Order Number 9230421

Markov partitions for hyperbolic toral automorphisms

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University of Washington, 1992

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Markov Partitions for Hyperbolic Toral Automorphisms

by

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A dissertation submitted in partial fulfillment of
the requirements for the degree of

Doctor of Philosophy

University of Washington

1992

Approved by______________________________
(Chairperson of Supervisory Committee)

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Program Authorized
to Offer Degree Mathematics

Date 6/1/92

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Date Jun 1, 1992

University of Washington

Abstract

Markov Partitions for Hyperbolic Toral Automorphisms

by Brenda L. Praggastis

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Mathematics

The study of the dynamical properties of hyperbolic toral automorphisms is simplified when the automorphisms are represented as shifts of finite type. The conventional method used to represent such an automorphism symbolically is to construct a Markov partition. The existence of Markov partitions for hyperbolic toral automorphisms is known. Because of the fractal nature of the boundary of such a partition an explicit construction is often difficult even in lower dimensions. Adler has suggested that such a construction may be possible by using digit expansions in powers of the automorphism. The basic idea is to generalize the correspondence between β-expansions, for β a Pisot number, and the β-shift. This is the motivation behind this thesis.

We generalize the notion of digit expansions by considering a subset $X \subset \mathbb{R}^n$ which is tiled by a periodically self similar tiling $\mathfrak{T}$ with expansion map ϕ. We show that every point in X has a digit expansion in powers of ϕ. The sequences of digits correspond to a path in a directed graph Γ. This is the content of Chapters 1 and 2.

In Chapter 3 we consider a hyperbolic automorphism ϕ of $\mathbb{R}^n$ which is invariant on the integer lattice. We suppose there is a periodically self similar tiling of the unstable eigenspace for ϕ. By applying the theory of Chapters 1 and 2 we construct digit expansions for the points in the unstable eigenspace in powers of ϕ. We then identify the unstable eigenspace modulo the integer lattice with points in a compact subset of $\mathbb{R}^n$ which we call Ω. We give necessary and sufficient conditions for Ω to be almost homeomorphic to the n-dimensional torus. We extend the idea of digit expansions to describe each point in Ω as a bi-infinite series in powers of ϕ. The

corresponding bi-infinite sequence of digits are in one-one correspondence with bi-infinite paths in a directed graph Γ.

In Chapter 4 we assume that Ω is almost homeomorphic to the n-dimensional torus. We show that the shift of finite type consisting of the bi-infinite paths in Γ is metrically similar to the dynamical system induced by ϕ on the n-dimensional torus.

We conclude in Chapter 5 with examples. In particular we indicate how the β-shift, for β Pisot, may be extended to a shift of finite type which represents a hyperbolic toral automorphism with eigenvalue β.

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ACKNOWLEDGMENTS

I would like to thank my thesis advisor Doug Lind for his encouragement and support as well as his time and counsel. I would like to thank Boris Solomyak for many helpful suggestions and counterexamples to some of my wilder ideas. And special thanks to my husband, Chris Praggastis. He has truly been the wind beneath my wings.

To my mother and the memory of my father.

INTRODUCTION

In 1969 Adler and Weiss showed that topological entropy is a complete invariant for metric equivalence of continuous ergodic automorphisms of the two-dimensional torus [3]. The method of proof used in their paper is to construct a partition of the 2-torus which satisfies certain properties. The partition is called a Markov partition. By assigning each element of the partition a symbol. it is possible to assign each point in the 2-torus a bi-infinite sequence of symbols which corresponds to the orbit of the point. The objective is to represent the continuous dynamical system as a symbolic one in such a way that periodicity and transitivity is preserved in the representation.

In 1970 Bowen showed that every Anosov diffeomorphism has a Markov partition [5]. In particular every hyperbolic toral automorphism has a Markov partition. His construction uses a recursive definition which deforms rectangles in the stable and unstable directions. While existence is shown, the proof does not indicate an efficient way to actually construct the partitions. Fortunately within the context of hyperbolic toral automorphisms we have so much structure that defining rectangles by a recursive deformation in *both* the stable and unstable directions is superfluous.

The purpose of this paper is to give a finite construction of Markov partitions for hyperbolic toral automorphisms. The approach is to consider self similar and periodically self similar tilings. The motivation for this comes from two sources. Firstly, self similar tilings have built in to them the desired properties that Bowen requires for his rectangles in the unstable direction. Secondly, there is a natural way to represent points in a tiled space using a symbolic system. The representations correspond to digit expansions. We show that it is possible to define digit expansions for points in the torus in powers of the given automorphism. Moreover the set of sequences of digits corresponds to a symbolic dynamical system which is metrically similar to the continous system defined by the automorphism.

In Chapters 1 and 2 we define self similar and periodically self similar tilings. We indicate how it is possible to represent points in a tiled space using digit expansions.

In Chapter 3 we consider a hyperbolic automorphism $\phi \in \mathrm{GL}(n, \mathbb{Z})$. We suppose there is a periodically self similar tiling of the unstable eigenspace for ϕ. We then identify modulo the integer lattice the unstable eigenspace for ϕ with a dense subset of a compact set Ω in $\mathbb{R}^n$. We indicate how the digit expansions used to represent points in the tiled space may be extended to representations for all points in Ω. We then give conditions on the tiling which insure that Ω is a fundamental region for a tiling of $\mathbb{R}^n$ mod $\mathbb{Z}^n$ and hence is almost homeomorphic to the n-torus.

In Chapter 4 we construct a Markov partition for ϕ when Ω is a fundamental region. We show that the partition we construct is consistent with the current notion of what it means to be a Markov partition. Finally in Chapter 5 we give examples and show that our construction generalizes previously known methods. We show that the construction of Adler and Weiss may be reproduced using tiling theory. We show that a similar construction by Bedford for hyperbolic automorphisms of the 3-torus also arises out of tiling theory. We conclude with a general construction which will produce a Markov partition for any ϕ whose characteristic polynomial is minimal for some Pisot number β.

Chapter 1

PERIODIC SELF SIMILAR TILINGS

We review the definitions and properties of tilings which will be used throughout this paper. In order to be consistent with the current work on the subject we will use the definitions established by Thurston [12] and Kenyon [8] whenever possible.

G-finite tilings.

Let X be a subset of $\mathbb{R}^n$ which is the closure of its interior.

Definition. A collection $\mathfrak{T}$ of compact subsets of X is a **tiling** if it satisfies the following properties.

(1) The union of the sets in $\mathfrak{T}$ is equal to X.

(2) Each set in $\mathfrak{T}$ is the closure of its interior.

(3) Each compact set in X intersects a finite number of sets in $\mathfrak{T}$. (In this case we say that $\mathfrak{T}$ is **locally finite.**)

(4) The interiors of the sets in $\mathfrak{T}$ are mutually disjoint. (Sets which exhibit this property will be called **almost disjoint.**)

The sets in $\mathfrak{T}$ are called **tiles.**

In general the tiles in $\mathfrak{T}$ may have an infinite number of shapes and sizes. We will restrict ourselves to tilings that only have a finite number of distinct tiles up to a translation in $\mathbb{R}^n$. Suppose G is a subgroup of $\mathbb{R}^n$.

Definition. A tiling $\mathfrak{T}$ of X is **G-finite** if there exists a finite partition $\{\mathfrak{T}_j\}_{j\in J}$ of $\mathfrak{T}$ such that if $T \in \mathfrak{T}_j$ then

$$\mathfrak{T}_j \subset \{T + g \colon g \in G\}.$$

The collection $\{\mathfrak{T}_j\}_{j\in J}$ is a **tile type partition** for $\mathfrak{T}$.

Suppose $\mathfrak{T}$ is a G-finite tiling of X with a tile type partition $\{\mathfrak{T}_j\}_{j\in J}$. If T^1, $T^2 \in \mathfrak{T}_j$ for some $j \in J$ then we will say that T^1 and T^2 are of the **same type.** This implies that there exists $g \in G$ such that $T^1 + g = T^2$. We say that T^2 is a **G-translate** of T^1. This does not mean that if T^1, $T^2 \in \mathfrak{T}$ and $g \in G$ such that $T^1 + g = T^2$ then T^1 and T^2 are of the same type. We will see in Chapter 2 that there will often exist tile type partitions which contain elements $\mathfrak{T}_{j_1}$ and $\mathfrak{T}_{j_2}$ such that every tile in $\mathfrak{T}_{j_1}$ is a G-translate of every tile in $\mathfrak{T}_{j_2}$. It follows that there are arbitrarily many different ways to define a tile type partition for a G-finite tiling. For that reason we will always specify when it is not clear from the context what tile type partition we are using. For now, let $\{\mathfrak{T}_j\}_{j\in J}$ define a tile type partition for $\mathfrak{T}$ and make the following definitions based on this partition.

Let $\mathfrak{P}$ be the set of all finite unions of tiles in $\mathfrak{T}$. Let $\mathfrak{B}$ be the set of all bounded subsets of X. Note that $\mathfrak{P} \subset \mathfrak{B}$.

Definition. Let P^1, $P^2 \in \mathfrak{P}$, then there is a finite collection of tiles $\{T^k\}_{k\in K}$ such that $P^1 = \cup_{k\in K} T^k$. The sets P^1 and P^2 have the **same pattern** if there exists $g \in G$ such that $P^1 + g = \cup_{k\in K} T^k + g = P^2$ and for each $k \in K$, $T^k + g$ is a tile in $\mathfrak{T}$ of the same type as T^k. In particular tiles of the same type have the same pattern. If $U \in \mathfrak{B}$ then there is a finite union of tiles $P \in \mathfrak{P}$ such that $U \subset P$. If $g \in G$ then U and $U + g$ have the same pattern if for some $P \in \mathfrak{P}$ such that $U \subset P$ we have that $P + g$ and P have the same pattern. If V is a subset of X and $U \in \mathfrak{B}$ then V **contains the pattern** of U if for some $g \in G$, $U + g \subset V$ and $U + g$ has the same pattern as U.

Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. For each $\delta > 0$ let

$$\mathfrak{P}_\delta = \{P \in \mathfrak{P}\colon \text{diameter}(P) < \delta\}.$$

If $\delta < \min\{\text{diameter}(T)\colon T \in \mathfrak{T}\}$ then $\mathfrak{P}_\delta = \emptyset$. If $\delta' > \delta > 0$ then $\mathfrak{P}_{\delta'} \supset \mathfrak{P}_\delta$. Also if $\delta > 0$ and $P \in \mathfrak{P}_\delta$ then $\mathfrak{P}_\delta$ contains all of the elements of $\mathfrak{P}$ with the same pattern as P.

Definition. A G-finite tiling $\mathfrak{T}$ with tile type partition $\{\mathfrak{T}_j\}_{j\in J}$ has a **finite number of local patterns** if for each $\delta > 0$ we have that $\mathfrak{P}_\delta$ contains a finite number of distinct patterns.

A slightly stronger restriction on $\mathfrak{T}$ is that $\mathfrak{T}$ has only a finite number of local patterns and these patterns are scattered uniformly throughout the tiling. Let $B_r(x)$ denote the open ball of radius r about x.

Definition. A G-finite tiling $\mathfrak{T}$ with tile type partition $\{\mathfrak{T}_j\}_{j\in J}$ is called **quasi-homogeneous** if for each $r > 0$ there exists $R = R(r) > 0$ such that every open ball of radius R contained in X contains the pattern of $B_r(x) \cap X$ for all $x \in X$.

Subdividing tilings.

Let ϕ be an expansive linear map on $\mathbb{R}^n$ such that $\phi X = X$. To be expansive means that the eigenvalues for ϕ all have modulus greater than 1. We wish to adapt a norm for $\mathbb{R}^n$ which reflects the expansiveness of ϕ. We will mimic the norm used by Lind in [9].

Let $\{\lambda_i\}_{i=1}^l$ be the eigenvalues of ϕ ordered so that

$$1 < |\lambda_1| \leq |\lambda_2| \leq \cdots \leq |\lambda_l|.$$

Choose ϱ and ζ such that

$$1 < \varrho < |\lambda_1| \text{ and } |\lambda_l| < \zeta.$$

Let $\|\cdot\|'$ be any norm on $\mathbb{R}^n$. For each $x \in \mathbb{R}^n$ define

$$\|x\|'' = \textstyle\sum_{k=0}^{\infty} \zeta^{-k} \|\phi^k x\|'.$$

Since $\zeta^{-1}\phi$ has eigenvalues of modulus less that 1 this series converges and defines a norm for $\mathbb{R}^n$. Note that

$$\|\phi x\|'' = \zeta \textstyle\sum_{k=1}^{\infty} \zeta^{-k} \|\phi^k x\|' \leq \zeta \|x\|''.$$

For each $x \in \mathbb{R}^n$ define

$$\|x\| = \textstyle\sum_{k=0}^{\infty} \varrho^k \|\phi^{-k} x\|''.$$

Since $\varrho\phi^{-1}$ has eigenvalues of modulus less than 1 this series converges and defines a norm for $\mathbb{R}^n$. Note that

$$\|\phi^{-1} x\| = \varrho^{-1} \textstyle\sum_{k=1}^{\infty} \varrho^k \|\phi^{-k} x\|'' \leq \varrho^{-1} \|x\|.$$

Moreover an easy check shows that

$$\|x\| < \varrho\|x\| \leq \|\phi x\| \leq \zeta\|x\|$$

and

$$\zeta^{-1}\|x\| \leq \|\phi^{-1}x\| \leq \varrho^{-1}\|x\| < \|x\|.$$

Let G be a subgroup of $\mathbb{R}^n$ such that $\phi G = G$.

Definition. A G-finite tiling $\mathfrak{T}$ of X with tile type partition $\{\mathfrak{T}_j\}_{j\in J}$ is **subdividing** with expansion map ϕ if

(1) for each $T \in \mathfrak{T}$, ϕT is the finite union of tiles, that is $\phi T \in \mathfrak{P}$, and

(2) if T and T' are tiles of the same type then ϕT and $\phi T'$ have the same pattern.

For each $T \in \mathfrak{T}_j$ we may think of the pattern of ϕT as defining a **subdivision rule** for $\mathfrak{T}_j$.

Example 1.1 Let ϕ be the expansive map on $\mathbb{R}$ given by multiplication by

$$\lambda = \frac{1+\sqrt{5}}{2}.$$

Note that λ is a zero of $x^2 - x - 1$. Let $X^+ = [0, \infty)$. We construct a $\mathbb{Z}[\lambda]$-finite subdividing tiling $\mathfrak{T}$ of X^+ with expansion map ϕ. Let $T_{\mathbf{A}} = [0,1]$ and $T_{\mathbf{B}} = [0, \lambda - 1]$. We will partition $\mathfrak{T}$ into two sets $\mathfrak{T}_{\mathbf{A}}$ and $\mathfrak{T}_{\mathbf{B}}$. The tiles in $\mathfrak{T}_{\mathbf{A}}$ will be $\mathbb{Z}[\lambda]$-translates of $T_{\mathbf{A}}$ and the tiles in $\mathfrak{T}_{\mathbf{B}}$ will be $\mathbb{Z}[\lambda]$-translates of $T_{\mathbf{B}}$. We define $\mathfrak{T}$ by inductively defining $\mathfrak{T}_{\mathbf{A}}$ and $\mathfrak{T}_{\mathbf{B}}$. Let $T_{\mathbf{A}} \in \mathfrak{T}_{\mathbf{A}}$. If $x \in \mathbb{Z}[\lambda]$ and $T_{\mathbf{A}} + x \in \mathfrak{T}_{\mathbf{A}}$ then

$$\begin{aligned}
\phi(T_{\mathbf{A}} + x) &= [0,\lambda] + \lambda x \\
&= ([0,1] + \lambda x) \cup ([1,\lambda] + \lambda x) \\
&= (T_{\mathbf{A}} + \lambda x) \cup (T_{\mathbf{B}} + 1 + \lambda x).
\end{aligned}$$

Let $T_{\mathbf{A}} + \lambda x \in \mathfrak{T}_{\mathbf{A}}$ and let $T_{\mathbf{B}} + 1 + \lambda x \in \mathfrak{T}_{\mathbf{B}}$. If $y \in \mathbb{Z}[\lambda]$ and $T_{\mathbf{B}} + y \in \mathfrak{T}_{\mathbf{B}}$ then

$$\begin{aligned}
\phi(T_{\mathbf{B}} + y) &= [0, \lambda^2 - \lambda] + \lambda y \\
&= [0,1] + \lambda y \\
&= T_{\mathbf{A}} + \lambda y.
\end{aligned}$$

Let $T_{\mathbf{A}} + \lambda y \in \mathfrak{T}_{\mathbf{A}}$. The tiling $\mathfrak{T}$ is a $\mathbb{Z}[\lambda]$-finite subdividing tiling of X^+ with expansion map ϕ and tile type partition $\{\mathfrak{T}_{\mathbf{A}}, \mathfrak{T}_{\mathbf{B}}\}$.

We can represent the subdivision rules for $\mathfrak{T}$ in terms of a substitution map. Let $\mathbf{A}$ represent a tile in $\mathfrak{T}_{\mathbf{A}}$ and $\mathbf{B}$ represent a tile in $\mathfrak{T}_{\mathbf{B}}$. Define a substitution map θ on the words in $\{\mathbf{A}, \mathbf{B}\}$ by letting $\theta(\mathbf{A}) = \mathbf{AB}$ and letting $\theta(\mathbf{B}) = \mathbf{A}$. We define θ on each finite or infinite string of symbols in $\{\mathbf{A}, \mathbf{B}\}$ by applying it to each symbol in the string. We see that $\mathfrak{T}$ is represented by the fixed word

$$w = \mathbf{ABAABABAABAAB}\ldots.$$

That is, $\theta w = w$.□

Example 1.1 illustrates a subdividing tiling which is *self similar.* We will define self similar tilings later. For now the main thing to note is that $T_{\mathbf{A}} \subset \phi T_{\mathbf{A}}$. It is possible to define subdivision rules for a G-finite tiling $\mathfrak{T}$ such that the image of each tile is a finite union of G-translates of tiles in $\mathfrak{T}$ but not a finite union of tiles in $\mathfrak{T}$. We illustrate this idea in the next example.

Example 1.2 Let $X^- = (-\infty, 0]$. Let ϕ be the expansive map defined in Example 1.1. We define two $\mathbb{Z}[\lambda]$-finite tilings $\mathfrak{T}^0$ and $\mathfrak{T}^1$ of X^-. The tiles in $\mathfrak{T}^0$ and $\mathfrak{T}^1$ will be $\mathbb{Z}[\lambda]$-translates of the intervals $T_{\mathbf{A}}$ and $T_{\mathbf{B}}$ from Example 1.1. The ϕ-image of a tile in $\mathfrak{T}^0$ will be the finite union of tiles in $\mathfrak{T}^1$. The ϕ-image of a tile in $\mathfrak{T}^1$ will be the finite union of tiles in $\mathfrak{T}^0$. We will partition $\mathfrak{T}^0$ into two sets $\mathfrak{T}^0_{\mathbf{A}}$ and $\mathfrak{T}^0_{\mathbf{B}}$. The tiles in $\mathfrak{T}^0_{\mathbf{A}}$ will be $\mathbb{Z}[\lambda]$-translates of $T_{\mathbf{A}}$ and the tiles in $\mathfrak{T}^0_{\mathbf{B}}$ will be $\mathbb{Z}[\lambda]$-translates of $T_{\mathbf{B}}$. Similarly, we will partition $\mathfrak{T}^1$ into two sets $\mathfrak{T}^1_{\mathbf{A}}$ and $\mathfrak{T}^1_{\mathbf{B}}$. We define $\mathfrak{T}^0$ and $\mathfrak{T}^1$ inductively by defining the tiles in each of the sets which partition them. Let $T_{\mathbf{A}} - 1 \in \mathfrak{T}^0_{\mathbf{A}}$ and $T_{\mathbf{B}} - (\lambda - 1) \in \mathfrak{T}^1_{\mathbf{B}}$. If $x \in \mathbb{Z}[\lambda]$ and $T_{\mathbf{A}} - 1 + x \in \mathfrak{T}^0_{\mathbf{A}}$ then as in Example 1.1

$$\phi(T_{\mathbf{A}} - 1 + x) = (T_{\mathbf{A}} - \lambda + \lambda x) \cup (T_{\mathbf{B}} + 1 - \lambda + \lambda x).$$

Let $T_{\mathbf{A}} - \lambda + \lambda x \in \mathfrak{T}^1_{\mathbf{A}}$ and let $T_{\mathbf{B}} + 1 - \lambda + \lambda x \in \mathfrak{T}^1_{\mathbf{B}}$. If $T_{\mathbf{A}} - 1 + x \in \mathfrak{T}^1_{\mathbf{A}}$ then let $T_{\mathbf{A}} - \lambda + \lambda x \in \mathfrak{T}^0_{\mathbf{A}}$ and let $T_{\mathbf{B}} + 1 - \lambda + \lambda x \in \mathfrak{T}^0_{\mathbf{B}}$. If $y \in \mathbb{Z}[\lambda]$ and $T_{\mathbf{B}} - (\lambda - 1) + y \in \mathfrak{T}^1_{\mathbf{B}}$ then as in Example 1.1

$$\begin{aligned} \phi(T_{\mathbf{B}} - (\lambda - 1) + y) &= T_{\mathbf{A}} - (\lambda^2 - \lambda) + \lambda y \\ &= T_{\mathbf{A}} - 1 + \lambda y. \end{aligned}$$

Let $T_{\mathbf{A}} - 1 + \lambda y \in \mathfrak{T}^0_{\mathbf{A}}$. If $T_{\mathbf{B}} - (\lambda - 1) + y \in \mathfrak{T}^0_{\mathbf{B}}$ then let $T_{\mathbf{A}} - 1 + \lambda y \in \mathfrak{T}^1_{\mathbf{A}}$.

The tilings $\mathfrak{T}^0$ and $\mathfrak{T}^1$ are $\mathbb{Z}[\lambda]$-finite tilings of X^-. The ϕ image of tiles in $\mathfrak{T}^0_{\mathbf{A}}$ subdivide in $\mathfrak{T}^1$ in the same pattern in $\mathfrak{T}^1$ as the ϕ image of tiles in $\mathfrak{T}^1_{\mathbf{A}}$ subdivide in $\mathfrak{T}^0$. We will say that $\mathfrak{T}^0$ and $\mathfrak{T}^1$ are *periodically subdividing.*

We use the substitution map on words in $\{\mathbf{A}, \mathbf{B}\}$ from Example 1.1 to represent the subdivision rules for the tilings $\mathfrak{T}^0$ and $\mathfrak{T}^1$. The tiling $\mathfrak{T}^0$ is represented by the infinite word

$$\ldots\mathbf{ABAABABAABABA} = w_0.$$

The tiling $\mathfrak{T}^1$ is represented by the infinite word

$$\ldots\mathbf{ABAABABAABAAB} = w_1.$$

Note that $\theta w_0 = w_1$ and $\theta^2 w_0 = w_0$.$\square$

Definition. A G-finite tiling $\mathfrak{T}$ of X with tile type partition $\{\mathfrak{T}_j\}_{j\in J}$ is **periodically subdividing** with expansion map ϕ and period p if it satisfies the following conditions.

(1) There is a **corresponding family** of G-finite tilings $\mathfrak{F}(\mathfrak{T}) = \{\mathfrak{T}^i\}_{i\in\mathbb{Z}_p}$ of X such that $\mathfrak{T} = \mathfrak{T}^0$ and if $T \in \mathfrak{T}^i$ then ϕT is the finite union of tiles in $\mathfrak{T}^{i+1}$ for each $i \in \mathbb{Z}_p$.

(2) Each $\mathfrak{T}^i \in \mathfrak{F}(\mathfrak{T})$ has a tile type partition $\{\mathfrak{T}^i_j\}_{j\in J}$ such that each tile in $\mathfrak{T}^i_j$ is a G-translate of a tile in $\mathfrak{T}_j$.

(3) For each $i \in \mathbb{Z}_p$, $j \in J$, and $T \in \mathfrak{T}^i_j$ the pattern of ϕT in $\mathfrak{T}^{i+1}$ depends only on j. That is the family $\mathfrak{F}(\mathfrak{T})$ is **uniformly subdividing.**

Note that for convenience we index a family of periodic subdividing tilings of period p with the group $\mathbb{Z}_p$ so that if $k \in \mathbb{Z}$ then $\mathfrak{T}^k = \mathfrak{T}^{k \bmod p}$. Let $i_1, i_2 \in \mathbb{Z}_p$. Suppose P is a finite union of tiles in $\mathfrak{T}^{i_1}$ and $g \in G$ such that $P + g$ is a finite union of tiles in $\mathfrak{T}^{i_2}$. We say that P has the **same pattern** in $\mathfrak{T}^{i_1}$ as $P + g$ has in $\mathfrak{T}^{i_2}$ if for each $T \in \mathfrak{T}^{i_1}_j$, $j \in J$, such that $T \subset P$ we have $T + g \in \mathfrak{T}^{i_2}_j$. Property (3) in the above definition implies that if $j \in J$ and $T^1 \in \mathfrak{T}^{i_1}_j$, $T^2 \in \mathfrak{T}^{i_2}_j$ then for all $N \geq 0$, $\phi^N T^1$ has the same pattern in $\mathfrak{T}^{i_1+N}$ as $\phi^N T^2$ has in $\mathfrak{T}^{i_2+N}$.

For a first reading the reader might just think in terms of subdividing tilings. Every periodic subdividing tiling $\mathfrak{T}$ with expansive map ϕ and period p is a subdividing tiling with expansive map ϕ^p.

Let $\mathfrak{T}$ be a periodic subdividing tiling of X with expansive map ϕ and period p. Let $\{\mathfrak{T}_j\}_{j\in J}$ be a tile type partition for $\mathfrak{T}$. Since $\mathfrak{T}$ has a finite number of tile types we may record the subdivision rules for $\mathfrak{T}$ in a finite manner. We will use a *graph* to indicate the subdivision rules for $\mathfrak{T}$. The generic definitions for graphs and the objects related to graphs given below are adapted from [1].

A **graph** Γ consists of a finite set of **vertices** $\mathcal{V}$ together with a finite set of **edges** $\mathcal{E}$. Each edge α in $\mathcal{E}$ starts at a vertex $s(\alpha)$ and ends at a vertex $t(\alpha)$. In particular all graphs used in this thesis are directed graphs. The edge α has a **source** $s(\alpha)$ and a **target** $t(\alpha)$. There may be more than one edge with the same source and target. If I is a set of consecutive integers in $\mathbb{Z}$ then a sequence (or finite sequence or bi-infinite sequence) of edges $\{\eta^k\}_{k\in I}$ is called a **path** in Γ if for each k, $k+1 \in I$ we have $t(\eta^k) = s(\eta^{k+1})$. If $\{\eta^k\}_{k=-N}^{\infty}$ is a path in Γ then $s(\eta^{-N})$ is the **source of the path.** If $\{\eta^k\}_{k=-\infty}^{M}$ is a path in Γ then $t(\eta^M)$ is the **target of the path.** A path $\{\eta^k\}_{k=-N}^{M}$ has both a source $s(\eta^{-N})$ and a target $t(\eta^M)$. A path $\{\eta^k\}_{k\in\mathbb{Z}}$ has neither a source nor a target. If I is a finite set of consecutive integers then the **length** of a path $\{\eta^k\}_{k\in I}$ is $|I|$.

If η and ϵ are two paths in Γ then we shall say that η equals ϵ if $\eta = \{\eta^k\}_{k\in I}$ and $\epsilon = \{\epsilon^k\}_{k\in I}$ for some consecutive sequence of integers I and for each $k \in I$ we have $\eta^k = \epsilon^k$. That is, the two paths must be indexed by the same set and each pair of edges η^k and ϵ^k with the same index are actually the same edge. If η is not equal to ϵ then we shall say that the paths are **distinct.**

Definition. The **subdivision graph** Γ for $\mathfrak{T}$ has a finite set of vertices $\mathcal{V}$ indexed by J and edges $\mathcal{E}$. If v_{j_1} and v_{j_2} are vertices in $\mathcal{V}$ then there are exactly $M_{j_1 j_2}$ edges with source v_{j_1} and target v_{j_2} if and only if for each $T \in \mathfrak{T}_{j_1}^i$ there are exactly $M_{j_1 j_2}$ distinct elements of $\mathfrak{T}_{j_2}^{i+1}$ contained in ϕT. Since $\mathfrak{F}(\mathfrak{T})$ is uniformly subdividing the number $M_{j_1 j_2}$ is well-defined for each pair j_1 and j_2.

Let Γ be the subdivision graph for $\mathfrak{T}$. For each $j_1, j_2 \in J$ let $\mathcal{E}_{j_1}^{j_2}$ be the subset of edges in $\mathcal{E}$ which have source v_{j_1} and target v_{j_2}. Let $\mathcal{E}_{j_1}$ be the subset of edges in $\mathcal{E}$ which have source v_{j_1}. Similarly let $\mathcal{E}^{j_2}$ be the subset of edges in $\mathcal{E}$ which have target

v_{j_2}. In this case we have $\mathcal{E}_{j_1}^{j_2} = \mathcal{E}_{j_1} \cap \mathcal{E}^{j_2}$. We will use the edges in $\mathcal{E}_{j_1}$ to index the tiles in ϕT for each $T \in \mathfrak{T}_{j_1}^i$. That is we will write

$$\phi T = \cup_{\alpha \in \mathcal{E}_{j_1}} T^\alpha$$

for the unique finite union of tiles found in ϕT such that if $t(\alpha) = v_{k_\alpha}$ then $T^\alpha \in \mathfrak{T}_{k_\alpha}^{i+1}$.

Let $\{T_j\}_{j \in J}$ be a representative set of tiles for $\{\mathfrak{T}_j\}_{j \in J}$ such that $T_j \in \mathfrak{T}_j$. Fix a point $x_j \in T_j$. For each $T_j + g \in \mathfrak{T}_j$ define

$$d(T_j + g) = x_j + g.$$

The point $d(T_j + g)$ is called a **positional point** for $T_j + g$ and $d(\mathfrak{T})$ is a **set of positional points for** $\mathfrak{T}$. Similarly, for each $T_j + g \in \mathfrak{T}_j^i$ define

$$d_i(T_j + g) = x_j + g.$$

Let $\{d_i\}_{i \in \mathbb{Z}_p}$ be a family of these maps for $\mathfrak{F}(\mathfrak{T})$. Suppose $T^0 \in \mathfrak{T}_{j_0}^0$ and $T^1 \in \mathfrak{T}_{j_1}^1$ such that $T^1 \subset \phi T^0$. We record the relative position of T^1 in ϕT^0 by noting the difference

$$d_1(T^1) - \phi d_0(T^0).$$

Let $T^2 \in \mathfrak{T}_{j_0}^i$. Then there exists a unique $T^3 \in \mathfrak{T}_{j_1}^{i+1}$ such that $T^3 \subset \phi T^2$ and

$$d_{i+1}(T^3) - \phi d_i(T^2) = d_1(T^1) - \phi d_0(T^0).$$

We have in this case

$$d_{i+1}(T^3) = \phi d_i(T^2) + (d_1(T^1) - \phi d_0(T^0)). \tag{1.1}$$

This is one way of saying that T^3 is in the same relative position in ϕT^2 as T^1 is in ϕT^0.

Definition. The **label map** L_d assigns to each edge $\alpha \in \mathcal{E}$ a vector in $\mathbb{R}^n$. If $T \in \mathfrak{T}_j^i$ and $\{T^\alpha\}_{\alpha \in \mathcal{E}_j}$ is the unique set of tiles in $\mathfrak{T}^{i+1}$ such that

$$\phi T = \cup_{\alpha \in \mathcal{E}_j} T^\alpha$$

then $L_d(\alpha) = d_{i+1}(T^\alpha) - \phi d_i(T)$ for each $\alpha \in \mathcal{E}_j$.

The pair (Γ, L_d) completely describes the subdivision rules for $\mathfrak{T}$. If $T \in \mathfrak{T}_j^i$ then for each $\alpha \in \mathcal{E}_j^k$ there is a unique tile $T' \in \mathfrak{T}_k^{i+1}$ such that $T' \subset \phi T$ and

$$d_{i+1}(T') = \phi d_i(T) + L_d(\alpha).$$

Definition. Let $\Theta_d : \mathcal{E} \to \mathcal{V} \times \mathcal{V} \times \mathbb{R}^n$ such that for each $\alpha \in \mathcal{E}$ we have

$$\Theta_d(\alpha) = (s(\alpha), t(\alpha), L_d(\alpha)).$$

The map Θ_d is a one-one assignment. For suppose α, α' are edges in $\mathcal{E}_j^k$ such that $\Theta_d(\alpha) = \Theta_d(\alpha')$. Then for each $T \in \mathfrak{T}_j$ there exists a unique $T^\alpha \in \mathfrak{T}_k^1$ such that

$$d_1(T^\alpha) = \phi d(T) + L_d(\alpha).$$

Similarly, there is a unique $T^{\alpha'} \in \mathfrak{T}_k^1$ such that

$$d_1(T^{\alpha'}) = \phi d(T) + L_d(\alpha').$$

But $L_d(\alpha) = L_d(\alpha')$ and $T^\alpha, T^{\alpha'} \in \mathfrak{T}_k^1$. So $T^\alpha = T^{\alpha'}$. But edges in $\mathcal{E}_j$ refer to distinct $\mathfrak{T}^1$ tiles in ϕT so $\alpha = \alpha'$. We find for each triple $(v_j, v_k, x) \in \Theta_d(\mathcal{E})$ there is a unique $\alpha \in \mathcal{E}$ such that $\Theta_d(\alpha) = (v_j, v_k, x)$ In this case we write $\Theta_d^{-1}(v_j, v_k, x) = \alpha$.

If $T \in \mathfrak{T}_0$ then there is a unique $T' \in \mathfrak{T}^{p-1}$ such that $\phi^{-1}(T) \subset T'$. If $T \in \mathfrak{T}_j$ and $T' \in \mathfrak{T}_k^{p-1}$ then there is a unique edge $\alpha(T) \in \mathcal{E}$ such that

$$\alpha(T) = \Theta_d^{-1}(v_k, v_j, d(T) - \phi d_{p-1}(T')).$$

Note that $T \in \mathfrak{T}_j$ if and only if $\alpha(T) \in \mathcal{E}^j$.

Example 1.3 (a) Let $\mathfrak{T}$ be the subdividing tiling of $X^+ = [0, \infty)$ with expansive map ϕ given in Example 1.1. For each $T \in \mathfrak{T}$ let $d(T)$ be the left endpoint of T. The subdivision graph for $\mathfrak{T}$ is in Figure 1.1 Each edge α is labeled with $L_d(\alpha)$.

(b) Let $\mathfrak{T}^0$ be the periodic subdividing tiling of $X^- = (-\infty, 0]$ given in Example 1.2. For each $T \in \mathfrak{T}^0$ let $d(T)$ be the right endpoint of T. the subdivision graph for $\mathfrak{T}^0$ is given in Figure 1.2. Each edge α is labeled with $L_d(\alpha)$. □

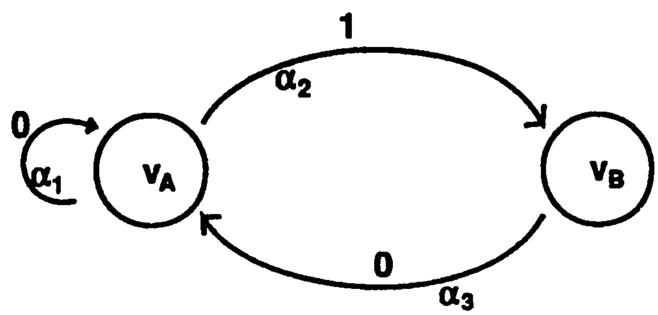

Figure 1.1: The subdivision graph for Example 1.1.

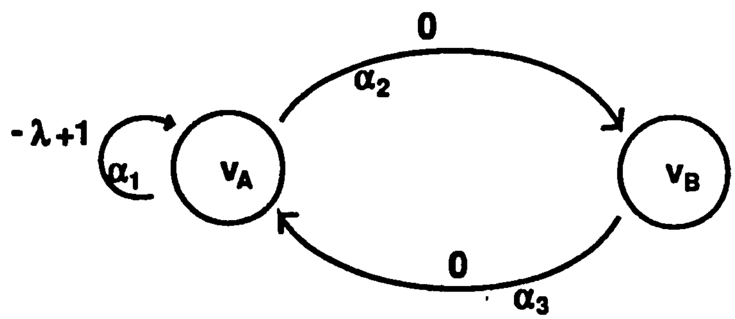

Figure 1.2: The subdivision graph for Example 1.2.

Mixing and self similar tilings.

Definition. A G-finite subdividing tiling $\mathfrak{T}$ with expansive map ϕ and tile type partition $\{\mathfrak{T}_j\}_{j\in J}$ is **mixing** if for each $T \in \mathfrak{T}$ and $P \in \mathfrak{P}$ there exists $N_0 \geq 0$ such that for all $N \geq N_0$ we have that $\phi^N(T)$ contains the pattern of P. If $\mathfrak{T}$ is a periodic subdividing tiling with period p, then $\mathfrak{T}$ is **p-mixing** if for for all $N \geq N_0$ we have that $\phi^{Np}(T)$ contains the pattern of P.

If $\mathfrak{T}$ is p-mixing then $\mathfrak{T}$ with expansive map ϕ^p is mixing. If $\mathfrak{T}$ is p-mixing then since $\mathfrak{T}$ has a finite number of tile types there exists $N_0 \geq 0$ such that for all $N \geq N_0$ and $T \in \mathfrak{T}$ we have that $\phi^{Np}(T)$ contains a tile of each tile type in $\mathfrak{T}$. Let Γ be the subdivision graph for $\mathfrak{T}$. Let M be the matrix $\{M_{j_1 j_2}\}_{j_1,j_2\in J}$ such that $M_{j_1 j_2}$ equals the number of edges in Γ with source v_{j_1} and target v_{j_2}. Then $M^{N_0 p} > 0$ so

M is aperiodic. It follows that Γ has no **stranded vertices**; that is, $\mathcal{E}_j$ and $\mathcal{E}^k$ are non-empty for all j, $k \in J$. We call M the **transition matrix** for $\mathfrak{T}$. Note that the entries in M depend only on the subdivision rules for $\mathfrak{T}$.

The following fact was also noted by Kenyon in [7].

Proposition 1.4 *Let $\mathfrak{T}$ be a G-finite subdividing tiling of X with expansive map ϕ and tile type partition $\{\mathfrak{T}_j\}_{j\in J}$. Then $\mathfrak{T}$ is quasi-homogeneous if and only if $\mathfrak{T}$ is mixing and has a finite number of local patterns.*

Proof. Suppose $\mathfrak{T}$ is quasi-homogeneous. Let $r > 0$. For each $P \in \mathfrak{P}_r$ let x_P be an element of P so that $P \subset B_r(x_P) \cap X$. By quasi-homogeneity there exists an $R > 0$ such that every ball of radius R which is contained in X contains the pattern of $B_r(x_P) \cap X$ for all $P \in \mathfrak{P}_r$. Since $\mathfrak{T}$ is locally finite each ball of radius R in X intersects a finite number of tiles, hence a finite number of elements of $\mathfrak{P}_r$. Since r was arbitrary $\mathfrak{T}$ has a finite number of local patterns. Let $T \in \mathfrak{T}$. Each tile in $\mathfrak{T}$ has nonempty interior so there exists $\delta > 0$ and $x \in T$ such that $B_\delta(x) \subset T$. Note that if $y \in B_\delta(x)$ then $\|\phi^N y - \phi^N x\| > \varrho^N\|y - x\|$ so $B_{\varrho^N\delta}(\phi^N x) \subset \phi^N T$ for all N. We choose N sufficiently large so that $\varrho^N\delta > R$. Then $\phi^N T$ contains the pattern of every element of $\mathfrak{P}_r$. Again since r was arbitrary $\mathfrak{T}$ is mixing.

Conversely, suppose that $\mathfrak{T}$ has a finite number of local patterns and that $\mathfrak{T}$ is mixing. Let $r > 0$. Let D be the maximum diameter of any tile. If $x \in X$ and $T \in \mathfrak{T}$ such that $T \cap B_r(x) \neq \emptyset$ then $T \subset B_{r+D}(x)$. So for each $x \in X$ there exists $P_x \in \mathfrak{P}_{2(r+D)}$ such that $B_r(x) \subset P_x$. Since $\mathfrak{T}$ has a finite number of local patterns $\mathfrak{P}_{2(r+D)}$ contains a finite number of distinct patterns. Since $\mathfrak{T}$ is mixing and has a finite number of tile types, there exists $N_0 \geq 0$ such that for all $N \geq N_0$ and $T \in \mathfrak{T}$ we have that $\phi^{Np}T$ contains the pattern of every element of $\mathfrak{P}_{2(r+d)}$ hence it contains the pattern of $B_r(x) \cap X$ for every $x \in X$. Note that every closed ball of radius D in X contains a tile. If $\|x - y\| \leq D$ then $\|\phi^N x - \phi^N y\| \leq \zeta^N D$. So every open ball of radius $\zeta^N D$ contains $\phi^N T$ for some tile T. Hence letting $N \geq N_0$ and $R = \zeta^N D$ we have that $\mathfrak{T}$ is quasi-homogeneous.□

Recall that quasi-homogeneity was defined for any G-finite tiling and is independent of the expansive map ϕ. It follows that if $\mathfrak{T}$ is p-mixing and has a finite number of local patterns then $\mathfrak{T}$ is quasi-homogeneous. Moreover, each $\mathfrak{T}^i \in \mathfrak{F}(\mathfrak{T})$ is quasi-homogeneous. For if $x \in X$ and $r > 0$ then the pattern of $B_r(x) \cap X$ in $\mathfrak{T}^i$ depends only on the pattern of $\phi^{-i}(B_r(x) \cap X)$ in $\mathfrak{T}$. By the quasi-homogeneity of $\mathfrak{T}$ there exists $R > 0$ such that every ball of radius R in X contains a subset with the same pattern in $\mathfrak{T}$ as $\phi^{-i}B_r(x) \cap X$ in $\mathfrak{T}$. Hence every ball of radius $\zeta^i R$ in X contains a subset with the same pattern in $\mathfrak{T}^i$ as $B_r(x) \cap X$. Since R depends only on r and not on x, $\mathfrak{T}^i$ is quasi-homogeneous.

Definition. A quasi-homogeneous subdividing tiling $\mathfrak{T}$ of X with expansive map ϕ is called a **self similar** tiling. A quasi-homogeneous periodic subdividing tiling $\mathfrak{T}$ of X with expansive map ϕ and period p is called a **periodic self similar** tiling.

Every periodic self similar tiling of X with expansive map ϕ and period p is a self similar tiling with expansive map ϕ^p.

A self similar tiling with expansive map ϕ has the property that if you apply ϕ to each of its tiles and subdivide the image of each tile according to the subdivision rules for $\mathfrak{T}$ then you get $\mathfrak{T}$ back again. The quasi-homogeneity guarantees that the tiling looks essentially the same over large areas. The mixing property indicates that in the small you may think of each tile as containing some preimage of every pattern found in X.

Example 1.5 (a) Let $\mathfrak{T}$ be the subdividing tiling of X^+ in Example 1.1. Let $\delta > 0$. There are only a finite number of ways to arrange almost disjoint $\mathbb{Z}[\lambda]$-translates of $T_{\mathbf{A}}$ and $T_{\mathbf{B}}$ in X^+ so that their union is connected and has diameter less than δ. So $\mathfrak{T}$ has a finite number of local patterns. Let P be a finite union of tiles in $\mathfrak{T}$. For some $N \geq 0$ we have $P \subset \phi^N T_{\mathbf{A}}$. Hence for all $T \in \mathfrak{T}_{\mathbf{A}}$ we have that $\phi^N T$ contains the pattern of P. If $T^1 \in \mathfrak{T}$ then ϕT^1 contains a tile in $\mathfrak{T}_{\mathbf{A}}$ so $\phi^{N+1}T^1$ contains the pattern of P. It follows from Proposition 1.4 that $\mathfrak{T}$ is self similar.

(b) Let $\mathfrak{T}^0$ be the subdividing tiling of X^- in Example 1.2. By the same argument as in (a), $\mathfrak{T}^0$ has a finite number of local patterns. If P is a finite union of tiles in $\mathfrak{T}^0$ then for some $N \geq 0$ we have $P \subset \phi^N(T_{\mathbf{A}} - 1)$. Hence we use the same argument as in (a) to show that $\mathfrak{T}^0$ is 2-mixing. It follows that $\mathfrak{T}^0$ and $\mathfrak{T}^1$ are periodically self similar tilings.□

The boundary of a self similar tiling.

Let $\mathfrak{T}$ be a self similar tiling of X with expansive map ϕ. The only topological restrictions on the tiles in $\mathfrak{T}$ are that they be compact and the closures of their interiors. The tiles need not be connected or even have piecewise smooth boundaries. The property of being self similar does force the boundary to have Lebesgue measure 0. For each tile T let ∂T denote the boundary of T. Let $\partial\mathfrak{T} = \cup_{T\in\mathfrak{T}}\partial T$ be the **boundary** of $\mathfrak{T}$. Let μ be Lebesgue measure on $\mathbb{R}^n$.

Proposition 1.6 *The Lebesgue measure of the boundary of $\mathfrak{T}$ is 0.*

Proof. Suppose $\mu(\partial\mathfrak{T}\ ) \neq 0$. Then since $\mathfrak{T}$ is locally finite for some tile T we have $\mu(\partial T) \neq 0$. Since $\mathfrak{T}$ contains only a finite number of tile types there exists

$$\delta\ = \max\left\{\frac{\mu(\partial T)}{\mu(T)} : T \in \mathfrak{T}\ \right\} < 1.$$

It follows that there exists a tile T_0 such that $\mu(\partial T_0) = \delta\mu(T_0)$, and for all $T' \in \mathfrak{T}$ we have $\mu(\partial T') \leq \delta\mu(T')$.

Note that if T^1 and T^2 are two distinct tiles then

$$\begin{aligned}
\mu(\partial(T^1 \cup T^2)) &\leq \mu(\partial T^1) + \mu(\partial T^2) - \mu(\partial T^1 \cap \partial T^2) \\
&\leq \delta\mu(T^1) + \delta\mu(T^2) - \mu(\partial T^1 \cap \partial T^2) \\
&= \delta\left(\mu(T^1 \cup T^2) + \mu(T^1 \cap T^2)\right) - \mu(\partial T^1 \cap \partial T^2) \\
&= \delta\mu(T^1 \cup T^2) + (\delta - 1)\mu(\partial T^1 \cap \partial T^2) \\
&\leq \delta\mu(T^1 \cup T^2)
\end{aligned}$$

since $T^1 \cap T^2 = \partial T^1 \cap \partial T^2$. Applying induction we find for any finite collection of distinct tiles $\{T^k\}_{k=1}^L$ that

$$\mu(\partial \cup_{k=1}^L T^k) \leq \delta\mu(\cup_{k=1}^L T^k).$$

We apply the mixing property of $\mathfrak{T}$ to choose N_0 such that for all $N \geq N_0$ we have $\mathrm{Int}(\phi^N T_0)$ contains a tile. Let P_N be the element of $\mathfrak{P}$ consisting of just those tiles in $\phi^N T_0$ which intersect the set $\phi^N(\partial T_0)$. Since ϕ is a homeomorphism $\phi^N(\partial T_0)$ is just the set $\partial(\phi^N T_0)$. We have $\mu(P_N) < \mu(\phi^N T_0)$ for all $N \geq N_0$ since $\phi^N T_0$ contains a tile which does not intersect its boundary. By the previous paragraph we have $\mu(\partial\phi^N T_0) \leq \delta\mu(P_N)$ since $\partial\phi^N T_0 \subset \partial P_N$. So

$$\begin{aligned}\delta &= \frac{|\det \phi^N| \cdot \mu(\partial T_0)}{|\det \phi^N| \cdot \mu(T_0)} \\ &= \frac{\mu(\phi^N \partial T_0)}{\mu(\phi^N T_0)} \\ &\leq \frac{\delta\mu(P_N)}{\mu(\phi^N T_0)} \\ &< \delta.\end{aligned}$$

The contradiction implies that $\mu(\partial T) = 0$ for each tile T.□

Tile maps and control points.

In [12] and [8] the tiles in a self similar tiling are assigned *control points.* The control points reflect the similarity properties of the tiling. By considering all ϕ-preimages of control points one may recover the tiling. By studying the behavior of control points under the map ϕ, Thurston and Kenyon are able to analyze the expansive maps associated to self similar tilings in terms of their eigenvalues. Kenyon also indicates that the control points are precisely the objects one creates when constructing a self similar tiling.

There are many ways to define control points. We will want our definition to apply to any periodic subdividing tiling. Let $\mathfrak{T}$ be a periodic subdividing tiling of X with expansive map ϕ and period p. Let $\{\mathfrak{T}_j\}_{j\in J}$ be the tile type partition for $\mathfrak{T}$. Let $\mathfrak{F}(\mathfrak{T})$ be the corresponding family of periodic subdividing tilings. We begin by defining a tile map for $\mathfrak{T}$.

Definition. A map $\gamma: \mathfrak{T} \to \mathfrak{T}^1$ is a **tile map** for $\mathfrak{T}$ if

(1) for each $T \in \mathfrak{T}$, $\gamma(T) \in \mathfrak{T}^1$ and $\gamma(T) \subset \phi T$, and

(2) for each $T \in \mathfrak{T}_j$, and $g \in G$ such that $T + g \in \mathfrak{T}_j$ we have $\gamma(T+g) = \gamma(T) + \phi g$.

Tile maps are constructed by fixing one tile $T_j \in \mathfrak{T}_j$ for each $j \in J$. Define $\gamma(T_j)$ to be some tile in $\mathfrak{T}^1$ contained in $\phi(T_j)$. For each $T_j + g \in \mathfrak{T}_j$ let $\gamma(T_j + g) = \gamma(T_j) + \phi g$.

Since $\mathfrak{F}(\mathfrak{T})$ is uniformly subdividing the tiles $\gamma(T_j)$ and $\gamma(T_j+g)$ have the same tile type in $\mathfrak{T}^1$. The tile map γ induces a tile map γ_i for each $\mathfrak{T}^i \in \mathfrak{F}(\mathfrak{T})$. If $T_j + g' \in \mathfrak{T}^i_j$ we define $\gamma_i(T_j+g') = \gamma(T_j) + \phi g' \in \mathfrak{T}^{i+1}$. Since $\mathfrak{F}(\mathfrak{T})$ is uniformly subdividing the tile $\gamma_i(T_j+g')$ has the same tile type in $\mathfrak{T}^{i+1}$ as $\gamma(T_j)$ has in $\mathfrak{T}^1$. If $T^1 \in \mathfrak{T}^{i_1}_j$ and $T^2 \in \mathfrak{T}^{i_2}_j$ such that $T^2 = T^1 + g$ then

$$\gamma_{i_2}(T^2) = \gamma_{i_1}(T^1) + \phi g.$$

For each $T \in \mathfrak{T}^i$ define $\gamma_i^2 = \gamma_{i+1} \circ \gamma_i(T)$. Likewise for each $k > 0$ define

$$\gamma_i^k(T) = \gamma_{i+k-1} \circ \gamma_{i+k-2} \circ \cdots \gamma_{i+1} \circ \gamma_i(T).$$

Note that $\gamma_i^k(T) \in \mathfrak{T}^{k+i}$. We have by the definition

$$\phi^{-k}\gamma_i^k(T) \subset T$$

for each $k \geq 0$. Since ϕ is expansive

$$\bigcap_{k=0}^{\infty} \phi^{-k}\gamma_i^k(T)$$

is a decreasing intersection of compact sets with diameters tending to 0. Hence there exists a unique point $c_i(T)$ contained in this intersection. We call $c_i(T)$ a **control point** for $\mathfrak{T}^i$ and $c_i(\mathfrak{T}^i)$ a set of control points for $\mathfrak{T}^i$. The definition for c_i depends on γ_i and hence on γ. Let $c = c_0$ so that $c(\mathfrak{T})$ is the set of control points for $\mathfrak{T}$ induced by γ.

Proposition 1.7 *Let $\{c_i\}_{i \in \mathbb{Z}_p}$ be the family of maps which assign to each $T \in \mathfrak{T}^i$ the unique point*

$$\{c_i(T)\} = \bigcap_{k=0}^{\infty} \phi^{-k}\gamma_i^k(T). \tag{1.2}$$

(1) If $T \in \mathfrak{T}^i$ then $\phi c_i(T) = c_{i+1}(\gamma_i(T))$.

(2) If $T^1 \in \mathfrak{T}^{i_1}_j$ and $T^2 \in \mathfrak{T}^{i_2}_j$ then

$$T^1 + c_{i_2}(T^2) - c_{i_1}(T^1) = T^2.$$

Proof. (1) Since 1.2 is a decreasing intersection

$$\{c_i(T)\} = \bigcap_{k=1}^{\infty} \phi^{-k}\gamma_i^k(T).$$

Hence

$$\begin{aligned}\{\phi c_i(T)\} &= \bigcap_{k=1}^{\infty} \phi^{1-k}\gamma_i^k(T)\\ &= \bigcap_{k=0}^{\infty} \phi^{-k}\gamma_{i+1}^k(\gamma_i(T))\\ &= \{c_{i+1}(\gamma_i(T))\}.\end{aligned}$$

(2) Let $g \in G$ such that $T^2 = T^1 + g$. Then

$$\{c_{i_2}(T^2)\} = \{c_{i_2}(T^1 + g)\} = \bigcap_{k=0}^{\infty} \phi^{-k}\gamma_{i_2}^k(T^1 + g).$$

By the definition of γ_{i_2}, we have $\gamma_{i_2}(T^1 + g) = \gamma_{i_1}(T^1) + \phi g$. Hence

$$\begin{aligned}\gamma_{i_2}^2(T^1 + g) &= \gamma_{i_2+1}(\gamma_{i_1}(T^1) + \phi g)\\ &= \gamma_{i_1+1}(\gamma_{i_1}(T^1)) + \phi^2 g\\ &= \gamma_{i_1}^2(T^1) + \phi^2 g.\end{aligned}$$

Applying induction we find

$$\gamma_{i_2}^k(T^1 + g) = \gamma_{i_1}^k(T^1) + \phi^k g.$$

So

$$\begin{aligned}\{c_{i_2}(T^2)\} &= \bigcap_{k=0}^{\infty} \phi^{-k}(\gamma_{i_1}^k(T^1) + \phi^k g)\\ &= \{c_{i_1}(T^1) + g\}.\end{aligned}$$

□

Corollary 1.8 *If* $T^1 \in \mathfrak{T}_j^{i_1}$ *and* $T^2 \in \mathfrak{T}_j^{i_2}$ *then* $c_{i_1}(T^1) = c_{i_2}(T^2)$ *if and only if* $T^1 = T^2$.

Note that if T^1, $T^2 \in \mathfrak{T}_j$ and $T^1 + g = T^2$ then $c(T^1) + g = c(T^2)$. So $c(\mathfrak{T})$ is a set of positional points for $\mathfrak{T}$. Let $L = L_c$ as defined on page 10. Since c depends only on γ we call L the **label map induced by** γ. Similarly, we let $\Theta = \Theta_c$ be the one-one assignment

$$\Theta(\alpha) = (s(\alpha), t(\alpha), L(\alpha))$$

for each edge $\alpha \in \mathcal{E}$.

Theorem 1.9 *Let $j_0 \in J$ and $T^0 \in \mathfrak{T}^i_{j_0}$.*

(1) For each $x \in T^0$ there exists a sequence of tiles $\{T^k\}_{k=1}^\infty$ in $\mathfrak{T}$ such that $T^k \in \mathfrak{T}^{i+k}$ and $\phi^k x \in T^k \subset \phi T^{k-1}$ for each $k \geq 1$. In this case

$$x = c_i(T^0) + \sum_{k=1}^{\infty} \phi^{-k}(c_{i+k}(T^k) - \phi c_{i+k-1}(T^{k-1})). \tag{1.3}$$

(2) For each $x \in T^0$, if $\{T^k\}_{k=1}^\infty$ is a sequence of tiles in $\mathfrak{T}$ satisfying (1) and for each $k \geq 1$, we have $T^k \in \mathfrak{T}^{i+k}_{j_k}$, $j_k \in J$, then there exists a unique path $\{\eta^k\}_{k=1}^\infty$ in Γ defined by

$$\eta^k = \Theta^{-1}(v_{j_{k-1}}, v_{j_k}, c_{i+k}(T^k) - \phi c_{i+k-1}(T^{k-1})).$$

In this case

$$x = c_i(T^0) + \sum_{k=1}^{\infty} \phi^{-k} L(\eta^k). \tag{1.4}$$

(3) If $\eta = \{\eta^k\}_{k=1}^\infty$ is a path in Γ with source v_{j_0} and $x \in T^0$ such that

$$x = c_i(T^0) + \sum_{k=1}^{\infty} \phi^{-k} L(\eta^k)$$

and

$$\Theta(\eta^k) = (v_{j_{k-1}}, v_{j_k}, L(\eta^k))$$

then there is a unique sequence of tiles $\{T^k\}_{k=1}^\infty$ satisfying (1) such that $T^k \in \mathfrak{T}^{i+k}_{j_k}$ and

$$c_{i+k}(T^k) = \phi c_{i+k}(T^{k-1}) + L(\eta^k).$$

In this case

$$\{x\} = \bigcap_{k=0}^{\infty} \phi^{-k} T^k.$$

(4) Suppose $\mathfrak{T}$ is a periodic self similar tiling. Then there exists a bound b such that for all $x \in T^0$ the number of sequences $\{T^k\}_{k=1}^{\infty}$ which satisfy (1) is less than or equal to b. Moreover, for μ-almost every $x \in T^0$ the sequence in (1) is uniquely defined.

Proof. (1) Since $x \in T^0 \in \mathfrak{T}^i_{j_0}$, there exists $T^1 \in \mathfrak{T}^{i+1}_{j_1}$ such that $\phi x \in T^1 \subset \phi T^0$ for some $j_1 \in J$. Applying induction we obtain a sequence of tiles $\{T^k\}_{k=1}^{\infty}$ such that $\phi^{k+1}x \in T^{k+1} \subset \phi T^k$, for each $k \geq 0$. And for some $j_k \in J$, we have $T^k \in \mathfrak{T}^{i+k}_{j_k}$ for each $k \geq 1$.

Let D be the maximum diameter of any tile. Since $x \in T^0$ we have

$$\|x - c_i(T^0)\| \leq D.$$

Similarly for each $N \geq 1$, we have $\phi^N x \in T^N$ so

$$\|\phi^N x - \phi^N c_i(T^0) - \textstyle\sum_{k=1}^{N} \phi^{N-k}(c_{i+k}(T^k) - \phi c_{i+k-1}(T^{k-1}))\| = \|\phi^N x - c_{i+N}(T^N)\| \leq D.$$

Recall that for each $y \in \mathbb{R}^n$, we have $\|\phi^{-N}y\| \leq \varrho^{-N}\|y\|$, so

$$\|x - c_i(T^0) - \textstyle\sum_{k=1}^{\infty} \phi^{-k}(c_{i+k}(T^k) - \phi c_{i+k-1}(T^{k-1}))\| \leq \varrho^{-N} D.$$

Letting N tend to infinity we prove (1).

(2) For each $k \geq 1$ we have $T^k \subset \phi T^{k-1}$ so there exists an edge $\eta^k \in \mathcal{E}^{j_k}_{j_{k-1}}$ such that $\eta^k = \Theta^{-1}(v_{j_{k-1}}, v_{j_k}, c_{i+k}(T^k) - \phi c_{i+k-1}(T^{k-1}))$. Since $t(\eta^k) = s(\eta^{k+1})$ for each $k \geq 1$ the sequence $\{\eta^k\}_{k=1}^{\infty}$ is a path in Γ with source $s(\eta^1) = v_{j_0}$. The map Θ^{-1} is well defined so $\{\eta^k\}_{k=1}^{\infty}$ is uniquely defined by $\{T^k\}_{k=1}^{\infty}$ Finally, we replace $c_i(T^k) - \phi c_{i+k-1}(T^{k-1})$ with $L(\eta^k)$ in 1.3 to obtain 1.4.

(3) We apply induction on k to construct $\{T^k\}_{k=1}^{\infty}$. Since $\Theta(\eta^1) = (v_{j_0}, v_{j_1}, L(\eta^1))$ there exists a unique tile $T^1 \subset \phi T^0$ such that $T^1 \in \mathfrak{T}^{i+1}_{j_1}$ and

$$c_{i+1}(T^1) = \phi c_i(T^0) + L(\eta^1).$$

We apply induction and suppose that $T^N \subset \phi T^{N-1}$, $T^N \in \mathfrak{T}^{i+N}_{j_N}$ and

$$c_{i+N}(T^N) = \phi c_{i+N-1}(T^{N-1}) + L(\eta^N).$$

Since $\Theta(\eta^{N+1}) = (v_{j_N}, v_{j_{N+1}}, L(\eta^{N+1}))$ there is a unique tile $T^{N+1} \subset \phi T^N$ such that $T^{N+1} \in \mathfrak{T}^{i+N+1}_{j_{N+1}}$ and

$$c_{i+N+1}(T^{N+1}) = \phi c_{i+N}(T^N) + L(\eta^{N+1}).$$

In this way we construct a unique sequence of tiles $\{T^k\}_{k=1}^{\infty}$. Since

$$\bigcap_{k=0}^{\infty} \phi^{-k} T^k$$

is a decreasing intersection of compact sets with diameters tending to 0 there exists a unique point $y \in \phi^{-k}T^k$ for all $k \geq 0$. In this case $\{T^k\}_{k=1}^{\infty}$ satisfies (1) for y and

$$y = c_i(T^0) + \sum_{k=1}^{\infty} \phi^{-k}(c_{i+k}(T^k) - \phi c_{i+k-1}(T^{k-1})).$$

By (2)

$$y = c_i(T^0) + \sum_{k=1}^{\infty} \phi^{-k} L(\eta^k) = x.$$

Hence

$$\{x\} = \bigcap_{k=0}^{\infty} \phi^{-k} T^k.$$

(4) Suppose $\mathfrak{T}$ is a periodic self similar tiling. Let $x \in T^0$ and suppose $\{T_1^k\}_{k=1}^{\infty}$ and $\{T_2^k\}_{k=1}^{\infty}$ are two distinct sequences of tiles which satisfy (1). Then for some minimal $N \geq 1$ we have $\phi^N x \in T_1^N \cap T_2^N$ and $T_1^N \neq T_2^N$. Since $T_1^{N+k} \subset \phi^k T_1^N$ and $T_2^{N+k} \subset \phi^k T_2^N$ for all $k \geq 1$ we have $\phi^{N+k} x \in T_1^{N+k} \cap T_2^{N+k}$ and $T_1^{N+k} \neq T_2^{N+k}$. It follows that the number of distinct sequences satisfying (1) is bounded by the number of tiles in $\mathfrak{T}^i$ which may share a point in common, for each i. Since $\mathfrak{T}^i$ is locally finite and quasi-homogeneous and $\mathfrak{F}(\mathfrak{T})$ is finite, there exists a uniform bound b on the number of tiles which may share a common point, for any of the tilings in $\mathfrak{F}(\mathfrak{T})$. Hence the number of distinct sequences which satisfy (1) is bounded by b.

If $x \in T^0$ and there are multiple sequences satisfying (1) then for some $k \geq 1$ we have

$$\phi^k x \in \partial \mathfrak{T}^{i+k}.$$

Hence if $x \notin \cup_{k=0}^{\infty} \phi^{-k} \partial \mathfrak{T}^{i+k}$ then there is a unique sequence of tiles satisfying (1). By Proposition 1.6, $\mu(\partial \mathfrak{T}^i) = 0$. Since ϕ is a linear map $\mu(\cup_{k=0}^{\infty} \phi^{-k} \partial \mathfrak{T}^{i+k}) = 0$. Hence for μ-almost every $x \in T^0$ there is a unique sequence of tiles satisfying (1).□

Corollary 1.10 *Let $j_0 \in J$ and $T \in \mathfrak{T}^i_{j_0}$. Then*

$$T - c_i(T) = \{\sum_{k=1}^{\infty} \phi^{-k} L(\eta^k) \colon \{\eta^k\}_{k=1}^{\infty} \text{ is a path in } \Gamma \text{ with source } v_{j_0}\}.$$

Corollary 1.11 *There exists a bound $\bar{b}$ such that for all $x \in \mathbb{R}^n$ if*

$$S_x = \{\eta = \{\eta^k\}_{k=1}^{\infty} \colon x = \sum_{k=1}^{\infty} \phi^{-k} L(\eta^k).\ \eta \text{ a path in } \Gamma\}$$

then $|S_x|$ is less than or equal to $\bar{b}$.

Proof. If $\eta = \{\eta^k\}_{k=1}^{\infty} \in S_x$ is a path in Γ with source v_j for some $j \in J$ then for any $i \in \mathbb{Z}_p$ and $T \in \mathfrak{T}^i_j$ we have

$$x + c_i(T) \in T$$

and η is a path satisfying Theorem 1.9(3). By (4) of the above theorem we have that there is a bound b on the number of such paths. Since there are $|J|$ tile types

$$|S_x| \leq b \cdot |J| = \bar{b}.$$

□

Example 1.12 Let $\mathfrak{T}$ be the tiling of X^+ in Example 1.1. Define a tile map $\gamma \colon \mathfrak{T} \to \mathfrak{T}$ as follows. If $x \in \mathbb{Z}[\lambda]$ and $T_{\mathbf{A}} + x \in \mathfrak{T}_{\mathbf{A}}$ then

$$\phi(T_{\mathbf{A}} + x) = (T_{\mathbf{A}} + \lambda\, x) \cup (T_{\mathbf{B}} + 1 + \lambda\, x).$$

Define $\gamma(T_{\mathbf{A}} + x) = T_{\mathbf{A}} + \lambda\, x$. If $y \in \mathbb{Z}[\lambda]$ and $T_{\mathbf{B}} + y \in \mathfrak{T}_{\mathbf{B}}$ then

$$\phi(T_{\mathbf{B}} + y) = T_{\mathbf{A}} + \lambda\, y.$$

Define $\gamma(T_{\mathbf{B}} + y) = T_{\mathbf{A}} + \lambda\, y$. It follows that $c(T_{\mathbf{A}} + x) = x$ and $c(T_{\mathbf{B}} + y) = y$. So the set of left endpoints of the tiles in $\mathfrak{T}$ form a set of control points. By examining Figure 1.1 we see that every $x \in T_{\mathbf{A}}$ has a representation of the form

$$x = \textstyle\sum_{k=1}^{\infty} \lambda^{-k} L(\eta^k)$$

where $L(\eta^k) \in \{0,1\}$ and $L(\eta^k) + L(\eta^{k+1}) \in \{0,1\}$ for all $k \geq 1$.□

The next proposition says that no point in X may have two finite paths of the same length and with the same source and target giving representations for it.

Proposition 1.13 *Suppose* $\eta = \{\eta^k\}_{k=1}^M$ *and* $\epsilon = \{\epsilon^k\}_{k=1}^M$ *are two paths in* Γ *with the same source* v_{j_0} *and target* v_{j_1}. *Then*

$$\sum_{k=1}^{M} \phi^{-k} L(\eta^k) = \sum_{k=1}^{M} \phi^{-k} L(\epsilon^k)$$

if and only if $\eta^k = \epsilon^k$ *for each* $1 \leq k \leq M$.

Proof. Without loss of generality we assume $\eta^1 \neq \epsilon^1$. For each $T \in \mathfrak{T}_{j_0}$ there exists distinct tiles T^1 and T^2 in $\mathfrak{T}^1$ contained in ϕT such that

$$c_1(T^1) = \phi c(T) + L(\eta^1)$$

and

$$c_1(T^2) = \phi c(T) + L(\epsilon^1).$$

Moreover, there exists distinct tiles T^3 and T^4 in $\mathfrak{T}_{j_1}^M$ such that $T^3 \subset \phi^{M-1} T^1 \subset \phi^M T$, and $T^4 \subset \phi^{M-1} T^2 \subset \phi^M T$ and

$$c_M(T^3) = \phi^M c(T) + \sum_{k=1}^{M} \phi^{M-k} L(\eta^k)$$

and

$$c_M(T^4) = \phi^M c(T) + \sum_{k=1}^{M} \phi^{M-k} L(\epsilon^k).$$

Hence $c_M(T^3) = c_M(T^4)$. By Corollary 1.8, $T^3 = T^4$. But T^3 and T^4 are distinct tiles. The contradiction gives the result.□

Chapter 2

GENERATING TILE MAPS

As we indicated in Chapter 1 there are many ways to define tile type partitions for G-finite tilings. If $\mathfrak{T}$ is a subdividing tiling with certain properties then if we say that $\{\mathfrak{T}_j\}_{j\in J}$ is a tile type partition for $\mathfrak{T}$ we implicitly mean that $\mathfrak{T}$ has those properties with the tile type partition $\{\mathfrak{T}_j\}_{j\in J}$. Not every partition of $\mathfrak{T}$ which is consistent with the definition for G-finite tilings will preserve the subdividing properties of $\mathfrak{T}$.

We are interested in being able to find different tile type partitions for $\mathfrak{T}$. Our motivation is that we would like each tile containing the origin to have 0 as its control point. We will then be able to apply Theorem 1.9 to construct digit expansions for every point in X in terms of the powers of ϕ. In order to define a set of control points which have this property we must be able to define a tile map which sends every tile containing the origin to another tile containing the origin. Suppose $T \in \mathfrak{T}_j$ contains the origin. It turns out that there is a unique $T' \in \mathfrak{T}^1$ contained in ϕT which also contains the origin. If T and T' have the same tile type in their respective tilings, but do not coincide as subsets of X then there is no way to define a tile map which has the properties we want. For if $c(T) = 0 = c(T')$ then by Corollary 1.8 we have $T = T'$.

Insplitting the tile types.

One way to construct a new tile type partition for $\mathfrak{T}$ is to perform an *insplitting* of the subdivision graph for $\mathfrak{T}$. This will define new subdivision rules for $\mathfrak{T}$ which preserve the subdivision properties that were true under the original rules. Insplitting is a procedure which takes a graph Γ and generates a new graph $\widehat{\Gamma}$ by splitting one set of edges sharing a common target into multiple sets and giving each of these sets a distinct target. In terms of the tiling, we will split a tile type into two or more sets and call each of these new sets a tile type.

Let $\mathfrak{T}$ be a periodic self similar tiling of X with expansion map ϕ and period p. Let $\{\mathfrak{T}_j\}_{j\in J}$ be a tile type partition for $\mathfrak{T}$ and $\mathfrak{F}(\mathfrak{T})$ the corresponding family of periodic self similar tilings. Let Γ be the subdivision graph for $\mathfrak{T}$. Let $\mathcal{V}$ be the vertex set for Γ indexed by J and $\mathcal{E}$ be the edge set for Γ.

The insplitting rule.

Let $j_0 \in J$ and split $\mathcal{E}^{j_0}$ into two sets $\mathcal{E}^{j_0^1}$ and $\mathcal{E}^{j_0^2}$. Let $\widehat{J} = J - \{j_0\} \cup \{j_0^1, j_0^2\}$. Let $\widehat{\Gamma}$ be the graph with vertex set $\widehat{\mathcal{V}}$ indexed by $\widehat{J}$ and edge set $\widehat{\mathcal{E}}$ defined as follows. Suppose α is an edge in $\mathcal{E}$, $i \in \{1,2\}$, and $j,\ k \in J$.

- If $\alpha \in \mathcal{E}^{j_0^i}$ and $s(\alpha) = v_{j_0}$ then let $\widehat{\alpha_1},\ \widehat{\alpha_2} \in \widehat{\mathcal{E}}$ such that

$$\widehat{\alpha_1} \in \widehat{\mathcal{E}}^{j_0^i}_{j_0^1} \text{ and } \widehat{\alpha_2} \in \widehat{\mathcal{E}}^{j_0^i}_{j_0^2}.$$

- If $\alpha\in \mathcal{E}^{j_0^i}$ and $s(\alpha) = v_k$ for some $k \neq j_0$ then let

$$\widehat{\alpha} \in \widehat{\mathcal{E}}^{j_0^i}_{k}.$$

- If $\alpha \in \mathcal{E}^k_{j_0}$ for some $k \neq j_0$ then let $\widehat{\alpha_1},\ \widehat{\alpha_2} \in \widehat{\mathcal{E}}$ such that

$$\widehat{\alpha_1} \in \widehat{\mathcal{E}}^{k}_{j_0^1} \text{ and } \widehat{\alpha_2} \in \widehat{\mathcal{E}}^{k}_{j_0^2}.$$

- If $\alpha \in \mathcal{E}^k_j$ for some $j,\ k \neq j_0$ then let

$$\widehat{\alpha} \in \widehat{\mathcal{E}}^{k}_{j}.$$

This procedure is called a **single insplitting of Γ** induced by the partition $\{\mathcal{E}^{j_0^1}, \mathcal{E}^{j_0^2}\}$. From the above construction we may compute a transition matrix for $\widehat{\Gamma}$, indexed by $\widehat{J} \times \widehat{J}$ given by

$$\left\{|\widehat{\mathcal{E}}^{j_2}_{j_1}|\right\}_{j_1, j_2 \in \widehat{J}}.$$

Let $i \in \{1,2\}$.

- If $j_1 \in \{j_0^1, j_0^2\}$ and $j_2 = j_0^i$ then

$$|\widehat{\mathcal{E}}^{j_2}_{j_1}| = |\mathcal{E}_{j_0} \cap \mathcal{E}^{j_0^i}|.$$

- If $j_1 \in J - \{j_0\}$ and $j_2 = j_0^i$ then

$$|\widehat{\mathcal{E}}_{j_1}^{j_2}| = |\mathcal{E}_{j_1} \cap \mathcal{E}^{j_0{}^i}|.$$

- If $j_1 \in \{j_0^1, j_0^2\}$ and $j_2 \in J - \{j_0\}$ then

$$|\widehat{\mathcal{E}}_{j_1}^{j_2}| = |\mathcal{E}_{j_0}^{j_2}|.$$

- If j_1, $j_2 \in J - \{j_0\}$ then

$$|\widehat{\mathcal{E}}_{j_1}^{j_2}| = |\mathcal{E}_{j_1}^{j_2}|.$$

It follows that if $\{\widehat{\mathfrak{T}}_j\}_{j \in \hat{J}}$ is a tile type partition for $\mathfrak{T}$ which induces a transition matrix equal to the transition matrix for $\widehat{\Gamma}$ then $\widehat{\Gamma}$ is the subdivision graph corresponding to $\mathfrak{T}$ with the partition $\{\widehat{\mathfrak{T}}_j\}_{j \in \hat{J}}$.

Let γ be a tile map for $\mathfrak{T}$ and L the label map for Γ induced by γ. If $T \in \mathfrak{T}_j^i$ and $T' \in \mathfrak{T}_k^{i-1}$ such that $\phi^{-1}T \subset T'$ then let

$$\alpha_i(T) = \Theta^{-1}(v_k, v_j, c_i(T) - \phi c_{i-1}(T')).$$

Let $j_0 \in J$ and let $\{\mathcal{E}^{j_0^1}, \mathcal{E}^{j_0^2}\}$ be a partition of $\mathcal{E}^{j_0}$. Let $\widehat{\Gamma}$ be obtained from Γ by a single insplitting induced by this partition. We construct a tile type partition $\{\widehat{\mathfrak{T}}_j\}_{j \in \hat{J}}$ for $\mathfrak{T}$ which induces the subdivision graph $\widehat{\Gamma}$.

Suppose $T^1 \in \mathfrak{T}_{j_0}^{k_1}$ and $T^2 \in \mathfrak{T}_{j_0}^{k_2}$ for some k_1, $k_2 \in \mathbb{Z}_p$ and

$$\alpha_{k_1}(T^1) \neq \alpha_{k_2}(T^2).$$

Then we may choose the partition of $\mathcal{E}^{j_0}$ so that if $\{\hat{\mathfrak{T}}_j^{k_1}\}_{j \in \hat{J}}$ and $\{\hat{\mathfrak{T}}_j^{k_2}\}_{j \in \hat{J}}$ are the corresponding tile type partitions for $\mathfrak{T}^{k_1}$ and $\mathfrak{T}^{k_2}$ then there exists j_1, $j_2 \in \hat{J}$ such that

$$T^1 \in \hat{\mathfrak{T}}_{j_1}^{k_1} \text{ and } T^2 \in \hat{\mathfrak{T}}_{j_2}^{k_2}$$

and $j_1 \neq j_2$. We do this as follows. Since $\alpha_{k_1}(T^1) \neq \alpha_{k_2}(T^2)$ there exists a partition of $\mathcal{E}^{j_0}$ such that $\alpha_{k_1}(T^1) \in \mathcal{E}^{j_0^1}$ and $\alpha_{k_2}(T^2) \in \mathcal{E}^{j_0^2}$. Let $\widehat{\Gamma}$ be the graph obtained from Γ by applying the insplitting rule using this partition. Let $\hat{J} = J - \{j_0\} \cup \{j_0^1, j_0^2\}$ as in the rule.

For each $\alpha \in \mathcal{E}$ there exists a corresponding edge or pair of edges in $\widehat{\mathcal{E}}$. If $\alpha \in \mathcal{E}_j^k$ and $k \neq j_0$ then the corresponding edges in $\widehat{\mathcal{E}}$ have target v_k. If $\alpha \in \mathcal{E}^{j_0^1}$ then the corresponding edge or edges in $\widehat{\mathcal{E}}$ have target $v_{j_0^1}$. If $\alpha \in \mathcal{E}^{j_0^2}$ then the corresponding edges in $\widehat{\mathcal{E}}$ have target $v_{j_0^2}$. For each $j \in J - \{j_0\}$ and $i \in \mathbb{Z}_p$ let

$$\hat{\mathfrak{T}}_j^i = \{T \in \mathfrak{T}^i : \alpha_i(T) \in \mathcal{E}^j\} = \mathfrak{T}_j^i.$$

Let

$$\hat{\mathfrak{T}}_{j_0^1}^i = \{T \in \mathfrak{T}^i : \alpha_i(T) \in \mathcal{E}^{j_0^1}\}$$

and

$$\hat{\mathfrak{T}}_{j_0^2}^i = \{T \in \mathfrak{T}^i : \alpha_i(T) \in \mathcal{E}^{j_0^2}\}.$$

Then $\{\hat{\mathfrak{T}}_{j_0^1}^i, \hat{\mathfrak{T}}_{j_0^2}^i\}$ is just a partition of $\mathfrak{T}_{j_0}^i$ and $\mathfrak{T}$ is a G-finite tiling with respect to the partition $\{\hat{\mathfrak{T}}_j^0\}_{j \in \hat{J}}$. It follows that the set of control points for $\mathfrak{T}^i$ with the tile type partition $\{\mathfrak{T}_j^i\}_{j \in J}$ is a set of positional points for $\mathfrak{T}^i$ in the new partition.

For each $T \in \mathfrak{T}_k^i$ let $\{T^\alpha\}_{\alpha \in \mathcal{E}_k}$ be the unique set of tiles in $\mathfrak{T}^{i+1}$ such that

$$\phi T = \cup_{\alpha \in \mathcal{E}_k} T^\alpha$$

and

$$c_{i+1}(T^\alpha) = \phi c_i(T) + L(\alpha(T^\alpha)).$$

This will define the subdivision rule for the tiles in $\mathfrak{T}^i$ of the same type as T in the new partition. For each $j' \in \hat{J}$ there exists $k \in J$ such that $\hat{\mathfrak{T}}_{j'}^i \subset \mathfrak{T}_k^i$. Hence for each $T \in \hat{\mathfrak{T}}_{j'}^i$ the pattern of ϕT in the partition $\{\hat{\mathfrak{T}}_j^{i+1}\}_{j \in \hat{J}}$ depends only on j'. Hence $\mathfrak{F}(\mathfrak{T})$ is uniformly subdividing with this new partition.

It follows that if P is a finite union of tiles in $\mathfrak{T}^i$ then the pattern of P with respect to the partition $\{\hat{\mathfrak{T}}_j^i\}_{j \in \hat{J}}$ depends only on the pattern of $\phi^{-1}P$ in $\mathfrak{T}^{i-1}$ with the partition $\{\mathfrak{T}_j^{i-1}\}_{j \in J}$. Since $\mathfrak{T}^{i-1}$ is quasi-homogeneous with the partition $\{\mathfrak{T}_j^{i-1}\}_{j \in J}$, $\mathfrak{T}^i$ is quasi-homogeneous with the partition $\{\hat{\mathfrak{T}}_j^i\}_{j \in \hat{J}}$. So $\mathfrak{T}^i$ is a periodic self similar tiling with the partition $\{\hat{\mathfrak{T}}_j^i\}_{j \in \hat{J}}$.

We show $\widehat{\Gamma}$ is the subdivision graph for $\mathfrak{T}$ corresponding to the partition $\{\widehat{\mathfrak{T}}_j\}_{j \in \hat{J}}$. We compute the transition matrix M for $\mathfrak{T}$ in terms of the subdivision rules determined by $\{\widehat{\mathfrak{T}}_j\}_{j \in \hat{J}}$. Let $j_1, j_2 \in \hat{J}$ and let $T \in \hat{\mathfrak{T}}_{j_1}$. Then $M_{j_1 j_2}$ is the number of tiles in $\hat{\mathfrak{T}}_{j_2}^1$ contained in ϕT. Let $T' \in \hat{\mathfrak{T}}_{j_2}^1$ such that $T' \subset \phi T$. Let $i \in \{1, 2\}$.

- We have $\alpha(T') \in \mathcal{E}_{j_0} \cap \mathcal{E}^{j_0^i}$ if and only if $j_1 \in \{j_0^1, j_0^2\}$ and $j_2 = j_0^i$. In this case
$$M_{j_1 j_2} = |\mathcal{E}_{j_0} \cap \mathcal{E}^{j_0^i}|.$$

- We have $\alpha(T') \in \mathcal{E}_j \cap \mathcal{E}^{j_0^i}$ for some $j \in J - \{j_0\}$ if and only if $j_1 \in J - \{j_0\}$ and $j_2 = j_0^i$. In this case
$$M_{j_1 j_2} = |\mathcal{E}_{j_1} \cap \mathcal{E}^{j_0^i}|.$$

- We have $\alpha(T') \in \mathcal{E}_{j_0} \cap \mathcal{E}^j$ for some $j \in J - \{j_0\}$ if and only if $j_1 \in \{j_0^1, j_0^2\}$ and $j_2 \in J - \{j_0\}$. In this case
$$M_{j_1 j_2} = |\mathcal{E}_{j_0}^{j^2}|.$$

- We have $\alpha(T') = \mathcal{E}_j \cap \mathcal{E}^k$ for some j, $k \in J - \{j_0\}$ if and only if $j_1, j_2 \in J - \{j_0\}$. In this case
$$M_{j_1 j_2} = |\mathcal{E}_{j_1}^{j_2}|.$$

Since the transition matrix for $\mathfrak{T}$ with the partition $\{\hat{\mathfrak{T}}_j\}_{j \in \hat{J}}$ equals the transition matrix for $\hat{\Gamma}$, we conclude that $\hat{\Gamma}$ is the subdivision graph for $\mathfrak{T}$ with the partition $\{\hat{\mathfrak{T}}_j\}_{j \in \hat{J}}$. Since $\alpha_{k_1}(T^1) \in \mathcal{E}^{j_0^1}$ we have $T^1 \in \hat{\mathfrak{T}}_{j_0^1}^{k_1}$. Since $\alpha_{k_2}(T^2) \in \mathcal{E}^{j_0^2}$ we have $T^2 \in \hat{\mathfrak{T}}_{j_0^2}^{k_2}$.

Example 2.1 Let $X = \mathbb{R}^2$ and I^2 be the unit square in $\mathbb{R}^2$ with vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$. Let
$$\mathfrak{T} = \{I^2 + z : z \in \mathbb{Z}^2\}$$
then $\mathfrak{T}$ is a $\mathbb{Z}^2$-finite tiling of X. Since every tile in $\mathfrak{T}$ is a $\mathbb{Z}^2$-translate of I^2 we let $\mathfrak{T}$ have one tile type $\mathfrak{T}_1$. Let $\phi: \mathbb{R}^2 \to \mathbb{R}^2$ be dilation by a factor of 2. We see that
$$\phi(I^2) = I^2 \cup (I^2 + (1,0)) \cup (I^2 + (1,1)) \cup (I^2 + (0,1)).$$

So $\mathfrak{T}$ is a subdividing tiling with expansion map ϕ and tile type partition $\{\mathfrak{T}_1\}$. Define $\gamma: \mathfrak{T} \to \mathfrak{T}$ by $\gamma(I^2) = I^2$. Then for all $z \in \mathbb{Z}^2$, $\gamma(I^2 + z) = I^2 + \phi z$. We find $c(I^2 + z) = z$. The subdivision graph Γ for $\mathfrak{T}$ is in Figure 2.1. Each edge α is labeled with $L(\alpha)$ where L is the label map induced by γ.

We split the edge set $\mathcal{E}$ into two sets. Let the edges labeled $(0,0)$ and $(1,1)$ be in one set and the edges labeled $(1,0)$ and $(0,1)$ be in the other set. Let $\hat{\Gamma}$ be the

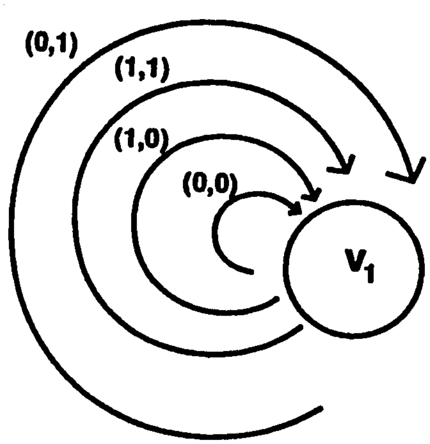

Figure 2.1: The subdivision graph for the tiling of the plane by unit squares.

graph obtained from Γ by a single insplitting induced by this partition. The graph $\widehat{\Gamma}$ is given in Figure 2.2. The tile type partition of $\mathfrak{T}$ corresponding to $\widehat{\Gamma}$ is

$$\hat{\mathfrak{T}}_1 = \{I^2 + (x,y): x,y \in \mathbb{Z}, \text{ and } x+y \text{ is even}\}$$

and

$$\hat{\mathfrak{T}}_2 = \{I^2 + (x,y): x,y \in \mathbb{Z}, \text{ and } x+y \text{ is odd}\}.$$

If we *color* the tiles in $\hat{\mathfrak{T}}_1$ white and the tiles in $\hat{\mathfrak{T}}_2$ red, then $\mathfrak{T}$ looks like an infinite checkerboard sitting on $\mathbb{R}^2$.□

Proposition 2.2 *Let i_1, $i_2 \in \mathbb{Z}_p$ and $j \in J$. Let $T_1 \in \mathfrak{T}_j^{i_1}$ and $T_2 \in \mathfrak{T}_j^{i_2}$. Suppose $T_1 \neq T_2$. Then there exists a tile type partition for $\mathfrak{T}$ in which T_1 and T_2 have distinct tile types.*

Proof. If $\alpha_{i_1}(T_1) \neq \alpha_{i_2}(T_2)$ then we are done by the preceeding remarks. We suppose now that $\alpha_{i_1}(T_1) = \alpha_{i_2}(T_2)$.

There exists a unique sequence of tiles $\{T_1^k\}_{k=0}^{\infty}$ such that $\phi^{-k}T_1 \subset T_1^k \in \mathfrak{T}^{i_1-k}$ and a unique sequence of tiles $\{T_2^k\}_{k=0}^{\infty}$ such that $\phi^{-k}T_2 \subset T_2^k \in \mathfrak{T}^{i_2-k}$. Since ϕ is expansive, for some $M_0 \geq 0$ we have $0 \in T_1^N \cap T_2^N$ for all $N \geq M_0$. We claim for some minimal $K_0 \geq 1$ we have

$$\alpha_{i_1-K_0}(T_1^{K_0}) \neq \alpha_{i_2-K_0}(T_2^{K_0}).$$

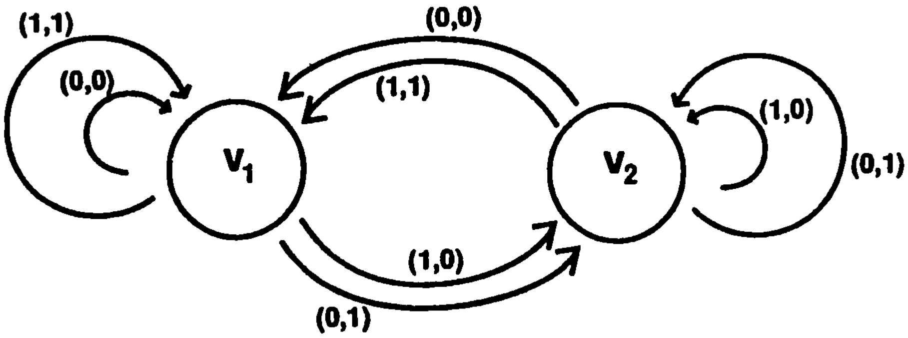

Figure 2.2: The subdivision graph for the checkerboard tiling of the plane.

Suppose for a moment this is true. Then the tile type of $T_1^{K_0}$ in $\mathfrak{T}^{i_1-K_0}$ is the same as the tile type of $T_2^{K_0}$ in $\mathfrak{T}^{i_2-K_0}$. Otherwise the source of $\alpha_{i_1-K_0+1}(T_1^{K_0-1})$ would be different from the source of $\alpha_{i_2-K_0+1}(T_2^{K_0-1})$ and

$$\alpha_{i_1-K_0+1}(T_1^{K_0-1}) \neq \alpha_{i_2-K_0+1}(T_2^{K_0-1})$$

contradicting the minimality of K_0.

By the remarks preceding this proposition we may construct a single insplitting of Γ to form $\widehat{\Gamma}^1$ and the resulting tile type partition for $\mathfrak{T}^{i_1-K_0}$ will give $T_1^{K_0}$ a different tile type than the tile type for $T_2^{K_0}$ in the new partition for $\mathfrak{T}^{i_2-K_0}$.

The control points for $\mathfrak{T}$ are positional points for $\mathfrak{T}$ so we may define a label map for $\widehat{\Gamma}^1$ in terms of the controls points. We define an edge assignment $\alpha_i^1 \colon \mathfrak{T}^i \to \widehat{\mathcal{E}}^1$ in terms of the label map for $\widehat{\Gamma}$. Since $T_1^{K_0}$ and $T_2^{K_0}$ have different tile types in their respective tilings there exists a minimal $K_1 < K_0$ such that

$$\alpha^1_{i_1-K_1}(T_1^{K_1}) \neq \alpha^1_{i_2-K_1}(T_2^{K_1}).$$

By applying induction we construct a finite sequence of insplittings to form $\widehat{\Gamma}^N$ with edge assignment $\alpha_i^N \colon \mathfrak{T}^i \to \mathcal{E}^N$. It follows that there is a decreasing sequence

$$0 < K_N < K_{N-1} < \cdots < K_1 < K_0$$

such that $\alpha^N_{i_1-K_N}(T_1^{K_N}) \neq \alpha^N_{i_2-K_N}(T_2^{K_N})$. Hence for some $N \geq 1$ we have

$$\alpha^N_{i_1}(T_1) \neq \alpha^N_{i_2}(T_2).$$

So we are in a position to define a tile type partition for $\mathfrak{T}$ which will force T_1 and T_2 to have different tile types in their respective tilings.

Why does K_0 exist? Suppose for all $k \geq 1$ we have

$$\alpha_{i_1-k}(T_1^k) = \alpha_{i_1-k}(T_2^k).$$

Then for all $N \geq 1$ we have $T_1^N \neq T_2^N$. Otherwise for some minimal $N_0 > 0$ we have $T_1^{N_0} = T_2^{N_0}$. Suppose this is the case and $T_1^{N_0} \in \mathfrak{T}_j^{i_1-N_0}$ and $T_2^{N_0} \in \mathfrak{T}_j^{i_2-N_0}$. Let $x_1 = \phi^{-N_0}c_{i_1}(T_1)$ and $x_2 = \phi^{-N_0}c_{i_2}(T_2)$. We apply the proof for Theorem 1.9 and find

$$\begin{aligned} x_1 &= c_{i_1-N_0}(T_1^{N_0}) + \sum_{i=1}^{N_0} \phi^{-i} L(\alpha_{i_1-N_0+i}(T_1^{N_0-i})) \\ &= c_{i_2-N_0}(T_2^{N_0}) + \sum_{i=1}^{N_0} \phi^{-i} L(\alpha_{i_2-N_0+i}(T_2^{N_0-i})) \\ &= x_2. \end{aligned}$$

So $c_{i_1}(T_1) = c_{i_2}(T_2)$. By Corollary 1.8 we have $T_1 = T_2$. But we assumed that $T_1 \neq T_2$.

So $\{T_1^k\}_{k=0}^{\infty}$ and $\{T_2^k\}_{k=0}^{\infty}$ are distinct sequences for which $T_1^k \neq T_2^k$ for any $k \geq 1$. Since $\alpha_{i_1-k+1}(T_1^{k-1}) = \alpha_{i_2-k+1}(T_2^{k-1})$ we have

$$c_{i_1-k+1}(T_1^{k-1}) - \phi c_{i_1-k}(T_1^k) = c_{i_2-k+1}(T_2^{k-1}) - \phi c_{i_2-k}(T_2^k).$$

So

$$c_{i_1-k+1}(T_1^{k-1}) - c_{i_2-k+1}(T_2^{k-1}) = \phi(c_{i_1-k}(T_1^k) - c_{i_2-k}(T_2^k))$$

for all $k \geq 0$. Hence

$$c_{i_1-k}(T_1^k) - c_{i_2-k}(T_2^k) = \phi^{-k}(c_{i_1}(T_1) - c_{i_2}(T_2))$$

for all $k \geq 0$. Since $T_1 \neq T_2$ we have $c_{i_1}(T_1) \neq c_{i_2}(T_2)$. Since ϕ is a homeomorphism

$$\{\phi^{-k}(c_{i_1}(T_1) - c_{i_2}(T_2))\}_{k=1}^{\infty}$$

is a sequence of distinct points converging to 0. But each tiling in $\mathfrak{F}(\mathfrak{T})$ is locally finite. For all $N \geq M_0$ we have T_1^N and T_2^N contain 0. So $\{c_{i_1-k}(T_1^k) - c_{i_2-k}(T_2^k)\}_{k\geq 0}$ contains a finite number of distinct differences and these are bounded away from zero. The contradiction implies for some K_0, $\alpha_{i_1-K_0}(T_1^{K_0}) \neq \alpha_{i_2-K_0}(T_2^{K_0})$. This proves the proposition.□

Generators and generating tile maps.

Definition. A **generator for** $\mathfrak{T}^i$ is a tile which contains the origin. A **generator for** $\mathfrak{F}(\mathfrak{T})$ is any generator for some $\mathfrak{T}^i \in \mathfrak{F}(\mathfrak{T})$.

There is a one to one correspondence between the generators in $\mathfrak{T}^i$ and the generators in $\mathfrak{T}^{i+1}$. For if T^1, T^2 are generators for $\mathfrak{T}^i$ then $0 \in \phi T^1 \cap \phi T^2$. Since ϕT^1 and ϕT^2 are almost disjoint, there are distinct generators in $\mathfrak{T}^{i+1}$ contained in ϕT^1 and ϕT^2. Applying induction we see the number of generators may stay the same or increase. But $\mathfrak{F}(\mathfrak{T})$ is a periodic family of tiles so the number of distinct generators must stay the same.

Since $\mathfrak{T}^i$ is locally finite, there are a finite number of generators for $\mathfrak{F}(\mathfrak{T})$. If T^1 is a generator for $\mathfrak{T}^{i_1}$ and T^2 is a generator for $\mathfrak{T}^{i_2}$ then we may assume by Proposition 2.2 that T^1 and T^2 have different tile types in their respective tilings or that they coincide as subsets in X and in tile type. It follows that there is a subset $J_0 \subset J$ which will index generators for $\mathfrak{F}(\mathfrak{T})$ by their tile types. If two generators coincide in X but have different tile types in their respective tilings then they will be distinguished by their indices.

Let $\{T_j\}_{j\in J_0}$ be the set of generators for $\mathfrak{F}(\mathfrak{T})$.

Proposition 2.3 *There exists a tile map γ for $\mathfrak{T}$ and $k_0 \geq 1$ such that if $\{\gamma_i\}_{i\in \mathbf{Z}_p}$ is the corresponding family of tile maps for $\mathfrak{F}(\mathfrak{T})$ then we have the following.*

(1) If $T_j \in \mathfrak{T}_j^i$ and $j \in J_0$ then $c_i(T_j) = 0$.

(2) For each $T \in \mathfrak{T}^i$ and $k \geq k_0$ we have that $\gamma_i^k(T)$ is a tile in $\mathfrak{T}^{i+k}$ of the same type as a generator for $\mathfrak{F}(\mathfrak{T})$.

(3) The set $\cup_{i\in \mathbf{Z}_p} c_i(\mathfrak{T}^i)$ is a subset of G.

Proof. We begin by defining γ on the tile types indexed by J_0. If $j \in J_0$ and $T_j \in \mathfrak{T}_j^i$ then there is a unique generator $T_{j'} \in \mathfrak{T}_{j'}^{i+1}$ such that $T_{j'} \subset \phi T_j$. Let $\gamma_i(T_j) = T_{j'}$. If $T_{j'}$ has the same tile type in $\mathfrak{T}^{i+1}$ as T_j has in $\mathfrak{T}^i$ then $j = j'$ and $\gamma_i(T_j) = T_j$. It follows that we may unambiguously define γ_i for each generator T_j in $\mathfrak{T}^i$ in such a way that $\gamma_i^k(T_j)$ is a generator for $\mathfrak{T}^{i+k}$. Hence $0 \in \gamma_i^k(T_j)$ for all $k \geq 0$ and $c_i(T_j) = 0$.

We partition $J - J_0$ by the lengths of paths between vertices in $\{v_j\}_{j \in J - J_0}$ and vertices in $\{v_j\}_{j \in J_0}$. Since $\mathfrak{T}$ is p-mixing, there exists k_0 such that every vertex in $\{v_j\}_{j \in J - J_0}$ is the source of a path containing no more than k_0 edges and with target in $\{v_j\}_{j \in J_0}$.

Let $J_1 \subset J - J_0$ index the set of vertices v_j such that v_j is the source of an edge with target in $\{v_j\}_{j \in J_0}$. Let $J_2 \subset J - (J_0 \cup J_1)$ index the set of vertices v_j such that v_j is the source of an edge with target in $\{v_j\}_{j \in J_1}$. We repeat this process until for some minimal k_0 we have $J = \cup_{i=0}^{k_0} J_i$.

For each $j \in J - J_0$ fix $T_j \in \mathfrak{T}_j$. If $j \in J_i$, $0 < i \leq k_0$, we know that $\phi(T_j)$ contains a tile $T \in \mathfrak{T}_{j'}^1$ such that $j' \in J_{i-1}$. Let $\gamma(T_j) = T$. This defines γ for all tile types and hence defines the family $\{\gamma_i\}_{i \in \mathbf{Z}_p}$.

Let $T \in \mathfrak{T}^i$. Then $\gamma_i^{k_0}(T)$ is a tile in $\mathfrak{T}^{i+k_0}$ of the same type as a generator for $\mathfrak{F}(\mathfrak{T})$. By Proposition 1.7, $\phi^{k_0} c_i(T) = c_{i+k_0}(\gamma_i^{k_0}(T))$. Let T_j be the generator for $\mathfrak{F}(\mathfrak{T})$ with the same tile type as $\gamma_i^{k_0}(T)$. Then if $T_j \in \mathfrak{T}^k$ we have

$$\phi^{k_0} c_i(T) = c_{i+k_0}(\gamma_i^{k_0}(T)) - c_k(T_j) \in G.$$

Hence $c_i(T) \in \phi^{-k_0} G = G$.□

Definition. A tile map satisfying Proposition 2.3 is a **generating tile map.**

Digit expansions for points in X.

We observe that in Theorem 1.9 every $x \in X$ has a representation of the form

$$x = c_i(T^0) + \sum_{k=1}^{\infty} \phi^{-k} L(\eta^k)$$

where $x \in T^0 \in \mathfrak{T}_{j_0}^i$ and $\{\eta^k\}_{k=1}^{\infty}$ is a path in Γ with source v_{j_0}, $j_0 \in J$. If T^0 is a generator for $\mathfrak{T}^i$ then $c_i(T^0) = 0$ and we would have a representation for x in powers of ϕ with digits in $L(\mathcal{E})$. Let γ be a generating tile map for $\mathfrak{T}$.

Theorem 2.4 *(1) For each $x \in X$ there exists a sequence of tiles $\{T^k\}_{k\in\mathbb{Z}}$ such that if $T^0 \in \mathfrak{T}^i$ then $T^k \in \mathfrak{T}^{i+k}$ and $\phi^k x \in T^k \subset \phi T^{k-1}$ for each $k \in \mathbb{Z}$. In this case*

$$x = \sum_{k=-\infty}^{\infty} \phi^{-k}(c_{i+k}(T^k) - \phi c_{i+k-1}(T^{k-1})). \tag{2.1}$$

(2) For each $x \in X$, if $\{T^k\}_{k\in\mathbb{Z}}$ is a sequence of tiles satisfying (1) and $T^k \in \mathfrak{T}^{i+k}_{j_k}$ then there exists a unique path $\{\eta^k\}_{k\in\mathbb{Z}}$ in Γ defined by

$$\eta^k = \Theta^{-1}(v_{j_{k-1}}, v_{j_k}, c_{i+k}(T^k) - \phi c_{i+k-1}(T^{k-1})).$$

In this case

$$x = \sum_{k=-\infty}^{\infty} \phi^{-k} L(\eta^k). \tag{2.2}$$

(3) For each $x \in X$, if $\{\eta^k\}_{k=-N}^{\infty}$ is a path in Γ with source in $\{v_j\}_{j\in J_0}$ such that

$$x = \sum_{k=-N}^{\infty} \phi^{-k} L(\eta^k)$$

then there exists a unique sequence of tiles $\{T^k\}_{k\in\mathbb{Z}}$ satisfying (1) such that if $T^k \in \mathfrak{T}^{i+k}_{j_k}$ then

$$\eta^k = \Theta^{-1}(v_{j_{k-1}}, v_{j_k}, c_{i+k}(T^k) - \phi c_{i+k-1}(T^{k-1}))$$

for each $k \geq -N$. In this case

$$\{x\} = \bigcap_{k\in\mathbb{Z}} \phi^{-k} T^k.$$

Moreover, there is a unique extension of $\{\eta^k\}_{k=-N}^{\infty}$ defined by

$$\eta^k = \Theta^{-1}(v_{j_{k-1}}, v_{j_k}, c_{i+k}(T^k) - \phi c_{i+k-1}(T^{k-1}))$$

for each $k < -N$, and

$$x = \sum_{-\infty}^{\infty} \phi^{-k} L(\eta^k).$$

(4) There exists a bound $\bar{b}$ such that for any $x \in X$ the number of sequences of tiles satisfying (1) is bounded by $\bar{b}$.

Proof. (1) By Theorem 1.9 for each $T^0 \in \mathfrak{T}^i$ and $x \in T^0$ there is a sequence of tiles $\{T^k\}_{k=1}^{\infty}$ such that for each $k \geq 1$, $T^k \in \mathfrak{T}^{i+k}$ and $\phi^k x \in T^k \subset \phi T^{k-1}$. In this case we have

$$x = c_i(T^0) + \sum_{k=1}^{\infty} \phi^{-k}(c_{i+k}(T^k) - \phi c_{i+k-1}(T^{k-1})).$$

For each $k < 1$ we find there is a unique tile $T^k \in \mathfrak{T}^{i+k}$ such that $\phi^k T^0 \subset T^k$. In this case $\phi^k x \in T^k \subset \phi T^{k-1}$ for each $k \in \mathbb{Z}$.

Since ϕ is expansive, for some $N > 0$ we have $T^{-N-1} = T_j$ for some $j \in J_0$. Hence for $k < -N$ we have $c_{i+k}(T^k) = 0$. Let $y = \phi^{-N-1}x \in T^{-N-1}$. Then $\{T^{-N-1+k}\}_{k=1}^{\infty}$ is a sequence of tiles satisfying Theorem 1.9 for y. Since $c_{-N-1+k}(T^{-N-1+k}) = 0$ for all $k < 1$ we have

$$y = \sum_{k=-\infty}^{\infty} \phi^{-k}(c_{-N-1+k}(T^{-N-1+k}) - \phi c_{-N-1+k-1}(T^{-N-1+k-1})).$$

Applying ϕ^{N+1} to both sides of the equation we obtain 2.1.

(2) Let $x \in X$ and $\{T^k\}_{k\in\mathbb{Z}}$ be a sequence of tiles satisfying (1). Suppose $T^k \in \mathfrak{T}_{j_k}^{i+k}$, for all $k \in \mathbb{Z}$. Define

$$\eta^k = \Theta^{-1}(v_{j_{k-1}}, v_{j_k}, c_{i+k}(T^k) - \phi c_{i+k-1}(T^{k-1})).$$

Since $s(\eta^k) = t(\eta^{k-1})$ for each $k \in \mathbb{Z}$, we have that $\{\eta^k\}_{k\in\mathbb{Z}}$ is a path in Γ. Since Θ^{-1} is well defined, $\{\eta^k\}_{k\in\mathbb{Z}}$ is uniquely defined. By replacing $c_{i+k}(T^k) - \phi c_{i+k-1}(T^{k-1})$ with $L(\eta^k)$ in equation 2.1 we obtain equation 2.2.

(3) Suppose $x \in X$ and $\{\eta^k\}_{k=-N}^{\infty}$ is a path in Γ with source in $\{v_j\}_{j\in J_0}$ such that

$$x = \sum_{k=-N}^{\infty} \phi^{-k} L(\eta^k).$$

Suppose $\Theta(\eta^k) = (v_{j_{k-1}}, v_{j_k}, L(\eta^k))$ for $k \geq -N$. Then

$$\phi^{-N-1}x = \sum_{k=1}^{\infty} \phi^{-k} L(\eta^{k-N-1}) \in T_j$$

for some generator T_j for $\mathfrak{T}^i$, $i \in \mathbb{Z}_p$. Let $T^{-N-1} = T_j$. We have that

$$\phi^{-N-1}x = c_i(T^{N-1}) + \sum_{k=1}^{\infty} \phi^{-k} L(\eta^{k-N-1}).$$

We may now apply Theorem 1.9 to construct the unique sequence $\{T^{-N-1+k}\}_{k=1}^{\infty}$ corresponding to $\{\eta^{-N-1+k}\}_{k=1}^{\infty}$. Then $T^{-N-1+k} \in \mathfrak{T}^{i+k}$ for each $k \geq 0$. For each $k < 1$ there is a unique tile $T^{-N-1+k} \in \mathfrak{T}^{i+k}$ such that $\phi^k T^{-N-1} \subset T^{-N-1+k}$. Moreover each T^{-N-1+k} is a generator for $\mathfrak{T}^{i+k}$, $k < 1$. We have

$$\{\phi^{-N-1}x\} = \bigcap_{k\in\mathbb{Z}} \phi^{-k}T^{-N-1+k}.$$

Hence

$$\{x\} = \bigcap_{k\in\mathbb{Z}} \phi^{-k}T^{k}.$$

By applying (2) to $\{T^k\}_{k\in\mathbb{Z}}$ we obtain the unique extension $\{\eta^k\}_{k\in\mathbb{Z}}$.

(4) This follows almost immediately from Corollary 1.11. For if $x \in X$ then for some $N \geq 0$ we have that $\phi^{-N}x$ belongs to a generator for $\mathfrak{F}(\mathfrak{T})$. We know that the number of paths $\{\eta^k\}_{k=1}^{\infty}$ which satisfy Theorem 1.9 for $\phi^{-N}x$ is bounded by $\bar{b}$. By applying (3) we know there is a unique sequence of tiles satisfying (1) for $\phi^{-N}x$ which corresponds to each path $\{\eta^k\}_{k=1}^{\infty}$. We have only to note that if $M > N$ then the family of paths $\{\eta^k\}_{k=1}^{\infty}$ which give representations for $\phi^{-M}x$ corresponds to the same family of sequences of tiles which satisfy (1) for $\phi^{-N}x$, if we permit ourselves to renumber the indices accordingly. For if $x \in T$ and $\{T^k\}_{k=1}^{\infty}$ is a sequence of tiles satisfying Theorem 1.9 for x then the extension defined in (1) given by $\{T^k\}_{k\in\mathbb{Z}}$ is uniquely determined.□

Suppose $x \in X$ and $\{T^k\}_{k\in\mathbb{Z}}$ is a sequence satisfying (1) for x. Suppose for all $k \in \mathbb{Z}$ we have that $T^k \in \mathfrak{T}^{i+k}$. We have that $x \in T^0$ and

$$c_i(T^0) = \sum_{k=-\infty}^{0} \phi^{-k}(c_{i+k}(T^k) - \phi c_{i+k-1}(T^{k-1})).$$

For some $N \geq 0$ we have T^{-N-1} is a generator for $\mathfrak{F}(\mathfrak{T})$. It follows that

$$c_{i+k}(T^k) - \phi c_{i+k-1}(T^{k-1}) = 0$$

for all $k < -N$ and

$$c_i(T^0) = \sum_{k=-N}^{0} \phi^{-k}(c_{i+k}(T^k) - \phi c_{i+k-1}(T^{k-1})).$$

It follows from (2) that if $T^{-N-1} \in \mathfrak{T}_j^{i-N-1}$, $j \in J_0$, then there exists a path $\{\eta^k\}_{k=-N}^0$ with source v_j and target v_{j_0} such that

$$c_i(T^0) = \sum_{k=-N}^{0} \phi^{-k} L(\eta^k).$$

On the other hand suppose $\{\eta^k\}_{k=-N}^0$ is a path in Γ with source in $\{v_j\}_{j\in J_0}$ and target v_{j_0}. If $s(\eta^{-N}) = v_{j_{-N-1}}$ and $T_{j_{-N-1}} \in \mathfrak{T}_{j_{-N-1}}^{i-1}$ then there exists a unique $T^0 \in \mathfrak{T}_{j_0}^{i+N}$ such that

$$\begin{aligned} c_{i+N}(T^0) &= \phi^{N+1} c_i(T_{j_{-N-1}}) + \sum_{k=-N}^{0} \phi^{-k} L(\eta^k) \\ &= \sum_{k=-N}^{0} \phi^{-k} L(\eta^k). \end{aligned}$$

We extend $\{\eta^k\}_{k=-N}^0$ to a path $\{\eta^k\}_{k=-N}^\infty$. Let $T^k = \gamma_{i+N}^k(T^0)$. Suppose $T^k \in \mathfrak{T}_{j_k}^{i+N+k}$, then define for each $k \geq 1$

$$\eta^k = \Theta^{-1}(v_{j_{k-1}}, v_{j_k}, 0).$$

By Theorem 2.4 there is a unique extension of $\{\eta^k\}_{k=-N}^\infty$ to $\{\eta^k\}_{k\in\mathbb{Z}}$ such that

$$c_{i+N}(T^0) = \sum_{k=-\infty}^{\infty} \phi^{-k} L(\eta^k).$$

Hence

$$\sum_{k=-\infty}^{0} \phi^{-k} L(\eta^k) \in c_{i+N}(\mathfrak{T}_{j_0}^{i+N}).$$

Let

$$\begin{aligned} \mathcal{S}_j = \{\eta = \{\eta^k\}_{k=-\infty}^0 &: \eta \text{ is a path in } \Gamma \text{ with target } v_j \\ &\text{such that there exists } N \geq 0 \text{ for which} \\ &s(\eta^k) \in \{v_j\}_{j\in J_0} \text{ for all } k \leq -N\}. \end{aligned}$$

By the above remarks

$$\cup_{i\in\mathbb{Z}_p} c_i(\mathfrak{T}_j^i) = \left\{ \sum_{k=-\infty}^{0} \phi^{-k} L(\eta^k) : \eta \in \mathcal{S}_j \right\}.$$

Let

$$\mathcal{N}(\mathcal{S}_j) = \left\{ \eta \in \mathcal{S}_j \colon \sum_{k=-\infty}^{0} \phi^{-k} L(\eta^k) \notin c(\mathfrak{T}_j) \right\}.$$

It follows from the above remarks that

Proposition 2.5

$$c(\mathfrak{T}_j) = \left\{ \sum_{k=-\infty}^{0} \phi^{-k} L(\eta^k) \colon \eta \in \mathcal{S}_j - \mathcal{N}(\mathcal{S}_j) \right\}.$$

To obtain digit expansions as in Theorem 2.4 it may not be necessary to define a tile map for which every generator has 0 as its control point. Suppose that γ is a generating tile map. Suppose $J_0' \subset J_0$ such that for each $j \in J_0'$ there exists $j' \in J_0'$ such that for all $T \in \mathfrak{T}_j^i$ we have $\gamma_i(T) \in \mathfrak{T}_{j'}^{i+1}$. In some sense we could think of $\{T_j\}_{j \in J_0'}$ as being invariant under the tile map γ. Suppose further that every $x \in X$ has an eventual preimage in one of the tiles in $\{T_j\}_{j \in J_0'}$. Then each $x \in X$ has a sequence of tiles $\{T^k\}_{k \in \mathbb{Z}}$ such that for some $N_0 \geq 0$ we have $T^{-N+k} \in \{T_j\}_{j \in J_0'}$ for all $k \leq 1$. Hence we could work with fewer tile types and still obtain digit expansions. To keep things simple we will always assume that γ is a generating tile map and that every generator has 0 as its control point. We will discuss the option of loosening the requirements on γ briefly in Chapter 5, Example 5.2.

Chapter 3

PERIODIC TILINGS OF $\mathbb{R}^n$

Let ϕ be a hyperbolic linear automorphism on $\mathbb{R}^n$ with matrix representation in $\mathrm{GL}(n,\mathbb{Z})$ and characteristic polynomial χ_ϕ irreducible over $\mathbb{Z}$. The map ϕ induces a hyperbolic automorphism $\widehat{\phi}$ on $\mathbb{R}^n$ mod $\mathbb{Z}^n$.

The irreducibility of χ_ϕ implies that χ_ϕ has no repeated zeroes and hence ϕ is diagonalizable. Since ϕ is hyperbolic we may order the eigenvalues for ϕ as $\{\lambda_i\}_{i=1}^n$ such that

$$|\lambda_1| \leq |\lambda_2| \leq \cdots |\lambda_l| < 1 < |\lambda_{l+1}| \leq \cdots |\lambda_n|.$$

There is a ϕ-invariant decomposition of $\mathbb{R}^n$ into spaces E_s and E_u such that the eigenvalues for $\phi|_{E_s}$ are $\{\lambda_i\}_{i=1}^l$ and the eigenvalues for $\phi|_{E_u}$ are $\{\lambda_i\}_{i=l+1}^n$. The space E_s is the **stable eigenspace** for ϕ and the space E_u is the **unstable eigenspace** for ϕ. Note that $\phi|_{E_s}^{-1}$ and $\phi|_{E_u}$ are both expansive maps. Hence as in Chapter 1 we will adapt a norm for $\mathbb{R}^n$ which reflects the expansive properties of $\phi|_{E_s}^{-1}$ and $\phi|_{E_u}$.

Choose ϱ and ζ so that

$$1 < \varrho < \min\{|\lambda_l^{-1}|, |\lambda_{l+1}|\} \text{ and } \zeta > \max\{|\lambda_n|, |\lambda_1^{-1}|\}.$$

By our discussion on page 5 we may define a norm $\|\cdot\|$ for $\mathbb{R}^n$ such that for each $x \in E_s$ we have

$$\|x\| < \varrho\|x\| \leq \|\phi^{-1}x\| \leq \zeta\|x\|$$

and

$$\zeta^{-1}\|x\| \leq \|\phi x\| \leq \varrho^{-1}\|x\| < \|x\|.$$

Likewise if $x \in E_u$ then

$$\|x\| < \varrho\|x\| \leq \|\phi x\| \leq \zeta\|x\|$$

and

$$\zeta^{-1}\|x\| \leq \|\phi^{-1}x\| \leq \varrho^{-1}\|x\| < \|x\|.$$

We give $\mathbb{R}^n$, E_s, and E_u the topologies induced by the norm $\|\cdot\|$.

Since χ_ϕ is irreducible over $\mathbb{Z}$, there is no ϕ-invariant subspace of $\mathbb{R}^n$ which intersects $\mathbb{Z}^n - \{0\}$. Let $\pi_s: \mathbb{R}^n \to E_s$ be projection along E_u to E_s. Let $\pi_u: \mathbb{R}^n \to E_u$ be projection along E_s to E_u. Then $\pi_s|_{\mathbb{Z}^n}$ and $\pi_u|_{\mathbb{Z}^n}$ are bijective $\mathbb{Z}$-linear maps.

Definition. Let $\zeta_u: \pi_u(\mathbb{Z}^n) \to \mathbb{Z}^n$ by $\zeta_u(\pi_u(z)) = z$. Let $\rho_s: \pi_u(\mathbb{Z}^n) \to \pi_s(\mathbb{Z}^n)$ by $\rho_s(\pi_u(z)) = -\pi_s(z)$.

We note that ζ_u and ρ_s are bijective $\mathbb{Z}$-linear maps which are related by

$$\rho_s(x) = x - \zeta_u(x)$$

for each $x \in \pi_u(\mathbb{Z}^n)$.

Periodic self similar tilings of E_u.

Let X_u be a subset of $E_u \cong \mathbb{R}^{n-l}$ which is the closure of its interior. We will make the following assumptions about X_u.

- $\phi X_u = \phi|_{E_u} X_u = X_u$.
- X_u mod $\mathbb{Z}^n$ is dense in $\mathbb{R}^n$ mod $\mathbb{Z}^n$.

Suppose that $\mathfrak{T}$ is a $\pi_u(\mathbb{Z}^n)$-finite periodic self similar tiling of X_u with expansive map ϕ and period p. Let $\mathfrak{F}(\mathfrak{T})$ be the corresponding family of tilings for $\mathfrak{T}$. By Proposition 2.3 there exists a tile type partition $\{\mathfrak{T}_j\}_{j\in J}$ for $\mathfrak{T}$ which forces the distinct generators in $\mathfrak{F}(\mathfrak{T})$ to be distinguished by their tile types. We apply Proposition 2.3 and let $\gamma: \mathfrak{T} \to \mathfrak{T}^1$ be a generating tile map so that

$$\cup_{i\in\mathbb{Z}_p} c_i(\mathfrak{T}^i) \subset \pi_u(\mathbb{Z}^n).$$

Let Γ be the subdivision graph for $\mathfrak{T}$. We continue with the notation of Chapter 2.

Definition. For each $j \in J$ let $\Omega_{u,j}$ be the subset of $\mathbb{R}^n$ given by

$$\Omega_{u,j} = \cup_{T\in\mathfrak{T}_j}(T - \zeta_u c(T)).$$

Since $\rho_s(c(T)) = c(T) - \zeta_u c(T)$ we have

$$T - \zeta_u c(T) = \rho_s c(T) \oplus (T - c(T))$$

where $\rho_s c(T) \in E_s$ and $T - c(T) \subset E_u$. By Corollary 1.10 for all $T \in \mathfrak{T}_j$ we have

$$T - c(T) = \left\{ \sum_{k=1}^{\infty} \phi^{-k} L(\eta^k) : s(\eta^1) = v_j,\ \{\eta^k\}_{k=1}^{\infty} \text{ is a path in } \Gamma \right\}.$$

By Proposition 2.5 and the remarks on page 37

$$\rho_s c(\mathfrak{T}_j) = \left\{ \sum_{k=-\infty}^{0} \phi^{-k} \rho_s L(\eta^k) : \{\eta^k\}_{k=-\infty}^{0} \in \mathcal{S}_j - \mathcal{N}(\mathcal{S}_j) \right\}.$$

Define

$$S_j = \left\{ \eta = \{\eta^k\}_{k\in\mathbf{Z}} : \{\eta^k\}_{k=-\infty}^{0} \in \mathcal{S}_j - \mathcal{N}(\mathcal{S}_j),\ \eta \text{ is a path in } \Gamma \right\}.$$

Then

$$\Omega_{u,j} = \left\{ \sum_{k=-\infty}^{0} \phi^{-k} \rho_s L(\eta^k) + \sum_{k=1}^{\infty} \phi^{-k} L(\eta^k) : \{\eta^k\}_{k\in\mathbf{Z}} \in S_j \right\}.$$

Since $\mathcal{E}$ is finite, if $x \in \Omega_{u,j}$ and $\eta \in S_j$ such that

$$x = \sum_{k=-\infty}^{0} \phi^{-k} \rho_s L(\eta^k) + \sum_{k=1}^{\infty} \phi^{-k} L(\eta^k)$$

then

$$\begin{aligned} \|x\| &\leq \sum_{k=-\infty}^{0} \varrho^k \|\rho_s L(\eta^k)\| + \sum_{k=1}^{\infty} \varrho^{-k} \|L(\eta^k)\| \\ &\leq \frac{2}{1+\varrho^{-1}} \max\{\|\rho_s L(\alpha)\|, \|L(\alpha)\| : \alpha \in \mathcal{E}\}. \end{aligned}$$

So $\Omega_{u,j}$ is bounded.

Definition.

- For each $j \in J$, let $\Omega_j = \mathrm{Clos}(\Omega_{u,j})$.
- Let $\Omega_u = \cup_{j\in J} \Omega_{u,j}$.
- Let $\Omega = \mathrm{Clos}(\Omega_u)$.

We see that Ω is a compact set. Since $X_u \bmod \mathbb{Z}^n = \Omega_u \bmod \mathbb{Z}^n$ and $X_u \bmod \mathbb{Z}^n$ is dense in $\mathbb{R}^n \bmod \mathbb{Z}^n$ we have that

$$\Omega \bmod \mathbb{Z}^n = \mathbb{R}^n \bmod \mathbb{Z}^n.$$

Hence

$$\bigcup_{z \in \mathbb{Z}^n} (\Omega + z) = \mathbb{R}^n.$$

It turns out that under special circumstances the collection

$$\{\Omega + z : z \in \mathbb{Z}^n\}$$

forms a periodic tiling of $\mathbb{R}^n$. This is the first step in constructing a Markov partition for $\phi \bmod \mathbb{Z}^n$.

We begin by taking advantage of a lot of foreknowledge and discuss the shift of finite type which will eventually represent the hyperbolic toral automorphism induced by ϕ.

The graph shift $(\Sigma_\Gamma, \sigma_\Gamma)$.

Definition. If Γ is a graph with edge set $\mathcal{E}$ then the **graph shift** Σ_Γ is the shift of finite type over the alphabet $\mathcal{E}$ specified by

$$\Sigma_\Gamma = \left\{\eta = \{\eta^k\}_{k \in \mathbb{Z}} \in \mathcal{E}^{\mathbb{Z}} : \eta \text{ is a path in } \Gamma\right\}.$$

The **shift operator**, $\sigma_\Gamma : \Sigma_\Gamma \to \Sigma_\Gamma$, is defined for each $\eta \in \Sigma_\Gamma$ and $k \in \mathbb{Z}$ so that

$$\sigma_\Gamma \eta^k = \eta^{k+1}.$$

We define a metric d_Γ on Σ_Γ such that if η, $\epsilon \in \Sigma_\Gamma$ and $k \in \mathbb{Z}$ then

$$d_\Gamma(\eta, \epsilon) = \frac{1}{1 + |k|}$$

if $|k|$ is minimal such that $\eta^k \neq \epsilon^k$. We give Σ_Γ the topology induced by the metric d_Γ. In this topology Σ_Γ is compact and σ_Γ is a homeomorphism.

For each $\alpha \in \mathcal{E}$ and $i \in \mathbb{Z}$ define

$$C_i(\alpha) = \{\eta \in \Sigma_\Gamma : \eta^i = \alpha\}.$$

Note that $C_i(\alpha) = \sigma_\Gamma{}^{-i} C_0(\alpha)$. For each finite path $\{\eta^k\}_{k=-N}^{M}$ in Γ define

$$C_0(\eta^{-N}\eta^{-N+1}\ldots\eta^M) = \cap_{k=0}^{N+M} C_k(\eta^{-N+k})$$

and for each $i \in \mathbb{Z}$

$$C_i(\eta^{-N}\eta^{-N+1}\cdots\eta^M) = \sigma_\Gamma{}^{-i} C_0(\eta^{-N}\eta^{-N+1}\cdots\eta^M).$$

We call the set $C_i(\eta^{-N}\ldots\eta^M)$ a **cylinder set** in Σ_Γ. The cylinder sets are both open and compact in the topology of Σ_Γ.

Let Σ_Γ be the graph shift induced by the subdivision graph Γ for $\mathfrak{T}$. For each $j \in J$ let

$$\bar{S}_j = \{\eta \in \Sigma_\Gamma : t(\eta^0) = v_j\}.$$

Note that

$$\Sigma_\Gamma = \cup_{j \in J} \bar{S}_j$$

where this is a disjoint union and $\bar{S}_j$ is compact in Σ_Γ.

Definition. Define $\psi : \Sigma_\Gamma \to \mathbb{R}^n$ for $\eta \in \Sigma_\Gamma$ by

$$\psi(\eta) = \sum_{k=-\infty}^{0} \phi^{-k} \rho_s L(\eta^k) + \sum_{k=1}^{\infty} \phi^{-k} L(\eta^k).$$

If η and ϵ are elements of Σ_Γ then

$$\sum_{k=-\infty}^{0} \|\phi^{-k}\rho_s L(\eta^k)\| + \sum_{k=-\infty}^{0} \|\phi^{-k}\rho_s L(\epsilon^k)\| + \sum_{k=1}^{\infty} \|\phi^{-k}\rho_s L(\eta^k)\| + \sum_{k=1}^{\infty} \|\phi^{-k}\rho_s L(\epsilon^k)\|$$

is a convergent series. If $N \in \mathbb{Z}$ and

$$d_\Gamma(\eta, \epsilon) \le \frac{1}{1 + |N|}$$

then

$$\begin{aligned}\|\psi(\eta)-\psi(\epsilon)\| &\le \sum_{k=-\infty}^{-N}\|\phi^{-k}\rho_s(L(\eta^k)-L(\epsilon^k))\| + \sum_{k=N}^{\infty}\|\phi^{-k}(L(\eta^k)-L(\epsilon^k))\| \\ &\le \frac{2\varrho^{-N}}{1+\varrho^{-1}}\max\{\|\rho_s L(\alpha)\|, \|L(\alpha)\|: \alpha\in\mathcal{E}\}.\end{aligned}$$

Hence ψ is a continuous map from Σ_Γ to $\mathbb{R}^n$. Note that $\psi(S_j)=\Omega_{u,j}$.

Proposition 3.1 *For each $j\in J$, S_j is dense in $\bar{S}_j$ and $\psi(\bar{S}_j)=\Omega_j$.*

Proof. Let $j\in J$ and $\eta\in\bar{S}_j$. For each $k\in\mathbb{Z}$ suppose

$$\Theta(\eta^k)=(v_{j_{k-1}}, v_{j_k}, L(\eta^k)).$$

For each $N>0$ choose

$$\{\epsilon_N^{k-N}\}_{k=-\infty}^{0}\in \mathcal{S}_{j_{-N}}-\mathcal{N}(\mathcal{S}_{j_{-N}})$$

and for each $k\ge 1$ let

$$\epsilon_N^{k-N}=\eta^{k-N}.$$

Then $\epsilon_N=\{\epsilon_N^k\}_{k\in\mathbb{Z}}\in S_j$ and

$$d_\Gamma(\epsilon_N,\eta)\le\frac{1}{1+N}.$$

Hence $\bar{S}_j=\mathrm{Clos}(S_j)$.

Since $\bar{S}_j$ is compact in Σ_Γ and ψ is continuous, $\psi(\bar{S}_j)$ is a compact subset of $\mathbb{R}^n$. Since S_j is dense in $\bar{S}_j$, $\Omega_{u,j}=\psi(S_j)$ is dense in $\psi(\bar{S}_j)$. Hence $\Omega_j=\psi(\bar{S}_j)$.□

Corollary 3.2 *The set $\cup_{j\in J}S_j$ is dense in Σ_Γ and*

$$\psi(\Sigma_\Gamma)=\Omega.$$

If $\{\Omega+z: z\in\mathbb{Z}^n\}$ is a tiling of $\mathbb{R}^n$ then we will think of Ω as representing the n-dimensional torus. We let Π be the quotient map from $\mathbb{R}^n$ to $\mathbb{R}^n$ mod $\mathbb{Z}^n$ and study the map

$$\Pi\circ\psi: (\Sigma_\Gamma,\sigma_\Gamma)\to(\Pi\Omega,\Pi\circ\phi).$$

For this reason we will continue our study of the properties of ψ.

Proposition 3.3 *For all $\eta \in \Sigma_\Gamma$ and $N \geq 0$ we have*

$$\psi({\sigma_\Gamma}^N \eta) = \phi^N \psi(\eta) - \phi^N \sum_{k=1}^{N} \phi^{-k} \zeta_u L(\eta^k)$$

and

$$\psi({\sigma_\Gamma}^{-N} \eta) = \phi^{-N} \psi(\eta) + \phi^{-N} \sum_{k=1-N}^{0} \phi^{-k} \zeta_u L(\eta^k).$$

Proof. Let $\eta \in \Sigma_\Gamma$ and $N \geq 0$. Then since ${\sigma_\Gamma}^N \eta^k = \eta^{k+N}$ we have

$$\begin{aligned}
\psi({\sigma_\Gamma}^N \eta) &= \sum_{k=-\infty}^{0} \phi^{-k} \rho_s L(\eta^{k+N}) + \sum_{k=1}^{\infty} \phi^{-k} L(\eta^{k+N}) \\
&= \sum_{k=-\infty}^{N} \phi^{-k+N} \rho_s L(\eta^k) + \sum_{k=N+1}^{\infty} \phi^{-k+N} L(\eta^k) \\
&= \sum_{k=-\infty}^{0} \phi^{-k+N} \rho_s L(\eta^k) + \sum_{k=1}^{\infty} \phi^{-k+N} L(\eta^k) - \zeta_u(\sum_{k=1}^{N} \phi^{-k+N} L(\eta^k)) \\
&= \phi^N \psi(\eta) - \phi^N \sum_{k=1}^{N} \phi^{-k} \zeta_u L(\eta^k).
\end{aligned}$$

Likewise we find

$$\begin{aligned}
\psi({\sigma_\Gamma}^{-N} \eta) &= \sum_{k=-\infty}^{0} \phi^{-k} \rho_s L(\eta^{k-N}) + \sum_{k=1}^{\infty} \phi^{-k} L(\eta^{k-N}) \\
&= \sum_{k=-\infty}^{-N} \phi^{-k-N} \rho_s L(\eta^k) + \sum_{k=1-N}^{\infty} \phi^{-k-N} L(\eta^k) \\
&= \sum_{k=-\infty}^{0} \phi^{-k-N} \rho_s L(\eta^k) + \sum_{k=1}^{\infty} \phi^{-k-N} L(\eta^k) + \zeta_u(\sum_{k=1-N}^{0} \phi^{-k-N} L(\eta^k)) \\
&= \phi^{-N} \psi(\eta) + \phi^{-N} \sum_{k=1-N}^{0} \phi^{-k} \zeta_u L(\eta^k).
\end{aligned}$$

□

Proposition 3.4 *(1) For each $\eta \in \bar{S}_j$ there exists $\bar{\eta} \in \bar{S}_j$ such that $\pi_s \psi(\eta) = \psi(\bar{\eta}) \in \Omega_j \cap E_s$.*

(2) $\phi \pi_s \Omega \subset \pi_s \Omega = \Omega \cap E_s$.

Proof. (1) Let $\eta \in \bar{S}_j$. Let $T \in \mathfrak{T}_j$. Suppose for each $k \geq 0$ we have $\gamma^k(T) \in \mathfrak{T}^k_{j_k}$. Define

$$\bar{\eta}^k = \Theta^{-1}(v_{j_{k-1}}, v_{j_k}, 0)$$

for all $k \geq 1$ and

$$\bar{\eta}^k = \eta^k$$

for all $k \leq 0$. Then

$$\psi(\bar{\eta}) = \sum_{k=-\infty}^{0} \phi^{-k}\rho_s L(\eta^k) = \pi_s\psi(\eta).$$

Since $t(\bar{\eta}) = v_j$, $\bar{\eta} \in \bar{S}_j$. So $\psi(\bar{\eta}) \in \Omega_j \cap E_s$.

(2) If $x \in \pi_s(\Omega)$ then $x \in \pi_s(\Omega_j)$ for some $j \in J$. Hence there exists $\eta \in \bar{S}_j$ such that $\psi(\eta) = x$ and $L(\eta^k) = 0$ for all $k \geq 1$. Hence by Proposition 3.3

$$\begin{aligned} \phi\psi(\eta) = \psi(\sigma_\Gamma\eta) + \zeta_u L(\eta^1) &= \sum_{k=-\infty}^{-1} \phi^{-k}\rho_s L(\eta^{k+1}) \\ &\in \pi_s\Omega. \end{aligned}$$

So $\phi\pi_s(\Omega) \subset \pi_s\Omega$. But

$$\pi_s(\Omega) = \cup_{j\in J}\pi_s(\Omega_j) = \cup_{j\in J}\Omega_j \cap E_s = \Omega \cap E_s.$$

□

Proposition 3.5 *The map ψ is boundedly finite to one.*

Proof. Let $x \in \Omega$ and $\{\eta_i\}_{i\in I} \subset \psi^{-1}(x)$ for some finite indexing set I. Without loss of generality we may assume that the paths η_i and $\eta_{i'}$ are distinct for each pair of distinct i, $i' \in I$. Hence there exists $N \geq 0$ such that the paths

$$\{\eta_i^k\}_{k=-N}^{\infty} \text{ and } \{\eta_{i'}^k\}_{k=-N}^{\infty}$$

are distinct for $i \neq i'$.

Since Ω is compact in $\mathbb{R}^n$ there exists a finite collection of $z \in \mathbb{Z}^n$ such that

$$\Omega \cap (\Omega + z) \neq \emptyset.$$

Hence there exists a bound b_1 such that for all $y \in \Omega$ the number of points in $y + \mathbb{Z}^n$ belonging to Ω is less than or equal to b_1. By Proposition 3.3 for each $i \in I$

$$\phi^{-N-1}x + \phi^{-N-1} \sum_{k=-N}^{0} \phi^{-k}\zeta_u L(\eta_i^k) = \psi({\sigma_\Gamma}^{-N-1}\eta_i) \in \Omega.$$

So for each $i \neq i'$ in I there exists $z \in \mathbb{Z}^n$ such that

$$\psi({\sigma_\Gamma}^{-N-1}\eta_i) = \psi({\sigma_\Gamma}^{-N-1}\eta_{i'}) + z.$$

Hence the number of distinct points in $\{\psi({\sigma_\Gamma}^{-N-1}\eta_i)\}_{i \in I}$ is less than or equal to b_1. Let $y \in \{\psi({\sigma_\Gamma}^{N-1}\eta_i)\}_{i \in I}$. Then

$$\pi_u(y) = \sum_{k=1}^{\infty} \phi^{-k} L({\sigma_\Gamma}^{-N-1}\eta_i^k).$$

By Corollary 1.11 there is a bound $\bar{b}$ on the number of paths which give a representation for $\pi_u(y)$. Hence the number of elements of $\{{\sigma_\Gamma}^{-N-1}\eta_i\}_{i \in I}$ which are mapped by ψ to the same point, is less than or equal to $\bar{b}$. It follows that $|I| \leq b_1\bar{b}$. Since x was arbitrary in Ω we have that $|\psi^{-1}(x)| \leq b_1\bar{b}$ for all $x \in \Omega$.□

Periodic tilings of $\mathbb{R}^n$ mod $\mathbb{Z}^n$.

Definition. A finite collection of compact sets $\{C_1, C_2, \ldots, C_k\}$ in $\mathbb{R}^n$ induces a periodic tiling of $\mathbb{R}^n$ mod $\mathbb{Z}^n$ if

(1) $(\cup_{i=1}^k C_i)$ mod $\mathbb{Z}^n = \mathbb{R}^n$ mod $\mathbb{Z}^n$,

(2) each C_i is the closure of its interior, and

(3) for all $z \in \mathbb{Z}^n$ if

$$(C_i + z) \cap \operatorname{Int}(C_j) \neq \emptyset$$

then $z = 0$ and $i = j$.

For each $j \in J$ we have that Ω_j is a compact set. It turns out that Ω_j is also the closure of its interior.

Proposition 3.6 *(1) For each $j \in J$, $\pi_s(\Omega_j)$ is the closure of its interior.*

(2) For each $j \in J$, Ω_j is the closure of its interior.

(3) Ω is the closure of its interior and $\pi_s(\Omega)$ is the closure of its interior.

Proof. Since

$$\bigcup_{z \in \mathbf{Z}^n} (\cup_{j \in J}(\Omega_j + z)) = \mathbb{R}^n$$

we may apply the Baire category theorem. That is for some $j \in J$ we have that Ω_j has non-empty interior. But

$$\Omega_j = \pi_s(\Omega_j) \oplus (T - c(T))$$

for any $T \in \mathfrak{T}_j$, so Ω_j has non-empty interior if and only if $\pi_s(\Omega_j)$ has non-empty interior. It follows that by showing (1) we immediately get (2). For each $j' \in J$, let $U_{j'} = \text{Int}(\pi_s \Omega_{j'})$. By the above remarks $U_j \neq \emptyset$. For each $x \in U_j$ there exists $\eta_x \in \bar{S}_j$ such that $\psi(\eta_x) = x$ and $L(\eta_x^k) = 0$ for $k \geq 1$. Since $\mathfrak{T}$ is p-mixing, for each $j' \in J$ there exists a path $\{\epsilon^k\}_{k=1}^{Np}$ in Γ such that $s(\epsilon^1) = j$ and $t(\epsilon^{Np}) = j'$. Let $\bar{\epsilon} \in \bar{S}_j$ be defined so that

$$\begin{aligned} \bar{\epsilon}^k = \eta_x^k \qquad & \text{for } k \leq 0, \\ \bar{\epsilon}^k = \epsilon^k \qquad & \text{for } 1 \leq k \leq Np, \text{ and} \\ L(\bar{\epsilon}^k) = 0 \qquad & \text{for } k > Np. \end{aligned}$$

Then

$$\begin{aligned} \phi^{Np}(\psi(\bar{\epsilon})) - \phi^{Np} \sum_{k=1}^{Np} \phi^{-k} \zeta_u L(\bar{\epsilon}^k) &= \psi({\sigma_\Gamma}^{Np} \bar{\epsilon}) \\ &= \sum_{k=-\infty}^{0} \phi^{-k} \rho_s L(\bar{\epsilon}^{k+Np}) + \sum_{k=1}^{\infty} \phi^{-k} L(\bar{\epsilon}^{k+Np}) \\ &= \sum_{k=-\infty}^{-Np} \phi^{-k} \rho_s L(\eta_x^{k+Np}) + \sum_{k=1-Np}^{0} \phi^{-k} \rho_s L(\epsilon^{k+Np}) \\ &= \phi^{Np} \psi(\eta_x) + \sum_{k=1-Np}^{0} \phi^{-k} \rho_s L(\epsilon^{k+Np}). \end{aligned}$$

Since $t(\epsilon^{Np}) = v_{j'}$, we have $\psi(\sigma_\Gamma{}^{Np}\bar{\epsilon}) \in \pi_s\Omega_{j'}$. Since x was arbitrary in U_j,

$$\phi^{Np}U_j + \sum_{k=1-Np}^{0} \phi^{-k}\rho_s L(\epsilon^{k+Np}) \subset U_{j'}.$$

So $U_{j'} \neq \emptyset$ for all $j' \in J$.

Let T_{j_0} be a generator for $\mathfrak{T}$. Then there is a unique generator T_{j_1} for $\mathfrak{T}$ such that

$$T_{j_0} = \gamma^p(T_{j_1}).$$

For each N let T_{j_N} be the unique generator for $\mathfrak{T}$ such that

$$T_{j_0} = \gamma^{Np}(T_{j_N}).$$

Then for $T \in \mathfrak{T}_{j_N}$, we have that $\phi^{Np}c(T) = c(\gamma^{Np}T) \in c(\mathfrak{T}_{j_0})$. Hence

$$\bigcup_{N=0}^{\infty} \phi^{Np}U_{j_N} \subset U_{j_0}.$$

Since $\phi|_{E_s}^{-1}$ is expansive, 0 belongs to the closure of U_{j_0}. Since T_{j_0} was an arbitrary generator for $\mathfrak{T}$, for all generators $T_j \in \mathfrak{T}_j$, we have that 0 belongs to the closure of U_j.

Let $j \in J$ and $T \in \mathfrak{T}_j$. Then for some generator T_{j_0} there exists an $N \geq 0$ such that $T \subset \phi^{Np}T_{j_0}$ in which case

$$c(T) = \phi^{Np}c(T_{j_0}) + c(T).$$

So for all $T' \in \mathfrak{T}_{j_0}$,

$$\phi^{Np}c(T') + c(T) \in c(\mathfrak{T}_j).$$

Hence $\phi^{Np}U_{j_0} + \rho_s c(T) \subset U_j$. Since 0 belongs to the closure of $\phi^{Np}U_{j_0}$ we have that $\rho_s c(T)$ belongs to the closure of U_j. But T was arbitrary in $\mathfrak{T}_j$ and

$$\pi_s\Omega_j = \text{Clos}\{\rho_s c(T): T \in \mathfrak{T}_j\}.$$

Hence $\pi_s\Omega_j \subset \text{Clos}(U_j)$. This proves (1) and (2).

Finally we note that $\Omega = \cup_{j\in J}\Omega_j$ and $\pi_s\Omega = \cup_{j\in J}\pi_s\Omega_j$. Since both unions are finite we obtain (3).□

Theorem 3.7 *The collection $\{\Omega_j\}_{j\in J}$ induces a periodic tiling of $\mathbb{R}^n$ mod $\mathbb{Z}^n$ if and only if for all $z \in \mathbb{Z}^n$ and $j \in J$ such that $X_u - z \cap Int(\Omega_j) \neq \emptyset$ we have that $z \in \zeta_u c(\mathfrak{T}_j)$.*

Proof. Suppose the sets Ω_j do induce a periodic tiling of $\mathbb{R}^n$ mod $\mathbb{Z}^n$. Let $z \in \mathbb{Z}^n$ and suppose $X_u - z \cap \text{Int}(\Omega_j) \neq \emptyset$ for some $j \in J$. Then for some $j' \in J$ and T in $\mathfrak{T}_{j'}$ we have that

$$(T - z) \cap \text{Int}(\Omega_j) \neq \emptyset.$$

Since $T \in \mathfrak{T}_{j'}$ we have that

$$(T - \zeta_u c(T)) \subset \Omega_{j'}.$$

Moreover,

$$(T - \zeta_u c(T) + (\zeta_u c(T) - z)) \cap \text{Int}(\Omega_j) \neq \emptyset$$

so

$$(\Omega_{j'} + (\zeta_u c(T) - z)) \cap \text{Int}(\Omega_j) \neq \emptyset.$$

By our hypothesis we have $\zeta_u c(T) = z$ and $j = j'$. Hence $z \in \zeta_u c(\mathfrak{T}_j)$.

Conversely, suppose $X_u - z \cap \text{Int}(\Omega_j) \neq \emptyset$ implies that $z \in c(\mathfrak{T}_j)$. Let $z \in \mathbb{Z}^n$ and $j, k \in J$ and suppose that

$$(\Omega_j + z) \cap \text{Int}(\Omega_k) \neq \emptyset.$$

Since $\Omega_{u,j}$ is dense in Ω_j for some $T \in \mathfrak{T}_j$ we have that

$$(T - \zeta_u c(T) + z) \cap \text{Int}(\Omega_k) \neq \emptyset.$$

Since T is the closure of its interior we have

$$(\text{Int}(T) - \zeta_u c(T) + z) \cap \text{Int}(\Omega_k) \neq \emptyset.$$

By the hypothesis $\zeta_u c(T) - z \in c(\mathfrak{T}_k)$. Hence for some $T' \in \mathfrak{T}_k$ we have

$$c(T') = \zeta_u c(T) - z.$$

Moreover,

$$(\text{Int}(T) - \zeta_u c(T) + z) \cap (T' - \zeta_u c(T')) \neq \emptyset.$$

Hence $\mathrm{Int}(T) \cap T' \neq \emptyset$. So $T = T'$ and $\zeta_u c(T) = \zeta_u c(T')$. It follows that $j = k$ and $z = 0$.□

In general Theorem 3.7 may or may not be useful to check if we have a periodic tiling of $\mathbb{R}^n$ mod $\mathbb{Z}^n$. The set X_u may not be connected so just checking if

$$X_u \cap \mathrm{Int}(\Omega_j) \neq \emptyset$$

may be difficult. Fortunately, the tilings that we will be interested in have more structure than the general case that we have been using.

Markov tilings

Suppose for some $j \in J$, there exists a tile T such that

$$T \in \cap_{i \in \mathbb{Z}_p} \mathfrak{T}_j^i.$$

Since $\mathfrak{F}(\mathfrak{T})$ is uniformly subdividing, the pattern of ϕT is the same in every $\mathfrak{T}^i \in \mathfrak{F}(\mathfrak{T})$. Hence there is a unique collection of tiles $\{T^\alpha\}_{\alpha \in \mathcal{E}_j}$ such that

$$\phi T = \cup_{\alpha \in \mathcal{E}_j} T^\alpha$$

and if $\alpha \in \mathcal{E}_j^{k_\alpha}$ then

$$T^\alpha \in \cap_{i \in \mathbb{Z}_p} \mathfrak{T}_{k_\alpha}^i.$$

It follows that the subset

$$\cup_{k=0}^{\infty} \phi^k T \subset X_u$$

is tiled in the same way by all of the tilings in $\mathfrak{F}(\mathfrak{T})$. For each $j \in J$ let

$$\mathcal{N}(\mathfrak{T}_j) = \{T \in \mathfrak{T}_j - \mathfrak{T}_j^i \colon \text{ for some } i \in \mathbb{Z}_p\}.$$

In other words $\mathcal{N}(\mathfrak{T}_j)$ is the set of all tiles in $\mathfrak{T}_j$ which do not belong to $\mathfrak{T}_j^i$ for some $i \in \mathbb{Z}_p$. Let

$$\mathcal{N}(\mathfrak{T}) = \cup_{j \in J} \mathcal{N}(\mathfrak{T}_j).$$

Let

$$\mathcal{N}(X_u) = \cup_{T \in \mathcal{N}(\mathfrak{T})} T.$$

If $\mathcal{N}(X_u) \neq X_u$ then $\mathcal{N}(\mathfrak{T}) \neq \mathfrak{T}$ and the portion of $\mathfrak{T}$ which is shared by each $\mathfrak{T}^i \in \mathfrak{F}(\mathfrak{T})$ is almost a self similar tiling of $X_u - \mathcal{N}(X_u)$ with respect to subdivisions and quasi-homogeneity. In particular, since $\cup_{k=0}^{\infty}\phi^k T$ is unbounded in X_u for any tile T, $X_u - \mathcal{N}(X_u)$ contains every bounded pattern found in $\mathfrak{T}$.

Definition. If $\{\Omega_j\}_{j\in J}$ yields a periodic tiling of $\mathbb{R}^n \bmod \mathbb{Z}^n$ and $\mathcal{N}(X_u) \neq X_u$ then we will call $\mathfrak{T}$ a **Markov tiling.**

Suppose $\mathfrak{T}$ is a Markov tiling. Then if $T \in \mathfrak{T}$ and $T \subset X - \mathcal{N}(X_u)$ then ϕT subdivides according to the rules for $\mathfrak{T}$ into tiles in $\mathfrak{T}$. If we could simply ignore $\mathcal{N}(X_u)$ then we could say $\mathfrak{T}$ is self similar on $X_u - \mathcal{N}(X_u)$. Of course

$$\phi(X_u - \mathcal{N}(X_u)) \subset X_u - \mathcal{N}(X_u)$$

and the inclusion is usually strict. So if $\mathcal{N}(X_u) \neq \emptyset$ then $\mathfrak{T}$ isn't really self similar.

The periodic self similar tilings that we will be using to construct Markov partitions will be Markov tilings. In general the tiled space X_u will be a semigroup. When X_u is a semigroup we can determine if $\{\Omega_j\}_{j\in J}$ induces a periodic tiling by considering the lattice points in $\mathbb{Z}^n$ which are close to X_u.

Definition. For each $i \in \mathbb{Z}_p$ let P_i be the union of the generators for $\mathfrak{T}^i$. Let $[P_i]$ be the set of $\pi_u(\mathbb{Z}^n)$-translates of P_i which have the same pattern as P_i in $\mathfrak{T}^i$.

If $P_i + g \in [P_i]$ then each tile in $P_i + g$ is of the same type as a generator for $\mathfrak{T}^i$. Moreover each tile in $P_i + g$ has g as its control point. For each $P \in [P_i]$ let $c_i(P)$ denote this control point, so that $P_i + c_i(P) = P$. Let $c_i[P_i]$ denote the set of all control points for the tiles in $P \in [P_i]$. It turns out that if the elements of $\mathbb{Z}^n$ which project onto $\cup_{i\in\mathbb{Z}_p} c_i[P_i]$ include all the elements of $\mathbb{Z}^n$ within a certain bounded region of X_u then we have a periodic tiling.

Definition. Let

$$X_s = \bigcup_{k=0}^{\infty} \phi^{-k}\pi_s\Omega.$$

Since $\pi_s\Omega = \{\sum_{k=-\infty}^{0} \phi^{-k}\rho_s L(\eta^k) : \eta \in \Sigma_\Gamma\}$ we have

$$X_s = \left\{ \sum_{k=-\infty}^{M} \phi^{-k}\rho_s L(\eta^k) : \eta \in \Sigma_\Gamma,\ M \in \mathbb{Z} \right\}.$$

Note that $\phi X_s = X_s$ and X_s is the closure of its interior.

Lemma 3.8 *For every $M \in \mathbb{Z}$ there exists $N \in \mathbb{Z}$ such that if $\eta \in \Sigma_\Gamma$, $M' \in \mathbb{Z}$, and*

$$\sum_{k=-\infty}^{M'} \phi^{-k}\rho_s L(\eta^k) \in \phi^N \pi_s \Omega$$

then $M' \leq M$.

Proof. Let $\eta \in \Sigma_\Gamma$ and $M' \in \mathbb{Z}$. Suppose

$$x = \sum_{k=-\infty}^{M'} \phi^{-k}\rho_s L(\eta^k).$$

If $M' \leq 0$ then $x \in \pi_s\Omega$. If $M' > 0$ then

$$x = \sum_{k=-\infty}^{0} \phi^{-k}\rho_s L(\eta^k) + \sum_{k=1}^{M'} \phi^{-k}\rho_s L(\eta^k) \in \Omega - \sum_{k=1}^{M'} \phi^k \zeta_u L(\eta^k).$$

Assume $M' > 0$ and let $z = \sum_{k=1}^{M'} \phi^{-k}\zeta_u L(\eta^k)$. By an application of Corollary 1.11 there is a bound on the number of finite paths in Γ given by $\{\eta^k\}_{k\in I}$ such that $\pi_u(z) = \sum_{k\in I} \phi^{-k} L(\eta^k)$. Hence there exists M_z such that $M' \leq M_z$.

Since Ω is compact there are at most a finite number of lattice points $z \in \mathbb{Z}^n$ such that

$$(\Omega - z) \cap \pi_s\Omega \neq \emptyset.$$

Hence there exists an $M_0 \geq 0$ such that if $\{\eta^k\}_{k=1}^{M'}$ is a path in Γ and

$$\left(\Omega - \sum_{k=1}^{M'} \phi^{-k}\zeta_u L(\eta^k)\right) \cap \pi_s\Omega \neq \emptyset$$

then $M' \leq M_0$. It follows that if $x \in \pi_s(\Omega)$ and $\eta \in \Sigma_\Gamma$, $M' \in \mathbb{Z}$ such that

$$x = \sum_{k=-\infty}^{M'} \phi^{-k}\rho_s L(\eta^k)$$

then $M' \leq M_0$. Moreover for all $N \in \mathbb{Z}$ if $x \in \phi^N \pi_s \Omega$ then $M' \leq M_0 - N$. Let $N = M_0 - M$.□

Lemma 3.9 *For each $N \geq 0$, $\phi^N c_i[P_i] \subset c_{i+N}[P_{i+N}]$.*

Proof. Let $P_i + g \in [P_i]$. If T_j is a generator for $\mathfrak{T}^i$ then $T_j + g \subset P_i + g$. Since $\gamma_i(T_j)$ is a generator for $\mathfrak{T}^{i+1}$, $\gamma_i T_j + \phi g$ is a tile in $\mathfrak{T}^{i+1}$ of the same type as a generator in $\mathfrak{T}^{i+1}$ and

$$\phi c_i(T_j + g) = c_{i+1}(\gamma_i(T_j)) + \phi g = \phi g.$$

There is a one-one correspondence between the generators in $\mathfrak{T}^i$ and there γ_i images. So

$$P_{i+1} + \phi g \subset \phi(P_i + g)$$

and $P_{i+1} + \phi g$ has the same pattern in $\mathfrak{T}^{i+1}$ as P_{i+1}. So $\phi g \in c[P_{i+1}]$. We apply induction and obtain the result.□

Theorem 3.10 *Suppose that X_u is a semigroup and $\mathcal{N}(X_u)$ is bounded. Suppose for all*

$$z \in -X_s \oplus X_u \cap \mathbb{Z}^n$$

there exists $N \geq 0$ such that $\phi^N z \in \cup_{i \in \mathbb{Z}_p} \zeta_u c_i[P_i]$. Then $\mathfrak{T}$ is a Markov tiling.

What this theorem essentially says is that if every translate of Ω by a lattice point in $\mathbb{Z}^n$ which is close to X_u intersects X_u in a set with the same pattern as P_i for some $i \in \mathbb{Z}_p$ then we have a periodic tiling of $\mathbb{R}^n$ mod $\mathbb{Z}^n$. It turns out that many tilings are constructed with this property built in. So checking if we have a Markov tiling becomes trivial.

Proof. The basic idea is to show that if for some $j, k \in J$ and $z \in \mathbb{Z}^n - \{0\}$ we have

$$\text{Int}(\Omega_j) \cap \text{Int}(\Omega_k) + z \neq \emptyset$$

then there exists $z' \in \mathbb{Z}^n$ such that

$$X_u - z' \cap \text{Int}(\pi_s(\Omega_j)) \neq \emptyset$$

and $z' \notin \zeta_u c(\mathfrak{T}_j)$. We construct a nonempty subset of $(-\text{Int}(\pi_s(\Omega_j)) \oplus X_u) \cap \mathbb{Z}^n$ in which every point is not in $\zeta_u c(\mathfrak{T}_j)$. We then use the hypothesis of the theorem to show that such a set cannot exist.

We begin by constructing an open subset U of X_s which contains 0 in its closure and $\phi U \subset U$. We will show if $z \in (-U \oplus X_u) \cap \mathbb{Z}^n$ then $z \in \cup_{i\in\mathbb{Z}_p}\zeta_u c_i[P_i]$. By hypothesis, every $z \in (-Xs \oplus X_u) \cap \mathbb{Z}^n$ is the eventual preimage of a point in $\cup_{i\in\mathbb{Z}_p}\zeta_u c_i[P_i]$. So for every such z there exists a finite path $\{\eta^k\}_{k=-N}^{M}$ in Γ with source in $\{v_j\}_{j\in J_0}$ such that

$$z = \sum_{k=-N}^{M} \phi^{-k}\zeta_u L(\eta^k).$$

By Lemma 3.8 there exists N_0 such that if $-\pi_s z \in \phi^{N_0}\pi_s\Omega$ then $M \leq 0$. Hence for all $z \in (-\varphi^{N_0}\pi_s\Omega \oplus X_u) \cap \mathbb{Z}^n$ we have $z \in \cup_{i\in\mathbb{Z}_p}\zeta_u c_i(\mathfrak{T}^i)$.

By hypothesis there are at most a finite number of control points which lie in $\mathcal{N}(X_u)$ belonging to any of the tilings . Hence there exists $N_1 \geq N_0$ such that if

$$z \in (-\phi^{N_1}\pi_s\Omega \oplus X_u) \cap \mathbb{Z}^n$$

we have $z \in \zeta_u c(\mathfrak{T} - \mathcal{N}(\mathfrak{T}))$.

For each $T \in \mathfrak{T} - \mathcal{N}(\mathfrak{T})$ let P_T be the union of the tiles in $\mathfrak{T}$ containing $c(T)$. Since $\mathfrak{T}$ is quasi-homogeneous the set of distinct patterns in

$$\{P_T : T \in \mathfrak{T} - \mathcal{N}(\mathfrak{T})\}$$

is finite. Hence there exists a finite collection of representatives $\{P_{T^k}\}_{k\in K}$ such that for every tile T in $\mathfrak{T} - \mathcal{N}(\mathfrak{T})$ there exists a $k \in K$ such that P_{T^k} has the same pattern as P_T. Since $\mathfrak{T}$ is locally finite there are only a finite number of control points in $\cup_{k\in K}P_{T^k}$. By hypothesis, for each $T \subset \cup_{k\in K}P_{T^k}$ there exists N_T such that

$$\phi^{N_T}c(T) \in \cup_{i\in\mathbb{Z}_p}c[P_i].$$

By Lemma 3.9 for all $N \geq 1$, $\phi^{N\cdot N_T}c(T) \in \cup_{i\in\mathbb{Z}_p}c[P_i]$. Let

$$N_2 = \text{l.c.m.}\{N_T : T \subset \cup_{k\in K}P_{T^k}\}.$$

Then for every $T \subset \cup_{k\in K}P_{T^k}$ we have

$$\phi^{N_2}c(T) \in \cup_{i\in\mathbb{Z}_p}c[P_i].$$

Let $T \in \mathfrak{T} - \mathcal{N}(\mathfrak{T})$. Then for some $k \in K$ we have that P_{T^k} has the same pattern as P_T. Hence $\phi^{N_2} P_T$ has the same pattern as $\phi^{N_2} P_{T^k}$. Since P_T contains every tile which contains $c(T)$, the set $\phi^{N_2} P_T$ contains every tile which contains $\phi^{N_2} c(T)$. Hence $P_i + \phi^{N_2} c(T) \subset \phi^{N_2} P_T$ for some $i \in \mathbb{Z}_p$ and $\phi^{N_2} c(T) \in \cup_{i \in \mathbb{Z}_p} c[P_i]$. Let $U = \phi^{N_1+N_2} \mathrm{Int}(\pi_s \Omega)$. If

$$z \in (-U \oplus X_u) \cap \mathbb{Z}^n$$

then $z \in \cup_{i \in \mathbb{Z}_p} \zeta_u c[P_i]$.

Since $\phi|_{E_u}$ is expansive, $X_u - \mathcal{N}(X_u)$ contains arbitrarily large connected components. If $T \in \mathfrak{T} - \mathcal{N}(\mathfrak{T})$ then choose N sufficiently large so that $T \subset \phi^N P_i$ for each $i \in \mathbb{Z}_p$. Then for all $P \in \cup_{i \in \mathbb{Z}_p} [P_i]$, $T + \phi^N c(P)$ is a tile in $\mathfrak{T}$ of the same type as T. By the quasi-homogeneity of $\mathfrak{T}$, each sufficiently large component of $X_u - \mathcal{N}(X_u)$ contains the pattern of $\phi^N P_i$ in its interior for some $i \in \mathbb{Z}_p$. Note that

$$c(T + \phi^N c(P)) = c(T) + \phi^N c(P).$$

It follows that

$$\|\rho_s c(T + \phi^N c(P)) - \rho_s c(T)\| \leq \varrho^{-N} \|\rho_s c(P)\|.$$

Hence for every $T \in \mathfrak{T} - \mathcal{N}(\mathfrak{T})$ there exists $T' \in \mathfrak{T} - \mathcal{N}(\mathfrak{T})$ of the same type as T and contained in an arbitrarily large connected component of $X_u - \mathcal{N}(X_u)$. Moreover, for any $\delta > 0$, we may choose T' so that

$$\|\rho_s c(T') - \rho_s c(T)\| < \delta.$$

If V is a bounded neighborhood of Ω then we may choose the connected component containing T' so that

$$(X_u - \mathcal{N}(X_u) - \zeta_u c(T')) \cap V = (E_u - \zeta_u c(T')) \cap V.$$

Suppose for some j, $k \in J$ and $z \in \mathbb{Z}^n - \{0\}$ we have

$$(\mathrm{Int}(\Omega_j) + z) \cap \mathrm{Int}(\Omega_k) \neq \emptyset.$$

Then for some $T \in \mathfrak{T}_k$ we have

$$(T - \zeta_u c(T)) \cap (\ \mathrm{Int}(\Omega_j) + z) \neq \emptyset.$$

By the above remarks we may choose T so that $T \notin \mathcal{N}(\mathfrak{T}_k)$ and

$$(X_u - \mathcal{N}(X_u) - \zeta_u c(T)) \cap (\Omega_j + z) = (E_u - \zeta_u c(T)) \cap (\Omega_j + z).$$

So

$$(X_u - \mathcal{N}(X_u) - \zeta_u c(T) - z) \cap \pi_s(\Omega_j) = \{\pi_s(-\zeta_u c(T) - z)\}.$$

Since

$$(X_u - \mathcal{N}(X_u) - \zeta_u c(T) - z) \cap \operatorname{Int}(\Omega_j) \neq \emptyset$$

and Ω_j is a rectangle,

$$\pi_s(-\zeta_u c(T) - z) \in \operatorname{Int}(\pi_s(\Omega_j)) \subset X_s.$$

Moreover z is within a bounded region of 0. By choosing the connected component of $X_u - \mathcal{N}(X_u)$ which contains T so that T lies within a large connected ball in $X_u - \mathcal{N}(X_u)$, we have that

$$c(T) + \pi_u(z) \in X_u - \mathcal{N}(X_u).$$

Also $c(T) + \pi_u(z) \notin c(\mathfrak{T}_j)$. For if not then

$$(X_u - \zeta_u c(T) - z) \cap \operatorname{Int}(\Omega_j) = \operatorname{Int}(T') - \zeta_u c(T')$$

for some $T' \in \mathfrak{T}_j$. Hence

$$(T - \zeta_u c(T) - z) \cap (\operatorname{Int}(T') - \zeta_u c(T')) \neq \emptyset$$

and $T \cap \operatorname{Int}(T') \neq \emptyset$. So $T = T'$. But then $z = 0$ and $j = k$.

It follows that there exists $j \in J$ and $z \in (-X_s \oplus (X_u - \mathcal{N}(X_u))) \cap \mathbb{Z}^n$ such that

$$(X_u - z) \cap \operatorname{Int}(\pi_s(\Omega_j)) \neq \emptyset$$

and $z \notin \zeta_u c(\mathfrak{T}_j)$.

Choose M_1 sufficiently large so that $\pi_u(z) \in \operatorname{Int}(\phi^{M_1} P_i)$ for each $i \in \mathbb{Z}_p$. Then for all $P \in \cup_{i \in \mathbb{Z}_p}[P_1]$ we have $\zeta_u \phi^{M_1} c(P) + z \notin \zeta_u c(\mathfrak{T}_j)$. Hence for all $w \in (-\phi^{M_1} U \oplus X_u) \cap \mathbb{Z}^n$, we have $w + z \notin \zeta_u c(\mathfrak{T}_j)$ and

$$((-\phi^{M_1} U \oplus X_u) + z) \cap \zeta_u c(\mathfrak{T}_j) = \emptyset. \tag{3.1}$$

Since $-\pi_s(z) \in \mathrm{Int}(\pi_s \Omega_j)$ and $0 \in \mathrm{Clos}(\phi^{M_1} U)$ there exists $T'' \in \mathfrak{T}_j - \mathcal{N}(\mathfrak{T}_j)$ such that

$$\rho_s c(T'') \in -\pi_s(z) + \phi^{M_1} U.$$

Choose M_2 sufficiently large so that

$$T'' \subset \phi^{M_2} P_i,$$

for all $i \in \mathbb{Z}_p$ and so that for all $P \in \cup_{i \in \mathbb{Z}_p} [P_i]$

$$\rho_s(\phi^{M_2} c(P) + c(T'')) \in \left((\phi^{M_1} U - \pi_s(z)) \cap \rho_s c(\mathfrak{T}_j) \right).$$

Since X_u is a semigroup, and $\pi_u(z) \in X_u$,

$$X_u + \pi_u(z) \subset X_u.$$

Since $\mathfrak{T}$ is quasi-homogeneous, there exists $P \in \cup_{i \in \mathbb{Z}_p} [P_i]$ such that

$$\phi^{M_2} P \subset X_u + \pi_u(z).$$

Hence $\phi^{M_2} c(P) + c(T'') \in X_u + \pi_u(z)$. So

$$\zeta_u \phi^{M_2} c(P) + \zeta_u c(T'') \in \left((-\phi^{M_1} U \oplus X_u) + z \right) \cap \zeta_u c(\mathfrak{T}_j)$$

contradicting equation 3.1. It follows that $z \in \zeta_u c(\mathfrak{T}_j)$. This proves the theorem.□

Corollary 3.11 *Suppose that X_u is a semigroup, $\mathcal{N}(X_u)$ is bounded, and $\mathfrak{T}$ has only one generator. If for all $z \in -X_s \oplus X_u \cap \mathbb{Z}^n$ we have $z \in \cup_{k=0}^{\infty} \phi^{-k} \zeta_u c(\mathfrak{T})$ then $\mathfrak{T}$ is a Markov tiling.*

Proof. This follows from the fact that γ is a generating tile map. If $\phi^N z \in \zeta_u c(\mathfrak{T})$ then $\phi^{(N+k_0)} z$ projects to the control point of a tile of the same type as the generator.□

At this point the reader might want to glance through Example 5.1 on page 79. This is a simple application of Corollary 3.11.

Proposition 3.12 *If $\mathcal{N}(X_u) \neq X_u$ then for all $T \in \mathfrak{T}$ and $\delta > 0$ there exists $T' \in \mathfrak{T} - \mathcal{N}(\mathfrak{T})$ such that T and T' have the same tile type and*

$$\|\rho_s c(T) - \rho_s c(T')\| < \delta.$$

Proof. This follows from the proof of Theorem 3.10. Since $\mathfrak{T}$ is self similar with expansion map ϕ^p and P_0 contains all of the generators for $\mathfrak{T}$ we have $T \subset \phi^{Np} P_0$ for some $N \geq 0$. Since $\mathcal{N}(X_u) \neq X_u$ we apply quasi-homogeneity to find a set $P \in [P_0]$ such that for any $N > 0$ we have $\phi^{Np} P \subset X_u - \mathcal{N}(X_u)$. We choose N sufficiently large so that $T + \phi^{Np} c(P)$ is a tile in $\mathfrak{T} - \mathcal{N}(\mathfrak{T})$ of the same type as T and at the same time $\|\rho_s \phi^{Np} c(P)\| < \delta$.□

This tells us that if $\mathfrak{T}$ is a Markov tiling then our definition of Ω_j could have been made just using tiles in $\mathfrak{T} - \mathcal{N}(\mathfrak{T})$. That is

$$\pi_s(\Omega_j) = \text{Clos}\{\rho_s c(T) : T \in \mathfrak{T}_j - \mathcal{N}(\mathfrak{T}_j)\}.$$

Chapter 4

SYMBOLIC REPRESENTATIONS

We have seen that if $\mathfrak{T}$ is a Markov tiling with expansion map ϕ, then we may *wrap* $\mathfrak{T}$ inside $\mathbb{R}^n$ mod $\mathbb{Z}^n$ in such a way that the tiles of the same type form almost disjoint rectangles $\{\Omega_j\}_{j\in J}$. This is the first step to construct a Markov partition for ϕ mod $\mathbb{Z}^n$. To obtain a Markov partition from $\{\Omega_j\}_{j\in J}$ we must have subdivision rules which reflect the behavior of ϕ mod $\mathbb{Z}^n$ on Ω mod $\mathbb{Z}^n$. The subdivision rules for the rectangles in Ω will agree with the subdivision rules for the tiles in $\mathfrak{T}$. The Markov partition will come from the intersections of sets in $\{\phi\Omega_j\}_{j\in J}$ and sets in $\{\Omega_j\}_{j\in J}$. Once we have a Markov partition for ϕ mod $\mathbb{Z}^n$ we will be able to represent the dynamical system ($\mathbb{R}^n$ mod $\mathbb{Z}^n, \phi$ mod $\mathbb{Z}^n$) as a shift of finite type.

Almost homeomorphic factor maps.

In [2] Adler and Marcus outline the essential requirements for a symbolic system to represent an abstract dynamical system. We review these ideas here. We then show that if $\mathfrak{T}$ is a Markov tiling then ψ may be extended to a map from Σ_Γ to $\mathbb{R}^n$ mod $\mathbb{Z}^n$ in such a way that this extension meets the requirements for a symbolic representation. The following definitions come from [2, pp. 5–6].

Definition. A dynamical system (X, f) is said to be **ergodically supported** if there exists an ergodic f-invariant probability measure μ_x which is positive on open sets. Such measures are called **ergodically supporting.** A subset $N \subset X$ is called **universally null** if it has measure zero with respect to all ergodically supporting measures.

Definition. If (X, f) and (Y, h) are ergodically supported systems and π is a map from Y into X then π is called an **almost homeomorphic factor map** of Y into X if

(1) π is onto,

(2) π is boundedly finite to one,

(3) π is continuous,

(4) $f \circ \pi = \pi \circ h$, and

(5) π maps $Y - \pi^{-1}(N)$ one to one onto $X - N$ for some f invariant universally null set N.

From now on we assume that $\mathfrak{T}$ is a Markov tiling of X_u and continue with the notation from Chapter 3. Define $\Pi\colon \mathbb{R}^n \to \mathbb{R}^n \bmod \mathbb{Z}^n$ to be the quotient map which identifies points in $\mathbb{R}^n$ modulo the integer lattice. Since Ω is a fundamental region for a periodic tiling of $\mathbb{R}^n \bmod \mathbb{Z}^n$ a set $U \subset \Pi\Omega$ is open if and only if $\Pi^{-1}U \cap \Omega$ is open in the relative topology of Ω. Let $\hat{\phi} : \Pi\Omega \to \Pi\Omega$ by $\hat{\phi}(\Pi x) = \Pi(\phi x)$. Let $\hat{\psi}\colon \Sigma_\Gamma \to \Pi\Omega$ by $\hat{\psi}(\eta) = \Pi\psi(\eta)$. Our main theorem for this chapter is the following.

Theorem. *The map $\hat{\psi}$ is an almost homeomorphic factor map from $(\Sigma_\Gamma, \sigma_\Gamma)$ to $(\Pi\Omega, \hat{\phi})$.*

To prove this we will begin by returning to Chapter 3 and recall the properties of ψ. From these it will follow that

(1) $\hat{\psi}$ is onto,

(2) $\hat{\psi}$ is boundedly finite to one,

(3) $\hat{\psi}$ is continuous, and

(4) $\hat{\phi} \circ \hat{\psi} = \hat{\psi} \circ \sigma_\Gamma$.

We then note that the non-doubly transitive points in $(\Pi\Omega, \hat{\phi})$ form a universally null set. We construct a partition for $\Pi\Omega$ which will yield a generating set for a sigma-algebra on $\Pi\Omega$ which is metrically equivalent to the Borel sigma algebra. We show that every point whose orbit misses the boundary of this partition has a unique preimage in Σ_Γ. Finally we show that the doubly transitive points have orbits which miss the boundary of the partition. We will call this partition a *Markov partition.*

The properties of $\hat{\psi}$.

Recall from the last chapter that ψ is a continuous surjection from Σ_Γ to Ω. Hence $\Pi \circ \psi$ is a continous surjection from Σ_Γ to $\Pi\Omega$.

Lemma 4.1 *For all $k \in \mathbb{Z}$*

$$\hat{\phi}^k \circ \hat{\psi} = \hat{\psi} \circ {\sigma_\Gamma}^k.$$

Proof. By Proposition 3.3 for all $\eta \in \Sigma_\Gamma$ and $N \geq 0$

$$\psi \circ {\sigma_\Gamma}^N(\eta) = \phi^N \circ \psi(\eta) - \phi^N \sum_{k=1}^{N} \phi^{-k} \zeta_u L(\eta^k)$$

and

$$\psi \circ {\sigma_\Gamma}^{-N}(\eta) = \phi^{-N} \circ \psi(\eta) + \phi^{-N} \sum_{k=1-N}^{0} \phi^{-k} \zeta_u L(\eta^k).$$

Since

$$\phi^{-N} \sum_{k=1-N}^{0} \phi^{-k} \zeta_u L(\eta^k) \text{ and } \phi^N \sum_{k=1}^{N} \phi^{-k} \zeta_u L(\eta^k) \in \mathbb{Z}^n$$

we have

$$\begin{aligned} \hat{\psi} \circ {\sigma_\Gamma}^k(\eta) &= \Pi(\psi \circ {\sigma_\Gamma}^k(\eta)) \\ &= \Pi(\phi^k \circ \psi(\eta)) \\ &= \hat{\phi}^k \circ \hat{\psi}(\eta) \end{aligned}$$

for all $k \in \mathbb{Z}$.□

Lemma 4.2 *$\hat{\psi}$ is boundedly finite to one.*

Proof. Let $x \in \Omega$ and consider the set $\hat{\psi}^{-1}(\Pi x)$. For each η in $\hat{\psi}^{-1}(\Pi x)$ we have $\psi(\eta) = x + g$ for some $g \in \pi_u(\mathbb{Z}^n)$. Since Ω is bounded, the number of $x + g$ which will lie in Ω is uniformly bounded. Since ψ is boundedly finite to one, the number of η which get mapped by ψ to the same $x + g$ is uniformly bounded. Hence the number of elements in $\hat{\psi}^{-1}(\Pi x)$ is uniformly bounded for all x.□

We now will consider the $\hat{\psi}$-images of the cylinder sets in Σ_Γ. These sets will form a generating set for a sigma-algebra which is metrically equivalent to the Borel algebra on $\Pi\Omega$.

Proposition 4.3 *Suppose N, $M \geq 0$ and $\{\eta^k\}_{k=-N}^{M}$ is a path in Γ with source v_{j_0} and target v_{j_1}. Then*

$$\begin{aligned}
&\psi(C_{-N}(\eta^{-N}\cdots\eta^M)) \\
&\quad = \left(\phi^{N+1}\Omega_{j_0} - \sum_{k=-N}^{0} \phi^{-k}\zeta_u L(\eta^k)\right) \bigcap \left(\phi^{-M}\Omega_{j_1} + \sum_{k=1}^{M} \phi^{-k}\zeta_u L(\eta^k)\right) \\
&\quad = \left(\pi_s\phi^{N+1}\Omega_{j_0} + \sum_{k=-N}^{0} \phi^{-k}\rho_s L(\eta^k)\right) \oplus \left(\pi_u\phi^{-M}\Omega_{j_1} + \sum_{k=1}^{M} \phi^{-k} L(\eta^k)\right),
\end{aligned}$$

and $\psi(C_{-N}(\eta^{-N}\cdots\eta^M))$ is the closure of its interior.

Proof. Suppose $\epsilon \in C_{-N}(\eta^{-N}\cdots\eta^M)$. Since $t(\sigma_\Gamma{}^{-N-1}\epsilon) = s(\eta^{-N}) = v_{j_0}$ we have $\sigma_\Gamma{}^{-N-1}\epsilon \in \bar{S}_{j_0}$ and by Proposition 3.3

$$\psi(\sigma_\Gamma{}^{-N-1}\epsilon) = \phi^{-N-1}\psi(\epsilon) + \phi^{-N-1}\sum_{k=-N}^{0} \phi^{-k}\zeta_u L(\epsilon^k).$$

Since $\epsilon^k = \eta^k$ for $-N \leq k \leq 0$ we may rewrite this and solve for $\psi(\epsilon)$ to obtain

$$\begin{aligned}
\psi(\epsilon) &= \phi^{N+1}\psi(\sigma_\Gamma{}^{-N-1}\epsilon) - \sum_{k=-N}^{0} \phi^{-k}\zeta_u L(\eta^k) \\
&\in \phi^{N+1}\Omega_{j_0} - \sum_{k=-N}^{0} \phi^{-k}\zeta_u L(\eta^k).
\end{aligned}$$

Since $t(\epsilon^M) = t(\eta^M) = v_{j_1}$ we have $\sigma_\Gamma{}^M\epsilon \in \bar{S}_{j_1}$ and by Proposition 3.3

$$\psi(\sigma_\Gamma{}^M\epsilon) = \phi^M\psi(\epsilon) - \phi^M\sum_{k=1}^{M} \phi^{-k}\zeta_u L(\epsilon^k).$$

Since $\epsilon^k = \eta^k$ for $1 \leq k \leq M$ we may rewrite this and solve for $\psi(\epsilon)$ to obtain

$$\begin{aligned}
\psi(\epsilon) &= \phi^{-M}\psi(\sigma_\Gamma{}^M\epsilon) + \sum_{k=1}^{M} \phi^{-k}\zeta_u L(\eta^k) \\
&\in \phi^{-M}\Omega_{j_1} + \sum_{k=1}^{M} \phi^{-k}\zeta_u L(\eta^k).
\end{aligned}$$

Hence

$$C_{-N}(\eta^{-N}\cdots\eta^{M}) \subset \left(\phi^{N+1}\Omega_{j_0} - \sum_{k=-N}^{0}\phi^{-k}\zeta_u L(\eta^k)\right)$$
$$\bigcap\left(\phi^{-M}\Omega_{j_1} + \sum_{k=1}^{M}\phi^{-k}\zeta_u L(\eta^k)\right).$$

Since Ω_{j_0} and Ω_{j_1} are rectangles in $E_s \oplus E_u$ we may rewrite the right side of this inclusion as

$$\left(\pi_s\phi^{N+1}\Omega_{j_0} + \sum_{k=-N}^{0}\phi^{-k}\rho_s L(\eta^k)\bigcap\pi_s\phi^{-M}\Omega_{j_1} - \sum_{k=1}^{M}\phi^{-k}\rho_s L(\eta^k)\right)$$
$$\oplus\left(\pi_u\phi^{N+1}\Omega_{j_0} - \sum_{k=-N}^{0}\phi^{-k}L(\eta^k)\bigcap\pi_u\phi^{-M}\Omega_{j_1} + \sum_{k=1}^{M}\phi^{-k}L(\eta^k)\right).$$

For all $T \in \mathfrak{T}_{j_0} - \mathcal{N}(\mathfrak{T}_{j_0})$ there exists $T' \in \mathfrak{T}_{j_1}$ such that

$$T' \subset \phi^{M+N+1}T$$

and $c(T') = \phi^{M+N+1}c(T) + \sum_{k=-N-M}^{0}\phi^{-k}L(\eta^{k+M})$. Hence

$$\phi^{-M}c(T') = \phi^{N+1}c(T) + \phi^{-M}\sum_{k=-N}^{M}\phi^{M-k}L(\eta^k)$$

and

$$\left(\phi^{-M}c(T') - \sum_{k=1}^{M}\phi^{-k}L(\eta^k)\right) = \left(\phi^{N+1}c(T) + \sum_{k=-N}^{0}\phi^{-k}L(\eta^k)\right).$$

So for all $T \in \mathfrak{T}_{j_0} - \mathcal{N}(\mathfrak{T}_{j_0})$

$$\left(\phi^{N+1}\rho_s c(T) + \sum_{k=-N}^{0}\phi^{-k}\rho_s L(\eta^k)\right) \in \left(\phi^{-M}\rho_s c(\mathfrak{T}_{j_1}) - \sum_{k=1}^{M}\phi^{-k}\rho_s L(\eta^k)\right).$$

By Proposition 3.12

$$\left(\phi^{N+1}\pi_s\Omega_{j_0} + \sum_{k=-N}^{0}\phi^{-k}\rho_s L(\eta^k)\right) \subset \left(\phi^{-M}\pi_s\Omega_{j_1} - \sum_{k=1}^{M}\phi^{-k}\rho_s L(\eta^k)\right).$$

Moreover,

$$\left(\phi^{N+1}\pi_u\Omega_{j_0} - \sum_{k=-N}^{0}\phi^{-k}L(\eta^k)\right)\bigcap\left(\phi^{-M}\pi_u\Omega_{j_1} + \sum_{k=1}^{M}\phi^{-k}L(\eta^k)\right)$$
$$= \left(\phi^{N+1}T - \phi^{N+1}c(T) - \sum_{k=-N}^{0}\phi^{-k}L(\eta^k)\right)\bigcap\left(\phi^{-M}T' - \phi^{-M}c(T') + \sum_{k=1}^{M}\phi^{-k}L(\eta^k)\right)$$
$$= \phi^{-M}T' - \phi^{M}c(T') + \sum_{k=1}^{M}\phi^{-k}L(\eta^k)$$
$$= \phi^{-M}\pi_u\Omega_{j_1} + \sum_{k=1}^{M}\phi^{-k}L(\eta^k).$$

So

$$\left(\phi^{N+1}\Omega_{j_0} - \sum_{k=-N}^{0}\phi^{-k}\zeta_u L(\eta^k)\right)\bigcap\left(\phi^{-M}\Omega_{j_1} + \sum_{k=1}^{M}\phi^{-k}\zeta_u L(\eta^k)\right)$$
$$= \left(\phi^{N+1}\pi_s\Omega_{j_0} + \sum_{k=-N}^{0}\phi^{-k}\rho_s L(\eta^k)\right)\oplus\left(\phi^{-M}\pi_u\Omega_{j_1} + \sum_{k=1}^{M}\phi^{-k}L(\eta^k)\right).$$

Since $\pi_s\Omega_{j_0}$ and $\pi_u\Omega_{j_1}$ are both the closure of their interiors, the sets given on both sides of the above equality are the closures of their interiors.

Suppose $\epsilon \in \Sigma_\Gamma$ and

$$\psi(\epsilon) \in \left(\phi^{N+1}\mathrm{Int}(\Omega_{j_0}) - \sum_{k=-N}^{0}\phi^{-k}\zeta_u L(\eta^k)\right)\bigcap\left(\phi^{-M}\mathrm{Int}(\Omega_{j_1}) + \sum_{k=1}^{M}\phi^{-k}\zeta_u L(\eta^k)\right).$$

Then

$$\phi^{-N-1}\psi(\epsilon) + \phi^{-N-1}\sum_{k=-N}^{0}\phi^{-k}\zeta_u L(\eta^k) \in \mathrm{Int}(\Omega_{j_0}).$$

By Proposition 3.3

$$\phi^{-N-1}\psi(\epsilon) = \psi({\sigma_\Gamma}^{-N-1}\epsilon) - \phi^{-N-1}\sum_{k=-N}^{0}\phi^{-k}\zeta_u L(\epsilon^k).$$

Hence

$$\psi({\sigma_\Gamma}^{-N-1}\epsilon) - \phi^{-N-1}\left(\sum_{k=-N}^{0}\phi^{-k}\zeta_u L(\epsilon^k) - \sum_{k=-N}^{0}\phi^{-k}\zeta_u L(\eta^k)\right) \in \mathrm{Int}(\Omega_{j_0}).$$

Let

$$z = \phi^{-N-1}\left(\sum_{k=-N}^{0}\phi^{-k}\zeta_u L(\epsilon^k) - \sum_{k=-N}^{0}\phi^{-k}\zeta_u L(\eta^k)\right).$$

If $t(\epsilon^{-N-1}) = v_{j_2}$ then

$$(\Omega_{j_2} - z) \cap \mathrm{Int}(\Omega_{j_0}) \neq \emptyset.$$

Since $\mathfrak{T}$ is a Markov tiling we must have $j_2 = j_0$ and $z = 0$. So

$$\sum_{k=-N}^{0}\phi^{-k}L(\eta^k) = \sum_{k=-N}^{0}\phi^{-k}L(\epsilon^k).$$

Moreover.

$$\phi^M\psi(\epsilon) - \phi^M\sum_{k=1}^{M}\phi^{-k}\zeta_u L(\eta^k) \in \mathrm{Int}(\Omega_{j_1}).$$

And by Proposition 3.3

$$\phi^M\psi(\epsilon) = \psi({\sigma_\Gamma}^M\epsilon) + \phi^M\sum_{k=1}^{M}\phi^{-k}\zeta_u L(\epsilon^k).$$

Hence

$$\psi({\sigma_\Gamma}^M\epsilon) + \phi^M\left(\sum_{k=1}^{M}\phi^{-k}\zeta_u L(\epsilon^k) - \sum_{k=1}^{M}\phi^{-k}\zeta_u L(\eta^k)\right) \in \mathrm{Int}(\Omega_{j_1}).$$

Let

$$z_1 = \phi^M\left(\sum_{k=1}^{M}\phi^{-k}\zeta_u L(\epsilon^k) - \sum_{k=1}^{M}\phi^{-k}\zeta_u L(\eta^k)\right).$$

If $t(\epsilon^M) = v_{j_2}$ then

$$\Omega_{j_2} + z_1 \cap \mathrm{Int}(\Omega_{j_1}) \neq \emptyset.$$

Again, since $\mathfrak{T}$ is a Markov tiling, $j_2 = j_1$ and $z_1 = 0$. So

$$\sum_{k=-N}^{M}\phi^{-k}L(\eta^k) = \sum_{k=-N}^{M}\phi^{-k}L(\epsilon^k)$$

and $s(\eta^{-N}) = s(\epsilon^{-N})$ and $t(\eta^M) = t(\epsilon^M)$. By Proposition 1.13

$$\{\epsilon^k\}_{k=-N}^{M} = \{\eta^k\}_{-N}^{M}.$$

Hence $\epsilon \in C_{-N}(\eta^{-N}\cdots\eta^M)$.

It follows that

$$\mathrm{Clos}\left(\left(\mathrm{Int}(\phi^{N+1}\Omega_{j_0}) - \sum_{k=-N}^{0} \phi^{-k}\zeta_u L(\eta^k)\right) \bigcap \left(\mathrm{Int}(\phi^{-M}\Omega_{j_1}) + \sum_{k=1}^{M} \phi^{-k}\zeta_u L(\eta^k)\right)\right)$$

$$\subset \psi C_{-N}(\eta^{-N} \cdots \eta^{M}).$$

This proves the result.□

This proposition tells us all we need to know about the ψ-images of the cylinder sets. By taking unions of sets of the form $C_{-N}(\eta^{-N} \cdots \eta^{M})$ we may find the ψ-image of any cylinder set.

Definition. For each $\alpha \in \mathcal{E}$ let $R_\alpha = \psi C_0(\alpha)$. Let $\mathcal{R} = \{R_\alpha : \alpha \in \mathcal{E}\}$, and $\Pi\mathcal{R} = \{\Pi R_\alpha : \alpha \in \mathcal{E}\}$.

By the proposition if $\alpha \in \mathcal{E}$ and $\Theta(\alpha) = (v_{j_0}, v_{j_1}, L(\alpha))$ then

$$\begin{aligned} R_\alpha &= (\phi\Omega_{j_0} - \zeta_u L(\alpha)) \cap \Omega_{j_1} \\ &= (\pi_s\phi\Omega_{j_0} + \rho_s L(\alpha)) \oplus \pi_u\Omega_{j_1}. \end{aligned}$$

Recall that our main objective now is to show that the $\hat{\psi}$ preimages of the doubly transitive points in $(\Pi\Omega, \hat{\phi})$ are unique. What we show is that the $\hat{\phi}$ orbit of the doubly transitive points lies within

$$\cup_{\alpha\in\mathcal{E}}\Pi(\mathrm{Int}(R_\alpha)).$$

We show that such points always have a unique preimage. We begin by showing the latter claim first.

Proposition 4.4 *(1) For each $\alpha \in \mathcal{E}$ the set $\Pi(\mathrm{Int}(R_\alpha))$ is dense in $\mathrm{Int}(\Pi R_\alpha)$ and*

$$\begin{aligned} \Pi R_\alpha &= \mathrm{Clos}\,(\Pi(\mathrm{Int}(R_\alpha))) \\ &= \mathrm{Clos}(\mathrm{Int}(\Pi R_\alpha)). \end{aligned}$$

(2) $\Pi\mathcal{R}$ forms an almost disjoint cover of $\Pi\Omega$.

(3) For each $\alpha \in \mathcal{E}$ and $\eta \in \Sigma_\Gamma$ we have $\hat{\psi}(\eta) \in \text{Int}(\Pi R_\alpha)$ only if $\eta^0 = \alpha$. Hence if $\hat{\psi}(\eta) \in \Pi(\text{Int}(R_\alpha))$ then $\eta^0 = \alpha$.

Proof. (1) By Proposition 4.3, $R_\alpha = \text{Clos}(\text{Int}(R_\alpha))$. Since Π is a quotient map, $\Pi R_\alpha = \text{Clos}(\text{Int}(\Pi R_\alpha))$. Moreover,

$$\Pi\text{Clos}(\text{Int}(R_\alpha)) \subset \text{Clos}(\Pi(\text{Int}(R_\alpha))) \subset \text{Clos}(\text{Int}(\Pi R_\alpha)).$$

(2) Let α, $\alpha' \in \mathcal{E}$ and $\Theta(\alpha) = (v_{j_\alpha}, v_{k_\alpha}, L(\alpha))$ and $\Theta(\alpha') = (v_{j_{\alpha'}}, v_{k_{\alpha'}}, L(\alpha'))$. If $\text{Int}(\Pi R_\alpha) \cap \text{Int}(\Pi R_{\alpha'}) \neq \emptyset$ then by (1)

$$\Pi(\text{Int}(R_\alpha)) \cap \Pi(\text{Int}(R_{\alpha'})) \neq \emptyset.$$

Hence for some $z \in \mathbb{Z}^n$

$$\text{Int}(\phi\Omega_{j_\alpha}) - \zeta_u L(\alpha) \cap \text{Int}(\phi\Omega_{j_{\alpha'}}) - \zeta_u L(\alpha') + z \neq \emptyset$$

and

$$\text{Int}(\Omega_{k_\alpha}) \cap \text{Int}(\Omega_{k_{\alpha'}}) + z \neq \emptyset.$$

Since $\mathfrak{T}$ is a Markov tiling we have $j_\alpha = j_{\alpha'}$, $k_\alpha = k_{\alpha'}$ and $z = 0$. Hence $L(\alpha) = L(\alpha')$ and $\Theta(\alpha) = \Theta(\alpha')$. So $\alpha = \alpha'$.

(3) If $\hat{\psi}(\eta) \in \text{Int}(\Pi R_\alpha)$ then $\Pi R_{\eta^0} \cap \text{Int}(\Pi R_\alpha) \neq \emptyset$. By (1) $\text{Int}(\Pi R_{\eta^0}) \cap \text{Int}(\Pi R_\alpha) \neq \emptyset$. Hence by (2) $\eta^0 = \alpha$.□

It follows from (3) above that if $x \in \Omega$ such that for all $k \in \mathbb{Z}$, $\Pi\phi^k x \in \Pi(\text{Int}(R_\alpha))$ for some $\alpha \in \mathcal{E}$ then Πx has a unique $\hat{\psi}$ preimage in Σ_Γ. We will show that the orbit of a doubly transitive point lies in

$$\cup_{\alpha \in \mathcal{E}} \Pi(\text{Int}(R_\alpha))$$

The natural way to approach this is to consider the boundary of R_α.

Definition. If $\alpha \in \mathcal{E}$ and $\Theta(\alpha) = (v_{j_0}, v_{j_1}, L(\alpha))$ then the **stable boundary** of R_α is

$$\partial^s R_\alpha = (\pi_s \phi\Omega_{j_0} + \rho_s L(\alpha)) \oplus \partial\pi_u\Omega_{j_1}$$

and the **unstable boundary** of R_α is

$$\partial^u R_\alpha = (\partial \pi_s \phi \Omega_{j_0} + \rho_s L(\alpha)) \oplus \pi_u \Omega_{j_1}.$$

If $x \in R_\alpha = (\pi_s \phi \Omega_{j_0} + \rho_s L(\alpha)) \oplus \pi_u \Omega_{j_1}$ but $x \notin \partial^s R_\alpha \cup \partial^u R_\alpha$ then

$$\pi_s(x) \in \text{Int}(\pi_s \phi \Omega_{j_0}) + \rho_s L(\alpha)$$

and

$$\pi_u(x) \in \text{Int}(\pi_u \Omega_{j_1}).$$

Hence $x \in \text{Int}(R_\alpha)$. So $\partial R_\alpha = \partial^s R_\alpha \cup \partial^u R_\alpha$. Let $\partial^s \mathcal{R} = \cup_{\alpha \in \mathcal{E}} \partial^s R_\alpha$ and $\partial^u \mathcal{R} = \cup_{\alpha \in \mathcal{E}} \partial^u R_\alpha$. Then $\partial \mathcal{R} = \cup_{\alpha \in \mathcal{E}} \partial R_\alpha = \partial^s \mathcal{R} \cup \partial^u \mathcal{R}$.

Proposition 4.5 *For each* $\alpha \in \mathcal{E}$

(1) $\Pi \phi \partial^s R_\alpha \subset \Pi \partial^s \mathcal{R}$ *and*

(2) $\Pi \phi^{-1} \partial^u R_\alpha \subset \Pi \partial^u \mathcal{R}$.

Proof. Suppose $\alpha \in \mathcal{E}$ and $\Theta(\alpha) = (v_{j_0}, v_{j_1}, L(\alpha))$. Then

$$\phi \partial^s R_\alpha = (\phi^2 \pi_s \Omega_{j_0} + \phi \pi_s L(\alpha)) \oplus \partial \phi \pi_u \Omega_{j_1}.$$

For $T \in \mathfrak{T}_{j_0} - \mathcal{N}(\mathfrak{T}_{j_0})$ we know there exists $T' \in \mathfrak{T}_{j_1}$ such that $T' \subset \phi T$ and $c(T') = \phi c(T) + L(\alpha)$. Hence as we have argued before

$$\left(\phi^2 \pi_s(\Omega_{j_0}) + \phi \pi_s L(\alpha)\right) \subset \phi \pi_s(\Omega_{j_1}).$$

For all $T' \in \mathfrak{T}_{j_1} - \mathcal{N}(\mathfrak{T}_{j_1})$ there is a unique collection of tiles $\{T^{\alpha'}\}_{\alpha' \in \mathcal{E}_{j_1}}$ in $\mathfrak{T}$ such that

$$\phi T' = \cup_{\alpha' \in \mathcal{E}_{j_1}} T^{\alpha'}.$$

And $c(T^{\alpha'}) = \phi c(T') + L(\alpha')$. Suppose for each $\alpha' \in \mathcal{E}_{j_1}$ we have

$$\Theta(\alpha') = (v_{j_1}, v_{j_{\alpha'}}, L(\alpha')).$$

Then

$$\begin{aligned}\phi\pi_u\Omega_{j_1} &= \phi(T' - c(T')) = \cup_{\alpha'\in\mathcal{E}_{j_1}} T^{\alpha'} - \phi c(T') \\ &= \cup_{\alpha'\in\mathcal{E}_{j_1}} (T^{\alpha'} - c(T^{\alpha'}) + L(\alpha')) \\ &= \cup_{\alpha'\in\mathcal{E}_{j_1}} \left(\pi_u(\Omega_{j_{\alpha'}}) + L(\alpha')\right).\end{aligned}$$

So $\partial\phi\pi_u\Omega_{j_1} \subset \cup_{\alpha'\in\mathcal{E}_{j_1}} \partial\pi_u(\Omega_{j_{\alpha'}} + L(\alpha'))$. Hence

$$\begin{aligned}&(\phi^2\pi_s\Omega_{j_0} + \phi\rho_s L(\alpha)) \oplus \partial\phi\pi_u\Omega_{j_1} \\ &\quad\subset \cup_{\alpha'\in\mathcal{E}_{j_1}} \left(\phi\pi_s\Omega_{j_1} \oplus \left(\partial\pi_u(\Omega_{j_{\alpha'}}) + L(\alpha')\right)\right) \\ &\quad= \cup_{\alpha'\in\mathcal{E}_{j_1}} \left(\left(\phi\pi_s\Omega_{j_1} + \rho_s L(\alpha') \oplus \partial\pi_u\Omega_{j_{\alpha'}}\right) + \zeta_u L(\alpha')\right).\end{aligned}$$

And

$$\phi\partial^s R_\alpha \subset \cup_{\alpha'\in\mathcal{E}_{j_1}} (\partial^s R_{\alpha'} + \zeta_u L(\alpha')).$$

Moreover,

$$\phi^{-1}\partial^u R_\alpha = (\partial\pi_s\Omega_{j_0} + \phi^{-1}\rho_s L(\alpha)) \oplus \phi^{-1}\pi_u\Omega_{j_1}.$$

For each $T \in \mathfrak{T}_{j_0} - \mathcal{N}(\mathfrak{T}_{j_0})$, $T \subset X_u - \phi\mathcal{N}(X_u)$, and $\alpha' \in \mathcal{E}^{j_0}$ there exists a $T^{\alpha'}$ in $\mathfrak{T} - \mathcal{N}(\mathfrak{T})$ such that $T \subset \phi T^{\alpha'}$ and $c(T) = \phi c(T^{\alpha'}) + L(\alpha')$. Suppose for each $\alpha' \in \mathcal{E}^{j_0}$ we have

$$\Theta(\alpha') = (v_{j_{\alpha'}}, v_{j_0}, L(\alpha')).$$

Then by Proposition 3.12

$$\pi_s\Omega_{j_0} + \phi^{-1}\rho_s L(\alpha) = \cup_{\alpha'\in\mathcal{E}^{j_0}} \left(\phi\pi_s(\Omega_{j_{\alpha'}}) + \rho_s L(\alpha')\right) + \phi^{-1}\rho_s L(\alpha).$$

So

$$\partial\pi_s\Omega_{j_0} + \phi^{-1}\rho_s L(\alpha) \subset \cup_{\alpha'\in\mathcal{E}^{j_0}} \left(\partial\phi\pi_s\Omega_{j_{\alpha'}} + \rho_s L(\alpha')\right) + \phi^{-1}\rho_s L(\alpha).$$

For all $T \in \mathfrak{T}_{j_0} - \mathcal{N}(\mathfrak{T}_{j_0})$ there exists $T' \in \mathfrak{T}_{j_1}$ such that $\phi^{-1}T' \subset T$ and $\phi^{-1}c(T') = c(T) + \phi^{-1}L(\alpha)$. So

$$\begin{aligned}\phi^{-1}\pi_u\Omega_{j_1} &= \phi^{-1}(T' - c(T')) \\ &= \phi^{-1}T' - c(T) - \phi^{-1}L(\alpha) \\ &\subset T - c(T) - \phi^{-1}L(\alpha) \\ &= \pi_u\Omega_{j_0} - \phi^{-1}L(\alpha).\end{aligned}$$

Hence

$$\begin{aligned}&(\partial\pi_s\Omega_{j_0} + \phi^{-1}\rho_s L(\alpha)) \oplus \phi^{-1}\pi_u\Omega_{j_1}\\ &\quad\subset \cup_{\alpha'\in\mathcal{E}^{j_0}} \left((\partial\phi\pi_s\Omega_{j_{\alpha'}} + \rho_s L(\alpha') + \phi^{-1}\rho_s L(\alpha)) \oplus (\pi_u\Omega_{j_0} - \phi^{-1}L(\alpha))\right)\\ &\quad= \cup_{\alpha'\in\mathcal{E}^{j_0}} \left(\partial\phi\pi_s\Omega_{j_{\alpha'}} + \rho_s L(\alpha') \oplus \pi_u\Omega_{j_0}\right) - \zeta_u\phi^{-1}L(\alpha).\end{aligned}$$

So

$$\phi^{-1}\partial^u R_\alpha \subset \cup_{\alpha'\in\mathcal{E}^{j_0}}\partial^u R_{\alpha'} - \zeta_u\phi^{-1}L(\alpha).$$

□

Corollary 4.6 *Let $\alpha \in \mathcal{E}$ and $\Theta(\alpha) = (v_{j_0}, v_{j_1}, L(\alpha))$ then*

(1) $\phi R_\alpha \subset \cup_{\alpha'\in\mathcal{E}_{j_1}}(R_{\alpha'} + \zeta_u L(\alpha'))$,

(2) $\phi^{-1}R_\alpha \subset \cup_{\alpha'\in\mathcal{E}^{j_0}} R_{\alpha'} - \zeta_u\phi^{-1}L(\alpha)$,

(3) $\phi^{-1}\pi_s R_\alpha = \cup_{\alpha'\in\mathcal{E}^{j_0}}\pi_s R_{\alpha'} + \rho_s\phi^{-1}L(\alpha)$,

(4) $\phi\pi_u R_\alpha = \cup_{\alpha'\in\mathcal{E}_{j_1}}(\pi_u R_{\alpha'} + L(\alpha'))$

where these are almost disjoint unions.

Theorem 4.7 *The map $\hat{\psi}$ is an almost homeomorphic factor map from $(\Sigma_\Gamma, \sigma_\Gamma)$ to $(\Pi\Omega, \hat{\phi})$.*

Proof. Let $\mathfrak{N} = \Pi(\cup_{k\in\mathbb{Z}}\phi^{-k}\partial\mathcal{R}) = \cup_{k\in\mathbb{Z}}\hat{\phi}^{-k}\Pi\partial\mathcal{R}$. If $x \in \Omega$ and $\Pi x \in \Pi\Omega - \mathfrak{N}$ then for all $k \in \mathbb{Z}$

$$\hat{\phi}^k\Pi x \notin \Pi\partial\mathcal{R}.$$

Hence if $y_k \in \Omega$ such that $\Pi y_k = \hat{\phi}^k\Pi x$ then $y_k \notin \partial\mathcal{R}$. So for all $k \in \mathbb{Z}$ there exists $\eta^k \in \mathcal{E}$ such that

$$\hat{\phi}^k\Pi x = \Pi y_k \in \Pi(\mathrm{Int}(R_{\eta^k})).$$

By Proposition 4.4 Πx has a unique $\hat{\psi}$ preimage $\{\eta^k\}_{k\in\mathbb{Z}}$.

Suppose Πx is doubly transitive. If for some $N \in \mathbb{Z}$ we have $\hat{\phi}^N\Pi x \in \Pi\partial\mathcal{R}$ then $\hat{\phi}^N\Pi x \in \Pi(\partial^s\mathcal{R} \cup \partial^u\mathcal{R})$. Hence by Proposition 4.5 Πx is not doubly transitive. It follows that the doubly transitive points lie in $\Pi\Omega - \mathfrak{N}$ and therefore have unique preimages. Since almost every point is doubly transitive (given any ergodic measure) $\mathfrak{N}$ is a universally null set.

With Lemmas 4.1 and 4.2 and the remarks at the beginning of the chapter we have shown that $\hat{\psi}$ satisfies the conditions for an almost homeomorphic factor map.□

Markov partitions

The conventional way to represent an abstract dynamical system with a symbolic dynamical system is by using a Markov partition. One definition, suggested by Adler, goes like this.

Let (X, f) be an ergodically supported abstract dynamical system, where X is a compact metric space with metric d and f a homeomorphism. Suppose f is expansive. This means there is a constant $c > 0$ such that if $x,\ y \in X$ and $d(f^k x, f^k y) \leq c$ for all $k \in \mathbb{Z}$ then $x = y$. A finite collection of compact sets $\mathcal{C} = \{C_0, C_1, \ldots, C_{M-1}\}$ is a **Markov partition** for f if it satisfies the following.

(1) $\mathcal{C}$ is a cover of X.

(2) Each $C_i \in \mathcal{C}$ is the closure of its interior.

(3) If $\text{Int}(C_i) \cap \text{Int}(C_j) \neq \emptyset$ then $i = j$.

Suppose $\{C_{i_k}\}_{k=-N}^{N}$ is a sequence in $\mathcal{C}$.

(4) $\mathcal{C}$ is a generating set. This means that diameter($f^N \text{Int}(C_{i_{-N}}) \cap \ldots \cap f^{-N} \text{Int}(C_{i_N})$) tends to 0 as N tends to infinity.

(5) If $\text{Int}(C_{i_k}) \cap f^{-1}\text{Int}(C_{i_{k+1}}) \neq \emptyset$ for $-N \leq k \leq N-1$ then

$$\cap_{k=-N}^{N} f^{-k} \text{Int}(C_{i_k}) \neq \emptyset.$$

This is the Markov condition.

(6) For each $C_i \in \mathcal{C}$ we have diameter$(C_i) < c/2$.

Suppose $\mathcal{C} = \{C_i\}_{i=0}^{M-1}$ is a Markov partition for f. Let A be the transition matrix for $\mathcal{C}$. Then A is the $0-1$ $M \times M$ matrix defined by

$$A_{ij} = 1 \text{ if } \mathrm{Int}(\phi C_i) \cap \mathrm{Int}(C_j) \neq \emptyset \text{ and}$$

$$A_{ij} = 0 \text{ otherwise.}$$

The **topological Markov shift** (Σ_A, σ_A) is a compact metric space (with metric d_A defined as before). Define

$$\Sigma_A = \{\eta \in \{0, 1, 2, \ldots, M-1\}^{\mathbb{Z}} : A_{\eta^k \eta^{k+1}} = 1 \text{ for all } k \in \mathbb{Z}\}.$$

The shift operator $\sigma_A : \Sigma_A \to \Sigma_A$ is defined as before so that for each $\eta \in \Sigma_A$, $\sigma_A \eta^k = \eta^{k+1}$.

Define $\pi : \Sigma_A \to X$ by

$$\pi(\eta) = \bigcap_{k=-\infty}^{\infty} f^{-k} C_{\eta^k}.$$

Theorem. (Adler) The map π is an almost homeomorphic factor map from (Σ_A, σ_A) to (X, f).

Adler has indicated that property (6) is used to show that π is boundedly finite to one. If $\mathcal{C}$ has some structure which insures that π is boundedly finite to one, then we may dismiss property (6).

Consider the set $\Pi\mathcal{R}$. We would like to show that $\Pi\mathcal{R}$ is a Markov partition for $\hat{\phi}$. By Proposition 4.4, $\Pi\mathcal{R}$ satisfies conditions (1), (2), (3).

Suppose α, $\alpha' \in \mathcal{E}$ and

$$\mathrm{Int}(\Pi R_\alpha) \cap \hat{\phi}^{-1} \mathrm{Int}(\Pi R_{\alpha'}) \neq \emptyset.$$

By Proposition 4.5 we have

$$\Pi(\mathrm{Int}(R_\alpha)) \cap \Pi(\mathrm{Int}(\phi^{-1} R_{\alpha'})) \neq \emptyset.$$

By Corollary 4.6 if $t(\alpha') = v_{j_1}$

$$\phi^{-1} R_{\alpha'} \subset \cup_{\alpha'' \in \mathcal{E}^{j_1}} R_{\alpha''} - \zeta_u \phi^{-1} L(\alpha').$$

So for some $\alpha'' \in \mathcal{E}^{j_1}$ and $z \in \mathbb{Z}^n$

$$(R_{\alpha''} - \zeta_u \phi^{-1} L(\alpha')) \cap (\mathrm{Int}(R_\alpha) + z) \neq \emptyset.$$

We've shown already that in this case $\alpha'' = \alpha$ and $\zeta_u \phi^{-1} L(\alpha') = z$. So

$$\mathrm{Int}(\Pi R_\alpha) \cap \hat{\phi}^{-1} \mathrm{Int}(\Pi R_{\alpha'}) \neq \emptyset$$

if and only if $\alpha\alpha'$ is a path in Γ. If $\eta \in \Sigma_\Gamma$ and

$$\hat{\psi}(\eta) \in \mathrm{Int}(\Pi R_\alpha) \cap \hat{\phi}^{-1} \mathrm{Int}(\Pi R_{\alpha'})$$

then $\eta^0 = \alpha$ and $\hat{\phi} \circ \hat{\psi}(\eta) \in \mathrm{Int}(\Pi R_{\alpha'})$ so $\hat{\psi}(\sigma_\Gamma \eta) \in \mathrm{Int}(\Pi R_{\alpha'})$ and $\eta^1 = \alpha'$. So

$$\mathrm{Int}(\pi R_\alpha) \cap \hat{\phi}^{-1} \mathrm{Int}(\pi R_{\alpha'})) \subset \hat{\psi} C_0(\alpha\alpha').$$

Conversely we have seen that

$$\hat{\psi} C_0(\alpha\alpha') \subset \Pi R_\alpha \cap \hat{\phi}^{-1} \Pi R_{\alpha'}.$$

Hence

$$\begin{aligned} \Pi R_\alpha \cap \hat{\phi}^{-1} \Pi R_{\alpha'} &= \mathrm{Clos}(\mathrm{Int}(\Pi R_\alpha) \cap \hat{\phi}^{-1} \Pi R_{\alpha'}) \\ &= \hat{\psi} C_0(\alpha\alpha'). \end{aligned}$$

It follows by induction that if $\{\Pi R_{\eta^k}\}_{k=-N}^{N}$ is a sequence in $\Pi\mathcal{R}$ such that

$$\cap_{k=-N}^{N} \hat{\phi}^{-k} \mathrm{Int}(\Pi R_{\eta^k}) \neq \emptyset$$

then $\{\eta^k\}_{k=-N}^{N}$ is a path in Γ and

$$\cap_{k=-N}^{N} \hat{\phi}^{-k} \Pi R_{\eta^k} = \hat{\psi} C_{-N}(\eta^{-N} \cdots \eta^N).$$

By Proposition 4.3 if $\Theta(\eta^k) = (v_{j_{k-1}}, v_{j_k}, L(\eta^k))$ then

$$\psi(C_{-N}(\eta^{-N} \cdots \eta^N)) = \left(\pi_s \phi^{N+1} \Omega_{j_{-N-1}} + \sum_{k=-N}^{0} \phi^{-k} \rho_s L(\eta^k) \right) \oplus \left(\pi_u \phi^{-N} \Omega_{j_N} + \sum_{k=1}^{N} \phi^{-k} L(\eta^k) \right).$$

So diameter($\psi(C_{-N}(\eta^{-N}\cdots\eta^{N}))) \leq 2\varrho^{-N}$diameter($\Omega$). And

$$\text{diameter}\left(\hat{\psi}(C_{-N}(\eta^{-N}\cdots\eta^{N}))\right) \leq \text{diameter}\left(\psi(C_{-N}(\eta^{-N}\cdots\eta^{N}))\right).$$

This shows that $\Pi\mathcal{R}$ is a generating set. Moreover if $\{\eta^k\}_{k=-N}^{N}$ is a path then

$$\cap_{k=-N}^{N}\hat{\phi}^{-k}\text{Int}(R_{\eta^k}) \neq \emptyset.$$

It follows from the above remarks that $\Pi\mathcal{R}$ satisfies (4) and(5).

But the sets in $\Pi\mathcal{R}$ need not be small so $\Pi\mathcal{R}$ need not satisfy(6). We have already shown that $\Pi\mathcal{R}$ has sufficient structure to insure that $\hat{\psi}$ is boundedly finite to one. We will show that $\hat{\psi} = \pi$. Let A be the transition matrix for the partition $\Pi\mathcal{R}$. Then as we have seen $\Sigma_A = \Sigma_\Gamma$. Note that since $\mathfrak{T}$ is p-mixing, A is aperiodic. If $\eta \in \Sigma_\Gamma$ then we compute

$$\bigcap_{k\in\mathbf{Z}} \hat{\phi}^{-k}\Pi R_{\eta^k}. \tag{4.1}$$

By our discussion

$$\text{diameter}(\cap_{k=-N}^{N}\hat{\phi}^{-k}\Pi R_{\eta^k}) \leq 2\varrho^{-N}\text{diameter}(\Omega).$$

So as N tends to infinity $\cap_{k=-N}^{N}\hat{\phi}^{-k}\Pi R_{\eta^k}$ tends to a unique point. But we know that $\hat{\psi}(\eta)$ belongs to (4.1). So $\pi(\eta) = \hat{\psi}(\eta)$. Hence $\pi\colon(\Sigma_\Gamma, \sigma_\Gamma) \to (\Pi\Omega, \hat{\phi})$ is well-defined and boundedly finite to one. So without reservation we call $\Pi\mathcal{R}$ a Markov partition for $\hat{\phi}$.

Metric similarity.

We conclude this chapter by showing that the map $\hat{\psi}$ is measure preserving when $\Pi\Omega$ is given Haar measure and Σ_Γ is given the Parry measure. Since $\hat{\psi}$ is an almost homeomorphic factor map this follows by an argument using entropy as in [3]. We will show this directly by computing the Haar measure of the $\hat{\psi}$ image of the cylinder sets.

We review the construction of the Parry measure for (Σ_A, σ_A). By the theory of non-negative matrices [11] there exists a positive eigenvalue λ for A such that λ is greater than the modulus of all other eigenvalues for A. Moreover there exist strictly positive row and column vectors $\mathbf{r}$ and $\mathbf{c}$ such that $\mathbf{r}A = \lambda A$ and $A\mathbf{c} = \lambda\,\mathbf{c}$. Since A is indexed by $\mathcal{E} \times \mathcal{E}$, $\mathbf{r}$ and $\mathbf{c}$ are indexed by $\mathcal{E}$. So

$$\mathbf{r}A = \left\{ \sum_{\alpha' \in \mathcal{E}} A_{\alpha'\alpha}\mathbf{r}_{\alpha'} \right\}_{\alpha \in \mathcal{E}} = \lambda\,\mathbf{r}$$

and

$$A\mathbf{c} = \left\{ \sum_{\alpha' \in \mathcal{E}} A_{\alpha\alpha'}\mathbf{c}_{\alpha'} \right\}_{\alpha \in \mathcal{E}} = \lambda\,\mathbf{c}.$$

Moreover $\mathbf{r}$ and $\mathbf{c}$ are unique up to multiples. We normalize $\mathbf{r}$ and $\mathbf{c}$ so that $\mathbf{rc} = 1$. Let $p = \{\mathbf{r}_\alpha \mathbf{c}_\alpha\}_{\alpha \in \mathcal{E}}$. Define a stochastic matrix P indexed by $\mathcal{E} \times \mathcal{E}$ so that

$$P_{\alpha\alpha'} = \frac{A_{\alpha\alpha'}\mathbf{c}_{\alpha'}}{\lambda \mathbf{c}_\alpha}.$$

Let Cyl be the sigma-algebra generated by the cylinder sets. We define a measure m for Cyl by defining m on the cylinder sets. For each $\alpha \in \mathcal{E}$ and $i \in \mathbb{Z}$ let

$$m(C_i(\alpha)) = p_\alpha.$$

If $\{\eta^k\}_{k=-M}^{N}$ is a path in Γ define

$$m(C_i(\eta^{-N} \cdots \eta^N)) = p_{\eta^{-N}} P_{\eta^{-N}\eta^{-N+1}} \cdots P_{\eta^{M-1}\eta^{M}}.$$

It follows by a bit of algebra that

$$m(C_i(\eta^{-N} \cdots \eta^M)) = \frac{\mathbf{r}_{\eta^{-N}}\mathbf{c}_{\eta^M}}{\lambda^{N+M}}$$

since $A_{\eta^{-N+k}\eta^{-N+k+1}} = 1$ for all $0 \le k \le M+N$. This defines the Parry measure for $(\Sigma_\Gamma, \sigma_\Gamma)$.

Let μ be Haar measure on $\Pi\Omega$. For each Borel set $U \subset \Omega$ the Lebesgue measure of U equals $\mu(\Pi U)$. The sets $\{\Pi U\}$ form the Borel sigma-algebra in $\Pi\Omega$. Consider the sigma-algebra $\hat{\psi}(Cyl)$ generated by the $\hat{\psi}$ images of the cylinder sets. By the discussion of the last section $\hat{\psi}(Cyl)$ is measure theoretically equivalent to the Borel sigma-algebra on $\Pi\Omega$. So if $\hat{\psi}$ is measure preserving on the cylinder sets, then $\hat{\psi}$ is measure preserving. We wish to show for each cylinder $C_i(\eta^{-N} \cdots \eta^M) \in Cyl$ that

$$\mu\hat{\psi}C_i(\eta^{-N} \cdots \eta^M) = mC_i(\eta^{-N} \cdots \eta^M).$$

For $N > 0$ let μ_N be Lebesgue measure on $\mathbb{R}^N$. Let Q be a measure preserving linear automorphism of $\mathbb{R}^n$ such that $QE_s \perp QE_u$. Then for any rectangle $A \oplus B \subset E_s \oplus E_u$ we have

$$\mu_n(A \oplus B) = \mu_n(QA \oplus QB) = \mu_l(QA) \cdot \mu_{n-l}(QB).$$

By Corollary 4.6 for each $\alpha \in \mathcal{E}$ such that $\Theta(\alpha) = (v_{j_0}, v_{j_1}, L(\alpha))$ we have

$$\phi^{-1}\pi_s R_\alpha = \cup_{\alpha' \in \mathcal{E}^{j_0}} \pi_s R_{\alpha'} + \rho_s \phi^{-1} L(\alpha)$$

and

$$\phi \pi_u R_\alpha = \cup_{\alpha' \in \mathcal{E}_{j_1}} (\pi_u R_{\alpha'} + L(\alpha'))$$

where these are almost disjoint unions. Since $\mu_n(\partial \mathcal{R}) = 0$ we have

$$0 = \mu_n(\partial^u R_\alpha) = \mu_l(\partial Q \pi_s R_\alpha) \cdot \mu_{n-l}(Q \pi_u R_\alpha)$$

so $\mu_l(\partial Q \pi_s R_\alpha) = 0$. We already know that $\mu_{n-l}(\partial Q \pi_u R_\alpha) = 0$ by Proposition 1.6. So

$$\begin{aligned} |\det \phi|_{E_s}^{-1}| \cdot \mu_l(Q\pi_s R_\alpha) &= \mu_l(Q\phi|_{E_s}^{-1} Q^{-1} Q \pi_s R_\alpha) \\ &= \sum_{\alpha' \in \mathcal{E}^{j_0}} \mu_l(Q\pi_s R_{\alpha'}) \\ &= \sum_{\alpha' \in \mathcal{E}} A_{\alpha'\alpha} \mu_l(Q\pi_s R_{\alpha'}). \end{aligned}$$

And

$$\begin{aligned} |\det \phi|_{E_u}| \cdot \mu_{n-l}(Q\pi_u R_\alpha) &= \mu_{n-l}(Q\phi|_{E_u} Q^{-1} Q \pi_u R_\alpha) \\ &= \sum_{\alpha' \in \mathcal{E}_{j_1}} \mu_{n-l}(Q\pi_u R_{\alpha'}) \\ &= \sum_{\alpha' \in \mathcal{E}} A_{\alpha\alpha'} \mu_{n-l}(Q\pi_u R_{\alpha'}). \end{aligned}$$

Since ϕ is hyperbolic $|\det \phi|_{E_s}^{-1}| = |\det \phi|_{E_u}| > 1$. It follows that $\{\mu_l(Q\pi_s R_\alpha)\}_{\alpha \in \mathcal{E}}$ is a strictly positive row eigenvector for A corresponding to $|\det \phi|_{E_s}^{-1}|$ and $\{\mu_{n-l}(Q\pi_u R_\alpha)\}_{\alpha \in \mathcal{E}}$ is a strictly positive column eigenvector for A corresponding to $|\det \phi|_{E_u}|$. By the theory of non-negative matrices

$$|\det \phi|_{E_u}| = |\det \phi|_{E_s}^{-1}| = \lambda\,.$$

Moreover,

$$1 = \mu_n(\Omega) = \sum_{\alpha\in\mathcal{E}} \mu_n(R_\alpha) = \sum_{\alpha\in\mathcal{E}} \mu_l(Q\pi_s R_\alpha)\cdot\mu_{n-1}(Q\pi_u R_\alpha).$$

So $\mu_n(R_\alpha) = p_\alpha$.

So without loss of generality, let $\mathbf{r} = \{\mu_l(Q\pi_s R_\alpha)\}_{\alpha\in\mathcal{E}}$ and $\mathbf{c} = \{\mu_{n-l}(Q\pi_u R_\alpha)\}_{\alpha\in\mathcal{E}}$. If $\{\eta^k\}_{k=-N}^M$ is a path in Γ then by Proposition 4.3 if $\Theta(\eta^k) = (v_{j_{k-1}}, v_{j_k}, L(\eta^k))$ then

$$\begin{aligned}
&\psi(C_{-N}(\eta^-N\cdots\eta^M)) \\
&= \left(\pi_s\phi^{N+1}\Omega_{j_{-N-1}} + \sum_{k=-N}^{0}\phi^{-k}\rho_s L(\eta^k)\right) \oplus \left(\pi_u\phi^{-M}\Omega_{j_M} + \sum_{k=1}^{M}\phi^{-k}L(\eta^k)\right) \\
&= \left(\phi^N\pi_s R_{\eta^{-N}} + \sum_{k=-N+1}^{0}\phi^{-k}\rho_s L(\eta^k)\right) \oplus \left(\phi^{-M}\pi_u R_{\eta^M} + \sum_{k=1}^{M}\phi^{-k}L(\eta^k)\right).
\end{aligned}$$

So

$$\begin{aligned}
\mu_n(\psi(C_{-N}(\eta^{-N}\cdots\eta^M))) &= \lambda^{-N}\mathbf{r}_{\eta^{-N}}\lambda^{-M}\mathbf{c}_{\eta^M} \\
&= \frac{\mathbf{r}_{\eta^{-N}}\cdot\mathbf{c}_{\eta^M}}{\lambda^{N+M}} \\
&= m(C_{-N}(\eta^{-N}\cdots\eta^M))
\end{aligned}$$

and $\mu(\hat{\psi}(C_{-N}(\eta^{-N}\cdots\eta^M))) = m(C_{-N}(\eta^{-N}\cdots\eta^M))$. Moreover, $\hat{\phi}$ is measure preserving so for all $i\in\mathbb{Z}$

$$\begin{aligned}
\mu(\hat{\psi}(C_{-N}(\eta^{-N}\cdots\eta^M))) &= \mu(\hat{\phi}^{-i-N}\hat{\psi}(C_{-N}(\eta^{-N}\cdots\eta^M))) \\
&= \mu(\hat{\psi}(\sigma_\Gamma{}^{-i-N}C_{-N}(\eta^{-N}\cdots\eta^M))) \\
&= \mu(\hat{\psi}(C_i(\eta^{-N}\cdots\eta^M))).
\end{aligned}$$

So $\hat{\psi}$ is measure preserving on the cylinder sets.

It follows that $\hat{\psi}$ is a one to one almost everywhere measure preserving map from $(\Sigma_\Gamma, \sigma_\Gamma)$ to $(\Pi\Omega, \hat{\phi})$ such that $\hat{\psi}\circ\sigma_\Gamma = \hat{\phi}\circ\hat{\psi}$. So $(\Sigma_\Gamma, \sigma_\Gamma)$ is metrically similar to $(\Pi\Omega, \hat{\phi})$.

Chapter 5

MARKOV TILINGS

Now that we know what to look for we begin our search for Markov partitions by looking for Markov tilings. We begin by doing a brief survey of the Markov partitions for two and three dimensional hyperbolic toral automorphisms which were previously known. We show that these constructions may be reproduced using Markov tilings. Next, we discuss Markov tilings which arise out of the β-shift for β a Pisot number. Finally, we give a general construction which produces tilings for automorphisms which have a Pisot number as an eigenvalue. Thurston and Kenyon have general constructions for tilings which may also yield Markov partitions. We will not discuss these here.

Hyperbolic automorphisms of the 2-torus.

Example 5.1 Let

$$\phi = \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}.$$

The map in $\mathbb{R}^2$ given by ϕ induces a hyperbolic automorphism of $\mathbb{R}^2 \bmod \mathbb{Z}^2$ with eigenvalues

$$\lambda_1 = \frac{3+\sqrt{5}}{2} \text{ and } \lambda_2 = \frac{3-\sqrt{5}}{2}.$$

The characteristic lines for ϕ are given by

$$L_1 \colon y = \lambda_2 x \text{ for } \lambda_1, \text{ and}$$

$$L_2 \colon y = \lambda_1 x \text{ for } \lambda_2.$$

So $E_u = L_1$ and $E_s = L_2$. Let $e_u = \pi_u(1,0)$ and $e_s = -\pi_s(1,0)$. We will tile the semigroup $X_u = [0,\infty)e_u$.

For each non-negative $n \in \mathbb{Z}$ let $g(n)$ be the unique element of $\mathbb{Z}$ such that

$$\pi_s(n, g(n)) \in (-\infty, 0]e_s, \text{ and}$$

$$\pi_s(n, g(n)+1) \in (0, \infty)e_s.$$

Then $(n, g(n))$ is just the point on the line $x = n$ which is *closest* to X_u but *below* it. Let

$$Z = \{z_n = (n, g(n)) : n \geq 0\}.$$

Note that $\phi Z = Z$. Let T_n be the line segment in E_u with endpoints $\pi_u z_n$ and $\pi_u z_{n+1}$. Then

$$\begin{aligned} T_0 &= [0,1]e_u \\ T_1 &= [1,2]e_u \\ T_2 &= [2, \lambda_1]e_u \\ T_3 &= [\lambda_1, \lambda_1 + 1]e_u \\ &\vdots \end{aligned}$$

Since the slope of L_1 is between 0 and 1, for every $n \geq 0$ we have

$$(n+1, g(n+1)) - (n, g(n)) \in \{(1,0), (1,1)\}.$$

Let

$$\mathfrak{T}_1 = \{T_n : g(n+1) - g(n) = 0\}$$

and

$$\mathfrak{T}_2 = \{T_n : g(n+1) - g(n) = 1\}.$$

Let $\mathfrak{T} = \mathfrak{T}_1 \cup \mathfrak{T}_2$. Then $\mathfrak{T}$ is a $\pi_u(Z^2)$-finite tiling of X_u with tile type partition $\{\mathfrak{T}_1, \mathfrak{T}_2\}$.

Moreover, $\mathfrak{T}$ is subdividing. We note that $T_0 \in \mathfrak{T}_1$ and

$$\begin{aligned} \phi T_0 &= [0, \lambda_1]e_u \\ &= T_0 \cup T_1 \cup T_2. \end{aligned}$$

Let $T_n \in \mathfrak{T}_1$. Then $T_n = T_0 + \pi_u z_n$ and

$$\phi T_n = (T_0 + \phi\pi_u z_n) \cup (T_1 + \phi\pi_u z_n) \cup (T_2 + \phi\pi_u z_n).$$

We must show

$$T_0 + \phi\pi_u z_n, \text{ and } T_1 + \phi\pi_u z_n \in \mathfrak{T}_1$$

and

$$T_2 + \phi\pi_u z_n \in \mathfrak{T}_2.$$

To do this we need only show

$$\phi z_n,\ \phi z_n + z_1,\ \phi z_n + z_2,\ \phi z_{n+1} \in Z.$$

Since $\phi Z \subset Z$ we have $\phi z_n, \phi z_{n+1} \in Z$. Consider the polygon Q with vertices $\pi_u z_n$, z_n, z_{n+1}, and $\pi_u z_{n+1}$. By hypothesis Q contains no lattice points in its interior. Hence ϕQ contains no lattice points in its interior. Consider the polygon Q' with vertices 0, z_1, z_2, and z_3. By hypothesis Q' contains no lattice points in its interior. Since $\phi z_{n+1} = \phi z_n + z_3$ it follows that the polygon with vertices

$$\phi\pi_u z_n,\ \phi z_n,\ \phi z_n + z_1,\ \phi z_n + z_2,\ \phi z_{n+1}, \text{ and } \phi\pi_u z_{n+1}$$

contains no lattice points in its interior. So $\phi z_n + z_1$ and $\phi z_n + z_2 \in Z$.

Similarly we note that $T_2 \in \mathfrak{T}_2$. The endpoints for T_2 are the projections of $z_2 = (2,0)$ and $z_3 = (3,1)$. So

$$\begin{aligned} \phi T_2 &= [2\lambda_1, \lambda_1^2]e_u \\ &= T_6 \cup T_7. \end{aligned}$$

Moreover, arguing as we did above, we find if $T_n \in \mathfrak{T}_2$ then $T_n = T_2 - 2e_u + \pi_u z_n$ and

$$\phi T_n = (T_6 - \phi(2e_u - \pi_u z_n)) \cup (T_7 - \phi(2e_u + \pi_u z_n)).$$

Moreover

$$T_6 - \phi(2e_u + \pi_u z_n) \in \mathfrak{T}_1, \text{ and}$$

$$T_7 - \phi(2e_u + \pi_u z_n) \in \mathfrak{T}_2.$$

So $\mathfrak{T}$ is subdividing. Since there are only a finite number of ways to arrange tiles in X_u within a bounded region, $\mathfrak{T}$ has a finite number of local patterns. Since $\mathfrak{T}$ has one generator T_0 and every tile contains T_0 in its image, $\mathfrak{T}$ is mixing. It follows that $\mathfrak{T}$ is a $\pi_u(\mathbb{Z}^n)$- finite self similar tiling of X_u with expansion map ϕ and tile type partition $\{\mathfrak{T}_1, \mathfrak{T}_2\}$.

Let $I_1 = [0,1]e_u$ and $I_2 = [0, \lambda_1 - 2]e_u$. If $T_n \in \mathfrak{T}_j$ then $T_n = I_j + \pi_u z_n$. The subdivision rules for $\mathfrak{T}$ may be summarized as follows.

$$\phi T_n = \begin{cases} (I_1 + \phi\pi_u z_n) \cup (I_1 + e_u + \phi\pi_u z_n) \cup (I_2 + 2e_u + \phi\pi_u z_n), & \text{if } T_n \in \mathfrak{T}_1 \\ (I_1 + \phi\pi_u z_n) \cup (I_2 + e_u + \phi\pi_u z_n), & \text{if } T_n \in \mathfrak{T}_2. \end{cases}$$

We picture the tiling in Figure 5.1. Note how the line segments in $\mathbb{R}^2$ project to tiles in X_u.

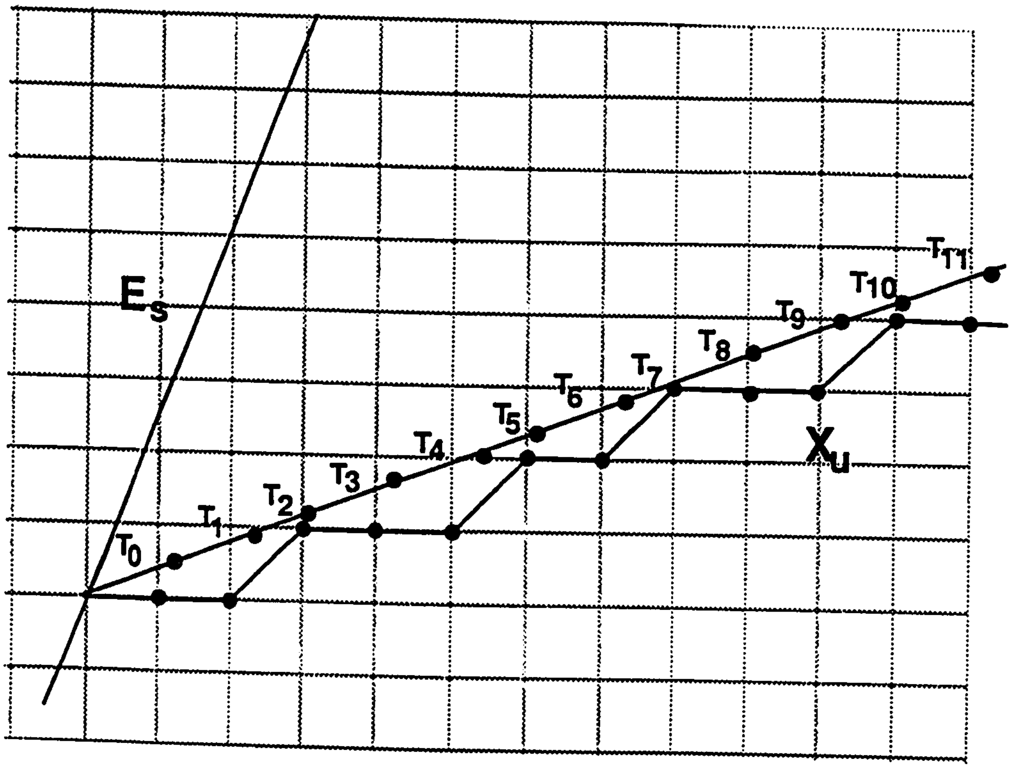

Figure 5.1: The tiling in Example 5.1.

Define $\gamma: \mathfrak{T} \to \mathfrak{T}$ by

$$\gamma(T_n) = I_1 + \phi\pi_u z_n$$

for each $T_n \in \mathfrak{T}$. Then $c(T_n) = \pi_u z_n$ and $\zeta_u c(T) = Z$. Moreover if

$$z \in (-\infty, 0]e_s \oplus X_u \cap \mathbb{Z}^2$$

then for some $N \geq 0$. $\phi^N z \in Z$. Note that

$$-\pi_s(\phi^N z) \in [0, \infty)e_s$$

so $X_s = [0,\infty)e_s$. It follows that by Corollary 3.11 Ω_1 and Ω_2 generate a periodic tiling of $\mathbb{R}^2$ mod $\mathbb{Z}^2$. The subdivision graph Γ for $\mathfrak{T}$ is given in Figure 5.2. The distinct edges are $\{\alpha_k\}_{k=1}^5$. Each edge α_k is also labeled with $L(\alpha_k)$.

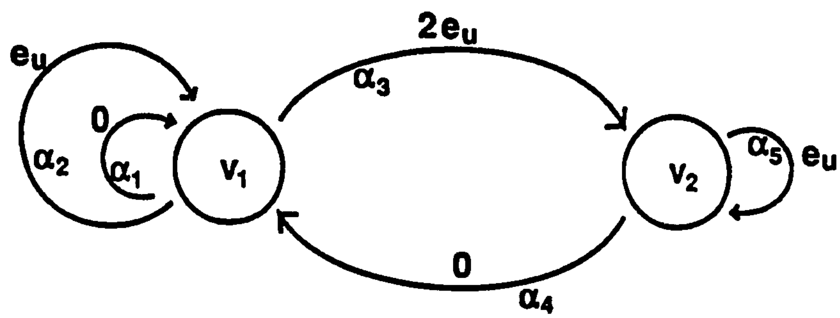

Figure 5.2: The subdivision graph for Example 5.1.

Let $\{R_{\alpha_k}\}_{k=1}^5$ be the rectangle partition of Ω which projects to a Markov partition for ϕ mod $\mathbb{Z}^2$. The partition is shown in Figure 5.3. The dots represent elements of $\mathbb{Z}^2$.□

The example given above comes from Adler and Weiss's paper [3]. It turns out that their constructions of Markov partitions for hyperbolic automorphisms of the 2-torus may be reproduced using tiling theory.

Let $\phi \in GL(2,\mathbb{Z})$ have an irreducible characteristic equation over $\mathbb{Z}$ and eigenvalues λ_1, λ_2 such that $|\lambda_1| > 1$ and $|\lambda_2| < 1$. Then λ_1 and λ_2 are real numbers. As in [3] we apply a conjugacy transformation of ϕ over $GL(n,\mathbb{Z})$ so that the characteristic line L_1 corresponding to λ_1 has positive slope less than 1 and the characteristic line L_2 corresponding to λ_2 has slope greater than 1. Let $e_u = \pi_u(1,0)$ and $e_s = -\pi_s(1,0)$. If $\lambda_2 > 0$ then we construct a tiling of E_u just as in the example. We project the lattice points closest to but below E_u onto E_u to form endpoints of the tiles.

For each $n \in \mathbb{Z}$ let $g(n) \in \mathbb{Z}$ such that

$$\pi_s(n, g(n)) \in (-\infty, 0]e_s, \text{ and}$$

$$\pi_s(n, g(n)+1) \in (0,\infty)e_s.$$

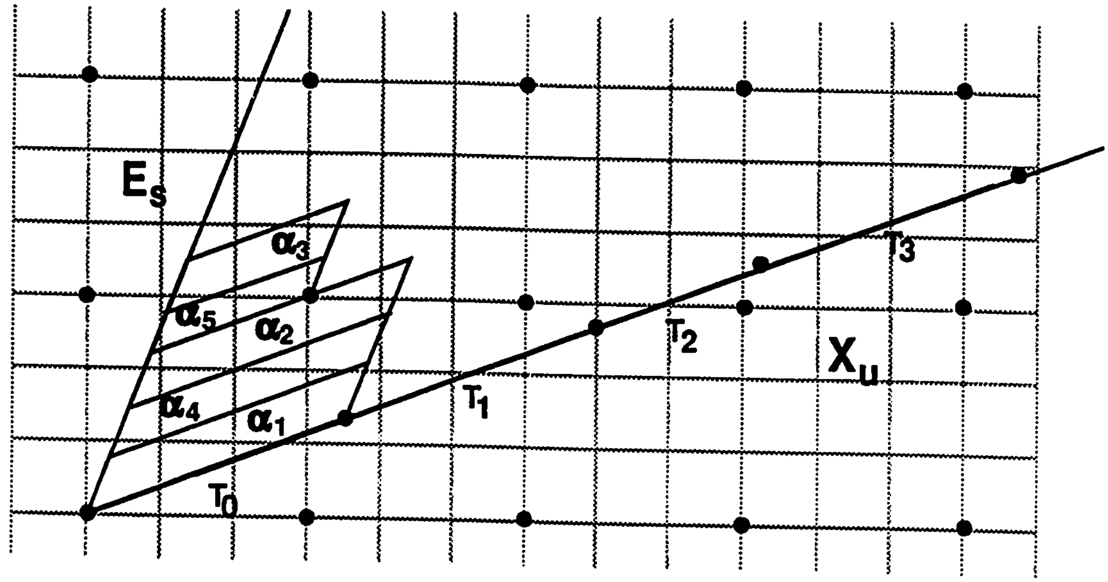

Figure 5.3: The rectangle partition for Example 5.1

Let

$$Z = \{(n, g(n)) \colon n \in \mathbb{Z}\}.$$

Let T_n be the line segment in E_u with endpoints $\pi_u z_n$ and $\pi_u z_{n+1}$. As in the example there are two tile types. Let

$$\mathfrak{T}_1 = \{T_n \colon g(n+1) - g(n) = 0\}$$

and

$$\mathfrak{T}_2 = \{T_n \colon g(n+1) - g(n) = 1\}.$$

Let $\mathfrak{T} = \mathfrak{T}_1 \cup \mathfrak{T}_2$. Then $\mathfrak{T}$ is a $\pi_u(\mathbb{Z}^2)$-finite self similar tiling of E_u with expansion map ϕ. This time $\mathfrak{T}$ has 2 generators, T_0 and T_{-1}. By the configuration of the lines L_1 and L_2 we know that $T_0 \in \mathfrak{T}_1$ and $T_{-1} \in \mathfrak{T}_2$. If $\lambda_1 > 0$ then $T_{-1} \subset \phi T_{-1}$ and $T_0 \subset \phi T_0$. In this case we define $\gamma(T_0) = T_0$ and $\gamma(T_{-1}) = T_{-1}$. Otherwise we define $\gamma(T_0) = T_{-1}$ and $\gamma(T_{-1}) = T_0$. In either case $c(T_0) = c(T_{-1}) = 0$. Moreover $\zeta_u c(\mathfrak{T})$ will be a subset of Z. Note that if z is an integer lattice point in $(-\infty, 0]e_s \oplus E_u$ then there exists an $N \geq 0$ such that $\phi^N z \in Z$. Moreover, if $\phi^N z$ is sufficiently close to E_u it will project to a common endpoint of a tile T_n in $\mathfrak{T}_1$ and a tile $T_{n-1} \in \mathfrak{T}_2$.

Hence by Theorem 3.10 the corresponding sets Ω_1 and Ω_2 induce a periodic tilings of $\mathbb{R}^2$ mod $\mathbb{Z}^2$. Note that the sets Ω_1 and Ω_2 correspond to the rectangles in [3].

If $\lambda_2 < 0$ then we construct a periodic self similar tiling of E_u with period 2. Let

$$Z^0 = \{z \in \mathbb{Z}^2 : \pi_s z \in (-1,1]\pi_s(0,1)\}.$$

Let

$$Z^1 = \{z \in \mathbb{Z}^2 : \pi_s z \in [-1,1)\pi_s(0,1)\}.$$

Then $\pi_u Z^0$ and $\pi_u Z^1$ are each a set of endpoints for a $\pi_u(\mathbb{Z}^2)$-finite tiling of E_u. Moreover, $\phi Z^0 \subset Z^1$ and $\phi Z^1 \subset Z^0$, for $(Z^0 - Z^1) \cup (Z^1 - Z^0) = \{(0,1),(0,-1)\}$. Let $\mathfrak{T}^0$ be the tiling of E_u with endpoints of tiles in $\pi_u Z^0$ and let $\mathfrak{T}^1$ be the tiling of E_u with endpoints of tiles in $\pi_u Z^1$. We note that outside the bounded set $[-1,1]e_u$ the tilings agree. At first we distinguish the tile types by length, we note there are three distinct lengths of tiles. We determine the subdivision rules for $\mathfrak{T}^0$ by looking outside $[-1,1]e_u$. We do an insplitting of the subdivision graph so that the tiles at the origin have 0 as a control point. Then the control points for $\mathfrak{T}^0$ are a subset of $\pi_u Z^0$. As in the example, for every $z \in \mathbb{Z}^2$ there exists $N \geq 0$ such that $\phi^N z$ is in Z^0. If $\phi^N z$ is sufficiently close to E_u then z projects to the endpoint of two tiles with the same pattern as the generators for $\mathfrak{T}^0$ or with the same pattern as the generators for $\mathfrak{T}^1$. It follows from Theorem 3.10 that the sets Ω_1 and Ω_2 form a periodic tiling of $\mathbb{R}^2$ mod $\mathbb{Z}^2$.

To illustrate this second case we consider another example.

Example 5.2 Let

$$\phi = \begin{pmatrix} -3 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then ϕ induces a hyperbolic automorphism of $\mathbb{R}^2$ mod $\mathbb{Z}^2$ with eigenvalues

$$\lambda_1 = \frac{-3-\sqrt{5}}{2} \text{ and } \lambda_2 = \frac{-3+\sqrt{5}}{2}.$$

The characteristic lines are the same as in Example 5.1. We will describe the tiles by the line segments with endpoints in $\mathbb{Z}^2$ which project to them.

Note that the line segments sketched in Figures 5.4 and 5.5 have endpoints differing by one of the vectors in $\{v_1 = (0,-1), v_2 = (1,2), v_3 = (1,1)\}$. Let I_1 be the

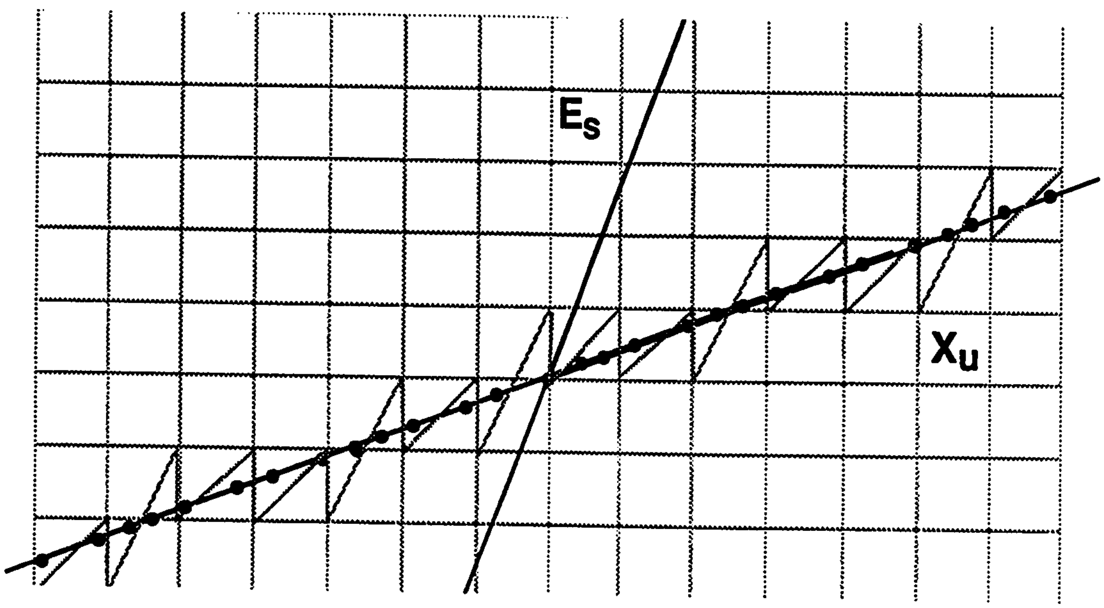

Figure 5.4: The tiling induced by Z^0 in Example 5.2.

line segment in E_u with endpoints 0 and $\pi_u v_1$. Let I_2 be the line segment in E_u with endpoints 0 and $\pi_u v_2$. Let I_3 be the line segment in E_u with endpoints 0 and $\pi_u v_3$. Then the tiles in $\mathfrak{T}^0$ and $\mathfrak{T}^1$ are $\pi_u\mathbb{Z}^2$-translates of the line segments I_1, I_2, and I_3.

Let $\mathfrak{T}_j^i = \{[a, b]e_u \in \mathfrak{T}^i : [0, b-a]e_u = I_j\}$. We define the subdivision rules by looking outside of $[-1, 1]e_u$ so there is no ambiguity in our choice. By using the same argument as in Example 5.1, we know that we may uniformly subdivide the image of all tiles in a type by determining how we subdivide one tile of each type. If $I_1 + x \in \mathfrak{T}_1^i$ then

$$\phi(I_1 + x) = (I_1 + \phi x) \cup (I_2 + \pi_u v_1 + \phi x) \cup (I_1 + \pi_u(v_1 + v_2) + \phi x)$$

and $I_1 + \phi x$, $I_1 + \pi_u(v_1 + v_2) + \phi x \in \mathfrak{T}_1^{i+1}$ while $I_2 + \pi_u v_1 + \phi x \in \mathfrak{T}_2^{i+1}$. If $I_2 + x \in \mathfrak{T}_2^i$ then

$$\phi(I_2 + x) = I_3 + \phi x \in \mathfrak{T}_3^{i+1}.$$

If $I_3 + x \in \mathfrak{T}_3^i$ then

$$\phi(I_3 + x) = (I_3 + \phi x) \cup (I_1 + \pi_u v_3 + \phi x) \cup (I_3 + \pi_u(v_3 + v_1) + \phi x)$$

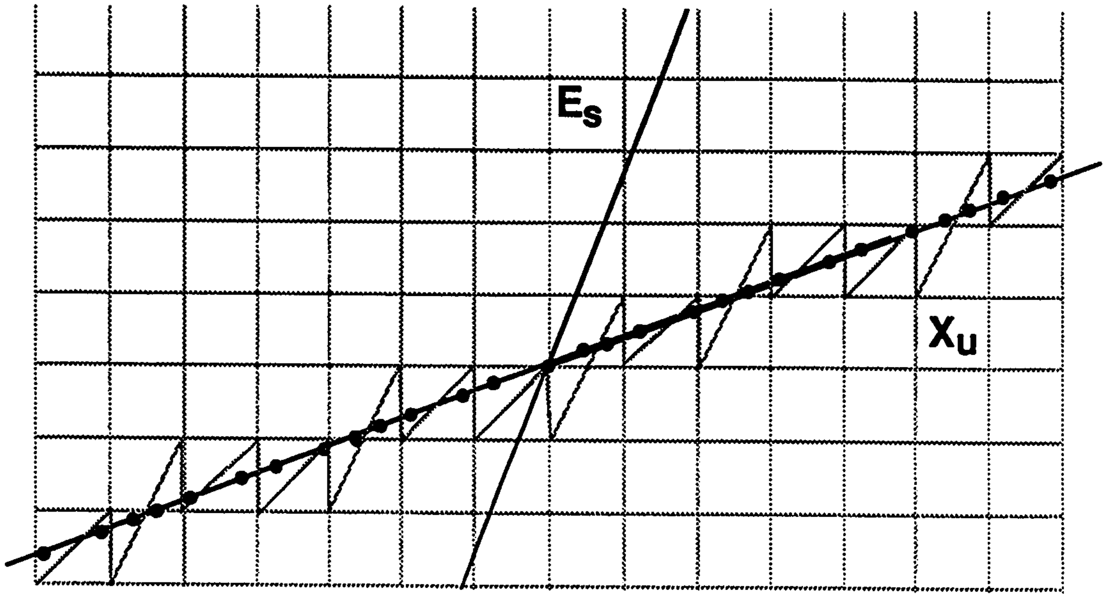

Figure 5.5: The tiling induced by Z^1 in Example 5.2.

and $I_3+\phi x$, $I_3+\pi_u(v_3+v_1)+\phi x \in \mathfrak{T}_3^{i+1}$ while $I_1+\pi_u v_3+\phi x \in \mathfrak{T}_1^{i+1}$. Let **A** represent a tile in $\mathfrak{T}_1^i$. Let **B** represent a tile in $\mathfrak{T}_2^i$. Let **C** represent a tile in $\mathfrak{T}_3^i$. Then the subdivision rules may be represented as substitution θ on words in $\{\mathbf{A},\mathbf{B},\mathbf{C}\}$. We find

$$\begin{aligned}\theta(\mathbf{A}) &= \mathbf{ABA}\\ \theta(\mathbf{B}) &= \mathbf{C}\\ \theta(\mathbf{C}) &= \mathbf{CAC}.\end{aligned}$$

The generators for $\mathfrak{T}^0$ are $I_1+\pi_u(0,1)$ and I_3. They may be represented in terms of their pattern as **AC**. The generators for $\mathfrak{T}^1$ are $I_3+\pi_u(-1,-1)$ and I_1. They may be represented in terms of their pattern as **CA**. Note that

$$\begin{aligned}\theta^2(\mathbf{A}) &= \mathbf{ABACABA}\\ \theta^2(\mathbf{B}) &= \mathbf{CAC}\\ \theta^2(\mathbf{C}) &= \mathbf{CACABACAC}.\end{aligned}$$

So the ϕ^2 image of every tile in $\mathfrak{T}^0$ contains the pattern **AC** and the ϕ^2 image of every tile in $\mathfrak{T}^1$ contains the pattern **CA**. Since every pattern is contained in the eventual image of either of these patterns, we see that $\mathfrak{T}^0$ is 2-mixing. As in Example 5.1. $\mathfrak{T}^0$ has a finite number of local patterns. So $\mathfrak{T}^0$ is a $\pi_u(\mathbb{Z}^2)$-finite periodic self similar tiling of E_u with expansion map ϕ and period 2.

Note that there is no way to define a tile map so that the origin is a control point. Let the left endpoint of each tile $T \in \mathfrak{T}^i$ be a positional point $d_i(T)$ for the tile. The subdivision graph Γ for $\mathfrak{T}^0$ is given in Figure 5.6 with each edge α labeled by $\zeta_u L_d(\alpha)$. Each vertex is labeled with the corresponding symbol which represents the tile type.

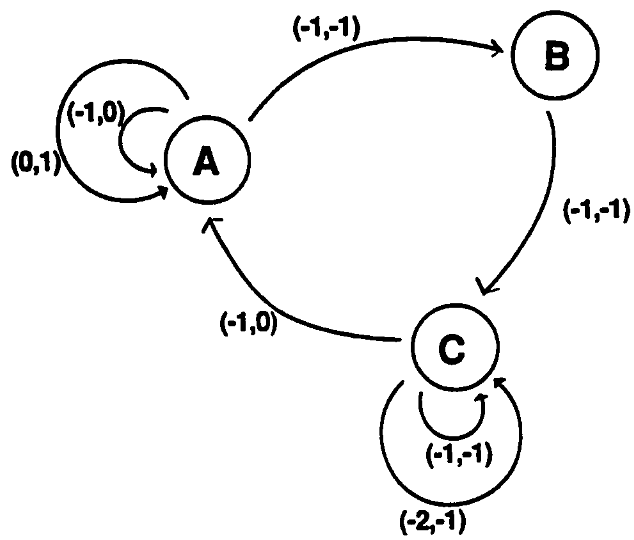

Figure 5.6: The graph Γ for Example 5.2.

We must perform an insplitting of Γ to get digit expansions. We start by letting $\mathcal{E}^{\mathbf{A}_1}$ be the edges given by

$$\Theta_d^{-1}(\mathbf{A}, \mathbf{A}, \pi_u(0,1))$$

and

$$\Theta_d^{-1}(\mathbf{C}, \mathbf{A}, \pi_u(-1,0))).$$

Let $\mathcal{E}^{\mathbf{A_2}}$ be the edge given by

$$\Theta_d^{-1}(\mathbf{A}, \mathbf{A}, \pi_u(-1,0)).$$

The new graph Γ^1 is given in Figure 5.7.

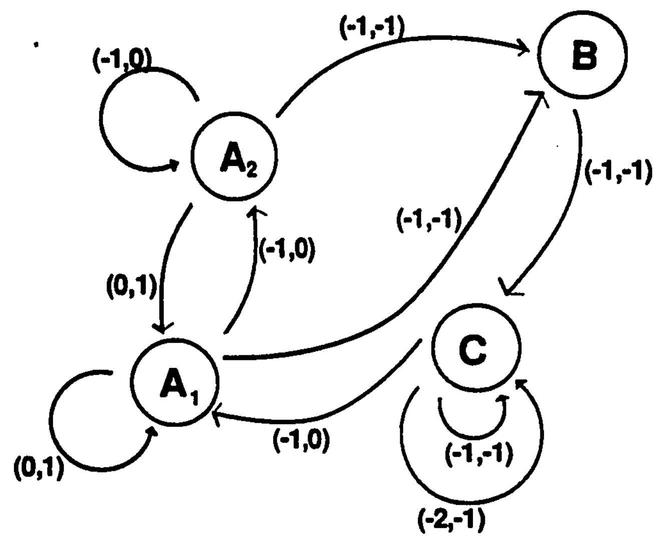

Figure 5.7: The graph Γ^1 for Example 5.2.

As we remarked at the end of Chapter 2 we could stop now. For every point in E_u has an eventual preimage in a generating tile of the type represented by $\mathbf{A}$. Moreover, the image of $\mathbf{A}$ has the pattern $\mathbf{ABA}$. If we let $\mathbf{A}_1$, $\mathbf{A}_2$ denote the tiles of type $\mathbf{A}$ in the new partition, then the ϕ image of $\mathbf{A}_i$ is $\mathbf{A}_2\mathbf{B}\mathbf{A}_1$. So we define $\gamma(\mathbf{A}_1) = \mathbf{A}_2$ and $\gamma(\mathbf{A}_2) = \mathbf{A}_1$. Then $c_i(\mathbf{A}_i) = 0$. We may fiddle with Theorem 3.10 to show we obtain a periodic tiling of $\mathbb{R}^2$ mod $\mathbb{Z}^2$. In fact this is precisely the periodic tiling used in [3] to form a Markov partition for ϕ mod $\mathbb{Z}^2$. To be consistent, however, we will insplit $\mathcal{E}^{\mathbf{C}}$ so that every generator has 0 as its control point.

Let $\mathcal{E}^{\mathbf{C_1}}$ be the edges in Γ^1 given by

$$\Theta_d^{-1}(\mathbf{B}, \mathbf{C}, \pi_u(-1, -1))$$

and

$$\Theta_d^{-1}(\mathbf{C}, \mathbf{C}, \pi_u(-1, -1)).$$

Let $\mathcal{E}^{C_2}$ be the edges in Γ^1 given by

$$\Theta_d^{-1}(C, C, \pi_u(-2, -1)).$$

The new graph Γ^2 is in Figure 5.8.

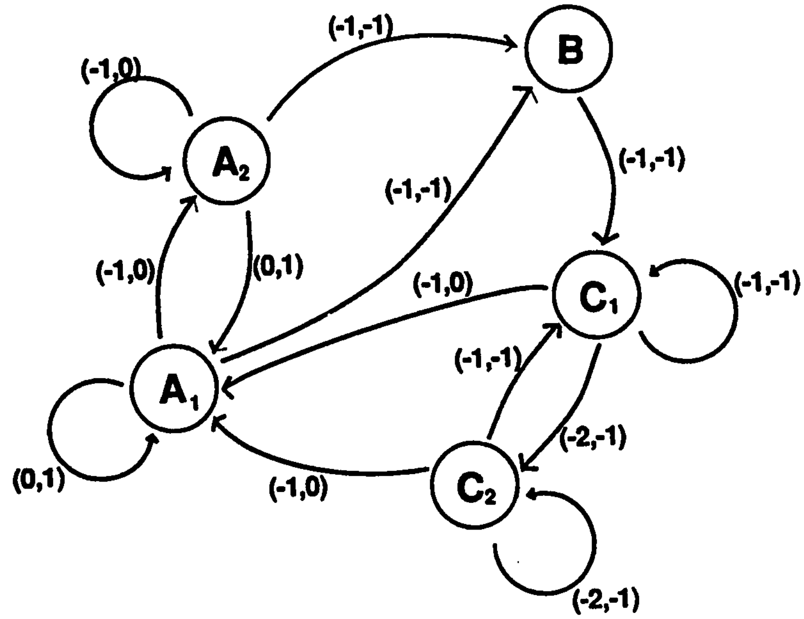

Figure 5.8: The graph Γ^2 for Example 5.2.

The periodic tiling of $\mathbb{R}^2$ mod $\mathbb{Z}^2$ is given in Figure 5.9. If we erased the line segments separating the rectangles corresponding to C_1 and C_2 then we would obtain the periodic tiling used in [3].□

Hyperbolic automorphisms of the 3-torus.

In this section we consider Bedford's construction in [4] of periodic tilings for $\mathbb{R}^3$ mod $\mathbb{Z}^3$ which induce Markov partitions.

Bedford considers hyperbolic automorphisms ϕ of the 3-torus satisfying the following conditions.

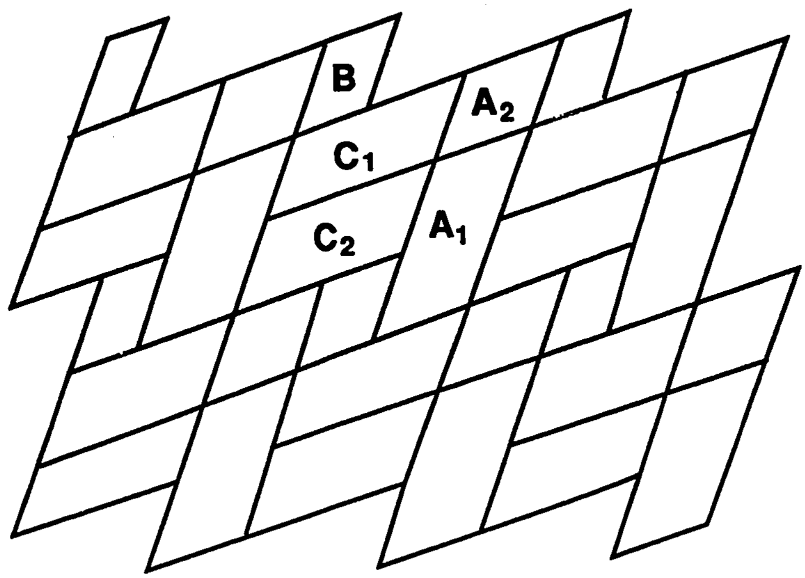

Figure 5.9: The periodic tiling from Example 5.2.

- ϕ^{-1} is given by a nonnegative matrix.
- We have $\dim(E_s) = 1$ and $\dim(E_u) = 2$.
- The contracting eigenvalue of ϕ is positive.
- For some E_u-neighborhood of the origin H, which is defined in the proof, $H \subset \phi H$.

While creating a periodic tiling of $\mathbb{R}^3$ mod $\mathbb{Z}^3$ he constructs a self similar tiling of E_u. The tiling he creates comes from a *stepped surface*.[4, p. 61] The conditions on ϕ insure that E_s intersects the interior of the positive cone in $\mathbb{R}^3$, while E_u intersects the positive cone only at the origin. Let $e_s = \pi_s(1,0,0)$. Let e_1, e_2, e_3 be the standard basis vectors for $\mathbb{R}^3$. Let I^3 be the unit cube with the origin and the three basis vectors as vertices.

We remove from $[0,\infty)e_s \oplus E_u$ any cube I^3+p, $p \in \mathbb{Z}^3$, which has interior points in E_u. The boundary of the remaining cubes in $[0,\infty)e_s \oplus E_u$ forms a stepped surface. It is made up of faces of the remaining cubes. For $i \neq j \neq k \neq i$, $i,j,k \in \{1,2,3\}$, let F_i be the square with vertices 0, e_j, e_k, and e_j+e_k. Then the stepped surface is composed of $\mathbb{Z}^3$ translates of F_1, F_2, F_3. The projection of each face of the stepped surface onto E_u gives a parallelogram tiling of E_u which is $\pi_u(\mathbb{Z}^3)$-finite and quasi-homogeneous. There are three tile types, given by the projections of the three different face types.

To make a self similar tiling a Dekking like construction is performed. That is, the boundary of each parallelogram tile P is deformed by using ϕ preimages of the boundaries of the parallelograms which ϕP intersects while keeping the vertices of ϕP fixed. This is done in a consistent way for all parallelograms of the same type. By repeating the procedure an infinite number of times the deformed boundaries of the parallelograms converge in the Hausdorf metric to a ϕ-invariant subset of E_u. Invariant in this sense means the ϕ image of the set is contained in the set. The new tiling has the property that the image of each tile subdivides into a finite number of tiles and the subdivision rule depends only on the tile type of the parallelogram which it came from. Let $\mathfrak{T}$ be the self similar tiling formed in this way. Then $\mathfrak{T}$ has three generators. The generators have different tile types as they come from the projection of the three faces F_1, F_2, F_3.

Between E_u and the stepped surface there are no integral lattice points. Every point in $[0,\infty)e_s \oplus E_u$ has an eventual ϕ image to a vertex on the stepped surface. Moreover, if z is a vertex of the stepped surface which is sufficiently close to E_u then $F_i + z$ is a face on the stepped surface for each $i \in \{1,2,3\}$. If γ is a generating tile map for $\mathfrak{T}$, then z projects to a control point for three tiles whose union has the same pattern as the generators for $\mathfrak{T}$. By Theorem 3.10 it follows that the self similar tiling is a Markov tiling of E_u.

This is all a bit ad hoc since Bedford's construction of $\mathfrak{T}$ relied on the fact that he already knew that he had a periodic tiling of $\mathbb{R}^3$ mod $\mathbb{Z}^3$. Recent work by Kenyon [7] indicates, however, that it may be possible to construct Markov tilings for E_u from stepped surfaces or similar structures without knowing ahead of time that the tiling will induce a periodic tiling in the higher dimensional space. (Kenyon uses a Delauney triangulation.) Moreover, such constructions should not depend on the position of the eigenspaces.

The β-shift.

Let β be a Pisot number. This means that β is an algebraic integer greater than 1 with Galois conjugates all having modulus less than 1. It is well known that there is a one-sided shift space which behaves like multiplication times β mod 1. Define $T_\beta: [0,1) \to [0,1)$ by

$$T_\beta(x) = \beta x - [\beta x],$$

where $[\beta x]$ denotes the greatest integer less than or equal to βx. For each $x \in [0,1)$ there is a well defined digit representation

$$x = \sum_{k=1}^{\infty} \beta^{-k} b_k$$

where

$$b_k = [\beta {T_\beta}^{k-1} x].$$

We call this the β-expansion of x. Moreover there is a sequence denoted by carry(β) = $a_1 a_2 a_3 \ldots$ such that $b_1 b_2 b_3 \ldots$ is lexicographically less than $a_1 a_2 a_3 \ldots$. We say $\{b_k\}_{k=1}^{\infty}$ is lexicographically less than $\{a_k\}_{k=1}^{\infty}$ if for some $N \geq 0$ we have $b_k = a_k$ for all $k < N$ and $b_N < a_N$. (If $N = 0$ then $b_1 < a_1$.) We call carry(β) the carry sequence for β as in [12] and note that carry(β) works for numbers in base β just as the sequence $999999\cdots$ works for numbers in base 10. In particular

$$1 = \sum_{k=1}^{\infty} \beta^{-k} a_k.$$

To find carry(β) we let a_1 be the greatest integer strictly less than β. Let a_2 be the greatest integer strictly less than $\beta(\beta - a_1)$. Let a_i be the greatest integer strictly less than $\beta^i - \sum_{k=1}^{i-1} \beta^{i-k} a_k$. Since β is Pisot, carry(β) is an eventually repeating sequence which has an infinite number of non-zero entries.

The β-shift Σ_β is the set of sequences in $\{0, 1, \ldots, [\beta]\}$ which are lexicographically less than or equal to carry(β). The shift operator is the one-sided shift σ_β such that

$$\sigma_\beta(b_1 b_2 b_3 \ldots) = b_2 b_3 b_4 \ldots.$$

Let $x \in [0,1)$ and let $\{b_k\}_{k=1}^{\infty} \in \Sigma_\beta$ such that

$$x = \sum_{k=1}^{\infty} \beta^{-k} b_k.$$

Then

$$\beta x = b_1 + \sum_{k=1}^{\infty} \beta^{-k} b_{k+1}.$$

If $\{b_{k+1}\}_{k=1}^{\infty}$ is lexicographically less than carry(β) then we have

$$T_\beta x = \sum_{k=1}^{\infty} \beta^{-k} b_{k+1}.$$

Otherwise we have $\sum_{k=1}^{\infty} \beta^{-k} b_{k+1} = 1$ and $\beta x = b_1 + 1$. In this case $T_\beta x = 0$.

Suppose β is a Pisot unit. That is, the product of β and its Galois conjugates is 1. Suppose $\mathbb{Z}[\beta]$ has dimension n. Let $\phi \in \mathrm{GL}(n, \mathbb{Z})$ be the companion matrix with characteristic polynomial equal to the minimal polynomial for β. Then ϕ induces a hyperbolic automorphism of $\mathbb{R}^n \bmod \mathbb{Z}^n$. Let e be a unit vector in $\mathbb{Z}^n$. Then $\mathbb{Z}[\phi]e = \mathbb{Z}^n$. Let $e_u = \pi_u \dot{e}$, then $E_u = \mathbb{R}e_u$. We construct a self similar tiling $\mathfrak{T}$ of $X_u = [0, \infty)e_u$. If

$$\text{carry}(\beta) = a_1 a_2 \ldots a_q (a_{q+1} \ldots a_{q+p})$$

(where $(a_{q+1} \ldots a_{q+p})$ is the repeating part) then $\mathfrak{T}$ will have $q+p$ sets in the tile type partition denoted by $\mathfrak{T}_j$, $j \in \{1, 2, 3, \ldots q+p\}$. Let T_1 be the line segment of E_u with endpoints 0 and e_u. Let $T_1 \in \mathfrak{T}_1$. Then $\phi T_1 = \beta T_1$ has endpoints 0 and βe_u. Let $T_1 + ke_u \in \mathfrak{T}_1$ for $k \in \{0, 1, 2, 3, \ldots a_1 - 1\}$. Let $[a_1, \beta]e_u \in \mathfrak{T}_2$. Let $T_2 = [0, \beta - a_1]e_u$. Then $T_2 + a_1 \in \mathfrak{T}_2$. We have $\phi(T_2 + a_1) = \beta T_2 + \beta a_1 = [a_1\beta, \beta^2]e_u$. Since a_2 is the greatest integer less than $\beta^2 - \beta a_1$ we let

$$T_1 + (a_1\beta + k)e_u \in \mathfrak{T}_1$$

for each $k \in \{0, \ldots, a_2 - 1\}$. We let $[a_1\beta + a_2, \beta^2]e_u \in \mathfrak{T}_3$. Let $T_3 = [0, \beta^2 - a_1\beta - a_2]e_u$. We repeat this procedure for $j \in \{3, 4, \ldots q+p\}$. Since carry($\beta$) is eventually repeating we see that

$$\beta^{q+p} - \beta^{q+p-1}a_1 - \cdots - a_{q+p} = \beta^q - a_1\beta^{q-1} - \cdots - a_q.$$

Note

$$T_{q+p} = [0, \beta^{q+p-1} - \sum_{k=1}^{q+p-1} \beta^{q+p-1-k} a_k]e_u$$

and

$$\begin{aligned}\beta(\beta^{q+p-1} - \sum_{k=1}^{q+p-1} \beta^{q+p-1-k} a_k) &= \beta^{q+p} - \sum_{k=1}^{q+p-1} \beta^{q+p-k} a_k \\ &= a_{q+p} + \beta^q - \sum_{k=1}^{q-1} \beta^{q-1-k} a_k.\end{aligned}$$

So if $T = [x, y]e_u \in \mathfrak{T}_{q+p}$ then ϕT is subdivided into a_{q+p} tiles in $\mathfrak{T}_1$ and 1 tile in $\mathfrak{T}_{q+1}$. That is, $T_1 + (\phi x + k)e_u \in \mathfrak{T}_1$ for $k \in \{0, \ldots, a_{q+p}\}$ and $T_{q+1} + (\phi x + a_{q+p}) \in \mathfrak{T}_{q+1}$.

This defines the subdivision rules for $\mathfrak{T}$ and the tile types. Since carry(β) has an infinite number of nonzero terms, every tile in $\mathfrak{T}$ has an eventual image which contains a tile in $\mathfrak{T}_1$. Since T_1 is the only generator for $\mathfrak{T}$, we have that $\mathfrak{T}$ is mixing. Since there are a finite number of tile types and line segments may be arranged in only a finite number of ways in a bounded region of X_u, $\mathfrak{T}$ has a finite number of local patterns. Moreover, the endpoints of tiles in $\mathfrak{T}$ lie in $\mathbb{Z}[\beta]e_u$. Hence the endpoints are the projection of points in $\mathbb{Z}[\phi]e = \mathbb{Z}^n$. So $\mathfrak{T}$ is a $\pi_u(\mathbb{Z}^n)$-finite self similar tiling of X_u with expansion map ϕ. Let γ be a generating tile map such that the left endpoint for each tile is its control point. Then $c(\mathfrak{T}) \subset \pi_u(\mathbb{Z}^n)$. the subdivision graph is given in Figure 5.10. Each edge α is labeled with $L(\alpha)$. Multiple edges from v_j to v_0 are indicated with a thick arrow and the range of the labels is given. Since every infinite path in the graph corresponds to an element of the β-shift, we'll call this the subdivision graph for the β-shift.

We are interested to know when $\mathfrak{T}$ is a Markov tiling. We note that $\mathfrak{T}$ is certainly a Markov tiling if every element of $\mathbb{Z}^n$ which projects to X_u has an eventual image which projects to a control point for $\mathfrak{T}$. Or equivalently, if every nonnegative element of $\mathbb{Z}[\beta]$ has a finite β-expansion.

Frougny and Solomyak have given sufficient conditions for the nonnegative elements of $\mathbb{Z}[\beta]$ to have finite β-expansions [6]. Let

$$x^n - d_1 x^{n-1} - \cdots - d_{n-1} x - d_n$$

be the minimal polynomial for β. If

$$d_1 \geq d_2 \geq \cdots \geq d_{n-1} \geq d_n = 1$$

then every nonnegative element of $\mathbb{Z}[\beta]$ has a finite β-expansion. So $\mathfrak{T}$ is a Markov

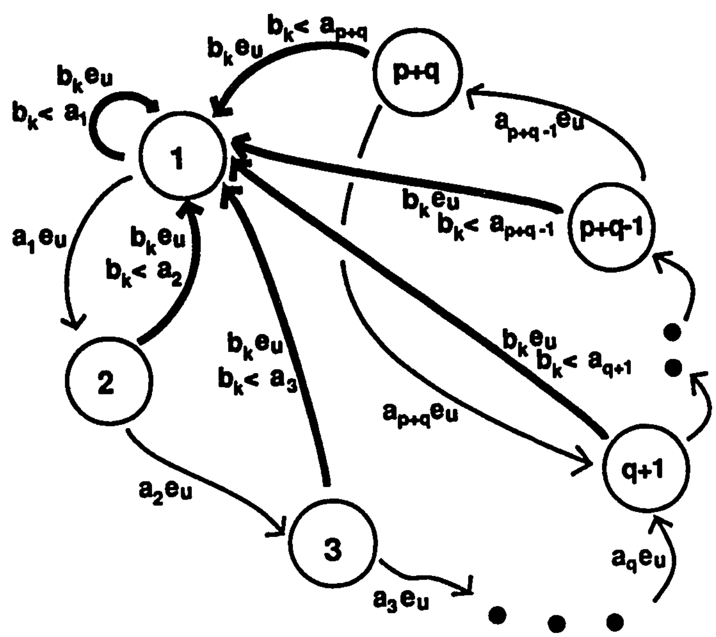

Figure 5.10: The subdivision graph for the β-shift.

tiling. In this case the two sided extension of the β-shift is a symbolic representation for $\phi \bmod \mathbb{Z}^n$.

Let $\phi \in \mathrm{GL}(n, \mathbb{Z})$ be a hyperbolic automorphism of $\mathbb{R}^n$ with characteristic polynomial χ_ϕ. Suppose χ_ϕ is the minimal polynomial for a Pisot number β. We apply the above construction to tile $X_u = [0, \infty)e_u$, for $e_u = \pi_u e$. The tiling $\mathfrak{T}$ is a $\pi_u(\mathbb{Z}^n)$-finite self similar tiling of X_u with expansion map ϕ. If $\mathbb{Z}[\phi]e = \mathbb{Z}^n$ and every non-negative element of $\mathbb{Z}[\beta]$ has a finite β-expansion then $\mathfrak{T}$ is a Markov tiling. If every non-negative element of $\mathbb{Z}[\beta]$ has a finite β-expansion but $\mathbb{Z}[\phi]e \neq \mathbb{Z}^n$ then $\mathfrak{T}$ induces a periodic tiling of $\mathbb{R}^n$ mod $\mathbb{Z}[\phi]e$.

Example 5.3 Let

$$\phi = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then ϕ has characteristic polynomial $x^3 - x^2 - x - 1$. The eigenvalues for ϕ are λ_1, λ_2, λ_3 where $\lambda_1 > 1$ and $\lambda_2 = \overline{\lambda_3}$ has modulus less than 1. Note that $x^3 - x^2 - x - 1$

satisfies the conditions of Frougny and Solomyak's theorem. Moreover $\mathbb{Z}[\phi]e = \mathbb{Z}^n$. So $\phi \bmod \mathbb{Z}^3$ is metrically similar to the two sided extension of the λ_1-shift. The carry sequence for λ_1 is (110). The subdivision graph is given in Figure 5.11. Each edge α is labeled with $L(\alpha)$.

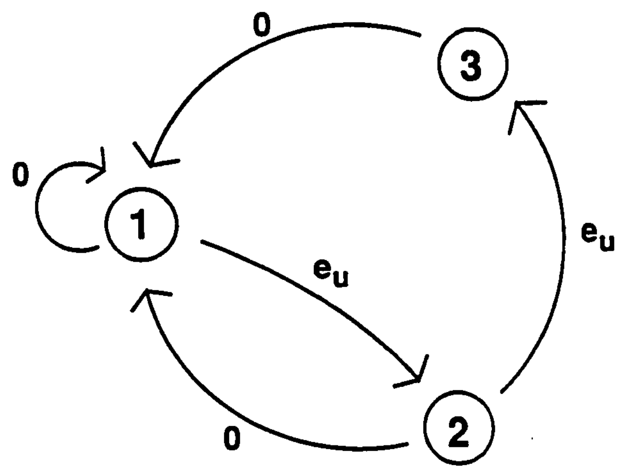

Figure 5.11: The subdivision graph for Example 5.3.

The sets $\pi_s(\Omega_1)$, $\pi_s(\Omega_2)$, $\pi_s(\Omega_3)$ which arise out of this tiling coincide with the basic tiles found by Rauzy in [10]. (It was by noting this fact that I got the idea for the approach used in this thesis.) We note also that $\pi_s(\Omega_1)$, $\pi_s(\Omega_2)$, $\pi_s(\Omega_3)$ are also translates of the three basic tiles used in Bedford's construction. We sketch the sets Ω_1, Ω_2, Ω_3 in Figure 5.12. The angles between the eigenvectors are distorted so that it is possible to see the rectangles. The bold lines indicate surfaces in front while the dotted lines indicate surfaces behind.□

Pisot numbers.

Let $\phi \in \mathrm{GL}(n, \mathbb{Z})$ be a hyperbolic automorphism of $\mathbb{R}^n$ with characteristic polynomial χ_ϕ. Suppose that χ_ϕ is the minimal polynomial for a Pisot number β. Let e be a unit vector in $\mathbb{Z}^n$. If $\mathbb{Z}[\phi]e$ is not equal to $\mathbb{Z}^n$ or if the nonnegative elements of $\mathbb{Z}[\beta]$ do not always have finite β-expansions then we must look for a self similar tiling different from the tiling given by the β-shift to construct a Markov partition for $\phi \bmod \mathbb{R}^n$.

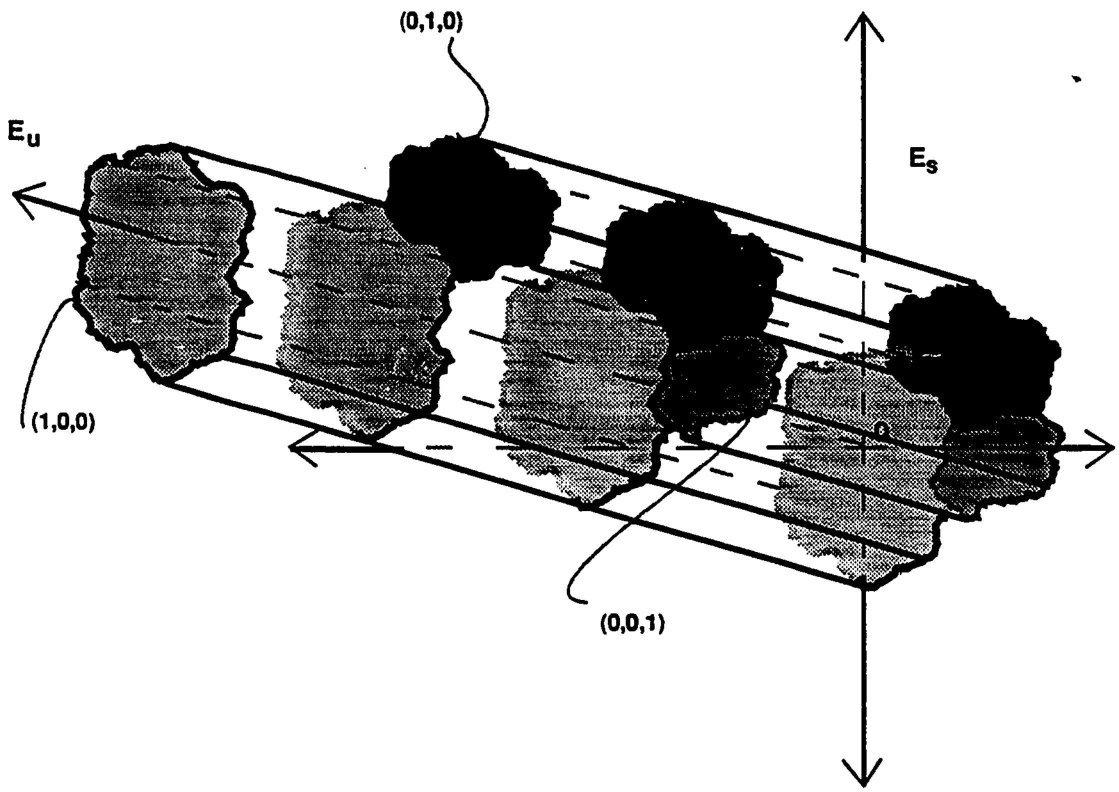

Figure 5.12: The periodic tiling in Example 5.3.

The following construction was suggested by Thurston. Let $e_u = \pi_u e$ and $\|\cdot\|$ be the norm defined in Chapter 3 which exhibits the expansive properties of $\phi|_{E_u}$ and $\phi|_{E_s}^{-1}$.

Proposition 5.4 *There exists a self similar tiling $\mathfrak{T}$ of $X_u = [0,\infty)e_u$ with expansion map ϕ $(= \phi|_{E_u})$ and a tile map $\gamma: \mathfrak{T} \to \mathfrak{T}$ satisfying the following conditions.*

(1) $\mathfrak{T}$ is a collection of connected sets in X_u of the form $[t_{i-1}, t_i]e_u$ for $i \geq 1$, $t_i \in [0,\infty)$ such that $t_0 = 0$ and $t_i < t_j$ for each $i < j$.

(2) $T_0 = [0, t_1]e_u$ is the only generator for $\mathfrak{T}$.

(3) For each $T = [t_{i-1}, t_i]e_u \in \mathfrak{T}$, $c(T) = t_{i-1}e_u$.

(4) *There exists a* $k_0 \geq 0$ *such that* $\gamma^k T$ *has the same tile type as* T_0 *for all* $k \geq k_0$, $T \in \mathfrak{T}$.

(5) $c(\mathfrak{T}) \subset \pi_u(\mathbb{Z}^n)$.

(6) $\cup_{k=0}^{\infty} \phi^{-k} c(T) = \pi_u[\mathbb{Z}^n \cap (E_s \oplus X_u)]$.

Proof. Let

$$U = \{x \in E_s : \|x\| < 1\}.$$

Let $C = U \oplus X_u$ and $Y = C \cap \mathbb{Z}^n$. Let $\hat{e_u}$ be an eigenvector for the transpose of ϕ such that $\hat{e_u} \cdot e_u = 1$. Since χ_ϕ is irreducible over $\mathbb{Z}$, $\mathbb{R}^n$ has no proper ϕ-invariant subspaces which intersect $\mathbb{Z}^n - \{0\}$. Hence the set Y is in one-one correspondence with the set $\hat{e_u} \cdot Y \subset [0, \infty)$. Let $Y = \{z_i\}_{i=0}^{\infty}$ such that $z_0 = 0$ and $\hat{e_u} \cdot z_i < \hat{e_u} \cdot z_j$ for all $i < j$. Note that $\phi Y \subset Y$. Since U is bounded there is a finite collection of vectors $V = \{v_\alpha\}_{\alpha=1}^{m}$ such that $z_i - z_{i-1} \in V$ for each $i \geq 1$. Each tile in $\mathfrak{T}$ will be congruent to a line segment in X_u of the form $[0, \hat{e_u} \cdot \phi^k v_\alpha] e_u$ for some bounded collection of k. The endpoints of the segments will be the projection along E_s of integral lattice points. Suppose for a moment that $w \in Y$, $v_\alpha \in V$, and

$$T = [\hat{e_u} \cdot w, \hat{e_u} \cdot (w + v_\alpha)] e_u \in \mathfrak{T}.$$

We wish to choose $k_\alpha > 0$ independent of w such that $\phi^{k_\alpha} w \in \phi C$ and $\phi^{k_\alpha} v_\alpha \in C$. Having chosen k_α we would let $\phi^N T$ be a tile in T for each $N < k_\alpha$. We will subdivide $\phi^{k_\alpha} T$ by translating $\phi^{k_\alpha} w$ to the origin. Suppose $\phi^{k_\alpha} v_\alpha = z_q \in Y$. Then

$$\phi^{k_\alpha} T = \cup_{i=1}^{q} [\hat{e_u} \cdot (\phi^{k_\alpha} w + z_{i-1}), \hat{e_u} \cdot (\phi^{k_\alpha} w + z_i)] e_u.$$

We let $[\hat{e_u} \cdot (\phi^{k_\alpha} w + z_{i-1}), \hat{e_u} \cdot (\phi^{k_\alpha} w + z_i)] e_u$ be a tile in $\mathfrak{T}$ for each $i \in \{1, \ldots, q\}$. At this point we could repeat the process on each of these new line segments. The problem with choosing k_α is that $\phi^{k_\alpha} w + z_i$ may or may not lie in C. In order to insure (6) we must be certain that the lattice points which determine the endpoints of our tiles do not get too far away from X_u. Since $\phi w + z_i \in \phi w + C$ and $\phi w \in \phi C$ we have

$$\|\pi_s(\phi w + z_i)\| < 1 + \delta$$

where $\delta = \sup\{\|x\|: x \in \phi U\} < 1$ (since $\phi|_{E_s}$ is strictly contracting in this norm). Let

$$U_1 = \{x \in E_s: \|x\| < 1 + \delta\}.$$

There exists a $k_1 > 0$ such that $\phi^{k_1} U_1 \subset \phi C$. Choose $k_\alpha \geq k_1$ so that $\phi^{k_\alpha} v_\alpha \in C$.

To insure that $\mathfrak{T}$ is self similar, we must have a tile T_0 such that $T_0 \subset \phi T_0$. Let $T_0 = [0, \hat{e_u} \cdot z_1] e_u$. To insure quasi-homogeneity we will want tiles of the same type as T_0 to be scattered uniformly throughout $\mathfrak{T}$. Let

$$J \times K = \{(0,0)\} \cup \{(\alpha, N)\}_{1 \leq \alpha \leq m, 0 \leq N < k_\alpha}.$$

We will use $J \times K$ to index a partition of $\mathfrak{T}$ into its tile types. Let $\{\mathfrak{T}_{(j,k)}\}_{(j,k) \in J \times K}$ be this partition.

We define $\mathfrak{T}$ inductively by defining the sets $\{\mathfrak{T}_{(j,k)}\}_{(j,k) \in J \times K}$. We will let $\mathfrak{T}_{(0,0)}$ be the set of tiles in $\mathfrak{T}$ of the same type as T_0. Suppose $T = [\hat{e_u} \cdot w, \hat{e_u} \cdot (w + z_1)] e_u \in \mathfrak{T}_{(0,0)}$. Suppose $\phi z_1 = z_p$. Then

$$\phi T = \cup_{i=1}^{p} [\hat{e_u} \cdot (\phi w + z_{i-1}), \hat{e_u} \cdot (\phi w + z_i)] e_u.$$

Let

$$[\hat{e_u} \cdot \phi w, \hat{e_u} \cdot (\phi w + z_1)] e_u \in \mathfrak{T}_{(0,0)}$$

and

$$[\hat{e_u} \cdot (\phi w + z_{i-1}), \hat{e_u} \cdot (\phi w + z_i)] e_u \in \mathfrak{T}_{(\alpha,0)}$$

for each $i \in \{2, \ldots, p\}$, $z_i - z_{i-1} = v_\alpha$. Suppose $T = [\hat{e_u} \cdot w, \hat{e_u} \cdot (w + v_\alpha)] e_u \in \mathfrak{T}_{(\alpha,0)}$. Then for each $N < k_\alpha$ let $\phi^N T \in \mathfrak{T}_{(\alpha,N)}$. We will subdivide $\phi^{k_\alpha} T$ as indicated above. That is, suppose $\phi^{k_\alpha} v_\alpha = z_q$. Then

$$\phi^{k_\alpha} T = \cup_{i=1}^{q} [\hat{e_u} \cdot (\phi^{k_\alpha} w + z_{i-1}), \hat{e_u} \cdot (\phi^{k_\alpha} w + z_i)] e_u.$$

Let

$$[\hat{e_u} \cdot \phi^{k_\alpha} w, \hat{e_u} \cdot (\phi^{k_\alpha} w + z_1)] e_u \in \mathfrak{T}_{(0,0)}$$

and

$$[\hat{e_u} \cdot (\phi^{k_\alpha} w + z_{i-1}), \hat{e_u} \cdot (\phi^{k_\alpha} w + z_i)] e_u \in \mathfrak{T}_{(\alpha',0)}$$

for each $i \in \{2, \ldots, q\}$, $z_i - z_{i-1} = v_{\alpha'}$.

If $T \in \mathfrak{T}_{(\alpha,N)}$, $0 < N < k_\alpha - 1$ then $\phi T \in \mathfrak{T}_{(\alpha,N+1)}$. If $T \in \mathfrak{T}_{(\alpha,k_\alpha-1)}$ then $T' = \phi^{-(k_\alpha-1)}T \in \mathfrak{T}_{(\alpha,0)}$ and we subdivide ϕT into the tiles found in $\phi^{k_\alpha}T'$, along with their tile types.

In this way we define a tiling $\mathfrak{T}$ of X_u. We wish to show that $\mathfrak{T}$ is a self similar tiling of X_u with expansion map ϕ. We see by the way that $\mathfrak{T}$ was defined that $\mathfrak{T}$ has a finite number of tile types. The way that the image of a tile in $\mathfrak{T}$ subdivides in $\mathfrak{T}$ depends only on its type. Let $k_0 = \max_{1\leq\alpha\leq m}\{k_\alpha\}$, then for every $T \in \mathfrak{T}$, $\phi^{k_0}T$ contains a tile in $\mathfrak{T}_{(0,0)}$. Since T_0 is the only generator for $\mathfrak{T}$, this insures that $\mathfrak{T}$ is quasi-homogeneous.

Next we wish to define a tile map $\gamma: \mathfrak{T} \to \mathfrak{T}$ such that γ satisfies properties (3), (4), (5), and (6). Suppose $T \in \mathfrak{T}_{(0,0)}$ then define $\gamma(T)$ to be the unique tile in ϕT which belongs to $\mathfrak{T}_{(0,0)}$. Suppose $T \in \mathfrak{T}_{(\alpha,N)}$. If $N < k_\alpha - 1$ let $\gamma(T) = \phi T$. If $N = k_\alpha - 1$ let $\gamma(T)$ be the unique tile in ϕT which belongs to $\mathfrak{T}_{(0,0)}$. Note that for all $T \in \mathfrak{T}$ we have $\gamma^k(T)$ is a tile of the same type as T_0, for all $k \geq k_0$.

The left endpoint of T is $c(T)$ for every $T \in \mathfrak{T}$. Clearly $c(\mathfrak{T}) \subset \pi_u(\mathbb{Z}^n)$. All that is left to show is that $\cup_{k=0}^{\infty}\phi^{-k}c(T) = \pi_u[\mathbb{Z}^n \cap (E_s \oplus X_u)]$. Recall

$$\sup\{\|x\|: x \in \phi U\} = \delta < 1.$$

Let

$$U_2 = \{x \in E_s: \|x\| < 1 - \delta\}.$$

Let $w \in Y \subset C$ then $\|\pi_s(\phi w)\| < \delta$. So if $x \in U_2$ then

$$\|x - \pi_s(\phi w)\| \leq \|x\| + \|\pi_s(\phi w)\| < 1.$$

In particular

$$U_2 \subset U + \pi_s(\phi w).$$

Suppose that $T = [\hat{e_u} \cdot w, \hat{e_u} \cdot (w + z_1)]e_u \in \mathfrak{T}_{(0,0)}$. Then $c(\mathfrak{T})$ contains the projection along E_s to E_u of every integral lattice point in

$$(U \oplus [0, \hat{e_u} \cdot \phi z_1]e_u) + \phi w$$

and

$$\pi_u(\mathbb{Z}^n \cap (U_2 \oplus \phi T)) \subset c(\mathfrak{T}).$$

Likewise, suppose that $T = [\hat{e_u} \cdot w, \hat{e_u} \cdot (w + \phi^N v_\alpha)]e_u \in \mathfrak{T}_{(\alpha,N)}$. Then $c(\mathfrak{T})$ contains the projection along E_s to E_u of every integral lattice point in

$$U \oplus [0, \hat{e_u} \cdot \phi^{k_\alpha} v_\alpha]e_u + \phi^{k_\alpha - N} w$$

and

$$\pi_u \left(\mathbb{Z}^n \cap (U_2 \oplus \phi^{k_\alpha - N} T) \right) \subset c(\mathfrak{T}).$$

Now suppose that $z \in \mathbb{Z}^n \cap (E_s \oplus X_u)$. There exists an L such that $\phi^L z \in U_2 \oplus X_u$. Suppose $\pi_u(\phi^L z) \in T \in \mathfrak{T}$. Then, as demonstrated above, there exist a $k \leq k_0$ such that

$$\phi^{L+k} z \in c(\mathfrak{T}).$$

This proves the proposition.□

By Corollary 3.11 the tiling $\mathfrak{T}$ defined above is a Markov tiling.

Epilogue

We conclude this chapter and this thesis with a brief summary. In Chapters 1 and 2 we established a natural correspondence between a **periodically subdividing tiling** and a **symbolic system.** The correspondence came by giving each point in the tiled space a **digit expansion.** The allowable strings of digits correspond to paths in a directed graph which we called the **subdivision graph** for the tiling. In Chapter 3 we considered the case when the tiled space is the **unstable eigenspace** for a hyperbolic automorphism ϕ of $\mathbb{R}^n$. We constructed a compact set Ω in $\mathbb{R}^n$ by identifying the points in the unstable eigenspace modulo the integer lattice with points in a bounded neighborhood of the origin. We established a natural correspondence between the points in Ω and the **graph shift** induced by the subdivision graph. We indicated conditions under which Ω is almost homeomorphic to the n-dimensional torus. In particular, we showed that if all integer lattice points within a bounded region of the tiled space project to a special set of **control points** then the set Ω is almost homeomorphic to the n-torus. We described a **Markov tiling** of a subset of the unstable eigenspace as being a type of tiling for which Ω is almost homeomorphic to the n-torus. In Chapter 4 we proved that if $\mathfrak{T}$ is a Markov tiling of the unstable eigenspace for ϕ then the graph shift induced by the subdivision graph for $\mathfrak{T}$ is metrically similar to the dynamical system ($\mathbb{R}^n \bmod \mathbb{Z}^n, \phi \bmod \mathbb{Z}^n$). In Chapter 5 we gave examples of Markov tilings and indicated how one might go about constructing them in special circumstances.

It is interesting to note that the Markov partitions constructed using the methods of this thesis, coincide with the Markov partitions constructed by Bowen in [5]. If $\mathcal{C}$ is a Markov partition for ϕ constructed using the methods in [5] then there is a Markov tiling which will generate $\mathcal{C}$. The unstable eigenspace E_u for ϕ is dense in $\mathbb{R}^n \bmod \mathbb{Z}^n$. Let $\partial^s\mathcal{C}$ be the stable boundary for $\mathcal{C}$. Let Y be the set of points in E_u which are identified modulo the integer lattice with points in $\partial^s\mathcal{C}$. Then Y is the union of the boundaries for a family $\mathfrak{F}(\mathfrak{T})$ of periodically self similar tilings of E_u. The tile types and subdivision rules for $\mathfrak{F}(\mathfrak{T})$ are determined by the rectangles in $\mathcal{C}$. Moreover, $E_u \bmod \mathbb{Z}^n$ intersects the interior of some rectangle in $\mathcal{C}$ in a set $T \bmod \mathbb{Z}^n$ such that $T \subset E_u$. The properties of a Markov partition guarantee that

$$\cup_{k=0}^{\infty} \phi^k T \bmod \mathbb{Z}^n$$

will lie in $\mathbb{R}^n \bmod \mathbb{Z}^n$ minus the unstable boundary of $\mathcal{C}$. In particular the tilings in $\mathfrak{F}(\mathfrak{T})$ coincide on $\cup_{k=0}^{\infty} \phi^k T$ in tile and tile type. It will follow that the tilings in $\mathfrak{F}(\mathfrak{T})$ are all Markov tilings of E_u with expansion map ϕ.

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