Interacting particle systems with partial annihilation through membranes

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Abstract

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This thesis studies the *hydrodynamic limit* and the *fluctuation limit* for a class of interacting particle systems in domains. These systems are introduced to model the transport of positive and negative charges in solar cells. However, they are general microscopic models that can describe a variety of macroscopic phenomena with coupled boundary conditions, such as the population dynamics of two segregated species under competition. Proving these two types of limits represents establishing the *functional law of large numbers* and the *functional central limit theorem*, respectively, for the time-trajectory of the particle densities. This also corresponds to the study of the behavior of the system at two different scales. We show that the hydrodynamic limit is a pair of deterministic measures whose densities solve a coupled nonlinear heat equations, while the fluctuation limit can be described by a Gaussian Markov process that solves a stochastic partial differential equation.

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DEDICATION

to my dearest parents, brother and sister, and to Coco.

NOTATION INDEX

We first list the notations that are adopted throughout this thesis. Additional notations that are used in each chapter will then be listed.

General Notation

$\mathbb{Z}$	set of all integers
$\mathbb{Z}_+$	$\{1, 2, 3, \cdots\}$ positive integers
$\mathbb{N}$	$\{0, 1, 2, \cdots\}$ non-negative integers
$\mathbb{R}$	set of all real numbers
$\mathcal{B}(E)$	Borel measurable functions on E
$\mathcal{B}_b(E)$	bounded Borel measurable functions on E
$\mathcal{B}^+(E)$	non-negative Borel measurable functions on E
$C(E)$	continuous functions on E
$C_b(E)$	bounded continuous functions on E
$C^+(E)$	non-negative continuous functions on E
$C_c(E)$	continuous functions on E with compact support
$W^{1,2}(D)$	$\{f \in L^2(D; dx) : \lvert\nabla f\rvert \in L^2(D; dx)\}$ Sobolev space of order $(1, 2)$
$D([0, \infty), E)$	space of *càdlàg* paths from $[0, \infty)$ to E equipped with the Skorokhod metric (see [4] or [35])
$\lVert \cdot \rVert$	uniform norm (unless otherwise stated)

$\mathcal{H}^m$	m-dimensional Hausdorff measure
$M_+(E)$ (or $M_{\geq 0}(E)$)	space of finite non-negative Borel measures on E with the weak topology
$M_{\leq 1}(E)$	$\{\mu \in M_+(E) : \mu(E) \leq 1\}$
$M_1(E)$ (or $\mathcal{P}(E)$)	$\{\mu \in M_+(E) : \mu(E) = 1\}$
$\{\mathcal{F}_t^X : t \geq 0\}$	filtration induced by the process $(X_t)_{t\geq 0}$, i.e. $\mathcal{F}_t^X = \sigma(X_s, s \leq t)$
$\mathbf{1}_x$	indicator function at x or the Dirac measure at x, depending on the context
$\xrightarrow{\mathcal{L}}$	convergence in law of random variables (or processes)
$\stackrel{\mathcal{L}}{=}$	equal in law
$:=$	is defined as
$\langle f, \mu \rangle$	$\int f(x)\, \mu(dx)$, where μ ia s measure and f is a function
$x \vee y$	$\max\{x, y\}$
$x \wedge y$	$\min\{x, y\}$

A constant C which depends only on D and T will sometimes be written as $C(D,T)$. The exact value of the constant may vary from line to line. We also use the following abbreviations:

a.s.	almost surely
càdlàg (or *r.c.l.l.*)	right continuous with left limits
CTRW	continuous time random walk
LDCT	Lebesque dominated convergence theorem
LHS	left hand side
local CLT	local central limit theorem
PDE	partial differential equation
RBM	reflected Brownian motion
RHS	right hand side
SPDE	stochastic partial differential equation
WLOG	without loss of generality
w.r.t.	with respect to

Definition 0.0.1. *A Borel subset E of $\mathbb{R}^d$ is called $\mathcal{H}^m$**-rectifiable** if E is a countable union of Lipschitz images of bounded subsets of $\mathbb{R}^m$ with $\mathcal{H}^m(E) < \infty$ (As usual, we ignore sets of $\mathcal{H}^m$ measure 0). Here $\mathcal{H}^m$ denotes the m-dimensional Hausdorff measure.*

Definition 0.0.2. *A **bounded Lipschitz domain** $D \subset \mathbb{R}^d$ is a bounded connected open set such that for any $\xi \in \partial D$, there exits $r_\xi > 0$ such that $B(\xi, r_\xi) \cap D$ is represented by $B(\xi, r_\xi) \cap \{(y', y^d) \in \mathbb{R}^d : \phi_\xi(y') < y^d\}$ for some coordinate system centered at ξ and a Lipschitz function ϕ_ξ with Lipschitz constant M_D, where $M_D > 0$ does not depend on ξ.*

Notation for Chapter 2

X	an $(\mathbf{a}, \rho)$-reflected diffusion (or an $(\mathcal{A}, \rho)$-reflected diffusion) in D
$X^{(\Lambda)}$	an $(\mathbf{a}, \rho)$-reflected diffusion killed upon hitting Λ, defined in (2.1.7)
$C_\infty(\overline{D} \setminus \Lambda)$	$\{f \in C(\overline{D}) : f \text{ vanishes on } \Lambda\}$
D^ε	square lattice of edge length ε that approximates D, see Section 2.2
∂D^ε	graph-boundary $\{x \in D^\varepsilon : v_\varepsilon(x) < 2d\}$, where $v_\varepsilon(x)$ is the degree of x in D^ε
I^ε	'ε-point approximation' of I constructed in Lemma 2.2.22
σ_ε	'discrete surface measure' constructed in Lemma 2.2.22

Process	Semigroup	Heat kernel	Measure	Generator	State space
$X(t)$	P_t	$p(t,x,y)$	ρ	$\mathcal{A}$	$\overline{D}$
$X^\varepsilon(t)$	P_t^ε	$p^\varepsilon(t,x,y)$	m_ε	$\mathcal{A}_\varepsilon$	D^ε

Notation for Chapter 3

$\eta_t^{\varepsilon,\pm}(x)$	number of living particles at $x \in D_\pm^\varepsilon$ at time t
$(\eta_t^\varepsilon)_{t\ge 0}$	process with generator $\mathfrak{L}^\varepsilon = \mathfrak{L}_0^\varepsilon + \mathfrak{K}^\varepsilon$ in Definition 3.1.1
$(\xi_t^0)_{t\ge0}$ and $(\eta_t^0)_{t\ge0}$	independent processes with generator $\mathfrak{L}_0^\varepsilon$
E^ε	$\mathbb{N}^{D_+^\varepsilon} \times \mathbb{N}^{D_-^\varepsilon}$, state space of $(\eta_t^\varepsilon)_{t\ge0}$, defined in (3.1.1)
$\mathfrak{X}_t^{N,\pm}(dz)$	$\frac{1}{N}\sum_{x\in D_\pm^\varepsilon} \eta_t^\pm(x)\mathbf{1}_x(dz)$, the normalized empirical measure in $\overline{D}_\pm$
$\mathfrak{E}$	$M_{\le1}(\overline{D}_+) \times M_{\le1}(\overline{D}_-)$, the state space of $(\mathfrak{X}_t^{N,+}, \mathfrak{X}_t^{N,-})_{t\ge0}$

Process	Semigroup	Heat kernel	Measure	Generator	State space
$X^\pm(t)$	$P_t^\pm$	$p^\pm(t,x,y)$	$\rho_\pm$	$\mathcal{A}^\pm$	$\overline{D}_\pm$
$X^{\varepsilon,\pm}(t)$	$P_t^{\varepsilon,\pm}$	$p^{\varepsilon,\pm}(t,x,y)$	$m_\varepsilon^\pm$	$\mathcal{A}_\varepsilon^\pm$	$D_\pm^\varepsilon$
$X_{(n,m)}(t)$	$P_t^{(n,m)}$	$p = p^{(n,m)}$	$\rho = \rho_{(n,m)}$	$\mathcal{A}^{(n,m)}$	$\overline{D}_+^n \times \overline{D}_-^m$
$X_{(n,m)}^\varepsilon(t)$	$P_t^{(n,m),\varepsilon}$	$p^\varepsilon = p^{(n,m),\varepsilon}$	$m_\varepsilon = m_\varepsilon^{(n,m)}$	$\mathcal{A}_\varepsilon^{(n,m)}$	$(D_+^\varepsilon)^n \times (D_-^\varepsilon)^m$

where in the above,

$$p^{(n,m)}(t,(\vec{r},\vec{s}),(\vec{r'},\vec{s'})) := \prod_{i=1}^{n} p^{+}(t,r_i,r_i') \prod_{j=1}^{m} p^{-}(t,s_j,s_j')$$

$$\rho_{(n,m)}(\vec{r},\vec{s}) := \prod_{i=1}^{n} \rho_{+}(r_i) \prod_{j=1}^{m} \rho_{-}(s_j)$$

$$p^{(n,m),\varepsilon}(t,(\vec{r},\vec{s}),(\vec{r'},\vec{s'})) := \prod_{i=1}^{n} p^{\varepsilon,+}(t,r_i,r_i') \prod_{j=1}^{m} p^{\varepsilon,-}(t,s_j,s_j')$$

$$m_{\varepsilon}^{(n,m)}(\vec{r},\vec{s}) := \prod_{i=1}^{n} m_{\varepsilon}^{+}(r_i) \prod_{j=1}^{m} m_{\varepsilon}^{-}(s_j).$$

Notation for Chapter 4

$C_\infty^{(n,m)}$	$\{\Phi \in C(\overline{D}_+^n \times \overline{D}_-^m) : \ \Phi$ vanishes outside $(\overline{D}_+ \setminus \Lambda_+)^n \times (\overline{D}_- \setminus \Lambda_-)^m\}$, see Subsection 4.5.2
$\langle \varphi,\, \mu^+(dx) \otimes \mu^-(dy) \rangle$	$\frac{1}{N^2}\sum_i \sum_j \varphi(x_i,y_j)$, when $\mu = (\frac{1}{N}\sum_i \mathbf{1}_{x_i},\ \frac{1}{N}\sum_j \mathbf{1}_{y_j})$
I^δ	$\{(x,y) \in D_+ \times D_- : \ \lvert x-z\rvert^2 + \lvert y-z\rvert^2 < \delta^2 \ \text{ for some } z \in I\}$,
c_{d+1}	volume of the unit ball in $\mathbb{R}^{d+1}$
ℓ_δ	annihilating potential functions in Assumption 4.0.9
$\underline{\mathbf{X}}_t^{(N)}$	configuration process defined in Subsection 4.1.1
S_N	$\cup_{m=1}^N \big(D_+^\partial(m) \times D_-^\partial(m)\big) \cup \{\partial\}$, the state space of $(\underline{\mathbf{X}}_t^{(N)})_{t\geq 0}$
$(\mathfrak{X}_t^{N,+},\, \mathfrak{X}_t^{N,-})$	empirical measure defined in Subsection 4.1.2
E_N	$\cup_{M=1}^N E_N^{(M)} \cup \{\mathbf{0}_*\}$, the state space of $(\mathfrak{X}_t^{N,+},\, \mathfrak{X}_t^{N,-})_{t\geq 0}$
$(\overline{\mathfrak{X}}_t^{N,+},\, \overline{\mathfrak{X}}_t^{N,-})$	empirical measure for reflected diffusions *without* annihilation
$\mathfrak{M}$	$M_{\leq 1}(\overline{D}_+ \setminus \Lambda_+) \times M_{\leq 1}(\overline{D}_- \setminus \Lambda_-)$, see Section 4.3
$D_\pm^\partial$	$\Big(\overline{D}_\pm \setminus \Lambda^\pm\Big) \cup \{\partial^\pm\}$, where $\partial^\pm$ is an isolated point of $\overline{D}_\pm$

Notation for Chapter 5

D^δ	$\{x \in D:\ dist(x, \partial D) < \delta\}$
$\mathfrak{X}^N_t$	empirical measure defined in (5.1.2)
$\{\phi_k\}$	complete orthonormal system of $\mathcal{A} := \frac{1}{2\rho}\nabla\cdot(\rho\,\mathbf{a}\nabla)$ in $L^2(D,\,\rho)$ consisting of Neumann eigenfunctions
λ_k	the eigenvalue corresponding to ϕ_k such that $\mathcal{A}\phi_k = -\lambda_k$
$\langle\phi,\,\psi\rangle_\rho$	$\int_D \phi(x)\psi(x)\,\rho(x)dx$, the inner product of $L^2(D,\,\rho(x)dx)$

Notation for Chapter 6

For I^δ, c_{d+1}, ℓ_δ, $(\mathfrak{X}^{N,+}_t,\,\mathfrak{X}^{N,-}_t)$, $(\overline{\mathfrak{X}}^{N,+}_t,\,\overline{\mathfrak{X}}^{N,-}_t)$ and $\langle\varphi,\,\mu^+(dx)\otimes\mu^-(dy)\rangle$, we follow the notation for Chapter 4.

$\mathcal{Z}^N := \mathcal{Y}^{N,+}\oplus\mathcal{Y}^{N,-}$	fluctuation process defined in (6.0.5)
$\mathbf{H}_{-\alpha}$	the Hilbert space $\{\mu^+\oplus\mu^-:\ \mu^\pm\in\mathcal{H}^\pm_{-\alpha}\}$ defined at the beginning of the chapter
C_0	$\lVert u_0^+\rVert \vee \lVert u_0^-\rVert$
$\{\phi^\pm_k\}$	complete orthonormal system of $\mathcal{A}^\pm := \frac{1}{2\rho_\pm}\nabla\cdot(\rho_\pm\,\mathbf{a}_\pm\nabla)$ in $L^2(D_\pm,\,\rho_\pm)$ consisting of Neumann eigenfunctions
$\lambda^\pm_k$	the eigenvalue corresponding to $\phi^\pm_k$ such that $\mathcal{A}^\pm\phi^\pm_k = -\lambda^\pm_k$
$\langle\phi,\,\psi\rangle_{\rho_\pm}$	$\int_{D_\pm}\phi(x)\psi(x)\,\rho_\pm(x)dx$, the inner product of $L^2(D_\pm,\,\rho_\pm(x)dx)$
$F^{(n,m)}_t = F^{N,(n,m)}_t$	(n,m)-correlation function at time t in Definition 6.3.1
$F^{(n,m),(p,q)}_{s,t} = F^{N,(n,m),(p,q)}_{s,t}$	generalized correlation function in Definition 6.4.1
$E^{(n,m),(p,q)}_{u,r}$	$F^{(n,m),(p,q)}_{u,u+r} - F^{(n,m)}_u\cdot F^{(p,q)}_{u+r}$ defined in (6.4.2)

Chapter 1
INTRODUCTION

This introduction serves as a motivation, trailer and pointer for the main results and key ideas that will be established in detail in this thesis. As can be seen from the titles, Chapter 3-6 correspond to subsections 1.2.1, 1.2.2, 1.3.1 and 1.3.2 in this introduction.

1.1 Motivation

Interacting particle systems is a family of mathematical models that are widely used in describing diverse phenomena, such as ecological systems [33], population dynamics [34, 56, 60], chemical reactions [55], super-conductivity [75], quantum dynamics [36], fluid dynamics [40], etc. A principal theme in investigating these phenomena is to **connect the microscopic mechanisms of the systems with the collective behaviors that emerge in the macroscopic scale** under suitable (space-time) scalings. The remarkable power of these models in illuminating this connection has long been recognized, but proving rigorous results is usually quite challenging.

My thesis studies a class of newly introduced interacting particle systems which are primarily designed as microscopic models for the transport of positive and negative charges in solar cells. However, these models are flexible and general enough to provide insight to a variety of natural phenomena, such as the population dynamics of two segregated species under competition. More precisely, two different but related stochastic interacting particle systems are introduced in Chapter 3 and Chapter 4.

Here is an informal description of the models. We model a solar cell by a domain in $\mathbb{R}^d$ that is divided into two adjacent sub-domains D_+ and D_- by an interface I, a $(d-1)$-dimensional hypersurface. Domains D_+ and D_- represent the hybrid medium that confine the positive and the negative charges, respectively. See Figure 1.1 for an illustration. At microscopic level, positive and negative charges are initially modeled by N independent reflected Brownian motions (or more generally, reflected diffusions) with drift on D_+ and on D_-, respectively [1]. These random motions model the transport of positive (respectively negative) charges under an applied electric potential field. Besides, there is a harvest region $\Lambda_\pm \subset \partial D_\pm \setminus I$ that absorbs (harvests) $\pm$ charges, respectively, whenever it is being visited. See Figure 1.1. Furthermore, these two types of particles annihilate each other in pairs at a certain rate when they come close to each other near the interface I. This interaction models the annihilation, trapping, recombination and separation phenomena of the charges.

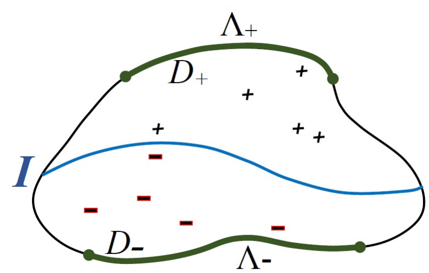


Figure 1.1: I = Interface, $\Lambda_\pm$ = Harvest sites

To connect the microscopic mechanism and the macroscopic evolution of the systems, we derive the *hydrodynamic limits* and the *fluctuation limits* for these models. Proving these two types of limits represents establishing the *functional law of large numbers* and the *functional central limit theorem*, respectively, for the *time trajectory* of the spatial densities of the particles in the systems. More precisely, we investigate the asymptotic behavior (when $N \to \infty$) of the empirical measure of positive and negative charges

$$\mathfrak{X}_t^{N,+}(dx) := \frac{1}{N} \sum_{\alpha:\, \zeta_\alpha > t} \mathbf{1}_{X_\alpha^+(t)}(dx) \quad \text{and} \quad \mathfrak{X}_t^{N,-}(dy) := \frac{1}{N} \sum_{\beta:\, \zeta_\beta > t} \mathbf{1}_{X_\beta^-(t)}(dy).$$

[1] In Chapter 3 (also in [16]), they are actually modeled by N biased random walks on lattices inside D_+ and D_- that serve as discrete approximation of reflected Brownian motions with drifts.

Here $\mathbf{1}_y(dx)$ stands for the Dirac measure concentrated at the point y and ζ_α for the life time of particle indexed by α. So $\zeta_\alpha > t$ (resp. $\zeta_\beta > t$) denotes the condition that the particle X^+_α (resp. X^-_β) is alive at time t. We show that, under a suitable scaling (to be explained in detail in each model) and an appropriate condition on the initial configuration, the pair of random measures $(\mathfrak{X}^{N,+}_t, \mathfrak{X}^{N,-}_t)$ converge in distribution to a deterministic pair of measures which are absolutely continuous with respect to the Lebesgue measure. Furthermore, the densities satisfy a system of partial differential equations (PDEs) that has non-linear boundary interaction terms at the interface. See Theorem 1.2.1 and Theorem 1.2.3 below for the statements of the hydrodynamic results. We further study the fluctuation of the empirical measure around the hydrodynamic limit and establish that the fluctuation limit is a continuous Gaussian Markov process that solves a stochastic partial differential equation (SPDE). The latter is a nonlinear version of the Langevin equation. See Theorem 6.1.2 below for the statement of the fluctuation result.

Interacting particle systems are exciting and active in mathematical research. We refer the reader to standard references such as [14, 50, 58] and the references therein for its history and a wealth of results. We will provide more specific literature review in each of the two main sections below and point out how our work fit into it. The first two sections (1.2 and 1.3) in this introduction are devoted to hydrodynamic limits and to fluctuation limits respectively. This summarizes the main results in this thesis. The last section (1.4) points to some applications and implications of the main results.

1.2 Hydrodynamic limits

The study of hydrodynamic limits dates back to the work of J. Clerk Maxell and L. Boltzmann, founders of the kinetic theory of gases. Later, D. Hilbert formulated the question of hydrodynamic limits as a mathematical problem, and presented it as an example in his sixth problem for the mathematical treatment of the axioms of physics. From the probabilistic or statistical point of view, proving hydrodynamic limits corresponds to establishing functional law of large numbers for the *evolution in time* of the empirical measure of some attributes (such as position, genetic type, spin type, etc.) of the individuals in the systems. It reveals fascinating connections between the microscopic stochastic systems and deterministic partial differential equations that describe the macroscopic pictures. It also provides approximations via stochastic models to some partial differential equations that are hard or impossible to solve directly.

Since Hilbert formulated his sixth problem in year 1900, there have been many different lines of research on stochastic particle systems. Various models were studied and different techniques were developed to establish hydrodynamic limits. Here we cite the *entropy method* [42] and the *relative entropy method* [75] as general methods, and mention the non-gradient techniques and attractiveness techniques [50], among many other important techniques. Unfortunately, these methods do not seem to work directly for our model due to the singular interactions near the interface. Among all the most extensively studied models, *reaction-diffusion systems* in [6, 23, 30, 53] and *Fleming-Viot type systems* in [11, 12, 41] are relatively close to ours. Nevertheless, our models distinguish themselves due to the coupling effect near the boundary which leads to nonlinear heat equations. So new approaches and techniques are called for to analyze these systems. Our models are non-equilibrium systems for which boundary effects are visible in the macroscopic scale.

1.2.1 Hydrodynamic limits for interacting random walks

In this subsection, we summarize the main results in Chapter 3 (and also that of [16]), which are the hydrodynamic result and the propagation of chaos result for a random walk version of the interacting diffusion models described in the introduction. To focus on the interaction near the interface I, we assume there is no harvest site in this model. We first describe the stochastic model, which we call "Annihilating Random Walk Model".

Heuristic description of the annihilating random walk model. We approximate $D_\pm$ by a square lattice $D^\varepsilon_\pm = D_\pm \cap \varepsilon\mathbb{Z}^d$ of edge length ε (as in Figure 1.2), where ε is an N-dependent parameter such that $N\varepsilon^d = 1$ and N is the initial number of particles in each of D^ε_+ and D^ε_-. Each particle in $D^\varepsilon_\pm$ performs biased random walk speeded up by a factor d/ε^2 (diffusive scaling). More precisely, the one-step transition probabilities is chosen in such a

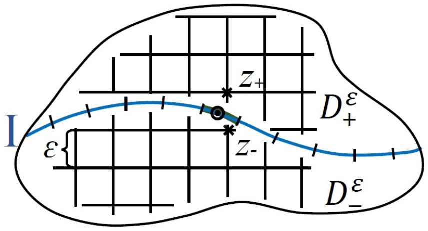


Figure 1.2: Annihilating random walk model

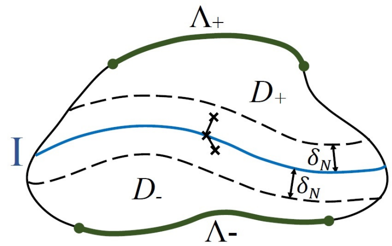


Figure 1.3: Annihilating diffusion model

way that the motion approximating a reflected Brownian motion with gradient drift $\frac{1}{2}\nabla(\log \rho_\pm)$, where $\rho_\pm$ is a positive continuously differentiable function on $\overline{D}_\pm$. Note that $\rho_\pm = 1$ corresponds to the particular case when there is no drift. When a pair of particles of different types are close to each other (which must happen near I, such as when they are at (z_+, z_-) in Figure 1.2), they annihilate each other at a rate of order λ/ε, where $\lambda > 0$ is a fixed parameter which is determined by the physical characteristics of the solar cell. Here, we say an event happens at rate r if the time of occurrence is an exponential random variable of parameter r (in particular, the probability of occurrence in a short amount of time t is $rt + o(t)$, where $o(t)/t \to 0$ as $t \to 0$). Overall and intuitively speaking, according to a random time clock which runs with a speed proportional to the number of pairs (one particle of each type) of distance ε, we annihilate a pair (picked uniformly among those pairs of distance less than ε) with an exponential rate of parameter λ/ε. The rigorous formulation of the particle system is captured by its generator $\mathfrak{L}^\varepsilon$ stated in Definition 3.1.1.

It is clear that the empirical measure $(\mathfrak{X}_t^{N,+}, \mathfrak{X}_t^{N,-})_{t\geq 0}$ is a continuous time Markov process with state space

$$\mathfrak{M} := M_{\geq 0}(\overline{D}_+) \times M_{\geq 0}(\overline{D}_-),$$

where $M_{\geq 0}(E)$ denotes the space of non-negative Borel sub-probability measures on E. In what follows, $\xrightarrow{\mathcal{L}}$ stands for convergence in law as $N \to \infty$, and $D([0,T], E)$ is the Skorokhod space of *càdlàq* paths in E (see, for example, [4, 35]). The following result on hydrodynamic limit is obtained in Chapter 3 (Theorem 3.3.1).

Theorem 1.2.1. ***(Functional Law of Large Numbers)*** *Under appropriate and mild condi-*

tions on the initial configuration $(\mathfrak{X}_0^{N,+},\, \mathfrak{X}_0^{N,-})$*, we have, for any* $T > 0$*,*

$$(\mathfrak{X}_t^{N,+},\, \mathfrak{X}_t^{N,-}) \xrightarrow{\mathcal{L}} (u_+(t,x)\rho_+(x)dx,\, u_-(t,y)\rho_-(y)dy) \quad \textit{in } D([0,T],\mathfrak{M}),$$

where (u_+, u_-) *is the solution of the following coupled nonlinear heat equations:*

$$\begin{cases} \dfrac{\partial u_+}{\partial t} = \dfrac{1}{2}\Delta u_+ + \dfrac{1}{2}\nabla(\log \rho_+)\cdot \nabla u_+ & \textit{on } (0,\infty)\times D_+ \\ \dfrac{\partial u_+}{\partial \vec{n}_+} = \dfrac{\lambda}{\rho_+}\, u_+ u_-\, \mathbf{1}_I & \textit{on } (0,\infty)\times \partial D_+ \end{cases} \tag{1.2.1}$$

and

$$\begin{cases} \dfrac{\partial u_-}{\partial t} = \dfrac{1}{2}\Delta u_- + \dfrac{1}{2}\nabla(\log \rho_-)\cdot \nabla u_- & \textit{on } (0,\infty)\times D_- \\ \dfrac{\partial u_-}{\partial \vec{n}_-} = \dfrac{\lambda}{\rho_-}\, u_+ u_-\, \mathbf{1}_I & \textit{on } (0,\infty)\times \partial D_-, \end{cases} \tag{1.2.2}$$

where $\vec{n}_\pm$ *is the inward unit normal vector field of* $D_\pm$ *and* $\mathbf{1}_I$ *is the indicator function on* I*.*

In words, u_+ satisfies the Neumann boundary condition on $\partial D_+ \setminus I$, and with boundary condition $\frac{\partial u_+}{\partial \vec{n}_+} = \frac{\lambda}{\rho_+} u_+ u_-$ on I. This nonlinear coupled boundary condition on I is a macroscopic phenomena which emerge from the local interactions. The above result tells us that for any fixed time $t > 0$, the probability distribution of a randomly picked particle in $D^\varepsilon_\pm$ at time t is close to $c_\pm(t)u_\pm(t,x)$ when N is large, where $c_\pm(t) = (\int_{D_\pm} u_\pm(t))^{-1}$ is a normalizing constant. In fact, the above convergence holds at the level of the path space. That is, the full trajectory (and hence the joint law at different times) of the particle profile converges to the deterministic scaling limit described by (1.2.1) and (1.2.2).

What about the joint distribution of more than one particle? The answer is provided by our second main result (Theorem 3.3.3) in Chapter 3, which says that propagation of chaos holds true for our system: when the number of particles tends to infinity, their positions appear to be independent of each other and hence the joint distribution can be factorized.

Theorem 1.2.2. *(**Propagation of chaos**) Suppose* n *and* m *unlabeled particles in* D^ε_+ *and* D^ε_- *respectively are chosen uniformly among the living particles at time* t*. Then, as* $N \to \infty$*,* *the probability joint distribution for their positions converges to*

$$c_{(n,m)}(t) \prod_{i=1}^{n} u_+(t, r_i) \prod_{j=1}^{m} u_-(t, s_j)$$

uniformly on compact sets, where $c_{(n,m)}(t)$ is a normalizing constant.

Scaling is an important and ubiquitous concept in stochastic interacting particle systems. The heuristic reason for the choice of the scaling λ/ε for the per-pair annihilation rate is to guarantee that, in the limit $N \to \infty$, a non-trivial proportion of particles is killed during the time interval $[0, t]$. Since diffusive particles typically spread out in space, the number of pairs near the interface is of order $N^2 \varepsilon^{d+1}$ (because there are $N\varepsilon$ particles in D_+ near I, and each of them is near to $N\varepsilon^d$ particles in D_-). Hence the expected number of pairs killed within t units of time is about $(N^2 \varepsilon^{d+1}) (\lambda t/\varepsilon) = \lambda N t$ when $t > 0$ is small (here we used the scaling $N\varepsilon^d = 1$). This implies that a non-trivial proportion of particles are annihilated in any open time interval and hence accounts for the boundary term in the hydrodynamic limit.

To establish hydrodynamic limit result (Theorem 1.2.1), we employ the classical tightness plus finite dimensional distribution approach. In fact, propagation of chaos (Theorem 1.2.2) is a crucial step in identifying the limit in Theorem 1.2.1. A key step in our proof of Theorem 1.2.2 is Theorem 3.4.7. The latter is the uniqueness of solution for the infinite system of equations satisfied by the correlation functions of the particles in the limit $N \to \infty$. Such infinite system of equations is sometimes called *BBGKY-hierarchy*[2] in statistical physics. Our limiting BBGKY hierarchy involves boundary terms on the interface, which is new to the literature. We show that the correlation functions of every subsequential limit of $(\mathfrak{X}^+, \mathfrak{X}^-)$ satisfies the same limiting BBGKY hierarchy. The crux of the difficulty is to establish the existence and uniqueness of solutions to this liming BBGKY hierarchy. With such uniqueness, we conclude that the propagation of chaos holds for our stochastic system, from which we can deduce the hydrodynamic limit result. Our proof of uniqueness involves a representation and manipulations

[2]BBGKY stands for N. N. Bogoliubov, Max Born, H. S. Green, J. G. Kirkwood, and J. Yvon, who derived this type of hierarchy of equations in the 1930s and 1940s in a series of papers.

of the hierarchy in terms of trees. This technique is related to but different from that in [36] which used Feynman diagrams. The techniques developed in Chapter 3 are potentially useful in the study of other stochastic models involving coupled differential equations.

In addition, two new tools for discrete approximation of random walks in domains are developed in this article. Namely, the local central limit theorem (local CLT) for reflected random walk on bounded Lipschitz domains (see Theorem 2.2.8 in Chapter 2) and the "discrete surface measure" (see Theorem 2.2.12 in Chapter 2). We believe these tools are potentially useful in many discrete schemes which involve reflected Brownian motions. Weak convergence of simple random walk on $D^{\varepsilon}_{\pm}$ to reflected Brownian motion (RBM in abbreviation) has been established for general bounded domains in [9] and [10]. However, we need a local convergence result which guarantees that the convergence rate is uniform up to the boundary. For this, we establish the local CLT. We further generalize the weak convergence result and the local limit theorem to deal with RBMs with gradient drift. The proof of the local CLT is based on a 'discrete relative isoperimetric inequality' which leads to the Poincaré inequality and the Nash inequality. The crucial point is that these two inequalities are uniform in ε (scaling of lattice size) and are invariant under the dilation of the domain $D \mapsto aD$.

1.2.2 Hydrodynamic limits for interacting diffusions

In this subsection, we summarize the main results in Chapter 4 (and also that of [17]). While the annihilating random walk model in Chapter 3 is more amenable to computer simulation, it is subject to technical restrictions associated with the discrete approximations of both the diffusions performed by the particles and the underlying domains $D_{\pm}$. Furthermore, that model does not consider harvest of charges. To address these issues, a new continuous state stochastic model is introduced and investigated in Chapter 4. This new model, which we refer to as the "Annihilating Diffusion Model" in this thesis, is different from the annihilating random walk model in three ways:

- the particles perform *reflected diffusions* on continuous state spaces rather than random walks over discrete state spaces,

- particles are absorbed (harvested) at some regions (harvest sites) away from the interface I, and

- the annihilation mechanism near I is different.

The annihilating diffusion model allows more flexibility in modeling the underlying spatial motions performed by the particles and in the study of their various properties. In particular, it is more convenient to work with when we study the fluctuation limit, which is the subject of Chapter 6 (also that of [19]).

Reflected diffusions are natural mathematical objects to study. After all, the random motions of the pollen grains observed by Robert Brown in year 1827 were reflected on the boundary of the water tank. Suppose $D \subset \mathbb{R}^d$ is a domain, ρ is a strictly positive function on D, and $\mathbf{a} = (a^{ij})$ is a symmetric and uniformly elliptic $d \times d$ matrix-valued function on D. An $(\mathbf{a}, \rho)$-reflected diffusion is a continuous strong Markov process with symmetrizing measure ρ and has infinitesimal generator $\mathcal{A} := \dfrac{1}{2\rho} \nabla \cdot (\rho\, \mathbf{a} \nabla)$. Intuitively, X behaves like a diffusion process associated to the second order elliptic differential operator $\mathcal{A}$ in the interior of D, and is instantaneously reflected at the boundary in the inward conormal direction $\vec{\nu} := \mathbf{a}\vec{n}$, where $\vec{n}$ is the inward unit normal. A special but very important case is when $\mathbf{a}$ is the identity matrix and $\rho = 1$, in which X is called a *reflected Brownian motion* (RBM). See Section 2.1 for details about reflected diffusions. We now describe the annihilating diffusion model.

Heuristic description of the annihilating diffusion model. Suppose, in addition to $D_\pm$ and the interface I, we are also given a harvest region $\Lambda_\pm \subset \partial D_\pm \setminus I$ that absorbs (harvests) $\pm$ charges (Figure 1.3). Let N be the initial number of particles in each of D_+ and D_-. Each particle in $D_\pm$ moves as an $(\mathbf{a}_\pm, \rho_\pm)$-reflected diffusion in $D_\pm$ and is absorbed upon hitting $\Lambda_\pm$. Moreover, when two particles of different types are of a small distance δ_N, they disappear with intensity $\lambda/(N\delta_N^{d+1})$, where $\lambda > 0$ is a fixed parameter. The reason for the choice of the above scaling is the same as that for the annihilating random walk model: to guarantee that a nontrivial proportion of particles are annihilated in any open time interval. The rigorous construction of the annihilating diffusion model is in Section 4.1.

Base on a strengthened result in geometric measure theory obtained in Lemma 4.5.5 in Chapter 4, we developed a new and direct approach to prove the following hydrodynamic limit result (Theorem 4.3.1) in Chapter 4.

Theorem 1.2.3. ***(Functional Law of Large Numbers)*** *Suppose* $\liminf_{N\to\infty} N\,\delta_N^d \in (0,\infty]$, $\delta_N \to 0$ *and* $(\mathfrak{X}_0^{N,+},\, \mathfrak{X}_0^{N,-})$ *converges in distribution. Then for any* $T>0$, *we have*

$$(\mathfrak{X}_t^{N,+},\, \mathfrak{X}_t^{N,-}) \xrightarrow{\mathcal{L}} (u_+(t,x)\rho_+(x)dx,\ u_-(t,y)\rho_-(y)dy) \quad in\ D([0,T],\mathfrak{M}),$$

where $(u_+,\, u_-)$ *is the solution of the coupled heat equations* $\frac{\partial u_\pm}{\partial t} = \mathcal{A}^\pm u_\pm$ *on* $D_\pm$, *with Dirichlet boundary condition* $u_\pm = 0$ *on* $\Lambda_\pm$ *and with the following nonlinear coupled boundary condition:*

$$\frac{\partial u_\pm}{\partial \vec{\nu}_\pm} = \frac{\lambda}{\rho_\pm} u_+ u_-\, \mathbf{1}_{\{I\}} \quad on\ \partial D_\pm \setminus \Lambda_\pm,$$

where $\vec{\nu}_\pm := \boldsymbol{a}_\pm \vec{n}_\pm$ *is the inward conormal vector field of* $\partial D_\pm$.

As an immediate application of Theorem 1.2.3, we get an analytic formula for the asymptotic mass of positive charges harvested during the time interval $[0,T]$, which is

$$1 - \int_{D_+} u_+(T,x)\rho_+(x)\,dx - \lambda \int_0^T \int_I u_+(s,z)u_-(s,z)\,d\sigma(z)\,ds.$$

The condition $\liminf_{N\to\infty} N\,\delta_N^d \in (0,\infty]$ is an upper bound for the rate at which the annihilations distance δ_N tends to 0. Such kind of condition is necessary by the following reason: The dimension of I is $d+1$ lower than that of $D_+ \times D_-$. So we can choose δ_N small enough so that particles of different types cannot 'see' each other in the limit $N\to\infty$, resulting a decoupled linear system of PDEs with Dirichlet boundary condition on $\Lambda_\pm$ and Neumann boundary condition on $\partial D_\pm \setminus \Lambda_\pm$. See Example 4.3.3 in Chapter 4 for a rigorous statement and proof. Note that Theorem 1.2.3 is valid for the case $\liminf_{N\to\infty} N\,\delta_N^d = \infty$ (a high density assumption). This is important because in the fluctuation result in Theorem 6.1.2, we require $\liminf_{N\to\infty} N\,\delta_N^{2d} \in (0,\infty]$.

Base on a strengthened version of [38, Theorem 3.2.39] from geometric measure theory, we develop a direct approach to establish the hydrodynamic limit. This approach avoids going through the delicate BBGKY hierarchy as described in the previous subsection 1.2.1. More precisely, [38, Theorem 3.2.39] asserts that

$$\lim_{\delta\to 0} \frac{\mathcal{H}^{2d}(I^\delta)}{c_{d+1}\,\delta^{d+1}} = \mathcal{H}^{d-1}(I), \tag{1.2.3}$$

where $I^\delta := \{(x,y)\in D_+\times D_- : |x-z|^2+|y-z|^2<\delta^2 \text{ for some } z\in I\}$, c_{d+1} is the volume of the unit ball in $\mathbb{R}^{d+1}$, and $\mathcal{H}^m$ is the m-dimensional Hausdorff measure. In Lemma 4.5.5, this has been strengthened to

$$\lim_{\delta\to 0}\int \ell_\delta(x,y) f(x,y)\,dxdy = \int_I f(z,z)\,d\mathcal{H}^{d-1}(z) \tag{1.2.4}$$

uniformly in f from any equi-continuous family in $C(\overline{D}_+\times\overline{D}_-)$, where $\ell_\delta(x,y) := \frac{1}{c_{d+1}\,\delta^{d+1}}\mathbf{1}_{I^\delta}(x,y)$.

This uniform convergence, together with the condition $\liminf_{N\to\infty} N\,\delta_N^{d} \in (0,\infty]$, lead us to the following key observation that

$$\lim_{\delta\to 0}\lim_{N\to\infty}\mathbb{E}\int_0^T \langle \ell_\delta(x,y),\,\mathfrak{X}_s^{N,+}\otimes\mathfrak{X}_s^{N,-}\rangle\,ds = \lim_{N\to\infty}\lim_{\delta\to 0}\mathbb{E}\int_0^T \langle \ell_\delta(x,y),\,\mathfrak{X}_s^{N,+}\otimes\mathfrak{X}_s^{N,-}\rangle\,ds, \tag{1.2.5}$$

where $\langle \ell_\delta(x,y),\,\mathfrak{X}_s^{N,+}\otimes\mathfrak{X}_s^{N,-}\rangle := \int_{\overline{D}_+}\int_{\overline{D}_-}\ell_\delta(x,y)\,\mathfrak{X}_s^{N,+}(dx)\,\mathfrak{X}_s^{N,-}(dy)$. This interchange of limit in turn allows us to characterize the mean of any subsequential limit of $(\mathfrak{X}^{N,+},\,\mathfrak{X}^{N,-})$ by comparing the integral equations satisfied by the hydrodynamic limit with its stochastic counterpart.

We remark that the condition $\liminf_{N\to\infty} N\,\delta_N^{d} \in (0,\infty]$ is not required to establish tightness of $(\mathfrak{X}^{N,+},\,\mathfrak{X}^{N,-})$, and it is not clear what are other possibilities for subsequential limits in the absence of this condition. Finally, we point out that $\int_0^t \langle \ell_\delta(x,y),\,\mathfrak{X}_s^{N,+}\otimes\mathfrak{X}_s^{N,-}\rangle\,ds$ quantifies the amount of interaction among the two types of particles, and is related (but different from) the *collision local time* introduced in [37].

1.3 Fluctuation limits

We next study the fluctuations for the annihilating diffusion model[3]. This provides a measure of extent by which the empirical measure deviates from the hydrodynamic limit and a rate of convergence for the hydrodynamic result in Theorem 1.2.3. Moreover, It enables us to do effective simulations in engineering situations where explicit solutions are not feasible (see (1.3.4)). While the hydrodynamic limits of our models are described by deterministic PDEs, their fluctuation limits are stochastic partial differential equations (SPDEs).

One of the earliest rigorous results about fluctuation limit was proven by Itô (in [47, 48]), who considered a system of independent and identically distributed (i.i.d.) Brownian motions in $\mathbb{R}^d$ and showed that the limit is a $\mathcal{S}'$-valued Gaussian process solving a generalized Langevin equation, where $\mathcal{S}'$ is the Schwartz space of tempered distributions. For particles living in domains, Sznitman [73] studied the fluctuations of a conservative system of reflected diffusions. Fluctuations of the reaction-diffusion systems on the cube $[0,1]^d$ with polynomial reaction terms were studied in [6, 31, 53, 54]. These fluctuation results are valid only for dimension $d \leq 3$. In Chapter 5, we extend the functional analytic framework of [53] to deal with more general domains and reflected diffusions killed by a time-dependent potential. Our fluctuation results hold for all dimensions $d \geq 1$ and the covariance structures of our fluctuation limits have boundary integral terms that capture the boundary interactions in the fluctuation level.

1.3.1 Fluctuations for Brownian particles with partial absorbtion

In this subsection, we summarize the main results in Chapter 5 (and also that of [18]. As a preliminary step to understand the fluctuation for the annihilating diffusion model, we consider a simpler model which consists of a single type of particles moving as independent RBMs in a

[3]We did not try to prove a fluctuation limit result for the annihilating random walk model because we anticipate that the limit is the same as that of the annihilating diffusion model. A proof similar to that of the annihilating diffusion model can most likely be applied for the annihilating random walk mode, but the class of suitable correlation functions should be a modified version of the *v-functions* used in [6] (in which fluctuation of a lattice model was studied).

bounded domain $D \subset \mathbb{R}^d$ which has a chance of being killed by a singular time dependent potential function $q(t,x)$ when it is near the boundary ∂D. More precisely, when there are N particles initially, the killing intensity for each particle is $q_N(t,x) := \frac{1}{\delta_N}\mathbf{1}_{D^{\delta_N}}(x)q(t,x)$, where $q(t,x)$ is a given time dependent non-negative continuous function and $D^{\delta} := \{x \in D:\ dist(x,\partial D) < \delta\}$. We coin this model the name 'Robin boundary model' since it can be easily shown that its hydrodynamic limit is the solution to the heat equation with Robin boundary condition $\frac{\partial u}{\partial \vec{n}} = qu$ on ∂D, where $\vec{n}$ is the inward unit normal vector field of D. See Theorem 5.1.4 for the full statement.

Despite its simplicity, the Robin boundary model poses new difficulties and leads to new results when we study its fluctuation. The fluctuation of the empirical measure $\mathfrak{X}^N$ at time t is defined by

$$\mathcal{Y}^N_t(\phi) := \sqrt{N}\,(\langle \mathfrak{X}^N_t, \phi\rangle - \mathbb{E}\langle \mathfrak{X}^N_t, \phi\rangle),$$

where $\langle \mathfrak{X}^N_t, \phi\rangle := \frac{1}{N}\sum_{\alpha:\zeta_\alpha>t}\phi(X_\alpha(t))$ is the integral of an observable (or test function) ϕ with respect to the measure $\mathfrak{X}^N_t$. Intuitively, when $\phi = \mathbf{1}_K$ is an indicator function of a subset $K \subset \overline{D}$, then $\langle \mathfrak{X}^N_t, \phi\rangle$ is the mass of particles in K (which is the number of particles in K divided by N). In this case, $\mathcal{Y}^N_t(\phi)$ is the fluctuation of the mass of particles in K at time t.

Even in this simple setting, it is nontrivial to obtain satisfactory answers to the following natural questions:

(1) What is the state space for $\mathcal{Y}^N_t$? This space should posses a topology which allows us to make sense of convergence of $\mathcal{Y}^N$, if it does converge. Observe that although $\mathcal{Y}^N$ acts on $L^2(D)$ linearly, it is not a bounded operator in general.

(2) Does $\mathcal{Y}^N$ converges? If so, what can we say about the limit?

We answer these two questions fully in Chapter 5 for symmetric reflected diffusions (not just for RBMs as being considered in this introduction chapter). It turns out that the state space of the process $(\mathcal{Y}^N_t)_{t\geq 0}$ is a Hilbert distribution space $\mathcal{H}$ which strictly contains $L^2(D)$.

Our fluctuation result (Theorem 5.1.5 in Chapter 5) for the Robin boundary model contains the convergence result and the properties of the limit, which is shown to be decomposable into an independent sum of a "transportation part" and a "white noise part" (see (5.1.7) below). The "transportation part" is governed by the evolution operator $\{Q_{s,t}\}_{s\leq t}$ generated on $C(\overline{D})$ by the backward PDE $\frac{\partial v}{\partial s} = -\frac{1}{2}\Delta v$ on $(0,t)\times D$ with Robin boundary condition $\frac{\partial v}{\partial \vec{n}} = qv$ on $(0,t)\times\partial D$. We denote by $\mathbf{U}_{(t,s)}$ the operator on $\mathcal{H}$ defined by $\langle \mathbf{U}_{(t,s)}\mu,\, \phi\rangle := \langle \mu,\, Q_{s,t}\phi\rangle$, where $\langle\,,\,\rangle$ is the dual paring extending the inner product on $L^2(D)$.

Theorem 1.3.1. ***(Functional central limit theorem)*** *Suppose $T>0$ and the initial positions of particles are i.i.d. with distribution $u_0(x)dx$ where $u_0\in C(\overline{D})$. Then $\mathcal{Y}^N \xrightarrow{\mathcal{L}} \mathcal{Y}$ in $D([0,\,T],\mathcal{H})$, where $\mathcal{Y}$ is the generalized Ornstein-Uhlenbeck process given by*

$$\mathcal{Y}_t \;=\; \mathbf{U}_{(t,0)}\mathcal{Y}_0 + \int_0^t \mathbf{U}_{(t,s)}\, dM_s. \tag{1.3.1}$$

Here M is a continuous square-integrable $\mathcal{H}$-valued Gaussian martingale with independent increments and covariance $\langle M(\phi)\rangle_t = \int_0^t \mathcal{E}_r(\phi,\phi)\,dr$, where $\mathcal{E}_r$ is the bilinear form on $\mathcal{H}$ defined by

$$\mathcal{E}_r(\phi,\psi) := \int_D \nabla\phi(x)\cdot\nabla\psi(x)\,u(r,x)\,dx + \int_{\partial D}\phi(x)\psi(x)\,q(r,x)\,u(r,x)\,d\sigma(x), \tag{1.3.2}$$

where u is the hydrodynamic limit of the Robin boundary model. $\mathcal{Y}_0$ is a centered Gaussian random variable independent of M. Moreover, $\mathcal{Y}$ is a continuous Gaussian Markov process.

Formally, $\mathcal{Y}$ is called the **evolution solution** of the following stochastic evolution equation (called the **Langevin equation**):

$$dY_t = \mathbf{A}_t^{(-\alpha)} Y_t\,dt + dM_t, \quad Y_0 = \mathcal{Y}_0, \tag{1.3.3}$$

where $\mathbf{A}_t^{(-\alpha)}$ is the 'generator' of $\mathbf{U}_{(t,s)}$.

Theorem 5.1.5 is called a fluctuation-dissipation theorem in statistical physics, since it quantifies how applied perturbations spread out in space-time. As an application of this theorem, (5.1.7) implies that for suitable ϕ, we have

$$\mathcal{Y}_t(\phi) = \mathcal{Y}_0(Q_{0,t}\phi) + \int_0^t \sqrt{\mathcal{E}_s(Q_{s,t}\phi)}\, dB_s. \tag{1.3.4}$$

where B_t is a standard Brownian motion independent of $\mathcal{Y}_0$. Therefore, we can simulate the evolution of the fluctuations with respect to an observable ϕ by using a *single* Brownian path.

The proof of Theorem 5.1.5 in Chapter 5 consists of the following 6 steps.

Step 1: $\mathcal{Y}^N$ satisfies the following stochastic integral equation

$$\mathcal{Y}_t^N = \mathbf{U}_{(t,0)}^N \mathcal{Y}_0^N + \int_0^t \mathbf{U}_{(t,s)}^N \, dM_s^N \quad \text{a.s.},$$

where $\mathbf{U}_{(t,s)}^N$ is an evolution system (see [24]) approximating $\mathbf{U}_{(t,s)}$.

Step 2: $M^N \xrightarrow{\mathcal{L}} M$ in $D([0,T],\mathcal{H})$.

Step 3: $\mathcal{Y}^N$ is tight in $D([0,T],\mathcal{H})$.

Step 4: $\mathbf{U}_{(t,0)}^N \mathcal{Y}_0^N \xrightarrow{\mathcal{L}} \mathbf{U}_{(t,0)}\mathcal{Y}_0$ in $D([0,T],\mathcal{H})$.

Step 5: $\int_0^t \mathbf{U}_{(t,s)}^N \, dM_s^N \xrightarrow{\mathcal{L}} \int_0^t \mathbf{U}_{(t,s)} \, dM_s$ in $D([0,T],\mathcal{H})$.

Step 6: All the stated properties for the fluctuation limit hold.

Note that $t \mapsto \int_0^t \mathbf{U}_{(t,s)} \, dM_s$ is not a martingale, even though $\theta \mapsto \int_0^\theta \mathbf{U}_{(t,s)} \, dM_s$ is a martingale for $\theta \in [0,t]$. The standard method based on Kotelenez's submartingale inequality [52] does not seem to work. This is because in our case $\mathbf{U}_{(t,s)}$ is not exponentially bounded; that is, there is no $\beta > 0$ so that the operator norm $\|\mathbf{U}_{(t,s)}\| \leq e^{\beta(t-s)}$ for $t \geq s$ (see [52]). In fact, we suspect it is not even a bounded operator on $\mathcal{H}_{-\alpha}$ due to the singular interaction near the boundary. Our approach is based on suitably extending the functional analytic framework of [53] and a direct analysis that uses heat kernel estimates and Dirichlet Form method.

1.3.2 Fluctuations for interacting diffusions

In this subsection, we summarize the main results in Chapter 6 (and also that of [19])[4]. The techniques developed in the Robin boundary model allow us to overcome some (but not all) challenges for the study of the fluctuation of the annihilating diffusion model. We need two new ingredients, namely the *asymptotic expansion of the correlation functions* and the *Boltzman-Gibbs principle*. More precisely, by considering an approximating BBGKY hierarchy and generalizing the approach of [31], we can show that the correlation functions have the structure

$$F_t^{N,(n,m)}(\vec{x},\vec{y}) \approx \prod_{i=1}^{n} u_+(t,x_i) \prod_{j=1}^{m} u_-(t,y_j) + \frac{1}{N}\, B_t^{N,(n,m)}(\vec{x},\vec{y}), \tag{1.3.5}$$

where $(u_+,\, u_-)$ is the hydrodynamic limit and $B_t^{N,(n,m)}$ is an *explicit* function. See Theorem 6.3.3 for the precise statement. This result implies propagation of chaos and allows explicit calculations of the covariance of the fluctuation process. On other hand, the Boltzman-Gibbs principle, first formulated mathematically and proven for some zero range processes in equilibrium in [8], says that the fluctuation fields of non-conserved quantities change on a time scale much faster than the conserved ones, hence in a time integral only the component along those fields of conserved quantities survives. Although this principle is proved to hold for few non-equilibrium situations (see [6] and the references therein), it is not known whether it holds in general. The validity of the principle for our annihilating diffusion model is far from obvious, since there is no conserved quantity.

We define $\mathcal{H}^+$ and $\langle\ ,\ \rangle_\pm$ just as in the Robin boundary model. We now consider the "direct sum" $\mathcal{Z}_t^N := \mathcal{Y}_t^{N,+} \oplus \mathcal{Y}_t^{N,-} \in \mathbf{H}$, where $\langle \mu^+ \oplus \mu^-,\ (\phi_+,\phi_-)\rangle := \langle \mu^+,\phi_+\rangle_+ + \langle \mu^-,\phi_-\rangle_-$, and $\mathbf{H} := \mathcal{H}^+ \oplus \mathcal{H}^-$. Our fluctuation result (Theorem 6.1.2 in Chapter 6) for the annihilating diffusion model also contains the convergence result and the properties of the limit, which is also shown to be decomposable into an independent sum of a "transportation part" and a "white noise part". The "transportation part" of the limit is now governed by the evolution semigroup

[4] To avoid unnecessary complications, we assume in this subsection that the harvest sites are empty sets and that the underlying particles perform RBMs.

$\{Q_{(s,t)}\}_{s\le t}$ generated on $C(\overline{D}_+)\times C(\overline{D}_-)$ by the coupled backward PDEs

$$\begin{cases} \dfrac{\partial v^+}{\partial s} = -\dfrac{1}{2}\Delta v^+ & \text{on } (0,t)\times D_+ \\ \dfrac{\partial v^+}{\partial \vec{n}_+} = (v^+ + v^-)\, u_-\, \mathbf{1}_{\{I\}} & \text{on } (0,t)\times \partial D_+ \\ \dfrac{\partial v^-}{\partial s} = -\dfrac{1}{2}\Delta v^- & \text{on } (0,t)\times D_- \\ \dfrac{\partial v^-}{\partial \vec{n}_-} = (v^+ + v^-)\, u_+\, \mathbf{1}_{\{I\}} & \text{on } (0,t)\times \partial D_- \end{cases} \tag{1.3.6}$$

which can be viewed as a linearization of the hydrodynamic limit in Theorem 1.2.3. As before, we define $\langle \mathbf{U}_{(t,s)}\mu,\ (\phi_+,\phi_-)\rangle := \langle \mu,\ Q_{s,t}(\phi_+,\phi_-)\rangle$ and state our fluctuation result of Chapter 6:

Theorem 1.3.2. ***(Functional central limit theorem)*** *Suppose* $\liminf_\infty N\,\delta_N^{2d} > 0$, $\delta_N \to 0$ *and the initial position of particles in each of* $\overline{D}_\pm$ *are i.i.d. with distribution* $u_0^\pm \in C(\overline{D}_\pm)$*. Then* $\mathcal{Z}^N \xrightarrow{\mathcal{L}} \mathcal{Z}$ *in* $D([0,\,T_0],\,\mathbf{H})$*, where* $\mathcal{Z}$ *is the generalized Ornstein-Uhlenbeck process*

$$\mathcal{Z}_t = \mathbf{U}_{(t,0)}\mathcal{Z}_0 + \int_0^t \mathbf{U}_{(t,s)}\, dM_s. \tag{1.3.7}$$

Here $T_0 := (\|u_0^+\|\vee\|u_0^-\|)^{-2}\,C$ *with* $\|\cdot\|$ *being the sup-norm and* $C>0$ *is a constant that depends only on* D*.* M *is a (unique in law) continuous square-integrable* $\mathbf{H}$*-valued Gaussian martingale with independent increments and covariance functional characterized by*

$$\begin{aligned} \langle M(\phi_+,\phi_-)\rangle_t \quad = \quad & \int_0^t \Big(\int_{D_+} |\nabla\phi_+(x)|^2\, u_+(s,x)\, dx + \int_{D_-} |\nabla\phi_-(y)|^2\, u_-(s,y)\, dy \\ & + \int_I (\phi_+(z)+\phi_-(z))^2\, u_+(s,z)u_-(s,z)\, d\sigma(z)\Big)\, ds, \end{aligned}$$

where $\langle M(\phi_+,\phi_-)\rangle_t$ *is the predictable quadratic variation of the real martingale* $M_t(\phi_+,\phi_-)$*, the pair* $(u_+,\,u_-)$ *is the hydrodynamic limit of the annihilating diffusion system given in Theorem 1.2.1. Moreover,* $\mathcal{Z}$ *is a continuous Gaussian Markov process.*

This result implies the co-existence of two effects at the level of fluctuation: the effect of diffusive transport and that of the interaction of the two types of particles. In other words, none of these two effects dominate the other in the limit. The basic outline for the proof of Theorem 6.1.2 is the same as that of Theorem 5.1.5 but with an additional step. Roughly speaking, Step 1 is now replaced by

$$\mathcal{Z}_t^N = \mathbf{U}_{(t,0)}^N \mathcal{Z}_0^N + \int_0^t \mathbf{U}_{(t,s)}^N \, dM_s^N + E_t^N,$$

where E_t^N is an extra error term. The additional step is to show that this error goes to zero in a suitable sense. This is true due to the choice of $\mathbf{U}_{(t,s)}^N$ which is facilitated by the Boltzman-Gibbs principle. An intuitive explanation for the validity of the Boltzman-Gibbs principle for our annihilating diffusion model is as follows: the high density assumption $\liminf_{N\to\infty} N\,\delta_N^{2d} > 0$ guarantees that the interaction near I changes the occupation number of the particles at a slow rate with respect to diffusion (which conserves the particle number). In other words, the particle number is conserved on the time scale that is relevant for the principle.

Remark 1.3.3. Note that (6.1.2) is only established for $t \in [0, T_0]$. We do not know if T_0 can be taken to be an arbitrary positive number. Such phenomenon also appears in [31] and [54]. We need this property to guarantee (1.3.5). It is interesting to investigate whether the theorem holds only on finite time interval or not.

1.4 Applications and implications

The most important implications of the hydrodynamic results and the fluctuation results are the connections, in two different scales of observations, between the microscopic mechanisms of the stochastic dynamics of the systems and the collective behaviors that emerge in the macroscopic scale. These two scales correspond, in the language of probability, to the level of law of large numbers and that of the central limit theorem respectively.

I now make the above assertions transparent by asking two questions. In our annihilating random walk model or annihilating diffusion model, let $K \subset D_+$ be an arbitrarily chosen subset (see Figure 1.4).

(1) What is a good estimate for the number of particles in K at time t ?

(2) What is the prediction error for the answer to question (1)?

If we view the system as the population dynamics of two interacting species with competition near a static boarder, then the above questions are about the number of a certain species in a region at a particular time, which is often the questions of interest in ecology.

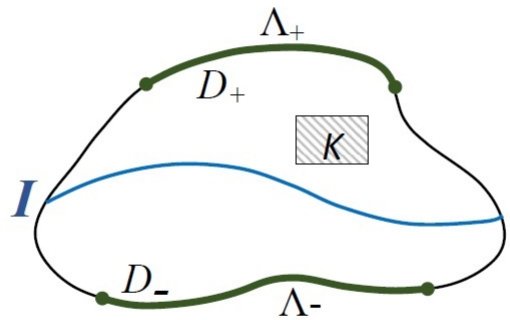


Figure 1.4: An arbitrary subset K

The answers to questions (1) and (2) are given by the hydrodynamic limits and the fluctuation limits respectively. More precisely, when N is large, the answer to question (1) is $N \int_K u_+(t,x)\,dx$. This is illustrated in Figure 1.5, in which the time-trajectories of $\int_K u_+(t,x)\,dx$ (the black smooth curve) and that of (Number of particles in $K)/N$ are drawn on the same figure for comparison. The blue curve is generated by a simulation with $N = 100$, $D_+ = (0,2)\times(0,1)$ and $D_- = (0,2)\times(0,-1)$ being rectangles and $K = (0.5,1.5)\times(0,25,0.75)$ being a smaller rectangle inside D_+. The simulation is in my webpage http://staff.washington.edu/louisfan. Our hydrodynamic results implies that the whole random blue curve is approximated by the deterministic black curve when N is large.

To answer question (2), we look at a more refined scale by magnifying the error in Figure 1.5 by a factor of $\sqrt{N}$. Figure 1.6 is what we obtain by doing so. Our fluctuation results imply that, when N is large, the random curve in Figure 1.6 has the law of a stochastic integral which is a Gaussian random variable at any time t. Moreover, the covariances of the values at different times are characterized. We also know the covariances of the fluctuations for different subsets

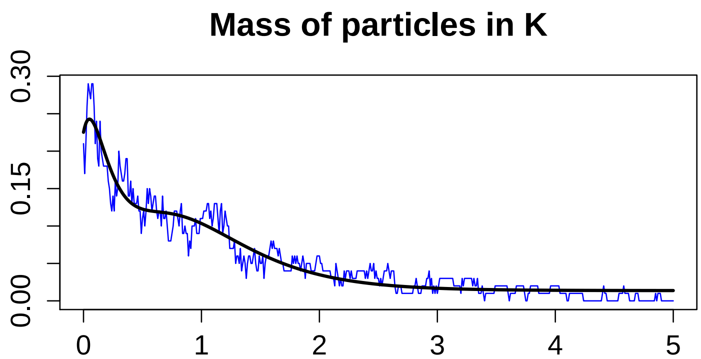


Figure 1.5: $\frac{\text{Number of particles in } K}{N}$ and $\int_K u_+(t,x)\,dx$

$K_1, K_2, K_3, \cdots, \subset D_\pm$. Thus, knowing the prediction error of one region (in either D_+ or D_-) will tell us more about the prediction error for every other region in D_+ and D_-. This fluctuation results also imply the co-existence of two effects at the level of fluctuation: the effect of diffusive transport and that of the interaction of the two types of particles. In other words, none of these two effects dominate the other in the limit: a fact which is unlikely to be obtained without computations.

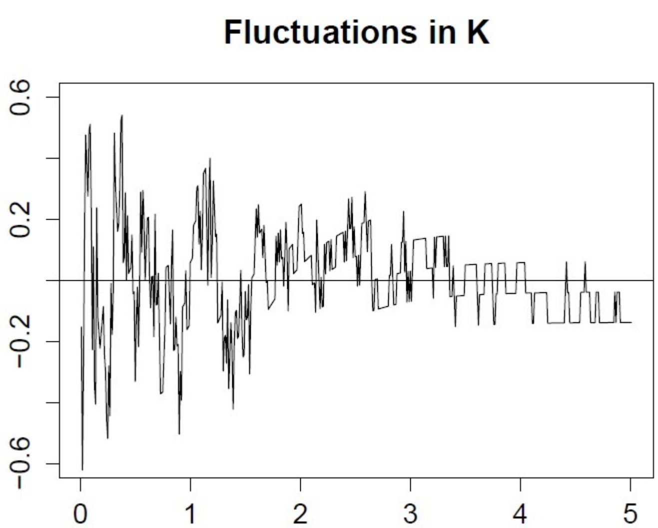


Figure 1.6: $\sqrt{N}\left(\frac{\text{Number of particles in } K}{N} - \int_K u_+(t,x)\,dx\right)$

Organization of this thesis:

Chapter 2 contains preliminaries such as basic properties of reflected diffusions and their discrete approximations. We will prove the local central limit theorem for RBMs with drifts and discuss its applications. In Chapter 3, we construct the annihilating random walk model, and then establish the hydrodynamic limit and the propagation of chaos result. In Chapter 4, we construct the annihilating diffusion model and establish its hydrodynamic limit result. Chapter 5 is about the fluctuation limit of the Robin boundary model. Chapter 6 is about the fluctuation limit of the annihilating diffusion model.

Chapter 2

REFLECTED DIFFUSIONS IN DOMAINS AND RANDOM WALK APPROXIMATIONS

In this chapter, D is always a bounded Lipschitz domain in $\mathbb{R}^d$. We outline some basic properties of symmetric reflected diffusions on D. We will also study reflected diffusion killed upon hitting a subset $\Lambda \subset \overline{D}$. This is fundamental to our understanding for the underlying motion of each particle of our interacting particle systems.

2.1 Reflected diffusions

Reflected diffusions are natural mathematical objects to study. After all, the random motions of the pollen grains observed by Robert Brown in year 1827 were reflected on the boundary of the water tank. Suppose $\rho \in W^{1,2}(D) \cap C(\overline{D})$ is a strictly positive function, and $\mathbf{a} = (a^{ij})$ is a symmetric, bounded, uniformly elliptic $d \times d$ matrix-valued function with $a^{ij} \in W^{1,2}(D)$ for each i, j. It is well known (see [2, 15]) that the bilinear form $(\mathcal{E},\, W^{1,2}(D))$ defined by

$$\mathcal{E}(f,g) := \frac{1}{2}\int_D \mathbf{a}\nabla f(x)\cdot\nabla g(x)\,\rho(x)\,dx := \frac{1}{2}\int_D \sum_{i,j=1}^{d} a^{ij}(x)\,\frac{\partial f}{\partial x_i}(x)\,\frac{\partial g}{\partial x_i}(x)\,\rho(x)\,dx \qquad (2.1.1)$$

is a regular Dirichlet form in $L^2(D,\, \rho(x)dx)$ and hence has an associated Hunt process X (unique in distribution). Furthermore, it can be checked (cf. Chapter 2 of [20]) that X is a continuous, irreducible, conservative strong Markov process with symmetrizing measure ρ and has infinitesimal generator

$$\mathcal{A} := \frac{1}{2\,\rho}\,\nabla\cdot(\rho\,\mathbf{a}\nabla) := \frac{1}{2\,\rho}\,\sum_{i,j=1}^{d}\frac{\partial}{\partial x_i}\Big(\rho\,a^{ij}\,\frac{\partial}{\partial x_j}\Big).$$

Definition 2.1.1. *Let* $(\boldsymbol{a},\,\rho)$ *and* X *be as in the preceding paragraph. We call* X *an* $(\boldsymbol{a},\,\rho)$***-reflected diffusion*** *or an* $(\mathcal{A},\,\rho)$***-reflected diffusion****. A special but important case is when* $\boldsymbol{a}$ *is the identity matrix, in which* X *is called a reflected Brownian motion with drift* $\frac{1}{2}\nabla(\log\rho)$*. If in addition* $\rho = 1$*, then* X *is called a* ***reflected Brownian motion (RBM)****.*

Intuitively, X behaves like a diffusion process associated to the second order elliptic differential operator $\mathcal{A}$ in the interior of D, and is instantaneously reflected at the boundary in the inward conormal direction $\vec{\nu} := \mathbf{a}\vec{n}$, where $\vec{n}$ is the inward unit normal. This can be seen via the *Skorokhod representation* which tells us precise pathwise properties of X:

$$X_t = X_0 + \int_0^t \beta(X_s)dB_s + \int_0^t \vec{b}(X_s)ds + \int_0^t \rho\,\mathbf{a}\,\vec{n}\,(X_s)\,dL_s \tag{2.1.2}$$

for all $t \geq 0$, $\mathbb{P}_x$-a.s. for all $x \in \overline{D}$. Here B is a d-dimensional Brownian motion martingale additive functional of X, β is the symmetric positive definite $d \times d$ matrix-valued function on D such that $\beta^2 = \mathbf{a} = [\vec{a_1}, \vec{a_2}, \cdots \vec{a_d}]$, $\vec{b} := \frac{1}{2}\left(\nabla\cdot\vec{a_i} + \mathbf{a}\nabla\log\rho\right)$, $\vec{n}$ is the *inward* unit normal vector field of ∂D and L is the positive additive continuous functional (PCAF) of X, called the *boundary local time*, associated with the measure $\sigma/2$. The Skorokhod representation (2.1.2) for symmetric reflected diffusions is proved in [15] for a general class of domains which contains Lipschitz domains.

2.1.1 Heat kernel estimates

It is well known (cf. [2, 44] and the references therein) that X has a transition density $p(t,x,y)$ with respect to the symmetrizing measure $\rho(x)dx$ (i.e., $\mathbb{P}_x(X_t \in dy) = p(t,x,y)\,\rho(y)dy$ and $p(t,x,y) = p(t,y,x)$), that p is locally Hölder continuous and hence $p \in C((0,\infty)\times\overline{D}\times\overline{D})$, and that we have the following Aronson type Gaussian estimates: for any $T > 0$, there are constants

$c_1 \geq 1$ and $c_2 \geq 1$ such that

$$\frac{1}{c_1 t^{d/2}} \exp\left(\frac{-c_2|y-x|^2}{t}\right) \leq p(t,x,y) \leq \frac{c_1}{t^{d/2}} \exp\left(\frac{-|y-x|^2}{c_2\, t}\right) \tag{2.1.3}$$

for every $(t,x,y) \in (0,T] \times \overline{D} \times \overline{D}$. The constants c_1 and c_2 depends on D, T, the ellipticity of $\mathbf{a}$ and the lower and upper bound of ρ.

Convention: Throughout this thesis, we suppress the dependence of any constant on the ellipticity of $\mathbf{a}$ and the lower and upper bound of ρ. For example, we will simply write $c_1 = c_1(D,T)$ and $c_2 = c_2(D,T)$ for (2.1.3).

Using (2.1.3) and the Lipschitz assumption for the boundary, we can check that

$$\sup_{x\in\overline{D}} \sup_{0<\delta\leq\delta_0} \frac{1}{\delta}\int_{D^\delta} p(t,x,y)\,dy \;\leq\; \frac{C_1}{\sqrt{t}} \quad \text{for } t\in(0,T] \quad \text{and} \tag{2.1.4}$$

$$\sup_{x\in\overline{D}} \int_{\partial D} p(t,x,y)\,\sigma(dy) \;\leq\; \frac{C_1}{\sqrt{t}} \quad \text{for } t\in(0,T], \tag{2.1.5}$$

where $C = C(d,D,T) > 0$, $\delta_0 = \delta_0(D) > 0$ are constants, $D^\delta := \{x \in D : \text{dist}(x,\partial D) < \delta\}$. In fact (2.1.5) follows from (2.1.4) via Lemma 5.3.1. On other hand, suppose $g \in \mathcal{B}_b([0,T]\times\partial D)$. Then for $t \in [0,T]$ and $x \in \overline{D}$, we have

$$\mathbb{E}^x\left[\int_0^t g(s,X_s)dL_s\right] = \frac{1}{2}\int_0^t \int_{\partial D} g(s,y)p(s,x,y)\,\rho(y)\,\sigma(dy)\,ds. \tag{2.1.6}$$

2.1.2 Reflected diffusions killed upon hitting a closed set $\Lambda \subset \overline{D}$

Now we consider an $(\mathbf{a},\rho)$-reflected diffusion *killed upon hitting a closed subset* Λ of $\overline{D}$. The results in this subsection hold, in particular, when Λ is an empty set (in which case we reduce to the study of reflected diffusion without killing) and when Λ is a subset of ∂D (this is the case for $\Lambda_\pm$ in figure 1.3). Define

$$X_t^{(\Lambda)} := \begin{cases} X_t, & t < T_\Lambda \\ \partial, & t \geq T_\Lambda, \end{cases} \tag{2.1.7}$$

where ∂ is a cemetery point and $T_\Lambda := \inf\{t > 0 : X_t \in \Lambda\}$ is the first hitting time of X on Λ. Since $\overline{D} \setminus \Lambda$ is open in $\overline{D}$, Theorem A.2.10 of [39] asserts that $X^{(\Lambda)}$ is a Hunt process on $(\overline{D} \setminus \Lambda) \cup \partial$ with transition function $P_t^\Lambda(x, A) = \mathbb{P}^x(X_t \in A,\, t < T_\Lambda)$. The characterization of the Dirichlet form of $X^{(\Lambda)}$ can be found in [20, Theorem 3.3.8] or [39, Theorem 4.4.2]; in particular, it implies that the semigroup $\{P_t^\Lambda\}_{t\geq 0}$ of $X^{(\Lambda)}$ is symmetric and strongly continuous on $L^2(\overline{D} \setminus \Lambda,\, \rho(x)dx)$. Clearly, $X^{(\Lambda)}$ has a transition density $p^{(\Lambda)}$ with respect to $\rho(x)dx$ (i.e. $P_t^\Lambda(x, dy) = p^{(\Lambda)}(t, x, y)\, \rho(y)\, dy$). Note that $p^{(\Lambda)}(t, x, y) \leq p(t, x, y)$ for all $x, y \in D$ and $t > 0$.

So far Λ is only assumed to be closed in $\overline{D}$. We will also need the following regularity assumption.

Definition 2.1.2. *$\Lambda \subset \overline{D}$ is said to be **regular** with respect to X if $\mathbb{P}^x(T_\Lambda = 0) = 1$ for all $x \in \Lambda$, where $T_\Lambda := \inf\{t > 0 : X_t \in \Lambda\}$.*

This regularity assumption implies that $p^{(\Lambda)}(t, x, y)$ is jointly continuous in x and y up to the boundary. In particular, $p^{(\Lambda)}(t, x, y)$ is continuous for x and y in a neighborhood of I. We now gather some basic properties of $p^{(\Lambda)}(t, x, y)$ for later use.

Proposition 2.1.3. *Let X be an $(\boldsymbol{a}, \rho)$-reflected diffusion defined in Definition 2.1.1, and $p^{(\Lambda)}(t, x, y)$ be the transition density, with respect to $\rho(x)dx$, of X^Λ defined in (2.1.7). Suppose Λ is closed and regular with respect to X. Then $p^{(\Lambda)}(t, x, y) \geq 0$ and $p^{(\Lambda)}(t, x, y) = p^{(\Lambda)}(t, y, x)$ for all $x, y \in \overline{D}$ and $t > 0$. Moreover, $p^{(\Lambda)}(t, x, y)$ can be extended to be jointly continuous on $(0, \infty) \times \overline{D} \times \overline{D}$. The last property implies that the semigroup $\{P_t^\Lambda\}_{t\geq 0}$ of X^Λ is strongly continuous on the Banach space*

$$C_\infty(\overline{D} \setminus \Lambda) := \{f \in C(\overline{D}) : f \text{ vanishes on } \Lambda\}$$

equipped with the uniform norm on $\overline{D}$. The domain of the Feller generator of $\{P_t^{(\Lambda)}\}_{t\geq 0}$, denoted by $Dom(\mathcal{A}^{(\Lambda)})$, is dense in $C_\infty(\overline{D} \setminus \Lambda)$.

Proof Define, for all $(t, x, y) \in (0, \infty) \times \overline{D} \times \overline{D}$,

$$q^{(\Lambda)}(t, x, y) := p(t, x, y) - r(t, x, y), \text{ where } r(t, x, y) := \mathbb{E}^x \left[p(t - T_\Lambda, X_{T_\Lambda}, y);\, t \geq T_\Lambda \right].$$

Using the fact that $x \mapsto \mathbb{P}^x(T_\Lambda < t)$ is lower semi-continuous (cf. Proposition 1.10 in Chapter II of [1]), it is easy to check that if Λ is closed and regular, then

$$\lim_{n \to \infty} \mathbb{P}^{x_n}(T_\Lambda < t) = 1 \tag{2.1.8}$$

whenever $t > 0$ and $x_n \in D$ converges to a point in Λ. Recall that $p(t, x, y)$ is symmetric in (x, y), has two-sided Gaussian estimates (2.1.3), and is jointly continuous on $(0, \infty) \times \overline{D} \times \overline{D}$. Using these properties of p together with (2.1.8), then applying the same argument of section 4 of Chapter II in [1], we have

(a) $q^{(\Lambda)}(t, x, y)$ is a density for the transition function X^Λ.

(b) $q^{(\Lambda)}(t, x, y) \geq 0$ and $q^{(\Lambda)}(t, x, y) = q^{(\Lambda)}(t, y, x)$ for all $x, y \in \overline{D}$ and $t > 0$.

(c) $q^{(\Lambda)}(t, x, y)$ is jointly continuous on $(0, \infty) \times \overline{D} \times \overline{D}$.

From (c), the semigroup $\{P_t^{(\Lambda)}\}$ of $X^{(\Lambda)}$ is strongly continuous by a standard argument. $C_\infty(\overline{D} \setminus \Lambda)$ is a Banach space since it is a closed subspace of $C(\overline{D})$. The Feller generator $Dom(\mathcal{A}^{(\Lambda)})$ of $\{P_t^{(\Lambda)}\}$ is dense in $C_\infty(\overline{D} \setminus \Lambda)$ because any $f \in C_\infty(\overline{D} \setminus \Lambda)$ is the strong limit $\lim_{t \downarrow 0} \frac{1}{t} \int_0^t P_s^{(\Lambda)} f ds$ in $C_\infty(\overline{D} \setminus \Lambda)$, and $\int_0^t P_s^{(\Lambda)} f ds \in Dom(\mathcal{A}^{(\Lambda)})$. □

2.1.3 Fundamental martingales for reflected diffusions

We will need the following collection of fundamental martingales, together with their quadratic variations, for reflected diffusions. Note that it holds for every $x \in \overline{D}$.

Lemma 2.1.4. *Suppose X^Λ is an $(\boldsymbol{a}, \rho)$-reflected diffusion in a bounded Lipschitz domain D killed upon hitting Λ. Suppose all assumptions in Proposition 2.1.3 hold, and f is in the domain* *of the Feller generator $Dom(\mathcal{A}^{(\Lambda)})$. Then we have*

$$M(t) := f(X^\Lambda(t)) - f(X^\Lambda(0)) - \int_0^t \mathcal{A}^{(\Lambda)} f(X^\Lambda(s))\, ds \tag{2.1.9}$$

is a $\mathcal{F}_t^{X^\Lambda}$-martingale that is bounded on each compact time interval and has predictable quadratic variation

$$\langle M \rangle_t := \int_0^t (\boldsymbol{a}\nabla f \cdot \nabla f)\,(X^\Lambda(s))\, ds$$

under $\mathbb{P}^x$ for any $x \in \overline{D}$. Moreover, if X_1 and X_2 are independent copies of X^Λ, and if M_i is the above M with X^Λ replaced by X_i, then the cross variation $\langle M_1,\, M_2\rangle_t = 0$.

Proof For $f \in Dom(\mathcal{A}^{(\Lambda)})$, $M(t)$ defined in (2.1.9) is an $\mathcal{F}_t^{X^\Lambda}$-martingale that is bounded on each compact time interval. Since D is bounded, f is clearly in the domain of the L^2-generator of X^Λ. Hence it follows from the Fukushima decomposition of $f(X_t^\Lambda)$ (see [20, Theorems 4.2.6 and 4.3.11] that $M(t)$ is a martingale additive functional of X^Λ of finite energy having quadratic variation $\langle M(t)\rangle_t = \int_0^t (\mathbf{a}\nabla f \cdot \nabla f)(X^\Lambda(s))ds$. If X_1 and X_2 are independent copies of X^Λ, then M_1 and M_2 are independent and so $\langle M_1, M_2\rangle = 0$. □

An immediate consequence of Lemma 2.1.4 is

$$\int_0^t P_s^\Lambda(\mathbf{a}\nabla f \cdot \nabla f)(x)\, ds = \mathbb{E}^x[M(t)^2] \le 8(\|f\|^2 + \|\mathcal{A}^{(\Lambda)} f\|^2\, t^2) \tag{2.1.10}$$

for $x \in \overline{D}$ and $t \ge 0$, where $\|\cdot\|$ is the uniform norm in $\overline{D}$.

Lemma 2.1.5. *Suppose X^Λ is an $(\boldsymbol{a}, \rho)$-reflected diffusion in a bounded Lipschitz domain D killed upon hitting a closed subset Λ of ∂D that is regular with respect to X. Then for any $T > 0$ and bounded measurable function ϕ on $\overline{D} \setminus \Lambda$, we have*

$$P_{T-s}^\Lambda \phi(X_s^\Lambda) \text{ is a } \mathcal{F}_s^{X^\Lambda}\text{-martingale for } s \in [0, T], \tag{2.1.11}$$

under $\mathbb{P}^x$ for any $x \in \overline{D}\backslash\Lambda$. Moreover, its quadratic variation is $\int_0^s \boldsymbol{a}\nabla P_{T-r}^\Lambda \phi \cdot \nabla P_{T-r}^\Lambda \phi(X^\Lambda(r))\, dr$.

Proof (2.1.11) follows from the Markov property of X^Λ. Denote by $\mathcal{L}^{(\Lambda)}$ the L^2-generator of $X^{(\Lambda)}$. Then for every $t \in [0,T)$, $P^\Lambda_{T-s}\phi \in Dom(\mathcal{L}^{(\Lambda)})$. It follows from the spectral representation of $\mathcal{L}^{(\Lambda)}$ that

$$\Big\| \frac{\partial P^\Lambda_{T-s}\phi}{\partial s} \Big\|_{L^2} = \| -\mathcal{L}^{(\Lambda)} P^\Lambda_{T-s}\phi\|_{L^2} \le \frac{\|\phi\|_{L^2}}{T-s}.$$

Thus $(s,x) \mapsto P^\Lambda_{T-s}\phi(x)$ for $s \in [0,T)$ and $x \in \overline{D} \setminus \Lambda$ is in the domain of the Dirichlet form for the space-time process $(s, X^{(\Lambda)}_s)$. By an application of the Fukushima decomposition in the context of time-dependent Dirichlet forms, one concludes that the quadratic variation of the martingale $s \mapsto P^\Lambda_{T-s}\phi(X^\Lambda_s)$ is $\int_0^s \mathbf{a}\nabla P^\Lambda_{T-r}\phi \cdot \nabla P^\Lambda_{T-r}\phi(X^\Lambda(r))\, dr$; see [65, Example 6.5.6]. □

2.1.4 Heat equation with Robin boundary condition

Observe that if we view u_- as a known function, then in each of the two hydrodynamic results Theorem 1.2.1 and Theorem 1.2.3, the functions u_+ satisfies a second order parabolic equation in D_+ with Robin boundary condition. This leads us to consider the following Robin boundary problem, where $g \in \mathcal{B}_b([0,\infty) \times \partial D)$ is arbitrary.

$$\begin{cases} \dfrac{\partial u}{\partial t} = \mathcal{A}u & \text{on } (0,\infty) \times D \\ \dfrac{\partial u}{\partial \vec{n}} = \dfrac{1}{\rho}\, gu & \text{on } (0,\infty) \times \partial D \\ u(0,\cdot) = \varphi & \text{on } D. \end{cases} \tag{2.1.12}$$

By Itó's formula and the Skorokhod representation for the $(\mathcal{A}, \rho)$-reflected diffusion X, we see that a classical solution of (2.1.12), should it exists, has the probabilistic representation

$$u(t,x) := \mathbb{E}^x \left[\varphi(X_t) e^{-\int_0^t g(t-s,X_s)dL_s} \right], \tag{2.1.13}$$

where L is the boundary local time of X.

First we show that the function u defined by (2.1.13) is continuous.

Lemma 2.1.6. *Suppose $\varphi \in \mathcal{B}_b(\overline{D})$, $g \in \mathcal{B}_b^+([0,T] \times \partial D)$ and u is defined by (2.1.13). Then $u \in C((0,T] \times \overline{D})$. Moreover, if $\varphi \in C(\overline{D})$, then $u \in C([0,T] \times \overline{D})$.*

Proof Observe that for any $r \in [0,t]$,

$$\begin{aligned} u(t,x) &= \mathbb{E}^x\left[\varphi(X_t)\, e^{-\int_r^t g(t-s,X_s)\,dL_s}\, e^{-\int_0^r g(t-s,X_s)\,dL_s}\right] \\ &= \mathbb{E}^x\left[\varphi(X_t)\, e^{-\int_r^t g(t-s,X_s)\,dL_s}\right] + \mathbb{E}^x\left[\varphi(X_t)\, e^{-\int_r^t g(t-s,X_s)\,dL_s}\left(e^{-\int_0^r g(t-s,X_s)\,dL_s} - 1\right)\right]. \end{aligned} \tag{2.1.14}$$

By Markov property, the first term is

$$\mathbb{E}^x\left[\mathbb{E}^{X_r}[\varphi(X_{t-r})e^{-\int_0^{t-r} g(t-r-s,X_s)dL_s}]\right] = \mathbb{E}^x[u(t-r,X_r)].$$

Since X has the strong Feller property (see [2]) and u is bounded, $x \mapsto \mathbb{E}^x[u(t-r,X_r)]$ is continuous on $\overline{D}$ for any fixed $t>0$ and $r \in (0,t)$.

The second term of (2.1.14) converges to zero uniformly on $(0,T] \times \overline{D}$, as $r \to 0$. This is because its absolute value is bounded by

$$\begin{aligned} & \|\varphi\|\mathbb{E}^x\left[1 - e^{-\int_0^r g(t-s,X_s)\,dL_s}\right] \\ \le\ & \|\varphi\|\mathbb{E}^x\left[\int_0^r g(t-s,X_s)\,dL_s\right] \quad \text{by mean-value theorem} \\ \le\ & \|\varphi\|\,\|g\|\,\frac{1}{2}\int_0^r \int_{\partial D} p(s,x,y)\,\rho(y)\,\sigma(dy)\,ds \quad \text{by (2.1.6)} \\ \le\ & \|\varphi\|\,\|g\|\,C\sqrt{r} \quad \text{by (2.1.5).} \end{aligned}$$

Hence, u is continuous in $x \in \overline{D}$.

By a similar calculation as in (2.1.14), we have

$$\begin{aligned} u(t+a,x) - u(t,x) &= \mathbb{E}^x[u(t,X_a) - u(t,x)] \\ &\quad + \mathbb{E}^x\left[\varphi(X_{t+a})e^{-\int_a^{t+a} g(t+a-s,X_s)dL_s}\left(e^{-\int_0^a g(t+a-s,X_s)\,dL_s} - 1\right)\right]. \end{aligned}$$

Hence,

$$\begin{aligned}
& |u(t+a,x)-u(t,x)| \\
\leq\ & \mathbb{E}^x[|u(t,X_a)-u(t,x)|] + \|\varphi\|\,\mathbb{E}^x\left[\int_0^a g(t+a-s,X_s)dL_s\right] \quad \text{by mean-value theorem} \\
\leq\ & \int_D |u(t,z)-u(t,x)|\,p(a,x,z)\,dz + \|\varphi\|\,\|g\|\,\frac{1}{2}\int_0^a\int_{\partial D} p(s,x,z)\,\sigma(dz)ds.
\end{aligned}$$

Both terms go to 0 uniformly in $x \in \overline{D}$ as a goes to 0. (In fact, the first term goes to 0 uniformly since the semigroup P_t is strongly continuous on $C(\overline{D})$. For the second term, $\int_0^a \int_{\partial D} p(s,x,z)\,\sigma(dz)ds \leq 2C_1\sqrt{a}+C_2 a$ also goes to 0 uniformly in x.) Hence u is continuous in $t \in (0,T]$ uniformly in $x \in \overline{D}$. Therefore, $u \in C((0,T]\times\overline{D})$. If $\varphi \in C(\overline{D})$, we can extend the above argument to show that $u \in C([0,T]\times\overline{D})$. □

Remark 2.1.7. In fact, one can allow g to be unbounded and show that the conclusion of Lemma 2.1.6 remains true if $g\sigma$ satisfies a Kato class condition:

$$\lim_{a\to 0}\sup_{x\in\overline{D}}\int_0^a\int_{\partial D} p(s,x,z)\,|g(t+a-s,z)|\,\sigma(dz)\,ds = 0.$$

□

Proposition 2.1.8. *Suppose $\varphi \in C(\overline{D})$ and $g \in \mathcal{B}_b^+([0,T]\times\partial D)$. Then*

$$u(t,x) := \mathbb{E}^x\left[\varphi(X_t)\,e^{-\int_0^t g(t-s,X_s)\,dL_s}\right]$$

is the unique element in $C([0,T]\times\overline{D})$ that satisfies the following integral equation:

$$u(t,x) = P_t\varphi(x) - \frac{1}{2}\int_0^t\int_{\partial D} p(t-r,x,y)g(r,y)u(r,y)\,\rho(y)\sigma(dy)\,dr. \tag{2.1.15}$$

Proof By Lemma 2.1.6, $u(t,x) := \mathbb{E}^x[\varphi(X_t)e^{-\int_0^t g(t-s,X_s)\,dL_s}]$ lies in $C([0,T]\times\overline{D})$. Moreover, by Markov property and (2.1.6) we have

$$
\begin{aligned}
u(t,x) &= \mathbb{E}^x[\varphi(X_t)] - \mathbb{E}^x[\varphi(X_t)\,(1 - e^{-\int_0^t g(t-s,X_s)\,dL_s})] \\
&= P_t\varphi(x) - \mathbb{E}^x\left[\varphi(X_t)\,e^{-\int_r^t g(t-s,X_s)\,dL_s}\Big|_{r=0}^{r=t}\right] \\
&= P_t\varphi(x) - \mathbb{E}^x\left[\varphi(X_t)\,\int_0^t g(t-r,X_r)\,e^{-\int_r^t g(t-s,X_s)\,dL_s}\,dL_r\right] \\
&= P_t\varphi(x) - \mathbb{E}^x\left[\int_0^t g(t-r,X_r)\,\mathbb{E}^{X_r}\left[\varphi(X_{t-r})\,e^{-\int_0^{t-r} g(t-r-s,X_s)dL_s}\right]\,dL_r\right] \\
&= P_t\varphi(x) - \mathbb{E}^x\left[\int_0^t g(t-r,X_r)\,u(t-r,X_r)\,dL_r\right] \\
&= P_t\varphi(x) - \frac{1}{2}\int_0^t\int_{\partial D} p(r,x,y)g(t-r,y)u(t-r,y)\,\rho(y)\sigma(dy)dr.
\end{aligned}
$$

Hence u satisfies the integral equation. It remains to prove uniqueness. Suppose $\tilde{u} \in C([0,T]\times\overline{D})$ also satisfies the integral equation. Then $w = u - \tilde{u} \in C([0,T]\times\overline{D})$ solves

$$
w(t,x) = -\frac{1}{2}\int_0^t\int_{\partial D} p(t-r,x,y)g(r,y)w(r,y)\,\rho(y)\sigma(dy)\,dr. \tag{2.1.16}
$$

By a Gronwall type argument and (2.1.5), we can show that $w = 0$. More precisely, let $\psi(s) = \sup_{x\in\overline{D}}|w(s,x)|$. Then by (2.1.5) we have

$$
\begin{aligned}
0 \le \psi(t) &\le \int_0^t \frac{C\,\psi(r)}{\sqrt{t-r}}\,dr \quad \forall\, t\in[0,T] \\
&= \frac{\partial}{\partial t}\int_0^t 2C\,\psi(r)\sqrt{t-r}\,dr.
\end{aligned}
$$

Integrating both sides with respect to t on an interval $[0,t_0]$, we have

$$
0 \le \int_0^{t_0}\psi(t)\,dt \le \int_0^{t_0} 2C\,\psi(r)\sqrt{t_0-r}\,dr.
$$

From this we have $\psi = 0$ on $[0,t_0]$, where $t_0 > 0$ is small enough so that $2C\sqrt{t_0} < 1$. Let

$\tilde{\psi}(t) = \psi(t+t_0)$, we can show that

$$0 \le \tilde{\psi}(t) \le \int_0^t \frac{C\,\tilde{\psi}(r)}{\sqrt{t-r}}\,dr \quad \forall\, t \in [0, T-t_0]$$

and repeat the above argument to obtain $\tilde{\psi} = 0$ on $[0, t_0]$ (i.e. $\psi = 0$ on $[0, 2t_0]$). Inductively, we obtain $\psi = 0$ on $[0, T]$. □

Definition 2.1.9. The function u defined by the probabilistic representation (2.1.13) (or equivalently in Proposition 2.1.8) is called a **probabilistic solution** of (2.1.12).

It can actually be shown that u is weakly differentiable and solve (2.1.12) in the distributional sense (see [21, Section 3]. However, our method only requires continuity of the solutions.

Probabilistic solution to a backward heat equation

Fix $t > 0$ and consider the following backward heat equation:

$$\begin{cases} \dfrac{-\partial v}{\partial s} = \mathcal{A}v - kv - h & \text{on } (0,t)\times D \\ \dfrac{\partial v}{\partial \vec{\nu}} = 0 & \text{on } (0,t)\times \partial D \\ v(t) = \varphi & \text{on } D, \end{cases} \tag{2.1.17}$$

where $\vec{\nu} := \mathbf{a}\vec{n}$ is the inward conormal direction, $\vec{n}$ is the unit inward normal, $\varphi \in L^2(D)$ is the terminal condition, $k(s,x) \in \mathcal{B}_b([0,\infty)\times D)$ is the killing potential and $h(s,x) \in \mathcal{B}_b([0,\infty)\times D)$ is an external perturbation.

By Itó's rule and the Skorokhod representation for the $(\mathbf{a}, \rho)$-reflected diffusion X on D (c.f. [15]), we see that a classical solution of (2.1.17), should it exists, has the probabilistic representation

$$v(s,x) := \mathbb{E}^x\left[\varphi(X_{t-s})e^{-\int_0^{t-s} k(s+r,X_r)dr} - \int_0^{t-s} h(s+\theta, X_\theta)e^{-\int_0^\theta k(s+r,X_r)dr}\,d\theta\right], \tag{2.1.18}$$

where L_t is the boundary local time of X.

By the same proof of Proposition 2.1.8, we have

Proposition 2.1.10. *Suppose $t > 0$ and $\varphi \in C(\overline{D})$. Then v defined in (2.1.18) is the unique element in $C([0,t] \times \overline{D})$ satisfying the integral equation*

$$v(s,x) = P_{t-s}\varphi(x) - \frac{1}{2}\int_0^{t-s} P_\theta \left(k(s+\theta)v(s+\theta) + h(s+\theta) \right)(x)\, d\theta. \qquad (2.1.19)$$

As in Definition 2.1.9, we introduce the following definition.

Definition 2.1.11. *The function v defined by (2.1.18) (or equivalently in Proposition 2.1.10) is called a* ***probabilistic solution*** *of (2.1.17).*

It can be shown that v is weakly differentiable and solve (2.1.17) in the distributional sense (see [21, Section 3]).

2.2 Random walk approximations

In this section, we will approximate D by a square lattice D^ε of edge length ε. We will construct a random walk X^ε on D^ε which approximates X, a RBM with gradient drift. Finally, we will prove that the transition kernel of X^ε enjoys two-sided Gaussian bound and is jointly Hölder continuous uniform in ε, and that p^ε converges to the transition kernel of X (the local CLT for RBMs with gradient drift in Theorem 2.2.8). We prove these properties of p^ε by establishing the 'discrete relative isoperimetric inequality' in Theorem 2.2.12.

Throughout this section, $\rho \in W^{1,2}(D) \cap C^1(\overline{D})$ is a given a strictly positive function and $X = (X_t)_{t \geq 0}$ denotes a RBM with drift $\frac{1}{2}\nabla(\log \rho)$ (recall Definition 2.1.1). We remark here that all properties of p^ε that are established in this section remains valid for general symmetric reflected diffusions. The proofs are the same, requiring only a more delicate construction for X^ε.

We first construct D^ε. Without loss of generality, we assume that the origin $0 \in D$. Let $\overline{\varepsilon \mathbb{Z}^d}$ be the union of all closed line segments joining nearest neighbors in $\varepsilon\mathbb{Z}^d$, and $(D^\varepsilon)^*$ the connected component of $D \cap \overline{\varepsilon\mathbb{Z}^d}$ that contains the point 0. Set $D^\varepsilon = (D^\varepsilon)^* \cap \varepsilon\mathbb{Z}^d$. We can view D^ε as the vertices of a graph whose edges coming from $(D^\varepsilon)^*$. We also denote the graph-boundary $\partial D^\varepsilon := \{x \in D^\varepsilon : v_\varepsilon(x) < 2d\}$, where $v_\varepsilon(x)$ is the degree of x in D^ε.

Next, we define X^ε to be a continuous time random walk (CTRW) on D^ε with exponential waiting time of parameter d/ε^2 and one step transition probabilities

$$p_{xy} := \frac{\mu_{xy}}{\sum_y \mu_{xy}},$$

where $\{\mu_{xy} : x, y \in D^\varepsilon\}$ are symmetric weights (conductances) to be constructed in two steps as follows: First, for every $x \in D^\varepsilon \setminus \partial D^\varepsilon$ and $i = 1, 2, \cdots, d$, define

$$\begin{aligned} \mu_{x,x+\varepsilon\vec{e_i}} &:= \left(1 + \frac{1}{2}\ln\frac{\rho(x+\varepsilon\vec{e_i})}{\rho(x)}\right)\left(\frac{\rho(x)+\rho(x+\varepsilon\vec{e_i})}{2}\right)\frac{\varepsilon^{d-2}}{2} \\ \mu_{x,x-\varepsilon\vec{e_i}} &:= \left(1 + \frac{1}{2}\ln\frac{\rho(x)}{\rho(x-\varepsilon\vec{e_i})}\right)\left(\frac{\rho(x)+\rho(x-\varepsilon\vec{e_i})}{2}\right)\frac{\varepsilon^{d-2}}{2}. \end{aligned}$$

Clearly, $\mu_{xy} = \mu_{yx}$ for all $x, y \in D^\varepsilon \setminus \partial D^\varepsilon$. Note that since ρ is in $C^1(\overline{D})$ and is strictly positive, when ε is sufficiently small, $\mu_{x,x+\varepsilon\vec{e_i}}$ and $\mu_{x,x-\varepsilon\vec{e_i}}$ are strictly positive for every $x \in D^\varepsilon \setminus \partial D^\varepsilon$ and $i = 1, 2, \cdots, d$. Second, we define

$$\mu_{xy} := \begin{cases} \mu_{yx}, & \text{if } x \in \partial D^\varepsilon,\ y \in D^\varepsilon \setminus \partial D^\varepsilon \\ \varepsilon^{d-2}/2, & \text{if } x, y \in \partial D^\varepsilon \text{ are adjacent in } D^\varepsilon. \end{cases}$$

Now $\mu_{xy} = \mu_{yx}$ for all $x, y \in D^\varepsilon$. We call X^ε the ε-**approximation** of X.

Remark 2.2.1. *A special but important case is when $\rho \equiv 1$. In this case, X is simply the reflected Brownian motion (RBM) on D, and X^ε is a simple random walk on the graph D^ε. It is proved in [9] that X^ε converges weakly to the RBM X as $\varepsilon \to 0$. We generalize this result to RBM with gradient drift in Theorem 2.2.20.*

Remark 2.2.2. *Note that the formula for $\mu_{x,x\pm\epsilon\vec{e_i}}$ is strictly positive for ϵ small enough (hence the conductances are well-defined). More precisely, note that $\rho = e^{2h}$ for some $h \in C^1(\overline{D})$, so* *the drift of X is ∇h with $\|\nabla h\| := \max_{1\le i\le d}\sup_{x\in\overline{D}}\left|\frac{\partial h}{\partial x_i}\right| < \infty$. By Mean value theorem on h, the first bracket in the formula for $\mu_{x,x+\epsilon\vec{e_i}}$ is*

$$1+\frac{1}{2}\ln\frac{\rho(x+\epsilon\vec{e_i})}{\rho(x)} = 1+h(x+\epsilon\vec{e_i})-h(x) = 1+\epsilon\frac{\partial h(\xi)}{\partial x_i}$$

which is strictly positive whenever $\epsilon < 1/\|\nabla h\|$ (we adopt the convention that $1/0=\infty$).

Remark 2.2.3. ***The heuristic reason of the above construction*** *for μ_{xy} is as follows: We need $\mathbb{E}_x[X_t] = \lim_{\epsilon\to 0}\mathbb{E}_x[X_t^\epsilon]$. When t is small, $\mathbb{E}_x[X_t] \approx \nabla h(X_t)\,t$ and $\mathbb{E}_x[X_t^\epsilon] \approx \frac{d}{\epsilon^2}\,\mathbb{E}_x[1 \text{ step}]\,t$. So we need*

$$\frac{\partial h(x)}{\partial x_i} \approx \frac{d}{\epsilon}\,\frac{\mu_{x,x+\epsilon\vec{e_i}}-\mu_{x,x-\epsilon\vec{e_i}}}{\mu(x)} \quad \text{for } 1\le i\le d.$$

We also need $\lim_{\epsilon\to 0}\mathcal{E}^{(\epsilon)}(f,f) = \mathcal{E}(f,f)$ for any $f\in C^1(\overline{D})$, where $\mathcal{E}^{(\epsilon)}$ and $\mathcal{E}$ are the Dirichlet forms of X^ε and X respectively. This is true if

$$\frac{\mu_{x,x+\epsilon\vec{e_i}}+\mu_{x,x-\epsilon\vec{e_i}}}{\epsilon^{d-2}} \approx \rho(x).$$

Assume further that $\mu(x) = \frac{d}{\epsilon^2}\,\epsilon^d = d\epsilon^{d-2}$ (since the holding time is $\frac{d}{\epsilon^2}$, we are assuming the symmetrizing measure is ϵ^d at $x\in D^\epsilon\setminus\partial D^\epsilon$). Solving (2.2.3) and (2.2.3), we have

$$\mu_{x,x+\epsilon\vec{e_i}} \approx \frac{\epsilon^{d-2}}{2}\left(\rho(x)+\epsilon\frac{\partial h(x)}{\partial x_i}\right).$$

Finally, the requirement that $\mu_{xy}=\mu_{yx}$ motivates our final definition for $\mu_{x,x+\epsilon\vec{e_i}}$. Here we took the advantage that the drift is of gradient form ∇h.

Here are some basic properties of X^ε. Clearly, X^ε is symmetric with respect to the measure m_ε defined by

$$m_\varepsilon(x) := \frac{\varepsilon^2}{d}\sum_y \mu_{xy}.$$

The stationary measure $\pi = \pi^{D^\varepsilon}$ of X^ε is given by $\pi(x) = m_\varepsilon(x)/m(D^\varepsilon)$, where $m(D^\varepsilon) :=$

$\sum_{x\in D^\varepsilon} m_\varepsilon(x)$. The Dirichlet form of X^ε in $l^2(m_\epsilon)$ is given by $(\mathcal{E}^{(\epsilon)}, l^2(m_\epsilon))$, where

$$\mathcal{E}^{(\epsilon)}(f,g) := \frac{1}{2}\sum_{x,y\in D^\epsilon}(f(y)-f(x))(g(y)-g(x))\,\mu_{xy}. \tag{2.2.1}$$

Since $\rho \in C^1(\overline{D})$, there exists a constant $C > 0$ such that

$$C^{-1} \leq \inf_x \frac{m_\varepsilon(x)}{\varepsilon^d} \leq \sup_x \frac{m_\varepsilon(x)}{\varepsilon^d} \leq C. \tag{2.2.2}$$

Moreover, $\lim_{\varepsilon\to 0} \frac{m_\varepsilon(x^\varepsilon)}{\varepsilon^d} = \rho(x)$ whenever $x^\varepsilon \in D^\varepsilon$ converges to $x \in D$.

The transition density p^ε of X^ε with respect to the measure m_ε is

$$p^\varepsilon(t,x,y) := \frac{\mathbb{P}^x(X_t^\varepsilon = y)}{m_\varepsilon(y)}, \quad t > 0,\ x,\ y \in D^\varepsilon. \tag{2.2.3}$$

Clearly, p^ε is strictly positive and is symmetric in x and y. We will explore the properties of p^ε in the next few subsections.

2.2.1 Discrete heat kernel p^ε and local central limit theorem

We will prove in the next few sections that the transition density p^ε enjoys two-sided Gaussian bound and is jointly Hölder continuous uniform in $\varepsilon \in (0, \varepsilon_0)$ for some $\varepsilon_0 > 0$, and that p^ε converges to p uniformly on compact subsets of $(0,\infty)\times\overline{D}\times\overline{D}$. In rigorous terms, we have the following four results. The important point is that the constants involved in these four results are uniform for ϵ small enough.

Theorem 2.2.4. *(Gaussian upper bound) There exist $C_k = C_k(d, D, \rho, T) > 0$, $k = 1, 2$, and $\varepsilon_0 = \varepsilon_0(d, D, \rho) \in (0, 1]$ such that for every $\varepsilon \in (0, \varepsilon_0)$ and $x, y \in D^\varepsilon$,*

$$p^\varepsilon(t,x,y) \leq \frac{C_1}{(\varepsilon \vee t^{1/2})^d} \exp\left(-C_2\frac{|x-y|^2}{t}\right) \quad \text{for } t \in [\varepsilon, T] \tag{2.2.4}$$

and

$$p^\varepsilon(t,x,y) \le \frac{C_1}{(\varepsilon \vee t^{1/2})^d} \exp\left(-C_2 \frac{|x-y|}{t^{1/2}}\right) \quad \textit{for } t \in (0,\varepsilon). \tag{2.2.5}$$

Observe that (2.2.4) implies that (2.2.5) also holds for $t \in [\varepsilon, T]$. As an application of the upper bound, we have an estimate for the exit time for a ball by a standard argument (see [2]) using the strong Markov property.

Corollary 2.2.5. *(Exit time estimate) For any $T > 0$, there exists $C = C(d, D, \rho, T)$ and $\epsilon_0 = \epsilon_0(d, D, \rho)$ such that*

$$\mathbb{P}^x\Big(\sup_{s \le t} |X^\epsilon_s - x| \ge \eta\Big) \le C \exp\left(t - \frac{\eta}{4\,(t^{\frac{1}{2}} \vee \epsilon)}\right) \tag{2.2.6}$$

for all $t \in (0,T]$, $x \in D^\epsilon$, $\eta > 0$ and $\epsilon \in (0, \epsilon_0)$, where $|y-x|$ is the Euclidean distance between x and y in $\mathbb{R}^d$.

Theorem 2.2.6. *(Gaussian lower bound) There exist $C_k = C_k(d, D, \rho, T) > 0$, $k = 1, 2$, and $\varepsilon_0 = \varepsilon_0(d, D, \rho) \in (0, 1]$ such that for every $\varepsilon \in (0, \varepsilon_0)$ and $x, y \in D^\varepsilon$,*

$$p^\varepsilon(t,x,y) \ge \frac{C_1}{(\varepsilon \vee t^{1/2})^d} \exp\left(-C_2 \frac{|x-y|^2}{t}\right) \quad \textit{for } t \in (0,T]. \tag{2.2.7}$$

Theorem 2.2.7. *(Hölder continuity) There exist positive constants $\gamma = \gamma(d, D, \rho)$, $\varepsilon_0(d, D, \rho)$ and $C(d, D, \rho)$ such that for all $\varepsilon \in (0, \varepsilon_0)$, we have*

$$|p^\varepsilon(t,x,y) - p^\varepsilon(t',x',y')| \le C\, \frac{(|t-t'|^{1/2} + |x-x'| + |y-y'|)^\gamma}{(t \wedge t')^{\sigma/2}\,(1 \wedge t \wedge t')^{d/2}}. \tag{2.2.8}$$

Theorem 2.2.8. *(Local CLT)*

$$\lim_{n\to\infty} \sup_{t\in[a,b]} \sup_{x,y\in\overline{D}} \Big| p^{(2^{-n})}(t,x,y) \;-\; p(t,x,y) \Big| = 0$$

for any compact interval $[a,b] \subset (0,\infty)$.

We now prove these four properties of p^ε in the next few subsections. The proofs are standard once we establish a discrete analogue of a relative isoperimetric inequality (Theorem 2.2.12) for bounded Lipschitz domains. The last subsection is about applications of these properties.

2.2.2 Discrete relative isoperimetric inequality

Note that any Lipschitz domain enjoys the uniform cone property and any bounded $\mathcal{H}^{d-1}$-rectifiable set has finite perimeter. Hence, by Corollary 3.2.3 (p.165) and Theorem 6.1.3 (p.300) of [61], we have the following relative isoperimetric inequality.

Proposition 2.2.9 (Relative isoperimetric inequality)**.** *Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $r \in (0,1]$. Then*

$$S(r,D) := \sup_{U \in \mathfrak{G}} \frac{|U|^{\frac{d-1}{d}}}{\sigma(\partial U \cap D)} < \infty, \tag{2.2.9}$$

where $\mathfrak{G}$ is the collection of open subsets $U \subset D$ such that $|U| \leq r|D|$ and $\partial U \cap D$ is $\mathcal{H}^{d-1}$-rectifiable set. Moreover, $S(r,D) = S(r,aD)$ for all $a > 0$.

In this subsection, we establish a discrete analogue for the relative isoperimetric inequality (Theorem 2.2.12).

Random walk on scaled graph aD^ε

We first consider the scaled graph $aD^\varepsilon = (aD)^{a\varepsilon}$, which is an approximation to the bounded Lipschitz domain aD by square lattice $a\varepsilon\mathbb{Z}^d$. Clearly the degrees of vertices are given by $v^{aD^\varepsilon}(ax) = v^{D^\varepsilon}(x)$. Define the function $\rho_{(aD)}$ on aD by $\rho_{(aD)}(ax) := \rho(x)$. Then define the CTRW X^{aD^ε} using $\rho_{(aD)}$ as we have done for X^ε using ρ.

The mean holding time of X^{aD^ε} is $(a\varepsilon)^2/d$. Clearly, the symmetrizing measure m^{aD^ε} and the stationary probability measure π^{aD^ε} have the scaling property $m^{aD^\varepsilon}(ax) = a^d\, m^{D^\varepsilon}(x)$ and $\pi^{aD^\varepsilon}(ax) = \pi^{D^\varepsilon}(x)$. Let $p^{a\varepsilon}_{aD}$ be the transition density of X^{aD^ε} with respect to the symmetrizing measure m^{aD^ε}. Then

$$a^d\, p^{a\varepsilon}_{aD}(a^2 t, ax, ay) = p^\varepsilon_D(t,x,y) \tag{2.2.10}$$

for every $t > 0$, $\varepsilon > 0$, $a > 0$ and $x, y \in D^\varepsilon$.

We will simply write m and π for the symmetrizing measure and the stationary probability measure when there is no ambiguity for the underlying graph.

An extension lemma

Following the notation of [67], we let G be a finite set, $K(x,y)$ be a Markov kernel on G and π the stationary measure of K. Note that a Markov chain on a finite set induces a natural graph structure as follows. Let $Q(e) := \frac{1}{2}(K(x,y)\pi(x) + K(y,x)\pi(y))$ for any $e = (x,y) \in G \times G$. Define the set of directed edges $E := \{e = (x,y) \in G \times G : \, Q(e) > 0\}$.

We use the following 2 different notions for the "boundary" of $A \subset G$:

$$\begin{aligned} \partial_e A &:= \{e = (x,y) \in E : \, x \in A, y \in G \setminus A \text{ or } y \in A, x \in G \setminus A\}, \\ \partial A &:= \{x \in A : \, \exists y \in G \setminus A \text{ such that } (x,y) \in E\}. \end{aligned}$$

Observe that each edge in $\partial_e A$ is counted twice. Set

$$Q(\partial_e A) := \frac{1}{2} \sum_{e \in \partial_e A} q(e) = \frac{1}{2} \sum_{x \in A,\, y \in G \setminus A} (K(x,y)\pi(x) + K(y,x)\pi(y)).$$

Definition 2.2.10. *For any $r \in (0,1)$, define*

$$S_\pi(r,G) := \sup_{\{A \subset G : \pi(A) \le r\}} \frac{2|A|^{(d-1)/d}}{|\partial_e A|} \quad \text{and} \quad \tilde{S}_\pi(r,G) := \sup_{\{A \subset G : \pi(A) \le r\}} \frac{\pi(A)^{(d-1)/d}}{Q(\partial_e A)}. \qquad (2.2.11)$$

We call $1/\tilde{S}_\pi(r,G)$ an ***isoperimetric constant*** *of the chain (K,π). It provides rich information about the geometric properties of G and the behavior of the chain (cf. [67]).*

In our case, $G = aD^\varepsilon$, $\pi(x) = \frac{m(x)}{m(aD^\varepsilon)}$ and $K(x,y) = p_{xy}$ in aD^ε, where $p_{x,y}$ is the one-step transition probabilities of X^{aD^ε} defined at the beginning of Section 2.2. For $a = 1$ and $A \subset D^\varepsilon$, we have

$$
\begin{aligned}
\partial_e A &= \{(x,y) \in A \times D^\varepsilon \setminus A \cup D^\varepsilon \setminus A \times A : \text{ the line segment } (x,y] \subset D\}, \\
\partial A &= \{x \in A : \exists y \in D^\varepsilon \setminus A \text{ such that } |x-y| = \varepsilon \text{ and the line segment } [x,y] \subset D\}, \\
\tilde{\partial} A &:= \{x \in A : \exists y \in \varepsilon\mathbb{Z}^d \text{ such that } |x-y| = \varepsilon \text{ and } (x,y] \cap \partial D \neq \emptyset\}, \\
\Delta A &:= \tilde{\partial} A \setminus \partial A.
\end{aligned}
$$

In the above notation, we have $\partial D^\varepsilon = \emptyset$, $\tilde{\partial} D^\varepsilon = \{x \in D^\varepsilon : v(x) < 2d\}$, $A \cap \tilde{\partial} D^\varepsilon = \tilde{\partial} A$ and $\tilde{\partial} D^\varepsilon = \Delta A \cup (\partial A \cap \tilde{\partial} D^\varepsilon) \cup (\tilde{\partial} D^\varepsilon \setminus A)$. See Figure 2.1 for an illustration.

We say that $A \subset D^\varepsilon$ is **grid-connected** if $\partial_e A_1 \cap \partial_e A_2 \neq \phi$ whenever $A = A_1 \cup A_2$. It is easy to check that A is grid-connected if and only if for every $x,\, y \in A$, there exists $\{x_1 = x,\, x_2,\, \cdots,\, x_{m-1},\, x_m = y\} \subset A$ such that each line segments $[x_j,\, x_{j+1}] \subset D$ and $|x_j - x_{j+1}| = \varepsilon$.

The following is a key lemma which allows us to derive the relative isoperimetric inequality for the discrete setting from that in the continuous setting, and hence leads us to Theorem 2.2.12.

Lemma 2.2.11. *(Extension of sub-domains) Let π_{srw} be the stationary measure of the simple random walk (SRW) on D^ε. For any $r \in (0,1)$, there exist positive constants $\varepsilon_1(d,D,r)$, $M_1(d,D,r)$ and $M_2(d,D,r)$ such that if $\varepsilon \in (0,\varepsilon_1)$, then for all grid-connected $A \subset D^\varepsilon$ with $\pi_{srw}(A) \le r$, we can find a connected open subset $U \subset D$ which contains A and satisfies:*

(a) $\partial U \cap D$ is $\mathcal{H}^{d-1}$-rectifiable,

(b) $|U| \le \frac{49r+1}{50}|D|$,

(c) $\varepsilon^d\,|A| \le M_1\,|U|$,

(d) $M_2\,\varepsilon^{d-1}|\partial A| \ge \sigma(\partial U \cap D)$.

Proof Since the proof for each $r \in (0,1)$ is the same, we just give a proof for the case $r = 1/2$.

For $x \in \varepsilon\mathbb{Z}^d$, let $U_x := \prod_{i=1}^d (x_i - \frac{\varepsilon}{2},\, x_i + \frac{\varepsilon}{2})$ be the cube in the dual lattice which contains x. Since A is grid-connected, we have $(W_1)^o$ is connected in $\mathbb{R}^d$, where $(W_1)^o$ is the interior of $W_1 := \cup_{x\in A}(\overline{U_x} \cap D)$. (See Figure 2.1 for an illustration.)

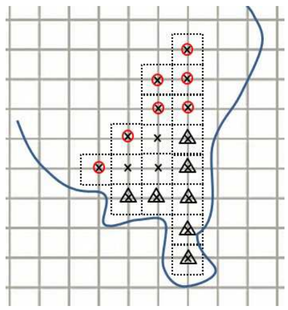


Figure 2.1: $W_1 := \cup_{x\in A}(\overline{U_x} \cap D)$

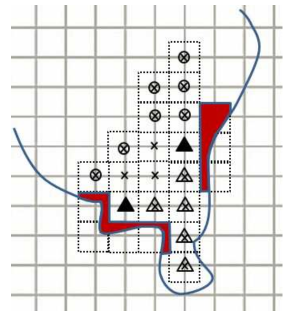


Figure 2.2: W_3 is the shaded part

Note that we cannot simply take $U = (W_1)^o$ because (d) may fail, for example when ΔA contributes too much to $\partial U \cap D$, i.e., when $(W_1)_\Delta := \partial W_1 \cap \left(\bigcup_{x\in\Delta A} \partial U_x\right)$ is large. However, $\Delta A \subset \tilde{\partial} D^\varepsilon$ is close to ∂D and so we can fill in the gaps between ΔA and ∂D to eliminate those contributions. In this process, we may create some extra pieces for $\partial U \cap D$, but we will show that those pieces are small enough. Following this observation, we will eventually take $U = (W_1 \cup W_2)^o$ where $W_2 \subset D_h$ for some small enough $h > 0$.

Since D is a bounded Lipschitz domains, we can choose $h > 0$ small enough so that $|D_h| < |D|/200$. Moreover, $\pi_{srw}(A) < 1/2$ implies $\varepsilon^d|A| \le \varepsilon^d|\partial(D^\varepsilon)| + m_{srw}(D^\varepsilon)/2$. So we can choose ε small enough so that $|W_1| \le \varepsilon^d|A| \le \frac{101}{200}|D|$. Hence U satisfies (b). By Lipschitz property again, there exists $M_1 > 0$ such that $|U_x \cap D| \ge |U_x|/M_1 = \varepsilon^d/M_1$ for any $x \in D^\varepsilon$. Hence (c) is satisfied.

It remains to construct W_2 in such a way that $W_2 \subset D_h$ for some small enough $h > 0$ (more precisely, for h small enough so that $|D_h| < |D|/200$) and that (a) and (d) are satisfied. We will construct W_2 in 3 steps:

Step 1: (Construct W_3 to seal the opening between ∂D and the subset of $(W_1)_\partial$ which are close to ∂D. See Figure 2.2.) Write $\Delta A = \Delta_1 A \cup \Delta_2 A$ where $\Delta_2 A := \Delta A \setminus \Delta_1 A$ and

$$\Delta_1 A := \{x \in \Delta A : \exists y \in \partial A \text{ such that } \max\{|x_i - y_i| : 1 \le i \le d\} = 1\}.$$

Points in $\Delta_1 A$ are marked in solid black in Figure 2.2. For $x \in \Delta_1 A$, consider the following closed cube centered at x:

$$T_x := \bigcup_{y \in \widehat{B}(x,\,10R\varepsilon)} \overline{U_y} \quad , \text{ where } R = \sqrt{d}(M+1)$$

Let Θ_x be the union of all connected components of $T_x \cap D$ whose closure intersects $\overline{U_x}$ and define

$$W_3 := \bigcup_{x \in \Delta_1 A} \Theta_x.$$

Step 2: (Fill in the gaps between ∂D and $(W_1)_\Delta$ near Δ_2. See Figure 2.3) Note that $\cup_{x\in\Delta_1 A}\partial U_x$ does not contribute to $\partial(W_1 \cup W_3) \cap D$. Let W_4 be the union of all connected components of $D \setminus (W_1 \cup W_3)$ whose closure intersects $\overline{U_x}$ for some $x \in \Delta_2 A$.

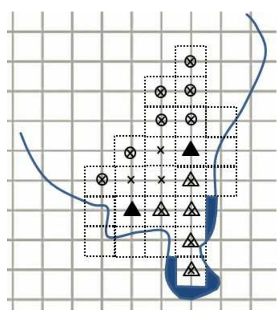


Figure 2.3: W_4 is the shaded part

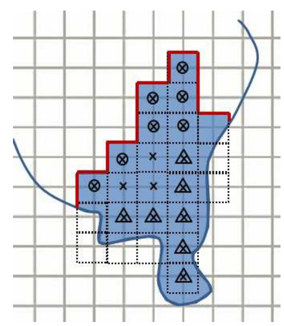


Figure 2.4: U is the shaded part

Step 3: Finally, take $W_2 := W_3 \cup W_4$, and set $U := (W_1 \cup W_3 \cup W_4)^o$. (See Figure 2.4.)

It is clear that U is connected and $\partial U \cap D \subset \bigcup_{x\in\varepsilon\mathbb{Z}^d} \partial U_x$ is piecewise linear, so (a) is satisfied. For any $W \subset D$, we have $\partial W \cap D = W_\partial \cup W_\Delta \cup W_\nabla$, where

$$W_\partial := \partial W \cap \left(\bigcup_{x\in\partial A} \partial U_x \right), \quad W_\Delta := \partial W \cap \left(\bigcup_{x\in\Delta A} \partial U_x \right) \text{ and } W_\nabla := \partial W \setminus \left(\bigcup_{x\in\partial A\cup\Delta A} \partial U_x \right).$$

Therefore, $\sigma(\partial W \cap D) \le \sigma(W_\partial) + \sigma(W_\Delta) + \sigma(W_\nabla)$ whenever the corresponding surface measures are defined. It is clear that by construction we have

- $(W_1)_\nabla = \emptyset$,

- $(W_1 \cup W_3)_\partial \subset (W_1)_\partial$, $(W_1 \cup W_3)_\Delta \subset (W_1)_{\Delta_2}$, $(W_1 \cup W_3)_\nabla \subset \bigcup_{x\in\Delta_1} \bigcup_{y \in \widehat{B}(x,10R\varepsilon)} \partial U_y$ where $(W_1)_{\Delta_2}$ is defined analogously as $(W_1)_\Delta$, with Δ replaced by Δ_2,

- $U_\partial \subset (W_1 \cup W_3)_\partial$, $U_\Delta = \emptyset$, $U_\nabla \subset (W_1 \cup W_3)_\nabla$.

Now $\sigma(U_\partial) \le \sigma((W_1)_\partial) \le |\partial A|\, 2d\varepsilon^{d-1}$. Moreover, each $x \in \partial A$ is adjacent to at most $3^d - 1$ points in $\Delta_1 A \cup \Delta A$, and for each $x \in \Delta_1 A$, there are at most $|\widehat{B}(10R\varepsilon)| \le (20R+1)^d$ cubes in T_x. So we have

$$\sigma(U_\nabla) \le \sigma((W_1 \cup W_3)_\nabla) \le (3^d - 1)|\partial A|\,(20R+1)^d\, 2d\varepsilon^{d-1}.$$

Hence (d) is satisfied.

Since $diam\,(T_x) < 20R\sqrt{d}\varepsilon$, we have $W_3 \subset D_{(20R\sqrt{d}+1)\varepsilon}$. To complete the proof, it suffices to show that $W_4 \subset D_{(10R)\varepsilon}$. This is equivalent to show that any curve in $D \setminus \overline{W_1 \cup W_3}$ starting from any point in $(W_1)_{\Delta_2}$ must lie in $D_{(10R)\varepsilon}$.

Let $\gamma[0,1]$ be an arbitrary continuous curve starting at an arbitrary point $p \in (W_1)_{\Delta_2}$ such that $\gamma(0,1) \subset D \setminus \overline{W_1}$ and $dist(\gamma(t), \partial D) > (10R)\varepsilon$ for some $t \in (0,1]$. Define $\Omega_{D^\varepsilon} := \left(\bigcup_{x\in D^\varepsilon} \overline{U_x}\right)^o \cap D$. Since $(W_1)_\Delta \subset \partial(\Omega_{D^\varepsilon}) \cap D \subset \bigcup_{z\in\tilde{\partial}D^\varepsilon} \partial U_z$ and $\sup_{z\in\tilde{\partial}D^\varepsilon} dist(z, \partial D) < \varepsilon$, the time when γ first exits $D \setminus \overline{\Omega_{D^\varepsilon}}$ must be less than t by continuity of γ. That is,

$$\tau := \inf \left\{ s > 0 :\, \gamma(s) \in \left(\bigcup_{z\in\partial D^\varepsilon \setminus A} \partial U_z \right) \cap D \right\} < t.$$

It suffices to show that $\gamma(0,\tau] \cap \Theta_x \neq \emptyset$ for some $x \in \Delta_1 A$. We do so by constructing a continuous curve $\tilde{\gamma}$ which is close to γ and passes through ∂U_x for some $x \in \Delta_1 A$.

Since $\sup_{s\in[0,\tau]} dist(\gamma(s), \partial D) < 2R\varepsilon$, we can choose ε small enough (depending only on D) and split $[0,\tau]$ into finitely many disjoint intervals I's so that the $4R\varepsilon$-tube of each $\gamma(I)$ lies in a coordinate ball $B_{(I)}$ of D. For $s \in I$, project $\gamma(s)$ vertically upward (along the d-th coordinate of $B_{(I)}$) onto $\partial(\Omega_{D^\varepsilon}) \cap D$ to obtain $\widehat{\gamma}(s)$. Note that $\widehat{\gamma}$ maybe discontinuous even in the interior of I. However, it is continuous on $[0,\tau]$ except possibly for finitely many points.

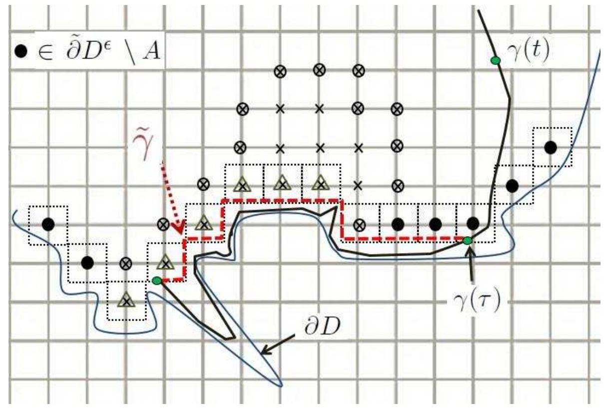


Figure 2.5: γ and a corresponding continuous $\tilde{\gamma} \subset \partial(\Omega_{D^\varepsilon}) \cap D$

Let $\{0 \le s_1 < s_2 < \cdots < s_m \le \tau\}$ be the collection of discontinuities for $\widehat{\gamma}([0,\tau])$. Then $0 < |\widehat{\gamma}(s_j-) - \widehat{\gamma}(s_j+)| \le 2R\varepsilon$ and we can connect $\widehat{\gamma}(s_j-)$ to $\widehat{\gamma}(s_j+)$ by a continuous curve $\beta_j : [0,1] \longrightarrow \partial(\Omega_{D^\varepsilon}) \cap D \cap B(\widehat{\gamma}(s_j-), 8R\varepsilon) \cap B(\widehat{\gamma}(s_j+), 8R\varepsilon)$.

Define $\tilde{\gamma} : [0, \tau+m] \longrightarrow \partial(\Omega_{D^\varepsilon}) \cap D$ to be the continuous curve obtained by concatenating $\widehat{\gamma}$ and $\{\beta_j : j = 1, 2, \cdots, m\}$ (See Figure 2.5). Then $\tilde{\gamma}(0) = \widehat{\gamma}(0) = p \in (W_1)_{\Delta_2}$ and $\tilde{\gamma}(\tau+m) = \widehat{\gamma}(\tau) = \gamma(\tau) \in \partial(\Omega_{D^\varepsilon}) \cap D \setminus (\partial W_1 \cap D)$. By the continuity of $\tilde{\gamma}$, there is some $t_* \in (0, \tau+m)$ such that $\tilde{\gamma}(t_*) \in (W_1)_{\Delta_1}$. (Roughly speaking, on $\tilde{\partial} D^\varepsilon$, $\Delta_2 A$ is separated from $\tilde{\partial} D^\varepsilon \setminus A$ by $\Delta_1 A$.)

Now for some $1 \le j \le m$, we have $\tilde{\gamma}(t_*)$ and $\gamma(s_j)$ are connected in $D \setminus \Omega_{D^\varepsilon} \subset D \setminus \overline{W_1}$, and

$$|\tilde{\gamma}(t_*) - \gamma(s_j)| \le |\tilde{\gamma}(t_*) - \widehat{\gamma}(s_j)| + |\widehat{\gamma}(s_j) - \gamma(s_j)| \le 8R\varepsilon + R\varepsilon.$$

Hence $\tilde{\gamma}(t_*) \in \partial U_x$ for some $x \in \Delta_1 A$. We therefore have $\gamma(s_j) \in \Theta_x$. The proof is now complete.

□

Discrete relative isoperimetric inequality

Let π_{srw} be the stationary measure of the simple random walk (SRW) on the graph under consideration and recall Definition 2.2.10.

Theorem 2.2.12 (Discrete relative isoperimetric inequality)**.** *For every $r \in (0,1)$, there exists*

$\widehat{S}_{srw} = \widehat{S}_{srw}(d, D, r) \in (0, \infty)$ *and* $\varepsilon_1 = \varepsilon_1(d, D, r) \in (0, \infty)$ *such that*

$$\sup_{\varepsilon \in (0,\varepsilon_1)} S_{srw}(r, D^{\varepsilon}) \leq \widehat{S}_{srw}, \text{ and}$$
$$\tilde{S}_{srw}(r, D^{\varepsilon}) \leq \frac{2d\,(m_{srw}(D^{\varepsilon}))^{1/d}}{\varepsilon}\widehat{S}_{srw} \quad \text{for every } \varepsilon \in (0, \varepsilon_1).$$

Proof We can also assume that A is grid-connected. This is because

$$\frac{|A|^{(d-1)/d}}{|\partial_e A|} \leq \frac{|A_1|^{(d-1)/d}}{|\partial_e A_1|} \vee \frac{|A_2|^{(d-1)/d}}{|\partial_e A_2|}$$

whenever $A = A_1 \cup A_2$ with $\partial_e A_1 \cap \partial_e A_2 = \phi$. From Lemma 2.2.11 and Proposition 2.2.9, we have

$$\sup_{\varepsilon \in (0,\varepsilon_1)} \sup_{\{A \subset D^{\varepsilon}: \pi(A) \leq r\}} \frac{|A|^{(d-1)/d}}{|\partial A|} \leq M_2\, M_1^{(d-1)/d}\, S\left(\frac{49r+1}{50},\, D\right).$$

We thus have the first inequality since $4d|\partial A| \geq |\partial_e A| \geq 2|\partial A|$. The second inequality follows from the first since $q(e) = \frac{(\varepsilon)^d}{2d\, m(D^{\varepsilon})}$. □

For the CTRW $X^{aD^{\varepsilon}}$ on aD^{ε}, we let π be the stationary measure. Observe that, because $\pi^{aD^{\varepsilon}}(aA) = \pi^{D^{\varepsilon}}(A)$ and $m(aD^{\varepsilon}) = a^d\, m(D^{\varepsilon})$, we have

$$S_{\pi}(r,\, aD^{\varepsilon}) = S_{\pi}(r,\, D^{\varepsilon}) \quad \text{and} \quad \tilde{S}_{\pi}(r,\, aD^{\varepsilon}) = \tilde{S}_{\pi}(r,\, D^{\varepsilon}) \tag{2.2.12}$$

for all $a > 0$ and $r > 0$. Hence we only need to consider the case $a = 1$. In view of Theorem 2.2.12 and (2.2.2), we have (taking $r = 1/2$)

Corollary 2.2.13. *There exist positive constants* $\widehat{S} = \widehat{S}(d, D, \rho)$, $\varepsilon_1 = \varepsilon_1(d, D, \rho)$ *and* $\widehat{C} = \widehat{C}(d, D, \rho)$ *such that*

$$\sup_{\varepsilon \in (0,\varepsilon_1)} S_{\pi}(1/2,\, D^{\varepsilon}) \leq \widehat{S}, \text{ and} \tag{2.2.13}$$
$$\tilde{S}_{\pi}(1/2,\, D^{\varepsilon}) \leq \frac{\widehat{C}}{\varepsilon}\widehat{S} \quad \text{for every } \varepsilon \in (0, \varepsilon_1). \tag{2.2.14}$$

As an immediate consequence of Corollary 2.2.13 and (2.2.12), we have the following Poincaré inequality.

Corollary 2.2.14. *(Poincaré inequality) There exist* $\varepsilon_1 = \varepsilon_1(d, D, \rho) > 0$ *such that*

$$\frac{|D|\, a^{d-2}}{16\, \widehat{C}^2\, \widehat{S}^2}\, \|f - \langle f\rangle_\pi\|^2_{l^2(\pi)} \leq \mathcal{E}^{a\varepsilon}_{aD}(f)$$

for all $f \in l^2(aD^\varepsilon, \pi)$, $\varepsilon \in (0, \varepsilon_1)$, $a > 0$. *Here* $\langle f\rangle_\pi := \sum f\,\pi$, $\widehat{C}$ *and* $\widehat{S}$ *are the same constants in Corollary 2.2.13, and* $\mathcal{E}^{a\varepsilon}_{aD}$ *is the Dirichlet form in* $l^2(m^{aD^\varepsilon})$ *of the CTRW* X^{aD^ε} *(see (2.2.1)).*

Proof By Corollary 2.2.13, the isoperimetric constant

$$\mathcal{I} := \inf_{\pi(A)\leq 1/2} \frac{Q(\partial A)}{\pi(A)} \geq 2^{1/d}\, \frac{1}{\tilde{S}} \geq 2^{1/d}\, \frac{\varepsilon}{\widehat{C}\,\widehat{S}}.$$

Hence, by the Cheeger's inequality (see [67, Lemma 3.3.7]),

$$\inf_f \frac{\mathcal{E}^{a\varepsilon}_{aD}(f)}{\|f - \langle f\rangle_\pi\|^2_{l^2(\pi)}} \geq \frac{d\, m(aD^\varepsilon)}{(a\varepsilon)^2}\, \frac{\mathcal{I}^2}{8} \geq \frac{|D|}{16}\, \frac{a^{d-2}}{\widehat{C}^2\, \widehat{S}^2}$$

when $\varepsilon > 0$ is small enough. □

The above Poincaré inequality already tells us a positive lower bound for the spectral gap of X^{aD^ε} and hence gives us an estimate for the mixing time. However, we will state a stronger result in Proposition 2.2.16 in the next subsection.

2.2.3 Nash's inequality and Poincaré inequality

The discrete relative isoperimetric inequality leads to the following two functional inequalities; namely, a Poincaré inequality and a Nash inequality that are uniform in ε and in scaling $D \mapsto aD$. The uniformity in scaling helps proving the near diagonal lower bound for p^ε.

Theorem 2.2.15. *(Nash's inequality and Poincaré inequality uniform in* ε *and in scaling) There* *exist* $\varepsilon_1 = \varepsilon_1(d, D, \rho) > 0$ *and* $C = C(d, D, \rho) > 0$ *such that*

$$\|f - \langle f\rangle_\pi\|_{l^2(\pi)}^{2(1+2/d)} \le 8\,\tilde S_\pi(1/2,\ D^\varepsilon)^2 \left(\frac{(a\,\varepsilon)^2}{d\,m(aD^\varepsilon)}\,\mathcal{E}_{aD}^{a\varepsilon}(f)\right) \|f\|_{l^1(\pi)}^{4/d} \tag{2.2.15}$$

$$\|f\|_{l^2(m)}^{2(1+2/d)} \le C\,\left(\mathcal{E}_{aD}^{a\varepsilon}(f) + (\widehat C\,\widehat S\,a)^{-2}\,\|f\|_{l^2(m)}^2\right)\,\|f\|_{l^1(m)}^{4/d} \tag{2.2.16}$$

for every $f \in l^2(aD^\varepsilon)$, $\varepsilon \in (0, \varepsilon_1)$ *and* $a \in (0, \infty)$, *where* $\widehat C$ *and* $\widehat S$ *are the same constants in Corollary 2.2.13;* $\langle f\rangle_\pi := \sum f\,\pi$ *and* $\mathcal{E}_{aD}^{a\varepsilon}$ *is the Dirichlet form in* $l^2(m^{aD^\varepsilon})$ *of the CTRW* X^{aD^ε} *(see (2.2.1)).*

Proof Note that $\dfrac{(a\,\epsilon)^2}{d\,m(aD^\epsilon)}\,\mathcal{E}_{aD}^{a\epsilon}(f)$ is the Dirichlet form of the unit speed CTRW with the same one-step transition probabilities as that of X^{aD^ϵ}. Hence (2.2.15) follows directly from [67, Theorem 3.3.11] and (2.2.14). For (2.2.16), let $R = (2^{\frac{1}{d}}\delta\,a\,\varepsilon)^{-1}$ with $\delta \ge (\widehat C\,\widehat S\,a)^{-1}$. For any nonempty subset $A \subset aD^\varepsilon$,

$$\frac{Q(\partial A) + \frac{1}{R}\pi(A)}{\pi(A)^{\frac{d-1}{d}}} \quad\ge\quad \frac{1}{\widehat S} \wedge \frac{1}{R}\left(\frac{1}{2}\right)^{\frac{1}{d}} \ge \left((\widehat C\,\widehat S)^{-1} \wedge a\,\delta\right)\,\varepsilon = (\widehat C\,\widehat S)^{-1}\,\varepsilon.$$

Hence,

$$\sup_{A\subset aD^\varepsilon} \frac{\pi(A)^{\frac{d-1}{d}}}{Q(\partial A) + \frac{1}{R}\pi(A)} \le \frac{\widehat C\,\widehat S}{\varepsilon}. \tag{2.2.17}$$

By [67, Theorem 3.3.10],

$$\|f\|_{l^2(\pi)}^{2(1+2/d)} \le 16\,(\frac{C}{\varepsilon})^2 \left(\frac{(a\,\varepsilon)^2}{d\,m(aD^\varepsilon)}\,\mathcal{E}_{aD}^{a\varepsilon}(f) + \frac{1}{8R^2}\|f\|_{l^2(\pi)}^2\right)\,\|f\|_{l^1(\pi)}^{4/d}.$$

Using the relations $\|f\|_{l^2(\pi)}^2 = (m(aD^\varepsilon))^{-1}\|f\|_{l^2(m)}^2$, $\|g\|_{l^1(\pi)} = (m(aD^\varepsilon))^{-1}\|f\|_{l^1(m)}$ and (2.2.2), we get the desired inequality (2.2.16). □

2.2.4 Mixing time

By the Poincaré inequalities in (2.2.15) and [67, Corollary 2.3.2], we obtain an estimate on the time needed to reach stationarity.

Proposition 2.2.16. *(Mixing time estimate) There exists* $C > 0$ *which depends only on* d *such that*

$$\left| p^{a\varepsilon}_{aD}(t,x,y) - \frac{1}{m(aD^\varepsilon)} \right| \le C \min\left\{ (a\,\widehat{C}\,\widehat{S})^d\, t^{-d/2},\ \frac{1}{(a\,\varepsilon)^d}\, \exp\left(\frac{-d\,t}{8\,(a\widehat{C}\widehat{S})^2} \right) \right\}$$

for every $t > 0$, $x,\, y \in aD^\varepsilon$, $\varepsilon \in (0, \varepsilon_1)$ *and* $a > 0$*. Here* $\widehat{C}$ *and* $\widehat{S}$ *are the constants in Corollary 2.2.13.*

Proof By (2.2.15) and Theorem 2.3.1 of [67], we have

$$\left| m(aD^\varepsilon)\, p^{a\varepsilon}_{aD}\left(\frac{(a\varepsilon)^2}{d} t, x, y \right) - 1 \right| \le \left(\frac{d\,(8\,\tilde{S}^2)}{2t} \right)^{d/2}$$

After simplification and using (2.2.14), we obtain the upper bound which is of order $t^{-d/2}$. On other hand, by Corollary 2.2.14 and [67, Lemma 2.1.4], we obtain the exponential term on the right hand side. □

2.2.5 Gaussian bounds and uniform Hölder continuity of p^ε

Equipped with the Nash inequality (2.2.16) and the Poincaré inequality (2.2.15), one can follow a now standard procedure (see, for example, [13] or [28]) to obtain two sided Gaussian estimates for p^ε. In the following, C_1, C_2 and ε_0 are positive constants which depends only on d, D, ρ and T.

More precisely, we only need the Nash inequality (2.2.16) and Davies' method to obtain the following Gaussian upper bound.

Theorem 2.2.17. *There exist constants* $C_i = C_i(d, D, \rho, T) > 0$*,* $i = 1, 2$*, and* $\varepsilon_0 = \varepsilon_0(d, D, \rho) \in$

$(0,1]$ *such that*

$$p_{aD}^{a\varepsilon}(t,x,y) \le \frac{C_1}{(a\varepsilon \vee t^{1/2})^d} \exp\left(\frac{C_2}{a^2}t - \frac{|y-x|^2}{(a\varepsilon)^2 \vee t}\right)$$

for every $t \ge a\varepsilon$, $\varepsilon \in (0,\varepsilon_0)$, $a>0$ *and* $x,y \in aD^\varepsilon$. *Moreover, the following weaker bound holds for* $t \in (0,T]$*:*

$$p_{aD}^{a\varepsilon}(t,x,y) \le \frac{C_1}{(a\varepsilon \vee t^{1/2})^d} \exp\left(\frac{C_2}{a^2}t - \frac{|y-x|}{a\varepsilon \vee t^{1/2}}\right).$$

In particular, this implies the upper bound in Theorem 2.2.4 which is the case when $a=1$.

We can then apply the Poincaré inequality (2.2.15) and argue as in section 3 of [28] to obtain the near diagonal lower bound.

Lemma 2.2.18.

$$p^\varepsilon(t,x,y) \ge \frac{C_2}{(\varepsilon \vee t^{1/2})^d}$$

for every $(t,x,y) \in (0,\infty) \times D^\varepsilon \times D^\varepsilon$ *with* $|x-y| \le C_1\, t^{1/2}$ *and* $\varepsilon \in (0,\varepsilon_0)$.

The Gaussian lower bound for p^ε in Theorem 2.2.6 then follows from the Lipschitz property of D and a well-known chaining argument (see, for example, page 329 of [71]). Therefore, we have the two-sided Gaussian bound for p^ε as stated in Theorem 2.2.4 and Theorem 2.2.6. It then follows from a standard 'oscillation' argument (cf. Theorem 1.31 in [72] or Theorem II.1.8 in [71]) that p^ε is Hölder continuous in (t,x,y), *uniformly in* ε. More precisely,

Theorem 2.2.19. *There exist positive constants* $\gamma = \gamma(d,D,\rho)$, $\varepsilon_0(d,D,\rho)$ *and* $C(d,D,\rho)$ *such that for all* $\varepsilon \in (0,\varepsilon_0)$*, we have*

$$|p^\varepsilon(t,x,y) - p^\varepsilon(t',x',y')| \le C\, \frac{(|t-t'|^{1/2} + \|x-x'\| + \|y-y'\|)^\gamma}{(t\wedge t')^{\sigma/2}\,[1 \wedge (t\wedge t')^{d/2}]}. \tag{2.2.18}$$

2.2.6 Proof of local central limit theorem

The following weak convergence result for RBM with drift is a natural generalization of [9, Theorem 3.3].

Theorem 2.2.20. *Let $D \subset \mathbb{R}^d$ be a bounded domain whose boundary ∂D has zero Lebesque measure. Suppose D also satisfies:*

$$C^1(\overline{D}) \text{ is dense in } W^{1,2}(D).$$

Suppose $\rho \in W^{1,2}(D) \cap C^1(\overline{D})$ is strictly positive. Then for every $T > 0$, as $k \to \infty$,

(i) $(X^{2^{-k}}, \mathbb{P}_m)$ *converges weakly to the stationary process* $(X, \mathbb{P}_\rho)$ *in the Skorokhod space* $D([0,T], \overline{D})$.

(ii) $(X^{2^{-k}}, \mathbb{P}_{x_k})$ *converges weakly to* $(X, \mathbb{P}_x)$ *in the Skorokhod space* $D([0,T], \overline{D})$ *whenever* x_k *converges to* $x \in D$.

Proof For (i), the proof follows from a direct modification of the proof of [9, Theorem 3.3]. Recall the definition of the one-step transition probabilities p_{xy}, defined in the paragraph that contains (2.2.1) and (2.2.1). Observe that, since $\rho \in C^1(\overline{D})$, approximations using Taylor's expansions in the proofs of [9, Lemma 3.1 and Lemma 2.2] continue to work with the current definition of p_{xy}. Thus we have

$$\begin{aligned} \lim_{k\to\infty} \mathcal{E}^{2^{-k}}(f,f) &= \frac{1}{2}\int_D |\nabla f(x)|^2 \rho(x)dx, \quad \forall f \in C^1(\overline{D}), \quad \text{and} \\ \lim_{k\to\infty} L^{(2^{-k})} f &= \frac{1}{2}\Delta f + \frac{1}{2}\nabla(\log\rho)\cdot\nabla f \quad \text{uniformly in } D, \quad \forall f \in C_c^\infty(D). \end{aligned}$$

The process X^ε has a Lévy system $(N^\varepsilon(x,dy), t)$, where for $x \in D^\varepsilon$,

$$N^\varepsilon(x,dy) = \frac{d}{\varepsilon^2} \sum_{z:\, z\leftrightarrow x} p_{xz}\delta_{\{z\}}(dy).$$

Following the same calculations as in the proof of [9, Theorem 3.3], while noting that [20, Theorem 6.6.9] (in place of [9, Theorem 1.1]) can be applied to handle general symmetric reflected diffusions as in our present case, we get part (i). Part (ii) follows from part (i) by a localization argument (cf. [10, Remark 3.7]). $\square$

We can now present the proof of the local CLT.

Proof of Theorem 2.2.8. For each $\varepsilon > 0$ and $t > 0$, we extend $p^\varepsilon(t, \cdot, \cdot)$ to $\overline{D} \times \overline{D}$ in such a way that p^ε is nonnegative and continuous on $(0, \infty) \times \overline{D} \times \overline{D}$, and that both the maximum and the minimum values are preserved on each cell in the grid $\varepsilon\mathbb{Z}^d$. This can be done in many ways, say by the interpolation described in [3], or by a sequence of harmonic extensions along the simplexes in the following steps:

(i) Extend $p^\epsilon(t, \cdot, \cdot)$ to $\epsilon\mathbb{Z}^d \times \epsilon\mathbb{Z}^d$ so that it is zero outside $D^\epsilon \times D^\epsilon$.

(ii) Extend to $\mathbb{R}^d \times \mathbb{R}^d$ by extending to a suitable harmonic function on each closed cube. More precisely, for any cube, we first extend to the edges (1-simplexes) harmonically using the values on the vertices, then to the 2-simplexes harmonically using the values on the 1-simplexes, etc, until we have the extended function on the closed cube (a d-simplex). The extended function is continuous on the closure of each cube because the boundary data in each stage is continuous. Moreover, it is unique by maximum principle. See the remark below for an explicit construction.

(iii) Restrict the extended function to $\overline{D} \times \overline{D}$.

Consider the family $\{t^{d/2}p^\varepsilon\}_\varepsilon$ of continuous functions on $(0, \infty) \times \overline{D} \times \overline{D}$. Theorem 2.2.4 and Theorem 2.2.19 give us uniform pointwise bound and equi-continuity respectively. By Arzela-Ascoli Theorem, it is relatively compact. i.e. for any sequence $\{\varepsilon_n\} \subset (0, 1]$ which decreases to 0, there is a subsequence $\{\varepsilon_{n'}\}$ and a continuous $q : (0, \infty) \times \overline{D} \times \overline{D} \longrightarrow [0, \infty)$ such that $p^{\varepsilon_{n'}}$ converges to q locally uniformly.

On other hand, by part (ii) of Theorem 2.2.20, if the original sequence $\{\varepsilon_k\}$ is a subsequence of $\{2^{-k}\}$, then $q = p$. More precisely, the weak convergence implies that for all $t > 0$,

$$\int_D \phi(y)p(t, x, y)dy = \int_D \phi(y)q(t, x, y)dy \quad \text{for all } \phi \in C_c(D) \text{ and } x \in D.$$

Then by the continuity of both p and q in the second coordinate, we have $q = p$ on $(0, \infty) \times D \times \overline{D}$. Since $p(t, \cdot, \cdot)$ and $q(t, \cdot, \cdot)$ are continuous on $\overline{D} \times \overline{D}$ (cf. [2]), we obtain $p = q$ on $(0, \infty) \times \overline{D} \times \overline{D}$.

In conclusion, we have p^ε converges to p locally uniformly through the sequence $\{\varepsilon_n = 2^{-n}; n \geq 1\}$. $\square$

Remark 2.2.21. *The harmonic extension described is in fact a polynomial on each cube. Fix a cube $U \subset \mathbb{R}^n$. WLOG, assume U is of unit length and the vertices $V = \{\vec{\alpha} \in \mathbb{R}^n : \alpha_i = 0 \text{ or } 1 \,\forall\, i\}$. 2 vertices are connected by an edge if and only if exactly one coordinate is different. Consider the $|V| \times |V| = 2^n \times 2^n$ matrix*

$$A = \left(\vec{\alpha}^{\vec{\beta}}\right) \quad , \text{ where } \vec{\alpha}^{\vec{\beta}} = \alpha_1^{\beta_1} \cdots \alpha_n^{\beta_n} \text{ and } 0^0 = 1$$

$\det(A) = 1$ *since* $\prod_{\vec{\alpha} \in V} \vec{\alpha}^{\sigma(\vec{\alpha})} = 1$ *if and only if σ is the identity permutation. Given any function h on V, we define the extended function H on $\overline{U}$ by*

$$H(\vec{x}) = \sum_{\vec{\beta} \in V} a(\vec{\beta})\, \vec{x}^{\vec{\beta}} \qquad \text{where } a = A^{-1} h$$

Then H is the unique function extending h from V to $\overline{U}$ which is harmonic on U and on each simplex on ∂U.

2.2.7 Applications: Discrete surface measure and discrete local time

To capture the boundary behavior of the random walk X^ε near the boundary in the discrete scheme, we need a discrete approximation of the surface measure σ on ∂D. The construction of I^ε and σ_ε in the following lemma is a key to our approximation scheme.

Lemma 2.2.22. *(Discrete surface measure) Suppose D is a bounded Lipschitz domain of $\mathbb{R}^d$. Let $I \subset \partial D$ be closed, connected and $\mathcal{H}^{d-1}$-rectifiable. Let $\varepsilon_j = 2^{-j}$ for $j \in \mathbb{N}$. Then there exist finite subsets $I^{(j)} = I^{\varepsilon_j}$ of I and functions $\sigma_{(j)} = \sigma_{\varepsilon_j} : I^{(j)} \to [\varepsilon^{d-1}/C, C\varepsilon^{d-1}]$ such that (a) and (b) below hold simultaneously:*

(a)

$$\sup_{x \in \overline{D}} \#\left(I^{(j)} \cap B(x, s)\right) \leq C \left(\frac{s}{\varepsilon_j} \vee 1\right)^{d-1} \qquad \forall\, s \in (0, \infty),\, j \in \mathbb{N}, \tag{2.2.19}$$

where $\#A$ denotes the number of elements in the finite set A, $B(x, s) = \{y \in \mathbb{R}^d : |y-x| < s\}$ is the ball with radius s centered at x, and C is a constant that depends only on D.

(b) For any equi-continuous and uniformly bounded family $\mathcal{F} \subset C(I)$,

$$\lim_{j\to\infty} \sup_{f\in\mathcal{F}} \Big| \sum_{I(j)} f\,\sigma_{(j)} - \int_I f\,d\sigma \Big| = 0. \tag{2.2.20}$$

Proof We can always split I into small pieces. The point is to guarantee that each piece is not too small, so that $\sigma_{(j)}/\varepsilon^{d-1} \geq C$ and that (2.2.19) holds. Since I is $\mathcal{H}^{d-1}$-rectifiable, we have

$$C^{-1}\,R^{d-1} \leq \sup_{x\in I} \mathcal{H}^{d-1}(I \cap B(x,R)) \leq C\,R^{d-1}$$

for $R \in (0,1]$, where C does not depend on R. Since I is closed, it is regular with dimension $d-1$ in the terminology of section 1 of [27]. Hence by [26] or section 2 of [27], we can build "dyadic cubes" for I. More precisely, there exists a family of partitions $\{\Delta_j\}_{j\in\mathbb{Z}}$ of I into "cubes" Q such that

(i) if $j \leq k$, $Q \in \Delta_j$ and $Q' \in \Delta_k$, then either $Q \cap Q' = \emptyset$ or $Q \subset Q'$;

(ii) if $Q \in \Delta_j$, then

$$C^{-1}\,2^j \leq \operatorname{diam}(Q) \leq C\,2^j \quad \text{and}$$

$$C^{-1}\,2^{j(d-1)} \leq \mathcal{H}^{d-1}(Q) \leq C\,2^{j(d-1)};$$

(iii)

$$\mathcal{H}^{d-1}(\{x \in Q :\, \operatorname{dist}(x,\, I \setminus Q) \leq r2^j\}) \leq C\,r^{1/C}\,2^{j(d-1)}$$

for all $Q \in \Delta_j$ and $r > 0$.

Here the constant C is independent of j, Q, or r. Note that $\mathcal{H}^{d-1}$ is the surface measure σ of ∂D and that property (iii) tells us that the cubes have relatively small boundary. In particular, (iii) implies $\sigma(\partial Q \cap I) = 0$ for all cube Q.

Suppose $\Delta_j = \left\{U_i^{(j)}\right\}_{i=1}^{k_j}$. We pick one point $z_i^{(j)}$ from each $U_i^{(j)}$ to form the set $I^{(j)}$. Finally, we define $\sigma_{(j)}(z_i^{(j)}) := \sigma(U_i^{(j)})$. It follows from (ii) that $\sigma_{(j)} \in [\varepsilon^{d-1}/C,\, C\varepsilon^{d-1}]$ for some C which depends only on D. The inequality (2.2.19) follows from $C^{-1}\,\varepsilon_j^{d-1} \leq \sigma\,(U_i^{(j)})$ and the Lipschitz property of ∂D. It remains to check (2.2.20).

Fix any $\eta > 0$. There exists $\lambda = \lambda(\eta) > 0$ such that $|f(x) - f(y)| < \eta$ whenever $|x - y| < \lambda$. Hence for j large enough (depending only on λ),

$$\Big|\int_I g\,d\sigma - \sum_{I^{(j)}} g\,\sigma_{(j)}\Big| = \left|\sum_i \Big(\int_{U_i^{(j)}} g\,d\sigma - g(z_i^{(j)})\sigma(U_i^{(j)})\Big)\right| \leq \eta \sum_i \sigma(U_i^{(j)}) = \eta\,\sigma(I).$$

The desired convergence (2.2.20) now follows. □

Remark 2.2.23. (2.2.20) implies that we have the weak convergence $\sum_{z\in I^{(j)}} \sigma_{(j)}\,\delta_z \to \sigma\big|_I$ on the space $M_+(I)$ of positive finite measure Borel measures on I. Here δ_z is the dirac delta measure at z, and $\sigma\big|_I$ is the surface measure restricted to I. (2.2.19) is a control on the number of points locally in I^{ε_j}. We call I^ε the 'ε-point approximation' of I and σ_ε the 'discrete surface measure' associated to I^ε. □

Remark 2.2.24. The above lemma remains true if I is the finite union of disjoint closed connected and $\mathcal{H}^{d-1}$-rectifiable subsets of ∂D. This enables us to deal with the more general case when I is possibly disconnected. □

The following uniform estimate is the discrete analog of (2.1.5).

Lemma 2.2.25. *There exist $C = C(d, D, \rho, T) > 0$ and $\varepsilon_0 = \varepsilon_0(d, D, \rho) > 0$ such that*

$$\sup_{x\in D^\varepsilon} \varepsilon^{d-1} \sum_{y\in\partial D^\varepsilon} p^\varepsilon(t, x, y) \leq \frac{C}{\varepsilon \vee t^{1/2}} \tag{2.2.21}$$

for all $t \in [0, T]$ and $\varepsilon \in (0, \varepsilon_0)$. Here ∂D^ε is the graph-boundary of D^ε, which is all the vertices in D^ε with degree less than $2d$.

Proof Fix $\theta \in [0, T]$. By the Gaussian upper bound in Theorem 2.2.4, we have

$$\begin{aligned}
& \sum_{y \in \partial D^\varepsilon} p^\varepsilon(\theta, x, y) \\
\leq\ & \frac{C_1}{(\varepsilon \vee \theta^{1/2})^d} \sum_{y \in \partial D^\varepsilon} \exp\left(\frac{-|y-x|}{\varepsilon \vee \theta^{1/2}}\right) \\
=\ & \frac{C_1}{(\varepsilon \vee \theta^{1/2})^d} \int_0^\infty \#\,|\{y \in D^\varepsilon : |f(y)| > r\}|\, dr \quad \text{by setting } f(y) = \mathbf{1}_{\partial D^\varepsilon}(y) \exp\left(\frac{-|y-x|}{\varepsilon \vee \theta^{1/2}}\right) \\
=\ & \frac{C_1}{(\varepsilon \vee \theta^{1/2})^d} \int_0^1 \#\,|\{\partial D^\varepsilon \cap B(x,\, (\varepsilon \vee \theta^{1/2})(-\ln r))\}|\, dr \quad (\text{since } f \leq 1) \\
=\ & \frac{C_1}{(\varepsilon \vee \theta^{1/2})^{d+1}} \int_0^\infty \#\,|\{\partial D^\varepsilon \cap B(x,\, s)\}| \exp\left(\frac{-s}{\varepsilon \vee \theta^{1/2}}\right) ds \quad (\text{where } s = (\varepsilon \vee \theta^{1/2})(-\ln r)) \\
\leq\ & \frac{C_1}{(\varepsilon \vee \theta^{1/2})^d} \vee \frac{C_2}{\varepsilon^{d-1}(\varepsilon \vee \theta^{1/2})^{d+1}} \int_0^\infty s^{d-1} \exp\left(\frac{-s}{\varepsilon \vee \theta^{1/2}}\right) ds \\
\leq\ & \frac{1}{\varepsilon^{d-1}} \left(\frac{C_1}{\varepsilon \vee \theta^{1/2}} \vee \frac{C_2}{\varepsilon \vee \theta^{1/2}} \int_0^\infty w^{d-1} e^{-w} dw\right) \quad (\text{where } w = \frac{s}{\varepsilon \vee \theta^{1/2}}).
\end{aligned}$$

Here C_i are all constants which depend only on d, D and T. Note that in the second last line, we used the fact that $\#\,|\{\partial D^\varepsilon \cap B(x,\, s)\}| \leq C((s/\varepsilon)^{d-1} \vee 1)$, which follows from Lemma 2.2.22. The proof is now complete. □

As an application of the local CLT, we have the following approximation for the local time $L^{(I)}$ of X on I.

Proposition 2.2.26. *(Discrete local time) Let D, I, I^ε and σ_ε be as in Lemma 2.2.22. For any function $f \in \mathcal{B}_b(D)$ that is continuous in a neighborhood of I, we have*

$$\mathbb{E}^x\left[\int_0^t f(X_s)\, dL_s^{(I)}\right] = \lim_{\varepsilon \to 0} \int_0^t \sum_{z \in I^\varepsilon} f(z)\, p^\varepsilon(s, x_\varepsilon, z_\varepsilon)\, \sigma_\varepsilon(z)\, ds \tag{2.2.22}$$

for any $x \in \overline{D}$ and any $\{x_\varepsilon\} \subset D^\varepsilon$ which converge to x. Here z_ε is any point in D^ε which is closest to $z \in I^\varepsilon$.

Proof By (2.1.6), we have $\mathbb{E}^x\left[\int_0^t f(X_s)\, dL_s^{(I)}\right] = \int_0^t \int_I f(z)\, p(s, x, z) \sigma(dz)\, ds$. By applying the local CLT (Theorem 2.2.8) and Lemma 2.2.22, we have

$$\int_I f(z)\, p(s,x,z)\sigma(dz) = \lim_{\varepsilon\to 0} \sum_{z\in I^\varepsilon} f(z)\, p^\varepsilon(s,x_\varepsilon,z_\varepsilon)\, \sigma_\varepsilon(z)$$

for any $s\in(0,t)$. The result then follows by Lemma 2.2.25 and LDCT. $\square$

Take I to be the whole boundary ∂D and construct $(\partial D)^\epsilon$ in Lemma 2.2.22. Define, for each any $\omega:\,[0,\infty)\to D^\epsilon$,

$$L^\epsilon_t(\omega) := \frac{1}{2}\int_0^t \sum_{z\in(\partial D)^\epsilon} \frac{\mathbf{1}_{\{\omega(s)=z_\varepsilon\}}}{m_\epsilon(z_\varepsilon)}\, \sigma_\epsilon(z)\, ds, \tag{2.2.23}$$

in which we pick exactly one $z_\varepsilon\in D^\epsilon$ which is nearest to z. Proposition 2.2.26 asserts that L^ϵ_t is a reasonable candidate for the discrete approximation of the boundary local time L_t of X. In fact, by the local CLT and a standard argument (using Fubini's theorem and Markov property), it is straightforward to show that that L^ϵ_t converges to L_t in any moment. This candidate for the local time will, in particular, give us a discrete approximation of the solution of the heat equation with Robin boundary condition (2.1.12), namely $\mathbb{E}^{x_\varepsilon}\left[\varphi(X^\varepsilon_t)e^{-\int_0^t g(t-s,\,X^\varepsilon_s)\,dL^\varepsilon_s}\right]$ (see Proposition 2.1.8).

Chapter 3

HYDRODYNAMIC LIMITS FOR INTERACTING RANDOM WALKS

In this chapter, we introduce a new stochastic reaction-diffusion system in which two families of random walks in two adjacent domains interact near an interface. Such a system can be used to model the transport of positive and negative charges in a solar cell or the population dynamics of two segregated species under competition. We show that in the macroscopic limit, the particle densities converge to the solution of a coupled nonlinear heat equations. For this, we first prove that propagation of chaos holds by establishing the uniqueness of a new BBGKY hierarchy. A local central limit theorem for reflected diffusions in bounded Lipschitz domains is also established as a crucial tool.

Besides the notation listed in Notation Index, we adopt the following assumption and notations throughout this chapter.

Assumption 3.0.27. *$D_\pm$ are given adjacent bounded Lipschitz domains in $\mathbb{R}^d$ such that $I := \overline{D}_+ \cap \overline{D}_- = \partial D_+ \cap \partial D_-$ is a finite union of disjoint connected $\mathcal{H}^{d-1}$-rectifiable sets, $\rho_\pm \in W^{(1,2)}(D_\pm) \cap C^1(\overline{D}_\pm)$ are given functions which are strictly positive, $\lambda > 0$ is a fixed parameter.*

An example to keep in mind is $D_+ = (0,1)^d$ and $D_- = (0,1)^{d-1} \times (0,-1)$ are two adjacent cubes, so the interface is $I = (0,1)^{d-1} \times \{0\}$.

We denote by $X^\pm$ a $(I_{d\times d}, \rho_\pm)$-reflected diffusion in $D_\pm$. The L^2-generator, semigroup and transition density (w.r.t. $\rho_\pm(x)dx$) of $X^\pm$ are denoted by $\mathcal{A}^\pm$, $(P_t^\pm)_{t\geq 0}$ and $p^\pm(t,x,y)$ respectively. For discrete approximations, we denote by $X^{\varepsilon,\pm}$ the $m_\varepsilon^\pm$-symmetric CTRW on the finite graph $D^\varepsilon_\pm$ which is the ε-approximation of $X^\pm$. The $L^2(m^\pm_\varepsilon)$-generator, semigroup and transition density (w.r.t. $m^\pm_\varepsilon$) of $X^{\varepsilon,\pm}$ are denoted by $\mathcal{A}^\pm_\varepsilon$, $(P^{\varepsilon,\pm}_t)_{t\geq 0}$ and $p^{\varepsilon,\pm}(t,x,y)$ respectively. We also denote

$$p^{(n,m)}(t,(\vec{r},\vec{s}),(\vec{r'},\vec{s'})) := \prod_{i=1}^n p^+(t,r_i,r'_i)\prod_{j=1}^m p^-(t,s_j,s'_j)$$

$$\rho_{(n,m)}(\vec{r},\vec{s}) := \prod_{i=1}^n \rho_+(r_i)\prod_{j=1}^m \rho_-(s_j)$$

$$p^{(n,m),\varepsilon}(t,(\vec{r},\vec{s}),(\vec{r'},\vec{s'})) := \prod_{i=1}^n p^{\varepsilon,+}(t,r_i,r'_i)\prod_{j=1}^m p^{\varepsilon,-}(t,s_j,s'_j)$$

$$m^{(n,m)}_\varepsilon(\vec{r},\vec{s}) := \prod_{i=1}^n m^+_\varepsilon(r_i)\prod_{j=1}^m m^-_\varepsilon(s_j).$$

3.1 Annihilating random walk model

Fix $\varepsilon = \varepsilon_j = 2^{-j}$ ($j \in \mathbb{N}$) and $N = 2^{jd}$ such that $N\varepsilon^d = 1$. Assume there are N "+" particles in D^ε_+ and N "−" particles in D^ε_- at $t = 0$. Each particle moves as an independent CTRW $X^{\varepsilon,\pm}$ in $D^\varepsilon_\pm$. Let I^ε be the finite subset of I defined in Lemma 2.2.22. For each $z \in I^\varepsilon$, pick an $z_+ \in D^\varepsilon_+$ and an $z_- \in D^\varepsilon_-$ which are closest to z (See Figure 3.1). A pair of particles of opposite charges at (z_+, z_-) is being killed with a certain rate to be explained. Note that for ε small enough, we have $\sup_{z\in I^\varepsilon} |z_\pm - z| \leq 2M\varepsilon$, where M is the Lipschitz constant of I.

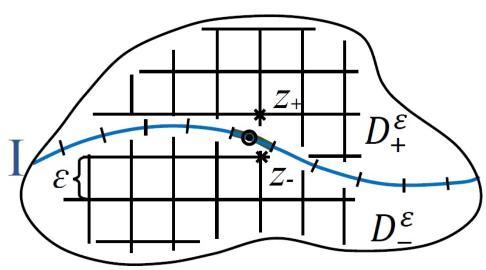


Figure 3.1: $z \in I^\varepsilon \subset I,\ z_\pm \in D^\varepsilon_\pm$

The state space of the particle system is the collection of configurations

$$E^\varepsilon := \left\{\eta^\varepsilon = (\eta^{\varepsilon,+}, \eta^{\varepsilon,-}) : \eta^{\varepsilon,\pm} : D^\varepsilon_\pm \to \mathbb{N}\right\}. \tag{3.1.1}$$

The state of the particle system at time t is a random element $\eta^\varepsilon_t = (\eta^{\varepsilon,+}_t, \eta^{\varepsilon,-}_t) \in E^\varepsilon$. Here $\eta^{\varepsilon,\pm}_t(x)$ stands for the number of "$\pm$" particles at $x \in D^\varepsilon_\pm$ at time t. We omit ε and N for convenience when there is no ambiguity. For example, we write η_t and $m(x)$ in place of η^ε_t and $m_\varepsilon(x)$ respectively. The function ξ such that $\xi(x) = 1$ and $\xi(y) = 0$ for $y \neq x$ is denoted as $\mathbf{1}_x$.

Definition 3.1.1. *η_t is defined to be the unique strong Markov process which has the generator $\mathfrak{L} = \mathfrak{L}^\varepsilon$ given by*

$$\mathfrak{L}^\varepsilon := \mathfrak{L}^\varepsilon_0 + \mathfrak{K}^\varepsilon, \tag{3.1.2}$$

where $\mathfrak{L}^\varepsilon_0$ is the generator of two families of independent random walks in D^ε_+ and D^ε_-, respectively, with no annihilation between them, namely

$$\begin{aligned} \mathfrak{L}^\varepsilon_0 f(\eta) \;:=\; & \frac{d}{\varepsilon^2} \sum_{x,y \in D^\varepsilon_+} \eta^+(x) p^+_{xy} \{f(\eta^+ - \mathbf{1}_x + \mathbf{1}_y, \eta^-) - f(\eta)\} \\ & + \frac{d}{\varepsilon^2} \sum_{x,y \in D^\varepsilon_-} \eta^-(x) p^-_{xy} \{f(\eta^+, \eta^- - \mathbf{1}_x + \mathbf{1}_y) - f(\eta)\} \end{aligned} \tag{3.1.3}$$

and $\mathfrak{K}^\varepsilon$ is the operator corresponding to annihilation between particles of opposite signs at the interface I^ε, namely

$$\mathfrak{K}^\varepsilon f(\eta) := \frac{\lambda}{\varepsilon} \sum_{z \in I^\varepsilon} \Psi_\varepsilon(z)\, \eta^+(z_+) \eta^-(z_-) \left\{f(\eta^+ - \mathbf{1}_{z_+}, \eta^- - \mathbf{1}_{z_-}) - f(\eta)\right\}, \tag{3.1.4}$$

where $p^\pm_{xy}$ is the one-step transition probabilities for the CTRW $X^{\varepsilon,\pm}$ on $D^\varepsilon_\pm$ (without any interaction) and

$$\Psi_\varepsilon(z) := \frac{\sigma_\varepsilon(z)}{\varepsilon^{d-1}} \frac{\varepsilon^{2d}}{m(z_+) m(z_-)} \tag{3.1.5}$$

with σ_ε and I^ε being constructed by Lemma 2.2.22.

The expression for $\mathfrak{K}^\varepsilon$ comes from the underlying assumptions of the model: First, the term $\eta^+(z_+)\eta^-(z_-)$ is combinatorial in nature. Since there are $\eta^+(z_+)\eta^-(z_-)$ pairs of particles at position (z_+, z_-), the chance of killing is proportional to the number of ways of selecting a pair of particles near the interface. Second, each pair of particles near I disappears at rate $(\lambda/\varepsilon)\,\Psi_\varepsilon(z)$ where λ is a parameter. Intuitively, in the limit, the amount of annihilation in a neighborhood of a point is proportional to the surface area of the interface I in that neighborhood. The scaling $1/\varepsilon$ is suggested by the observation that there are about $1/\varepsilon$ "layers" starting from the interface I, so that the chance for a particle to arrive near I is of order ε. $\Psi_\varepsilon(z)$ is comparable to 1 and can be viewed as a normalizing constant with respect to the lattice. This choice (3.1.5) is justified in the proof of Theorem 3.4.5.

3.2 Coupled heat equations with non-linear boundary condition

In this subsection, we provide suitable notion of solutions for the coupled PDE (1.2.1) and (1.2.2), and then prove the existence and uniqueness of the solution. These are motivated by Proposition 2.1.8.

Proposition 3.2.1. *For $T>0$, consider the Banach space $\Lambda_T = C([0,T]\times\overline{D}_+)\times C([0,T]\times\overline{D}_-)$ with norm $\|(u,v)\| := \|u\| + \|v\|$. Suppose $u_+(0) = f \in C(\overline{D}_+)$ and $u_-(0) = g \in C(\overline{D}_-)$. Then there is a unique element $(u_+, u_-) \in \Lambda_T$ which satisfies the coupled integral equation*

$$\begin{cases} u_+(t,x) = P_t^+ f(x) - \dfrac{\lambda}{2}\displaystyle\int_0^t \int_I p^+(t-r,x,z)\,[u_+(r,z)u_-(r,z)]\,\rho_+(y)d\sigma(z)\,dr \\ u_-(t,y) = P_t^- g(y) - \dfrac{\lambda}{2}\displaystyle\int_0^t \int_I p^-(t-r,y,z)\,[u_+(r,z)u_-(r,z)]\,\rho_-(y)d\sigma(z)\,dr. \end{cases} \tag{3.2.1}$$

Moreover, (u_+, u_-) satisfies

$$\begin{cases} u_+(t,x) = \mathbb{E}^x[\, f(X_t^+)\, e^{-\lambda\int_0^t u_-(t-s,X_s^+)\, dL_s^+}\,] \\ u_-(t,y) = \mathbb{E}^y[\, g(X_t^-)\, e^{-\lambda\int_0^t u_+(t-s,X_s^-)\, dL_s^-}\,], \end{cases} \tag{3.2.2}$$

where $L^{\pm}$ is the boundary local time of $X^{\pm}$ on the interface I.

Proof Define the operator S on Λ_T by $S(u,v) = (S^+v, S^-u)$, where

$$S^+v(t,x) = \mathbb{E}^x\left[f(X_t^+)\, e^{-\lambda\int_0^t v(t-s,X_s^+)\,dL_s^+}\right] \quad \text{for } (t,x)\in[0,T]\times\overline{D}_+,$$

$$S^-u(t,y) = \mathbb{E}^y\left[g(X_t^-)\, e^{-\lambda\int_0^t u(t-s,X_s^-)\,dL_s^-}\right] \quad \text{for } (t,y)\in[0,T]\times\overline{D}_-.$$

Lemma 2.1.6 implies that S maps into Λ_T. Moreover, for $(t,x)\in[0,T]\times\overline{D}_+$,

$$\begin{aligned}
|(S^+v_1 - S^+v_2)(t,x)| &= \left|\mathbb{E}^x\left[f(X_t^+)\left(e^{-\lambda\int_0^t v_1(t-s,X_s)dL_s^+} - e^{-\lambda\int_0^t v_2(t-s,X_s^+)dL_s^+}\right)\right]\right| \\
&\le \|f\|\mathbb{E}^x\left[\left|\lambda\int_0^t v_1(t-s,X_s)dL_s^+ - \lambda\int_0^t v_2(t-s,X_s^+)\,dL_s^+\right|\right] \\
&= \|f\|\lambda\,\mathbb{E}^x\left[\int_0^t |v_1(t-s,X_s^+) - v_2(t-s,X_s^+)|\,dL_s^+\right] \\
&\le \lambda\,\|f\|\,\|v_1-v_2\|\,\mathbb{E}^x[L_t^+] \\
&= \lambda\,\|f\|\,\|v_1-v_2\|\,\frac{1}{2}\int_0^t\int_I p^+(s,x,y)\,\rho_+(y)\sigma(dy)ds \\
&\le C_1\lambda\sqrt{T}\|f\|\,\|v_1-v_2\|.
\end{aligned}$$

A similar result holds for $S^-u_1 - S^-u_2$. Hence,

$$\begin{aligned}
\|S(u_1,v_1) - S(u_2,v_2)\| &= \|S^+v_1 - S^+v_2\| + \|S^-u_1 - S^-u_2\| \\
&\le C_1\lambda\sqrt{T}\|u_0\|\,\|v_1-v_2\| + C_2\lambda\sqrt{T}\|v_0\|\,\|u_1-u_2\| \\
&\le \gamma\|(u_1,v_1)-(u_2,v_2)\|
\end{aligned}$$

for some $\gamma<1$ when T is small enough.

Hence there is a $T_0>0$ such that $S:\Lambda_{T_0}\to\Lambda_{T_0}$ is a contraction map. By Banach fixed point theorem, there is a unique element $(u^\star,v^\star)\in\Lambda_{T_0}$ such that $(u^\star,v^\star) = S(u^\star,v^\star)$. By Proposition 2.1.8, $(u^\star,v^\star)$ is the unique weak solution to the coupled PDE on $[0,T_0]$.

Repeat the above argument, with $u_0(\cdot)$ replaced by $u^\star(T_0,\cdot)$, and $v_0(\cdot)$ replaced by $v^\star(T_0,\cdot)$.

We see that, since $\|u^\star(T_0,\cdot)\| \le \|u_0\|$, $\|v^\star(T_0,\cdot)\| \le \|v_0\|$ and $C_i\,(i=1,2)$ are the same, we can extend the solution of the coupled PDE uniquely to $[T_0, 2T_0]$. Iterating the argument, we have for any $T>0$, the coupled PDE has a unique weak solution in Λ_T. Invoke Proposition 2.1.8 once more, we obtain the desired implicit probabilistic representation (3.2.2).

Finally, by using Markov property as in the proof of Proposition 2.1.8, we see that (3.2.2) and (3.2.1) are equivalent. □

As in Definition 2.1.9, we introduce the following definition.

Definition 3.2.2. *The pair of functions* (u_+, u_-) *satisfying equation* (3.2.1) *is called a* ***probabilistic solution*** *of* (1.2.1) *and* (1.2.2).

3.3 Main results: rigorous statements

In this paper, we always assume the scaling $N\varepsilon^d = 1$ holds for simplicity, so that the interacting random walk model is parameterized by a single parameter N which is the initial number of particles in each of D_+^ε and D_-^ε. More precisely, for each fixed N, we set $\varepsilon = N^{-1/d}$ and let $(\eta_t^\varepsilon)_{t\ge 0}$ be a Markov process having generator $\mathfrak{L}^\varepsilon$ defined in (3.1.2) and having initial distribution satisfies $\sum_{x\in D_+^\varepsilon} \eta_0^{\varepsilon,+}(x) = \sum_{y\in D_-^\varepsilon} \eta_0^{\varepsilon,-}(y) = N$. We define the empirical measures

$$\mathfrak{X}_t^{N,\pm}(dz) := \frac{1}{N}\sum_{x\in D_\pm^\varepsilon} \eta_t^{\varepsilon,\pm}(x)\mathbf{1}_x(dz).$$

It is clear that $(\mathfrak{X}_t^{N,+}, \mathfrak{X}_t^{N,-})_{t\ge 0}$ is a continuous time Markov process (inheriting from that of η_t) with state space

$$\mathfrak{E} := M_{\le 1}(\overline{D}_+) \times M_{\le 1}(\overline{D}_-),$$

where $M_{\le 1}(E)$ denotes the space of non-negative Borel measures on E with mass at most 1. $M_{\le 1}(E)$ is a closed subset of $M_+(E)$, where the latter denotes the space of finite non-negative

Borel measures on E equipped with the following metric:

$$\|\mu - \nu\| := \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|\langle \mu, \phi_k \rangle - \langle \nu, \phi_k \rangle|}{1 + |\langle \mu, \phi_k \rangle - \langle \nu, \phi_k \rangle|}, \tag{3.3.1}$$

where $\{\phi_k : k \geq 1\}$ is any countable dense subset of $C(E)$. The topology induced by this metric is equivalent to the weak topology (i.e. $\|\mu_n - \mu\| \to 0$ if and only if $\langle \mu_n, f \rangle \to \langle \mu, f \rangle$ for all $f \in C(E)$). Under this metric, $M_+(\overline{D})$ is a complete separable metric space, hence so are $\mathfrak{E}$ and the Skorokhod space $D([0,T], \mathfrak{E})$ (see e.g. Theorem 3.5.6 of [35]). Recall that $\xrightarrow{\mathcal{L}}$ stands for convergence in law as $N \to \infty$. Here is our first main result.

Theorem 3.3.1. ***(Hydrodynamic limit)*** *Suppose Assumption 3.0.27 holds and the sequence of initial configurations η_0^{ε} satisfies the following conditions:*

(i) $\mathfrak{X}_0^{N,\pm} \xrightarrow{\mathcal{L}} u_0^{\pm}(z)dz$ *in* $M_{\leq 1}(\overline{D}_{\pm})$*, where* $u_0^{\pm} \in C(\overline{D}_{\pm})$*.*

(ii) $\overline{\lim}_{N\to\infty} \sup_{z \in D_{\pm}^{\varepsilon}} \mathbb{E}[(\eta_0^{\varepsilon,\pm}(z))^2] < \infty$.

Then for any $T > 0$, as $\varepsilon \to 0$ along the sequence $\varepsilon_j = 2^{-j}$, we have

$$(\mathfrak{X}^{N,+},\, \mathfrak{X}^{N,-}) \xrightarrow{\mathcal{L}} (\nu^+,\, \nu^-) \in D([0,T],\, \mathfrak{E}),$$

where (ν^+, ν^-) is the deterministic element in $C([0,T], \mathfrak{E})$ such that

$$(\nu_t^+(dx),\, \nu_t^-(dy)) = (u_+(t,x)\, \rho_+(x)dx,\, u_-(t,y)\, \rho_-(y)dy)$$

for all $t \in [0,T]$, and (u_+, u_-) is the probabilistic solution of the coupled PDEs (1.2.1) and (1.2.2) with initial value (u_0^+, u_0^-).

Theorem 3.3.1 gives the limiting probability distribution of one particle randomly picked in $D_{\pm}^{\varepsilon}$ at time t. This is the 1-particle distribution in the terminology of statistical physics.

Question: What is the limiting joint distribution of more than one particle?

Before stating the answer, we need to introduce a standard tool in the study of stochastic particle systems: the notion of correlation functions[1]. Recall that the state space of $\eta^\varepsilon = (\eta^\varepsilon_t)_{t\geq 0}$ is E^ε defined in (3.1.1). We denote by

$$\Omega^\varepsilon_{n,m} := \Big\{\xi = (\xi^+, \xi^-) \in E^\varepsilon : |\xi^+| := \sum_x \xi^+(x) = n,\ |\xi^-| := \sum_y \xi^-(y) = m\Big\}$$

the set of configurations with n and m particles in D^ε_+ and D^ε_- respectively. We then define $A : E^\varepsilon \times E^\varepsilon \to \mathbb{R}$ in such a way that whenever $\xi \in \Omega^\varepsilon_{n,m}$,

$$A(\xi, \eta) := A^+(\xi^+, \eta^+)A^-(\xi^-, \eta^-) := \prod_{x\in D_+} A^{\eta^+(x)}_{\xi^+(x)} \prod_{x\in D_-} A^{\eta^-(x)}_{\xi^-(x)}, \tag{3.3.2}$$

where $n \mapsto A^n_k$ is the Poisson polynomial[2] of order k, namely $A^n_0 := 1$ and $A^n_k := n(n-1)\cdots(n-k+1)$ for $k \geq 1$ (in particular, $A^n_k = 0$ for $k > n$). Note that A^n_k is the number of permutations of k objects chosen from n distinct objects. So $A(\xi, \eta)$ is the total number of possible site to site pairings between labeled particles having configuration ξ with a subset of labeled particles having configuration η. An alternative representation of (3.3.2) will be given in (3.4.5).

Convention: For $(\vec{r}, \vec{s}) \in (D^\varepsilon_+)^n \times (D^\varepsilon_-)^m$ and $\eta \in \Omega^\varepsilon_{N,M}$, we define $A((\vec{r}, \vec{s}),\, \eta)$ to be $A(\xi, \eta)$ with $\xi = (\sum_i \delta_{r_i}, \sum_j \delta_{s_j})$.

Definition 3.3.2. *Let $\mathbb{P}^\eta$ is the law of a process with generator $\mathfrak{L}^\varepsilon$ and initial distribution η satisfying the two conditions specified in Theorem 3.3.1. For all $t \geq 0$, we define*

$$\gamma^\varepsilon(\xi, t) := \gamma^{\varepsilon,\,(n,m)}(\xi, t) := \frac{\varepsilon^{d(n+m)}}{\alpha_\varepsilon(\xi)}\, \mathbb{E}^\eta[A(\xi, \eta_t)] \tag{3.3.3}$$

for all $\xi \in \Omega^\varepsilon_{n,m}$, where

$$\alpha_\varepsilon(\xi) := m_\varepsilon(\vec{r}, \vec{s}) := \prod_{i=1}^n m^+_\varepsilon(r_i) \prod_{j=1}^m m^-_\varepsilon(s_j). \tag{3.3.4}$$

[1] More precisely, we will be using correlation functions for *unlabeled* particles. We refer the readers to [57] for the relation between labeled and unlabeled correlation functions.

[2] The notation A^n_k is suggested by the fact that $\mathbb{E}[A^\varrho_k] = \theta^k$ when ϱ is a Poisson random variable with mean θ.

when $\xi = (\sum_i \delta_{r_i}, \sum_j \delta_{s_j})$. *By convention, we also have* $\gamma^\varepsilon((\vec{r},\vec{s}),t) := \gamma^\varepsilon(\xi,t)$. *Note that* γ^ε *depends on the initial configuration of* η.

Intuitively, suppose we randomly pick n and m living particles in D_+ and D_- respectively at time t, then $(\vec{r},\vec{s}) \mapsto \gamma^{\varepsilon,(n,m)}((\vec{r},\vec{s}),\, t)$ is the joint probability density function for their positions, up to a normalizing constant. Therefore, it is natural that $\gamma^{\varepsilon,(n,m)}$ defined by (3.3.3) is called the (n,m)-**particle correlation function**.

The next is our second main result, **propagation of chaos**, for our system. It says that when the number of particles tends to infinity, they appears to be independent of each other. Mathematically, the correlation function factors out in the limit $N \to \infty$.

Theorem 3.3.3. *(Propagation of Chaos) Under the same condition as in Theorem 3.3.1, for all* $n,\, m \in \mathbb{N}$ *and any compact interval* $[a,b] \subset (0,\infty)$,

$$\lim_{\varepsilon\to 0} \sup_{(\vec{r},\vec{s})\in \overline{D}_+^n\times\overline{D}_-^m} \sup_{t\in[a,b]} \left|\gamma^\varepsilon((\vec{r},\vec{s}),t) - \prod_{i=1}^n u_+(t,r_i)\prod_{j=1}^m u_-(t,s_j)\right| = 0\,,$$

where (u_+,u_-) *is the weak solution of the coupled PDE.*

To investigate the intensity of killing near the interface, we define $J^{N,\pm} \in D([0,\infty), M_+(\overline{D}_\pm))$ by

$$J_t^{N,+}(A) \;:=\; \varepsilon^{d-1}\sum_{z\in I^\varepsilon}\Psi(z)\,\eta_t^+(z_+)\eta_t^-(z_-)\,\mathbf{1}_A(z_+) \quad \text{for } A\subset \overline{D}_+, \tag{3.3.5}$$

$$J_t^{N,-}(B) \;:=\; \varepsilon^{d-1}\sum_{z\in I^\varepsilon}\Psi(z)\,\eta_t^+(z_+)\eta_t^-(z_-)\,\mathbf{1}_B(z_-) \quad \text{for } B\subset \overline{D}_-. \tag{3.3.6}$$

Clearly, $\langle J_t^{N,+},1\rangle = \langle J_t^{N,-},1\rangle$, which measures the number of encounters of the two types of particles near I. An immediately corollary of Theorem 3.3.3 is the following, which is what we need to identify the limit of $(\mathfrak{X}^{N,+},\, \mathfrak{X}^{N,-})$.

Corollary 3.3.4. *For any fixed $t \in (0,\infty)$ and $\phi \in C(\overline{D}_\pm)$, we have*

$$\begin{aligned}
\lim_{N\to\infty} \mathbb{E}[\langle J_t^{N,\pm}, \phi\rangle] &= \frac{1}{2}\int_I u_+(t,y)u_-(t,y)\,\phi(y)\,\sigma(dy), \\
\lim_{N\to\infty} \mathbb{E}[(\langle J_t^{N,\pm}, \phi\rangle)^2] &= \left(\frac{1}{2}\int_I u_+(t,y)u_-(t,y)\,\phi(y)\,\sigma(dy)\right)^2, \\
\lim_{N\to\infty} \mathbb{E}[\langle \mathfrak{X}_t^{N,\pm}, \phi\rangle] &= \langle u_\pm(t),\, \phi\rangle_{\rho_\pm}, \\
\lim_{N\to\infty} \mathbb{E}[(\langle \mathfrak{X}_t^{N,\pm}, \phi\rangle)^2] &= \left(\langle u_\pm(t),\, \phi\rangle_{\rho_\pm}\right)^2,
\end{aligned}$$

where $\langle u_\pm(t),\, \phi\rangle_{\rho_\pm} := \int_{D_\pm} u_\pm(t,y)\,\phi(y)\,\rho_\pm(y)\,dy$.

Proof We only need to apply Theorem 3.3.3 for the cases $(n,m)=(1,1)$ and $(n,m)=(1,0)$. By definition,

$$\gamma^\varepsilon(\mathbf{1}_r,t) = \frac{\varepsilon^d}{m^+(r)}\,\mathbb{E}^\eta[\eta_t^+(r)] \quad \text{and} \quad \gamma^\varepsilon(\mathbf{1}_r+\mathbf{1}_s,t) = \frac{\varepsilon^{2d}}{m^+(r)m^-(s)}\,\mathbb{E}^\eta[\eta_t^+(r)\eta_t^-(s)].$$

Using (2.2.2) and Lemma 2.2.22, we get the first two equations via Theorem 3.3.3. Using (2.2.2) and the assumption that $\rho_\pm \in C(\overline{D}_\pm)$, we have the last two equations again by Theorem 3.3.3. □

Remark 3.3.5. (Conditions on η_0) The two conditions for the initial configuration η_0 in Theorem 3.3.1 are mild and natural. They are satisfied, for example, when each particle has the same random initial distribution $\frac{u_0^\pm(z)}{\sum_{D_\pm} u_0^\pm}$. Condition (ii) guarantees that, asymptotically, there is no "blow up" of number of particles at any site. More precisely, this technical condition is imposed so that we have

$$\sup_{t\geq 0}\mathbb{E}\left[\langle 1,\, J_t^{N,+}\rangle^2\right] \leq C < \infty \quad \text{for sufficiently large } N. \tag{3.3.7}$$

The above can be easily checked by comparing with the process $\overline{\eta}$ that has no annihilation (i.e. $\overline{\eta}$ has generator $\mathfrak{L}_0^\varepsilon$). Alternatively, we can use the comparison result (3.4.11) to prove (3.3.7). □

Remark 3.3.6. (Generalization) We can generalize our results in a number of ways by the same argument. For example, the initial number of particles in D_+ and D_- can be different, the condition $N\varepsilon^d = 1$ can be relaxed to $\lim_{N\to\infty} N\varepsilon^d \to 1$ where ε depends on N. The annihilation constant λ can be replaced by a space and time dependent function $\lambda(t,x) \in C([0,\infty)\times I)$. The diffusion coefficients in D_+ and D_- can be different. The condition "$\mathfrak{X}_0^{N,\pm}$ has mass one for all N" can be replaced by "the mass of $\mathfrak{X}_0^{N,\pm}$ is uniformly bounded in N". More generally, the same method can be extended to deal with similar models with more than two types of particles. □

The remaining part of this paper is devoted to the proof of Theorem 3.3.1 and Theorem 3.3.3. We first prove Theorem 3.3.3 because the proof of Theorem 3.3.1 relies on Theorem 3.3.3.

3.4 Propagation of Chaos

3.4.1 Duality

The starting point of our analysis is the discrete integral equation for γ^ε in Lemma 3.4.2. At the heart of its proof is the dual relation in Lemma 3.4.1, which says that the two independent processes $\xi^0 = (\xi^0_t)_{t\geq 0}$ and $\eta^0 = (\eta^0_t)_{t\geq 0}$ of independent ransom walks *with no interaction* are dual to each other with respect to the function $\frac{A(\xi,\eta)}{\alpha_\varepsilon(\xi)}$, where $\xi,\, \eta \in E^\varepsilon$. Such kind of dual formula for the whole grid $\mathbb{Z}^d$ appeared in [7] and in Chapter 15 of [14].

Lemma 3.4.1. *(Duality for independent processes) Let* $\xi^0 = (\xi^0_t)_{t\geq 0}$ *and* $\eta^0 = (\eta^0_t)_{t\geq 0}$ *be independent continuous time Markov processes on* E^ε *with generator* $\mathfrak{L}^\varepsilon_0$ *defined in Definition 3.1.1. Then we have*

$$\mathbb{E}\left[\frac{A(\xi^0_t,\eta^0_0)}{\alpha_\varepsilon(\xi^0_t)}\right] = \mathbb{E}\left[\frac{A(\xi^0_0,\eta^0_t)}{\alpha_\varepsilon(\xi^0_0)}\right] \quad \textit{for every } t\geq 0. \tag{3.4.1}$$

Proof Assume $\xi^0_0 \in \Omega^\varepsilon_{n,m}$ and $\eta^0_0 \in \Omega^\varepsilon_{N,M}$. Then we have $\xi^0_t \in \Omega^\varepsilon_{n,m}$ and $\eta^0_t \in \Omega^\varepsilon_{N,M}$ for all $t\geq 0$. Without loss of generality, we may assume $N\geq n\geq 1$ and $M\geq m\geq 1$ as otherwise both sides inside expectations of (3.4.1) are zero by the definition of $A(\xi,\eta)$.

Denote U the map that sends $(\vec{r},\vec{s}) \in (D^\varepsilon_+)^n \times (D^\varepsilon_-)^m$ to $(\sum_i \delta_{r_i}, \sum_j \delta_{s_j}) \in \Omega^\varepsilon_{n,m}$ for any (n,m). We first focus on D_+ in Step 1 and Step 2 below.

Step 1. For any $\vec{r} \in (D^\varepsilon_+)^n$ and $\eta^+ \in \Omega^\varepsilon_{N,0}$, fix some $\vec{x}^+ = (x^+_1, \ldots, x^+_N) \in U^{-1}(\eta^+)$. Then by the definition (3.3.2) of A,

$$A^+(\vec{r}, \eta^+) = \sharp \left\{ \vec{i} : \vec{x}^+_{\vec{i}} = \vec{r} \right\}, \tag{3.4.2}$$

where n-tuples $\vec{i} := (i_1, \cdots, i_n)$ consist of distinct positive integers in the set $\{1, 2, \cdots, N\}$, $\vec{x}^+_{\vec{i}} := (x^+_{i_i}, \cdots, x^+_{i_n})$ and $\sharp S$ denotes the number of elements in the finite set S.

Step 2. Denote by $\mathbb{P}^{\eta^+}_0$ the law of the *unlabeled* process $(\eta^0_t)_{t\geq 0}$ starting from $\eta^+ \in \Omega^\varepsilon_{N,0}$ and has generator $\mathfrak{L}^\varepsilon_0$. Let $\vec{x}^+ = (x^+_1, \cdots, x^+_N) \in U^{-1}(\eta^+)$, and $\vec{X}^{+,\varepsilon}_t := (X^{+,\varepsilon}_1(t), \cdots, X^{+,\varepsilon}_N(t))$ be independent CTRWs in D^ε_+ starting from $\vec{x}$, whose law will be denoted as $\mathbb{P}^{\vec{x}^+}$. Then by (3.4.2), we have

$$\mathbb{E}^{\eta^+}_0[A(\vec{r}, \eta^0_t)] = \mathbb{E}[\,\sharp\,\{n\text{-tuples } \vec{i}:\ \vec{X}^{+,\varepsilon}_{\vec{i}}(t) = \vec{r}\}\,] = \sum_{\vec{i}:\ n\text{-tuples}} \mathbb{P}^{\vec{x}^+_{\vec{i}}}(\vec{X}^{+,\varepsilon}_{\vec{i}}(t) = \vec{r}). \tag{3.4.3}$$

where $\mathbb{P}^{\vec{x}^+_{\vec{i}}}$ is the law of $\{\vec{X}^{+,\varepsilon}_{\vec{i}}(t); t \geq 0\}$. Denote by $p^\varepsilon(\theta, \vec{z}.\vec{w})$ the transition density of n independent CTRWs in D^ε_+. By Chapman-Kolmogorov equation, we have for any $\theta \in [0, t]$,

$$\mathbb{P}^{\vec{x}^+_{\vec{i}}}(\vec{X}^{+,\varepsilon}(t) = \vec{r}\,) = \sum_{\vec{z}\in(D^\varepsilon_+)^n} p^\varepsilon(\theta, \vec{x}^+_{\vec{i}}, \vec{z}) p^\varepsilon(t-\theta, \vec{z}, \vec{r})\, m(\vec{z})\, m(\vec{r}).$$

Putting this into (3.4.3), we have

$$\begin{aligned}
\mathbb{E}^{\eta^+}_0[A(\vec{r}, \eta^0_t)] &= m_\varepsilon(\vec{r}) \sum_{\vec{z}} \sum_{\vec{i}} \mathbb{P}^{\vec{x}^{+,\varepsilon}_{\vec{i}}}(\vec{X}^+_{\vec{i}}(\theta) = \vec{z}\,)\, p^\varepsilon(t-\theta, \vec{z}, \vec{r}) \\
&= m_\varepsilon(\vec{r}) \sum_{\vec{z}} \mathbb{E}^{\eta^+}_0[A(\vec{z}, \eta^0_\theta)]\, p^\varepsilon(t-\theta, \vec{z}, \vec{r}) \quad \text{by (3.4.3) again} \\
&= m_\varepsilon(\vec{r}) \sum_{\vec{z}} \mathbb{E}^{\eta^+}_0[A(\vec{z}, \eta^0_\theta)]\, p^\varepsilon(t-\theta, \vec{r}, \vec{z}) \quad \text{by symmetry of } p^\varepsilon \\
&= m_\varepsilon(\vec{r})\, \mathbb{E}\left[\frac{A(\vec{Y}^{+,\varepsilon}_{t-\theta}, \eta^0_\theta)}{m_\varepsilon(\vec{Y}^{+,\varepsilon}_{t-\theta})}\right],
\end{aligned} \tag{3.4.4}$$

where $\mathbb{E}$ is the expectation corresponding the probability measure under which the coordinate processes of $\{\vec{Y}_t^{+,\varepsilon}; t \geq 0\}$ are independent CTRWs with $\vec{Y}_0^{+,\varepsilon} = \vec{r}$ and are independent of $(\eta_t^0)_{t\geq 0}$.

Step 3. Now we work on $D_+ \times D_-$. For any $(\vec{r}, \vec{s}) \in (D_+^\varepsilon)^n \times (D_-^\varepsilon)^m$ and $\eta = (\eta^+, \eta^-) \in \Omega_{N,M}^\varepsilon$, take $\vec{x} = (x_1^+, \cdots, x_N^+, x_1^-, \cdots, x_M^-) \in U^{-1}(\eta)$. As in step 1, we have

$$A((\vec{r}, \vec{s}), \eta) = \sharp \left\{ (\vec{i}, \vec{j}) : (\vec{x}_{\vec{i}}^+, \vec{x}_{\vec{j}}^-) = (\vec{r}, \vec{s}) \right\}, \tag{3.4.5}$$

where $\vec{i}$ runs over all n-tuples $\vec{i} := (i_1, \cdots, i_n)$ consisting of distinct positive integers in the set $\{1, 2, \cdots, N\}$, and $\vec{j}$ over all m-tuples $\vec{j} := (j_1, \cdots, j_m)$ consisting of distinct positive integers in the set $\{1, 2, \cdots, M\}$.

Denote by $\mathbb{P}_0^\eta$ the law of the *unlabeled* process $(\eta_t^0)_{t\geq 0}$ starting from $\eta \in \Omega_{N,M}^\varepsilon$ and has generator $\mathfrak{L}_0^\varepsilon$. Since all processes on D_+^ε are independent of those on D_-^ε, we can proceed as in step 2 (via (3.4.5)) to obtain

$$\mathbb{E}_0^\eta[A((\vec{r}, \vec{s}), \eta_t^0)] = m_\varepsilon(\vec{r}, \vec{s})\, \mathbb{E}\left[\frac{A(\vec{Y}_{t-\theta}^\varepsilon, \eta_\theta^0)}{m_\varepsilon(\vec{Y}_{t-\theta}^\varepsilon)} \right] \quad \text{for } \theta \in [0, t], \tag{3.4.6}$$

where $\vec{Y}^\varepsilon := (Y_1^{+,\varepsilon}, \cdots, Y_n^{+,\varepsilon}, Y_1^{-,\varepsilon}, \cdots, Y_m^{-,\varepsilon})$ is independent of η^0 with $\vec{Y}_0^\varepsilon = (\vec{r}, \vec{s})$, and its components are mutually independent CTRWs on $D_\pm^\varepsilon$, respectively. The proof is now complete by taking $\theta = 0$. $\square$

We now formulate the discrete integral equations that we need. Recall the definition of $\mathfrak{K}^\varepsilon$ from (3.1.4) and the definition of $\gamma^\varepsilon((\vec{r}, \vec{s}), t)$ from (3.3.3).

Lemma 3.4.2. *(Discrete integral equation for γ^ε) For any $\varepsilon > 0$, $t > 0$, $(\vec{r}, \vec{s}) \in (D_+^\varepsilon)^n \times (D_-^\varepsilon)^m$,* *non-negative integers n, m and initial distribution η_0, we have*

$$\gamma^\varepsilon((\vec{r},\vec{s}),t) = \sum_{(\vec{r'},\vec{s'})} \gamma^\varepsilon((\vec{r'},\vec{s'}),0)\, p^\varepsilon(t,(\vec{r},\vec{s}),(\vec{r'},\vec{s'}))\, m(\vec{r'},\vec{s'}) + \int_0^t \sum_{(\vec{r'},\vec{s'})} p^\varepsilon(t-s,(\vec{r},\vec{s}),(\vec{r'},\vec{s'}))\, \mathbb{E}[\mathfrak{K}^\varepsilon A((\vec{r'},\vec{s'}),\eta_s)]\, \varepsilon^{d(n+m)}\, ds, \tag{3.4.7}$$

where $\mathfrak{K}^\varepsilon$ acts on the η-variable of $A((\vec{r},\vec{s}),\eta)$.

Proof Starting from (3.4.1), we can obtain Lemma 3.4.2 by 'integration by parts' as follows.

Let $\mathbb{P}_{(\xi^0)}$ and $\mathbb{P}_{(\eta^0)}$ be the laws of ξ^0 and η^0 respectively. (3.4.1) is equivalent to saying that for any ξ and η, we have

$$\mathbb{E}_{(\xi^0)}\left[\frac{A(\xi^0_w,\eta)}{\alpha_\varepsilon(\xi^0_w)}\,\Big|\, \xi^0_0=\xi\right] = \mathbb{E}_{(\eta^0)}\left[\frac{A(\xi,\eta^0_w)}{\alpha_\varepsilon(\xi)}\,\Big|\, \eta^0_0=\eta\right] \quad \text{for every } w\geq 0. \tag{3.4.8}$$

Taking $w=t-s$, we see that (3.4.8) is in turn equivalent to

$$F^{(\xi)}_s(\eta) := P^{(\xi^0)}_{t-s}\left(\frac{A(\cdot,\eta)}{\alpha_\varepsilon(\cdot)}\right)(\xi) = P^{(\eta^0)}_{t-s}\left(\frac{A(\xi,\cdot)}{\alpha_\varepsilon(\xi)}\right)(\eta) =: G^{(\eta)}_s(\xi) \quad \text{for every } s\in[0,t] \text{ and } t\geq 0, \tag{3.4.9}$$

where $P^{(\xi^0)}_t$ and $P^{(\eta^0)}_t$ are the transition semigroup of ξ^0 and η^0, respectively, and they act on the ξ and η variables in $\frac{A(\xi,\eta)}{\alpha_\varepsilon(\xi)}$, respectively. Therefore, with $\mathfrak{L}^\varepsilon_0$ acting on the η variable, we have

$$\frac{\partial}{\partial s}\, F^{(\xi)}_s(\eta) = \frac{\partial}{\partial s}\, G^{(\eta)}_s(\xi) = -\mathfrak{L}^\varepsilon_0\, P^{(\eta^0)}_{t-s}\left(\frac{A(\xi,\cdot)}{\alpha_\varepsilon(\xi)}\right)(\eta) = -\mathfrak{L}^\varepsilon_0\, F^{(\xi)}_s(\eta). \tag{3.4.10}$$

Recall that η_t is the configuration process of our interacting system with generator $\mathfrak{L}^\varepsilon_0+\mathfrak{K}^\varepsilon$ (see Definition 3.1.1). Fix ξ and consider the function $(s,\eta)\mapsto F_s(\eta):=F^{(\xi)}_s(\eta)$. We have

$$M_s := F_s(\eta_s) - F_0(\eta_0) - \int_0^s \left(\frac{\partial F_r}{\partial r} + \mathfrak{L}^\varepsilon_0 F_r + \mathfrak{K}^\varepsilon F_r\right)(\eta_r)\, dr$$

is a $\mathcal{F}^\eta_s$-martingale for $s\in[0,t]$. By (3.4.10) and the fact that $E^\eta[M_t]=E^\eta[M_0]=0$, where $\mathbb{P}^\eta$

is the law of $(\eta_t)_{t\geq 0}$ starting from η, we have

$$0 = \mathbb{E}^\eta\left[\frac{A(\xi,\eta_t)}{\alpha_\varepsilon(\xi)}\right] - P_t^{(\xi^0)}\left(\frac{A(\cdot,\eta)}{\alpha_\varepsilon(\cdot)}\right)(\xi) - \int_0^t \mathbb{E}^\eta\left[\mathfrak{K}^\varepsilon\, P_{t-r}^{(\xi^0)}\left(\frac{A(\cdot,\eta_r)}{\alpha_\varepsilon(\cdot)}\right)(\xi)\right] dr$$

for all ξ and η. This is equivalent to the stated equation in the lemma.

□

It is clear that $\mathfrak{K}^\varepsilon A(\xi,\eta) \leq 0$. Hence, as an immediate consequence of Lemma 3.4.2, we have the following comparison result:

$$\gamma^\varepsilon((\vec{r},\vec{s}),t) \leq \sum_{(\vec{r'},\vec{s'})} \gamma^\varepsilon((\vec{r'},\vec{s'}),0)\, p^\varepsilon(t,(\vec{r},\vec{s}),(\vec{r'},\vec{s'}))\, m(\vec{r'},\vec{s'}) \tag{3.4.11}$$

for all $t > 0$ and $(\vec{r},\vec{s}) \in (D_+^\varepsilon)^n \times (D_-^\varepsilon)^m$.

3.4.2 Annihilation operator

For any $\xi = (\xi^+,\xi^-) \in E^\varepsilon$, we let $\xi^+_{(x)} = \xi^+(x)\mathbf{1}_x$, the element that has only $\xi^+(x)$ number of particles at x, and none elsewhere. Similarly, we denote $\xi^-(y)\mathbf{1}_y$ by $\xi^-_{(y)}$. Set $\xi_{(x,y)} = (\xi^+(x)\mathbf{1}_x, \xi^-(y)\mathbf{1}_y)$, the element that has only $\xi^+(x)$ number of particles at x, $\xi^-(y)$ number of particles at y, and none elsewhere.

Lemma 3.4.3. *Let $\mathfrak{K}^\varepsilon$ be the operator defined in* (3.1.4) *and acts on the η-variable of $A(\xi,\eta)$. Then*

$$\mathfrak{K}^\varepsilon A(\xi,\eta) = \sum_{z\in I^\varepsilon} A(\xi - \xi_{(z_+,z_-)},\, \eta)\cdot \mathfrak{K}^\varepsilon A(\xi_{(z_+,z_-)},\eta). \tag{3.4.12}$$

Moreover, if $\xi \in \Xi := \{\xi :\ \xi^{\pm}(z_{\pm}) \le 1$ *for every* $z \in I^{\varepsilon}\}$*, then*

$$\begin{aligned}
\mathfrak{K}^{\varepsilon} A(\xi,\eta) \;=\; & -\frac{\lambda}{\varepsilon} \sum_{z \in I^{\varepsilon}:\, \xi^{+}(z_{+})=1} \Psi_{\varepsilon}(z)\, A(\xi + \mathbf{1}_{(0,z_{-})}, \eta) && (3.4.13)\\
& -\frac{\lambda}{\varepsilon} \sum_{z \in I^{\varepsilon}:\, \xi^{-}(z_{-})=1} \Psi_{\varepsilon}(z)\, A(\xi + \mathbf{1}_{(z_{+},0)}, \eta) && (3.4.14)\\
& -\frac{\lambda}{\varepsilon} \sum_{z \in I^{\varepsilon}:\, \xi(z_{+},z_{-})=(1,1)} \Psi_{\varepsilon}(z)\, A(\xi, \eta). && (3.4.15)
\end{aligned}$$

Proof Observe that $A(\xi - \xi_{(x,y)}, \eta)\, A(\xi_{(x,y)}, \eta) = A(\xi, \eta)$. Consequently

$$\begin{aligned}
& \frac{\lambda}{\varepsilon}\Psi_{\varepsilon}(z)\eta^{+}(z_{+})\eta^{-}(z_{-})\left(A(\xi,\, \eta - \mathbf{1}_{(z_{+},z_{-})}) - A(\xi,\eta)\right)\\
= \;& \frac{\lambda}{\varepsilon}\Psi_{\varepsilon}(z)\eta^{+}(z_{+})\eta^{-}(z_{-})A(\xi - \xi_{(z_{+},z_{-})}, \eta)\left(A(\xi_{(z_{+},z_{-})}, \eta - \mathbf{1}_{(z_{+},z_{-})}) - A(\xi_{(z_{+},z_{-})}, \eta)\right)\\
= \;& A(\xi - \xi_{(z_{+},z_{-})}, \eta)\mathfrak{K}^{\varepsilon} A(\xi_{(z_{+},z_{-})}, \eta).
\end{aligned}$$

Thus (3.4.12) holds. On other hand,

$$\mathfrak{K}^{\varepsilon} A(\xi_{(z_{+},z_{-})}, \eta) = \Psi_{\varepsilon}(z)\, \frac{\lambda}{\varepsilon}\, \eta^{+}(z_{+})\eta^{-}(z_{-}) \times \begin{cases} -1, & \text{if } \xi(z_{+},z_{-}) = (1,0) \text{ or } (0,1)\\ 1 - \eta^{+}(z_{+}) - \eta^{-}(z_{-}), & \text{if } \xi(z_{+},z_{-}) = (1,1) \end{cases}.$$

Observe also that for $x \in D^{\varepsilon}_{+}$ and $y \in D^{\varepsilon}_{-}$,

$$A\left(\xi - \xi_{(x,y)},\, \eta\right)\, \eta^{+}(x)\eta^{-}(y) = A\left(\xi - \xi_{(x,y)} + \mathbf{1}_{(x,y)},\, \eta\right)$$

and

$$\begin{aligned}
& A(\xi - \xi_{(x,y)})\, \eta^{+}(x)^{2}\, \eta^{-}(y)\\
= \;& A\left(\xi - \xi_{(x,y)}, \eta\right)\, \left(\eta^{+}(x)^{2} - \eta^{+}(x) + \eta^{+}(x)\right)\, \eta^{-}(y)\\
= \;& A(\xi - \xi_{(x,y)}, \eta)\, A(2\mathbf{1}_{x}, \eta^{+})\, \eta^{-}(y) + A(\xi - \xi_{(x,y)}, \eta)\, \eta^{+}(x)\, \eta^{-}(y)\\
= \;& A(\xi - \xi_{(x,y)} + 2\mathbf{1}_{(x,0)} + \mathbf{1}_{(0,y)}, \eta) + A(\xi, \eta).
\end{aligned}$$

Similarly,

$$A(\xi - \xi_{(x,y)}, \eta)\, \eta^+(x)\eta^-(y)^2 = A(\xi - \xi_{(x,y)} + \mathbf{1}_{(x,0)} + 2\mathbf{1}_{(0,y)}, \eta) + A(\xi, \eta).$$

From the above calculations and (3.4.12), we see that for $\xi \in \Xi$,

$$\begin{aligned}
\mathfrak{K}^\varepsilon A(\xi,\eta) \quad = \quad & -\frac{\lambda}{\varepsilon} \sum_{z\in I^\varepsilon:\, \xi(z_+,z_-)=(1,0)} \Psi_\varepsilon(z)\, A(\xi + \mathbf{1}_{(0,z_-)},\, \eta) \\
& -\frac{\lambda}{\varepsilon} \sum_{z\in I^\varepsilon:\, \xi(z_+,z_-)=(0,1)} \Psi_\varepsilon(z)\, A(\xi + \mathbf{1}_{(z_+,0)},\, \eta) \\
& -\frac{\lambda}{\varepsilon} \sum_{z\in I^\varepsilon:\, \xi(z_+,z_-)=(1,1)} \Psi_\varepsilon(z) \left(A(\xi,\eta) + A(\xi + \mathbf{1}_{(0,z_-)}, \eta) + A(\xi + \mathbf{1}_{(z_+,0)}, \eta)\right),
\end{aligned}$$

which gives the desired formula. □

3.4.3 Uniform bound and equi-continuity for correlation functions

We extend to define $\gamma^{\varepsilon,(n,m)}(\cdot,t)$ continuously on $\overline{D}_+^n \times \overline{D}_-^m$ while preserving the supremum and the infinmum in each small ε-cube. We can accomplish this by the interpolation described in [3] or [72], or by a sequence of harmonic extensions along simplexes with increasing dimensions (described in the proof of Theorem 2.2.8 in Chapter 2). Recall that the definition of $\gamma^{\varepsilon,(n,m)}(\cdot,t)$ depends on the initial configuration η_0 of the interacting random walks (see Definition 3.3.2), which satisfies the two conditions in Theorem 3.3.1.

Theorem 3.4.4. *There exists $\varepsilon_0 > 0$ such that for any $(n,m) \in \mathbb{N}\times\mathbb{N}$, the family of functions $\{\gamma^\varepsilon((\vec{r},\vec{s}),t)\}_{\varepsilon\in(0,\varepsilon_0)}$ is uniformly bounded and equi-continuous on $\overline{D}_+^n \times \overline{D}_-^m \times (0,\infty)$, which is uniform in the initial configuration η_0 that satisfies the two conditions in Theorem 3.3.1.*

Proof We first prove uniform boundedness. By (3.4.11) and the Gaussian upper bound in

Theorem 2.2.4, we have

$$\begin{aligned}\gamma^\varepsilon((\vec{r},\vec{s}),t) &\leq \sum_{(\vec{r'},\vec{s'})} \gamma^\varepsilon((\vec{r'},\vec{s'}),0)\, p^\varepsilon(t,(\vec{r},\vec{s}),(\vec{r'},\vec{s'}))\, m(\vec{r'},\vec{s'}) \\ &\leq \left(\frac{C}{t^{d/2}}\right)^{n+m} \sum_{(\vec{r'},\vec{s'})\in \overline{D}_+^n\times \overline{D}_-^m} A\left((\vec{r'},\vec{s'}),\eta_0\right)\varepsilon^{d(n+m)} \quad \text{whenever } \varepsilon\in(0,\varepsilon_0).\end{aligned}$$

Since the initial distribution $\eta_0 = (\eta_0^+,\eta_0^-)$ has the property that $\sum_{x\in D_+^\varepsilon}\eta_0^+(x) = \sum_{y\in D_-^\varepsilon}\eta_0^-(y) = \varepsilon^{-d}$, we have

$$\begin{aligned}&\sum_{(\vec{r},\vec{s})\in \overline{D}_+^n\times \overline{D}_-^m} A((\vec{r},\vec{s}),\eta_0)\varepsilon^{d(n+m)} \\ =& \left(\sum_{\vec{r}\in\overline{D}_+^n} A^+(\vec{r},\eta_0^+)\right)\left(\sum_{\vec{s}\in\overline{D}_-^m} A^-(\vec{s},\eta_0^-)\right)\varepsilon^{d(n+m)} \\ \leq& \left(\sum_{\vec{r}\in\overline{D}_+^n}\prod_{i=1}^n \eta_0^+(r_i)\right)\left(\sum_{\vec{s}\in\overline{D}_-^m}\prod_{j=1}^m \eta_0^-(s_j)\right)\varepsilon^{d(n+m)} \qquad \text{since } A_k^n\leq n^k \\ \leq& \prod_{i=1}^n\left(\sum_{r_i\in\overline{D}_+}\eta_0^+(r_i)\varepsilon^d\right)\prod_{j=1}^m\left(\sum_{s_j\in\overline{D}_-}\eta_0^-(s_j)\varepsilon^d\right) = 1. \end{aligned} \tag{3.4.16}$$

Thus there exist $\varepsilon_0 = \varepsilon_0(d,D,\rho)$ and $C = C(d,D,\rho)>0$ such that for all $t\in(0,\infty)$, $(n,m)\in\mathbb{N}\times\mathbb{N}$ and $\varepsilon\in(0,\varepsilon_0)$,

$$\sup_{\varepsilon\in(0,\varepsilon_0)}\sup_{\xi\in\Omega_{n,m}^\varepsilon}\gamma^\varepsilon(\xi,t)\leq\left(\frac{C}{t^{d/2}}\right)^{n+m}. \tag{3.4.17}$$

We next show that both terms on the right hand side of (3.4.7) are equi-continuous. Recall that we can rewrite the equation (3.4.7) as

$$\gamma^\varepsilon((\vec{r},\vec{s}),t) = F^\varepsilon((\vec{r},\vec{s}),t) + G^\varepsilon((\vec{r},\vec{s}),t),$$

where

$$
\begin{aligned}
F^\varepsilon((\vec{r},\vec{s}),t) &:= \sum_{(\vec{r'},\vec{s'})} \gamma^\varepsilon((\vec{r'},\vec{s'}),0)\, p^\varepsilon(t,(\vec{r},\vec{s}),(\vec{r'},\vec{s'}))\, m(\vec{r'},\vec{s'}),\\
G^\varepsilon((\vec{r},\vec{s}),t) &:= \int_0^t \sum_{(\vec{r'},\vec{s'})} p^\varepsilon(t-s,(\vec{r},\vec{s}),(\vec{r'},\vec{s'}))\, \mathbb{E}[\mathfrak{K}^\varepsilon A((\vec{r'},\vec{s'}),\eta_s)]\, \varepsilon^{d(n+m)}\, ds.
\end{aligned}
$$

Now let $(\vec{r},\vec{s})$, $(\vec{p},\vec{q}) \in (D^\varepsilon_+)^n \times (D^\varepsilon_-)^m$ and $0 < t < \ell \leq \infty$. For the first term,

$$
\begin{aligned}
& \Big|F^\varepsilon((\vec{r},\vec{s}),t) - F^\varepsilon((\vec{p},\vec{q}),\ell)\Big| \\
= & \Big| \sum_{(\vec{r'},\vec{s'})} \Big(p^\varepsilon(t,(\vec{r},\vec{s}),(\vec{r'},\vec{s'})) - p^\varepsilon(\ell,(\vec{p},\vec{q}),(\vec{r'},\vec{s'}))\Big)\ \mathbb{E}[A((\vec{r'},\vec{s'}),\eta_0)]\varepsilon^{d(n+m)}\Big| \\
\leq & \left(\sup_{(\vec{r'},\vec{s'})} \big|p^\varepsilon(t,(\vec{r},\vec{s}),(\vec{r'},\vec{s'})) - p^\varepsilon(\ell,(\vec{p},\vec{q}),(\vec{r'},\vec{s'}))\big|\right) \mathbb{E}_0\left[\sum_{(\vec{r'},\vec{s'})} A((\vec{r'},\vec{s'}),\eta_0)\varepsilon^{d(n+m)}\right] \\
\leq & \sup_{(\vec{r'},\vec{s'})} \big|p^\varepsilon(t,(\vec{r},\vec{s}),(\vec{r'},\vec{s'})) - p^\varepsilon(\ell,(\vec{p},\vec{q}),(\vec{r'},\vec{s'}))\big|,
\end{aligned}
$$

where we have used (3.4.16) in the last line. By the uniform Hölder continuity of $p^\varepsilon(t,(\vec{r},\vec{s}),(\vec{r'},\vec{s'}))$ (see Theorem 2.2.19 below) and the fact that $p^\pm(t,x,y) \in C((0,\infty)\times \overline{D}_\pm \times \overline{D}_\pm)$, we see that $\{F^\varepsilon\}$ is equi-continuous at $((\vec{r},\vec{s}),t)$. For the second term, note that

$$
G^\varepsilon((\vec{p},\vec{q}),\ell) - G^\varepsilon((\vec{r},\vec{s}),t) = \int_t^\ell H^{(1)}(s)\,ds + \int_0^t H^{(2)}(s)\,ds, \tag{3.4.18}
$$

where

$$
H^{(1)}(s) := \sum_{(\vec{r'},\vec{s'})} p^\varepsilon(\ell-s,(\vec{p},\vec{q}),(\vec{r'},\vec{s'}))\, \mathbb{E}[\mathfrak{K}^\varepsilon A((\vec{r'},\vec{s'}),\eta_s)]\, \varepsilon^{d(n+m)} \text{ and}
$$

$$
H^{(2)}(s) := \sum_{(\vec{r'},\vec{s'})} \Big[p^\varepsilon(\ell-s,(\vec{r},\vec{s}),(\vec{r'},\vec{s'})) - p^\varepsilon(t-s,(\vec{p},\vec{q}),(\vec{r'},\vec{s'}))\Big]\ \mathbb{E}[\mathfrak{K}^\varepsilon A((\vec{r'},\vec{s'}),\eta_s)]\, \varepsilon^{d(n+m)}.
$$

In the remaining, we will show that G^ε is equi-continuous. We first deal with $H^{(1)}$ in (3.4.18).

As in (3.4.16), we have

$$\sum_{(\vec{r'},\vec{s'})} p^{\varepsilon}(\theta_1,(\vec{p},\vec{q}),(\vec{r'},\vec{s'}))\, A((\vec{r'},\vec{s'}),\eta_{\theta_2})\,\varepsilon^{d(n+m)} \tag{3.4.19}$$
$$\le \prod_{i=1}^{n}\left(\sum_{r'_i} p^{\varepsilon}(\theta_1,p_i,r'_i)\eta^{+}_{\theta_2}(r'_i)\varepsilon^{d}\right)\prod_{j=1}^{m}\left(\sum_{s'_j} p^{\varepsilon}(\theta_1,q_j,s'_j)\eta^{-}_{\theta_2}(s'_j)\varepsilon^{d}\right).$$

On other hand, using (3.4.11), the Chapman Kolmogorov equation and assumption (ii) for η_0, in this order, we have

$$\sup_{\theta_1,\,\theta_2>0}\ \sup_{a\in D^{\varepsilon}_{+}} \mathbb{E}\left[\left(\sum_{x\in D^{\varepsilon}_{+}} p^{\varepsilon}(\theta_1,a,x)\eta^{+}_{\theta_2}(x)\varepsilon^{d}\right)^{2}\right]\le C \tag{3.4.20}$$

for large enough N, where $C>0$ is a constant.

$$\sum_{(\vec{r'},\vec{s'})} p^{\varepsilon}(\theta_1,(\vec{p},\vec{q}),(\vec{r'},\vec{s'}))\left|\mathbb{E}\left[\mathfrak{K}^{\varepsilon}A((\vec{r'},\vec{s'}),\eta_{\theta_2})\right]\right|\varepsilon^{d(n+m)}$$
$$\le \mathbb{E}\left[\frac{\lambda}{\varepsilon}\sum_{z\in I}\eta^{+}_{\theta_2}(z_+)\eta^{-}_{\theta_2}(z_-)\sum_{(\vec{r'},\vec{s'})} p^{\varepsilon}(\theta_1,(\vec{p},\vec{q}),(\vec{r'},\vec{s'}))\,2A((\vec{r'},\vec{s'}),\eta_{\theta_2})\,\varepsilon^{d(n+m)}\right]$$
$$\le 2\mathbb{E}\left[\langle 1,\,J^{N}_{\theta_2}\rangle\prod_{i=1}^{n}\left(\sum_{r'_i} p^{\varepsilon}(\theta_1,p_i,r'_i)\eta^{+}_{\theta_2}(r'_i)\varepsilon^{d}\right)\prod_{j=1}^{m}\left(\sum_{s'_j} p^{\varepsilon}(\theta_1,q_j,s'_j)\eta^{-}_{\theta_2}(s'_j)\varepsilon^{d}\right)\right]$$
$$\le C\quad\text{uniformly for } \theta_1>0,\ \theta_2>0,\ (\vec{p},\vec{q})\in(D^{\varepsilon}_{+})^{n}\times(D^{\varepsilon}_{-})^{m}\text{ and }\varepsilon>0\text{ small enough.}$$

We have used (3.4.19) for the second inequality. The last inequality follows from Hölder's inequality, (3.3.7) and (3.4.20). Therefore for any (n,m),

$$\sup_{\theta_1,\,\theta_2>0}\ \sup_{(\vec{p},\vec{q})\in(D^{\varepsilon}_{+})^{n}\times(D^{\varepsilon}_{-})^{m}}\ \sum_{(\vec{r'},\vec{s'})} p^{\varepsilon}(\theta_1,(\vec{p},\vec{q}),(\vec{r'},\vec{s'}))\left|\mathbb{E}\left[\mathfrak{K}^{\varepsilon}A((\vec{r'},\vec{s'}),\eta_{\theta_2})\right]\right|\varepsilon^{d(n+m)}\le C \tag{3.4.21}$$

for large enough N, where $C>0$ is a constant. Hence $\int_t^{\ell}|H^{(1)}(s)|\,ds\le C(\ell-t)\to 0$ as $\ell\to t$, uniformly for $(\vec{p},\vec{q})$, $s \in (t,\ell)$ and ε small enough. Finally, we deal with $H^{(2)}$. For any $h \in (0,t)$, we have

$$\left|\int_0^t H^{(2)}(s)\,ds\right| \le \int_0^{t-h} |H^{(2)}(s)|ds + \int_{t-h}^t |H^{(2)}(s)|ds.$$

By (3.4.21), we have $\int_{t-h}^t |H^{(2)}(s)|ds \le C\,h$. By the Hölder continuity of p^ε (cf. Theorem 2.2.19),

$$\begin{aligned}
\int_0^{t-h} |H^{(2)}(s)|ds \;&\le\; \int_0^{t-h} \sup_{(\vec{r'},\vec{s'})} \left|p^\varepsilon(\ell - s,(\vec{r},\vec{s}),(\vec{r'},\vec{s'})) - p^\varepsilon(t-s,(\vec{p},\vec{q}),(\vec{r'},\vec{s'}))\right| \\
&\qquad \mathbb{E}\left[\frac{\lambda}{\varepsilon}\sum_{z\in I}\eta_s^+(z_+)\eta_s^-(z_-)\sum_{(\vec{r'},\vec{s'})} 2A((\vec{r'},\vec{s'}),\eta_s)\,\varepsilon^{d(n+m)}\right] ds \\
&\le\; (t-h)\,C\frac{|\ell-t|^{\sigma_1} + \|(\vec{r},\vec{s})-(\vec{p},\vec{q})\|^{\sigma_2}}{h^{\sigma_3}} \quad \text{for sufficiently small } \varepsilon > 0,
\end{aligned}$$

where σ_i $(i=1,2,3)$ are positive constants. Since $h \in (0,t)$ is arbitrary, we see that $\left|\int_0^t H^{(2)}(s)\,ds\right| \to 0$ as $|\ell - t| + \|(\vec{r},\vec{s}) - (\vec{p},\vec{q})\| \to 0$, uniformly for small enough $\varepsilon > 0$. Hence G^ε is equi-continuous at an arbitrary $((\vec{r},\vec{s}),t) \in (D_+^\varepsilon)^n \times (D_-^\varepsilon)^m \times (0,\infty)$. □

From Theorem 3.4.4 and a diagonal selection argument, it follows that for any sequence $\varepsilon_k \to 0$ there is a subsequence along which γ^ε converges on $\overline{D}_+^n \times \overline{D}_-^m \times (0,T)$, uniformly on the compacts, to some $\gamma^{(n,m)} \in C(\overline{D}_+^n \times \overline{D}_-^m \times (0,T))$, for every $(n,m) \in \mathbb{N}\times\mathbb{N}$. Our goal is to show that

$$\gamma_t^{(n,m)}(\vec{r},\vec{s}) = \prod_{i=1}^n u_+(t,r_i)\prod_{j=1}^m u_-(t,s_j).$$

We will achieve this by first showing that both $\Phi_t^{(n,m)} := \prod_{i=1}^n u_+(t,r_i)\prod_{j=1}^m u_-(t,s_j)$ and $\Gamma = \{\gamma^{(n,m)}\}$ satisfy the same an infinite hierarchy of equations, and then establishing uniqueness of the hierarchy.

3.4.4 Limiting hierarchy

Note that $D_+^n \times D_-^m$ is a bounded Lipschitz domain in $\mathbb{R}^{(n+m)d}$, and that the boundary $\partial(D_+^n \times D_-^m)$ contains the disjoint union $\cup_{i=1}^n \partial_+^i \bigcup \cup_{j=1}^m \partial_-^j$ where

$$\partial_+^i := \left(D_+ \times \cdots \times \overset{i^{th}}{(\partial D_+ \cap I)} \times \cdots \times D_+ \right) \times D_-^m, \tag{3.4.22}$$

$$\partial_-^j := D_+^n \times \left(D_- \times \cdots \times \overset{j^{th}}{(\partial D_- \cap I)} \times \cdots \times D_- \right). \tag{3.4.23}$$

We define the function $\rho = \rho_{(n,m)} : D_+^n \times D_-^m \to \mathbb{R}$ by $\rho(\vec{r}, \vec{s}) := \prod_{i=1}^n \rho_+(r_i) \prod_{j=1}^m \rho_-(s_j)$. We also denote $p(t, (\vec{r}, \vec{s}), (\vec{r'}, \vec{s'})) := \prod_{i=1}^n p^+(t, r_i, r_i') \prod_{j=1}^m p^-(t, s_j, s_j')$, where $p^{\pm}$ is the transition density of the reflected diffusion $X^{\pm}$ on $\overline{D}_{\pm}$ with respect to the measure $\rho_{\pm}(x)dx$. We now characterize the subsequential limits of $\{\gamma^{\varepsilon}\}_{\varepsilon > 0}$.

Theorem 3.4.5. *Let η_0^{ε} be a sequence of initial configurations that satisfy the two conditions in Theorem 3.3.1, with $\varepsilon = N^{-1/d}$; that is, their corresponding empirical measures $\mathfrak{X}_0^{N,\pm}$ converges weakly to $u_0^{\pm}(z)dz$ in $M_{\leq 1}(\overline{D}_{\pm})$ for some $u_0^{\pm} \in C(\overline{D}_{\pm})$ and*

$$\overline{\lim_{\varepsilon \to 0}} \sup_{z \in D_{\pm}^{\varepsilon}} \mathbb{E}\left[\eta_0^{\varepsilon, \pm}(z)^2 \right] < \infty. \tag{3.4.24}$$

Denote by $\Gamma^{\varepsilon} = \{\gamma^{\varepsilon, (n,m)}; t \geq 0, n, m \in \mathbb{N}\}$ the correlation functions for the interacting random walks with initial configuration η_0^{ε}. Let $\Gamma = \{\gamma_t^{(n,m)}; t \geq 0, n, m \in \mathbb{N}\}$ be any subsequential limit (as $\varepsilon \to 0$) of $\Gamma^{\varepsilon} = \{\gamma^{\varepsilon, (n,m)}; t \geq 0, n, m \in \mathbb{N}\}$. Then the following infinite system of hierarchical *equations holds:*

$$\begin{aligned}\gamma_t^{(n,m)}(\vec{r},\vec{s}) &= \int_{D_+^n\times D_-^m} \Phi^{(n,m)}(\vec{a},\vec{b})\, p(t,(\vec{r},\vec{s}),(\vec{a},\vec{b}))\,\rho(\vec{a},\vec{b})\,d(\vec{a},\vec{b}) \\ &\quad -\frac{\lambda}{2}\int_0^t \left(\sum_{i=1}^n \int_{\partial_+^i} \gamma_\theta^{(n,m+1)}(\vec{a},(\vec{b},a_i))\,p(t-\theta,(\vec{r},\vec{s}),(\vec{a},\vec{b}))\,\frac{\rho(\vec{a},\vec{b})}{\rho_+(a_i)}\,d\sigma_{(n,m)}(\vec{a},\vec{b})\right. \\ &\qquad \left. +\sum_{j=1}^m \int_{\partial_-^j} \gamma_\theta^{(n+1,m)}((\vec{a},b_j),\vec{b}))\,p(t-\theta,(\vec{r},\vec{s}),(\vec{a},\vec{b}))\,\frac{\rho(\vec{a},\vec{b})}{\rho_-(b_j)}\,d\sigma_{(n,m)}(\vec{a},\vec{b})\right) d\theta, \end{aligned} \tag{3.4.25}$$

where $d(\vec{a},\vec{b})$ is the Lebesgue measure on $\mathbb{R}^{n+m}$, $\sigma_{(n,m)}$ is the surface measure of $\partial(D_+^n \times D_-^m)$ and $\Phi^{(n,m)}(\vec{a},\vec{b}) := \prod_{i=1}^n u_0^+(a_i)\prod_{j=1}^m u_0^-(b_j)$.

Remark 3.4.6. (i) The equation expresses $\gamma_t^{(n,m)}$ as an integral in time involving $\gamma^{(n,m+1)}$ and $\gamma^{(n+1,m)}$, thus forming a coupled chain of equations. In statistical physics, it is sometimes called the BBGKY hierarchy[3]. It describes the evolution of the limiting (n,m)-particle correlation functions and hence the dynamics of the particles.

(ii) By Proposition 2.1.6, (3.4.25) is equivalent to

$$\gamma_t^{(n,m)}(\vec{r},\vec{s}) = \mathbb{E}^{(\vec{r},\vec{s})}\left[\Phi^{(n,m)}(X_{(n,m)}(t)) - \lambda\int_0^t (\Upsilon\gamma_s)^{(n,m)}(X_{(n,m)}(t-s))\,dL_s^{(n,m)}\right]. \tag{3.4.26}$$

Here $L^{(n,m)}$ is boundary local time of $X_{(n,m)}$, the symmetric reflected diffusion on $D_+^n \times D_-^m$ corresponding to $(I_{(n+m)d\times(n+m)d},\, \rho_{(n,m)})$, and $(\Upsilon v)^{(n,m)}$ is a function on $\partial(D_+^n \times D_-^m)$ defined as

$$(\Upsilon v)^{(n,m)}(\vec{r},\vec{s}) := \begin{cases} v^{(n,m+1)}(\vec{r},(\vec{s},r_i))\frac{\rho_{(n,m)}(\vec{r},\vec{s})}{\rho_+(r_i)}, & \text{if } (\vec{r},\vec{s})\in\partial_+^i\,; \\ v^{(n+1,m)}((\vec{r},s_j),\vec{s}))\frac{\rho_{(n,m)}(\vec{r},\vec{s})}{\rho_-(s_j)}, & \text{if } (\vec{r},\vec{s})\in\partial_-^j\,; \\ 0, & \text{otherwise}\,. \end{cases}$$

Observe that the coordinate processes of $X_{(n,m)}$ consist of n independent copies of reflected

[3]BBGKY stand for N. N. Bogoliubov, Max Born, H. S. Green, J. G. Kirkwood, and J. Yvon, who derived this type of hierarchy of equations in the 1930s and 1940s in a series papers.

diffusions in $\overline{D}_+$ and m independent copies of reflected diffusions in $\overline{D}_-$.

(iii) It is easy to check by using (ii) and Proposition 2.1.8 that

$$\widetilde{\gamma}_t^{(n,m)}(\vec{r},\vec{s}) := \prod_{i=1}^n u_+(t,r_i) \prod_{j=1}^m u_-(t,s_j)$$

is a solution of (3.4.25), where (u_+, u_-) is the weak solution of the coupled PDEs (1.2.1)-(1.2.2) with initial value (u_0^+, u_0^-).

Proof of Theorem 3.4.5. Recall that $\Xi := \{\xi : \xi_\pm(z_\pm) \le 1 \text{ for every } z \in I^\varepsilon\}$. We can rewrite (3.4.7) as

$$\begin{aligned}
\gamma^\varepsilon((\vec{r},\vec{s}),t) &= \sum_{(\vec{a},\vec{b})} \gamma^\varepsilon((\vec{a},\vec{b}),0)\, p^\varepsilon(t,(\vec{r},\vec{s}),(\vec{a},\vec{b}))\, m(\vec{a},\vec{b}) \\
&\quad + \int_0^t \sum_{(\vec{a},\vec{b})\notin\Xi} p^\varepsilon(t-s,(\vec{r},\vec{s}),(\vec{a},\vec{b}))\, \mathbb{E}[\mathfrak{K}^\varepsilon A((\vec{a},\vec{b}),\eta_s)]\, \varepsilon^{d(n+m)}\, ds \\
&\quad + \int_0^t \sum_{(\vec{a},\vec{b})\in\Xi} p^\varepsilon(t-s,(\vec{r},\vec{s}),(\vec{a},\vec{b}))\, \mathbb{E}[\mathfrak{K}^\varepsilon A((\vec{a},\vec{b}),\eta_s)]\, \varepsilon^{d(n+m)}\, ds. \quad (3.4.27)
\end{aligned}$$

Fix any $(n,m) \in \mathbb{N}\times\mathbb{N}$, $t > 0$ and $(\vec{r},\vec{s}) \in (D_+^\varepsilon)^n \times (D_-^\varepsilon)^m$. By a simple counting argument and condition (3.4.24) for η_0^ε, we see that the first term in (3.4.27) equals

$$\begin{aligned}
&\mathbb{E}^{\eta_0^\varepsilon}\left[\sum_{(\vec{a},\vec{b})\in(D_+^\varepsilon)^n\times(D_-^\varepsilon)^m} p^\varepsilon(t,(\vec{r},\vec{s}),(\vec{a},\vec{b})) \prod_{i=1}^n \eta^+(a_i) \prod_{j=1}^m \eta^+(b_j)\right] + o(N) \\
=\ &\mathbb{E}^{\eta_0^\varepsilon}\left[\prod_{i=1}^n \langle \mathfrak{X}_0^{N,+},\, p^\varepsilon(t,r_i,\cdot)\rangle \prod_{j=1}^m \langle \mathfrak{X}_0^{N,-},\, p^\varepsilon(t,s_j,\cdot)\rangle\right] + o(N),
\end{aligned}$$

which converges to $\mathbb{E}^{(\vec{r},\vec{s})}[\Phi^{(n,m)}(X_{(n,m)}(t))]$ by Theorem 2.2.8 and assumption (i) for the initial distributions. Here $\mathbb{P}^{(\vec{r},\vec{s})}$ is the probability measure for $X_{(n,m)}$ starting at $(\vec{r},\vec{s})$.

We now prove that the second term in (3.4.27) tends to 0 as $\varepsilon \to 0$. The integrand with respect to ds is at most

$$\mathbb{E}\left[\frac{\lambda}{\varepsilon}\sum_{z\in I}\eta_\theta^+(z_+)\eta_\theta^-(z_-)\sum_{(\vec{a},\vec{b})\notin\Xi}p^\varepsilon(t-\theta,(\vec{r},\vec{s}),(\vec{a},\vec{b}))\,2A((\vec{a},\vec{b}),\eta_\theta)\,\varepsilon^{d(n+m)}\right]. \tag{3.4.28}$$

Note that $\{(\vec{a},\vec{b})\notin\Xi\}$ is a subset of

$$\bigcup_{w\in I}\left[\left(\bigcup_{k=2}^{n}\{(\vec{a},\vec{b}):\,\vec{a}(w_+)=k\}\right)\bigcup\left(\bigcup_{\ell=2}^{m}\{(\vec{a},\vec{b}):\,\vec{b}(w_-)=\ell\}\right)\right], \tag{3.4.29}$$

and that for fixed $w\in I$ and $k\in\{2,\cdots,n\}$, we further have

$$\{(\vec{a},\vec{b}):\,\vec{a}(w_+)=k\}=\bigcup_{\substack{i_1,\cdots,i_k\\ \text{distinct}}}\{(\vec{a},\vec{b}):\,a_{i_1}=\cdots=a_{i_k}=w_+\}.$$

Now we restrict the sum over $\{(\vec{a},\vec{b})\notin\Xi\}$ in (3.4.28) to the subset $\{(\vec{a},\vec{b}):\,a_{i_1}=\cdots=a_{i_k}=w_+\}$, where $w\in I$, $k\in\{2,\cdots,n\}$ and $(i_1,\cdots,i_k)$ are fixed. Moreover, we denote $(a_1,\cdots,a_k)$ by $\vec{a_k}$ and $(a_{k+1},\cdots,a_n)$ by $\vec{a}\setminus\vec{a_k}$. Then

$$\begin{aligned}
&\mathbb{E}\left[\frac{\lambda}{\varepsilon}\sum_{z\in I}\eta_\theta^+(z_+)\eta_\theta^-(z_-)\sum_{\{(\vec{a},\vec{b}):\,a_{i_1}=\cdots=a_{i_k}=w_+\}}p^\varepsilon(t-\theta,(\vec{r},\vec{s}),(\vec{a},\vec{b}))\,2A((\vec{a},\vec{b}),\eta_\theta)\,\varepsilon^{d(n+m)}\right]\\
\le\;& p^\varepsilon(t-\theta,(r_1,\cdots,r_k),(w_+,\cdots,w_+))\,\varepsilon^{kd}\sum_{(\vec{a}\setminus\vec{a_k},\,\vec{b})}p^\varepsilon(t-\theta,\vec{r}\setminus\vec{r_k},\vec{a}\setminus\vec{a_k})p(t-\theta,\vec{s},\vec{b})\varepsilon^{d(n+m-k)}\\
&\cdot\frac{\lambda}{\varepsilon}\,\mathbb{E}\left[\sum_{z\in I}\eta_\theta^+(z_+)\eta_\theta^-(z_-)\,2A((\vec{a},\vec{b}),\eta_\theta)\right]\\
\le\;& \frac{C\,\varepsilon^{kd}}{(t-\theta)^{kd/2}}\,\frac{\lambda}{\varepsilon}\,\#|I^\varepsilon|\,\sup_{(\vec{a},\vec{b})}\mathbb{E}[\eta_\theta^+(z_+)\eta_\theta^-(z_-)\,2A((\vec{a},\vec{b}),\eta_\theta)]\\
\le\;& \frac{\lambda\,\varepsilon^{(k-1)d}}{(t-\theta)^{kd/2}}\,C\qquad\text{where } C=C(n,m,\theta,d,D_\pm)\\
\le\;& \frac{\lambda\,\varepsilon^{d}}{(t-\theta)^{kd/2}}\,C=O(\varepsilon^d)\qquad\text{since } k\ge 2.
\end{aligned}$$

The second to the last inequality above follows from the bound $\#|I^\varepsilon| \le C\,\varepsilon^{-(d-1)}$ (see Lemma 2.2.22) and the uniform upper bound (3.4.17). Repeat the above argument for the other subsets of $\{(\vec{a},\vec{b}) \notin \Xi\}$ and use the fact $\#|I^\varepsilon| \le C\,\varepsilon^{-(d-1)}$ again (for $w \in I$ in (3.4.29)), we have, for any $\theta \in (0,t)$, (3.4.28) is of order ε and hence converges to 0 uniformly for $(\vec{r},\vec{s})$, as $\varepsilon \to 0$. The second term in (3.4.27) then converges to 0, by (3.4.21) and LDCT.

For the third term in (3.4.27), we split the integrand with respect to $d\theta$ into three terms corresponding to (3.4.13), (3.4.14) and (3.4.15) respectively. The term corresponding to (3.4.13) equals

$$
\begin{aligned}
& -\frac{\lambda}{\varepsilon} \sum_{(\vec{a},\vec{b})\in\Xi} p^\varepsilon(t-s,(\vec{r},\vec{s}),(\vec{a},\vec{b})) \sum_{\substack{z\in I^\varepsilon \\ \vec{a}(z_+)=1}} \Psi(z)\, A((\vec{a},\,(\vec{b},z_-)),\,\eta_\theta)\,\varepsilon^{d(n+m)} \\
= \;& -\frac{\lambda}{\varepsilon} \sum_{z\in I^\varepsilon} \sum_{\substack{(\vec{a},\vec{b})\in\Xi \\ \vec{a}(z_+)=1}} p^\varepsilon(t-s,(\vec{r},\vec{s}),(\vec{a},\vec{b}))\,\Psi(z)\, A((\vec{a},\,(\vec{b},z_-)),\,\eta_\theta)\,\varepsilon^{d(n+m)} \\
= \;& -\frac{\lambda}{\varepsilon} \sum_{z\in I^\varepsilon} \Psi(z) \sum_{i=1}^{n} \sum_{\substack{(\vec{a},\vec{b})\in\Xi \\ a_i=z_+}} p^\varepsilon(t-s,(\vec{r},\vec{s}),(\vec{a},\vec{b}))\, A((\vec{a},\,(\vec{b},z_-)),\,\eta_\theta)\,\varepsilon^{d(n+m)} \\
= \;& -\frac{\lambda}{\varepsilon} \sum_{z\in I^\varepsilon} \Psi(z) \sum_{i=1}^{n} p^\varepsilon(t-s,r_i,z_+) \sum_{(\vec{a}\setminus a_i,\vec{b})\in\Xi} p^\varepsilon(t-s,(\vec{r}\setminus r_i,\vec{s}),(\vec{a}\setminus a_i,\vec{b})) \\
& \qquad \times \frac{m((\vec{a},\,(\vec{b},z_-))}{\varepsilon^{d(n+m+1)}}\,\gamma^\varepsilon((\vec{a},\,(\vec{b},z_-)),\,\theta)\,\varepsilon^{d(n+m)} \\
= \;& -\lambda \sum_{i=1}^{n} \sum_{(\vec{a}\setminus a_i,\vec{b})\in\Xi} p^\varepsilon(t-s,(\vec{r}\setminus r_i,\vec{s}),(\vec{a}\setminus a_i,\vec{b}))\, m(\vec{a}\setminus a_i,\vec{b}) \\
& \qquad \times \sum_{z\in I^\varepsilon} \sigma_\varepsilon(z)\, p^\varepsilon(t-s,r_i,z_+)\,\gamma^\varepsilon((\vec{a},\,(\vec{b},z_-)),\,\theta).
\end{aligned}
$$

By Theorem 2.2.8 and Lemma 2.2.22,

$$
\lim_{\varepsilon\to 0} \sum_{z\in I^\varepsilon} \sigma_\varepsilon(z)\, p^\varepsilon(t-s,r_i,z_+)\,\gamma^\varepsilon((\vec{a},\,(\vec{b},z_-)),\,\theta) = \int_I p(t-s,r_i,z)\,\gamma_\theta(\vec{a},(\vec{b},z))\,d\sigma(z)
$$

and the convergence is uniform for $r_i \in D_+^\varepsilon$. Therefore, by applying Theorem 2.2.8 again, the term corresponding to (3.4.13) converges to

$$-\lambda \sum_{i=1}^{n} \int_{\partial_+^i} \gamma_\theta(\vec{a}, (\vec{b}, a_i))\, p(t-\theta, (\vec{r}, \vec{s}), (\vec{a}, \vec{b}))\, \frac{\rho_{(n,m)}(\vec{r}, \vec{s})}{\rho_+(r_i)}\, d\vec{b}\, da_1 \cdots d\sigma(a_i) \cdots da_n.$$

We repeat the same argument for the term corresponding to (3.4.14). Moreover, note that the term corresponding to (3.4.15) will not contribute to the limit as $\varepsilon \to 0$, by the same argument we used for the second term in (3.4.27). Therefore, the integrand of the second term in (3.4.27) converges to

$$\begin{aligned}
&-\lambda \sum_{i=1}^{n} \int_{\partial_+^i} \gamma_\theta(\vec{a}, (\vec{b}, a_i))\, p(t-\theta, (\vec{r}, \vec{s}), (\vec{a}, \vec{b}))\, \frac{\rho_{(n,m)}(\vec{a}, \vec{b})}{\rho_+(a_i)}\, d\vec{b}\, da_1 \cdots d\sigma(a_i) \cdots da_n \\
&-\lambda \sum_{j=1}^{m} \int_{\partial_-^j} \gamma_\theta((\vec{a}, b_j), \vec{b}))\, p(t-\theta, (\vec{r}, \vec{s}), (\vec{a}, \vec{b}))\, \frac{\rho_{(n,m)}(\vec{a}, \vec{b})}{\rho_-(b_j)}\, d\vec{a}\, db_1 \cdots d\sigma(b_j) \cdots db_m.
\end{aligned}$$

The integral for $\theta \in (0, t)$ in the third term in (3.4.27) then converges to the desired quantity, by (3.4.21) and LDCT. The proof is complete. □

In view of Remark 3.4.6(iii), the proof of Theorem 3.3.3 (Propagation of Chaos) will be complete once we establish the uniqueness of the solution of the limiting hierarchy (3.4.25). This will be accomplished in Theorem 3.4.7 in the next subsection.

3.4.5 Uniqueness of infinite hierarchy

Uniqueness of BBGKY hierarchy is an important issue in statistical physics. For example, it is a key step in the derivation of the cubic non-linear Schrödinger equation from the quantum dynamics of many body systems obtained in [36]. Our BBGKY hierarchy (3.4.25) is new to the literature and the proof of its uniqueness involves a representation and manipulations of the hierarchy in terms of trees. The technique is related but different from that in [36], which used the Feynman diagrams.

Note that, by Theorem 3.4.5, $\gamma_t^{(n,m)}$ can be extended continuously to $t = 0$. Uniqueness of solution for the hierarchy will be established on a subset of the space

$$C([0,T],\mathcal{D}) := \bigoplus_{(n,m)\in\mathbb{N}\times\mathbb{N}} C([0,T],\, \overline{D}_+^n \times \overline{D}_-^m)$$

equipped with the product topologies induced by the uniform norm $\|\cdot\|_{(T,n,m)}$ on $[0,T]\times\overline{D}_+^n\times\overline{D}_-^m$.

Theorem 3.4.7. *(Uniqueness of the infinite hierarchy) Given any $T>0$. Suppose $\beta_t = \{\beta_t^{(n,m)}\}\in C([0,T],\mathcal{D})$ is a solution to the infinite hierarchy (3.4.25) with zero initial condition (i.e. $\beta_0=\Phi=0$) and satisfies $\|\beta_t^{(n,m)}\|_{(T,n,m)}\le C^{n+m}$ for some $C\ge 0$. Then we have $\|\beta_t^{(n,m)}\|_{(T,n,m)}=0$ for every $n,m\in\mathbb{N}$.*

The remaining of this subsection is devoted to give a proof of this theorem.

Convention in this subsection: $\beta=\{\beta^{(n,m)}\}$ will always denote the functions stated in Theorem 3.4.7. For notational simplicity, we will also assume $\lambda=2$ and $\rho_\pm=1$. The proof for the general case is the same. We will also drop T from the notation $\|\beta_t^{(n,m)}\|_{(T,n,m)}$.

It is convenient to rewrite the infinite hierarchy (3.4.25) in a more compact form as

$$\gamma_t^{(n,m)} = P_t^{(n,m)}\Phi^{(n,m)} - \int_0^t P_{t-s}^{(n,m)}\left(\sum_{i=1}^n V_i^+\gamma_s^{(n,m+1)} + \sum_{j=1}^m V_j^-\gamma_s^{(n+1,m)}\right) ds\ , \qquad (3.4.30)$$

where $V_i^+\gamma^{(n,m+1)}$ is a measure concentrated on ∂_+^i defined as

$$\begin{aligned} V_i^+\gamma^{(n,m+1)} &:= \gamma^{(n,m+1)}(\vec{a},(\vec{b},a_i))\, d\sigma_{(n,m)}\Big|_{\partial_+^i}(\vec{a},\vec{b}) \\ &= \gamma^{(n,m+1)}(\vec{a},(\vec{b},a_i))\, d\sigma\Big|_I(a_i)\, d(\vec{a}\setminus a_i)\, d\vec{b}. \end{aligned}$$

Here $\sigma_{(n,m)}\Big|_{\partial_+^i}$ is the surface measure of $\partial(D_+^n\times D_-^m)$ restricted to ∂_+^i. Similarly, $V_j^-\gamma^{(n+1,m)}$ is a measure concentrating on ∂_-^j defined as

$$\begin{aligned} V_j^- \gamma^{(n+1,m)} &:= \gamma^{(n+1,m)}((\vec{a}, b_j), \vec{b})\, d\sigma_{(n,m)}\Big|_{\partial_-^j} (\vec{a}, \vec{b}) \\ &= \gamma^{(n+1,m)}((\vec{a}, b_j), \vec{b})\, d\sigma\Big|_I (b_j)\, d\vec{a}\, d(\vec{b} \setminus b_j). \end{aligned}$$

Duhamel tree expansion

We now describe the infinite hierarchy in detail. It is natural and illustrative to represent the infinite hierarchy in terms of a tree structure, with the 'root' at the top and the 'leaves' at the bottom. Fix two positive integers n and m. We construct a sequence of finite trees $\{\mathbb{T}_N^{(n,m)} : N = 0, 1, 2, \cdots\}$ recursively as follows.

1. $\mathbb{T}_0^{(n,m)}$ is the root, with label (n, m).

2. $\mathbb{T}_1^{(n,m)}$ is constructed from $\mathbb{T}_0^{(n,m)}$ by attaching $n + m$ new vertices (call them leaves of $\mathbb{T}_1^{(n,m)}$) to it. More precisely, we attach $n + m$ new vertices to the root by drawing n '$+$' edges and m '$-$' edges from the root. Those new leaves drawn by the '$+$' edges are labeled $(n, m + 1)$, while those drawn by the '$-$' edges are labeled $(n + 1, m)$. We also label the edges as $\{+_i\}_{i=1}^n$ and $\{-_j\}_{j=1}^m$ (See Figure 3.2).

3. When $N = 2$, we view each of the $n + m$ leaves of $\mathbb{T}_1^{(n,m)}$ as a 'root' (with a new label, being either $(n, m + 1)$ or $(n + 1, m)$), and then attach new leaves (leaves of $\mathbb{T}_2^{(n,m)}$) to it by drawing '$\pm$' edges. Hence there are $(m + n)(m + n + 1)$ new leaves, coming from $n^2 + m(n + 1)$ new '$+$' edges and $n(m + 1) + m^2$ new '$-$' edges.

4. Having drawn $\mathbb{T}_{N-1}^{(n,m)}$, we construct $\mathbb{T}_N^{(n,m)}$ by attaching new edges and new leaves from each leaf of $\mathbb{T}_{N-1}^{(n,m)}$ by the same construction, viewing a leaf of $\mathbb{T}_{N-1}^{(n,m)}$ as a 'root'.

In $\mathbb{T}_N^{(n,m)}$, the root is connected to each leaf by a unique path $\vec{\theta} = (\theta_1, \theta_2, \cdots, \theta_N)$ formed by the '$\pm$' edges. Moreover, such a path passes through a sequence of labels formed by the leaves of $\mathbb{T}_k^{(n,m)}$ $(k = 1, 2, \cdots, N)$. We denote these labels by $\vec{l}(\vec{\theta}) = (l_1(\vec{\theta}), l_2(\vec{\theta}), \cdots, l_N(\vec{\theta}))$.

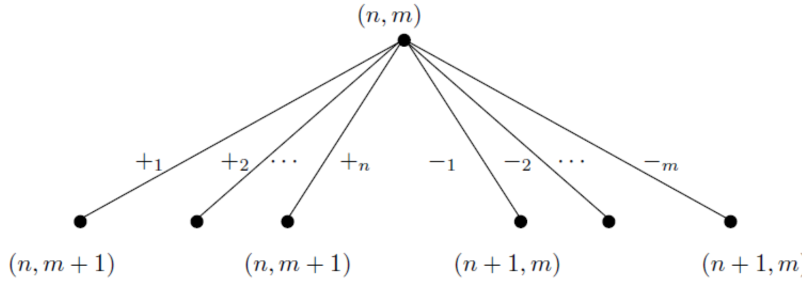


Figure 3.2: $\mathbb{T}_1^{(n,m)}$

For example, when $(n,m) = (2,5)$, $N = 3$ and the path is $\vec{\theta} = (+_1, -_6, -_5)$. Then $\vec{l}(\vec{\theta}) = ((2,6), (3,6), (4,6))$ and the path connects the root to a leaf of $\mathbb{T}_3^{(2,5)}$ with label $(4,6)$. Note that the label is not one-to-one. For example, $\vec{l}(+_1, -_6, -_5) = \vec{l}(+_2, -_6, -_4)$.

For mnemonic reason, we use the same notation $\mathbb{T}_N^{(n,m)}$ to denote the collection of paths that connect the root to a leaf in $\mathbb{T}_N^{(n,m)}$. By induction, the total number of paths (or the total number of leaves) is

$$(n+m)(n+m+1)\cdots(n+m+N-1) = \frac{(n+m+N-1)!}{(n+m-1)!}. \tag{3.4.31}$$

Iterating (3.4.30) N times gives

$$\begin{aligned} \beta_t^{(n,m)} &= -\int_{t_2=0}^{t} P_{t-t_2}^{(n,m)} \left(\sum_{i=1}^{n} V_i^+ \beta_{t_2}^{(n,m+1)} + \sum_{j=1}^{m} V_j^- \beta_{t_2}^{(n+1,m)} \right) dt_2 \\ &= \cdots \\ &= (-1)^N \int_{t_2=0}^{t} \int_{t_3=0}^{t_2} \cdots \int_{t_{N+1}=0}^{t_N} dt_2 \cdots dt_{N+1} \\ &\qquad \sum_{\vec{\theta} \in \mathbb{T}_N^{(n,m)}} P_{t-t_2}^{(n,m)}\, V_{\theta_1}\, P_{t_2-t_3}^{l_1(\vec{\theta})}\, V_{\theta_2}\, P_{t_3-t_4}^{l_2(\vec{\theta})}\, V_{\theta_3} \cdots P_{t_N-t_{N+1}}^{l_{N-1}(\vec{\theta})}\, V_{\theta_N}\, \beta_{t_{N+1}}^{l_N(\vec{\theta})}. \end{aligned} \tag{3.4.32}$$

where V_{θ_i} (for $i = 1, 2, \cdots, N$) is defined by $V_{+_i} = V_i^+$ and $V_{-_j} = V_j^-$. For example, if $(n, m) = (2, 5)$, $N = 3$ and the path is $\vec{\theta} = (+_1, -_6, -_5)$, then

$$P^{(n,m)} V_{\theta_1} P^{l_1(\vec{\theta})} V_{\theta_2} P^{l_2(\vec{\theta})} V_{\theta_3} \beta^{l_3(\vec{\theta})} = P^{(2,5)} V_1^+ P^{(2,6)} V_6^- P^{(3,6)} V_5^- \beta^{(4,6)}.$$

Telescoping via Chapman-Kolmogorov equation

By (3.4.31), the right hand side of (3.4.32) is a sum of $(n+m)(n+m+1)\cdots(n+m+N-1)$ terms of multiple integrals. We will apply the bound $\|\beta_t^{(p,q)}\|_{(p,q)} \leq C^{p+q}$ to each term, and then simplify the integrand using Chapman-Kolmogorov equation.

We demonstrate this for the twelve terms for the case $(n, m, N) = (1, 2, 2)$. The twelve terms on the right hand side of (3.4.32) are

$$\begin{aligned} P_{t-t_2}^{(1,2)} \Big\{ \quad & V_1^+ P_{t_2-t_3}^{(1,3)} \left(V_1^+ \beta_{t_3}^{(1,4)} + (V_1^- + V_2^- + V_3^-)\, \beta_{t_3}^{(2,3)} \right) \\ + \quad & V_1^- P_{t_2-t_3}^{(2,2)} \left((V_1^+ + V_2^+)\, \beta_{t_3}^{(2,3)} + (V_1^- + V_2^-)\, \beta_{t_3}^{(3,2)} \right) \\ + \quad & V_2^- P_{t_2-t_3}^{(2,2)} \left((V_1^+ + V_2^+)\, \beta_{t_3}^{(2,3)} + (V_1^- + V_2^-)\, \beta_{t_3}^{(3,2)} \right) \Big\}. \end{aligned} \tag{3.4.33}$$

The first four terms came from the leftmost leaf of the previous level, we group them together to obtain, for $(x, y_1, y_2) \in \overline{D}_+ \times \overline{D}_-^2$,

$$\begin{aligned} & \left| P_{t-t_2}^{(1,2)} V_1^+ P_{t_2-t_3}^{(1,3)} \left(V_1^+ \beta_{t_3}^{(1,4)} + (V_1^- + V_2^- + V_3^-)\, \beta_{t_3}^{(2,3)} \right) (x,\, y_1,\, y_2) \right| \\ \leq \quad & C^5 \int d\sigma(x')\, dy_1'\, dy_2'\, p^{(1,2)}(t - t_2,\, (x, y_1, y_2), (x', y_1', y_2')) \\ & \quad \left(\int d\sigma(a)\, db_1\, db_2\, db_3 + \int da\, d\sigma(b_1)\, db_2\, db_3 + \int da\, db_1\, d\sigma(b_2)\, db_3 + \int da\, db_1\, db_2\, d\sigma(b_3) \right) \\ & \quad p^{(1,3)}(t_2 - t_3,\, (x', y_1', y_2', x'), (a, b_1, b_2, b_3)) \\ = \quad & C^5 \int d\sigma(x')\, p^+(t - t_2, x, x') \Big(\int d\sigma(a)\, p^+(t_2 - t_3, x', a) + \int d\sigma(b_1)\, p^-(t - t_3, y_1, b_1) \\ & \quad + \int d\sigma(b_2)\, p^-(t - t_3, y_2, b_2) + \int d\sigma(b_3)\, p^-(t_2 - t_3, x', b_3) \Big). \end{aligned}$$

Note the telescoping effect upon using the Chapman-Kolmogorov equation for the middle two terms in the last equality above gives rise to $t - t_3$ rather than $t_2 - t_3$.

We apply (2.1.5) to obtain

$$\begin{aligned}
&\left\| P^{(1,2)}_{t-t_2} V_1^+ P^{(1,3)}_{t_2-t_3} \left(V_1^+ \, \beta^{(1,4)}_{t_3} + (V_1^- + V_2^- + V_3^-)\, \beta^{(2,3)}_{t_3} \right) \right\|_{(1,2)} \\
&\le C^5 \, \frac{C_+}{\sqrt{t-t_2}} \left(\frac{C_+}{\sqrt{t_2 - t_3}} + \frac{2\,C_-}{\sqrt{t-t_3}} + \frac{C_-}{\sqrt{t_2-t_3}} \right),
\end{aligned}$$

where $C_\pm = C(D_\pm, T)$ are positive constants. Repeat the above argument for the remaining eight terms of (3.4.33), we obtain

$$\begin{aligned}
\|\beta^{(1,2)}_t\|_{(1,2)} \le C^5 \int_{t_2=0}^{t} \int_{t_3=0}^{t_2} & \frac{C_+}{\sqrt{t-t_2}} \left(\frac{C_+}{\sqrt{t_2-t_3}} + \frac{2\,C_-}{\sqrt{t-t_3}} + \frac{C_-}{\sqrt{t_2-t_3}} \right) \\
&+ \frac{2\,C_-}{\sqrt{t-t_2}} \left(\frac{C_+}{\sqrt{t-t_3}} + \frac{C_+}{\sqrt{t_2-t_3}} + \frac{C_-}{\sqrt{t-t_3}} + \frac{C_-}{\sqrt{t_2-t_3}} \right).
\end{aligned} \tag{3.4.34}$$

The key is to visualize the twelve terms on the right as 12 paths of $\mathbb{T}_2^{(1,2)}$ *with the edges relabeled.* We denote this relabeled tree by $\mathbb{S}_2^{(1,2)}$ (See Figure 3.3, ignoring the five leaves in $\mathbb{S}_3^{(1,2)}$ at the moment). More precisely, since all twelve terms on the right are of the form $\frac{C_\pm}{\sqrt{t_p - t_2}} \frac{C_\pm}{\sqrt{t_q - t_3}}$, we only need to record the indexes (p, q) and the $\pm$ sign. For example, the first four terms can be represented by

$$(+_1 \, +_2, \; +_1 \, -_1, \; +_1 \, -_1, \; +_1 \, -_2).$$

Each $+_1 \, -_1$ corresponds to $\frac{C_+}{\sqrt{t-t_2}} \frac{C_-}{\sqrt{t-t_3}}$ and hence it appears twice. In $\mathbb{S}_2^{(1,2)}$, these four paths are formed by a $+_1$ edge followed by four edges with labels $\{+_2, \, -_1, \, -_1, \, -_2\}$.

In general, we obtain $\mathbb{S}_N^{(n,m)}$ by relabeling the edges of $\mathbb{T}_N^{(n,m)}$, while keeping the labels for the vertices and the $\pm$ sign for the edges. The relabeling of edges are performed as follows:

1. At level 1, we assign the number '1' to all the edges connected to the root. Hence we have n '$+_1$' edges and m '$-_1$' edges, rather than the labels $\{+_i\}_{i=1}^n$ and $\{-_j\}_{j=1}^m$ (See Figure 3.4 for $\mathbb{S}_1^{(n,m)}$).

2. At level $k \ge 2$, consider the set $\Lambda^+ := \{+_1, \cdots, +_1, +_2, +_3, \cdots, +_k\}$ in which we have

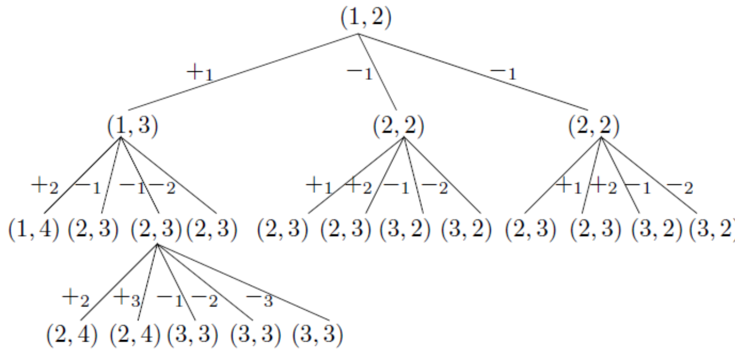


Figure 3.3: $\mathbb{S}_2^{(1,2)}$ together with 5 leaves in $\mathbb{S}_3^{(1,2)}$

n copies of $+_1$ (hence there are $n+k-1$ elements in Λ^+, in which n of them are $+_1$). Similarly, let $\Lambda^- := \{-_1, \cdots, -_1, -_2, -_3, \cdots, -_k\}$ in which we have m copies of $-_1$. For an arbitrary leaf ξ of $\mathbb{T}_{k-1}^{(n,m)}$, let R^ξ be the labels of (the edges of) the path from the root to ξ in $\mathbb{S}_{k-1}^{(n,m)}$, counting with multiplicity. Finally, the collection of new labels of the edges below ξ, denoted by L^ξ, is chosen in such a way that

$$\Lambda^+ \cup \Lambda^- = R^\xi \cup L^\xi \qquad \text{(counting multiplicity).}$$

Since $|R^\xi| = k-1$ and $|L^\xi| = n+m+k-1$ (again, counting multiplicity), the cardinalities of the two sides match:

$$(n+k-1) + (m+k-1) = (k-1) + (n+m+k-1).$$

Induction shows that $R^\xi \subset \Lambda^+ \cup \Lambda^-$ and the choice for L^ξ is unique. For example, for leaf $\xi = (1,3)$ of $\mathbb{T}_1^{(1,2)}$, $R^\xi = \{+_1\}$, $\Lambda^+ := \{+_1, +_2\}$ and $\Lambda^- := \{-_1, , -_1, -_2\}$. So $L^\xi = \{+_2, -_1, , -_1, -_2\}$, which gives the new labels to the edges below ξ; see Figure 3.3.

As a further illustration, we continue to 'grow' $\mathbb{S}_2^{(1,2)}$ (see Figure 3.3) by adding suitably labeled edges to leaves of $\mathbb{S}_2^{(1,2)}$. Precisely, let ξ be a leaf of $\mathbb{S}_2^{(1,2)}$.

- If $R^\xi = \{+_1, +_2\}$, then $L^\xi = \{+_3, -_1, -_1, -_2, -_3\}$ (this is the case for the leftmost leaf,

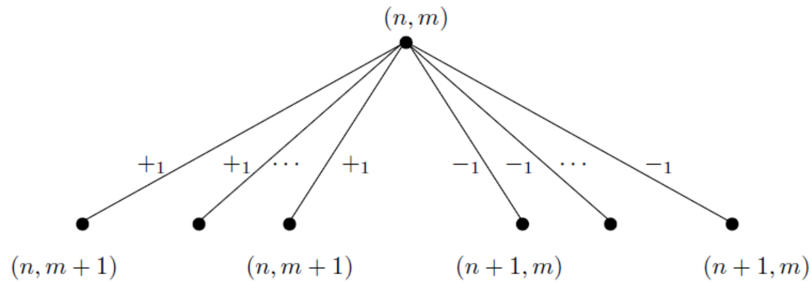


Figure 3.4: $\mathbb{S}_1^{(n,m)}$

which has label $(1, 4)$)

- If $R^\xi = \{+_1,\, -_1\}$, then $L^\xi = \{+_2,\, +_3,\, -_1,\, -_2,\, -_3\}$ (shown in Figure 3.3).
- If $R^\xi = \{+_1,\, -_2\}$, then $L^\xi = \{+_2,\, +_3,\, -_1,\, -_1,\, -_3\}$.
- If $R^\xi = \{-_1,\, +_1\}$, then $L^\xi = \{+_2,\, +_3,\, -_1,\, -_2,\, -_3\}$.
- If $R^\xi = \{-_1,\, +_2\}$, then $L^\xi = \{+_1,\, +_3,\, -_1,\, -_2,\, -_3\}$.
- If $R^\xi = \{-_1,\, -_1\}$, then $L^\xi = \{+_1,\, +_2,\, +_3,\, -_2,\, -_3\}$.
- If $R^\xi = \{-_1,\, -_2\}$, then $L^\xi = \{+_1,\, +_2,\, +_3,\, -_1,\, -_3\}$.

For mnemonic reason, we use the same notation $\mathbb{S}_N^{(n,m)}$ to denote the collection of paths from the root to the leaves of $\mathbb{S}_N^{(n,m)}$. Any such path is represented by the *ordered* (new) labels of the edges. We now 'forget' the sign of the edges and only record the integer labels. For example, the path $(-_1,\, +_1,\, -_2,\, +_3)$ is replaced by $(1,\, 1,\, 2,\, 3)$.

Using the hypothesis $\|\beta_t^{(n,m)}\|_{(n,m)} \leq C^{n+m}$ (of Theorem 3.4.7) and applying Chapman-Kolmogorov equation to (3.4.32), and then applying (2.1.5), we obtain the following lemma by the same argument that we used to obtain (3.4.34).

Lemma 3.4.8.

$$\|\beta_t^{(n,m)}\|_{(n,m)} \leq C^{n+m+N}\ (C_+ \vee C_-)^N\, I_N^{(n,m)}(t),$$

where

$$I_N^{(n,m)}(t) := \int_{t_2=0}^{t} \cdots \int_{t_{N+1}=0}^{t_N} \sum_{\vec{\upsilon}\in \mathbb{S}_N^{(n,m)}} \frac{1}{\sqrt{(t_{\upsilon_1}-t_2)\,(t_{\upsilon_2}-t_3)\cdots(t_{\upsilon_N}-t_{N+1})}} dt_2\cdots dt_{N+1}.$$

Our goal is to show that $I_N^{(n,m)}(t) \le (C\,t)^{N/2}$ for some $C = C(n,m) > 0$. This will imply $\|\beta_t^{(n,m)}\|_{(n,m)} = 0$ for $t > 0$ small enough. Clearly we have

$$\begin{aligned} I_N^{(n,m)}(t) &\le \frac{(n+m+N-1)!}{(n+m-1)!} \int_{t_2=0}^{t} \cdots \int_{t_{N+1}=0}^{t_N} \frac{1}{\sqrt{(t-t_2)\,(t_2-t_3)\cdots(t_N-t_{N+1})}} dt_2\cdots dt_{N+1} \\ &= \frac{(n+m+N-1)!}{(n+m-1)!}\,\frac{(\pi\,t)^{N/2}}{\Gamma(\frac{N+2}{2})}. \end{aligned}$$

Unfortunately, this crude bound is asymptotically larger than $(Ct)^{N/2}$ for any $C > 0$.

Comparison with a 'dominating' tree

Note that

$$\begin{aligned} I_3^{(1,2)}(t) \;\le\; & \int_{t_2=0}^{t}\int_{t_3=0}^{t_2}\int_{t_4=0}^{t_3} \left(\frac{3}{\sqrt{t-t_2}}\right)\left(\frac{2}{\sqrt{t_2-t_3}}+\frac{2}{\sqrt{t-t_3}}\right) \\ & \left(\frac{2}{\sqrt{t_3-t_4}}+\frac{2}{\sqrt{t_2-t_4}}+\frac{1}{\sqrt{t-t_4}}\right) dt_2\,dt_3\,dt_4. \end{aligned}$$

This is obtained by comparing the labels in $\mathbb{S}_3^{(1,2)}$ with a 'dominating' labeling, in which the labels of the edges below *every* leaf of $\mathbb{S}_2^{(1,2)}$ is $\{3, 3, 2, 2, 1\}$ (the $\pm$ sign is discarded). This trick enables us to group the terms at each level.

For the general case, let ξ be an arbitrary leaf of $\mathbb{S}_{k-1}^{(n,m)}$. Note that in L^ξ, each of the integers 2, 3, $\cdots$, k appears at most twice and the integer 1 appears at most $n+m$ times. We compare L^ξ with the 'dominating' label $\tilde{L}^\xi$ defined below:

Level k	$\lvert\tilde{L}^\xi\rvert = \lvert L^\xi\rvert$	$\tilde{L}^\xi$
1	$n+m$	$1,\ 1,\ 1, \cdots, 1$
2	$n+m+1$	$2,\ 2,\ 1,\ 1,\ 1, \cdots, 1$
3	$n+m+2$	$3,\ 3,\ 2,\ 2,\ 1,\ 1,\ 1, \cdots, 1$
$\cdots$	$\cdots$	$\cdots$
$n+m-1$	$2(n+m)-2$	$n+m-1,\ n+m-1,\ n+m-2, \cdots, 3,\ 3,\ 2,\ 2,\ 1,\ 1$
$n+m$	$2(n+m)-1$	$n+m,\ n+m,\ n+m-1,\ n+m-1, \cdots, 3,\ 3,\ 2,\ 2,\ 1$
$n+m+1$	$2(n+m)$	$n+m+1,\ n+m+1,\ n+m,\ n+m, \cdots, 3,\ 3,\ 2,\ 2$
$n+m+2$	$2(n+m)+1$	$n+m+2,\ n+m+2,\ n+m+1, \cdots, 3,\ 3,\ 2$
$\cdots$	$\cdots$	$\cdots$
N	$n+m+N-1$	$N,\ N,\ N-1,\ N-1, \cdots, c,\ b,\ a$

In the last row, if $n+m+N-1$ is even, then $a = b = (N-n-m+3)/2$ and $c = b+1$; if $n+m+N-1$ is odd, then $a = (N-n-m+2)/2$ and $b = c = a+1$.

We can now group the terms in each level k as a sum of k terms to obtain

$$\begin{aligned} I_N^{(n,m)}(t) \quad \leq \quad & \int_{t_2=0}^{t} \int_{t_3=0}^{t_2} \cdots \int_{t_{N+1}=0}^{t_N} \left(\frac{n+m}{\sqrt{t-t_2}} \right) \left(\frac{2}{\sqrt{t_2-t_3}} + \frac{n+m-1}{\sqrt{t-t_3}} \right) \\ & \left(\frac{2}{\sqrt{t_3-t_4}} + \frac{2}{\sqrt{t_2-t_4}} + \frac{n+m-2}{\sqrt{t-t_4}} \right) \cdots \\ & \left(\frac{2}{\sqrt{t_{n+m}-t_{n+m+1}}} + \cdots + \frac{2}{\sqrt{t_2-t_{n+m+1}}} + \frac{1}{\sqrt{t-t_{n+m+1}}} \right) \\ & \prod_{k=n+m+1}^{N} \left(\frac{2}{\sqrt{t_k-t_{k+1}}} + \frac{2}{\sqrt{t_{k-1}-t_{k+1}}} + \cdots + \frac{2}{\sqrt{t_2-t_{k+1}}} \right) dt_2 \cdots dt_{N+1}. \end{aligned}$$

In the last term, we have used the observation that when $k > n+m$, the smallest element in $\tilde{L}^\xi$ is at least 2 and so the sum stops before reaching $1/\sqrt{t-t_{k+1}}$. From this and simple estimates like

$$\frac{2}{\sqrt{t_3-t_4}} + \frac{2}{\sqrt{t_2-t_4}} + \frac{n+m-2}{\sqrt{t-t_4}} \leq \frac{2(n+m)}{3} \left(\frac{1}{\sqrt{t_3-t_4}} + \frac{1}{\sqrt{t_2-t_4}} + \frac{1}{\sqrt{t-t_4}} \right),$$

we have derived the following

Lemma 3.4.9. *For any* (n,m), N *and* $0 \le t_{N+1} \le t_N \le \cdots \le t_2 \le t$, *we have*

$$\sum_{\vec{\upsilon}\in \mathbb{S}_N^{(n,m)}} \frac{1}{\sqrt{(t_{\upsilon_1}-t_2)\,(t_{\upsilon_2}-t_3)\cdots(t_{\upsilon_N}-t_{N+1})}} \le \frac{(n+m)^{(n+m)}}{(n+m)!}\, 2^N \prod_{i=2}^{N+1}\Bigg(\sum_{j=1}^{i-1}\frac{1}{\sqrt{t_j-t_i}}\Bigg).$$

In particular,

$$I_N^{(n,m)}(t) \le \frac{(n+m)^{(n+m)}}{(n+m)!}\, 2^N\, J_N(t),$$

where

$$J_N(t) := \int_0^t\int_0^{t_2}\cdots\int_0^{t_N}\prod_{i=2}^{N+1}\left(\sum_{j=1}^{i-1}\frac{dt_i}{\sqrt{t_j-t_i}}\right).$$

Estimating J_N

Our goal in this section is show that $J_N(t) \le (C\,t)^{N/2}$ for some $C>0$. Our proof relies on the following recursion formula pointed out to us by David Speyer:

$$J_N(t) = \sum_{k=1}^{N}\frac{1}{k!}\sum_{\substack{n_2+n_3+\cdots+n_{k+1}=N\\ n_2,n_3,\cdots,n_{k+1}\ge 1}}\prod_{j=2}^{k+1}\int_0^t\frac{J_{n_j-1}(t_j)}{\sqrt{t-t_j}}\,dt_j. \tag{3.4.35}$$

We assume (3.4.35) for now and use it to establish the following lemma. The proof of (3.4.35) will be given immediately after it.

Lemma 3.4.10. $J_N(t)$ *is homogeneous in the sense that*

$$J_N(t) = J_N\cdot t^{N/2} \quad \textit{where } J_N := J_N(1). \tag{3.4.36}$$

Moreover,

$$2^N \le J_N \le \frac{(N+1)^N\,\pi^{N/2}}{(N+1)!}. \tag{3.4.37}$$

Proof $J_N(t) = J_N\cdot t^{N/2}$ is obvious from (3.4.35) after a change of variable. Let $\mathcal{M}_N$ be the collection of functions $f : \{2, 3, \cdots, N+1\} \to \{1, 2, \cdots, N\}$ satisfying $f(i) < i$. We can rewrite $J_N(t_1)$ as

$$J_N(t_1) = \sum_{f \in \mathcal{M}_N} \int_{t_2=0}^{t_1} \int_{t_3=0}^{t_2} \cdots \int_{t_{N+1}=0}^{t_N} \prod_{i=2}^{N+1} \frac{dt_i}{\sqrt{t_{f(i)} - t_i}}. \tag{3.4.38}$$

This is a sum of $N!$ terms. When we put $t_1 = 1$, the smallest term is

$$\int_{t_2=0}^{1} \int_{t_3=0}^{t_2} \cdots \int_{t_{N+1}=0}^{t_N} \prod_{j=2}^{N+1} \frac{dt_j}{\sqrt{1 - t_j}} = \frac{1}{N!} \left(\int_0^1 \frac{1}{\sqrt{1-s}}\, ds \right)^N = \frac{2^N}{N!}.$$

Hence we have the lower bound $2^N \leq J_N$. Unfortunately, the largest term is exactly

$$\int_{t_2=0}^{1} \int_{t_3=0}^{t_2} \cdots \int_{t_{N+1}=0}^{t_N} \prod_{i=1}^{N} \frac{dt_j}{\sqrt{t_{j-1} - t_j}} = \frac{\pi^{N/2}}{\Gamma(\frac{N+2}{2})}.$$

which grows faster than C^N for any $C \in (0, \infty)$. Hence for the upper bound, we will employ the recursion formula (3.4.35). We apply the homogeneity to the right hand side of (3.4.35) to obtain

$$J_N = \sum_{k=1}^{N} \frac{1}{k!} \sum_{\substack{n_2+n_3+\cdots+n_{k+1}=N \\ n_2,\cdots,n_{k+1} \geq 1}} \prod_{j=2}^{k+1} J_{n_j-1} \int_0^1 \frac{t_j^{(n_j-1)/2}}{\sqrt{1-t_j}}\, dt_j.$$

The integrals are now simple one dimensional and can be evaluated:

$$J_N = \sum_{k=1}^{N} \frac{1}{k!} \sum_{\substack{n_2+n_3+\cdots+n_{k+1}=N \\ n_2,\cdots,n_{k+1} \geq 1}} \prod_{j=2}^{k+1} \left(J_{n_j-1} \cdot \frac{\sqrt{\pi}\, \Gamma((n_j+1)/2)}{\Gamma((n_j+2)/2)} \right). \tag{3.4.39}$$

Since $\frac{\sqrt{\pi}\,\Gamma((n_j+1)/2)}{\Gamma((n_j+2)/2)} \leq \sqrt{\pi}$, we have $J_N \leq K_N$, where K_N is defined by the recursion

$$K_N = \sum_{k=1}^{N} \frac{1}{k!} \sum_{\substack{n_2+n_3+\cdots+n_{k+1}=N \\ n_2,\cdots,n_{k+1} \geq 1}} \prod_{j=2}^{k+1} \left(K_{n_j-1} \cdot \sqrt{\pi} \right)$$

with $K_0 = 1$. The generating function $\phi(x) := \sum_{N=0}^{\infty} K_N x^N$ of K_N clearly satisfies $\phi(x) = \exp(\sqrt{\pi}x\phi(x))$. We thus see that $\phi(x) = W(-\sqrt{\pi}x)/(-\sqrt{\pi}x)$, where W is the Lambert W function. By Lagrange inversion Theorem (see Theorem 5.4.2 of [69]), $W(z) = \sum_{k=1}^{\infty}(-k)^{k-1}z^k/k!$ (for $|z| < 1/e$). Hence by comparing coefficients in the series expansion of $\phi(x)$, we have $K_N = \frac{(N+1)^N \pi^{N/2}}{(N+1)!}$ as desired. □

Remark 3.4.11. By Stirling's formula, $\frac{(N+1)^N \pi^{N/2}}{(N+1)!} \sim (\sqrt{\pi}e)^N$ (where $a(N) \sim b(N)$ means $\lim_{N\to\infty} \frac{a(N)}{b(N)} = 1$). Hence $J_N \leq C^N$ for some $C > 0$. Monte Carlo simulations suggests that $J_N \sim \pi^N$. The recursion (3.4.39) also makes it clear that J_N's are all in $\mathbb{Q}[\pi]$ (polynomials in π with rational coefficients) and makes it easy to compute them recursively. This is because $\frac{\sqrt{\pi}\,\Gamma((n_j+1)/2)}{\Gamma((n_j+2)/2)}$ is rational if n_j is odd and is a rational multiple of π if n_j is even. For example, $J_1 = 2$, $J_2 = 2 + \pi$ and $J_3 = 4 + \frac{10\pi}{3}$. □

We now turn to the proof the recursion formula (3.4.35) which is restated in the following lemma.

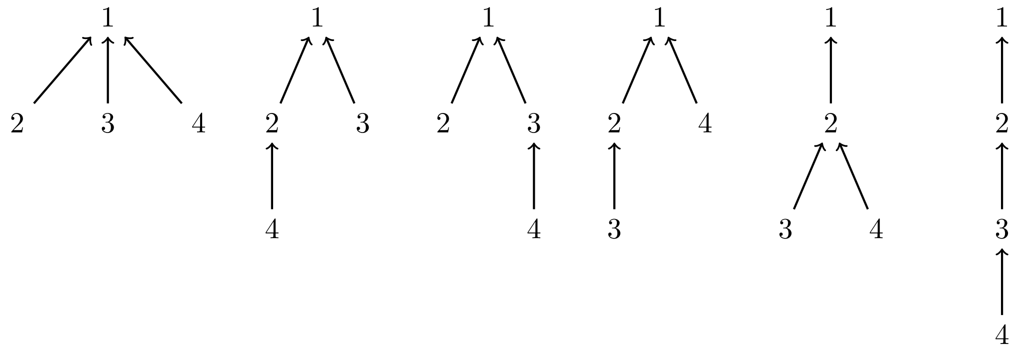
Lemma 3.4.12.

$$J_N(t) = \sum_{k=1}^{N} \frac{1}{k!} \sum_{\substack{n_2+n_3+\cdots+n_{k+1}=N \\ n_2,n_3,\cdots,n_{k+1}\geq 1}} \prod_{j=2}^{k+1} \int_0^t \frac{J_{n_j-1}(t_j)}{\sqrt{t-t_j}}\, dt_j \tag{3.4.40}$$

provided that we set $J_0(t) = 1$.

Proof The proof is based on standard combinatorial methods for working on sums over planar rooted trees (see [69]).

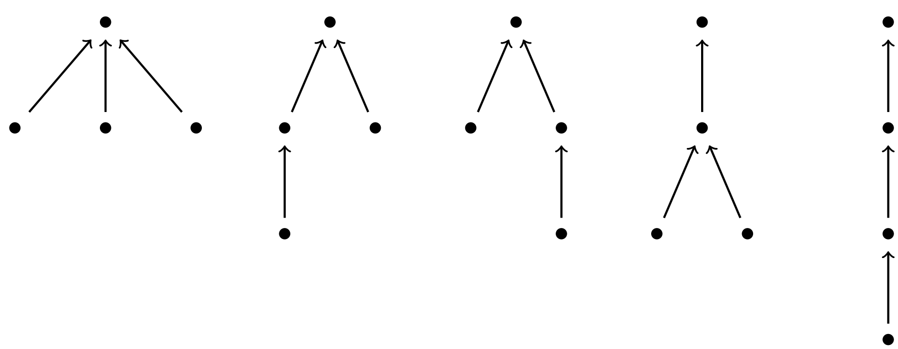
Step 1: Summing over labeled trees. Recall (3.4.38). There are $N!$ elements in $\mathcal{M}_N$. We can visualize each of them as a rooted tree with vertex set $\{1, 2, \cdots, N+1\}$ and a directed edge from i to $f(i)$ for each i. For example, the 6 elements of $\mathcal{M}_3$ can be represented by



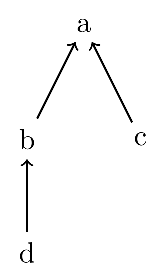
The trees are drawn so that arrows point upwards and the children of a given vertex are listed from left to right. Note that the second and forth tree of the list are the same up to relabeling the vertices. The idea is to group terms in (3.4.38) like this together. First, we rewrite (3.4.38) in terms of trees. Let $\mathcal{D}_N$ be the set of 'decreasing trees', which are trees whose vertices are labeled by $\{1, 2, \cdots, N+1\}$ and such that $i < j$ whenever there is an edge $i \leftarrow j$. Then

$$J_N(t_1) = \sum_{T \in \mathcal{D}_N} \int_{t_1 \geq t_2 \geq \cdots \geq t_{N+1} \geq 0} \prod_{(i \leftarrow j) \in Edge(T)} \frac{dt_j}{\sqrt{t_i - t_j}}. \tag{3.4.41}$$

Step 2: Summing over unlabeled trees. A planar tree is a rooted unlabeled tree where, for each vertex, the children of that vertex are ordered. We draw a planar tree so that its children are ordered from left to right. Here are the 5 planar rooted trees on 3+1 vertices:



Let $\mathcal{T}_k$ be the set of planar rooted trees with k vertices. In general, there are $\frac{(2N)!}{N!(N+1)!}$ (the Catalan number) elements in $\mathcal{T}_{N+1}$, see exercise 6.19 in [69]. We now group all the integrals in (3.4.41) with the same planar tree. For example, two different labeled trees (the second and forth in our list of labeled trees) both give the same unlabeled planar tree (which is the second in the above list). We redraw this unlabeled planar tree T_0 below and attach letters $\{a, b, c, d\}$ to T_0 for later use.



The integrands corresponding to the second and forth labeled trees are

$$\frac{dt_2\, dt_3\, dt_4}{\sqrt{(t_1 - t_2)(t_2 - t_4)(t_1 - t_3)}} \quad \text{and} \quad \frac{dt_2\, dt_3\, dt_4}{\sqrt{(t_1 - t_2)(t_2 - t_3)(t_1 - t_4)}}.$$

They are the same to

$$\frac{dt_b\, dt_c\, dt_d}{\sqrt{(t_a - t_b)(t_a - t_c)(t_b - t_d)}},$$

once we relabel the variables by the vertices of T_0. That is, $(1, 2, 3, 4) \to (a, b, c, d)$ for the first term and $(1, 2, 3, 4) \to (a, b, d, c)$ for the second.

We now go back and keep track of the bounds of integration. In the first integral, they are $t_a \geq t_b \geq t_c \geq t_d$ and, in the second integral, they are $t_a \geq t_b \geq t_d \geq t_c$. We can group these together as

$$t_a \geq t_b, \quad t_a \geq t_c, \quad t_b \geq t_d, \quad t_b > t_c,$$

which is the same as $t_a \geq t_b \geq t_c$ and $t_b \geq t_d$.

In general, the inequality constraints we have are of two types. First, whenever we have an edge $u \leftarrow v$, we get the inequality $t_u \geq t_v$. Second, if v and w are children of u with v to the left of w, then $t_v \geq t_w$. Let $P(T, t_1)$ be the polytope cut out by these inequalities where t_1 is the variable at the root. We have proved

$$J_N(t_1) = \sum_{T \in \mathcal{T}_{N+1}} \int_{P(T,t_1)} \prod_{(u \leftarrow v) \in Edge(T)} \frac{dt_v}{\sqrt{t_u - t_v}}, \tag{3.4.42}$$

Step 3: Grouping terms for which the root has degree k. We abbreviate

$$\omega(T, t_1) := \prod_{(u \leftarrow v) \in Edge(T)} \frac{dt_v}{\sqrt{t_u - t_v}}$$

if T has more than one vertex (otherwise $\omega(T, t_1) := 1$). Then (3.4.42) translates into

$$J_N(t_1) = \sum_{T \in \mathcal{T}_{N+1}} \int_{P(T,t_1)} \omega(T, t_1). \tag{3.4.43}$$

Fix an integer k and let T be a tree whose root has degree k. Removing the root leaves behind k children, denoted in chronical order by $t_2, \cdots t_{k+1}$, and k planar subtrees T_j having t_j as its root for $2 \leq j \leq k+1$. Then

$$\int_{P(T,t_1)} \omega(T, t_1) = \int_{t_1 \geq t_2 \geq \cdots \geq t_{k+1} \geq 0} \prod_{j=2}^{k+1} \frac{dt_j}{\sqrt{t_1 - t_j}} \int_{P(T_j,t_j)} \omega(T_j, t_j).$$

Hence, group together the terms where the root has degree k, we have

$$J_N(t_1) \quad = \quad \sum_{k=1}^{N} \sum_{T_2, \cdots, T_{k+1}} \int_{t_1 \geq t_2 \geq \cdots \geq t_{k+1} \geq 0} \prod_{j=2}^{k+1} \frac{dt_j}{\sqrt{t_1 - t_j}} \int_{P(T_j,t_j)} \omega(T_j, t_j). \tag{3.4.44}$$

Here the summation conditions include that $\sum_{j=2}^{k+1} |T_j| = N$ and each different ordering of $(T_2, T_3, \cdots, T_{k+1})$ are considered to be different, where $|T_j|$ is the number of vertices in T_j. This abbreviation applies whenever $\sum_{T_2, \cdots, T_{k+1}}$ appears.

On other hand, we have by applying (3.4.43) to each $J_{n_j-1}(t_j)$ below that

$$\begin{aligned}
&\sum_{k=1}^{N}\frac{1}{k!}\sum_{\substack{n_2+n_3+\cdots+n_{k+1}=N\\ n_2,\cdots,n_{k+1}\geq 1}}\prod_{j=2}^{k+1}\int_0^{t_1}\frac{J_{n_j-1}(t_j)}{\sqrt{t_1-t_j}}\,dt_j\\
=\;&\sum_{k=1}^{N}\frac{1}{k!}\sum_{\substack{n_2+n_3+\cdots+n_{k+1}=N\\ n_2,\cdots,n_{k+1}\geq 1}}\prod_{j=2}^{k+1}\sum_{|T_j|=n_j}\int_0^{t_1}\frac{dt_j}{\sqrt{t_1-t_j}}\int_{P(T_j,t_j)}\omega(T_j,t_j)\\
=\;&\sum_{k=1}^{N}\frac{1}{k!}\sum_{T_2,\cdots,T_{k+1}}\prod_{j=2}^{k+1}\int_0^{t_1}\frac{dt_j}{\sqrt{t_1-t_j}}\int_{P(T_j,t_j)}\omega(T_j,t_j)\\
=\;&\sum_{k=1}^{N}\frac{1}{k!}\sum_{T_2,\cdots,T_{k+1}}\int_{[0,t_1]^{k+1}}\prod_{j=2}^{k+1}\frac{dt_j}{\sqrt{t_1-t_j}}\int_{P(T_j,t_j)}\omega(T_j,t_j). \qquad (3.4.45)
\end{aligned}$$

Step 4: Identifying the integrals. It remains to show that (3.4.44) is equal to (3.4.45). Let $\mathcal{S}_k$ denote the space of permutations of $\{2, 3, \cdots, k+1\}$. Then the right hand side of (3.4.45) is equal to

$$\begin{aligned}
&\sum_{k=1}^{N}\frac{1}{k!}\sum_{T_2,\cdots,T_{k+1}}\sum_{\sigma\in\mathcal{S}_k}\int_{t_1\geq t_{\sigma(2)}\geq\cdots\geq t_{\sigma(k+1)}\geq 0}\prod_{j=2}^{k+1}\frac{dt_{\sigma(j)}}{\sqrt{t_1-t_{\sigma(j)}}}\int_{P(T_{\sigma(j)},t_{\sigma(j)})}\omega(T_{\sigma(j)},t_{\sigma(j)})\\
=\;&\sum_{k=1}^{N}\frac{1}{k!}\sum_{\sigma\in\mathcal{S}_k}\sum_{T_2,\cdots,T_{k+1}}\int_{t_1\geq s_2\geq\cdots\geq s_{k+1}\geq 0}\prod_{j=2}^{k+1}\frac{ds_j}{\sqrt{t_1-s_j}}\int_{P(T_{\sigma(j)},s_j)}\omega(T_{\sigma(j)},s_j)\\
=\;&\sum_{k=1}^{N}\frac{1}{k!}\sum_{\sigma\in\mathcal{S}_k}\sum_{T_2,\cdots,T_{k+1}}\int_{t_1\geq s_2\geq\cdots\geq s_{k+1}\geq 0}\prod_{j=2}^{k+1}\frac{ds_j}{\sqrt{t_1-s_j}}\int_{P(T_j,s_j)}\omega(T_j,s_j)\\
=\;&\sum_{k=1}^{N}\sum_{T_2,\cdots,T_{k+1}}\int_{t_1\geq t_2\geq\cdots\geq t_{k+1}\geq 0}\prod_{j=2}^{k+1}\frac{dt_j}{\sqrt{t_1-t_j}}\int_{P(T_j,t_j)}\omega(T_j,t_j),
\end{aligned}$$

which is $J_N(t)$ by (3.4.44). This completes the proof of the lemma. □

Proof of uniqueness

Proof of Theorem 3.4.7. By Lemma 3.4.8, Lemma 3.4.9 and Lemma 3.4.10, we have

$$\|\beta_t^{(n,m)}\|_{(n,m)} \le C_1(n,m)\, C_2(D_+, D_-, T)^N\, t^{N/2} \tag{3.4.46}$$

for all $t \in [0,T]$ and $N \in \mathbb{N}$. This implies that there is a constant $\tau > 0$ so that $\|\beta_t^{(n,m)}\|_{(n,m)} = 0$ for $t \le \tau$ and for all $(n,m) \in \mathbb{N} \times \mathbb{N}$. Note that $\tilde{\beta}_t := \beta_{\tau+t}$ also satisfies the hierarchy (3.4.30), and that $\tilde{\beta}_0 = 0$. Using the hypothesis $\|\beta_t^{(n,m)}\|_{(T,n,m)} \le C^{n+m}$, we can extend to obtain $\|\beta_t^{(n,m)}\|_{(n,m)} = 0$ for $t \in [0,T]$. □

3.5 Hydrodynamic Limits

With all the machinery developed, we can now prove Theorem 3.3.1 by a standard procedure in this section. The reader is suggested to recall the conditions of Theorem 3.3.1 which are assumed for the rest of this section.

3.5.1 Martingales

Since $\eta_t = \eta_t^\varepsilon$ has a finite state space, we know that for all bounded function $F : \mathbb{R}_+ \times E^\varepsilon \to \mathbb{R}$ that is smooth in the first coordinate with $\sup_{(s,x)} \left|\frac{\partial F}{\partial s}(s,x)\right| < C < \infty$, we have two $\mathfrak{F}_t^\eta$-martingales below:

$$M(t) := F(t,\eta_t) - F(0,\eta_0) - \int_0^t \frac{\partial F}{\partial s}(s,\eta_s) + \mathfrak{L}F(s,\cdot)(\eta_s)ds \tag{3.5.1}$$

and

$$N(t) := M(t)^2 - \int_0^t \mathfrak{L}(F^2(s,\cdot))(\eta_s) - 2F(s,\eta_s)\mathfrak{L}F(s,\cdot)(\eta_s)ds, \tag{3.5.2}$$

where $\mathfrak{L} = \mathfrak{L}^\varepsilon$ is the generator defined in (3.1.2). See Lemma 5.1 (p.330) of [50] or Proposition 4.1.7 of [35] for a proof. We will use this fact to construct some important martingales in Lemma 3.5.1 below.

In general, suppose $X = (X_t)_{t \ge 0}$ is a CTRW in a finite state space E, whose one step transition probability is p_{xy} and mean holding time at x is $h(x)$. Its infinitesimal generator of X is the discrete operator

$$\mathcal{A}f(x) := \frac{1}{h(x)} \sum_{y\in E} p_{xy}(f(y) - f(x)).$$

The formal adjoint $\mathcal{A}^*$ of $\mathcal{A}$ is defined by

$$\mathcal{A}^* f(x) := \sum_{y\in E} \Big(\frac{1}{h(y)} p_{yx} f(y) - \frac{1}{h(x)} p_{xy} f(x) \Big).$$

It can be easily checked that

$$\langle f,\, \mathcal{A}^* g\rangle = \langle \mathcal{A}f,\, g\rangle, \quad \text{where } \langle f, g\rangle := \frac{1}{N}\sum_{x\in E} f(x)g(x). \tag{3.5.3}$$

We denote by $\mathcal{A}^{\pm}_{\varepsilon}$ the generator of the CTRW $X^{\pm,\varepsilon}$ on $D^{\varepsilon}_{\pm}$, respectively, and by $\mathcal{A}^{*,\pm}_{\varepsilon}$ the corresponding formal adjoint. In our case, $h(x) = h_\varepsilon(x) = \varepsilon^2/d$ for all x. We can check that if $f \in C^2(D_\pm)$, then

$$\lim_{\varepsilon\to 0} \mathcal{A}^{\pm}_{\varepsilon} f(x^\varepsilon) = \mathcal{A}^{\pm} f(x) \text{ whenever } x^\varepsilon \in D^{\varepsilon}_{\pm} \text{ converges to } x \in D_\pm. \tag{3.5.4}$$

Lemma 3.5.1. *For any $\phi \in \mathcal{B}_b(D_+)$,*

$$M(t) := M^{+,N}_{\phi}(t) := \langle \phi,\, \mathfrak{X}^{N,+}_t\rangle - \langle \phi,\, \mathfrak{X}^{N,+}_0\rangle - \int_0^t \langle \mathcal{A}^{+}_{\varepsilon}\phi,\, \mathfrak{X}^{N,+}_s\rangle ds + \lambda \int_0^t \langle J^{N,+}_s,\, \phi\rangle\, ds \tag{3.5.5}$$

is an $\mathfrak{F}^{\eta}_{t}$-martingale for $t \geq 0$, where $J^{N,+}$ is the measure-valued process defined by (3.3.5). Moreover, if $\phi \in C^1(\overline{D}_+)$, then there is a constant $C > 0$ independent of N so that for every $T > 0$,

$$\mathbb{E}\Big[\sup_{t\in[0,T]} M^2(t) \Big] \leq \frac{C\,T}{N}. \tag{3.5.6}$$

Similar statements hold for $\mathfrak{X}^{N,-}$.

Proof The lemma follows by applying (3.5.1) and (3.5.2) to the function

$$F(s,\eta) := f(\eta) := \frac{1}{N}\sum_{x\in D_+} \phi^+(x)\eta^+(x) \quad , (s,\eta) \in [0,\infty)\times E^\varepsilon.$$

We spell out the details here for completeness. Observe that $f(\eta_t) = \langle \phi, \mathfrak{X}_t^{N,+}\rangle$. Fix $x_0 \in D_+^\varepsilon$, and define $\eta_{x_0}^+$ to be the function from E^ε to $\mathbb{R}$ which maps η to $\eta^+(x_0)$. Then by the definition of $\mathfrak{L} = \mathfrak{L}^\varepsilon$ in (3.1.2),

$$\mathfrak{L}\eta_{x_0}^+(\eta) = \left(\mathcal{A}_\varepsilon^{*,+}\eta^+\right)(x_0) - \sum_{\{z\in I^\varepsilon:\, z_+ = x_0\}} \frac{\lambda}{\varepsilon}\,\Psi(z)\,\eta_t^+(z_+)\eta_t^-(z_-)\,. \tag{3.5.7}$$

Similarly, for $y_0 \in D_-^\varepsilon$, we have

$$\mathfrak{L}\eta_{y_0}^-(\eta) = \left(\mathcal{A}_\varepsilon^{*,-}\eta^-\right)(y_0) - \sum_{\{z\in I^\varepsilon:\, z_- = y_0\}} \frac{\lambda}{\varepsilon}\,\Psi(z)\,\eta_t^+(z_+)\eta_t^-(z_-). \tag{3.5.8}$$

Hence, by linearity of $\mathfrak{L}$, (3.5.7) and then (3.5.3), we have

$$\begin{aligned}
\mathfrak{L}f(\eta) &= \frac{1}{N}\sum_{x\in D_+} \phi(x)(\mathfrak{L}\eta_x^+)(\eta) \\
&= \frac{1}{N}\sum_{x\in D_+} \phi(x)(\mathcal{A}_\varepsilon^{*,+}\eta^+(x)) - \frac{\lambda}{N\varepsilon}\sum_{z\in I^\varepsilon} \Psi(z)\,\eta^+(z_+)\eta^-(z_-)\,\phi(z_+) \\
&= \frac{1}{N}\sum_{x\in D_+} \eta^+(x)(\mathcal{A}_\varepsilon^{+}\phi(x)) - \frac{\lambda}{N\varepsilon}\sum_{z\in I^\varepsilon} \Psi(z)\,\eta^+(z_+)\eta^-(z_-)\,\phi(z_+).
\end{aligned}$$

Hence

$$\mathfrak{L}f(\eta_s) = \langle \mathcal{A}_\varepsilon^+\phi, \mathfrak{X}_s^{N,+}\rangle - \frac{\lambda}{N\,\varepsilon^d}\,\langle J_s^{N,+},\, \phi\rangle \tag{3.5.9}$$

and $M(t)$ is an $\mathfrak{F}_t^\eta$-martingale by (3.5.1). Next, we compute $\mathbb{E}[\langle M\rangle_t]$. Note that

$$\mathfrak{L}(f^2)(\eta) = \frac{1}{N^2}\sum_{a\in D_+^\varepsilon}\sum_{b\in D_+^\varepsilon} \phi^+(a)\phi^+(b)\,\mathfrak{L}(\eta_a\eta_b)(\eta),$$

where $\mathfrak{L}(\eta_a\eta_b)$ can be computed explicitly using (3.1.2). Hence from (3.5.2), we can check that

$$\mathbb{E}[M^2(t)] = \mathbb{E}[\langle M\rangle_t] = \mathbb{E}\left[\int_0^t \mathfrak{L}(f^2)(\eta_s) - 2f(\eta_s)\mathfrak{L}f(\eta_s)\,ds\right] = \int_0^t \mathbb{E}[g(\eta_r)]dr,$$

where

$$\begin{aligned} g(\eta) &= \frac{1}{N^2}\left(\sum_{y,z\in D_+} \eta^+(z)h^{-1}(z)p_{zy}(\phi(y)-\phi(z))^2 + \frac{\lambda}{\varepsilon}\sum_{z\in I^\varepsilon} \Psi(z)\,\eta(z_+)\eta^-(z_-)\,(-\phi(z_+))^2\right) \\ &\leq \frac{1}{N^2}\left(\varepsilon^2\|\nabla\phi\|^2\sum_{x\in D_+}\eta(x)h^{-1}(x) + \frac{\lambda}{\varepsilon}\,\|\phi\|^2\sum_{z\in I^\varepsilon}\Psi(z)\,\eta^+(z_+)\eta^-(z_-)\right) \\ &\leq \frac{d\,\|\nabla\phi\|^2}{N} + \frac{\lambda\,\|\phi\|^2}{N^2}\left(\frac{1}{\varepsilon}\sum_{z\in I^\varepsilon}\Psi(z)\,\eta^+(z_+)\eta^-(z_-)\right). \end{aligned}$$

After taking expectation for $g(\eta_r)$, the first term is the last display is of order at most $1/N$ since $\phi \in C^1(\overline{D}_+)$, while the second term inside the bracket is of order at most $1/N$, uniformly in $r \in [0,t]$, by (3.3.7). Hence $\mathbb{E}[M^2(t)] \leq \frac{C}{N}$ for some $C = C(\phi, d, D_\pm, \lambda)$. Doob's maximal inequality then gives (3.5.6). □

Remark 3.5.2. From the second term of (3.5.9), we see that if the parameter of the killing time is of order λ/ε, then we need $N\varepsilon^d$ to be comparable to 1. □

3.5.2 Tightness

The following simple observation is useful for proving tightness when the transition kernel of the process has a singularity at $t=0$. It says that we can break down the analysis of the fluctuation of functionals of a process on $[0,T]$ into two parts. One part is near $t=0$, and the other is away from $t=0$ where we have a bound for a higher moment. Its proof, which is based on the Prohorov's theorem, is simple and is omitted.

Lemma 3.5.3. *Let $\{Y_N\}$ be a sequence of real valued processes such that $t \mapsto \int_0^t Y_N(r)\,dr$ is continuous on $[0,T]$ a.s., where $T \in [0,\infty)$. Suppose the following holds:*

(i) There exists $q > 1$ such that $\overline{\lim}_{N\to\infty} \mathbb{E}[\int_h^T |Y_N(r)|^q \, dr] < \infty$ for any $h > 0$,

(ii) $\lim_{\alpha\searrow 0} \overline{\lim}_{N\to\infty} \mathbb{P}(\int_0^\alpha |Y_N(r)| \, dr > \varepsilon_0) = 0$ for any $\varepsilon_0 > 0$.

Then $\left\{ \int_0^t Y_N(r) \, dr; t \in [0,t] \right\}_{N\in\mathbb{N}}$ is tight in $C([0,T],\mathbb{R})$.

Here is our tightness result for $\{(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})\}$. We need Lemma 3.5.3 in the proof mainly because we do not know if $\overline{\lim}_{N\to\infty} \mathbb{E} \int_0^T \langle \mathcal{A}_\varepsilon^+ \varphi_+, \mathfrak{X}_s^{N,+} \rangle^2 \, ds$ is finite or not.

Theorem 3.5.4. *The sequence $\{(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})\}$ is relatively compact in $D([0,T],\mathfrak{E})$ and any subsequential limit of the laws of $\{(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})\}$ carries on $C([0,T],\mathfrak{E})$. Moreover, for all $\varphi_\pm \in C^2(\overline{D}_\pm)$,*

$$\left\{ \int_0^t \langle J_s^{N,+}, \varphi_\pm \rangle \, ds \right\}, \quad \left\{ \int_0^t \langle \mathcal{A}_\varepsilon^+ \varphi_+, \mathfrak{X}_s^{N,+} \rangle \, ds \right\} \text{ and } \left\{ \int_0^t \langle \mathcal{A}_\varepsilon^- \varphi_-, \mathfrak{X}_s^{N,-} \rangle \, ds \right\}$$

are all tight in $C([0,T],\mathbb{R})$.

Proof We write $\mathfrak{X}^\pm$ in place of $\mathfrak{X}^{N,\pm}$ for convenience. By Stone-Weierstrass Theorem, $C^2(\overline{D}_\pm)$ is dense in $C(\overline{D}_\pm)$ in uniform topology. It suffices to check that $\{(\langle \mathfrak{X}^+, \phi^+ \rangle, \langle \mathfrak{X}^-, \phi^- \rangle)\}$ is relatively compact in $D([0,T],\mathbb{R}^2)$ for all $\phi^\pm \in C^2(\overline{D}_\pm)$ (cf. Proposition 1.7 (p.54) of [50]) for this weak tightness criterion). By Prohorov's theorem (see, for example, Theorem 1.3 and Remark 1.4 of [50]), $\{(\langle \mathfrak{X}^+, \phi^+ \rangle, \langle \mathfrak{X}^-, \phi^- \rangle)\}$ is relatively compact in $D([0,T],\mathbb{R}^2)$ if (1) and (2) below holds:

(1) For all $t \in [0,T]$ and $\varepsilon_0 > 0$, there exists a compact set $K(t,\varepsilon_0) \subset \mathbb{R}^2$ such that

$$\sup_N \mathbb{P}\left((\langle \mathfrak{X}_t^+, \phi^+ \rangle, \langle \mathfrak{X}_t^-, \phi^- \rangle) \notin K(t,\varepsilon_0) \right) < \varepsilon_0;$$

(2) For all $\varepsilon_0 > 0$,

$$\lim_{\delta\to 0} \overline{\lim_{N\to\infty}} \mathbb{P}\left(\sup_{\substack{|t-s|<\delta \\ 0\le s,t\le T}} \left| \Big(\langle \mathfrak{X}_t^+, \phi^+ \rangle, \langle \mathfrak{X}_t^-, \phi^- \rangle \Big) - \Big(\langle \mathfrak{X}_s^+, \phi^+ \rangle, \langle \mathfrak{X}_s^-, \phi^- \rangle \Big) \right|_{\mathbb{R}^2} > \varepsilon_0 \right) = 0.$$

We first check (1). Since $\phi^{\pm}$ is bounded on $\overline{D}_{\pm}$ and $|\langle \mathfrak{X}_t^+, 1\rangle| \le 1$ for all $t \in [0,\infty)$, we can always take $K = [-\|\phi^+\|, \|\phi^+\|] \times [-\|\phi^-\|, \|\phi^-\|]$.

To verify (2), since $|(x_1,y_1)-(x_2,y_2)|_{\mathbb{R}^2} \le |x_1-x_2| + |y_1-y_2|$, we only need to focus on $\mathfrak{X}^+$. By Lemma 3.5.1,

$$\Big|\langle \phi, \mathfrak{X}_t^+\rangle - \langle \phi, \mathfrak{X}_s^+\rangle\Big| = \Big| \int_s^t \langle \mathcal{A}_\varepsilon^+\, \phi, \mathfrak{X}_r^+\rangle dr - \int_s^t \frac{\lambda}{N\,\varepsilon^d}\, \langle J_r^{N,+},\, \phi\rangle\, dr + (M_\phi(t) - M_\phi(s)) \Big|. \quad (3.5.10)$$

So we only need to verify (2) with $\langle \phi, \mathfrak{X}_t^+\rangle - \langle \phi, \mathfrak{X}_s^+\rangle$ replaced by each of the 3 terms on RHS of the above equation (3.5.10).

For the first term of (3.5.10), we apply Lemma 3.5.3 for the case $q = 2$ and $Y_N(r) = \langle \mathcal{A}_\varepsilon^+\, \phi,\, \mathfrak{X}_r^+\rangle$. Since $\phi \in C^2(\overline{D}_+)$, we have

$$\sup_{x \in D^\varepsilon \setminus \partial D^\varepsilon} |\mathcal{A}_\varepsilon^+ \phi(x)| \le C(\phi) \text{ and } \sup_{x \in \partial D^\varepsilon} |\varepsilon\, \mathcal{A}_\varepsilon^+ \phi(x)| \le C(\phi)$$

for some constant $C(\phi)$ which only depends on ϕ. Using Lemma 2.2.25, we have

$$\mathbb{E}\left[\langle |\mathcal{A}_\varepsilon^+ \phi|,\, \mathfrak{X}_r^{N,+}\rangle\right] \le \frac{1}{N}\sum_{i=1}^N \sum_{D^\varepsilon} |\mathcal{A}_\varepsilon^+ \phi(\cdot)| p^{\varepsilon,+}(r, x_i, \cdot) m_\varepsilon(\cdot) \le C_1(d, D_+, \phi) + \frac{C_2(d, D_+, \phi)}{\varepsilon \vee r^{1/2}},$$

which is in $L^1[0,T]$ as a function in r. This implies hypothesis (ii) of Lemma 3.5.3, via the Chebyshev's inequality. Hypothesis (i) of Lemma 3.5.3 can be verified easily using the upper bound (3.4.17) for the correlation function, or by direct comparison to the process without annihilation:

$$\begin{aligned}
&\mathbb{E}\left[\langle |\mathcal{A}_\varepsilon \phi|,\, \mathfrak{X}_r^+\rangle^2\right] \\
\le\ & \left(\frac{1}{N}\sum_{i=1}^N \mathbb{E}[\mathcal{A}_\varepsilon \phi(X_r^i)]\right)^2 + \frac{1}{N^2}\sum_{i=1}^N \mathbb{E}\left[(\mathcal{A}_\varepsilon \phi)^2(X_r^i)\right] - \frac{1}{N^2}\sum_{i=1}^N \left(\mathbb{E}[\mathcal{A}_\varepsilon \phi(X_r^i)]\right)^2 \\
\le\ & C(d, D, \phi)\left(1 + \frac{1}{\sqrt{r}} + \frac{1}{r}\right).
\end{aligned}$$

For the second term of (3.5.10), by (3.3.7) we have $\overline{\lim}_{N\to\infty} \mathbb{E}\left[\int_0^T \langle 1,\, J_r^N\rangle^2\, dr\right] < \infty$. Hence (2) holds for this term by Lemma 3.5.3.

For the third term of (3.5.10), by Chebyshev's inequality, Doob's maximal inequality and Lemma 3.5.1, we have

$$\begin{aligned}\mathbb{P}\left(\sup_{|t-s|<\delta} |M_\phi(t) - M_\phi(s)| > \varepsilon_0\right) &\leq \frac{1}{\varepsilon_0^2}\,\mathbb{E}\left[\left(\sup_{|t-s|<\delta} |M_\phi(t) - M_\phi(s)|\right)^2\right] \\ &\leq \frac{1}{\varepsilon_0^2}\,\mathbb{E}\left[\left(2\sup_{t\in[0,T]} |M_\phi(t)|\right)^2\right] \\ &\leq 16\mathbb{E}[M_\phi(T)^2] \leq \frac{C}{N}.\end{aligned}$$

We have proved that (2) is satisfied. Hence $\{(\mathfrak{X}^{N,+},\, \mathfrak{X}^{N,-})\}$ is relatively compact. Using (2) and the metric of $\mathfrak{E}$, we can check that any subsequential limit L^* of the laws of $\{(\mathfrak{X}^{N,+},\, \mathfrak{X}^{N,-})\}$ concentrates on $C([0,\infty), \mathfrak{E})$. □

Remark 3.5.5. In general, to prove tightness for $(X_n,\, Y_n)$ in $D([0,T], A\times B)$, it is NOT enough to prove tightness separately for (X_n) and (Y_n) in $D([0,T], A)$ and $D([0,T], B)$ respectively. (However, the latter condition implies tightness in $D([0,T], A) \times D([0,T], B)$ trivially). See Exercise 22(a) in Chapter 3 of [35]. For example, $\left(\mathbf{1}_{[1+\frac{1}{n},\infty)},\, \mathbf{1}_{[1,\infty)}\right)$ converges in $D_{\mathbb{R}}[0,\infty) \times D_{\mathbb{R}}[0,\infty)$ but not in $D_{\mathbb{R}^2}[0,\infty)$. The reason is that the two processes can jump at different times ($t = 1$ and $t = 1 + \frac{1}{n}$) that become identified in the limit (only one jump at $t = 1$); this can be avoided if one of the two processes is C-tight (i.e. has only continuous limiting values), which is satisfied in our case since $\mathfrak{X}^{N,+}$ and $\mathfrak{X}^{N,-}$ turns out to be both C-tight. □

Remark 3.5.6. Even without condition (ii) of Theorem 3.3.1 for η_0, we can still verify hypothesis (i) of Lemma 3.5.3. Actually, applying (3.5.5) to suitable test functions, we have

$$\lim_{\alpha\to 0} \overline{\lim_{N\to\infty}}\, \mathbb{E}\left[\int_0^\alpha \langle J_s^{N,+},\, 1\rangle\, ds\right] = 0. \tag{3.5.11}$$

□

3.5.3 Identifying the limit

Suppose $(\mathfrak{X}^{\infty,+},\, \mathfrak{X}^{\infty,-})$ is a subsequential limit of $(\mathfrak{X}^{N,+},\, \mathfrak{X}^{N,-})$, say the convergence is along the subsequence $\{N'\}$. By the Skorokhod representation Theorem, the continuity of the limit in t and [35, Theorem 3.10.2], there exists a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ such that

$$\lim_{N'\to\infty} \sup_{t\in[0,T]} \left\| (\mathfrak{X}_t^{+,N'},\, \mathfrak{X}_t^{-,N'}) - (\mathfrak{X}_t^{\infty,+},\, \mathfrak{X}_t^{\infty,-}) \right\|_{\mathfrak{E}} = 0 \quad \mathbb{P}\text{-a.s,} \tag{3.5.12}$$

Hence we have for any $t > 0$ and $\phi \in C(\overline{D}_+)$,

$$\lim_{N'\to\infty} \mathbb{E}[\langle \mathfrak{X}_t^{+}, \phi\rangle] = \mathbb{E}[\langle \mathfrak{X}_t^{+,\infty}, \phi\rangle] \quad \text{and} \quad \lim_{N'\to\infty} \mathbb{E}[(\langle \mathfrak{X}_t^{+}, \phi\rangle)^2] = \mathbb{E}[(\langle \mathfrak{X}_t^{+,\infty}, \phi\rangle)^2].$$

Combining with Corollary 3.3.4, we have

$$\langle \mathfrak{X}_t^{+,\infty}, \phi\rangle = \langle u_+(t),\, \phi\rangle_{\rho_+} \quad \mathbb{P}\text{-a.s. for every } t \geq 0 \text{ and for } \phi.$$

Here we have used the simple fact that if $\mathbb{E}[X] = (\mathbb{E}[X^2])^{1/2} = a$, then $X = a$ a.s.

Suppose $\{\phi_k\}$ is a countable dense subset of $C(\overline{D}_+)$. Then for every $t \geq 0$,

$$\langle \mathfrak{X}_t^{\infty,+}, \phi_k\rangle = \langle u_+(t),\, \phi_k\rangle_{\rho_+} \quad \text{for every } k \geq 1 \ \mathbb{P}\text{-a.s.}$$

Since $\mathfrak{X}^{\infty,+} \in C((0,\infty), M_+(\overline{D}_+))$, we can pass to rational numbers to obtain

$$\langle \mathfrak{X}_t^{\infty,+}, \phi_k\rangle = \langle u_+(t),\, \phi_k\rangle_{\rho_+} \quad \text{for every } t \geq 0 \text{ and } k \geq 1 \ \mathbb{P}\text{-a.s.}$$

Hence,

$$\mathfrak{X}_t^{\infty,+}(dx) = u_+(t,x)\, \rho_+(x) dx \quad \text{for every } t \geq 0 \quad \mathbb{P}\text{-a.s.}$$

Similarly,

$$\mathfrak{X}_t^{\infty,-}(dy) = u_-(t,y)\, \rho_-(y) dy \quad \text{for every } t \geq 0 \quad \mathbb{P}\text{-a.s.}$$

In conclusion, any subsequential limit is the dirac delta measure

$$\delta_{u_+(t,x)dx,u_-(t,y)dy} \in M_1(D([0,\infty),\mathfrak{E})).$$

This together with Theorem 3.5.4 completes the proof of Theorem 3.3.1. $\square$

Chapter 4

HYDRODYNAMIC LIMITS FOR INTERACTING DIFFUSIONS

In this chapter, we construct the annihilating diffusion model described informally in the Introduction (Chapter 1) and establish the functional law of large numbers for this new system, thereby extending the hydrodynamic limit in Chapter 3 (and that in [16]) to reflected diffusions in domains with mixed-type boundary conditions, which include absorption (harvest of electric charges). We employ a new and direct approach that circumvents the delicate BBGKY hierarchy.

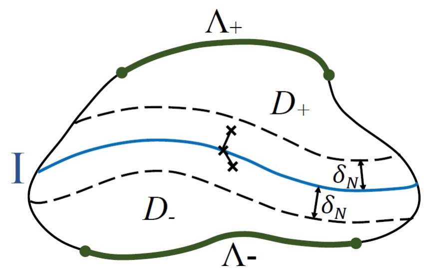


Figure 4.1: Annihilating diffusion model

Assumptions and notations

Recall that before being annihilated by a particle of the opposite kind near I, a particle in $D_\pm$ performs a reflected diffusion with absorption on $\Lambda_\pm \subset \partial D_\pm \setminus I$. If a particle is absorbed (in $\Lambda_\pm$) rather than annihilated (near I), it is considered to be harvested.

The following assumptions are in force throughout this chapter.

Assumption 4.0.7. *(Geometric setting) Suppose D_+ and D_- are given adjacent bounded Lipschitz domains in $\mathbb{R}^d$ such that $I := \overline{D}_+ \cap \overline{D}_- = \partial D_+ \cap \partial D_-$ is $\mathcal{H}^{d-1}$-rectifiable. $\Lambda_\pm$ is a closed subset of $\overline{D}_\pm \setminus I$ which is regular with respect to the $(\boldsymbol{a}_\pm,\, \rho_\pm)$-reflected diffusion $X^\pm$, where $\rho_\pm \in W^{(1,2)}(D_\pm) \cap C(\overline{D}_\pm)$ is a given strictly positive function, $\boldsymbol{a}_\pm = (a^{ij}_\pm)$ is a symmetric, bounded, uniformly elliptic $d \times d$ matrix-valued function such that $a^{ij}_\pm \in W^{1,2}(D_\pm)$ for each $i,\, j$.*

Assumption 4.0.8. *(Parameter of annihilation) Suppose $\lambda \in C_+(I)$ is a given non-negative continuous function on I. Let $\widehat{\lambda} \in C(\overline{D}_+ \times \overline{D}_-)$ be an arbitrary but fixed extension of λ in the sense that $\widehat{\lambda}(z,z) = \lambda(z)$ for all $z \in I$. (Such $\widehat{\lambda}$ always exists.)*

Assumption 4.0.9. *(The annihilation potential) We choose $\{\ell_\delta : \delta > 0\} \subset C_+(\overline{D}_+ \times \overline{D}_-)$ in such a way that $\ell_\delta(x,y) \le \frac{\widehat{\lambda}(x,y)}{c_{d+1}\,\delta^{d+1}} \mathbf{1}_{I^\delta}(x,y)$ on $D_+ \times D_-$ and*

$$\lim_{\delta \to 0} \left\| \ell_\delta - \frac{\widehat{\lambda}}{c_{d+1}\,\delta^{d+1}} \mathbf{1}_{I^\delta} \right\|_{L^2(D_+ \times D_-)} = 0, \tag{4.0.1}$$

where $I^\delta := \{(x,y) \in D_+ \times D_- :\, |x-z|^2 + |y-z|^2 < \delta^2 \text{ for some } z \in I\}$ and c_{d+1} is the volume of the unit ball in $\mathbb{R}^{d+1}$. See (1.2.3) in Introduction for the motivation for the definition of ℓ_δ.

Assumption 4.0.9 is natural in view of (1.2.3). Intuitively, if N is the initial number of particles, then $\delta = \delta_N$ is the annihilation distance and I^{δ_N} controls the frequency of interactions. As remarked in the introduction, we need to assume that the annihilation distance δ_N does not shrink too fast. This is formulated in Assumption 4.0.10 below:

Assumption 4.0.10. *(The annihilation distance) $\liminf_{N\to\infty} N\,\delta_N^d \in (0,\infty]$, where $\{\delta_N\} \subset (0,\infty)$ converges to 0 as $N \to \infty$.*

Convention: To simplify notation, we suppress $\Lambda_\pm$ and write $X^\pm$ in place of $X^{\Lambda_\pm}$ for a $(\mathbf{a}_\pm,\, \rho_\pm)$-reflected diffusions on $D_\pm$ killed upon hitting $\Lambda_\pm$. We also use $p^\pm(t,x,y)$, $P^\pm_t$ and $\mathcal{A}^\pm$ to denote, respectively, the transition density w.r.t. $\rho_\pm$, the semigroup associated to $p^\pm(t,x,y)$ and the $C_\infty(\overline{D}_\pm \setminus \Lambda_\pm)$-generator (called the Feller generator) of $X^\pm = X^{\Lambda_\pm}$. Under Assumption

5.1.1, $X^{\pm}$ is a Hunt (hence strong Markov) process on

$$D^{\partial}_{\pm} := \left(\overline{D}_{\pm} \setminus \Lambda^{\pm}\right) \cup \{\partial^{\pm}\},$$

where $\partial^{\pm}$ is the cemetery point for $X^{\pm}$ (see Proposition 2.1.3).

4.1 Annihilating diffusion model

In this section, we fix $N \in \mathbb{N}$ and construct the configuration process $\underline{\mathbf{X}}^{(N)}$ and the normalized empirical measure process $(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})$ for our annihilating diffusion system. In the construction, we will label (rather than annihilate) pairs of particles to keep track of the annihilated particles. This provides a *coupling* of our annihilating particle system and the corresponding system without annihilation.

Let $m \in \{1, 2, \cdots, N\}$ (in fact, m can be any positive integer). Starting with m points in each of D^{∂}_{+} and D^{∂}_{-}, we perform the following construction:

Let $\{X_i^{\pm} = X_i^{\Lambda_{\pm}}\}_{i=1}^m$ be $(\mathbf{a}_{\pm}, \rho_{\pm})$-reflected diffusions on $D_{\pm}$ killed upon hitting $\Lambda_{\pm}$, starting from the given points in $D^{\partial}_{\pm}$. These $2m$ processes are constructed to be mutually independent. In case $X_i^{\pm}$ starts at the cemetery point $\partial^{\pm}$, we have $X_i^{\pm}(t) = \partial^{\pm}$ for all $t \geq 0$. Let $\{R_k\}_{k=1}^m$ be i.i.d. exponential random variables with parameter one which are independent of $\{X_i^{+}\}_{i=1}^m$ and $\{X_i^{-}\}_{j=1}^m$.

Define the first time of labeling (or annihilation) to be

$$\tau_1 := \inf \left\{ t \geq 0 : \frac{1}{2N} \int_0^t \sum_{i=1}^m \sum_{j=1}^m \ell_{\delta_N}(X_i^{+}(s), X_j^{-}(s))\, ds \geq R_1 \right\}. \tag{4.1.1}$$

In the above, $\ell_{\delta_N}(x, y) = 0$ if either $x = \partial^{+}$ or $y = \partial^{-}$. Hence particles absorbed at $\Lambda_{\pm}$ do not contribute to rate of labeling (or annihilation). At τ_1, we label exactly one pair in $\{(i, j)\}$

according to the probability distribution given by

$$\frac{\ell_{\delta_N}(X_i^+(\tau_1-), X_j^-(\tau_1-))}{\sum_{p=1}^m \sum_{q=1}^m \ell_{\delta_N}(X_p^+(\tau_1-), X_q^-(\tau_1-))} \quad \text{assigned to } (i,j).$$

Denote (i_1, j_1) to be the labeled pair at τ_1 (think of the labeled pair as begin removed due to annihilation of the corresponding particles).

We repeat this labeling procedure using the remaining unlabeled $2(m-1)$ particles. Precisely, for $k = 2, 3, \cdots, m$, we define

$$\tau_k := \inf \left\{ t \geq 0 : \frac{1}{2N} \int_{\tau_1+\cdots+\tau_{k-1}}^{\tau_1+\cdots+\tau_{k-1}+t} \sum_{i \notin \{i_1,\cdots,i_{l-1}\}} \sum_{j \notin \{j_1,\cdots,j_{l-1}\}} \ell_{\delta_N}(X_i^+(s), X_j^-(s))\, ds \geq R_k \right\}.$$

At $\sigma_k := \tau_1 + \tau_2 + \cdots + \tau_{k-1} + \tau_k$, the k-th time of labeling (annihilation), we label exactly one pair (i_k, j_k) in $\{\, (i,j) : i \notin \{i_1, \cdots, i_{k-1}\},\ j \notin \{j_1, \cdots, j_{k-1}\} \,\}$ according to the probability distribution given by

$$\frac{\ell_{\delta_N}(X_i^+(\sigma_k-), X_i^-(\sigma_k-))}{\sum_{i \notin \{i_1,\cdots,i_{k-1}\}} \sum_{j \notin \{j_1,\cdots,j_{k-1}\}} \ell_{\delta_N}(X_i^+(\sigma_k-), X_i^-(\sigma_k-))} \quad \text{assigned to } (i,j).$$

We will study the evolution of the *unlabeled (or surviving)* particles, which is described in detail below.

4.1.1 Configuration process

We denote $D_\pm^\partial(m)$ the space of unordered m-tuples of elements in $D_\pm^\partial := \left(\overline{D}_\pm \setminus \Lambda^\pm\right) \cup \{\partial^\pm\}$. The **configuration space** for the particles is defined as

$$S_N := \cup_{m=1}^N \left(D_+^\partial(m) \times D_-^\partial(m)\right) \cup \{\partial\}, \tag{4.1.2}$$

where ∂ is a cemetery point (different from $\partial^\pm$).

We define $\underline{\mathbf{X}}_t^{(N)} \in S_N$ to be the following unordered list of (the position of) *unlabeled* (surviving) particles at time t. That is,

$$\underline{\mathbf{X}}_t^{(N)} := \begin{cases} \big(\{X_1^+(t),\cdots,X_m^+(t)\},\ \{X_1^-(t),\cdots,X_m^-(t)\}\big), & \text{if } t\in[0,\,\sigma_1=\tau_1);\\ \big(\{X_i^+(t)\}_{i\notin\{i_1,\cdots,i_{k-1}\}},\ \{X_j^-(t)\}_{j\notin\{j_1,\cdots,j_{k-1}\}}\big), & \text{if } t\in[\sigma_{k-1},\,\sigma_k),\ \text{for } k=2,3,\cdots,m;\\ \partial, & \text{if } t\in[\sigma_m,\,\infty). \end{cases}$$

By definition, $\underline{\mathbf{X}}_t^{(N)} \in D_+^\partial(m-k+1)\times D_-^\partial(m-k+1)$ when $t\in[\sigma_{k-1},\,\sigma_k)$, and $\underline{\mathbf{X}}_t^{(N)}=\partial$ if and only if all particles are labeled (annihilated) at time t (in particular, none of them is absorbed at $\Lambda^\pm$). We call $\underline{\mathbf{X}}^{(N)}=(\underline{\mathbf{X}}_t^{(N)})_{t\geq 0}$ the **configuration process**.

Denote $(\Omega,\mathcal{F},\wp)$ the ambient probability space on which the above random objects $\{X_i^+\}_{i=1}^m$, $\{X_i^-\}_{j=1}^m$, $\{R_i\}_{i=1}^m$ and $\{(i_1,j_1),\cdots,(i_m,j_m)\}$ are defined. For any $z\in S_N$, we define $\mathbb{P}^z$ to be the conditional measure $\wp\big(\,\cdot\,|\,\underline{\mathbf{X}}_0^{(N)}=z\big)$. From the construction, we have

Proposition 4.1.1. $\{\underline{\mathbf{X}}^{(N)}\}$ *is a strong Markov processes under* $\{\mathbb{P}^z:\ z\in S_N\}$.

The key is to note that the choice of (i_k,j_k) depends only on the value of $\underline{\mathbf{X}}_{\sigma_k-}^{(N)}$, and that

$$\tau_{k+1}=\inf\{t\geq 0:\ A_t^{(k)}>R_{k+1}\},\quad \text{where } A_t^{(k)}=\frac{1}{2N}\int_{\sigma_k}^{\sigma_k+t}\sum_{i=1}\sum_{j=1}\ell_{\delta_N}(X_i^+(s),X_j^-(s))\,ds.$$

Hence $\underline{\mathbf{X}}^{(N)}$ is obtained through a patching procedure reminiscent to that of Ikeda, Nagasawa and Watanabe [46]. The proof is standard and is left to the reader.

4.1.2 Normalized empirical process

Next, we consider $E_N := \cup_{M=1}^N E_N^{(M)}\cup\{\mathbf{0}_*\}$, where

$$E_N^{(M)} := \left\{\left(\frac{1}{N}\sum_{i=1}^M \mathbf{1}_{x_i},\ \frac{1}{N}\sum_{j=1}^M \mathbf{1}_{y_j}\right):\ x_i\in D_+^\partial,\ y_j\in D_-^\partial\right\}$$

and $\mathbf{0}_*$ is an abstract point isolated from $\cup_{M=1}^{N} E_N^{(M)}$. We define the **normalized empirical measure** $(\mathfrak{X}^{N,+},\, \mathfrak{X}^{N,-})$ by

$$(\mathfrak{X}_t^{N,+},\, \mathfrak{X}_t^{N,-}) := U_N(\underline{\mathbf{X}}_t^{(N)}), \tag{4.1.3}$$

where $U_N : S_N \to E_N$ is the canonical map given by $U_N(\partial) := \mathbf{0}_*$ and

$$U_N :\ (\underline{x}, \underline{y}) = (x_1, \cdots, x_m,\, y_1, \cdots, y_m) \mapsto \left(\frac{1}{N} \sum_{i=1}^{m} \mathbf{1}_{x_i},\ \frac{1}{N} \sum_{j=1}^{m} \mathbf{1}_{y_j} \right)$$

For comparison, we also consider the empirical measure for the independent reflected diffusions *without* annihilation:

$$(\overline{\mathfrak{X}}^{N,+},\, \overline{\mathfrak{X}}^{N,-}) := \left(\frac{1}{N} \sum_{i=1}^{m} \mathbf{1}_{X_i^+(t)}\, ,\ \frac{1}{N} \sum_{j=1}^{m} \mathbf{1}_{X_j^-(t)} \right). \tag{4.1.4}$$

For any $\mu \in E_N$, we define $\mathbb{P}^{\mu}$ to be the conditional measure $\wp\big(\ \cdot \mid (\mathfrak{X}_0^{N,+},\, \mathfrak{X}_0^{N,-}) = \mu\big)$. From Proposition 4.1.2, we have

Proposition 4.1.2. *$\{(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})\}$ is a strong Markov processes under $\{\mathbb{P}^{\mu} :\ \mu \in E_N\}$.*

4.2 Coupled heat equations with non-linear boundary condition

Denote by $C_\infty([0,T]; \overline{D} \setminus \Lambda)$ the space of continuous functions on $[0,T]$ taking values in $C_\infty(\overline{D} \setminus \Lambda) := \{f \in C(\overline{D}) :\ f \text{ vanishes on } \Lambda\}$. We equip the Banach space $C_\infty([0,T]; \overline{D}_+ \setminus \Lambda_+) \times C_\infty([0,T]; \overline{D}_- \setminus \Lambda_-)$ with norm $\|(u,v)\| := \|u\| + \|v\|$, where $\|\cdot\|$ is the uniform norm. Using a probabilistic representation and the Banach fixed point theorem in the same way as we did in the proof of the existence and uniqueness result for the PDE in Proposition 3.2.1 in Chapter 3, we have the following:

Proposition 4.2.1. *Let $T > 0$ and $u_0^{\pm} \in C_\infty(\overline{D}_\pm \setminus \Lambda_\pm)$. Then there is a unique element*

$(u_+, u_-) \in C_\infty([0,T]; \overline{D}_+ \setminus \Lambda_+) \times C_\infty([0,T]; \overline{D}_- \setminus \Lambda_-)$ *that satisfies the coupled integral equation*

$$\begin{cases} u_+(t,x) = P_t^{\Lambda^+} u_0^+(x) - \frac{1}{2}\int_0^t \int_I p^{\Lambda^+}(t-r,x,z)[\lambda(z)u_+(r,z)u_-(r,z)]d\sigma(z)\,dr \\ u_-(t,y) = P_t^{\Lambda^-} u_0^-(y) - \frac{1}{2}\int_0^t \int_I p^{\Lambda^-}(t-r,y,z)[\lambda(z)u_+(r,z)u_-(r,z)]d\sigma(z)\,dr. \end{cases} \tag{4.2.1}$$

Moreover, (u_+, u_-) *satisfies*

$$\begin{cases} u_+(t,x) = \mathbb{E}^x\Big[\, u_0^+(X_t^{\Lambda^+}) \exp\Big(- \displaystyle\int_0^t (\lambda \cdot u_-)(t-s,\, X_s^{\Lambda^+})\, dL_s^{I,+}\Big)\Big] \\ u_-(t,y) = \mathbb{E}^y\Big[\, u_0^-(X_t^{\Lambda^-}) \exp\Big(- \displaystyle\int_0^t (\lambda \cdot u_+)(t-s,\, X_s^{\Lambda^-})\, dL_s^{I,-}\Big)\Big], \end{cases} \tag{4.2.2}$$

where $L^{I,\pm}$ *is the boundary local time of* $X^{\Lambda^\pm}$ *on the interface* I*, i.e. the positive continuous additive functional having Revuz measure* $\sigma|_I$*, the surface measure* σ *restricted to* I*.*

Definition 4.2.2. We call the unique solution $(u_+,\, u_-) \in C_\infty([0,T]; \overline{D}_+ \setminus \Lambda_+) \times C_\infty([0,T]; \overline{D}_- \setminus \Lambda_-)$ of (4.2.1) the **probabilistic solution** to the following coupled PDEs starting from (u_0^+, u_0^-):

$$\begin{cases} \dfrac{\partial u_+}{\partial t} = \mathcal{A}^+ u_+ & \text{on } (0,\infty) \times D_+ \\ \quad u_+ = 0 & \text{on } (0,\infty) \times \Lambda_+ \\ \dfrac{\partial u_+}{\partial \vec{\nu_+}} = \dfrac{\lambda}{\rho_+}\, u_+ u_-\, \mathbf{1}_{\{I\}} & \text{on } (0,\infty) \times \partial D_+ \setminus \Lambda_+ \end{cases} \tag{4.2.3}$$

and

$$\begin{cases} \dfrac{\partial u_-}{\partial t} = \mathcal{A}^- u_- & \text{on } (0,\infty) \times D_- \\ \quad u_- = 0 & \text{on } (0,\infty) \times \Lambda_- \\ \dfrac{\partial u_-}{\partial \vec{\nu_-}} = \dfrac{\lambda}{\rho_-}\, u_+ u_-\, \mathbf{1}_{\{I\}} & \text{on } (0,\infty) \times \partial D_- \setminus \Lambda_-, \end{cases} \tag{4.2.4}$$

where $\vec{\nu_\pm} := \mathbf{a}_\pm \vec{n}_\pm$ is the inward conormal vector field on $\partial D_\pm$. Here $\mathbf{1}_{\{I\}}$ is the indicator function of I.

It can be shown that the pair of continuous functions (u_+, u_-) satisfying (3.2.1) is weakly differentiable and satisfies the PDEs (4.2.3)-(4.2.4) in the distributional sense (see [21, Section

3]). However, our method only requires continuity of the solutions.

4.3 Main result: rigorous statement

Denote by $M_{\leq 1}(\overline{D}_\pm \setminus \Lambda_\pm)$ the space of non-negative Borel measures on $\overline{D}_\pm \setminus \Lambda_\pm$ with mass at most 1 and set

$$\mathfrak{M} := M_{\leq 1}(\overline{D}_+ \setminus \Lambda_+) \times M_{\leq 1}(\overline{D}_- \setminus \Lambda_-),$$

equipped with the topology of weak convergence. Regard $\mathbf{1}_{\partial^\pm}$ as $\mathbf{0}^\pm$ and $\mathbf{0}_*$ as $(\mathbf{0}^+, \mathbf{0}^-)$, where $\mathbf{0}^\pm$ is the zero measure on $\overline{D}_\pm$, respectively. Then $E_N \subset \mathfrak{M}$ for all N, and the processes $(\mathfrak{X}^{N,+},\, \mathfrak{X}^{N,-})$ have sample paths in $D([0,\infty), \mathfrak{M})$, the Skorokhod space of *càdlàg* paths in $\mathfrak{M}$. We can now rigorously state our main result.

Theorem 4.3.1. ***(Hydrodynamic Limit)*** *Suppose that Assumptions 5.1.1 to 4.0.9 hold. If as $N \to \infty$, $(\mathfrak{X}_0^{N,+},\, \mathfrak{X}_0^{N,-}) \xrightarrow{\mathcal{L}} (u_+^0(x)\rho_+(x)dx,\, u_-^0(y)\rho_-(y)dy)$ in $\mathfrak{M}$, where $u_\pm^0 \in C_\infty(\overline{D}_\pm \setminus \Lambda_\pm)$, then*

$$(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-}) \xrightarrow{\mathcal{L}} (u_+(t,x)\rho_+(x)dx,\, u_-(t,y)\rho_-(y)dy) \quad \text{in } D([0,T], \mathfrak{M})$$

for any $T > 0$, where $(u_+,\, u_-)$ is the probabilistic solution of (4.2.3)-(4.2.4) *with initial value (u_+^0, u_-^0).*

Remark 4.3.2. $\mathfrak{M}$ is in fact a Polish space. Let $\{f_n; n \geq 1\}$ and $\{g_n; n \geq 1\}$ be sequences of continuous functions with $|f_n| \leq 1$ and $|g_n| \leq 1$ whose linear span are dense in $C_\infty(\overline{D}_+ \setminus \Lambda_+)$ and $C_\infty(\overline{D}_- \setminus \Lambda_-)$, respectively. For $\mu = (\mu_+, \mu_-)$ and $\nu = (\nu_+, \nu_-)$ in $\mathfrak{M}$, define

$$\varrho(\mu, \nu) := \sum_{n=1}^{\infty} 2^{-n} \left(\left| \int_{\overline{D}_+} f_n(x)(\mu_+ - \nu_+)(dx) \right| + \left| \int_{\overline{D}_-} g_n(y)(\mu_- - \nu_-)(dy) \right| \right).$$

It is well known that $\mathfrak{M}$ is a complete separable metric space under the metric ϱ. □

As mentioned right after Theorem 1.2.3 (which is Theorem 4.3.1) in the Introduction, an assumption on the rate at which δ_N tends to zero, such as Assumption 4.0.10, is *necessary* for Theorem 4.3.1 to hold. Below is a counter-example.

Example 4.3.3. Suppose that $\{X_i^+(t)\}_{i=1}^\infty$ and $\{X_j^-(t)\}_{j=1}^\infty$ are RBMs on $\overline{D}_+$ and $\overline{D}_-$, respectively, and they are all mutually independent. Note that X_i^+ and X_j^- never meet in the sense that

$$\mathbb{P}\Big(X_i^+(t) = X_j^-(t) \text{ for some } t \in [0,\infty) \text{ and } i,\, j \in \{1,2,3,\cdots\} \Big) = 0. \tag{4.3.1}$$

This implies that there exists $\{\delta_N\}$ so that $\sum_{N=1}^\infty \alpha_N < \infty$, where

$$\alpha_N := \mathbb{P}\Big((X_i^+(t),\, X_j^-(t)) \in I^{\delta_N} \text{ for some } t \in [0,\infty) \text{ and } i,\, j \in \{1,2,\cdots,N\} \Big). \tag{4.3.2}$$

Hence by Borel-Cantelli lemma, we know that with probability 1, there will be no annihilation for the particle system (which occurs only when a pair of particles are in I^{δ_N}) when N is sufficiently large. In this case, $(\mathfrak{X}_t^{N,+},\, \mathfrak{X}_t^{N,-})$ converges to $(P_t^+ u_0^+(x)dx,\, P_t^- u_0^-(y)dy)$ in distribution in $D([0,T],\mathfrak{M})$ instead, provided that $(\mathfrak{X}_0^{N,+},\, \mathfrak{X}_0^{N,-})$ converges to $(u_0^+(x)dx,\, u_0^-(y)dy)$ in distribution in $\mathfrak{M}$.

Question. We will see from Theorem 4.4.4 below that the tightness of $(\mathfrak{X}_t^{N,+},\, \mathfrak{X}_t^{N,-})$ holds without Assumption 4.0.10. Can we characterize all limit points of $(\mathfrak{X}_t^{N,+},\, \mathfrak{X}_t^{N,-})$ without Assumption 4.0.10? Is $\liminf_{N\to\infty} N\,\delta_N^d \in (0,\infty]$ the sharpest condition for Theorem 4.3.1 to hold?

We end this section by providing a *key idea of the proof.* Recall that, as mentioned in the introduction, the interchange of limit (1.2.5) allows us to characterize the mean of any subsequential limit of $(\mathfrak{X}^{N,+},\, \mathfrak{X}^{N,-})$ by comparing the integral equations satisfied by the hydrodynamic limit with its stochastic counterpart. To see why this is true, we look at the case when $\Lambda_\pm$ are empty and the reflected diffusions are all RBMs for simplicity. Let $(\mathfrak{X}^{\infty,+},\, \mathfrak{X}^{\infty,-})$ be an arbitrary subsequential limit of $(\mathfrak{X}^{N,+},\, \mathfrak{X}^{N,-})$. We argue that for any $\phi \in C(\overline{D}_+)$, we have

$$\mathbb{E}\Big[\langle \phi,\, \mathfrak{X}_t^{\infty,+}\rangle\Big] = \langle \phi,\, u_+(t)\rangle. \tag{4.3.3}$$

Taking $L^2(D_+)$ inner product with ϕ on both sides of the integral equation satisfied by u_+ yields

$$\langle \phi,\, u_+(t)\rangle - \langle \phi,\, P_t^+ f(x)\rangle = -\frac{\lambda}{2}\int_0^t \int_I P_{t-r}^+\phi(z)\, u_+(r,z)\, u_-(r,z)\, d\sigma(z)\, dr. \tag{4.3.4}$$

On other hand, by the Dynkin's formula,

$$\mathbb{E}\left[\langle \phi, \mathfrak{X}_t^{N,+}\rangle\right] - \mathbb{E}\left[\langle P_t^+\phi, \mathfrak{X}_0^{N,+}\rangle\right] = -\frac{\lambda}{2}\int_0^t \mathbb{E}\left[\langle \ell_{\delta_N}\, P_{t-r}^+\phi, \mathfrak{X}_r^{N,+}\otimes \mathfrak{X}_r^{N,-}\rangle\right]\, dr. \tag{4.3.5}$$

The challenge is to compare the right hand sides of the above two equations, which involves a nonlinear term. However, (1.2.5) guarantees that we can pass the limit through the integral (along some subsequence of $N\to\infty$) to deduce from (3.1.4)

$$\mathbb{E}^\infty[\langle \phi,\, \mathfrak{X}_t^{\infty,+}\rangle] - \langle P_t^+\phi,\, \mathfrak{X}_0^{\infty,+}\rangle = -\frac{\lambda}{2}\int_0^t \lim_{\varepsilon\to 0}\mathbb{E}^\infty\left[\langle \ell_\varepsilon\, P_{t-r}^+\phi_+,\, \mathfrak{X}_r^{\infty,+}\otimes \mathfrak{X}_r^{\infty,-}\rangle\right]\, dr. \tag{4.3.6}$$

Now (4.3.3) follows from a standard Gronwall argument by comparing (4.3.4) with (4.3.6). Using a similar argument, we can identify the second moment of any subsequential limit, and hence characterize any subsequential limit of $(\mathfrak{X}^{N,+},\, \mathfrak{X}^{N,-})$. Together with the tightness result, we obtain Theorem 4.3.1.

The rest of the paper is devoted to the proof of Theorem 4.3.1. Recall that Assumptions 5.1.1 to 4.0.9 are in force.

4.4 Martingales and tightness

In this subsection, we present some key martingales that are used to establish tightness of $(\mathfrak{X}^{N,+},\, \mathfrak{X}^{N,-})$. More martingales related to the time dependent process $(t,\, (\mathfrak{X}_t^{N,+},\, \mathfrak{X}_t^{N,-}))$ will be given in subsection 4.5.2.

Martingales for annihilating diffusion model

Theorem 4.4.1. *Fix any positive integer N. Suppose $F \in C_b(E_N)$ is a bounded continuous function and $G \in \mathcal{B}(E_N)$ is a Borel measurable function on E_N such that*

$$\overline{M}_t := F(\overline{\mathfrak{X}}_t^{N,+}, \overline{\mathfrak{X}}_t^{N,-}) - \int_0^t G(\overline{\mathfrak{X}}_s^{N,+}, \overline{\mathfrak{X}}_s^{N,-})\, ds$$

is an $\mathcal{F}_t^{(\overline{\mathfrak{X}}^{N,+}, \overline{\mathfrak{X}}^{N,-})}$-martingale under $\mathbb{P}^\mu$ for any $\mu \in E_N$. Then

$$M_t := F(\mathfrak{X}_t^{N,+}, \mathfrak{X}_t^{N,-}) - \int_0^t (G + KF)(\mathfrak{X}_s^{N,+}, \mathfrak{X}_s^{N,-})\, ds$$

is a $\mathcal{F}_t^{(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})}$-martingale under $\mathbb{P}^\mu$ for any $\mu \in E_N$, where

$$KF(\nu) := -\frac{1}{2N} \sum_{i=1}^{M} \sum_{j=1}^{M} \ell_{\delta_N}(x_i, y_j) \Big(F(\nu) \; - \; F\big(\nu^+ - N^{-1}\mathbf{1}_{\{x_i\}},\, \nu^- - N^{-1}\mathbf{1}_{\{y_j\}}\big) \Big) \tag{4.4.1}$$

whenever $\nu = \Big(\frac{1}{N} \sum_{i=1}^M \mathbf{1}_{\{x_i\}},\, \frac{1}{N} \sum_{j=1}^M \mathbf{1}_{\{y_j\}} \Big) \in E_N^{(M)}$, and $KF(\boldsymbol{0}_*) := 0$.

Remark 4.4.2. (i) Theorem 4.4.1 indicates the infinitesimal generator of $(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})$ on $C_b(E_N)$ is given by $\overline{L} + K$, where $\overline{L}$ is the infinitesimal generator of $(\overline{\mathfrak{X}}^{N,+}, \overline{\mathfrak{X}}^{N,-})$ on $C_b(E_N)$. Note that G is merely assumed to be Borel measurable, the above provides us with a broader class of martingales (such as $N_t^{(\phi_+, \phi_-)}$ in the proof of Corollary 4.4.3) than from using the $C_b(E_N)$-generator.

(ii) Theorem 4.4.1 can be generalized to deal with time-dependent functions $F_s \in C_b(E_N)$ $(s \geq 0)$. See Theorem 4.5.8 in subsection 4.5.2. □

Proof of Theorem 4.4.1. We adopt the abbreviation $\mathfrak{X} := (\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})$ when there is no confusion. In particular, we write $\mathcal{F}_t^{\mathfrak{X}}$ in place of $\mathcal{F}_t^{(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})}$. By Markov property for $\mathfrak{X}$, it suffices to show that for all $t \geq 0$ and $\nu \in E_N$,

$$\mathbb{E}^\nu\Big[F(\mathfrak{X}_t) - F(\mathfrak{X}_0) - \int_0^t (G + KF)(\mathfrak{X}_s)\,ds\Big] = 0. \tag{4.4.2}$$

The idea is to spit the time interval $[0,t]$ into pieces according to the jumping times of $F(\mathfrak{X}_s)$ ($s \in [0,t]$) caused by annihilation (excluding the jumps caused by absorbtion at the harvest sites $\Lambda^\pm$), then apply $\overline{M}$ in each piece and take into account the jump distributions.

Suppose $\nu = (\nu^+, \nu^-) = (\frac{1}{N}\sum_{i=1}^m \mathbf{1}_{x_i}, \frac{1}{N}\sum_{j=1}^m \mathbf{1}_{y_j}) \in E_N^{(m)}$. Recall that $\sigma_i := \tau_1 + \cdots \tau_i$ ($i = 1, 2, \cdots, m$) is the time of the i-th labeling (annihilation) of particles. write

$$F(\mathfrak{X}_t) - F(\mathfrak{X}_0) = \sum_{i=0}^m \Big(F(\mathfrak{X}_{(t\wedge\sigma_{i+1})-}) - F(\mathfrak{X}_{t\wedge\sigma_i})\Big) + \sum_{j=1}^m \Big(F(\mathfrak{X}_{t\wedge\sigma_j}) - F(\mathfrak{X}_{(t\wedge\sigma_j)-})\Big), \tag{4.4.3}$$

where $\sigma_0 := 0$, $\sigma_{m+1} := \infty$ and $\mathfrak{X}_{s-} := \lim_{r\nearrow s}\mathfrak{X}_r$. Hence it suffices to show that

$$\begin{aligned} \mathbb{E}^\nu\Big[F(\mathfrak{X}_{(t\wedge\sigma_{i+1})-}) - F(\mathfrak{X}_{t\wedge\sigma_i}) - \int_{t\wedge\sigma_i}^{t\wedge\sigma_{i+1}} G(\mathfrak{X}_s)\,ds\Big] &= 0 \quad \text{and} && (4.4.4)\\ \mathbb{E}^\nu\Big[F(\mathfrak{X}_{t\wedge\sigma_j}) - F(\mathfrak{X}_{(t\wedge\sigma_j)-}) - \int_{t\wedge\sigma_{j-1}}^{t\wedge\sigma_j} KF(\mathfrak{X}_s)\,ds\Big] &= 0 && (4.4.5) \end{aligned}$$

for $i \in \{0, 1, 2, \cdots, m\}$ and $j \in \{1, 2, \cdots, m\}$.

The left hand side of (4.4.4) equals

$$\begin{aligned} &\mathbb{E}^\nu\Bigg[\mathbb{E}^\nu\Big[F(\mathfrak{X}_{(t\wedge\sigma_{i+1})-}) - F(\mathfrak{X}_{t\wedge\sigma_i}) - \int_{t\wedge\sigma_i}^{t\wedge\sigma_{i+1}} G(\mathfrak{X}_s)\,ds\Big|\,\mathcal{F}^{\mathfrak{X}}_{t\wedge\sigma_i}\Big]\Bigg]\\ =\;& \mathbb{E}^\nu\Bigg[\mathbb{E}^{\mathfrak{X}_{\sigma_i}}\Big[F(\mathfrak{X}_{(t\wedge\sigma_{i+1}-\sigma_i)-}) - F(\mathfrak{X}_0) - \int_0^{t\wedge\sigma_{i+1}-\sigma_i} G(\mathfrak{X}_s)\,ds\Big]\mathbf{1}_{t>\sigma_i}\Bigg]\\ =\;& \mathbb{E}^\nu\Bigg[\mathbb{E}^{\mathfrak{X}_{\sigma_i}}\Big[F(\mathfrak{X}_{((t-\sigma_i)\wedge\tau_{i+1})-}) - F(\mathfrak{X}_0) - \int_0^{(t-\sigma_i)\wedge\tau_{i+1}} G(\mathfrak{X}_s)\,ds\Big]\mathbf{1}_{t>\sigma_i}\Bigg]. \end{aligned}$$

The first equality follows from the strong Markov property of $\mathfrak{X}$ (applied to the stopping time σ_i) and the fact that the expression inside the expectation vanishes when $t \leq \sigma_i$. Note that σ_i is regarded as a constant w.r.t. the expectation $\mathbb{E}^{\mathfrak{X}_{\sigma_i}}$, because $\mathcal{F}^{\mathfrak{X}}_{\sigma_i}$ contains the sigmaalgebra generated by σ_i. The second equality follows from the easy fact that $(t \wedge \sigma_{i+1}) - \sigma_i = (t - \sigma_i) \wedge (\sigma_{i+1} - \sigma_i) = (t - \sigma_i) \wedge \tau_{i+1}$ on $t > \sigma_i$. Therefore, to establish (4.4.4), it is enough to show that for any $\eta \in E_N$ and $w \geq 0$, we have

$$\mathbb{E}^\eta \Big[F(\mathfrak{X}_{(w\wedge\tau)-}) - F(\mathfrak{X}_0) - \int_0^{w\wedge\tau} G(\mathfrak{X}_s)\, ds \Big] = 0, \tag{4.4.6}$$

where τ is the time of the first annihilation for $\mathfrak{X}$ starting from η (i.e. $\tau = \tau_1$ under $\mathbb{P}^\eta$ where τ_1 is defined by (4.1.1)).

(4.4.6) obviously holds if η is the zero measure since both sides vanish. Suppose $\eta \in E_N^{(n)}$. Observe that τ is a stopping time for $\tilde{\mathcal{F}}_t^{\overline{\mathfrak{X}}} := \sigma(\mathcal{F}_t^{\overline{\mathfrak{X}}}, \{R_i;\, 1 \leq i \leq n\})$ and that $\overline{M}_t$ is a $\tilde{\mathcal{F}}_t^{\overline{\mathfrak{X}}}$-martingale under $\mathbb{P}^\eta$ since $\{R_i\}$ is independent of $\overline{\mathfrak{X}}$ under $\mathbb{P}^\eta$. Hence, by the optional sampling theorem, (4.4.6) is true, and so is (4.4.4).

Following the same arguments as above, the left hand side of (4.4.5) equals

$$\mathbb{E}^\nu \Bigg[\mathbb{E}^{\mathfrak{X}_{\sigma_{j-1}}} \Big[F(\mathfrak{X}_{(t-\sigma_{j-1})\wedge\tau_j}) - F(\mathfrak{X}_{((t-\sigma_{j-1})\wedge\tau_j)-}) + \int_0^{(t-\sigma_{j-1})\wedge\tau_j} KF(\mathfrak{X}_s)\, ds \Big] \mathbf{1}_{t>\sigma_{j-1}} \Bigg],$$

where σ_{j-1} is regarded as a constant w.r.t. the expectation $\mathbb{E}^{\mathfrak{X}_{\sigma_{j-1}}}$. Therefore, (4.4.5) holds if for any $\eta \in E_N$ and $\theta \geq 0$, we have

$$\mathbb{E}^\eta \Big[F(\mathfrak{X}_{\theta\wedge\tau}) - F(\mathfrak{X}_{(\theta\wedge\tau)-}) - \int_0^{\theta\wedge\tau} KF(\mathfrak{X}_s)\, ds \Big] = 0, \tag{4.4.7}$$

where τ is the time of the first killing for $\mathfrak{X}$ starting from η.

Suppose $\eta = (\frac{1}{N}\sum_{i=1}^n \mathbf{1}_{x_i}, \frac{1}{N}\sum_{j=1}^n \mathbf{1}_{y_j}) \in E_N^{(n)}$ and $\mathfrak{X}_{\tau-} = (\frac{1}{N}\sum_{i=1}^n \mathbf{1}_{X_i^+(\tau-)}, \frac{1}{N}\sum_{j=1}^n \mathbf{1}_{X_j^-(\tau-)})$, where $\{X_k^\pm : k = 1, \cdots, n\}$ are reflected diffusions killed upon hitting $\Lambda^\pm$ in the construction of $\mathfrak{X}$. At time τ, one pair of particles among $\{(X_i^+, \mathfrak{X}_j^-) : 1 \leq i, j \leq n\}$ is labeled (annihilated), where the pair $(X_i^+, \mathfrak{X}_j^-)$ is chosen to be labeled (annihilated) with probability $\frac{\ell_{\delta_N}(X_i^+(\tau-), X_j^-(\tau-))}{\sum_{p=1}^n\sum_{q=1}^n \ell_{\delta_N}(X_p^+(\tau-), X_q^-(\tau-))}$. Hence

$$
\begin{aligned}
& \mathbb{E}^\eta\Big[\, F(\mathfrak{X}_{(\theta\wedge\tau)-}) - F(\mathfrak{X}_{\theta\wedge\tau})\,\Big] && (4.4.8)\\
= \ & \mathbb{E}^\eta\bigg[\mathbb{E}^\eta\Big[\, F(\mathfrak{X}_{\tau-}) - F(\mathfrak{X}_{\tau})\Big|\, \mathcal{F}^{\mathfrak{X}}_{\tau-}\,\Big]\,;\,\tau<\theta\bigg]\\
= \ & \mathbb{E}^\eta\bigg[\,\sum_{i=1}^n\sum_{j=1}^n \frac{\ell_{\delta_N}(X_i^+(\tau-),\,X_j^-(\tau-))}{\sum_{p=1}^n\sum_{q=1}^n \ell_{\delta_N}(X_p^+(\tau-),\,X_q^-(\tau-))} && (4.4.9)\\
& \qquad \Big(F(\mathfrak{X}_{\tau-})\ -\ F\Big(\mathfrak{X}_{\tau-} - (\frac{1}{N}\mathbf{1}_{X_i^+(\tau-)},\frac{1}{N}\mathbf{1}_{X_j^-(\tau-)})\Big)\Big)\,;\,\tau<\theta\bigg]\\
= \ & \mathbb{E}^\eta\bigg[\,\frac{-(2N)\,KF(\mathfrak{X}_{\tau-})}{\sum_{p=1}^n\sum_{q=1}^n \ell_{\delta_N}(X_p^+(\tau-),\,X_q^-(\tau-))}\,;\,\tau<\theta\bigg]\\
= \ & \mathbb{E}^\eta\Big[\int_0^\theta -KF(\mathfrak{X}_s)\,ds\Big].
\end{aligned}
$$

The last equality follows from the fact that

$$\tau = \inf\Big\{t\ge 0:\ \frac{1}{2N}\int_0^t\sum_{p=1}^n\sum_{q=1}^n \ell_{\delta_N}(X_p^+(s),\,X_q^-(s))\,ds\ \ge R\Big\},$$

where R is an independent exponential random variable of parameter 1 under $\mathbb{P}^\eta$ (see Proposition 2.2 of [22] for a rigorous proof). Hence (4.4.7) is established and the proof is complete. □

The following corollary is the key to the tightness of $(\mathfrak{X}^{N,+},\mathfrak{X}^{N,-})$. Recall that $\mathcal{A}^\pm$ is the Feller generator of the diffusion $X^\pm = X^{\Lambda_\pm}$ on $\overline{D}_\pm\setminus\Lambda_\pm$, respectively.

Corollary 4.4.3. *Fix any positive integer N. For any $\phi_\pm\in Dom(\mathcal{A}^\pm)$, we have*

$$
\begin{aligned}
M_t^{(\phi_+,\phi_-)} \ := \ & \langle\phi_+,\mathfrak{X}_t^{N,+}\rangle + \langle\phi_-,\mathfrak{X}_t^{N,-}\rangle\\
& - \int_0^t \langle\mathcal{A}^+\phi_+,\,\mathfrak{X}_s^{N,+}\rangle + \langle\mathcal{A}^-\phi_-,\,\mathfrak{X}_s^{N,-}\rangle - \frac{1}{2}\langle\ell_{\delta_N}(\phi_+ + \phi_-),\,\mathfrak{X}_s^{N,+}\otimes\mathfrak{X}_s^{N,-}\rangle\,ds
\end{aligned}
$$

is an $\mathcal{F}_t^{(\mathfrak{X}^{N,+},\mathfrak{X}^{N,-})}$*-martingale under* $\mathbb{P}^\mu$ *for any* $\mu \in E_N$*, where*

$$\langle f(x,y),\, \mu^+(dx) \otimes \mu^-(dy)\rangle := \frac{1}{N^2}\sum_i \sum_j f(x_i,y_j) \quad \textit{whenever } \mu = (N^{-1}\sum_i \mathbf{1}_{x_i},\, N^{-1}\sum_j \mathbf{1}_{y_j}).$$

Moreover, $M_t^{(\phi_+,\phi_-)}$ *has predictable quadratic variation*

$$\begin{aligned}\langle M^{(\phi_+,\phi_-)}\rangle_t &= \frac{1}{N}\int_0^t \Big(\langle \boldsymbol{a}_+\nabla\phi_+\cdot\nabla\phi_+,\, \mathfrak{X}_s^{N,+}\rangle + \langle \boldsymbol{a}_-\nabla\phi_-\cdot\nabla\phi_-,\, \mathfrak{X}_s^{N,-}\rangle \\ &\qquad + \frac{1}{2}\langle \ell_{\delta_N}(\phi_+ + \phi_-)^2,\, \mathfrak{X}_s^{N,+}\otimes\mathfrak{X}_s^{N,-}\rangle\Big)ds \end{aligned} \tag{4.4.10}$$

and $\sup_{t\in[0,T]}\mathbb{E}^\mu[(M_t^{(\phi_+,\phi_-)})^2] \le \frac{C}{N}$ *for some constant* C *that is independent of* N *and* μ*.*

Proof From Lemma 2.1.4, we have the following two $\mathcal{F}_t^{(\overline{\mathfrak{X}}^{N,+},\overline{\mathfrak{X}}^{N,-})}$-martingales for $\phi_\pm \in Dom(\mathcal{A}^\pm)$:

$$\begin{aligned}\overline{M}_t^{(\phi_+,\phi_-)} &:= \langle\phi_+,\overline{\mathfrak{X}}_t^{N,+}\rangle + \langle\phi_-,\overline{\mathfrak{X}}_t^{N,-}\rangle - \int_0^t \langle\mathcal{A}^+\phi_+,\,\overline{\mathfrak{X}}_s^{N,+}\rangle + \langle\mathcal{A}^-\phi_-,\,\overline{\mathfrak{X}}_s^{N,-}\rangle\, ds \quad \text{and} \\ \overline{N}_t^{(\phi_+,\phi_-)} &:= (\langle\phi_+,\overline{\mathfrak{X}}_t^{N,+}\rangle + \langle\phi_-,\overline{\mathfrak{X}}_t^{N,-}\rangle)^2 \\ &\quad -\int_0^t 2\Big(\langle\phi_+,\overline{\mathfrak{X}}_s^{N,+}\rangle + \langle\phi_-,\overline{\mathfrak{X}}_s^{N,-}\rangle\Big)\Big(\langle\mathcal{A}^+\phi_+,\overline{\mathfrak{X}}_s^{N,+}\rangle + \langle\mathcal{A}^-\phi_-,\overline{\mathfrak{X}}_s^{N,-}\rangle\Big) \\ &\qquad + \frac{1}{N}\Big(\langle\mathbf{a}_+\nabla\phi_+\cdot\nabla\phi_+,\,\overline{\mathfrak{X}}_s^{N,+}\rangle + \langle\mathbf{a}_-\nabla\phi_-\cdot\nabla\phi_-,\,\overline{\mathfrak{X}}_s^{N,-}\rangle\Big)\, ds.\end{aligned}$$

Note that $F_1(\mu) = F_1(\mu^+,\mu^-) := \langle\phi_+,\mu^+\rangle + \langle\phi_-,\mu^-\rangle$ is a function in $C(E_N)$, with the convention that $\phi_\pm(\partial^\pm) := 0$ and $F_1(\mathbf{0}_*) := 0$. A direct calculations shows that

$$KF_1(\mu) = \frac{-1}{2}\langle\ell_{\delta_N}(\phi_+ + \phi_-),\, \mu^+\otimes\mu^-\rangle$$

Therefore, by Theorem 4.4.1, $M_t^{(\phi_+,\phi_-)}$ is an $\mathcal{F}_t^{(\mathfrak{X}^{N,+},\mathfrak{X}^{N,-})}$-martingale. Similarly, $F_2(\mu) := (\langle\phi_+,\mu^+\rangle + \langle\phi_-,\mu^-\rangle)^2 \in C(E_N)$ and

$$KF_2(\mu) = -\left(\langle\phi_+,\mu^+\rangle + \langle\phi_-,\mu^-\rangle\right)\langle\ell_{\delta_N}(\phi_+ + \phi_-),\, \mu^+\otimes\mu^-\rangle + \frac{1}{2N}\langle\ell_{\delta_N}(\phi_+ + \phi_-)^2,\, \mu^+\otimes\mu^-\rangle.$$

Hence Theorem 4.4.1 asserts that

$$\begin{aligned}
N_t^{(\phi_+,\phi_-)} \quad := \quad & \Big(\langle \phi_+, \mathfrak{X}_t^{N,+}\rangle + \langle \phi_-, \mathfrak{X}_t^{N,-}\rangle\Big)^2 \\
& - \int_0^t 2\Big(\langle \phi_+, \mathfrak{X}_s^{N,+}\rangle + \langle \phi_-, \mathfrak{X}_s^{N,-}\rangle\Big)\Big(\langle \mathcal{A}^+\phi_+, \mathfrak{X}_s^{N,+}\rangle + \langle \mathcal{A}^-\phi_-, \mathfrak{X}_s^{N,-}\rangle\Big) \\
& \quad + \frac{1}{N}\Big(\langle \mathbf{a}_+\nabla\phi_+\cdot\nabla\phi_+,\, \mathfrak{X}_s^{N,+}\rangle + \langle \mathbf{a}_-\nabla\phi_-\cdot\nabla\phi_-,\, \mathfrak{X}_s^{N,-}\rangle\Big) \\
& \quad - (\langle \phi_+, \mathfrak{X}_s^{N,+}\rangle + \langle \phi_-, \mathfrak{X}_s^{N,-}\rangle)\langle \ell_{\delta_N}(\phi_+ + \phi_-),\, \mathfrak{X}_s^{N,+}\otimes\mathfrak{X}_s^{N,-}\rangle \\
& \quad + \frac{1}{2N}\langle \ell_{\delta_N}(\phi_+ + \phi_-)^2,\, \mathfrak{X}_s^{N,+}\otimes\mathfrak{X}_s^{N,-}\rangle \, ds
\end{aligned}$$

is an $\mathcal{F}_t^{(\mathfrak{X}^{N,+},\mathfrak{X}^{N,-})}$-martingale. Since $\Big(M_t^{(\phi_+,\phi_-)}\Big)^2 - N_t^{(\phi_+,\phi_-)}$ is equal to the right hand side of (4.4.10), which is a continuous process of finite variation, it has to be $[M^{(\phi_+,\phi_-)}]_t$. This proves (4.4.10). Therefore,

$$\begin{aligned}
& \mathbb{E}^\mu[(M_t^{(\phi_+,\phi_-)})^2] = \mathbb{E}^\mu\Big[\,[M^{(\phi_+,\phi_-)}]_t\,\Big] \\
\leq \; & \frac{1}{N}\bigg(\int_0^t \langle P_s^+\big(\mathbf{a}_+\nabla\phi_+\cdot\nabla\phi_+\big),\, \mathfrak{X}_0^{N,+}\rangle\, ds + \int_0^t \langle P_s^-\big(\mathbf{a}_-\nabla\phi_-\cdot\nabla\phi_-\big),\, \mathfrak{X}_0^{N,-}\rangle\, ds \\
& \qquad + \frac{1}{2}\,\|(\phi_+ + \phi_-)^2\| \int_0^t \langle \ell_{\delta_N},\, \mathfrak{X}_s^{N,+}\otimes\mathfrak{X}_s^{N,-}\rangle\bigg)\, ds \\
\leq \; & \frac{1}{N}\bigg(8\big(\|\phi_+\|^2 + \|\mathcal{A}^+\phi_+\|^2\, t^2\big) + 8\big(\|\phi_-\|^2 + \|\mathcal{A}^-\phi_-\|^2\, t^2\big) \\
& \qquad + \frac{1}{2}\,\|(\phi_+ + \phi_-)^2\| \int_0^t \langle \ell_{\delta_N},\, \mathfrak{X}_s^{N,+}\otimes\mathfrak{X}_s^{N,-}\rangle\bigg)\, ds,
\end{aligned}$$

where we have used (2.1.10) in the last inequality. Finally, we show that

$$\sup_{\mu\in E_N}\int_0^t \mathbb{E}^\mu[\langle \ell_{\delta_N},\, \mathfrak{X}_s^{N,+}\otimes\mathfrak{X}_s^{N,-}\rangle] \leq 1. \tag{4.4.11}$$

Let $(\tilde{\mathfrak{X}}^{N,+},\, \tilde{\mathfrak{X}}^{N,-})$ be the normalized empirical measure corresponding to the case $\Lambda_\pm$ being empty sets. By applying the martingale $M_t^{(\phi_+,\phi_-)}$ to the case $\Lambda_\pm$ being empty sets and $\phi_\pm = 1$

(now 1 is in the domain of the Feller generator), we have

$$\int_0^t \mathbb{E}[\langle \ell_{\delta_N}, \tilde{\mathfrak{X}}^{N,+}_s \otimes \tilde{\mathfrak{X}}^{N,-}_s \rangle]\, ds = \Big(\langle 1,\, \tilde{\mathfrak{X}}^{N,+}_0 \rangle - \mathbb{E}[\langle 1,\, \tilde{\mathfrak{X}}^{N,+}_t \rangle] \Big) \le 1.$$

We then obtain (4.4.11) by a coupling of $(\mathfrak{X}^{N,+},\, \mathfrak{X}^{N,-})$ and $(\tilde{\mathfrak{X}}^{N,+},\, \tilde{\mathfrak{X}}^{N,-})$. The idea is that $(\tilde{\mathfrak{X}}^{N,+},\, \tilde{\mathfrak{X}}^{N,-})$ dominates $(\mathfrak{X}^{N,+},\, \mathfrak{X}^{N,-})$. This coupling can be constructed by labeling (rather than killing) particles which hit $\Lambda_\pm$, using the same method of subsection 4.1.1. Hence we obtain the desired bound for $\mathbb{E}^\mu[(M^{(\phi_+,\phi_-)}_t)^2]$. □

Tightness

The proof of tightness for $(\mathfrak{X}^{N,+},\, \mathfrak{X}^{N,-})$ is non-trivial because $\mathbb{E}\left[\langle \ell_{\delta_N},\, \mathfrak{X}^{N,+}_s \otimes \mathfrak{X}^{N,-}_s \rangle^2\right]$ blows up near $s = 0$ in such a way that $\lim_{N\to\infty} \int_0^t \mathbb{E}\left[\langle \ell_{\delta_N},\, \mathfrak{X}^{N,+}_s \otimes \mathfrak{X}^{N,-}_s \rangle^2\right] ds = \infty$. To deal with this singularity at $s = 0$, we will use Lemma 3.5.3 in Chapter 3.

Here is our tightness result for $(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})$. Note that it does not require Assumption 4.0.10.

Theorem 4.4.4. *(Tightness) Suppose $\{\delta_N\}$ tends to 0. Then $\{(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})\}$ is tight in $D([0,T], \mathfrak{M})$ and any of subsequential limits is carried on $C_{\mathfrak{M}}[0,T]$. Moreover, $\{J_N\}$ is tight in $C([0,T])$, where $J_N(t) := \int_0^t \langle \ell_{\delta_N},\, \mathfrak{X}^{N,+}_s \otimes \mathfrak{X}^{N,-}_s \rangle\, ds$.*

Proof Recall from Remark 4.3.2 that $\mathfrak{M}$ is a complete separable metric space. Since $Dom(\mathcal{A}^\pm)$ is dense in $C_\infty(\overline{D}_\pm \setminus \Lambda_\pm)$, we only need to check a "weak tightness criteria" (cf. Proposition 1.7 of [50]), i.e. it suffices to check that $\{(\langle \phi_+, \mathfrak{X}^{N,+} \rangle,\, \langle \phi_-, \mathfrak{X}^{N,-} \rangle)\}_N$ is tight in $D([0,T], \mathbb{R}^2)$ for any $\phi_\pm \in Dom(\mathcal{A}^\pm)$. By Prohorov's theorem (see Theorem 1.3 and Remark 1.4 of [50]), $\{(\langle \phi_+, \mathfrak{X}^{N,+} \rangle,\, \langle \phi_-, \mathfrak{X}^{N,-} \rangle)\}_N$ is tight in $D([0,T], \mathbb{R}^2)$ if the following two properties (a) and (b) hold:

(a) For all $t \in [0,T]$ and $\varepsilon_0 > 0$, there exists a compact set $K(t, \varepsilon_0) \subset \mathbb{R}^2$ such that

$$\sup_N \mathbb{P}\Big((\langle \phi_+, \mathfrak{X}^{N,+}_t \rangle,\, \langle \phi_-, \mathfrak{X}^{N,-}_t \rangle) \notin K(t, \varepsilon_0) \Big) < \varepsilon_0.$$

(b) For all $\varepsilon_0 > 0$,

$$\lim_{\gamma\to 0}\overline{\lim_{N\to\infty}}\,\mathbb{P}\left(\sup_{\substack{|t-s|<\gamma\\ 0\le s,t\le T}}\left|\Big(\langle\phi_+,\mathfrak{X}_t^{N,+}\rangle,\,\langle\phi_-,\mathfrak{X}_t^{N,-}\rangle\Big)-\Big(\langle\phi_+,\mathfrak{X}_s^{N,+}\rangle,\,\langle\,\phi_-,X_s^{N,-}\rangle\Big)\right|_{\mathbb{R}^2}>\varepsilon_0\right)=0.$$

Property (a) is true since we can always take $K = [\,-\|\phi_+\|,\,\|\phi_+\|\,]\times[\,-\|\phi_-\|,\,\|\phi_-\|\,]$. To verify property (b), we only need to focus on $\mathfrak{X}^{N,+}$. Note that (writing $\phi=\phi_+$ for simplicity) by Corollary 4.4.3, we have

$$\langle\phi,\mathfrak{X}_t^{N,+}\rangle-\langle\phi,\mathfrak{X}_s^{N,+}\rangle=\int_s^t\langle\mathcal{A}^+\phi,\,\mathfrak{X}_r^{N,+}\rangle\,dr-\frac{1}{2}\int_s^t\langle\ell_{\delta_N}\,\phi,\,\mathfrak{X}_r^{N,+}\otimes\mathfrak{X}_r^{N,-}\rangle\,dr+(M_N(t)-M_N(s)),\tag{4.4.12}$$

where $M_N(t)$ is a martingale. So we only need to verify (b) with $\langle\phi,\mathfrak{X}_t^{N,+}\rangle-\langle\phi,\mathfrak{X}_s^{N,+}\rangle$ replaced by each of the three terms on the right hand side of (4.4.12).

The first term of (4.4.12) is obvious since $\langle\mathcal{A}^+\phi,\,\mathfrak{X}_r^{N,+}\rangle\le\|\mathcal{A}^+\phi\|$. For the third term of (4.4.12), recall that $\lim_{N\to\infty}\mathbb{E}\left[M_N(t)^2\right]=0$ by Corollary 4.4.3. Hence, by applying Chebyshev's inequality and then Doob's maximal inequality, we see that (b) is satisfied by the third term of (4.4.12).

For the second term of (4.4.12), we show that

$$\lim_{\gamma\to 0}\overline{\lim_{N\to\infty}}\,\mathbb{P}\left(\sup_{\substack{|t-s|<\gamma\\ 0\le s,t\le T}}\int_s^t\langle\ell_{\delta_N},\,\mathfrak{X}_r^{N,+}\otimes\mathfrak{X}_r^{N,-}\rangle\,dr>\varepsilon_0\right)=0.\tag{4.4.13}$$

Observe that, since $\langle\ell_{\delta_N},\,\mathfrak{X}_r^{N,+}\otimes\mathfrak{X}_r^{N,-}\rangle$ is non-negative, it suffices to prove (4.4.13) for the dominating case where $\Lambda_\pm$ are empty. We now prove this together with the tightness of $\{J_N\}$ at one stroke by applying Lemma 3.5.3 to the special case $q=2$ and $Y_N(r)=\langle\ell_{\delta_N},\,\mathfrak{X}_r^{N,+}\otimes\mathfrak{X}_r^{N,-}\rangle$.

Using the Gaussian upper bound (2.1.3) for the heat kernel of the reflected diffusions, we have

$$\overline{\lim_{N\to\infty}}\int_h^T\mathbb{E}\Big[\langle\ell_{\delta_N},\,\overline{\mathfrak{X}}_s^{N,+}\otimes\overline{\mathfrak{X}}_s^{N,-}\rangle^2\Big]\,ds\le C(d,D_+,D_-)\|\rho_+\|\,\|\rho_-\|\int_h^T s^{-2d}\,ds<\infty.$$

The hypothesis (i) of Lemma 3.5.3 is therefore satisfied, since $(\overline{\mathfrak{X}}^{N,+},\, \overline{\mathfrak{X}}^{N,-})$ dominates $(\mathfrak{X}^{N,+},\, \mathfrak{X}^{N,-})$.

It remains to verify hypothesis (ii) of Lemma 3.5.3, that is, to prove that for any $\varepsilon_0 > 0$, $\lim_{\alpha\to 0} \overline{\lim}_{N\to\infty} \mathbb{P}(J_n(\alpha) > \varepsilon_0) = 0$. By Corollary 4.4.3 again, for any $\phi \in Dom(\mathcal{A}^+)$, we have

$$\frac{1}{2}\int_0^t \langle \ell_{\delta_N}\phi,\, \mathfrak{X}_s^{N,+} \otimes \mathfrak{X}_s^{N,-}\rangle\, ds = \langle \phi, \mathfrak{X}_0^{N,+}\rangle - \langle \phi, \mathfrak{X}_t^{N,+}\rangle + \int_0^t \langle \mathcal{A}^+\phi,\, \mathfrak{X}_s^{N,+}\rangle\, ds + M_N(t), \quad (4.4.14)$$

where $M_N(t)$ is a martingale and $\lim_{N\to\infty} \mathbb{E}\left[(M_N(t))^2\right] = 0$ for all $t > 0$. Note that the left hand side of (4.4.14) is comparable to $J_N(t)$ whenever we pick $\phi \in Dom(\mathcal{A}^+)$ in such a way that $\ell_{\delta_N}\phi \approx \ell_{\delta_N}$. The idea is to pick $\phi \approx \mathbf{1}_{(D_+)_r}$, then let $r \to 0$ to bound $J_N(t)$ from above. Here $\mathbf{1}_{(D_+)_r}$ is the set of points in D_+ whose distance from the boundary is less than r. More specifically, for any $r > 0$, let $\psi_r \in C(\overline{D}_+)$ be such that $\psi_r = 1$ on $(D_+)_r$, $\psi_r = 0$ on $D_+ \setminus (D_+)_{2r}$ and $0 \le \psi \le 1$. Let $\phi_r \in Dom(\mathcal{A}^+) \cap C^+(\overline{D}_+)$ be such that $\|\phi_r - \psi_r\| = o(r)$. Such ϕ_r exists since $Dom(\mathcal{A}^+)$ is dense in $C(\overline{D}_+)$. Then (4.4.14) implies

$$\begin{aligned} 0 &\le J_N(\alpha) \\ &\le \left| \int_0^\alpha \langle \ell_{\delta_N} - \ell_{\delta_N}\phi_r,\, \mathfrak{X}_s^{N,+} \otimes \mathfrak{X}_s^{N,-}\rangle\, ds \right| + \langle \phi_r, \mathfrak{X}_0^{N,+}\rangle - \langle \phi_r, \mathfrak{X}_\alpha^{N,+}\rangle + \|\mathcal{A}^+\phi_r\|\, \alpha + |M_N(\alpha)| \\ &\le o(r) J_N(\alpha) + \langle \phi_r, \mathfrak{X}_0^{N,+}\rangle + \|\mathcal{A}^+\phi_r\|\, \alpha + |M_N(\alpha)| \quad \text{whenever } r > 2\delta_N. \end{aligned}$$

This is because when $r > 2\delta_N$, $\phi_r(x)$ is close to 1 on $(D_+)_{\delta_N}$. Hence we have, for $r > 2\delta_N$,

$$(1 - o(r))\, J_N(\alpha) \le \langle \phi_r, \mathfrak{X}_0^{N,+}\rangle + \|\mathcal{A}^+\phi_r\|\, \alpha + |M_N(\alpha)|.$$

From this, we have

$$\lim_{\alpha\to 0} \overline{\lim_{N\to\infty}}\, \mathbb{P}(J_N(\alpha) > 3\varepsilon_0) \le \overline{\lim_{N\to\infty}}\, \mathbb{P}\left(\langle \phi_r, \mathfrak{X}_0^{N,+}\rangle > \varepsilon_0(1 - o(r))\right).$$

Note that $0 \le \phi_r \le \mathbf{1}_{(D_+)_{2r}} + o(r)$. So for $r > 0$ small enough,

$$\mathbb{P}\Big(\langle \phi_r, \mathfrak{X}_0^{N,+}\rangle > \varepsilon_0(1 - o(r))\Big) \le \mathbb{P}(\langle \mathbf{1}_{(D_+)_{2r}}, \mathfrak{X}_0^{N,+}\rangle > \varepsilon_0/2).$$

Moreover, since $\mathfrak{X}_0^{N,+} \xrightarrow{\mathcal{L}} u_0^+(x)dx$ with $u_0^+ \in C(\overline{D})$, we have

$$\lim_{r\to 0} \overline{\lim_{N\to\infty}} \mathbb{P}(\langle \mathbf{1}_{(D_+)_{2r}}, \mathfrak{X}_0^{N,+}\rangle > \varepsilon_0/2) = 0.$$

Hence the second hypothesis of Lemma 3.5.3 is verified. We have shown that (ii) is true. Thus $(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})$ is relatively compact. Property (ii) above also tells us that any subsequential limit has law concentrated on $C([0,\infty),\mathfrak{M})$ (detail can be found in the proof of Theorem 3.5.4 in Chapter 3). $\square$

4.5 Characterization of the mean and the variance

Recall that we have already established tightness of $\{(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-}); N \geq 1\}$ in Theorem 4.4.4. Hence any subsequence has a further subsequence which converges in distribution in $D([0,T],\mathfrak{M})$. Let $\mathbb{P}^\infty$ be the law of an arbitrary subsequential limit $(\mathfrak{X}^{\infty,+}, \mathfrak{X}^{\infty,-})$. Then $\mathbb{P}^\infty((\mathfrak{X}^{\infty,+}, \mathfrak{X}^{\infty,-}) \in C([0,\infty),\mathfrak{M})) = 1$ by Theorem 4.4.4. Our goal is to show that

$$(\mathfrak{X}^{\infty,+}, \mathfrak{X}^{\infty,-}) = (u_+(t,x)\rho_+(x)dx,\, u_-(t,y)\rho_-(y)dy) \quad \mathbb{P}^\infty - a.s.$$

An immediate question is whether $\mathfrak{X}^{\infty,+}$ and $\mathfrak{X}^{\infty,-}$ have densities with respect to the Lebesque measure. For this, we can compare $(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})$ with $(\overline{\mathfrak{X}}^{N,+}, \overline{\mathfrak{X}}^{N,-})$ to get an affirmative answer. The construction in subsection 4.1.1 provides a natural coupling between $\{(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})\}$ and $\{(\overline{\mathfrak{X}}^{N,+}, \overline{\mathfrak{X}}^{N,-})\}$. We summarize some preliminary information about $(\mathfrak{X}^{\infty,+}, \mathfrak{X}^{\infty,-})$ in the following lemma. Its proof is straightforward and is omitted.

Lemma 4.5.1.

$$\mathbb{P}^\infty\Big(\langle \mathfrak{X}_t^{\infty,+}, \phi_+\rangle_{\rho_+} \leq \langle P_t^+ u_0^+, \phi_+\rangle_{\rho_+} \text{ and } \langle \mathfrak{X}_t^{\infty,-}, \phi_-\rangle_{\rho_-} \leq \langle P_t^- u_0^-, \phi_-\rangle_{\rho_-} \\ \text{for } t \geq 0 \text{ and } \phi_\pm \in C_\infty(\overline{D}_\pm \setminus \Lambda_\pm)\Big) = 1.$$

In particular, both $\mathfrak{X}_t^{\infty,+}$ *and* $\mathfrak{X}_t^{\infty,-}$ *are absolutely continuous with respect to the Lebesque measure for* $t \geq 0$. *Moreover,* $(\mathfrak{X}_t^{\infty,+}, \mathfrak{X}_t^{\infty,-}) = (v_+(t,x)\rho_+(x)dx, v_-(t,y)\rho_-(y)dy)$ *for some* $v_\pm(t) \in \mathcal{B}_b(D_\pm)$ *with* $v_+(t,x) \leq P_t^+ u_0^+(x)$ *and* $v_-(t,y) \leq P_t^- u_0^-(y)$ *for a.e.* $(x,y) \in D_+ \times D_-$.

The characterization $(\mathfrak{X}^{\infty,+}, \mathfrak{X}^{\infty,-})$ will be accomplished by the following result of 'mean-variance analysis':

Proposition 4.5.2. *For all* $\phi_\pm \in C_\infty(\overline{D}_\pm \setminus \Lambda_\pm)$ *and* $t \geq 0$, *we have*

$$\mathbb{E}^\infty[\langle v_\pm(t), \phi_\pm \rangle_{\rho_\pm}] = \langle u_\pm(t), \phi_\pm \rangle_{\rho_\pm}, \tag{4.5.1}$$

$$\mathbb{E}^\infty[\langle v_\pm(t), \phi_\pm \rangle^2_{\rho_\pm}] = \langle u_\pm(t), \phi_\pm \rangle^2_{\rho_\pm}. \tag{4.5.2}$$

where $v_\pm$ *is the density of* $\mathfrak{X}^{\infty,\pm}$, *w.r.t.* $\rho_\pm(x)dx$, *stated in Lemma 4.5.1.*

The goal and the remaining part of this last section is to prove Proposition 4.5.2.

4.5.1 Results about Minkowski content

We first look at a single domain and strengthen some results from Geometric Measure Theory.

Minkowski content for ∂D

Lemma 4.5.3. *Let* $D \subset \mathbb{R}^d$ *be a bounded Lipschitz domain. If* $\mathcal{F} \subset C(\overline{D})$ *is an equi-continuous and uniformly bounded family of functions on* $\overline{D}$, *then*

$$\lim_{\varepsilon \to 0} \sup_{f \in \mathcal{F}} \left| \frac{1}{\varepsilon} \int_{D_\varepsilon} f(x)dx - \int_{\partial D} f(x)\sigma(dx) \right| = 0.$$

Proof The result holds trivially when $d = 1$, by the uniform continuity of f. We will only consider $d \geq 2$. The idea is to cut ∂D into small pieces so that f is almost constant in each piece, and then apply (1.2.3) in each piece.

Fix $\eta > 0$. There exists $\delta > 0$ such that $|f(x) - f(y)| < \eta$ whenever $|x - y| \leq \delta$. Since D is bounded and Lipschitz (or by a more general result by G. David in [26] or [27, Section 2]), we can reduce to local coordinates to obtain a partition $\{Q_i\}_{i=1}^N$ of ∂D in such a way that for any i, Q_i is the Lipschitz image of a bounded subset of $\mathbb{R}^{d-1}$ (hence it is $(\mathcal{H}^{d-1})$-rectifiable), $diam(Q_i) \le \delta$ and ∂Q_i is $(\mathcal{H}^{d-2})$-rectifiable. Here ∂Q_i is the boundary of Q_i with respect to the topology induced by ∂D.

Let $(Q_i)_\varepsilon := \{x \in D : dist(x, Q_i) < \varepsilon\}$ and $(\partial Q_i)_\varepsilon := \{x \in D : dist(x, \partial Q_i) < \varepsilon\}$. Since $\{(Q_i)_\varepsilon \setminus (\partial Q_i)_\varepsilon\}_{i=1}^N$ are disjoint and $\cup_{i=1}^N (Q_i)_\varepsilon \setminus (\partial Q_i)_\varepsilon \subset D_\varepsilon \subset \cup_{i=1}^N (Q_i)_\varepsilon$, we have

$$\Big|\sum_{i=1}^N \int_{(Q_i)_\varepsilon} f\,dx - \int_{D_\varepsilon} f\,dx\Big| \le \sum_{i=1}^N \int_{(\partial Q_i)_\varepsilon} |f|\,dx. \tag{4.5.3}$$

Therefore, we have

$$\begin{aligned}
&\Big|\frac{1}{\varepsilon}\int_{D_\varepsilon} f\,dx - \int_{\partial D} f\,d\sigma\Big| \\
\le\ & \Big|\frac{1}{\varepsilon}\int_{D_\varepsilon} f\,dx - \frac{1}{\varepsilon}\sum_{i=1}^N \int_{(Q_i)_\varepsilon} f\,dx\Big| + \Big|\frac{1}{\varepsilon}\sum_{i=1}^N \int_{(Q_i)_\varepsilon} f\,dx - \int_{\partial D} f\,d\sigma\Big| \\
\le\ & \frac{1}{\varepsilon}\sum_{i=1}^N \int_{(\partial Q_i)_\varepsilon} |f|\,dx + \sum_{i=1}^N \Big|\frac{1}{\varepsilon}\int_{(Q_i)_\varepsilon} f\,dx - \int_{Q_i} f\,d\sigma\Big| \quad \text{by (4.5.3)} \\
\le\ & \sum_{i=1}^N \left(\|f\|\,\frac{|(\partial Q_i)_\varepsilon|}{\varepsilon} + \Big|\frac{1}{\varepsilon}\int_{(Q_i)_\varepsilon} f - f(\xi_i)\,dx\Big| + |f(\xi_i)|\,\Big|\frac{|(Q_i)_\varepsilon|}{\varepsilon} - \sigma(Q_i)\Big| + \Big|\int_{Q_i} f - f(\xi_i)\,d\sigma\Big|\right) \\
\le\ & \eta \sum_{i=1}^N \left(\frac{|(Q_i)_\varepsilon|}{\varepsilon} + \sigma(Q_i)\right) + \|f\| \sum_{i=1}^N \left(\frac{|(\partial Q_i)_\varepsilon|}{\varepsilon} + \Big|\frac{|(Q_i)_\varepsilon|}{\varepsilon} - \sigma(Q_i)\Big|\right).
\end{aligned}$$

Since ∂Q_i and $(Q_i)_\varepsilon$ are $(\mathcal{H}^{d-2})$-rectifiable and $(\mathcal{H}^{d-1})$-rectifiable, respectively, [38, Theorem 3.2.39] tells us that

$$\lim_{\varepsilon\to 0} \frac{|(\partial Q_i)_\varepsilon|}{c_2\varepsilon^2} = \mathcal{H}^{d-2}(\partial Q_i) \quad \text{and} \quad \lim_{\varepsilon\to 0} \frac{|(Q_i)_\varepsilon|}{\varepsilon} = \mathcal{H}^{d-1}(Q_i),$$

where $c_m := |\{x \in \mathbb{R}^m : |x| < 1\}|$. Thus,

$$\varlimsup_{\varepsilon\to 0} \Big|\frac{1}{\varepsilon}\int_{D_\varepsilon} f\,dx - \int_{\partial D} f\,d\sigma\Big| \le 2\eta \sum_i \sigma(Q_i) = 2\sigma(\partial D)\,\eta.$$

Since $\eta > 0$ is arbitrary and the above estimate is uniform over $f \in \mathcal{F}$, we get the desired result. $\square$

By the same proof of Lemma 5.3.1, we obtain the following stronger result.

Lemma 4.5.4. *Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $k \in \mathbb{N}$. If $\mathcal{F} \subset C(\overline{D}^k)$ is an equi-continuous and uniformly bounded family of functions, then*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^k} \int_{(D^\varepsilon)^k} f(z_1, \cdots, z_k)\, dz_1 \cdots dz_k = \int_{(\partial D)^k} f(z_1, \cdots, z_k)\, \sigma(dz_1) \cdots \sigma(dz_1)$$

uniformly for $f \in \mathcal{F}$, where $D^\varepsilon := \{x \in D : \ dist(x, \partial D) < \varepsilon\}$ is the ε-neighborhood of ∂D in D and σ is the surface measure on ∂D.

Minkowski content for $\{(z, z) : \ z \in I\}$

Now we prove analogous results for the interface I for our annihilation model.

Lemma 4.5.5. *Under our geometric setting in Assumption 5.1.1, if $\mathcal{F} \subset C(\overline{D}_+ \times \overline{D}_-)$ is an equi-continuous and uniformly bounded family of functions on $\overline{D}_+ \times \overline{D}_-$, then*

$$\lim_{\delta \to 0} \sup_{f \in \mathcal{F}} \Big| (c_{d+1}\, \delta^{d+1})^{-1} \int_{I^\delta} f(x, y)\, dx\, dy - \int_I f(z, z)\, d\sigma(z) \Big| = 0.$$

Proof By the same argument as in the proof of Lemma 5.3.1, we can construct a nice partition $\{Q_i\}_{i=1}^N$ of I and apply [38, Theorem 3.2.39 (p. 275)]. The only essential difference is that now we require $\partial Q_i \setminus \partial I$ to be $(\mathcal{H}^{d-2})$−rectifiable, where ∂I is the boundary of I with respect to the topology induced by ∂D_+, or equivalently by ∂D_-. Moreover, instead of (4.5.3), we now have

$$\Big| \sum_{i=1}^N \int_{(Q_i)_\delta} f\, dxdy - \int_{I^\delta} f\, dxdy \Big| \le \sum_{i=1}^N \int_{(\partial Q_i \setminus \partial I)_\delta} |f|\, dxdy. \tag{4.5.4}$$

Note that we do not need any assumption on ∂I. $\square$

By the same proof of Lemma 5.3.1, we obtain the following stronger result.

Lemma 4.5.6. *Suppose Assumptions 5.1.1, 4.0.8 and 4.0.9 hold. Suppose $k \in \mathbb{N}$ and $\mathcal{F} \subset C\big((\overline{D}_+ \times \overline{D}_-)^k\big)$ is an equi-continuous and uniformly bounded family of functions on $(\overline{D}_+ \times \overline{D}_-)^k$. Then as $\delta \to 0$, we have*

$$\int_{(x_1,y_1)\in D_+\times D_-} \cdots \int_{(x_k,y_k)\in D_+\times D_-} f(x_1,y_1,\cdots,x_k,y_k) \prod_{i=1}^k \ell_\delta(x_i,y_i)\; d(x_1,y_1,\cdots,x_k,y_k)$$
$$\to \quad \int_{z_1\in I} \cdots \int_{z_k \in I} f(z_1,z_1,\cdots,z_k,z_k)\,) \prod_{i=1}^k \lambda(z_i)\, d\sigma(z_1)\cdots d\sigma(z_k)$$

uniformly for $f \in \mathcal{F}$.

Remark 4.5.7. Following the same proof as above, clearly we can strengthen Lemma 4.5.5 and Lemma 4.5.6 by only requiring $\mathcal{F}$ to be equi-continuous and uniformly bounded on a neighborhood of the interface I. We can also generalize Lemma 5.3.1 and Lemma 4.5.4 to deal with $\int_J f(x)\, d\sigma(x)$ for any closed $\mathcal{H}^{d-1}$-rectifiable subset of J of ∂D (rather than the whole boundary ∂D), and by requiring $\mathcal{F}$ to be equi-continuous and uniformly bounded on a neighborhood of J.

4.5.2 Martingales for space-time processes

In this subsection, we collect some integral equations satisfied by $(\mathfrak{X}^{N,+},\, \mathfrak{X}^{N,-})$ that will be used later to identify the limit. These integral equations can be viewed as the Dynkins' formulae for our annihilating diffusion system, and will be proved rigorously by considering suitable martingales associated with the process $(t,\, (\mathfrak{X}^{N,+}_t,\, \mathfrak{X}^{N,-}_t))$.

Theorem 4.5.8. *Let $T > 0$, and $f_s \in C_b(E_N)$ and $g_s \in \mathcal{B}(E_N)$ for $s \in [0,T]$. Suppose*

$$\overline{M}_s := f_s(\overline{\mathfrak{X}}^{N,+}_s, \overline{\mathfrak{X}}^{N,-}_s) - \int_0^s g_r(\overline{\mathfrak{X}}^{N,+}_r, \overline{\mathfrak{X}}^{N,-}_r)\, dr$$

is a $\mathcal{F}_s^{(\overline{\mathfrak{X}}^{N,+},\overline{\mathfrak{X}}^{N,-})}$-martingale for $s \in [0,T]$, under $\mathbb{P}^\mu$ for any $\mu \in E_N$. Then

$$M_s := f_s(\mathfrak{X}^{N,+}_s, \mathfrak{X}^{N,-}_s) - \int_0^s (g_r + Kf_r)(\mathfrak{X}^{N,+}_r, \mathfrak{X}^{N,-}_r)\, dr$$

is a $\mathcal{F}_r^{(\mathfrak{X}^{N,+},\mathfrak{X}^{N,-})}$*-martingale for* $s \in [0,T]$*, under* $\mathbb{P}^\mu$ *for any* $\mu \in E_N$*, where the operator* K *is given by (4.4.1).*

This is a time-dependent version of Theorem 4.4.1. A proof can be obtained by following the same argument in the proof of Theorem 4.4.1, but now to the time dependent process $(t,\,(\mathfrak{X}_t^{N,+},\mathfrak{X}_t^{N,-}))$. The detail is left to the reader.

Consider $X_{(n,m)} := (X_1^+,\cdots,\,X_n^+,\;\mathfrak{X}_1^-,\cdots,\,X_m^-) \in (D_+^\partial)^n \times (D_-^\partial)^m$, which consists of independent copies of $X^\pm$'s. The transition density of $X_{(n,m)}$ w.r.t. $\rho_{(n,m)}$ is $p^{(n,m)}$, where

$$p^{(n,m)}(t,(\vec{x},\vec{y}),(\vec{x'},\vec{y'})) := \prod_{i=1}^n p^+(t,x_i,x_i')\prod_{j=1}^m p^-(t,y_j,y_j')$$

$$\rho_{(n,m)}(\vec{x},\vec{y}) := \prod_{i=1}^n \rho_+(x_i)\prod_{j=1}^m \rho_-(y_j).$$

The semigroup of $X_{(n,m)}$, denoted by $P_t^{(n,m)}$, is strongly continuous on

$$C_\infty^{(n,m)} := \left\{\Phi \in C(\overline{D}_+^n \times \overline{D}_-^m):\ \Phi \text{ vanishes outside } (\overline{D}_+ \setminus \Lambda_+)^n \times (\overline{D}_- \setminus \Lambda_-)^m\right\} \tag{4.5.5}$$

Clearly, $C_\infty^{(1,0)} = C_\infty(\overline{D}_+ \setminus \Lambda_+)$ and $C_\infty^{(0,1)} = C_\infty(\overline{D}_- \setminus \Lambda_-)$.

Corollary 4.5.9. *Let* n *and* m *be any non-negative integers,* $T > 0$ *be any positive number and* $\Phi \in C_\infty^{(n,m)}$*. Consider the function* $f:\,[0,T]\times E_N \to \mathbb{R}$ *defined as follows:* $f(s,\boldsymbol{0}_*) := 0$ *and for an arbitrary element* $\mu \in E_N \setminus \{\boldsymbol{0}_*\}$*, we can write* $\mu = (\frac{1}{N}\sum_{i\in A_+}\mathbf{1}_{x_i},\,\frac{1}{N}\sum_{j\in A_-}\mathbf{1}_{y_j})$ *for some index sets* A_+ *and* A_-*, then*

$$f(s,\,\mu) \;:=\; \sum_{\substack{i_1,\cdots,i_n\\ \textit{distinct}}}\;\sum_{\substack{j_1,\cdots,j_m\\ \textit{distinct}}} P_{T-s}^{(n,m)}\Phi(x^{i_1},\cdots,x^{i_n},\,y^{j_1},\cdots,y^{j_m}),$$

where the first summation is on the collection of all n*-tuples* $(i_1,\cdots,i_n)$ *chosen from distinct elements of* A_+*, the second summation is on the collection of all* m*-tuples* $(j_1,\cdots,j_m)$ *chosen* *from distinct elements of A_-. Then we have*

$$f\big(s,\,(\mathfrak{X}^{N,+}_s,\mathfrak{X}^{N,-}_s)\big)-\int_0^s Kf(r,\,\cdot)(\mathfrak{X}^{N,+}_r,\mathfrak{X}^{N,-}_r)\,dr$$

is a $\mathcal{F}_s^{(\mathfrak{X}^{N,+},\mathfrak{X}^{N,-})}$-martingale for $s\in[0,T]$, under $\mathbb{P}^\nu$, for any $\nu\in E_N$.

Proof Clearly, $f(s,\cdot)\in C_b(E_N)$ for $s\in[0,T]$. By Lemma 2.1.5, we have $f(s,\overline{\mathfrak{X}}_s)$ is a $\mathcal{F}_s^{\overline{\mathfrak{X}}}$-martingale for $s\in[0,T]$ for all $T\geq 0$. Hence we can take g_r to be constant zero and f_r to be $f(r,\cdot)$ in Theorem 4.5.8 to finish the proof. □

As an immediate consequence, we obtain the Dynkin's formula for our system: For $0\leq t\leq T$, we have

$$\mathbb{E}\Big[f\big(T,\,(\mathfrak{X}^{N,+}_T,\mathfrak{X}^{N,-}_T)\big)-f\big(t,\,(\mathfrak{X}^{N,+}_t,\mathfrak{X}^{N,-}_t)\big)-\int_t^T Kf(r,\,\cdot)(\mathfrak{X}^{N,+}_r,\mathfrak{X}^{N,-}_r)\,dr\Big]=0 \tag{4.5.6}$$

Corollary 4.5.9 is the key to obtain the system of equations satisfied by the correlation functions of the particles in the annihilating diffusion system. This system of equations, usually called BBGKY hierarchy, will be formulated in the forthcoming paper [19]. The specific integral equations that we need to identify subsequential limits of $\{(\mathfrak{X}^{N,+},\mathfrak{X}^{N,-})\}$ are stated in the following lemmas. These equations are a part of the BBGKY hierarchy.

Lemma 4.5.10. *For any $\phi_\pm\in C_\infty(\overline{D}_\pm\setminus\Lambda_\pm)$ and $0\leq t\leq T<\infty$, we have*

$$\begin{aligned}&\mathbb{E}\left[\langle\phi_+,\,\mathfrak{X}^{N,+}_T\rangle+\langle\phi_-,\,\mathfrak{X}^{N,-}_T\rangle\right]-\mathbb{E}\left[\langle P^+_{T-t}\phi_+,\,\mathfrak{X}^{N,+}_t\rangle+\langle P^-_{T-t}\phi_-,\,\mathfrak{X}^{N,-}_t\rangle\right] \\ =\;&-\frac{1}{2}\int_t^T\mathbb{E}\left[\,\langle\ell_{\delta_N}\,(P^+_{T-r}\phi_++P^-_{T-r}\phi_-),\,\mathfrak{X}^{N,+}_r\otimes\mathfrak{X}^{N,-}_r\rangle\,\right]\,dr\end{aligned} \tag{4.5.7}$$

and

$$\begin{aligned}&\mathbb{E}\left[\langle\phi_+,\,\mathfrak{X}^{N,+}_T\rangle^2\right]-\mathbb{E}\left[\langle P^+_{T-t}\phi_+,\,\mathfrak{X}^{N,+}_t\rangle^2\right] \\ =\;&-\int_t^T\mathbb{E}\left[\,\langle P^+_{T-r}\phi_+,\mathfrak{X}^{N,+}_r\rangle\,\langle\ell_{\delta_N}\,(P^+_{T-r}\phi_+),\,\mathfrak{X}^{N,+}_r\otimes\mathfrak{X}^{N,-}_r\rangle\,\right]\,dr++o(N),\end{aligned} \tag{4.5.8}$$

where $o(N)$ is a term which tends to zero as $N \to \infty$. A similar formula for (4.5.8) holds for $\mathfrak{X}^{N,-}$.

Proof Since $Dom(\mathcal{A}^{\pm})$ is dense in $C_\infty(\overline{D}_\pm \setminus \Lambda_\pm)$. Therefore, it suffices to check the lemma for $\phi_\pm \in Dom(\mathcal{A}^{\pm})$.

Identity (4.5.7) follows directly from Corollary 4.5.9 by taking $f(s,\mu) = \langle P^+_{T-s}\phi_+,\, \mu^+\rangle + \langle P^-_{T-s}\phi_-,\, \mu^-\rangle$.

For (4.5.8), we can apply Lemma 2.1.5 and Theorem 4.5.8, with $f_s(\mu) = \langle P^+_{T-s}\phi_+, \mu^+\rangle^2$ and $g_s(\mu) = \frac{1}{N}\langle \mathbf{a}_+\nabla P^+_{T-s}\phi_+ \cdot \nabla P^+_{T-s}\phi_+,\, \mu^+\rangle$, to obtain

$$
\begin{aligned}
& \mathbb{E}\left[\langle \phi_+,\, \mathfrak{X}^{N,+}_T\rangle^2\right] - \mathbb{E}[\langle P^+_{T-t}\phi_+,\, \mathfrak{X}^{N,+}_t\rangle^2\Big] \\
= \quad & -\int_t^T \mathbb{E}\left[\, \langle P^+_{T-r}\phi_+, \mathfrak{X}^{N,+}_r\rangle\, \langle \ell_{\delta_N}\, (P^+_{T-r}\phi_+),\, \mathfrak{X}^{N,+}_r \otimes \mathfrak{X}^{N,-}_r\rangle\right] dr \\
& + \frac{1}{2N}\int_t^T \mathbb{E}\left[2\mathbf{a}_+\nabla P^+_{T-s}\phi_+ \cdot \nabla P^+_{T-s}\phi_+,\, \mathfrak{X}^{N,+}_r\rangle + \langle \ell_{\delta_N}\, (P^+_{T-r}\phi_+)^2,\, \mathfrak{X}^{N,+}_r \otimes \mathfrak{X}^{N,-}_r\rangle\right] dr.
\end{aligned}
$$

Note that the term with a factor $\frac{1}{N}$ converges to zero as $N \to \infty$. This can be proved by the same argument for the bound of the quadratic variation $\mathbb{E}^\mu[(M^{(\phi_+,\phi_-)}_t)^2]$ in Corollary 4.4.3. Hence we have (4.5.8). □

We now derive the integral equations satisfied by the integrands (with respect to dr) on the right hand side of (4.5.7) and (4.5.8). The integrand (with respect to dr) of the right hand side of (4.5.8) is of the form

$$
\langle \phi,\, \mu^+\rangle\langle \varphi,\, \mu^+ \otimes \mu^-\rangle = \frac{1}{N^3}\Big(\sum_i\sum_j \phi(x_i)\varphi(x_i,y_j) + \sum_\ell\sum_{i\neq\ell}\sum_j \phi(x_\ell)\varphi(x_i,y_j)\Big),
$$

where $\varphi \in \mathcal{B}(\overline{D}_+ \times \overline{D}_-)$, $\phi = \phi_+ \in \mathcal{B}(\overline{D}_+)$ and $\mu = (\frac{1}{N}\sum_i \mathbf{1}_{x_i},\, \frac{1}{N}\sum_j \mathbf{1}_{y_j}) \in E_N$. We define

$$
\begin{aligned}
& P_t^{(*)}(\langle \phi,\, \mu^+\rangle\langle \varphi,\, \mu^+\otimes\mu^-\rangle) \qquad (4.5.9)\\
:= \;& \frac{1}{N^3}\Big(\sum_i\sum_j P_t^{(1,1)}(\phi\varphi)(x_i,y_j) + \sum_\ell \sum_{i\neq \ell}\sum_j P_t^{(2,1)}(\phi\varphi)(x_\ell,x_i,y_j)\Big)\\
= \;& \langle P_t^{(2,1)}(\phi\varphi)(x_1,x_2,y),\; \mu^+(dx_1)\otimes\mu^+(dx_2)\otimes\mu^-(dy)\rangle\\
& +\frac{1}{N}\,\langle\, P_t^{(1,1)}(\phi\varphi)(x,y) - P_t^{(2,1)}(\phi\varphi)(x,x,y),\; \mu^+(dx)\otimes\mu^-(dy)\,\rangle,
\end{aligned}
$$

In $P_t^{(1,1)}(\phi\varphi)$, we view $\phi\varphi$ as the function of two variables $(a,b)\mapsto \phi(a)\varphi(a,b)$; in $P_t^{(2,1)}(\phi\varphi)$, we view $\phi\varphi$ as the function of three variables $(a_1,a_2,b)\mapsto\phi(a_1)\varphi(a_2,b)$. The definition of $P_t^{(*)}$ is motivated by the fact that $f(s,\mu) := P_{T-s}^{(*)}\langle\phi_+\varphi,\,\mu^+\otimes\mu^+\otimes\mu^-\rangle$ is of the same form as the function in Corollary 4.4.3.

Lemma 4.5.11. *For any $\varphi\in C_\infty^{(1,1)}$, $\phi_\pm \in C_\infty(\overline{D}_\pm\setminus\Lambda_\pm)$ and $0\le t\le T<\infty$, we have*

$$
\begin{aligned}
& \mathbb{E}[\langle\varphi,\, \mathfrak{X}_T^{N,+}\otimes\mathfrak{X}_T^{N,-}\rangle] - \mathbb{E}[\langle P_{T-t}^{(1,1)}\varphi,\, \mathfrak{X}_t^{N,+}\otimes\mathfrak{X}_t^{N,-}\rangle]\\
= \;& -\frac{1}{2}\int_t^T \mathbb{E}\Big[\Big\langle\, \ell_{\delta_N}(x,y)\,\Big(\langle F_r(x,\cdot),\, \mathfrak{X}_r^{N,-}\rangle + \langle F_r(\cdot,y),\,\mathfrak{X}_r^{N,+}\rangle - \frac{1}{N}F_r(x,y)\Big),\\
& \qquad \mathfrak{X}_r^{N,+}(dx)\otimes\mathfrak{X}_r^{N,-}(dy)\Big\rangle\Big]\,dr \qquad (4.5.10)
\end{aligned}
$$

and

$$
\begin{aligned}
& \mathbb{E}[\langle\phi_+,\,\mathfrak{X}_T^{N,+}\rangle\langle\varphi,\,\mathfrak{X}_T^{N,+}\otimes\mathfrak{X}_T^{N,-}\rangle] - \mathbb{E}\left[P_{T-t}^{(*)}\,\langle\phi_+\varphi,\,\mathfrak{X}_t^{N,+}\otimes\mathfrak{X}_t^{N,+}\otimes\mathfrak{X}_t^{N,-}\rangle\right]\\
= \;& -\frac{1}{2}\int_t^T\mathbb{E}\Big[\Big\langle\,\ell_{\delta_N}(x,y)\,\Big(\langle H_r(x,\cdot,\cdot),\,\mathfrak{X}_r^{N,+}\otimes\mathfrak{X}_r^{N,-}\rangle\\
& +\langle H_r(\cdot,x,\cdot),\,\mathfrak{X}_r^{N,+}\otimes\mathfrak{X}_r^{N,-}\rangle + \langle H_r(\cdot,\cdot,y),\,\mathfrak{X}_r^{N,+}\otimes\mathfrak{X}_r^{N,+}\rangle\\
& -\frac{1}{N}\left[\langle 2H_r(x,x,\cdot),\,\mathfrak{X}_r^{N,-}\rangle + \langle H_r(\cdot,x,y),\,\mathfrak{X}_r^{N,+}\rangle + \langle H_r(x,\cdot,y),\,\mathfrak{X}_r^{N,+}\rangle\right]\\
& +\frac{1}{N}\left[\langle G_r(x,\cdot),\,\mathfrak{X}_r^{N,-}\rangle + \langle G_r(\cdot,y),\,\mathfrak{X}_r^{N,+}\rangle - \langle H_r(\cdot,\cdot,y),\,\mathfrak{X}_r^{N,+}\rangle\right]\\
& +\frac{1}{N^2}\left[2H_r(x,x,y) - G_r(x,y)\right]\Big),\,\mathfrak{X}_r^{N,+}(dx)\otimes\mathfrak{X}_r^{N,-}(dy)\Big\rangle\Big]\,dr, \qquad (4.5.11)
\end{aligned}
$$

where $F_r = P^{(1,1)}_{T-r}\varphi$, $G_r = P^{(1,1)}_{T-r}(\phi_+\varphi)$ *and* $H_r = P^{(2,1)}_{T-r}(\phi_+\varphi)$. *A similar formula for (4.5.11) holds for* $\mathbb{E}\left[\langle \phi_-,\, \mathfrak{X}^{N,-}_T\rangle\, \langle \varphi,\, \mathfrak{X}^{N,+}_T \otimes \mathfrak{X}^{N,-}_T\rangle\right]$.

Proof We first prove (4.5.10). Consider, for $s \in [0,T]$, $f_s(\mu) = f(s,\mu) := \langle P^{(1,1)}_{T-s}\varphi,\, \mu^+ \otimes \mu^-\rangle$. Then (4.5.10) follows from Corollary 4.4.3 by directly calculating $\mathbb{E}[K(f_r)(\mathfrak{X}^{N,+}_r, \mathfrak{X}^{N,-}_r)]$ as follows: If $U_N(\vec{x},\vec{y}) = \mu$ where $(\vec{x},\vec{y}) \in E^{(m)}_N$, then

$$
\begin{aligned}
-Kf_r(\mu) &= \frac{1}{2N}\sum_{i=1}^{m}\sum_{j=1}^{m} \ell_{\delta_N}(x_i,y_j)\left(f_r(\mu) - f_r(\mu^+ - \frac{1}{N}\mathbf{1}_{\{x_i\}},\, \mu^- - \frac{1}{N}\mathbf{1}_{\{y_j\}})\right)\\
&= \frac{1}{2N}\sum_{i=1}^{m}\sum_{j=1}^{m} \ell_{\delta_N}(x_i,y_j)\left(\frac{1}{N^2}\left(\sum_l F_r(x_i,y_l) + \sum_k F_r(x_k,y_j) - F_r(x_i,y_j)\right)\right)\\
&= \frac{1}{2N}\sum_{i=1}^{m}\sum_{j=1}^{m} \ell_{\delta_N}(x_i,y_j)\left(\frac{1}{N}\langle F_r(x_i),\mu^-\rangle + \frac{1}{N}\langle F_r(y_j),\mu^+\rangle - \frac{1}{N^2}F_r(x_i,y_j)\right)\\
&= \frac{1}{2}\langle \ell_{\delta_N}(\langle F_r,\mu^-\rangle + \langle F_r,\mu^+\rangle - N^{-1}F_r),\, \mu^+\otimes\mu^-\rangle.
\end{aligned}
$$

For (4.5.11), we choose $f_s(\mu) := P^{(*)}_{T-s}\langle \phi_+\varphi,\, \mu^+\otimes\mu^+\otimes\mu^-\rangle$ instead and follow the same argument as above. The expression on the right hand side of (4.5.11) follows from the observation that, for fixed (i,j), we have

$$
\begin{aligned}
&N^3\left(g_r(\mu) - g_r(\mu^+ - \frac{1}{N}\mathbf{1}_{\{x_i\}},\, \mu^- - \frac{1}{N}\mathbf{1}_{\{y_j\}})\right)\\
= \;& \sum_q\sum_\ell H_r(x_i,x_q,y_\ell) + \sum_p\sum_\ell H_r(x_p,x_i,y_\ell) + \sum_p\sum_q H_r(x_p,x_q,y_j)\\
&- \sum_\ell H_r(x_i,x_i,y_\ell) - \sum_p H_r(x_p,x_i,y_j) - \sum_q H_r(x_i,x_q,y_j) + H_r(x_i,x_i,y_j)\\
&+ \sum_\ell G_r(x_i,y_\ell) + \sum_p G_r(x_p,y_j) - G_r(x_i,y_j)\\
&- \sum_\ell H_r(x_i,x_i,y_\ell) - \sum_p H_r(x_p,x_p,y_j) + H_r(x_i,x_i,y_j).
\end{aligned}
$$

The above expression can be obtained by using the Inclusion-Exclusion Principle. □

The next two sections will be devoted to the proof of (4.5.1) and (4.5.2), respectively.

4.5.3 First moment

The goal of this subsection is to prove (4.5.1) in Proposition 4.5.2. The following key lemma allows us to interchange limits. This is a crucial step in our characterization of $(\mathfrak{X}^{\infty,+}, \mathfrak{X}^{\infty,-})$, and is the step where Assumption 4.0.10 that $\liminf_{N\to\infty} N\,\delta_N^d \in (0,\infty]$ is used.

Lemma 4.5.12. *Suppose Assumption 4.0.10 holds. Then for any $t>0$ and any $\phi\in C_\infty^{(1,1)}$, as $\varepsilon\to 0$, each of $\mathbb{E}^\infty\left[\langle \ell_\varepsilon\phi,\, v_+(t)\rho_+\otimes v_-(t)\rho_-\rangle\right]$ and $\mathbb{E}\left[\langle \ell_\varepsilon\phi,\, \mathfrak{X}_t^{N,+}\otimes\mathfrak{X}_t^{N,-}\rangle\right]$ converges uniformly in $N\in\mathbb{N}$ and in any initial distributions $\{(\mathfrak{X}_0^{N,+},\mathfrak{X}_0^{N,-})\}$. Moreover,*

$$A^\phi(t) := \lim_{\varepsilon\to 0}\mathbb{E}\left[\langle \ell_\varepsilon\phi,\, v_+(t)\rho_+\otimes v_-(t)\rho_-\rangle\right] = \lim_{N'\to\infty}\lim_{\varepsilon\to 0}\mathbb{E}\left[\langle \ell_\varepsilon\phi,\, \mathfrak{X}_t^{N',+}\otimes\mathfrak{X}_t^{N',-}\rangle\right]$$

for any subsequence $\{N'\}$ along which $\{(\mathfrak{X}^{N,+},\mathfrak{X}^{N,-})\}_N$ converges to $(\mathfrak{X}^{\infty,+},\mathfrak{X}^{\infty,-})$ in distribution in $D([0,T],\mathfrak{M})$. Furthermore, $|A^\phi(t)|\le \|\phi\|\,\|P_t^+f\|\,\|P_t^-g\|\,\|\rho_+\|\,\|\rho_-\|\,\sigma(\partial I)$.

Proof Since $\rho_\pm\in C(\overline{D}_\pm)$ and is strictly positive, for notational simplicity, we assume without loss of generality that $\rho_\pm=1$. (The general case can be proved in the same way.) Recall from (4.5.10) that for any $\varphi\in C_\infty^{(1,1)}$, $\phi_\pm\in C_\infty(\overline{D}_\pm\setminus\Lambda_\pm)$ and $0\le s\le t<\infty$, we have

$$\begin{aligned}&\mathbb{E}\left[\langle \varphi,\, \mathfrak{X}_t^{N,+}\otimes\mathfrak{X}_t^{N,-}\rangle\right]-\mathbb{E}\left[\langle P_{t-s}^{(1,1)}\varphi,\, \mathfrak{X}_s^{N,+}\otimes\mathfrak{X}_s^{N,-}\rangle\right] \qquad (4.5.12)\\ =\;& -\frac{1}{2}\int_s^t\mathbb{E}\left[\left\langle \ell_{\delta_N}\left(\langle P_{t-r}^{(1,1)}\varphi,\, \mathfrak{X}_r^{N,-}\rangle+\langle P_{t-r}^{(1,1)}\varphi,\, \mathfrak{X}_r^{N,+}\rangle-\frac{1}{N}P_{t-r}^{(1,1)}\varphi\right),\, \mathfrak{X}_r^{N,+}\otimes\mathfrak{X}_r^{N,-}\right\rangle\right]dr.\end{aligned}$$

Note that $\ell_\varepsilon\phi\in C_\infty^{(1,1)}$ for ε small enough since I is disjoint from $\Lambda_\pm$. We fix $s\in(0,t)$. Putting

$\ell_{\varepsilon_1}\phi$ and $\ell_{\varepsilon_2}\phi$, respectively, in the place of φ in (4.5.12) and then subtract, we have

$$\begin{aligned}
&\Theta := \Big| \mathbb{E}[\langle \ell_{\varepsilon_1}\phi,\, \mathfrak{X}_t^{N,+}\otimes\mathfrak{X}_t^{N,-}\rangle] - \mathbb{E}[\langle \ell_{\varepsilon_2}\phi,\, \mathfrak{X}_t^{N,+}\otimes\mathfrak{X}_t^{N,-}\rangle] \Big| \qquad (4.5.13)\\
= &\ \Big| \mathbb{E}\left[\langle F_s,\, \mathfrak{X}_s^{N,+}\otimes\mathfrak{X}_s^{N,-}\rangle\right] - \frac{1}{2}\int_s^t \mathbb{E}\Big[\Big\langle\, \ell_{\delta_N}\Big(\langle F_r,\, \mathfrak{X}_r^{N,-}\rangle + \langle F_r,\, \mathfrak{X}_r^{N,+}\rangle - \frac{1}{N}F_r \Big),\, \mathfrak{X}_r^{N,+}\otimes\mathfrak{X}_r^{N,-}\Big\rangle \Big]\, dr \Big| \\
\le &\ \mathbb{E}\Big[\langle |F_s|,\, \overline{\mathfrak{X}}_s^{N,+}\otimes\overline{\mathfrak{X}}_s^{N,-}\rangle\Big] + \frac{1}{2}\int_s^t \mathbb{E}\Big[\Big\langle\, \ell_{\delta_N}\langle |F_r|,\, \overline{\mathfrak{X}}_r^{N,-}\rangle,\, \overline{\mathfrak{X}}_r^{N,+}\otimes\overline{\mathfrak{X}}_r^{N,-}\Big\rangle \Big] \\
&\qquad + \frac{1}{2}\mathbb{E}\Big[\Big\langle\, \ell_{\delta_N}\langle |F_r|,\, \overline{\mathfrak{X}}_r^{N,+}\rangle,\, \overline{\mathfrak{X}}_r^{N,+}\otimes\overline{\mathfrak{X}}_r^{N,-}\Big\rangle \Big] + \frac{1}{2N}\mathbb{E}\Big[\Big\langle\, |\ell_{\delta_N} F_r|,\, \overline{\mathfrak{X}}_r^{N,+}\otimes\overline{\mathfrak{X}}_r^{N,-}\Big\rangle \Big]\, dr \\
\le &\ \big\| P_s^{(1,1)}(|F_s|) \big\| + \frac{1}{2}\int_s^t (A_1 + A_2 + A_3)\, dr,
\end{aligned}$$

where $F_r := P_{t-r}^{(1,1)}(\ell_{\varepsilon_1}\phi - \ell_{\varepsilon_2}\phi)$, $A_1 := \Big\| P_r^{(1,1)}\Big(\ell_{\delta_N} P_r^{(0,1)}(|F_r|)\Big) \Big\|$, $A_2 := \Big\| P_r^{(1,1)}\Big(\ell_{\delta_N} P_r^{(1,0)}(|F_r|)\Big) \Big\|$, and $A_3 := \frac{1}{N}\Big\| P_r^{(1,1)}\Big(|\ell_{\delta_N} F_r|\Big) \Big\|$.

Clearly $\big\| P_s^{(1,1)}(|F_s|) \big\| \le \|F_s\|$. By applying Lemma 4.5.5 to the equi-continuous and uniformly bounded family

$$\{(x,y)\mapsto \phi(x)\, p(t-s,(a,b),(x,y)):\ (a,b)\in \overline{D}_+\times\overline{D}_-\} \subset C_\infty^{(1,1)} \subset C(\overline{D}_+\times\overline{D}_-),$$

we see that $\|F_s\|$ converges to zero uniformly for $N\in\mathbb{N}$ and for any initial configuration, as ε_1 and ε_2 both tend to zero.

By definition of A_1, (1.2.3), the Gaussian upper bound estimate (2.1.3) for the transition density p of the reflected diffusion, we have

$$\begin{aligned}
A_1 &= \sup_{(a,b)} \int_{\overline{D}_+}\int_{\overline{D}_-} \ell_{\delta_N}(x,y) \left(\sup_y P_r^-(|F_r|)(x,y) \right) p(r,(a,b),(x,y))\, dxdy \\
&\le \left(\sup_{(x,y)} P_r^-(|F_r|)(x,y) \right) \frac{C(d,D_+,D_-)}{s^d} \quad \text{if } N \ge N(d,D_+,D_-).
\end{aligned}$$

Using this bound, we have

$$
\begin{aligned}
& \int_s^t A_1 dr \\
\leq\; & \frac{C}{s^d} \int_s^t \sup_{(x,y)} P_r^- \left(|P_{t-r}^{(1,1)} (\ell_{\varepsilon_1}\phi - \ell_{\varepsilon_2}\phi)\,| \right)(x,y)\, dr \\
=\; & \frac{C}{s^d} \int_0^{t-s} \sup_{(x,y)} P_{t-w}^- \left(|P_w^{(1,1)} (\ell_{\varepsilon_1}\phi - \ell_{\varepsilon_2}\phi)\,| \right)(x,y)\, dw \\
=\; & \frac{C}{s^d} \int_0^{t-s} \Big(\sup_{(x,y)} \int_{D_-} \Big| \int_{D_+}\int_{D_-} (\ell_{\varepsilon_1}\phi - \ell_{\varepsilon_2}\phi)(\tilde{x},\tilde{y}) p(w,(x,b),(\tilde{x},\tilde{y}))\, d\tilde{x}\, d\tilde{y} \Big| \, p^-(t-w,y,b)\, db \Big)\, dw \\
\leq\; & \frac{C}{s^d} \left(\int_0^\alpha \frac{2C}{\sqrt{w}}\, t^{-d/2}\, dw + \int_\alpha^{t-s} \left\| P_w^{(1,1)} (\ell_{\varepsilon_1}\phi - \ell_{\varepsilon_2}\phi) \right\| dw \right). \qquad (4.5.14)
\end{aligned}
$$

The last inequality holds for any $\alpha \in (0, t-s)$. This is because

$$
\begin{aligned}
& \sup_{(x,y)} \int_{D_-}\int_{D_+}\int_{D_-} \ell_\varepsilon(\tilde{x},\tilde{y})\, p(w,(x,b),(\tilde{x},\tilde{y}))\, d\tilde{x}\, d\tilde{y}\, p^-(t-w,y,b)\, db \\
=\; & \sup_{(x,y)} \int_{D_-}\int_{D_+} \ell_\varepsilon(\tilde{x},\tilde{y})\, p^+(w,x,\tilde{x})\, p^-(t,y,\tilde{y})\, d\tilde{x}\, d\tilde{y} \\
& \qquad \text{by Chapman-Kolmogorov equation for } p^- \\
\leq\; & \frac{2C(d,D_+,D_-,T)}{\sqrt{w}}\, t^{-d/2} \quad \text{by applying the bound (2.1.4) on } D_+.
\end{aligned}
$$

Hence, from (4.5.14), by letting $\alpha \downarrow 0$ suitably and applying Lemma 4.5.5 to the equi-continuous and uniformly bounded family

$$
\Big\{ (x,y) \mapsto \phi(x)\, p(w,(a,b),(x,y)) :\; (a,b) \in \overline{D}_+ \times \overline{D}_-,\; w \in [\alpha, t-s] \Big\} \subset C(\overline{D}_+ \times \overline{D}_-),
$$

we see that $\int_s^t A_1\, dr$ converges to 0 as ε_1 and ε_2 tends to 0 uniformly for N large enough. The same conclusion hold for $\int_s^t A_2\, dr$ by the same argument.

So far we have not used the Assumption 4.0.10 of $\liminf_{N\to\infty} N\, \delta_N^d \in (0,\infty]$. We now use this assumption to show that $\int_s^t A_3\, dr$ tends to 0 uniformly for N large enough, as ε_1 and ε_2

tend to 0. By a change of variable $r \mapsto t-w$,

$$\begin{aligned}\int_s^t A_3 dr &\le \int_0^{t-s} \sup_{(a,b)} \int_{D_+}\int_{D_-} p(t-w,(a,b),(x,y))\, \frac{1}{N}\ell_{\delta_N}(x,y) \left| P_w^{(1,1)}\left(\ell_{\varepsilon_1}\phi - \ell_{\varepsilon_2}\phi\right)(x,y)\right| dxdy\, dw \\ &\le \frac{2C_1}{s^{d/2}\, t^{d/2}} \int_0^{\alpha} \frac{1}{\sqrt{w}}\, dw + \frac{C_2}{N\, s^d} \int_{\alpha}^{t-s} \left\| P_w^{(1,1)}\left(\ell_{\varepsilon_1}\phi - \ell_{\varepsilon_2}\phi\right)\right\| dw.\end{aligned}$$

The last inequality holds for any $\alpha \in (0, t-s)$, where $C_1 = C_1(d, D_+, D_-, T, \phi)$ and $C_2 = C_2(d, D_+, D_-)$. This is because for $\varepsilon > 0$,

$$\begin{aligned}&\sup_{(a,b)} \int\int \left(\int\int p(w,(x,y),(\tilde{x},\tilde{y}))\, \ell_\varepsilon(\tilde{x},\tilde{y})\, d\tilde{x}\, d\tilde{y}\right) p(t-w,(a,b),(x,y))\, \frac{1}{N}\ell_{\delta_N}(x,y)\, dxdy \\ \le\ & \frac{|I^\varepsilon|}{c_{d+1}\varepsilon^{d+1}} \sup_{(a,b)} \sup_{(\tilde{x},\tilde{y})} \frac{1}{c_{d+1} N \delta_N^{d+1}} \int_{D_+^{\delta_N}} \int_{D_-\cap B(x,\delta_N)} p(w,(x,y),(\tilde{x},\tilde{y})) p(t-w,(a,b),(x,y))\, dydx \\ \le\ & \frac{|I^\varepsilon|}{c_{d+1}\varepsilon^{d+1}} \frac{C(d,D_-)}{t^{d/2}} \sup_{a} \sup_{\tilde{x}} \frac{1}{c_{d+1} N \delta_N^{d+1}} \int_{D_+^{\delta_N}} p^+(w,x,\tilde{x}) p^+(t-w,a,x)\, dx \\ \le\ & \frac{|I^\varepsilon|}{c_{d+1}\varepsilon^{d+1}} \frac{C(d,D_-)}{t^{d/2}} \frac{C(d,D_+)}{s^{d/2}} \sup_{\tilde{x}} \frac{1}{c_{d+1} N \delta_N^{d+1}} \int_{D_+^{\delta_N}} p^+(w,x,\tilde{x})\, dx \\ & \qquad \text{by the Gaussian upper bound (2.1.3) for } p^+ \\ \le\ & \frac{|I^\varepsilon|}{c_{d+1}\varepsilon^{d+1}} \frac{C(d,D_+,D_-)}{s^{d/2}\, t^{d/2}} \frac{1}{\sqrt{w}} \qquad \text{for } N \ge N(d, D_+), \\ & \quad \text{by the assumption } \liminf_{N\to\infty} N\,\delta_N^d \in (0,\infty] \text{ and the bound (2.1.4) on } D_+.\end{aligned}$$

In conclusion, we have shown that $\left\{\mathbb{E}[\langle \ell_\varepsilon\phi,\, \mathfrak{X}_t^{N,+} \otimes \mathfrak{X}_t^{N,-}\rangle]\right\}_{\varepsilon>0}$ is a Cauchy family and converges as $\varepsilon \to 0$ to a number in $[-\infty, \infty]$. Furthermore, the convergence is uniformly for N large enough and for any initial configuration. On other hand, since $\{(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})\}_N$ converges in distribution to a continuous process to $(v_+(\cdot,x)dx,\, v_-(\cdot,y)dy)$ and $(\mu^+,\mu^-) \mapsto \langle \ell_\varepsilon\phi,\, \mu^+\otimes\mu^-\rangle$ is a bounded continuous function on $\mathfrak{M}$, we have

$$\mathbb{E}^\infty\left[\langle \ell_\varepsilon\phi,\, v_+(t)\otimes v_-(t)\rangle\right] = \lim_{N'\to\infty} \mathbb{E}\left[\langle \ell_\varepsilon\phi,\, \mathfrak{X}_t^{N',+} \otimes \mathfrak{X}_t^{N',-}\rangle\right]$$

for all $t \ge 0$. Hence the proof for the convergence of $\lim_{\varepsilon\to 0}\mathbb{E}^\infty\left[\langle \ell_\varepsilon\phi,\, v_+(t)\otimes v_-(t)\rangle\right]$ is the same. Finally, the bound for $|A^\phi(t)|$ follows directly from Lemma 4.5.1 and Lemma 4.5.5. This bound also tells us that $A^{\phi}(t)$ actually lies in $\mathbb{R}$. $\square$

From the above lemma, we immediately have

Corollary 4.5.13. *Suppose that Assumption 4.0.10 holds and $\{N'\}$ is any subsequence along which $\{(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})\}_N$ converges to $(\mathfrak{X}^{\infty,+}, \mathfrak{X}^{\infty,-})$ in distribution in $D([0,T], \mathfrak{M})$. Then for $\phi \in C_\infty(\overline{D}_+ \setminus \Lambda_+) \cup C_\infty(\overline{D}_- \setminus \Lambda_-)$, we have*

$$\lim_{N'\to\infty} \mathbb{E}[\langle \ell_{\delta_{N'}}\phi,\, \mathfrak{X}_r^{N',+} \otimes \mathfrak{X}_r^{N',-}\rangle] = A^{\phi}(r) \quad \textit{for } r > 0, \textit{ and}$$

$$\lim_{N'\to\infty} \int_s^t \mathbb{E}\left[\langle \ell_{\delta_{N'}}\phi,\, \mathfrak{X}_r^{N',+} \otimes \mathfrak{X}_r^{N',-}\rangle\right] dr = \int_s^t A^{\phi}(r)\, dr \quad \textit{for } 0 < s \le t < \infty. \tag{4.5.15}$$

Question. It is an interesting question if one can strengthen (4.5.15) to include $s = 0$.

We can now present our proof for (4.5.1) by applying a Gronwall type argument to (4.5.19).

Proof of (4.5.1). Without loss of generality, we continue to assume $\rho_\pm = 1$. Recall from (4.5.7) that for $\phi_+ \in C_\infty(\overline{D}_+ \setminus \Lambda_+)$ and $0 < s \le t < \infty$, we have

$$\mathbb{E}\left[\langle \phi_+, \mathfrak{X}_t^{N,+}\rangle\right] - \mathbb{E}\left[\langle P_{t-s}^+\phi_+, \mathfrak{X}_s^{N,+}\rangle\right] = -\frac{1}{2}\int_s^t \mathbb{E}\left[\langle \ell_{\delta_N} P_{t-r}^+\phi_+, \mathfrak{X}_r^{N,+} \otimes \mathfrak{X}_r^{N,-}\rangle\right] dr. \tag{4.5.16}$$

By (4.5.15), we can let $N \to \infty$ to obtain

$$\mathbb{E}^\infty[\langle \phi_+,\, v_+(t)\rangle] - \mathbb{E}^\infty[\langle P_{t-s}^+\phi_+,\, v_+(s)\rangle] = -\frac{1}{2}\mathbb{E}^\infty\left[\int_s^t A^{P_{t-r}^+\phi_+}(r)\, dr\right] \tag{4.5.17}$$

for $0 < s \le t < \infty$. Now let $s \to 0$. By the uniform bound for (v_+, v_-) given by Lemma 4.5.1, the continuity of $(v_+(s), v_-(s))$ in s and Lebesgue dominated convergence theorem, we obtain

$$\mathbb{E}^\infty[\langle \phi_+,\, v_+(t)\rangle] - \langle P_t^+\phi_+,\, u_0^+\rangle = -\frac{1}{2}\int_0^t \lim_{\varepsilon\to 0}\mathbb{E}^\infty\left[\langle \ell_\varepsilon\, P_{t-r}^+\phi_+,\, v_+(r) \otimes v_-(r)\rangle\right] dr. \tag{4.5.18}$$

Using the first equation in (4.2.1) in the definition of (u_+, u_-), the above equation (4.5.18) also holds if we replace (v_+, v_-) by (u_+, u_-). On subtraction, we get

$$\begin{aligned} & \big\langle \phi_+,\, u_+(t) - \mathbb{E}^\infty[v_+(t)]\big\rangle \qquad (4.5.19)\\ = \;& -\frac{1}{2}\int_0^t \lim_{\varepsilon\to 0}\int_{D_-}\int_{D_+} \ell_\varepsilon(x,y)\, P^+_{t-r}\phi_+(x)\,\Big(u_+(r,x)u_-(r,y) - \mathbb{E}^\infty[v_+(r,x)v_-(r,y)]\Big)\, dxdy\, dr. \end{aligned}$$

The above equation holds for $\phi_+ \in C_\infty(\overline{D}_+ \setminus \Lambda_+)$ (and since ρ_+ has support in the entire domain $\overline{D}_+$), so we have

$$\begin{aligned} & u_+(t) - \mathbb{E}^\infty[v_+(t)] \qquad (4.5.20)\\ = \;& -\frac{1}{2}\int_0^t \lim_{\varepsilon\to 0}\int_{D_-}\int_{D_+} \ell_\varepsilon(x,y)\, p^+(t-r,x,\cdot)\,\Big(u_+(r,x)u_-(r,y) - \mathbb{E}^\infty[v_+(r,x)v_-(r,y)]\Big)\, dxdy\, dr \end{aligned}$$

almost everywhere in D_+.

Let $w_\pm(t) := u_\pm(t) - \mathbb{E}^\infty[v_\pm(t)] \in \mathcal{B}_b(D_\pm)$ and $\|w_\pm(r)\|_\pm$ be the L^∞ norm in $D_\pm$. Then by the a.s. bound of $v_\pm$ in Lemma 4.5.1 and a simple use of triangle inequality, we have $\|u_+(r,x)u_-(r,y) - \mathbb{E}^\infty[v_+(r,x)v_-(r,y)]\| \le (\|u_0^+\|\,\|w_-(r)\| + \|u_0^-\|\,\|w_+(r)\|)$. On other hand,

$$\begin{aligned} & \int_{D_-}\int_{D_+} \ell_\varepsilon(x,y)\, p^+(t-r,x,a)\, dxdy \qquad (4.5.21)\\ = \;& \frac{1}{c_{d+1}\varepsilon^{d+1}} \int_{I^\varepsilon} p^+(t-r,x,a)\, dxdy \\ \le \;& \frac{1}{c_{d+1}\varepsilon^{d+1}} \int_{D_+^\varepsilon}\int_{B(x,\varepsilon)\cap D_-^\varepsilon} p^+(t-r,x,a)\, dydx \\ \le \;& \frac{|B(x,\varepsilon)\cap D_-^\varepsilon|}{c_{d+1}\varepsilon^{d+1}} \int_{D_+^\varepsilon} p^+(t-r,x,a)\, dx \\ \le \;& \frac{C(d,D_+)}{\sqrt{t-r}} + \tilde{C}(d,D_+) \quad \text{uniformly for } a \in \overline{D}_+, \text{ for } \varepsilon < \varepsilon(d,D_+). \end{aligned}$$

Using these observations, it is easy to check that (4.5.20) implies

$$\|w_+(t)\|_+ \le \int_0^t (\|u_0^+\|\,\|w_-(r)\| + \|u_0^-\|\,\|w_+(r)\|)\, \frac{C(d,D_+,T)}{\sqrt{t-r}}\, dr. \qquad (4.5.22)$$

By the same argument, we have

$$\|w_-(t)\|_- \leq \int_0^t \left(\|u_0^+\|\,\|w_-(r)\| + \|u_0^-\|\,\|w_+(r)\|\right) \frac{C(d, D_-, T)}{\sqrt{t-r}}\, dr. \tag{4.5.23}$$

Adding (4.5.22) and (4.5.23), we have, for $C = C(\|u_0^+\|, \|u_0^-\|, d, D_+, D_-, T)$,

$$\|w_+(t)\|_+ + \|w_-(t)\|_- \leq C \int_0^t \left(\|w_-(r)\| + \|w_+(r)\|\right) \frac{1}{\sqrt{t-r}}\, dr. \tag{4.5.24}$$

By a "Gronwall type" argument (cf. the proof of Proposition 2.1.8), we have $\|w_+(t)\|_+ + \|w_-(t)\|_- = 0$ for all $t \in [0, T]$. Since $T > 0$ is arbitrary, we have $\|w_+(t)\|_+ + \|w_-(t)\|_- = 0$ for all $t \geq 0$. This completes the proof for (4.5.1). $\square$

4.5.4 Second moment

In this subsection, we give a proof for (4.5.2) in Proposition 4.5.2. We start with a key lemma that is analogous to Lemma 4.5.12.

Lemma 4.5.14. *Suppose Assumption 4.0.10 holds. Then for $t > 0$ and $\phi \in C_\infty(\overline{D}_+ \setminus \Lambda_+)$, as $\varepsilon \to 0$, each of* $\mathbb{E}^\infty\left[\langle \phi,\, v_+(t)\rangle_{\rho_+} \langle \ell_\varepsilon \phi,\, v_+(t)\rho_+ \otimes v_-(t)\rho_-\rangle\right]$ *and* $\mathbb{E}\left[\langle \phi,\, \mathfrak{X}_t^{N,+}\rangle \langle \ell_\varepsilon \phi,\, \mathfrak{X}_t^{N,+} \otimes \mathfrak{X}_t^{N,-}\rangle\right]$ *converges uniformly for $N \in \mathbb{N}$ and for any initial distributions $\{(X_0^{N,+}, \mathfrak{X}_0^{N,-})\}$. Moreover, we have*

$$\begin{aligned} B^\phi(t) &:= \lim_{\varepsilon \to 0} \mathbb{E}^\infty\left[\langle \phi,\, v_+(t)\rangle_{\rho_+} \langle \ell_\varepsilon \phi,\, v_+(t)\rho_+ \otimes v_-(t)\rho_-\rangle\right] \\ &= \lim_{N' \to \infty} \lim_{\varepsilon \to 0} \mathbb{E}\left[\langle \phi,\, \mathfrak{X}_t^{N',+}\rangle \langle \ell_\varepsilon \phi,\, \mathfrak{X}_t^{N',+} \otimes \mathfrak{X}_t^{N',-}\rangle\right] \in \mathbb{R} \end{aligned}$$

for any subsequence $\{N'\}$ along which $\{(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})\}_N$ converges to $(\mathfrak{X}^{\infty,+}, \mathfrak{X}^{\infty,-})$ in distribution in $D([0,T], \mathfrak{M})$. Similar results hold for $\phi \in C_\infty(\overline{D}_- \setminus \Lambda_-)$, but with $\langle \phi,\, v_-(t)\rangle_{\rho_-}$ and $\langle \phi,\, \mathfrak{X}_t^{N,-}\rangle$ in place of $\langle \phi,\, v_+(t)\rangle_{\rho_+}$ and $\langle \phi,\, \mathfrak{X}_t^{N,+}\rangle$ respectively.

Proof The proof follows the same strategy as that of Lemma 4.5.12, based on (4.5.11) rather than (4.5.10). We only provide the main steps. Without loss of generality, assume $\phi = \phi_+ \in$

$C_\infty(\overline{D}_+ \setminus \Lambda_+)$ and $\rho_\pm = 1$.

Suppose $t > 0$ and $s \in (0,t)$ are fixed. Then (4.5.11) implies that

$$\begin{aligned}
\Theta \;&:=\; \Big| \mathbb{E}\Big(\langle \phi,\, \mathfrak{X}_t^{N,+}\rangle \langle \ell_{\varepsilon_1}\phi,\, \mathfrak{X}_t^{N,+}\otimes \mathfrak{X}_t^{N,-}\rangle - \langle \phi,\, \mathfrak{X}_t^{N,+}\rangle \langle \ell_{\varepsilon_2}\phi,\, \mathfrak{X}_t^{N,+}\otimes \mathfrak{X}_t^{N,-}\rangle \Big) \Big| \qquad (4.5.25)\\
&\le\; \Big| \mathbb{E}\Big(P_{t-s}^{(*)} \langle \phi(x_1)(\ell_{\varepsilon_1}(x_2,y) - \ell_{\varepsilon_2}(x_2,y))\phi(x_2),\; \mathfrak{X}_t^{N,+}(dx_1)\otimes \mathfrak{X}_t^{N,+}(dx_2)\otimes \mathfrak{X}_t^{N,-}(dy)\rangle \Big) \Big|\\
&\quad + \frac{1}{2}\int_s^t \mathbb{E}\Big[\Big\langle\, \ell_{\delta_N}(x,y) \Big(\langle H_r(x,\cdot,\cdot),\, \mathfrak{X}_r^{N,+}\otimes \mathfrak{X}_r^{N,-}\rangle\\
&\qquad\qquad + \langle H_r(\cdot,x,\cdot),\, \mathfrak{X}_r^{N,+}\otimes \mathfrak{X}_r^{N,-}\rangle + \langle H_r(\cdot,\cdot,y),\, \mathfrak{X}_r^{N,+}\otimes \mathfrak{X}_r^{N,+}\rangle\\
&\qquad\qquad + \frac{1}{N}\left[\langle 2H_r(x,x,\cdot),\, \mathfrak{X}_r^{N,-}\rangle + \langle H_r(\cdot,x,y),\, \mathfrak{X}_r^{N,+}\rangle + \langle H_r(x,\cdot,y),\, \mathfrak{X}_r^{N,+}\rangle\right]\\
&\qquad\qquad + \frac{1}{N}\left[\langle G_r(x,\cdot),\, \mathfrak{X}_r^{N,-}\rangle + \langle G_r(\cdot,y),\, \mathfrak{X}_r^{N,+}\rangle + \langle H_r(\cdot,\cdot,y),\, \mathfrak{X}_r^{N,+}\rangle\right]\\
&\qquad\qquad + \frac{1}{N^2}\left[2H_r(x,x,y) - G_r(x,y)\right]\Big),\, \mathfrak{X}_r^{N,+}(dx)\otimes \mathfrak{X}_r^{N,-}(dy) \Big\rangle \Big]\, dr,
\end{aligned}$$

where

$$\begin{aligned}
G_r \;&:=\; \Big| P_{t-r}^{(1,1)}\Big(\phi^2(x)\; (\ell_{\varepsilon_1}(x,y) - \ell_{\varepsilon_2}(x,y)) \Big) \Big| \in C_\infty^{(1,1)} \subset C(\overline{D}_+ \times \overline{D}_-) \quad \text{and}\\
H_r \;&:=\; \Big| P_{t-r}^{(2,1)}\Big(\phi(x_1)\phi(x_2)\; (\ell_{\varepsilon_1}(x_2,y_1) - \ell_{\varepsilon_2}(x_2,y_1)) \Big) \Big| \in C_\infty^{(2,1)} \subset C(\overline{D}_+^2 \times \overline{D}_-).
\end{aligned}$$

In the formula for G_r above, $P_{t-r}^{(1,1)}(\varphi(x,y)) \in C(\overline{D}_+ \times \overline{D}_-)$ is defined as

$$(a,b) \mapsto \int_{\overline{D}_+\times\overline{D}_-} p^{(1,1)}(t-r,\,(a,b),\,(x,y))\,dxdy.$$

The expression $P_{t-r}^{(2,1)}(\varphi(x,y))$ is defined in a similar way.

Comparison with $(\overline{\mathfrak{X}}^{N,+}, \overline{\mathfrak{X}}^{N,-})$ then yields

$$\Theta \le (1+\frac{1}{N})\,\|H_s\| + \frac{1}{N}\,\|G_s\| + \int_s^t \left(\sum_{i=1}^{9} A_i + B_1 + B_2 \right) dr, \qquad (4.5.26)$$

where, with abbreviations that will be explained,

$$
\begin{aligned}
A_1 &:= \left\| P_r^{(1,1)} \left(\ell_{\delta_N}(x,y) \left\| P_r^{(1,1)} H_r(x,\cdot,\cdot) \right\| \right) \right\| \\
A_2 &:= \left\| P_r^{(1,1)} \left(\ell_{\delta_N}(x,y) \left\| P_r^{(1,1)} H_r(\cdot,x,\cdot) \right\| \right) \right\| \\
A_3 &:= \left\| P_r^{(1,1)} \left(\ell_{\delta_N}(x,y) \left\| P_r^{(2,0)} H_r(\cdot,\cdot,y) \right\| \right) \right\| \\
A_4 &:= \frac{2}{N} \left\| P_r^{(1,1)} \left(\ell_{\delta_N}(x,y) \left\| P_r^{(0,1)} H_r(x,x,\cdot) \right\| \right) \right\| \\
A_5 &:= \frac{1}{N} \left\| P_r^{(1,1)} \left(\ell_{\delta_N}(x,y) \left\| P_r^{(1,0)} H_r(\cdot,x,y) \right\| \right) \right\| \\
A_6 &:= \frac{1}{N} \left\| P_r^{(1,1)} \left(\ell_{\delta_N}(x,y) \left\| P_r^{(1,0)} H_r(x,\cdot,y) \right\| \right) \right\| \\
A_7 &:= \frac{1}{N} \left\| P_r^{(1,1)} \left(\ell_{\delta_N}(x,y) \left\| P_r^{(0,1)} G_r(x,\cdot) \right\| \right) \right\| \\
A_8 &:= \frac{1}{N} \left\| P_r^{(1,1)} \left(\ell_{\delta_N}(x,y) \left\| P_r^{(1,0)} G_r(\cdot,y) \right\| \right) \right\| \\
A_9 &:= \frac{1}{N} \left\| P_r^{(1,1)} \left(\ell_{\delta_N}(x,y) \left\| P_r^{(1,0)} H_r(\cdot,\cdot,y) \right\| \right) \right\| \\
B_1 &:= \frac{2}{N^2} \left\| P_r^{(1,1)} \left(\ell_{\delta_N}(x,y) H_r(x,x,y) \right) \right\| \\
B_2 &:= \frac{1}{N^2} \left\| P_r^{(1,1)} \left(\ell_{\delta_N}(x,y) G_r(x,y) \right) \right\|
\end{aligned}
$$

In the above, the first $P_r^{(1,1)}$ acts on the (x,y) variable, while the second $P_r^{(i,j)}$ in each A_i acts on the ' $\cdot$ ' variable. Beware of the difference between $P_r^{(2,0)} H_r(\cdot,\cdot,y)$ and $P_r^{(1,0)} H_r(\cdot,\cdot,y)$ in A_3 and A_9 respectively. In fact, $P_r^{(2,0)} H_r(\cdot,\cdot,y)$ is defined as the function on $\overline{D}_+^2$ which maps (a_1,a_2) to $\int_{D_+^2} p^{(2,0)}(r,\,(a_1,a_2),\,(x_1,x_2))\,H_r(x_1,x_2,y)\,d(x_1,x_2)$, while $P_r^{(1,0)} H_r(\cdot,\cdot,y)$ is defined as the function on $\overline{D}_+$ which maps a_1 to $\int_{D_+} p^{(1,0)}(r,\,a_1,\,x)\,H_r(x,x,y)\,dx$.

The rest of the proof goes in the same way as that for Lemma 4.5.12. For example, note that

$$
\begin{aligned}
\|H_s\| \;=\; \sup_{(a_1,a_2,b_1)} \Bigg| \int_{D_+^2 \times D_-} & \phi(x_1)\phi(x_2)\,\left(\ell_{\varepsilon_1}(x_2,y_1) - \ell_{\varepsilon_2}(x_2,y_1)\right) \\
& p^{(2,1)}(t-s,(a_1,a_2,b_1),(x_1,x_2,y_1))\,d(x_1,x_2,y_1) \Bigg|.
\end{aligned}
$$

By applying Lemma 4.5.5 to the equi-continuous and uniformly bounded family

$$\left\{(x_1,x_2,y)\mapsto \phi(x_1)\phi(x_2)\,p^{(2,1)}(t-s,(a_1,a_2,b),(x_1,x_2,y)):\ (a_1,a_2,b)\in \overline{D}_+^2\times\overline{D}_-\right\}\subset C(\overline{D}_+^2\times\overline{D}_-),$$

we see that $\|H_s\|$ converges to zero uniformly for N large enough and for any initial configuration, as ε_1 and ε_2 both tend to zero. The integral term with respect to dr can be estimated as in the proof of Lemma 4.5.12, using the bound (2.1.4), Lemma 4.5.5 and Assumption 4.0.10 that $\liminf_{N\to\infty} N\,\delta_N^d\in(0,\infty]$.

We have shown that $\left\{\mathbb{E}\left[\langle\phi,\,\mathfrak{X}_t^{N,+}\rangle\langle\ell_\varepsilon\phi,\,\mathfrak{X}_t^{N,+}\otimes\mathfrak{X}_t^{N,-}\rangle\right]\right\}_{\varepsilon>0}$ is a Cauchy family which converges, as $\varepsilon\to 0$, uniformly for N large enough and for any initial configuration. Hence $B^\phi(t)$ in the statement of the lemma exists in $[-\infty,\infty]$. Finally, we have $B^\phi(t)\in\mathbb{R}$ since $|B^\phi(t)|<\infty$ by Lemma 4.5.1 and Lemma 4.5.5. □

From the above lemma, we immediately obtain

Corollary 4.5.15. *Suppose Assumption 4.0.10 holds and $\{N'\}$ is a subsequence along which $\{(\mathfrak{X}^{N,+},\mathfrak{X}^{N,-})\}$ converges to $(\mathfrak{X}^{\infty,+},\mathfrak{X}^{\infty,-})$ in distribution in $D([0,T],\mathfrak{M})$. Then for $\phi\in C_\infty(\overline{D}_+\setminus\Lambda_+)$,*

$$\lim_{N'\to\infty}\mathbb{E}\left[\langle\phi,\,\mathfrak{X}_r^{N',+}\rangle\langle\ell_{\delta_{N'}}\phi,\,\mathfrak{X}_r^{N',+}\otimes\mathfrak{X}_r^{N',-}\rangle\right]=B^\phi(r)\quad \textit{for } r>0,\quad \textit{and}$$

$$\lim_{N'\to\infty}\int_s^t\mathbb{E}\left[\langle\phi,\,\mathfrak{X}_r^{N',+}\rangle\langle\ell_{\delta_{N'}}\phi,\,\mathfrak{X}_r^{N',+}\otimes\mathfrak{X}_r^{N',-}\rangle\right]dr=\int_s^t B^\phi(r)\,dr\quad \textit{for } 0<s\le t<\infty. \tag{4.5.27}$$

We are now ready to give the

Proof of (4.5.2). As before, without loss of generality we assume $\rho_\pm=1$. Recall from (4.5.8) that for $\phi=\phi_+\in C_\infty(\overline{D}_+\setminus\Lambda_+)$ and $0<s\le t<\infty$, we have

$$\begin{aligned}&\mathbb{E}[\langle\phi,\mathfrak{X}_t^{N,+}\rangle^2]-\mathbb{E}[\langle P_{t-s}^+\phi,\mathfrak{X}_s^{N,+}\rangle^2] \qquad (4.5.28)\\ =\ &-\frac{1}{2}\int_s^t\mathbb{E}[\,\langle P_{t-r}^+\phi,\mathfrak{X}_r^{N,+}\rangle\,\langle\ell_{\delta_N}\,(P_{t-r}^+\phi),\,\mathfrak{X}_r^{N,+}\otimes\mathfrak{X}_r^{N,-}\rangle\,]\,dr+o(N).\end{aligned}$$

Letting $N' \to \infty$ in (4.5.27), we get

$$\mathbb{E}^\infty[\langle \phi,\, v_+(t)\rangle^2] - \mathbb{E}^\infty[\langle P^+_{t-s}\phi,\, v_+(s)\rangle^2 = -\frac{1}{2}\int_s^t B^{P^+_{t-r}\phi}(r)\, dr$$

for $0 < s \le t < \infty$. Now let $s \to 0$. By the uniform bound for (v_+, v_-) given by Lemma 4.5.1, the continuity of $(v_+(s), v_-(s))$ in s and the Lebesgue dominated convergence theorem, we obtain

$$\mathbb{E}^\infty\left[\langle \phi_+,\, v_+(t)\rangle^2\right] - \langle P^+_t\phi,\, u^+_0\rangle^2 = -\frac{1}{2}\int_0^t \lim_{\varepsilon\to 0}\mathbb{E}^\infty\left[\langle P^+_{t-r}\phi,\, v_+(r)\rangle\langle \ell_\varepsilon P^+_{t-r}\phi,\, v_+(r)\otimes v_-(r)\rangle\right] dr. \tag{4.5.29}$$

Using the definition of (u_+, u_-), the above equation (4.5.29) also holds if we replace (v_+, v_-) by (u_+, u_-). On subtraction, we get

$$\begin{aligned}\mathbb{E}^\infty[\langle \phi,\, v_+(t)\rangle^2] - \langle \phi, u_+(t)\rangle^2 \;&=\; \frac{1}{2}\int_0^t \lim_{\varepsilon\to 0}\mathbb{E}^\infty\Big[\langle P^+_{t-r}\phi, u_+(r)\rangle\,\langle \ell_\varepsilon P^+_{t-r}\phi,\, u_+(r)\otimes u_-(r)\rangle \\ &\quad -\langle P^+_{t-r}\phi,\, v_+(r)\rangle\,\langle \ell_\varepsilon P^+_{t-r}\phi,\, v_+(r)\otimes v_-(r)\rangle\Big]\, dr. \end{aligned} \tag{4.5.30}$$

The left hand side of (4.5.30) equals $\mathbb{E}^\infty[\langle \phi,\, v_+(t) - u_+(t)\rangle^2]$ because $\mathbb{E}^\infty[\langle \phi,\, v_+(t)\rangle] = \langle \phi, u_+(t)\rangle$. Since $\mathbb{E}^\infty[\langle \ell_\varepsilon P^+_{t-r}\phi,\, v_+(r)\otimes v_-(r)\rangle] = \langle \ell_\varepsilon P^+_{t-r}\phi,\, u_+(r)\otimes u_-(r)\rangle$, the integrand in the right hand side of (4.5.30) with respect to dr equals

$$\lim_{\varepsilon\to 0}\mathbb{E}^\infty\left[\langle \ell_\varepsilon P^+_{t-r}\phi,\, v_+(r)\otimes v_-(r)\rangle\left(\langle P^+_{t-r}\phi,\, u_+(r) - v_+(r)\rangle\right)\right] \le C\,\mathbb{E}^\infty\left[\left|\langle P^+_{t-r}\phi,\, u_+(r) - v_+(r)\rangle\right|\right].$$

The constant $C = C(\phi, f, g, D_+, D_-)$ above arises from the uniform bound for $v(r)$ in Lemma 4.5.1 and the bound (2.1.4). Hence we have

$$\mathbb{E}^\infty[\langle \phi,\, v_+(t) - u_+(t)\rangle^2] \le C\int_0^t \mathbb{E}^\infty\left[\left|\langle P^+_{t-r}\phi,\, u_+(r) - v_+(r)\rangle\right|\right] dr.$$

Letting $w_+(t) = u_+(t) - v_+(t)$, we obtain

$$\mathbb{E}^\infty[\langle \phi,\, w_+(t)\rangle^2] \le C \int_0^t \mathbb{E}^\infty[\langle P^+_{t-r}\phi,\, w_+(r)\rangle^2]\, dr. \tag{4.5.31}$$

We can then deduce by a "Gronwall-type" argument that $\mathbb{E}^\infty[\langle \phi,\, w_+(t)\rangle^2] = 0$ for all $t \ge 0$. In fact, by Fubinni's theorem, the left hand side of (4.5.31) equals

$$\int_{D_+}\int_{D_+} \phi(x_1)\phi(x_2)\, \mathbb{E}^\infty[w_+(t,x_1)w_+(t,x_2)]\, dx_1\, dx_2, \tag{4.5.32}$$

and the integrand with respect to dr of the right hand side of (4.5.31) is

$$\int_{D_+}\int_{D_+} \phi(a_1)\phi(a_2) \int_{D_+}\int_{D_+} p^+(t-r,x_1,a_1)p^+(t-r,x_2,a_2)\, \mathbb{E}^\infty[w_+(t,x_1)w_+(t,x_2)]\, dx_1 dx_2 da_1 da_2.$$

Hence for a.e. $a_1, a_2 \in D_+$, we have

$$\begin{aligned} & \mathbb{E}^\infty[w_+(t,a_1)w_+(t,a_2)] \\ \le \;& C \int_0^t \int_{D_+}\int_{D_+} p^+(t-r,x_1,a_1)p^+(t-r,x_2,a_2)\, \mathbb{E}^\infty[w_+(t,x_1)w_+(t,x_2)]\, dx_1\, dx_2\, dr. \end{aligned}$$

Let $\overline{f}(t) := \sup_{(a_1,a_2)\in \overline{D}_+^2} \left|\mathbb{E}^\infty[w_+(t,a_1)w_+(t,a_2)]\right|$, then the above equation asserts that $\overline{f}(t) \le C\int_0^t \overline{f}(r)\, dr$. Note that $\overline{f}(r) \in L^1[0,t]$ since it is bounded. Hence by Gronwall's lemma, we have $\overline{f}(t) = 0$ for all $t \ge 0$. This together with (4.5.32) yields $\mathbb{E}^\infty[\langle \phi,\, w_+(t)\rangle^2] = 0$. Hence $\mathbb{E}^\infty[\langle \phi,\, v_+(t)\rangle^2] = \langle \phi,\, u_+(t)\rangle^2$. The same holds for v_-. This completes the proof for (4.5.2). The proof of Proposition 4.5.2 is complete. $\square$

4.6 Proof of hydrodynamic limit result

Proof Tightness of $\{(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})\}$ was proved in Theorem 4.4.4. It remains to identify any subsequential limit. We conclude from (4.5.1) and (4.5.2) that

$$\langle \mathfrak{X}_t^{\infty,+},\, \phi_+\rangle = \langle u_+(t),\, \phi_+\rangle_{\rho_+} \quad \text{and} \quad \langle \mathfrak{X}_t^{\infty,-},\, \phi_-\rangle = \langle u_-(t),\, \phi_-\rangle_{\rho_-} \quad \mathbb{P}^{\infty}\text{-a.s.}$$

for any fixed $t > 0$ and $\phi_\pm \in C_\infty(\overline{D}_\pm \setminus \Lambda_\pm)$. Recall that $(\mathfrak{X}^{\infty,+},\, \mathfrak{X}^{\infty,-}) \in C([0,\infty), \mathfrak{M})$ by Theorem 4.4.4 and that $C_\infty(\overline{D}_\pm \setminus \Lambda_\pm)$ is separable. Hence through rational numbers and a countable dense subset of $C_\infty(\overline{D}_\pm \setminus \Lambda_\pm)$ to strengthen the previous statement to

$$\mathbb{P}^{\infty}\left((\mathfrak{X}_t^{\infty,+},\, \mathfrak{X}_t^{\infty,-}) = (u_+(t,x)\rho_+(x)dx,\, u_-(t,y)\rho_-(y)dy) \in \mathfrak{M} \quad \text{for every } t \geq 0 \right) = 1.$$

This completes the proof of Theorem 4.3.1. □

Chapter 5

FLUCTUATIONS FOR BROWNIAN PARTICLES WITH PARTIAL ABSORBTION

We rigorously derive non-equilibrium space-time fluctuation for the particle density of a system of reflected diffusions in bounded Lipschitz domains in $\mathbb{R}^d$. The particles are independent and are killed by a time-dependent potential which is asymptotically proportional to the boundary local time. We generalize the functional analytic framework introduced by Kotelenez [53, 54] to deal with time-dependent perturbations. Our proof relies on Dirichlet form method rather than the machineries derived from Kotelenez's sub-martingale inequality. Our result holds for any symmetric reflected diffusion, for any bounded Lipschitz domain and for any dimension $d \geq 1$.

5.1 Robin boundary model

The goal of this chapter is to develop a machinery to overcome some difficulties that arise in the study of fluctuations for systems of reflected diffusions (such as reflected Brownian motions) with a singular type of time-dependent killing potential. The primary examples are the systems of annihilating diffusions introduced in [16] and [17], which can be used to model the transport of positive and negative charges in solar cells or the population dynamics of two segregated species under competition. The model in [17] consists of two families of reflected diffusions confined in two adjacent domains, say two adjacent rectangles $(0,2)\times(0,1)$ and $(0,2)\times(-1,0)$, respectively. These two families of particles (positive and negative charges respectively) annihilate each other at a certain rate when they come close to each other near the interface $(0,2)\times\{0\}$. This interaction models the annihilation, trapping, recombination and separation phenomena of the charges. From the viewpoint of the positive charges, they are themselves reflected diffusions in $(0,2)\times(0,1)$ subject to killing by a time-dependent random potential.

In this chapter, we focus our attention to a one-type particle model which consists of i.i.d. reflected diffusions killed by a deterministic time-dependent potential near the boundary. The following assumption on reflected diffusions is in force throughout this paper:

Assumption 5.1.1. *Suppose $D\subset\mathbb{R}^d$ is a bounded Lipschitz domain, $\rho\in W^{1,2}(D)\cap C(\overline{D})$ is a strictly positive function, $\boldsymbol{a}=(a^{ij})$ is a symmetric, bounded, uniformly elliptic $d\times d$ matrix-valued function such that $a^{ij}\in W^{1,2}(D)$ for each i, j. Here $C(\overline{D})$ denotes the space of continuous functions on $\overline{D}$ and $W^{1,2}(D):=\{f\in L^2(D):\ |\nabla f|\in L^2(D)\}$ denotes the usual Sobolev space of order $(1,2)$.*

Under Assumption 5.1.1, it is well known (see [2, 15]) that the bilinear form $(\mathcal{E},\,W^{1,2}(D))$ defined by

$$\mathcal{E}(f,g)=\frac{1}{2}\int_D \mathbf{a}(x)\nabla f(x)\cdot\nabla g(x)\,\rho(x)\,dx=\frac{1}{2}\int_D\sum_{i,j=1}^d a^{ij}(x)\,\frac{\partial f}{\partial x_i}(x)\,\frac{\partial g}{\partial x_i}(x)\,\rho(x)\,dx \tag{5.1.1}$$

is a regular Dirichlet form in $L^2(D,\,\rho(x)dx)$ and hence has an associated Hunt process X (unique in distribution). Furthermore, X is a continuous strong Markov process with symmetrizing measure ρ and has infinitesimal generator

$$\mathcal{A}:=\frac{1}{2\,\rho}\,\nabla\cdot(\rho\,\mathbf{a}\nabla):=\frac{1}{2\,\rho}\,\sum_{i,j=1}^d\frac{\partial}{\partial x_i}\Big(\rho\,a^{ij}\,\frac{\partial}{\partial x_j}\Big)$$

Intuitively, X behaves like a diffusion process associated to the second order elliptic differential operator $\mathcal{A}$ in the interior of D, and is instantaneously reflected at the boundary in the inward *conormal* direction $\vec{\nu}=\mathbf{a}\vec{n}$, where $\vec{n}$ is the unit inward normal vector field on ∂D. See Chen [15] for the Skorokhod representation for X, which tells us some precise pathwise properties of X. We call X an $(\mathbf{a},\,\rho)$-**reflected diffusion** or an $(\mathcal{A},\,\rho)$-**reflected diffusion**. A special but very important case is when $\mathbf{a}$ is the identity matrix and $\rho = 1$, in which X is called a **reflected Brownian motion (RBM)**. Next, we make the following assumption about the killing potential throughout this chapter.

Assumption 5.1.2. *(Killing potential) Suppose $q(t, x)$ is a given non-negative bounded function on $[0, \infty) \times \overline{D}$ such that $q(t, \cdot) \in C(\overline{D})$ for all $t \geq 0$. Suppose also that δ_N is a sequence of positive numbers which converges to zero and denote $q_N(t, x) = \delta_N^{-1} \mathbf{1}_{D^{\delta_N}}(x) q(t, x)$, where $D^\delta = \{x \in D : \, dist(x, \partial D) < \delta\}$. See Figure 5.1.*

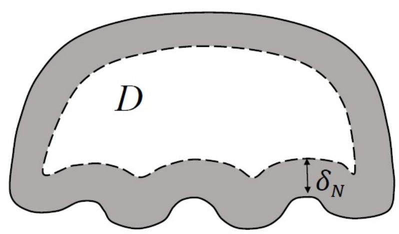


Figure 5.1: D^{δ_N} is the shaded region

Our particle system is parameterized by $N \in \mathbb{N}$, the initial number of particles. The function q_N plays the role of a time-dependent killing potential. This killing potential is singular in the sense that $\delta_N^{-1} \mathbf{1}_{D^{\delta_N}}(x)$ converges weakly to the surface measure σ which is singular with respect to Lebesque measure. More precisely, for $N \in \mathbb{N}$, we let $\{X_i\}_{i=1}^N$ be independent $(\mathbf{a}, \rho)$-reflected diffusions in D and $\{R_i\}_{i=1}^N$ be independent exponential random variables with mean one. The normalized empirical measure of the particles *alive* is defined as:

$$\mathfrak{X}_t^N(dz) := \frac{1}{N} \sum_{\{i : \, t < \zeta_i^{(N)}\}} \mathbf{1}_{X_i(t)}(dz), \text{ where} \tag{5.1.2}$$

$$\zeta_i^{(N)} = \inf \left\{ t \geq 0 : \frac{1}{2} \int_0^t q_N(s, X_i(s)) ds \geq R_i \right\}. \tag{5.1.3}$$

Note that $\mathfrak{X}_t^N$ is a random measure on $\overline{D}$. Moreover, $\mathfrak{X}^N = (\mathfrak{X}_t^N)_{t \geq 0}$ is a strong Markov process in $M_+(\overline{D})$, the space of finite non-negative Borel measures on $\overline{D}$ equipped with weak topology, and $\mathfrak{X}^N$ has sample paths in the Skorokhod space $D([0, \infty), M_+(\overline{D}))$ almost surely.

Remark 5.1.3. Let $\{Z_i^{(N)}\}_{i=1}^N$ be independent **sub-processes** (cf. [20]) of reflected diffusions killed by the potential q_N. That is,

$$Z_i^{(N)}(t) := \begin{cases} X_i(t), & t < \zeta_i^{(N)} \\ \partial, & t \geq \zeta_i^{(N)}, \end{cases}$$

where ∂ is an isolated point of $\overline{D}$. Then $\mathfrak{X}_t^N(dz)$ defined in (5.1.2) is equal to $\frac{1}{N}\sum_{i=1}^N \mathbf{1}_{Z_i^{(N)}(t)}(dz)$ if we view $\mathbf{1}_\partial$ as the zero measure.

We coin this model the name **Robin boundary model** due to the following hydrodynamic result. In what follows, $\xrightarrow{\mathcal{L}}$ denotes convergence in law and $\stackrel{\mathcal{L}}{=}$ denotes equal in law.

Theorem 5.1.4. ***(Functional Law of Large Numbers)*** *Suppose Assumptions 5.1.1 and 5.1.2 hold. Suppose $\{\mathfrak{X}_0^N\} \xrightarrow{\mathcal{L}} u_0(x)\rho(x)\,dx$ in $M_+(\overline{D})$, where $u_0 \in C(\overline{D})$. Then*

$$\mathfrak{X}_t^N(dx) \xrightarrow{\mathcal{L}} u(t,x)\rho(x)\,dx \quad \text{in } D([0,\infty), M_+(\overline{D})),$$

where $u \in C([0,\infty) \times \overline{D})$ is the probabilistic solution [1] *to the heat equation $\frac{\partial u}{\partial t} = \mathcal{A}u$ with Robin boundary condition $\frac{\partial u}{\partial \vec{\nu}}(t,x) = q(t,x)u(t,x)/\rho(x)$ on $(0,\infty) \times \partial D$ and initial condition $u(0,\cdot) = u_0$.*

The proof of Theorem 5.1.4 is an elementary law of large numbers argument involving the calculation of two moments. Since it is much easier than Theorem 4.3.1 in Chapter 4 (see also [17]), we omit it here and refer the reader to that chapter.

5.1.1 Main result

Our object of study in this chapter is the **fluctuation process** $\mathcal{Y}^N = (\mathcal{Y}_t^N)_{t\geq 0}$ defined by

$$\langle \mathcal{Y}_t^N, \phi\rangle := N^{1/2}(\langle \mathfrak{X}_t^N, \phi\rangle - \mathbb{E}\langle \mathfrak{X}_t^N, \phi\rangle) \quad t \geq 0,\ \phi \in L^2(D), \tag{5.1.4}$$

[1] By [17], u has the probabilistic representation $\mathbb{E}^x\left[u_0(X_t)\exp\left(-\int_0^t q(t-s,X_s)\,dL_s\right)\right]$ where L_t is the boundary local time of X.

where $\langle \mathfrak{X}_t^N, \phi \rangle := \frac{1}{N} \sum_{\{i:\, t<\zeta_i^{(N)}\}} \phi(X_i(t))$ is the integral of an observable (or test function) ϕ with respect to the measure $\mathfrak{X}_t^N$. Even in this simple setting, answers to the following natural questions are non-trivial.

(i) What is the state space for $\mathcal{Y}_t^N$? This space should posses a topology which allows us to make sense of convergence of $\mathcal{Y}^N$, if it does converge. Observe that although $\mathcal{Y}^N$ acts on $L^2(D)$ linearly, it is not a bounded operator in general.

(ii) Does $\mathcal{Y}^N$ converge? If so, what can we say about its limit?

The answer for question (i) is given by Lemma 5.4.1. It says that the process $(\mathcal{Y}_t^N)_{t\geq 0}$ has sample paths in $D([0,\infty), \mathcal{H}_{-\alpha})$ for $\alpha > 0$ large enough, where $\mathcal{H}_{-\alpha}$ is a Hilbert space of distributions that strictly contains $L^2(D,\, \rho(x)dx)$. See subsection 5.2.1 for the precise construction of $\mathcal{H}_{-\alpha}$, which can be identified with the dual of the Sobolev space $W^{\alpha/2,2}(D)$ of fractional order.

The answer for question (ii) is given by Theorem 5.1.5, the main result of this chapter. Theorem 5.1.5 contains 2 parts: the convergence result and the properties of the limit. The limit is shown to be decomposable into an independent sum of a "transportation part" and a "white noise part" (see (5.1.7) below). The 'transportation part' is governed by the evolution operators $\{Q_{s,t}\}_{s\leq t}$ generated on $C(\overline{D})$ by the backward PDE $\frac{\partial v}{\partial s} = -\mathcal{A}v$ on $(0,t)\times D$ with Robin boundary condition $\frac{\partial v}{\partial \vec{n}} = qv/\rho$ on $(0,t)\times \partial D$. More precisely, for $0 \leq s \leq t$ and $\phi \in L^2(D)$, we define

$$\begin{aligned} Q_{s,t}\phi(x) \quad &:= \quad \mathbb{E}\left[\phi(X_t)\, \exp\left(-\int_s^t q(r, X_r)\, dL_r\right) \Big|\, X_s = x\right] \\ &= \quad \mathbb{E}\left[\phi(X_{t-s})\, \exp\left(-\int_0^{t-s} q(s+r, X_r)\, dL_r\right) \Big|\, X_0 = x\right]. \end{aligned} \tag{5.1.5}$$

Define

$$\mathbf{U}_{(t,s)}\mu(\phi) := \mu(Q_{s,t}\phi) \tag{5.1.6}$$

for $\alpha > 0$, $\mu \in \mathcal{H}_{-\alpha}$ and $\phi \in L^2(D)$ whenever it is well defined (i.e. $Q_{s,t}\phi \in \mathcal{H}_\alpha$); see Theorem 5.1.5 and Remark 5.2.1. For simplicity, denote by $\langle \phi,\, \psi \rangle_\rho := \int_D \phi(x)\psi(x)\, \rho(x)dx$ the inner product of $L^2(D,\, \rho(x)dx)$. We can now formulate our main result.

Theorem 5.1.5. ***(Functional Central Limit Theorem)*** *Suppose that Assumptions 5.1.1 and 5.1.2 hold and that the initial positions of particles are i.i.d with distribution $u_0(x)\rho(x)\,dx$, where $u_0 \in C(\overline{D})$. Then for any $\alpha > d+2$ and $T > 0$, $\mathcal{Y}^N$ converges to $\mathcal{Y}$ in distribution as $N \to \infty$ in the Skorokhod space $D([0,\, T], \mathcal{H}_{-\alpha})$, where $\mathcal{Y}$ is the generalized Ornstein-Uhlenbeck process taking values in $D([0,\, T], \mathcal{H}_{-\alpha})$ given by*

$$\mathcal{Y}_t \stackrel{\mathcal{L}}{=} \mathbf{U}_{(t,0)}\mathcal{Y}_0 + \int_0^t \mathbf{U}_{(t,s)}\, dM_s. \tag{5.1.7}$$

In the above, M is a (unique in distribution) continuous, $\tilde{\mathcal{F}}_t$-adapted, square integrable, $\mathcal{H}_{-\alpha}$-valued Gaussian martingale with independent increments and covariance functional characterized by

$$\tilde{\mathbb{E}}\left[\langle M_t, \phi\rangle^2\right] = \int_0^t \left(\langle \boldsymbol{a}\nabla\phi \cdot \nabla\phi,\, u(s)\rangle_\rho + \int_{\partial D} \phi^2(z)u(s,z)q(s,z)\rho(z)\, d\sigma(z) \right) ds, \quad \phi \in \mathcal{H}_\alpha, \tag{5.1.8}$$

defined on a complete probability space with right continuous filtration $(\tilde{\Omega},\, \tilde{\mathcal{F}},\, \tilde{\mathcal{F}}_t,\, \tilde{\mathbb{P}})$, where the function $u(s,x)$ is given by Theorem 5.1.4. $\mathcal{Y}_0$ is the centered Gaussian random variable with covariance

$$\tilde{\mathbb{E}}\left[\mathcal{Y}_0(\phi)\mathcal{Y}_0(\psi)\right] = \langle \phi\,\psi,\, u_0\rangle_\rho - \langle \phi, u_0\rangle_\rho \langle \psi, u_0\rangle_\rho \quad \textit{for } \phi, \psi \in \mathcal{H}_\alpha,$$

defined on the same probability space as M and is independent of M. Moreover, $\mathcal{Y}$ is a continuous Gaussian Markov process which is unique in distribution, and $\mathcal{Y}$ has a version in $C^\gamma([0,\infty),\, \mathcal{H}_{-\alpha})$ (i.e. Hölder continuous with exponent γ) for any $\gamma \in (0, 1/2)$.

Remark 5.1.6. (i) In (5.1.7), $\int_0^t \mathbf{U}_{(t,s)}\, dM_s$ is the stochastic integral with respect to the Hilbert space valued martingale M (cf. [62]). In the Appendix, we prove that it is well-defined. For the convenience of the reader, we also stated the precise definition of Hilbert space valued continuous Gaussian processes with independent increment. The existence and uniqueness of M is given in Theorem 5.4.6. Furthermore, for $\alpha > d+2$, both $\mathbf{U}_{(t,0)}\mathcal{Y}_0$ and

$\int_0^t \mathbf{U}_{(t,s)}\, dM_s$ live in $\mathcal{H}_{-\alpha}$ (i.e. they extend to be continuous functionals on $\mathcal{H}_\alpha$).

(ii) Roughly speaking, $\mathcal{Y}$ solves the following stochastic evolution equation (called the **Langevin equation**) in the weak sense:

$$dY_t = \mathbf{A}_t^{(-\alpha)} Y_t\, dt + dM_t, \quad Y_0 = \mathcal{Y}_0, \tag{5.1.9}$$

where $\mathbf{A}_t^{(-\alpha)}$ is the generator of $\{\mathbf{U}_{(t,s)}\}_{t\geq s}$ in the Hilbert space $\mathcal{H}_{-\alpha}$.

(iii) Define a bilinear forms $\mathcal{E}_s^{(q)}$ on $L^2(D,\, \rho(x)dx) \cap L^2(\partial D, d\sigma)$ by

$$\mathcal{E}_s^{(q)}(\phi, \psi) := \langle \mathbf{a}\nabla\phi \cdot \nabla\psi,\, u(s)\rangle_\rho + \int_{\partial D} \phi\, \psi\, u(s)\, q(s)\, \rho\, d\sigma \tag{5.1.10}$$

and $\mathcal{E}_s^{(q)}(\phi) := \mathcal{E}_s^{(q)}(\phi, \phi)$ for $s \geq 0$. Now (5.1.8) reads as $\tilde{\mathbb{E}}[\langle M_t, \phi\rangle^2] = \int_0^t \mathcal{E}_s^{(q)}(\phi)\, ds$. As an immediate application of (5.1.7), for all fixed $\phi \in \mathcal{H}_\alpha$ with $\alpha > d+2$, we have

$$\mathcal{Y}_t(\phi) \overset{\mathcal{L}}{=} \mathcal{Y}_0(Q_{0,t}\phi) + \int_0^t \sqrt{\mathcal{E}_s^{(q)}(Q_{s,t}\phi)}\, dB_s^{(\phi)} \quad \text{in } D([0,\, T], \mathbb{R}), \tag{5.1.11}$$

where $B^{(\phi)}$ is a standard Brownian motion independent of $\mathcal{Y}_0$. Therefore, we can simulate the evolution (in time t) of the fluctuations of the particle density with respect to an observable ϕ by running a Brownian motion.

(iv) When D is a cube (such as when $d = 1$), Theorem 5.1.5 holds with $\alpha > d/2 + 2$ in place of $\alpha > d + 2$, since we have a stronger unform upper bound for eigenfunctions, namely $\sup_\ell \|\phi_\ell\| < C(d, D)$. □

Remark 5.1.7. (i) When $q = 0$, Theorem 5.1.5 in particular gives the fluctuation result for independent reflecting Brownian motions in bounded Lipschitz domains.

(ii) (Killing by local time) The measure $q_N(t,x)\, dx$ clearly converges weakly to $q(t,x)d\sigma(x)$ when $N \to \infty$, where σ denotes the surface measure on ∂D. The positive additive continuous functional (see the Appendix of [20]) of X_i associated to $q(t,x)d\sigma(x)$ is $2\int_0^t q(s, X_i(s))dL_s^{(i)}$, where $L_t^{(i)}$ is the boundary local time of X_i. Hence it is natural to ask: what if $\{X_i\}$ are killed by $2\int_0^t q(s, X_i(s))dL_s^{(i)}$ (which no longer depends on N) rather than by a potential function q_N on the strip D^{δ_N}? It turns out that, with little extra effort, one can show that Theorem 5.1.4 and Theorem 5.1.5 both remain valid if we replace the definition of $\zeta_i^{(N)}$ in (5.1.3) by

$$\zeta_i^{(N)} = \zeta_i := \inf\left\{t \geq 0 :\ 2\int_0^t q(s, X_i(s))dL_s^{(i)} \geq R_i\right\}. \tag{5.1.12}$$

See subsection 5.4.7 for details. □

One of the earliest rigorous results about fluctuation limit was proven by Itô [47, 48], who considered a system of independent and identically distributed (i.i.d.) Brownian motions in $\mathbb{R}^d$ and showed that the limit is a $\mathcal{S}'$-valued Gaussian process solving a Langevin equation, where $\mathcal{S}'$ is the Schwartz space of tempered distributions. Fluctuation limits for stochastic particle systems in domains are very limited. Sznitman [73] studied the fluctuations of a conservative system of diffusions with normal reflected boundary conditions on smooth domains. Fluctuations of the reaction-diffusion systems on the cube $[0, 1]^d$ with linear or quadratic reaction terms were studied in [6, 31, 53, 54]. These fluctuation results are valid only for dimension $d \leq 3$.

5.1.2 Outline of proof

We prove Theorem 5.1.5 through the following six steps.

Step 1: $\mathcal{Y}^N$ satisfies the following stochastic integral equation

$$\mathcal{Y}_t^N = \mathbf{U}_{(t,0)}^N \mathcal{Y}_0^N + \int_0^t \mathbf{U}_{(t,s)}^N \, dM_s^N \text{ a.s.},$$

where $\mathbf{U}_{(t,s)}^N$ is an evolution system approximating $\mathbf{U}_{(t,s)}$; see Theorem 5.4.3.

Step 2: $M^N \xrightarrow{\mathcal{L}} M$ in $D([0,T], \mathcal{H}_{-\alpha})$; see Theorem 5.4.6.

Step 3: $\mathcal{Y}^N$ is tight in $D([0,T], \mathcal{H}_{-\alpha})$; see Theorem 5.4.7.

Step 4: $\mathbf{U}_{(t,0)}^N \mathcal{Y}_0^N \xrightarrow{\mathcal{L}} \mathbf{U}_{(t,0)} \mathcal{Y}_0$ in $D([0,T], \mathcal{H}_{-\alpha})$; see Theorem 5.4.8.

Step 5: $\int_0^t \mathbf{U}^N_{(t,s)}\, dM^N_s \xrightarrow{\mathcal{L}} \int_0^t \mathbf{U}_{(t,s)}\, dM_s$ in $D([0,T],\mathcal{H}_{-\alpha})$; see Theorem 5.4.9.

Step 6: All the stated properties for the fluctuation limit hold; see Theorem 5.4.11.

The main difficulty is in establishing the convergence in Step 5. Note that $t \mapsto \int_0^t \mathbf{U}_{(t,s)}\, dM_s$ is not a martingale. The standard method based on Kotelenez's submartingale inequality [52] does not seem to work. This is because in our case $\mathbf{U}_{(t,s)}$ is not exponentially bounded; that is, there is no $\beta > 0$ so that the operator norm $\|\mathbf{U}_{(t,s)}\| \le e^{\beta(t-s)}$ for $t \ge s$ (see [52]). In fact, we suspect it is not even a bounded operator on $\mathcal{H}_{-\alpha}$ due to the singular interaction near the boundary. To overcome this difficulty, we need first to make sense of the expression $\int_0^t \mathbf{U}_{(t,s)}\, dM_s$, which is done in Subsection 5.4.4. Our approach is then based on suitably extending the functional analytic framework of [53] and a direct analysis that uses heat kernel estimates and Dirichlet Form method.

5.2 Functional analytic framework

Our method to study the fluctuation is functional analytic, with the mathematical framework being the calculus of evolution equations on Hilbert spaces (see, for example, [25, 43, 45]). As remarked in [53], this approach yields a useful representation of the limiting process (the generalized Ornstein-Uhlenbeck process) as the mild solution of a stochastic partial differential equation (SPDE), which yields uniqueness and Gaussian property for free. It also tells us the smallest Hilbert space in which the generalized Ornstein-Uhlenbeck process lives.

Conventions and notations:

In this chapter, we use := as a way of definition. For $a, b \in \mathbb{R}$, $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$. We use abbreviation r.c.l.l. for right continuous having left limits, and $\|\cdot\|$ to denote the supremum norm in $\overline{D}$. Even though the constants appearing in the article may depend on $\mathbf{a}$ or ρ given in Assumption 5.1.1, we will not mention this dependence explicitly. For example, we use $C(d, D)$ to denote a constant which depends only on d and D (and possibly on $\mathbf{a}$ or ρ). The exact value of the constant may vary from line to line.

5.2.1 Hilbert space $\mathcal{H}_\gamma$

Recall that $\mathcal{A} = \frac{1}{2\rho}\nabla\cdot(\rho\,\mathbf{a}\nabla)$ denotes the $L^2(D,\,\rho(x)dx)$-generator for an $(\mathbf{a},\,\rho)$-reflected diffusion. Clearly, $\mathcal{A}$ is a self-adjoint, non-positive operator on $L^2(D,\,\rho(x)dx)$. Together with the fact that D is bounded, we see that $\mathcal{A}$ has a discrete spectrum in $\mathcal{H}_0$. Let ϕ_k be a complete orthonormal system (CONS) of eigenvectors of $\mathcal{A}$ in $\mathcal{H}_0$ with eigenvalues $-\lambda_k$, where $0<\lambda_1\le\lambda_2\le\lambda_3\le\cdots$. Note that the linear span of $\{\phi_k\}$ is dense in $L^2(D;\rho dx)$. We define, for $\alpha\in(-\infty,\infty)$,

$$\mathcal{H}_\alpha := \text{ the closure of the linear span of } \{\phi_k\} \text{ with respect to the inner product } \langle\,,\,\rangle_\alpha, \tag{5.2.1}$$

where $\langle\phi,\psi\rangle_\alpha := \langle(I-\mathcal{A})^\alpha\phi,\,\psi\rangle_\rho$. Here I is the identity operator on $\mathcal{H}_0 = L^2(D;\rho dx)$ and $(I-\mathcal{A})^\alpha$ is the α-th power (defined through spectral representation) of the positive self-adjoint operator $I-\mathcal{A}$. In particular, $\langle\,,\,\rangle_0 = \langle\,,\,\rangle_\rho$ by definition.

Note that $(\mathcal{H}_\alpha,\,\langle\,,\,\rangle_\alpha)$ is a real separable Hilbert space and that $\mathcal{H}_\beta\subset\mathcal{H}_\alpha$ when $\beta>\alpha$. Moreover, $\mathcal{H}_\alpha$ and $\mathcal{H}_{-\alpha}$ are dual to each other. Equip $\Phi := \cap_{\alpha\ge0}\mathcal{H}_\alpha$ with the locally convex topology defined by the set of norms $\{|\varphi|_\alpha := \langle\varphi,\varphi\rangle_\alpha^{1/2}:\ \varphi\in\Phi,\,\alpha\in[0,\infty)\}$. Let Φ' be the strong dual of Φ. Identifying $\mathcal{H}_0$ with its dual $\mathcal{H}_0'$, we obtain the chain of dense continuous inclusions

$$\Phi\subset\mathcal{H}_\alpha\subset\mathcal{H}_0=\mathcal{H}_0'\subset\mathcal{H}_{-\alpha}\subset\Phi',\quad \alpha\in[0,\infty). \tag{5.2.2}$$

Moreover, for $\beta\in\mathbb{R}$, we have

$$h_k^{(\beta)} := (1+\lambda_k)^{-\beta/2}\,\phi_k \quad \text{is a CONS for } \mathcal{H}_\beta. \tag{5.2.3}$$

Hence, $\langle\phi,\psi\rangle_\beta = \sum_{k\ge1}\langle\phi\,,\,h_k^{(-\beta)}\rangle\langle\psi\,,\,h_k^{(-\beta)}\rangle$ for $\phi,\,\psi\in\mathcal{H}_\beta$ and

$$\mathcal{H}_\beta = \Big\{\mu\in\Phi':\ \sum_{k\ge1}\langle\mu\,,\,h_k^{(-\beta)}\rangle^2<\infty\Big\}, \tag{5.2.4}$$

where $\langle\,,\,\rangle$ denotes the dual paring extending $\langle\,,\,\rangle_\rho$.

Remark 5.2.1. When $\alpha > 0$, $\mathcal{H}_\alpha$ can be identified with the fractional Sobolev space $W^{\alpha/2,2}(D)$ on D. This is because for $\alpha \geq 0$, $\mathcal{H}_\alpha = (I - \mathcal{A})^{-\alpha/2} L^2(D, \rho dx)$. Since D is a bounded Lipschitz domain, it is known that $\mathcal{H}_\alpha = W^{\alpha/2,2}(D)$ when $\alpha = 1$ (see [15]) and hence for every integer $\alpha \geq 1$. It follows by interpolation that $\mathcal{H}_\alpha = W^{\alpha/2,2}(D)$ for every $\alpha > 0$. When $\alpha < 0$, $\mathcal{H}_\alpha$ can be identified as the dual space of $\mathcal{H}_{-\alpha}$. □

5.2.2 Weyl's law and eigenfunction estimates

For a general bounded Lipschitz domain $D \subset \mathbb{R}^d$, the Weyl's asymptotic law for the Neumann eigenvalues holds (see [63]). That is, the number of eigenvalues (counting their multiplicities) less than or equal to x, denoted by $\sharp\{k : \lambda_k \leq x\}$, satisfies

$$\lim_{x\to\infty} \frac{\sharp\{k : \lambda_k \leq x\}}{x^{d/2}} = C \quad \text{for some constant } C = C(d, D) > 0. \tag{5.2.5}$$

Lemma 5.2.2. *There exists $C = C(d, D) > 0$ such that for all integers $k \geq 1$ we have*

$$\|\phi_k\| \leq C\,\lambda_k^{d/4} \quad \textit{and} \quad \int_{\partial D} \phi_k^2\, d\sigma \leq C\,(\lambda_k + 1). \tag{5.2.6}$$

Proof By Cauchy-Schwartz inequality, Chapman Kolmogorov equation and then the Gaussian upper bound, we have

$$\begin{aligned} |\phi_k(x)| &= e^{\lambda_k t}|P_t\phi_k(x)| \leq e^{\lambda_k t}\|\phi_k\|_{L^2(\rho)}\sqrt{p(2t, x, x)} \\ &\leq e^{\lambda_k t}\,\frac{C(d, D)}{t^{d/4}} \quad \text{for } t \leq 1/\lambda_1. \end{aligned}$$

Taking $t = 1/\lambda_k$ yields the first inequality in (5.2.6).

Recall that the Dirichlet form $(\mathcal{E}, Dom(\mathcal{E}))$ (in $L^2(D, \rho(x)dx)$) for the $(\mathcal{A}, \rho)$-reflected diffusion X is regular (since D has Lipschitz boundary (cf. [2])) and that the surface measure σ is smooth. Hence by Theorem 2.1 of [70], we have the following generalized trace theorem:

$$\int_{\partial D} f(x)^2\,\sigma(dx) \le \|G_\beta \sigma\| \left(\mathcal{E}(f,f) + \beta \int_D f^2(x)\,dx \right) \tag{5.2.7}$$

for any $f \in Dom(\mathcal{E})$ and $\beta > 0$, where $G_\beta\sigma(x) := \int_0^\infty \int_{\partial D} e^{-\beta t} p(t,x,y)\,\sigma(dy)dt$. Note that $\|G_\beta\sigma\| < \infty$ by (2.1.5) and the fact that $p(t,x,y)$ converges to $1/\int_D \rho(x)dx$ as $t \to \infty$ uniformly for $(x,y) \in \overline{D} \times \overline{D}$ exponentially fast (by eigenfunction expansion). Hence, taking $\beta = 1$, we obtain the second inequality in (5.2.6). □

5.2.3 Hilbert-Schmidt Operators

Hilbert-Schmidt operators appear naturally in stochastic analysis in infinite dimensions. The main properties of these operators can be found in standard references (e.g. [43]). We now recall the main definitions.

Definition 5.2.3. *Let $X = (X_t)_{t\ge 0}$ be an $\mathcal{H}_{-\alpha}$-valued process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We say X is (centered)* ***Gaussian*** *if $\{X_t(\phi) : \phi \in \mathcal{H}_\alpha,\, t \in [0,\infty)\}$ form a (centered) Gaussian system. That is, $(X_{t_1}(\phi_1), \cdots, X_{t_k}(\phi_k))$ is a (centered) Gaussian vector in $\mathbb{R}^k$ for any $k \in \mathbb{N}$, any $\{t_i\}_{i=1}^k \subset [0,\infty)$ and any $\{\phi_i\}_{i=1}^k \subset \mathcal{H}_\alpha$. We say X is* ***continuous*** *if $t \mapsto X_t$ is continuous $\mathbb{P}$-a.s. X is said to be* ***square-integrable*** *if $\mathbb{E}[|X_t|_{-\alpha}^2] < \infty$ for all $t \ge 0$. Finally, we say $\mathfrak{X}$ has* ***independent increments*** *if for any $0 \le s < t$ and $\phi \in \mathcal{H}_\alpha$, the real random variable $X_t(\phi) - X_s(\phi)$ is independent of the σ-field generated by $\{X_r(\psi) : 0 \le r \le s,\, \psi \in \mathcal{H}_\alpha\}$.*

Suppose X and Y are real separable Hilbert spaces with inner product $\langle\, ,\rangle_X$ and $\langle\, ,\rangle_Y$ (we simply write $\langle\, ,\rangle$ when there is no confusion for which Hilbert space we are considering). The class of bounded linear operators from X to Y will be denoted by $L(X,Y)$ ($L(X)$ for short when $X = Y$). It is well known that $A \in L(X,Y)$ is **compact** (i.e. the range of the unit sphere in X is relatively compact in Y) if and only if there exist orthonormal systems (ONS for short) $\{e_n\} \subset X$, $\{f_n\} \subset Y$ and a sequence of real numbers $a_n \to 0$ such that A has the representation

$$Ax = \sum_{n \geq 1} a_n \langle x, e_n \rangle f_n \quad \text{for all } x \in X. \tag{5.2.8}$$

Definition 5.2.4. *1. $A \in L(X, Y)$ is said to be **Hilbert-Schmidt** (denoted by $A \in L_2(X, Y)$) if A has the representation (5.2.8) with $\sum_{n \geq 1} a_n^2 < \infty$. In this case, the **Hilbert-Schmidt norm** of A is defined to be*

$$\|A\|_2 := \left(\sum_{n \geq 1} a_n^2 \right)^{1/2} = \left(\sum_{n \geq 1} |Ae_n|^2 \right)^{1/2}$$

Note that $\|A\|_2$ is independent of the choice of the ONS $\{e_n\} \subset X$.

*2. The **Trace** of $A \in L(X)$ is*

$$Tr(A) := \sum_{n \geq 1} \langle Ae_n, e_n \rangle$$

Note that $Tr(A)$ is independent of the choice of the ONS $\{e_n\} \subset X$.

The following lemma is equivalent to the statement that $(\Phi_{imb}, \mathcal{H}_\beta, \mathcal{H}_\gamma)$ is an abstract Wiener space if $\beta > d/2 + \gamma$ (cf. [68]).

Lemma 5.2.5. *For any $\beta, \gamma \in \mathbb{R}$ with $\beta > \gamma + d/2$, the imbedding $\Phi_{imb} : \mathcal{H}_\beta \to \mathcal{H}_\gamma$ is Hilbert-Schmidt.*

Proof We want to show that $\sum_k \left| \Phi_{imb} \left(h_k^{(\beta)} \right) \right|_\gamma^2 < \infty$. The left hand side equals

$$\sum_k (1 + \mu_k)^{-\beta} |\phi_k|_\gamma^2 = \sum_k (1 + \lambda_k)^{-\beta + \gamma}.$$

By Weyl's formula (5.2.5), the latter quantity is finite if and only if

$$\int_1^\infty (1 + x)^{-\beta + \gamma} x^{d/2 - 1} \, dx < \infty.$$

This is true if and only if $\beta - \gamma > d/2$. $\square$

5.3 Preliminaries

5.3.1 Minkowski content for ∂D

By the same proof of [17, Lemma 7.1], we obtain the following result.

Lemma 5.3.1. *Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $k \in \mathbb{N}$. If $\mathcal{F} \subset C(\overline{D}^k)$ is an equi-continuous and uniformly bounded family of functions, then*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^k} \int_{(D^\varepsilon)^k} f(z_1, \cdots, z_k)\, dz_1 \cdots dz_k = \int_{(\partial D)^k} f(z_1, \cdots, z_k)\, \sigma(dz_1) \cdots \sigma(dz_1)$$

uniformly for $f \in \mathcal{F}$, where $D^\varepsilon := \{x \in D : \, dist(x, \partial D) < \varepsilon\}$ is the ε-neighborhood of ∂D in D and σ is the surface measure on ∂D.

By a simple modification of the same proof, we can strengthen the above lemma as follows.

Lemma 5.3.2. *Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain, I be a $\mathcal{H}^{d-1}$-rectifiable closed subset of ∂D and $k \in \mathbb{N}$. If $\mathcal{F} \subset \mathcal{B}(D^k)$ is an equi-continuous and uniformly bounded family of functions on an open neighborhood of I^k, then*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^k} \int_{(I^\varepsilon)^k} f(z_1, \cdots, z_k)\, dz_1 \cdots dz_k = \int_{I^k} f(z_1, \cdots, z_k)\, \sigma(dz_1) \cdots \sigma(dz_1)$$

uniformly for $f \in \mathcal{F}$, where $I^\varepsilon := \{x \in D : \, dist(x, I) < \varepsilon\}$ is the ε-neighborhood of I in D.

The following is about a convergence result uniform in the shrinking rate of $\delta = \delta_N$. It is used to guarantee that δ_N can be any sequence (which converges to zero) in the proof of Lemma 5.4.10.

Lemma 5.3.3. *Suppose $\{\mathfrak{X}_0^N\} \xrightarrow{\mathcal{L}} u_0(x)\rho(x)\, dx$ in $M_+(\overline{D})$ as $N \to \infty$, where $u_0 \in C(\overline{D})$. Let $\{\varphi_N(r) : r \geq 0, N \in \mathbb{N}\}$ be a family of non-negative continuous functions on $\overline{D}$ such that $\sup_N \sup_{r \geq 0} \|\varphi_N(r)\| < \infty$. For any $\delta_N \to 0$, $T > 0$ and $p \geq 1$, we have*

$$\lim_{N \to \infty} \mathbb{E}\left[\left(\sup_{t \in [0,T]} \Big| \int_0^t \langle \varphi_N(r)\delta_N^{-1}\mathbf{1}_{D^{\delta_N}}, \, \mathfrak{X}_r^N \rangle - \langle \varphi_N(r)\delta_N^{-1}\mathbf{1}_{D^{\delta_N}}, \, u(r) \rangle_\rho \, dr \Big| \right)^p \right] = 0, \qquad (5.3.1)$$

where $\mathbf{1}_{D^{\delta_N}}$ *is the indicator function on* D^{δ_N}*.*

Proof Let $H_N(t) := \int_0^t \langle \varphi_N(r)\delta_N^{-1}\mathbf{1}_{D^{\delta_N}}, \mathfrak{X}_r^N\rangle\, dr$ and $G_N(t) := \int_0^t \langle \varphi_N(r)\delta_N^{-1}\mathbf{1}_{D^{\delta_N}}, u(r)\rangle_\rho\, dr$. It can be shown, by a standard argument and using Lemma 5.3.1, that for any $T > 0$,

$$H_N(t) - G_N(t) \xrightarrow{\mathcal{L}} 0 \quad \text{in } C([0,T],\mathbb{R}).$$

In particular, by the metric of $C([0,T],\mathbb{R})$ and the deterministic nature of the limit, we have

$$\sup_{t\in[0,T]} |H_N(t) - G_N(t)| \to 0 \quad \text{both in law and in probability.}$$

On other hand, since $H_N(t)$ and $G_N(t)$ are increasing, we have

$$\limsup_{N\to\infty} \mathbb{E}\left[\Big(\sup_{t\in[0,T]} |H_N(t) - G_N(t)|\Big)^p\right] \le \limsup_{N\to\infty} 2^p \Big(\mathbb{E}\left[\, H_N^p(T)\,\right] + G_N^p(T)\Big).$$

Furthermore, we can check that $\limsup_{N\to\infty} \mathbb{E}[H_N^p(T) + G_N^p(T)] < \infty$. Denote by $\mathcal{P}(\overline{D})$ the collection of sub-probability measures on $\overline{D}$. Comparing with the process without killing (i.e. replacing the sub-processes $Z_i^{(N)}$ by the reflected diffusions X_i in the definition of $\mathfrak{X}^N$), we have by (2.1.4)

$$\sup_{\mu\in\mathcal{P}(\overline{D})} \mathbb{E}_\mu\left[H_N(t)\right] \le \|\varphi_N\| \sup_{x\in\overline{D}} \mathbb{E}_x \int_0^t \mathbf{1}_{D^{\delta_N}}(X_1(r))dr = \|\varphi_N\| \sup_{x\in\overline{D}} \int_0^t \int_{D^{\delta_N}} p(r,x,y)dydr \le C_1\, t^{1/2},$$

where C_1 is a positive constant independent of N and t. Let $f(r) := \langle \delta_N^{-1}\mathbf{1}_{D^{\delta_N}}, \mathfrak{X}_r^N\rangle$. Then for any positive integer k, by Fubinni's theorem and the Markov property, we have for any initial distribution μ of $\mathfrak{X}_0^N$,

$$\mathbb{E}_\mu[H_N^k(T)] = k!\,\mathbb{E}\int_{0\le r_1\le r_2\le\cdots\le r_k\le T} f(r_1)f(r_2)\ldots f(r_k)dr_1dr_2\cdots dr_k \le k!\,(C_1T^{1/2})^k.$$

It in particular implies that, under the assumption $\{\mathfrak{X}_0^N\} \xrightarrow{\mathcal{L}} u_0(x)\rho(x)\,dx$ in $M_+(\overline{D})$,

$$\limsup_{N\to\infty} \mathbb{E}[H_N^k(T)] \le \|u_0\|\|\rho\|\, k!\,(C_1 T^{1/2})^k.$$

A similar argument yields $\limsup_{N\to\infty} \mathbb{E}[G_N^k(T)] < \infty$ for any positive integer k. Hence, by interpolation, we have $\limsup_{N\to\infty} \mathbb{E}[H_N^p(T) + G_N^p(T)] < \infty$ for all $p \ge 1$.

The uniform integrability implied by $\limsup_{N\to\infty} \mathbb{E}\Big[\Big(\sup_{t\in[0,T]} |H_N(t) - G_N(t)|\Big)^p\Big] < \infty$, together with the convergence $\sup_{t\in[0,T]} |H_N(t) - G_N(t)| \to 0$ in probability, guarantee (see, e.g. Theorem 5.2 in [32, Chapter 4]) that the lemma is true. □

5.3.2 Estimates for evolution semigroups $Q^N_{(s,t)}$ and $Q_{(s,t)}$

Recall the definition of $Q_{(s,t)}$ and $\mathbf{U}_{(t,s)}$ in (5.1.5) and (5.1.6), respectively. For any fixed $t > 0$ and $\phi \in C(\overline{D})$, $v(s,x) := Q_{(s,t)}\phi(x)$ is the unique element in $C([0,t]\times\overline{D})$ satisfying the integral equation

$$v(s,x) = P_{t-s}\phi(x) - \frac{1}{2}\int_0^{t-s}\int_{\partial D} p(\theta,x,y)\,q(s+\theta,y)\,v(s+\theta,y)\,\rho(y)\,d\sigma(y)\,d\theta; \qquad (5.3.2)$$

see [17, Proposition 4.1]. We call v the **probabilistic solution** of the backward equation

$$\begin{cases} \dfrac{\partial v}{\partial s} = -\mathcal{A}v & \text{on } (0,t)\times D \\ \dfrac{\partial v}{\partial \vec{\nu}} = \dfrac{q\,v}{\rho} & \text{on } (0,t)\times\partial D \\ v(t) = \phi & \text{on } D. \end{cases} \qquad (5.3.3)$$

Analogous to the definition of $Q_{(s,t)}$ and $\mathbf{U}_{(t,s)}$, we define

$$\begin{aligned} Q^N_{s,t}\phi(x) &:= \mathbb{E}\left[\phi(X_t)\,\exp\left(-\int_s^t q_N(r,X_r)\,dr\right)\Big|\,X_s = x\right] \\ &= \mathbb{E}\left[\phi(X_{t-s})\,\exp\left(-\int_0^{t-s} q_N(s+r,X_r)\,dr\right)\Big|\,X_0 = x\right] \end{aligned} \qquad (5.3.4)$$

and

$$\mathbf{U}^N_{(t,s)}\mu(\phi) := \mu(Q^N_{s,t}\phi) \tag{5.3.5}$$

for $\alpha > 0$, $\mu \in \mathcal{H}_{-\alpha}$ and $\phi \in L^2(D)$ whenever it is well defined (i.e. $Q^N_{s,t}\phi \in \mathcal{H}_\alpha$). Then $v_N(s,x) := Q^N_{(s,t)}\phi(x)$ is the unique element in $C([0,t]\times\overline{D})$ satisfying the integral equation

$$v_N(s,x) = P_{t-s}\phi(x) - \frac{1}{2}\int_0^{t-s} P_\theta\left(q_N(s+\theta)v_N(s+\theta)\right)(x)\,d\theta, \quad 0\le s\le t, \tag{5.3.6}$$

provided that $\phi \in C(\overline{D})$. Here $\{P_t; t\geq 0\}$ is the transition semigroup of X in $L^2(D,\,\rho(x)dx)$ (i.e. $P_tf(x) = \mathbb{E}^x[f(X_t)] = \int_D f(y)p(t,x,y)\rho(y)\,dy$). As before, v_N is called the **probabilistic solution** of the backward equation

$$\begin{cases} \dfrac{\partial v_N}{\partial s} = -\frac{1}{2}\Delta v_N + q_N\, v_N & \text{on } (0,t)\times D \\ \dfrac{\partial v_N}{\partial \vec{\nu}} = 0 & \text{on } (0,t)\times \partial D \\ v_N(t) = \phi & \text{on } D \end{cases} \tag{5.3.7}$$

Remark 5.3.4. It can be shown (cf. [20]), using the Markov property of the reflected diffusion X, that each $Z = Z^i$ (described in Remark 5.1.3) is a time-inhomogeneous Markov process on $\overline{D}\cup\{\Delta^{(i)}\}$ with $(Q^N_{s,t})_{s\le t}$ being its transition operator: $Q^N_{s,t}f(x) = \mathbb{E}[f(Z_t)|Z_s = x]$, with the convention that $f(\Delta) = 0$. Besides, (5.3.7) is the Kolmogorov's backward equation for Z and (5.3.4) is the probabilistic representation of the solution to (5.3.7). □

The following uniform convergence and uniform bound are useful in many places of this chapter.

Lemma 5.3.5. *For all $\phi \in C(\overline{D})$ and $0\le s\le t$, we have*

$$\lim_{N\to\infty} Q^N_{s,t}\phi = Q_{s,t}\phi \quad \textit{uniformly on } \overline{D} \quad \textit{and} \tag{5.3.8}$$

$$\sup_N |Q^N_{s,t}\phi(x)| \vee |Q_{s,t}\phi(x)| \le P_{t-s}|\phi|(x) \le \|\phi\| \quad \textit{for } x \in \overline{D}. \tag{5.3.9}$$

Proof Estimates (5.3.9) follows immediately from (5.3.4), (5.1.5) and the non-negativity of q. For (5.3.8), note that

$$\begin{aligned}
&\left|Q^N_{s,t}\phi(x) - Q_{s,t}\phi(x)\right| = \left|\mathbb{E}_x\left[\phi(X_{t-s})\left(e^{-\int_0^{t-s} q_N(s+r,X_r)\,dr} - e^{-\int_0^{t-s} q(s+r,X_r)\,dL_r}\right)\right]\right| \\
\le\ & \|\phi\|\,\mathbb{E}_x\left[\left|\int_0^{t-s} q_N(s+r,X_r)\,dr - \int_0^{t-s} q(s+r,X_r)\,dL_r\right|^2\right] \\
=\ & 2\,\|\phi\| \int_{r_1=0}^{t-s}\int_{r_2=r_1}^{t-s}\left(\int_D\int_D q_N(s+r_1,z_1)q_N(s+r_2,z_2)p(r_1,x,z_1)p(r_2-r_1,z_1,z_2)\rho(z_1)\rho(z_2)\,dz_1\,dz_2\right. \\
& -2\int_D\int_{\partial D} q_N(s+r_1,z_1)q(s+r_2,z_2)p(r_1,x,z_1)p(r_2-r_1,z_1,z_2)\rho(z_1)\rho(z_2)\,dz_1\,d\sigma(z_2) \\
& \left.+\int_{\partial D}\int_{\partial D} q(s+r_1,z_1)q(s+r_2,z_2)p(r_1,x,z_1)p(r_2-r_1,z_1,z_2)\rho(z_1)\rho(z_2)\,d\sigma(z_1)\,d\sigma(z_2)\right)dr_1\,dr_2,
\end{aligned}$$

which converges to zero uniformly for $x \in \overline{D}$ by Lemma 5.3.1. □

Remark 5.3.6. While the non-negativity of q easily implies that Q has the contraction property (5.3.9), we may lose this property for $\mathbf{U}$ because intuitively the killing effect induces a jump in the system and hence can increase the fluctuation. □

The following gradient convergence is the cornerstone in Step 5 of the proof the main theorem. Its proof is based on the inequality $\mathcal{E}(P_t f) \le (2e\,t)^{-1}\,\|f\|_\rho^2$ (see the Appendix of [20]).

Lemma 5.3.7. *For any $0 \le s \le t$ and $\phi \in C(\overline{D})$, we have*

$$\lim_{N\to\infty} \mathcal{E}\left(Q^N_{(s,t)}\phi - Q_{(s,t)}\phi\right) = 0. \tag{5.3.10}$$

where $\mathcal{E}$ is the Dirichlet form of the $(\mathcal{A},\,\rho)$-reflected diffusion defined in (5.1.1) and $\mathcal{E}(u) := \mathcal{E}(u,u)$.

Proof From (5.3.2) and (5.3.6), we have

$$
\begin{aligned}
& Q^N_{(s,t)}\phi(x) - Q_{(s,t)}\phi(x) \\
= & \int_0^{t-s} \int_{\partial D} p(\theta,x,y) q(s+\theta,y) Q_{(s+\theta,t)}\phi(y)\rho(y) d\sigma(y) \; - \; P_\theta(q_N(s+\theta) Q^N_{(s+\theta,t)}\phi)(x)\, d\theta \\
= & \int_0^{t-s} P_\theta \left(q(s+\theta) Q_{(s+\theta,t)}\phi\, \sigma \; - \; q_N(s+\theta) Q^N_{(s+\theta,t)}\phi \right)(x)\, d\theta \\
= & \int_0^{t-s} P_\theta \left(h_N^{(s,t)}(\theta) \right)(x)\, d\theta,
\end{aligned}
$$

where $h_N^{(s,t)}(\theta)$ is the signed Borel measure $q(s+\theta,y)Q_{(s+\theta,t)}\phi(y)\,\rho(y)\sigma(dy) - q_N(s+\theta,y)Q^N_{(s+\theta,t)}\phi(y)\,\rho(y)dy$ and $P_\theta\mu(x) := \int_{\overline{D}} p(\theta,x,y)\,\mu(dy)$ for any measure μ on $\overline{D}$.

On the other hand, by spectral decomposition, $\mathcal{E}(P_t f) \le (2e\,t)^{-1}\, \|f\|_\rho^2$ (see the Appendix of [20]), where $\|\cdot\|_\rho$ is the $L^2(D,\, \rho(x)dx)$-norm. Hence

$$
\begin{aligned}
\sqrt{\mathcal{E}\Big(Q^N_{(s,t)}\phi(x) - Q_{(s,t)}\phi(x) \Big)} &= \sqrt{\mathcal{E}\Big(\int_0^{t-s} P_\theta \left(h_N^{(s,t)}(\theta) \right)(x)\, d\theta \Big)} \\
&\le \int_0^{t-s} \sqrt{\mathcal{E}\left(P_\theta \left(h_N^{(s,t)}(\theta) \right) \right)}\, d\theta \\
&= \int_0^{t-s} \sqrt{\mathcal{E}\Big(P_{\theta/2} P_{\theta/2} \left(h_N^{(s,t)}(\theta) \right) \Big)}\, d\theta \\
&\le \int_0^{t-s} \sqrt{\frac{1}{e\,\theta} \left\| P_{\theta/2} \left(h_N^{(s,t)}(\theta) \right) \right\|_\rho^2}\, d\theta. \qquad (5.3.11)
\end{aligned}
$$

We now show that the last quantity in (5.3.11) converges to zero as $N \to \infty$. Note that for each $\theta \in (0,\, t-s)$, the semigroup property yields

$$
\left\| P_{\theta/2} \left(h_N^{(s,t)}(\theta) \right) \right\|_\rho^2 = \int_{\overline{D}} \Big(P_\theta h_N^{(s,t)}(\theta) \Big)(x)\, h_N^{(s,t)}(\theta)(dx) \to 0 \quad \text{as } N \to \infty
$$

by Lemma 5.3.1 and the uniform convergence (5.3.8). By the uniform bounds (2.1.4) and (5.3.9), for N large enough which depends only on D (hence independent of θ), we have $\left\| P_\theta h_N^{(s,t)}(\theta) \right\| \le$

$\|q\|\,\|\phi\|\,\frac{C(d,D)}{\sqrt{\theta}}$ and

$$\begin{aligned}\left|P_\theta h_N^{(s,t)}(\theta)\right|(\overline{D}) &= \int_{\partial D} q(s+\theta,y)Q_{(s+\theta,t)}\phi(y)\,\rho(y)\,\sigma(dy) \;+\; \int_D q_N(s+\theta,y)Q^N_{(s+\theta,t)}\phi(y)\,\rho(y)\,dy \\ &\leq C(d,D)\,\|q\|\,\|\phi\|.\end{aligned}$$

Hence the last quantity in (5.3.11) converges to zero as $N \to \infty$ by the Lebesgue dominated convergence theorem and the fact that

$$\|P_{\theta/2}\mu\|_\rho^2 = \int_{\overline{D}} P_\theta\mu(x)\,\mu(dx) \leq \|P_\theta\mu\| \cdot |\mu|(\overline{D}),$$

where $|\mu|$ is the total variation measure of the signed measure μ. □

The following equality will be used in Lemma 5.3.9, which explore the continuity in time for both $Q_{s,t}$ and $Q^N_{s,t}$.

Lemma 5.3.8.

$$\int_0^s \cdots \int_0^{s_k} \frac{1}{\sqrt{(s-s_2)(s_2-s_3)\cdots(s_k-s_{k+1})}}\,ds_{k+1}\cdots ds_2 = \frac{\pi^{k/2}}{\Gamma\left(\frac{k+2}{2}\right)}\,s^{k/2}.$$

Proof Denote the integral on the left hand side as V_k. For any $a \in (0,\infty)$,

$$\int_0^x \frac{y^a}{\sqrt{x-y}}\,dy = \frac{\sqrt{\pi}\,\Gamma(1+a)}{\Gamma(3/2+a)}\,x^{1/2+a}$$

Using this, we can iterate it to obtain $V_k = \int_0^t c_k\, s^{k/2}\,ds$, where

$$c_1 = 2 \quad \text{and} \quad c_{k+1} = c_k\,\frac{\sqrt{\pi}\,\Gamma(1+k/2)}{\Gamma(3/2+k/2)} \text{ for } k \geq 2.$$

□

Lemma 5.3.9. *There exists a constant $c > 0$ such that for any $0 \leq s \leq t \leq T$ and $k \geq 1$,*

$$\sup_{r\in[0,s]} \left\|Q_{(r,t)}\phi_k \;-\; Q_{(r,s)}\phi_k\right\| \leq c\,\|\phi_k\|\,\left(\lambda_k(t-s) + C\,\|q\|\,(t-s)^{1/2}\right),$$

where $C = C(d, D, T)$ is the same constant in (2.1.5). Furthermore, there exists $N_0 = N_0(D)$ such that for $N \geq N_0$, the above inequality holds with $\{Q^N_{s,t}\}$ in replace of $\{Q_{s,t}\}$.

Proof The proof will follow from a Grownwall type argument and the evolution property of the operators $\{Q_{(s,t)}\}_{s\leq t}$. By (5.3.2), for any $0 \leq r \leq s \leq t$ and k, we have

$$\begin{aligned}
&\left|Q_{(r,t)}\phi_k(x) - Q_{(r,s)}\phi_k(x)\right| \\
\leq\ & \left|e^{-\lambda_k(t-r)}\phi_k(x) - e^{-\lambda_k(s-r)}\phi_k(x)\right| \\
&+\frac{1}{2}\Big|\int_{s-r}^{t-r}\int_{\partial D} p(\theta,x,y)q(r+\theta,y)Q_{(r+\theta,t)}\phi_k(y)\,\rho(y)\,d\sigma(y)\,d\theta\Big| \\
&+\frac{1}{2}\Big|\int_0^{s-r}\int_{\partial D} p(\theta,x,y)q(r+\theta,y)\left(Q_{(r+\theta,t)}\phi_k - Q_{(r+\theta,s)}\phi_k\right)(y)\,\rho(y)\,d\sigma(y)\,d\theta\Big|.
\end{aligned}$$

Now we fix k, fix $0 \leq s \leq t$ and define $f(r) := \left\|Q_{(r,t)}\phi_k - Q_{(r,s)}\phi_k\right\|$ for $r \in [0, s]$. Then the above estimate, together with (2.1.5) and (5.3.9), implies that

$$f(r) \leq A + B\int_0^{s-r}\frac{f(r+\theta)}{\sqrt{\theta}}\,d\theta \quad \text{for } r \in [0,s], \tag{5.3.12}$$

where $A = \lambda_k\|\phi_k\|(t-s) + \|q\|\,C(d,D,T)\|\phi_k\|(t-s)^{1/2}$ and $B = \frac{1}{2}C(d,D,T)\|q\|$.

Rewriting (5.3.12) as $f(r) \leq A + B\int_r^s \frac{f(w)}{\sqrt{w-r}}\,dw$ and keep iterating yields

$$\begin{aligned}
f(r) &\leq A + AB\int_{w_1=r}^{s}\frac{1}{\sqrt{w_1-r}} + AB^2\int_{w_1=r}^{s}\int_{w_2=w_1}^{s}\frac{1}{\sqrt{(w_1-r)(w_2-w_1)}} \\
&\quad +AB^3\int_{w_1=r}^{s}\int_{w_2=w_1}^{s}\int_{w_3=w_2}^{s}\frac{1}{\sqrt{(w_1-r)(w_2-w_1)(w_3-w_2)}} + \cdots \\
&= A\sum_{k=0}^{\infty}B^k\,a_k\,(s-r)^{k/2}, \text{ where } a_k = \frac{\pi^{k/2}}{\Gamma((k+2)/2)} \text{ by Lemma 5.3.8 in Appendix} \\
&\leq \frac{c}{2}A\sum_{k=0}^{\infty}B^k\,(s-r)^{k/2} \quad \text{for some absolute constant } c>0 \\
&\leq c\,A \qquad \text{if} \quad |B\sqrt{s-r}| \leq 1/2
\end{aligned}$$

Note that when $B > 0$, $|B\sqrt{s-r}| \leq 1/2$ holds if and only if $s - \frac{1}{4B^2} \leq s \leq s + \frac{1}{4B^2}$. (The case $B=0$ is trivial since then $q=0$.) When $0 \le r < s - 1/(4B^2)$, by the evolution property and the contraction property (5.3.2), we have

$$\begin{aligned} \left\|Q_{(r,t)}\phi_k - Q_{(r,s)}\phi_k\right\| &= \left\|Q_{(r,\,s-1/(4B^2))}\left(Q_{(s-1/(4B^2),\,t)}\phi_k - Q_{(s-1/(4B^2),\,s)}\phi_k\right)\right\| \\ &\le \left\|Q_{(s-1/(4B^2),\,t)}\phi_k - Q_{(s-1/(4B^2),\,s)}\phi_k\right\| \le c\,A \end{aligned}$$

The above arguments clearly hold with $\{Q^N_{s,t}\}$ in replace of $\{Q_{s,t}\}$, if we use (2.1.4) instead of (2.1.5). This completes the proof of the lemma. □

The next lemma is a key estimate that we need to establish Theorem 5.4.9. Recall from (5.1.10) that

$$\mathcal{E}^{(q)}_r(\phi,\psi) := \langle \mathbf{a}\nabla\phi\cdot\nabla\psi,\, u(s)\rangle_\rho + \int_{\partial D}\phi\,\psi\,u(s)\,q(s)\,\rho\,d\sigma,\ \mathcal{E}^{(q)}_r(\phi) := \mathcal{E}^{(q)}_r(\phi,\phi). \qquad (5.3.13)$$

In view of (5.3.25), we also define

$$\mathcal{E}^{(q),N}_s(\phi,\psi) := \langle \mathbf{a}\nabla\phi\cdot\nabla\psi + q_N(s)\phi\psi,\, \mathfrak{X}^N_s\rangle,\ \mathcal{E}^{(q),N}_s(\phi) := \mathcal{E}^{(q),N}_s(\phi,\phi). \qquad (5.3.14)$$

Lemma 5.3.10. *For all integers $k \ge 1$ and $0 \le s \le t \le T$, we have*

$$\int_s^t \mathcal{E}^{(q)}_r(Q_{(r,t)}\phi_k)\,dr \le C\,\|u_0\|\,(1\vee\|q\|)^2\,(\lambda_k + \|\phi_k\|^2)\,(t-s), \qquad (5.3.15)$$

$$\int_0^s \mathcal{E}^{(q)}_r(Q_{(r,t)}\phi_k - Q_{(r,s)}\phi_k)\,dr \le C\,\|u_0\|\,(1\vee\|q\|)^4\,(\lambda_k^2 + \|\phi_k\|^2 + \|\phi_k\|^2\lambda_k^2)\,(t-s), \qquad (5.3.16)$$

where $C = C(d,D,T) > 0$ is a constant. Moreover, these two inequalities remain valid if we replace $Q_{r,t}$ by $Q^N_{r,t}$ and $\mathcal{E}^{(q)}_r$ by $\mathcal{E}^{(q),N}_r$ at the same time.

Proof For the first inequality, note that

$$0 \le \mathcal{E}^{(q)}_r(Q_{(r,t)}\phi_k) \le \|u_0\|\left(\mathcal{E}(Q_{(r,t)}\phi_k) + \sigma(\partial D)\,\|q\|\,\|\rho\|\,\|\phi_k^2\|\right). \qquad (5.3.17)$$

Moreover, by the integral equation (5.3.2), we have

$$\begin{aligned}\mathcal{E}(Q_{(r,t)}\phi_k) &\leq 2\,\mathcal{E}(P_{t-r}\phi_k) + 2\,\mathcal{E}\Big(\frac{1}{2}\int_0^{t-r} P_\theta\Big[H^{(r,t)}(\theta)\Big](x)\,d\theta\Big) \\ &= 2\lambda_k\,e^{-2(t-r)\lambda_k} + \mathcal{E}\Big(\int_0^{t-r} P_\theta\Big[H^{(r,t)}(\theta)\Big]\,d\theta\Big), \end{aligned} \tag{5.3.18}$$

where $H^{(r,t)}(\theta)$ is the signed Borel measure $q(r+\theta,y)Q_{(r+\theta,t)}\phi_k(y)\,\rho(y)\,\sigma(dy)$ and $P_\theta\mu(x) := \int_{\overline{D}} p(t,x,y)\,\mu(dy)$ for any measure μ on $\overline{D}$.

By the same argument as that in the proof of Lemma 5.3.7, we have

$$\begin{aligned}\mathcal{E}\Big(\int_0^{t-r} P_\theta\Big[H^{(r,t)}(\theta)\Big]\,d\theta\Big) &\leq \left(\int_0^{t-r}\sqrt{\frac{1}{e\,\theta}\Big\|P_{\theta/2}H^{(r,t)}(\theta)\Big\|_\rho^2}\,d\theta\right)^2 \\ &\leq \left(\int_0^{t-r}\sqrt{\frac{1}{e\,\theta}\Big\|P_\theta H^{(r,t)}(\theta)\Big\|\,|H^{(r,t)}(\theta)|(\overline{D})}\,d\theta\right)^2 \\ &\leq \left(\int_0^{t-r} C(d,D,T)\,\|q\|\,\|\phi_k\|\,\theta^{-3/4}\,d\theta\right)^2 \\ &\leq C(d,D,T)\,\|q\|^2\,\|\phi_k\|^2\,(t-r)^{1/2}. \end{aligned} \tag{5.3.19}$$

Now we put (5.3.19) into (5.3.18) and then put the result into (5.3.17) to obtain

$$\mathcal{E}_r^{(q)}(Q_{(r,t)}\phi_k) \leq \|u_0\|\,\left(2\lambda_k e^{-2(t-r)\lambda_k} + C(d,D,T)\left(\|q\|^2\,\|\phi_k\|^2\,(t-r)^{1/2} + \|q\|\,\|\phi_k^2\|\right)\right). \tag{5.3.20}$$

By integration, we obtain

$$\int_s^t \mathcal{E}_r^{(q)}(Q_{(r,t)}\phi_k)\,dr \leq C(d,D,T)\,\|u_0\|\,\left(\|\phi_k\|^2\,\|q\|^2\,(t-s)^{3/2} + (\lambda_k + \|\phi_k^2\|\,\|q\|)(t-s)\right)$$

which implies (5.3.15).

The second inequality in the lemma can be dealt with in a similar way. More precisely, we have as in (5.3.17),

$$
\begin{aligned}
0 &\leq \mathcal{E}_r^{(q)}(Q_{(r,t)}\phi_k - Q_{(r,s)}\phi_k) \\
&\leq \|u_0\| \left(\mathcal{E}(Q_{(r,t)}\phi_k - Q_{(r,s)}\phi_k) + \sigma(\partial D)\,\|q\|\,\|\rho\|\,\left\|Q_{(r,t)}\phi_k - Q_{(r,s)}\phi_k\right\|^2\right)
\end{aligned} \tag{5.3.21}
$$

and

$$
\begin{aligned}
& \mathcal{E}(Q_{(r,t)}\phi_k - Q_{(r,s)}\phi_k) \\
\leq\ & 2\left(e^{-(t-r)\lambda_k} - e^{-(s-r)\lambda_k}\right)^2 \mathcal{E}(\phi_k) \\
& +2\,\mathcal{E}\Big(\int_{s-r}^{t-r}\int_{\partial D} p(\theta,x,y)q(r+\theta,y)Q_{(r+\theta,t)}\phi_k(y)\,d\sigma(y)\,d\theta\Big) \\
& +2\,\mathcal{E}\Big(\int_{0}^{s-r}\int_{\partial D} p(\theta,x,y)q(r+\theta,y)\left(Q_{(r+\theta,t)}\phi_k - Q_{(r+\theta,s)}\phi_k\right)(y)\,d\sigma(y)\,d\theta\Big) \\
\leq\ & 2\left(e^{-(t-r)\lambda_k} - e^{-(s-r)\lambda_k}\right)^2 \lambda_k \\
& +C(d,D,T)\,\|q\|^2\,\|\phi_k\|^2\,(t-s)\left(\frac{1}{\sqrt{s-r}} - \frac{1}{\sqrt{t-r}}\right) \\
& +C(d,D,T)\,\|q\|^2\left(\sup_{r\in[0,s-\theta]}\left\|Q_{(r+\theta,t)}\phi_k - Q_{(r+\theta,s)}\phi_k\right\|\right)^2 (s-r)^{1/2} \\
\leq\ & 2\left(e^{-(t-r)\lambda_k} - e^{-(s-r)\lambda_k}\right)^2 \lambda_k \\
& +C(d,D,T)\,\|q\|^2\,\|\phi_k\|^2\,(t-s)\left[\left(\frac{1}{\sqrt{s-r}} - \frac{1}{\sqrt{t-r}}\right) + (\lambda_k^2 + \|q\|^2)(s-r)^{1/2}\right]
\end{aligned} \tag{5.3.22}
$$

In the second last inequality, we have applied the same argument that we used to obtain (5.3.19). In the last inequality, we have used Lemma 5.3.9.

Now we put (5.3.22) into (5.3.21) and then apply Lemma 5.3.9 to obtain

$$\begin{aligned}
&\int_0^s \mathcal{E}_r^{(q)}(Q_{(r,t)}\phi_k - Q_{(r,s)}\phi_k)\,dr \\
\leq\;& \|u_0\| \bigg((1-e^{-(t-s)\lambda_k})^2(1-e^{-2s\lambda_k}) + C(d,D,T)\|q\|^2\,\|\phi_k\|^2\,(t-s)^{3/2} \\
&\quad + C(d,D,T)\|q\|^2 \big(\sup_{r\in[0,s]} \big\|Q_{(r+\theta,t)}\phi_k - Q_{(r+\theta,s)}\phi_k\big\|\big)^2 s^{3/2} \\
&\quad + C(d,D,T)\|q\| \big(\sup_{r\in[0,s]} \big\|Q_{(r+\theta,t)}\phi_k - Q_{(r+\theta,s)}\phi_k\big\|\big)^2 s \bigg) \\
\leq\;& \|u_0\|\,\lambda_k^2(t-s)^2(1\wedge 2s\lambda_k) \\
&+ C(d,D,T)\,\|u_0\|\,\|\phi_k\|^2\Big(\|q\|^2\,(t-s)^{3/2} + (\|q\|^2+1)\left(\lambda_k^2(t-s)^2 + \|q\|^2(t-s)\right)\Big) \\
\leq\;& C(d,D,T)\,\|u_0\|\,\Big(\lambda_k^2(t-s)^2 + \|\phi_k\|^2\,\|q\|^2\lambda_k^2(t-s)^2 + \|\phi_k\|^2\,(\|q\|^2+\|q\|^4)(t-s)\Big).
\end{aligned}$$

This implies (5.3.16).

Using (2.1.4) instead of (2.1.5), we see that the above arguments remain valid if we replace $Q_{r,t}$ by $Q^N_{r,t}$ and $\mathcal{E}_r^{(q)}$ by $\mathcal{E}_r^{(q),N}$. This completes the proof of the lemma. □

Remark 5.3.11. From the proof above, there exists $N_0 = N_0(D)$ such that, for $0 \leq r \leq t \leq T$ and $N \geq N_0$, inequalities (5.3.20) and (5.3.22) remain valid if we replace $Q_{r,t}$ by $Q^N_{r,t}$ and $\mathcal{E}_r^{(q)}$ by $\mathcal{E}_r^{(q),N}$. □

5.3.3 Martingales

With Λ being an empty set in Lemma 2.1.4, we obtain the following key martingales that we need for the study of $\mathfrak{X}^N$. The proof is the same as that for [17, Corollary 6.4] so it is omitted here.

Lemma 5.3.12. *Fix any positive integer N. For any $\phi \in Dom^{Feller}(\mathcal{A})$, we have under $\mathbb{P}^\mu$ for* *any* $\mu \in E_N$*,*

$$M_t^\phi := \langle \phi, \mathfrak{X}_t^N \rangle - \langle \phi, \mathfrak{X}_0^N \rangle - \int_0^t \langle \mathcal{A}\phi - q_N(s)\,\phi,\, \mathfrak{X}_s^N \rangle\, ds \quad \textit{and} \tag{5.3.23}$$

$$N_t^\phi := \langle \phi, \mathfrak{X}_t^N \rangle^2 - \langle \phi, \mathfrak{X}_0^N \rangle^2 - \int_0^t \frac{1}{N} \langle \boldsymbol{a}\nabla\phi \cdot \nabla\phi,\, \mathfrak{X}_s^N \rangle + 2\langle \phi, \mathfrak{X}_s \rangle\, \langle \mathcal{A}\phi, \mathfrak{X}_s^N \rangle - 2\langle q_N\phi, \mathfrak{X}_s^N \rangle \langle \phi, \mathfrak{X}_s^N \rangle + \frac{1}{N} \langle q_N \phi^2, \mathfrak{X}_s^N \rangle\, ds \tag{5.3.24}$$

are $\mathcal{F}_t^{\mathfrak{X}^N}$*-martingales under* $\mathbb{P}^\mu$ *for any* $\mu \in E_N$*. Moreover,* M_t^ϕ *has predictable quadratic variation*

$$\langle M^\phi \rangle_t = \frac{1}{N} \int_0^t \langle \boldsymbol{a}\nabla\phi \cdot \nabla\phi + q_N(s)\phi^2,\, \mathfrak{X}_s^N \rangle\, ds. \tag{5.3.25}$$

From (5.3.25), (2.1.4) and Lemma 5.3.1, we have for all $T > 0$,

$$\mathbb{E}^\mu[(M_t^\phi)^2] \leq \frac{1}{N} \Big(8(\|\phi\|^2 + \|\mathcal{A}\phi\|^2\, t^2) + \|\phi^2\, q\|\, C(d, D, T) t^{1/2} \Big) \quad \text{for } t \in [0, T]. \tag{5.3.26}$$

5.4 Non-equilibrium fluctuations

In this section, we present the proof of Theorem 5.1.5, the main result of this chapter. Throughout this section, Assumptions 5.1.1 and 5.1.2 (Killing potential) are in force. The initial distributions of the particles are assumed to be i.i.d with distribution $u_0(x)\rho(x)dx$ for some $u_0 \in C(\overline{D})$.

5.4.1 Langevin equation

This subsection represents Step 1 towards the proof of Theorem 5.1.5 mentioned at the end of the Introduction. Recall that $\mathcal{Y}_t^N$ is the fluctuation process defined by (5.1.4). We first answer question (i) raised at the beginning of this chapter.

Lemma 5.4.1. *Whenever* $\alpha > d/2$*, we have* $\mathcal{Y}_t^N \in \mathcal{H}_{-\alpha}$ *for* $t > 0$ *and* $N \geq 1$*.*

Proof Since our system is an i.i.d. sequence of sub-processes $Z_i^{(N)}$ (see Remark 5.1.3), we easily obtain

$$\mathbb{E}\left[\langle \mathcal{Y}_t^N,\, \phi\rangle^2\right] = \mathrm{Var}(\phi(Z_i^{(N)})) \leq \mathbb{E}\left[\phi(Z_1^N(t))^2\right] \leq \langle P_t\phi^2,\, u_0\rangle \leq \|u_0\|\, \langle \phi^2,\, 1\rangle. \tag{5.4.1}$$

Hence for $\alpha > d/2$ and $t \geq 0$, by (5.2.3) and (5.2.5),

$$\mathbb{E}\left[|\mathcal{Y}_t^N|_{-\alpha}^2\right] = \sum_k \mathbb{E}\left[\langle \mathcal{Y}_t^N,\, h_k^{(\alpha)}\rangle^2\right] \leq \|u_0\| \sum_k (1+\lambda_k)^{-\alpha} < \infty. \tag{5.4.2}$$

Then $\mathcal{Y}_t^N \in \mathcal{H}_{-\alpha}$ a.s. □

Remark 5.4.2. *The condition $\alpha > d/2$ in the above lemma is sharp since, in view of* (5.2.5), $\mathbb{E}\left[|\mathcal{Y}_t^N|_0^2\right] = \infty$ *when $u_0 = 1$, $q = 0$ and $\alpha \leq d/2$.* □

Unlike $\mathbf{U}_{(t,s)}$, we can check that $\{\mathbf{U}_{(t,s)}^N\}_{t\geq s}$ is a strongly continuous evolution system on $\mathcal{H}_\gamma$ with generator $\{\mathbf{A}^{(\gamma)} + \mathbf{B}_t^{(N)}\}_{t\geq 0}$, where $\mathbf{A}^{(\gamma)}\mu(\phi) = \mu(\mathcal{A}\phi)$ and $\mathbf{B}_t^{(N)}\mu(\phi) = \mu(q_N(t)\phi)$; see [24]. Using the fact that $\mathcal{A}\phi_\ell = -\lambda_k\, \phi_k$, we have $\left|\mathbf{A}^{(\gamma)}\mu\right|_\gamma^2 = \sum_k (1+\lambda_k)^\gamma \lambda_k^2 \langle \mu, \phi_k\rangle^2$, which is finite if and only if (by Weyl's law) $|\mu|_{\gamma+2}^2$ is finite. Hence $Dom(\mathbf{A}^{(\gamma)}) = \mathcal{H}_{\gamma+2}$. Since q_N is bounded for each fixed N, we have, as operators on $\mathcal{H}_\gamma$,

$$Dom(\mathbf{A}^{(\gamma)} + \mathbf{B}_t^{(N)}) = Dom(\mathbf{A}^{(\gamma)}) = \mathcal{H}_{\gamma+2} \quad \text{for all } N \geq 1. \tag{5.4.3}$$

Moreover,

$$\left|\mathbf{U}_{(t,s)}^N \mu\right|_\gamma^2 \leq e^{(t-s)\beta_N} |\mu|_\gamma^2 \quad \text{for some } \beta_N > 0. \tag{5.4.4}$$

The next result says that $\mathcal{Y}^N$ solves a **stochastic evolution equation** in $\mathcal{H}_{-\alpha}$

$$dY_t = (\mathbf{A}^{(-\alpha)} + \mathbf{B}_t^{(N)})Y_t\, dt + dM_t^N, \quad Y_0 = \mathcal{Y}_0^N. \tag{5.4.5}$$

Theorem 5.4.3. *Suppose $\alpha > d \vee (d/2+1)$. For large enough N, there exists a r.c.l.l. square-integrable $\mathcal{H}_{-\alpha}$-valued martingale $M^N = (M_t^N)_{t\geq 0}$ such that $\mathcal{Y}^N$ satisfies the following two equivalent statements:*

(i) ***(Weak solution)*** *For any $\phi \in \mathcal{H}_{-\alpha+2}$ and $t \geq s \geq 0$, we have $\mathbb{P}$-a.s.*

$$\langle \mathcal{Y}_t^N, \phi\rangle_{-\alpha} = \langle \mathcal{Y}_s^N, \phi\rangle_{-\alpha} + \int_s^t \langle (\mathbf{A}^{(-\alpha)} + \mathbf{B}_r^{(N)})\mathcal{Y}_r^N, \phi\rangle_{-\alpha}\, dr + \langle M_t^N - M_s^N, \phi\rangle_{-\alpha} \tag{5.4.6}$$

(ii) ***(Evolution solution)*** *For $t \geq s \geq 0$, we have $\mathbb{P}$-a.s.*

$$\mathcal{Y}_t^N = \mathbf{U}_{(t,s)}^N \mathcal{Y}_s^N + \int_s^t \mathbf{U}_{(t,r)}^N\, dM_r^N \qquad \text{in } \mathcal{H}_{-\alpha}. \tag{5.4.7}$$

Moreover, M^N has bounded jumps and, for every $\phi \in \mathcal{H}_\alpha$, $M^N(\phi)$ is a real-valued square-integrable martingale with $M_t^N(\phi) - M_{t-}^N(\phi) = \langle \mathfrak{X}_t^N - \mathfrak{X}_{t-}^N, \phi\rangle$ and predictable quadratic variation

$$\langle M^N(\phi)\rangle_t = \int_0^t \langle \boldsymbol{a}\nabla\phi \cdot \nabla\phi \,+\, q_N(s)\phi^2,\ \mathfrak{X}_s^N\rangle\, ds. \tag{5.4.8}$$

Remark 5.4.4. Here $\int_0^{\cdot} \mathbf{U}_{(t,s)}^N\, dM_s^N$ is the stochastic integral of the operator-valued function $s \mapsto \mathbf{U}_{(t,s)}^N$ with respect to M^N on $[0,t]$. Its construction and its basic properties can be found in the monograph [62] of M. Metivier and J. Pellaumail (See also the book by P. Protter [66] for a more recent and comprehensive treatment for stochastic integration which used the same approach). Be aware that $t \mapsto \int_0^t \mathbf{U}_{(t,s)}^N\, dM_s^N$ is *not* a martingale. However, since M^N has a r.c.l.l. version and by (5.4.4), we have $\int_0^t \mathbf{U}_{(t,s)}^N\, dM_s^N$ has a r.c.l.l. version by the submartingale type inequality of Kotelenez (cf. [52]). □

Proof (i) and (ii) assert that $\mathcal{Y}^N$ is a **weak solution** and an **evolution solution** of (5.4.5), respectively. Since $Dom(\mathbf{A}^{(-\alpha)}) = \mathcal{H}_{-\alpha+2}$ is dense in $\mathcal{H}_{-\alpha}$, these two notion of solutions are equivalent by variation of constant (see Section 2.1.2 of [43]). So it suffices to prove (i).

By Lemma 5.3.12, for every $\phi \in Dom^{Feller}(\mathcal{A})$,

$$\langle \mathcal{Y}_t^N, \phi\rangle = \langle \mathcal{Y}_0^N, \phi\rangle + \int_0^t \langle \mathcal{Y}_s^N, \mathcal{A}\phi - q_N(s)\phi\rangle\, ds + M_t^N(\phi), \tag{5.4.9}$$

where $M_t^N(\phi)$ is a real valued $\mathcal{F}_t^{\mathfrak{X}^N}$-martingale with quadratic variation given by (5.4.8)

Note that in view of (5.2.6), each eigenfunction ϕ_k is bounded and continuous on $\overline{D}$ and hence is in the Feller generator of $\mathcal{A}$. By Doob's inequality, (5.2.3), (5.4.8) and the fact that $\mathbb{E}\langle \phi, \mathfrak{X}_s^N \rangle \leq \langle P_s|\phi|,\, u_0\rangle$, we have

$$
\begin{aligned}
& \sum_k \mathbb{E}\left[\sup_{[0,T]} \Big(M_t^N(h_k^{(\alpha)}) \Big)^2\right] \\
\leq \;& C(T) \sum_k \int_0^T \mathbb{E}\left[\langle \mathbf{a}\nabla h_k^{(\alpha)} \cdot \nabla h_k^{(\alpha)} \;+\; q_N(s) \left(h_k^{(\alpha)}\right)^2,\; \mathfrak{X}_s^N\rangle\right] \, ds \\
= \;& C(T) \sum_k (1+\lambda_k)^{-\alpha} \int_0^T \langle \mathbf{a}\nabla \phi_k \cdot \nabla \phi_k + q_N(s)\phi_k^2,\; P_s u_0\rangle_\rho \, ds.
\end{aligned}
$$

Recall that $\int_{\partial D} \phi_k(x)^2 \, \sigma(dx) \leq C(d,D)(\lambda_k+1)$ by (5.2.6). Hence

$$
\begin{aligned}
& \limsup_{N\to\infty} \sum_k \mathbb{E}\left[\sup_{[0,T]} \Big(M_t^N(h_k^{(\alpha)}) \Big)^2\right] \\
\leq \;& C(T)\, \|u_0\| T \sum_k (1+\lambda_k)^{-\alpha} \Big(\mathcal{E}(\phi_k) + C(d,D)\|q\|(\lambda_k+1) \Big) \\
= \;& C(T)\, \|u_0\| T \sum_k (1+\lambda_k)^{-\alpha} \Big(\lambda_k + C(d,D)\|q\|(\lambda_k+1) \Big) \\
\leq \;& C(d,D,T)\, \|u_0\|\, (1 \vee \|q\|) \sum_k \frac{1}{(1+\lambda_k)^{\alpha-1}} \qquad (5.4.10)
\end{aligned}
$$

which by (5.2.5) is finite if and only if $\alpha > d/2+1$. Hence for $\alpha > d/2+1$, there is $N_0 \geq 1$ so that for every $N \geq N_0$,

$$
c_N := \sum_k \mathbb{E}\left[\sup_{[0,T]} \Big(M_t^N(h_k^{(\alpha)}) \Big)^2\right] < \infty. \qquad (5.4.11)
$$

For $\phi \in \mathcal{H}_\alpha$, $\phi = \sum_{k=1}^\infty a_k h_k^{(\alpha)}$, where $a_k = \langle \phi, h_k^{(\alpha)}\rangle_\alpha$. Define $M_t^N(\phi) = \sum_{k=1}^\infty a_k M_t^N(h_k^{(\alpha)})$, which is well defined in view of (5.4.11). Moreover, by the Doob's maximal inequality, $M_t^N(\phi)$ is the L^2 and uniform limit in $t \in [0,T]$ of $\sum_{k=1}^j a_k M_t^N(h_k^{(\alpha)})$. Hence $M^N(\phi)$ is a real-valued r.c.l.l. square-integrable martingale with

$$\mathbb{E}\left[(M_T^N(\phi))^2\right] \le c_N \sum_{k=1}^{\infty} a_k^2 = c_N \|\phi\|_\alpha^2. \tag{5.4.12}$$

Thus $\langle M^N, \phi\rangle := M^N(\phi)$ with $\phi \in \mathcal{H}_\alpha$ determines a r.c.l.l. square-integrable $\mathcal{H}_{-\alpha}$-valued martingale M^N. On other hand,

$$\begin{aligned}
\sup_{t\in[0,\infty)} |M_t^N - M_{t-}^N|_{-\alpha}^2 &= \sup_{t\in[0,\infty)} \sum_k (1+\lambda_k)^{-\alpha} \left(M_t^N(\phi_k) - M_{t-}^N(\phi_k)\right)^2 \\
&= \sup_{t\in[0,\infty)} \sum_k (1+\lambda_k)^{-\alpha}\, N \left(\mathfrak{X}_t^N(\phi_k) - \mathfrak{X}_{t-}^N(\phi_k)\right)^2 \\
&\le \frac{1}{N} \sum_k (1+\lambda_k)^{-\alpha} \|\phi_k\|^2 \\
&\le C/N \qquad \text{by (5.2.5), (5.2.6) and the assumption } \alpha > d.
\end{aligned}$$

This in particular implies that M_t^N has bounded jumps.

Finally, since $Dom(\mathbf{A}^{(-\alpha)}) = \mathcal{H}_{-\alpha+2}$, (5.4.6) follows from (5.4.9) provided that $\alpha > d/2+1$. This completes the proof. □

5.4.2 Convergence of M^N and tightness of $\mathcal{Y}^N$

This subsection represents Step 2 and Step 3 towards the proof of Theorem 5.1.5. By Prohorov's theorem, a sequence of $\mathcal{H}_{-\alpha}$-processes $\{R_N\}$ is tight in $D([0,T], \mathcal{H}_{-\alpha})$ provided that it satisfies the two conditions below:

(1) For all $t \in [0,T]$ and $\varepsilon_0 > 0$, there exists $K > 0$ such that

$$\overline{\lim_{N\to\infty}}\, \mathbb{P}\left(|R_N(t)|_{-\alpha}^2 > K\right) < \varepsilon_0 \tag{5.4.13}$$

(2) For all $\varepsilon_0 > 0$, as $\delta \to 0$ we have

$$\overline{\lim_{N\to\infty}} \mathbb{P}\left(\sup_{\substack{|t-s|<\delta \\ 0\le s,t\le T}} \left| R_N(t) - R_N(s) \right|_{-\alpha}^2 > \varepsilon_0 \right) \to 0 \tag{5.4.14}$$

Moreover, (5.4.14) implies that any limit point has its law concentrates on $C([0,T], \mathcal{H}_{-\alpha})$. The following "weak tightness criterion" can be easily checked by using (5.4.13), (5.4.14), the Chebyshev's inequality, the metric of $\mathcal{H}_{-\alpha}$.

Lemma 5.4.5. *Suppose $\{R_N; N \geq 1\}$ is a sequence of $\mathcal{H}_{-\alpha}$-processes for some $\alpha \in \mathbb{R}$ such that for any $\varepsilon_0 > 0$,*

$$\overline{\lim_{N\to\infty}} \mathbb{P}\left(\sup_{t\in[0,T]} \sum_{|k|>K} \langle R_N(t),\, h_k^{(\alpha)} \rangle^2 > \varepsilon_0 \right) \to 0 \quad \textit{as } K \to \infty. \tag{5.4.15}$$

Then the tightness of $\{R_N\}$ in $D([0,T], \mathcal{H}_{-\alpha})$ follows from the tightness of the one-dimensional processes $\{\langle R_N,\, h_k^{(\alpha)} \rangle\}_{N\geq 1}$ (for all $k \in \mathbb{N}^d$).

The following result is Step 2 towards the proof of Theorem 5.1.5.

Theorem 5.4.6. *When $\alpha > d \vee (d/2+1)$, the square-integrable martingale M^N in Theorem 5.4.3 converges to M in distribution in $D([0,\infty), \mathcal{H}_{-\alpha})$ as $N \to \infty$, where M is the (unique in distribution) continuous $\mathcal{H}_{-\alpha}$-valued square-integrable Gaussian martingale with independent increments and covariance functional characterized by* (5.1.8).

Proof We first prove the existence and uniqueness of M. Recall the bilinear forms $\mathcal{E}_t^{(q)}$ defined by (5.1.10). Fix $\alpha > d \vee (d/2+1)$ and define a self-adjoint operator $A(t)$ on $\mathcal{H}_{-\alpha}$ by

$$\langle A(t)\varphi^*,\, \psi^* \rangle_{-\alpha} = \int_0^t \mathcal{E}_s^{(q)}(J(\varphi^*),\, J(\psi^*))\, ds, \tag{5.4.16}$$

where $J : \mathcal{H}_{-\alpha} \to \mathcal{H}_\alpha$ denote the Riesz representation, i.e. for $\varphi^* \in \mathcal{H}_{-\alpha}$ and $\psi \in \mathcal{H}_\alpha$, we have $\langle \varphi^*,\, \psi \rangle = \langle \psi,\, J(\varphi^*) \rangle_\alpha$. Then $A(t)$ is a self-adjoint compact operator on the Hilbert space $\mathcal{H}_{-\alpha}$

of finite trace because

$$\sum_k \langle A(t)h_k^{(-\alpha)},\, h_k^{(-\alpha)}\rangle_{-\alpha} = \sum_k \int_0^t \mathcal{E}_s^{(q)}(h_k^{(\alpha)},\, h_k^{(\alpha)})ds < \infty$$

by a calculation similar to (5.4.10). Moreover, $\langle A(t)\varphi^*,\, \varphi^*\rangle_{-\alpha}$ is a positive-definite quadratic functional of φ^* for every t, and is continuous and increasing in t for every φ^*.

Hence (cf. [47] for a proof using Kolmogorov's extension theorem) there is a unique (in distribution) $\mathcal{H}_{-\alpha}$-valued Gaussian process M on some probability space $(\tilde{\Omega},\, \tilde{\mathcal{F}},\, \tilde{\mathbb{P}})$ with independent increments, continuous sample paths, and characteristic functional

$$\tilde{\mathbb{E}} \exp\left(i\,\langle M_t,\, \varphi^*\rangle_{-\alpha}\right) = \exp\left(-\frac{1}{2}\langle A(t)\varphi^*,\, \varphi^*\rangle_{-\alpha}\right). \tag{5.4.17}$$

The tightness of $\{M^N\}$ and continuity of any limit are implied by Lemma 5.4.5 and (5.4.10). Hence we only need to identify any subsequential limit. Observe that $\mathbb{P}$-a.s. we have

$$\sup_{t\in[0,T]} \left|M_t^N(\phi) - M_{t-}^N(\phi)\right| = \sup_{t\in[0,T]} \sqrt{N}\left|\mathfrak{X}_t^N(\phi) - \mathfrak{X}_{t-}^N(\phi)\right| \le \frac{1}{\sqrt{N}}\, \|\phi\| \to 0 \tag{5.4.18}$$

and that by Theorem 5.1.4, the quadratic variation (5.4.8) of $M_t^N(\phi)$ converges to the deterministic quantity (5.1.8) in probability for any $t \geq 0$. These two observations imply, by a standard functional central limit theorem for semi-martingales (see, e.g., [59]), that $\{M^N(\phi)\}$ converges to $M(\phi)$ in distribution in $D([0,T],\mathbb{R})$ for any $\phi \in Dom^{Feller}(\mathcal{A})$. Finally, since $\mathcal{H}_\alpha$ has a countable dense subset in $Dom^{Feller}(\mathcal{A})$ (for example, the linear span of eigenfunctions), and since any subsequential limit of M^N is continuous in t, we know that the subsequential limit is indeed M. The proof is now complete. □

Here is Step 3 towards the proof of Theorem 5.1.5.

Theorem 5.4.7. *The sequence of processes* $\{\mathcal{Y}^N\}$ *is tight in* $D([0,T],\, \mathcal{H}_{-\alpha})$ *whenever* $\alpha > d \vee (d/2+2)$. *Moreover, any subsequential limit has a continuous version.*

Proof We first verify (5.4.15) for Y^N. By (5.4.9), we have

$$\mathbb{E}\Big[\sup_{[0,T]}\langle \mathcal{Y}_t^N,\,\phi\rangle^2\Big] \le C(T)\,\mathbb{E}\left[\langle \mathcal{Y}_0^N,\phi\rangle^2 + \int_0^T \langle \mathcal{Y}_s^N,\mathcal{A}\phi\rangle^2 ds + \Big(\int_0^T \langle \mathcal{Y}_s^N,\,q_N(s)\phi\rangle\,ds\Big)^2 + \sup_{[0,T]}\big(M_t^N(\phi)\big)^2\right].$$

Observe that we have treated the second term and the third term (which involve q_N) in the right hand side in a different way. This is because $\int_0^T \mathbb{E}\langle \mathcal{Y}_s^N,\,q_N(s)\rangle^2\,ds$ tends to infinity when q and u_0 are strictly positive. The first two terms in the right hand side can be estimated using the fact $\mathbb{E}[\langle \mathcal{Y}_s^N,\phi\rangle^2] \le \|u_0\|\,\langle \phi^2, 1\rangle$ proved in (5.4.1). The martingale term can be estimated as in (5.4.10). For the third term which involve q_N, using the fact that $(\int_s^t f(r)\,dr)^2 = 2\int_s^t\int_u^t f(u)f(v)\,dv\,du$ and (2.1.4), we can check that

$$\mathbb{E}\left[\left(\int_s^t \langle \mathcal{Y}_r^N,\,q_N(r)\phi\rangle\,dr\right)^2\right] \le C(D,T)\,\|\phi\|^2\,(t-s)^{3/2} \quad \text{for } N \ge N_0(D). \tag{5.4.19}$$

Combining the above calculations, we have

$$\varlimsup_{N\to\infty}\sum_{k>K}\mathbb{E}\left[\sup_{[0,T]}\langle \mathcal{Y}_t^N,\,h_k^{(\alpha)}\rangle^2\right] \le C(D,T)\,\|u_0\|\sum_{k>K}\frac{1+\lambda_k^2+\|\phi_k^2\|+\lambda_k}{(1+\lambda_k)^\alpha}$$

which, by (5.2.6) and Weyl's law (5.2.5), tends to 0 as $K\to\infty$, provided that $\alpha > d\vee(d/2+2)$. We conclude by Chebyshev's inequality that (5.4.15) for $\mathcal{Y}^N$ (in place of R_N) is satisfied if $\alpha > d\vee(d/2+2)$.

By Lemma 5.4.5, it remains to verify that the one-dimensional processes $\{\langle \mathcal{Y}^N,\,\phi_k\rangle; N\ge 1\}$ (for all $k\in\mathbb{N}$) are tight. Since $\mathbb{E}[\langle \mathcal{Y}_t^N,\,\phi\rangle^2] \le \|u_0\|\,\langle \phi^2,\,1\rangle$ by (5.4.1), it is enough to show that

$$\varlimsup_{N\to\infty}\mathbb{P}\left(\sup_{\substack{|t-s|<\delta\\ 0\le s,t\le T}}\big|\langle \mathcal{Y}_t^N,\,\phi_k\rangle - \langle \mathcal{Y}_s^N,\,\phi_k\rangle\big| > \varepsilon_0\right) \to 0 \quad \text{as } \delta\to 0 \tag{5.4.20}$$

for any $k\in\mathbb{N}$. Note that (5.4.20) together with (5.4.15) for $\mathcal{Y}^N$ imply that any subsequential limit of $\{\mathcal{Y}^N; N\ge 1\}$ has a continuous version. Since $\mathcal{A}\phi_k$ is uniformly bounded and $\widehat{M}_t^N(\phi_k)$ defined by (5.4.9) converge in $D([0,T],\mathbb{R})$ as $N\to\infty$ by Theorem 5.4.6, it remains to show that

$$\overline{\lim_{N\to\infty}}\, \mathbb{P}\left(\sup_{\substack{|t-s|<\delta \\ 0\le s,t\le T}} \Big| \int_s^t \langle \mathcal{Y}_r^N,\, q_N(r)\phi_k\rangle\, dr \Big| > \varepsilon_0 \right) \to 0 \quad \text{as } \delta \to 0. \tag{5.4.21}$$

For this, note that even though $\int_0^T \mathbb{E}\langle q_N(s),\, \mathcal{Y}_s^N\rangle^2\, ds$ tends to infinity when q and u_0 are strictly positive, we have

$$\mathbb{E}\left[\left(\int_s^t \langle \mathcal{Y}_r^N,\, q_N(r)\phi\rangle\, dr \right)^2 \right] \le C(T,D)\,(t-s)^{3/2} \quad \text{for } N \ge N_0(D). \tag{5.4.22}$$

This can be checked by using the fact that $(\int_s^t f(r)\,dr)^2 = 2\int_s^t \int_u^t f(u)f(v)\,dvdu$. Hence we have (5.4.21). See, for example, Problem 4.11 in Chapter 2 of [49]. □

5.4.3 Convergence of transportation part

The goal of this subsection is to prove the following result, which is Step 4 towards the proof of Theorem 5.1.5.

Theorem 5.4.8. *For $\alpha > d+2$, as $N \to \infty$*

$$\mathbf{U}^N_{(t,0)}\mathcal{Y}_0^N \xrightarrow{\mathcal{L}} \mathbf{U}_{(t,0)}\mathcal{Y}_0 \quad \textit{in } C([0,T], \mathcal{H}_{-\alpha}). \tag{5.4.23}$$

Moreover, $\mathbf{U}_{(t,0)}\mathcal{Y}_0$ has a version in $C^\gamma([0,T],\, \mathcal{H}_{-\alpha})$ for any $\gamma \in (0,1/2)$.

Proof **(i) Continuity of the limit.** We first prove that $\mathbf{U}_{(\cdot,0)}\mathcal{Y}_0$ has a version in $C^\gamma([0,T],\, \mathcal{H}_{-\alpha})$ for any $\gamma \in (0,1/2)$. Precisely, we will show that for $\alpha > d+2$ and $n \in \mathbb{N}$,

$$\mathbb{E}\left[\left| \mathbf{U}_{(t,0)}\mathcal{Y}_0 - \mathbf{U}_{(s,0)}\mathcal{Y}_0 \right|_{-\alpha}^{2n} \right] \le C\, \|u_0\|^n\, \left((t-s)^{2n} + \|q\|^{2n}(t-s)^n \right) \quad \text{whenever } 0 \le s \le t \le T, \tag{5.4.24}$$

where $C = C(n,d,D,T,\alpha) > 0$ is a constant independent of s and t. By Kolmogorov continuity criteria, (5.4.24) implies the desired Hölder continuity.

From Lemma 5.3.9, we have

$$\begin{aligned}\mathbb{E}\langle \mathbf{U}_{(t,0)}\mathcal{Y}_0 - \mathbf{U}_{(s,0)}\mathcal{Y}_0,\ \phi_k\rangle^2 &\leq \|u_0\|\,\langle\,\big(Q_{0,t}\phi_k - Q_{(0,s)}\phi_k\big)^2\,,\,1\rangle\\ &\leq \|u_0\|\,C(d,D,T)\,\|\phi_k\|^2\,\big(\lambda_k^2(t-s)^2 + \|q\|^2\,(t-s)\big)\,.\end{aligned}$$

Using the Gaussian property of $\langle \mathbf{U}_{(t,0)}\mathcal{Y}_0 - \mathbf{U}_{(s,0)}\mathcal{Y}_0,\ \phi\rangle$, the above inequality and the simple fact $(a+b)^n \leq 2^n(a^n+b^n)$, we have

$$\begin{aligned}&\mathbb{E}\left[\langle\,\mathbf{U}_{(t,0)}\mathcal{Y}_0 - \mathbf{U}_{(s,0)}\mathcal{Y}_0,\ \phi_k\,\rangle^{2n}\right] = (2n-1)!!\,\left(\mathbb{E}\left[\langle \mathbf{U}_{(t,0)}\mathcal{Y}_0 - \mathbf{U}_{(s,0)}\mathcal{Y}_0,\ \phi\rangle^2\right]\right)^n\\ \leq\ &(2n-1)!!\,2^n\,C^n(d,D,T)\,\|u_0\|^n\,\|\phi_k\|^{2n}\,\big(\lambda_k^{2n}(t-s)^{2n} + \|q\|^{2n}\,(t-s)^n\big)\,.\end{aligned}$$

Therefore, using Hölder inequality $(\sum_i a_ib_i)^n \leq (\sum_i a_i^{n/(n-1)})^{n-1}(\sum_i b_i^n)$ for non-negative numbers a_i and b_i, we have for any $\beta \in (0,\alpha]$,

$$\begin{aligned}&\mathbb{E}\left[\big|\mathbf{U}_t\mathcal{Y}_0 - \mathbf{U}_s\mathcal{Y}_0\big|_{-\alpha}^{2n}\right]\\ =\ &\mathbb{E}\left[\Big(\sum_k (1+\lambda_k)^{-\alpha}\,\langle \mathbf{U}_t\mathcal{Y}_0 - \mathbf{U}_s\mathcal{Y}_0,\ \phi_k\rangle^2\Big)^n\right]\\ \leq\ &\left(\sum_k (1+\lambda_k)^{-\frac{\beta n}{n-1}}\right)^{n-1}\left(\sum_k (1+\lambda_k)^{-(\alpha-\beta)n}\,\mathbb{E}\left[\langle \mathbf{U}_t\mathcal{Y}_0 - \mathbf{U}_s\mathcal{Y}_0,\ \phi_k\rangle^{2n}\right]\right)\\ \leq\ &C(n,d,D,T)\,\|u_0\|^n\left(\sum_k \frac{1}{(1+\lambda_k)^{\frac{\beta n}{n-1}}}\right)^{n-1}\\ &\quad\cdot\left((t-s)^{2n}\sum_k \frac{\|\phi_k\|^{2n}\,\lambda_k^{2n}}{(1+\lambda_k)^{(\alpha-\beta)n}} + \|q\|^{2n}\,(t-s)^n\sum_k \frac{\|\phi_k\|^{2n}}{(1+\lambda_k)^{(\alpha-\beta)n}}\right).\end{aligned}$$

From (5.2.6), it follows that (5.4.25) holds true once we choose $\beta \in \left(\frac{d(n-1)}{2n},\ \alpha - \frac{d}{2} - 2 - \frac{d}{2n}\right)$. This choice of β is possible if and only if $\alpha > d+2$. Hence the proof of (5.4.24) is complete.

(ii) Tightness. Next, we show that $\{\mathbf{U}^N_{(\cdot,0)}\mathcal{Y}^N_0\}$ is tight in $C([0,T],\mathcal{H}_{-\alpha})$. Let $\psi = Q^N_{0,t}\phi_k -$

$Q^N_{0,s}\phi_k$ and $\{x_i\}_{i=1}^N$ be i.i.d. with distribution $u_0(x)\rho(x)dx$. Then

$$\begin{aligned}
&\mathbb{E}\langle \mathbf{U}^N_{(t,0)}\mathcal{Y}^N_0 - \mathbf{U}^N_{(s,0)}\mathcal{Y}^N_0,\, \phi_k\rangle^4 = \mathbb{E}\langle \mathcal{Y}^N_0,\, \psi\rangle^4 \\
= \;& N^2\,\mathbb{E}\Big[\Big(\frac{1}{N}\sum_{i=1}^N \Big(\psi(x_i)-\mu_\psi\Big)\Big)^4\Big] \qquad \text{(where } \mu_\psi := \langle \psi,\, u_0\rangle_\rho \text{ is the mean of each term)} \\
= \;& \frac{1}{N}\,\mathbb{E}[(\psi(x_1)-\mu_\psi)^4] + \Big(\mathbb{E}[(\psi(x_1)-\mu_\psi)^2]\Big)^2 \\
\leq \;& C(d,D,T)\,\|u_0\|^2\,\|\phi_k\|^4\,\Big(\lambda_k^4(t-s)^4 + \|q\|^4\,(t-s)^2\Big) \quad \text{(by Lemma 5.3.9).}
\end{aligned}$$

Using Hölder inequality $(\sum_i a_ib_i)^n \leq (\sum_i a_i^{n/(n-1)})^{n-1}(\sum_i b_i^n)$ as in step (i) above (with $n=2$ here), we obtain

$$\sup_{N>1}\mathbb{E}\Big[\big|\mathbf{U}^N_{(t,0)}\mathcal{Y}^N_0 - \mathbf{U}^N_{(s,0)}\mathcal{Y}^N_0\big|^4_{-\alpha}\Big] \leq C\,\|u_0\|^2\,\big((t-s)^4 + \|q\|^4(t-s)^2\big) \tag{5.4.25}$$

whenever $0\leq s\leq t\leq T$ and $\alpha > d+2$, where $C = C(\alpha,d,D,T)>0$. Inequality (5.4.25) implies the tightness we need, in view of the Kolmogorov-Centov tightness criteria (see [35, Theorem 3.8.8]).

(iii) Convergence of finite dimensional distributions. To finish the proof of Theorem 5.4.8, it remains to show that for any $n\in\mathbb{N}$ and $0\leq t_1\leq\cdots\leq t_n<\infty$, we have

$$\Big(\mathbf{U}^N_{(t_1,0)}\mathcal{Y}^N_0,\,\cdots,\,\mathbf{U}^N_{(t_n,0)}\mathcal{Y}^N_0\Big) \xrightarrow{\mathcal{L}} \big(\mathbf{U}_{(t_1,0)}\mathcal{Y}_0,\,\cdots,\,\mathbf{U}_{(t_n,0)}\mathcal{Y}_0\big) \quad \text{in } (\mathcal{H}_{-\alpha})^n \tag{5.4.26}$$

as $N\to\infty$.

For this, it suffices to show that for any $\psi_1,\cdots,\psi_n\in\mathcal{C}\subset C(\overline{D})$,

$$\Big(\langle\mathbf{U}^N_{(t_1,0)}\mathcal{Y}^N_0,\,\psi_1\rangle,\,\cdots,\,\langle\mathbf{U}^N_{(t_n,0)}\mathcal{Y}^N_0,\,\psi_n\rangle\Big) \xrightarrow{\mathcal{L}} \big(\langle\mathbf{U}_{(t_1,0)}\mathcal{Y}_0,\,\psi_1\rangle,\,\cdots,\,\langle\mathbf{U}_{(t_n,0)}\mathcal{Y}_0,\,\psi_n\rangle\big) \quad \text{in } \mathbb{R}^n, \tag{5.4.27}$$

where $\mathcal{C}$ denotes the linear span of the eigenfunctions $\{\phi_k\}$. This is because $\mathcal{C}$ is dense in $\mathcal{H}_\alpha$ and the Borel σ-field in $(\mathcal{H}_{-\alpha})^n$ is generated by the finite dimensional sets.

We first prove (5.4.27) when $n=1$. For notational simplicity, write t and ψ for t_1 and ψ_1.

Note that $\left\{\langle \mathbf{U}^N_{(t,0)}\mathcal{Y}^N_0,\, \psi\rangle\right\}$ is tight in $\mathbb{R}$ since by (5.3.9),

$$\sup_N \mathbb{E}\left[\langle \mathbf{U}^N_t \mathcal{Y}^N_0,\, \psi\rangle^2\right] = \sup_N \langle \left(Q^N_{0,t}\psi\right)^2,\, u_0\rangle - \langle Q^N_{0,t}\psi,\, u_0\rangle^2 \leq \|u_0\| \sup_N \langle (Q^N_{0,t}\psi)^2,\, 1\rangle < \infty.$$

Suppose Z is a subsequential limit of $\langle \mathbf{U}^N_{(t,0)}\mathcal{Y}^N_0,\, \psi\rangle$. We claim that $Z \overset{\mathcal{L}}{=} \langle \mathbf{U}_{(t,0)}\mathcal{Y}_0,\, \psi\rangle$. This is due to the following two facts: $\langle \mathcal{Y}^N_0,\, Q_{0,t}\psi\rangle \xrightarrow{\mathcal{L}} \langle \mathcal{Y}_0,\, Q_{0,t}\psi\rangle$ (by the standard central limit theorem) and $\lim_{N\to 0} \mathbb{E}\big|\langle \mathcal{Y}^N_0,\, Q^N_{0,t}\psi\rangle - \langle \mathcal{Y}^N_0,\, Q_{0,t}\psi\rangle\big| = 0$, which follows from

$$\begin{aligned}
\mathbb{E}\left[\big|\langle \mathcal{Y}^N_0,\, Q^N_{0,t}\psi\rangle - \langle \mathcal{Y}^N_0,\, Q_{0,t}\psi\rangle\big|^2\right] &= \langle (Q^N_{0,t}\phi - Q_{0,t}\psi)^2,\, u_0\rangle - \langle Q^N_{0,t}\psi - Q_{0,t}\psi,\, u_0\rangle^2 \\
&\leq \|u_0\|\, \langle (Q^N_{0,t}\psi - Q_{0,t}\psi)^2,\, 1\rangle \to 0 \quad \text{by (5.3.8).}
\end{aligned}$$

In fact, the second fact implies that $\langle \mathcal{Y}^{N'}_0,\, Q^{N'}_{0,t}\psi\rangle - \langle \mathcal{Y}^{N'}_0,\, Q_{0,t}\psi\rangle \to 0$ a.s. along some subsequence N', and so by the Lebesgue dominated convergence theorem, $\mathbb{E}F(\langle \mathcal{Y}^{N'}_0,\, Q^{N'}_{0,t}\psi\rangle) - \mathbb{E}F(\langle \mathcal{Y}^{N'}_0,\, Q_{0,t}\psi\rangle) \to 0$ for any bounded continuous function F.

The proof of (5.4.27) for general $n \in \mathbb{N}$ is the same as that for $n = 1$, using the standard multidimensional central limit theorem. So we get the desired (5.4.26).

The proof of Theorem 5.4.8 is now complete. □

5.4.4 $\int_0^t \mathbf{U}_{(t,s)} dM_s$ is well defined

As mentioned earlier, we have to make sure that $\mathbf{U}_{(t,s)}$ (for $s \in [0,t]$) lies within the class of integrands with respect to M. We will follow the construction of stochastic integrals with respect to Hilbert space valued r.c.l.l. square-integrable martingales in [62]. See [25, 43, 66] for more comprehensive and recent treatments.

We denote by $M^2_c([0,\infty), \mathcal{H}_{-\alpha})$ the class of continuous square-integrable $\mathcal{H}_{-\alpha}$-valued martingales with zero initial value. Fix $\alpha > d \vee (d/2+1)$ and recall from Theorem 5.4.6 that $M \in M^2_c([0,\infty), \mathcal{H}_{-\alpha})$ is Gaussian, has independent increments and covariance

$$\tilde{\mathbb{E}}\left[\langle M_s, \phi\rangle \langle M_t, \psi\rangle\right] = \int_0^{s\wedge t} \mathcal{E}^{(q)}_r(\phi,\psi)\, dr, \tag{5.4.28}$$

where $\mathcal{E}_r^{(q)}$ is the bilinear form on $\mathcal{H}_{-\alpha}$ defined in (5.1.10). We will omit the filtration when there is no ambiguity. For example, we simply say that M is adapted rather than $\tilde{\mathcal{F}}_t$-adapted since it is defined on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_t, \tilde{\mathbb{P}})$. For $T \in (0,\infty]$, denote by $\mathfrak{P}_{[0,T]}$ the σ-field of predictable sets on $\tilde{\Omega} \times [0,T]$. That is, the smallest σ-field making all adapted processes with left continuous paths measurable (c.f. p.156 of [66] or section 1.7 [62]). When $T = \infty$, we write $\mathfrak{P}$ for $\mathfrak{P}_{[0,\infty)}$.

By a direct calculation,

$$[[M]]_t := \sum_k \int_0^t \mathcal{E}_r^{(q)}(h_k^{(\alpha)})\, dr \tag{5.4.29}$$

is the unique continuous, adapted and increasing real process such that $\left|M_t\right|_{-\alpha}^2 - [[M]]_t$ is a real martingale (cf. Remark 2.2 in [43]). $[[M]]$ is called the **real increasing process** associated to M. Besides, the operators $Q_s : H_{-\alpha} \to H_{-\alpha}$ (for $s \geq 0$) defined by

$$\big\langle Q_s(h_i^{(-\alpha)}),\, h_j^{(-\alpha)}\big\rangle_{-\alpha} := \frac{\mathcal{E}_s^{(q)}(h_i^{(\alpha)},\, h_j^{(\alpha)})}{\sum_k \mathcal{E}_s^{(q)}(h_k^{(\alpha)})} \tag{5.4.30}$$

is called the **characteristic operator process** associated to M. Clearly, Q_s is a non-negative operator on $H_{-\alpha}$ with $Tr(A) = 1$ where 'Tr' means 'Trace'. As a remark, the operator-valued process $\langle\langle M\rangle\rangle_t := \int_0^t Q_s\, d[[M]]s$ (in the sense of Bochner's integral) is called the **operator increasing process associated to** M and plays an analogous role as the quadratic variation of real-valued martingales (see Theorem 2.3 in Chapter 1 of [43] for its basic properties).

Following [62], the class of possible integrands for the stochastic integral with M_t as integrator (on the interval $[0,T]$) can be defined as follows: On the space of $\mathfrak{P}_{[0,T]}$-simple $L(\mathcal{H}_{-\alpha})$-valued processes, we define a scalar product

$$(A,B) := \tilde{E}\left[\int_0^T Tr(A\, Q_s\, B^*)\, d[[M]]_s\right], \tag{5.4.31}$$

where B^* is the adjoint of the operator B. The completion of the $\mathfrak{P}_{[0,T]}$-simple $L(\mathcal{H}_{-\alpha})$-valued processes with respect to the scalar product in (5.4.31), denoted by $\Lambda^2(\mathcal{H}_{-\alpha}, \mathfrak{P}_{[0,T]}, M)$, is the desired class of integrands. It is worth noting that (c.f. p.171 [62]) $\Lambda^2(\mathcal{H}_{-\alpha}, \mathfrak{P}_{[0,T]}, M)$ contains processes whose values may be unbounded operators.

By section 1.3 of [43], $\Lambda^2(\mathcal{H}_{-\alpha}, \mathfrak{P}_{[0,T]}, M)$ contains the class of all processes $(\Phi_t)_{t\in[0,T]}$ such that

(i) Φ_t is a linear operator (not necessarily bounded) from $\sqrt{Q_t}\,\mathcal{H}_{-\alpha}$ to $\mathcal{H}_{-\alpha}$ such that $\Phi_t\sqrt{Q_t} \in L_2(\mathcal{H}_{-\alpha})$ is Hilbert-Schmidt for all $t \in [0,T]$ a.s.,

(ii) $\Phi_t\sqrt{Q_t}$ is $\mathfrak{P}|_{\tilde{\Omega}\times[0,T]}$-measurable (i.e. predictable), and

(iii) $\mathbb{E}\left[\int_0^T \|\Phi_t\sqrt{Q_t}\|_2^2\, d[[M]]_t\right] < \infty$ where $\|\cdot\|_2$ is the Hilbert-Schmidt norm.

Now for any $t > 0$, the deterministic process $(U_{(t,\theta)})_{\theta\in[0,t]}$ lies in the class of integrands with respect to M. This is because on one hand

$$\begin{aligned}
\|U_{(t,\theta)}\sqrt{Q_\theta}\|_2^2 &= Tr\left(U_{(t,\theta)}Q_\theta U^*_{(t,\theta)}\right) \qquad \text{the trace of } U_{(t,\theta)}Q_\theta U^*_{(t,\theta)}\\
&= \sum_k \langle U_{(t,\theta)}Q_\theta U^*_{(t,\theta)}(h_k^{(-\alpha)}),\, h_k^{(-\alpha)}\rangle_{-\alpha}\\
&= \sum_k \langle Q_\theta U^*_{(t,\theta)}(h_k^{(-\alpha)}),\, U^*_{(t,\theta)}(h_k^{(-\alpha)})\rangle_{-\alpha}\\
&= \frac{\sum_k \mathcal{E}_\theta^{(q)}(Q_{(\theta,t)}h_k^{(\alpha)})}{\sum_i \mathcal{E}_\theta^{(q)}(h_i^{(\alpha)})}
\end{aligned}$$

which is finite provided that u_0 is not identically zero; and on the other hand, by Lemma 5.3.10,

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t \|U_{(t,\theta)}\sqrt{Q_\theta}\|_2^2\, d[[M]]_\theta\right]\\
=\;& \sum_k \int_0^t \mathcal{E}_\theta^{(q)}(Q_{(\theta,t)}h_k^{(\alpha)})\, d\theta\\
\leq\;& C(d,D,T)\,\|u_0\| \left(t\sum_k \frac{\lambda_k + \|\phi_k^2\|}{(1+\lambda_k)^\alpha} + \|q\|\, t^{3/2}\sum_k \frac{\|\phi_k\|^2}{(1+\lambda_k)^\alpha}\right) \quad \text{for } t\in[0,T]\\
<\;& \infty \quad \text{if } \alpha > d \vee (d/2+1).
\end{aligned}$$

We conclude that for any fixed $t \geq 0$, $\left\{\int_0^s U_{(t,\theta)}\, dM_\theta; s\in[0,t]\right\}$ is a continuous, adapted square-integrable $\mathcal{H}_{-\alpha}$-valued martingale with $\mathbb{E}\left[\left|\int_0^s U_{(t,\theta)}\, dM_\theta\right|_{-\alpha}^2\right] = \sum_k \int_0^s \mathcal{E}_\theta^{(q)}(Q_{(\theta,t)}h_k^{(\alpha)})\, d\theta$.

In particular, putting $s = t$, we have that $\int_0^t U_{(t,\theta)}\, dM_\theta$ is a well defined $\tilde{\mathcal{F}}_t$-measurable $\mathcal{H}_{-\alpha}$-valued random variable with finite second moment. Moreover, since M is centered Gaussian with independent increments, $\int_0^t U_{(t,\theta)}\, dM_\theta$ is also centered Gaussian.

5.4.5 Convergence of stochastic integrals

Our goal in this subsection is to prove the following result, which corresponds to Step 5 towards the proof of Theorem 5.1.5.

Theorem 5.4.9. *For $\alpha > d+2$ and $T > 0$, as $N \to \infty$*

$$\int_0^t \mathbf{U}^N_{(t,s)} dM^N_s \xrightarrow{\mathcal{L}} \int_0^t \mathbf{U}_{(t,s)} dM_s \quad \text{in } D([0,T], \mathcal{H}_{-\alpha}). \tag{5.4.32}$$

Moreover, $\int_0^t \mathbf{U}_{(t,s)} dM_s$ has a version in $C^\gamma([0,T],\, \mathcal{H}_{-\alpha})$ for any $\gamma \in (0, 1/2)$.

First, we need the following lemma which is the key for establishing finite dimensional convergence. Lemma 5.3.10 also plays a crucial role in the proof of Theorem 5.4.9. Recall from (5.3.13) and (5.3.14) that

$$\begin{aligned} \mathcal{E}^{(q)}_r(\phi, \psi) &:= \langle \mathbf{a}\nabla\phi \cdot \nabla\psi,\, u(s)\rangle_\rho + \int_{\partial D} \phi\, \psi\, u(s)\, q(s)\, \rho\, d\sigma,\ \mathcal{E}^{(q)}_r(\phi) := \mathcal{E}^{(q)}_r(\phi,\phi), \text{ and} \\ \mathcal{E}^{(q),N}_s(\phi,\psi) &:= \langle \mathbf{a}\nabla\phi\cdot\nabla\psi + q_N(s)\phi\psi,\, \mathfrak{X}^N_s\rangle,\ \mathcal{E}^{(q),N}_s(\phi) := \mathcal{E}^{(q),N}_s(\phi,\phi). \end{aligned}$$

Lemma 5.4.10. *For $0 \le a \le b \le T$, $i = \sqrt{-1}$ and $\phi \in C(\overline{D})$, as $N \to \infty$, we have*

$$\mathbb{E}\Big[\exp\Big(i\Big\langle \int_a^b \mathbf{U}^N_{(T,s)} dM^N_s,\, \phi\Big\rangle\Big)\Big|\mathcal{F}^N_a\Big] \quad \text{converges in } L^1(\mathbb{P}) \text{ to}$$
$$\exp\Big(-\frac{1}{2}\int_a^b \mathcal{E}^{(q)}_s(Q_{(s,T)}\phi)ds\Big) = \mathbb{E}\Big[\exp\Big(i\Big\langle \int_a^b \mathbf{U}_{(T,s)} dM_s,\, \phi\Big\rangle\Big)\Big].$$

Proof **(i)** Fix $T > 0$ and $\phi \in \mathcal{H}_\alpha$. Then

$$K_t = K^N_t := \Big\langle \int_0^t \mathbf{U}^N_{(T,s)} dM^N_s,\, \phi\Big\rangle \quad \text{is a martingale for } t \in [0,T].$$

Let $\triangle K_r := K_r - K_{r-}$ denote the jump of K at time r. Then by (5.4.18) and (5.3.9),

$$\sup_{r\in[0,T]} |\triangle K_r| \leq \sup_{s\in[0,T]} \|Q^N_{s,T}\phi\|/\sqrt{N} \leq \|\phi\|/\sqrt{N}. \tag{5.4.33}$$

Moreover, by Theorem 5.4.3, the dual predictable projection $\langle K\rangle$ of the quadratic variation $[K]$ of K is

$$\langle K\rangle_t = \int_0^t \mathcal{E}^{(q),N}_s(Q^N_{(s,T)}\phi)ds \qquad \text{for } t\in[0,T], \tag{5.4.34}$$

By a similar argument as that for $H_N(t)$ in the proof of Lemma 5.3.3 (using an inequality in Remark 5.3.11), we have $\limsup_{N\to\infty}\mathbb{E}\big[\langle K\rangle^k_T\big] < \infty$ for every integer $k\geq 1$. Observe that $n_t := [K]_t - \langle K\rangle_t$ is a purely discontinuous martingale with jumps $\Delta n_t := n_t - n_{t-} = (\Delta K_t)^2$. It follows from (5.4.33) that $\mathbb{E}[n^2_T] = \mathbb{E}\left[\sum_{0<s\leq T}(\Delta n_s)^2\right] \leq \|\phi\|^2 N^{-1}\,\mathbb{E}[K]_T = \|\phi\|^2 N^{-1}\,\mathbb{E}\langle K\rangle_T$. Hence

$$\mathbb{E}\left[[K]^2_T\right] = \mathbb{E}\left[(\langle K\rangle_T + n_T)^2\right] \leq 2\mathbb{E}\left[\langle K\rangle^2_T + n^2_T\right] \leq 2\mathbb{E}\langle K\rangle_T + 2\|\phi\|^2 N^{-1}\,\mathbb{E}\langle K\rangle_T \tag{5.4.35}$$

which is uniformly bounded in N, by Lemma 5.3.10.

Let $f(r) := e^{ir}$, $g(r) := \mathcal{E}^{(q)}_r(Q_{r,T}\phi)$, and $g_N(r) := \mathcal{E}^{(q),N}_r(Q^N_{r,T}\phi)$. Fix $a\in[0,T]$, and set $h_N(t) := \mathbb{E}\Big[f(K_t - K_a)|\mathcal{F}^N_a\Big]$ and $h(t) := \exp\Big(-\frac{1}{2}\int_a^t g(r)dr\Big)$. Note that $h(t) = 1 - \frac{1}{2}\int_a^t h(r)g(r)dr$. We claim that

$$h_N(t) = 1 - \frac{1}{2}\int_a^t h_N(r)g(r)dr + \varepsilon_N(t) \;\text{ with }\; \sup_{t\in[a,T]}|\varepsilon_N(t)| \to 0 \;\text{ in }\; L^1(\mathbb{P}). \tag{5.4.36}$$

By Gronwall's inequality, the above equations yield

$$|h_N(t) - h(t)| \leq \left(\sup_{t\in[a,T]}|\varepsilon_N(t)|\right)\exp\left(\frac{1}{2}\int_a^t g(r)dr\right)$$

and hence $h_N(t)\to h(t)$ in $L^1(\mathbb{P})$ as $N\to\infty$. On other hand, since M is Gaussian with independent increment, $\big\langle \int_a^b \mathbf{U}_{(c,s)}dM_s,\,\phi\big\rangle$ is Gaussian with variance $\int_a^b \mathcal{E}^{(q)}_s(Q_{(s,T)}\phi)ds$ (see subsection

5.4.4 in the Appendix). Thus we have $\exp\left(-\frac{1}{2}\int_a^b \mathcal{E}_s^{(q)}(Q_{(s,T)}\phi)ds\right) = \mathbb{E}\left[\exp\left(i\langle \int_a^b \mathbf{U}_{(T,s)}dM_s,\, \phi\rangle\right)\right]$. This proves Lemma 5.4.10 once the claim (5.4.36) is verified. We now prove (5.4.36) in the next two steps.

(ii) By Itó's formula (see, e.g., Theorem 36 in [66, Chapter II]),

$$\begin{aligned} f(K_t) &= 1 + \int_{0+}^t f'(K_{r-})dK_r + \frac{1}{2}\int_{0+}^t f''(K_{r-})d[K]_r \\ &\quad + \sum_{0<r\le t}\left(f(K_r) - f(K_{r-}) - f'(K_{r-})\triangle K_r - \frac{1}{2}f''(K_{r-})(\triangle K_r)^2\right). \end{aligned} \tag{5.4.37}$$

Hence for $t \in [a, T]$,

$$\begin{aligned} \mathbb{E}\left[f(K_t)\middle|\mathcal{F}_a^N\right] &= f(K_a) + \mathbb{E}\left[\frac{1}{2}\int_{a+}^t f''(K_{r-})d[K]_r\middle|\mathcal{F}_a^N\right] \\ &\quad + \mathbb{E}\left[\sum_{0<r\le t}\left(f(K_r) - f(K_{r-}) - f'(K_{r-})\triangle K_r - \frac{1}{2}f''(K_{r-})(\triangle K_r)^2\right)\middle|\mathcal{F}_a^N\right] \\ &= f(K_a) - \frac{1}{2}\mathbb{E}\left[\int_{a+}^t f(K_{r-})g(r)dr\middle|\mathcal{F}_a^N\right] + \varepsilon_N^{(1)}(t) + \varepsilon_N^{(2)}(t), \end{aligned}$$

where

$$\begin{aligned} \varepsilon_N^{(1)}(t) &:= \frac{1}{2}\mathbb{E}\left[\int_{a+}^t f(K_{r-})(g(r) - g_N(r))dr\middle|\mathcal{F}_a^N\right] \quad \text{and} \\ \varepsilon_N^{(2)}(t) &:= \mathbb{E}\left[\sum_{0<r\le t}\left(f(K_r) - f(K_{r-}) - f'(K_{r-})\triangle K_r - \frac{1}{2}f''(K_{r-})(\triangle K_r)^2\right)\middle|\mathcal{F}_a^N\right]. \end{aligned}$$

We have used (5.4.34) and the fact that $f'' = -f$ in the last equality.

Dividing both sides by $f(K_a)$, the above calculations give

$$h_N(t) = 1 - \frac{1}{2}\int_{a+}^t h_N(r)g(r)dr + \frac{\varepsilon_N^{(1)}(t) + \varepsilon_N^{(2)}(t)}{f(K_a)}. \tag{5.4.38}$$

Since $|f| = 1$ and $|e^{ia} - 1 - ia + a^2/2| \leq |a|^3/6$, we have by (5.4.33)

$$\left|\varepsilon_N^{(2)}(t)\right| \leq \frac{1}{6}\,\mathbb{E}\Big[\sum_{0<r\leq T} |\triangle K_r|^3 \Big| \mathcal{F}_a^N\Big] \leq \frac{\|\phi\|}{6\sqrt{N}}\,\mathbb{E}\Big[\sum_{0<r\leq T} (\triangle K_r)^2 \Big| \mathcal{F}_a^N\Big] \leq \frac{\|\phi\|}{6\sqrt{N}}\,\mathbb{E}\left[[K]_T \Big| \mathcal{F}_a^N\right].$$

Since $\mathbb{E}[K]_T = \int_0^T \mathbb{E}[g_N(s)]ds \to \int_0^T g(s)ds$, we get $\lim_{N\to\infty} \mathbb{E}\left[\sup_{t\in[a,T]} |\varepsilon_N^{(2)}(t)|\right] = 0$.

For $\varepsilon_N^{(1)}$, we let $\psi(r) := Q_{(r,T)}\phi$ and $\psi_N(r) := Q^N_{(r,T)}\phi$ for simplification. Since $|f| = 1$, triangle inequality gives

$$\begin{aligned}
& 2\sup_{t\in[a,T]} \left|\varepsilon_N^{(1)}(t)\right| \\
\leq\ & \mathbb{E}\Big[\int_{a+}^T \big|\langle \mathbf{a}\nabla\psi\cdot\nabla\psi,\, u(r)\rangle_\rho - \langle \mathbf{a}\nabla\psi\cdot\nabla\psi,\, \mathfrak{X}_r^N\rangle\big|\, dr \,\Big|\, \mathcal{F}_a^N\Big] \\
& + \mathbb{E}\Big[\int_{a+}^T \big|\langle \mathbf{a}\nabla\psi\cdot\nabla\psi - \mathbf{a}\nabla\psi_N\cdot\nabla\psi_N,\, \mathfrak{X}_r^N\rangle\big|\, dr \,\Big|\, \mathcal{F}_a^N\Big] \\
& + \int_{a+}^T \Big| \int_{\partial D} \psi^2 q(r) u(r)\,\rho\, d\sigma - \langle \psi_N^2 q_N(r),\, u(r)\rangle_\rho \Big|\, dr \\
& + \sup_{t\in[a,T]} \left|\mathbb{E}\Big[\int_{a+}^t f(K_{r-})\Big(\langle \psi_N^2 q_N(r),\, u(r)\rangle_\rho - \langle \psi_N^2 q_N(r),\, \mathfrak{X}_r^N\rangle\Big)\, dr \,\Big|\, \mathcal{F}_a^N\Big]\right|. \quad (5.4.39)
\end{aligned}$$

The expectation of the first term on the right hand side of (5.4.39) tends to zero by the hydrodynamic result (Theorem 5.1.4). The expectation of the second term is at most

$$\begin{aligned}
& \mathbb{E}\int_{a+}^T \big\langle \big|\mathbf{a}\nabla\psi\cdot\nabla\psi - \mathbf{a}\nabla\psi_N\cdot\nabla\psi_N\big|,\, \mathfrak{X}_r^N\big\rangle\, dr \\
\leq\ & \int_{a+}^T \big\langle P_r\big(\big|\mathbf{a}\nabla\psi\cdot\nabla\psi - \mathbf{a}\nabla\psi_N\cdot\nabla\psi_N\big|\big),\, u_0\big\rangle_\rho\, dr \\
\leq\ & \|u_0\| \int_{a+}^T \big\langle \big|\mathbf{a}\nabla\psi\cdot\nabla\psi - \mathbf{a}\nabla\psi_N\cdot\nabla\psi_N\big|,\, 1\big\rangle_\rho\, dr \\
=\ & \|u_0\| \int_{a+}^T \big\langle \big|\mathbf{a}\nabla(\psi - \psi_N)\cdot\nabla(\psi+\psi_N)\big|,\, 1\big\rangle_\rho\, dr \quad \text{by symmetry of } \mathbf{a} \\
\leq\ & \|u_0\| \int_{a+}^T \sqrt{\mathcal{E}(\psi-\psi_N)\,\mathcal{E}(\psi+\psi_N)}\, dr \quad \text{by Cauchy-Schwarz inequality}.
\end{aligned}$$

This last quantity tends to zero as $N\to\infty$ by Lemma 5.3.7 and Lebesgue dominated convergence theorem.

The third term (which is deterministic) on the right hand side of (5.4.39) converges to zero as $N \to \infty$ by Lemma 5.3.1 and the uniform convergence (5.3.8).

(iii) It remains to show that the forth (and last) term on the right hand side of (5.4.39) converges to zero in $L^1(\mathbb{P})$. This term can be written as

$$\sup_{t\in[a,T]} \left| \mathbb{E}\Big[\int_{a+}^{t} f(K_{r-})\, dH_N(r) - \int_{a+}^{t} f(K_{r-})\, dG_N(r) \,\Big|\, \mathcal{F}_a^N \Big] \right|, \tag{5.4.40}$$

where

$$H_N(t) := \int_{a+}^{t} \langle \psi_N^2 q_N(r),\, \mathfrak{X}_r^N \rangle\, dr \quad \text{and} \quad G_N(t) := \int_{a+}^{t} \langle \psi_N^2 q_N(r),\, u(r) \rangle_\rho\, dr.$$

We have by Lemma 5.3.3

$$\lim_{N\to\infty} \mathbb{E}\left[\Big(\sup_{t\in[0,T]} |H_N(t) - G_N(t)| \Big)^p \right] = 0 \quad \text{for every } p \geq 1. \tag{5.4.41}$$

In view of (5.4.37) and (5.4.33), it suffice to show (5.4.40) converges to zero in $L^1(\mathbb{P})$ with $f(K_t)$ replaced by $\tilde{f}(K_t) := 1 + \int_{0+}^{t} f'(K_{r-}) dK_r + \frac{1}{2}\int_{0+}^{t} f''(K_{r-}) d[K]_r$. Furthermore, since $H_N(t)$ and $G_N(t)$ have bounded variations, by an integration by parts (see, e.g., Corollary 2 in [66, Chapter II]), we have

$$\begin{aligned} \int_{a+}^{t} \tilde{f}(K_{r-})\, dH_N(r) &= \tilde{f}(K_t)\, H_N(t) - \int_{a+}^{t} H_N(r)\, d\tilde{f}(K_r) \quad \text{and} \\ \int_{a+}^{t} \tilde{f}(K_{r-})\, dG_N(r) &= \tilde{f}(K_t)\, G_N(t) - \int_{a+}^{t} G_N(r)\, d\tilde{f}(K_r). \end{aligned}$$

On subtraction, it suffices to show

$$\mathbb{E}\Big[\sup_{t\in[a,T]} \big| \tilde{f}(K_t) \big(H_N(t) - G_N(t) \big) \big| \,\Big|\, \mathcal{F}_a^N \Big] \quad \text{and} \tag{5.4.42}$$

$$\sup_{t\in[a,T]} \left|\mathbb{E}\Big[\int_{a+}^{t} \Big(H_N(r) - G_N(r)\Big)\, f(K_r)\, d[K]_r \,\Big|\, \mathcal{F}_a^N\Big]\right| \tag{5.4.43}$$

both converge to zero in $L^1(\mathbb{P})$.

Since $|f| = 1$, $|e^{ia} - 1 - ia + a^2/2| \le |a|^3/6$, we have by (5.4.33) and (5.4.37),

$$\sup_{t\in[a,T]} |\tilde{f}(K_{t-})| \le \sup_{t\in[a,T]} \Big(|f(K_{t-})| + |f(K_{t-}) - \tilde{f}(K_{t-})|\Big) \le 1 + \frac{\|\phi\|\,[K]_T}{6\sqrt{N}}.$$

Hence (5.4.42) converges to zero in $L^1(\mathbb{P})$ by (5.4.35) and (5.4.41). Finally, the expectation of (5.4.43) is at most

$$\mathbb{E}\left[\sup_{t\in[a,T]} |(H_N(r) - G_N(r))|\,([K]_T - [K]_a)\right] \le \left(\mathbb{E}\left[\sup_{t\in[a,T]} (H_N(r) - G_N(r))^2\right]\right)^{1/2} \left(\mathbb{E}\left[[K]_T^2\right]\right)^{1/2},$$

which goes to 0 as $N \to \infty$ by (5.4.35) and (5.4.41). Hence by (5.4.39), $\sup_{t\in[a,T]} |\varepsilon_N^{(1)}(t)| \to 0$ in $L^1(\mathbb{P})$. We then conclude from (5.4.38) that (5.4.36) holds. The proof of the lemma is now complete. □

We can now present the proof of Theorem 5.4.9.

Proof of Theorem 5.4.9.

For notational convenience, we let $J_N(t) := \int_0^t \mathbf{U}_{(t,r)}^N \, dM_r^N$ and $J(t) := \int_0^t \mathbf{U}_{(t,r)}\, dM_r$.

(i) Continuity of the limit. In the Appendix, we checked that $J(t)$ is a well-defined $\mathcal{H}_{-\alpha}$-valued Gaussian random variable. We now prove that $J(\cdot)$ has a version in $C^\gamma([0,T], \mathcal{H}_{-\alpha})$ for any $\gamma \in (0, 1/2)$. By Kolmogorov continuity criteria, it suffices to show that for $\alpha > d + 2$ and $n \in \mathbb{N}$,

$$\mathbb{E}\Big[\,|J(t) - J(s)|_{-\alpha}^{2n}\,\Big] \le C\,(t-s)^n \quad \text{whenever } 0 \le s \le t \le T, \tag{5.4.44}$$

where $C = C(n, d, D, T, \alpha)\,\|u_0\|^n (1 \vee \|q\|)^{4n} > 0$ is a constant.

Note that for $\phi \in C(\overline{D})$,

$$\langle J(t) - J(s),\ \phi \rangle = \Big\langle \int_s^t \mathbf{U}_{(t,r)} dM_r,\ \phi \Big\rangle + \Big\langle \int_0^s \mathbf{U}_{(t,r)} - \mathbf{U}_{(s,r)}\, dM_r,\ \phi \Big\rangle,$$

which, as the sum of two independent centered Gaussian variable, is a centered Gaussian random variable with variance

$$V_s^t(\phi_k) := \int_s^t \mathcal{E}_r^{(q)}(Q_{(r,t)}\phi_k)\, dr + \int_0^s \mathcal{E}_r^{(q)}(Q_{(r,t)}\phi_k - Q_{(r,s)}\phi_k)\, dr.$$

By Lemma 5.3.10, we have

$$\begin{aligned} & \mathbb{E}\left[\langle J(t) - J(s)\,,\ \phi_k \rangle^{2n}\right] = (2n-1)!!\ \left(V_s^t(\phi_k)\right)^n \\ \leq\ & C(n,d,D,T)\, \|u_0\|^n (1 \vee \|q\|)^{4n}\, (1 \vee \lambda_k)^{2n}\, \|\phi_k\|^{2n}\, (t-s)^n \end{aligned}$$

for any $0 \leq s \leq t \leq T$ and $k \in \mathbb{N}$. Applying Hölder inequality $(\sum_i a_i b_i)^n \leq (\sum_i a_i^{n/(n-1)})^{n-1} (\sum_i b_i^n)$, we have for any $\beta \in (0, \alpha]$,

$$\begin{aligned} & \mathbb{E}\left[|J(t) - J(s)|_{-\alpha}^{2n}\right] \\ =\ & \mathbb{E}\left[\left(\sum_k (1+\lambda_k)^{-\alpha} \langle J(t) - J(s)\,,\ \phi_k \rangle^2\right)^n\right] \\ \leq\ & \left(\sum_k (1+\lambda_k)^{-\frac{\beta n}{n-1}}\right)^{n-1} \left(\sum_k (1+\lambda_k)^{-(\alpha-\beta)n}\, \mathbb{E}\left[\langle J(t) - J(s)\,,\ \phi_k \rangle^{2n}\right]\right) \\ \leq\ & C \left(\sum_k \frac{1}{(1+\lambda_k)^{\frac{\beta n}{n-1}}}\right)^{n-1} \left(\sum_k \frac{(1 \vee \lambda_k)^{2n}\, \|\phi_k\|^{2n}}{(1+\lambda_k)^{(\alpha-\beta)n}}\right) (t-s)^n. \end{aligned}$$

It follows from (5.2.6) that (5.4.45) holds true if we choose $\beta \in \left(\frac{d(n-1)}{2n},\ \alpha - \frac{d}{2} - 2 - \frac{d}{2n}\right)$. This choice of β is possible if and only if $\alpha > 2 + d$.

(ii) Tightness. We will show that there exists $N_0 = N_0(D)$ such that for $\alpha > d+2$,

$$\sup_{N>N_0} \mathbb{E}\left[|J_N(t) - J_N(s)|_{-\alpha}^4\right] \le C\,(t-s)^2 \tag{5.4.45}$$

whenever $0 \le s \le t \le T$, where $C = C(d, D, T, \alpha, \|u_0\|, \|q\|) > 0$ is a constant independent of N, s and t. By the Kolmogorov-Centov tightness criteria (see [35, Theorem 3.8.8]), (5.4.45) implies tightness of $\{J_N\}_{N\ge1}$ in $D([0,T], \mathcal{H}_{-\alpha})$.

Using Hölder inequality $(\sum_i a_ib_i)^n \le (\sum_i a_i^{n/(n-1)})^{n-1}(\sum_i b_i^n)$ (with $n=2$) and the condition $\alpha > d+2$ as in step (i) above, it suffices to show that

$$\sup_{N\ge N_0} \mathbb{E}\left[\langle\, J_N(t) - J_N(s)\,,\ \phi_k\,\rangle^4\right] \le C\,(1\vee\lambda_k)^4\,\|\phi_k\|^4\,(t-s)^2 \tag{5.4.46}$$

for any $0 \le s \le t \le T$ and $k \in \mathbb{N}$, where $N_0 = N_0(D)$ and $C = C(d, D, T, \|u_0\|, \|q\|)$.

We now prove (5.4.46) by first writing

$$J_N(t) - J_N(s) = \Big(\int_0^s \mathbf{U}^N_{(t,r)} - \mathbf{U}^N_{(s,r)}\, dM^N_r\Big) + \int_s^t \mathbf{U}^N_{(t,r)} dM^N_r.$$

Fix ϕ_k and $s \le t$. Observe that

$$\Gamma_w := \Big\langle \int_0^w \mathbf{U}^N_{(t,r)} - \mathbf{U}^N_{(s,r)}\, dM^N_r,\ \phi_k \Big\rangle$$

is a martingale for $w \in [0,s]$. As in (5.4.33), the jump size $\triangle\Gamma_w := \Gamma_w - \Gamma_{w-}$ satisfies

$$\sup_{w\in[0,s]} |\triangle\Gamma_w| \le \sup_{r\in[0,s]} \|Q^N_{r,t}\phi_k - Q^N_{r,s}\phi_k\|/\sqrt{N}.$$

Moreover, by Theorem 5.4.3, the dual predictable projection $\langle\Gamma\rangle$ of the quadratic variation $[\Gamma]$ of Γ is

$$\langle\Gamma\rangle_w = \int_0^w \mathcal{E}^{(q),N}_r(Q^N_{(r,t)}\phi_k - Q^N_{(r,s)}\phi_k)\, dr \qquad \text{for } w \in [0,s],$$

where $\mathcal{E}^{(q),N}_r(\phi,\psi) := \langle \mathbf{a}\nabla\phi\cdot\nabla\psi + q_N(r)\phi\psi,\, \mathfrak{X}^N_r\rangle$ and $\mathcal{E}^{(q),N}_r(\phi) := \mathcal{E}^{(q),N}_r(\phi,\phi)$. Therefore, by

Burkholder-Davis-Gundy inequality for discontinuous martingales (see the remark after Theorem 74 in Chapter IV of [66]), we have $\mathbb{E}[\,\Gamma_s^4\,] \leq \bar{c}\,\mathbb{E}[\,[\Gamma]_s^2\,]$ for some absolute constant $\bar{c}$. Hence, argue as in (5.4.35), and then by Lemma 5.3.10 and Lemma 5.3.9, we obtain

$$\begin{aligned} &\mathbb{E}[\,\Gamma_s^4\,] \leq \bar{c}\,\mathbb{E}[\,[\Gamma]_s^2\,] \\ \leq\;& 2\bar{c}\left(\mathbb{E}[\,\langle\Gamma\rangle_s^2\,] + \frac{\sup_{r\in[0,s]}\|Q_{r,t}^N\phi_k - Q_{r,s}^N\phi_k\|^2}{N}\,\mathbb{E}[\langle\Gamma\rangle_s]\right) \\ \leq\;& 2\bar{c}\,\mathbb{E}[\,\langle\Gamma\rangle_s^2\,] + C\,\frac{\|\phi_k\|^2\Big(\lambda_k^2(t-s)^2+(t-s)\Big)}{N}\Big(\lambda_k^2+\|\phi_k\|^2+\lambda_k^2\|\phi_k\|^2\Big)(t-s). \end{aligned}$$

where $C = C(D,T,\|u_0\|,\|q\|) > 0$ Estimating $\mathbb{E}[\,\langle\Gamma\rangle_s^2\,]$ by the argument we used for $H_N^k(t)$ in the proof of Lemma 5.3.3 (via an inequality in Remark 5.3.11), we see that

$$\mathbb{E}\left[\Big\langle \int_0^s \mathbf{U}_{(t,r)}^N - \mathbf{U}_{(s,r)}^N\,dM_r^N,\ \phi_k\Big\rangle^4\right] = \mathbb{E}[(\Gamma_s)^4]$$

is bounded above by the RHS of (5.4.46) for $N \geq N_0(D)$.

Similarly, by consider the martingale

$$\Theta_w := \Big\langle \int_s^{s+w} \mathbf{U}_{(t,r)}^N dM_r^N,\ \phi_k\Big\rangle, \quad w\in[0,t-s];$$

and by using Lemma 5.3.10, we can check that $\mathbb{E}\Big[\langle \int_s^t \mathbf{U}_{(t,r)}^N\,dM_r^N,\ \phi_k\rangle^4\Big] = \mathbb{E}[(\Theta_{t-s})^4]$ is bounded above by the RHS of (5.4.46) for $N \geq N_0(D)$.

(iii) Convergence of finite dimensional distributions. As in the proof of Theorem 5.4.8, it suffices to show that as $N\to\infty$,

$$(\langle J_N(t_1),\,\psi_1\rangle,\ldots,\langle J_N(t_n),\,\psi_n\rangle) \xrightarrow{\mathcal{L}} (\langle J(t_1),\,\psi_1\rangle,\ldots,\langle J(t_n),\,\psi_n\rangle) \quad \text{in } \mathbb{R}^n \tag{5.4.47}$$

for any $n\in\mathbb{N}$, $0\leq t_1\leq\cdots\leq t_n<\infty$ and $\{\psi_j\}_{j=1}^n \subset C(\overline{D})$.

For $n=1$, fix $t\geq 0$ and $\phi\in C(\overline{D})$. Note that $\theta\mapsto\langle\int_0^\theta \mathbf{U}_{(t,s)}^N dM_s^N,\,\phi\rangle$ is a martingale for $\theta\in[0,t]$, with jumps size at most $\sup_{\theta\in[0,t]}|M_\theta^N(Q_{(\theta,t)}^N\phi) - M_{\theta-}^N(Q_{(\theta,t)}^N\phi)| \leq \|\phi\|/\sqrt{N}$, by (5.3.9) and (5.4.18). Hence by the functional central limit theorem for real-valued martingales (see [59]),

$$\Big\{\langle \int_0^\theta \mathbf{U}^N_{(t,s)} dM^N_s,\, \phi\rangle;\ \theta\in[0,t]\Big\} \xrightarrow{\mathcal{L}} \Big\{\langle \int_0^\theta \mathbf{U}_{(t,s)} dM_s,\, \phi\rangle;\ \theta\in[0,t]\Big\} \quad \text{in } D([0,t],\mathbb{R}) \tag{5.4.48}$$

as $N\to\infty$.

For an integer $n>1$, (5.4.47) follows from Lemma 5.4.10 and the towering property

$$\mathbb{E}Z = \mathbb{E}\mathbb{E}[Z|\mathcal{F}_{t_1}] = \mathbb{E}\mathbb{E}[\mathbb{E}[Z|\mathcal{F}_{t_2}]|\mathcal{F}_{t_1}] = \cdots \quad \text{for } 0\le t_1\le t_2\le t_3\le\cdots.$$

We illustrate this for the case $n=3$; the proof for the general case is the same. The Fourier transform

$$\begin{aligned}\mathbb{E}\Big[\exp\Big(i\sum_{k=1}^3 a_k\langle J^N_{t_k},\,\psi_k\rangle\Big)\Big] \;=\; &\mathbb{E}\Big[\exp\Big(i\sum_{j=1}^3 a_j\Big\langle\int_0^{t_1}\mathbf{U}^N_{(t_j,s)}dM^N_s,\,\psi_j\Big\rangle\Big)\\ &\cdot\mathbb{E}\Big[\exp\Big(i\sum_{j=2}^3 a_j\Big\langle\int_{t_1}^{t_2}\mathbf{U}^N_{(t_j,s)}dM^N_s,\,\psi_j\Big\rangle\Big)\\ &\cdot\mathbb{E}\Big[\exp\Big(ia_3\Big\langle\int_{t_2}^{t_3}\mathbf{U}^N_{(t_3,s)}dM^N_s,\,\phi_3\Big\rangle\Big)\Big|\mathcal{F}_{t_2}\Big]\Big|\mathcal{F}_{t_1}\Big]\Big].\end{aligned}$$

We then apply Lemma 5.4.10 three times successively, starting from the inner most term involving $\mathcal{F}_{t_2}$. Hence we have convergence (5.4.47).

The proof of the lemma is complete. □

5.4.6 Characterization of $\mathcal{Y}$

Let $\mathcal{Y}$ be any subsequential limit of $\mathcal{Y}^N$. By Theorem 5.4.7, $\mathcal{Y}$ has a continuous version in $\mathcal{H}_{-\alpha}$ for every $\alpha > d\vee(d/2+2)$. It follows from Theorems 5.4.3, 5.4.8 and 5.4.9 that we have

$$\mathcal{Y}_t \overset{\mathcal{L}}{=} \mathbf{U}_{(t,0)}\mathcal{Y}_0 + \int_0^t \mathbf{U}_{(t,s)}\, dM_s, \quad \text{in } D([0,\,T],\mathcal{H}_{-\alpha}). \tag{5.4.49}$$

Theorem 5.4.11. *The limiting process $\mathcal{Y}$ is a continuous Gaussian Markov process that is unique in distribution. Moreover, $\mathcal{Y}$ has a version in $C^{\gamma}([0,\infty),\mathcal{H}_{-\alpha})$ for $\gamma\in(0,1/2)$.*

Proof Since M is Gaussian, $\int_0^t \mathbf{U}_{(t,s)}\,dM_s$ is a Gaussian process by the construction of the stochastic integral. On the other hand, $\mathbf{U}_{(t,0)}\mathcal{Y}_0$ is a Gaussian process and is independent of $\int_0^t \mathbf{U}_{(t,s)}\,dM_s$ since M has independent increments. Therefore $\mathcal{Y}_t$, as the sum of two independent Gaussian processes, is a Gaussian process.

The Markov property of $\mathcal{Y}$ is basically due to the independent increments of the differentials; see section 5.6 of [66]. For reader's convenience, we give a proof that $\mathcal{Y}$ is a Markov process with respect to its own filtration $\mathcal{F}^{\mathcal{Y}}_t:=\sigma(\mathcal{Y}_r:\, r\le t)=\sigma(\langle\mathcal{Y}_r,\phi\rangle:\, r\le t,\,\phi\in\mathcal{H}_\alpha)$. We in particular have from (5.4.49) that for $s\le t$,

$$\mathcal{Y}_t\overset{\mathcal{L}}{=}\mathbf{U}_{(t,s)}\mathcal{Y}_s+\int_s^t \mathbf{U}_{(t,r)}\,dM_r\quad\text{ in }\mathcal{H}_{-\alpha}. \tag{5.4.50}$$

Together with the fact that M has independent increments, we have

$$\operatorname{Cov}(\langle\mathcal{Y}_s,\phi\rangle,\ \langle\mathcal{Y}_t,\psi\rangle)=\operatorname{Cov}(\langle\mathcal{Y}_s,\phi\rangle,\ \langle\mathbf{U}_{(t,s)}\mathcal{Y}_s,\psi\rangle) \tag{5.4.51}$$

for all $s\le t$ and $\phi,\,\psi\in\mathcal{H}_\alpha$. To show that $\mathcal{Y}$ is Markov, note that (5.4.51) together with the fact that $\mathbf{U}_{(t,s)}\mathcal{Y}_s\in\mathcal{F}^{\mathcal{Y}}_s$ yield $\mathbb{E}[F(\mathcal{Y}_t)\big|\mathcal{F}^{\mathcal{Y}}_s]=F\big(\mathbf{U}_{(t,s)}\mathcal{Y}_s\big)$ for all $F\in C_b(\mathcal{H}_{-\alpha})$. Using (5.4.50) and the fact that $\mathbf{U}_{(t,s)}\mathcal{Y}_s\in\sigma(\mathcal{Y}_s)$, we obtain $\mathbb{E}[F(\mathcal{Y}_t)\big|\mathcal{Y}_s]=F\big(\mathbf{U}_{(t,s)}\mathcal{Y}_s\big)$ for all $F\in C_b(\mathcal{H}_{-\alpha})$. This shows that $\mathcal{Y}$ is Markov.

The Hölder continuity of $\mathcal{Y}$ follows immediately from Theorem 5.4.8 and Theorem 5.4.9. □

The proof of Theorem 5.1.5 is now complete. □

5.4.7 Appendix: Reflected diffusions killed by local time

Suppose now, instead of being killed by q_N, that $Z_i^{(N)}=Z_i$ is the subprocess of X_i killed by $2\int_0^t q(s,X_i(s))dL^{(i)}_s$ *for all* N. In Remark 5.1.7(ii), we claimed that Theorem 5.1.4 and Theorem 5.1.5 remain valid. The claim that Theorem 5.1.4 remains true is easy to be verified. We now provide some details to support the claim that Theorem 5.1.5 remains valid.

By the same proof of Lemma 5.3.12, we have the following:

Lemma 5.4.12. *Fix any positive integer N. For any $\phi \in Dom^{Feller}(\mathcal{A})$, we have under $\mathbb{P}^\mu$ for any $\mu \in E_N$,*

$$M_t^\phi \quad := \quad \langle \phi, \mathfrak{X}_t^N \rangle - \langle \phi, \mathfrak{X}_0^N \rangle - \int_0^t \langle \mathcal{A}\phi, \mathfrak{X}_s^N \rangle \, ds + \frac{1}{N} \sum_{i=1}^N \int_0^t q(s, Z_i(s)) \phi(Z_i(s)) \, dL_s^i$$

is an $\mathcal{F}_t^{\mathfrak{X}^N}$-martingale under $\mathbb{P}^\mu$ for any $\mu \in E_N$. Moreover, M_t^ϕ has predictable quadratic variation

$$\langle M^\phi \rangle_t = \frac{1}{N} \left[\int_0^t \langle \boldsymbol{a} \nabla \phi \cdot \nabla \phi, \, \mathfrak{X}_s^N \rangle \, ds \, + \, \frac{1}{N} \sum_{i=1}^N \int_0^t q(s, Z_i(s)) \phi^2(Z_i(s)) \, dL_s^i \right].$$

Moreover, (5.3.26) still holds for this new martingale.

Starting from the above lemma, we just need slight modifications in the proof of Theorem 5.1.5. It is easier in this case since now we have $Q^N = Q$ and $\mathbf{U}^N = \mathbf{U}$. Note that in the proof of Lemma 5.4.10, the expressions $H_N(t)$ and $G_N(t)$ in (5.4.40) should be replaced by, respectively,

$$\frac{1}{N} \sum_{i=1}^N \int_0^t \psi^2(Z_r^i) q(r, Z_r^i) \, dL_r^i \quad \text{and} \quad \int_0^t \int_{\partial D} \psi^2(z) q(r,z) \, u(r,z) \rho(z) \, d\sigma(z) \, dr.$$

In addition, we should also use the following lemma rather than Lemma 5.3.3.

Lemma 5.4.13. *Let $\{\phi(r) \, : \, r \geq 0\} \subset C(\overline{D})$ be such that $\sup_{r \geq 0} \|\phi(r)\| < \infty$. For any $p \geq 1$, we have*

$$\lim_{N \to \infty} \mathbb{E} \left[\left(\sup_{t \in [0,T]} \Big| \sum_{i=1}^N \int_0^t \phi(r, Z_r^i) \, dL_r^i \, - \, \int_0^t \int_{\partial D} \phi(r,z) \, u(r,z) \, \rho(z) \, d\sigma(z) \, dr \Big| \right)^p \right] = 0.$$

The proof of Lemma 5.4.13 is the same as that of Lemma 5.3.3.

Chapter 6

FLUCTUATIONS FOR INTERACTING DIFFUSIONS

In this chapter, we prove the fluctuation limit for the annihilating diffusion model introduced in Chapter 4. To focus on the fluctuation effect caused by the interaction on the interface I, we assume the harvest sites are empty in this chapter. All other assumptions in Chapter 4 remain valid in this chapter. As a reminder, we have the following assumptions.

Assumption 6.0.14. *(Geometric setting) $D_\pm$ are given adjacent bounded Lipschitz domains in $\mathbb{R}^d$ such that $I := \overline{D}_+ \cap \overline{D}_- = \partial D_+ \cap \partial D_-$ is a finite union of disjoint connected $\mathcal{H}^{d-1}$-rectifiable sets. $\rho_\pm \in W^{(1,2)}(D_\pm) \cap C(\overline{D}_\pm)$ are given strictly positive functions, $\boldsymbol{a}_\pm = (a^{ij}_\pm)$ are symmetric, bounded, uniformly elliptic $d \times d$ matrix-valued functions such that $a^{ij}_\pm \in W^{1,2}(D_\pm)$ for each i, j.*

We also denote

$$\mathcal{A}^\pm := \frac{1}{2\,\rho_\pm}\,\nabla \cdot (\rho_\pm\,\mathbf{a}_\pm \nabla) := \frac{1}{2\,\rho_\pm} \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(\rho_\pm\, a^{ij}_\pm \frac{\partial}{\partial x_j}\Big)$$

to be the infinitesimal generator of an $(\mathbf{a}_\pm,\, \rho_\pm)$-reflected diffusion.

An example to keep in mind is when $D_+ = (0,1)^d$ and $D_- = (0,1)^{d-1} \times (0,-1)$ are two adjacent unit cubes, the functions $\rho_\pm = 1$ are constants and $\mathbf{a}_\pm$ are the identity matrices. The interface is then $I = (0,1)^{d-1} \times \{0\}$, and we have $\mathcal{A}^\pm = \frac{1}{2}\Delta$.

Assumption 6.0.15. *(Parameter of annihilation) Suppose $\lambda \in C_+(I)$ is a given non-negative continuous function on I. Let $\widehat{\lambda} \in C(\overline{D}_+ \times \overline{D}_-)$ be an arbitrary extension of λ in the sense that $\widehat{\lambda}(z,z) = \lambda(z)$ for all $z \in I$. Note that such $\widehat{\lambda}$ always exists.*

Assumption 6.0.16. *(The annihilation potential) We choose* $\{\ell_\delta : \delta > 0\} \subset C_+(\overline{D}_+ \times \overline{D}_-)$ *in such a way that* $\ell_\delta(x,y) \le \frac{\widehat{\lambda}(x,y)}{c_{d+1}\,\delta^{d+1}}\mathbf{1}_{I^\delta}(x,y)$ *on* $D_+ \times D_-$ *and*

$$\lim_{\delta\to 0}\left\|\ell_\delta - \frac{\widehat{\lambda}}{c_{d+1}\,\delta^{d+1}}\mathbf{1}_{I^\delta}\right\|_{L^2(D_+\times D_-)} = 0, \tag{6.0.1}$$

where $I^\delta := \{(x,y) \in D_+ \times D_- : |x-z|^2 + |y-z|^2 < \delta^2 \text{ for some } z \in I\}$ *and* c_{d+1} *is the volume of the unit ball in* $\mathbb{R}^{d+1}$.

See (1.2.3) in Introduction (Chapter 1) for the motivation for the definition of ℓ_δ. Intuitively, if N is the initial number of particles, then $\delta = \delta_N$ is the annihilation distance and I^δ controls the frequency of interactions. As remarked in the Introduction (Chapter 1), we need to assume that the annihilation distance δ_N does not shrink too fast. This is formulated in Assumptions 6.0.17 and 6.1.1.

Assumption 6.0.17. *(Annihilation distance for functional LLN)* $\liminf_{N\to\infty} N\,\delta_N^d \in (0,\infty]$, *where* $\{\delta_N\} \subset (0,\infty)$ *converges to 0 as* $N \to \infty$.

With the above assumptions, we let $(\mathfrak{X}^{N,+},\, \mathfrak{X}^{N,-})$ be the normalized empirical measures for the annihilating diffusion system defined in (4.1.3) in Chapter 4. The main result of Chapter 4, Theorem 4.3.1, implies that we have

Theorem 6.0.18. ***(Hydrodynamic Limit)*** *Suppose Assumptions 6.0.14 to 6.0.17 in the above hold. If* $(\mathfrak{X}_0^{N,+},\, \mathfrak{X}_0^{N,-}) \xrightarrow{\mathcal{L}} (u_0^+(x)\rho_+(x)dx,\, u_0^-(y)\rho_-(y)dy)$ *in* $M_+(\overline{D}_+)\times M_+(\overline{D}_-)$ *as* $N \to \infty$, *where* $u_0^\pm \in C(\overline{D}_\pm)$, *then*

$$(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-}) \xrightarrow{\mathcal{L}} (u_+(t,x)\rho_+(x)dx,\, u_-(t,y)\rho_-(y)dy) \quad \text{in } D([0,T],\, M_+(\overline{D}_+) \times M_+(\overline{D}_-))$$

for any $T > 0$, *where* $(u_+,\, u_-)$ *is the probabilistic solution (see Remark 6.0.19) of the following* *coupled heat equations*

$$\begin{cases} \dfrac{\partial u_+}{\partial t} = \mathcal{A}^+ u_+ & \textit{on } (0,\infty)\times D_+ \\ \dfrac{\partial u_+}{\partial \vec{\nu_+}} = \dfrac{\lambda}{\rho_+}\, u_+ u_- \,\mathbf{1}_{\{I\}} & \textit{on } (0,\infty)\times \partial D_+ \end{cases} \tag{6.0.2}$$

and

$$\begin{cases} \dfrac{\partial u_-}{\partial t} = \mathcal{A}^- u_- & \textit{on } (0,\infty)\times D_- \\ \dfrac{\partial u_-}{\partial \vec{\nu_-}} = \dfrac{\lambda}{\rho_-}\, u_+ u_- \,\mathbf{1}_{\{I\}} & \textit{on } (0,\infty)\times \partial D_-, \end{cases} \tag{6.0.3}$$

with initial value (u_0^+, u_0^-), *where* $\vec{\nu}_\pm := \boldsymbol{a}_\pm \vec{n}_\pm$ *is the inward conormal vector field of* $\partial D_\pm$.

Remark 6.0.19. The notion of probabilistic solution in Theorem 6.0.18 follows that of Definition 4.2.2 in Chapter 4. Precisely, $(u_+,\, u_-)$ is the unique element in $C([0,\infty)\times\overline{D}_+)\times C([0,\infty)\times\overline{D}_-)$ satisfying

$$\begin{cases} u_+(t,x) = \mathbb{E}^x\Big[\, u_0^+(X_t^+)\, \exp\Big(-\displaystyle\int_0^t (\lambda\, u_-)(t-s, X_s^+)\, dL_s^+\Big)\Big] \\ u_-(t,y) = \mathbb{E}^y\Big[\, u_0^-(X_t^-)\, \exp\Big(-\displaystyle\int_0^t (\lambda\, u_+)(t-s, X_s^-)\, dL_s^-\Big)\Big], \end{cases} \tag{6.0.4}$$

where $L^\pm$ is the boundary local time of the reflected diffusion $X^\pm$ on the interface I. Proposition 3.2.1 in Chapter 3 guarantees the validity of the previous assertion. In this chapter, $(u_+,\, u_-)$ always denote the probabilistic solution of the coupled PDEs (also known as hydrodynamic limit) in Theorem 6.0.18. □

The fluctuation field in $\overline{D}_\pm$ is defined by

$$\mathcal{Y}_t^{N,\pm}(\phi_\pm) := \sqrt{N}\,\big(\langle \mathfrak{X}_t^{N,\pm}, \phi_\pm\rangle - \mathbb{E}\langle \mathfrak{X}_t^{N,\pm}, \phi_\pm\rangle\big) \quad \text{for } \phi_\pm \in L^2(D_\pm),$$

where $\langle \mathfrak{X}_t^{N,\pm}, \phi_\pm\rangle$ is the integral of an observable (or test function) $\phi_\pm \in L^2(D_\pm)$ with respect to the measure $\mathfrak{X}_t^{N,\pm}$. Intuitively, if $\phi_+ = \mathbf{1}_K$ is an indicator function of a subset $K \subset \overline{D}_+$, then $\langle \mathfrak{X}_t^{N,+}, \phi_+\rangle$ is the mass of particles in K (which is the number of particles in K divided by N).

In this case, $\mathcal{Y}_t^{N,+}(\phi_+)$ is the fluctuation of the mass of particles in K at time t. Our object of study in this chapter is the **fluctuation process** defined by

$$\mathcal{Z}^N := \mathcal{Y}^{N,+} \oplus \mathcal{Y}^{N,-} = (\mathcal{Y}_t^{N,+} \oplus \mathcal{Y}_t^{N,-})_{t\geq 0}, \tag{6.0.5}$$

where $\mathcal{Y}_t^{N,+} \oplus \mathcal{Y}_t^{N,-}(\phi_+, \phi_-) := \mathcal{Y}_t^{N,+}(\phi_+) + \mathcal{Y}_t^{N,-}(\phi_-)$.

Remark 6.0.20. We do not lose any information (in terms of finite dimensional distributions) by considering $\mathcal{Y}^{N,+} \oplus \mathcal{Y}^{N,-}$ rather than $(\mathcal{Y}^{N,+}, \mathcal{Y}^{N,-})$. This is because the distribution of

$$\Big(\big(\mathcal{Y}_{t_1}^{N,+}(f_1), \mathcal{Y}_{t_1}^{N,-}(g_1)\big), \; \cdots, \; \big(\mathcal{Y}_{t_k}^{N,+}(f_k), \mathcal{Y}_{t_k}^{N,-}(g_k)\big) \Big) \in (\mathbb{R}^2)^k,$$

is determined by that of

$$\Big(\mathcal{Z}_{t_1}^N, \mathcal{Z}_{t_2}^N, \cdots, \mathcal{Z}_{t_k}^N \Big) \in (\mathbf{H}_{-\alpha})^k,$$

where $k \in \mathbb{N}$, $\{f_i\} \subset \mathcal{H}_\alpha^+$ and $\{g_j\} \subset \mathcal{H}_\alpha^-$ are arbitrary. □

As in Chapter 5, it is nontrivial to describe the state space of $\mathcal{Z}^N$ in which weak convergence makes sense. For this, we adopt the functional analytic setting developed in Chapter 5 to each of D_+ and D_-.

Functional analytic framework

We denote by $\{\phi_k^\pm\}$ the complete orthonormal system (CONS) of

$$\mathcal{A}^\pm := \frac{1}{2\,\rho_\pm} \nabla \cdot (\rho_\pm \, \mathbf{a}_\pm \nabla)$$

in $\mathcal{H}_0^\pm := L^2(D_\pm, \rho_\pm)$ consisting of Neumann eigenfunctions, and $-\lambda_k^\pm$ the eigenvalue corresponding to $\phi_k^\pm$ (i.e. $\mathcal{A}^\pm \phi_k^\pm = -\lambda_k^\pm \phi_k^\pm$), with $0 < \lambda_1^\pm \leq \lambda_2^\pm \leq \lambda_3^\pm \leq \cdots$. Moreover, for $\gamma \in \mathbb{R}$, $\mathcal{H}_\gamma^\pm$ is the separable Hilbert space with inner product $\langle \, , \, \rangle_\gamma^\pm$ constructed as in Chapter 5, which has CONS $\{h_k^{(\gamma),\pm} := (1 + \lambda_k^\pm)^{-\gamma/2} \, \phi_k^\pm\}$.

Now for $\alpha \geq 0$ and $(\mu^+,\, \mu^-) \in \mathcal{H}^+_{-\alpha} \times \mathcal{H}^-_{-\alpha}$, we define $\mu^+ \oplus \mu^-$ by

$$\mu^+ \oplus \mu^- \,(\phi_+, \phi_-) := \langle \mu^+, \phi_+ \rangle_+ + \langle \mu^-, \phi_- \rangle_-,$$

where $\langle \, , \, \rangle_\pm$ is the dual paring extending $\langle \, , \, \rangle^\pm_0$. Equip $\mathbf{H}_{-\alpha} := \{\mu^+ \oplus \mu^- :\ \mu^\pm \in \mathcal{H}^\pm_{-\alpha}\}$ with the inner product

$$\langle \mu^+ \oplus \mu^-,\, \nu^+ \oplus \nu^- \rangle_{-\alpha} := \langle \mu^+, \nu^+ \rangle^+_{-\alpha} + \langle \mu^-, \nu^- \rangle^-_{-\alpha}.$$

Then $\mathbf{H}_{-\alpha}$ is a separable Hilbert space which has CONS $\{(h_k^{(-\alpha),+},\ 0)\} \cup \{(0,\ h_k^{(-\alpha),-})\}$ and hence has norm given by

$$|\mu^+ \oplus \mu^-|^2_{-\alpha} := \sum_k \left(\frac{1}{(1+\lambda_k^+)^\alpha} \langle \mu^+, \phi_k^+ \rangle_+^2 + \frac{1}{(1+\lambda_k^-)^\alpha} \langle \mu^-, \phi_k^- \rangle_-^2 \right). \tag{6.0.6}$$

Remark 6.0.21. As a matter of fact, $\mathbf{H}_{-\alpha}$ is equal to the set of linear functionals on $\mathcal{H}^+_\alpha \times \mathcal{H}^-_\alpha$, where $\mathcal{H}^+_\alpha \times \mathcal{H}^-_\alpha$ is equipped with the natural linear structure inherited from $\mathcal{H}^\pm_\alpha$. □

For a general bounded Lipschitz domain $D \subset \mathbb{R}^d$, the Weyl's asymptotic law for the Neumann eigenvalues holds (see [63]). That is, the number of eigenvalues (counting their multiplicities) less than or equal to x, denoted by $\sharp\{k:\ \lambda_k \leq x\}$, satisfies

$$\lim_{x\to\infty} \frac{\sharp\{k:\ \lambda_k \leq x\}}{x^{d/2}} = C \qquad \text{for some constant } C = C(d, D) > 0. \tag{6.0.7}$$

Moreover, we have the following bounds for the eigenfunctions proved in (5.2.6) in Chapter 5:

$$\|\phi_k\| \leq C\, \lambda_k^{d/4} \quad \text{ and } \quad \int_{\partial D} \phi_k^2 \, d\sigma \leq C\, (\lambda_k + 1) \tag{6.0.8}$$

for some $C = C(d, D) > 0$.

The following lemma tells us the space in which the fluctuation processes $\mathcal{Z}^N$ live.

Lemma 6.0.22. *Suppose that Assumption 6.1.1 holds and that the initial position of particles in each of $\overline{D}_\pm$ are i.i.d with distribution $u_0^\pm \in C(\overline{D}_\pm)$. Then for any $\alpha > d$, $t \geq 0$ and $N \geq 1$*

we have $\mathcal{Z}_t^N \in \mathbf{H}_{-\alpha}$.

Proof Fix any integer $N \geq 1$ and $t \geq 0$. We have, by definition,

$$\mathbb{E}[\,(\mathcal{Y}_t^{N,+}(\phi))^2\,] = N\left(\mathbb{E}[\langle \phi, \mathfrak{X}_t^{N,+}\rangle^2] - \left(\mathbb{E}[\langle \phi, \mathfrak{X}_t^{N,+}\rangle]\right)^2\right) \leq N\,\|\phi\|^2.$$

Using the definition of the norm $|\cdot|_{-\alpha}$ is defined in (6.0.6), we have $\mathbb{E}[|\mathcal{Z}_t^N|_{-\alpha}^2] < \infty$ provided that

$$\sum_{k\geq 1}\left(\frac{\|\phi_k^+\|^2}{(1+\lambda_k^+)^\alpha} + \frac{\|\phi_k^-\|^2}{(1+\lambda_k^-)^\alpha}\right) < \infty,$$

which is true if $\alpha > d$, using the Weyl's law (6.0.7) and the bound (6.0.8). □

Suppose the initial position of particles in each of $\overline{D}_\pm$ are i.i.d with distribution $u_0^\pm \in C(\overline{D}_\pm)$. It is easy to check that if $\alpha > d/2$, then $\mathcal{Z}_0^N \in \mathbf{H}_{-\alpha}$; furthermore,

$$\mathcal{Z}_0^N \xrightarrow{\mathcal{L}} \mathcal{Z}_0 := \mathcal{Y}_0^+ \oplus \mathcal{Y}_0^- \quad \text{in } \mathbf{H}_{-\alpha}, \tag{6.0.9}$$

where $\mathcal{Y}_0^\pm$ is the centered Gaussian random variable in $\mathcal{H}_{-\alpha}^\pm$ with covariance

$$\tilde{\mathbb{E}}\,[\mathcal{Y}_0^\pm(\phi)\mathcal{Y}_0^\pm(\psi)] = \langle \phi\,\psi,\, u_0^\pm\rangle_{\rho_\pm} - \langle \phi, u_0^\pm\rangle_{\rho_\pm}\,\langle \psi, u_0^\pm\rangle_{\rho_\pm}.$$

Here $\langle \phi,\, \psi\rangle_{\rho_\pm} := \int_{D_\pm}\phi(x)\psi(x)\,\rho_\pm(x)dx$ is the inner product of $L^2(D_\pm,\, \rho_\pm(x)dx)$. Our main goal in this chapter is to show that the sequence of processes $\{(\mathcal{Z}_t^N)_{t\geq 0}\}$ converges as $N \to \infty$, and to characterize the limit.

6.1 Main result and key idea of proof

For our fluctuation result (Theorem 6.1.2) to hold, we need the following assumption on $\{\delta\}$ which is stronger than Assumption 6.0.17: roughly speaking, we require δ to decrease at a slower rate so that the fluctuations in D_+ propagate through D_-. This is a high density assumption for the particles.

Assumption 6.1.1. *(Annihilation distance for functional CLT)* $\liminf_{N\to\infty} N\,\delta_N^{2d} \in (0,\infty]$, *where* $\{\delta_N\} \subset (0,\infty)$ *converges to 0 as* $N \to \infty$.

Before stating the fluctuation result, we first define an evolution operator (see [24]) $\{Q_{s,t}\}_{s\le t}$ as follows: Fix any $\phi_\pm \in C(\overline{D}_\pm)$ and $t > 0$. Consider the following system of backward heat equations for $(v^+(s,x),\, v^-(s,y))$ (for $s \in (0,t)$) with terminal data $v^\pm(t) = \phi_\pm$ and with *nonlinear and coupled* boundary conditions:

$$\begin{cases} -\dfrac{\partial v^+(s,x)}{\partial s} = \mathcal{A}^+ v^+(s,x) & \text{on } (0,t)\times D_+ \\ \dfrac{\partial v^+(s,z)}{\partial \vec{n}_+} = \lambda(z)\Big(v^+(s,z)+v^-(s,z)\Big)\,u_-(s,z)\,\rho_-(z)\,\mathbf{1}_{\{I\}} & \text{on } (0,t)\times \partial D_+ \\ -\dfrac{\partial v^-(x,y)}{\partial s} = \mathcal{A}^- v^-(s,y) & \text{on } (0,t)\times D_- \\ \dfrac{\partial v^-(s,z)}{\partial \vec{n}_-} = \lambda(z)\Big(v^+(s,z)+v^-(s,z)\Big)\,u_+(s,z)\,\rho_+(z)\,\mathbf{1}_{\{I\}} & \text{on } (0,t)\times \partial D_-, \end{cases} \tag{6.1.1}$$

where $(u_+,\, u_-)$ is the hydrodynamic limit in Theorem 6.0.18, $\vec{n}_\pm$ is the inward unit normal of $D_\pm$ and $\mathbf{1}_{\{I\}}$ is the indicator function on the interface I. Let $Q_{s,t}(\phi_+,\phi_-) := (v^+(s), v^-(s))$ be the solution[1] for (6.1.1) and define

$$(\mathbf{U}_{(t,s)}\mu)(\phi_+,\phi_-) := \mu\Big(Q_{s,t}(\phi_+,\phi_-)\Big)$$

for $\alpha \ge 0$, $\mu \in \mathbf{H}_{-\alpha}$ and (ϕ_+,ϕ_-) whenever $Q_{s,t}(\phi_+,\phi_-) \in \mathcal{H}^+_\alpha \times \mathcal{H}^-_\alpha$.

We are now in the position to state our main result in this chapter.

Theorem 6.1.2. ***(Fluctuation limit)*** *Suppose that Assumptions 6.0.14 to 6.0.16 hold, and that Assumption 6.1.1 holds. Suppose the initial position of particles in* $\overline{D}_\pm$ *are i.i.d with distribution* $u_0^\pm \in C(\overline{D}_\pm)$. *Then for any* $T > 0$, *there exists a constant* $C = C(D_+, D_-, T) > 0$ *such that*

$$\mathcal{Z}^N \xrightarrow{\mathcal{L}} \mathcal{Z} \quad \textit{in } D([0,\, T_0],\, \mathbf{H}_{-\alpha})$$

[1] See Proposition 6.2.9 for the existence and uniqueness of solution for (6.1.1) in $C([0,t]\times\overline{D}_+)\times C([0,t]\times\overline{D}_-)$.

for $\alpha > d+2$, *where* $T_0 := T \wedge (\|u_0^+\| \vee \|u_0^-\|)^{-2}\, C$ *and* $\mathcal{Z}$ *is the generalized Ornstein-Uhlenbeck process given by*

$$\mathcal{Z}_t \overset{\mathcal{L}}{=} \mathbf{U}_{(t,0)}\mathcal{Z}_0 + \int_0^t \mathbf{U}_{(t,s)}\, dM_s \quad in\ D([0,\, T_0],\ \mathbf{H}_{-\alpha}). \tag{6.1.2}$$

In (6.1.2), M *is a (unique in distribution) continuous, square integrable,* $\mathbf{H}_{-\alpha}-$*valued Gaussian martingale with independent increments and covariance functional characterized by*

$$\begin{aligned}\langle M^N(\phi_+,\phi_-)\rangle_t \;=\; \int_0^t \Bigg(& \langle \boldsymbol{a}_+\nabla\phi_+\cdot\nabla\phi_+,\ u_+(s)\rangle_{\rho_+} + \langle \boldsymbol{a}_-\nabla\phi_-\cdot\nabla\phi_-,\ u_-(s)\rangle_{\rho_-} \\ & + \int_I \lambda\,(\phi_+ + \phi_-)^2\, u_+(s)u_-(s)\,\rho_+\,\rho_-\, d\sigma\Bigg)\, ds, \end{aligned} \tag{6.1.3}$$

where $\langle M(\phi_+,\phi_-)\rangle_t$ *is the predictable quadratic variation of the real martingale* $M_t(\phi_+,\phi_-)$*, the pair* $(u_+(s),\, u_-(s))$ *is the hydrodynamic limit given by Theorem 6.0.18, and* $\mathcal{Z}_0 := \mathcal{Y}_0^+ \oplus \mathcal{Y}_0^-$ *is the centered Gaussian random variable in (6.0.9) defined on the same probability space* $(\tilde{\Omega},\, \tilde{\mathcal{F}},\, \tilde{\mathcal{F}}_t,\, \tilde{\mathbb{P}})$ *as* M*, with* $\{M,\, \mathcal{Y}_0^+,\, \mathcal{Y}_0^-\}$ *being independent.*

Remark 6.1.3. Observe that the representation (6.1.2) of $\mathcal{Z}$ tells us that $\mathcal{Z}$ is the sum of two independent Gaussian processes, hence is Gaussian itself. The covariance structure of $\mathcal{Z}$ is completely characterized; hence the distribution of $\mathcal{Z}$ in $D([0,\, T_0],\, \mathbf{H}_{-\alpha})$ is uniquely determined. Moreover, the coupled PDE (6.1.1) describes the 'transportation' for the fluctuation limit $\mathcal{Z}$, and M defined above describes the 'driving noise'. Formally, (6.1.1) is obtained from (6.2.9), and (6.1.3) is obtained from (6.1.5), both by letting $N \to \infty$. □

As mentioned in Remark 6.1.3, the limiting process $\mathcal{Z}$ is a Gaussian. Moreover, we obtain the following properties for the limiting process:

Theorem 6.1.4. ***(Properties of*** Z***)*** *The fluctuation limit* $\mathcal{Z}$ *in Theorem 6.1.2 is a continuous Gaussian Markov process which is uniquely determined in distribution, and* $\mathcal{Z}$ *has a version in* $C^\gamma([0,T_0],\, \mathbf{H}_{-\alpha})$ *(i.e. Hölder continuous with exponent* γ*) for any* $\gamma \in (0, 1/2)$*.*

We omit the proof here, since it follows from that of Theorem 5.4.11 in Chapter 5 and the estimates that we develop in the preliminary section. Roughly speaking, the Makov property follows from the evolution property of $\mathbf{U}_{(t,s)}$ and the independent increments of the differentials. In particular, the exponent of the Hölder continuity for $\mathcal{Z}$ follows from Lemma 6.5.3 and Theorem 6.5.5.

Remark 6.1.5. (i) Observe that the limiting process $\mathcal{Z} = \mathcal{Y}^+ \oplus \mathcal{Y}^-$ for some process $\mathcal{Y}^\pm$ in $\mathcal{H}^\pm_{-\alpha}$ when α is large enough, since it has state space in $\mathbf{H}_{-\alpha}$. Theorem 6.1.2 implies that $\{\mathcal{Y}^+_t(\phi_+) + \mathcal{Y}^-_t(\phi_-) : \ t \geq 0, \phi_+ \in \mathcal{H}^+_\alpha, \phi_- \in \mathcal{H}^-_\alpha\}$ is a Gaussian system. Since we can choose $\phi_\pm$ to be identically 0, we can strengthen the previous statement to be:

$$\big(\mathcal{Y}^+_{s_1}(\phi^+_1), \cdots, \mathcal{Y}^+_{s_k}(\phi^+_k), \mathcal{Y}^-_{t_1}(\phi^-_1), \cdots, \mathcal{Y}^-_{t_\ell}(\phi^-_\ell)\big)$$

is a centered Gaussian vector in $\mathbb{R}^{k+\ell}$ for any $k, \ell \in \mathbb{N}$, $\{s_i\}_{i=1}^k \subset [0,T]$, $\{t_j\}_{j=1}^\ell \subset [0,T]$, $\{\phi^+_i\}_{i=1}^k \subset \mathcal{H}^+_\alpha$ and $\{\phi^-_j\}_{j=1}^\ell \subset \mathcal{H}^-_\alpha$.

(ii) Moreover, M can be decomposed as

$$M \overset{\mathcal{L}}{=} M^+ \oplus M^- \; in \, C([0,T_0], \mathbf{H}_{-\alpha}),$$

where $M^\pm = (M^\pm_t)_{t\geq 0}$ is a continuous $\mathcal{H}^\pm_{-\alpha}$-valued Gaussian martingale with independent increment and with covariance functionals

$$\begin{aligned}
\langle M^+(\phi)\rangle_t &= \int_0^t \langle \mathbf{a}_+ \nabla\phi \cdot \nabla\phi, \ u_+(r)\rangle_{\rho_+} + \int_I \lambda\, \phi^2\, u_+(r) u_-(r)\, \rho_+\, \rho_-\, d\sigma\, dr, \\
\langle M^-(\psi)\rangle_t &= \int_0^t \langle \mathbf{a}_- \nabla\psi \cdot \nabla\psi, \ u_-(r)\rangle_{\rho_-} + \int_I \lambda\, \psi^2\, u_+(r) u_-(r)\, \rho_+\, \rho_-\, d\sigma\, dr \quad \text{and} \\
\mathbb{E}[M^+_s(\phi)\, M^-_t(\psi)] &= \int_0^{s\wedge t} \int_I \lambda\, \phi\psi\, u_+(r) u_-(r)\, \rho_+\, \rho_-\, d\sigma\, dr.
\end{aligned}$$

□

Idea of the proof

Our starting point for the study of fluctuation is the following result proved in Chapter 4. Let us recall it here for the convenience of the readers.

Lemma 6.1.6. *For any $\phi_\pm \in Dom(\mathcal{A}^\pm)$, we have*

$$\begin{aligned}
&\langle \phi_+, \mathfrak{X}_t^{N,+}\rangle + \langle \phi_-, \mathfrak{X}_t^{N,-}\rangle - \langle \phi_+, \mathfrak{X}_0^{N,+}\rangle + \langle \phi_-, \mathfrak{X}_0^{N,-}\rangle \\
&- \int_0^t \langle \mathcal{A}^+\phi_+,\, \mathfrak{X}_s^{N,+}\rangle + \langle \mathcal{A}^-\phi_-,\, \mathfrak{X}_s^{N,-}\rangle - \langle \ell_{\delta_N}(\phi_+ + \phi_-),\, \mathfrak{X}_s^{N,+}\otimes \mathfrak{X}_s^{N,-}\rangle \, ds
\end{aligned}$$

is a $\mathcal{F}_t^{(\mathfrak{X}^{N,+},\mathfrak{X}^{N,-})}$-martingale with predictable quadratic variation

$$\begin{aligned}
\frac{1}{N}\int_0^t &\langle \boldsymbol{a}_+\nabla\phi_+\cdot\nabla\phi_+,\, \mathfrak{X}_s^{N,+}\rangle + \langle \boldsymbol{a}_-\nabla\phi_-\cdot\nabla\phi_-,\, \mathfrak{X}_s^{N,-}\rangle \\
&+\langle \ell_{\delta_N}(\phi_+ + \phi_-)^2,\, \mathfrak{X}_s^{N,+}\otimes \mathfrak{X}_s^{N,-}\rangle \, ds.
\end{aligned}$$

Here $\langle \varphi(x,y),\, \mu^+(dx)\otimes\mu^-(dy)\rangle := \frac{1}{N^2}\sum_i\sum_j \varphi(x_i,y_j)$ when $\mu = (\frac{1}{N}\sum_i \mathbf{1}_{x_i},\, \frac{1}{N}\sum_j \mathbf{1}_{y_j})$.

Recall that $\mathcal{Z}_t^N := \mathcal{Y}_t^{N,+}\oplus\mathcal{Y}_t^{N,-}$. Hence Lemma 6.1.6 reads as

$$\mathcal{Z}_t^N - \mathcal{Z}_0^N = \int_0^t \mathbf{A}\mathcal{Z}_s^N - K_s^N \, ds + M_t^N, \tag{6.1.4}$$

where

$$\begin{aligned}
\mathbf{A}\mu(\phi_+,\phi_-) &:= \mu(\mathcal{A}^+\phi_+,\, \mathcal{A}^-\phi_-), \\
K_s^N(\phi_+,\phi_-) &:= \sqrt{N}\Big(\langle \ell_{\delta_N}(\phi_+ + \phi_-),\, \mathfrak{X}_s^{N,+}\otimes\mathfrak{X}_s^{N,-}\rangle - \mathbb{E}[\langle \ell_{\delta_N}(\phi_+ + \phi_-),\, \mathfrak{X}_s^{N,+}\otimes\mathfrak{X}_s^{N,-}\rangle] \Big).
\end{aligned}$$

and $M_t^N(\phi_+,\phi_-)$ is a real valued $\mathcal{F}_t^{(\mathfrak{X}^{N,+},\, \mathfrak{X}^{N,-})}$-martingale with predictable quadratic variation

$$\begin{aligned}
\langle M^N(\phi_+,\phi_-)\rangle_t &= \int_0^t \langle \mathbf{a}_+\nabla\phi_+\cdot\nabla\phi_+,\, \mathfrak{X}_s^{N,+}\rangle + \langle \mathbf{a}_-\nabla\phi_-\cdot\nabla\phi_-,\, \mathfrak{X}_s^{N,-}\rangle \\
&\qquad +\langle \ell_{\delta_N}(\phi_+ + \phi_-)^2,\, \mathfrak{X}_s^{N,+}\otimes\mathfrak{X}_s^{N,-}\rangle \, ds.
\end{aligned} \tag{6.1.5}$$

Key step in the proof: The key is to rewrite (6.1.4) as

$$\mathcal{Z}_t^N - \mathcal{Z}_0^N = \int_0^t (\mathbf{A} - \mathbf{B}_s^N)\mathcal{Z}_s^N \, ds + \int_0^t (\mathbf{B}_s^N \mathcal{Z}_s^N - K_s^N)\, ds + M_t^N, \tag{6.1.6}$$

in which

$$\mathbf{B}_s^N \mu(\phi_+, \phi_-) := \mu\big(\langle \ell_{\delta_N}(\phi_+ + \phi_-),\, f_s^{N,-}\rangle_{\rho_-},\, \langle \ell_{\delta_N}(\phi_+ + \phi_-),\, f_s^{N,+}\rangle_{\rho_+}\big) \tag{6.1.7}$$

is inserted to, roughly speaking, project K_s^N onto the image of $\mathcal{Z}_s^N$. Here $(f^+, f^-) := (f^{N,+}, f^{N,-})$ is defined to be the unique element in $C([0,\infty)\times \overline{D}_+) \times C([0,\infty)\times \overline{D}_-)$ satisfying the coupled integral equations

$$\begin{cases} f_t^+(x) = P_t^+ u_0^+(x) - \displaystyle\int_0^t P_{t-r}^+\Big(f_r^+(\cdot) \int_{D_-} \ell_{\delta_N}(\cdot, y) f_r^-(y) dy\Big)(x)\, dr \\ f_t^-(y) = P_t^- u_0^-(y) - \displaystyle\int_0^t P_{t-r}^-\Big(f_r^-(\cdot) \int_{D_+} \ell_{\delta_N}(x, \cdot) f_r^+(x) dx\Big)(y)\, dr, \end{cases} \tag{6.1.8}$$

where $P_{t-r}^{\pm}$ acts on the dot variable. The existence and uniqueness of (f^+, f^-) can be checked by the same fixed point argument as that for (u_+, u_-) in Proposition 3.2.1. We will show (in Lemma 6.2.2) that $(f^{N,+},\, f^{N,-})$ converges to $(u_+,\, u_-)$ as $N \to \infty$. Intuitively, both $(f^{N,+}, f^{N,-})$ and (u_+, u_-) are approximations to $(\mathfrak{X}^{N,-},\, \mathfrak{X}^{N,-})$, but $(f^{N,+}, f^{N,-})$ is a better one.

The hardest part is to show that, in an appropriate sense,

$$\int_0^t (\mathbf{B}_s^N \mathcal{Z}_s^N - K_s^N)\, ds \to 0 \text{ when } N \to \infty;$$

that is, we can replace $\int K_s^N \, ds$ by $\int \mathbf{B}_s^N \mathcal{Z}_s^N \, ds$ in (6.1.4). This is basically step 6 in the 'Outline of proof' below.

We discovered the formula (6.1.7) of $\mathbf{B}_s^N$ by, roughly speaking, projecting K_s^N onto the image of $\mathcal{Z}_s^N$. This inspiration comes from the well-known *Boltzman-Gibbs principle* in mathematical physics. The principle says that the fluctuation fields of non-conserved quantities change on a time scale much faster than the conserved ones, hence in a time integral only the component along those fields of conserved quantities survive. This idea leads us to reasonably hope that $\int_0^t (\mathbf{B}_s^N \mathcal{Z}_s^N - K_s^N)\,ds \to 0$, which is confirmed in Theorem 6.5.6. Analytically, the proof of $\int_0^t (\mathbf{B}_s^N \mathcal{Z}_s^N - K_s^N)\,ds \to 0$ stems from a 'magical cancelation' (see (6.4.21) and (6.4.22) in the proof of Theorem 6.5.6) for the first *two* terms of the asymptotic expansion of the correlation functions.

The Boltzman-Gibbs principle was first formulated mathematically and proven for some zero range processes in equilibrium in [8]. Although this principle is proved to hold for few non-equilibrium situations (see [6] and the references therein), it is not known whether it holds in general. The validity of the principle for our annihilating diffusion model is far from obvious, since there is no conserved quantity. An intuitive explanation for the validity here is as follows: the high density assumption (Assumption 6.1.1) guarantees that the interaction near I changes the occupation number of the particles at a slow rate with respect to diffusion (which conserves the particle number). In other words, the particle number is conserved on the time scale that is relevant for the principle. Hence we are not far away from equilibrium fluctuation.

For our annihilating diffusion model, the transportation component of $\mathcal{Z}$ in Theorem 6.1.2, described by (6.1.1), can be view as a 'linearization' of the hydrodynamic equation in Theorem 6.0.18. This is consistent in spirit with the results of [6, 31, 53, 54], in which the transportation component of the fluctuations of certain stochastic particle systems around the chemical reaction equation $\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + R(u)$ were shown to be $\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + R'(u)$, where $R(u)$ is a polynomial in u and $R'(u)$ is its derivative. A bit more precisely, the fluctuation limit $\mathcal{Y}$ weakly solves the stochastic partial differential equation (called a **Langevin equation**):

$$d\mathcal{Y}_t = \Big(\frac{1}{2}\Delta\mathcal{Y}_t + R'(u(t))\mathcal{Y}_t\Big)\,dt + dM_t,$$

where $u(t,x)$ solves the chemical reaction equation, $R'(u)$ is viewed as a multiplicative operator, M is a Gaussian martingale with independent increment and covariance structure $\mathbb{E}[(M_t(\phi))^2] = \int_0^t \langle |\nabla\phi|^2,\, u(s)\rangle + \langle \phi^2,\, |R(u(s))|\rangle\,ds$. Here $\langle\cdot\,,\,\cdot\rangle$ is the L^2 inner product in the spatial variable and $|R(u)|$ is the polynomial obtained by putting an absolute sign to each coefficient in $R(u)$.

Outline of proof: We prove Theorem 6.1.2 through the following seven steps.

Step 1 $\mathcal{Z}^N$ satisfies the following stochastic integral equation:

$$\mathcal{Z}_t^N = \mathbf{U}_{(t,0)}^N \mathcal{Z}_0^N + \int_0^t \mathbf{U}_{(t,s)}^N \, dM_s^N + \int_0^t \mathbf{U}_{(t,s)}^N (\mathbf{B}_s^N \mathcal{Z}_s^N - K_s^N) \, ds \ \text{ a.s.},$$

where $\mathbf{U}_{(t,s)}^N$ is the evolution system generated by $\mathbf{A} - \mathbf{B}_s^N$; see Theorem 6.5.2.

Step 2 $\mathcal{Z}^N$ is tight in $D([0,T_0], \mathbf{H}_{-\alpha})$; see Theorem 6.5.1.

Step 3 $M^N \xrightarrow{\mathcal{L}} M$ in $D([0,T_0], \mathbf{H}_{-\alpha})$; see Theorem 6.5.4.

Step 4 $\mathbf{U}_{(t,0)}^N \mathcal{Z}_0^N \xrightarrow{\mathcal{L}} \mathbf{U}_{(t,0)} \mathcal{Z}_0$ in $D([0,T_0], \mathbf{H}_{-\alpha})$; see Lemma 6.5.3.

Step 5 $\int_0^t \mathbf{U}_{(t,s)}^N \, dM_s^N \xrightarrow{\mathcal{L}} \int_0^t \mathbf{U}_{(t,s)} \, dM_s$ in $D([0,T_0], \mathbf{H}_{-\alpha})$; see Theorem 6.5.5.

Step 6 $\int_0^t \mathbf{U}_{(t,s)}^N (\mathbf{B}_s^N \mathcal{Z}_s^N - K_s^N) \, ds \ \to \ 0$ in $D([0,T_0], \mathbf{H}_{-\alpha})$; see Theorem 6.5.6.

This rough outline is the same as that for the Robin model in Chapter 5. In fact, with all the preliminary estimates in section 2, and with the asymptotic expansion of the correlation functions (Theorem 6.3.3) proved in section 3, all the steps except Step 2 and Step 6 can be treated using the method in Chapter 5.

Convention: To avoid unnecessary complications, we assume, from now on, that $\lambda = \widehat{\lambda} = 1$ and that the underlying motion of the particles are reflected Brownian motions (i.e. $\rho_\pm = 1$ and $\mathbf{a}_\pm$ are identity matrices). However, our arguments work for general symmetric reflected diffusions as in Chapter 5 and for any continuous functions $\lambda(z) \in C(I)$ as in Chapter 4. We also write $\ell(x,y)$ in place of $\ell_{\delta_N}(x,y)$ for simplicity. The constant C_0 is always equal to $C_0 := \|u_0^+\| \vee \|u_0^-\|$. The natural filtration $\mathcal{F}_t^{(\mathfrak{X}^{N,+},\, \mathfrak{X}^{N,-})}$ of the annihilating diffusion process will be abbreviated as $\mathcal{F}_t^N$. Assumptions 6.0.14 to 6.0.17 are in force throughout the whole chapter, and we will indicate explicitly whenever Assumption 6.1.1 is invoked.

6.2 Preliminaries

6.2.1 Minkowski content

We will make extensive use of the following result about Minkowski content of the interface I. It is proved in Lemma 4.5.6 in Chapter 4 and is restated here for the convenience of the reader.

Lemma 6.2.1. *Suppose Assumptions 6.0.14, 6.0.15 and 6.0.16 hold. Suppose $k \in \mathbb{N}$ and $\mathcal{F} \subset C\big((\overline{D}_+ \times \overline{D}_-)^k\big)$ is an equi-continuous and uniformly bounded family of functions on $(\overline{D}_+ \times \overline{D}_-)^k$. Then as $\delta \to 0$, we have*

$$\int_{(x_1,y_1)\in D_+\times D_-} \cdots \int_{(x_k,y_k)\in D_+\times D_-} f(x_1,y_1,\cdots,x_k,y_k) \prod_{i=1}^{k} \ell_\delta(x_i,y_i)\, d(x_1,y_1,\cdots,x_k,y_k)$$

$$\to \int_{z_1\in I} \cdots \int_{z_k\in I} f(z_1,z_1,\cdots,z_k,z_k)\,) \prod_{i=1}^{k} \lambda(z_i)\, d\sigma(z_1)\cdots d\sigma(z_k)$$

uniformly for $f \in \mathcal{F}$.

6.2.2 Three sets of coupled equations

Recall that $(f^+, f^-) = (f^{N,+}, f^{N,-})$ is the deterministic pair solving (6.1.8). In this subsection, we will construct two more coupled integral equations that is the core in the study of fluctuations of the annihilating diffusion system. For each $N \in \mathbb{N}$, the solutions of them will be denoted by $(G^N,\, G^{N,+},\, G^{N,-})$ and $(g^{N,+},\, g^{N,-})$ respectively. We will suppress the notation N and write (g^+, g^-) in place of $(g^{N,+},\, g^{N,-})$, etc.

We first prove that (f^+, f^-) is a good approximation to $(u_+,\, u_-)$.

Lemma 6.2.2. $|f^{N,\pm}|$ *is uniformly bounded above by $\|u_0^{\pm}\|$. Moreover, For each $t \geq 0$, we have $f_t^{N,\pm} \to u_\pm(t)$ uniformly on $\overline{D}_\pm$, as $N \to \infty$.*

Proof Clearly, $\sup_{(t,x)} \sup_N |f^{N,\pm}(t,x)| \leq \|u_0^{\pm}\|$. This can be seen, for example, by the probabilistic representations of $(f_t^+(x),\, f_t^-(y))$ given by

$$\begin{cases} f_t^+(x) = \mathbb{E}^x\Big[\, u_0^+(X_t^+)\, \exp\Big(- \displaystyle\int_0^t \int_{D_-} \ell(X_s^+, y)\, f_{t-s}^-(y)\, dy\, ds \Big) \Big] \\ f_t^-(y) = \mathbb{E}^y\Big[\, u_0^-(X_t^-)\, \exp\Big(- \displaystyle\int_0^t \int_{D_+} \ell(x, X_s^-)\, f_{t-s}^+(x)\, dx\, ds \Big) \Big]. \end{cases} \tag{6.2.1}$$

We now show that $\{(f_t^{N,+},\, f_t^{N,-})\}_{N\geq 1}$ is an equi-continuous sequence in $C(\overline{D}_+) \times C(\overline{D}_-)$. This can be achieved by using Lemma 6.2.1 and the continuity of $p^{\pm}(t,x,y)$. Precisely, from (6.1.8) we have

$$\begin{aligned} & |f_t^+(x_1) - f_t^+(x_2)| \\ = \;& \left| \int_0^t \int_{D_+} \Big(p^+(t-r, x_1, z) - p^+(t-r, x_2, z) \Big) \left(f_r^+(x) \int_{D_-} \ell(z,y) f_r^-(y)\, dy \right) dz\, dr \right| \\ \leq \;& \|u_0^+\|\, \|u_0^-\| \int_0^t \int_{z\in D_+} \int_{y \in D_-} \ell(z,y) \Big| p^+(t-r, x_1, z) - p^+(t-r, x_2, z) \Big|\, dy\, dz\, dr \\ \to \;& \|u_0^+\|\, \|u_0^-\| \int_0^t \int_I \Big| p^+(t-r, x_1, z) - p^+(t-r, x_2, z) \Big|\, d\sigma(z)\, dr \quad (\text{as } N \to \infty) \end{aligned}$$

which is less than $\|u_0^+\|\, \|u_0^-\|\, C\, |x_1 - x_2|^{\gamma}$ for some $\gamma > 0$ and $C = C(D_+,\, t)$ by the Hölder continuity of the transition density p^+. A similar calculation applies to f_t^-. Hence $\{(f_t^{N,+},\, f_t^{N,-})\}_{N\geq 1}$ is equi-continuous for any $t > 0$.

Finally, by comparing the probabilistic representations of $(u_+(t,x),\, u_-(t,y))$ in (6.0.4) and that of $(f_t^+(x),\, f_t^-(y))$ in (6.2.1), we can check that any subsequential limit of $f_t^{N,+}$ is equal to $u_+(t)$ by using Lemma 6.2.1. See the proof of Lemma 5.3.8 in Chapter 5 for more detail. □

Next, we define $(G,\, G^+,\, G^-) = (G^N,\, G^{N,+},\, G^{N,-})$ to be the unique solution in $C([0,\infty) \times$

$\overline{D}_+ \times \overline{D}_-) \times C([0,\infty) \times \overline{D}_+ \times \overline{D}_+) \times C([0,\infty) \times \overline{D}_- \times \overline{D}_-)$ to the coupled integral equations.

$$\begin{aligned} G_t(x,y) &= -\int_0^t P^{(1,1)}_{t-r}\Big\{ G_r(\tilde{x},\tilde{y}) \left(\int_{D_-} \ell(\tilde{x},w) f_r^-(w)dw + \int_{D_+} \ell(z,\tilde{y}) f_r^+(z)dz \right) \\ &\quad + \int_{D_+} G_r^+(\tilde{x},z)\ell(z,\tilde{y}) f_r^-(w)\,dz + \int_{D_-} G_r^-(\tilde{y},w)\ell(\tilde{x},w) f_r^+(\tilde{x})\,dw \\ &\quad - f_r^+(\tilde{x}) f_r^-(\tilde{y})\ell(\tilde{x},\tilde{y}) \Big\}(x,y)\,dr, \end{aligned}$$

$$\begin{aligned} G_t^+(x_1,x_2) &= -\int_0^t P^{(2,0)}_{t-r}\Big\{ G_r^+(\tilde{x_1},\tilde{x_2}) \int_{D_-} [\ell(\tilde{x_1},w) + \ell(\tilde{x_2},w)] f_r^-(w)dw \\ &\quad + \int_{D_-} f_r^+(\tilde{x_1})\ell(\tilde{x_1},w) G_r(\tilde{x_2},w) + f_r^+(\tilde{x_2})\ell(\tilde{x_2},w) G_r(\tilde{x_1},w)\,dw \Big\}(x,y)\,dr \end{aligned}$$

and

$$\begin{aligned} G_t^-(y_1,y_2) &= -\int_0^t P^{(0,2)}_{t-r}\Big\{ G_r^-(\tilde{y_1},\tilde{y_2}) \int_{D_+} [\ell(z,\tilde{y_1}) + \ell(z,\tilde{y_2})] f_r^+(z)dz \\ &\quad + \int_{D_+} f_r^-(\tilde{y_1})\ell(z,\tilde{y_1}) G_r(z,\tilde{y_2}) + f_r^-(\tilde{y_2})\ell(z,\tilde{y_2}) G_r(z,\tilde{y_1})\,dz \Big\}(x,y)\,dr, \end{aligned}$$

where the semigroup $P_t^{(i,j)}$ acts on the variables with a ‘tilde’.

Remark 6.2.3. It is clear from the definition that $G^\pm$ is symmetric; that is, $G^+(x_1,x_2) = G^+(x_2,x_1)$ and $G^-(y_1,y_2) = G^-(y_2,y_1)$. The term $f_r^+(\tilde{x})f_r^-(\tilde{y})\ell(\tilde{x},\tilde{y})$ in the equation for G guarantees that (G, G^+, G^-) cannot be constantly zero, even though they are zero when $t = 0$. This non-negative term contributes to the creation of fluctuation near the I.

Finally, $(g^+, g^-) = (g^{N,+}, g^{N,-})$ is defined to be the unique solution in $C([0,\infty) \times \overline{D}_+) \times C([0,\infty) \times \overline{D}_-)$ to the following coupled integral equations:

$$\begin{aligned} g_t^+(x) &= -\int_0^t P^+_{t-r}\Big\{ \int_{D_-} \ell(\tilde{x},w)\,[\, g_r^+(\tilde{x}) f_r^-(w) + g_r^-(w) f_r^+(\tilde{x}) + G_r(\tilde{x},w)\,]\,dw \Big\}(x)\,dr \\ g_t^-(y) &= -\int_0^t P^-_{t-r}\Big\{ \int_{D_+} \ell(z,\tilde{y})\,[\, g_r^+(z) f_r^-(\tilde{y}) + g_r^-(\tilde{y}) f_r^+(z) + G_r(z,\tilde{y})\,]\,dz \Big\}(y)\,dr, \end{aligned}$$

where the semigroups P^+_{t-r} and P^-_{t-r} act on $\tilde{x}$ and $\tilde{y}$ respectively.

Remark 6.2.4. The function $(G^N,\, G^{N,+},\, G^{N,-})$ and $(g^{N,+},\, g^{N,-})$ appear in the second order term in the asymptotic expansion of the correlation functions in Theorem 6.3.3. Their definitions are motivated and justified by the hierarchy (6.3.8). It turns out that, as in Corollary 6.3.6, the covariance structure of $\mathcal{Z}^N$ involves $(G^N,\, G^{N,+},\, G^{N,-})$ but not $(g^{N,+},\, g^{N,-})$. □

Although for fixed N, the supremum norms for G and $G^{\pm}$ are finite, these norms blow up when $N \to \infty$ (unlike [31, 53, 54])[2]. Fortunately, we still have the following bounds.

Lemma 6.2.5. *For any $T > 0$, there exist $C = C(D_+, D_-, T) > 0$ and an integer $N_0 = N_0(D_+, D_-)$ such that*

$$\int \ell(\tilde{x}, y)|G_t(x,y)|\, d(x,\tilde{x},y) + \int \ell(x,\tilde{y})|G_t(x,y)|\, d(x,\tilde{y},y) \leq (C_0\, C)^2 \sqrt{t} \tag{6.2.2}$$

$$\int \ell(x_1,\tilde{y})|G^+_t(x_1,x_2)|\, d(x_1,\tilde{y},x_2) \leq (C_0\, C)^3\, t \tag{6.2.3}$$

$$\int \ell(\tilde{x},y_1)|G^-_t(y_1,y_2)|\, d(y_1,\tilde{x},y_2) \leq (C_0\, C)^3\, t \tag{6.2.4}$$

$$\int |G_t(x,y)|\, d(x,y) \leq (C_0\, C)^2\, t \tag{6.2.5}$$

$$\int |G^+_t(x_1,x_2)|\, d(x_1,x_2) \leq (C_0\, C)^3\, t^{3/2} \tag{6.2.6}$$

$$\int |G^-_t(y_1,y_2)|\, d(y_1,y_2) \leq (C_0\, C)^3\, t^{3/2} \tag{6.2.7}$$

for all $t \in [0,\, T \wedge (C_0\, C)^{-2}]$ and $N \geq N_0$.

Proof Since each of G, G^+ and G^- is the probabilistic solution of a heat equation, they have

[2] In fact, we can check, using the probabilistic representation of G (cf. the proof of the lemma below) and a simple exit time estimate, that $G \to \infty$ as $N \to \infty$ on the set $\{(z,z) : z \in I\}$, provided $\inf u_0^{\pm} > 0$.

the following probabilistic representations (c.f. Proposition 3.2.1 of Chapter 3):

$$
\begin{aligned}
G_t(x,y) &= \int_{\theta=0}^{t} \mathbb{E}^{(x,y)}\Big[\Big(f^+_{t-\theta}(X_\theta)f^-_{t-\theta}(Y_\theta)\ell(X_\theta,Y_\theta) - \int_{D_+} G^+_{t-\theta}(X_\theta,z)\ell(z,Y_\theta)f^-_{t-\theta}(Y_\theta)dz \\
&\qquad - \int_{D_-} G^-_{t-\theta}(Y_\theta,w)\ell(X_\theta,w)f^+_{t-\theta}(X_\theta)dw\Big) \\
&\qquad \cdot \exp\Big(-\int_{s=0}^{\theta}\int_{D_-}\ell(X_s,w)f^-_{t-s}(w)dw + \int_{D_+}\ell(z,Y_s)f^+_{t-s}(z)dz\,ds\Big)\Big]\,d\theta,
\end{aligned}
$$

$$
\begin{aligned}
G^+_t(x_1,x_2) &= -\int_{\theta=0}^{t} \mathbb{E}^{(x_1,x_2)}\Big[\Big(f^+_{t-\theta}(X^1_\theta)\int_{D_-}\ell(X^1_\theta,w)G_{t-\theta}(X^2_\theta,w)dw \\
&\qquad + f^+_{t-\theta}(X^2_\theta)\int_{D_-}\ell(X^2_\theta,w)G_{t-\theta}(X^1_\theta,w)dw\Big) \\
&\qquad \cdot \exp\Big(-\int_{s=0}^{\theta}\int_{D_-}[\ell(X^1_s,w)+\ell(X^2_s,w)]f^-_{t-s}(w)dw\,ds\Big)\Big]\,d\theta,
\end{aligned}
$$

$$
\begin{aligned}
G^-_t(y_1,y_2) &= -\int_{\theta=0}^{t} \mathbb{E}^{(y_1,y_2)}\Big[\Big(f^-_{t-\theta}(Y^1_\theta)\int_{D_+}\ell(z,Y^1_\theta)G_{t-\theta}(z,Y^2_\theta)dz \\
&\qquad + f^-_{t-\theta}(Y^2_\theta)\int_{D_+}\ell(z,Y^2_\theta)G_{t-\theta}(z,Y^1_\theta)dz\Big) \\
&\qquad \cdot \exp\Big(-\int_{s=0}^{\theta}\int_{D_+}[\ell(z,Y^1_s)+\ell(z,Y^2_s)]f^+_{t-s}(z)dz\,ds\Big)\Big]\,d\theta,
\end{aligned}
$$

where $\{X,\, X^1,\, X^2\}$ are independent RBMs on D_+ and $\{Y,\, Y^1,\, Y^2\}$ are independent RBMs on D_-, and the six processes are independent. $\mathbb{E}^{(x,y)}$ is the expectation w.r.t. the law of (X,Y) starting at (x,y), etc.

Since $f^\pm \geq 0$ and $\|f^\pm\| \leq \|u^\pm_0\| \leq C_0$, the three formulae above give rise to the following point-wise bounds:

$$\begin{aligned}
|G_t(x,y)| &\leq C_0 \int_{\theta=0}^{t} \mathbb{E}^{(x,y)} \Big[C_0\, \ell(X_\theta, Y_\theta) + \Big| \int_{D_+} G^+_{t-\theta}(X_\theta, z)\ell(z, Y_\theta) dz \Big| \\
&\qquad + \Big| \int_{D_-} G^-_{t-\theta}(Y_\theta, w)\ell(X_\theta, w) dw \Big| \Big], \\
|G^+_t(x_1,x_2)| &\leq C_0 \int_{\theta=0}^{t} \mathbb{E}^{(x_1,x_2)} \Big[\Big| \int_{D_-} G_{t-\theta}(X^2_\theta, w)\ell(X^1_\theta, w) dw \Big| + \Big| \int_{D_-} G_{t-\theta}(X^1_\theta, w)\ell(X^2_\theta, w) dw \Big| \Big], \\
|G^-_t(y_1,y_2)| &\leq C_0 \int_{\theta=0}^{t} \mathbb{E}^{(y_1,y_2)} \Big[\Big| \int_{D_+} G_{t-\theta}(z, Y^2_\theta)\ell(z, Y^1_\theta) dz \Big| + \Big| \int_{D_+} G_{t-\theta}(z, Y^1_\theta)\ell(z, Y^2_\theta) dz \Big| \Big].
\end{aligned}$$

Plug in the bound for $|G^+|$ and $|G^-|$ into that of $|G|$, we have

$$\begin{aligned}
|G_t(x,y)| &\leq C_0^2\, \mathbb{E}^{(x,y)} \int_{\theta=0}^{t} \ell(X_\theta, Y_\theta) \,+\, C_0^2\, \mathbb{E}^{(x,y)} \int_{\theta=0}^{t} \int_{\alpha=0}^{t-\theta} \int_{z\in D_+} \int_{w\in D_-} \Big\{ \\
&\quad \ell(z, Y_\theta)\, \mathbb{E}^{(X_\theta, z)} \Big(\Big| G_{t-\theta-\alpha}(X^2_\alpha, w) \Big| \ell(X^1_\alpha, w) + \Big| G_{t-\theta-\alpha}(X^1_\alpha, w) \Big| \ell(X^2_\alpha, w) \Big) \\
&\quad + \ell(X_\theta, w)\, \mathbb{E}^{(Y_\theta, w)} \Big(\Big| G_{t-\theta-\alpha}(z, Y^2_\alpha) \Big| \ell(z, Y^1_\alpha) + \Big| G_{t-\theta-\alpha}(z, Y^1_\alpha) \Big| \ell(z, Y^2_\alpha) \Big) \Big\}.
\end{aligned} \tag{6.2.8}$$

Define

$$\phi(t) := \phi^{(N)}(t) := \int \ell(\tilde{x}, y) |G_t(x,y)|\, d(x, \tilde{x}, y) + \int \ell(x, \tilde{y}) |G_t(x,y)|\, d(x, \tilde{y}, y),$$

which is being thought of as an approximation to

$$\int_{x\in D_+} \int_{y\in I} |G_t(x,y)|\, dx\, d\sigma(y) + \int_{x\in I} \int_{y\in D_-} |G_t(x,y)|\, d\sigma(x)\, dy.$$

Simplifying the RHS of (6.2.8) using Chapman Kolmogorov equation and then applying

(2.1.4), we obtain, for $N \geq N_0(D_\pm)$,

$$
\begin{aligned}
\phi(t) &\leq (C_0\, C)^2 \left(\sqrt{t} + \int_{\theta=0}^{t} \int_{\alpha=0}^{t-\theta} \Big(\frac{1}{\theta\, \alpha} + \frac{1}{\sqrt{(\theta+\alpha)\,\alpha}} \Big)\, \phi(t-\theta-\alpha) \right) \\
&= (C_0\, C)^2 \left(\sqrt{t} + (\pi+2) \int_{\alpha=0}^{t} \phi(t-\alpha) \right) \quad \text{by Fubinni's Theorem.}
\end{aligned}
$$

By Gronwall's inequality,

$$\phi(t) \leq (C_0\, C)^2 \sqrt{t}\, \exp\left((C_0\, C)^2\, t\right)$$

for all $t \in [0, T]$ and $N \geq N_0$. Hence the first inequality in Lemma 6.2.5 are established. The remaining inequalities in the lemma then follow by the same argument, using point-wise upper bound for $|G|$ and $|G^{\pm}|$ we obtained. $\square$

Remark 6.2.6. It can be showed that $(G^N,\, G^{N,+},\, G^{N,-})$ converges uniformly on compact subsets of $[0, \infty) \times (\overline{D}_+ \times \overline{D}_- \setminus \mathfrak{I})$, $[0, \infty) \times (\overline{D}_+ \times \overline{D}_+ \setminus I \times I)$ and $[0, \infty) \times (\overline{D}_- \times \overline{D}_- \setminus I \times I)$ respectively, where $\mathfrak{I} := \{(z, z) \in \overline{D}_+ \times \overline{D}_- :\ z \in I\}$. Furthermore, the limit $(G^\infty,\, G^{\infty,+},\, G^{\infty,-})$ is the unique continuous solution to the couple integral equations below:

$$
\begin{aligned}
G^\infty_t(x, y) \;=\; &- \int_0^t \int_I \int_{D_-} p(t-r, (x, y), (z, \tilde{y}))\, (G^\infty_r(z, \tilde{y}) f^-_r(z) + G^{\infty,-}_r(\tilde{y}, z) f^+_r(z))\, d\tilde{y}\, d\sigma(z) \\
&+ \int_I \int_{D_+} p(t-r, (x, y), (\tilde{x}, z))\, (G^\infty_r(\tilde{x}, z) f^+_r(z) + G^{\infty,+}_r(\tilde{x}, z) f^-_r(z))\, d\tilde{x}\, d\sigma(z) \\
&- \int_I p(t-r, (x, y), (z, z)) f^+_r(z) f^-_r(z)\, d\sigma(z)\; dr,
\end{aligned}
$$

$$
\begin{aligned}
G^{\infty,+}_t(x_1, x_2) \;=\; &- \int_0^t \int_I \int_{D_+} p(t-r, (x_1, x_2), (z, \tilde{x}))\, (G^{\infty,+}_r(z, \tilde{x}) f^-_r(z) + G^\infty_r(\tilde{x}, z) f^+_r(z)) \\
&+ p(t-r, (x_1, x_2), (\tilde{x}, z))\, (G^{\infty,+}_r(\tilde{x}, z) f^-_r(z) + G^\infty_r(\tilde{x}, z) f^+_r(z))\, d\tilde{x}\, d\sigma(z)\, dr,
\end{aligned}
$$

$$G_t^{\infty,-}(y_1,y_2) = -\int_0^t \int_I \int_{D_-} p(t-r,(y_1,y_2),(z,\tilde{y}))\,(G_r^{\infty,-}(z,\tilde{y})f_r^+(z)+G_r^{\infty}(z,\tilde{y})f_r^-(z))$$
$$+\,p(t-r,(y_1,y_2),(\tilde{y},z))\,(G_r^{\infty,-}(\tilde{y},z)f_r^+(z)+G_r^{\infty}(z,\tilde{y})f_r^-(z))\,d\tilde{y}\,d\sigma(z)\,dr.$$

6.2.3 Evolution operators $\mathbf{U}^N_{(t,s)}$ and $\mathbf{U}_{(t,s)}$

The reader is recommended to recall the notion of probabilistic solution to backward heat equations in Proposition 2.1.10 in Chapter 2.

Operators $Q^N_{s,t}$ and $\mathbf{U}^N_{(t,s)}$

We fix $N \in \mathbb{N}$ and consider the following coupled backward PDE for $(v_N^+,\, v_N^-)$, with Neumann boundary conditions and terminal data $v_N^{\pm}(t) = \phi_\pm \in L^2(D_\pm)$:

$$\begin{cases} -\dfrac{\partial v_N^+}{\partial s} = \dfrac{1}{2}\Delta v_N^+ - \langle \ell(v_N^+ + v_N^-),\, f^-\rangle_- & \text{on } (0,t)\times D_+ \\ -\dfrac{\partial v_N^-}{\partial s} = \dfrac{1}{2}\Delta v_N^- - \langle \ell(v_N^+ + v_N^-),\, f^+\rangle_+ & \text{on } (0,t)\times D_-, \end{cases} \tag{6.2.9}$$

where $(f^+,\, f^-) = (f^{N,+},\, f^{N,-})$ is defined in (6.1.8) and $\langle \phi,\, \psi\rangle_\pm := \int_{D_\pm} \phi\,\psi$.

Note that each of the two equations in (6.2.9) is of the form (2.1.17) in Chapter 2 because we can rewrite

$$\langle \ell(v^+ + v^-),\, f^-\rangle_- \quad \text{as} \quad k^+v^+ + h^+ := \langle \ell,\, f^-\rangle_-\, v^+ + \langle \ell v^-,\, f^-\rangle_- \quad \text{and} \tag{6.2.10}$$

$$\langle \ell(v^+ + v^-),\, f^+\rangle_+ \quad \text{as} \quad k^-v^- + h^- := \langle \ell,\, f^+\rangle_+\, v^- + \langle \ell v^+,\, f^+\rangle_+\,. \tag{6.2.11}$$

By the same proof as that of Proposition 3.2.1 in Chapter 3, we have the following analogous result to Proposition 2.1.10.

Proposition 6.2.7. *For $N \in \mathbb{N}$ large enough, $t > 0$ and $\phi_\pm \in C(\overline{D}_\pm)$. There is a unique element $(v^+, v^-) = (v^{N,+},\, v^{N,-})$ in $C([0,t]\times \overline{D}_+) \times C([0,t]\times \overline{D}_-)$ which satisfies the following* *coupled integral equations:*

$$\begin{aligned} v^+(s,x) &= P^+_{t-s}\phi_+(x) - \frac{1}{2}\int_0^{t-s} P^+_\theta\left(k^+(s+\theta)v^+(s+\theta)+h^+(s+\theta)\right)(x)\,d\theta \\ v^-(s,y) &= P^-_{t-s}\phi_-(y) - \frac{1}{2}\int_0^{t-s} P^-_\theta\left(k^-(s+\theta)v^-(s+\theta)+h^-(s+\theta)\right)(y)\,d\theta, \end{aligned}$$

where $k^\pm$ and $h^\pm$ (which are functions indexed by N) are defined in (6.2.10) and (6.2.11). Moreover, (v^+, v^-) has the following probabilistic representations:

$$\begin{aligned} v^+(s,x) &= \mathbb{E}\left[\phi_+(X^+_{t-s})e^{-\int_0^{t-s}k^+(s+r,X^+_r)dr} - \int_0^{t-s} h^+(s+\theta, X^+_\theta)e^{-\int_0^\theta k^+(s+r,X^+_r)dr}\,d\theta \,\middle|\, X^+_0 = x\right] \\ v^-(s,y) &= \mathbb{E}\left[\phi_-(X^-_{t-s})e^{-\int_0^{t-s}k^-(s+r,X^-_r)dr} - \int_0^{t-s} h^-(s+\theta, X^-_\theta)e^{-\int_0^\theta k^-(s+r,X^-_r)dr}\,d\theta \,\middle|\, X^-_0 = y\right]. \end{aligned}$$

We call this $(v^+, v^-) = (v^{N,+},\, v^{N,-})$ the ***probabilistic solution*** *of the coupled PDE (6.2.9) with Neumann boundary conditions and terminal data $\phi_\pm$.*

Definition 6.2.8. *For $0 \le s \le t$ and $\phi_\pm \in C(\overline{D}_\pm)$, we define*

$$Q^N_{s,t}(\phi_+, \phi_-) := (v^{N,+}(s),\, v^{N,-}(s))$$

to be the probabilistic solution given by Proposition 6.2.7. Clearly, $Q^N_{s,t} : C(\overline{D}_+)\times C(\overline{D}_-) \to C(\overline{D}_+)\times C(\overline{D}_-)$ and $Q^N_{s,u}\circ Q^N_{u,t} = Q^N_{s,t}$ for $0 \le s \le u \le t$. Now we define

$$\langle \mathbf{U}^N_{(t,s)}\mu,\ (\phi_+,\phi_-)\rangle := \langle \mu,\ Q^N_{s,t}(\phi_+,\phi_-)\rangle \tag{6.2.12}$$

for $\alpha \ge 0$, $\mu \in \mathbf{H}_{-\alpha}$ and (ϕ_+, ϕ_-) whenever $Q^N_{s,t}(\phi_+,\phi_-) \in \mathcal{H}^+_\alpha \times \mathcal{H}^-_\alpha$.

Operators $Q_{s,t}$ and $\mathbf{U}_{(t,s)}$

Formally, if we let $N \to \infty$ in (6.2.9), we obtain

$$\begin{cases} -\dfrac{\partial v^+}{\partial s} = \dfrac{1}{2}\Delta v^+ - (v^+ + v^-)\, u_-\, d\sigma\big|_I & \text{on } (0,t)\times D_+ \\ -\dfrac{\partial v^-}{\partial s} = \dfrac{1}{2}\Delta v^- - (v^+ + v^-)\, u_+\, d\sigma\big|_I & \text{on } (0,t)\times D_-, \end{cases} \tag{6.2.13}$$

where (u_+, u_-) is the hydrodynamic limit. Observe that this equation is equivalent to (6.1.1) with $\lambda = 1$ and $\mathcal{A}^\pm = \frac{1}{2}\Delta$. Note the difference between this coupled PDEs and that for the hydrodynamic limit.

Let us see what do we obtain by letting $N \to \infty$ in the integral equations in Proposition 6.2.7. Recall that $k^\pm$, $h^\pm$, $f^\pm$, and $v^\pm$ are functions indexed by N. Heuristically we have

$$\begin{aligned} & P_\theta^+ \left(k^+(s+\theta)v^+(s+\theta) + h^+(s+\theta)\right)(x) \\ = & \int_{D_+} p^+(\theta,x,z)\left(\langle \ell,\, f^-\rangle_-(s+\theta,z)v^+(s+\theta,z) + \langle \ell v^-,\, f^-\rangle_-(s+\theta,z)\right) dz \\ \to & \int_I p^+(\theta,x,z)\left[v^+(s+\theta,z) + v^-(s+\theta,z)\right] u_-(s+\theta,z)\, d\sigma(z) \\ := & \ \partial_\theta^+\Big(\left[v^+(s+\theta) + v^-(s+\theta)\right] u_-(s+\theta) \Big)(x). \end{aligned}$$

The abbreviation in the last term is based on the notation $\partial_\theta^\pm \varphi(x) := \int_I p^\pm(\theta,x,z)\,\varphi(z)\,d\sigma(z)$. The following result is analogous to Proposition 6.2.7 and can be proved as in the same way.

Proposition 6.2.9. *Fix $t > 0$ and $\phi_\pm \in C(\overline{D}_\pm)$. There is a unique element (v^+, v^-) in $C([0,t]\times\overline{D}_+)\times C([0,t]\times\overline{D}_-)$ which satisfies the following coupled integral equations:*

$$\begin{aligned} v^+(s,x) &= P_{t-s}^+\phi_+(x) - \frac{1}{2}\int_0^{t-s} \partial_\theta^+\Big(\left[v^+(s+\theta) + v^-(s+\theta)\right] u_-(s+\theta) \Big)(x)\, d\theta \\ v^-(s,y) &= P_{t-s}^-\phi_-(y) - \frac{1}{2}\int_0^{t-s} \partial_\theta^-\Big(\left[v^+(s+\theta) + v^-(s+\theta)\right] u_+(s+\theta) \Big)(y)\, d\theta, \end{aligned}$$

where $\partial_\theta^\pm \varphi(x) := \int_I p^\pm(\theta,x,z)\,\varphi(z)\,d\sigma(z)$. Moreover, (v^+, v^-) has the following probabilistic *representations:*

$$
\begin{aligned}
v^+(s,x) &= \mathbb{E}\Big[\phi_+(X^+_{t-s})e^{-\int_0^{t-s} u_-(s+r,X^+_r)dL^+_r} \\
&\qquad - \int_0^{t-s} (v^- \cdot u_-)(s+\theta, X^+_\theta)\, e^{-\int_0^\theta u_-(s+r,X^+_r)dL^+_r}\, dL^+_\theta \,\Big|\, X^+_0 = x\Big] \\
v^-(s,y) &= \mathbb{E}\Big[\phi_-(X^-_{t-s})e^{-\int_0^{t-s} u_+(s+r,X^-_r)dL^-_r} \\
&\qquad - \int_0^{t-s} (v^+ \cdot u_+)(s+\theta, X^-_\theta)\, e^{-\int_0^\theta u_+(s+r,X^-_r)dL^-_r}\, dL^-_\theta \,\Big|\, X^-_0 = y\Big],
\end{aligned}
$$

where $L^\pm_t$ is the boundary local time of the RBM $X^\pm$ on I. We call this (v^+, v^-) the ***probabilistic solution*** *of the coupled PDE (6.2.13) with terminal data $\phi_\pm$.*

We stress that the above formula makes sense. For example, $\int_0^{t-s} u_-(s+r, X^+_r)dL^+_r$ is well defined since the value of u_- at $(s+r, X^+_r)$ is picked up only when X^+ hits I (which is a subset of $\overline{D}_-$).

Definition 6.2.10. *For $0 \le s \le t$ and $\phi_\pm \in C(\overline{D}_\pm)$, we define*

$$
Q_{s,t}(\phi_+, \phi_-) := (v^+(s),\, v^-(s))
$$

to be the probabilistic solution given by Proposition 6.2.9. Clearly, $Q_{s,t} : C(\overline{D}_+) \times C(\overline{D}_-) \to C(\overline{D}_+) \times C(\overline{D}_-)$ and $Q_{s,u} \circ Q_{u,t} = Q_{s,t}$ for $0 \le s \le u \le t$. To stress the dependence in t, we sometimes write $Q_{s,t}(\phi_+, \phi_-)$ as $(v^+_t(s),\, v^-_t(s))$ for (ϕ_+, ϕ_-) fixed. Now we define, for $\alpha > 0$ and $\mu \in \mathcal{H}^+_{-\alpha} \oplus \mathcal{H}^-_{-\alpha}$,

$$
\langle \mathbf{U}_{(t,s)}\mu,\ (\phi_+, \phi_-)\rangle := \langle \mu,\ Q_{s,t}(\phi_+, \phi_-)\rangle. \tag{6.2.14}
$$

Some key estimates

On the space $C(\overline{D}_+) \times C(\overline{D}_-)$, we let $(\psi_+, \psi_-) - (\phi_+, \phi_-) = (\psi_+ - \phi_+,\ \psi_- - \phi_-)$ and denote by $\|(\psi_+, \psi_-)\| := \|\psi_+\| + \|\psi_-\|$ the sum of the sup-norm of its components. The following uniform bound and uniform convergence are useful in many places in this Chapter.

Lemma 6.2.11. *For all $\phi_\pm \in C(\overline{D}_\pm)$ and $0 \le s \le t \le T$, we have*

$$\sup_{0\le s\le t\le T}\left(\sup_{N\ge N_0}\|Q^N_{s,t}(\phi_+,\phi_-)\| \vee \|Q_{s,t}(\phi_+,\phi_-)\|\right) \le \hat{c}\,\|(\phi_+,\phi_-)\| \tag{6.2.15}$$

for some positive integer $N_0 = N_0(D_+, D_-)$ and $\hat{c} = \hat{c}(d, D_+, D_-, T) > 0$. Moreover,

$$\lim_{N\to\infty}\left\|Q^N_{s,t}(\phi_+,\phi_-) - Q_{s,t}(\phi_+,\phi_-)\right\| = 0. \tag{6.2.16}$$

Proof Recalling the probabilistic representations of Q^N and Q in Proposition 6.2.7 and Proposition 6.2.9 respectively, we see that (6.2.15) follows from the non-negativity of $f^{N,\pm}$ and $u_\pm$. To prove (6.2.16), we fix $t < T$ and let $(v^{N,+}(s),\, v^{N,-}) := Q^N_{s,t}(\phi_+,\phi_-)$ and $(v^+(s),\, v^-) := Q_{s,t}(\phi_+,\phi_-)$ for $s \in [0,t]$. We look at the RHSs of the integral equations satisfied by $v^{N,+}(s)$ and $v^+(s)$, in Proposition 6.2.7 and Proposition 6.2.9, respectively. The proof is the same as that of Lemma 6.2.2, with the uniform bound for $f^{N,\pm}$ replaced by the bound (6.2.15). □

Lemma 6.2.12. *There exists a constant $\overline{c} > 0$ such that for any $0 \le s \le t \le T$ and $k \in \mathbb{N}$, we have*

$$\begin{aligned} &\sup_{r\in[0,s]}\left\|Q_{(r,t)}(\phi^+_k,0) \,-\, Q_{(r,s)}(\phi^+_k,0)\right\| \\ \le\;& (\hat{c}\vee 1)\,\overline{c}\,\|\phi^+_k\|\left(\lambda^+_k\,(t-s) + \hat{c}\,(c^+ + c^-)\,(\|u^+_0\|\vee\|u^-_0\|)\,(t-s)^{1/2}\right) \end{aligned}$$

and

$$\begin{aligned} &\sup_{r\in[0,s]}\left\|Q_{(r,t)}(0,\phi^-_k) \,-\, Q_{(r,s)}(0,\phi^-_k)\right\| \\ \le\;& (\hat{c}\vee 1)\,\overline{c}\,\|\phi^-_k\|\left(\lambda^-_k\,(t-s) + \hat{c}\,(c^+ + c^-)\,(\|u^+_0\|\vee\|u^-_0\|)\,(t-s)^{1/2}\right). \end{aligned}$$

Here $c^\pm = C(d, D_\pm, T)$ is the constant in (2.1.5) applied to $D_\pm$ and $\hat{c} = \hat{c}(d, D_+, D_-, T)$ is the constant in (6.2.15). Furthermore, there exists $N_0 = N_0(D_+, D_-)$ such that for $N \ge N_0$, the above two inequalities remain valid with $\{Q^N_{s,t}\}$ in replace of $\{Q_{s,t}\}$ and $c^\pm = C(d, D_\pm, T)$ being *the constant in (2.1.4).*

Proof We fix $(\phi_k^+, 0)$ and only prove the first inequality, since the second inequality follows from the same proof. Recall the definition of $Q_{r,t}(\phi_k^+, 0) \in C(\overline{D}_+) \times C(\overline{D}_+)$ in Definition 6.2.10. Suppose $Q_{r,t}(\phi_k^+, 0) = (v_t^+(r),\, v_t^-(r))$. Then

$$
\begin{aligned}
v_t^+(r,x) &= P_{t-s}^+ \phi_k^+(x) - \frac{1}{2}\int_0^{t-r} \partial_\theta^+ \Big([v_t^+(r+\theta) + v_t^-(r+\theta)]\, u_-(r+\theta) \Big)(x)\, d\theta \\
v_t^-(r,y) &= 0 - \frac{1}{2}\int_0^{t-r} \partial_\theta^- \Big([v_t^+(r+\theta) + v_t^-(r+\theta)]\, u_+(r+\theta) \Big)(y)\, d\theta,
\end{aligned}
$$

where $\partial_\theta^\pm \varphi(x) := \int_I p^\pm(\theta, x, z)\, \varphi(z)\, d\sigma(z)$. Hence for any $0 \le r \le s \le t$, we have

$$
\begin{aligned}
& |v_t^+(r,x) - v_s^+(r,x)| \\
\le \; & \Big| \Big(e^{-\lambda_k^+(t-r)} - e^{-\lambda_k^+(s-r)} \Big)\, \phi_k^+(x) \Big| \\
& + \frac{1}{2}\Big| \int_{s-r}^{t-r} \partial_\theta^+ \Big([v_t^+(r+\theta) + v_t^-(r+\theta)]\, u_-(r+\theta) \Big)(x)\, d\theta \Big| \\
& + \frac{1}{2}\Big| \int_0^{s-r} \partial_\theta^+ \Big([(v_t^+ - v_s^+)(r+\theta) + (v_t^- - v_s^-)(r+\theta)]\, u_-(r+\theta) \Big)(x)\, d\theta \Big|.
\end{aligned}
$$

A similar inequality holds for $|v_t^-(r,y) - v_s^-(r,y)|$, which has 2 terms instead of 3 term on the RHS. View $0 \le s \le t$ as fixed and define, for $r \in [0,s]$,

$$
f(r) := \big\| Q_{(r,t)}(\phi_k^+, 0) \, - \, Q_{(r,s)}(\phi_k^+, 0) \big\| = \|(v_t^+ - v_s^+)(r)\| + \|(v_t^- - v_s^-)(r)\|.
$$

Then the above estimates, together with (2.1.5) and (6.2.15), implies that

$$
f(r) \le A + B \int_0^{s-r} \frac{f(r+\theta)}{\sqrt{\theta}}\, d\theta \quad \text{for } r \in [0,s], \tag{6.2.17}
$$

where $B = (\|u_0^+\| \vee \|u_0^-\|)\,(c^+ + c^-)$ and

$$A = \lambda_k^+ \,\|\phi_k^+\|\,(t-s) + \hat{c}\,(c^+ + c^-)\,(\|u_0^+\| \vee \|u_0^-\|)\,\|\phi_k^+\|\,(t-s)^{1/2}.$$

Iterating (6.2.17) as in the proof of Lemma 5.3.9 in Chapter 5, we have

$$\begin{aligned} f(r) &\leq A + AB\int_{w_1=r}^{s} \frac{1}{\sqrt{w_1-r}} + AB^2\int_{w_1=r}^{s}\int_{w_2=w_1}^{s} \frac{1}{\sqrt{(w_1-r)(w_2-w_1)}} \\ &\quad + AB^3\int_{w_1=r}^{s}\int_{w_2=w_1}^{s}\int_{w_3=w_2}^{s} \frac{1}{\sqrt{(w_1-r)(w_2-w_1)(w_3-w_2)}} + \cdots \\ &= A\sum_{k=0}^{\infty} B^k\, a_k\,(s-r)^{k/2}, \quad \text{where} \quad a_k = \frac{\pi^{k/2}}{\Gamma((k+2)/2)} \\ &\leq \frac{\overline{c}}{2}\,A\sum_{k=0}^{\infty} B^k\,(s-r)^{k/2} \quad \text{for some absolute constant } \overline{c} > 0 \\ &\leq \overline{c}\,A \qquad \text{if} \quad |B\sqrt{s-r}| \leq 1/2. \end{aligned}$$

Hence,

$$\sup_{r\in[0\vee s-1/(4B^2),\, s]} f(r) \,\leq\, \overline{c}\,A. \tag{6.2.18}$$

(The case $B = 0$ is trivial.) We can then extend (6.2.18) to take care of the case $0 \leq r < s - 1/(4B^2)$. Namely, by the evolution property and (6.2.15), we have

$$\begin{aligned} &\left\|Q_{(r,t)}(\phi_k^+,0) \,-\, Q_{(r,s)}(\phi_k^+,0)\right\| \\ &= \left\|Q_{(r,\,s-1/(4B^2))}\left(Q_{(s-1/(4B^2)),\,t)}(\phi_k^+,0) \,-\, Q_{(s-1/(4B^2)),\,s)}(\phi_k^+,0)\right)\right\| \\ &\leq \hat{c}\left\|Q_{(s-1/(4B^2)),\,t)}(\phi_k^+,0) \,-\, Q_{(s-1/(4B^2)),\,s)}(\phi_k^+,0)\right\| \leq \hat{c}\,\overline{c}\,A. \end{aligned}$$

The proof is complete. $\square$

Unlike Chapter 5, we need to analyse the correlation functions more deeply. This will be developed in the next two section.

6.3 Asymptotic expansion for correlation functions $F_t^{N,(n,m)}$

Definition 6.3.1. *Fix $N \in \mathbb{N}$ and consider the annihilating diffusion system. For $n, m \in \mathbb{N}$ and $t \geq 0$, we define the (n,m)-**correlation function at time** t, $F_t^{(n,m)} = F_t^{N,(n,m)}$, by*

$$\int_{D_+^n \times D_-^m} \Phi(\vec{x},\vec{y})\, F_t^{(n,m)}(\vec{x},\vec{y})\, d(\vec{x},\vec{y}) = \mathbb{E}\left[\Phi_{(n,m)}(t)\right]$$

for all $\Phi \in C(\overline{D}_+^n \times \overline{D}_-^m)$, where

$$\Phi_{(n,m)}(t) := \frac{1}{N^{(n)}\, N^{(m)}} \sum_{\substack{i_1,\cdots i_n \\ distinct}}^{\sharp_t} \sum_{\substack{j_1,\cdots j_m \\ distinct}}^{\sharp_t} \Phi(X_t^{i_1}, \cdots, X_t^{i_n}, Y_t^{j_1}, \cdots, Y_t^{j_m}), \tag{6.3.1}$$

$\sharp_t$ is the number of particles alive at time $t \in [0,\infty)$ in each of $\overline{D}_\pm$ and $N^{(n)} := N(N-1)\cdots(N-n+1)$ is the number of permutations of n objects chosen from N objects, $N^{(0)} := 1$.

Example 6.3.2. *For example, we have*

$$\mathbb{E}[\langle \phi, \mathfrak{X}_t^{N,+}\rangle] = \int_{D_+} \phi(x)\, F_t^{(1,0)}(x) \quad and \quad \mathbb{E}[\langle \ell, \mathfrak{X}_t^{N,+} \otimes \mathfrak{X}_t^{N,-}\rangle] = \int_{D_+ \times D_-} \ell(x,y)\, F_t^{(1,1)}(x,y).$$

Intuitively, if we randomly pick n and m living particles in D_+ and D_- respectively at time t, then $F_t^{(n,m)}(\vec{x},\vec{y})$ is the probability joint density function for their positions. Note that $F_t^{(n,m)}$ is defined for almost all $(\vec{x},\vec{y}) \in D_+^n \times D_-^m$, and that it depends on both N and the initial configurations $(\mathfrak{X}_0^{N,+}, \mathfrak{X}_0^{N,-})$. We will see, via the BBGKY hierarchy (6.3.5) which will be proved below, that $F_t^{(n,m)} \in C(\overline{D}_+^n \times \overline{D}_-^m)$ for $t > 0$. We can also replace $N^{(n)}$ by N^n (cf. Dittrich [31] and Lang and Xanh [57]). This because we will be interested in the behavior of $F^{n,m}$ when $N \to \infty$, and for any n,

$$\frac{N^{(n)}}{N^n} = (1 - \frac{1}{N})(1 - \frac{2}{N}) \cdots (1 - \frac{n-1}{N}) \nearrow 1 \text{ as } N \to \infty.$$

It is natural, base on the annihilating random walk model in Chapter 3, to expect that we have **propagation of chaos**, which says that when the number of particles tends to infinity, the particles will appear to be independent from each other. More precisely, we expect to have

$$\lim_{N\to\infty} F_t^{(n,m)}(\vec{x},\vec{y}) = \prod_{i=1}^{n} u_+(t,x_i) \prod_{j=1}^{m} u_-(t,y_j). \tag{6.3.2}$$

This will be implied by a more exact asymptotic behavior of the $F^{(n,m)}$, namely Theorem 6.3.3, which is a key ingredient for the study of fluctuation. Our method is an extension of the approach of [31].

Theorem 6.3.3. *Suppose that* $F_0^{(n,m)}(\vec{x},\vec{y}) = \prod_{i=1}^{n} u_0^+(x_i) \prod_{j=1}^{m} u_0^-(y_j)$ *(this implies that the N particles are initially independently Poisson distributed) and that* $C_0 := \|u_0^+\| \vee \|u_0^-\|$*. Then for all $T > 0$, there exists $C = C(D_+, D_-, T) > 0$ and an integer $N_0(D_+, D_-)$ such that for $0 \le t \le T \wedge (C_0\, C)^{-2}$ and $N \ge N_0$, the correlation function has decomposition*

$$F_t^{(n,m)} = A_t^{(n,m)} + \frac{1}{N} B_t^{(n,m)} + \frac{1}{N} C_t^{(n,m)} \tag{6.3.3}$$

with

$$\|A_t^{(n,m)} - F_t^{(n,m)}\|_{(n,m)} \le \frac{(C_0\, C)^{n+m}\sqrt{t}}{N\,\delta_N^d} \quad \text{and} \quad \|C_t^{(n,m)}\|_{(n,m)} \le \frac{(C_0\, C)^{n+m}\, t}{N\,\delta_N^{2d}}, \tag{6.3.4}$$

where

$$\begin{aligned} A_t^{(n,m)}(\vec{x},\vec{y}) &:= \prod_{i=1}^{n} f_t^+(x_i) \prod_{j=1}^{m} f_t^-(y_j)\,, \\ B_t^{(n,m)}(\vec{x},\vec{y}) &:= -A_t^{(n,m)}(\vec{x},\vec{y}) \Bigg(\sum_{i=1}^{n} \frac{g_t^+(x_i)}{f_t^+(x_i)} + \sum_{j=1}^{m} \frac{g_t^-(y_j)}{f_t^-(y_j)} + \sum_{i=1}^{n}\sum_{j=1}^{m} \frac{G_t(x_i,y_j)}{f_t^+(x_i) f_t^-(y_j)} \\ &\qquad + \sum_{i<p}^{n} \frac{G_t^+(x_i,x_p)}{f_t^+(x_i) f_t^+(x_p)} + \sum_{j<q}^{m} \frac{G_t^-(y_j,y_q)}{f_t^-(y_j) f_t^-(y_q)} \Bigg). \end{aligned}$$

Proof The key point of our method is to compare three hierarchies (6.3.5,) (6.3.7) and (6.3.8) in step 1 below:

Step 1: BBGKY hierarchy for the correlation functions.

Apply Dynkin's formula to (see Corollary 4.5.9 in Chapter 4) the functional

$$(s,\,(\mathfrak{X}_s^{N,+},\mathfrak{X}_s^{N,-}))\mapsto \frac{1}{N^{(n)}\,N^{(m)}}\sum_{\substack{i_1,\cdots,i_n\\ \text{distinct}}}^{\sharp_s}\ \sum_{\substack{j_1,\cdots,j_m\\ \text{distinct}}}^{\sharp_s} P_{t-s}\Phi(X_s^{i_1},\cdots,X_s^{i_n},\,Y_s^{j_1},\cdots,Y_s^{j_m})\ \ ,\ s\in[0,t]$$

yields

$$F_t^{(n,m)}=P_t^{(n,m)}F_0^{(n,m)}-\int_0^t P_{t-s}^{(n,m)}\left(V^+F_s^{(n,m+1)}+V^-F_s^{(n+1,m)}+\frac{Q}{N}F_s^{(n,m)}\right)\,ds, \qquad (6.3.5)$$

where $V^+=\sum_{i=1}^n V_{+_i}$, $V^-=\sum_{j=1}^m V_{-_j}$ are operators, $V_{+_i}F^{(n,m+1)}$, $V_{-_j}F^{(n+1,m)}$ and $QF^{(n,m)}$ are functions on $\overline{D}_+^n\times\overline{D}_-^m$ defined by

$$\begin{aligned}
V_{+_i}F^{(n,m+1)}(\vec{x},\vec{y}) &:= \int_{D_-}\ell(x_i,y)\,F^{(n,m+1)}(\vec{x},\,(\vec{y},y))\,dy\\
V_{-_j}F^{(n+1,m)}(\vec{x},\vec{y}) &:= \int_{D_+}\ell(x,y_j)\,F^{(n+1,m)}((\vec{x},x),\,\vec{y})\,dx\\
QF^{(n,m)}(\vec{x},\vec{y}) &:= \left(\sum_{i=1}^n\sum_{j=1}^m \ell(x_i,y_j)\right)\,F^{(n,m)}(\vec{x},\vec{y}).
\end{aligned}$$

Note that Q is a multiplication operator, so it is natural to denote $Q^{(n,m)}$ to be the function $Q^{(n,m)}(\vec{x},\vec{y})=\sum_{i=1}^n\sum_{j=1}^m\ell(x_i,y_j)$. Note also that the above is a finite sum since $F^{(n,m)}=0$ when $n\vee m>N$. The system of equation (6.3.5) is usually called *BBGKY hierarchy.* [3]

On other hand, it can be easily verified that $A_t^{(n,m)}$ solves

$$A_t^{(n,m)}=P_t^{(n,m)}A_0^{(n,m)}-\int_0^t P_{t-s}^{(n,m)}\left(V^+A_s^{(n,m+1)}+V^-A_s^{(n+1,m)}\right)\,ds, \qquad (6.3.7)$$

[3]We can also view (6.3.5) as the 'variation of constant' and $F^{(n,m)}$ as the probabilistic solution (cf. Definition 2.1.9 in Chapter 2) for the following heat equation on $D_+^n\times D_-^m$ with Neumann boundary condition:

$$\frac{\partial F_t^{(n,m)}}{\partial t}=\frac{1}{2}\,\Delta F_t^{(n,m)}-\left(V^+F_t^{(n,m+1)}+V^-F_t^{(n+1,m)}+\frac{Q}{N}F_t^{(n,m)}\right). \qquad (6.3.6)$$

and that we have chosen $B_t^{(n,m)}$ in such a way that $\alpha_t^{(n,m)} := A_t^{(n,m)} + \frac{B_t^{(n,m)}}{N}$ solves

$$\alpha_t^{(n,m)} = P_t^{(n,m)} F_0^{(n,m)} - \int_0^t P_{t-s}^{(n,m)} \left(V^+ \alpha_s^{(n,m+1)} + V^- \alpha_s^{(n+1,m)} + \frac{Q}{N} A_s^{(n,m)} \right) ds. \qquad (6.3.8)$$

Step 2: Duhamel expansion for $N(A_t^{(n,m)} - F_t^{(n,m)})$ in terms of a tree.

Since $F_0^{(n,m)} = A_0^{(n,m)}$ by assumption, by repeatedly iterating (6.3.5) and (6.3.7), we have

$$\begin{aligned}
& N(A_t^{(n,m)} - F_t^{(n,m)}) \\
= \quad & -\int_{t_2=0}^t P_{t-t_2}^{(n,m)} Q F_{t_2}^{(n,m)} \\
& + \int_{t_2=0}^t \int_{t_3=0}^{t_2} P_{t-t_2}^{(n,m)} \left(\sum_{i=1}^n V_{+i} P_{t_2-t_3}^{(n,m+1)} Q F_{t_3}^{(n,m+1)} + \sum_{j=1}^m V_{-j} P_{t_2-t_3}^{(n+1,m)} Q F_{t_3}^{(n+1,m)} \right) \\
& - \cdots \\
& + (-1)^M \int_{t_2=0}^t \int_{t_3=0}^{t_2} \cdots \int_{t_{M+1}=0}^{t_M} \\
& \qquad \sum_{\vec{\theta} \in \mathbb{T}_{M-1}^{(n,m)}} P_{t-t_2}^{(n,m)} V_{\theta_1} P_{t_2-t_3}^{l_1(\vec{\theta})} V_{\theta_2} P_{t_3-t_4}^{l_2(\vec{\theta})} V_{\theta_3} \cdots P_{t_M-t_{M+1}}^{l_{M-1}(\vec{\theta})} Q F_{t_{M+1}}^{l_{M-1}(\vec{\theta})} \\
& + \cdots ,
\end{aligned}$$

where $\mathbb{T}_{M-1}^{(n,m)}$ and $(l_1(\vec{\theta}), l_2(\vec{\theta}), \cdots, l_N(\vec{\theta}))$ are the tree and the labels defined in Subsection 3.4.5 in Chapter 3.

Replacing $F_{t_{M+1}}^{l_{M-1}(\vec{\theta})}$ by the constant function 1 in the $M-$th iterated integral above, we define the following function on $\overline{D}_+^n \times \overline{D}_-^m$:

$$\Theta_M^{(n,m)}(t) := \int_{t_2=0}^t \int_{t_3=0}^{t_2} \cdots \int_{t_{M+1}=0}^{t_M} \sum_{\vec{\theta} \in \mathbb{T}_{M-1}^{(n,m)}} P_{t-t_2}^{(n,m)} V_{\theta_1} P_{t_2-t_3}^{l_1(\vec{\theta})} V_{\theta_2} P_{t_3-t_4}^{l_2(\vec{\theta})} V_{\theta_3} \cdots P_{t_M-t_{M+1}}^{l_{M-1}(\vec{\theta})} Q\, 1. \qquad (6.3.9)$$

Step 3: Bounding $\|\Theta_M^{(n,m)}(t)\|_{(n,m)}$.

We now bound $\|\Theta_M^{(n,m)}(t)\|_{(n,m)}$ by employing our method developed in Subsection 3.4.5 in Chapter 3. For the convenience of the reader, we summarize the key steps.

Note that $\Theta_M^{(n,m)}(t)$ is a sum of $(n{+}m)(n{+}m{+}1)\cdots(n{+}m{+}M{-}2)$ terms of multiple integrals. Following Subsection 3.4.5, we simplify (or telescope) each integrand by Chapman-Kolmogorov equation, and then apply (2.1.5) to obtain

$$\begin{aligned}\|\Theta_M^{(n,m)}(t)\|_{(n,m)} \;\; &\leq \;\; \frac{1}{\delta_N^d}\, C^M\,(n+m+M-1) \\ &\qquad \int_{t_2=0}^{t}\cdots\int_{t_{M+1}=0}^{t_M}\sum_{\vec{\upsilon}\in\mathbb{S}_M^{(n,m)}}\frac{1}{\sqrt{(t_{\upsilon_1}-t_2)\,(t_{\upsilon_2}-t_3)\cdots(t_{\upsilon_M}-t_{M+1})}}\end{aligned}$$

for all $0\leq t\leq T$ and $N\geq N_0(D_+,D_-)$, where $C=C(D_+,D_-,T)>0$ and $\mathbb{S}_M^{(n,m)}$ is a relabeled tree of $\mathbb{T}_M^{(n,m)}$ defined in Subsection 3.4.5 of Chapter 3. Lemma 3.4.9 and Lemma 3.4.10 in Chapter 3 yield

$$\begin{aligned}&\int_{t_2=0}^{t}\cdots\int_{t_{M+1}=0}^{t_M}\sum_{\vec{\upsilon}\in\mathbb{S}_M^{(n,m)}}\frac{1}{\sqrt{(t_{\upsilon_1}-t_2)\,(t_{\upsilon_2}-t_3)\cdots(t_{\upsilon_M}-t_{M+1})}} \\ \leq\;\; &\frac{(n+m)^{(n+m)}}{(n+m)!}\,2^M\int_{t_2=0}^{t}\cdots\int_{t_{M+1}=0}^{t_M}\prod_{i=2}^{M+1}\left(\sum_{j=1}^{i-1}\frac{1}{\sqrt{t_j-t_i}}\right) \\ \leq\;\; &c^{n+m+M}\,t^{M/2},\end{aligned}$$

where c is an absolute constant. Therefore we have

$$\|\Theta_M^{(n,m)}(t)\|_{(n,m)}\leq\frac{1}{\delta_N^d}\,C^{n+m+M}\,t^{M/2} \tag{6.3.10}$$

for all $0\leq t\leq T$ and $N\geq N_0(D_+,D_-)$, where $C=C(D_+,D_-,T)>0$.

Step 4: Upper bound for $N\,\|A_t^{(n,m)}-F_t^{(n,m)}\|_{(n,m)}$.

Since $\|F^{(p,q)}\|_{(p,q)}\leq C_0^{p+q}$, and since the sum of the two components in $l_{M-1}(\vec{\theta})$ is $n{+}m{+}M{-}1$

for any $\vec{\theta} \in \mathbb{T}_{M-1}^{(n,m)}$, Step 2 and Step 3 yields

$$\begin{aligned} N \, \|A_t^{(n,m)} - F_t^{(n,m)}\|_{(n,m)} &\leq \sum_{M=1}^{\infty} C_0^{n+m+M-1} \, \|\Theta_M^{(n,m)}(t)\|_{(n,m)} \\ &\leq \frac{1}{\delta_N^d} \frac{C_0^{n+m} \, C^{n+m+1}\sqrt{t}}{1 - C_0 \, C\sqrt{t}} \\ &\leq \frac{1}{\delta_N^d} \, (C_0 \, C)^{n+m} \sqrt{t} \end{aligned} \tag{6.3.11}$$

for all $t \in [0, T \wedge (C_0 \, C)^{-2})$ and $N \geq N_0$.

Step 5: Upper bound for $C^{(n,m)} := N \left(F_t^{(n,m)} - A_t^{(n,m)} - \frac{B_t^{(n,m)}}{N} \right)$**.**

Iterating $C_t^{(n,m)} := N \left(F_t^{(n,m)} - A_t^{(n,m)} - \frac{B_t^{(n,m)}}{N} \right)$ as in step 2, we have

$$\begin{aligned} C_t^{(n,m)} &= \int_{t_2=0}^{t} P_{t-t_2}^{(n,m)} Q(A_{t_2}^{(n,m)} - F_{t_2}^{(n,m)}) \\ &\quad + \int_{t_2=0}^{t} \int_{t_3=0}^{t_2} P_{t-t_2}^{(n,m)} \Bigg(\sum_{i=1}^{n} V_{+_i} P_{t_2-t_3}^{(n,m+1)} Q(A_{t_3}^{(n,m+1)} - F_{t_3}^{(n,m+1)}) \\ &\qquad\qquad + \sum_{j=1}^{m} V_{-_j} P_{t_2-t_3}^{(n+1,m)} Q(A_{t_3}^{(n+1,m)} - F_{t_3}^{(n+1,m)}) \Bigg) \\ &\quad + \cdots \\ &= \sum_{M=1}^{\infty} \int_{t_2=0}^{t} \int_{t_3=0}^{t_2} \cdots \int_{t_{M+1}=0}^{t_M} \\ &\qquad \sum_{\vec{\theta} \in \mathbb{T}_{M-1}^{(n,m)}} P_{t-t_2}^{(n,m)} \, V_{\theta_1} \, P_{t_2-t_3}^{l_1(\vec{\theta})} \, V_{\theta_2} \, P_{t_3-t_4}^{l_2(\vec{\theta})} \, V_{\theta_3} \cdots P_{t_M-t_{M+1}}^{l_{M-1}(\vec{\theta})} \, Q\,(A_{t_{M+1}}^{l_{M-1}(\vec{\theta})} - F_{t_{M+1}}^{l_{M-1}(\vec{\theta})}). \end{aligned}$$

Hence

$$\begin{aligned}
\|C_t^{(n,m)}\|_{(n,m)} &\leq \sum_{M=1}^{\infty} \frac{2C_0^{n+m+M-1}\, C_1^{n+m+M}\sqrt{t}}{N\,\delta_N^d}\, \|\Theta_M^{(n,m)}(t)\|_{(n,m)} \quad \text{by (6.3.11)} \\
&\leq \sum_{M=1}^{\infty} \frac{2C_0^{n+m+M-1}\, C_1^{n+m+M}\sqrt{t}}{N\,\delta_N^d}\, \frac{1}{\delta_N^d}\, C_1^{n+m+M}\, t^{M/2} \quad \text{by (6.3.10)} \\
&\leq \frac{(C_0\, C)^{n+m}\, t}{N\,\delta_N^{2d}}
\end{aligned}$$

for all $t \in [\,0,\ (C_0\, C)^{-2}\,]$ and $N \geq N_0(D_\pm)$. □

It follows from Theorem 6.3.3 that we have

Corollary 6.3.4. *(a) (**Propagation of chaos**) Suppose Assumption 6.0.17 holds. Then for any $T > 0$ and any (n, m), we have*

$$\lim_{N\to\infty} \|(A_t^{(n,m)} - F_t^{(n,m)})\|_{(n,m)} = 0$$

uniformly for $t \in [0,\, T]$.

(b) Suppose, furthermore, that Assumption 6.1.1 holds. Then for any $T > 0$ and any (n, m), we have

$$\lim_{N\to\infty} \|C_t^{(n,m)}\|_{(n,m)} = 0$$

uniformly for $t \in [0,\, T \wedge (C_0\, C)^{-2}]$.

Proof Suppose $\liminf_{N\to\infty} N\,\delta_N^d \in (0, \infty]$. We have shown that the following upper bound of the series expansion of $\|(A_t^{(n,m)} - F_t^{(n,m)})\|_{(n,m)}$ (in Step 2 of the proof of Theorem 6.3.3) converges uniformly in N.

$$\begin{aligned}
&\|A_t^{(n,m)} - F_t^{(n,m)}\| \\
\leq \quad & \int_{t_2=0}^{t} \frac{1}{N} \left\| P_{t-t_2}^{(n,m)} Q F_{t_2}^{(n,m)} \right\| \\
&+ \int_{t_2=0}^{t} \int_{t_3=0}^{t_2} \frac{1}{N} \left\| P_{t-t_2}^{(n,m)} \left(\sum_{i=1}^{n} V_{+_i} P_{t_2-t_3}^{(n,m+1)} Q F_{t_3}^{(n,m+1)} + \sum_{j=1}^{m} V_{-_j} P_{t_2-t_3}^{(n+1,m)} Q F_{t_3}^{(n+1,m)} \right\| \right) \\
&+ \cdots
\end{aligned}$$

We can check that the integrand (w.r.t. $dt_2\, dt_3 \cdots$) for each term converges to zero by Lemma 4.5.6 in Chapter 4. Hence each term converges to zero as $N \to \infty$. Therefore, the whole series converges to zero and we obtained part (a).

The proof for part (b) is the same, using the series expansion of $C_t^{(n,m)}$ in Step 4 of the proof of Theorem 6.3.3. □

Remark 6.3.5. From part (b) of Corollary 6.3.4, we have

$$F_t^{(n,m)} = A_t^{(n,m)} + \frac{1}{N} B_t^{(n,m)} + \frac{o(N)}{N}, \tag{6.3.12}$$

where $o(N)$ is a term which tends to zero uniformly for $t \in [0,\, T \wedge (C_0\, C)^{-2}]$.

The following corollary of Theorem 6.3.3 gives us a point-wise bound for the difference between $F^{(n+p,m+q)}$ and $F^{(n,m)} \cdot F^{(p,q)}$.

Corollary 6.3.6. *For any $T > 0$ and any non-negative integers n, m, p, q, we have*

$$\begin{aligned}
& N \left| F_t^{(n+p,m+q)}(\vec{x}, \vec{z},\, \vec{y}, \vec{w}) \;-\; F_t^{(n,m)}(\vec{x}, \vec{y}) \cdot F_t^{(p,q)}(\vec{z}, \vec{w}) \right| \\
\leq \quad & C_0^{n+m+p+q-2} \left| \sum_{i,\,l} G_t(x_i, w_l) + \sum_{k,\,j} G_t(z_k, y_j) + \sum_{i,\,k} G_t^{+}(x_i, z_k) + \sum_{j,\,l} G_t^{-}(y_j, w_l) \right| \\
& + \frac{(C_0\, C)^{n+m+p+q}\, t}{N\, \delta^{2d}}
\end{aligned}$$

whenever $0 \le t \le T \wedge (C_0\, C)^{-2}$ *and* $N \ge N_0(D_+, D_-)$.

Proof By Theorem 6.3.3 and the shorthand $F^{(n,m)} = F_t^{(n,m)}(\vec{x}, \vec{y})$, we have

$$\begin{aligned}
& N \left[F^{(n+p,m+q)} - F^{(n,m)} \cdot F^{(p,q)} \right] \\
= \; & N \left[\left(A + \frac{B+C}{N}\right)^{(n+p,m+q)} - \left(A + \frac{B+C}{N}\right)^{(n,m)} \cdot \left(A + \frac{B+C}{N}\right)^{(p,q)} \right] \\
= \; & \left(B^{(n+p,m+q)} - A^{(n,m)} B^{(p,q)} - A^{(p,q)} B^{(n,m)} \right) \\
& + \left(C^{(n+p,m+q)} - A^{(n,m)} C^{(p,q)} - A^{(p,q)} C^{(n,m)} - \frac{(B+C)^{(n,m)}(B+C)^{(p,q)}}{N} \right).
\end{aligned}$$

It is remarkable that all terms involving g^+ and g^- cancel out in $B^{(n+p,m+q)} - A^{(n,m)} B^{(p,q)} - A^{(p,q)} B^{(n,m)}$ and we have control over all the remaining terms via the bounds (6.3.4) in Theorem 6.3.3. In fact,

$$\begin{aligned}
& B^{(n+p,m+q)}(\vec{x}, \vec{z}, \vec{y}, \vec{w}) - A^{(n,m)}(\vec{x}, \vec{y}) B^{(p,q)}(\vec{z}, \vec{w}) - A^{(p,q)}(\vec{z}, \vec{w}) B^{(n,m)}(\vec{x}, \vec{y}) \\
= \; & -A^{(n+p,m+q)}(\vec{x}, \vec{z}, \vec{y}, \vec{w}) \Bigg(\sum_{i=1}^{n} \sum_{l=1}^{q} \frac{G(x_i, w_l)}{f^+(x_i) f^-(w_l)} + \sum_{k=1}^{p} \sum_{j=1}^{m} \frac{G(z_k, y_j)}{f^+(z_k) f^-(y_j)} \\
& + \sum_{i=1}^{n} \sum_{k=1}^{p} \frac{G^+(x_i, z_k)}{f^+(x_i) f^+(z_k)} + \sum_{j=1}^{m} \sum_{l=1}^{q} \frac{G^-(y_j, w_l)}{f^-(y_j) f^-(w_l)} \Bigg). \qquad (6.3.13)
\end{aligned}$$

The result now follows from the fact that $\|f^\pm\| \le \|u_0^\pm\| \le C_0$ and (6.3.4). □

Remark 6.3.7. (Generalizing to the case $\|F_0^{(n,m)} - A_0^{(n,m)}\|_{(n,m)} \neq 0$) In Theorem 6.3.3, we have assumed the initial error $e_N^{(n,m)} := \|F_0^{(n,m)} - A_0^{(n,m)}\|_{(n,m)}$ to be zero for all n, m and N. In fact we can weaken this condition by requiring $e_N^{(n,m)} \to 0$ fast enough as $N \to \infty$. This can be quantified by taking into account the contributions of the terms $F_0^{(n,m)} - A_0^{(n,m)}$ in the difference between (6.3.5) and (6.3.7) in Step 2. □

6.4 Generalized correlation functions $F_{s,t}^{N,(n,m),(p,q)}$

The proof for Step 2 (Tightness) and Step 6 (Boltzman-Gibbs Principle) for Theorem 6.1.2 require analysis not only for the correlation function at a fixed time t, but also for the joint probability distributions of the particles at two different times $s \le t$.

Definition 6.4.1. *For $n, m, p, q \in \mathbb{N}$ and $0 \le s \le t$, we define the* ***generalized correlation functions*** $F_{s,t}^{(n,m),(p,q)} = F_{s,t}^{N,(n,m),(p,q)}$ *by*

$$\int \Phi(\vec{x}, \vec{y})\, \Psi(\vec{z}, \vec{w})\, F_{s,t}^{(n,m),(p,q)}(\vec{x}, \vec{y},\ \vec{z}, \vec{w})\, d(\vec{x}, \vec{y}, \vec{z}, \vec{w}) = \mathbb{E}\left[\Phi_{(n,m)}(s)\ \Psi_{(p,q)}(t)\right] \tag{6.4.1}$$

for all $\Phi \in C(\overline{D}_+^n \times \overline{D}_-^m)$ and $\Psi \in C(\overline{D}_+^p \times \overline{D}_-^q)$. Here $\Phi_{(n,m)}$ is defined in (6.3.1) and $\Psi_{(p,q)}$ is defined in the same way.

Example 6.4.2. *For example, we have*

$$\mathbb{E}[\langle \phi, \mathfrak{X}_s^{N,+} \rangle\, \langle \psi, \mathfrak{X}_t^{N,+} \rangle] = \int_{D_+^2} \phi(x)\psi(z)\, F_{s,t}^{(1,0)(1,0)}(x, z)\, d(x, z) \qquad \textit{and}$$

$$\mathbb{E}\left[\langle \ell,\ \mathfrak{X}_u^{N,+} \otimes \mathfrak{X}_u^{N,-} \rangle\, \langle \ell,\ \mathfrak{X}_v^{N,+} \otimes \mathfrak{X}_v^{N,-} \rangle\right] = \int \ell(x, y)\ell(z, w)\, F_{u,v}^{(1,1)(1,1)}(x, y, z, w)\, d(x, y, z, w)$$

To compare $F_{u,u+r}^{(n,m),(p,q)}$ and $F_u^{(n,m)} \cdot F_{u+r}^{(p,q)}$, we also define

$$E_{u,r}^{(n,m),(p,q)}(\vec{x}, \vec{y},\ \vec{z}, \vec{w}) := F_{u,u+r}^{(n,m),(p,q)}(\vec{x}, \vec{y},\ \vec{z}, \vec{w}) - F_u^{(n,m)}(\vec{x}, \vec{y}) \cdot F_{u+r}^{(p,q)}(\vec{z}, \vec{w}) \tag{6.4.2}$$

6.4.1 A technical lemma towards tightness

The following lemma is the key and hardest part in the proof of the tightness result for $\mathcal{Z}^N$, Theorem 6.5.1.

Lemma 6.4.3. *Suppose Assumption 6.1.1 holds. For any $T > 0$, there exists $C = C(D_+, D_-, T) >$*

0 and $N_0 = N_0(D_+, D_-)$ so that we have

$$\mathbb{E}\left[\left(\sqrt{N}\int_a^b \langle \ell\,\varphi_r,\ \mathfrak{X}_r^{N,+}\otimes\mathfrak{X}_r^{N,-}\rangle - \mathbb{E}[\langle \ell\,\varphi_r,\ \mathfrak{X}_r^{N,+}\otimes\mathfrak{X}_r^{N,-}\rangle]\,dr\right)^2\right] \leq C\,\|\varphi\|^2\,(b-a)^{3/2}$$

whenever $0 \leq a \leq b \leq T_0 := T\wedge(C_0\,C)^{-2}$, for any $N > N_0$ and any bounded function $\varphi_t(x,y)$ on $[0,\,T_0]\times D_+\times D_-$ with uniform norm $\|\varphi\|$.

A direct calculation suggests that the $L^2(\mathbb{P})$ norm of

$$\sqrt{N}\left(\langle \ell,\ \mathfrak{X}_r^{N,+}\otimes\mathfrak{X}_r^{N,-}\rangle - \mathbb{E}[\langle \ell,\ \mathfrak{X}_r^{N,+}\otimes\mathfrak{X}_r^{N,-}\rangle]\right)$$

blows up in the order of $1/\delta$ (for $r > 0$), due to the fact that $\int \ell(x,y_1)\ell(x,y_2)\,d(x,y_1,y_2)$ is of order $1/\delta$. Hence we need to look into the generalized correlation functions.

Proof **Step 1: Write LHS in terms of the generalized correlation functions.**

Note that $(\int_a^b f(r)\,dr)^2 = 2\int_{u=a}^b\int_{v=u}^b f(u)f(v) = 2\int_{u=a}^b\int_{t=0}^{b-u} f(u)f(t)$ by Fubinni's Theorem followed by the change of variable $t = v-u$. Hence Lemma 6.4.3 is implied by

$$\int_{u=a}^b\int_{t=0}^{b-u}\int_{\substack{(x,y)\in D_+\times D_-\\ (\tilde{x},\tilde{y})\in D_+\times D_-}} N\,\ell(x,y)\ell(\tilde{x},\tilde{y})\,E_{u,t}^{(1,1)(1,1)}(x,y,\,\tilde{x},\tilde{y})\ \leq C\,(b-a)^{3/2},$$

where $E_{u,t}^{(1,1)(1,1)}$ is defined in (6.4.2). The ideas is to first obtain a 'variation of constant' formula for $E_{u,t}$ via the Dynkin's formula; then iterate the formula to obtain a series expansion of $E_{u,t}$ in terms of $E_{u,0}$; finally we estimate $E_{u,0}$ and each term of the series.

Step 2: Estimate $|E_t^{(1,1)}|$ in terms of $\{E_0^{(p,q)}\}$.

Apply Dynkin's formula as in (6.3.5) yields

$$\begin{aligned} F_{u,u+r}^{(n,m),(p,q)} &= P_r^{(p,q)}F_{u,u}^{(n,m),(p,q)} \\ &\quad - \int_0^r P_{r-\theta}^{(p,q)}\left(V^+F_{u,u+\theta}^{(n,m),(p,q+1)} + V^-F_{u,u+\theta}^{(n,m),(p+1,q)} + \frac{Q}{N}F_{u,u+\theta}^{(n,m),(p,q)}\right)\,d\theta, \end{aligned} \tag{6.4.3}$$

where $P_t^{(p,q)}$, V^+, V^- and Q are operators defined as before and act on the $(\vec{z},\vec{w})$ variables.

Fix $u \geq 0$, (n,m) and $(\vec{x},\vec{y}) \in \overline{D}_+^n \times \overline{D}_-^m$, and then write

$$E_r^{(p,q)}(\vec{z},\vec{w}) := E_{u,r}^{(n,m),(p,q)}(\vec{x},\vec{y},\ \vec{z},\vec{w}) \quad \text{for notational simplicity.}$$

Then (6.4.3) yields

$$E_r^{(p,q)} = P_r^{(p,q)} E_0^{(p,q)} - \int_0^r P_{r-\theta}^{(p,q)} \left(V^+ E_\theta^{(p,q+1)} + V^- E_\theta^{(p+1,q)} + \frac{Q}{N} E_\theta^{(p,q)} \right) d\theta, \tag{6.4.4}$$

where $P_t^{(p,q)}$, V^+, V^- and Q are operators defined before, acting on the $(\vec{z},\vec{w})$ variables. In other words, $(t,\ (\vec{z},\vec{w})) \mapsto E_t^{(p,q)}(\vec{z},\vec{w})$ is the mild solution of

$$\begin{cases} \dfrac{\partial E}{\partial t} = \dfrac{1}{2}\Delta E - \dfrac{Q}{N} E - \left(V^+ E^{(p,q+1)} + V^- E^{(p+1,q)} \right) & \text{on } (0,\infty) \times D_+^p \times D_-^q, \\ \dfrac{\partial E}{\partial \vec{n}} = 0 & \text{on } (0,\infty) \times \partial(D_+^p \times D_-^q), \\ E_0(\cdot) = F_{u,u}^{(n,m),(p,q)}(\vec{x},\vec{y},\ \cdot) - F_u^{(n,m)}(\vec{x},\vec{y}) \cdot F_u^{(p,q)}(\cdot) & \text{on } D_+^p \times D_-^q. \end{cases}$$

It can be shown (see Proposition 3.2.1 in Chapter 3 for a proof) that the following probabilistic representation holds true for $E = E^{(p,q)}$:

$$E_t(\vec{z},\vec{w}) = \mathbb{E}^{\vec{z},\vec{w}} \left[E_0(X_t) e^{-\int_0^t k(X_s)ds} - \int_0^t g(t-\theta, X_\theta) e^{-\int_0^\theta k(X_s)ds} d\theta \right], \tag{6.4.5}$$

where $k = \frac{Q^{(p,q)}}{N}$, $g(t) = V^+ E_t^{(p,q+1)} + V^- E_t^{(p+1,q)}$ and X_t is the RBM in $D_+^p \times D_-^q$ starting at $(\vec{z},\vec{w})$. From this, the triangle inequality and the non-negativity of $k = \frac{Q^{(p,q)}}{N}$, we have

$$\left| E_t^{(p,q)} \right| \leq P_t^{(p,q)}(|E_0^{(p,q)}|) + \int_0^t P_{t-t_2}^{(p,q)} \left(\left| V^+ E_{t_2}^{(p,q+1)} + V^- E_{t_2}^{(p+1,q)} \right| \right) dt_2.$$

It then follows that almost everywhere in $D_+^p \times D_-^q$, we have

$$\left| E_t^{(p,q)} \right| \leq P_t^{(p,q)}(|E_0^{(p,q)}|) + \int_0^t P_{t-t_2}^{(p,q)} \left(V^+ |E_{t_2}^{(p,q+1)}| + V^- |E_{t_2}^{(p+1,q)}| \right) dt_2. \tag{6.4.6}$$

Now we iterate (6.4.6) to obtain

$$\begin{aligned}
\left|E_t^{(p,q)}\right| \quad &\le \quad P_t^{(p,q)}|E_0^{(p,q)}| \\
&\quad + \int_{t_2=0}^{t} P_{t-t_2}^{(p,q)} \left(V^+ P_{t_2}^{(p,q+1)}|E_0^{(p,q+1)}| + V^- P_{t_2}^{(p+1,q)}|E_0^{(p+1,q)}| \right) \\
&\quad + \int_{t_2=0}^{t} \int_{t_3=0}^{t_2} P_{t-t_2}^{(p,q)} \Bigg(V^+ P_{t_2-t_3}^{(p,q+1)} \left(V^+ P_{t_3}^{(p,q+2)}|E_0| + V^- P_{t_3}^{(p+1,q+1)}|E_0| \right) \\
&\qquad\qquad + V^- P_{t_2-t_3}^{(p+1,q)} \left(V^+ P_{t_3}^{(p+1,q+1)}|E_0| + V^- P_{t_3}^{(p+2,q)}|E_0| \right) \Bigg) \\
&\quad + \cdots \\
&= \sum_{M=0}^{\infty} \int_{t_2=0}^{t} \int_{t_3=0}^{t_2} \cdots \int_{t_{M+1}=0}^{t_M} \\
&\qquad \sum_{\vec{\theta} \in \mathbb{T}_M^{(n,m)}} P_{t-t_2}^{(n,m)} \, V_{\theta_1} \, P_{t_2-t_3}^{l_1(\vec{\theta})} \, V_{\theta_2} \, P_{t_3-t_4}^{l_2(\vec{\theta})} \, V_{\theta_3} \cdots P_{t_M-t_{M+1}}^{l_{M-1}(\vec{\theta})} \, V_{\theta_N} \, |E_0^{l_M(\vec{\theta})}|. \qquad (6.4.7)
\end{aligned}$$

From this inequality and the triangle inequality, we have, for any $u \ge 0$, $(n,m) = (1,1)$ and $(x,y) \in D_+ \times D_-$,

$$\begin{aligned}
&\int \ell \left|E_t^{(1,1)}\right| := \int_{(z,w)\in D_+ \times D_-} \ell(z,w) \left|E_t^{(1,1)}(x,y,z,w)\right| \\
\le \quad & \int \Psi^{(root)} \left|E_0^{(1,1)}\right| \\
&+ \int_0^t \left(\int \Psi^{(+_1)} \left|E_0^{(1,2)}\right| + \int \Psi^{(-_1)} \left|E_0^{(2,1)}\right| \right) dt_2 \\
&+ \int_0^t \int_0^{t_2} \Bigg(\int \Psi^{(+_1,+_2)} \left|E_0^{(1,3)}\right| + \int \Psi^{(+_1,-_1)} \left|E_0^{(2,2)}\right| + \int \Psi^{(+_1,-_2)} \left|E_0^{(2,2)}\right| \\
&\qquad + \int \Psi^{(-_1,+_1)} \left|E_0^{(2,2)}\right| + \int \Psi^{(-_1,+_2)} \left|E_0^{(2,2)}\right| + \int \Psi^{(-_1,-_2)} \left|E_0^{(3,1)}\right| \Bigg) \, dt_3 \, dt_2 \\
&+ \cdots \\
= \quad & \sum_{M=0}^{\infty} \int_{t_2=0}^{t} \int_{t_3=0}^{t_2} \cdots \int_{t_{M+1}=0}^{t_M} \left(\sum_{\vec{\theta} \in \mathbb{T}_M^{(1,1)}} \int \Psi^{\vec{\theta}} \left|E_0^{l_M(\vec{\theta})}\right| \right) dt_{M+1} \cdots dt_3 \, dt_2, \qquad (6.4.8)
\end{aligned}$$

where the integral sign for $E_0^{(p,q)}$ is on the set $D_+^p \times D_-^q$,

$$\begin{aligned}
\Psi^{(root)}(z,w) &:= P_t^{(1,1)}\ell(z,w) \\
\Psi^{(+_1)}(z,w_1,w_2) &:= P_{t_2}^{(1,2)}\left((P_{t_1-t_2}^{(1,1)}\ell)(a_1,b_1)\cdot\ell(a_1,b_2)\right)(z,w_1,w_2) \\
\Psi^{(-_1)}(z_1,z_2,w) &:= P_{t_2}^{(2,1)}\left((P_{t_1-t_2}^{(1,1)}\ell)(a_1,b_1)\cdot\ell(a_2,b_1)\right)(z_1,z_2,w).
\end{aligned}$$

Inductively, $\Psi^{(\vec{\theta},+_i)} \in C(\overline{D}_+^p \times \overline{D}_-^{q+1})$ and $\Psi^{(\vec{\theta},-_j)} \in C(\overline{D}_+^{p+1} \times \overline{D}_-^q)$ are obtained from $\Psi^{\vec{\theta}}$ as follows: if $\Psi^{\vec{\theta}}(\vec{z},\vec{w}) = P_{t_{M+1}}^{(p,q)}F(\vec{z},\vec{w})$, then

$$\begin{aligned}
\Psi^{(\vec{\theta},+_i)}(\vec{z},(\vec{w},w_{q+1})) &:= P_{t_{M+2}}^{(p,q+1)}\left((P_{t_{M+1}-t_{M+2}}^{(p,q)}F)(\vec{a},\vec{b})\;\ell(a_i,b_{q+1})\right)(\vec{z},(\vec{w},w_{q+1})) \quad \text{and} \\
\Psi^{(\vec{\theta},-_j)}((\vec{z},z_{p+1}),\vec{w}) &:= P_{t_{M+2}}^{(p+1,q)}\left((P_{t_{M+1}-t_{M+2}}^{(p,q)}F)(\vec{a},\vec{b})\;\ell(a_{p+1},b_j)\right)((\vec{z},z_{p+1}),\vec{w}).
\end{aligned}$$

Step 3: Estimate $E_0^{(p,q)} = F_{u,u}^{(1,1),(p,q)} - F_u^{(1,1)} \cdot F_u^{(p,q)}$.

For any $\Psi \in C(\overline{D}_+^p \times \overline{D}_-^q)$, by Definition 6.4.1 we have

$$\begin{aligned}
&\int_{D_+^{p+1}\times D_-^{q+1}} \ell(x,y)\,\Psi(\vec{z},\vec{w})\,F_{u,u}^{(1,1),(p,q)}(x,y,\,\vec{z},\vec{w})\,d(x,y,\,\vec{z},\vec{w}) \\
=\;& \mathbb{E}\left[\left(\frac{1}{N^2}\sum_i^{\sharp_u}\sum_j^{\sharp_u}\ell(X_u^i,\,Y_u^j)\right)\left(\frac{1}{N^{(p)}\,N^{(q)}}\sum_{\substack{k_1,\cdots k_p\\ \text{distinct}}}^{\sharp_u}\;\sum_{\substack{l_1,\cdots l_q\\ \text{distinct}}}^{\sharp_u}\Psi(X_u^{k_1},\cdots,X_u^{k_p},\,Y_u^{l_1},\cdots,Y_u^{l_q})\right)\right] \\
=\;& \frac{N^{(p+1)}N^{(q+1)}}{N^2N^{(p)}N^{(q)}}\int_{D_+^{p+1}\times D_-^{q+1}}\ell(x,y)\,\Psi(\vec{z},\vec{w})\,F_u^{(p+1,q+1)}((x,\vec{z}),\,(y,\vec{w})) \\
&+\frac{N^{(q+1)}}{N^2\,N^{(q)}}\sum_{i=1}^{p}\int_{D_+^{p}\times D_-^{q+1}}\ell(z_i,y)\,\Psi(\vec{z},\vec{w})\,F_u^{(p,q+1)}(\vec{z},\,(y,\vec{w})) \\
&+\frac{N^{(p+1)}}{N^2\,N^{(p)}}\sum_{j=1}^{q}\int_{D_+^{p+1}\times D_-^{q}}\ell(x,w_j)\,\Psi(\vec{z},\vec{w})\,F_u^{(p+1,q)}((x,\vec{z}),\,\vec{w}) \\
&+\frac{1}{N^2}\sum_{i=1}^{p}\sum_{j=1}^{q}\int_{D_+^{p}\times D_-^{q}}\ell(z_i,w_j)\,\Psi(\vec{z},\vec{w})\,F_u^{(p,q)}(\vec{z},\,\vec{w}).
\end{aligned} \tag{6.4.9}$$

This connects $F_{u,u}^{(1,1),(p,q)}$ to $F_u^{(p+1,q+1)}$ and we have more knowledge (such as Theorem 6.3.3) for the latter.

Furthermore, we use the simple fact that $\int f\,|g| = \int \tilde{f}\, g$ where $\tilde{f}(x) = \begin{cases} f(x) & , \text{ if } g(x) \geq 0 \\ -f(x) & , \text{ if } g(x) < 0. \end{cases}$

Therefore, for any $\Psi \in C_+(\overline{D}_+^p \times \overline{D}_-^q)$, we have

$$\begin{aligned}
&\int_{D_+^{p+1}\times D_-^{q+1}} \ell(x,y)\,\Psi(\vec{z},\vec{w})\left|E_0^{(p,q)}(x,y,\vec{z},\vec{w})\right| d(x,y,\ \vec{z},\vec{w}) \\
\leq\ & \frac{(N-p)(N-q)}{N^2}\int_{D_+^{p+1}\times D_-^{q+1}} \ell\,\Psi\left|F_u^{(p+1,q+1)} - F_u^{(1,1)}\cdot F_u^{(p,q)}\right| \\
&+\left|\frac{(N-p)(N-q)}{N^2}-1\right|\left(\int_{D_+\times D_-}\ell\,F_u^{(1,1)}\right)\left(\int_{D_+^p\times D_-^q}\Psi\,F_u^{(p,q)}\right) \\
&+\frac{N-q}{N^2}\sum_{i=1}^{p}\int_{D_+^p\times D_-^{q+1}} \ell(z_i,y)\,\Psi(\vec{z},\vec{w})\,F_u^{(p,q+1)}(\vec{z},\ (y,\vec{w})) \\
&+\frac{N-p}{N^2}\sum_{j=1}^{q}\int_{D_+^{p+1}\times D_-^{q}} \ell(x,w_j)\,\Psi(\vec{z},\vec{w})\,F_u^{(p+1,q)}((x,\vec{z}),\ \vec{w}) \\
&+\frac{1}{N^2}\sum_{i=1}^{p}\sum_{j=1}^{q}\int_{D_+^p\times D_-^q} \ell(z_i,w_j)\,\Psi(\vec{z},\vec{w})\,F_u^{(p,q)}(\vec{z},\ \vec{w}) \\
\leq\ & \int_{D_+^{p+1}\times D_-^{q+1}} \ell\,\Psi\left|F_u^{(p+1,q+1)} - F_u^{(1,1)}\cdot F_u^{(p,q)}\right| \\
&+\frac{p+q}{N}\left(\int_{D_+\times D_-}\ell\,F_u^{(1,1)}\right)\left(\int_{D_+^p\times D_-^q}\Psi\,F_u^{(p,q)}\right) \\
&+\frac{1}{N}\sum_{i=1}^{p}\int_{D_+^p\times D_-^{q+1}} \ell(z_i,y)\,\Psi(\vec{z},\vec{w})\,F_u^{(p,q+1)}(\vec{z},\ (y,\vec{w})) \\
&+\frac{1}{N}\sum_{j=1}^{q}\int_{D_+^{p+1}\times D_-^{q}} \ell(x,w_j)\,\Psi(\vec{z},\vec{w})\,F_u^{(p+1,q)}((x,\vec{z}),\ \vec{w}) \\
&+\frac{1}{N^2}\sum_{i=1}^{p}\sum_{j=1}^{q}\int_{D_+^p\times D_-^q} \ell(z_i,w_j)\,\Psi(\vec{z},\vec{w})\,F_u^{(p,q)}(\vec{z},\ \vec{w}).
\end{aligned}$$

On other hand, by Corollary 6.3.6,

$$
\begin{aligned}
&N \left| F_u^{(p+1,q+1)}(x,y,\vec{z},\vec{w}) - F_u^{(1,1)}(x,y) \cdot F_u^{(p,q)}(\vec{z},\vec{w}) \right| \qquad (6.4.10)\\
\leq\ & C_0^{p+q} \left(\sum_{i=1}^{p} G_u(z_i,y) + \sum_{j=1}^{q} G_u(x,w_j) + \sum_{i=1}^{p} G_u^{+}(x,z_i) + \sum_{j=1}^{q} G_u^{-}(y,w_j) \right) + \frac{(C_0\, C)^{p+q+2}\, u}{N\, \delta_N^{2d}}.
\end{aligned}
$$

Combining with the calculation just before the previous inequality, we obtain

$$
\begin{aligned}
&N \int_{D_+^{p+1}\times D_-^{q+1}} \ell(x,y)\, \Psi(\vec{z},\vec{w}) \left| E_0^{(p,q)}(x,y,\vec{z},\vec{w}) \right| \, d(x,y,\ \vec{z},\vec{w}) \qquad (6.4.11)\\
\leq\ & C_0^{p+q} \int_{D_+^{p+1}\times D_-^{q+1}} \ell(x,y)\, \Psi(\vec{z},\vec{w}) \left(\sum_{i=1}^{p} G_u(z_i,y) + \sum_{j=1}^{q} G_u(x,w_j) + \sum_{i=1}^{p} G_u^{+}(x,z_i) + \sum_{j=1}^{q} G_u^{-}(y,w_j) \right)\\
&+ \frac{(C_0\, C)^{p+q+2}\, u}{N\, \delta_N^{2d}} \int_{D_+^{p}\times D_-^{q}} \Psi(\vec{z},\vec{w})\\
&+ C_0^{p+q+1} \int_{D_+^{p}\times D_-^{q+1}} \sum_{i=1}^{p} \ell(z_i,y)\, \Psi(\vec{z},\vec{w})\\
&+ C_0^{p+q+1} \int_{D_+^{p+1}\times D_-^{q}} \sum_{j=1}^{q} \ell(x,w_j)\, \Psi(\vec{z},\vec{w})\\
&+ \frac{C_0^{p+q}}{N} \int_{D_+^{p}\times D_-^{q}} \sum_{i=1}^{p}\sum_{j=1}^{q} \ell(z_i,w_j)\, \Psi(\vec{z},\vec{w}).
\end{aligned}
$$

Step 4: Final estimates.

We now put $\Psi = \Psi^{\vec{\theta}}$ into inequality (6.4.11) for each $\Psi^{\vec{\theta}}$ that appears in (6.4.8) at the end of Step 2. Specifically, by (6.4.8) and (6.4.11) respectively, we have

$$
\begin{aligned}
&N \int_{(x,y)\in D_+\times D_-} \int_{(\tilde{x},\tilde{y})\in D_+\times D_-} \ell(x,y)\, \ell(\tilde{x},\tilde{y}) \left| E_t^{(1,1)}(x,y,\ \tilde{x},\tilde{y}) \right|\\
\leq\ & \sum_{M=0}^{\infty} \int_{t_2=0}^{t} \cdots \int_{t_{M+1}=0}^{t_M} \sum_{\vec{\theta}\in \mathbb{T}_M^{(1,1)}} \left[N \int_{(x,y)} \ell(x,y) \int_{(\vec{z},\vec{w})\in D_+^{p}\times D_-^{q}} \Psi^{\vec{\theta}}(\vec{z},\vec{w}) \left| E_{u,0}^{(1,1),\, l_M(\vec{\theta})}(x,y,\vec{z},\vec{w}) \right| \right]\\
\leq\ & \sum_{M=0}^{\infty} \int_{t_2=0}^{t} \cdots \int_{t_{M+1}=0}^{t_M} \sum_{\vec{\theta}\in \mathbb{T}_M^{(1,1)}} \left[\sum_{i=1}^{5} \Theta_i^{l_M(\vec{\theta})}\left(\Psi^{\vec{\theta}}\right) \right], \qquad (6.4.12)
\end{aligned}
$$

where in the first inequality, the integration over the variables $(\vec{z}, \vec{w})$ is on $D_+^p \times D_-^q$ where $l_M(\vec{\theta}) = (p, q)$; in the second inequality, $\Theta_i^{l_M(\vec{\theta})}\Big(\Psi\Big)$ is the i-th term that appear on the RHS of (6.4.11).

We will estimate each of the five terms ($i = 1, 2, 3, 4, 5$) on the RHS of (6.4.12) separately. The arguments are the same for all of them. We first consider the term for $i = 2$. This term is

$$\begin{aligned} & \sum_{M=0}^{\infty} \int_{t_2=0}^{t} \cdots \int_{t_{M+1}=0}^{t_M} \sum_{\vec{\theta} \in \mathbb{T}_M^{(1,1)}} \Big[\Theta_2^{l_M(\vec{\theta})}\Big(\Psi^{\vec{\theta}}\Big) \Big] \\ = & \sum_{M=0}^{\infty} \int_{t_2=0}^{t} \cdots \int_{t_{M+1}=0}^{t_M} \sum_{\vec{\theta} \in \mathbb{T}_M^{(1,1)}} \Big[\frac{(C_0\, C)^{M+4}\, u}{N\, \delta_N^{2d}} \int_{D_+^p \times D_-^q} \Psi^{\vec{\theta}}(\vec{z}, \vec{w}) \Big], \end{aligned} \tag{6.4.13}$$

where we have used the fact that the sum of the two components of $l_M(\vec{\theta})$ is $M + 2$ (i.e. $p + q = M + 2$). Using the same argument of Step 3 in the proof of Theorem 6.3.3, we have, for each $M \geq 1$,

$$\begin{aligned} & \int_{t_2=0}^{t} \cdots \int_{t_{M+1}=0}^{t_M} \sum_{\vec{\theta} \in \mathbb{T}_M^{(1,1)}} \int_{D_+^p \times D_-^q} \Psi^{\vec{\theta}}(\vec{z}, \vec{w}) \\ \leq & \int_{t_2=0}^{t} \cdots \int_{t_{M+1}=0}^{t_M} \sum_{\vec{\upsilon} \in \mathbb{S}_M^{(1,1)}} \frac{C^M}{\sqrt{(t_{\upsilon_1} - t_2)\,(t_{\upsilon_2} - t_3) \cdots (t_{\upsilon_M} - t_{M+1})}} \\ \leq & \; C^M\, t^{M/2} \end{aligned}$$

for $N \geq N(D_+, D_-)$, where $C = C(D_+, D_-, T) > 0$. This inequality implies that (6.4.13) is at most

$$\frac{u}{N\, \delta_N^{2d}}\, (C_0\, C)^4 \sum_{M=0}^{\infty} (C_0\, C)^M\, t^{M/2} \leq \frac{C_0^4\, C\, u}{N \delta_N^{2d}}$$

when $0 \leq t \leq (C_0\, C)^{-2}$ and N is large enough, where $C = C(D_+, D_-, T) > 0$.

For $i = 1$, we only need to invoke Lemma 6.2.5 and then use the same argument for $i = 2$.

The term on the RHS of (6.4.12) for $i = 1$ is at most

$$\frac{C_0^4\, C\, \sqrt{u}}{\sqrt{t}} + \frac{C_0^5\, C\, u}{\sqrt{t}}.$$

For $i = 3$, the term on the RHS of (6.4.12) is equal to

$$\sum_{M=0}^{\infty} \int_{t_2=0}^{t} \cdots \int_{t_{M+1}=0}^{t_M} \sum_{\vec{\theta} \in \mathbb{T}_M^{(1,1)}} \left[C_0^{M+3} \int_{D_+^p \times D_-^{q+1}} \sum_{i=1}^{p} \ell(z_i, y)\, \Psi^{\vec{\theta}}(\vec{z}, \vec{w}) \right]. \qquad (6.4.14)$$

By the same argument as that for $i = 2$, we have, for each $M \geq 1$,

$$\begin{aligned}
& \int_{t_2=0}^{t} \cdots \int_{t_{M+1}=0}^{t_M} \sum_{\vec{\theta} \in \mathbb{T}_M^{(1,1)}} \int_{D_+^p \times D_-^{q+1}} \sum_{i=1}^{p} \ell(z_i, y)\, \Psi^{\vec{\theta}}(\vec{z}, \vec{w}) \\
\leq\ & C^M \int_{t_2=0}^{t} \cdots \int_{t_M=0}^{t_{M-1}} \int_{t_{M+1}=0}^{t_M} \Big(\frac{M+1}{\sqrt{t_{M+1}}}\Big) \sum_{\vec{\upsilon} \in \mathbb{S}_M^{(1,1)}} \frac{1}{\sqrt{(t_{\upsilon_1} - t_2)\,(t_{\upsilon_2} - t_3) \cdots (t_{\upsilon_M} - t_{M+1})}} \\
\leq\ & C^M \int_{t_2=0}^{t} \cdots \int_{t_M=0}^{t_{M-1}} (M+1) \sum_{\vec{\upsilon} \in \mathbb{S}_{M-1}^{(1,1)}} \frac{1}{\sqrt{(t_{\upsilon_1} - t_2)\,(t_{\upsilon_2} - t_3) \cdots (t_{\upsilon_{M-1}} - t_M)}} \\
& \qquad \cdot \Big(\int_{t_{M+1}=0}^{t_M} \frac{M+1}{\sqrt{(t_M - t_{M+1})\, t_{M+1}}}\, dt_{M+1} \Big) \\
=\ & C^M\, (M+1)^2\, \pi \int_{t_2=0}^{t} \cdots \int_{t_M=0}^{t_{M-1}} \sum_{\vec{\upsilon} \in \mathbb{S}_{M-1}^{(1,1)}} \frac{1}{\sqrt{(t_{\upsilon_1} - t_2)\,(t_{\upsilon_2} - t_3) \cdots (t_{\upsilon_{M-1}} - t_M)}} \\
\leq\ & C^M\, t^{(M-1)/2},
\end{aligned}$$

where we have used the facts that $\int_0^{t_M} \frac{1}{\sqrt{s(t_M - s)}}\, ds = \pi$ and that $\upsilon_M \leq M$. The extra factor $(M+1)$ in the second inequality comes from the number of children (in $\mathbb{S}_M^{(1,1)}$) for each leaf in $\mathbb{S}_{M-1}^{(1,1)}$. Therefore, (6.4.14) is at most

$$C_0^3 \sum_{M=0}^{\infty} (C_0\, C)^M\, t^{(M-1)/2} \leq \frac{C_0^3\, C}{\sqrt{t}}.$$

The term for $i = 4$ is symmetric to that of $i = 3$, hence the upper bound is of the same form.

Finally, the term for $i = 5$ can be compared to the term for $i = 2$ directly, since $\sup_{(x,y)} \frac{\ell(x,y)}{N} \leq \delta^{d-1} \leq 1$ under Assumption 6.1.1 and hence we can ignore the factor $\ell(z_i, w_j)$. Therefore, the term for $i = 5$ is at most $C_0^4\, C$.

From the above five estimates for the RHS of (6.4.12), it follows that

$$
\begin{aligned}
& N \int_{(x,y)\in D_+\times D_-} \int_{(\tilde{x},\tilde{y})\in D_+\times D_-} \ell(x,y)\, \ell(\tilde{x},\tilde{y}) \left| E_t^{(1,1)}(x,y,\, \tilde{x},\tilde{y}) \right| \\
\leq \;\; & C \left(\frac{C_0^4 \sqrt{u}}{\sqrt{t}} + \frac{C_0^5\, u}{\sqrt{t}} + \frac{C_0^4\, u}{N\delta_N^{2d}} + \frac{C_0^3}{\sqrt{t}} + C_0^4 \right)
\end{aligned}
$$

for $N \geq N(D_+, D_-)$, where $C = C(D_+, D_-, T) > 0$. Therefore the assertion in Step 1 is valid under Assumption 6.1.1. The proof of the lemma is complete. □

6.4.2 A technical lemma towards Boltzman-Gibbs principle

Our goal for this subsection is to prove the following lemma, which is an indicator of the validity of the Boltzman-Gibbs principle for our annihilating diffusion model. It is instructive to compare the statement of Lemma 6.4.4 below with that of Lemma 6.4.3.

Lemma 6.4.4. *Suppose Assumption 6.1.1 holds. For any $T > 0$, there exists $C = C(D_+, D_-, T) > 0$, $N_0 = N_0(D_+, D_-)$ and positive constants $\{C_N\}$ satisfying $\lim_{N\to\infty} C_N = 0$ such that*

$$
\mathbb{E}\Big[\Big| \int_0^t \mathcal{Z}_s^N \Big(\langle \ell\varphi_s,\, f_s^- \rangle_-,\, \langle \ell\varphi_s,\, f_s^+ \rangle_+ \Big) \; - \; \sqrt{N} \Big(\langle \ell\varphi_s,\, \otimes_s \rangle - \mathbb{E}[\langle \ell\varphi_s,\, \otimes_s \rangle] \Big)\, ds \Big|^2 \Big] \leq C_N\, \|\varphi\|^2\, t^{3/2}
$$

whenever $0 \leq t \leq T_0 := T \wedge (C_0\, C)^{-2}$, for any $N > N_0$ and any bounded function $\varphi_t(x,y)$ on $[0,\, T_0] \times D_+ \times D_-$ with uniform norm $\|\varphi\|$. Here $\otimes_s := \mathfrak{X}_s^{N,+} \otimes \mathfrak{X}_s^{N,-}$ in abbreviation.

Proof The proof follows from the same argument that we used for Lemma 6.4.3. Namely, we first write the LHS in terms of the generalized correlation functions (more specifically in terms of $E_{u,r} = E_{u,r}^{(n,m),(p,q)}$ defined in (6.4.2)); we then bound $E_{u,r}$ in terms of $E_{u,0}$ via (6.4.7); finally we estimate $E_{u,0}$. However, unlike Lemma 6.4.3, the LHS here *vanishes* in the limit due to a 'magical cancelations' of the first two terms in the asymptotic expansion of the correlation functions. See (6.4.21) and (6.4.22) in the proof below.

Step 1: Abbreviations and notations.

To avoid unnecessary complications, we assume $\varphi_s = 1$ in the proof. The general case follows from a routine modification. By the fact (which follows from Fubinni's theorem and a change of variable $r = v - u$)

$$\Big(\int_0^t h(s)\,ds\Big)^2 = 2\int_{u=0}^t \int_{v=u}^t h(u)h(v)\,dv\,du = 2\int_{u=0}^t \int_{r=0}^{t-u} h(u)h(r)\,dr\,du,$$

we have

$$\begin{aligned}
&\mathbb{E}\Big[\Big|\int_0^t \mathcal{Z}_s^N\Big(\langle \ell,\, f_s^-\rangle_-,\, \langle \ell,\, f_s^+\rangle_+\Big) \,-\, \sqrt{N}\Big(\langle \ell,\, \otimes_s\rangle - \mathbb{E}[\langle \ell,\, \otimes_s\rangle]\Big)\,ds\Big|^2\Big] \\
= \;& N\,\mathbb{E}\Big[\Big|\int_0^t \langle \alpha_s,\, \mathfrak{X}_s^{N,+}\rangle + \langle \beta_s,\, \mathfrak{X}_s^{N,-}\rangle - \langle \ell,\, \otimes_s\rangle \,-\, \mathbb{E}[\langle \alpha_s,\, \mathfrak{X}_s^{N,+}\rangle + \langle \beta_s,\, \mathfrak{X}_s^{N,-}\rangle - \langle \ell,\, \otimes_s\rangle]\,ds\Big|^2\Big] \\
= \;& N\,\mathbb{E}\Big[\Big|\int_0^t (\eta_s - \xi_s) \,-\, \mathbb{E}[\eta_s - \xi_s]\,ds\Big|^2\Big] \\
= \;& 2N\int_0^t\int_u^t \mathbb{E}[(\eta_u - \xi_u)(\eta_v - \xi_v)] \,-\, \mathbb{E}[\eta_u - \xi_u]\cdot\mathbb{E}[\eta_v - \xi_v]\,dv\,du,
\end{aligned} \tag{6.4.15}$$

where we have introduced the abbreviations

$$\alpha_s := \langle \ell,\, f_s^-\rangle_-,\ \beta_s := \langle \ell,\, f_s^+\rangle_+,\ \eta_s := \langle \alpha_s,\, \mathfrak{X}_s^{N,+}\rangle + \langle \beta_s,\, \mathfrak{X}_s^{N,-}\rangle \text{ and } \xi_s := \langle \ell,\, \otimes_s\rangle. \tag{6.4.16}$$

Note that we have, for example,

$$\begin{aligned}
\mathbb{E}[\eta_s] \;&=\; \int_{D_+} \alpha_s(x)\,F_s^{(1,0)}(x)\,dx + \int_{D_-} \beta_s(y)\,F_s^{(0,1)}(y)\,dy \\
&=\; \int_{D_+\times D_-} \ell(x,y)\,f_s^-(y)\,F_s^{(1,0)}(x) \,+\, \ell(x,y)\,f_s^+(x)\,F_s^{(0,1)}(y)\,dx\,dy.
\end{aligned}$$

Step 2: Write LHS in terms of correlation functions.

Direct calculation yields

$$
\begin{aligned}
& \mathbb{E}[(\eta_u-\xi_u)(\eta_v-\xi_v)] = \mathbb{E}[\eta_u\eta_v - \eta_v\xi_u - \eta_u\xi_v + \xi_u\xi_v] \\
= \; & \int_{D_+^2} \alpha_u(x_1)\alpha_v(x_2)\, F_{u,v}^{(10)(10)}(x_1,x_2) + \int_{D_+\times D_-} \alpha_u(x_1)\beta_v(y_2)\, F_{u,v}^{(10)(01)}(x_1,y_2) \\
& + \int_{D_+\times D_-} \alpha_u(x_2)\beta_v(y_1)\, F_{u,v}^{(01)(10)}(x_2,y_1) + \int_{D_-^2} \beta_u(y_1)\beta_v(y_2)\, F_{u,v}^{(01)(01)}(y_1,y_2) \\
& - \int_{D_+^2\times D_-} \alpha_v(x_2)\,\ell(x_1,y_1)\, F_{u,v}^{(11)(10)}((x_1,y_1),x_2) - \int_{D_+\times D_-^2} \beta_v(y_2)\,\ell(x_1,y_1)\, F_{u,v}^{(11)(01)}((x_1,y_1),y_2) \\
& - \int_{D_+^2\times D_-} \alpha_u(x_1)\,\ell(x_2,y_2)\, F_{u,v}^{(10)(11)}(x_1,(x_2,y_2)) - \int_{D_+\times D_-^2} \beta_u(y_1)\,\ell(x_2,y_2)\, F_{u,v}^{(01)(11)}(y_1,(x_2,y_2)) \\
& + \int_{D_+^2\times D_-^2} \ell(x_1,y_1)\,\ell(x_2,y_2)\, F_{u,v}^{(11)(11)}((x_1,y_1),(x_2,y_2)).
\end{aligned}
$$

Computing $\mathbb{E}[\eta_u-\xi_u]\cdot\mathbb{E}[\eta_v-\xi_v]$ in the same way, then using the definition of α_s and β_s in (6.4.16), we can rewrite the integrand in (6.4.15) as follows.

$$
\begin{aligned}
& \mathbb{E}[(\eta_u-\xi_u)(\eta_v-\xi_v)] - \mathbb{E}[\eta_u-\xi_u]\cdot\mathbb{E}[\eta_v-\xi_v] \qquad (6.4.17) \\
= \; & \int_{D_+^2\times D_-^2} \ell(x_1,y_1)\,\ell(x_2,y_2) \Big\{ f_u^-(y_1)f_v^-(y_2)\left[F_{u,v}^{(10)(10)}(x_1,x_2) - F_u^{(10)}(x_1)F_v^{(10)}(x_2)\right] \\
& \quad + f_u^-(y_1)f_v^+(x_2)\left[F_{u,v}^{(10)(01)}(x_1,y_2) - F_u^{(10)}(x_1)F_v^{(01)}(y_2)\right] \\
& \quad + f_u^+(x_1)f_v^-(y_2)\left[F_{u,v}^{(01)(10)}(y_1,x_2) - F_u^{(01)}(y_1)F_v^{(10)}(x_2)\right] \\
& \quad + f_u^+(x_1)f_v^+(x_2)\left[F_{u,v}^{(01)(01)}(y_1,y_2) - F_u^{(01)}(y_1)F_v^{(01)}(y_2)\right] \\
& \quad - f_v^-(y_2)\left[F_{u,v}^{(11)(10)}((x_1,y_1),x_2) - F_u^{(11)}(x_1,y_1)F_v^{(10)}(x_2)\right] \\
& \quad - f_v^+(x_2)\left[F_{u,v}^{(11)(01)}((x_1,y_1),y_2) - F_u^{(11)}(x_1,y_1)F_v^{(01)}(y_2)\right] \\
& \quad - f_u^-(y_1)\left[F_{u,v}^{(10)(11)}(x_1,(x_2,y_2)) - F_u^{(10)}(x_1)F_v^{(11)}(x_2,y_2)\right] \\
& \quad - f_u^+(x_1)\left[F_{u,v}^{(01)(11)}(y_1,(x_2,y_2)) - F_u^{(01)}(y_1)F_v^{(11)}(x_2,y_2)\right] \\
& \quad + \left[F_{u,v}^{(11)(11)}((x_1,y_1),(x_2,y_2)) - F_u^{(11)}(x_1,y_1)F_v^{(11)}(x_2,y_2)\right] \Big\}.
\end{aligned}
$$

Note that each of the nine terms can be written in terms of

$$E_{u,r}^{(n,m),(p,q)}(\vec{x},\vec{y},\ \vec{z},\vec{w}) := F_{u,u+r}^{(n,m),(p,q)}(\vec{x},\vec{y},\ \vec{z},\vec{w}) - F_u^{(n,m)}(\vec{x},\vec{y}) \cdot F_{u+r}^{(p,q)}(\vec{z},\vec{w})$$

defined in (6.4.2), where $r = v - u$. We split these nine terms into three groups $\Lambda_1(u,v) + \Lambda_2(u,v) + \Lambda_3(u,v)$, where $\Lambda_1(u,v)$ consists of the first, third and fifth terms; $\Lambda_2(u,v)$ consists of the second, forth and sixth terms; and $\Lambda_3(u,v)$ consists of the last three terms. That is,

$$\begin{aligned}\Lambda_1(u,v) \quad := \quad \int_{D_+^2\times D_-^2} \ell(x_1,y_1)\,\ell(x_2,y_2)\Big\{ & f_u^-(y_1)f_v^-(y_2)\,E_{u,r}^{(10)(10)}(x_1,x_2) \\ & +f_u^+(x_1)f_v^-(y_2)\,E_{u,r}^{(01)(10)}(y_1,x_2) \\ & -f_v^-(y_2)\,E_{u,r}^{(11)(10)}((x_1,y_1),x_2)\Big\},\end{aligned} \qquad (6.4.18)$$

$$\begin{aligned}\Lambda_2(u,v) \quad := \quad \int_{D_+^2\times D_-^2} \ell(x_1,y_1)\,\ell(x_2,y_2)\Big\{ & f_u^-(y_1)f_v^-(y_2)\,E_{u,r}^{(10)(10)}(x_1,x_2) \\ & +f_u^-(y_1)f_v^+(x_2)\,E_{u,r}^{(10)(01)}(x_1,y_2) \\ & +f_u^+(x_1)f_v^+(x_2)\,E_{u,r}^{(01)(01)}(y_1,y_2) \\ & -f_v^+(x_2)\,E_{u,r}^{(11)(01)}((x_1,y_1),y_2)\Big\}\end{aligned} \qquad (6.4.19)$$

and

$$\begin{aligned}\Lambda_3(u,v) \quad := \quad \int_{D_+^2\times D_-^2} \ell(x_1,y_1)\,\ell(x_2,y_2)\Big\{ & -f_u^-(y_1)\,E_{u,r}^{(10)(11)}(x_1,(x_2,y_2)) \\ & -f_u^+(x_1)\,E_{u,r}^{(01)(11)}(y_1,(x_2,y_2)) \\ & +E_{u,r}^{(11)(11)}((x_1,y_1),(x_2,y_2))\Big\}.\end{aligned} \qquad (6.4.20)$$

Step 3: Cancellations. To illustrate the 'magical cancellations' mentioned at the beginning of the proof, we first provide details of these cancellations for Λ_3.

Note that we can bound $E_{u,r}$ in terms of $E_{u,0}$ via (6.4.7). Consider the first among the three terms in Λ_3 with $E_{u,r}$ replaced $E_{u,0}$. We apply (6.4.9) to write $F_{u,u}^{(10)(11)}$ in terms of $F_u^{(21)}$ plus a lower order term. This gives

$$\begin{aligned}
& -\int_{D_+^2\times D_-^2} \ell(x_1,y_1)\,\ell(x_2,y_2)\,f_u^-(y_1)\,E_{u,0}^{(10)(11)}(x_1,(x_2,y_2)) \\
= & -\int_{D_+^2\times D_-^2} \ell(x_1,y_1)\,\ell(x_2,y_2)\,f_u^-(y_1)\left(F_u^{(21)}(x_1,x_2,y_2) - F_u^{(10)}(x_1)F_u^{(11)}(x_2,y_2)\right) \\
& -\frac{1}{N}\int_{D_+\times D_-^2} \ell(x,y_1)\,\ell(x,y_2)\,f_u^-(y_1)\,F_u^{(11)}(x,y_2).
\end{aligned}$$

Similarly, when $r=0$, the second term and the third term in Λ_2 are, respectively,

$$\begin{aligned}
& -\int_{D_+^2\times D_-^2} \ell(x_1,y_1)\,\ell(x_2,y_2)\,f_u^+(x_1)\,E_{u,0}^{(01)(11)}(x_2,(y_1,y_2)) \\
= & -\int_{D_+^2\times D_-^2} \ell(x_1,y_1)\,\ell(x_2,y_2)\,f_u^+(x_1)\left(F_u^{(12)}(x_2,y_1,y_2) - F_u^{(01)}(y_1)F_u^{(11)}(x_2,y_2)\right) \\
& -\frac{1}{N}\int_{D_+^2\times D_-} \ell(x_1,y)\,\ell(x_2,y)\,f_u^+(x_1)\,F_u^{(11)}(x_2,y)
\end{aligned}$$

and

$$\begin{aligned}
& \int_{D_+^2\times D_-^2} \ell(x_1,y_1)\,\ell(x_2,y_2)\,E_{u,0}^{(11)(11)}((x_1,y_1),(x_2,y_2)) \\
= & \int_{D_+^2\times D_-^2} \ell(x_1,y_1)\,\ell(x_2,y_2)\left(F_u^{(22)}(x_1,x_2,y_1,y_2) - F_u^{(11)}(x_1,y_1)F_u^{(11)}(x_2,y_2)\right) \\
& +\frac{1}{N}\int_{D_+\times D_-^2} \ell(x,y_1)\,\ell(x,y_2)\,F_u^{(12)}(x,y_1,y_2) \\
& +\frac{1}{N}\int_{D_+^2\times D_-} \ell(x_1,y)\,\ell(x_2,y)\,F_u^{(21)}(x_1,x_2,y) \\
& +\frac{1}{N^2}\int_{D_+\times D_-} \ell^2(x,y)\,F_u^{(11)}(x,y).
\end{aligned}$$

Now we add up the three equations above. The sum of the lower order terms is, by Theorem 6.3.3 or (6.3.12), of order $o(N)/N$ (i.e. a term which tends to zero even if we multiply it by N) uniformly for $u \in [0,t]$. On other hand, the sum of the leading terms is, by Theorem 6.3.3

again, equal to

$$\int_{D_+^2\times D_-^2} \ell(x_1,y_1)\,\ell(x_2,y_2)\Big\{ -\, f_u^-(y_1)\Big(F_u^{(21)}(x_1,x_2,y_2)-F_u^{(10)}(x_1)F_u^{(11)}(x_2,y_2)\Big)$$
$$-\,f_u^+(x_1)\Big(F_u^{(12)}(x_2,y_1,y_2)-F_u^{(01)}(y_1)F_u^{(11)}(x_2,y_2)\Big)$$
$$+\,F_u^{(22)}(x_1,x_2,y_1,y_2)-F_u^{(11)}(x_1,y_1)F_u^{(11)}(x_2,y_2)\Big\}$$

$$\begin{aligned}
= \;& \frac{1}{N}\int_{D_+^2\times D_-^2} \ell(x_1,y_1)\,\ell(x_2,y_2)\Big\{ \\
& \quad -f_u^-(y_1)\Big(B_u^{(21)}(x_1,x_2,y_2)-A_u^{(10)}(x_1)B_u^{(11)}(x_2,y_2)-B_u^{(10)}(x_1)A_u^{(11)}(x_2,y_2)\Big) \\
& \quad -f_u^+(x_1)\Big(B_u^{(12)}(x_2,y_1,y_2)-A_u^{(01)}(y_1)B_u^{(11)}(x_2,y_2)-B_u^{(01)}(y_1)A_u^{(11)}(x_2,y_2)\Big) \\
& \quad +B_u^{(22)}(x_1,x_2,y_1,y_2)-A_u^{(11)}(x_1,y_1)B_u^{(11)}(x_2,y_2)-B_u^{(11)}(x_1,y_1)A_u^{(11)}(x_2,y_2)\Big\} \\
& +o(N)/N \\
= \;& \frac{-1}{N}\int_{D_+^2\times D_-^2} \ell(x_1,y_1)\,\ell(x_2,y_2)\Big\{ \\
& \quad -f_u^-(y_1)\Big(G_u(x_1,y_2)f_u^+(x_2)+G_u^+(x_1,x_2)f_u^-(y_2)\Big) \\
& \quad -f_u^+(x_1)\Big(G_u(x_2,y_1)f_u^+(x_1)+G_u^-(y_1,y_2)f_u^+(x_2)\Big) \\
& \quad +\Big(G_u(x_1,y_2)f_u^+(x_2)f_u^-(y_1)+G_u(x_2,y_1)f_u^+(x_1)f_u^-(y_2)\Big) \\
& \quad +\Big(G_u^+(x_1,x_2)f_u^-(y_1)f_u^-(y_2)+G_u^-(y_1,y_2)f_u^+(x_1)f_u^+(x_2)\Big)\Big\} \\
& +o(N)/N \quad \text{by (6.3.13)} \\
= \;& o(N)/N. \qquad (6.4.21)
\end{aligned}$$

The two $o(N)/N$ terms are the same and can be kept track of via the computation in the proof of Corollary 6.3.6. Note that ALL terms involving G, G^+, G^- cancel out in the last equality. The cancellation in (6.4.21), together with the cancellation for the lower order terms, are the 'magical cancellations' mentioned at the beginning of the proof.

The same type of 'magical cancellations' occur for each of Λ_1 and Λ_2 by the same reasons.

In short, applying (6.4.9) and (6.3.13) to each of the six terms in $\Lambda_1 + \Lambda_2$, we see that the sum of these six terms when $r = 0$ is, up to an additive error of order $o(N)/N$ which is uniform for $u \in [0, t]$, equal to

$$
\begin{aligned}
&\frac{1}{N} \int_{D_+^2 \times D_-^2} \ell(x_1, y_1)\, \ell(x_2, y_2) \Big\{ \\
&\qquad f_u^-(y_1) f_v^-(y_2) \Big(B_u^{(20)}(x_1, x_2) - A_u^{(10)}(x_1) B_u^{(10)}(x_2) - B_u^{(10)}(x_1) A_u^{(10)}(x_2) \Big) \\
&\qquad + f_u^-(y_1) f_v^+(x_2) \Big(B_u^{(11)}(x_1, y_2) - A_u^{(10)}(x_1) B_u^{(01)}(y_2) - B_u^{(10)}(x_1) A_u^{(01)}(y_2) \Big) \\
&\qquad + f_u^+(x_1) f_v^-(y_2) \Big(B_u^{(11)}(x_2, y_1) - A_u^{(01)}(y_1) B_u^{(10)}(x_2) - B_u^{(01)}(y_1) A_u^{(10)}(x_2) \Big) \\
&\qquad + f_u^+(x_1) f_v^+(x_2) \Big(B_u^{(02)}(y_1, y_2) - A_u^{(01)}(y_1) B_u^{(01)}(y_2) - B_u^{(01)}(y_1) A_u^{(01)}(y_2) \Big) \\
&\qquad - f_v^-(y_2) \Big(B_u^{(21)}(x_1, x_2, y_1) - A_u^{(11)}(x_1, y_1) B_u^{(10)}(x_2) - B_u^{(11)}(x_1, y_1) A_u^{(10)}(x_2) \Big) \\
&\qquad - f_v^+(x_2) \Big(B_u^{(12)}(x_1, y_1, y_2) - A_u^{(11)}(x_1, y_1) B_u^{(01)}(y_2) - B_u^{(11)}(x_1, y_1) A_u^{(01)}(y_2) \Big) \Big\}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{-1}{N} \int_{D_+^2 \times D_-^2} \ell(x_1, y_1)\, \ell(x_2, y_2) \Big\{ f_u^-(y_1) f_v^-(y_2)\, G_u^+(x_1, x_2) \\
&\qquad + f_u^-(y_1) f_v^+(x_2)\, G_u(x_1, y_2) \\
&\qquad + f_u^+(x_1) f_v^-(y_2)\, G_u(x_2, y_1) \\
&\qquad + f_u^+(x_1) f_v^+(x_2)\, G_u^-(y_1, y_2) \\
&\qquad - f_v^-(y_2) \Big(G_u(x_2, y_1) f_u^+(x_1) + G_u^+(x_1, x_2) f_u^-(y_1) \Big) \\
&\qquad - f_v^+(x_2) \Big(G_u(x_1, y_2) f_u^-(y_1) + G_u^-(y_1, y_2) f_u^+(x_1) \Big) \Big\} \\
= \quad & 0. \qquad (6.4.22)
\end{aligned}
$$

Observe that on the RHS of $\Lambda_3(u, v)$ in (6.4.20), if we view u and (x_1, y_1) as fixed variables, then

$$\begin{aligned}\Upsilon_r^{(p,q)}(x_2,y_2) \;&:=\; -f_u^-(y_1)\,E_{u,r}^{(10)(pq)}(x_1,(x_2,y_2)) - f_u^+(x_1)\,E_{u,r}^{(01)(pq)}(y_1,(x_2,y_2)) \\ &\quad + E_{u,r}^{(11)(pq)}((x_1,y_1),(x_2,y_2))\end{aligned}$$

satisfies

$$\Upsilon_r^{(p,q)} = P_r^{(p,q)}\Upsilon_0^{(p,q)} - \int_0^r P_{r-\theta}^{(p,q)}\left(V^+\Upsilon_\theta^{(p,q+1)} + V^-\Upsilon_\theta^{(p+1,q)} + \frac{Q}{N}E_\theta^{(p,q)}\right)\,d\theta \tag{6.4.23}$$

since $E_r^{(p,q)}$ satisfies (6.4.4). That is, $\{\Upsilon^{(p,q)}\}$ and $\{E^{(p,q)}\}$ solve the same hierarchy of equations, but the initial condition $\Upsilon_0^{(p,q)}$ is of smaller order of magnitude $o(N)/N$, by the above cancellations. Following the same argument that we used for Lemma 6.4.3, with $\Upsilon_r^{(p,q)}$ in place of $E_r^{(p,q)}$, while keeping track of these $o(N)$ terms. we obtain

$$N\int_0^t\int_u^t \Lambda_3(u,v)\,dv\,du \le o(N)\,t^{3/2} \tag{6.4.24}$$

whenever $0 \le t \le T_0 := T\wedge(C_0\,C)^{-2}$ and $N > N_0$. By the same argument, (6.4.24) holds with Λ_3 replaced by either Λ_1 or Λ_2.

Recall that the integrand of (6.4.15) is $\Lambda_1+\Lambda_2+\Lambda_3$. The proof is complete. □

6.5 Proof of the main theorem

With all the results developed in the previous sections, the proof of Theorem 6.1.2 is ready to be summarized in this section. Recall Steps 1-6 in the outline of proof at the beginning to this chapter. We will establish tightness of $\{\mathcal{Z}^N\}$ (which is Step 2) and then identify any subsequential limit through Steps 1, 3, 4, 5 and 6. Note that for Steps 1, 3, 4 and 5, we do not need to go into the analysis of correlation functions; hence the results for these steps are for arbitrary time interval rather than for a short time interval as in Steps 2 and 6.

The following is Step 2 in the outline of proof for Theorem 6.1.2. Note that we do not need any estimate about the evolution systems $\mathbf{U}^N_{(t,s)}$ and $\mathbf{U}_{(t,s)}$ for this step. The key of the proof is Lemma 6.4.3.

Theorem 6.5.1. *(Tightness) Suppose Assumption 6.1.1 holds and $\alpha > d \vee (d/2+2)$. For any $T > 0$, there exists $C = C(D_+, D_-, T) > 0$ such that $\{\mathcal{Z}^N\}$ is tight in $D([0,\, T_0],\, \mathbf{H}_{-\alpha})$, where $T_0 := T \wedge (\|u_0^+\| \vee \|u_0^-\|)^{-2}\, C$. Moreover, any subsequential limit has a continuous version.*

Proof We first prove the following one dimensional tightness result: For any $\phi_\pm \in C(\overline{D}_\pm)$ fixed (such as eigenfunctions), $\{(\mathcal{Y}^{N,+}(\phi_+),\, \mathcal{Y}^{N,-}(\phi_-))\}_N$ is tight in $D([0,T_0],\mathbb{R}^2)$. For this, it suffices to show $\{Z_N := \mathcal{Y}^{N,+}(\phi_+) + \mathcal{Y}^{N,-}(\phi_-)\}_N$ is tight in $D([0,T],\mathbb{R})$ for any fixed $\phi_\pm \in \mathcal{H}_\pm$ (cf. problem 22 in Chapter 3 of [35]). By Prohorov's Theorem. It suffices to show that

(1) for all $t \in [0,T_0]$ and $\epsilon_0 > 0$, there exists $K \in (0,\infty)$ s.t.

$$\overline{\lim_{N\to\infty}}\, \mathbb{P}\left(|Z_N(t)| > K\right) < \epsilon_0, \quad \text{and that}$$

(2) for all $\epsilon_0 > 0$, we have

$$\lim_{\delta\to 0}\overline{\lim_{N\to\infty}}\, \mathbb{P}\left(\sup_{\substack{|t-s|<\delta\\ 0\le s,t\le T}} \big|Z_N(t) - Z_N(s)\big| > \epsilon_0\right) = 0$$

By (6.1.4) and (6.1.5), we have $Z_N := \mathcal{Y}^{N,+}(\phi_+) + \mathcal{Y}^{N,-}(\phi_-)$ satisfies

$$\begin{aligned} Z_N(t) - Z_N(s) \;&=\; \int_s^t \mathcal{Y}^{N,+}_r(\tfrac{1}{2}\Delta\phi_+) + \mathcal{Y}^{N,-}_r(\tfrac{1}{2}\Delta\phi_-)\, dr \\ &\quad -\sqrt{N}\int_s^t \langle \ell(\phi_+ + \phi_-),\, \mathfrak{X}^{N,+}_r \otimes \mathfrak{X}^{N,-}_r\rangle - \mathbb{E}[\langle \ell(\phi_+ + \phi_-),\, \mathfrak{X}^{N,+}_r \otimes \mathfrak{X}^{N,-}_r\rangle]\, dr \\ &\quad + M_N(t) - M_N(s) \end{aligned} \tag{6.5.1}$$

for $0 \le s \le t$, where $M_N(t)$ is a real valued $\mathcal{F}_t^{(\mathfrak{X}^{N,+},\, \mathfrak{X}^{N,-})}$-martingale with quadratic variation

$$\int_0^t \langle |\nabla\phi_+|^2,\, \mathfrak{X}^{N,+}_s\rangle + \langle |\nabla\phi_-|^2,\, \mathfrak{X}^{N,-}_s\rangle + \langle \ell(\phi_+ + \phi_-)^2,\, \mathfrak{X}^{N,+}_s \otimes \mathfrak{X}^{N,-}_s\rangle\, ds. \tag{6.5.2}$$

(1) is implied by the fact that $\sup_{N\geq N_0(D)} \mathbb{E}[(Z_N(t))^2] < \infty$ for all $t \in [0,T_0]$ and $\alpha > d$. This fact can be proved as follows: By definition of the correlation functions, the covariance

$$
\begin{aligned}
& \mathbb{E}\left[\mathcal{Y}_t^{N,+}(\phi)\,\mathcal{Y}_t^{N,+}(\psi)\right] \qquad (6.5.3)\\
= \;& N\left(\mathbb{E}[\langle\phi,\mathfrak{X}_t^{N,+}\rangle\langle\psi,\mathfrak{X}_t^{N,+}\rangle] - \mathbb{E}[\langle\phi,\mathfrak{X}_t^{N,+}\rangle]\mathbb{E}[\langle\psi,\mathfrak{X}_t^{N,+}\rangle]\right)\\
= \;& N\bigg(\int\int \phi(x_1)\psi(x_2)F_t^{(2,0)}(x_1,x_2)\,dx_1dx_2 + \frac{1}{N}\int(\phi\psi)(x)F_t^{(1,0)}(x)dx\\
& \quad - \int\phi(x_1)F_t^{(1,0)}(x_1)dx_1\cdot\int\psi(x_2)F_t^{(1,0)}(x_2)dx_2\bigg)\\
= \;& \int(\phi\psi)(x)f_t^+(x)dx + N\int\int\phi(x_1)\psi(x_2)\left(F_t^{(2,0)}(x_1,x_2) - F_t^{(1,0)}(x_1)F_t^{(1,0)}(x_2)\right)dx_1dx_2.
\end{aligned}
$$

By Theorem 6.3.3 and Lemma 6.2.5, the absolute value of the last quantity in (6.5.3) is bounded above by

$$
\begin{aligned}
& C_0\int|(\phi\psi)(x)|dx + \|\phi\|\,\|\psi\|\left(\int\int|G^+(x_1,x_2)|\,dx_1dx_2 + \frac{(C_0\,C)^2\,t}{N\,\delta^{2d}}\right)\\
\leq\;& C\,\|\phi\|\,\|\psi\|\left(C_0 + C_0^3\,t^{3/2} + \frac{C_0^2\,t}{N\,\delta^{2d}}\right)\\
\leq\;& C\,\|\phi\|\,\|\psi\|\,(C_0\vee 1)
\end{aligned}
$$

for all $0\leq t\leq T\wedge(C_0\,C)^{-2}$ and $N\geq N_0(D)$, where $C_0 := \|u_0^+\|\vee\|u_0^-\|$ and $C = C(D_+,D_-,T)$. In particular, $\mathbb{E}[(\mathcal{Y}_t^{N,+}(\phi_k^+))^2]\leq C\,\|\phi_k^+\|^2$. Similarly, we have $\mathbb{E}[(\mathcal{Y}_t^{N,-}(\phi_k^-))^2]\leq C\,\|\phi_k^-\|^2$. Therefore, when $\alpha > d$, we have $\mathbb{E}[(Z_N(t))^2] < \infty$ for all $t\in[0,T_0]$ and $N\geq N_0(D)$ (as in the proof of Lemma 6.0.22). Hence (1) is satisfied.

It remains to show that (2) holds with $Z_N(t) - Z_N(s)$ replaced by each of the three terms on the RHS of (6.5.1). For the first term, (2) holds by Chebyshev's inequality, Holder's inequality and (6.5.3). For the second term, (2) holds by Lemma 6.4.3. For the third term, namely $M_N(t) - M_N(s)$, we have (2) holds upon applying Chebyshev's inequality, Doob's maximal inequality and the explicit expression for the quadratic variation (6.5.2). Hence we have one dimensional tightness for fixed $\phi_\pm\in C(\overline{D}_\pm)$.

Following the same proof of Theorem 5.4.7 in Chapter 5, we complete the proof by using the definition (6.0.6) of the metric of $\mathbf{H}_{-\alpha}$ and the condition on α. □

We identify any subsequential limit of $\{\mathcal{Z}^N\}$ for the rest of this section. Steps 1, 3, 4 and 5 follow from the method developed in Chapter 5, via the estimates for $\mathbf{U}^N_{(t,s)}$ and $\mathbf{U}_{(t,s)}$ that we developed. We now present the precise statements that we obtain.

Starting from Lemma 6.1.6 and the argument that leads to (6.1.6) at the beginning of this chapter, we can follow the proof of Theorem 5.4.3 in Chapter 5 to obtain Step 1.

Theorem 6.5.2. *Suppose $\alpha > d \vee (d/2+1)$. For all N large enough, there exists a càdlàg square integrable $\mathbf{H}_{-\alpha}$-valued $\mathcal{F}^N_t$-martingale $M^N = (M^N_t)_{t\geq 0}$ such that*

$$\mathcal{Z}^N_t = \mathbf{U}^N_{(t,0)}\mathcal{Z}^N_0 + \int_0^t \mathbf{U}^N_{(t,s)}\, dM^N_s + \int_0^t \mathbf{U}^N_{(t,s)}(\mathbf{B}^N_s \mathcal{Z}^N_s - K^N_s)\, ds \qquad \text{for } t\geq 0, \quad \mathbb{P}-a.s, \tag{6.5.4}$$

where $\mathbf{U}^N_{(t,s)}$ is defined in Definition 6.2.8 and $\mathcal{F}^N_t$ is the natural filtration of the annihilating diffusion process. Moreover, M^N has bounded jumps and predictable quadratic variation given by (6.1.5).

As a remark, equation (6.5.4) is equivalent to (6.1.4) by variation of constant (see Section 2.1.2 of [43]). For Step 4, it can be checked that we have the following, as in Theorem 5.4.8 in Chapter 5.

Lemma 6.5.3. *For $\alpha > d+2$ and $T>0$, we have*

$$\mathbf{U}^N_{(t,0)}\mathcal{Z}^N_0 \xrightarrow{\mathcal{L}} \mathbf{U}_{(t,0)}\mathcal{Z}_0 \quad \text{in } D([0,T],\mathbf{H}_{-\alpha}).$$

Moreover, $\mathbf{U}_{(t,0)}\mathcal{Z}_0$ has a version in $C^\gamma([0,T],\, \mathbf{H}_{-\alpha})$ for any $\gamma\in(0,1/2)$.

By Theorem 6.0.18 and Lemma 6.2.1, we can check that the quadratic variation of M^N converges in probability to the deterministic quantity (6.1.3). Hence, by a standard functional central limit theorem for semi-martingales (see, e.g., [59]), we have for any $\phi_\pm \in Dom^{Feller}(\mathcal{A}^\pm)$ fixed, $\{M^N(\phi_+,\phi_-)\}$ converges in distribution in $D([0,T],\mathbb{R})$ to a continuous Gaussian martingale with independent increments and covariance functional (6.1.3). In fact, following the proof of Theorem 5.4.6 in Chapter 5, we obtain Step 3.

Theorem 6.5.4. *When $\alpha > d \vee (d/2+1)$, the square-integrable martingale $\{M^N\}$ in Theorem 6.5.2 converges to M in distribution in $D([0,T], \mathbf{H}_{-\alpha})$ for any $T > 0$, where M is the (unique in distribution) continuous square-integrable $\mathbf{H}_{-\alpha}$-valued Gaussian martingale with independent increments and covariance functional characterized by (6.1.3).*

With Lemma 6.2.1, we can check, as in Chapter 5, that the expression $\int_0^t \mathbf{U}_{(t,s)} dM_s$ is well-defined. That is $\mathbf{U}_{(t,s)}$ (for $s \in [0,t]$) lies within the class of integrands with respect to M. Furthermore, following the same proof for Theorem 5.4.9 in Chapter 5, we obtain Step 5.

Theorem 6.5.5. *For $\alpha > d+2$ and $T > 0$, we have*

$$\int_0^t \mathbf{U}^N_{(t,s)} dM^N_s \xrightarrow{\mathcal{L}} \int_0^t \mathbf{U}_{(t,s)} dM_s \quad \text{in } D([0,\, T], \mathbf{H}_{-\alpha}) \tag{6.5.5}$$

Moreover, $\int_0^t \mathbf{U}_{(t,s)} dM_s$ has a version in $C^\gamma([0,T],\, \mathbf{H}_{-\alpha})$ for any $\gamma \in (0, 1/2)$.

Finally, we complete Step 6 towards the proof of Theorem 6.1.2. The key is Lemma 6.4.4.

Theorem 6.5.6. *(Boltzman-Gibbs principle) Suppose $\alpha > d+2$ and Assumption 6.1.1 holds. For any $T > 0$, there exists $C = C(D_+, D_-, T) > 0$ such that*

$$\int_0^t \mathbf{U}^N_{(t,s)} (\mathbf{B}^N_s \mathcal{Z}^N_s - K^N_s)\, ds \xrightarrow{\mathcal{L}} 0 \quad \text{in } D([0,\, T_0],\, \mathbf{H}_{-\alpha}), \tag{6.5.6}$$

where $T_0 := T \wedge (C_0\, C)^{-2}$, the operator $\mathbf{U}^N_{(t,s)}$ is defined in (6.2.8),

$$\begin{aligned} \mathbf{B}^N_s \mu(\phi_+, \phi_-) &:= \mu\big(\langle \ell(\phi_+ + \phi_-),\, f^-_s \rangle_-,\, \langle \ell(\phi_+ + \phi_-),\, f^+_s \rangle_+ \big) \quad \textit{and} \\ K^N_s(\phi_+, \phi_-) &:= \sqrt{N}\Big(\langle \ell_{\delta_N}(\phi_+ + \phi_-),\, \mathfrak{X}^{N,+}_s \otimes \mathfrak{X}^{N,-}_s \rangle - \mathbb{E}[\langle \ell(\phi_+ + \phi_-),\, \mathfrak{X}^{N,+}_s \otimes \mathfrak{X}^{N,-}_s \rangle] \Big). \end{aligned}$$

Proof Observe that $\alpha > d$ (we will need $\alpha > d+2$ later in the proof) guarantees, base on Weyl's law (6.0.7) and (6.0.8), that

$$\sum_{k\geq 1}\left(\frac{\|\phi_k^+\|^2}{(1+\lambda_k^+)^\alpha}+\frac{\|\phi_k^-\|^2}{(1+\lambda_k^-)^\alpha}\right)<\infty.$$

Using the definition of the norm $|\cdot|_{-\alpha}$ is defined in (6.0.6), the uniform bound (6.2.15) and Lemma 6.4.4, we have the following: For any $T>0$, there exists a constant $C=C(D_+,D_-,T)>0$, an integer $N_0=N_0(D_+,D_-)$ and positive constants $\{C_N\}$ satisfying $\lim_{N\to\infty}C_N=0$ such that

$$\mathbb{E}\Big[\Big|\int_0^t \mathbf{U}^N_{(t,s)}(\mathbf{B}^N_s\mathcal{Z}^N_s-K^N_s)\,ds\Big|^2_{-\alpha}\Big]\leq C_N\,t^{3/2} \tag{6.5.7}$$

whenever $0\leq t\leq T_0:=T\wedge(C_0\,C)^{-2}$ and $N>N_0$. In particular, we have, for $\alpha>d$,

$$\lim_{N\to\infty}\sup_{t\in[0,\,T_0]}\mathbb{E}\Big[\Big|\int_0^t \mathbf{U}^N_{(t,s)}(\mathbf{B}^N_s\mathcal{Z}^N_s-K^N_s)\,ds\Big|^2_{-\alpha}\Big]=0. \tag{6.5.8}$$

Theorem 6.5.6 follows from (6.5.8) and tightness for the process

$$\mathbf{e}_N(t):=\int_0^t \mathbf{U}^N_{(t,s)}(\mathbf{B}^N_s\mathcal{Z}^N_s-K^N_s)\,ds$$

in $D([0,\,T_0],\,\mathbf{H}_{-\alpha})$ which can be verified by the same argument that we used for $\mathcal{Z}$ in the proof of Theorem 6.5.1. Precisely, by (6.5.4), we have almost surely,

$$\mathbf{e}_N(t)=\mathcal{Z}^N_t-\mathbf{U}^N_{(t,0)}\mathcal{Z}^N_0-\int_0^t\mathbf{U}^N_{(t,s)}\,dM^N_s\qquad\text{for } t\geq 0.$$

Each of the three terms on the RHS is C-tight (i.e. has only continuous limits) in $D([0,\,T_0],\,\mathbf{H}_{-\alpha})$ by Theorem 6.5.1, Lemma 6.5.3 and Theorem 6.5.5 respectively, provided that $\alpha>d+2$. Hence $\mathbf{e}_N$ is tight in $D([0,\,T_0],\,\mathbf{H}_{-\alpha})$.

□

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