\cong \mathbb{R}\mathrm{Hom}_\Lambda(C, \mathbb{R}\mathrm{Hom}_\Lambda(I(\mathbb{N}), I(\mathbb{N}))) \\ &\cong \mathbb{R}\mathrm{Hom}_\Lambda(C, I(I(\mathbb{N}))) \\ &\cong \mathbb{R}\mathrm{Hom}_\Lambda(C, \Lambda). \end{aligned}$$

We have

$$\pi_0(\mathbb{R}\mathrm{Hom}_\Lambda(C, \Lambda)) \cong [C, \Lambda]_0.$$

The module $\Lambda$ is self-injective, because $\Lambda$ is a $P$-algebra [Mar83, Thm. 13.12], so $[C, \Lambda]_0$ is homotopy classes of degree zero chain maps from $C$ to $\Lambda$. There are nontrivial such classes of maps.

$$\begin{array}{ccccccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & \Lambda & \xrightarrow{x_1} & \Lambda & \xrightarrow{x_1 x_2} & \Lambda & \xrightarrow{x_2 x_3} & \Lambda & \longrightarrow & \cdots \\ & & {\scriptstyle 0}\downarrow & & {\scriptstyle 1}\downarrow & & {\scriptstyle 0}\downarrow & & & & & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & \Lambda & \longrightarrow & 0 & \longrightarrow & \cdots & & & & \end{array}$$

Therefore $I(C \wedge I(\mathbb{N})) \neq 0$, and $C \wedge I(\mathbb{N}) \neq 0$. □

In Section 5 of [DP08], the authors asks if every object is Bousfield equivalent to the direct sum of its homology groups. The last two results show that this is not true. This was also shown recently in [IK11, 4.8].

Let $\mathcal{M}$ denote the subcategory of $D(\Lambda)$ of all modules. In the following lemma, we think of a Bousfield class $\langle X \rangle$ as the localizing subcategory of $X$-acyclics.

**Lemma 6.1.3.**

$$\mathcal{M} \cap \langle \mathsf{Tel} \rangle = \mathcal{M} \cap \langle I(\mathbb{N}) \rangle.$$

**Proof.** Since $\langle I(\mathbb{N})\rangle$ is minimum among nonzero Bousfield classes, we know $\langle I(\mathbb{N})\rangle \leq \langle \mathsf{Tel}\rangle$, so we already have the $\subseteq$ direction. We will show that if $M$ is a module in $D(\Lambda)$ and $M \wedge I(\mathbb{N}) = 0$, then $M \wedge \mathsf{Tel} = 0$.

In [KM95, Thm. 4.7] the authors construct a strongly convergent Eilenberg-Moore spectral sequence in the category of ($\mathbb{Z}$-graded, so unbounded) DG-modules over a differential graded algebra. In order to use this spectral sequence, we must temporarily neglect the grading on $\Lambda$. More specifically, let $\widetilde{\Lambda}$ be the same ring as $\Lambda$ but ungraded, and let $D(\widetilde{\Lambda})$ be the derived category of ungraded modules over $\widetilde{\Lambda}$. Let $F : D(\Lambda) \to D(\widetilde{\Lambda})$ be the forgetful functor, and write $F(X) = \widetilde{X}$. Then we can consider $\widetilde{\Lambda}$ as a differential graded algebra concentrated in chain degree zero, and DG-modules $X$ and $Y$ over $\widetilde{\Lambda}$ are just chain complexes of ungraded $\widetilde{\Lambda}$-modules, thus represent objects in $D(\widetilde{\Lambda})$. Then the spectral sequence in [KM95, Thm. 4.7] becomes

$$E^2_{p,q} = \bigoplus_{m+n=q} \mathrm{Tor}_p^{\widetilde{\Lambda}}(\pi_m(X), \pi_n(Y)) \Longrightarrow \pi_{p+q}(X \wedge Y).$$

Now suppose $M$ is a module in $D(\Lambda)$ such that $M \wedge I(\mathbb{N}) = 0$ in $D(\Lambda)$. This says that given a projective resolution $P$ of $M$ in $D(\Lambda)$, we have $P \otimes I(\mathbb{N})$ acyclic, so $\widetilde{P \otimes I(\mathbb{N})}$ is acyclic. This implies $\widetilde{P} \otimes \widetilde{I(\mathbb{N})}$ is acyclic as well. But if we neglect the grading on $P$ and $M$, then $\widetilde{P}$ is still a projective resolution of $\widetilde{M}$, and we get $\widetilde{M} \wedge \widetilde{I(\mathbb{N})} \cong \widetilde{P} \otimes \widetilde{I(\mathbb{N})}$ acyclic in $D(\widetilde{\Lambda})$. Letting $X = \widetilde{M}$ and $Y = \widetilde{\mathsf{Tel}}$, the spectral sequence $E^2$ page becomes

$$E^2_{p,q} = \bigoplus_{m+n=q} \mathrm{Tor}_p^{\widetilde{\Lambda}}(\pi_m(\widetilde{M}), \pi_n(\widetilde{\mathsf{Tel}})) = \mathrm{Tor}_p^{\widetilde{\Lambda}}(\widetilde{M}, \widetilde{I(\mathbb{N})}) = \pi_p(\widetilde{M} \wedge \widetilde{I(\mathbb{N})}).$$

This collapses to zero, so we must have $\pi_{p+q}(\widetilde{M} \wedge \widetilde{\mathsf{Tel}}) = 0$ for all $p$ and $q$. By the same argument as above, any projective resolution of $M$ in $D(\Lambda)$ giving $M \wedge \mathsf{Tel} \neq 0$ would also give $\widetilde{M} \wedge \widetilde{\mathsf{Tel}} \neq 0$, so we can conclude that $M \wedge \mathsf{Tel} = 0$ in $D(\Lambda)$, as desired.

□

**Theorem 6.1.4.** *In $D(\Lambda)$, there are objects that are not Bousfield equivalent to any module. Specifically, every $I(\mathbb{N})$-acyclic object that is not $\mathsf{Tel}$-acyclic cannot be Bousfield equivalent to a module.*

**Proof.** Suppose, towards a contradiction, that every object $Y$ in $D(\Lambda)$ is Bousfield equivalent to some module, $M_Y$. Take $X$ with $X \wedge I(\mathbb{N}) = 0$. Then $M_X \wedge I(\mathbb{N}) = 0$. Using Lemma 6.1.3, this says that

$$M_X \in \mathcal{M} \cap \langle I(\mathbb{N}) \rangle = \mathcal{M} \cap \langle \mathsf{Tel} \rangle.$$

Thus $M_X \wedge \mathsf{Tel} = 0$, so $X \wedge \mathsf{Tel} = 0$.

This implies that $\langle I(\mathbb{N}) \rangle \geq \langle \mathsf{Tel} \rangle$. Since we already have $\langle I(\mathbb{N}) \rangle \leq \langle \mathsf{Tel} \rangle$, we conclude that $\langle I(\mathbb{N}) \rangle = \langle \mathsf{Tel} \rangle$. But this contradicts Lemma 6.1.2. $\square$

Note that the proof of the theorem used only that $\pi_n(\mathsf{Tel}) \cong I(\mathbb{N})$ for all $n$, as graded $\Lambda$-modules, and did not require a specific shift, $\pi_n(\mathsf{Tel}) \xrightarrow{\sim} I(\mathbb{N})[r]$. Therefore the theorem holds either with the $\mathsf{Tel}$ constructed in this Case, where each $n_i = 2$, or with the $\mathsf{Tel}$ constructed in Case 2, where the $n_i$ are arbitrary.

**Question 6.1.5.** *We have shown that $\langle I(\mathbb{N}) \rangle < \langle \mathsf{Tel} \rangle$, so that there are $I(\mathbb{N})$-acyclics that are not $\mathsf{Tel}$-acyclic. Can we construct such an object more explicitly?*

One natural candidate may be $\mathsf{Tel}$ itself. However, the theorem tells us nothing, since $\mathsf{Tel}$ is $\mathsf{Tel}$-acyclic. This follows from Corollary 4.12 in [DP08], which shows that $I(\mathbb{N}) \wedge I(\mathbb{N}) = 0$. If we set $X = Y = \mathsf{Tel}$ in the spectral sequence of Lemma 6.1.3, then $I(\mathbb{N}) \wedge I(\mathbb{N}) = 0$ implies $\mathsf{Tel} \wedge \mathsf{Tel} = 0$. Without Theorem 6.1.4, it's not clear how to ascertain whether $\mathsf{Tel}$ is or is not Bousfield equivalent to a module.

**Question 6.1.6.** *To what extent can this argument and result be applied to other derived categories, or to the stable homotopy category $\mathcal{S}$?*

*6.1.2 Proof of Proposition 6.1.1.*

Our goal is to show that for all $n \in \mathbb{Z}$,

$$\pi_n(\mathsf{Tel}) \cong \mathsf{colim}\left[\pi_n(C) \longrightarrow \pi_n(\Sigma^2 C) \longrightarrow \cdots\right] \cong I(\mathbb{N})[2].$$

For concreteness, we will compute $\pi_{-2}(\mathsf{Tel})$, and then indicate the general case. We will split the computation into several lemmas.

Because of the shift, we are trying to compute

$$\pi_{-2}(\mathsf{Tel}) \cong \mathsf{colim}\left[\pi_{-2}(C) \longrightarrow \pi_{-4}(C) \longrightarrow \pi_{-6}(C) \longrightarrow \cdots\right].$$

We have

$$\pi_{-2}(C) = \frac{\ker(x_2x_3)}{\mathrm{im}(x_1x_2)}[-8] \cong \frac{(x_2, x_3)}{(x_1x_2)}[-8], \text{ and generally } \pi_{-n}(C) \cong \frac{(x_n, x_{n+1})}{(x_{n-1}x_n)}[4-3\cdot 2^n], \text{ for } n \geq 2.$$

Define

$$M_{-2} = \frac{(x_3)}{(x_3) \cap (x_2, x_4, x_5, x_6, ...)}[-8],$$

and in general

$$M_{-n} = \frac{(x_{n+1})}{(x_{n+1}) \cap (x_n, x_{n+2}, x_{n+3}, x_{n+4}, ...)}[4 - 3 \cdot 2^n],$$

and (omitting module shifts and simplifying denominators for readability) consider the collection of maps

$$\frac{(x_{n+1})}{(x_n, x_{n+2}, x_{n+3}, ...)} = M_{-n} \xrightarrow{x_n x_{n+3}} M_{-n-2} = \frac{(x_{n+3})}{(x_{n+2}, x_{n+4}, x_{n+5}, ...)}.$$

**Lemma 6.1.7.**

$$\pi_{-2}(\mathsf{Tel}) \cong \mathsf{colim}\left[\frac{(x_2, x_3)}{(x_1x_2)} \xrightarrow{x_2x_5} \frac{(x_4, x_5)}{(x_3x_4)} \xrightarrow{x_4x_7} \frac{(x_6, x_7)}{(x_5x_6)} \xrightarrow{x_6x_9} \cdots\right]$$

$$\cong \mathsf{colim}\left[M_{-2} \xrightarrow{x_2x_5} M_{-4} \xrightarrow{x_4x_7} M_{-6} \xrightarrow{x_6x_9} \cdots\right].$$

**Proof.** This uses the universal property of $\mathsf{colim}$. For all $n \geq 2$, we have surjective projection maps

$$\psi_{-n} : \pi_{-n}(C) \cong \frac{(x_n, x_{n+1})}{(x_{n-1}x_n)} \longrightarrow \frac{(x_{n+1})}{(x_n, x_{n+2}, x_{n+3}, x_{n+4}, \dots)} = M_{-n}.$$

These maps are compatible with the colimit maps; one can check that the following square commutes for all $r$.

$$\begin{array}{ccc} \frac{(x_r, x_{r+1})}{(x_{r-1}x_r)} & \xrightarrow{\;x_r x_{r+3}\;} & \frac{(x_{r+2}, x_{r+3})}{(x_{r+1}x_{r+2})} \\ {\scriptstyle proj}\big\downarrow & & \big\downarrow{\scriptstyle proj} \\ \frac{(x_{r+1})}{(x_r, x_{r+2}, x_{r+3}, \dots)} & \xrightarrow{\;x_r x_{r+3}\;} & \frac{(x_{r+3})}{(x_{r+2}, x_{r+4}, x_{r+5}, \dots)} \end{array}$$

Thus we get maps $\pi_{-n}(C) \to \mathsf{colim} M_i$, which induce $\Psi : \mathsf{colim}\pi_i(C) \to \mathsf{colim} M_i$. We will show that $\Psi$ is surjective and injective.

surjectivity: We will use standard properties of colimits (see e.g. [Mar83, App. 1.2, Prop. 7]). Take $\tilde{x} \in \mathsf{colim} M_i$. So $\tilde{x}$ is represented by $x \in M_{-r}$ for some $r$. Since $\psi_{-r}$ is surjective, we can pick a $y \in \pi_{-r}(C)$ such that $\psi_{-r}(y) = x$. By the definition of a colimit, this factors through $\Psi$. So, letting $\tilde{y}$ be the image of $y$ in $\mathsf{colim}\pi_i(C)$, we get $\Psi(\tilde{y}) = \tilde{x}$.

injectivity: Suppose $\Psi(\tilde{y}) = 0$. Then $\tilde{y}$ is represented by $y \in \pi_{-r}(C)$ for some $r$. We have a commuting diagram

$$\begin{array}{ccc} \pi_{-r}(C) & \longrightarrow & \mathsf{colim}\pi_i(C) \\ {\scriptstyle \psi_{-r}}\big\downarrow & & \big\downarrow{\scriptstyle \Psi} \\ M_{-r} & \longrightarrow & \mathsf{colim} M_i \end{array}.$$

Therefore $x = \psi_{-r}(y) \in M_{-r}$ maps to zero in $\mathsf{colim} M_i$. This means that either $x = 0$, or $x$ becomes zero eventually in the sequence $M_{-r} \to M_{-r-2} \to M_{-r-4} \to \cdots$. Suppose that $x$ becomes zero at $M_{-r-s}$, where it could be that $s = 0$. The following square commutes.

$$\begin{array}{ccc} \pi_{-r}(C) & \longrightarrow & \pi_{-r-s}(C) \\ \downarrow{\scriptstyle \psi_{-r}} & & \downarrow{\scriptstyle \psi_{-r-s}} \\ M_{-r} & \longrightarrow & M_{-r-s} \end{array} .$$

Since $\psi_{-r}(y) = x$, this implies that the image of $y$ in $\pi_{-r-s}(C)$, call it $z$, maps to zero in $M_{-r-s}$.

If $z = 0$, then we're done - this implies that $\tilde{y} = 0$. So consider the case that $z \neq 0$, but $\psi_{-r-s}(z) = 0$. Now, $\psi_{-r-s}$ is the map

$$\pi_{-r-s}(C) \cong \frac{(x_{r+s}, x_{r+s+1})}{(x_{r+s-1}x_{r+s})} \longrightarrow \frac{(x_{r+s+1})}{(x_{r+s}, x_{r+s+2}, x_{r+s+3}, x_{r+s+4}, ...)}.$$

Therefore $z \in (x_{r+s}, x_{r+s+2}, x_{r+s+3}, x_{r+s+4}, ...)$. But from $\pi_{-r-s}(C)$, the maps encountered in $\mathsf{colim}\pi_i(C)$ are precisely $x_{r+s}, x_{r+s+2}, x_{r+s+3}, x_{r+s+4}, ...$, so we are guaranteed that eventually $z$ will be sent to zero. This implies that $\tilde{y} = 0$, so $\Psi$ is injective.

□

**Lemma 6.1.8.**

$$\mathsf{colim}\left[M_{-2} \xrightarrow{x_2x_5} M_{-4} \xrightarrow{x_4x_7} M_{-6} \xrightarrow{x_6x_9} \cdots\right]$$

$$\cong \mathsf{colim}\left[I\left(\frac{k[x_1]}{(x_i^2)}\right)[2] \hookrightarrow I\left(\frac{k[x_1,x_2,x_3]}{(x_i^2)}\right)[2] \hookrightarrow I\left(\frac{k[x_1,x_2,x_3,x_4,x_5]}{(x_i^2)}\right)[2] \hookrightarrow \cdots\right].$$

**Proof.** First consider $M_{-4} = \frac{(x_5)}{(x_4,x_6,x_7,...)}[-44]$. As a $\Lambda$-module, this has generator $x_5$, in degree $-44 + 2^5 = -12$, and top degree element $x_1x_2x_3x_5$, in degree $-44 + 2^1 + 2^2 + 2^3 + 2^5 = 2$.

Let $\overline{x_i}$ denote the dual of $x_i$. As a $\Lambda$-module, $I\left(\frac{k[x_1,x_2,x_3]}{(x_i^2)}\right)[2]$ is generated by $\overline{x_1x_2x_3}$, in degree $2 - 2^1 - 2^2 - 2^3 = -12$, and has top degree element $\overline{1}$, in degree 2. In fact, we can define a $\Lambda$-isomorphism from

$$\frac{(x_5)}{(x_4,x_6,x_7,...)}[-44] \longrightarrow I\left(\frac{k[x_1,x_2,x_3]}{(x_i^2)}\right)[2],$$

by sending $x_5 \mapsto \overline{x_1x_2x_3}$.

Similarly, for all $n \geq 2$, we have $\Lambda$-isomorphisms

$$M_{-n} = \frac{(x_{n+1})}{(x_n, x_{n+2}, x_{n+3}, x_{n+4}, \ldots)}[4 - 3 \cdot 2^n] \longrightarrow I\left(\frac{k[x_1, x_2, \ldots, x_{n-1}]}{(x_i^2)}\right)[2],$$

defined by sending

$$x_{n+1} \mapsto \overline{x_1 x_2 \cdots x_{n-1}}.$$

The degree of $x_{n+1}$ is $4 - 3 \cdot 2^n + 2^{n+1} = 4 - 2^n$, and the degree of $\overline{x_1 x_2 \cdots x_{n-1}}$ is

$$2 - \left(2^1 + \cdots + 2^{n-1}\right) = 2 - (2^n - 2) = 4 - 2^n.$$

Now, we will show that the maps among the $M_i$'s become inclusions among the duals. First, consider an example.

$$\begin{array}{ccc} M_{-2} = \frac{(x_3)}{(x_2, x_4, x_5, \ldots)}[-8] & \xrightarrow{x_2 x_5} & \frac{(x_5)}{(x_4, x_6, x_7, \ldots)}[-44] = M_{-4}\, . \\ \cong \downarrow & & \downarrow \cong \\ I\left(\frac{k[x_1]}{(x_i^2)}\right)[2] & \longrightarrow & I\left(\frac{k[x_1, x_2, x_3]}{(x_i^2)}\right)[2] \end{array}$$

In the bottom left, the generator $\overline{x_1}$ goes up to the generator $x_3$, then right to $x_2 x_3 x_5$, which gets sent down to

$$x_2 x_3 . (\overline{x_1 x_2 x_3}) = \overline{x_1}$$

in the bottom right, and all these maps are degree zero.

In general, we have

$$\begin{array}{ccc} M_{-r} = \frac{(x_{r+1})}{(x_r, x_{r+2}, x_{r+3}, \ldots)}[4 - 3 \cdot 2^r] & \xrightarrow{x_r x_{r+3}} & \frac{(x_{r+3})}{(x_{r+2}, x_{r+4}, x_{r+5}, \ldots)}[4 - 3 \cdot 2^{r-2}] = M_{-r-2}\, . \\ \cong \downarrow & & \downarrow \cong \\ I\left(\frac{k[x_1, x_2, \ldots, x_{r-1}]}{(x_i^2)}\right)[2] & \longrightarrow & I\left(\frac{k[x_1, x_2, \ldots, x_{r+1}]}{(x_i^2)}\right)[2] \end{array}$$

The generator in the bottom left is $\overline{x_1 x_2 \cdots x_{r-1}}$, which is sent up to the generator $x_{r+1}$, then over to $x_r x_{r+1} x_{r+3} = (x_r x_{r+1}).x_{r+3}$. This gets sent down to

$$(x_r x_{r+1}).\overline{x_1 x_2 \cdots x_{r+1}} = \overline{x_1 x_2 \cdots x_{r-1}}.$$

This shows that each map becomes the natural degree-zero inclusion under the isomorphisms just described. $\square$

**Lemma 6.1.9.**

$$\mathsf{colim}\left[I\left(\frac{k[x_1]}{(x_i^2)}\right)[2] \hookrightarrow I\left(\frac{k[x_1,x_2,x_3]}{(x_i^2)}\right)[2] \hookrightarrow I\left(\frac{k[x_1,x_2,x_3,x_4,x_5]}{(x_i^2)}\right)[2] \hookrightarrow \cdots\right]$$

$$\cong I\left(\mathsf{lim}\left[\cdots \to \frac{k[x_1,x_2,x_3,x_4,x_5]}{(x_i^2)}[-2] \xrightarrow{proj} \frac{k[x_1,x_2,x_3]}{(x_i^2)}[-2] \xrightarrow{proj} \frac{k[x_1]}{(x_i^2)}\right][-2]\right) \cong I(\mathbb{N})[2].$$

**Proof.** Let $V_j = \frac{k[x_1,x_2,\ldots,x_j]}{(x_i^2)}[-2]$; it's clear that $I(V_j) = I\left(\frac{k[x_1,x_2,\ldots,x_j]}{(x_i^2)}\right)[2]$. Since these are locally finite, we have $I(I(V_j)) \cong V_j$ for all $j$. The definition of a sequential colimit gives a certain exact sequence

$$\coprod I(V_j) \xrightarrow{G} \coprod I(V_j) \longrightarrow (\mathsf{colim} I(V_j)) \longrightarrow 0,$$

which dualizes to an exact sequence

$$0 \longrightarrow I(\mathsf{colim} I(V_j)) \longrightarrow \prod I(I(V_j)) \xrightarrow{I(G)} \prod I(I(V_j)).$$

One can check that in fact $I(G)$ is the map used in the definition of the sequential limit, so we have

$$\mathsf{lim} V_j \cong I(\mathsf{colim} I(V_j)).$$

Since $\mathsf{lim} V_j \cong \Lambda$, this shows that $\mathsf{colim} I(V_j)$ is the thing that dualizes to $\Lambda$. In other words $\mathsf{colim} I(V_j) \cong I(\mathbb{N})$. $\square$

**Proof of Proposition 6.1.1.** Combining the three previous lemmas, we have an isomorphism $\tau : \pi_{-2}(\mathsf{Tel}) \xrightarrow{\sim} I(\mathbb{N})[2]$. Because the map $f : C \to \Sigma^2 C$ has degree two, and sequential colimits are determined by their long-term behavior, it's easy to see that $\pi_i(\mathsf{Tel}) \cong \pi_{-2}(\mathsf{Tel}) \cong I(\mathbb{N})[2]$ for all even $i$.

Additionally, a computation of $\pi_{-3}(\mathsf{Tel})$, for example, would proceed as above, but with all indices incremented/decremented by one. The result is the same: $\pi_{-3}(\mathsf{Tel}) \cong \pi_i(\mathsf{Tel}) \cong I(\mathbb{N})$ for all odd $i$, and we have checked that this gives the same module shift, 2. Therefore, $\tau$ induces $\pi_i(\mathsf{Tel}) \cong I(\mathbb{N})[2]$ for all $i$. $\square$

### *6.1.3* ***CASE 2****: In the definition of* $\Lambda$*, let* $n_i$ *be any fixed integers, each* $\geq 2$*.*

Define $m_i = n_i - 1$ for all $i$. In this case, we basically just change all maps from $x_i$ to $x_i^{m_i}$. Our goal is an isomorphism $\tau : \pi_n \xrightarrow{\sim} I(\mathbb{N})[r]$ for all $n \in \mathbb{Z}$ and some fixed $r$. However, the (finite) constant $r$ depends on the $n_i$ in a nontrivial way, and we have not worked out a general formula for this relationship. Furthermore, in this generality we can no longer insist on $\deg(x_i) = 2^i$, but rather must define $\deg(x_i)$ inductively, depending on $n_i$ values. So we have decided not to keep track of the module shifts, in this subsection and remaining sections. The reader should, however, bear in mind that we are implicitly shifting modules as needed, so that all maps have degree zero.

Now define the chain complex $C$ and the chain map $f : C \to \Sigma^2 C$

$$
\begin{array}{ccccccccccccccc}
 & & & & 0 & \xrightarrow{0} & \Lambda & \xrightarrow{x_1^{m_1}} & \Lambda & \xrightarrow{x_1^{m_1}x_2^{m_2}} & \Lambda & \xrightarrow{x_2^{m_2}x_3^{m_3}} & \Lambda & \xrightarrow{x_3^{m_3}x_4^{m_4}} & \cdots \\
 & & & & {\scriptstyle \Sigma^s 0}\downarrow & & \downarrow{\scriptstyle x_3^{m_3}} & & \downarrow{\scriptstyle x_1^{m_1}x_4^{m_4}} & & \downarrow{\scriptstyle x_2^{m_2}x_5^{m_5}} & & \downarrow{\scriptstyle x_3^{m_3}x_6^{m_6}} & & \\
0 & \xrightarrow[0]{} & \Lambda & \xrightarrow[x_1^{m_1}]{} & \Lambda & \xrightarrow[x_1^{m_1}x_2^{m_2}]{} & \Lambda & \xrightarrow[x_2^{m_2}x_3^{m_3}]{} & \Lambda & \xrightarrow[x_3^{m_3}x_4^{m_4}]{} & \Lambda & \xrightarrow[x_4^{m_4}x_5^{m_5}]{} & \Lambda & \xrightarrow[x_5^{m_5}x_6^{m_6}]{} & \cdots
\end{array}
$$

Specifically, $C_n = \Lambda$ for all $n \leq 0$ and $C_n = 0$ for $n > 0$, and

$$
d_{-n} = \begin{cases} x_n^{m_n} x_{n+1}^{m_{n+1}} & n \geq 1 \\ x_1^{m_1} & n = 0 \\ 0 & n < 0 \end{cases} \quad \text{and} \quad f_{-n} = \begin{cases} x_n^{m_n} x_{n+3}^{m_{n+3}} & n \geq 1 \\ x_3^{m_3} & n = 0 \\ 0 & n < 0 \end{cases} .
$$

(Another possibility that suggests itself is choosing $x_n x_{n+1}^{m_{n-1}}$ for the $n \geq 1$ differential. However, this does not yield good isomorphisms with duals.)

It's clear that $C$ is a chain complex, and $f$ is a chain map, as long as the degrees work out. For this, we're going to need fractional degrees probably. We can take $|x_1|$, $|x_2|$, and $|x_3|$ to be anything, then set

$$|x_4| = \frac{m_2|x_2| + 2m_3|x_3| - 2m_1|x_1|}{m_4}, \text{ and in general}$$

$$|x_n| = \frac{m_{n-2}|x_{n-2}| + 2m_{n-1}|x_{n-1}| - 2m_{n-3}|x_{n-3}|}{m_n} \text{ for all } n \geq 4.$$

We will compute $\pi_{-2}(\mathsf{Tel})$, to show in what ways this case is different than the previous. We have

$$\pi_{-n}(C) = \frac{\ker(x_n^{m_n} x_{n+1}^{m_{n+1}})}{\mathrm{im}(x_{n-1}^{m_{n-1}} x_n^{m_n})} \cong \frac{(x_n, x_{n+1})}{(x_{n-1}^{m_{n-1}} x_n^{m_n})}, \text{ for } n \geq 2.$$

Just as before, define

$$M_{-n} = \frac{(x_{n+1})}{(x_n, x_{n+2}, x_{n+3}, x_{n+4}, ...)}.$$

Then we claim

$$\begin{aligned}
\pi_{-2}(\mathsf{Tel}) &\cong \mathsf{colim}\left[\frac{(x_2,x_3)}{(x_1^{m_1}x_2^{m_2})} \xrightarrow{x_2^{m_2}x_5^{m_5}} \frac{(x_4,x_5)}{(x_3^{m_3}x_4^{m_4})} \xrightarrow{x_4^{m_4}x_7^{m_7}} \frac{(x_6,x_7)}{(x_5^{m_5}x_6^{m_6})} \xrightarrow{x_6^{m_6}x_9^{m_9}} \cdots\right] \\
&\cong_1 \mathsf{colim}\left[M_{-2} \xrightarrow{x_2^{m_2}x_5^{m_5}} M_{-4} \xrightarrow{x_4^{m_4}x_7^{m_7}} M_{-6} \xrightarrow{x_6^{m_6}x_9^{m_9}} \cdots\right] \\
&\cong_2 \mathsf{colim}\left[I\left(\frac{k[x_1,x_3]}{(x_3^{m_3},x_i^{n_i})}\right) \hookrightarrow I\left(\frac{k[x_1,x_2,x_3,x_5]}{(x_5^{m_5},x_i^{n_i})}\right) \hookrightarrow I\left(\frac{k[x_1,x_2,x_3,x_4,x_5,x_7]}{(x_7^{m_7},x_i^{n_i})}\right) \hookrightarrow \cdots\right] \\
&\cong_3 I\left(\mathsf{lim}\left[\cdots \to \frac{k[x_1,x_2,x_3,x_4,x_5,x_7]}{(x_7^{m_7},x_i^{n_i})} \xrightarrow{proj} \frac{k[x_1,x_2,x_3,x_5]}{(x_5^{m_5},x_i^{n_i})} \xrightarrow{proj} \frac{k[x_1,x_3]}{(x_3^{m_3},x_i^{n_i})}\right]\right) \\
&\cong I(\Lambda) = I(\mathbb{N}).
\end{aligned}$$

<u>Proof of $\cong_1$:</u>

This is just like in Case 1.

<u>Proof of $\cong_2$:</u>

This is basically the same as Case 1. Consider $M_{-4} = \frac{(x_5)}{(x_4,x_6,x_7,...)}$. As a $\Lambda$-module, this has generator $x_5$, and top degree element $x_1^{m_1}x_2^{m_2}x_3^{m_3}x_5^{m_5}$.

On the other hand, $I\left(\frac{k[x_1,x_2,x_3,x_5]}{(x_5^{m_5},x_i^{n_i})}\right)$ has generator

$$\overline{x_1^{m_1}x_2^{m_2}x_3^{m_3}x_5^{m_5-1}},$$

and top degree element $\overline{1}$. It's not hard to see that we get a $\Lambda$-isomorphism because we send the generator of one to the generator of the other.

An argument similar to that in Case 1 shows that the maps among $M_i$'s induce inclusions among duals.

<u>Proof of $\cong_3$:</u>

This is the same as in Case 3.

In conclusion, we have $\pi_{-2}(\mathsf{Tel}) \cong I(\mathbb{N})$. Just as in Case 1, we in fact have $\pi_i(\mathsf{Tel}) \cong I(\mathbb{N})$ for all $i$.

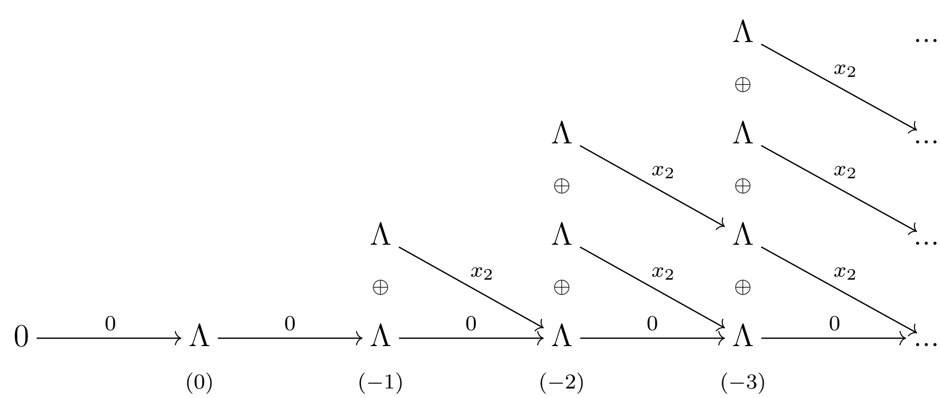
As noted after the proof of Theorem 6.1.4, that theorem applies in this Case as well.

## 6.2 *An object* Tel *with* $\pi_n(\mathsf{Tel}) \cong I(S)$ *for all* $n$*, where* $S \subseteq \mathbb{N}$ *is arbitrary.*

We'll work assuming that in the definition of $\Lambda$, $n_i = 2$ for all $i$, for simplicity. Where necessary we'll indicate how to alter the construction for the general case. As before, we will construct a specific $\Lambda$-module isomorphisms $\tau : \pi_n(\mathsf{Tel}) \to I(S)[r]$ for all $n$, and some fixed $r$. However, unlike in Subsection 6.1.1, we will not calculate the value $r$. Module shifts, required to make all maps degree zero, are implicit.

### 6.2.1 ***CASE 1***: *Suppose* $S = \mathbb{N} - \{2\}$.

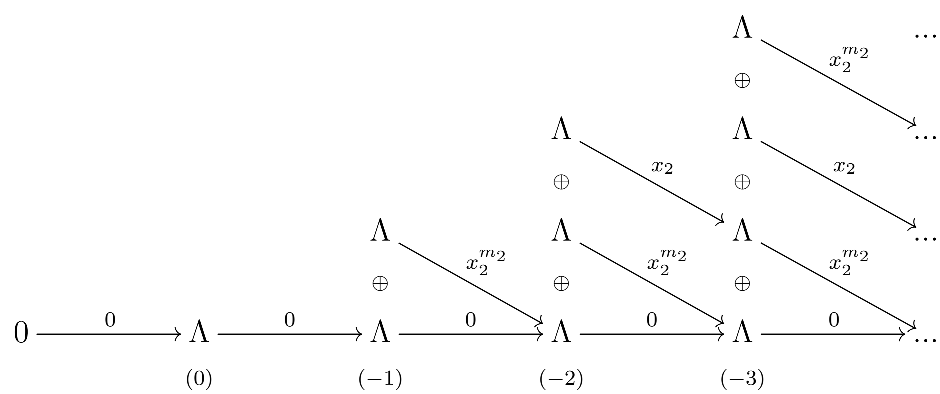
Define a chain complex $D_2$ as in the following picture, concentrated in non-positive degrees.



We will call such a chain complex a *sheet*. In degree $-n$, the sheet $D_2$ is a direct sum of $n+1$ copies of $\Lambda$. The differential $d_{-n} : (D_2)_{-n} \to (D_2)_{-n-1}$ is described by an $(n+2)\times(n+1)$ matrix with entry $(i,j)$ entry $x_2$ if $i-j=2$, and zero otherwise. For example, this matrix for $d_{-3}$ is

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ x_2 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & 0 & x_2 & 0 \end{bmatrix}.$$

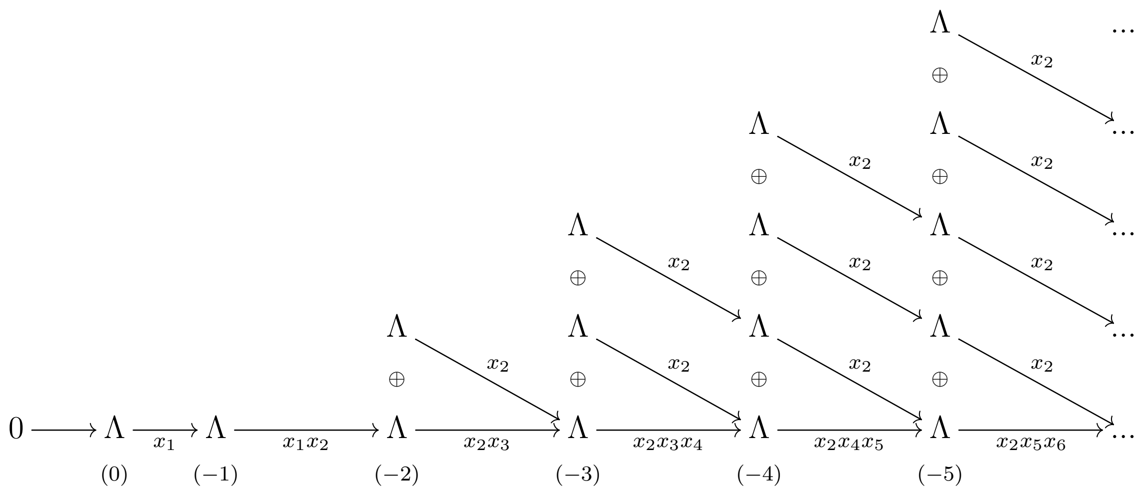
Note: in the general case, where not all $n_i = 2$, we would alternate maps $x_2^{m_2}$ and $x_2$:




Now define a chain complex $B$, similar to the one used in Section 6.1, but with the map $x_2$ added into each differential after degree $-2$:

$$B: \quad 0 \to \Lambda \xrightarrow{x_1} \Lambda \xrightarrow{x_1x_2} \Lambda \xrightarrow{x_2x_3} \Lambda \xrightarrow{x_2x_3x_4} \Lambda \xrightarrow{x_2x_4x_5} \Lambda \xrightarrow{x_2x_5x_6} \Lambda \longrightarrow \cdots$$

We will call $B$ the *seam*, and denote its differentials by $d^B_{-n}$. The chain complex we want to work with, $C$, will have $(C)_n = (\Sigma^{-2}D_2)_n \oplus (B)_n$, with a linking bottom map $x_2$. So C is




We can describe the differential $d_{-n} : C_{-n} \to C_{-n-1}$ using an $(n+1) \times n$ matirx with entries

$$(d_{-n})_{ij} = \begin{cases} x_2 & \text{if } i - j = 2, \\ d^B_{-n} & \text{if } (i,j) = (n+1, n) \\ 0 & \text{else} \end{cases}$$

For example, in the case of $-3$ and $-5$, these are

$$[d_{-3}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x_2 & 0 & 0 \\ 0 & x_2 & x_2x_3x_4 \end{bmatrix} \quad \text{and} \quad [d_{-5}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ x_2 & 0 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 & 0 \\ 0 & 0 & x_2 & 0 & 0 \\ 0 & 0 & 0 & x_2 & x_2x_5x_6 \end{bmatrix}.$$

Before we define the chain map $f : C \to \Sigma^2 C$, let's consider some homotopy groups of $C$. For example,

$$\begin{aligned}\pi_{-3}(C) &= \frac{\ker(x_2x_3x_4)}{\operatorname{im}(x_2x_3)\oplus\operatorname{im}(x_2)} \oplus \ker(x_2)\oplus\ker(x_2)\\ &\cong \frac{(x_2,x_3,x_4)}{(x_2x_3,x_2)} \oplus (x_2)\oplus(x_2)\\ &\cong \frac{(x_3,x_4)}{(x_2,x_2x_3)} \oplus (x_2)\oplus(x_2).\end{aligned}$$

On the other hand,

$$\begin{aligned}\pi_{-5}(C) &= \frac{\ker(x_2x_5x_6)}{\operatorname{im}(x_2x_4x_5)\oplus\operatorname{im}(x_2)} \oplus \frac{\ker(x_2)}{\operatorname{im}(x_2)} \oplus \frac{\ker(x_2)}{\operatorname{im}(x_2)} \oplus \ker(x_2)\oplus\ker(x_2)\\ &\cong \frac{(x_2,x_5,x_6)}{(x_2,x_2x_4x_5)} \oplus \frac{(x_2)}{(x_2)} \oplus \frac{(x_2)}{(x_2)} \oplus (x_2)\oplus(x_2)\\ &\cong \frac{(x_5,x_6)}{(x_2,x_2x_4x_5)} \oplus 0 \oplus 0 \oplus (x_2)\oplus(x_2).\end{aligned}$$

Now we define $f : C \to \Sigma^2 C$. It maps the seam to the seam, and the sheet to the sheet. The maps along the seam are exactly the same maps as $f$ in Section 6.1:

$$f_{-n}|_{\text{seam}} = \begin{cases} x_n x_{n+3} & n \geq 1 \\ x_3 & n = 0 \\ 0 & n < 0 \end{cases}$$

The remaining maps, on the sheet $D_2$, are defined as needed to make $f$ a chain map. On the next page is a picture of what this means, looking at (chain complex) degrees $-1$ to $-4$. For $n \geq 1$, we can also describe $f_{-n} : C_{-n} \to (\Sigma^2 C)_{-n} = C_{-n-2}$ using an $(n+2) \times n$ matrix with entries

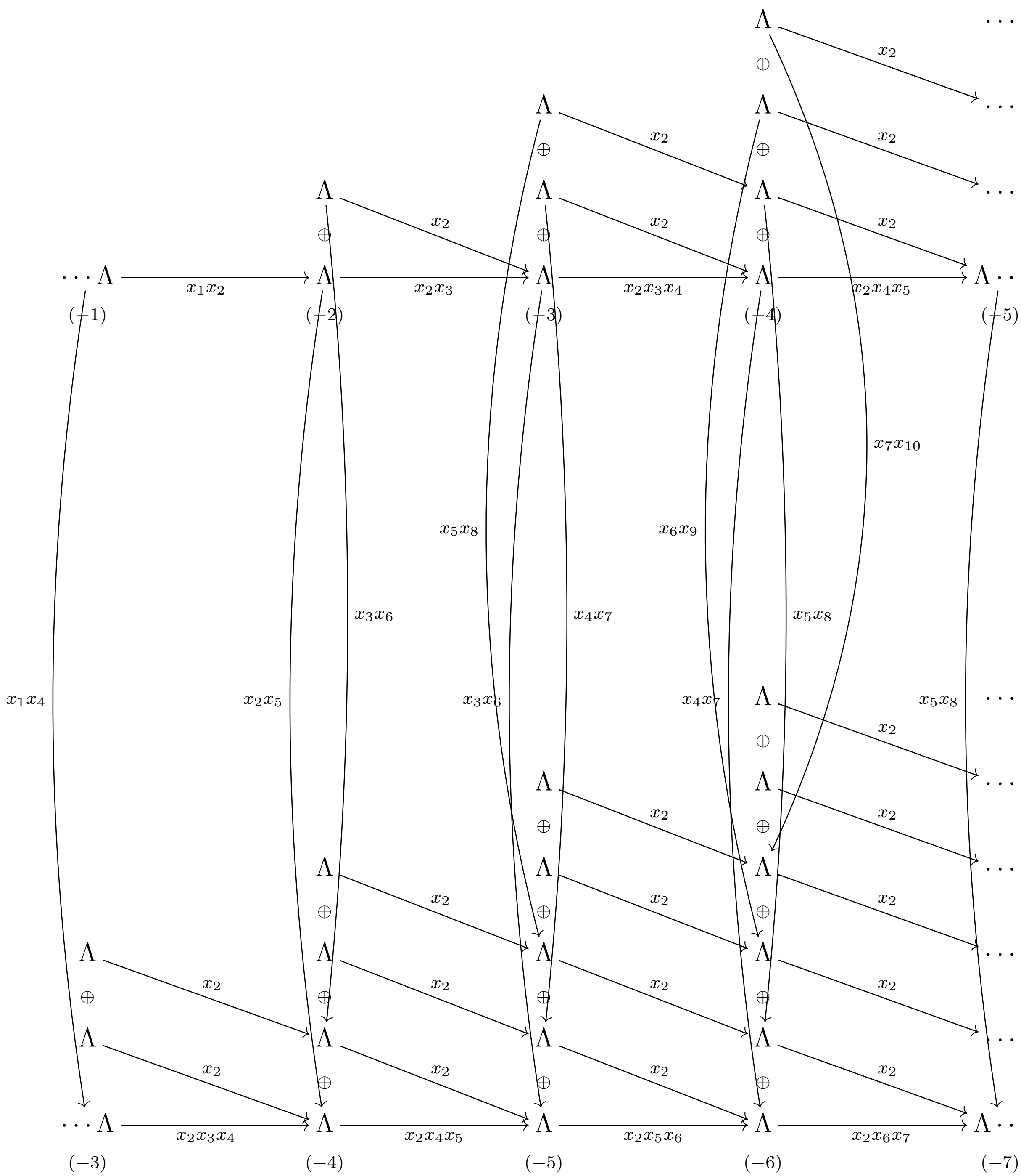
$$(f_{-n})_{ij} = \begin{cases} x_{2n-j}x_{2n-j+3} & \text{if } i - j = 2 \\ 0 & \text{else} \end{cases}.$$

At $-3$ and $-4$ this is

$$[f_{-3}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x_5x_8 & 0 & 0 \\ 0 & x_4x_7 & 0 \\ 0 & 0 & x_3x_6 \end{bmatrix} \quad \text{and} \quad [f_{-4}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ x_7x_{10} & 0 & 0 & 0 \\ 0 & x_6x_9 & 0 & 0 \\ 0 & 0 & x_5x_8 & 0 \\ 0 & 0 & 0 & x_4x_7 \end{bmatrix}.$$

Using matrix multiplication, we can check that $f$ is a chain map. For example, using the above matrix examples, we see that $f_{-4} \circ d_{-3} = d_{-5} \circ f_{-3}$.

Just as in Section 6.1, we can define the module degrees of each $x_i$ inductively, when building $f$ along the seam; the components of $f$ on the sheet won't cause any problems. Note that the module degrees will end up different than what we had in Section 6.1.



x_2
x_1x_2
x_2x_3
x_2x_3x_4
x_2x_4x_5
(−1)
(−2)
(−3)
(−4)
(−5)
x_7x_10
x_5x_8
x_6x_9
x_3x_6
x_4x_7
x_5x_8
x_1x_4
x_2x_5
x_3x_6
x_4x_7
x_5x_8
x_2x_3x_4
x_2x_4x_5
x_2x_5x_6
x_2x_6x_7
(−3)
(−4)
(−5)
(−6)
(−7)

Next we look at how homotopy groups map under $f$. For example, the induced map $f : \pi_{-3}(C) \to \pi_{-5}(C) \to \pi_{-7}(C)$ looks like

$$
\begin{array}{ccccc}
 & & & & (x_2) \\
 & & & & \oplus \\
 & & & & (x_2) \\
 & & & & \oplus \\
 & & (x_2) & \xrightarrow{x_7x_{10}} & 0 \\
 & & \oplus & & \oplus \\
 & & (x_2) & \xrightarrow{x_6x_9} & 0 \\
 & & \oplus & & \oplus \\
(x_2) & \xrightarrow{x_5x_8} & 0 & \xrightarrow{x_7x_{10}} & 0 \\
\oplus & & \oplus & & \oplus \\
(x_2) & \xrightarrow{x_4x_7} & 0 & \xrightarrow{x_6x_9} & 0 \\
\oplus & & \oplus & & \oplus \\
\frac{(x_3,x_4)}{(x_2,x_2x_3)} & \xrightarrow{x_3x_6} & \frac{(x_5,x_6)}{(x_2,x_2x_4x_5)} & \xrightarrow{x_5x_8} & \frac{(x_7,x_8)}{(x_2,x_2x_6x_7)}.
\end{array}
$$

As this suggests, in computing $\mathsf{colim}\pi_i(C)$, we find that the sheet makes no contribution. Colimits commute with direct sums, and $f$ maps the sheet into the interior of the sheet, which is exact. So we have, for example,

$$\pi_{-3}(\mathsf{Tel}) \cong \mathsf{colim}\left[\frac{(x_3,x_4)}{(x_2,x_2x_3)} \xrightarrow{x_3x_6} \frac{(x_5,x_6)}{(x_2,x_2x_4x_5)} \xrightarrow{x_5x_8} \frac{(x_7,x_8)}{(x_2,x_2x_6x_7)} \longrightarrow \cdots\right].$$

This colimit is the same as the one in Section 6.1, except for the addition of the $x_2$ terms in the denominators. A careful application of the methods from Section 6.1

gives $\Lambda$-isomorphisms

$$\begin{aligned}
\pi_{-3}(\mathsf{Tel}) &\cong \mathsf{colim}\left[\frac{(x_3,x_4)}{(x_2,x_2x_3)} \xrightarrow{x_3x_6} \frac{(x_5,x_6)}{(x_2,x_2x_4x_5)} \xrightarrow{x_5x_8} \frac{(x_7,x_8)}{(x_2,x_2x_6x_7)} \longrightarrow \cdots\right] \\
&\cong \mathsf{colim}\left[\frac{(x_4)}{(x_2,x_3,x_5,x_6,x_7,...)} \xrightarrow{x_3x_6} \frac{(x_6)}{(x_2,x_5,x_7,x_8,...)} \xrightarrow{x_5x_8} \frac{(x_8)}{(x_2,x_7,x_9,x_{10},...)} \longrightarrow \cdots\right] \\
&\cong \mathsf{colim}\left[I\left(\frac{k[x_1,x_2]}{(x_2,x_i^2)}\right) \hookrightarrow I\left(\frac{k[x_1,x_2,x_3,x_4]}{(x_2,x_i^2)}\right) \hookrightarrow I\left(\frac{k[x_1,x_2,x_3,x_4,x_5,x_6]}{(x_2,x_i^2)}\right) \hookrightarrow \cdots\right] \\
&\cong I\left(\mathsf{lim}\left[\cdots \to \frac{k[x_1,x_2,x_3,x_4,x_5,x_6]}{(x_2,x_i^2)} \xrightarrow{proj} \frac{k[x_1,x_2,x_3,x_4]}{(x_2,x_i^2)} \xrightarrow{proj} \frac{k[x_1,x_2]}{(x_2,x_i^2)}\right]\right) \\
&\cong I\left(\frac{\Lambda}{(x_2)}\right) = I(\Lambda(\mathbb{N}-\{2\})) = I(\mathbb{N}-\{2\}).
\end{aligned}$$

Similar computations give $\pi_i(\mathsf{Tel}) \cong I(\mathbb{N}-\{2\})$ for all $i$.

### *6.2.2* ***CASE 2****: Suppose $S \subseteq \mathbb{N}$ is arbitrary.*

What we did in Case 1, with $S^c = \{2\}$, was basically this: we constructed an expanding chain complex $D_2$, and beginning at degree $-2$, sewed it into a chain complex $B$ that was similar to the one from Section 6.1. The linking maps between the sheet and seam were all $x_2$, and the chain complex $B$ also had $x_2$ added to its differentials from degree $-2$ on. Now, for arbitrary $S \subseteq \mathbb{N}$, we will just sew on a sheet $D_i$, in the same way, for every $i \in S^c$. Because each $D_i$ is sewn on at degree $-i$, at each degree the seam along $B$ has only finitely many linking maps and the chain complex has only finitely many copies of $\Lambda$.

We make this precise. Define the sheet $D_i$ to be a chain complex identical to $D_2$ in Case 1, but with maps $x_i$ instead of $x_2$. Define

$$D = \bigoplus_{i \in S^c} \Sigma^{-i} D_i.$$

The different sheets of $D$ do not interact with each other.

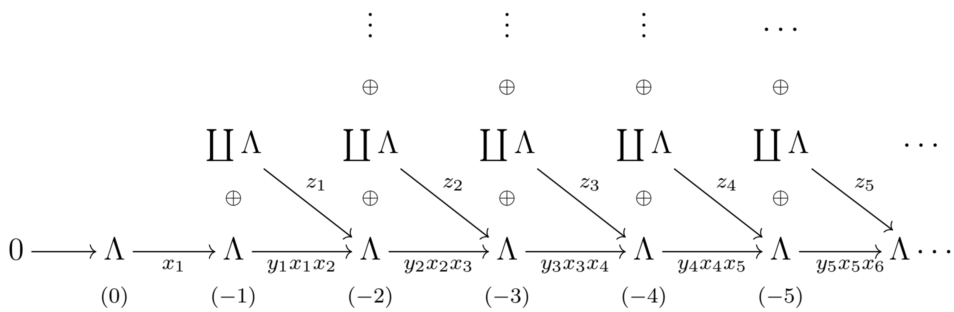
Define $S_i = S^c \cap \{0,1,2,3,...,i\}$, and for $n > 0$ define $y_n \in \Lambda$ and $z_n : \coprod \Lambda \to \Lambda$ by

$$y_n = \prod_{j \in S_{n-1}} x_j, \text{ and } z_n = \coprod_{j \in S_n} x_j.$$

Let $B$ be the chain complex

$$B: \quad 0 \to \Lambda \xrightarrow{x_1} \Lambda \xrightarrow{y_1x_1x_2} \Lambda \xrightarrow{y_2x_2x_3} \Lambda \xrightarrow{y_3x_3x_4} \Lambda \xrightarrow{y_4x_4x_5} \Lambda \xrightarrow{y_5x_5x_6} \Lambda \longrightarrow \cdots .$$

Then the chain complex $C$ we're interested in has $(C)_n = (B)_n \oplus (D)_n$, with linking bottom maps $z_i$:




Now, the definition of $y_i$ and $z_i$ are just right, to allow the sewing to proceed as in Case 1. For example, if $4 \in S^c$, then starting at degree $-4$, the sheet $D_4$, with maps $x_4$, will be sewn in. The difference between $z_3$ nd $z_4$ will be that $z_4$ (and $z_5$, $z_6$, etc.) has a direct summand linking the sheet $D_4$ into the chain complex $B$. And the differentials in $B$ will change: the difference between the differential at degree $-4$ and at degree $-5$ (and $-6$, $-7$, etc.) will be that from $-5$ on there is a fixed term $x_4$ added to the monomial.

The next step is to define $f : C \to \Sigma^2 C$. The seam maps to the seam and each sheet $D_i$ maps to $\Sigma^2 D_i$. Along the seam, $f$ is defined just as in Case 1 and in Section 6.1.

The sheet components of $f$ are defined as needed to make $f$ a chain map. Each sheet of $D$ is independent except for its seam at $B$, so nothing tricky happens when we fill in these maps; it proceeds just like in Case 1. In order for the degrees to work out, we can, as always, work inductively out along the seam. (Note that each $x_i$, for $i \geq 4$, appears in a seam map before it shows up in the differentials in a $y$ map, and filling in the sheet components of $f$ causes no issues with the map degrees.)

The homotopy groups along the seam $B$ are what we're really interested in. As in Case 1, on each sheet $D_i$, the chain map $f$ will move us to a spot in $\Sigma^2 D_i$ that is exact; thus in $\mathsf{colim}\pi_i(C)$ the contribution of these sheets will be zero. Along the seam, the contribution is, for example at degree $-2$,

$$\frac{\ker(y_2x_2x_3)}{\mathrm{im}(z_1)\oplus\mathrm{im}(y_1x_1x_2)} \cong \frac{(x_2,x_3,x_i : i\in S_1)}{(y_1x_1x_2, x_i : i \in S_1)} \cong \frac{(x_2,x_3)}{(y_1x_1x_2, x_i : i\in S_1)}.$$

Thus we have

$$\pi_{-2}(\mathsf{Tel}) \cong \mathsf{colim}\left[\frac{(x_2,x_3)}{(y_1x_1x_2,x_i : i\in S_1)} \xrightarrow{x_2x_5} \frac{(x_4,x_5)}{(y_3x_3x_4, x_i : i\in S_3)} \xrightarrow{x_4x_7} \frac{(x_6,x_7)}{(y_5x_5x_6, x_i : i \in S_5)} \longrightarrow \cdots\right].$$

Note that at each step we're modding out more elements of $S^c$. Now using the same sorts of arguments as before, which we won't detail, we get the following $\Lambda$-isomorphisms.

$$\begin{aligned}\pi_{-2}(\mathsf{Tel}) &\cong \mathsf{colim}\left[\frac{(x_2,x_3)}{(y_1x_1x_2,x_i : i\in S_1)} \xrightarrow{x_2x_5} \frac{(x_4,x_5)}{(y_3x_3x_4, x_i : i\in S_3)} \longrightarrow \cdots\right]\\
&\cong \mathsf{colim}\left[\frac{(x_3)}{(x_i : i\in S_1, x_2, x_4, x_5, x_6, ...)} \xrightarrow{x_2x_5} \frac{(x_5)}{(x_i : i\in S_3, x_4, x_6, x_7, x_8, ...)} \longrightarrow \cdots\right]\\
&\cong \mathsf{colim}\left[I\left(\frac{k[x_1]}{(x_j^2, x_i : i\in S_1)}\right) \to I\left(\frac{k[x_1,x_2,x_3]}{(x_j^2, x_i : i \in S_3)}\right) \to I\left(\frac{k[x_1,x_2,x_3,x_4,x_5]}{(x_j^2, x_i : i\in S_5)}\right) \to \cdots\right]\\
&\cong I\left(\mathsf{lim}\left[\cdots \to \frac{k[x_1,x_2,x_3,x_4,x_5]}{(x_j^2,x_i : i\in S_5)} \longrightarrow \frac{k[x_1,x_2,x_3]}{(x_j^2,x_i : i\in S_3)} \longrightarrow \frac{k[x_1]}{(x_j^2, x_i : i\in S_1)}\right]\right)\\
&\cong I\left(\frac{\Lambda}{(x_i : i\in S^c)}\right) = I(\Lambda(S)) = I(S).\end{aligned}$$

One subtlety is that, once we switch to vector space duals, the maps are not inclusions any more. For example, if $h$ is the map

$$I\left(\frac{k[x_1,x_2,x_3]}{(x_j^2, x_i : i\in S_3)}\right) \xrightarrow{h} I\left(\frac{k[x_1,x_2,x_3,x_4,x_5]}{(x_j^2,x_i : i\in S_5)}\right),$$

then we want to take

$$h = \prod_{i\in S_5\setminus S_3} x_i.$$

To see this, consider the square

$$\begin{array}{ccc}
\frac{(x_5)}{(x_i : i\in S_3, x_4, x_6, x_7, x_8, \ldots)} & \xrightarrow{x_4 x_7} & \frac{(x_7)}{(x_i : i\in S_5, x_6, x_8, x_9, x_{10}, \ldots)} \\
\cong \big\downarrow & & \big\downarrow \cong \\
I\left(\frac{k[x_1,x_2,x_3]}{(x_j^2, x_i : i\in S_3)}\right) & \xrightarrow{h} & I\left(\frac{k[x_1,x_2,x_3,x_4,x_5]}{(x_j^2, x_i : i\in S_5)}\right)
\end{array} .$$

In the bottom left, the generator is $\left(\prod_{i\in S_3} x_i\right).\overline{x_1x_2x_3}$, where the period signifies the $\Lambda$-action. This is sent up to the generator $x_5$, which is sent to $x_4x_5x_7 = (x_4x_5).x_7$, which is sent down to

$$(x_4x_5).\left[\left(\prod_{i\in S_5} x_i\right).\overline{x_1x_2x_3x_4x_5}\right] = \left(\prod_{i\in S_5} x_i\right).\overline{x_1x_2x_3}.$$

This shows that $h$ is $\prod_{i\in S_5\setminus S_3} x_i$. When we dualize (or un-dualize) to consider $\mathsf{lim}$, $h$ dualizes to be exactly what is needed to make maps like

$$\frac{k[x_1,x_2,x_3,x_4,x_5]}{(x_j^2, x_i : i\in S_5)} \longrightarrow \frac{k[x_1,x_2,x_3]}{(x_j^2, x_i : i\in S_3)}$$

good $\Lambda$-module maps.

In conclusion, we've shown that $\pi_{-2}(\mathsf{Tel}) \cong I(S)$. As in earlier arguments, this can be extended to show that in fact $\pi_i(\mathsf{Tel}) \cong I(S)$ for all $i$. And this construction could be generalized, without too much difficulty, to the case where some $n_i > 2$.

### 6.3 An object $\mathsf{Tel}$ with $\pi_n(\mathsf{Tel})$ periodic, with values $\{I(S_1), \ldots I(S_p)\}$ for arbitrary subsets $S_i \subseteq \mathbb{N}$.

The following construction will also make it clear that, by inserting zero maps in the right places, the periodic homology can have zeros as well as arbitrary $I(S)$'s.

We'll assume all $n_i = 2$ in the definition of $\Lambda$; the general case doesn't introduce any subtleties that haven't already been addressed.

### 6.3.1 ***CASE 1***: *Suppose each* $S_i = \mathbb{N}$.

Our goal here is to construct a chain complex $C$, with a degree $p$ chain map $f : C \to \Sigma^p C$, such that the telescope $\mathsf{Tel} = \mathsf{colim}(C \to \Sigma^p C \to \Sigma^{2p} C \to \cdots)$ has $\pi_i(\mathsf{Tel}) \cong I(\mathbb{N})$ for all $i$. We'll assume that $p \geq 3$.

The chain complex $C$ will be the same as in Case 1 of Section 6.1, that is

$$0 \longrightarrow \underset{(0)}{\Lambda} \xrightarrow[x_1]{} \underset{(-1)}{\Lambda} \xrightarrow[x_1x_2]{} \underset{(-2)}{\Lambda} \xrightarrow[x_2x_3]{} \underset{(-3)}{\Lambda} \xrightarrow[x_3x_4]{} \underset{(-4)}{\Lambda} \xrightarrow[x_4x_5]{} \underset{(-5)}{\Lambda} \xrightarrow[x_5x_6]{} \dots .$$

For the chain map $f$, define

$$f_{-i} = \begin{cases} x_i x_{i+p+1} \left( \prod_{i+2 \leq j \leq i+p-1} x_j \right) & i \geq 1 \\ x_{p+1} \left( \prod_{2 \leq j \leq p-1} x_j \right) & i = 0 \\ 0 & i < 0 \end{cases}$$

In order for $f$ to be a chain map, the degrees of each $x_i$ must work out. It's possible to define these degrees inductively. We can choose $|x_1|, |x_2|, ..., |x_{i+p}|$ arbitrarily, and then $|x_{i+p+1}|$ is determined by the square in which it first shows up, etc.

To show that this works, we'll work out the case when $p = 3$. The chain map $f : C \to \Sigma^3 C$ is

$$\begin{array}{ccccccccccccccccccc}
 & & & & & & 0 & \xrightarrow{0} & \Lambda & \xrightarrow{x_1} & \Lambda & \xrightarrow{x_1x_2} & \Lambda & \xrightarrow{x_2x_3} & \Lambda & \xrightarrow{x_3x_4} & \Lambda & \xrightarrow{x_4x_5} & \Lambda \longrightarrow \cdots \\
 & & & & & & {\scriptstyle \Sigma^s 0}\downarrow & & \downarrow{\scriptstyle x_4x_2} & & \downarrow{\scriptstyle x_1x_3x_5} & & \downarrow{\scriptstyle x_2x_4x_6} & & \downarrow{\scriptstyle x_3x_5x_7} & & \downarrow{\scriptstyle x_4x_6x_8} & & \downarrow{\scriptstyle x_5x_7x_9} \\
0 & \xrightarrow[0]{} & \Lambda & \xrightarrow[x_1]{} & \Lambda & \xrightarrow[x_1x_2]{} & \Lambda & \xrightarrow[x_2x_3]{} & \Lambda & \xrightarrow[x_3x_4]{} & \Lambda & \xrightarrow[x_4x_5]{} & \Lambda & \xrightarrow[x_5x_6]{} & \Lambda & \xrightarrow[x_6x_7]{} & \Lambda & \xrightarrow[x_7x_8]{} & \Lambda \longrightarrow \cdots
\end{array}$$

Notice that we can take $|x_1|, ..., |x_4|$ to be anything, and then set $|x_5|$ so that the degrees work out, and then set $|x_6|$, $|x_7|$, etc.

From this we compute, for example,

$$\pi_{-1}(\mathsf{Tel}) \cong \mathsf{colim} \left[ \frac{(x_1, x_2)}{(x_1)} \xrightarrow{x_1x_3x_5} \frac{(x_4, x_5)}{(x_3x_4)} \xrightarrow{x_4x_6x_8} \frac{(x_7, x_8)}{(x_6x_7)} \xrightarrow{x_7x_9x_{11}} \cdots \right].$$

In Section 6.1, we showed this sequential colimit was isomorphic to $\mathsf{colim} M_i$, where each $M_i$ was "the things in $\pi_i(C)$ that survive in the sequential colimit." The maps

$f_i$ have been chosen so that the same $M_i$ from Section 6.1 will work here. So, taking $M_{-n} = \frac{(x_{n+1})}{(x_n, x_{n+2}, x_{n+3}, x_{n+4}, \ldots)}$, we have

$$\begin{aligned}\pi_{-1}(\mathsf{Tel}) &\cong \mathsf{colim}\left[\frac{(x_1,x_2)}{(x_1)} \xrightarrow{x_1x_3x_5} \frac{(x_4,x_5)}{(x_3x_4)} \xrightarrow{x_4x_6x_8} \frac{(x_7,x_8)}{(x_6x_7)} \xrightarrow{x_7x_9x_{11}} \cdots\right]\\ &\cong \mathsf{colim}\left[\frac{(x_2)}{(x_1,x_3,x_4,\ldots)} \xrightarrow{x_1x_3x_5} \frac{(x_5)}{(x_4,x_6,x_7,\ldots)} \xrightarrow{x_4x_6x_8} \frac{(x_8)}{(x_7,x_9,x_{10},\ldots)} \xrightarrow{x_7x_9x_{11}} \cdots\right].\end{aligned}$$

Just as in Section 6.1, we can construct $\Lambda$-isomorphisms from each $M_i$ to a vector space dual, and check that under these isomorphisms, the above maps become inclusions. So in this example we get

$$\begin{aligned}\pi_{-1}(\mathsf{Tel}) &\cong \mathsf{colim}\left[I(k) \hookrightarrow I\left(\frac{k[x_1,x_2,x_3]}{(x_i^2)}\right) \hookrightarrow I\left(\frac{k[x_1,x_2,x_3,x_4,x_5,x_6]}{(x_i^2)}\right) \hookrightarrow \cdots\right]\\ &\cong I\left(\mathsf{lim}\left[\cdots \to \frac{k[x_1,x_2,x_3,x_4,x_5,x_6]}{(x_i^2)} \xrightarrow{proj} \frac{k[x_1,x_2,x_3]}{(x_i^2)} \xrightarrow{proj} k\right]\right)\\ &\cong I(\Lambda) = I(\mathbb{N}).\end{aligned}$$

As in Section 6.1, similar computations show that $\pi_i(\mathsf{Tel}) \cong I(\mathbb{N})$ for all $i$.

### 6.3.2 ***CASE 2***: *The subsets $S_1, \ldots, S_p \subseteq \mathbb{N}$ are arbitrary.*

We'll be using, in part, the chain map $f : C \to \Sigma^p C$ from Case 1, but with a different chain complex $C$. Note that, because of the way colimits work, for all $k \in \mathbb{Z}$ we have

$$\pi_{-j+pk}(\mathsf{Tel}) \cong \pi_{-j}(\mathsf{Tel}) \cong \mathsf{colim}\left[\pi_{-j}(C) \to \pi_{-j-p}(C) \to \pi_{-j-2p}(C) \to \cdots\right].$$

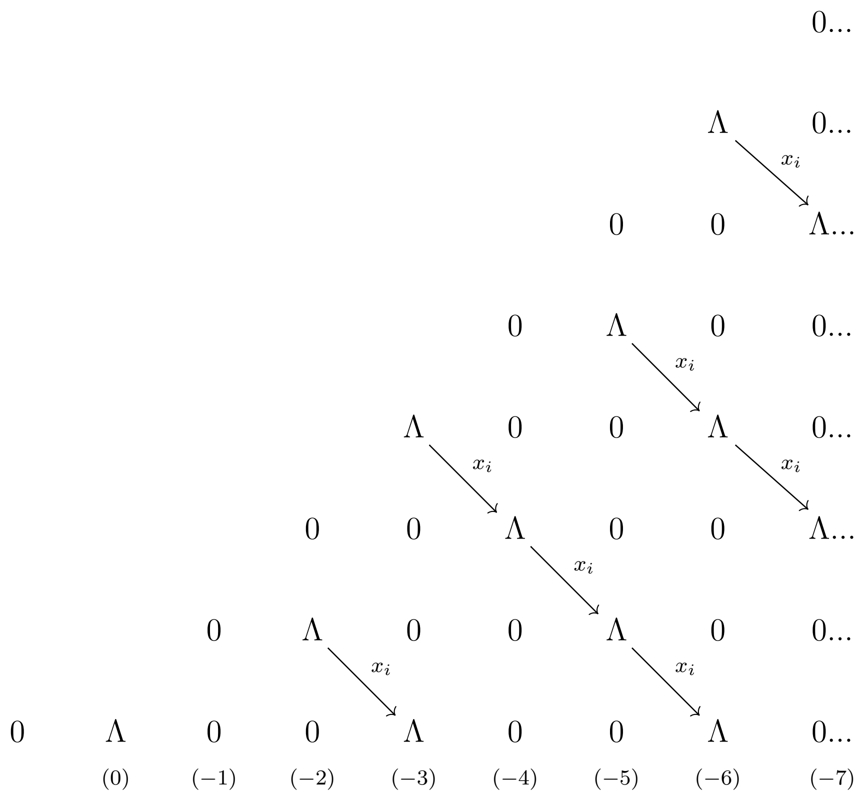
As in Section 6.2, we'll be starting with the basic chain complex $C$ from Section 6.1, and then adding to the monomials in its differentials. We'll also be sewing in sheets $D_i$, a few at a time. But everything will be periodic with period $p$, and the different places along that period will be independent of each other. Because of this, each sheet will be more sparse (of period $p$), and we will be sewing them in differently - in groups of up to $p$ at a time.

It is easiest to show an example, to see how the work of Section 6.2 and Section 6.3 Case 1 can be combined. So we will show the case when $p = 3$.

Let $R, S, T \subseteq \mathbb{N}$ be arbitrary subsets of $\mathbb{N}$. Our goal is thus to construct an object $\mathsf{Tel}$ with

$$\pi_i(\mathsf{Tel}) \cong \begin{cases} I(R) & i \equiv 0 \pmod 3 \\ I(S) & i \equiv 1 \pmod 3 \\ I(T) & i \equiv 2 \pmod 3 \end{cases}$$

For any $i \in \mathbb{N}$, define $D_i$ as the sparse sheet




Define

$$R_j = R^c \cap \{0, 1, 2, ..., j\},$$

$$S_j = S^c \cap \{0, 1, 2, ..., j\}, \text{ and}$$

$$T_j = T^c \cap \{0, 1, 2, ..., j\}.$$

We'll need to sew in sheets to account for $R$, $S$, and $T$, but for simplicity we'll just show how to do it for $R$. Adding in $S$ and $T$ will be similar, just shifted.

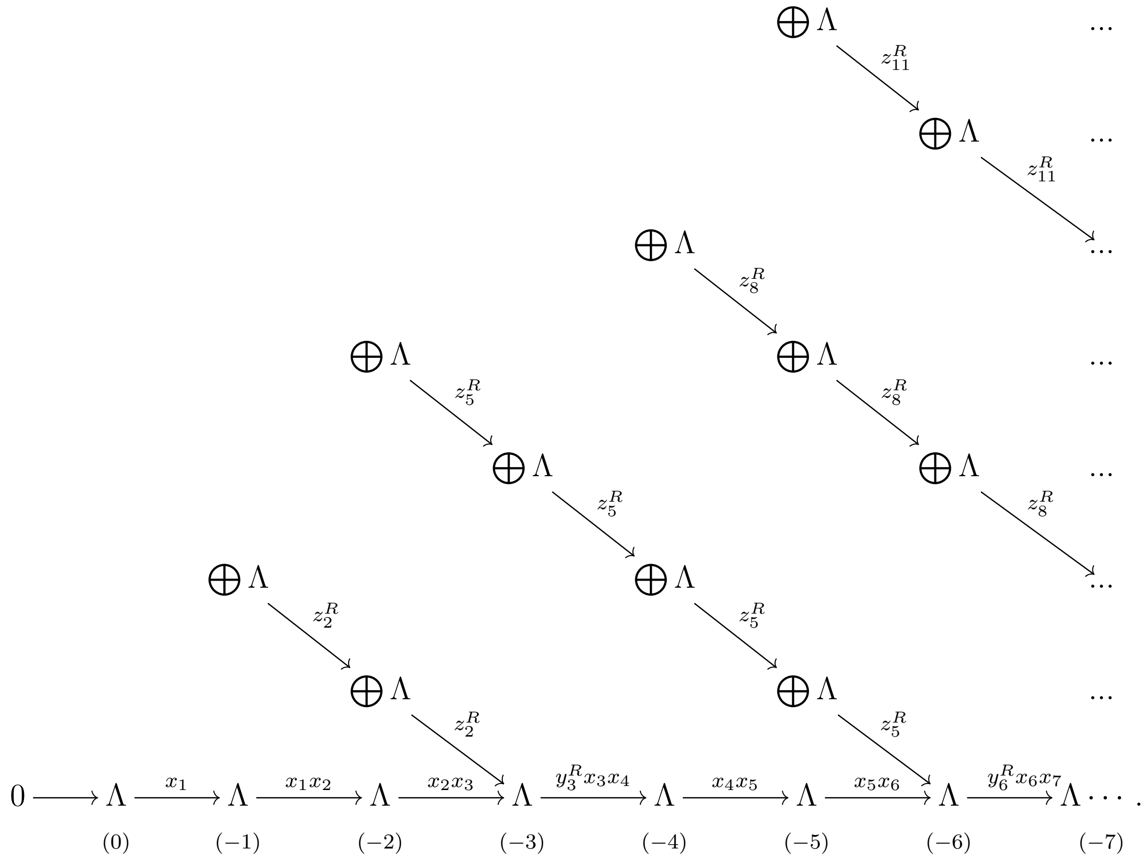
In Section 6.2, we sewed in each sheet $D_i$ at degree $-i$. This was somewhat arbitrary, and was just done to guarantee that at any degree there were only finitely many terms in the monomial in the differential along the seam. Since we're now working with period $p$, we need to sew in the sheets with period $p$, but also only a few at a time. So define

$$D^R = \left( \bigoplus_{i \in R_3} \Sigma^{-2} D_i \right) \oplus \left( \bigoplus_{i \in R_6 \setminus R_3} \Sigma^{-5} D_i \right) \oplus \left( \bigoplus_{i \in R_9 \setminus R_6} \Sigma^{-8} D_i \right) \oplus \cdots .$$

Then define

$$z^R_{3i-1} = \prod_{j \in R_{3i}} x_j, \text{ and } y^R_{3i} = \prod_{j \in R_{3i}} x_j.$$

The seam will be the chain complex from Section 6.1, call it $B$, but with some $y$ maps added to the differentials. We link $D^R$ to this seam using the $z$ maps. So, ignoring $D^S$ and $D^T$ for now, consider the chain complex $C$ with $(C)_n = (B)_n \oplus (D^R)_n$. This looks like




Let the differentials along the seam be denoted $d_{-i}$. Before we define the map $f : C \to \Sigma^3 C$, consider a few homotopy groups of $C$. The sheets $D_i$ contribute nontrivially at their edge, but once inside the sheet the maps are exact. Along the seam, consider $\pi_{-3}$:

$$\frac{\ker(y_3^R x_3 x_4)}{\operatorname{im}(d_{-2}) \oplus \operatorname{im}(z_2^R)} \cong \frac{(x_3, x_4, x_i : i \in R_3)}{\operatorname{im}(d_{-2}) \oplus (x_i : i \in R_3)} \cong \frac{(x_3, x_4)}{\operatorname{im}(d_{-2}) \oplus (x_i : i \in R_3)}.$$

In the above picture, ignoring the contributions of $S$ and $T$, we've written $\operatorname{im}(d_{-2}) = (x_2 x_3)$, but in reality we'd have $d_{-2} = y_2^T x_2 x_3$. We don't know what $y_2^T$ is, but it won't matter. The key observation is that $d_{-2}$ will always contain the term $x_3$. In general, since we're just adding $y$ maps to the differentials of the old chain complex $B$, $d_{-i}$ will always contain a term $x_{i+1}$.

Likewise, the contribution of the seam to $\pi_{-6}(C)$ is

$$\frac{\ker(y_6^R x_6 x_7)}{\operatorname{im}(d_{-5}) \oplus \operatorname{im}(z_5^R)} = \frac{(x_6, x_7, x_i : i \in R_6)}{\operatorname{im}(d_{-5}) \oplus (x_i : i \in R_6)} = \frac{(x_6, x_7)}{\operatorname{im}(d_{-5}) \oplus (x_i : i \in R_6)}.$$

Now, we define the chain map $f : C \to \Sigma^3 C$, which maps the seam to the seam and each sheet to itself. The restriction of $f$ to the seam is the same map as in Case 1 of this section. Namely,

$$f_{-n}|\text{seam} = \begin{cases} x_n x_{n+p+1} \left( \prod_{n+2 \le j \le n+p-1} x_j \right) & n \ge 1 \\ x_{p+1} \left( \prod_{2 \le j \le p-1} x_j \right) & n = 0 \\ 0 & n < 0 \end{cases} .$$

These maps will still commute, but only if we agree to adjust the (module) degrees of the maps $x_i$ appropriately. For example, at the (chain) degree $-3$ and $-4$ this looks like

$$\begin{array}{ccccccc} \cdots \longrightarrow & \Lambda & \xrightarrow{y_3^R x_3 x_4} & \Lambda & \longrightarrow \cdots \\ & \downarrow{\scriptstyle g_{-3}=x_3x_5x_7} & & \downarrow{\scriptstyle g_{-4}=x_4x_6x_8} & \\ \cdots \longrightarrow & \Lambda & \xrightarrow{y_6^R x_6 x_7} & \Lambda & \longrightarrow \cdots \end{array} .$$

Both directions around this square are zero, and we can adjust $|x_8|$ to make the degrees work, since this is the first time that $x_8$ shows up as we build $g$ out.

Once we have $f$ along the seam, the components of $f$ on each sheet can be filled in as required to make everything commute. There will be no trouble doing this - everything commutes on the nose.

Now we compute $\mathsf{colim}\pi_i(C)$. As in Section 6.2, the sheets $D_i$ contribute nothing, since the map $f$ moves into the interior of each sheet, where maps are exact. All that remains is the contribution along the seam. For example, we get

$$\pi_{-3}(\mathsf{Tel}) \cong \mathsf{colim} \left[ \frac{(x_3, x_4)}{\operatorname{im}(d_{-2}) \oplus (x_i : i \in R_3)} \xrightarrow{x_3x_5x_7} \frac{(x_6, x_7)}{\operatorname{im}(d_{-5}) \oplus (x_i : i \in R_6)} \xrightarrow{x_6x_8x_{10}} \cdots \right].$$

This looks similar to Case 1 of this Section. As we've done before, we can use the universal property of the colimit to show that this is isomorphic to $\mathsf{colim} M_i$, where each $M_i$ is "the things from the $i$th step that survive in the limit." Again, the key observation is that, for example, whatever $d_{-5}$ is, it contains a term $x_6$, and because the seam map $f_{-6}$ also has a $x_6$ term, so $\mathrm{im}(d_{-5})$ gets sent to zero in $M_{-5}$. We get

$$\begin{aligned}
&\mathsf{colim}\left[\frac{(x_3,x_4)}{\mathrm{im}(d_{-2})\oplus(x_i : i\in R_3)} \xrightarrow{x_3x_5x_7} \frac{(x_6,x_7)}{\mathrm{im}(d_{-5})\oplus(x_i : i\in R_6)} \xrightarrow{x_6x_8x_{10}} \cdots\right]\\
&\cong \mathsf{colim}\left[\frac{(x_4)}{(x_i : i\in R_3, x_3, x_5, x_6, x_7, \ldots)} \xrightarrow{x_3x_5x_7} \frac{(x_7)}{(x_i : i\in R_6, x_6, x_8, x_9, x_{10}, \ldots)} \xrightarrow{x_6x_8x_{10}} \cdots\right].
\end{aligned}$$

Now the same reasoning used in Section 6.2 shows that this is isomorphic to

$$\begin{aligned}
&\mathsf{colim}\left[I\left(\frac{k[x_1,x_2]}{(x_j^2, x_i : i\in R_3)}\right) \to I\left(\frac{k[x_1,x_2,x_3,x_4,x_5]}{(x_j^2, x_i : i\in R_6)}\right) \to \cdots\right]\\
&\cong I\left(\mathsf{lim}\left[\cdots \to \frac{k[x_1,x_2,x_3,x_4,x_5]}{(x_j^2, x_i : i\in R_6)} \longrightarrow \frac{k[x_1,x_2]}{(x_j^2, x_i : i\in R_3)}\right]\right)\\
&\cong I\left(\frac{\Lambda}{(x_i : i\in R^c)}\right) = I(\Lambda(R)) = I(R).
\end{aligned}$$

Thus we have $\pi_{-3}(\mathsf{Tel}) \cong \pi_i(\mathsf{Tel}) \cong I(R)$, for all $i \equiv 0 \pmod 3$.

To take $S$ and $T$ into consideration, we would build $D^S$ and $D^T$ just like $D^R$, lined up to hit the seam at the appropriate places. We've shown in the above analysis that these three elements operate independently, so modulo certain shifts in indices, everything we said above will work for $S$ and $T$. We thereby consider our period three Tel constructed. It should be clear how this construction could be altered for the case of a general period $p$.

## *6.4 Dualizing*

In this section, we dualize all the above constructions. In each of the cases above, we've defined the object

$$\mathsf{Tel} = \mathsf{colim}\left(C \xrightarrow{f} \Sigma^r C \xrightarrow{\Sigma^r f} \Sigma^{2r} C \longrightarrow \cdots\right),$$

with variations on the chain complex $C$ and the map $f$.

As discussed in Definition 6.0.1, we define a contravariant functor on $D(\Lambda)$ by $I(X) = \mathbb{R}\mathrm{Hom}_k(X, k)$. Dwyer and Palmieri [DP08, 3.4] show this definition also gives

$$[Y, IX]_0 \cong \mathrm{Hom}_k(\pi_0(X \wedge Y), k).$$

For each different construction done above, we can define the corresponding microscope to be

$$\mathsf{Mic} = \mathsf{lim}\left(\cdots \longrightarrow I(\Sigma^r C) \xrightarrow{I(f)} I(C)\right).$$

To show the homology groups of the $\mathsf{Mic}$ objects are what we want them to be, we'll use the fact that

$$\pi_{-r} \circ I \cong I \circ \pi_r.$$

This can be shown directly.

$$\pi_{-r} \circ I(X) = [S^{-r}, I(X)]_0 \cong \mathrm{Hom}_k(\pi_0(\Sigma^{-r} X), k)$$

$$\cong \mathrm{Hom}_k(\pi_r(X), k) = I \circ \pi_r(X).$$

This shows, for example, that the complex $I(C)$ is concentrated in non-negative chain degrees.

For the categorical definition of the sequential colimit $\mathsf{Tel}$, we have a triangle

$$\coprod C \xrightarrow{G} \coprod C \longrightarrow (\mathsf{colim}\, C),$$

and by applying the exact functor $I(-)$ we get a triangle

$$\prod (IC) \xleftarrow{I(G)} \prod I(C) \longleftarrow I(\mathsf{colim}\, C).$$

The map $I(G)$ is exactly the map used in the defining triangle for sequential limits, so this shows that

$$I(\mathsf{colim}\, C) = \mathsf{lim}(IC).$$

In other words, $I(\mathsf{Tel}) = \mathsf{Mic}$.

For simplicity, consider the $\mathsf{Tel}$ with $\pi_n(\mathsf{Tel}) \cong I(\mathbb{N})$ for all $n$. The other cases are similar, since $I(S)$ is locally finite for all $S \subseteq \mathbb{N}$. Then we have the microscope object $\mathsf{Mic} = I(\mathsf{Tel})$, and

$$\pi_n(\mathsf{Mic}) = \pi_n(I(\mathsf{Tel})) \cong I \circ \pi_{-n}(\mathsf{Tel}) \cong I(I(\mathbb{N})) \cong \Lambda, \text{ for all } n.$$

In conclusion, by dualizing we can construct microscope objects with homology groups $\Lambda$, or $\Lambda(S)$, or periodic $\Lambda(S_i)$'s and zeros, for arbitrary $S, S_i \subseteq \mathbb{N}$. The next proposition shows that these microscope objects do not necessarily behave as expected. Recall that in Lemma 6.1.2 we showed that the $\mathsf{Tel}$ object with $\pi_n(\mathsf{Tel}) \cong I(\mathbb{N})$ for all $n$ had $\langle \mathsf{Tel} \rangle \neq \langle I(\mathbb{N}) \rangle$.

**Proposition 6.4.1.** *Let* $\mathsf{Mic}$ *have* $\pi_n(\mathsf{Mic}) \cong \Lambda$ *for all* $n$. *Then* $\langle \mathsf{Mic} \rangle = \langle \Lambda \rangle$.

**Proof.** As before, let $\mathcal{M}$ denote the subcategory of $D(\Lambda)$ of all modules, and think of a Bousfield class $\langle X \rangle$ as the localizing subcategory of $X$-acyclics. First, we'll show that

$$\mathcal{M} \cap \langle \mathsf{Mic} \rangle = \mathcal{M} \cap \langle \Lambda \rangle = \{0\}.$$

Because $\langle \mathsf{Mic} \rangle \leq \langle \Lambda \rangle$, we only need to show

$$\mathcal{M} \cap \langle \mathsf{Mic} \rangle \subseteq \mathcal{M} \cap \langle \Lambda \rangle.$$

For this, use the same spectral sequence as in Lemma 6.1.3, but with $A = M$ a module, and $B = \mathsf{Mic}$.

The $E^2_{p,q}$ term is $\pi_p(M \wedge \oplus_{i \in \mathbb{Z}} \Sigma^i \Lambda)$, so the page collapses to just a copy of $M$ at each spot along the $p = 0$ line. This converges to $\pi_{p+q}(M \wedge \mathsf{Mic})$, so we conclude that if $M \wedge \mathsf{Mic} = 0$, then $M = M \wedge \Lambda = 0$. This proves the desired containment.

Therefore $\mathsf{Mic} \wedge I(\mathbb{N}) \neq 0$, because $I(\mathbb{N}) = I(\mathbb{N}) \wedge \Lambda \neq 0$.

Now, suppose that $a\langle \mathsf{Mic} \rangle \geq \langle I(\mathbb{N}) \rangle$, where $a(-)$ is the complementation operator defined on Bousfield classes in the Introduction. Applying $a(-)$ gives $\langle \mathsf{Mic} \rangle \leq a\langle I(\mathbb{N}) \rangle$, which implies $\mathsf{Mic} \wedge I(\mathbb{N}) = 0$. This is false, but we know every nonzero Bousfield class in $D(\Lambda)$ is greater than or equal to $\langle I(\mathbb{N})\rangle$. Thus we conclude that $a\langle \mathsf{Mic}\rangle = 0$. Then $a\langle \mathsf{Mic}\rangle = 0 = a\langle\Lambda\rangle$ implies that $\langle \mathsf{Mic}\rangle = \langle\Lambda\rangle$, as desired. □

### 6.5 *Experiential context*

The majority of this material is from the fall of 2010 and the winter of 2011, in Seattle. The key idea was to take advantage of the non-Noetherian nature of $\Lambda$ in some way. The insight was that this could be done with a self-map with a nontrivial colimit, because $\Lambda$ has infinitely many generators. This key idea and insight appeared only after a month or so of colimit calculations, but after this I was able to streamline calculations to, in a sense, most elegantly capture this non-Noetherian essence of $\Lambda$.

It was a surprise to recognize that the homology groups of the colimit might be $I(\mathbb{N})$. It was clear from the construction that I had made several arbitrary choices, and in thinking about these I realized I might be able to generalize to $I(S)$, for arbitrary $S \subseteq \mathbb{N}$, and then to periodic homology. Then, in working out the details of the proofs I refined the exposition of the construction to include sheets, and then staggered sheets.

The best way to think of this $\mathsf{Tel}$ construction is to use the concise image of sheets being sewn, in a staggered fashion, into a seam. This image didn't appear until I had almost finished the construction. Because the construction was so elaborate, with so many levels of countable infinity (within $\Lambda$, along the seam of the chain complex $C$, in the sheets sewn into $C$ corresponding to $S^c$, and following the colimit), it was essential to develop metaphors and images, like sheets (or a patchwork umbrella for $I(\mathbb{N})$, or a sort of Whac-a-Mole game when thinking about the $M_i$).

The development of the ideas on this chapter relied on my use of a floor-to-ceiling whiteboard, as seen in the *Cuculetsu* video. The large empty space provided room for all the fine details of the construction but allowed me to step back and maintain important perspective on the bigger picture. More generally, while in Seattle my life as a mathematician has hinged on my ability to both experience mathematics up close from the inside, and step back to contemplate mathematical culture from without.

# Chapter 7

## THE DEPENDENT CO-ARISING OF MATH AND MATHEMATICIANS

In this chapter, we present an account of the interdependence and mutual conditioning of the content of mathematics and the culture of mathematicians. On the one hand, we consider mathematics as a body of knowledge – all the definitions, ideas, theorems, and areas of study – as it exists among the minds of mathematicians and in the mathematical literature. On the other hand, we take a sociological perspective, and consider the community of mathematicians – their behaviors, philosophies, cultural norms, and tacit knowledge.

We view these two aspects of mathematics – the content and the context – through the lens of the Buddhist concept of *paticca samuppada*. One of the central teachings of Buddhism, *paticca samuppada* is commonly translated as dependent co-arising, or dependent origination, or simply interconnectedness. It is a law of non-linear causation [Mac91, Ch.5]. Rather than identifying objects as cause or effect, it recognizes apparently differentiated objects as co-causes and co-effects of each other.

One of the ancient Buddhist scriptures, the Anguttara Nikaya, describes *paticca samuppada* in the following passage [Bhi12].

> "When this is, that is.
> From the arising of this comes the arising of that.
> When this isn't, that isn't.
> From the cessation of this comes the cessation of that."

Approaching the content and context of mathematics from this perspective allows us to respect the two independent domains – one rigorous and one sociological – while still acknowledging and investigating their integration. The discipline of mathematics, as documented in textbooks and journals, has an elegant internal coherence. It seems to exemplify precision, objectivity, consistency, and universality. Contrastingly, the community of mathematicians holds a place in human society and history, just as any other community in the present or past. As a subculture, its characteristics can be described in generalizing terms, and analyzed with conjectural social theories.

But the rigorous and precise modes of reasoning found in mathematics cannot be applied to a sociological study about mathematicians. And conversely, non-rigorous meta-theories and oversimplifications have very limited utility and scope in the realm of mathematics. This has exacerbated the tendency to disconnect the content and the context of mathematics from one another. Practicing mathematicians spend little time trying to understand their cultural norms[1]. Sociologists of mathematics, and to an extent even mathematics education specialists, are out of touch with the real, lived mathematical experience.

Mathematics and the community of mathematicians have dependently co-arisen. We can, and must, respect their individual domains. But we also can, and must, explore their interdependence. In this chapter, we will repeatedly transit the divide. In each section, our method is as follows. We will exhibit a sociological observation $(A)$ about the culture of mathematics, then propose characteristics $(B)$ of mathematics itself that may have caused or encouraged this property. We will discuss how aspect $(A)$ of how we do math, in turn, has affected the development $(C)$ of mathematics. At the same time, we'll present contrasting characteristics $(B', B'')$ of mathematics, and show how they suggest alternative ways $(A', A'')$ of doing mathematics.

We conclude each section with suggestions of how changing the way we do math-

[1]The most notable exception is topologist Raymond Wilder's inimitable [Wil81].

ematics might change what mathematics is. This final critique is in line with the concept of *paticca samuppada*. In Buddhism, self-reflection yields an awareness of the dependent origination of self and other. This carries a strong prescriptive and ethical imperative: by changing oneself, we change everything.

## *7.1 Learning based on problem-solving*

**Observation 7.1.1.** The dominant mode of learning mathematics, and generating new mathematics through research, is problem-solving. An ability to solve problems based on given material is often equated with understanding that material.

This observation can be made in almost any ordinary mathematics classroom. After the teacher presents material, the student must work on problems. Tests and quizzes are usually collections of problems to be solved. This contrasts with science classrooms, where experiments and explorations are more common, or humanities classrooms, where understanding comes from discussing and articulating interpretations, verbally or through writing.

This method is reiterated in research mathematics. The most straightforward, and likely most common, research question to ask is, "What concrete unsolved problems can I work on?"

We propose that this aspect of the doing of mathematics arises, in part, from the precise, universal, abstract, and easily symbolized nature of mathematics.

**Observation 7.1.2.** The content of mathematics is precise, universal, abstract, and easily symbolized.

Because mathematics is precise and universal, it is easy for us to talk about it clearly and agree. Discrete problems can be formulated and discussed unambiguously. Because mathematical concepts lend themselves to concrete symbolization, we can also easily present problems in a succinct and objective expression. The same cannot be said about problems in ethics or literary theory.

Furthermore, mathematics is abstract; the referents of mathematical discourse exist mainly in pure thought. Any mathematical understanding is necessarily a very personal understanding. When posed a question about, say, complex vector spaces, one must internalize the problem in a way that allows application of very personalized imagery and modes of reasoning. The problem is solved in this intimate mental space, and then the solution must be impersonalized and articulated in a clear and objective response. When two mathematicians work together to solve a problem, the abstract nature of mathematics introduces ambiguity into the imagery and metaphors that are being used while working together.

This contrasts with less abstract disciplines. For example, a problem in molecular biology will have concrete representations, suggested by physical reality, that supports a more universal vocabulary of imagery and metaphors.

To work with this abstract mathematics, then, a mathematician must learn to navigate back and forth between the concise objective expressions and the fantasic subjective interpretations.

Problem-solving is well-suited to this navigation. When learning mathematics, problem-solving trains us to take a concrete question, internalize it, use whatever personal modes of reasoning necessary to solve it, then externalize the solution into a concrete answer [Sch94]. Problem-solving in research allows us to discretize the development of a field. We can collectively advance, without having to ever communicate the ambiguous, non-linear, and contingent mental processes that found our understanding.

This cultural characteristic - a reliance on problem-solving - shapes the development of mathematics itself. We proceed by asking, “What can we do?” Specific unsolved problems are tackled and solved, and new problems present themselves.

Throughout history this has been, perhaps, the dominant force in the development of new mathematics. But there are alternative approaches, as Felix Browder discussed in his retiring address as president of the American Mathematical Society in 2002 [Bro02]. We can ask, instead, "What *should* we do? How *should* mathematics grow?" The most famous example of this are Hilbert's problems, delivered in part to the International Congress of Mathematicians in 1900. Many of these problems are vague or ill-posed, and are not as much problems to be solved, as they are declarations of what mathematicians should be interested in.

Research "programs" also serve as long-term proposals for guiding research. The Erlangen Program revolutionized our understanding of non-Euclidean geometries, and centered on a manifesto written by Felix Klein in 1872. The Langlands Program has been extremely successful at focusing research on a family of problems in algebraic number theory and representation theory.

### *7.1.1 Discussion.*

By changing how we do mathematics, we could change what mathematics is. This does not mean removing the precision and objectivity of mathematics, as much as more thoroughly acknowledging the imprecise or intuitive aspects of mathematics. In a widely debated 1993 article in the Bulletin of the AMS, Arthur Jaffe and Frank Quinn proposed doing precisely this [JQ93]. They observe that mathematical journals only present completed proofs – solved problems – and propose establishing a public forum for conjectural or incomplete mathematics. This would provide a space to present vague, or ill-posed, questions and speculations. Most likely, this would result in wider discussion of long-term research directions and a de-emphasis of concrete problems.

The Bulletin also published comments on this proposal [Aea94], which were mostly positive, and Jaffe and Quinn's responses to the comments [JQ94].

In recent years, the online research forum MathOverflow has grown in popularity. It has strict rules encouraging well-posed, pertinent research questions, but also includes tags for 'soft-question' and 'big-picture.' To date, these tags rank 18th and 72nd in popularity, respectively.

The physicist and mathematician Freeman Dyson has colorfully promoted the importance of both problem-solving and large-scale vision.

> Some mathematicians are birds, others are frogs. Birds fly high in the air and survey broad vistas of mathematics out to the far horizon. They delight in concepts that unify our thinking and bring together diverse problems from different parts of the landscape. Frogs live in the mud below and see only the flowers that grow nearby. They delight in the details of particular objects, and they solve problems one at a time...
>
> Mathematics needs both birds and frogs. Mathematics is rich and beautiful because birds give it broad visions and frogs give it intricate details. Mathematics is both great art and important science, because it combines generality of concepts with depth of structures. It is stupid to claim that birds are better than frogs because they see farther, or that frogs are better than birds because they see deeper. The world of mathematics is both broad and deep, and we need birds and frogs working together to explore it. [Dys09]

We believe contemporary mathematical practice promotes a skew towards frogs. And as mathematics advances, and the number of fields of mathematical specialty grows, it becomes more difficult to be, or train, birds. This puts mathematics on a trajectory towards further fragmentation. We should consider the merits of and methods for altering this trajectory.

## 7.2 *Knowledge transmission methods*

**Observation 7.2.1.** Mathematical knowledge is transmitted in two distinct modes: at first through a teacher-student relationship, and later through a master-apprentice relationship.

Through high school and college, and the beginning of mathematics graduate school, the dominant mode of mathematics education is the classroom, with one teacher and many students. The goal is comprehension of, or at least familiarity with, a certain body of knowledge that has been deemed valuable to becoming a successful member of society. This body of knowledge is old and well-established, so the focus for the student is on digestion, and the focus for the teacher is on efficient pedagogy.

The final years of graduate school consist almost entirely of one-on-one interaction between advisor and advisee. The goal – for the advisee, advisor, and institution – is to train the advisee to be a productive and helpful member of the mathematical community. This entails nurturing an ability to produce original research. But a successful advisee must also learn how to communicate mathematics and participate within the community. He or she must learn the cultural norms of mathematical society. This training is done through a master-apprentice relationship, relying more on learning through imitation than learning through successful pedagogy.

This separation is apparent in every doctoral program in academia, but the contrast is particularly stark with mathematics.

In science classes, even young children are exposed to doing new science experiments in class, whereby they imitate the process of a researcher. It is rare for a math class to explore concepts in a way that imitates the discovery-based interplay between formulation of definition, statement of conjecture, and clarification of proof that math researchers rely on. Imre Lakatos has proposed classroom methods that elucidate this aspect of creative mathematics [Lak76], but he acknowledges that even in these exercises, very rarely are students actually generating new mathematics.

In humanities classes, the content is often more current and less objective, so that already by middle school classroom discussions can mirror the intellectual debates that are sustained among humanities scholars. Furthermore, the personal nature of humanities classes encourages more personal one-on-one exchanges between teacher and student.

We propose that this separation, in modes of mathematical instruction, is caused in part by a duality in the nature of math itself.

**Observation 7.2.2.** Lower mathematics is egalitarian, functional, and old. Higher mathematics strongly resembles art.

By lower mathematics, we mean the content that is commonly taught in classrooms. One of the most striking characteristics of lower mathematics is how useful it is in describing and understanding the natural world. As recognized in Wigner's famous essay [Wig85], there is much uncertainty as to why mathematics is so useful, but no doubt that it is. For this reason, some of the oldest, stalest math is taught to as wide an audience as possible. Mathematics education and performance is valued highly in all modernized societies, and often used as a measure of international standing [NCE12].

On the other hand, it is also apparent that higher math – graduate level material and beyond – is less clearly useful. The results are more recent, so it is natural that any applications are less developed or pervasive. The level of abstraction of much of contemporary mathematics permits significant freedom of thought and imagination, within structure, so as to resemble art. Indeed, a successful research mathematician must have a nuanced intuition, of what questions to ask and what conjectures to make. We must develop a skill at discerning the essence of an idea or question, and condensing it into a concise statement or definition. We require an ability to juggle the ambiguity of various mental images and conceptual metaphors. And a mathematician must have some aesthetic sense of a good proof, or an elegant argument.

All these skills are difficult to impart. Educators have yet to figure out how to teach the art of mathematics, or any art. However, for centuries, if not millenia, this type of nuanced knowledge has been passed on through master and apprentice relationships. Like medieval blacksmiths or classical Indian musicians, mathematicians and other scholars use mentorship to teach their way of life.

This duality in mathematical instruction has had an impact on the development of mathematics. The societal importance placed on lower mathematics, taught in the classroom, has spurred the application of mathematics throughout disciplines. Biologists, economists, and engineers continue to find applications for mathematics, thanks to considerable mathematical literacy. This in turn provides a stable context for the pursuit of higher, as-yet-unapplied mathematics.

Centuries of master-apprentice relationships have left a mark on mathematics itself. Because masters transmit not just content, but also a philosophy and value system, different fields of mathematics have different cultural flavors. For much of the development of mathematics, content has been localized geographically, around schools of study and thought centered on a rough lineage. The dispute surrounding Newton and Leibniz was not just about priority of discovery, but also philosophy and notation. Bourbaki was famously principled in its values and mathematical aesthetic, and one can hardly extricate their formal results from their philosophical perspective. The Italian school of algebraic geometry adopted and handed down, for over half a century, an unrigorous style of proof that eventually backfired and caused many of the results to collapse or be disproved.

Another fascinating example of the localization of content and philosophy around masters and schools is described by Hersh and John-Steiner [HJS11]. They discuss the University of Göttingen, Germany, in the 1880s. Felix Klein established a strong research program, interested in finding applications for mathematics; he sought to contrast with the approach of Weierstrass, Frobenius, and Kronecker, in Berlin, who were focused on pure math. Göttingen became known for its inclusive atmosphere, encouraging collaboration. With Klein, David Hilbert carried on this tradition, working to attract visiting scholars, and famously inviting Emmy Noether to teach. As described in [HJS11], Richard Courant, a student of Hilbert's, sought to relocate both the academic rigor and the philosophy of Göttingen, when he created The Courant Institute in New York City.

### *7.2.1 Discussion.*

If we embrace this dual nature of mathematics instruction, and the dual nature of mathematics that it points to, then we might do mathematics differently. Without challenging the importance of traditional math education, or the validity of mathematical applications, we can also more thoroughly engage the aspects of math that resemble art. As Borel [Bor83] points out, mathematicians make aesthetic judgements all the time, when creating and communicating proofs. Tymoczko [Tym93] investigates the nature of these often overlooked judgements, and the problems inherent in a more explicit codification of mathematical aesthetics. Sinclair [Sin11] shows how these aesthetic judgements are already guiding the development of mathematics. In [Tao07], Fields Medalist Terence Tao lists over 20 potential aspects of "good" mathematics (e.g. elegant, creative, useful, strong, deep, intuitive) and observes "there is the remarkable phenomenon that good mathematics in one of the above senses tends to beget more good mathematics in many of the other senses as well."

A more artistic approach to doing math would acknowledge lectures, or any mathematical communication, as a performance with an audience. It would promote self-awareness and a study of mathematical process, as discussed in Chapter 3. It might include experiential components into the exposition of papers and proofs, as demonstrated in the "Experiential context" sections of Chapters 2, 4, and 6.

Tymoczko [Tym93] gives another image of what this might look like:

> If we can follow out this flight of fancy a bit more, we can begin to see the mathematics lecture hall, be it a classroom or a conference, as if it were a concert hall or small parlor where some have gathered to attend a performance. The (performing) mathematician presents a proof, recreates for the audience the lived work of a discovery composed by himself or another. The audience, whether it is faculty or students, we imagine listening for the pure enjoyment of the piece. In presenting the proof, the mathematician is also functioning as a critic, not only in his initial selection of the proof, but in his organization and emphasis. If we are very bold, or very foolish, we might even glimpse the possibility of a more humanistic mathematics moving a little more slowly, being a little more patient with itself, a little more grateful for the pearls that have been obtained.

## *7.3 Platonism, Formalism, Humanism*

**Observation 7.3.1.** The dominant philosophy among mathematicians is a combination of Platonism and Formalism. Mathematics is commonly identified with proof and rigor. Mathematical proofs and exposition are terse, dense, and presented with minimal commentary and human context.

**Observation 7.3.2.** Mathematical ideas and truths seem to enjoy a universality, objectivity, and absoluteness outside of human minds. The abstract and precise nature of mathematics lends itself to symbolization and formalization.

The prevalence of mathematical applications, and the apparent universality of statements such as "2 + 2 = 4" clearly promote a Platonic view of mathematical truth in which mathematics exists absolutely, "out there," independent of our minds.

David Hilbert had a “conviction that mathematics can and must provide truth and certainty ‘or where else are we to find it?’” [DHM95]. Mathematics has become identified with that which can be reasoned about objectively, and proofs are the ideal expression of perfect logic. Thus, making proofs seems to be at the center of what mathematicians do.

There was a time when mathematical arguments were vague, and lacked rigor by today’s standards. The discovery of non-Euclidean geometries, and the desire in analysis to make sense of the continuous versus the discrete, led to a drive towards rigor and precision in mathematical arguments. Mathematical logic was symbolized, and Formalism blossomed in the early 20th century. The abstract nature of mathematical ideas encouraged articulation in a formal language of symbols.

Conversely, the rigorous formalization of mathematics allowed for significant development of new areas of mathematics. Abstract algebra, category theory, and number theory represent a departure from a mathematics that was tied to and motivated by real-world questions. Consequently, starting in the 20th century mathematics became even more abstract, with every field embracing this freedom [Wil81].

The majority of proofs give priority to rigor and deduction, following the cleanest, straightest line of argument from hypotheses to conclusion. According to the Platonist, and the average working mathematician, proofs are a chance to express an objective, eternally true argument [DH86]. Therefore it is preferable to aim for conciseness and inevitability, while avoiding much commentary or subjective contingencies.

The consequence of centering our discipline on terse, human-less proofs that follow deductive logic, is that concepts are often obscured by rigor. Key ideas or insights are hidden or not clearly articulated. A reader must often struggle to re-personalize the logic. Mathematics continues to seem independent of humans, and obscure. According to Byers [Bye07], “...teaching theoretical mathematics is often identified with communicating its formal structure. Understanding lies behind the formal structure but is not captured by that structure. The formal structure is blind to the ideas in mathematics and, as a result, the teacher may feel that it is not his job to communicate the ideas."

Platonism and Formalism are prevalent and valid because they encapsulate definite aspects of the mathematical experience. However, as some mathematicians, philosophers, and math educators have recognized, they don't satisfactorily capture all aspects (see e.g. [Thu94, Her97, Cor03, App95]). For several decades, there has been a countermovement, aiming to recenter mathematics around mathematical ideas and mathematical practice. This is usually referred to as *humanistic mathematics*.

As White argues [Whi47], while it is true that mathematics exists independent of any one human mind, this does not mean that it exists completely independent of all human minds. Rather, it is simply an aspect of human culture, created by a community of thinkers and doers over millenia, guided by interactions with their environment.

Humanistic writers bemoan the current overemphasis on rigor and deduction. Cellucci [Cel06] points out:

> "the method of mathematics... does not start from axioms which are given once and for all and are used to prove any theorem, nor does it proceed forward from axioms to theorems, but proceeds backwards from problems to hypotheses. Thus proof does not begin from axioms that are not themselves proved. Unlike axioms, hypotheses are not given from the start, but are the very goal of the investigation. They are never definitive, but liable to be replaced by other hypotheses...
>
> ...the logic of mathematics is not deductive logic but a broader logic, dealing with non-deductive (inductive, analogical, metaphorical, metonymical, etc.) inferences in addition to deductive inferences."

Recently, Byers [Bye07] has investigated the essential role that ambiguity, contradiction, and paradox play in the development of mathematics. He argues for prioritizing mathematical ideas over rigor. Pragmatically, this means reorganizing how we communicate mathematics to emphasize the key ideas. Byers believes proofs should give a sense of "what's really going on," and *why* a statement is true. He sees proofs, and lectures, as a conversation with an audience, the telling of a story with the goal of convincing and explaining.

Mathematical ideas, as Byers points out, have both an objective dimension and a subjective dimension. A mathematical statement is a precise, objective idea that can be communicated and unambiguously connected with other ideas. But it is also an experience, with attendant imagery and intuition. The internal experience of ideas has an "objective subjectivity" or intersubjectivity.

Lakoff and Núñez [LN00] have tackled this subjective dimension head-on, attempting to establish the field of *mathematical idea analysis.* They systematically analyze the imagery and conceptual metaphors that we use to reason about a range of mathematical concepts. Their conclusion is that, despite the apparently internal and subjective nature of this reasoning, it is intersubjective. There is substantial overlap and commonality between people. Furthermore, these images and metaphors are drawn from everyday human experiences, such as moving forward and backward, or the schema of inside-boundary-outside.

### *7.3.1 Discussion.*

Doing more humanistic mathematics will result in mathematics being more humanistic. Rather than focusing on promoting or deconstructing Platonism and Formalism, we see benefit in acknowledging their validity but moving further. Renert [Ren11] has analyzed the development of mathematics, from pre-formal to formal, through the lens of integral theory [Wil06, McI07]. Formalism developed as a necessary and successful evolution from non-rigorous mathematics. But now our overemphasis on rigor, and on the Platonic dimensions of the mathematical experience, are hindering future development. Renert agrees that the next step for mathematicians involves embracing the intersubjectivity of mathematical ideas, and reconceptualizing proof, or lecture, as a communication between humans.

Adopting this perspective has implications for mathematical communication. In lecture, or in written communication, one should maintain rigor, but focus on ideas and include a human context. In the remainder of this section, we analyze this dissertation itself from this perspective.

Chapters 2, 4, and 6 are written in a mostly conventional style, although with increased attention to illuminating the key ideas.

Chapter 2 is organized to follow the unfolding of hypotheses on the rings and ring map $f : R \to S$. In the first sections, with fewer hypotheses, the results are less significant or interesting. The most interesting results come in Section 2.4 from the hypothesis that $f$ is surjective and $S$ is Noetherian. Then the following Section 2.5, assuming $R$ is also Noetherian, is a sort of denouement, as there is almost too much structure. The take-home message is that we are able to study non-Noetherian behavior via Noetherian structure.

By organizing Chapter 2 in this way, we underscore what hypotheses affect which behavior, while giving the development of the ideas and results a conceptually pleasant and instructionally helpful arc. A more concise, deductive organization might have began with Section 2.4, and hidden earlier results within proofs or lemmas.

Chapter 4 begins with computations, based on results from Chapter 2, and builds towards the main structure theorem result, Theorem 4.2.5, that proves a splitting of the Bousfield lattices. The exposition of this chapter follows the paper [DP08], of which it is clearly an extension. But it also mirrors the development in Chapter 2, and thus is presented as an application of those results. By building up to it, Theorem 4.2.5 arrives on a foundation of examples and smaller results. We believe this aids in grounding understanding. The more formal approach would likely begin immediately with Theorem 4.2.5, and omit computations.

Chapter 6 is narrative. We slowly construct a range of Tel objects, each one slightly more general than the previous. Each construction introduces a new subtlety – first sheets, then staggered sheets, then staggered sparse sheets. As the complexity of the construction increases, we increase the complexity of the narrative, including more diagrams, then matrix descriptions. For each Tel constructed, we provide explicit, rigorous definitions of objects and maps, with hard-to-decipher subscripts, as well as examples for specific subscripts. This demonstration-by-example, which would most likely be omitted in a more formal exposition, aids considerably in capturing what we are doing. Furthermore, in addition to rigorous descriptions of the construction, we interweave more intuitive language, like "sew in different sheets to the seam, at staggered intervals." This loose language also helps to capture what is really going on. Note that the most formal exposition of this chapter would proceed in the reverse direction; it would begin with the final, most general construction, and simply describe the earlier examples as special cases.

The "Experiential context" sections included at the end of each of Chapters 2, 4, and 6 are less conventional. They contain personal details that are virtually never included in conventional mathematical exposition. Some of these details, e.g. where and when the results were found, are perhaps too subjective to be useful. But we also describe the key insights and organizing principles (as perceived by the author), mental imagery used to reason about the concepts, and a narrative of the development of the ideas. This meta-data is relevant. Furthermore, providing these experiential contexts promotes the view that mathematics is a human endeavor and mathematical truths are contingent on a community of practitioners.

A more radically humanistic expository style would thoroughly interweave the rigorous mathematics with the experiential contexts, mental imagery, and historical and personal narratives of discovery/creation. We have chosen to maintain separation, to emphasize that we do not wish to remove rigor, only augment and re-prioritize it.

Of course, in addition to a mildly unconventional exposition of the rigorous mathematical results of Chapters 2, 4, and 6, this dissertation incorporates metamathematical content in Chapters 3, 5, and 7. Once we identify rigor as one component of mathematics, but not the only one, we are more able to engage in metamathematical discourse. This chapter discusses mathematics and the culture of mathematics together. Chapter 5 illustrates the potential that math-art collaboration provides, for examining the mathematical experience and making novel contributions to the content and practice of mathematics. Chapter 3 provides evidence that a humanistic approach to mathematics is a fruitful one.

Taken as a whole, this dissertation is an example of what a more humanistic mathematics might look like. Rigorous mathematical results are presented and augmented with human context, then placed side-by-side with substantial metamathematical contributions, within a framework that reveals and sustains the many dimensions of the mathematical experience.

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