# THE "HOT SPOTS" PROBLEM
# IN PLANAR DOMAINS WITH ONE HOLE

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**Abstract**. There exists a planar domain with piecewise smooth boundary and one hole such that the second eigenfunction for the Laplacian with Neumann boundary conditions attains its maximum and minimum inside the domain.

## 1. Introduction.

We will be concerned with bounded planar domains with piecewise smooth boundaries. The Laplacian with Neumann boundary conditions in such a domain has a discrete spectrum (see, e.g., [BB1]). Recall that the first eigenvalue is equal to 0 and let $\lambda$ denote the second eigenvalue. Our main result is the following.

**Theorem 1.1**. *There exists a planar domain $D$ with one hole, such that the second Neumann eigenvalue is simple (i.e., the subspace of $L^2$ corresponding to $\lambda$ is one-dimensional) and the corresponding eigenfunction $\varphi$ satisfies*

$$\inf_{x\in D} \varphi(x) < \inf_{x\in \partial D} \varphi(x) \leq \sup_{x\in \partial D} \varphi(x) < \sup_{x\in D} \varphi(x). \tag{1.1}$$

The "hot spots" conjecture of J. Rauch, proposed in 1974, states, roughly speaking, that the second Neumann eigenfunction attains its maximum on the boundary of a Euclidean domain. The conjecture is false at this level of generality (see [BB2] and [BW]) but

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it is true for some classes of domains (see [A], [AB], [BB1], [JN], [K], [P]). The counterexample given in [BW] is a planar domain with two holes and it suggests, in the intuitive sense, that any planar domain where the hot spots conjecture fails must have at least two holes. Theorem 1.1 shows that this is not true. Theorem 1.1 is also a small step towards understanding of the "hot spots" problem for domains with no holes. The first part of the following version of the "hot spots" conjecture was stated by Kawohl [K] while the second part is our own.

**Conjecture 1.2**. *(i) The second Neumann eigenfunction attains its maximum on the boundary if $D$ is any convex domain in $\mathbf{R}^n$, for any $n \geq 1$.*

*(ii) The second Neumann eigenfunction attains its maximum on the boundary if $D$ is a simply connected planar domain.*

Counterexamples to the "hot spots" conjecture presented in [BW] and [BB2] involved domains with bizarre shapes (the shapes were unusual for technical reasons). The counterexample given in this paper is rather simple (see Fig. 1 in the next section) so it shows that the hot spots conjecture fails in some "ordinary" domains.

It was pointed out in [BB2] that it would be rather easy to construct a two-dimensional manifold with a boundary (see Figs. 2.1 and 2.2 in [BB2]) based on the same idea as that in [BW], with the property that both maximum and minimum of the second Neumann eigenfunction lie inside the manifold. The example given in this article is much harder, from the intuitive point of view, because a similar distortion of the example (i.e., a two-dimensional manifold of a similar shape) would not be any easier to deal with than the planar domain itself.

One of the goals of this paper is to develop new techniques for studying the "hot spots" problem. Many of the articles cited above converted the "hot spots" problem for eigenfunctions with Neumann boundary conditions to a mixed boundary problem, by cutting the domain into two subdomains along the nodal line (i.e., zero line) for the second Neumann eigenfunction (the nodal line becomes a part of the boundary with the Dirichlet boundary conditions). This technique proved to be very fruitful and we will apply it in this paper. However, when the geometry of the domain is not very simple, it is either hard to find the location of the nodal line or to incorporate the nodal line into the argument. The main part of the proof of Theorem 1.1 will be based on cutting the domain along a level line of the second eigenfunction. This modification makes it necessary to develop arguments that are more quantitative than qualitative in nature, as compared to the existing proofs.

Of course, our proofs will include many ideas from the existing literature, for example, [BB1] and [BW].

We will now briefly describe the idea of the proof of Theorem 1.1. The domain $D$ depicted in Fig. 1 (see the next section) has two axes of symmetry. First we will show that $\varphi$ is symmetric with respect to one of them and antisymmetric with respect to the other one. Hence, it is enough to analyze the upper right quarter of the domain; let us call this subdomain $D_1$. The set $D_1$ is a very thin "tube" of slightly variable width. The point $(0,0)$ lies on the boundary of $D_1$ and it is enough to show that $\varphi$ is strictly larger at $(0,0)$ than at any point in $\partial D \cap \partial D_1$. The point $(0,0)$ is the most distant point from the other end of the tube $D_1$, in the sense that a reflected Brownian motion in $D_1$ starting from $(0,0)$ will take longer (on average) to reach the other end of $D_1$ than a reflected Brownian motion starting from any other point of $\partial D \cap \partial D_1$. This probabilistic statement can be translated into an estimate needed for the proof of (1.1).

**2. Proofs.**

Our proofs will rely to large extent on techniques developed in [BB1] and other papers. We will be brief at many places to keep this article short. We ask the reader to consult [BB1] and other articles cited below for more details.

An open disc with center $x$ and radius $r$ will be denoted $B(x,r)$. We will identify points $x \in \mathbf{R}^2$ with vectors $\overrightarrow{(0,0),x}$ and complex numbers $x = re^{i\theta}$. The angle between $x = r_x e^{i\theta_x}$ and $y = r_y e^{i\theta_y}$, i.e., $\theta_x - \theta_y$, will be denoted $\angle(x,y)$. We will write $\angle(x)$ instead of $\angle(x,(1,0))$, i.e., $\angle(x)$ will denote the angle formed by the vector $x$ with the positive horizontal semi-axis. We will use the convention that $\angle(x,y) \in (-\pi,\pi]$. For any process $Z_t$ we will denote the hitting time of a set $A$ by $T_A^Z$, i.e., $T_A^Z = \inf\{t \geq 0 : Z_t \in A\}$. The superscript will be dropped if no confusion may arise.

Our definition of a domain $D \subset \mathbf{R}^2$ satisfying (1.1) will involve a parameter $\varepsilon \in (0,1/4)$. The value of $\varepsilon$ will be chosen later and should be thought of as a very small number; it will be suppressed in the notation. Let $A_1$ be a convex polygonal domain with the consecutive vertices $(0,-\varepsilon), (0,\varepsilon), (1,2\varepsilon), (2,\varepsilon_0), (2,-\varepsilon_0)$ and $(1,-2\varepsilon)$, where $\varepsilon_0 \in (0,\varepsilon)$. The value of the parameter $\varepsilon_0$ will be specified later. Let $C_1$ be a polygonal Jordan arc inside

$$(B((2,-1),1+2\varepsilon_0) \setminus B((2,-1),1+\varepsilon_0/2)) \cap \{(x_1,x_2) : x_1 \geq 2, x_2 \geq -1\},$$

with endpoints $(2,\varepsilon_0)$ and $(3+\varepsilon_0,-1)$, and such that for any line segments $\overline{x,y}, \overline{y,z} \subset C_1$

we have $|\angle(y-x,z-y)| \le \varepsilon_0$. Similarly, let $C_2$ be a polygonal Jordan arc inside

$$(B((2,-1),1-\varepsilon_0/2) \setminus B((2,-1),1-2\varepsilon_0)) \cap \{(x_1,x_2) : x_1 \ge 2, x_2 \ge -1\},$$

with endpoints $(2,-\varepsilon_0)$ and $(3-\varepsilon_0,-1)$, and such that for any line segments $\overline{x,y}, \overline{y,z} \subset C_2$ we have $|\angle(y-x,z-y)| \le \varepsilon_0$. Let $A_2$ be an open domain whose boundary consists of $C_1$, $C_2$, and line segments $\overline{(2,\varepsilon_0),(2,-\varepsilon_0)}$ and $\overline{(3+\varepsilon_0,-1),(3-\varepsilon_0,-1)}$. Let $A_3 = A_1 \cup A_2$, let $A_4$ be the symmetric image of $A_3$ with respect to the line $\{(x_1,x_2) : x_2 = -1\}$, and let $A_5$ and $A_6$ be the symmetric images of $A_3$ and $A_4$ with respect to $\{(x_1,x_2) : x_1 = 0\}$. Finally we let $D$ be the interior of the closure of $A_3 \cup A_4 \cup A_5 \cup A_6$. A schematic drawing of $D$ is presented in Fig. 1. The polygonal lines $C_1$ and $C_2$ are very close to circular arcs so they are represented graphically as such. A substantial part of the argument will be focused on a subdomain $D_1$ of $D$ depicted in Fig. 1.

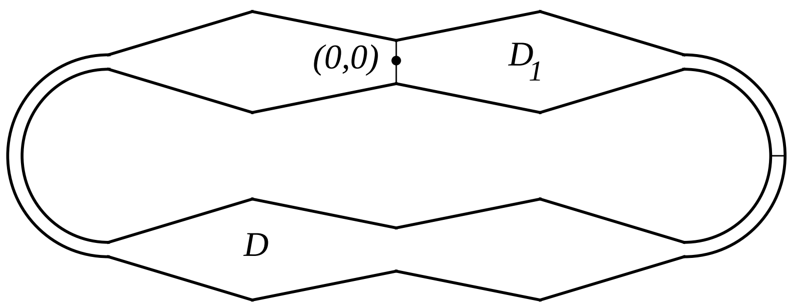


Figure 1. (Drawing not to scale.)

We will now review a few basic facts about reflected Brownian motion and "synchronous" couplings. Let $\mathbf{n}(x)$ denote the unit inward normal vector at $x \in \partial D$. Let $W$ be standard planar Brownian motion, $x,y \in \overline{D}$, and consider the following Skorohod equations,

$$X_t = x + W_t + \int_0^t \mathbf{n}(X_s) dL_s^X, \tag{2.1}$$

$$Y_t = y + W_t + \int_0^t \mathbf{n}(Y_s) dL_s^Y. \tag{2.2}$$

Here $L^X$ is the local time of $X_t$ on $\partial D$, i.e., a non-decreasing continuous process which does not increase when $X$ is in $D$: $\int_0^\infty \mathbf{1}_D(X_t) dL_t^X = 0$, a.s. Equation (2.1) has a unique pathwise solution $(X_t, L_t^X)$ such that $X_t \in \overline{D}$ for all $t \ge 0$ (see [LS]). The "reflected Brownian motion" $X$ is a strong Markov process. The same remarks apply to (2.2), so $(X,Y)$ is also strong Markov. We will call $(X,Y)$ a "synchronous coupling." Note that on any interval $(s,t)$ such that $X_u \in D$ and $Y_u \in D$ for all $u \in (s,t)$, we have $X_u - Y_u = X_s - Y_s$ for all $u \in (s,t)$.

Recall that $\lambda$ denotes the second Neumann eigenfunction in $D$.

**Lemma 2.1**. *For any $c_1 > 0$ there exists $c_2 > 0$ such that if $\varepsilon_0 \le c_2 \varepsilon$ then $\lambda \le c_1$ and $\lambda$ is simple.*

**Proof**. Let $r = \varepsilon_0/(2\varepsilon - \varepsilon_0)$ and note that the point $y \stackrel{\text{df}}{=} (2+r, 0)$ lies at the intersection of straight lines passing through the line segments $\overline{(1,2\varepsilon),(2,\varepsilon_0)}$ and $\overline{(1,-2\varepsilon),(2,-\varepsilon_0)}$. Let $K = \{x = (x_1,x_2) \in D : |x_1| \le 2, x_2 > -1\}$, $K_1 = B(y,2r) \cap K$, $K_2 = \partial B(y,1/2) \cap K$ and $K_3 = \partial B(y,1) \cap K$. Let $X$ be a reflected Brownian motion in $D$ with $X_0 \in K_2$. Let $T_0 = 0$, and for $k \ge 1$ let

$$\begin{aligned} S_k &= \inf\{t \ge T_{k-1} : X_t \in K_1 \cup K_3\}, \\ T_k &= \inf\{t \ge S_k : X_t \in K_2\}. \end{aligned}$$

Let $R_t = \text{dist}(X_t, y)$ and note that if $X$ is between $K_1$ and $K_3$, the process $R$ is a 2-dimensional Bessel process because the normal reflection of $X$ on $\partial D$ has no effect on $R$. It follows that for any $p_1 < 1$, there exists $r_0 > 0$ so small that if $r \le r_0$ then

$$P(X_{S_k} \in K_3 \mid \mathcal{F}_{T_{k-1}}) = \frac{\log(1/2) - \log(2r)}{\log 1 - \log(2r)} \ge p_1.$$

Moreover, for some $t_0 > 0$ not depending on $r$, $P(X_{S_k} \in K_3, S_k - T_{k-1} > t_0 \mid \mathcal{F}_{T_{k-1}}) \ge p_1$.

Let $z = (-2-r, 0)$, $K_1' = (B(y,2r) \cup B(z,2r)) \cap K$, $K_2' = (\partial B(y,1/2) \cup \partial B(z,1/2)) \cap K$, $K_3' = (\partial B(y,1) \cup \partial B(z,1)) \cap K$, $T_0' = 0$, and for $k \ge 1$ let

$$\begin{aligned} S_k' &= \inf\{t \ge T_{k-1}' : X_t \in K_1' \cup K_3'\}, \\ T_k' &= \inf\{t \ge S_k' : X_t \in K_2'\}. \end{aligned}$$

By symmetry, $P(X_{S_k'} \in K_3', S_k' - T_{k-1}' > t_0 \mid \mathcal{F}_{T_{k-1}'}) \ge p_1$. By the repeated use of the strong Markov property,

$$P(T^X_{K_1'} \ge k t_0 \mid X_0 \in K_2') \ge P\Big(\bigcap_{1 \le j \le k} \{X_{S_j'} \in K_3', S_j' - T_{j-1}' > t_0\} \mid X_0 \in K_2'\Big) \ge p_1^k.$$

Let $D_- = \{x = (x_1,x_2) \in D : x_2 < -1\}$ and $A = \partial D_- \cap D$. Let $u(t,x)$ be the heat equation solution in $D$ with the Neumann boundary conditions and the initial condition $u(0,x) = 1$ for $x \in D_-$ and $u(0,x) = 0$ otherwise. Note that $u$ can be represented probabilistically as $u(t,x) = P(X_t \in D_- \mid X_0 = x)$. By the strong Markov property applied at $T_A^X$ and symmetry, $P(X_t \in D_- \mid T_A^X < t) = 1/2$, so for $x \in K_2'$ and large $t$,

$$\begin{aligned}
u(t,x) &= P(X_t \in D_- \mid X_0 = x) \\
&= P(X_t \in D_-, T_A^X < t \mid X_0 = x) \\
&= (1/2) P(T_A^X < t \mid X_0 = x) \\
&= 1/2 - (1/2) P(T_A^X \geq t \mid X_0 = x) \\
&\leq 1/2 - (1/2) P(T_{K_1'}^X \geq t \mid X_0 = x) \\
&\leq 1/2 - (1/2) p_1^{t/(2t_0)} = 1/2 - (1/2) e^{(\log p_1/(2t_0))t}.
\end{aligned}$$

Since $p_1$ can be made arbitrarily close to 1 by making $r$ small, $-\log p_1/(2t_0) > 0$ can be arbitrarily close to 0. By symmetry, $u(t,x)$ converges to $1/2$ as $t \to \infty$. By Proposition 2.1 of [BB1], $\sup_{x\in D} |u(t,x) - 1/2| \leq c_3 e^{-\lambda t}$ for large $t$. Hence, $\lambda \leq -\log p_1/(2t_0)$ and we see that for any $c_1 > 0$ we have $\lambda \leq c_1$, provided $r$ is sufficiently small. If $c_2 < 1$ and $\varepsilon_0 \leq c_2 \varepsilon$ then $r = \varepsilon_0/(2\varepsilon - \varepsilon_0) \leq c_2$, so $\lambda \leq c_1$ if we assume that $c_2$ is small. This proves the first claim of the lemma.

The assertion that $\lambda$ is simple is totally analogous to the claims proved in Sections 4 and 5 of [BW]. The proofs in [BW] are based on the fact that the domain has a bottleneck and they extend easily to our domain $D$. We leave the details to the reader. □

We will assume from now on that $\varepsilon_0$ and $\varepsilon$ are such that $\lambda < 1$ and $\lambda$ is simple. Recall that the "nodal line" is the set of points $x$ such that $\varphi(x) = 0$. We will use the phrase "nodal line" even if the set of zeros of $\varphi$ is not connected.

**Lemma 2.2**. *For any $\varepsilon > 0$ there is $\varepsilon_1 \in (0, \varepsilon)$ such that if $\varepsilon_0 \in (0, \varepsilon_1)$ then the following is true. The eigenfunction $\varphi$ is symmetric with respect to the vertical axis and antisymmetric with respect to the line $\{(x_1, x_2) : x_2 = -1\}$, i.e., for any $(x_1, -1 + x_2) \in D$, we have $\varphi(x_1, -1+x_2) = \varphi(-x_1, -1+x_2) = -\varphi(x_1, -1-x_2) = -\varphi(-x_1, -1-x_2)$. It follows that the nodal line is $\{(x_1, x_2) \in D : x_2 = -1\}$.*

**Proof**. The function $\varphi_1(x_1, x_2) \stackrel{\text{df}}{=} \varphi(x_1, x_2) + \varphi(-x_1, x_2)$ is an eigenfunction corresponding to $\lambda$. If $\varphi_1$ is identically equal to zero then $\varphi$ is antisymmetric with respect to the vertical axis. If $\varphi_1$ is not identically equal to zero then it is a constant multiple of $\varphi$ (because $\lambda$ is simple) and it follows that $\varphi$ is symmetric with respect to the vertical axis. A similar argument shows that either $\varphi$ is antisymmetric with respect to $\{(x_1,x_2) : x_2 = -1\}$ or it is symmetric with respect to this line.

An argument similar to that in Section 4 of [BW] shows that for any fixed $\varepsilon$, the nodal line cannot intersect the set $\{(x_1,x_2) \in D : |x_1| \le 1\}$ if $\varepsilon_0$ is sufficiently small. By the Courant Nodal Line Theorem ([CH]), the nodal line divides $D$ into two connected components. These facts taken together with the symmetries described in the first paragraph of the proof imply that the nodal line must be equal to $\{(x_1,x_2) \in D : x_2 = -1\}$. It follows that $\varphi$ is antisymmetric with respect to $\{(x_1,x_2) : x_2 = -1\}$ and it is symmetric with respect to the vertical axis. $\square$

The next lemma is a prelude to a theorem on geometric properties of "mirror" couplings, to be defined later. The lemma is concerned with convergence of a sequence of processes to the reflected Brownian motion—we start with the construction of this sequence. Suppose that $W$ is a planar Brownian motion, $x$ is a point in the upper half-plane $D_* \stackrel{\text{df}}{=} \{(y_1,y_2) \in \mathbf{R}^2 : y_2 > 0\}$, and $c_1 < \infty$ is a constant. For every fixed $\delta > 0$, we will construct a process $X^\delta$ inductively. Let $X^{\delta,1}$ be the reflected Brownian motion in $D_*$, starting from $x$ and driven by $W$, in the sense of (2.1). Let $T_0 = 0$, and $T_1 \ge T_0$ be a stopping time such that $X^{\delta,1}_{T_1} \in \partial D_*$ a.s. Let $V_1$ be a random variable satisfying $|V_1| \le c_1\delta^2$, a.s. For the induction step, suppose that the process $X^{\delta,j}$ is defined, $T_j$ is a stopping time for $X^{\delta,j}$ such that $T_j \ge T_{j-1}$, $X^{\delta,j}_{T_j} \in \partial D_*$ a.s., and $V_j$ is a random variable satisfying $|V_j| \le c_1\delta^2$, a.s. We define $\{X^{\delta,j+1}_t, t \ge T_j\}$ as the reflected Brownian motion driven by $\{W_t, t \ge T_j\}$, starting at $X^{\delta,j}_{T_j} + (V_j, \delta)$. Then we choose any $X^{\delta,j+1}$-stopping time $T_{j+1} \ge T_j$ such that $X^{\delta,j+1}_{T_{j+1}} \in \partial D_*$ a.s., and a random variable $V_{j+1}$ satisfying $|V_{j+1}| \le c_1\delta^2$, a.s. The process $X^\delta$ is defined by $X^\delta_t = X^{\delta,j}_t$ for $t \in [T_{j-1}, T_j)$. It is elementary to see that $T_j \to \infty$ a.s., so $X^\delta_t$ is well defined for all $t \ge 0$ a.s.

**Lemma 2.3**. *The processes $X^\delta$ converge in distribution to the reflected Brownian motion in $D_*$ as $\delta \to 0$.*

**Proof**. Let us denote coordinates of processes as follows, $W = (\widetilde{W}, \widehat{W})$ and $X^\delta = (\widetilde{X}^\delta, \widehat{X}^\delta)$. Note that $\widehat{X}^\delta_t = \widehat{W}_t + L^\delta_t$, where $L^\delta$ is a non-decreasing process. It is elementary to prove that $L^\delta$ converge to a process $L$ as $\delta \to 0$, on every time interval $[0, t_0]$, and the process $L$ is non-decreasing, continuous and does not increase when $\widehat{W}_t + L_t > 0$. By the uniqueness of the Skorohod decomposition, $\widehat{W}_t + L_t$ is a one-dimensional reflected

Brownian motion. Hence, $\widehat{X}^\delta$ converge to the Shorohod transform (in the sense of (2.1)) of $\widehat{W}$.

Fix a time interval $[0, t_0]$ and let $N = N(\delta)$ be the number of $j$ with $T_j \le t_0$. Note that $L^\delta_{t_0} \ge N\delta$ and the jumps of $L^\delta$ occur only when $X^\delta$ approaches $\partial D_*$. Since $\{W_t, t \in [0, t_0]\}$ has a bounded diameter a.s., it follows that there exists a random variable $N_0 < \infty$ such that $N \le N_0/\delta$ for every $\delta \in (0, 1)$, a.s. This implies that that $\sum_{j \le N} |V_j| \le c_1 N_0 \delta$. Since this random quantity converges to 0 in distribution, as $\delta \to 0$, $\widetilde{X}^\delta$ converge to $\widetilde{W}$. $\square$

We will now review some properties of "mirror couplings" for reflected Brownian motions which are relevant to our arguments. These aspects of mirror couplings were originally developed in [BK] and later applied in [BB1] and [BB2]. Our review is borrowed from [BB2].

We start with the mirror coupling of two Brownian motions in $\mathbf{R}^2$. Suppose that $x, y \in \mathbf{R}^2$ are symmetric with respect to a line $M$. Let $X$ be a Brownian motion starting from $x$ and let $Y_t$ be the mirror image of $X_t$ with respect to $M$ for $t \le T^X_M$. We let $Y_t = X_t$ for $t > T^X_M$. The process $Y$ is a Brownian motion starting from $y$. The pair $(X, Y)$ is a "mirror coupling" of Brownian motions.

Next we turn to the mirror coupling of reflected Brownian motions in a half-plane $D_*$, starting from $x, y \in D_*$. Let $M$ be the line of symmetry for $x$ and $y$. The case when $M$ is parallel to $\partial D_*$ is essentially a one-dimensional problem, so we focus on the case when $M$ intersects $\partial D_*$. By performing rotation and translation, if necessary, we may suppose that $D_*$ is the upper half-plane and $M$ passes through the origin. We will write $x = (r^x, \theta^x)$ and $y = (r^y, \theta^y)$ in polar coordinates. The points $x$ and $y$ are at the same distance from the origin so $r^x = r^y$. Suppose without loss of generality that $\theta^x < \theta^y$. We first generate a 2-dimensional Bessel process $R_t$ starting from $r^x$. Then we generate two coupled one-dimensional processes on the "half-circle" as follows. Let $\widetilde{\Theta}^x_t$ be a 1-dimensional Brownian motion starting from $\theta^x$. Let $\widetilde{\Theta}^y_t = -\widetilde{\Theta}^x_t + \theta^x + \theta^y$. Let $\Theta^x_t$ be reflected Brownian motion on $[0, \pi]$, constructed from $\widetilde{\Theta}^x_t$ by the means of the Skorokhod equation. Thus $\Theta^x_t$ solves the stochastic differential equation $d\Theta^x_t = d\widetilde{\Theta}^x_t + dL_t$, where $L_t$ is a continuous process that changes only when $\Theta^x_t$ is equal to 0 or $\pi$ and $\Theta^x_t$ is always in the interval $[0, \pi]$. The process $\Theta^x_t$ is constructed in such a way that the difference $\Theta^x_t - \widetilde{\Theta}^x_t$ is constant on every interval of time on which $\Theta^x_t$ does not hit 0 or $\pi$. The analogous reflected process obtained from $\widetilde{\Theta}^y_t$ will be denoted $\widehat{\Theta}^y_t$. Let $\tau^\Theta$ be the smallest $t$ with $\Theta^x_t = \widehat{\Theta}^y_t$. Then we let $\Theta^y_t = \widehat{\Theta}^y_t$ for $t \le \tau^\Theta$ and $\Theta^y_t = \Theta^x_t$ for $t > \tau^\Theta$. We define a "clock" by $\sigma(t) = \int_0^t R_s^{-2} ds$. Then

$X_t = (R_t, \Theta^x_{\sigma(t)})$ and $Y_t = (R_t, \Theta^y_{\sigma(t)})$ are reflected Brownian motions in $D_*$ with normal reflection—one can prove this using the same ideas as in the discussion of the skew-product decomposition for 2-dimensional Brownian motion presented in [IMK]. Moreover, $X$ and $Y$ behave like free Brownian motions coupled by the mirror coupling as long as they are both strictly inside $D_*$. The processes will stay together after the first time they meet. We call $(X, Y)$ a "mirror coupling" of reflected Brownian motions.

The two processes $X$ and $Y$ in the upper half-plane remain at the same distance from the origin. Suppose now that $D_*$ is an arbitrary half-plane, and $x$ and $y$ belong to $D_*$. Let $M$ be the line of symmetry for $x$ and $y$. Then an analogous construction yields a pair of reflected Brownian motions starting from $x$ and $y$ such that the distance from $X_t$ to $M \cap \partial D_*$ is always the same as for $Y_t$. Let $M_t$ be the line of symmetry for $X_t$ and $Y_t$. Note that $M_t$ may move, but only in a continuous way, while the point $M_t \cap \partial D_*$ will never move. We will call $M_t$ the *mirror* and the point $H = M_t \cap \partial D_*$ will be called the *hinge*. The absolute value of the angle between the mirror and the normal vector to $\partial D_*$ at $H$ can only decrease.

The next level of generality is to consider a mirror coupling of reflected Brownian motions in a polygonal domain $D$. For the first rigorous construction of a mirror coupling in a domain with piecewise $C^2$-boundary see [AB]. Earlier applications of mirror couplings in such domains lacked full justification. A technical problem that prevents us from generalizing the mirror coupling construction in a half-plane given above to polygons is that it may occur, with positive probability, that the two processes are on two different line segments in the boundary of the domain at the same time (proving this claim does not seem to be trivial; we omit the proof because it is not needed in this article). Suppose that $(X, Y)$ is a mirror coupling in a polygonal domain $D$ and consider an interval $[t_0, t_1]$ such that for every $t \in [t_0, t_1]$, either $X_t \notin \partial D$ or $Y_t \notin \partial D$. Let $I$ be the edge of $\partial D$ which is hit first by one of the particles after time $t_0$. Let $K$ be the straight line containing $I$. Since the process which hits $I$ does not "feel" the shape of $\partial D$ except for the direction of $I$, it follows that the two processes will remain at the same distance from the hinge $H_t = M_t \cap K$. The mirror $M_t$ can move but the hinge $H_t$ will remain constant as long as $I$ remains the side of $\partial D$ where the reflection takes place. The hinge $H_t$ will jump when the reflection location moves from $I$ to another edge of $\partial D$. The hinge $H_t$ may from time to time lie outside $\partial D$, if $D$ is not convex.

Our arguments will be based in part on the analysis of all possible movements of the "mirror" $M_t$. If $D$ is a polygonal domain and only one of the processes is on the boundary of $D$ at time $t_0$, then the possible movements of the mirror on a small time interval $[t_0, t_0 + \Delta t]$ are described in the above paragraph. We cannot apply the same analysis to the case when both processes are on the boundary of $D$ at time $t_0$ so we will provide an alternative approach in Lemma 2.4 below.

With probability one, reflected Brownian motion never visits any vertices of the union of polygons $\partial D$, so we will assume that whenever $X_t \in \partial D$ then $X_t$ lies on a single edge of $\partial D$.

Suppose that $X_t \in \partial D$ and let $K_{X,t}$ be the line containing the edge of $\partial D$ to which $X_t$ belongs. We will be interested only in the case when $M_t$ is not perpendicular to $K_{X,t}$. Consider any other straight line $I$ intersecting $M_t$ at a single point $x$. If $M_.$ turns around the hinge $H_t = M_t \cap K_{X,t}$ so that the (smaller) angle between $M_.$ and $K_{X,t}$ increases, i.e., the two lines become "more perpendicular," the intersection point of $M_.$ and $I$ will move into one of the half-lines $I \setminus \{x\}$; we will denote the closure of this half-line $I^{X,t}$. Let $K_{Y,t}$ and $I^{Y,t}$ be defined in an analogous way relative to $Y$. If for some $t$ both processes belong to $\partial D$ then the above definitions can be applied to $I = K_{X,t}$ and $I = K_{Y,t}$, so $K_{X,t}^{Y,t}$ and $K_{Y,t}^{X,t}$ are well defined.

**Lemma 2.4**. *Suppose that $(X, Y)$ is a mirror coupling of reflected Brownian motions in $D$. With probability one, for every $t \geq 0$ such that $X_t, Y_t \in \partial D$ and $M_t$ is not perpendicular to any of the lines $K_{X,t}$ and $K_{Y,t}$, there exists $a = a(t) > 0$ such that for $s \in [t, t + a]$, we have $M_s \cap K_{X,s} \in K_{X,t}^{Y,t}$ and $M_s \cap K_{Y,s} \in K_{Y,t}^{X,t}$.*

**Proof**. Suppose that $\mathrm{dist}(X_0, Y_0) = r_0 > 0$ and fix an arbitrarily small $r \in (0, r_0)$. Consider a $\delta \in (0, r/100)$. First we will modify the mirror coupling $(X, Y)$ as follows.

Let $T_1 = \inf\{t \geq 0 : X_t, Y_t \in \partial D\}$ and let $H_t^X = M_t \cap K_{X,t}$. Let $X_{T_1}^{\delta,1}$ be the point in $A_1 \stackrel{\mathrm{df}}{=} D \cap \partial B(H_{T_1}^X, \mathrm{dist}(X_{T_1}, H_{T_1}^X))$ whose distance from $K_{X,t}$ is $\delta \wedge \mathrm{diam}(A_1)$. Let $Y_{T_1}^{\delta,1} = Y_{T_1}$ and let $\{(X_t^{\delta,1}, Y_t^{\delta,1}), t \geq T_1\}$ be a mirror coupling in $D$ starting from $(X_{T_1}^{\delta,1}, Y_{T_1}^{\delta,1})$ at time $T_1$ but otherwise independent of $\{(X_t, Y_t), t \in [0, T_1]\}$. Let $T_2 = \inf\{t \geq T_1 : X_t^{\delta,1}, Y_t^{\delta,1} \in \partial D\}$. We continue the construction by induction. Suppose that $(X_t^{\delta,j}, Y_t^{\delta,j})$ and $T_{j+1}$ have been defined. Then we let $M_t^j$ denote the mirror for $X_t^{\delta,j}$ and $Y_t^{\delta,j}$, and $H_t^{X,j} = M_t^j \cap K_{X^{\delta,j},t}$. We define $X_{T_{j+1}}^{\delta,j+1}$ to be the point in $A_{j+1} \stackrel{\mathrm{df}}{=} D \cap \partial B(H_{T_{j+1}}^{X,j}, \mathrm{dist}(X_{T_{j+1}}^{\delta,j}, H_{T_{j+1}}^{X,j}))$ whose distance from $K_{X^{\delta,j},T_{j+1}}$ is $\delta \wedge \mathrm{diam}(A_{j+1})$. We also let $Y_{T_{j+1}}^{\delta,j+1} = Y_{T_{j+1}}^{\delta,j}$ and $\{(X_t^{\delta,j+1}, Y_t^{\delta,j+1}), t \geq T_{j+1}\}$ be a mirror coupling in $D$ starting from $(X_{T_{j+1}}^{\delta,j+1}, Y_{T_{j+1}}^{\delta,j+1})$ at time $T_{j+1}$ but otherwise independent of $(X^{\delta,k}, Y^{\delta,k})$, $k = 1, 2, \ldots, j$. It is easy to see that $\sup_j T_j \to \infty$ as $\delta \to 0$ in probability, because reflected Brownian motion does not hit vertices of $D$.

Let $U_r = \inf_j \inf\{t \geq T_j : \text{dist}(X_t^{\delta,j}, Y_t^{\delta,j}) \leq r\}$. Let $X_t^\delta = X_t$ for $t \in [0, T_1)$, $X_t^\delta = X_t^{\delta,j}$ for $t \in [T_j \wedge U_r, T_{j+1} \wedge U_r)$, and $X_t^\delta = X_t^{\delta,k}$ for $t \geq U_r$, where $k$ is such that $U_r \in [T_k, T_{k+1})$. We define $Y^\delta$ in a similar way. Note that $Y^\delta$ is a reflected Brownian motion in $D$ so, trivially, $Y^{1/n}$ converge in distribution to the reflected Brownian motion in $D$ as $n \to \infty$.

Before time $U_r$, the distance between $X^\delta$ and $Y^\delta$ is bounded below by $r$ so simple geometry shows that the jumps of $X^\delta$ at times $T_j < U_r$ satisfy the assumptions of Lemma 2.3. That lemma and a localization argument show that $X^{1/n}$ converge in distribution to a reflected Brownian motion. By passing to a subsequence, if necessary, we see that $(X^{1/n}, Y^{1/n})$ converge in distribution to $(X^*, Y^*)$, where $X^*$ and $Y^*$ are reflected Brownian motions in $D$. It follows easily from the definition of $(X^\delta, Y^\delta)$ that $(X^*, Y^*)$ is a mirror coupling on every interval $[t_0, t_1]$ such that neither $X_t^*$ nor $Y_t^*$ visit $\partial D$ for $t \in [t_0, t_1]$. By the uniqueness of the mirror coupling proved in [AB], it follows that $(X^*, Y^*)$ has the same distribution as $(X, Y)$.

Let $M_t^\delta$, $K_{\delta,X,t}$, etc., be defined relative to $(X^\delta, Y^\delta)$ in the same way as $M_t$, $K_{X,t}$, etc., have been defined for $(X, Y)$. The jumps of $X^\delta$ have been chosen so that $M_t^\delta \cap K_{\delta,X,t}$ moves in one direction along $K_{\delta,X,t}$, and the same holds for $M_t^\delta \cap K_{\delta,Y,t}$ and $K_{\delta,Y,t}$, as long as $X^\delta$ and $Y^\delta$ are reflecting on the same two edges of $\partial D$. It is not hard to see that this property is preserved under the passage to the limit in distribution and that it implies the statement in the lemma. □

From now on, we will restrict our attention to the domain $D_1 \stackrel{\text{df}}{=} \{(x_1, x_2) \in D : x_1 > 0, x_2 > -1\}$. Let $\partial^d D_1 = \{(x_1, x_2) \in \partial D_1 : x_2 = -1\}$, $\partial^n D_1 = \partial D_1 \setminus \partial^d D_1$, $\partial^\ell D_1 = \{(z_1, z_2) \in \partial D_1 : z_1 = 0\}$, and $\partial^s D_1 = \partial D_1 \setminus (\partial^d D_1 \cup \partial^\ell D_1)$.

Consider the restriction of $\varphi$ to $D_1$ normalized so that $\varphi(x) > 0$ in $D_1$. By Lemma 2.2, $\varphi$ is an eigenfunction for the Laplacian in $D_1$ with the following boundary conditions:

(2.3) Dirichlet boundary conditions on $\partial^d D_1$, and Neumann boundary conditions on $\partial^n D_1$.

Since $\varphi(x) > 0$ for $x \in D_1$, $\varphi$ is the first eigenfunction in $D_1$ with the boundary conditions (2.3). By Lemma 2.2, it will suffice to show that $\varphi(0,0) > \sup_{x \in \partial^s D_1} \varphi(x)$ to prove Theorem 1.1.

For $z \in D_1$, let $\rho(z)$ denote the infimum of lengths of Jordan arcs contained in $D_1$ and joining $z$ with $\partial^d D_1$.

**Lemma 2.5**. *Suppose that $x, y \in \overline{D}_1$, $x = (x_1, x_2)$, $y = (y_1, y_2)$, and the line of symmetry $M$ for $x$ and $y$ does not intersect $\partial^\ell D_1$. Assume that one of the following conditions holds, (i) $\angle(y - x) \in [-\pi/4, \pi/4]$ and $x_1 \le 2$, or (ii) $\angle(y - x, x - (2, -1)) \in [-3\pi/4, -\pi/4]$ and $x_1 \ge 2$. Then $\varphi(x) \ge \varphi(y)$.*

**Proof**. Let $u(t, z)$ be the heat equation solution in $D_1$ with the boundary conditions (2.3) and the initial condition $u(0, z) = 1$ for all $z$. Suppose that $(X, Y)$ is a mirror coupling of reflected Brownian motions in $D_1$ with $(X_0, Y_0) = (x, y)$. The following representation of the heat equation solution is well known, $u(t, x) = P(T^X_{\partial^d D_1} > t)$ and $u(t, y) = P(T^Y_{\partial^d D_1} > t)$. Suppose that we can show that $T^Y_{\partial^d D_1} \le T^X_{\partial^d D_1}$, a.s. Then $u(t, y) \le u(t, x)$ for all $t \ge 0$ and the eigenfunction expansion applied for large $t$ shows that $\varphi(y) \le \varphi(x)$ (see Proposition 2.1 and the proof of Theorem 3.3 in [BB1]). Hence, it will suffice to show that $T^Y_{\partial^d D_1} \le T^X_{\partial^d D_1}$, a.s.

Recall the definition of $\rho(z)$ stated before the lemma. It is enough to show that $\rho(Y_t) \le \rho(X_t)$ for all $t$. Recall that $M_t$ denotes the mirror, i.e., the line of symmetry for $X_t$ and $Y_t$. Suppose that one of the conditions (i) or (ii) in the statement of the lemma is satisfied by $x$ and $y$. The rules for the possible movements of $M_t$ described before and in Lemma 2.4 imply that as long as $Y_t \in D_2 \stackrel{\text{df}}{=} \{(z_1, z_2) \in D_1 : z_1 \ge 1/2\}$, the mirror $M_t$ has a tendency to intersect $\partial D_1$ at angles closer to the right angle than the initial angle, assuming that the parameter $\varepsilon$ in the definition of $D$ is small. The proof of the last claim is somewhat tedious but totally elementary so it is left to the reader. We conclude that $M_t$ cannot turn to the point that $\rho(Y_t) > \rho(X_t)$, as long as $Y_t \in D_2$.

It remains to analyze possible motions of $M_t$ when $Y_t \notin D_2$. Then one of the processes may reflect on $\partial^\ell D_1$ while the other is not too far from $\partial^\ell D_1$. We will show that $M_t$ never intersects $\partial^\ell D_1$. Note that if $Y_t \notin D_2$, $\rho(Y_t) \le \rho(X_t)$ and one of the processes reflects on $\partial^s D_1$ then the hinge will stay at a fixed point on $\partial^s D_1$ and the mirror will move in such a way that its other intersection point with $\partial D_1$ will not touch $\partial^\ell D_1$.

Suppose that $Y_t \notin D_2$, $\rho(Y_t) \le \rho(X_t)$, $M_t$ does not intersect $\partial^\ell D_1$, and one of the processes (necessarily $X$) reflects on $\partial^\ell D_1$. Then the hinge lies outside $\overline{D}_1$. Since both processes $X_t$ and $Y_t$ must be in $\overline{D}_1$, the geometry of this domain makes it impossible for $M_t$ to turn closer to the horizontal direction than $\pi/8$ (actually, this lower bound is closer to $\pi/4$, if $\varepsilon$ is small). This implies that the relation $\rho(Y_t) \le \rho(X_t)$ will remain in force if the reflection point belongs to $\partial^\ell D_1$. Finally, Lemma 2.4 can be used to show that the above analysis, based on the assumption that only one process at a time reflects on the boundary, remains valid when we consider the situation when both processes reflect at the same time.

We conclude that $\rho(Y_t) \le \rho(X_t)$ for all $t \ge 0$ and this completes the proof. $\square$

**Lemma 2.6**. *Let* $a = \sup_{(x_1,x_2)\in D_1, x_1=1} \varphi(x)$, $\Gamma = \{x \in D_1 : \varphi(x) = a\}$, $r_1 = \inf_{(x_1,x_2)\in\Gamma} x_1$, *and* $r_2 = \sup_{(x_1,x_2)\in\Gamma} x_1$. *Then for small* $\varepsilon$ *we have* $1 - 2\varepsilon \le r_1 \le r_2 \le 1$, *and* $\inf_{(x_1,x_2)\in\overline{D}_1, x_1\le 1/2} \varphi(x) \ge \sup_{(x_1,x_2)\in\overline{D}_1, x_1 \ge r_1} \varphi(x)$.

**Proof**. It follows easily from Lemma 2.5 that $\angle(\nabla\varphi) \in [3\pi/4, 5\pi/4]$ for $x = (x_1, x_2) \in D_1$ with $1/4 \le x_1 \le 3/2$, assuming that $\varepsilon$ is small. This and simple geometry imply the lemma. $\square$

Recall the definition of $\rho(x)$ stated before Lemma 2.5.

**Lemma 2.7**. *Let* $\Gamma_1$ *and* $\Gamma_2$ *denote the two connected components of* $\partial^s D_1$. *If* $x, y \in \Gamma_1$ *and* $\rho(x) > \rho(y)$ *then* $\varphi(x) > \varphi(y)$. *A similar statement holds for* $\Gamma_2$.

**Proof**. Suppose that $x, y \in \Gamma_1$ and $\rho(x) > \rho(y)$. By the proof of Lemma 2.5, $\varphi(x) \ge \varphi(y)$. In fact, this is all we need to prove Theorem 1.1 but we will show that the inequality is strict because the proof is short and easy. Suppose that $\varphi(x) = \varphi(y)$. It is easy to see that one can find a non-empty open set $A \subset D_1$ such that for any $z \in A$, the pair $(x, z)$ satisfies the assumptions of Lemma 2.5, and the same holds for the pair $(z, y)$. By Lemma 2.5, $\varphi(x) \ge \varphi(z) \ge \varphi(y)$. Since we have assumed that $\varphi(x) = \varphi(y)$, we see that $\varphi(x) = \varphi(z)$ for all $z \in A$. The remark following Corollary (6.31) in [F] may be applied to the operator $\Delta + \lambda$ to conclude that the eigenfunctions are real analytic and therefore they cannot be constant on an open set unless they are constant on the whole domain $D$. This contradiction completes the proof. $\square$

We will now define a coupling $(X, Y)$ of reflected Brownian motions in $D_1$ with $X_0 = (0, 0)$ and $Y_0 = (0, \varepsilon)$. The mechanism of the coupling will change, as time goes on, depending on the outcome of some events. Let

$$A_1 = \{(x_1, x_2) \in D_1 : 0 < x_1 < \varepsilon, x_2 > 7\varepsilon/10\},$$
$$A_2 = \{(x_1, x_2) \in \partial A_1 : x_1 = \varepsilon, 7\varepsilon/10 \le x_2 \le 8\varepsilon/10\},$$
$$A_3 = \partial A_1 \cap D_1,$$
$$A_4 = \{(x_1, x_2) \in D_1 : 0 < x_1 < \varepsilon, -3\varepsilon/10 < x_2 < \varepsilon/10\}.$$

Let $\{(X^1_t, Y^1_t), t \geq 0\}$ be a synchronous coupling of reflected Brownian motions in $D_1$ with $X^1_0 = (0,0)$ and $Y^1_0 = (0,\varepsilon)$ (see (2.1)-(2.2)). Let

$$
\begin{aligned}
S_0 &= T^{Y^1}_{A_3} \wedge T^{X^1}_{\partial A_4 \cap D_1}, \\
G_0 &= \{Y^1_{T^{Y^1}_{A_3}} \in A_2, T^{Y^1}_{A_3} \leq T^{X^1}_{\partial A_4 \cap D_1}\}.
\end{aligned}
$$

We let $(X_t, Y_t) = (X^1_t, Y^1_t)$ for $t \leq S_0$. If $G_0$ does not occur we let $\{(X_t, Y_t), t \geq S_0\}$ be a mirror coupling starting from $(X^1_{S_0}, Y^1_{S_0})$, but otherwise independent from $\{(X_t, Y_t), t \in [0, S_0]\}$. Let $S^0_1 = S^0_2 = T^{Y^1}_{A_3}$ and for integer $j \in [0, 2/\varepsilon]$,

$$
\begin{aligned}
A^j_5 &= \{(x_1, x_2) \in D_1 : j\varepsilon < x_1 < (j+2)\varepsilon, 6\varepsilon/10 < x_2 < 9\varepsilon/10\}, \\
A^j_6 &= \{(x_1, x_2) \in \partial A^j_5 : x_1 = (j+2)\varepsilon, 7\varepsilon/10 < x_2 < 8\varepsilon/10\}, \\
S^{j+1}_1 &= \inf\{t \geq S^j_2 : Y^1_t \in A^j_6\}, \\
S^{j+1}_2 &= \inf\{t \geq S^j_2 : Y^1_t \in \partial A^j_5\}, \\
F_j &= \{S^{j+1}_1 \leq S^{j+1}_2\}.
\end{aligned}
$$

Fix some $c_* \in (0,1)$ whose value will be chosen later, let $j_0$ be the integer part of $c_*/\varepsilon$ and $F_* = \bigcap_{0 \leq j \leq j_0} F_j$.

If $G_0$ holds and there exists $j \leq j_0$ such that $F_j$ does not occur then we let $j_1$ be the smallest $j$ with this property, $(X_t, Y_t) = (X^1_t, Y^1_t)$ for $t \in [S_0, S^{j_1+1}_2]$, and $\{(X_t, Y_t), t \geq S^{j_1+1}_2\}$ be a mirror coupling starting from $(X^1(S^{j_1+1}_2), Y^1(S^{j_1+1}_2))$, but otherwise independent of $\{(X_t, Y_t), t \in [0, S^{j_1+1}_2]\}$. Let

$$
\begin{aligned}
A_7 &= \{(x_1, x_2) \in D_1 : j_0\varepsilon < x_1 < (j_0+3)\varepsilon, x_2 > -5\varepsilon/10\}, \\
A_8 &= \{(x_1, x_2) \in \partial A_7 \cap \partial D_1 : (j_0+1)\varepsilon \leq x_1 \leq (j_0+2)\varepsilon\}, \\
S_3 &= \inf\{t \geq S^{j_0+1}_2 : X^1_t \in \partial A_7\}, \\
G_1 &= \{X^1_{S_3} \in A_8\}.
\end{aligned}
$$

If $G_0$ and $F_*$ hold then we let $(X_t, Y_t) = (X^1_t, Y^1_t)$ for $t \in [S_0, S_3]$. We let $\{(X^2_t, Y^2_t), t \geq S_3\}$ be a mirror coupling starting from $(X^1_{S_3}, Y^1_{S_3})$, but otherwise independent of the process

$\{(X_t, Y_t), t \in [0, S_3]\}$. Let

$$\begin{aligned}
A_9 &= \{(x_1, x_2) \in D_1 : (j_0 - 100)\varepsilon < x_1 < (j_0 + 100)\varepsilon, x_2 > 0\}, \\
A_{10} &= \{(x_1, x_2) \in \partial A_9 : x_1 = (j_0 - 100)\varepsilon \text{ or } x_1 = (j_0 + 100)\varepsilon\}, \\
S_4 &= \inf\{t \ge S_3 : X_t^2 \in A_{10}\}, \\
S_5 &= \inf\{t \ge S_3 : Y_t^2 \in A_{10}\}, \\
S_6 &= \inf\{t \ge S_3 : X_t^2 \in \partial A_9\}, \\
S_7 &= \inf\{t \ge S_3 : Y_t^2 \in \partial A_9\}, \\
S_8 &= S_6 \wedge S_7, \\
G_2 &= \{S_4 \le S_5 \wedge S_6\} \cup \{S_5 \le S_4 \wedge S_7\}.
\end{aligned}$$

If $G_0$ and $F_*$ hold then we let $(X_t, Y_t) = (X_t^2, Y_t^2)$ for $t \in [S_3, S_8]$. If $G_0$ and $F_*$ hold but $G_2$ does not occur then we let $(X_t, Y_t) = (X_t^2, Y_t^2)$ for $t \ge S_8$. We let $\{(X_t^3, Y_t^3), t \ge S_8\}$ be a pair of reflected Brownian motions in $D_1$ starting from $(X_{S_8}^2, Y_{S_8}^2)$, independent from each other and independent from $\{(X_t, Y_t), t \in [0, S_8]\}$. Let

$$\begin{aligned}
A_{11} &= \{(x_1, x_2) \in D_1 : (j_0 - 50)\varepsilon < x_1 < (j_0 + 50)\varepsilon \text{ or } x_1 < \varepsilon\}, \\
A_{12} &= \{(x_1, x_2) \in D_1 : x_1 = 1\}, \\
A_{13} &= \{(x_1, x_2) \in D_1 : x_1 \le 1/2\}, \\
S_9 &= \inf\{t \ge S_8 : X_t^3 \in A_{11}\}, \\
S_{10} &= \inf\{t \ge S_8 : Y_t^3 \in A_{11}\}, \\
S_{11} &= \inf\{t \ge S_8 : Y_t^3 \in A_{12}\}, \\
S_{12} &= S_9 \wedge S_{10}, \\
G_3 &= \{S_{11} < S_{12}, X_{S_{11}} \in A_{13}\}.
\end{aligned}$$

If $G_0 \cap F_* \cap G_2$ holds then we let $(X_t, Y_t) = (X_t^3, Y_t^3)$ for $t \in [S_8, S_{12}]$ and we let $\{(X_t, Y_t), t \ge S_{12}\}$ be a mirror coupling starting from $(X_{S_{12}}^3, Y_{S_{12}}^3)$ but otherwise independent from $\{(X_t, Y_t), t \in [0, S_{12}]\}$.

**Lemma 2.8**. *Let $\Gamma$ be the curve defined in Lemma 2.6 and let $(X, Y)$ be the coupling of reflected Brownian motions defined before this lemma. There exist $c_1, \varepsilon_1 > 0$ such that for $\varepsilon \in (0, \varepsilon_1)$ we have $P(T_\Gamma^X < T_\Gamma^Y) \le e^{-c_1/\varepsilon}$.*

**Proof**. The pair $(X, Y)$ is not a mirror coupling but we can still define the "mirror" $M_t$ for $(X_t, Y_t)$ as the line of symmetry for these processes.

Let $K_t = (K_t^1, K_t^2)$ be that of intersection points of the mirror $M_t$ with $\partial D_1$ which satisfies $\angle(K_t - X_t, Y_t - X_t) \geq 0$. First we will show that $K_t^2 \geq 0$ for all $t \leq T_\Gamma^X \wedge T_\Gamma^Y$, a.s., that is, the "left" (looking from $X_t$ towards $Y_t$) intersection point of $M_t$ with $\partial D_1$ cannot cross $\partial^\ell D_1$ below $(0,0)$.

We will start by analyzing possible movements of $(X,Y)$. We will use the following convention, introduced in Lemma 2.3, to denote coordinates of processes: $X_t = (\widetilde{X}_t, \widehat{X}_t)$, and similarly for other processes. We will show that $\widetilde{X}_t \leq \widetilde{Y}_t$ for $t \in [0, S_0]$. Suppose that there is $t_0 \in [0, S_0]$ with $\widetilde{X}_{t_0} > \widetilde{Y}_{t_0}$ and let $t_1 = \sup\{t < t_0 : \widetilde{X}_t \leq \widetilde{Y}_t\}$. By continuity of reflected Brownian paths, $\widetilde{X}_{t_1} = \widetilde{Y}_{t_1}$. Let $W$ be the Brownian motion driving $X$ and $Y$, in the sense of (2.1)-(2.2). Since $\widetilde{X}_t > \widetilde{Y}_t \geq 0$ for $t \in [t_1, t_0]$, $X$ is not reflecting on this interval, so $\widetilde{X}_{t_0} - \widetilde{X}_{t_1} = \widetilde{W}_{t_0} - \widetilde{W}_{t_1}$. The horizontal component of the vector of reflection for $Y$ is non-negative so $\widetilde{Y}_{t_0} - \widetilde{Y}_{t_1} \geq \widetilde{W}_{t_0} - \widetilde{W}_{t_1}$. We see that $\widetilde{Y}_{t_0} - \widetilde{Y}_{t_1} \geq \widetilde{X}_{t_0} - \widetilde{X}_{t_1}$ and this contradicts the facts that $\widetilde{X}_{t_0} > \widetilde{Y}_{t_0}$ and $\widetilde{X}_{t_1} = \widetilde{Y}_{t_1}$. Hence, $\widetilde{X}_t \leq \widetilde{Y}_t$ for $t \in [0, S_0]$. This implies that $K_t^2 \geq 0$ for $t \in [0, S_0]$.

Recall the definitions of $j_1$ and $j_0$ from the construction of $(X,Y)$. On the interval $[S_0, S_2^{j_1+1}]$, processes $X$ and $Y$ do not hit the boundary of $D_1$, so the mirror is translated but not rotated and the constraints on the positions of $X$ and $Y$ are such that it is easy to see that $K_t^2 \geq 0$ for $t \in [S_0, S_2^{j_1+1}]$.

Suppose that $G_0 \cap F_*$ holds. Then only $Y$ can be reflecting on the interval $[S_2^{j_0+1}, S_3]$, so $\widetilde{Y}_t - \widetilde{X}_t$ is non-decreasing on $[S_2^{j_0+1}, S_3]$. It follows that $\widetilde{X}_t \leq \widetilde{Y}_t$ for $t \in [S_2^{j_0+1}, S_3]$ and $K_t^2 \geq 0$ on this interval.

If $G_0 \cap F_* \cap G_1 \cap G_2$ holds then $\widetilde{X}_t \leq \widetilde{Y}_t$ for $t \in [S_3, S_{12}]$, so $K_t^2 \geq 0$ on this interval.

Suppose that $K_t^2 = 0$ for some $t \leq T_\Gamma^X \wedge T_\Gamma^Y$ and let $U = \inf\{t \geq 0 : K_t^2 = 0\}$. The above analysis covers all cases when $X$ and $Y$ are not mirror-coupled. In other words, if $U$ exists then $\{(X_t, Y_t), t \geq U\}$ is a mirror coupling. It is not hard to see that an even stronger statement holds—for some $U_1 < U$, $\{(X_t, Y_t), t \geq U_1\}$ is a mirror coupling.

Suppose that $K_U^1 > 2$. This means that $\angle(Y_t - X_t)$ must have changed its value from $\pi/2$ at time $t = 0$, to 0 or $\pi$ at some time $T_0 \leq T_\Gamma^X \wedge T_\Gamma^Y$, and then take a value less then $-\pi/4$ at time $U$. Such a change of $\angle(Y_t - X_t)$ between $T_0$ and $U$ is impossible, by the argument given in the proof of Lemma 2.5. Next suppose that $K_U = (0,0)$. If $\angle(Y_U - X_U) = \pi/2$ then, by symmetry and uniqueness of the mirror coupling, $\angle(Y_t - X_t) = \pi/2$ and $M_t = M_U$ for all $t \in [U, T_\Gamma^X \wedge T_\Gamma^Y]$. Hence, in this case, $K_t^2 \geq 0$ for all $t \leq T_\Gamma^X \wedge T_\Gamma^Y$.

Suppose that $K_U = (0,0)$ and $\angle(Y_U - X_U) < \pi/2$. Note that at time $U$, at least one of the processes must be on the boundary of $D_1$ (otherwise the mirror is not moving). In the present case, geometry shows that $X_U \in \partial D_1$ and $Y_U \notin \partial D_1$. Hence, for some $U_2 < U$ and all $t \in [U_2, U]$, $Y_t \notin \partial D_1$. This implies that the only process that can reflect on $\partial D_1$ on the interval $[U_2, U]$ is $X$. However, such reflection could only push $K_t^2$ up, so $K_t^2 \le 0$ for $t \in [U_2, U]$, a contradiction with the definition of $U$.

Now assume that $K_U = (0,0)$ and $\angle(Y_U - X_U) > \pi/2$. The point $v \stackrel{\text{df}}{=} (-1,0)$ lies at the intersection of lines containing the two line segments comprising $J \stackrel{\text{df}}{=} \{(x_1,x_2) \in \partial D_1 : 0 < x_1 < 1\}$. Let $Z_t$ be the intersection point of $M_t$ with $\{(x_1,x_2) : x_1 = -1\}$. Note that the introductory arguments in this proof showed not only that $U$ cannot occur when $X$ and $Y$ are not mirror-coupled, but also that $M_t$ passes above $v$ (i.e., $Z_t > 0$) for $t \in [0, U_1]$. Since $M_U$ passes below $v$ (i.e., $Z_U < 0$), there must be a time $U_3 \in (U_1, U)$ such that either $M_{U_3}$ is vertical and $\angle(Y_t - X_t) = 0$ or $v \in M_{U_3}$. In the first case, we have $\angle(Y(T_\Gamma^X \wedge T_\Gamma^Y) - X(T_\Gamma^X \wedge T_\Gamma^Y)) \in [-\pi/4, \pi/4]$, by the argument given in the proof of Lemma 2.5. In the second case, let $U_4 = \inf\{t \ge U_1 : Z_t < 0\}$. By continuity, $v \in M_{U_4}$. At time $U_4$, at least one of the processes must be on the boundary. Since $U_4 < U$, $X_{U_4} \notin \partial D_1$. On a small interval $[U_4, U_5]$, only $Y$ can reflect on the boundary of $\partial D_1$. But this reflection will either leave $Z_t$ unchanged (if $Y$ is reflecting on $J$) or it will push $Z_t$ up (if $Y$ is reflecting on $\partial^\ell D_1$), and this contradicts the definition of $U_4$. This completes the proof of the claim that $K_t^2 \ge 0$ for all $t \le T_\Gamma^X \wedge T_\Gamma^Y$.

Let $C$ be the part of $\{(x_1,x_2) \in \partial D_1 : 0 \le x_1 \le 1, x_2 > 0\}$ that lies to the left of $\Gamma$, and $V = T_C^X$. Standard estimates show that $P(V \ge T_\Gamma^X) \le e^{-c_1/\varepsilon}$ for some $c_1 > 0$. If $V < T_\Gamma^X$ and $V \ge T_\Gamma^Y$ then $T_\Gamma^X \ge T_\Gamma^Y$. Suppose that $V < T_\Gamma^X$ and $V < T_\Gamma^Y$. Then we have three possibilities. First, $X_V = Y_V$. This implies that $T_\Gamma^X = T_\Gamma^Y$. The second possibility is that $\angle(Y_V - X_V) > \pi$. This implies the existence of a time $U_6 < V$ such that $K_{U_6} = (0,0)$, which is impossible by the first part of the proof. Finally, suppose that $\angle(Y_V - X_V) < \pi$. Then in fact $\angle(Y_V - X_V) < \pi/8$, for small $\varepsilon$. This implies that $T_\Gamma^X \ge T_\Gamma^Y$, by an argument similar to that in the proof of Lemma 2.5, because $\{(X_t, Y_t), t \ge V\}$ is necessarily a mirror coupling, except for the interval $[S_8, S_{12}]$, where the processes are independent but well separated. We conclude that $T_\Gamma^X < T_\Gamma^Y$ only if $\{V \ge T_\Gamma^X\}$ occurs. Since the probability of this event is bounded by $e^{-c_1/\varepsilon}$, the lemma follows. ◻

**Lemma 2.9**. *There exist $c_2, \varepsilon_1 > 0$ such that for $\varepsilon \in (0, \varepsilon_1)$ and $t \ge 1$ we have $P(t \le T_\Gamma^X < T_\Gamma^Y) \le e^{-c_2 t/\varepsilon^2}$.*

**Proof**. Recall the set $C$ from the proof of Lemma 2.8 and let $C_0$ be the part of $D_1$ to the left of $\Gamma$. Let $Q_j$ be the event that $X$ does not hit $C$ during the time interval $[j\varepsilon^2, (j+1)\varepsilon^2]$.

It is easy to see that $P(Q_j \mid X_{j\varepsilon^2} \in C_0, \mathcal{F}_{j\varepsilon^2}) < p_1 < 1$, where $p_1$ is independent of $j$. By the Markov property applied at times $j\varepsilon^2$, $P(\{T_\Gamma^X \geq t\} \cap \bigcap_{j \leq t/\varepsilon^2} Q_j) \leq p_1^{t/\varepsilon^2} = e^{-c_2 t/\varepsilon^2}$, for some $c_2 > 0$. If one of the events $Q_j^c$ does happen, an argument similar to that in the proof of Lemma 2.8 shows that $T_\Gamma^X \geq T_\Gamma^Y$. □

**Lemma 2.10**. *Let $C_1 = \{(x_1, x_2) \in D_1 : x_1 \leq 1/2\}$. For any $c_3 > 0$ there exist $c_* \in (0,1)$ (used in the construction of the coupling $(X, Y)$) and $\varepsilon_1 > 0$ such that for $\varepsilon \in (0, \varepsilon_1)$ we have $P(T_\Gamma^Y < T_\Gamma^X, X_{T_\Gamma^Y} \in C_1) \geq e^{-c_3/\varepsilon}$.*

**Proof**. We will argue that for some constants $c_j$ independent of $\varepsilon$ we have the following bounds for the probabilities of events defined in the construction of the coupling $(X, Y)$,

$$P(G_0) \geq c_4, \tag{2.4}$$

$$P\left(F_j \mid G_0 \cap \bigcap_{k<j} F_k\right) \geq c_5, \qquad 0 \leq j \leq j_0, \tag{2.5}$$

$$P\left(G_1 \mid G_0 \cap \bigcap_{0 \leq j \leq j_0} F_k\right) \geq c_6, \tag{2.6}$$

$$P\left(G_2 \mid G_0 \cap G_1 \cap \bigcap_{0 \leq j \leq j_0} F_k\right) \geq c_7 \varepsilon, \tag{2.7}$$

$$P\left(G_3 \mid G_0 \cap G_1 \cap G_2 \cap \bigcap_{0 \leq j \leq j_0} F_k\right) \geq c_8 \varepsilon^2. \tag{2.8}$$

Let $W$ be the Brownian motion driving $(X, Y)$, in the sense of (2.1)-(2.2), on the interval $[0, S_0]$. Recall the notation $W = (\widetilde{W}, \widehat{W})$. By the support theorem (see Theorem I (6.6) in [B]) for the planar Brownian motion and scaling, the following event $G_0^*$ has probability greater than $c_4 > 0$, independent of $\varepsilon$.

($G_0^*$) The Brownian motion $W$ goes from $(0,0)$ to $B((0, -0.25\varepsilon), 0.01\varepsilon)$ before touching the boundary of $\{(x_1, x_2) \in \mathbf{R}^2 : |x_1| < 0.02\varepsilon, -0.26\varepsilon < x_2 < 0.01\varepsilon\}$ in less than $\varepsilon^2$ units of time, and then goes to $B((2\varepsilon, -0.25\varepsilon), 0.01\varepsilon)$ without hitting the boundary of $\{(x_1, x_2) \in \mathbf{R}^2 : -0.01\varepsilon < x_1 < 3\varepsilon, -0.26\varepsilon < x_2 < -0.24\varepsilon\}$, in another time interval of $\varepsilon^2$ units or less.

Let $T_*$ be the time needed to complete the movements described in $G_0^*$. We will argue that if $G_0^*$ holds then so does $G_0$. Note that $\widehat{X} = \widehat{W}$ on $[0, T_*]$. We have already shown that $\widetilde{X}_t \leq \widetilde{Y}_t$ for $t \in [0, S_0]$ in the proof of Lemma 2.8, so it remains to show that $S_0 \leq T_*$ and $Y_{T^Y_{A_3}} \in A_2$. Since $\widehat{W}_t \leq 0.01\varepsilon$ for $t \in [0, T_*]$, we have $\widehat{Y}_t \in [\widehat{W}_t - 0.01\varepsilon, \widehat{W}_t]$ for $t \in [0, T_*]$. The reflection vector for $Y$ is either horizontal or pointing down, at an angle not greater than $\varepsilon$ with the vertical. Since $\widetilde{W}_t \geq -0.02\varepsilon$ for $t \in [0, T_*]$, this implies that $\widetilde{Y}_t \in [\widetilde{W}_t, \widetilde{W}_t + 0.01\varepsilon^2 + 0.02\varepsilon]$ for $t \in [0, T_*]$. Now easy geometry shows that $X$ and $Y$ are transforms of $W$ that satisfy $G_0$. This proves (2.4).

The support theorem for the planar Brownian motion (i.e., without reflection) easily yields (2.5) and (2.6).

If $G_0 \cap G_1 \cap \bigcap_{0 \leq j \leq j_0} F_k$ holds then $\text{dist}(X_{S_3}, Y_{S_3}) \geq c_9 \varepsilon^2$ because $X$ is located to the left of $Y$ (in the sense of the first coordinate) at time $S_2^{j_0+1}$. The event $G_2$ will occur if $Y$ moves above the horizontal axis, about $100\varepsilon$ units to the right before moving $c_9\varepsilon^2$ units to the left. By the "gambler's ruin" estimate, we obtain (2.7).

Recall the point $v \stackrel{\text{df}}{=} (-1, 0)$ that lies at the intersection of lines containing the two line segments comprising $\{(x_1, x_2) \in \partial D_1 : 0 < x_1 < 1\}$. Let $R^X_t = \text{dist}(X_t, v)$ and $R^Y_t = \text{dist}(Y_t, v)$. As long as $X$ and $Y$ stay inside $\{(x_1, x_2) \in D_1 : \varepsilon \leq x_1 \leq 1 - \varepsilon\}$, $R^X_t$ and $R^Y_t$ are 2-dimensional Bessel processes because the reflection has no effect on the distance of $X$ or $Y$ from $v$. It is standard to show the the 2-dimensional Bessel process $R^Y_t$ starting about $c_{10} + 100\varepsilon$ units from 0 will reach the value $1 + \varepsilon$ at a time $T_0 \in (1/2, 1)$, before hitting the level $c_{10} + 50\varepsilon$, with probability exceeding $c_{11}\varepsilon$. The other 2-dimensional Bessel process, $R^X$, starting about $c_{10} - 100\varepsilon$ units from 0, will stay in the interval $(1/4, 1/2)$ during the time interval $(1/2, 1)$, before hitting levels $c_{10} - 50\varepsilon$ or $\varepsilon$, with probability exceeding $c_{12}\varepsilon$. By independence of $R^X$ and $R^Y$ on $[S_8, S_{12}]$, we obtain (2.8).

Recall that $j_0$ is the integer part of $c_*/\varepsilon$ and $F_* = \bigcap_{0 \leq j \leq j_0} F_j$. It follows from (2.4)-(2.8) and the repeated application of the strong Markov property that

$$P(G_0 \cap F_* \cap G_1 \cap G_2 \cap G_3) \geq c_{13}\varepsilon^3 c_5^{c_*/\varepsilon}. \tag{2.9}$$

For any fixed $c_3 > 0$, the right hand side of (2.9) is greater than $e^{-c_3/\varepsilon}$ if $c_*$ and $\varepsilon$ are sufficiently small. If the event in (2.9) occurs then $T^Y_\Gamma < T^X_\Gamma$ and $X_{T^Y_\Gamma} \in C_1$, so the lemma follows. □

**Proof of Theorem 1.1**. Lemmas 2.2 and 2.7 show that it will suffice to prove that $\varphi(0,0) > \varphi(0, -\varepsilon) \vee \varphi(0, \varepsilon)$. We will only show that $\varphi(0,0) > \varphi(0, \varepsilon)$ because the claim that $\varphi(0,0) > \varphi(0, -\varepsilon)$ can be proved in a completely analogous way, by symmetry.

Let $(X,Y)$ be the coupling constructed before Lemma 2.6, $S = T_\Gamma^X \wedge T_\Gamma^Y$ and let $U$ be the coupling time, i.e., $U = \inf\{t \geq 0 : X_t = Y_t\}$. Let $u(t,x) = \varphi(x)e^{\lambda t}$ and note that $u$ is a solution to the heat equation with the Neumann boundary conditions on $\partial^n D_1$ and Dirichlet boundary conditions on $\partial^d D_1$. Since $X$ and $Y$ are reflected Brownian motions in $D_1$, we have the following probabilistic representation of $u$, for bounded stopping times $T \leq T^X_{\partial^d D_1}$,

$$u(0,(0,0)) = Eu(X_T, T) = E\varphi(X_T)e^{\lambda T}.$$

Let $D_2 = \{(x_1,x_2) \in D_1 : x_1 \leq 1\}$ and let $\mu$ be the first eigenvalue for the Laplacian in $D_2$ with Neumann boundary conditions on $\partial^n D_1 \cap \partial D_2$ and Dirichlet boundary conditions elsewhere. It is easy to see that $\mu > \lambda$. We have

$$P(S > t) \leq P(T^X_{\partial D_2 \cap D_1} > t) \leq c_1 e^{-\mu t}.$$

This and the fact that the eigenfunction $\varphi$ is bounded (because $D$ is Lipschitz) imply that random variables $\varphi(X_{S\wedge n})e^{\lambda(S\wedge n)}$ are dominated by a random variable with an exponential tail. Hence, we can use the fact that $u(0,(0,0)) = E\varphi(X_{S\wedge n})e^{\lambda(S\wedge n)}$ and the dominated convergence theorem to prove that $u(0,(0,0)) = E\varphi(X_S)e^{\lambda S}$. Similarly, $u(0,(0,\varepsilon)) = E\varphi(Y_S)e^{\lambda S}$. We have

$$\begin{aligned} u(0,(0,0)) - u(0,(0,\varepsilon)) &= E\varphi(X_S)e^{\lambda S} - E\varphi(Y_S)e^{\lambda S} \\ &= E(\varphi(X_S)e^{\lambda S} - \varphi(Y_S)e^{\lambda S})\mathbf{1}_{\{S>U\}} \\ &= E(\varphi(X_S) - \varphi(Y_S))e^{\lambda S}\mathbf{1}_{\{T_\Gamma^X < T_\Gamma^Y\}} + E(\varphi(X_S) - \varphi(Y_S))e^{\lambda S}\mathbf{1}_{\{T_\Gamma^X > T_\Gamma^Y\}}. \end{aligned}$$

Let $a_1 = \sup_{x=(x_1,x_2)\in D_1, x_1 \geq 1-2\varepsilon, y\in\Gamma} \varphi(x) - \varphi(y)$ and recall from Lemma 2.6 that $a_1 \leq a_2 \overset{\text{df}}{=} \inf_{x=(x_1,x_2)\in D_1, x_1\leq 1/2, y\in\Gamma} \varphi(x) - \varphi(y)$. By the proof of Lemma 2.8, $\varphi(X_S) - \varphi(Y_S) \geq -a_1$ if $T_\Gamma^X < T_\Gamma^Y$. By Lemmas 2.1, 2.8 and 2.9,

$$\begin{aligned} &E(\varphi(X_S) - \varphi(Y_S))e^{\lambda S}\mathbf{1}_{\{T_\Gamma^X < T_\Gamma^Y\}} \\ &\quad = E(\varphi(X_S) - \varphi(Y_S))e^{\lambda S}\mathbf{1}_{\{T_\Gamma^X < T_\Gamma^Y, S\leq 1\}} \\ &\qquad + \sum_{k\geq 1} E(\varphi(X_S) - \varphi(Y_S))e^{\lambda S}\mathbf{1}_{\{T_\Gamma^X < T_\Gamma^Y, S\in[k,k+1]\}} \\ &\quad \geq -a_1 e^{\lambda} e^{-c_1/\varepsilon} - \sum_{k\geq 1} a_1 e^{\lambda(k+1)} e^{-c_2 k/\varepsilon^2}. \end{aligned}$$

For some $c_4$ and small $\varepsilon$, this is greater than $-a_1 c_4 e^{-c_1/\varepsilon}$. According to Lemma 2.10, one can choose $c_*$ and $c_3$ such that for small $\varepsilon$,

$$E(\varphi(X_S) - \varphi(Y_S))e^{\lambda S}\mathbf{1}_{\{T_\Gamma^X > T_\Gamma^Y\}} \geq a_2 e^{-c_3/\varepsilon} > a_1 c_4 e^{-c_1/\varepsilon}.$$

Hence $u(0,(0,0)) - u(0,(0,\varepsilon)) > 0$ and $\varphi(0,0) > \varphi(0,\varepsilon)$. □

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