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SIXTH ORDER CHARGE RENORMALIZATION CONSTANT

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SIXTH ORDER CHARGE RENORMALIZATION CONSTANT

by

HOWARD EDWARD BRANDT

A thesis submitted in partial fulfillment
of the requirements for the degree of

DOCTOR OF PHILOSOPHY

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Approved by Marshall Baker
(Chairman of Supervisory Committee)

Department Physics
(Department Faculty sponsoring candidate)

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We have carefully read the dissertation entitled "Sixth Order Charge Renormalization Constant" submitted by Mr. Howard E. Brandt in partial fulfillment of the requirements of the degree of Doctor of Philosophy and recommend its acceptance. In support of this recommendation we present the following joint statement of evaluation to be filed with the dissertation.

Mr. Howard E. Brandt has made a calculation of the divergent part of the charge renormalization constant to sixth order in the fine structure constant. His calculation has been carried out in the Feynman gauge and is found to agree with a calculation previously performed by Rosner in the generalized Landau gauge, thus affording a reliable check of this result. This check is necessary in view of the enormous complication of such calculations and the delicate cancellation of infinities that are involved. The calculation of the divergent part of the charge renormalization constant to high order in perturbation theory is of great importance in our understanding of the basic properties of Quantum Electrodynamics and has direct bearing on the construction of a finite theory.

We believe that Mr. Brandt's work is a significant contribution to our understanding of Quantum Electrodynamics.

DISSERTATION READING COMMITTEE: Lowell S Brown

Marshall Baker

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Signature Howard E. Brandt

Date May 12, 1970

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Prof. Josef Jauch, Prof. Joseph Weber, Dr. David Cutler, Dr. Joseph Becker, Dr. Carl Jung, and the Kingfisher were brilliant lights during a difficult period of psychic individuation.

When the Eternal first made Light,
A myriad eyes sprang out to look,
And hearing ears and seeing eyes,
Once more a mighty choral took:
"All glory to the God of Light!"

--- Dr. Carl Gustav Jung *

* C. G. Jung, Symbols of Transformation, An Analysis of the Prelude to a Case of Schizophrenia, Bollingen Series XX, Princeton University Press (1956).

Chapter 1

INTRODUCTION

Quantum electrodynamics is to date the most successful theory in physics. There are no phenomena occuring under prescribed conditions for which this theory should provide an explanation, and where at least a qualitative explanation has not been found. The quantitative successes of quantum electrodynamical theory are legion.[1-18] Over the years it has almost inevitably met every experimental challenge with refined quantitative success. Its range of application has extended from 10^{-16} to 10^{11} meters, i.e. from the nuclear to the planetary domain. It is conjectured with well founded reason that all the diverse phenomena of chemical, electrical, biological, and psychophysical systems find their explanation ultimately in terms of this theory.

In the planetary domain, satellites have measured the magnetic field of the earth out to 10^{11} meters, verifying the r^{-3} fall off, thereby substantiating a vanishing photon mass. It is to be understood however that quantum mechanics plays no essential role in electrodynamics at large distances. Quantum electrodynamics, the quantum field theory of the electromagnetic interaction of charged particles, is a covering theory of classical electrodynamics.

The basic symmetry properties of the theory have surmounted diverse tests. Lorentz and translational invariance display their validity by the facts that high energy accelerators work, standard relativistic kinematics works, and cosmic ray shower theory explains the basic properties of high energy showers.

The success of the simple relation $E^2 = p^2 + m^2$ cannot be underestimated. The discrete symmetries P and T have been checked by the limits placed on the electron electric dipole moment. TCP has been tested by the ratios of lifetimes and masses of negative and positive muons.

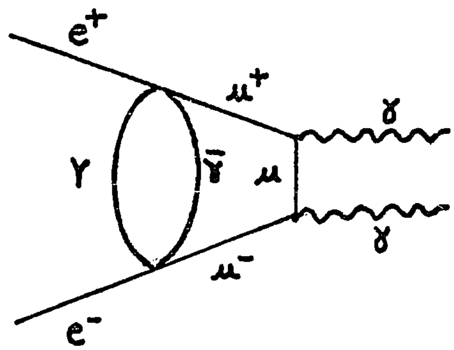
In the atomic domain, the separation between the singlet and triplet 1s states of positronium is in good agreement with experiment. The most discriminating tests of quantum electrodynamics have been those of the hyperfine and fine structure in hydrogenic atoms, and precise determinations of electron and muon anomolous magnetic moments.[17-18] These tests fall in the realm of low energy, essentially static tests of quantum electrodynamics, with the qualification of course that virtual momenta are involved. Using the latest value of the fine structure constant, obtained from the a.c. Josephson effect in superconductors, the theoretical ground state hyperfine splitting in hydrogen is $1420.04023(1 + \delta N) \pm 0.0057$ MHz, where δN is a small proton polarization correction. The experimental value is 1420.405751786417 MHz ($\pm 1.2/10^{13}$), which is consistent with the expected small value of δN. The $2S_{1/2}$-$2P_{1/2}$ separation in hydrogen, more commonly known as the Lamb shift, has passed successfully through many vicissitudes of reconciliation between theory and experimant. Prior to this year, the accepted theoretical value was $1057.555 \pm .086$ MHz. Discrepan-with experiments over the past two decades have ranged from $0.09 \pm .06$ to $0.34 \pm .07$ MHz. The Lamb shift remained the only test of quantum electrodynamics in serious disagreement with theory.

However it was thought unlikely that the disagreement reflected any breakdown of the theory, and it was thought more probable that a misapplication of theory was responsible. The very recent work of S. Brodsky and T. Appelquist confirms this view.[19] On recalculation of the fourth order radiative corrections to the free electron vertex, they obtained a value in disagreement with previous calculation, implying a new theoretical value for the Lamb shift in hydrogen, increased by 0.35 ± .07 MHz and putting theory in very good agreement with the results of recent experiments.

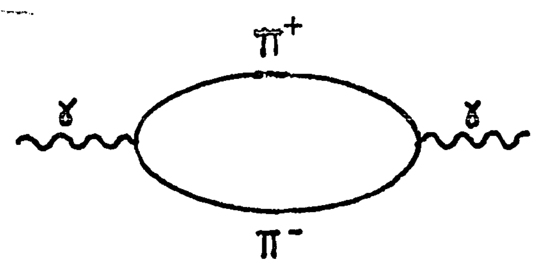
The other basic test of quantum electrodynamics is that of the anomalous magnetic moment of the electron. Theory and experiment agree at the 70 ppm level and differ only in the eighth significant figure. This is a critical test of the theory, since the experiment can be idealized as a measurement of the electron in isolation from other dynamics. Resolution of the discrepancy rests upon further theoretical and experimental work currently in progress.

In the domain of high energies, all predictions of Born amplitudes for electron-electron and electron-positron scattering are confirmed in detail by numerous experiments in the energy and momentum transfers accessible by current accelerators. This confirmation can be roughly parameterized in terms of a regulator mass exceeding several Bev. The most interesting high energy tests of quantum electrodynamics await the results of colliding beam scattering and annihilation experiments.

However, to conclude from all of this that all is well with quantum electrodynamics would be grossly naive. Firstly, this theory is necessarily incomplete, having a delimited domain of validity at best. At large distances it is to be expected that gravitation gets in the way. At small distances the weak interaction intrudes. For example, consider the following weak correction to electron-positron pair annihilation:



which contains virtual muons and neutrinos. This amplitude becomes important at 300 Bev in the center of mass. Ordinary quantum electrodynamics cannot be valid at higher energies. Furthermore all charged particles ultimately contribute to the vacuum polarization; consider for example the following strong interaction contribution:



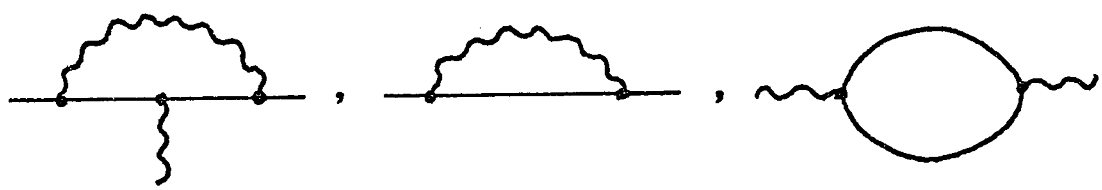
Also, the theory appears to tell us nothing about the magnitude of the charge and mass of the electron. In these senses the theory is incomplete.

But aside from the question of completeness, quantum electrodynamics suffers from more serious theoretical maladies. Unfortunately no one has found a way to solve the Schwinger-Dyson equations[20-21]; therefore all calculations have of necessity resorted to perturbation theory, and it is here that the theory is plagued by the appearance of certain infinite quantities. The pessimism which these infinities have induced can be seen at the close of Dirac's book[2]: "The difficulties, being of a profound character, can be removed only by some drastic change in the foundations of the theory, probably a change as drastic as the passage from Bohr's orbit theory to the present quantum mechanics"; or again in Schwinger's book[3]: "It seems that we have reached the limits of the quantum theory of measurement, which asserts the possibility of instantaneous observations without reference to specific agencies. The resulting appearance of divergences, and contradictions, serves to deny the basic measurement hypothesis. We conclude that a convergent theory cannot be formulated consistently within the framework of present space-time concepts." However, in Feynman's report to the Twelfth Solvay Congress[15], he pragmatically advocates Wheeler's principle of radical conservatism by which one would perserveringly continue to investigate canonical quantum electrodynamics under the assumption that though incomplete this theory is exact; the inference being that the maladies of the canonical theory are purely mathematical; that if exact non-perturbative solutions could be found, the maladies would vanish; that these infinities might arise merely from an unjustified use of perturbation theory.

Such pragmatism has been eminently successful. It is a redeeming fact that only the renormalized counterparts of the infinite quantities appear in the calculation of observables. The efforts of Dyson, Feynman, Schwinger, Tomonaga, and others gave the theory a systematic calculational scheme, including an incredibly mathematically heuristic renormalization procedure.[3] The prescription proceeds in two steps. First the infinite is separated from the finite observable parts of matrix elements. The second step consists in a proof that the infinite parts can be combined with the two phenomenological constants of the theory, namely the mass and charge of the electron. The divergences are thus eliminated by a redefinition of mass and charge. This procedure of mass and charge renormalization works, but is mathematically heuristic and aesthetically repugnant.

In the early days of renormalization theory it was suggested by Pauli[22] that the infinities occuring in the successive orders of perturbation theory might be peculiar to perturbation theory itself, and that if all terms in the series were summed, the infinities might cancel. He suggested that the magnitude of the bare fine structure constant might be obtained from the requirement that such compensation occur. This was the germ of an idea which has been explored over the past few decades in the work of M. Gell-Mann, F. Low, R. Jost, M. Luttinger, G. Källén[23-25], and more recently by M. Baker, K. Johnson, R. Willey, J. Rosner, and R. Abdellatif.[26-31] The present thesis addresses itself to a particular aspect of this development.

There are three serious divergences occuring in quantum electrodynamics, namely Z_1, Z_2, δm, and Z_3^{-1}; associated with the electron-photon vertex, the electron self energy, and the vacuum polarization respectively.[32] To second order these are represented by the following Feynman graphs:



respectively. Upon application of the Feynman rules, each of these graphs is found to be divergent. Ward's identity requires $Z_1 = Z_2$.[33] The work of Baker, Johnson, and Willey has shown that if Z_3 is nonvanishing, then the divergence in Z_1 and Z_2 can be eliminated by a suitable choice of gauge.[20] In particular, both Fermion self energy and vertex parts can be made finite to each order in the bare fine structure constant α_0, by working with a photon propagator of the form:

$$\mathcal{D}_{\mu\nu}(K) = \left(g_{\mu\nu} - \lambda \frac{K_\mu K_\nu}{K^2}\right)\mathcal{D}(K^2)\,, \tag{1}$$

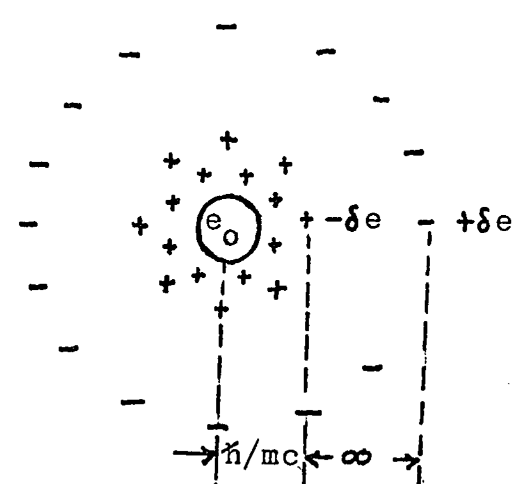
where the gauge constant λ is expanded as a power series in α_0 :

$$\lambda = \sum_{n=0}^{\infty} c_n \alpha_0^n \,. \tag{2}$$

Here $c_0 = 1$ and $c_1 = -3/8\pi$. Further these authors show that the divergence in the self energy, δm, can be eliminated by avoiding perturbation theory, again assuming Z_3 nonvanishing, and with the further requirement that the bare mass, m_0, be vanishing; the electron mass must be totally dynamical in origin. In a subsequent paper they show that nonvanishing Z_3 requires vanishing bare mass.[28]

The divergence in Z_3^{-1}, i.e. vanishing charge renormalization constant Z_3, is the remaining outstanding problem upon which all of these conjectures, regarding the true finiteness of canonical quantum electrodynamics, rest. However, as will be shown below, it is promising that this divergence too can be eliminated. This possibility is contrary to the well known Källén conjecture that at least one of the renormalization constants must be infinite.[25] Källén admits however that the mathematical rigour of his argument is not great, and that it is certainly possible that a singular solution of the equations with finite renormalization constants could exist, where certain formal interchanges of orders of integration implicit in his proof would not be allowed.[16] With this in mind we seek a finite Z_3^{-1}.

Intuitively the charge renormalization arises as follows: Imagine a bare charge e_o in vacuum. The electron polarizes the vacuum, surrounding itself by a neutral cloud of electrons and positrons as depicted in the figure below.



Some of these, having a net charge e, are repelled to infinity, leaving a net charge, $-\delta e$ in the part of the cloud bound to the test body, within a distance of the order of the Compton wavelength of the electron, $\hbar/mc$. Thus the observed charge e is $e_0 - \delta e$, the renormalized charge. Historically, the charge renormalization constant is taken to be $Z_3 = e/e_0 = 1 - \delta e/e_0$.

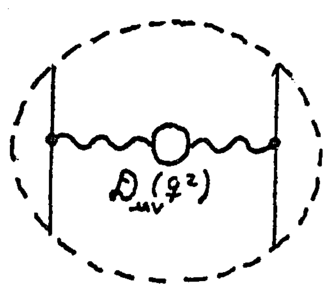
Mathematically, Z_3 is defined to be the residue at the pole of the unrenormalized photon propagator $\mathcal{D}_{\mu\nu}$.[34] Thus the Lehmann spectral form for $\mathcal{D}_{\mu\nu}$ is

$$\mathcal{D}_{\mu\nu}(K^2) = \left(g_{\mu\nu} - \frac{K_\mu K_\nu}{K^2}\right)\mathcal{D}(K^2)\,, \tag{3}$$

where

$$\mathcal{D}(K^2) = \frac{Z_3}{K^2+i\epsilon} + \int_0^\infty dM^2 \frac{\sigma(M^2)}{K^2-M^2+i\epsilon} \quad . \tag{4}$$

Any amplitude in the theory can be written in perturbation theory as the sum of Feynman graphs composed of connected Fermion and photon propagators, an insertion of which is shown below:



,

the photon line always connecting two electron and/or positron lines. Using the Feynman rules, the contribution of this graph to the amplitude A is:

$$A = \ldots e_0 \gamma_\mu \ldots e_0 \gamma_\nu D_{\mu\nu}(q^2) \ldots \tag{5}$$

where the dots denote the integrals, traces, and other propagators occuring in the graph. Here e_0 is the bare charge of the electron. The experimental charge e is determined by the interaction between the electrons and/or positrons at large distances. This corresponds to small q. Consider then the limit of A as $q^2 \to 0$, using Eqs.(4, 5):

$$\lim_{q^2 \to 0} A = \ldots e_0 \gamma_\mu \ldots e_0 \gamma_\mu \frac{Z_3}{q^2} \ldots$$

$$= \ldots (e_0 Z_3^{1/2}) \gamma_\mu \ldots (e_0 Z_3^{1/2}) \gamma_\mu \frac{1}{q^2} \ldots$$

The observed charge e is therefore taken to be

$$e = e_0 Z_3^{1/2} \; . \tag{6}$$

As an example, the amplitude for Compton scattering of a very low frequency photon by a free electron reduces to the well known Thompson formula if the identification, Eq.(6), is made.[35] In this way the charge renormalization is experimentally defined such that the low frequency Compton scattering reduces to the Thompson formula. Alternatively Z_3 can be defined by the requirement that the Coulomb law for two static electrons hold at large distances.[36] This leads again to Eq.(6). Källén has shown that the charge renormalization is uniquely defined as a consequence of charge conservation.[37]

Jost and Luttinger first calculated Z_3 to fourth order in perturbation theory, obtaining:[24]

$$Z_3^{-1} = \frac{\alpha_0}{2\pi}\left(\frac{2}{3} + \frac{\alpha_0}{2\pi} + O(\alpha_0^2)\right) \ln \frac{M^2}{m^2} + (\text{ terms finite as } M \rightarrow \infty). \quad (7)$$

Here α_0 is the bare fine structure constant, m is the mass of the electron, and M is an infinite cutoff mass. Thus Z_3^{-1} was thought to be infinite, violating Pauli's conjecture of compensation; both terms in the coefficient of the logarithmic divergence are positive, prohibiting a cancellation which would otherwise give a finite result. However to have assumed that higher order terms are also positive was premature, e.g. see Schweber's book[22] , as we shall see.

A hallmark in finite quantum electrodynamics was a paper by Gell-Mann and Low in 1954.[23] In an investigation of quantum electrodynamics at small distances, they showed that Z_3^{-1} is finite if a certain eigenvalue equation $\Psi(e_0^2) = 0$ is satisfied. The function Ψ was related to the coefficient of a logarithmic divergence in the vacuum polarization.

Recently, two further steps have been made in the calculation of Z_3. Baker, Johnson, and Willey have shown that if Z_3 is calculated from the vacuum polarization, neglecting self energy insertions of internal photon lines, then the divergent part of Z_3^{-1} is a simple logarithmic divergence:

$$(Z_3^{-1}) \quad = f(\alpha_0) \ln \frac{M^2}{m^2} \qquad (8)$$

to any order in perturbation theory.[27] In this model, Z_3^{-1} is then finite for those values of α_0 for which $f(\alpha_0) = 0$. If internal photon self energy insertions are included, other terms will appear in Eq.(8) which will be higher powers of logarithms, viz.

$$\left(Z_3^{-1}\right) = f(\alpha_0)\ln\frac{M^2}{m^2} + \sum_{n=1}^{\infty} g_n(\alpha_0)\left(\ln\frac{M^2}{m^2}\right)^n . \tag{9}$$

In a subsequent paper[28] these authors show that if $f(\alpha_0) = 0$ has a real positive root for which $f'(\alpha_0) = -\epsilon/\alpha_0 < 0$ with $\epsilon > 0$, then $g_n(\alpha_0) = 0$ also, or equivalently the Gell-Mann and Low function $\Psi = 0$. The detailed arguments are sufficiently complex that the interested reader must be referred to the references. An excessively brief argument runs as follows: If the above conditions on f are satisfied, then the leading correction to the photon propagator for large k^2 is $(m^2/k^2)^\epsilon/k^2$, which is negligible compared to the main term $1/k^2$. Therefore if one calculates f neglecting all photon self energy insertions, and the resultant f satisfies the above conditions, then the finiteness of Z_3^{-1} in the full theory is guaranteed. This then justifies the neglect of photon self energy insertions, thereby simplifying the calculation.

The second recent advance has been the calculation by Rosner[29-30] of the sixth order contribution to $f(\alpha_0)$. Rosner performed the calculation in the generalized Landau gauge, Eqs.(1,2), obtaining the following simple result:

$$f(\alpha_0) = \frac{\alpha_0}{2\pi}\left[\frac{2}{3} + \frac{\alpha_0}{2\pi} - \frac{1}{4}\left(\frac{\alpha_0}{2\pi}\right)^2\right] . \tag{10}$$

This result is encouraging because the sixth order contribution is negative, providing the possibility at least of a vanishing $f(\alpha_0)$ and thus a finite Z_3^{-1}. The object of this thesis is to recalculate Rosner's sixth order result, however using the Feynman gauge, i.e. Eq.(1) with $\lambda = 0$; in order to test the gauge invariance of the calculation, and more importantly, to check Rosner's result, because of its importance regarding the possibility of a finite Z_3^{-1} and consequently a finite canonical quantum electrodynamics. The result is

$$f^{(6)}(\alpha_0) = -\frac{1}{4}\left(\frac{\alpha_0}{2\pi}\right)^3$$

which is in perfect agreement with Rosner.

Chapter 2

THE FUNDAMENTAL FUNCTION $f(\alpha_0)$

In the following we shall obtain a general expression for the divergent part of Z_3^{-1}, expressed in terms of the vacuum polarization tensor. As mentioned in the introduction, Eq.(4), Z_3 is defined to be the residue at the pole of the factor $D(k^2)$ of the unrenormalized photon propagator at $k^2 = 0$. Thus

$$Z_3 = \lim_{K^2 \to 0} K^2 D(K^2). \tag{11}$$

The unrenormalized photon propagator in a general gauge is given by

$$D_{\mu\nu}(K^2) = (g_{\mu\nu} - K_\mu K_\nu / K^2) D(K^2) + K_\mu K_\nu f(K^2)/K^4 . \tag{12}$$

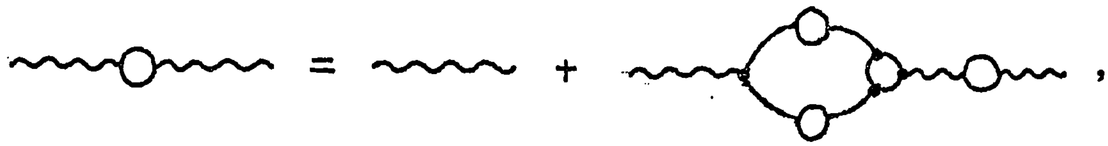
From the general theory of quantum electrodynamics we know that the function $D_{\mu\nu}(K)$ obeys the following equation:[38]

$$D_{\mu\nu}(K) = D^{(0)}_{\mu\nu}(K) - D^{(0)}_{\mu\lambda}(K)\, \Pi_{\lambda\sigma}(K)\, D_{\sigma\nu}(K) . \tag{13}$$

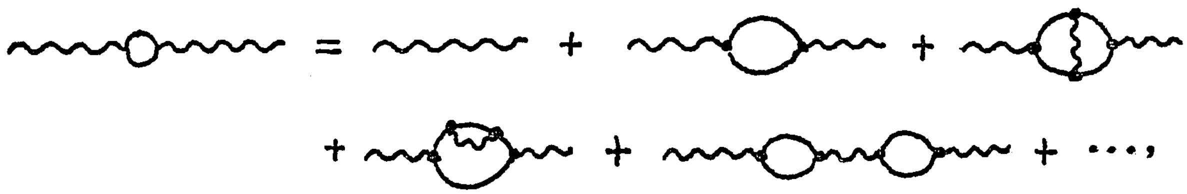
Here $D^{(0)}_{\mu\nu}(K)$ is the free photon propagator and $\Pi_{\lambda\sigma}(K)$ is the vacuum polarization tensor. $D^{(0)}_{\mu\nu}(K)$ has the following form in a general gauge:

$$D^{(0)}_{\mu\nu}(K) = \left(g_{\mu\nu} - \frac{K_\mu K_\nu}{K^2}\right)\Big/K^2 + K_\mu K_\nu f_0(K^2)/K^4 . \tag{14}$$

Diagramatically Eq.(13) is

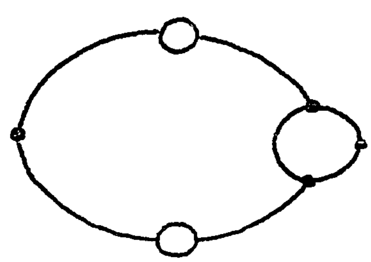


or expanding in α_0 :



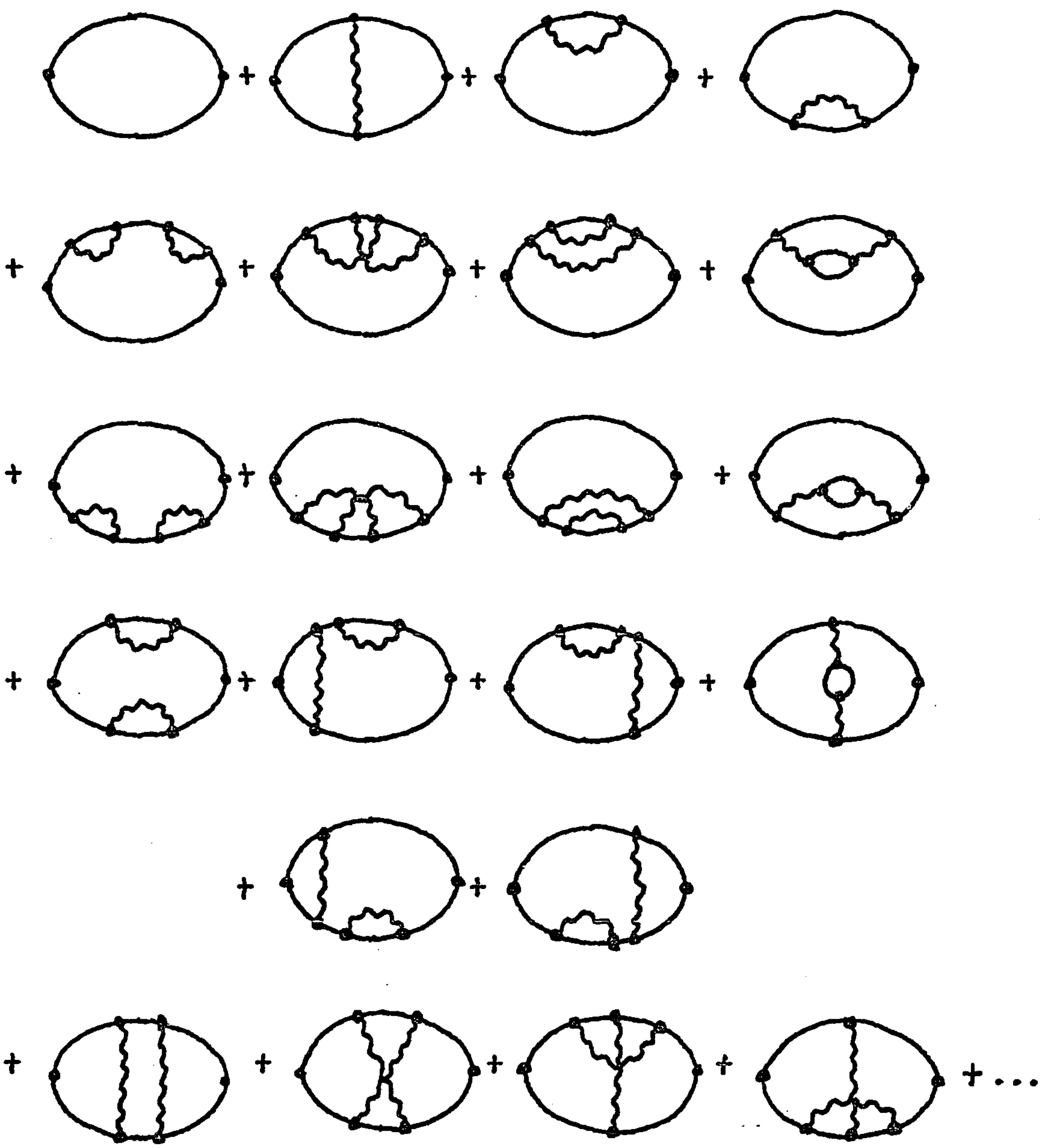
where  ,  ,  , ,  , and 

denote the full photon propagator, the free photon propagator, the full electron propagator, the free electron propagator, the full vertex, and the bare vertex respectively. The above perturbation expansion is seen to include all amplitudes for the propagation of a photon between two points in space-time. The vacuum polarization tensor $\Pi^{\mu\nu}$ is represented diagramatically as

$\Pi^{\mu\nu} =$  .

which expanded to sixth order in e_0 is:

$\Pi_{\mu\nu} =$



Gauge invariance requires that

$$K_\mu \Pi^{\mu\nu} = \Pi^{\nu\mu} K_\mu = 0,$$

and hence $\Pi^{\mu\nu}$ assumes the following form:[39]

$$\Pi^{\mu\nu} = (K^2 g_{\mu\nu} - K_\mu K_\nu)\rho(K^2), \tag{15}$$

where ρ is some function of k^2. We now put Eqs.(12, 14, 15) in Eq.(13) and contract indices, obtaining

$$3D(K^2) + f(K^2)/K^2 = 3/K^2 + f_0(K^2)/K^2 - 3\rho(K^2)D(K^2) \ . \tag{16}$$

Therefore

$$f(K^2) = f_0(K^2) \tag{17}$$

and

$$D(K^2) = \frac{1}{K^2(1+\rho(K^2))} \ . \tag{18}$$

Further, putting Eq.(18) in Eq.(11),

$$Z_3 = \lim_{K^2 \to 0} \frac{1}{1+\rho(K^2)} = (1+\rho(0))^{-1} ,$$

or

$$Z_3^{-1} - 1 = \rho(0) \ . \tag{19}$$

Contracting indices in Eq.(15), we obtain

$$\rho(K^2) = \frac{1}{3} \frac{\Pi_{\mu\mu}(K^2)}{K^2} \ . \tag{20}$$

Inserting Eq.(20) in Eq.(19) we get

$$Z_3^{-1} - 1 = \lim_{K^2 \to 0} \frac{1}{3} \frac{\Pi_{\mu\mu}(K^2)}{K^2} \ . \tag{21}$$

We then expand $\Pi_{\mu\mu}(K^2)$ about $k^2 = 0$:

$$\Pi_{\mu\mu}(K^2) = \Pi_{\mu\mu}(0) + \Pi'_{\mu\mu}(K^2)\Big|_{K^2=0} K^2 + \ldots \ . \tag{22}$$

Putting Eq.(22) in Eq.(21), then

$$Z_3^{-1} - 1 = \frac{1}{3} \Pi'(K^2)\Big|_{K^2=0} \ . \tag{23}$$

We then observe that

$$\frac{\partial^2}{\partial K_\alpha \partial K^\alpha} \Pi_{\mu\mu}(K^2)\Big|_{K=0} = \frac{\partial}{\partial K_\alpha}\left(\Pi'(K^2)2K_\alpha\right)\Big|_{K=0} = \left(\Pi'' 4K^2 + 8\Pi'\right)\Big|_{K=0}$$

$$= 8\Pi'(K^2)\Big|_{K=0} \quad ,$$

so that

$$\Pi'(K^2)\Big|_{K=0} = \frac{1}{8}\frac{\partial^2}{\partial K_\alpha \partial K^\alpha} \Pi_{\mu\mu}(K^2)\Big|_{K=0} \quad . \tag{24}$$

Inserting Eq.(24) in Eq.(23), then

$$Z_3^{-1} - 1 = \frac{1}{24}\frac{\partial^2}{\partial K_\alpha \partial K^\alpha} \Pi_{\mu\mu}(K^2)\Big|_{K=0} \quad . \tag{25}$$

From the functional equations for the Green's functions, we know that[40]

$$\Pi_{\mu\nu}(K) = i e_0^2 \,\mathrm{Tr}\, \gamma_\mu \int \frac{d^4p}{(2\pi)^4} S(p+\tfrac{K}{2})\Gamma_\nu(p+\tfrac{K}{2}, p-\tfrac{K}{2}) S(p-\tfrac{K}{2}) \; . \tag{26}$$

Here $S(p)$ is the unrenormalized electron Green's function. Its zero-order perturbation theory value is given by

$$S^{(0)}(p) = \frac{1}{i\gamma p + m} \quad . \tag{27}$$

$\Gamma_\nu(p,p')$ is the unrenormalized vertex function. To lowest order it is

$$\Gamma_\nu^{(0)}(pp') = \gamma_\nu \; . \tag{28}$$

The Feynman rules are as follows:

$$\left.\begin{array}{ll}
\text{Electron or positron line:} & (i\gamma p + m)^{-1} \\
\text{Photon line in Feynman gauge:} & g_{\mu\nu}/K^2 \\
\text{Vertex:} & \gamma_\mu \\
\text{Pair of vertices:} & i e_0^2 \\
\text{Loop:} & (2\pi)^{-4} d^4 K
\end{array}\right\} \quad (29)$$

We will calculate the divergent part of Z_3^{-1} using Eq.(25), which we will write as

$$Z_3^{-1} - 1 = i e_0^2 \int \frac{d^4 p}{(2\pi)^4} \frac{\sigma(p^2, m^2)}{p^4} \quad . \qquad (30)$$

Comparing Eqs.(25, 26) with Eq.(30), then

$$\sigma(p^2, m^2) = \frac{p^4}{24} \frac{\partial^2}{\partial K_\alpha \partial K^\alpha} \, Tr \, \gamma_\mu S\left(p+\frac{K}{2}\right) \Gamma^\mu\left(p+\frac{K}{2}, p-\frac{K}{2}\right) S\left(p-\frac{K}{2}\right) \Big|_{K=0} \cdot \qquad (31)$$

Eq.(30) says $\sigma(p^2, m^2)$ is dimensionless and therefore it must be a function of p^2/m^2, since p and m are the only parameters that enter in Eq.(31).

We now invoke the Johnson–Baker–Willey theorem as discussed in the introduction, Eq.(8). As justified there, we drop internal photon self energy insertions in Eq.(31). Comparing Eq.(8) with Eq.(30) we conclude that $\lim_{p^2 \to \infty} \sigma(p^2, m^2)$ is finite, so that the single logarithm in Eq.(8) arises only from the integration over p in Eq.(30); thus

$$f(\alpha_o) = -\frac{1}{2}\left(\frac{\alpha_o}{2\pi}\right) \lim_{p^2\to\infty} \sigma(p^2; m^2) \ . \tag{32}$$

We now rotate contours in Eq.(30), so that all integrals are over Euclidean variables, then

$$\left.\begin{aligned} \int d^4p &= \frac{i}{2}\int p^2 dp^2 d\Omega_p \\ \int d\Omega_p &= 2\pi^2 \end{aligned}\right\} \tag{33}$$

so Eq.(30) becomes

$$Z_3^{-1} - 1 = -\left(\frac{\alpha_o}{2\pi}\right)\frac{1}{2}\int \frac{dp^2}{p^2}\,\sigma(p^2; m^2) \ , \tag{34}$$

where

$$\alpha_o \equiv \frac{e_o^2}{4\pi} \ . \tag{35}$$

Comparing Eqs.(34, 8, 31), then

$$f(\alpha_o) = -\frac{1}{2}\left(\frac{\alpha_o}{2\pi}\right)\lim_{p^2\to\infty}\sigma(p^2/m^2) = -\frac{1}{2}\left(\frac{\alpha_o}{2\pi}\right)\lim_{m^2\to 0}\sigma(p^2/m^2) , \tag{36}$$

or

$$f(\alpha_o) = -\frac{p^4}{48}\left(\frac{\alpha_o}{2\pi}\right) Tr \frac{\partial^2}{\partial k_\alpha \partial k^\alpha}\gamma_\mu S(p+\tfrac{k}{2})\Gamma^\mu(p+\tfrac{k}{2}, p-\tfrac{k}{2}) S(p-\tfrac{k}{2})\Big|_{\substack{k=0\\ m=0}} \ . \tag{37}$$

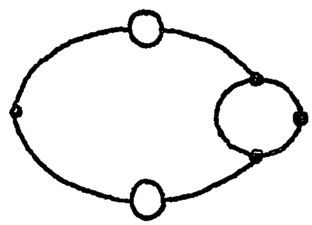
From Eq.(37) we may calculate $f(\alpha_o)$ to any order in perturbation theory. The fact that we can set m = 0 in Eq.(37) produces a major mathematical simplification.

Chapter 3

GENERAL EXPRESSION FOR $f(\alpha_0)$ TO SIXTH ORDER

The sixth order contribution $f^{(6)}(\alpha_0)$ to $f(\alpha_0)$ is to be calculated in the Feynman gauge. In Eq.(37) we employ the symbolic notation:

$$\gamma S \Gamma S \equiv \gamma G \Gamma = \tag{38}$$



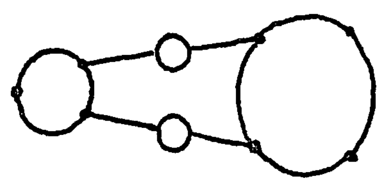
.

From Eq.(37) we see that to get the sixth order contribution to $f(\alpha_0)$, we must work with the fourth order $(\gamma G \Gamma)^{(4)}$. Clearly

$$(\gamma G \Gamma)^{(4)} = \gamma G^{(4)} \gamma + \gamma G^{(2)} \Gamma^{(2)} + \gamma G^{(0)} \Gamma^{(4)}. \tag{39}$$

Quantum electrodynamics tells us that[41]

$$\Gamma = \gamma + K G \Gamma = \gamma + \Gamma G K = \bullet + \tag{40}$$



,

where K is the Bethe-Salpeter kernal for electron-positron scattering, and the integrations are implicit in the notation. Therefore

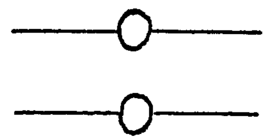
$$\Gamma^{(2)} = K^{(2)} G^{(0)} \gamma \tag{41}$$

and

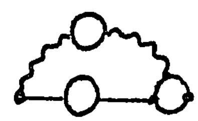
$$\Gamma^{(4)} = K^{(4)} G^{(0)} \gamma + K^{(2)} G^{(0)} K^{(2)} G^{(0)} \gamma + K^{(2)} G^{(2)} \gamma \tag{42}$$

$$= \gamma G^{(0)} K^{(4)} + \gamma G^{(0)} K^{(2)} G^{(0)} K^{(2)} + \gamma G^{(2)} K^{(2)}. \tag{43}$$

We must obtain expressions for $G^{(2)}$ and $G^{(4)}$ to insert in Eq.(39). Symbolically

$$G = S(p+\tfrac{K}{2})\,S(p-\tfrac{K}{2}) \equiv S_{\uparrow}S_{\downarrow} =$$  . (44)

Furthermore quantum electrodynamics gives S^{-1} in terms of the self energy Σ of the electron[42]; taking $m_o = 0$, as prescribed in Eq.(37),

$$S^{-1} = i\gamma P - \Sigma = i\gamma P\left(1 - \frac{1}{i\gamma P}\Sigma\right) = (\text{———})^{-1} -$$  (45)

$$\equiv S_{(0)}^{-1}\left(1 + B^{(2)}(p^2) + B^{(4)}(p^2) + \ldots\right) , \tag{46}$$

where

$$B^{(2)}(p^2) = -\frac{1}{i\gamma P}\Sigma^{(2)} , \tag{47}$$

and

$$B^{(4)}(p^2) = -\frac{1}{i\gamma P}\Sigma^{(4)} . \tag{48}$$

From Eq.(46), then

$$S = S^{(0)}\left(1 - B^{(2)}(p^2) - B^{(2)2}(p^2) - B^{(4)}(p^2) + \ldots\right) . \tag{49}$$

Putting Eq.(49) in Eq.(44), then

$$G^{(2)} = -b^{(2)}G^{(0)} , \tag{50}$$

and

$$G^{(4)} = -b^{(4)}G^{(0)} \quad , \tag{51}$$

where

$$b^{(2)} \equiv B_{\uparrow}^{(2)} + B_{\downarrow}^{(2)} \equiv B^{(2)}(p+\tfrac{K}{2}) + B^{(2)}(p-\tfrac{K}{2}) \ , \tag{52}$$

and

$$b^{(4)} \equiv B_{\uparrow}^{(4)} + B_{\downarrow}^{(4)} - B_{\uparrow}^{(2)2} - B_{\uparrow}^{(2)}B_{\downarrow}^{(2)} - B_{\downarrow}^{(2)2} \ . \tag{53}$$

Inserting Eqs.(41, 42, 50, and 51) in Eq.(39) we obtain

$$\begin{aligned}(\gamma G\Gamma)^{(4)} = & -b^{(4)}\gamma G^{(0)}\gamma - b^{(2)}\gamma G^{(0)}K^{(2)}G^{(0)}\gamma + \gamma G^{(0)}K^{(4)}G^{(0)}\gamma \\ & + \gamma G^{(0)}K^{(2)}G^{(0)}K^{(2)}G^{(0)}\gamma + \gamma G^{(0)}K^{(2)}G^{(2)}\gamma \ .\end{aligned} \tag{54}$$

Defining

$$\frac{\partial^2}{\partial K_\alpha \partial K^\alpha}(\gamma G\Gamma)^{(4)} \equiv (\gamma G\Gamma)''^{(4)} \ , \tag{55}$$

then Eq.(54) gives

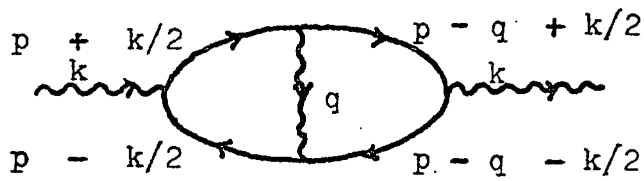
$$\begin{aligned}(\gamma G\Gamma)''^{(4)} = & \underset{(1)}{-b^{(4)}\gamma G''^{(0)}\gamma} \underset{(2)}{-b^{(2)}\gamma G''^{(0)}K^{(2)}G^{(0)}\gamma} \underset{(3)}{-2b^{(2)}\gamma G'^{(0)}K'^{(2)}G^{(0)}\gamma} \\ & \underset{(4)}{-2b^{(2)}\gamma G'^{(0)}K^{(2)}G'^{(0)}\gamma} \underset{(5)}{-b^{(2)}\gamma G^{(0)}K''^{(2)}G^{(0)}\gamma} \underset{(6)}{-2b^{(2)}\gamma G^{(0)}K'^{(2)}G'^{(0)}\gamma} \underset{(7)}{-b^{(2)}\gamma G^{(0)}K^{(2)}G''^{(0)}\gamma} \\ & \underset{(8)}{+\gamma G''^{(0)}\Gamma^{(4)}} \underset{(9)}{+2\gamma G'^{(0)}K'^{(4)}G^{(0)}\gamma} \underset{(10)}{+2\gamma G'^{(0)}K^{(4)}G'^{(0)}\gamma} \underset{(11)}{+\gamma G^{(0)}K''^{(4)}G^{(0)}\gamma} \underset{(12)}{+2\gamma G^{(0)}K'^{(4)}G'^{(0)}\gamma} \\ & \underset{(13)}{+\Gamma^{(4)}G''^{(0)}\gamma} \underset{(14)}{+b^{(2)}\gamma G^{(0)}K^{(2)}G''^{(0)}\gamma} \underset{(15)}{+2\gamma G'^{(0)}K'^{(2)}G^{(0)}K^{(2)}G^{(0)}\gamma} \underset{(16)}{+2\gamma G'^{(0)}K^{(2)}G'^{(0)}K^{(2)}G^{(0)}\gamma} \\ & \underset{(17)}{+2\gamma G'^{(0)}K^{(2)}G^{(0)}K'^{(2)}G^{(0)}\gamma} \underset{(18)}{+2\gamma G'^{(0)}K^{(2)}G^{(0)}K^{(2)}G'^{(0)}\gamma} \underset{(19)}{+\gamma G^{(0)}K''^{(2)}G^{(0)}K^{(2)}G^{(0)}\gamma} \\ & \underset{(20)}{+2\gamma G^{(0)}K'^{(2)}G'^{(0)}K^{(2)}G^{(0)}\gamma} \underset{(21)}{+2\gamma G^{(0)}K'^{(2)}G^{(0)}K'^{(2)}G^{(0)}\gamma} \underset{(22)}{+2\gamma G^{(0)}K'^{(2)}G^{(0)}K^{(2)}G'^{(0)}\gamma} \\ & \underset{(23)}{+\gamma G^{(0)}K^{(2)}G''^{(0)}K^{(2)}G^{(0)}\gamma} \underset{(24)}{+2\gamma G^{(0)}K^{(2)}G'^{(0)}K'^{(2)}G^{(0)}\gamma} \underset{(25)}{+2\gamma G^{(0)}K^{(2)}G'^{(0)}K^{(2)}G'^{(0)}\gamma}\end{aligned}$$

$$\underset{(26)}{+\gamma G^{(0)}K^{(2)}G^{(0)}K''^{(2)}G^{(0)}\gamma} \; \underset{(27)}{+\, 2\gamma G^{(0)}K^{(2)}G^{(0)}K'^{(2)}G'^{(0)}\gamma} \; \underset{(28)}{+\, 2\gamma G'^{(0)}K'^{(2)}G^{(2)}\gamma}$$

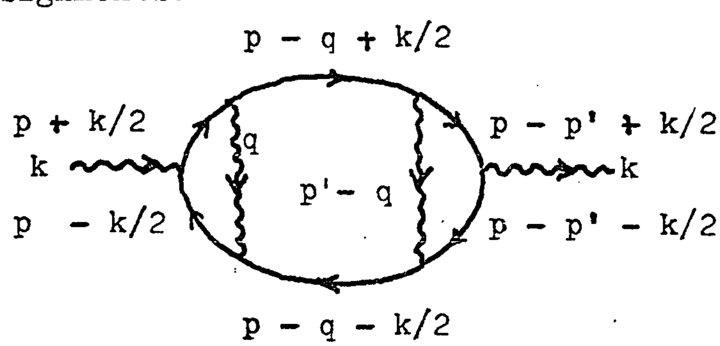
$$\underset{(29)}{+\, 2\gamma G'^{(0)}K^{(2)}G'^{(2)}\gamma} \; \underset{(30)}{+\,\gamma G^{(0)}K''^{(2)}G^{(2)}\gamma} \; \underset{(31)}{+\,2\gamma G^{(0)}K'^{(2)}G'^{(2)}\gamma} \; \underset{(32)}{+\,\gamma G^{(0)}K^{(2)}G''^{(2)}\gamma}$$

$$\underset{(33)}{-b'^{(4)}\gamma G'^{(0)}\gamma} \; \underset{(34)}{-\, b'^{(2)}(\gamma G^{(0)}K^{(2)}G^{(0)}\gamma)'} \; \underset{(35)}{-\, b''^{(4)}\gamma G^{(0)}\gamma} \; \underset{(36)}{-\, b''^{(2)}\gamma G^{(0)}K^{(2)}G^{(0)}\gamma} \, . \qquad (56)$$

We label terms with an (n) beneath, for purpose of identification. We make the following momentum assignments in terms (3, 5, 6, 28):



These terms vanish because, for example in the third term $K^{(2)}$ does not depend on K and therefore $K'^{(2)} = 0$. Similarly terms (15, 17, 19, 20, 21, 22, 24, 26, 27) are seen to vanish with the following momentum assignments:



The momentum k with respect to which we differentiate will always be routed through Fermion lines only.

We invoke Ward's identity[43]:

$$\Gamma_\mu(p,p) = \frac{1}{i}\frac{\partial}{\partial p^\mu} S^{-1}(p) \quad . \tag{57}$$

Putting Eq.(46) in Eq.(57) then

$$\Gamma_\mu(p,p) = \gamma_\mu\left(B^{(2)}(p^2) + B^{(4)}(p^2) + \dots\right) + 2p_\mu \gamma p\left(B'^{(2)}(p^2) + B'^{(4)}(p^2) + \dots\right) \quad . \tag{58}$$

Also using Eq.(50),

$$G''^{(2)} = \left(-b^{(2)}G^{(0)}\right)'' = -b''^{(2)}G^{(0)} - 2b'^{(2)}G'^{(0)} - b^{(2)}G''^{(0)} \quad . \tag{59}$$

Inserting Eqs.(52, 53, 58, 59) in Eq.(56), then

$$\begin{aligned}
(\gamma G\Gamma)''^{(4)}_{|k=0} = & -2B^{(4)}(p^2)\gamma G''^{(0)}\gamma + 3B^{(2)2}(p^2)\gamma G''^{(0)}\gamma - 2B^{(2)}(p^2)\gamma G''^{(0)}\left(B^{(2)}(p^2)\gamma + 2p\gamma p B'^{(2)}(p^2)\right) \\
& -4B^{(2)}(p^2)\gamma G'^{(0)}K^{(2)}G'^{(0)}\gamma - 2B^{(2)}(p^2)\left(B^{(2)}(p^2) + 2p\gamma p B'^{(2)}(p^2)\right)G''^{(0)}\gamma + 2\gamma G'^{(0)}K'^{(4)}G^{(0)}\gamma \\
& + 2\gamma G'^{(0)}K^{(4)}G'^{(0)}\gamma + \gamma G^{(0)}K''^{(4)}G^{(0)}\gamma + 2\gamma G^{(0)}K'^{(4)}G'^{(0)}\gamma + 2B^{(2)}(p^2)\left[B^{(2)}(p^2)\gamma + 2p\gamma p B'^{(2)}(p^2)\right]G''^{(0)}\gamma \\
& + 2\gamma G'^{(0)}K^{(2)}G'^{(0)}K^{(2)}G^{(0)}\gamma + 2\gamma G'^{(0)}K^{(2)}G^{(0)}K^{(2)}G'^{(0)}\gamma \\
& + \left(B^{(2)}(p^2) + 2p\gamma p B'^{(2)}(p^2)\right)G''^{(0)}\left(B^{(2)}(p^2)\gamma + 2p\gamma p B'^{(2)}(p^2)\right) + 2\left(B^{(2)}(p^2)\gamma + 2p\gamma p B'^{(2)}(p^2)\right)G'^{(0)}K^{(2)}G'^{(0)}\gamma \\
& + 2\gamma G'^{(0)}K^{(2)}G'^{(2)}\gamma + \left(B^{(2)}(p^2)\gamma + 2p\gamma p B'^{(2)}(p^2)\right)G''^{(2)}\gamma - b'^{(4)}\gamma G'^{(0)}\gamma \\
& - b'^{(2)}(p^2)\left(\gamma G^{(0)}K^{(2)}G^{(0)}\gamma\right)' - b'^{(4)}(p^2)\gamma G^{(0)}\gamma - b''^{(2)}\gamma G^{(0)}\left(B^{(2)}(p^2)\gamma + 2p\gamma p B'^{(2)}(p^2)\right) .
\end{aligned} \tag{60}$$

Further Eqs.(52, 53) give

$$b'^{(2)}_{|k=0} = 0 \quad , \tag{61}$$

$$b''^{(2)}_{|k=0} = 2B''^{(2)}(p^2)p^2 + 4B'^{(2)}(p^2) \quad , \tag{62}$$

$$b'^{(4)}_{|k=0} = 0 \quad , \tag{63}$$

and

$$b''^{(4)}_{|k=0} = 2p^2B''^{(4)}(p^2) + 4B'^{(4)}(p^2) - 6p^2B^{(2)}(p^2)B''^{(2)}(p^2) - 12B^{(2)}(p^2)B'^{(2)}(p^2) - 2p^2B'^{(2)}(p^2) \quad . \tag{64}$$

We define

$$4\gamma G'^{(0)}\bar{K}^{(4)}G'^{(0)}\gamma \equiv 4\gamma G'^{(0)}\Gamma_{\mu}^{(2)}(pp)\, G'^{(0)}\gamma\,\gamma^{\mu}/q^2\ . \tag{65}$$

Putting Eqs.(61–6) and adding and subtracting Eq.(65) in Eq.(60), we obtain:

$$(\gamma G\Gamma)''^{(4)}_{|k=0} = 2\gamma G'^{(0)}K^{(2)}G^{(0)}K'^{(2)}G'^{(0)}\gamma + 2\gamma G'^{(0)}[K^{(4)} - 2\bar{K}^{(4)}]G'^{(0)}\gamma$$

$$+ 4\gamma G'^{(0)}K'^{(4)}G^{(0)}\gamma + \gamma G^{(0)}K''^{(4)}G^{(0)}\gamma + 4\gamma G'^{(0)}\Gamma_{\mu}^{(2)}(pp)\, G'^{(0)}\gamma\gamma_{\mu}/q^2$$

$$-2B'^{(2)}(p^2)\,\gamma G'^{(0)}K^{(2)}G'^{(0)}\gamma \;\; + 4B'^{(2)}(p^2)\,\gamma p G'^{(0)}K^{(2)}G'^{(0)}\gamma p$$

$$+ 2\gamma G'^{(0)}K^{(2)}G'^{(0)}K^{(2)}G^{(0)}\gamma$$

$$+ (4B'^{(4)}(p^2) + 4p^2 B'^{(2)2}(p^2) - 4B^{(2)}(p^2)B'^{(2)}(p^2))\,\gamma p G''^{(0)}\gamma p$$

$$+ 2\gamma G'^{(0)}K^{(2)}G'^{(2)}\gamma + (-2b''^{(2)}B^{(2)}(p^2) - b''^{(4)})\gamma G^{(0)}\gamma$$

$$- 4b''^{(2)}B'^{(2)}(p^2)\,\gamma p G^{(0)}\gamma p\ . \tag{66}$$

We combine the eighth and tenth terms of Eq.(66) as follows. Using Eq.(41, 52, 58) we see that

$$Tr\,\gamma G'^{(0)}K^{(2)}G'^{(0)}K^{(2)}G^{(0)}\gamma = Tr\,\gamma G'^{(0)}(p)\,K^{(2)}(q)\,G'^{(0)}(p-q)\big(B^{(2)}(p-q)\gamma + 2(p-q)B'^{(2)}(p-q)\gamma(p-q)\big)$$

$$= \frac{1}{2}\,Tr\,\gamma G'^{(0)}(p)\,K^{(2)}(q)\,G'^{(0)}(p-q)\,b^{(2)}(p-q)\gamma$$

$$+ 2\,Tr\,\gamma(p-q)G'^{(0)}(p)\,K^{(2)}(q)\,G'^{(0)}(p-q)\,\gamma(p-q)\,B'^{(2)}(p-q)$$

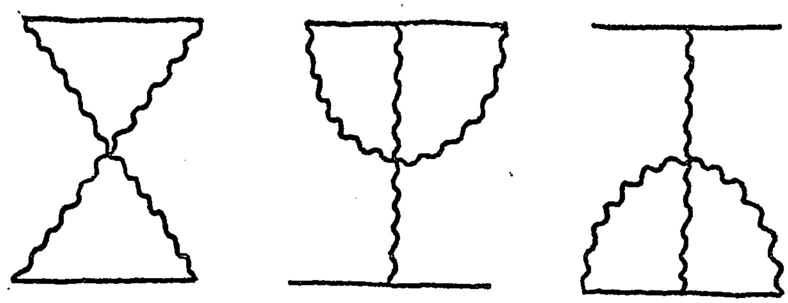
$$= -\frac{1}{2}\,Tr\,\gamma G'^{(0)}K^{(2)}G'^{(2)}\gamma\ , \tag{67}$$

where in the last step the second term vanishes since

$$G'^{(0)}(p-q)\gamma(p-q) = \frac{1}{2pq}\left[\gamma_\mu\gamma(p-q)\gamma_\alpha - \gamma_\alpha\gamma(p-q)\gamma_\mu\right](p-q)^\alpha = 0 \ .$$

As prescribed above we omit internal photon self energy insertions in $K^{(4)}$. The contributions to $K^{(4)}$ are the following, which we denote by $K_b^{(4)}$, $K_c^{(4)}$, and $K_d^{(4)}$ respectively:

$K^{(4)} =$



(68)

$$= \quad K_b^{(4)} \quad + \quad K_c^{(4)} \quad + \quad K_d^{(4)} \ .$$

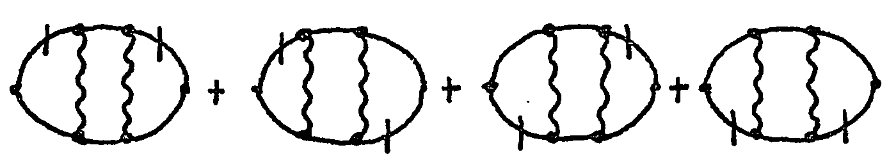
Because of the symmetric routing of k in Eq.(37), the contributions of $K_c^{(4)}$ and $K_d^{(4)}$ are equal, so effectively

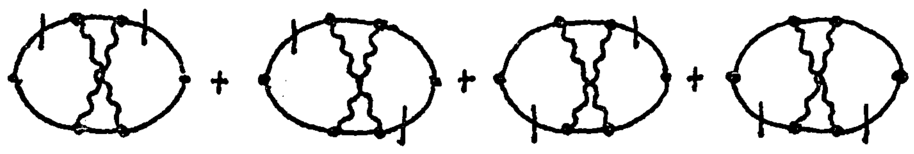
$$K^{(4)} = K_b^{(4)} + 2K_c^{(4)} \ . \tag{69}$$

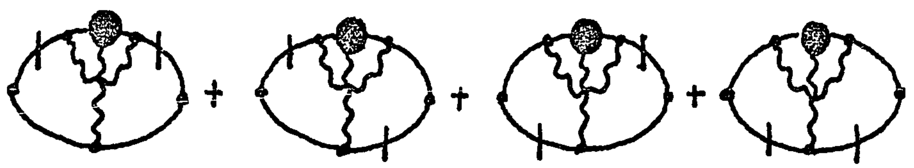
In the second term of Eq.(66) we will combine $-2\bar{K}^{(4)}$ with $2K_c^{(4)}$, which will lead to a finite result for this tensor. Putting Eq.(67, 69) in Eq.(66), then

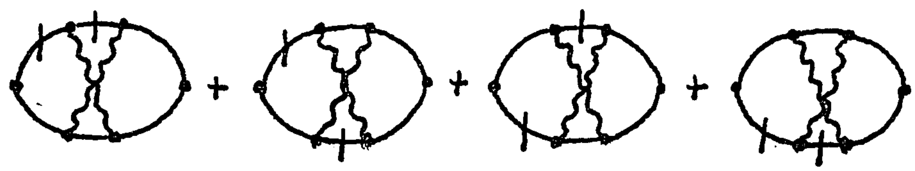
$$
\begin{aligned}
Tr(\gamma G \Gamma)''^{(4)}_{|k=0} = {} & Tr\, 2\gamma G'^{(0)} K^{(2)} G'^{(0)} K^{(2)} G'^{(0)} \gamma + 2 Tr\, \gamma G'^{(0)} K_b^{(4)} G'^{(0)} \gamma \\
& + Tr\, 4\gamma G'^{(0)} [K_c^{(4)} - \bar{K}^{(4)}] G'^{(0)} \gamma + Tr\, 4\gamma G'^{(0)} K_b'^{(4)} G^{(0)} \gamma \\
& + Tr\, 8\gamma G'^{(0)} K_c'^{(4)} G^{(0)} \gamma + Tr\, \gamma G^{(0)} K_b''^{(4)} G^{(0)} \gamma + Tr\, 2\gamma G^{(0)} K_c''^{(4)} G^{(0)} \gamma \\
& + Tr\, 4\gamma G'^{(0)} \Gamma'^{(2)}_{\mu}(p,p)\, G'^{(0)} \gamma \gamma^{\mu}/q^2 \ - 2B'^{(2)}(p^2)\, Tr\, \gamma G'^{(0)} K^{(2)} G'^{(0)} \gamma \\
& + 4B'^{(2)}(p^2)\, Tr\, \gamma p G'^{(0)} K^{(2)} G'^{(0)} \gamma p \ + Tr\, \gamma G'^{(0)} K^{(2)} G'^{(2)} \gamma \\
& + (4B'^{(4)}(p^2) + 4p^2 B'^{(2)\,2}(p^2) - 4B^{(2)}(p^2) B'^{(2)}(p^2))\, Tr\, \gamma p G''^{(0)} \gamma p \\
& + (-2b''^{(2)}(p^2) B^{(2)}(p^2) - b''^{(4)}(p^2))\, Tr\, \gamma G^{(0)} \gamma - 4b''^{(2)}(p^2) B'^{(2)}(p^2)\, Tr\, \gamma p G'^{(0)} \gamma p \\
\equiv {} & \sum_{i=1}^{14} T_i \, , \qquad (70)
\end{aligned}
$$

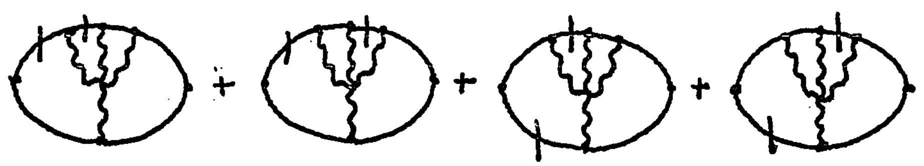
where the T_i, $i = 1, 2, \ldots, 14$, are given by the respective terms of Eq.(70). In terms of Feynman graphs, the T_i are:

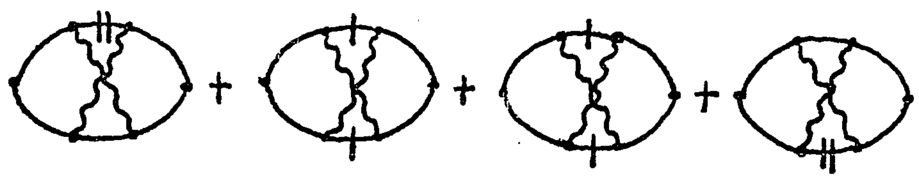
$\frac{1}{2} T_1 =$  ,

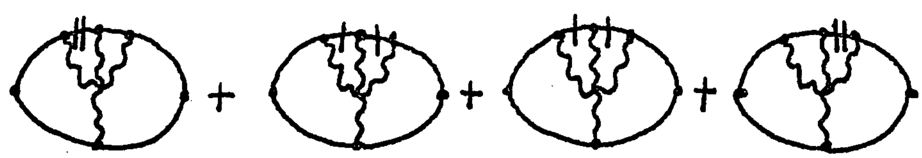
$\frac{1}{2} T_2 =$ 

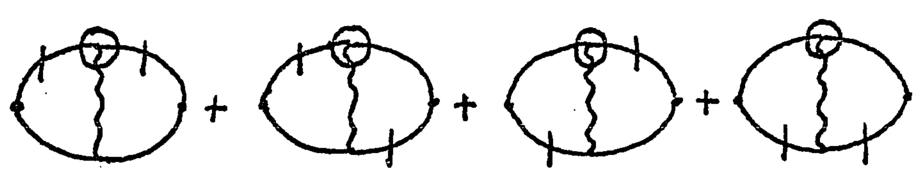
$\frac{1}{4} T_3 =$ 

$\frac{1}{4} T_4 =$ 

$\frac{1}{8} T_5 =$ 

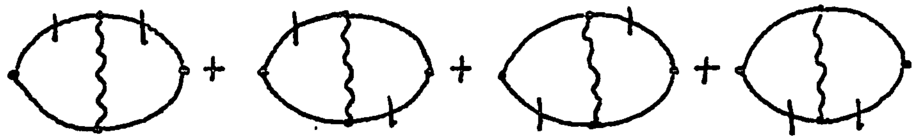
$T_6 =$ 

$\frac{1}{2} T_7 =$ 

$\frac{1}{4} T_8 =$ 

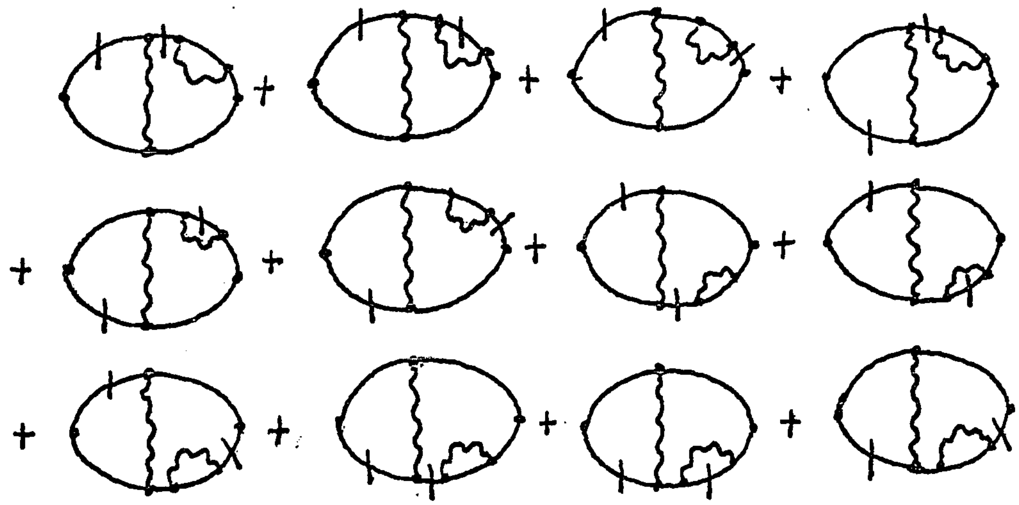
$$-\frac{1}{2}\left(B''^{(2)}(p^2)\right)^{-1} T_9 = \frac{1}{4}\left(B'^{(2)}(p^2)\, p^\mu p^\nu\right)^{-1} T_{10}$$

=



,

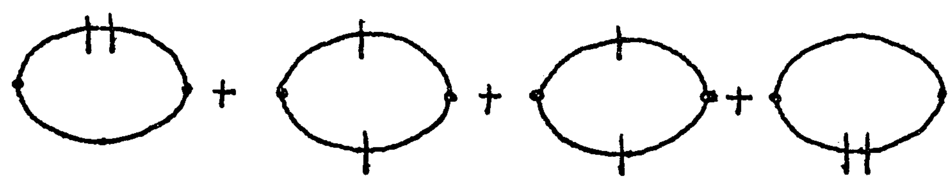
T_{11} =



,

$$\left[\left(4B'^{(4)}(p^2) + 4p^2 B'^{2\,(2)}(p^2) - 4B^{(2)}(p^2)\, B'^{(2)}(p^2)\right) p^\mu p^\nu\right]^{-1} T_{12}$$

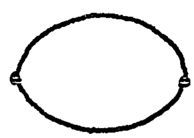
=



,

$$\left[-2\, b''^{(2)}(p^2)\, B^{(2)}(p^2) - b''^{(4)}(p^2)\right]^{-1} T_{13} = -\frac{1}{4}\left(b''^{(2)}(p^2)\, B'^{(2)}(p^2)\, p^\mu p^\nu\right)^{-1} T_{14}$$

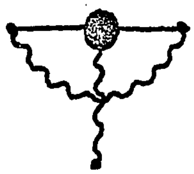
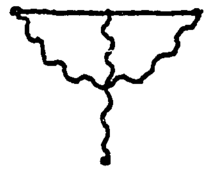
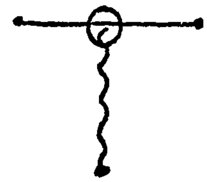
=



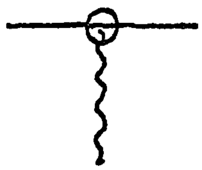
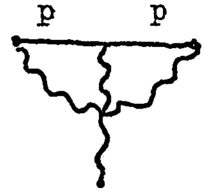
.

Here the slashes on Fermion lines denote $\partial/\partial K_\alpha$ of that line,

and

 $\equiv$  $-$  ,

where

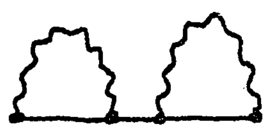
 $\equiv \Gamma_{\mu}^{(2)}(p,p) =$ 

.

Also

$\frac{\gamma p}{i} B^{(2)}(p^2) =$  ,

and

$\frac{\gamma p}{i} B^{(4)}(p^2) =$  $+$  $+$  ,

again omitting photon self energy insertions. $b^{(2)}(p^2)$ and $b^{(4)}(p^2)$ are given by Eqs.(52, 53) in terms of $B^{(2)}(p^2)$ and $B^{(4)}(p^2)$. In the following sections we calculate all fourteen T_i respectively. We begin with the calculation of T_9 to outline the calculational procedure used in the calculation of all T_i, and relate T_9 to the well established Jost-Luttinger result.

Chapter 4

CALCULATIONAL PROCEDURE, THE JOST-LUTTINGER RESULT, AND T_9

The calculation of T_9 demonstrates all of the essential mathematics involved in all parts of the calculation, and is furthermore a good term to begin with since it is simply related to the Jost-Luttinger result[44]:

$$f^{(4)}(\alpha_0) = \left(\frac{\alpha_0}{2\pi}\right)^2 = -\frac{p^4}{48}\left(\frac{\alpha_0}{2\pi}\right) 2\, Tr\, \gamma G'^{(0)} K^{(2)} G'^{(0)} \gamma \ , \tag{71}$$

or

$$f^{(4)}(\alpha_0) = \frac{p^4}{48}\left(\frac{\alpha_0}{2\pi}\right)\left(B^{(2)}\right)^{-1} T_9 \ , \tag{72}$$

where we have used Eq.(70),

$$T_9 = -2 B^{(2)}(p^2)\, Tr\, \gamma G'^{(0)} K^{(2)} G'^{(0)} \gamma . \tag{73}$$

To be perfectly clear regarding our notation, in Eq.(73),

$$\begin{aligned} Tr\, \gamma G'^{(0)} K^{(2)} G'^{(0)} \gamma = Tr \int \frac{d^4 q}{(2\pi)^4} \Big[& \gamma_\mu \frac{\partial}{\partial K^\alpha} S^{(0)}(p+\tfrac{K}{2}) \gamma^\lambda \frac{\partial}{\partial K_\alpha} S^{(0)}(p-q+\tfrac{K}{2}) \gamma^\mu S^{(0)}(p-q) \gamma^\nu S^{(0)}(p)\, D^{(0)}_{\lambda\nu}(q) \\ & + \gamma_\mu \frac{\partial}{\partial K^\alpha} S^{(0)}(p+\tfrac{K}{2}) \gamma^\lambda S^{(0)}(p-q) \gamma^\mu \frac{\partial}{\partial K_\alpha} S^{(0)}(p-q-\tfrac{K}{2}) \gamma^\nu S^{(0)}(p)\, D^{(0)}_{\lambda\nu}(q) \\ & + \gamma_\mu S^{(0)}(p) \gamma^\lambda \frac{\partial}{\partial K^\alpha} S^{(0)}(p-q+\tfrac{K}{2}) \gamma^\mu S^{(0)}(p-q) \gamma^\nu \frac{\partial}{\partial K_\alpha} S^{(0)}(p-\tfrac{K}{2})\, D^{(0)}_{\lambda\nu}(q) \\ & + \gamma_\mu S^{(0)}(p) \gamma^\lambda S^{(0)}(p-q) \gamma^\mu \frac{\partial}{\partial K^\alpha} S^{(0)}(p-q-\tfrac{K}{2}) \gamma^\nu \frac{\partial}{\partial K_\alpha} S^{(0)}(p-\tfrac{K}{2})\, D^{(0)}_{\lambda\nu}(q) \Big]_{\substack{K=0 \\ N=0}} . \end{aligned}$$

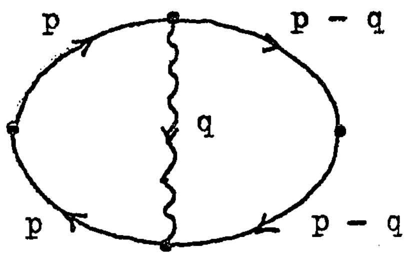
Proceeding then, we first note that

$$i^2 \gamma G'^{(0)} = (\gamma_\alpha \gamma P \gamma_\mu - \gamma_\mu \gamma P \gamma_\alpha)/2P^4 \;, \tag{74}$$

and

$$i^2 G'^{(0)} \gamma = (\gamma_\mu \gamma P \gamma_\alpha - \gamma_\alpha \gamma P \gamma_\mu)/2P^4 \,. \tag{75}$$

Using Eqs.(74, 75) and the Feynman rules, Eq.(29), with the following momentum assignments:



,

then Eq.(73) becomes

$$T_9 = -2B^{(2)}(p^2)\, Tr \int \frac{d^4q}{(2\pi)^4} \frac{ie_0^2(\gamma_\alpha \gamma P \gamma_\mu - \gamma_\mu \gamma P \gamma_\alpha)\gamma_\beta(\gamma_\mu \gamma(p-q)\gamma_\alpha - \gamma_\alpha \gamma(p-q)\gamma_\mu)\gamma_\beta}{2p^4 i^2 q^2\, 2(p-q)^4 i^2} \;. \tag{76}$$

Let k = p − q, then

$$T_9 = -2B^{(2)}(p^2)\, Tr \int \frac{d^4k}{(2\pi)^4} \frac{ie_0^2(\gamma_\alpha \gamma P \gamma_\mu - \gamma_\mu \gamma P \gamma_\alpha)\gamma_\beta(\gamma_\mu \gamma K \gamma_\alpha - \gamma_\alpha \gamma K \gamma_\mu)\gamma_\beta}{4p^4 (p-K)^2 K^4} \;. \tag{77}$$

Using the trace theorems of Appendix 1,

$$Tr(\gamma_\alpha \gamma P \gamma_\mu - \gamma_\mu \gamma P \gamma_\alpha)\gamma_\beta(\gamma_\mu \gamma K \gamma_\alpha - \gamma_\alpha \gamma K \gamma_\mu)\gamma_\beta = 192\, p \cdot K \;. \tag{78}$$

Putting Eq.(78) in Eq.(77) then

$$T_9 = \frac{96 B^{(2)}(p^2) e_0^2}{p^4} I_1 \ , \tag{79}$$

where

$$I_1 = \int \frac{d^4K}{(2\pi)^4} \frac{1}{i} \frac{p\cdot K}{(p-K)^2 K^4} \ . \tag{80}$$

To evaluate I_1 we first perform a Wick rotation to convert the four momentum vectors to four dimensional Euclidean vectors, thereby affording the expansion of the momentum-difference denominators in Chebyshev polynomials. Thus using Eq.(33) and Appendix 2 to expand $(p - k)^{-2} p\cdot k$ in Chebyshev polynomials, then

$$I_1 = \int \frac{d^4K}{(2\pi)^4} \frac{1}{iK^4} \sum_{n=0}^{\infty} \begin{pmatrix} 11 \\ pK \end{pmatrix}_n C_n(pK) \ , \tag{81}$$

where $\begin{pmatrix} 11 \\ pK \end{pmatrix}_n$ are Chebyshev coefficients and $C_n(pk)$ denotes an nth order Chebyshev polynomial in the angle between p and k. Employing the Chebyshev coefficients and the orthogonality property of Appendix 2,

$$I_1 = \int \frac{d^4K}{(2\pi)^4} \frac{1}{iK^4} \sum_{n=0}^{\infty} \frac{1}{2} \left[\left(\frac{P}{K} + \frac{K}{P} \right) \left\langle \frac{P}{K} \right\rangle^{n+1} - \delta_{n0} \right] C_n(PK)$$

$$= \frac{2\pi^2 i}{(2\pi)^4} \int K^3 dK \, \frac{d\Omega_K}{2\pi^2} \frac{1}{iK^4} \sum_{n=0}^{\infty} \frac{1}{2} \left[\left(\frac{P}{K} + \frac{K}{P} \right) \left\langle \frac{P}{K} \right\rangle^{n+1} - \delta_{n0} \right] C_n(PK)\, C_0(PK)$$

$$= \frac{1}{16\pi^2} \int \frac{dK}{K} \left[\left(\frac{P}{K} + \frac{K}{P} \right) \left\langle \frac{P}{K} \right\rangle - 1 \right]$$

$$= \frac{1}{16\pi^2} \left\{ \int_0^P \frac{dK}{K} \left[\left(\frac{P}{K} + \frac{K}{P} \right) \frac{K}{P} - 1 \right] + \int_P^{\infty} \frac{dK}{K} \left[\left(\frac{P}{K} + \frac{K}{P} \right) \frac{P}{K} - 1 \right] \right\}$$

$$= \frac{1}{16\pi^2} \, .$$

The properties of Chebyshev polynomials, including Chebyshev coefficients, have been checked and found to agree with the literature.[45]

Putting Eq.(82) in Eq.(79) and using the definition

$$B^{(2)} \equiv \frac{3\alpha_0}{8\pi} = \frac{3}{32} \frac{e_0^2}{\pi^2} \, , \tag{83}$$

then

$$T_9 = \frac{64}{p^4} B^{(2)} B^{(2)}(p^2) \, . \tag{84}$$

Finally, putting Eq.(84) in Eq.(72), and again using Eq.(83), we obtain

$$f^{(4)}(\alpha_0) = \left(\frac{\alpha_0}{2\pi} \right)^2 \, , \tag{85}$$

which is the Jost-Luttinger result.

Chapter 5

CALCULATION OF T_1

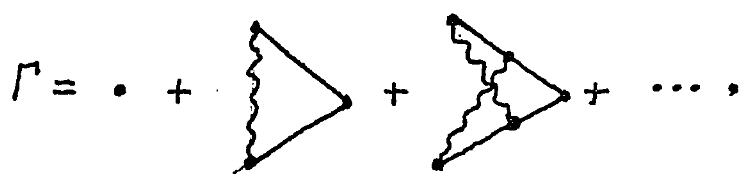
By definition, Eq.(70),

$$T_1 = Tr\, 2\gamma G'^{(0)} K^{(2)} G^{(0)} K^{(2)} G'^{(0)} \gamma \, . \tag{85}$$

Using Eq.(41) and noting that k does not pass through $K^{(2)}$ then

$$K^{(2)} G'^{(0)} \gamma = \frac{\partial}{\partial k_\alpha} \Gamma_\mu^{(2)}\left(p+\frac{k}{2}, p-\frac{k}{2}\right)\Big/_{k=0} \, . \tag{86}$$

The only terms which contribute to $\Gamma_\mu(p+\frac{k}{2}, p-\frac{k}{2})$ with m = 0 are those containing an odd number of gamma-matrices, viz.



the successive terms containing 1, 3, and 9 gamma-matrices respectively. Hence by Lorentz invariance

$$\Gamma_\mu\left(p+\frac{k}{2}, p-\frac{k}{2}\right) = G_{\mu\lambda}(p,k)\gamma_\lambda + G^5_{\mu\lambda}(p,k)\gamma_\lambda \gamma_5 \, , \tag{87}$$

where $G_{\mu\lambda}(p,k)$ and $G^5_{\mu\lambda}(p,k)$ are tensor and pseudotensor functions respectively of p and k. Further CPT invariance requires that

$$\Gamma_\mu(p,p') = \gamma_5 \Gamma_\mu(-p,-p')\gamma_5 \, . \tag{88}$$

C invariance alone implies

$$-\Gamma_\mu^T(-p,-p') = C^{-1}\Gamma_\mu(p',p)\, C \, , \tag{89}$$

where Γ_μ^T is the transposed matrix, and the charge conjugation matrix C satisfies

$$C^{-1}\gamma_\mu C = -\gamma_\mu^T , \tag{90}$$

and

$$C^{-1}\gamma_5 C = \gamma_5^T , \tag{91}$$

where

$$\gamma_5^+ = -\gamma_5 , \tag{92}$$

and

$$\gamma_5^2 = -1 . \tag{93}$$

Then PT invariance gives the requirement

$$\Gamma_\mu(p,p') = -\gamma_5 C^{-1} \Gamma_\mu^T(p',p) C \gamma_5 . \tag{94}$$

Putting Eq.(87) in Eq.(94) and using Eqs.(92, 93), we obtain the conditions:

$$G_{\mu\lambda}(p,-K) = G_{\mu\lambda}(p,K) , \tag{95}$$

and

$$G^5_{\mu\lambda}(p,-K) = -G^5_{\mu\lambda}(p,K) . \tag{96}$$

Now from Eq.(87)

$$\frac{\partial}{\partial K_\alpha}\Gamma^\mu\left(p+\frac{K}{2}, p-\frac{K}{2}\right)\Big|_{K=0} = \lim_{\Delta K_\alpha \to 0}\left\{\frac{\left[G_{\mu\lambda}(p,K+\frac{\Delta K}{2}) - G_{\mu\lambda}(p,K-\frac{\Delta K}{2})\right]\gamma_\lambda}{\Delta K_\alpha} + \frac{\left[G^5_{\mu\lambda}(p,K+\frac{\Delta K}{2}) - G^5_{\mu\lambda}(p,K-\frac{\Delta K}{2})\right]\gamma_\lambda\gamma_5}{\Delta K_\alpha}\right\}\Big|_{K=0} ,$$

or employing Eqs.(95, 96)

$$\frac{\partial}{\partial K_\alpha}\Gamma^\mu\left(p+\frac{K}{2}, p-\frac{K}{2}\right)\Big|_{K=0} = \lim_{\Delta K_\alpha = 0}\left\{\frac{\left[G_{\mu\lambda}(p,\frac{\Delta K}{2}) - G_{\mu\lambda}(p,\frac{\Delta K}{2})\right]\gamma_\lambda}{\Delta K_\alpha} + \frac{\left[G^5_{\mu\lambda}(p,\frac{\Delta K}{2}) + G^5_{\mu\lambda}(p,\frac{\Delta K}{2})\right]\gamma_\lambda\gamma_5}{\Delta K_\alpha}\right\}$$

$$= \frac{\partial}{\partial K_\alpha} G^5_{\mu\lambda}(p,K)\Big|_{K=0}\gamma_\lambda\gamma_5 . \tag{97}$$

But $\frac{\partial}{\partial K_\alpha} G^5_{\mu\lambda}(p,K)$ is a pseudotensor function of p. The only candidate is proportional to $\epsilon_{\alpha\mu\lambda\kappa} p_\kappa / p^2$. Therefore

$$\frac{\partial}{\partial K_\alpha} \Gamma^\mu(p+\frac{K}{2}, p-\frac{K}{2})\Big|_{K=0} = g^{(1)} \epsilon_{\alpha\mu\lambda\kappa} \frac{p_\kappa \gamma_\lambda \gamma_5}{p^2}$$

$$= \frac{g^{(2)}(\gamma_\alpha \gamma p \gamma_\mu - \gamma_\mu \gamma p \gamma_\alpha)}{2p^2} , \tag{98}$$

where $g^{(1)}$ and $g^{(2)}$ are constants. Putting Eq.(98) in Eq.(86), then

$$K^{(2)} G'^{(0)} \gamma = \frac{g^{(2)}}{2p^2} (\gamma_\alpha \gamma p \gamma_\mu - \gamma_\mu \gamma p \gamma_\alpha) . \tag{99}$$

Using Eqs.(99, 75), then

$$G^{(0)} K^{(2)} G'^{(0)} \gamma = g^{(2)} G^{(0)} G'^{(0)} p^2 \tag{100}$$

$$= -g^{(2)} G'^{(0)} \gamma . \tag{101}$$

Again using Eq.(99), then

$$K^{(2)} G^{(0)} K^{(2)} G'^{(0)} \gamma = -g^{(2)} K^{(2)} G'^{(0)} \gamma$$

$$= -\frac{g^{2}_{(2)}}{2p^2} (\gamma_\alpha \gamma p \gamma_\mu - \gamma_\mu \gamma p \gamma_\alpha) . \tag{102}$$

Putting Eq.(102, 75) in Eq.(85), then

$$T_1 = \frac{g^{2}_{(2)}}{2p^6} Tr (\gamma_\alpha \gamma p \gamma_\mu - \gamma_\mu \gamma p \gamma_\alpha)(\gamma_\alpha \gamma p \gamma_\mu - \gamma_\mu \gamma p \gamma_\alpha) . \tag{103}$$

Using the relations given in Appendix 1, we obtain

$$Tr (\gamma_\alpha \gamma p \gamma_\mu - \gamma_\mu \gamma p \gamma_\alpha)(\gamma_\alpha \gamma p \gamma_\mu - \gamma_\mu \gamma p \gamma_\alpha) = -96 p^2 . \tag{104}$$

Putting Eq.(104) in Eq.(103), then

$$T_1 = -\frac{48g^{(2)2}}{p^4} \; . \tag{105}$$

To determine $g^{(2)}$ we project it out of Eq.(99), using the projection operator $\mathcal{P}$:

$$\mathcal{P} \equiv \frac{1}{48} Tr(\gamma_\mu \gamma P \gamma_\alpha - \gamma_\alpha \gamma P \gamma_\mu) \ldots , \tag{106}$$

such that

$$\begin{aligned} \mathcal{P} K^{(2)} G'^{(0)} \gamma &= \frac{g^{(2)}}{96p^2} Tr(\gamma_\mu \gamma P \gamma_\alpha - \gamma_\alpha \gamma P \gamma_\mu)(\gamma_\alpha \gamma P \gamma_\mu - \gamma_\mu \gamma P \gamma_\alpha) \\ &= g^{(2)} , \end{aligned} \tag{107}$$

where we have used Eq.(104). Hence

$$g^{(2)} = \frac{1}{48} Tr(\gamma_\mu \gamma P \gamma_\alpha - \gamma_\alpha \gamma P \gamma_\mu) K^{(2)} G'^{(0)} \gamma , \tag{108}$$

or using Eq.(75) and the Feynman rules

$$g^{(2)} = \frac{ie_0^2}{48} \int \frac{d^4p'}{(2\pi)^4} \frac{Tr(\gamma_\mu \gamma P \gamma_\alpha - \gamma_\alpha \gamma P \gamma_\mu)\gamma_\kappa(\gamma_\alpha \gamma P' \gamma_\mu - \gamma_\mu \gamma P' \gamma_\alpha)\gamma_\kappa}{2p'^4(p-p')^2} \; . \tag{109}$$

We now perform a Wick rotation, using Eq.(33). Then using Eq.(35),

$$g^{(2)} = \frac{1}{48}\left(\frac{\alpha_0}{2\pi}\right)\int p'^3 dp' \int \frac{d\Omega_{p'}}{2\pi^2} \frac{Tr(\gamma_\mu \gamma P \gamma_\alpha - \gamma_\alpha \gamma P \gamma_\mu)\gamma_\kappa(\gamma_\mu \gamma P' \gamma_\alpha - \gamma_\alpha \gamma P' \gamma_\mu)\gamma_\kappa}{2p'^4(P-p')^2} \; . \tag{110}$$

Using Eq.(78) in Eq.(110), then

$$g^{(2)} = 2\left(\frac{\alpha_0}{2\pi}\right)\int\frac{dp'}{p'}\int\frac{d\Omega_{p'}}{2\pi^2}\,\frac{p\cdot p'}{(p-p')^2} \quad . \tag{111}$$

We now perform an expansion in Chebyshev polynomials, using Appendix 2:

$$\begin{aligned}
g^{(2)} &= 2\left(\frac{\alpha_0}{2\pi}\right)\int\frac{dp'}{p'}\int\frac{d\Omega_{p'}}{2\pi^2}\,pp'\sum_{n=0}^{\infty}\left(\frac{1\,1}{pp'}\right)_n C_n(pp') \\
&= 2\left(\frac{\alpha_0}{2\pi}\right)\int\frac{dp'}{p'}\,pp'\sum_{n=0}^{\infty}\frac{1}{2pp'}\left\{\left(\frac{p}{p'}+\frac{p'}{p}\right)\left\langle\frac{p}{p'}\right\rangle^{n+1}-\delta_{n0}\right\}\int\frac{d\Omega_{p'}}{2\pi^2}\,C_0(pp')\,C_n(pp') \\
&= \frac{\alpha_0}{2\pi}\int\frac{dp'}{p'}\sum_{n=0}^{\infty}\left\{\left(\frac{p}{p'}+\frac{p'}{p}\right)\left\langle\frac{p}{p'}\right\rangle^{n+1}-\delta_{n0}\right\}\delta_{n0} \\
&= \frac{\alpha_0}{2\pi}\int\frac{dp'}{p'}\left[\left(\frac{p}{p'}+\frac{p'}{p}\right)\left\langle\frac{p}{p'}\right\rangle-1\right] \\
&= \frac{\alpha_0}{2\pi}\int\frac{dp'}{p'}\left\langle\frac{p'}{p}\right\rangle^2 = \frac{\alpha_0}{2\pi}\left\{\int_0^p\frac{dp'}{p'}\left(\frac{p'}{p}\right)^2+\int_p^{\infty}\frac{dp'}{p'}\left(\frac{p}{p'}\right)^2\right\} \\
&= \frac{\alpha_0}{2\pi} \quad .
\end{aligned} \tag{112}$$

Finally putting Eq.(112) in Eq.(105), then

$$T_1 = -\frac{48}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \quad . \tag{113}$$

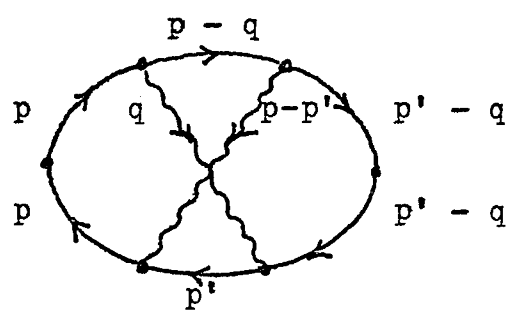
Chapter 6

CALCULATION OF T_2

By definition, Eq.(70):

$$T_2 = 2\,Tr\ \gamma G'^{(0)} K_b^{(4)} G'^{(0)} \gamma \ . \tag{114}$$

We make the following momentum assignments in Eq.(114):



Using Eqs.(74, 75) and the Feynman rules in Eq.(114), then

$$T_2 = 2\,Tr \frac{1}{j^2} \frac{(\gamma_\alpha \gamma p \gamma_\mu - \gamma_\mu \gamma p \gamma_\alpha)}{2p^4} \int \frac{d^4 q}{(2\pi)^4} \int \frac{d^4 p'}{(2\pi)^4} (ie_0^2)^2 \gamma_\mu \frac{1}{i\gamma(p-q)} \gamma_\rho \frac{1}{j^2} \left[\gamma_\mu \gamma(p'-q)\gamma_\alpha \right. \tag{115}$$
$$\left. - \gamma_\alpha \gamma(p'-q)\gamma_\mu\right] \frac{1}{2(p'-q)^4} \gamma_\mu \frac{1}{i\gamma p'} \gamma_\rho \frac{1}{(p-p')^2 q^2} \ .$$

Then rotating contours, using Eq.(33), Eq.(115) becomes:

$$T_2 = \frac{1}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \int_0^\infty dp' \int_0^\infty dq \int \frac{d\Omega_{p'}}{2\pi^2} \int \frac{d\Omega_q}{2\pi^2} \frac{-p'q\,Tr(\gamma_\alpha \gamma p \gamma_\mu - \gamma_\mu \gamma p \gamma_\alpha)\gamma_\mu \gamma(p-q)\gamma_\rho \gamma_\mu \gamma(p'-q)\gamma_\alpha \gamma_\mu \gamma p' \gamma_\rho}{(p-q)^2 (p'-q)^4 (p-p')^2} \ . \tag{116}$$

Using Appendix 1, we obtain:

$$Tr(\gamma_\alpha \gamma p \gamma_\mu - \gamma_\mu \gamma p \gamma_\alpha)\gamma_\mu \gamma(p-q)\gamma_\rho \gamma_\mu \gamma(p'-q)\gamma_\alpha \gamma_\mu \gamma p' \gamma_\rho = 384\,p\cdot(p'-q)\,p'\cdot(p-q) \ . \tag{117}$$

Putting Eq.(117) in Eq.(116), then

$$T_2 = -\frac{1}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \int_0^\infty dp' \int_0^\infty dq \int \frac{d\Omega_{p'}}{2\pi^2} \int \frac{d\Omega_q}{2\pi^2} \; \frac{384\, p'q\left[p'\cdot p\; p\cdot(p'-q) - p'q\; p\cdot(p'-q)\right]}{(p-q)^2(p'-q)^4(p-p')^2} \; . \tag{118}$$

T_2 is symmetric in p' and q, therefore the second term gives no contribution to T_1 since

$$p'q\; p'\cdot q\; p\cdot(p'-q) \xrightarrow[p'\leftrightarrow q]{} -p'q\; p'\cdot q\; p\cdot(p'-q) \; ,$$

so

$$T_2 = \frac{-384}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \int_0^\infty dp' \int_0^\infty dq \int \frac{d\Omega_{p'}}{2\pi^2} \int \frac{d\Omega_q}{2\pi^2} \; \frac{p'q\; p'\cdot p\; p\cdot(p'-q)}{(p-q)^2(p'-q)^4(p-p')^2} \; . \tag{119}$$

We now show that T_2 is symmetric in p and p'. Since T_2 is independent of the direction of p, we multiply it by $1 = \int d\Omega_p/2\pi^2$ with resulting symmetry in angles. We then perform the angular integrations, obtaining the following form:

$$T_2 = \frac{1}{p^4}\int_0^\infty \frac{dp'}{p'} \int_0^\infty \frac{dq}{q} f(p,p',q) = \frac{1}{p^4}\int_0^\infty \frac{dp'}{p'} \int_0^\infty \frac{dq}{q}\, g\!\left(\frac{p'}{p}, \frac{q}{p}\right) , \tag{120}$$

since $p^4 T_2$ is proportional to f, which is dimensionless. Let

$$y = \frac{p'}{p} \; , \quad x = \frac{q}{p} \; .$$

Then

$$\frac{dp'\,dq}{p'q} = \frac{dy\,dx}{y\,x} \; ,$$

and T_2 becomes

$$T_2 = \frac{1}{p^4}\int_0^\infty \frac{dy}{y}\int_0^\infty \frac{dx}{x}\, g(y,x) \,.$$

Now let

$$y = \frac{p}{p'} \,, \qquad x = \frac{q}{p'} \,.$$

Then

$$\frac{dy\,dx}{y\,x} = -\,\frac{dp'\,dq}{p'q} \,,$$

and

$$\begin{aligned} y=0 &\Rightarrow p'=\infty \,, \\ y=\infty &\Rightarrow p'=0 \,, \\ x=0 &\Rightarrow q=0 \,, \\ x=\infty &\Rightarrow q=\infty \,, \end{aligned}$$

so

$$\begin{aligned} T_2 &= -\frac{1}{p^4}\int_\infty^0 \frac{dp'}{p'}\int_0^\infty \frac{dq}{q}\, g\!\left(\frac{p}{p'},\frac{q}{p'}\right) \\ &= \frac{1}{p^4}\int_0^\infty \frac{dp'}{p'}\int_0^\infty \frac{dq}{q}\, g\!\left(\frac{p}{p'},\frac{q}{p'}\right) . \end{aligned} \tag{121}$$

Comparing Eq.(120, 121) we conclude that we may symmetrize the integrand of T_1 in p and p'. Letting $p' \rightarrow p$ and $p \rightarrow p'$ in Eq.(119), then

$$T_2 = -\frac{384}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2\int_0^\infty \frac{dq}{q}\int_0^\infty \frac{dp'}{p'}\int \frac{d\Omega_{p'}}{2\pi^2}\int \frac{d\Omega_q}{2\pi^2}\, \frac{p^2q^2\, p\cdot q\, p'\cdot(p-q)}{(p-q)^4(p'-q)^2(p-p')^2} \,, \tag{122}$$

or

$$T_2 = -\frac{384}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2\int_0^\infty dq\int_0^\infty dp'\int \frac{d\Omega_{p'}}{2\pi^2}\int \frac{d\Omega_q}{2\pi^2}\, \frac{q\,p'p^4\cos^2 pp' - q^2p'p^3\cos pp'\cos p'q}{(p-q)^4(p'-q)^2(p-p')^2} \,. \tag{123}$$

We now expand Eq.(123) in Chebyshev polynomials, using Appendix 2,

$$T_2 = -\frac{384}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \int_0^\infty dq \int_0^\infty dp' \left(qp'p^4 \begin{bmatrix} 21 & 01 & 02 \\ pp', & p'q, & pq \end{bmatrix} - q^2 p' p^3 \begin{bmatrix} 11 & 11 & 02 \\ pp', & p'q, & pq \end{bmatrix} \right)$$

$$= -\frac{384}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \sum_{n=0}^\infty \frac{1}{n+1} \int_0^\infty dq \int_0^\infty dp' \left(qp'p^4 \binom{21}{pp'}_n \binom{01}{p'q}_n \binom{02}{pq}_n - q^3 p' p^3 \binom{11}{pp'}_n \binom{11}{p'q}_n \binom{02}{pq}_n \right)$$

$$\equiv -\frac{384}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \sum_{n=0}^\infty \frac{1}{n+1} \int_0^\infty dq \int_0^\infty dp' \, g_n(p'q)$$

$$= -\frac{384}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \sum_{n=0}^\infty \frac{1}{n+1} \left(\int_{p'<q} dp' dq \, g_n(p'q) + \int_{p'>q} dp' dq \, g_n(p'q) \right)$$

$$= -\frac{384}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \sum_{n=0}^\infty \frac{1}{n+1} \int_{p'<q} dp' dq \left(g_n(p'q) + g_n(qp') \right)$$

$$\equiv -\frac{384}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \sum_{n=0}^\infty \frac{1}{n+1} \int_{p'<q} dp' dq \, g_n(\underline{p'q}) \equiv \sum_{n=0}^\infty T_{2n} \quad . \tag{124}$$

Using Appendix 2, we have

$$g_0(p'q) = \frac{qp'p^4}{4pp'}\left(\frac{p}{p'}+\frac{p'}{p}\right)\left\langle\frac{p}{p'}\right\rangle^2 \frac{1}{p'q}\left\langle\frac{p'}{q}\right\rangle \frac{1}{pq|p^2-q^2|}\left\langle\frac{p}{q}\right\rangle$$

$$- q^2 p' p^3 \frac{1}{2pp'}\left\langle\frac{p}{p'}\right\rangle^2 \frac{1}{2p'q}\left\langle\frac{p'}{q}\right\rangle^2 \frac{1}{pq|p^2-q^2|}\left\langle\frac{p}{q}\right\rangle$$

$$= \frac{1}{|p^2-q^2|}\left(\frac{p^3}{4p'^2q} + \frac{p}{4q} - \frac{p}{4p'}\left\langle\frac{p'}{q}\right\rangle\right)\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle \quad . \tag{125}$$

Define

$$g_{n>0}(p'q) = \bar{g}_n(p'q) + \delta_1(p'q)\,\delta_{1n} \quad , \tag{126}$$

and let

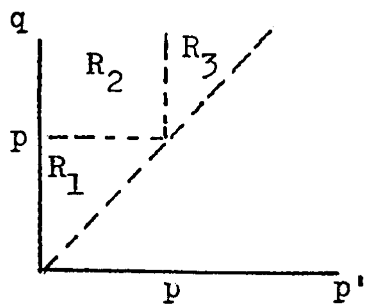
$$g_n(\underline{p'q}) = \begin{cases} g_0(\underline{p'q}) + \frac{1}{2}\,\delta_1(\underline{p'q}) \equiv \bar{g}_0(\underline{p'q}) & , \quad n=0 \\ \bar{g}_n(p'q) & , \quad n>0 \end{cases} \quad . \tag{127}$$

Using Appendix 2, then

$$\delta_1(p'q) = -q\,p'\,p^4\,\frac{1}{4pp'}\,\frac{1}{p'q}\left\langle\frac{p'}{q}\right\rangle^2 \frac{2}{pq\,|p^2-q^2|}\left\langle\frac{p}{q}\right\rangle^2$$

$$= -\frac{1}{2\,|p^2-q^2|}\,\frac{p^2}{qp'}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 \; . \tag{128}$$

We divide the region of integration in Eq.(124) into three regions R_1, R_2, and R_3:



R_1: $p' < q < p$

R_2: $p' < p < q$

R_3: $p < p' < q$

Putting Eqs.(125, 128) in Eq.(127), symmetrizing in p' and q, and evaluating $\bar{g}_0(p'q)$ in regions R_1, R_2, and R_3 respectively, we obtain:

$$\bar{g}_0(\underline{p'q}\,R_1) = \frac{1}{4}\,\frac{p'}{qp^2} + \left(\frac{1}{4}\,\frac{p'q}{p^2} - \frac{1}{4}\,\frac{p'^3}{q^3}\right)\frac{1}{|p^2-p'^2|} \quad , \tag{129}$$

$$\bar{g}_0(\underline{p'q}\,R_2) = \frac{1}{2}\,\frac{p^2p'}{q^5} + \frac{1}{4}\,\frac{p'}{q^3} \quad , \tag{130}$$

and

$$\bar{g}_0(\underline{p'q}\,R_3) = \left(\frac{1}{4}\frac{p^6}{p'^3q^3} - \frac{1}{4}\frac{p^4p'}{q^5}\right)\frac{1}{|q^2-p^2|} - \frac{1}{4}\frac{p^4}{p'q^5}\,. \tag{131}$$

We evaluate T_{2o} in regions R_1, R_2, and R_3:

$$T_{20} = T_{20}(R_1) + T_{20}(R_2) + T_{20}(R_3)\,. \tag{132}$$

Putting Eqs.(129, 130, 131) in Eq.(124), then

$$T_{20}(R_1) = -\frac{384}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2\left(\int_0^p dq\int_0^q dp'\frac{p'}{4qp^2} + \int_0^p dp'\int_{p'}^p dq\left(\frac{1}{4}\frac{p'q}{p^2} - \frac{1}{4}\frac{p'^3}{q^3}\right)\frac{1}{(p^2-p'^2)}\right)$$

$$= -\frac{384}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2\left(-\frac{1}{16}\right)\,, \tag{133}$$

$$T_{20}(R_2) = -\frac{384}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2\int_p^\infty dq\int_0^p dp'\left(\frac{1}{2}\frac{p^2p'}{q^5} + \frac{1}{4}\frac{p'}{q^3}\right)$$

$$= -\frac{384}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2\left(\frac{1}{8}\right)\,, \tag{134}$$

and

$$T_{20}(R_3) = -\frac{384}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2\left(\int_p^\infty dq\int_p^q dp'\left(\frac{1}{4}\frac{p^6}{p'^3q^3} - \frac{1}{4}\frac{p^4p'}{q^5}\right)\frac{1}{(q^2-p^2)} + \int_p^\infty dp'\int_{p'}^\infty dq\left(-\frac{1}{4}\frac{p^4}{p'q^5}\right)\right)$$

$$= -\frac{384}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2\frac{1}{64}\,. \tag{135}$$

Putting Eqs.(133-135) in Eq.(132), then

$$T_{20} = -\frac{5}{64}\left(\frac{384}{p^4}\right)\left(\frac{\alpha_0}{2\pi}\right)^2. \tag{136}$$

Using Appendix 2 in Eq.(124),

$$\bar{g}^{n}(p'q) = q p' p^{4} \frac{1}{4pp'}\left(\frac{p}{p'} + \frac{p'}{p}\right)^{2}\left\langle\frac{p}{p'}\right\rangle^{n+1} \frac{1}{pq}\left\langle\frac{p'}{q}\right\rangle^{n+1} \frac{n+1}{pq\,|p^{2}-q^{2}|}\left\langle\frac{p}{q}\right\rangle^{n+1}$$

$$- q^{2} p' p^{3} \frac{1}{2pp'}\left(\frac{p}{p'} + \frac{p'}{p}\right)\left\langle\frac{p}{p'}\right\rangle^{n+1} \frac{1}{2p'q}\left(\frac{p'}{q} + \frac{q}{p'}\right)\left\langle\frac{p'}{q}\right\rangle^{n+1} \frac{n+1}{pq\,|p^{2}-q^{2}|}\left\langle\frac{p}{q}\right\rangle^{n+1}$$

$$= \frac{n+1}{|p^{2}-q^{2}|}(p^{2}-q^{2})\left\langle\frac{p}{p'}\right\rangle^{n+1}\left\langle\frac{p'}{q}\right\rangle^{n+1}\left\langle\frac{p}{q}\right\rangle^{n+1}\left(\frac{p^{2}}{4qp'^{3}} + \frac{1}{4qp'}\right). \qquad (137)$$

Then

$$\bar{g}^{n}(\underline{p'q}\,R_{1}) = (n+1)\left(\frac{p'^{2n-1}}{4qp^{2n}} + \frac{p'^{2n+1}}{2qp^{2n+2}} + \frac{p'^{2n+1}}{4p^{2n}q^{3}}\right), \qquad (138)$$

$$\bar{g}^{n}(\underline{p'q}\,R_{2}) = (n+1)\left(-\frac{p'^{2n-1}p^{2}}{4q^{2n+3}} + \frac{p'^{2n+1}p^{2}}{4q^{2n+5}}\right), \qquad (139)$$

and

$$\bar{g}^{n}(\underline{p'q}\,R_{3}) = (n+1)\left(-\frac{p^{2n+4}}{4p'^{3}q^{2n+3}} - \frac{p^{2n+2}}{2p'q^{2n+3}} - \frac{p^{2n+4}}{4p'q^{2n+5}}\right). \qquad (140)$$

Putting Eq.(138) in Eq.(124), then

$$\sum_{n=1}^{\infty} T_{2n}(R_{1}) = -\frac{384}{p^{4}}\left(\frac{\alpha_{0}}{2\pi}\right)^{2}\sum_{n=1}^{\infty}\frac{1}{4}\int_{0}^{p}dq\int_{0}^{q}dp'\left(\frac{p'^{2n-1}}{qp^{2n}} + \frac{2p'^{2n+1}}{qp^{2n+2}} + \frac{p'^{2n+1}}{p^{2n}q^{3}}\right)$$

$$= -\frac{384}{p^{4}}\left(\frac{\alpha_{0}}{2\pi}\right)^{2}\left(\frac{3}{16}\zeta(2) - \frac{1}{16}\right), \qquad (141)$$

where

$$\zeta(n) \equiv \sum_{K=1}^{\infty}\frac{1}{K^{n}} \qquad (142)$$

is a Riemann zeta function.[46] Putting Eq.(139) in Eq.(124),

$$\sum_{n=1}^{\infty} T_{2n}(R_2) = -\frac{384}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \sum_{n=1}^{\infty} -\frac{1}{4}\int_{p}^{\infty} dq \int_{0}^{p} dp' \left(\frac{p'^{\,2n-1}p^2}{q^{2n+3}} - \frac{p'^{\,2n+1}p^2}{q^{2n+5}} \right)$$

$$= \frac{384}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2\left(\frac{1}{32}\right) . \tag{143}$$

Putting Eq.(140) in Eq.(124)

$$\sum_{n=1}^{\infty} T_{2n}(R_3) = -\frac{384}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \sum_{n=1}^{\infty} -\frac{1}{4}\int_{p}^{\infty} dp' \int_{p'}^{\infty} dq \left(\frac{p^{2n+4}}{p'^3 q^{2n+3}} + \frac{2p^{2n+2}}{p' q^{2n+3}} + \frac{p^{2n+4}}{p' q^{2n+5}} \right)$$

$$= -\frac{384}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2\left(-\frac{3}{16}\zeta(2) + \frac{11}{64}\right) . \tag{144}$$

Combining Eqs.(136, 141, 143, 144), then

$$T_2 = -\frac{96}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 . \tag{145}$$

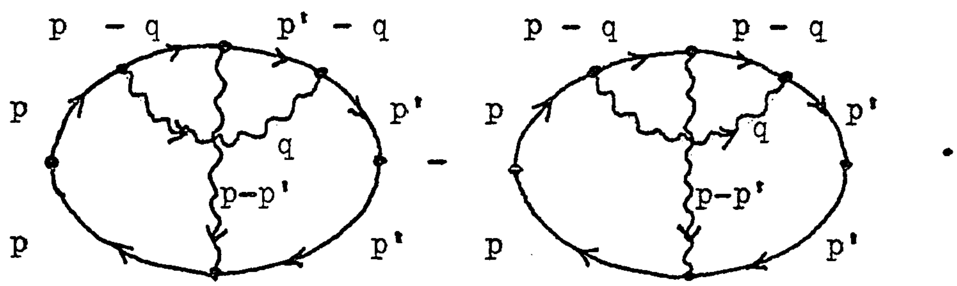
Chapter 7

CALCULATION OF T_3

By definition, Eq.(70),

$$T_3 = 4\,\mathrm{Tr}\,\gamma G'^{(0)}\left[K_c^{(4)} - \bar{K}^{(4)}\right]G'^{(0)}\gamma. \tag{146}$$

We employ Eq.(65) and make the following momentum assignments in Eq.(146):



Using Eqs.(74, 75) and the Feynman rules in Eq.(146), then

$$T_3 = 4\,\mathrm{Tr}\frac{1}{i^2}\,\frac{\gamma_\alpha\gamma p\gamma_\mu - \gamma_\mu\gamma p\gamma_\alpha}{2p^4}\int\frac{d^4q}{(2\pi)^4}\int\frac{d^4p'}{(2\pi)^4}\,(ie_0^2)^2\Big[\gamma_\mu\frac{1}{i\gamma(p-q)}\gamma_\rho\frac{1}{i\gamma(p'-q)}\gamma_\mu$$

$$-\gamma_\mu\frac{1}{i\gamma(p-q)}\gamma_\rho\frac{1}{i\gamma(p-q)}\gamma_\mu\Big]\,\frac{(\gamma_\mu\gamma p'\gamma_\alpha - \gamma_\alpha\gamma p'\gamma_\mu)\gamma_\rho}{i^2\,2p^4\quad q^2(p-p')^2}\,. \tag{147}$$

Rotating contours, using Eq.(33), Eq.(147) becomes

$$T_3 = -\frac{2}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2\int\frac{dp'}{p'}\int\frac{dq}{q}\int\frac{d\Omega_{p'}}{2\pi^2}\int\frac{d\Omega_q}{2\pi^2}\,(N_1^2N_2^2N_3^2)^{-2}\,\mathrm{Tr}\Big\{(\gamma_\alpha\gamma p\gamma_\mu - \gamma_\mu\gamma p\gamma_\alpha)$$

$$\cdot(N_1^2N_2^2N_3^2q^2\gamma_\mu\gamma N_1\gamma_\rho\gamma N_2\gamma_\mu - N_2^4N_3^2q^2\gamma_\mu\gamma N_1\gamma_\rho\gamma N_1\gamma_\mu)$$

$$\cdot\gamma_\mu\gamma p\gamma_\alpha\gamma_\rho\Big\}. \tag{148}$$

where

$$N_1 = p - q \ , \ N_2 = p' - q , \quad N_3 = p - p' . \tag{149}$$

Consider the numerator of Eq.(148):

$$\eta \equiv N_1^2 N_2^2 N_3^2 q^2 \, Tr\,(\gamma_\alpha \gamma P \gamma_\mu - \gamma_\mu \gamma P \gamma_\alpha) \gamma_\kappa \gamma N_1 \gamma_\rho \gamma N_2 \gamma_\kappa \gamma_\mu \gamma P' \gamma_\alpha \gamma_\rho$$

$$- N_2^4 N_3^2 q^2 \, Tr\,(\gamma_\alpha \gamma P \gamma_\mu - \gamma_\mu \gamma P \gamma_\alpha) \gamma_\kappa \gamma N_1 \gamma_\rho \gamma N_1 \gamma_\kappa \gamma_\mu \gamma P' \gamma_\alpha \gamma_\rho \ . \tag{150}$$

From Eq.(149),

$$N_2 = - N_3 + N_1 \ . \tag{151}$$

Inserting Eq.(151) in the first term of Eq.(150) inside the trace,

$$\eta = N_1^2 N_2^2 N_3^2 q^2 \, Tr\,(\gamma_\alpha \gamma P \gamma_\mu - \gamma_\mu \gamma P \gamma_\alpha) \gamma_\kappa \gamma N_1 \gamma_\rho (-\gamma N_3) \gamma_\kappa \gamma_\mu \gamma P \gamma_\alpha \gamma_\rho$$

$$+ q^2 N_2^2 N_3^2 (N_1^2 - N_2^2) \, Tr\,(\gamma_\alpha \gamma P \gamma_\mu - \gamma_\mu \gamma P \gamma_\alpha) \gamma_\kappa \gamma N_1 \gamma_\rho \gamma N_1 \gamma_\kappa \gamma_\mu \gamma P' \gamma_\alpha \gamma_\rho . \tag{152}$$

From Eq.(149) we observe that

$$N_1^2 - N_2^2 = p \cdot N_1 + p \cdot N_2 - p' \cdot N_1 - p' \cdot N_2 \ . \tag{153}$$

Putting Eq.(153) in Eq.(152) and using Eq.(149), then

$$\eta = N_1^2 N_2^2 N_3^2 q^2 \, Tr\,(\gamma_\alpha \gamma P \gamma_\mu - \gamma_\mu \gamma P \gamma_\alpha) \gamma_\kappa \gamma N_1 \gamma_\rho \gamma (p' - p) \gamma_\kappa \gamma_\mu \gamma P' \gamma_\alpha \gamma_\rho$$

$$+ q^2 N_2^2 N_3^2 (p \cdot N_1 + p \cdot N_2 - p' \cdot N_1 - p' \cdot N_2) \tag{154}$$

$$\cdot \, Tr\,(\gamma_\alpha \gamma P \gamma_\mu - \gamma_\mu \gamma P \gamma_\alpha) \gamma_\kappa \gamma N_1 \gamma_\rho \gamma N_1 \gamma_\kappa \gamma_\mu \gamma P' \gamma_\alpha \gamma_\rho \ .$$

Using Appendix 1 to expand the trace, Eq.(154) becomes:

$$\eta = 128\, q^2 N_1^2 N_2^2 N_3^2 (-p'^2 p^2 + p'^2 p\cdot q + (p\cdot p')^2 - p\cdot q\, p\cdot p')$$

$$+128\, q^2 N_2^2 N_3^2 \left[-p'\cdot N_1 (p\cdot N_1)^2 - p\cdot N_2\, p'\cdot N_1\, p\cdot N_1 + p\cdot N_1 (p'\cdot N_1)^2 + p'\cdot N_2\, p'\cdot N_1\, p\cdot N_1 \right] . \quad (155)$$

Inserting η , Eq.(155), as defined in Eq.(150), in Eq.(148), then

$$T_3 = -\frac{256}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \left(T_3^1 + T_3^2\right) , \quad (156)$$

where

$$T_3^1 = \int_0^\infty \frac{dq}{q} \int_0^\infty \frac{dp'}{p'} \int \frac{d\Omega_{p'}}{2\pi^2} \int \frac{d\Omega_q}{2\pi^2}\; \frac{q^2 N_1^2 N_2^2 N_3^2 (-p'^2 p^2 + p'^2 p\cdot q + (p\cdot p')^2 - p\cdot q\, p\cdot p')}{(N_1^2 N_2^2 N_3^2)^2} , \quad (157)$$

and

$$T_3^2 = \int_0^\infty \frac{dq}{q} \int_0^\infty \frac{dp'}{p'} \int \frac{d\Omega_{p'}}{2\pi^2} \int \frac{d\Omega_q}{2\pi^2}\; \frac{q^2 N_2^2 N_3^2 (-p'\cdot N_1 (p\cdot N_1)^2 - p\cdot N_2\, p'\cdot N_1\, p\cdot N_1 + p\cdot N_1 (p'\cdot N_1)^2 + p'\cdot N_2\, p'\cdot N_1\, p\cdot N_1)}{(N_1^2 N_2^2 N_3^2)^2} . \quad (158)$$

We expand Eq.(157) in Chebyshev polynomials, using Appendix 2:

$$T_3^1 = \int_0^\infty dq \int_0^\infty dp' \Big(-q p' p^2 \begin{bmatrix} 01 & 01 & 01 \\ pp', & p'q, & pq \end{bmatrix} + q^2 p' p \begin{bmatrix} 01 & 01 & 11 \\ pp', & p'q, & pq \end{bmatrix}$$

$$+ q p' p^2 \begin{bmatrix} 21 & 01 & 01 \\ pp', & p'q, & pq \end{bmatrix} - q^2 p^2 \begin{bmatrix} 11 & 01 & 11 \\ pp', & p'q, & pq \end{bmatrix} \Big) , \quad (159)$$

or

$$T_3' = \sum_{n=0}^{\infty} \frac{1}{n+1} \int_0^{\infty} dq \int_0^{\infty} dp' \Big\{ -q p' p^2 \begin{pmatrix} 0 & 1 \\ p & p' \end{pmatrix}_n \begin{pmatrix} 0 & 1 \\ p' & q \end{pmatrix}_n \begin{pmatrix} 0 & 1 \\ p & q \end{pmatrix}_n + q^2 p' p \begin{pmatrix} 0 & 1 \\ p & p' \end{pmatrix}_n \begin{pmatrix} 0 & 1 \\ p' & q \end{pmatrix}_n \begin{pmatrix} 1 & 1 \\ p & q \end{pmatrix}_n$$

$$+ q p' p^2 \begin{pmatrix} 2 & 1 \\ p & p' \end{pmatrix}_n \begin{pmatrix} 0 & 1 \\ p' & q \end{pmatrix}_n \begin{pmatrix} 0 & 1 \\ p & q \end{pmatrix}_n - q^2 p^2 \begin{pmatrix} 1 & 1 \\ p & p' \end{pmatrix}_n \begin{pmatrix} 0 & 1 \\ p' & q \end{pmatrix}_n \begin{pmatrix} 1 & 1 \\ p & q \end{pmatrix}_n \Big\}$$

$$\equiv \sum_{n=0}^{\infty} \frac{1}{n+1} \int_0^{\infty} dq \int_0^{\infty} dp' \, g_n(p'q)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n+1} \int_{p'<q} dp' dq \, \big(g_n(p'q) + g_n(qp') \big)$$

$$\equiv \sum_{n=0}^{\infty} \frac{1}{n+1} \int_{p'<q} dp' dq \, g_n(\underline{p'q})$$

$$= \sum_{n=0}^{\infty} T_{3n} \quad . \tag{160}$$

From Appendix 2 and Eq.(160), we have

$$g_0(p'q) = -\frac{1}{p'q} \left\langle \frac{p}{p'} \right\rangle \left\langle \frac{p'}{q} \right\rangle \left\langle \frac{p}{q} \right\rangle + \frac{1}{2p'p} \left\langle \frac{p}{p'} \right\rangle \left\langle \frac{p'}{q} \right\rangle \left\langle \frac{p}{q} \right\rangle^2$$

$$+ \frac{p}{4qp'^2} \left\langle \frac{p}{p'} \right\rangle^2 \left\langle \frac{p'}{q} \right\rangle \left\langle \frac{p}{q} \right\rangle + \frac{1}{4pq} \left\langle \frac{p}{p'} \right\rangle^2 \left\langle \frac{p'}{q} \right\rangle \left\langle \frac{p}{q} \right\rangle$$

$$- \frac{1}{4p'^2} \left\langle \frac{p}{p'} \right\rangle^2 \left\langle \frac{p'}{q} \right\rangle \left\langle \frac{p}{q} \right\rangle^2 \quad . \tag{161}$$

Define

$$g_{n>0}(p'q) = \bar{g}_n(p'q) + \delta_1(p'q)\,\delta_{1n} \quad , \tag{162}$$

and let

$$g_n(\underline{p'q}) = \begin{cases} g_0(\underline{p'q}) + \frac{1}{2}\delta_1(\underline{p'q}) \equiv \bar{g}_0(\underline{p'q}) \,, & n=0 \\ \bar{g}_n(\underline{p'q}) & , n>0 \end{cases} \quad . \tag{163}$$

From Eq.(160) and Appendix 2,

$$\delta_1(p'q) = -q p' p \frac{1}{4pp'} \frac{1}{p'q} \left\langle \frac{p'}{q} \right\rangle^2 \frac{1}{pq} \left\langle \frac{p}{q} \right\rangle^2 . \tag{164}$$

Putting Eqs.(161, 164) in Eq.(160), then

$$\begin{aligned} \bar{g}_0(\underline{p'q}) = & -\frac{1}{p'q}\left\langle\frac{p}{p'}\right\rangle\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle + \frac{1}{2pp'}\left\langle\frac{p}{p'}\right\rangle\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle^2 + \frac{p}{4qp'^2}\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle \\ & + \frac{1}{4pq}\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle - \frac{1}{4p'^2}\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle^2 - \frac{1}{8p'q}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 \\ & - \frac{1}{p'q}\left\langle\frac{p}{q}\right\rangle\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{p'}\right\rangle + \frac{1}{2pq}\left\langle\frac{p}{q}\right\rangle\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{p'}\right\rangle^2 + \frac{p}{4p'q^2}\left\langle\frac{p}{q}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{p'}\right\rangle \\ & + \frac{1}{4pp'}\left\langle\frac{p}{q}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{p'}\right\rangle - \frac{1}{4q^2}\left\langle\frac{p}{q}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{p'}\right\rangle^2 - \frac{1}{8p'q}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{p'}\right\rangle^2 . \end{aligned} \tag{165}$$

We divide the region of integration into three regions, as we did for T_2:

R_1 : $p' < q < p$,

R_2 : $p' < p < q$,

R_3 : $p < p' < q$,

and evaluate T_{30} in these regions:

$$T'_{30} = \sum_{i=1}^{3} T'_{30}(R_i) = \sum_{i=1}^{3} \int_{R_i} dp'\,dq\, \bar{g}_0(\underline{p'q}) \quad . \tag{166}$$

Putting Eq.(165) in Eq.(166), then

$$T_{30}'(R_1)=\int_0^p dq\int_0^q dp'\left(-\frac{13}{8}\frac{p'}{qp^2}+\frac{p'q}{2p^4}+\frac{p'^3}{2qp^4}-\frac{p'^3}{8q^3p^2}\right)=-\frac{21}{64}\,, \tag{167}$$

$$T_{30}'(R_2)=\int_p^\infty dq\int_0^p dp'\left(-\frac{5}{4}\frac{p'}{q^3}+\frac{5}{8}\frac{p'^3}{p^2q^3}+\frac{p'p^2}{8q^5}-\frac{1}{4}\frac{p'^3}{q^5}\right)=-\frac{15}{64}\,, \tag{168}$$

and

$$T_{30}'(R_3)=\int_p^\infty dp'\int_{p'}^\infty dq\left(-\frac{5}{8}\frac{p^2}{p'q^3}-\frac{p'p^2}{8q^5}\right)=-\frac{11}{64}\,. \tag{169}$$

Further, from Eq.(160) and Appendix 2,

$$\bar{g}^n(p'q)=\left(\frac{q}{2p'p^2}+\frac{p'}{4qp^2}-\frac{q}{4p'^3}-\frac{1}{4p'q}-\frac{q}{4p^2p'}\right)\left\langle\frac{p}{p'}\right\rangle^{n+1}\left\langle\frac{p'}{q}\right\rangle^{n+1}\left\langle\frac{p}{q}\right\rangle^{n+1}, \tag{170}$$

so that

$$\begin{aligned}\sum_{n=1}^\infty T_{3n}'(R_1)&=\sum_{n=1}^\infty\frac{1}{n+1}\int_0^p dq\int_0^q dp'\left(\frac{qp'^{2n+1}}{2p^{2n+4}}+\frac{p'^{2n+3}}{2qp^{2n+4}}-\frac{qp'^{2n-1}}{4p^{2n+2}}-\frac{p'^{2n+1}}{2qp^{2n+2}}-\frac{p'^{2n+3}}{4q^3p^{2n+2}}\right)\\&=-\frac{1}{8}\zeta(3)+\frac{1}{8}\,,\end{aligned} \tag{171}$$

$$\begin{aligned}\sum_{n=1}^\infty T_{3n}'(R_2)&=\sum_{n=1}^\infty\frac{1}{n+1}\int_p^\infty dq\int_0^p dp'\left(\frac{p'^{2n+1}}{2p^2q^{2n+1}}+\frac{p'^{2n+3}}{2p^2q^{2n+3}}-\frac{p'^{2n-1}}{4q^{2n+1}}-\frac{p'^{2n+1}}{2q^{2n+3}}-\frac{p'^{2n+3}}{4q^{2n+5}}\right)\\&=-\frac{1}{8}\zeta(3)+\frac{9}{64}\,,\end{aligned} \tag{172}$$

and

$$\begin{aligned}\sum_{n=1}^\infty T_{3n}'(R_3)&=\sum_{n=1}^\infty\frac{1}{n+1}\int_p^\infty dp'\int_{p'}^\infty dq\left(\frac{p^{2n}}{2p'q^{2n+1}}+\frac{p'p^{2n}}{2q^{2n+3}}-\frac{p^{2n+2}}{4p'^3q^{2n+1}}-\frac{p^{2n+2}}{2p'q^{2n+3}}-\frac{p'p^{2n+2}}{4q^{2n+5}}\right)\\&=-\frac{1}{8}\zeta(3)+\frac{7}{32}\,,\end{aligned} \tag{173}$$

where we have used Eq.(142). Putting Eqs.(167-169, 171-173) in Eq.(160), then

$$T_3^1 = -\frac{3}{8}\zeta(3) - \frac{1}{4} \ . \tag{174}$$

Further, expanding Eq.(158) we obtain:

$$\begin{aligned} T_3^2 = \int_0^\infty dq \int_0^\infty dp' \int \frac{d\Omega_{p'}}{2\pi^2} \int \frac{d\Omega_q}{2\pi^2} \Big[& -qp^5 \cos pp' + 3p^4q^2 \cos pp' \cos pq \\ & -2q^3p^3 \cos pp' \cos^2 pq + q^2p^4 \cos p'q - 3q^3p^3 \cos pq \cos p'q \\ & +2q^4p^2 \cos^2 pq \cos p'q + q^2p'p^3 \cos^2 pp' \cos pq - 2q^2p^3p' \cos pp' \cos p'q \\ & +2q^3p^2p' \cos pp' \cos p'q \cos pq + q^3p^2p' \cos^2 p'q - q^2p'p^3 \cos pq \cos^2 pp' \\ & -2q^4p'p \cos pq \cos^2 p'q + qp'^2p^3 \cos pp' - q^2p'^2p^2 \cos pp' \cos pq - q^2p^2p'^2 \cos p'q \\ & + q^3p'^2p \cos p'q \cos pq + q^3p'p^2 \cos^2 p'q \Big] \Big/ (p-q)^4 (p-p')^2 (p'-q)^2 . \end{aligned} \tag{175}$$

Expanding Eq.(175) in Chebyshev polynomials, using Appendix 2, then

$$
\begin{aligned}
T_3^2 = \sum_{n=0}^{\infty} \frac{1}{n+1} \int_0^{\infty} dq \int_0^{\infty} dp' \Big(& -q p^5 \binom{1\,1}{p\,p'}_n \binom{0\,1}{p'\,q}_n \binom{0\,2}{p\,q}_n + 3p^4 q^2 \binom{1\,1}{p\,p'}_n \binom{0\,1}{p'\,q}_n \binom{1\,2}{p\,q}_n \\
& -2q^3p^3 \binom{1\,1}{p\,p'}_n \binom{0\,1}{p'\,q}_n \binom{2\,2}{p\,q}_n + q^2p^4 \binom{0\,1}{p\,p'}_n \binom{1\,1}{p\,q}_n \binom{0\,2}{p\,q}_n \\
& -3q^3p^3 \binom{0\,1}{p\,p'}_n \binom{1\,1}{p'\,q}_n \binom{1\,2}{p\,q}_n + 2q^4p^2 \binom{0\,1}{p\,p'}_n \binom{1\,1}{p'\,q}_n \binom{2\,2}{p\,q}_n \\
& +q^2p'p^3 \binom{2\,1}{p\,p'}_n \binom{0\,1}{p'\,q}_n \binom{1\,2}{p\,q}_n - 2q^2p^3p' \binom{1\,1}{p\,p'}_n \binom{1\,1}{p'\,q}_n \binom{0\,2}{p\,q}_n \\
& +2q^3p^2p' \binom{1\,1}{p\,p'}_n \binom{1\,1}{p'\,q}_n \binom{1\,2}{p\,q}_n + q^3p^2p' \binom{0\,1}{p\,p'}_n \binom{2\,1}{p'\,q}_n \binom{0\,2}{p\,q}_n \\
& -q^2p'p^3 \binom{2\,1}{p'\,q}_n \binom{0\,1}{p'\,q}_n \binom{1\,2}{p\,q}_n - 2q^4pp' \binom{0\,1}{p\,p'}_n \binom{2\,1}{p'\,q}_n \binom{1\,2}{p\,q}_n \\
& +qp'^2p^3 \binom{1\,1}{p\,p'}_n \binom{0\,1}{p'\,q}_n \binom{0\,2}{p\,q}_n - q^2p^2p'^2 \binom{1\,1}{p\,p'}_n \binom{0\,1}{p'\,q}_n \binom{1\,2}{p\,q}_n \\
& -q^2p^2p'^2 \binom{0\,1}{p\,p'}_n \binom{1\,1}{p'\,q}_n \binom{0\,2}{p\,q}_n + q^3p'^2p \binom{0\,1}{p\,p'}_n \binom{1\,1}{p'\,q}_n \binom{1\,2}{p\,q}_n \\
& +q^3p'p^2 \binom{0\,1}{p\,p'}_n \binom{2\,1}{p'\,q}_n \binom{0\,2}{p\,q}_n \Big)
\end{aligned}
$$

$$
\equiv \sum_{n=0}^{\infty} \frac{1}{n+1} \int_{p'<q} dp'\, dq\, g_n(\underline{p'q}) \equiv \sum_{n=0}^{\infty} T_{3n} \;. \tag{176}
$$

Again we break up $g_0(p'q)$ as in Eqs.(162–163). Then using Eq.(176) and Appendix 2:

$$
\begin{aligned}
\bar{g}^0(\underline{p'q}\,R_1) = \; & \frac{p'q}{p^4} - \frac{3}{8}\frac{p'q^3}{p^6} - \frac{1}{2}\frac{p'}{qp^2} + \frac{1}{4}\frac{p'^3}{qp^4} \\
& + \frac{1}{2}\frac{p'^3}{p^2q^3} - \frac{1}{4}\frac{p'^5}{q^3p^4} - \frac{3}{8}\frac{p'^3q}{p^6} \;,
\end{aligned} \tag{177}
$$

$$\bar{g}^{0}(\underline{p'q}\,R_2)=\frac{1}{8}\frac{p'}{p^2q}-\frac{1}{2}\frac{p^2p'}{q^5}+\frac{3}{4}\frac{p'^3}{q^5}-\frac{1}{4}\frac{p'^5}{p^2q^5}-\frac{3}{8}\frac{p'^3}{p^2q^3}+\frac{1}{2}\frac{p'}{q^3}\ , \tag{178}$$

and

$$\bar{g}^{0}(\underline{p'q}\,R_3)=-\frac{1}{4}\frac{p^4}{p'^3q^3}+\frac{1}{8}\frac{p^2}{p'^3q}+\frac{3}{8}\frac{p^2}{p'q^3}-\frac{1}{4}\frac{p^4}{p'q^5}+\frac{1}{4}\frac{p'p^2}{q^5}\ . \tag{179}$$

Breaking up T_{30} as in Eq.(166), then

$$T_{30}^{2}(R_1)=\int_0^p dq\int_0^q dp'\,\bar{g}^{0}(\underline{p'q}\,R_1)=\frac{4}{64}-\frac{1}{24}\quad , \tag{180}$$

$$T_{30}^{2}(R_2)=\int_p^\infty dq\int_0^p dp'\,\bar{g}^{0}(\underline{p'q}\,R_2)=\frac{1}{16}\ln\frac{M}{p}+\frac{5}{96}\ , \tag{181}$$

and

$$T_{30}^{2}(R_3)=\int_p^\infty dp'\int_{p'}^\infty dq\,\bar{g}^{0}(\underline{p'q}\,R_3)=\frac{1}{16}\ln\frac{M}{p}+\frac{3}{64}\quad , \tag{182}$$

where M is an infinite cutoff. Again using Eq.(176) and Appendix 2,

$$\bar{g}^{n}(p'q)=\frac{\left\langle\frac{p}{p'}\right\rangle^{n+1}\left\langle\frac{p'}{q}\right\rangle^{n+1}\left\langle\frac{p}{q}\right\rangle^{n+1}}{|p^2-q^2|}\left(\left\langle\frac{p}{q}\right\rangle-\left\langle\frac{p}{q}\right\rangle^{-1}\right)\left(\frac{p^3}{4p'^3}-\frac{pq^2}{2p'^3}+\frac{q^4}{4p'^3p}\right)\ , \tag{183}$$

so

$$\begin{aligned}\bar{g}^{n}(\underline{p'q}\,R_1)=&-\frac{p'^{2n-1}}{4qp^{2n}}+\frac{qp'^{2n-1}}{2p^{2n+2}}-\frac{q^3p'^{2n-1}}{4p^{2n+4}}\\&-\frac{1}{4}\frac{p'^{2n+1}}{q^3p^{2n}}+\frac{p'^{2n+3}}{2q^3p^{2n+2}}-\frac{p'^{2n+5}}{4q^3p^{2n+4}}\ ,\end{aligned} \tag{184}$$

$$\begin{aligned}\bar{g}^{n}(\underline{p'q}\,R_2)=&-\frac{p^2p'^{2n-1}}{4q^{2n+3}}+\frac{p'^{2n-1}}{2q^{2n+1}}-\frac{p'^{2n-1}}{4p^2q^{2n-1}}\\&-\frac{p^2p'^{2n+1}}{4q^{2n+5}}+\frac{p'^{2n+3}}{2q^{2n+5}}-\frac{p'^{2n+5}}{4p^2q^{2n+5}}\ ,\end{aligned} \tag{185}$$

and

$$\bar{g}^{n}(\underline{p'q}\,R_3) = -\frac{p^{2n+4}}{4p'^3q^{2n+3}} + \frac{p^{2n+2}}{2p'^3q^{2n+1}} - \frac{p^{2n}}{4p'^3q^{2n-1}}$$
$$-\frac{p^{2n+4}}{4p'q^{2n+5}} + \frac{p'p^{2n+2}}{2q^{2n+5}} - \frac{p'^3p^{2n}}{4q^{2n+5}} \quad . \tag{186}$$

Therefore

$$\sum_{n=1}^{\infty} T_{3n}^{2}(R_1) = \sum_{n=1}^{\infty}\frac{1}{n+1}\int_0^p dq\int_0^q dp'\,\bar{g}^{n}(\underline{p'q}\,R_1) = -\frac{1}{48} \quad , \tag{187}$$

$$\sum_{n=1}^{\infty} T_{3n}^{2}(R_2) = \sum_{n=1}^{\infty}\frac{1}{n+1}\int_p^\infty dq\int_0^p dp'\,\bar{g}^{n}(\underline{p'q}\,R_2) = -\frac{1}{16}\ln\frac{M}{p} + \frac{1}{24} \quad , \tag{188}$$

and

$$\sum_{n=1}^{\infty} T_{3n}^{2}(R_3) = \sum_{n=1}^{\infty}\frac{1}{n+1}\int_p^\infty dp'\int_{p'}^\infty dq\,\bar{g}^{n}(\underline{p'q}\,R_3) = -\frac{1}{16}\ln\frac{M}{p} + \frac{3}{64} \quad . \tag{189}$$

Then adding Eqs.(180-182) and Eqs.(187-189),

$$T_3^2 = \frac{3}{16} \quad . \tag{190}$$

Finally putting Eqs.(174, 190) in Eq.(156), we obtain

$$T_3 = \frac{96\,\zeta(3) + 16}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \quad . \tag{191}$$

Note that the logarithmic divergences conspired to cancel, as they ultimately must.

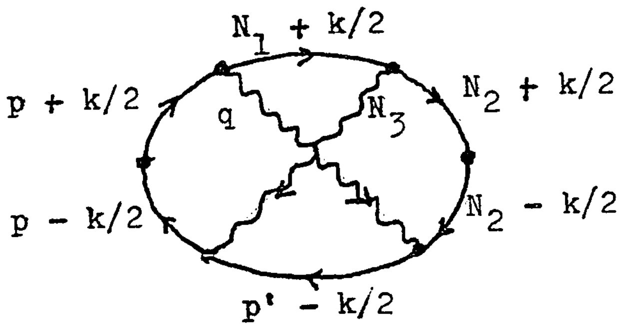
Chapter 8

CALCULATION OF T_4

By definition, Eq.(70),

$$T_4 = 4\,\mathrm{Tr}\,\gamma G'^{(0)} K_6'^{(4)} G^{(0)} \gamma \ . \tag{192}$$

We make the following momentum assignments in Eq.(192):



$$N_1 = p - q$$
$$N_2 = p' - q \tag{193}$$
$$N_3 = p - p'$$

We first note that

$$\frac{\partial}{\partial K_\alpha}\left(\frac{1}{\gamma(p+\frac{k}{2})}\right)_{/k=0} = \frac{1}{2}\,\frac{\gamma_\alpha}{p^2} - \frac{\gamma p\, p_\alpha}{p^4} = -\frac{1}{\gamma p}\,\frac{\gamma_\alpha}{2}\,\frac{1}{\gamma p} \ . \tag{194}$$

Then using Eqs.(74, 194) and the Feynman rules in Eq.(192),

$$T_4 = 4(ic_0^2)^2 \int \frac{d^4q}{(2\pi)^4} \int \frac{d^4p'}{(2\pi)^4}\, \mathrm{Tr}\, \frac{(\gamma_\alpha \gamma p \gamma_\mu - \gamma_\mu \gamma p \gamma_\alpha)}{i^2\, 2p^4} \Big[\gamma_\lambda \frac{1}{i}\left(\frac{\gamma_\alpha}{2N_1^2} - \frac{N_{1\alpha}\gamma N_1}{N_1^4}\right)$$

$$\cdot \gamma_\nu \frac{1}{iN_2} \gamma_\mu \frac{1}{iN_2} \gamma_\lambda \frac{1}{i\gamma p'} \frac{\gamma_\nu}{N_3^2 q^2} + \gamma_\lambda \frac{1}{iN_1} \gamma_\nu \frac{1}{iN_2} \gamma_\mu \frac{1}{iN_2}\left(-\frac{\gamma_\alpha}{2p'^2} + \frac{p'_\alpha \gamma p'}{p'^4}\right) \frac{\gamma_\nu}{N_3^2 q^2}\Big]$$

$$\propto \int \frac{d^4q}{(2\pi)^4} \int \frac{d^4p'}{(2\pi)^4}\, t_4 \ , \tag{195}$$

where

$$t_4 \equiv \frac{t_{11}+t_{12}}{2N_1^2N_2^4p'^2N_3^2q^2} + \frac{t_{21}+t_{22}}{N_1^4N_2^4p'^2N_3^2q^2} + \frac{t_{31}+t_{32}}{N_1^4N_2^2p'^2N_3^2q^2} - \frac{t_{41}+t_{42}}{N_1^2N_2^4p'^4N_3^2q^2} \quad , \tag{196}$$

and

$$t_{11} = Tr\,\gamma_\alpha \gamma p \gamma_\mu \gamma_\lambda \gamma_\alpha \gamma_\nu \gamma N_2 \gamma_\mu \gamma N_2 \gamma_\lambda \gamma p' \gamma_\nu = 64N_2^2\, p\cdot p' , \tag{197}$$

$$t_{12} = -Tr\,\gamma_\mu \gamma p \gamma_\alpha \gamma_\lambda \gamma_\alpha \gamma_\nu \gamma N_2 \gamma_\mu \gamma N_2 \gamma_\lambda \gamma p' \gamma_\nu = -64N_2^2\, p\cdot p' , \tag{198}$$

$$t_{21} = Tr\,\gamma_\alpha \gamma p \gamma_\mu \gamma_\lambda N_{1\alpha} \gamma N_1 \gamma_\nu \gamma N_2 \gamma_\mu \gamma N_2 \gamma_\lambda \gamma p' \gamma_\nu = 64N_2^2 N_1\cdot p' N_1\cdot p , \tag{199}$$

$$t_{22} = -Tr\,\gamma_\mu \gamma p \gamma_\alpha \gamma_\lambda N_{1\alpha} \gamma N_1 \gamma_\nu \gamma N_2 \gamma_\mu \gamma N_2 \gamma_\lambda \gamma p' \gamma_\nu = -64N_2^2 N_1\cdot p\, N_1\cdot p' \quad , \tag{200}$$

$$t_{31} = Tr\,\gamma N_1 \gamma p \gamma_\mu \gamma_\lambda \gamma N_1 \gamma_\nu \gamma N_2 \gamma_\mu \gamma N_2 \gamma_\lambda \gamma p' \gamma_\nu = 64N_2^2 N_1\cdot p' N_1\cdot p \quad , \tag{201}$$

$$t_{32} = -Tr\,\gamma_\mu \gamma p \gamma_\alpha \gamma_\lambda N_{1\alpha} \gamma N_1 \gamma_\nu \gamma N_2 \gamma_\mu \gamma N_2 \gamma_\lambda \gamma p' \gamma_\nu = -64N_2^2 N_1\cdot p' N_1\cdot p , \tag{202}$$

$$t_{41} = Tr\,\gamma p' \gamma p \gamma_\mu \gamma_\lambda \gamma N_1 \gamma_\nu \gamma N_2 \gamma_\mu \gamma N_2 \gamma_\lambda \gamma p' \gamma_\nu = 64N_2^2 N_1\cdot p'\, p'\cdot p , \tag{203}$$

$$t_{42} = -Tr\,\gamma_\mu \gamma p \gamma_\alpha \gamma_\lambda \gamma N_1 \gamma_\nu \gamma N_2 \gamma_\mu \gamma N_2 \gamma_\lambda p'_\alpha \gamma p' \gamma_\nu = -64N_2^2 N_1\cdot p'\, p'\cdot p , \tag{204}$$

where we have used Appendix 1 to expand the traces. Putting Eqs.(197–204) in Eq.(196), then

$$t_4 = 0 \quad , \tag{205}$$

so that from Eq.(195) we conclude:

$$T_4 = 0 \quad . \tag{206}$$

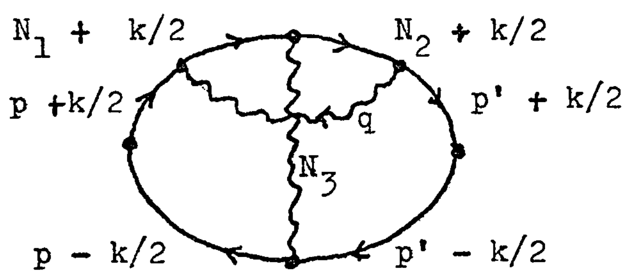
Chapter 9

CALCULATION OF T_5

By definition, Eq.(70),

$$T_5 = 8\,Tr\,\gamma G'^{(0)} K_c'^{(4)} G^{(0)} \gamma \; . \tag{207}$$

We make the following momentum assignments:



$$N_1 = p - q$$
$$N_2 = p' - q \tag{208}$$
$$N_3 = p - p'$$

From Eq.(74) we see that

$$(\gamma G'^{(0)})_{\alpha\mu} = \frac{1}{j^2}\,\frac{\gamma_\alpha \gamma p \gamma_\mu - \gamma_\mu \gamma p \gamma_\alpha}{2p^4} = -(\gamma G'^{(0)})_{\mu\alpha} \tag{209}$$

is antisymmetric in μ and α. We proceed to show that $T_5 = 0$, by showing that $K_c'^{(4)} G^{(0)}\gamma$ has a vanishing antisymmetric part. We write the antisymmetric part as

$$(K_c'^{(4)} G^{(0)}\gamma)_{\underset{\vee}{\mu\alpha}} = \frac{1}{2}\left[(K_c'^{(4)} G^{(0)}\gamma)_{\mu\alpha} - (K_c'^{(4)} G^{(0)}\gamma)_{\alpha\mu}\right] . \tag{210}$$

Using Eq.(194) and the Feynman rules,

$$K_c'^{(4)} G^{(0)}\gamma \propto \gamma_\rho \left(\frac{\gamma_\alpha}{2N_1^2} - \frac{N_{1\alpha}\gamma N_1}{N_1^4}\right)\gamma_\lambda \gamma N_2 \gamma_\rho \gamma p' \gamma_\mu \gamma p' \gamma_\lambda \frac{1}{N_2^2}$$

$$+ \gamma_\rho \frac{\gamma N_1}{N_1^2} \gamma_\lambda \left(\frac{\gamma_\alpha}{2N_2^2} - \frac{N_{2\alpha}\gamma N_2}{N_2^4}\right)\gamma_\rho \gamma p' \gamma_\mu \gamma p' \gamma_\lambda \; . \tag{211}$$

The second and third terms of Eq.(211) are symmetric in μ and α , so putting Eq.(211) in Eq.(210), then

$$(K_c^{\prime(4)} G^{(0)} \gamma)_{\underset{V}{\mu\alpha}} \propto \gamma_\nu \gamma_\alpha \gamma_\lambda \gamma N_2 \gamma_\nu \gamma P' \gamma_\mu \gamma P' \gamma_\lambda - \gamma_\nu \gamma_\mu \gamma_\lambda \gamma N_2 \gamma_\nu \gamma P' \gamma_\alpha \gamma P' \gamma_\lambda$$

$$+ \gamma_\nu \gamma N_1 \gamma_\lambda \gamma_\alpha \gamma_\nu \gamma P' \gamma_\mu \gamma P' \gamma_\lambda - \gamma_\nu \gamma N_1 \gamma_\lambda \gamma_\mu \gamma_\nu \gamma P' \gamma_\alpha \gamma P' \gamma_\lambda \ , \tag{212}$$

or employing Appendix 1,

$$(K_c^{\prime(4)} G^{(0)} \gamma)_{\underset{V}{\mu\alpha}} \propto -2 \gamma N_2 \gamma_\lambda \gamma_\alpha \gamma P' \gamma_\mu \gamma P' \gamma_\lambda + 2 \gamma N_2 \gamma_\lambda \gamma_\mu \gamma P' \gamma_\alpha \gamma P' \gamma_\lambda$$

$$-2 \gamma_\alpha \gamma_\lambda \gamma N_1 \gamma P' \gamma_\mu \gamma P' \gamma_\lambda + 2 \gamma_\mu \gamma_\lambda \gamma N_1 \gamma P' \gamma_\alpha \gamma P' \gamma_\lambda$$

$$\propto -\gamma N_2 (\gamma P' \gamma_\alpha \gamma P' \gamma_\mu + \gamma_\mu \gamma P' \gamma_\alpha \gamma P') + \gamma N_2 (\gamma P' \gamma_\mu \gamma P' \gamma_\alpha + \gamma_\alpha \gamma P' \gamma_\mu \gamma P')$$

$$- \gamma_\alpha (\gamma P' \gamma N_1 \gamma P' \gamma_\mu + \gamma_\mu \gamma P' \gamma N_1 \gamma P')$$

$$+ \gamma_\mu (\gamma P' \gamma N_1 \gamma P' \gamma_\alpha + \gamma_\alpha \gamma P' \gamma N_1 \gamma P')$$

$$\propto \gamma N_2 (\gamma P' \gamma_\mu \gamma P' \gamma_\alpha + \gamma_\alpha \gamma P' \gamma_\mu \gamma P' - \gamma P' \gamma_\alpha \gamma P' \gamma_\mu - \gamma_\mu \gamma P' \gamma_\alpha \gamma P')$$

$$+ \gamma_\mu \gamma P' \gamma N_1 \gamma P' \gamma_\alpha + \gamma_\mu \gamma_\alpha \gamma P' \gamma N_1 \gamma P' - \gamma_\alpha \gamma P' \gamma N_1 \gamma P' \gamma_\mu - \gamma_\alpha \gamma_\mu \gamma P' \gamma N_1 \gamma P' , \tag{213}$$

or using the gamma commutation relations, Appendix 1, and noting for example that $P_\alpha \gamma_\mu$ is symmetric in α and μ ,

$$(K_c^{\prime(4)} G^{(0)} \gamma)_{\underset{V}{\mu\alpha}} \propto \gamma N_2 (\gamma P' \gamma_\mu \gamma P' \gamma_\alpha - \gamma P' \gamma_\mu \gamma P' \gamma_\alpha - \gamma P' \gamma_\alpha \gamma P' \gamma_\mu - \gamma P' \gamma_\alpha \gamma P' \gamma_\mu)$$

$$+ \gamma_\mu \gamma P' \gamma N_1 \gamma P' \gamma_\alpha - \gamma_\mu \gamma P' \gamma N_1 \gamma P' \gamma_\alpha - \gamma_\alpha \gamma P' \gamma N_1 \gamma P' \gamma_\mu - \gamma_\alpha \gamma P' \gamma N_1 \gamma P' \gamma_\mu)$$

$$= 0 \ . \tag{214}$$

Therefore

$$T_5 = 0 . \tag{215}$$

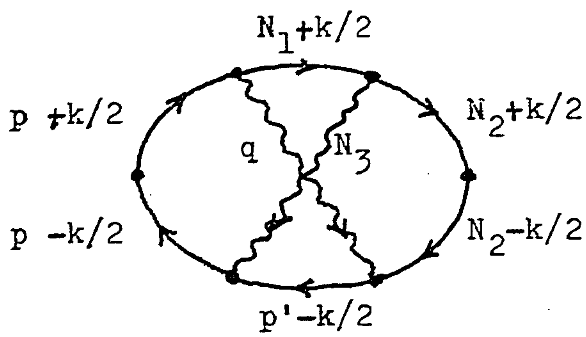
Chapter 10

CALCULATION OF T_6

By definition, Eq.(70),

$$T_6 = Tr\, \gamma G^{(0)} K_b^{''(4)} G^{(0)} \gamma \quad . \tag{216}$$

We make the following momentum assignments in Eq.(216):



$$\begin{aligned} N_1 &= p - q \\ N_2 &= p' - q \\ N_3 &= p - p' \end{aligned} \tag{217}$$

Then using the Feynman rules, Eq.(216) becomes:

$$T_6 = 2\,Tr\, \gamma_\mu \frac{1}{i\gamma p} \int \frac{d^4p'}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} (ie_0^2)^2 \gamma_\lambda \Big[\frac{1}{i\gamma N_1} \gamma_\mu \frac{1}{i\gamma N_2} \gamma_\mu \frac{1}{i\gamma N_2} \gamma_\lambda \frac{\partial^2}{\partial K_\alpha^2} \Big(\frac{1}{i\gamma (p' - \frac{K}{2})} \Big)$$

$$\cdot \gamma_\mu \frac{1}{i\gamma p} \frac{1}{q^2 N_3^2} + \frac{\partial}{\partial K_\alpha} \Big(\frac{1}{i\gamma (N_1 + \frac{K}{2})} \Big) \gamma_\mu \frac{1}{i\gamma N_2} \gamma_\mu \frac{1}{i\gamma N_2} \gamma_\lambda \frac{\partial}{\partial K_\alpha} \Big(\frac{1}{i\gamma (p' - \frac{K}{2})} \Big) \Big] \; . \tag{218}$$

We note that

$$\frac{\partial^2}{\partial K_\alpha^2} \Big(\frac{1}{\gamma (p + \frac{K}{2})} \Big)_{/K=0} = - \frac{\gamma p}{p^4} \; . \tag{219}$$

Using Eq.(194, 219) and rotating contours, then Eq.(218) becomes:

$$T_6 = T_6^1 + T_6^2 \; , \tag{220}$$

where

$$T_6^1 = \frac{2}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \int \frac{dp'}{p'} \int \frac{dq}{q} \int \frac{d\Omega_{p'}}{2\pi^2} \int \frac{d\Omega_q}{2\pi^2} \frac{q^2 N_1^2 N_3^2 \, Tr\, \gamma p \gamma_\mu \gamma p \gamma_\lambda \gamma N_1 \gamma_\mu \gamma N_2 \gamma_\mu \gamma N_2 \gamma_\lambda \gamma p' \gamma_\mu}{(N_1^2 N_2^2 N_3^2)^2} \quad , \tag{221}$$

and

$$T_6^2 = \frac{1}{2p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \int \frac{dp'}{p'} \int \frac{dq}{q} \int \frac{d\Omega_{p'}}{2\pi^2} \int \frac{d\Omega_q}{2\pi^2} \frac{q^2 N_3^2 \, Tr\, \gamma p \gamma_\mu \gamma p \gamma_\mu \gamma N_1 \gamma_\alpha \gamma N_1 \gamma_\rho \gamma N_2 \gamma_\mu \gamma N_2 \gamma_\mu \gamma p' \gamma_\alpha \gamma p' \gamma_\rho}{(N_1^2 N_2^2 N_3^2)^2} . \tag{222}$$

Using Appendix 1 to expand the trace, Eq.(221) becomes:

$$T_6^1 = -\frac{256}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \int_0^\infty \frac{dp'}{p'} \int_0^\infty \frac{dq}{q} \int \frac{d\Omega_{p'}}{2\pi^2} \int \frac{d\Omega_q}{2\pi^2} \frac{q^2 N_2^2 N_3^2 p \cdot N_2 (p' \cdot N_1)^2}{(N_1^2 N_2^2 N_3^2)^2} \quad , \tag{223}$$

or using Eq.(217),

$$T_6^1 = -\frac{256}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \int_0^\infty dp' \int_0^\infty dq \int \frac{d\Omega_{p'}}{2\pi^2} \int \frac{d\Omega_q}{2\pi^2} \Big(q p^3 p'^2 \cos^3 pp' - 2q^2 p'^2 p^2 \cos^2 pp' \cos p'q$$

$$+ q^3 p p'^2 \cos pp' \cos^2 p'q - q^2 p' p^3 \cos pq \cos^2 pp'$$

$$+ 2q^3 p^2 p' \cos pp' \cos pq \cos p'q - q^4 p' p \cos pq \cos^2 p'q \Big) \frac{1}{(p-q)^4 (p'-q)^2 (p-p')^2} \tag{224}$$

Using Appendix 2 to expand Eq.(224), then

$$T_6' = -\frac{256}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \sum_{n=0}^{\infty} \frac{1}{n+1} \int dq \int dp' \Big[q p^3 p'^2 \binom{2\,1}{pp'}_n \binom{0\,1}{p'q}_n \binom{0\,2}{pq}_n$$

$$- 2 q^2 p'^2 p^2 \binom{2\,1}{pp'}_n \binom{1\,1}{p'q}_n \binom{0\,2}{pq}_n + q^3 p' p \binom{1\,1}{pp'}_n \binom{2\,1}{p'q}_n \binom{0\,2}{pq}_n$$

$$- q^2 p' p^3 \binom{0\,1}{pp'}_n \binom{0\,1}{p'q}_n \binom{1\,2}{pq}_n + 2 q^3 p^2 p' \binom{1\,1}{pp'}_n \binom{1\,1}{p'q}_n \binom{1\,2}{pq}_n$$

$$- q^4 p' p \binom{0\,1}{pp'}_n \binom{2\,1}{p'q}_n \binom{1\,2}{pq}_n \Big]$$

$$= -\frac{256}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \sum_{n=0}^{\infty} \frac{1}{n+1} \int_{p'<q} dp'\, dq\, g_n(\underline{p'q})$$

$$\equiv \sum_{n=0}^{\infty} T_{6n}' \, . \tag{225}$$

We define

$$g_{n>0}(p'q) = \bar{g}_n(p'q) + \delta_1(p'q)\delta_{1n} + \delta_2(p'q)\delta_{2n} \, , \tag{226}$$

and

$$g_n(p'q) = \begin{cases} g_0(p'q) + \frac{1}{2}\delta_1(p'q) + \frac{1}{3}\delta_2(p'q) \equiv \bar{g}_0(p'q), & n=0 \\ \bar{g}_n(p'q) & , n>0 \end{cases} \, . \tag{227}$$

Then from Eq.(225) and Appendix 2,

$$\bar{g}^0(p'q) = \frac{1}{|p^2-q^2|}\Big(\frac{1}{8}\frac{p}{q}\left\langle\frac{p}{p'}\right\rangle^4\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle + \frac{1}{4}\frac{p}{q}\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle + \frac{1}{4}\frac{p}{p'}\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle$$

$$-\frac{1}{4}\frac{p'}{p}\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle + \frac{1}{8}\frac{p'}{p}\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle + \frac{1}{8}\frac{q^2}{p'p}\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle$$

$$-\frac{1}{4}\frac{p^2}{p'^2}\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle^2 - \frac{1}{4}\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle^2 + \frac{1}{2}\frac{q}{p'}\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2$$

$$-\frac{1}{4}\frac{q}{p}\left\langle\frac{p}{p'}\right\rangle\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 - \frac{1}{4}\frac{q^3}{p'^2p}\left\langle\frac{p}{p'}\right\rangle\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 - \frac{1}{8}\frac{p^2}{p'q}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2$$

$$-\frac{1}{8}\frac{p'}{q}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 + \frac{1}{4}\frac{p'}{q}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 + \frac{1}{4}\frac{q}{p'}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2$$

$$-\frac{1}{8}\frac{q}{p'}\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 - \frac{1}{8}\frac{p'q}{p^2}\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 + \frac{1}{8}\frac{p^2}{qp'}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2$$

$$+\frac{1}{8}\frac{q}{p'}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 + \frac{1}{16}\frac{p}{p'}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^3 - \frac{1}{16}\frac{p}{p'}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle$$

$$+\frac{1}{8}\frac{q}{p'}\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 + \frac{1}{8}\frac{q^3}{p^2p'}\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 + \frac{1}{16}\frac{q^2}{pp'}\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^3$$

$$-\frac{1}{16}\frac{q^2}{pp'}\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p}{q}\right\rangle - \frac{1}{8}\frac{p}{q}\left\langle\frac{p'}{q}\right\rangle^3\left\langle\frac{p}{q}\right\rangle^3\Big). \qquad (228)$$

Evaluating Eq.(228) in regions R_1, R_2, and R_3,

$$\bar{g}^0(\underline{p'q}\,R_1) = \frac{3}{8}\frac{p'q}{p^4} - \frac{1}{16}\frac{p'q^3}{p^6} - \frac{1}{16}\frac{p'}{qp^2} - \frac{1}{16}\frac{p'^3q}{p^6} - \frac{3}{8}\frac{p'^3}{p^4q} + \frac{1}{4}\frac{p'^5}{p^4q^3} - \frac{1}{16}\frac{p'^3}{p^2q^3}, \qquad (229)$$

$$\bar{g}^0(\underline{p'q}\,R_2) = \frac{1}{8}\frac{p'^3p^2}{q^7} + \frac{1}{8}\frac{p'}{q^3} + \frac{1}{16}\frac{p'p^2}{q^5} + \frac{1}{16}\frac{p'}{qp^2} - \frac{3}{8}\frac{p'^3}{q^5} + \frac{1}{4}\frac{p'^5}{q^5p^2} - \frac{1}{4}\frac{p'^3}{q^3p^2}, \qquad (230)$$

and

$$\bar{g}^{0}(\underline{p'q}R_3)=\frac{1}{16}\frac{p^2}{p'^3q}+\frac{1}{8}\frac{p^4}{p'^3q^3}-\frac{3}{16}\frac{p'p^2}{q^5}+\frac{1}{8}\frac{p'^3p^2}{q^7}-\frac{1}{4}\frac{p^2}{p'q^3}+\frac{1}{8}\frac{p^4}{q^5p'}\,. \tag{231}$$

Putting Eqs.(229-231) in Eq.(225), then

$$T_{60}^{\,1}(R_1)=-\frac{256}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2\int_0^p dq\int_0^q dp'\,\bar{g}^{0}(\underline{p'q}R_1)=-\frac{256}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2\frac{1}{6(64)}\;, \tag{232}$$

$$T_{60}^{\,1}(R_2)=-\frac{256}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2\int_p^\infty dq\int_0^p dp\,\bar{g}^{0}(\underline{p'q}R_2)=-\frac{256}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2\frac{1}{32}\ln\frac{M}{p}\,, \tag{233}$$

and

$$T_{60}^{\,1}(R_3)=-\frac{256}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2\int_p^\infty dp'\int_{p'}^\infty dq\,\bar{g}^{0}(\underline{p'q}R_3)=-\frac{256}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2\left(\frac{1}{32}\ln\frac{M}{p}-\frac{13}{6(32)}\right)\,. \tag{234}$$

Adding Eqs.(232-234), then

$$T_{60}^{\,1}=-\frac{256}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2\left(-\frac{25}{6(64)}+\frac{1}{16}\ln\frac{M}{p}\right). \tag{235}$$

From Eq.(225) and Appendix 2,

$$\bar{g}_n(p'q)=\left\langle\frac{p}{p'}\right\rangle^{n+1}\left\langle\frac{p'}{q}\right\rangle^{n+1}\left\langle\frac{p}{q}\right\rangle^{n+1}\left[\frac{p^2q^2}{|p^2-q^2|}\left(\frac{1}{8p'q}-\frac{q}{8p^2p'}-\frac{q}{8p'^3}+\frac{q^3}{p^2p'^3}\right)(n+1)\right.$$
$$\left.+\frac{\left\langle\frac{p}{q}\right\rangle-\left\langle\frac{p}{q}\right\rangle^{-1}}{|p^2-q^2|}\left(-\frac{p^3}{8p'^3}+\frac{q^2p}{4p'^3}-\frac{q^4}{8p'^3p}\right)\right]. \tag{236}$$

Evaluating Eq.(236) in regions R_1, R_2, and R_3,

$$\bar{g}_n(\underline{p'q}\,R_1) = (n+1)\Big[\frac{1}{4}\frac{p'^{2n+1}}{q p^{2n+2}} - \frac{1}{8}\frac{q p'^{2n+1}}{p^{2n+4}} - \frac{1}{8}\frac{q p'^{2n-1}}{p^{2n+2}} + \frac{1}{8}\frac{q^3 p'^{2n-1}}{p^{2n+4}}$$

$$- \frac{1}{8}\frac{p'^{2n+3}}{q p^{2n+4}} - \frac{1}{8}\frac{p'^{2n+3}}{q^3 p^{2n+2}} + \frac{1}{8}\frac{p'^{2n+5}}{q^3 p^{2n+4}}\Big]$$

$$+ \frac{1}{8}\frac{p'^{2n-1}}{q p^{2n}} - \frac{1}{4}\frac{q p'^{2n-1}}{p^{2n+2}} + \frac{1}{8}\frac{q^3 p'^{2n-1}}{p^{2n+4}}$$

$$+ \frac{1}{8}\frac{p'^{2n+1}}{q^3 p^{2n}} - \frac{1}{4}\frac{p'^{2n+3}}{q^3 p^{2n+2}} + \frac{1}{8}\frac{p'^{2n+5}}{q^3 p^{2n+4}} \quad , \tag{237}$$

$$\bar{g}_n(\underline{p'q}\,R_2) = (n+1)\Big[\frac{1}{8}\frac{p'^{2n+1}}{p^2 q^{2n+1}} + \frac{1}{8}\frac{p'^{2n-1}}{q^{2n+1}} - \frac{1}{8}\frac{p'^{2n-1}}{p^2 q^{2n-1}}$$

$$- \frac{1}{8}\frac{p'^{2n+3}}{p^2 q^{2n+3}} - \frac{1}{8}\frac{p'^{2n+3}}{q^{2n+5}} + \frac{1}{8}\frac{p'^{2n+5}}{p^2 q^{2n+5}}\Big]$$

$$+ \frac{1}{8}\frac{p^2 p'^{2n-1}}{q^{2n+3}} - \frac{1}{4}\frac{p'^{2n-1}}{q^{2n+1}} + \frac{1}{8}\frac{p'^{2n-1}}{p^2 q^{2n-1}}$$

$$+ \frac{1}{8}\frac{p'^{2n+1} p^2}{q^{2n+5}} - \frac{1}{4}\frac{p'^{2n+3}}{q^{2n+5}} + \frac{1}{8}\frac{p'^{2n+5}}{p^2 q^{2n+5}} \quad , \tag{238}$$

and

$$\bar{g}_n(\underline{p'q}\,R_3) = (n+1)\Big[-\frac{1}{4}\frac{p^{2n+2}}{p' q^{2n+3}} + \frac{1}{8}\frac{p^{2n}}{p' q^{2n+1}} + \frac{1}{8}\frac{p^{2n+2}}{p'^3 q^{2n+1}}$$

$$- \frac{1}{8}\frac{p^{2n}}{p'^3 q^{2n-1}} + \frac{1}{8}\frac{p' p^{2n}}{q^{2n+3}} + \frac{1}{8}\frac{p' p^{2n+2}}{q^{2n+5}} - \frac{1}{8}\frac{p'^3 p^{2n}}{q^{2n+5}}\Big]$$

$$+ \frac{1}{8}\frac{p^{2n+4}}{p'^3 q^{2n+3}} - \frac{1}{4}\frac{p^{2n+2}}{p'^3 q^{2n+1}} + \frac{1}{8}\frac{p^{2n}}{p'^3 q^{2n-1}} + \frac{1}{8}\frac{p^{2n+4}}{p' q^{2n+5}}$$

$$- \frac{1}{4}\frac{p' p^{2n+2}}{q^{2n+5}} - \frac{1}{4}\frac{p' p^{2n+2}}{q^{2n+5}} + \frac{1}{8}\frac{p'^3 p^{2n}}{q^{2n+5}} \quad . \tag{239}$$

Putting Eqs.(237-239) in Eq.(225), then

$$\sum_{n=1}^{\infty} T_{6n}^{1}(R_1) = -\frac{256}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \sum_{n=1}^{\infty}\frac{1}{n+1}\int_0^p dq \int_0^q dp' \bar{q}^n (p' q R_1) = -\frac{256}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2\left(\frac{1}{32}\zeta(2) - \frac{1}{24}\right), \tag{240}$$

$$\sum_{n=1}^{\infty} T_{6n}^{1}(R_2) = -\frac{256}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \sum_{n=1}^{\infty}\frac{1}{n+1}\int_p^{\infty} dq \int_0^p dp' \bar{q}^n (p' q R_2) = -\frac{256}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2\left(\frac{1}{32}\ln\frac{M}{p} + \frac{5}{6(32)}\right), \tag{241}$$

and

$$\sum_{n=1}^{\infty} T_{6n}^{1}(R_3) = -\frac{256}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \sum_{n=1}^{\infty}\frac{1}{n+1}\int_p^{\infty} dp' \int_{p'}^{\infty} dq\, \bar{q}^n (p' q R_3) = -\frac{256}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2\left(-\frac{1}{32}\zeta(2) - \frac{1}{32}\ln\frac{M}{p} + \frac{3}{32}\right). \tag{242}$$

Adding Eq.(235) and Eqs.(240-242), then

$$T_6^{1} = 0 . \tag{243}$$

Using Appendix 1 to evaluate the trace in Eq.(222),

we obtain:

$$T_6^{2} = -\frac{256}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \int_0^{\infty}\frac{dq}{q}\int_0^{\infty}\frac{dp'}{p'}\int\frac{d\Omega_{p'}}{2\pi^2}\int\frac{d\Omega_q}{2\pi^2}\, \frac{q^2[p\cdot(p'-q)]^2[p'\cdot(p-q)]^2}{(p-p')^2(p'-q)^4(p-q)^4},$$

or

$$\begin{aligned} T_6^{2} = -\frac{256}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \int_0^{\infty} dq \int_0^{\infty} dp' \int\frac{d\Omega_{p'}}{2\pi^2}\int\frac{d\Omega_q}{2\pi^2}\Big(& q p^4 p'^3 \cos^4 pp' - 2p^3 p'^3 q^2 \cos^3 pp' \cos p'q \\ & + p^2 p'^3 q^3 \cos^2 pp' \cos^2 p'q - 2p^4 p'^2 q^2 \cos^3 pp' \cos pq + 4p^3 p'^2 q^3 \cos^2 pp' \cos pq \cos p'q \\ & - 2p^2 p'^2 q^4 \cos pp' \cos pq \cos^2 p'q + p^4 p' q^3 \cos^2 pq \cos^2 pp' - 2p' p^3 q^4 \cos pp' \cos p'q \cos^2 pq \\ & + p^2 q^5 p' \cos^2 pq \cos^2 p'q \Big)\frac{1}{(p-p')^2(p'-q)^4(p-q)^4} . \end{aligned} \tag{244}$$

Using Appendix 2 to expand Eq.(244) in Chebyshev polynomials, we get:

$$
\begin{aligned}
T_6^2 = -\frac{256}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \sum_{n=0}^{\infty}\frac{1}{n+1}\int dp'dq \Big\{ & p^4p'^3q\binom{4\,1}{p\,p'}_n\binom{0\,2}{p'q}_n\binom{0\,2}{p\,q}_n \\
& -2p^3p'^3q^2\binom{3\,1}{p\,p'}_n\binom{1\,2}{p'q}_n\binom{0\,2}{p\,q}_n + p^2p'^3q^3\binom{2\,1}{p\,p'}_n\binom{2\,2}{p'q}_n\binom{0\,2}{p\,q}_n \\
& -2p^4p'^2q^2\binom{3\,1}{p\,p'}_n\binom{0\,2}{p'q}_n\binom{1\,2}{p\,q}_n + 4p^3p'^2q^3\binom{2\,1}{p\,p'}_n\binom{1\,2}{p'q}_n\binom{1\,2}{p\,q}_n \\
& -2p^2p'^2q^4\binom{1\,1}{p\,p'}_n\binom{2\,2}{p'q}_n\binom{1\,2}{p\,q}_n + p^4p'q^3\binom{2\,1}{p\,p'}_n\binom{0\,2}{p'q}_n\binom{2\,2}{p\,q}_n \\
& -2p'p^3q^4\binom{1\,1}{p\,p'}_n\binom{1\,2}{p'q}_n\binom{2\,2}{p\,q}_n + p^2q^5p'\binom{0\,1}{p\,p'}_n\binom{2\,2}{p'q}_n\binom{2\,2}{p\,q}_n \Big\} \\
\equiv \sum_{n=0}^{\infty}\frac{1}{n+1}\int_{p'<q} dp'dq\, g_n(\underline{p'q}) & \\
\equiv \sum_{n=0}^{\infty} T_{6n}^2\ . & \qquad (245)
\end{aligned}
$$

From Eq.(245) and Appendix 2, we have

$$
\begin{aligned}
g_0(p'q) = \frac{1}{|q^2-p'^2||p^2-q^2|}\Bigg(&\frac{1}{16}\frac{p^2p'}{q}\left\langle\frac{p}{p'}\right\rangle^5\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle + \frac{3}{16}\frac{p^2p'}{q}\left\langle\frac{p}{p'}\right\rangle^3\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle \\
&+\frac{1}{8}\frac{p^2p'}{q}\left\langle\frac{p}{p'}\right\rangle\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle - \frac{1}{4}pp'\left\langle\frac{p}{p'}\right\rangle^4\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle \\
&-\frac{1}{2}pp'\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle + \frac{3}{16}qp\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle^3\left\langle\frac{p}{q}\right\rangle \\
&+\frac{1}{16}pq\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle + \frac{3}{16}\frac{p'^2q}{p}\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle^3\left\langle\frac{p}{q}\right\rangle \\
&+\frac{1}{16}\frac{p'^2q}{p}\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle - \frac{1}{4}p^2\left\langle\frac{p}{p'}\right\rangle^4\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle^2 \\
&-\frac{1}{2}p^2\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle^2 + \frac{p^2q}{p'}\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 \\
&+p'q\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 - \frac{3}{4}q^2\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle^3\left\langle\frac{p}{q}\right\rangle^2 \\
&-\frac{1}{4}q^2\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle^2 + \frac{3}{16}\frac{p^3q}{p'^2}\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle^3 \\
&+\frac{1}{16}\frac{p^3q}{p'^2}\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle + \frac{3}{16}pq\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle^3 \\
&+\frac{1}{16}pq\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle - \frac{3}{4}\frac{pq^2}{p'}\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^3 \\
&-\frac{1}{4}\frac{pq^2}{p'}\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle + \frac{9}{16}\frac{q^3}{p'}\left\langle\frac{p}{p'}\right\rangle\left\langle\frac{p'}{q}\right\rangle^3\left\langle\frac{p}{q}\right\rangle^3 \\
&+\frac{3}{16}\frac{q^3}{p'}\left\langle\frac{p}{p'}\right\rangle\left\langle\frac{p'}{q}\right\rangle^3\left\langle\frac{p}{q}\right\rangle + \frac{3}{16}\frac{q^3}{p'}\left\langle\frac{p}{p'}\right\rangle\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle^3 \\
&+\frac{1}{16}\frac{q^3}{p'}\left\langle\frac{p}{p'}\right\rangle\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle\Bigg) , \qquad (246)
\end{aligned}
$$

and

$$\delta^1(p'q) = \frac{1}{|p'^2-q^2||p^2-q^2|}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2\Big[-\frac{p^2p'}{q}-\frac{1}{2}\frac{p^2q}{p'}-\frac{3}{2}\frac{q^3}{p'}-\frac{1}{2}p'q$$

$$+\left(\left\langle\frac{p'}{q}\right\rangle-\left\langle\frac{p'}{q}\right\rangle^{-1}\right)\left(-\frac{1}{4}p^2-\frac{3}{4}q^2-\frac{1}{4}pq\left\langle\frac{p}{q}\right\rangle+\frac{1}{4}pq\left\langle\frac{p}{q}\right\rangle^{-1}\right)$$

$$+\left(\left\langle\frac{p}{q}\right\rangle-\left\langle\frac{p}{q}\right\rangle^{-1}\right)\left(-\frac{1}{4}pp'-\frac{3}{4}\frac{pq^2}{p'}\right)\Big], \tag{247}$$

$$\delta^2(p'q) = \frac{1}{|p'^2-q^2||p^2-q^2|}\left\langle\frac{p'}{q}\right\rangle^3\left\langle\frac{p}{q}\right\rangle^3\Big[\frac{9}{16}\frac{pp'^2}{q}+\frac{9}{16}\frac{p^3}{q}+\frac{9}{4}pq$$

$$+\frac{3}{8}pp'\left(\left\langle\frac{p'}{q}\right\rangle-\left\langle\frac{p'}{q}\right\rangle^{-1}\right)+\frac{3}{8}p^2\left(\left\langle\frac{p}{q}\right\rangle-\left\langle\frac{p}{q}\right\rangle^{-1}\right)\Big], \tag{248}$$

and

$$\delta^3(p'q) = \frac{1}{|p'^2-q^2||p^2-q^2|}\left(-\frac{p^2p'}{q}\left\langle\frac{p'}{q}\right\rangle^4\left\langle\frac{p}{q}\right\rangle^4\right). \tag{249}$$

Then evaluating $\bar{g}_0(\underline{p'q})$ in regions R_1, R_2, and R_3,

$$\bar{g}^0(\underline{p'q}\,R_1) = \frac{5}{16}\frac{p'}{p^2q}+\frac{3}{16}\frac{p'q}{p^4}-\frac{11}{16}\frac{p'^3}{p^4q}+\frac{9}{16}\frac{p'^5}{q^3p^4}-\frac{1}{8}\frac{p'^3}{q^3p^2}$$

$$+\frac{1}{|p^2-p'^2|}\left(\frac{1}{16}\frac{q^3p'}{p^4}-\frac{3}{16}\frac{qp'}{p^2}+\frac{3}{16}\frac{p'^3}{p^2q}-\frac{1}{16}\frac{p'^3}{q^3}\right), \tag{250}$$

$$\bar{g}^0(\underline{p'q}\,R_2) = \frac{3}{16}\frac{p'}{q^3}+\frac{3}{8}\frac{p'p^2}{q^5}-\frac{1}{4}\frac{p'^3p^4}{q^9}-\frac{3}{16}\frac{p'^3}{q^3p^2}-\frac{11}{16}\frac{p'^3}{q^5}$$

$$+\frac{1}{16}\frac{p'^3p^2}{q^7}+\frac{9}{16}\frac{p'^5}{p^2q^5}+\frac{1}{16}\frac{p^4p'}{q^7}, \tag{251}$$

and

$$\bar{g}^{0}(\underline{p'q}R_3) = \frac{1}{8}\frac{p^4}{p'q^5} + \frac{1}{4}\frac{p^2}{p'q^3} + \frac{1}{16}\frac{p^4p'}{q^7} - \frac{1}{4}\frac{p^4p'^3}{q^9} - \frac{3}{16}\frac{p^2p'}{q^5}$$

$$+ \frac{1}{|p^2-q^2|}\left(-\frac{1}{16}\frac{p^6}{p'^3q^3} - \frac{3}{16}\frac{p^2p'}{q^3} + \frac{1}{16}\frac{p^4p'^3}{q^7} + \frac{3}{16}\frac{p^4}{p'q^3}\right) .$$

Therefore

$$T_{60}^{2}(R_1) = -\frac{256}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2\left[\int_0^p dp'\int_{p'}^p dq\frac{1}{(p^2-p'^2)}\left(\frac{1}{16}\frac{q^3p'}{p^4} - \frac{3}{16}\frac{qp'}{p^2} + \frac{3}{16}\frac{p'^3}{p^2q} - \frac{1}{16}\frac{p'^3}{q^3}\right)\right.$$

$$\left. + \int_0^p dq\int_0^q dp'\left(\frac{5}{16}\frac{p'}{p^2q} + \frac{3}{16}\frac{p'q}{p^4} - \frac{11}{16}\frac{p'^3}{p^4q} + \frac{9}{16}\frac{p'^5}{q^3p^4} - \frac{1}{8}\frac{p'^3}{q^3p^2}\right)\right]$$

$$= -\frac{256}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2\left[\frac{7}{256} + \frac{5}{128} + \int_0^p dp'\left(-\frac{7}{64}p' - \frac{1}{64}\frac{p'^5}{p^4} + \frac{1}{8}\frac{p'^3}{p^2} + \frac{3}{16}\frac{p'^3}{p^2}\ln\frac{p}{p'}\right)\frac{1}{(p^2-p'^2)}\right], \tag{253}$$

$$T_6^{2}(R_2) = -\frac{256}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2\int_p^\infty dq\int_0^p dp'\left(\frac{3}{16}\frac{p'}{q^3} + \frac{3}{8}\frac{p'p^2}{q^5} - \frac{1}{4}\frac{p'^3p^4}{q^9} - \frac{3}{16}\frac{p'^3}{q^3p^2}\right.$$

$$\left. - \frac{11}{16}\frac{p'^3}{q^5} + \frac{1}{16}\frac{p'^3p^2}{q^7} + \frac{9}{16}\frac{p'^5}{p^2q^5} + \frac{1}{16}\frac{p^4p'}{q^7}\right)$$

$$= -\frac{256}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2\left(\frac{9}{128} + \frac{23}{384} - \frac{61}{12(64)}\right), \tag{254}$$

and

$$T_{60}^{2}(R_3) = -\frac{256}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2\left[\int_p^\infty dq\int_q^\infty dp'\frac{1}{(p^2-q^2)}\left(-\frac{1}{16}\frac{p^6}{p^3q^3}-\frac{3}{16}\frac{p^2p'}{q^3}+\frac{1}{16}\frac{p^4p'^3}{q^7}+\frac{3}{16}\frac{p^4}{p'q^3}\right)\right.$$

$$\left.+\int_p^\infty dp'\int_{p'}^\infty dq\left(\frac{1}{8}\frac{p^4}{p'q^5}+\frac{1}{4}\frac{p^2}{p'q^3}+\frac{1}{16}\frac{p^4p'}{q^7}-\frac{1}{4}\frac{p^4p'^3}{q^9}-\frac{3}{16}\frac{p^2p'}{q^5}\right)\right]$$

$$= -\frac{256}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2\left[\frac{11}{12(64)}+\frac{7}{4(64)}+\int_p^\infty dq\left(\frac{1}{32}\frac{p^6}{q^5}+\frac{5}{64}\frac{p^4}{q^3}-\frac{3}{32}\frac{p^2}{q}\right.\right.$$

$$\left.\left.-\frac{1}{64}\frac{p^8}{q^7}+\frac{3}{16}\frac{p^4}{q^3}\ln\frac{q}{p}\right)\frac{1}{p^2-q^2}\right]. \quad (255)$$

In Eq.(255) let

$$q = \frac{p^2}{p'},$$

then

$$T_{60}^{2}(R_3) = -\frac{256}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2\left[\frac{11}{12(64)}+\frac{7}{4(64)}+\int_0^p dp'\left(\frac{3}{32}p'-\frac{1}{32}\frac{p'^5}{p^4}-\frac{5}{64}\frac{p'^3}{p^2}\right.\right.$$

$$\left.\left.+\frac{1}{64}\frac{p'^7}{p^6}-\frac{3}{16}\frac{p'^3}{p^2}\ln\frac{p}{p'}\right)\frac{1}{p^2-p'^2}\right]. \quad (256)$$

We now add Eqs.(253, 254, 256), obtaining

$$T_{60}^{2} = -\frac{256}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2\left[\frac{7}{256}+\frac{5}{128}+\frac{9}{128}+\frac{23}{384}-\frac{61}{24(64)}+\frac{11}{12(64)}+\frac{7}{256}\right.$$

$$\left.+\int_0^p dp'\left(-\frac{1}{64}\frac{p'}{p^2}-\frac{1}{64}\frac{p'^3}{p^4}-\frac{1}{64}\frac{p'^5}{p^6}+\frac{3}{64}\frac{p'^3}{p^4}\right)\right]$$

$$= -\frac{256}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2\left[\frac{7}{256}+\frac{5}{128}+\frac{9}{128}+\frac{23}{384}-\frac{61}{12(64)}+\frac{11}{12(64)}+\frac{7}{256}-\frac{1}{384}\right], \quad (257)$$

or

$$T_{60}^{2} = -\frac{256}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \frac{5}{32} \,. \tag{258}$$

From Eq.(245) and Appendix 2:

$$\bar{g}_n(p'q) = \bar{g}_{n2}(p'q) + \bar{g}_{n1}(p'q) + \bar{g}_{n0}(p'q) \,, \tag{259}$$

where

$$\bar{g}_{n2}(p'q) = \frac{(n+1)^2}{16 q p'^3 p^2} \frac{[p'^2-q^2]^2[p^2-q^2]^2}{|p'^2-q^2||p^2-q^2|} \left\langle \frac{p}{p'} \right\rangle^{n+1} \left\langle \frac{p'}{q} \right\rangle^{n+1} \left\langle \frac{p}{q} \right\rangle^{n+1} , \tag{260}$$

$$\bar{g}_{n1}(p'q) = (n+1) \frac{\left\langle \frac{p}{p'} \right\rangle^{n+1} \left\langle \frac{p'}{q} \right\rangle^{n+1} \left\langle \frac{p}{q} \right\rangle^{n+1}}{|p'^2-q^2||p^2-q^2|} \Bigg\{ \left(-\frac{1}{8}\frac{p'^2}{p^2} + \frac{1}{4}\frac{q^2}{p^2} - \frac{1}{8}\frac{q^4}{p'^2 p^2}\right)(p^2-q^2)\left(\left\langle \frac{p'}{q} \right\rangle - \left\langle \frac{p'}{q} \right\rangle^{-1}\right)$$

$$+ \left(-\frac{1}{8}\frac{p^3}{p'^3} + \frac{1}{4}\frac{pq^2}{p'^3} - \frac{1}{8}\frac{q^4}{pp'^3}\right)\left(\left\langle \frac{p}{q} \right\rangle - \left\langle \frac{p}{q} \right\rangle^{-1}\right)(p'^2-q^2) \Bigg\} , \tag{261}$$

and

$$\bar{g}_{n0}(p'q) = \frac{\left\langle \frac{p}{p'} \right\rangle^{n+1} \left\langle \frac{p'}{q} \right\rangle^{n+1} \left\langle \frac{p}{q} \right\rangle^{n+1}}{4|p'^2-q^2||p^2-q^2|} \left(-\frac{pq^3}{p'^2} - \frac{q^3}{p} + qp + \frac{q^5}{p'^2 p}\right)\left(\left\langle \frac{p'}{q} \right\rangle - \left\langle \frac{p'}{q} \right\rangle^{-1}\right)\left(\left\langle \frac{p}{q} \right\rangle - \left\langle \frac{p}{q} \right\rangle^{-1}\right) . \tag{262}$$

Or evaluating in regions R_1, R_2, and R_3:

$$\bar{g}_{n2}(\underline{p'q}\, R_1) = \frac{(n+1)^2}{16}\left(\frac{q p'^{2n-1}}{p^{2n+2}} - \frac{q^3 p'^{2n-1}}{p^{2n+4}} + \frac{q p'^{2n+1}}{p^{2n+4}} - \frac{p'^{2n+3}}{q p^{2n+4}} - \frac{p'^{2n+3}}{q^3 p^{2n+2}} + \frac{p'^{2n+5}}{q^3 p^{2n+4}} \right) , \tag{263}$$

$$\bar{g}_{n2}(\underline{p'q}\,R_2) = \frac{(n+1)^2}{16}\left(\frac{p'^{\,2n-1}}{p^2 q^{2n-1}} - \frac{p'^{\,2n-1}}{q^{2n+1}} - \frac{p'^{\,2n+1}}{p^2 q^{2n+1}} + \frac{2p'^{\,2n+1}}{q^{2n+3}} - \frac{p'^{\,2n+3}}{p^2 q^{2n+3}} - \frac{p'^{\,2n+3}}{q^{2n+5}} + \frac{p'^{\,2n+5}}{p^2 q^{2n+5}}\right), \tag{264}$$

$$\bar{g}_{n2}(\underline{p'q}\,R_3) = \frac{(n+1)^2}{16}\left(\frac{p^{2n}}{p'^{\,3} q^{2n-1}} - \frac{p^{2n+2}}{p'^{\,3} q^{2n+1}} - \frac{p^{2n}}{p' q^{2n+1}} + \frac{p' p^{2n}}{q^{2n+3}} - \frac{p'^{\,3} p^{2n}}{q^{2n+5}} + \frac{p' p^{2n+2}}{q^{2n+5}}\right), \tag{265}$$

$$\bar{g}_{n1}(\underline{p'q}\,R_1) = (n+1)\left(-\frac{1}{8}\frac{p'^{\,2n+3}}{q p^{2n+4}} - \frac{1}{8}\frac{p'^{\,2n+1} q}{p^{2n+4}} - \frac{1}{8}\frac{p'^{\,2n-1}}{q p^{2n}} + \frac{1}{4}\frac{p'^{\,2n-1} q}{p^{2n+2}} + \frac{1}{8}\frac{p'^{\,2n+1}}{q^3 p^{2n}} - \frac{1}{4}\frac{p'^{\,2n+3}}{q^3 p^{2n+2}} + \frac{1}{4}\frac{p'^{\,2n+5}}{q^3 p^{2n+4}}\right), \tag{266}$$

$$\bar{g}_{n1}(\underline{p'q}\,R_2) = (n+1)\Big(\frac{3}{8}\frac{p'^{2n+1}}{p^2 q^{2n+1}} - \frac{1}{8}\frac{p^2 p'^{2n-1}}{q^{2n+3}} + \frac{1}{4}\frac{p'^{2n-1}}{q^{2n+1}}$$
$$- \frac{3}{8}\frac{p'^{2n+3}}{p^2 q^{2n+3}} - \frac{1}{4}\frac{p'^{2n-1}}{p^2 q^{2n-1}} - \frac{1}{4}\frac{p'^{2n+3}}{q^{2n+5}}$$
$$+ \frac{1}{4}\frac{p'^{2n+5}}{p^2 q^{2n+5}} + \frac{1}{8}\frac{p^2 p'^{2n+1}}{q^{2n+5}}\Big) \, , \tag{267}$$

$$\bar{g}_{n1}(\underline{p'q}\,R_3) = (n+1)\Big(\frac{1}{8}\frac{p' p^{2n}}{q^{2n+3}} + \frac{1}{8}\frac{p^{2n}}{p' q^{2n+1}} + \frac{1}{4}\frac{p^{2n+2}}{p'^3 q^{2n+1}} - \frac{1}{4}\frac{p^{2n}}{p'^3 q^{2n-1}}$$
$$- \frac{1}{8}\frac{p^{2n+4}}{p'^3 q^{2n+3}} - \frac{1}{4}\frac{p' p^{2n+2}}{q^{2n+5}} + \frac{1}{8}\frac{p^{2n+4}}{p' q^{2n+5}}\Big) \, , \tag{268}$$

$$\bar{g}_{n0}(\underline{p'q}\,R_1) = \frac{1}{4}\Big(-\frac{q p'^{2n-1}}{p^{2n+2}} - \frac{q p'^{2n+1}}{p^{2n+4}} + \frac{2 p'^{2n+1}}{q p^{2n+2}} + \frac{q^3 p'^{2n-1}}{p^{2n+4}}$$
$$- \frac{p'^{2n+3}}{q^3 p^{2n+2}} - \frac{p'^{2n+3}}{q p^{2n+4}} + \frac{p'^{2n+5}}{q^3 p^{2n+4}}\Big) \, , \tag{269}$$

$$\bar{g}_{n0}(\underline{p'q}\,R_2) = \frac{1}{4}\Big(-\frac{p'^{2n-1}}{q^{2n+1}} - \frac{p'^{2n+1}}{p^2 q^{2n+1}} + 2\frac{p'^{2n+1}}{q^{2n+3}} + \frac{p'^{2n-1}}{p^2 q^{2n-1}}$$
$$- \frac{p'^{2n+3}}{q^{2n+5}} - \frac{p'^{2n+3}}{p^2 q^{2n+3}} + \frac{p'^{2n+5}}{p^2 q^{2n+5}}\Big) \, , \tag{270}$$

and

$$\bar{g}_{n0}(p'q\,R_3) = \frac{1}{4}\Big(-\frac{p^{2n+2}}{p'^3 q^{2n+1}} - \frac{p^{2n}}{p' q^{2n+1}} + \frac{2p^{2n+2}}{p' q^{2n+3}} + \frac{p^{2n}}{p'^3 q^{2n-1}}$$
$$- \frac{p' p^{2n+2}}{q^{2n+5}} - \frac{p' p^{2n}}{q^{2n+3}} + \frac{p'^3 p^{2n}}{q^{2n+5}}\Big) \, . \tag{271}$$

Putting Eqs.(263-271) in Eq.(259), then

$$\sum_{n=1}^{\infty} T_{6n}^{2}(R_1) = -\frac{256}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \sum_{n=1}^{\infty}\frac{1}{n+1}\int_0^p dp' \int_{p'}^{p} dq \sum_{i=0}^{2} \bar{g}_{ni}\,(\underline{p'q}\,R_1)$$

$$= -\frac{256}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2\left(\frac{1}{8}\zeta(3) - \frac{3}{64}\zeta(2) - \frac{1}{16}\right) , \tag{272}$$

$$\sum_{n=1}^{\infty} T_{6n}^{2}(R_2) = -\frac{256}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \sum_{n=1}^{\infty}\frac{1}{n+1}\int_p^{\infty} dq \int_0^{p} dp' \sum_{i=0}^{2} \bar{g}_{ni}\,(\underline{p'q}\,R_2)$$

$$= -\frac{256}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2\left(\frac{1}{8}\zeta(3) - \frac{35}{4(64)}\right) , \tag{273}$$

and

$$\sum_{n=1}^{\infty} T_{6n}^{2}(R_3) = -\frac{256}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \sum_{n=1}^{\infty}\frac{1}{n+1}\int_p^{\infty} dp' \int_{p'}^{\infty} dq \sum_{i=0}^{2} \bar{g}_{ni}\,(\underline{p'q}\,R_3)$$

$$= -\frac{256}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2\left(\frac{1}{8}\zeta(3) + \frac{3}{64}\zeta(2) - \frac{61}{4(64)}\right) . \tag{274}$$

Adding Eqs.(258, 272-274),

$$T_6^2 = -\frac{96}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2\left(\zeta(3) - \frac{3}{4}\right) . \tag{275}$$

Finally, adding Eqs.(243, 275), then

$$T_6 = -\frac{96}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2\left(\zeta(3) - \frac{3}{4}\right) . \tag{276}$$

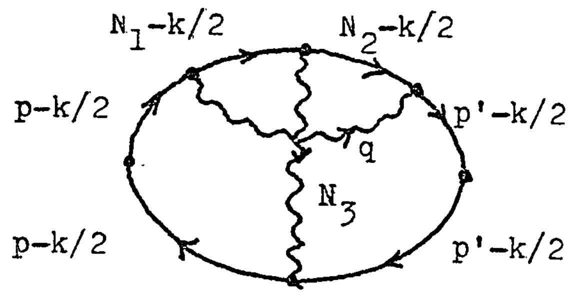
Chapter 11

CALCULATION OF T_7

By definition, Eq.(70),

$$T_7 = 2\,Tr\,\gamma G^{(0)} K_c^{\prime\prime(4)} G^{(0)} \gamma \quad . \tag{277}$$

We make the following momentum assignments:



$$\begin{aligned} N_1 &= p - q \\ N_2 &= p' - q \\ N_3 &= p - p' \end{aligned} \tag{278}$$

Using the Feynman rules, then

$$\begin{aligned} T_7 = 2\,Tr\,\gamma_\mu \frac{1}{i\gamma p}\int\frac{d^4p'}{(2\pi)^4}\int\frac{d^4q}{(2\pi)^4}(ie_0^2)^2\gamma_\mu\Big[\frac{\partial^2}{\partial K_\alpha^2}\Big(\frac{1}{i\gamma(N_1+\frac{K}{2})}\Big)\gamma_\lambda\frac{1}{i\gamma N_2} + 2\frac{\partial}{\partial K_\alpha}\Big(\frac{1}{i\gamma(N_1+\frac{K}{2})}\Big) \\ \cdot\gamma_\lambda\frac{\partial}{\partial K_\alpha}\Big(\frac{1}{i\gamma(N_2+\frac{K}{2})}\Big) + \frac{1}{i\gamma N_1}\gamma_\lambda\frac{\partial^2}{\partial K_\alpha^2}\Big(\frac{1}{i\gamma(N_2+\frac{K}{2})}\Big)\Big]\gamma_\mu\frac{1}{i\gamma p'}\gamma_\mu\frac{1}{i\gamma p'}\gamma_\lambda\frac{1}{i\gamma p}\frac{1}{q^2N_3^2} \quad . \end{aligned} \tag{279}$$

The first and third terms are equal by symmetry in p and p', as shown in Eq.(121). Further, using Eqs.(194, 219), Eq.(279) becomes

$$T_7 = T_7^1 + T_7^2 , \tag{280}$$

where

$$T_7^1 = \frac{4}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \int \frac{dp'}{p'} \int \frac{dq}{q} \int \frac{d\Omega_{p'}}{2\pi^2} \int \frac{d\Omega_q}{2\pi^2} \frac{q^2 N_2^2 N_3^2 \, Tr\, \gamma p \gamma_\mu \gamma p \gamma_\mu \gamma N_1 \gamma_\lambda \gamma N_2 \gamma_\mu \gamma p' \gamma_\mu \gamma p \gamma_\lambda}{(N_1^2 N_2^2 N_3^2)^2} , \tag{281}$$

and

$$T_7^2 = -\frac{1}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \int \frac{dp'}{p'} \int \frac{dq}{q} \int \frac{d\Omega_{p'}}{2\pi^2} \int \frac{d\Omega_q}{2\pi^2} \frac{q^2 N_3^2 \, Tr\, \gamma p \gamma_\mu \gamma p \gamma_\mu \gamma N_1 \gamma_\alpha \gamma N_1 \gamma_\lambda \gamma N_2 \gamma_\alpha \gamma N_2 \gamma_\mu \gamma p' \gamma_\mu \gamma p' \gamma_\lambda}{(N_1^2 N_2^2 N_3^2)^2} . \tag{282}$$

Using Appendix 1 to expand the trace, Eq.(281) becomes

$$T_7^1 = -\frac{128}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \int_0^\infty \frac{dq}{q} \int_0^\infty \frac{dp'}{p'} \int \frac{d\Omega_{p'}}{2\pi^2} \int \frac{d\Omega_q}{2\pi^2} \frac{q^2 N_2^2 N_3^2}{(N_1^2 N_2^2 N_3^2)^2} \Big[p\cdot p' p\cdot N_2 \, p'\cdot N_1 - 2 p'^2 p\cdot N_1 p\cdot N_2$$

$$- 2p^2 p'\cdot N_1 p'\cdot N_2 + p^2 p'^2 N_1\cdot N_2 \Big] , \tag{283}$$

or using Eq.(278) and expanding in Chebyshev polynomials,

$$T_7' = -\frac{128}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \sum_{n=0}^{\infty} \frac{1}{n+1} \int_0^\infty dq \int_0^\infty dp' \Big[4 q p'^2 p^3 \begin{pmatrix} 3 & 1 \\ p & p' \end{pmatrix}_n \begin{pmatrix} 0 & 1 \\ p' & q \end{pmatrix}_n \begin{pmatrix} 0 & 2 \\ p & q \end{pmatrix}_n$$

$$- 4 q^2 p'^2 p^2 \begin{pmatrix} 2 & 1 \\ p & p' \end{pmatrix}_n \begin{pmatrix} 1 & 1 \\ p' & q \end{pmatrix}_n \begin{pmatrix} 0 & 2 \\ p & q \end{pmatrix}_n - 4 q^2 p' p^3 \begin{pmatrix} 2 & 1 \\ p & p' \end{pmatrix}_n \begin{pmatrix} 0 & 1 \\ p' & q \end{pmatrix}_n \begin{pmatrix} 1 & 2 \\ p & q \end{pmatrix}_n$$

$$+ 4 q^3 p' p^2 \begin{pmatrix} 1 & 1 \\ p & p' \end{pmatrix}_n \begin{pmatrix} 1 & 1 \\ p' & q \end{pmatrix}_n \begin{pmatrix} 1 & 2 \\ p & q \end{pmatrix}_n - 3 q p'^2 p^3 \begin{pmatrix} 1 & 1 \\ p & p' \end{pmatrix}_n \begin{pmatrix} 0 & 1 \\ p' & q \end{pmatrix}_n \begin{pmatrix} 0 & 2 \\ p & q \end{pmatrix}_n$$

$$+ q^2 p' p^3 \begin{pmatrix} 0 & 1 \\ p & p' \end{pmatrix}_n \begin{pmatrix} 0 & 1 \\ p' & q \end{pmatrix}_n \begin{pmatrix} 1 & 2 \\ p & q \end{pmatrix}_n + 2 q^2 p'^2 p^2 \begin{pmatrix} 1 & 1 \\ p & p' \end{pmatrix}_n \begin{pmatrix} 0 & 1 \\ p' & q \end{pmatrix}_n \begin{pmatrix} 1 & 2 \\ p & q \end{pmatrix}_n$$

$$- 2 q^3 p' p^2 \begin{pmatrix} 0 & 1 \\ p & p' \end{pmatrix}_n \begin{pmatrix} 0 & 1 \\ p' & q \end{pmatrix}_n \begin{pmatrix} 2 & 2 \\ p & q \end{pmatrix}_n + 2 q^2 p' p^3 \begin{pmatrix} 1 & 1 \\ p & p' \end{pmatrix}_n \begin{pmatrix} 1 & 1 \\ p' & q \end{pmatrix}_n \begin{pmatrix} 0 & 2 \\ p & q \end{pmatrix}_n$$

$$+ q^2 p'^2 p^2 \begin{pmatrix} 0 & 1 \\ p & p' \end{pmatrix}_n \begin{pmatrix} 1 & 1 \\ p' & q \end{pmatrix}_n \begin{pmatrix} 0 & 2 \\ p & q \end{pmatrix}_n - 2 q^3 p' p^2 \begin{pmatrix} 0 & 1 \\ p & p' \end{pmatrix}_n \begin{pmatrix} 2 & 1 \\ p' & q \end{pmatrix}_n \begin{pmatrix} 0 & 2 \\ p & q \end{pmatrix}_n$$

$$+ q^3 p' p^2 \begin{pmatrix} 0 & 1 \\ p & p' \end{pmatrix}_n \begin{pmatrix} 0 & 1 \\ p' & q \end{pmatrix}_n \begin{pmatrix} 0 & 2 \\ p & q \end{pmatrix}_n \Big]$$

$$\equiv -\frac{128}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \sum_{n=0}^{\infty} \frac{1}{n+1} \int_{p'<q} dp'\, dq\; g_n(\underline{p'q})$$

$$\equiv \sum_{n=0}^{\infty} T_{7n}' \; . \tag{284}$$

We define

$$g_{n>0}(p'q) = \bar{g}_n(p'q) + \delta_1(p'q)\,\delta_{1n} + \delta_2(p'q)\,\delta_{2n} \; ,$$

and

$$g_n(p'q) = \begin{cases} g_0(p'q) + \frac{1}{2}\delta_1(p'q) + \frac{1}{3}\delta_2(p'q) \equiv \bar{g}_0(p'q) & , n=0 \\ \bar{g}_n(p'q) & , n>0 \end{cases} \tag{285}$$

From Appendix 2 and Eq.(284), we have

$$\begin{aligned} \bar{g}_0(p'q) = {} & \frac{1}{2}\frac{p}{q}\left\langle\frac{p}{p'}\right\rangle^4\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle + \frac{p}{q}\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle - \frac{1}{2}\frac{p}{p'}\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle \\ & - \frac{1}{2}\frac{p'}{p}\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle - \frac{p^2}{p'}\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle - \left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle \\ & + \frac{q}{p'}\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 - \frac{3}{2}\frac{p}{q}\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle + \frac{p}{p'}\left\langle\frac{p}{p'}\right\rangle\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle^2 \\ & + \left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle^2 - \frac{3}{2}\frac{q}{p'}\left\langle\frac{p}{p'}\right\rangle\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle^3 - \frac{1}{2}\frac{q}{p'}\left\langle\frac{p}{p'}\right\rangle\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle \\ & + \frac{1}{2}\frac{p}{p'}\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle + \frac{1}{2}\left\langle\frac{p}{p'}\right\rangle\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle - \frac{1}{2}\left\langle\frac{p}{p'}\right\rangle\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle \\ & - \frac{1}{2}\frac{q^2}{p'^2}\left\langle\frac{p}{p'}\right\rangle\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle + \frac{q}{p'}\left\langle\frac{p}{p'}\right\rangle\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle - \frac{1}{2}\frac{p^2}{p'q}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 \\ & - \frac{1}{2}\frac{p'}{q}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 + \frac{1}{2}\frac{p'}{q}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 + \frac{1}{2}\frac{q}{p'}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 \\ & + \frac{1}{2}\frac{p^2}{qp'}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 + \frac{1}{2}\frac{q}{p'}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 + \frac{1}{4}\frac{p}{p'}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^3 \\ & - \frac{1}{4}\frac{p}{p'}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle + \frac{1}{2}\frac{q}{p'}\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 - \frac{1}{2}\frac{p}{q}\left\langle\frac{p'}{q}\right\rangle^3\left\langle\frac{p}{q}\right\rangle^3 . \end{aligned} \tag{286}$$

We divide the region of integration into three regions:

R_1: $p' < q < p$

R_2: $p' < p < q$

R_3: $p < p' < q$

Eq.(286) gives

$$\bar{g}^{\circ}(\underline{p'q}\,R_1) = \frac{p'q}{p^4} - \frac{1}{4}\frac{p'}{p^2 q} - \frac{1}{2}\frac{p'^3}{q p^4} - \frac{1}{4}\frac{p'^3}{q^3 p^2} + \frac{1}{2(p^2-p'^2)}\left(\frac{p'q^3}{p^4} - \frac{p'q}{p^2}\right) , \tag{287}$$

$$\bar{g}^{\circ}(\underline{p'q}\,R_2) = \frac{p'^3 p^2}{2q^7} + \frac{1}{4}\frac{p' p^2}{q^5} + \frac{3}{4}\frac{p'^3}{q^3 p^2} , \tag{288}$$

and

$$\bar{g}^{\circ}(\underline{p'q}\,R_3) = \frac{1}{2}\frac{p^4}{q^3 p'^3} + \frac{1}{2}\frac{p^4}{p' q^5} - \frac{1}{2}\frac{p^2 p'}{q^5} + \frac{3}{4}\frac{p^2}{p' q^3} . \tag{289}$$

Putting Eqs.(287-289) in Eq.(284), then

$$T'_{70}(R_1) = \Bigg(\int_0^p dq \int_0^q dp' \left(\frac{p'q}{p^4} - \frac{1}{4}\frac{p'}{p^2 q} - \frac{1}{2}\frac{p'^3}{q p^4} - \frac{1}{4}\frac{p'^3}{q^3 p^2}\right.$$
$$\left. + \int_0^p dp' \int_{p'}^p dq \frac{1}{2(p^2-p'^2)}\left(\frac{p'q^3}{p^4} - \frac{p'q}{p^2}\right)\right)\Bigg)\left(-\frac{128}{p^4}\right)\left(\frac{\alpha_0}{2\pi}\right)^2 = -\frac{1}{32}\left(-\frac{128}{p^4}\right)\left(\frac{\alpha_0}{2\pi}\right)^2, \tag{290}$$

$$T'_{70}(R_2) = -\frac{128}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \int_p^\infty dq \int_0^p dp'\, \bar{g}_0(\underline{p'q}\,R_2) = \frac{7}{48}\left(-\frac{128}{p^4}\right)\left(\frac{\alpha_0}{2\pi}\right)^2 , \tag{291}$$

and

$$T'_{70}(R_3) = -\frac{128}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \int_p^\infty dp' \int_{p'}^\infty dq\, \bar{g}_0(\underline{p'q}\,R_3) = \left(\frac{11}{48} + \frac{1}{32}\right)\left(-\frac{128}{p^4}\right)\left(\frac{\alpha_0}{2\pi}\right)^2 . \tag{292}$$

Adding Eqs.(290–292), then

$$T_{70} = \frac{9}{24}\left(-\frac{128}{p^4}\right)\left(\frac{\alpha_0}{2\pi}\right)^2 . \tag{293}$$

Further, from Eq.(284) and Appendix 2:

$$\bar{g}^n(\underline{p'q}) = \left\langle\frac{p}{p'}\right\rangle^{n+1}\left\langle\frac{p}{q}\right\rangle^{n+1}\left\langle\frac{p'}{q}\right\rangle^{n+1}\left(\frac{1}{2}\frac{p^2}{p'^3q} + \frac{1}{p'q} - \frac{1}{2}\frac{q}{p'^3} + \frac{1}{2}\frac{q}{p^2p'} - \frac{1}{2}\frac{p'}{qp^2}\right.$$
$$\left. + \frac{1}{2}\frac{p^2}{q^3p'} - \frac{1}{2}\frac{p'}{q^3} + \frac{1}{2}\frac{p'}{p^2q} - \frac{1}{2}\frac{q}{p'p^2}\right) . \tag{294}$$

Evaluating Eq.(294) in regions R_1, R_2, and R_3

$$\bar{g}^n(\underline{p'q}\,R_1) = \frac{1}{2}\frac{p'^{2n-1}}{qp^{2n}} + \frac{p'^{2n+1}}{qp^{2n+2}} - \frac{1}{2}\frac{qp'^{2n-1}}{p^{2n+2}} + \frac{1}{2}\frac{p'^{2n+1}}{q^3p^{2n}} - \frac{1}{2}\frac{p'^{2n+3}}{q^3p^{2n+2}} , \tag{295}$$

$$\bar{g}^n(\underline{p'q}\,R_2) = \frac{1}{2}\frac{p^2p'^{2n-1}}{q^{2n+3}} + \frac{p'^{2n+1}}{q^{2n+3}} - \frac{1}{2}\frac{p'^{2n-1}}{q^{2n+1}} + \frac{1}{2}\frac{p^2p'^{2n+1}}{q^{2n+5}} - \frac{1}{2}\frac{p'^{2n+3}}{q^{2n+5}} , \tag{296}$$

and

$$\bar{g}^n(\underline{p'q}\,R_3) = \frac{1}{2}\frac{p^{2n+4}}{p'^3q^{2n+3}} + \frac{p^{2n+2}}{p'q^{2n+3}} - \frac{1}{2}\frac{p^{2n+2}}{p'^3q^{2n+1}} + \frac{1}{2}\frac{p^{2n+4}}{p'q^{2n+5}} - \frac{1}{2}\frac{p'p^{2n+2}}{q^{2n+5}} . \tag{297}$$

Putting Eqs.(295–297) in Eq.(284), then

$$\sum_{n=1}^{\infty} T^1_{7n}(R_1) = -\frac{128}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \sum_{n=1}^{\infty}\frac{1}{n+1}\int_0^p dq\int_0^q dp'\,\bar{g}_n(\underline{p'q}\,R_1)$$
$$= -\frac{128}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2\left(\frac{1}{4}\zeta(3) - \frac{3}{16}\right) , \tag{298}$$

$$\sum_{n=1}^{\infty} T'_{7n}(R_2) = -\frac{128}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \sum_{n=1}^{\infty} \frac{1}{n+1} \int_p^{\infty} dq \int_0^p dp'\, \bar{g}_n(\underline{p'q}\, R_2)$$

$$= -\frac{128}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \left(\frac{1}{4}\zeta(3) - \frac{9}{32}\right), \tag{299}$$

and

$$\sum_{n=1}^{\infty} T'_{7n}(R_3) = -\frac{128}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \sum_{n=1}^{\infty} \frac{1}{n+1} \int_p^{\infty} dp' \int_{p'}^{\infty} dq\, \bar{g}_n(\underline{p'q}\, R_3)$$
$$= -\frac{128}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \left(\frac{1}{4}\zeta(3) - \frac{9}{32}\right). \tag{300}$$

Putting Eqs.(293, 298-300) in Eq.(284), then

$$T'_7 = -\frac{96}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \left(\zeta(3) - \frac{1}{2}\right). \tag{301}$$

Let us now expand the trace in Eq.(282):

$$T \equiv Tr\, \gamma p \gamma_\mu \gamma p \gamma_\mu \gamma N_1 \gamma_\alpha \gamma N_1 \gamma_\lambda \gamma N_2 \gamma_\alpha \gamma N_2 \gamma_\mu \gamma p' \gamma_\mu \gamma p' \gamma_\lambda . \tag{302}$$

Using Appendix 1, Eq.(302) becomes

$$T = -8\, Tr\, \gamma N_1 \gamma N_2 \gamma_\lambda \gamma N_1 \gamma N_2 \gamma p \gamma p' \gamma_\lambda \gamma p \gamma p'. \tag{303}$$

We now employ an identity of E. Caianiello.[47] The identity is:

$$\gamma_\alpha \gamma A \gamma B \gamma C \gamma D \gamma^\alpha = Tr(\gamma A \gamma B \gamma C \gamma D), \tag{304}$$

if A, B, C, and D are not all linearly independent. Since p, p', and q are the only linearly independent vectors in the problem, we can use Eq.(233) to reduce Eq.(232) to

$$T = -8\left(Tr\, \gamma N_1 \gamma N_2 \gamma p \gamma p'\right)^2 . \tag{305}$$

Putting Eq.(305) in Eq.(282), then

$$
\begin{aligned}
T_7^2 = {} & \frac{128}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \int_0^\infty \frac{dq}{q} \int_0^\infty \frac{dp'}{p'} \int \frac{d\Omega_{p'}}{2\pi^2} \int \frac{d\Omega_q}{2\pi^2} \frac{q^2}{N_1^4 N_2^4 N_3^4} \Big[4p^4 p'^4 \cos^4 pp' - 8p^4 p'^3 q \cos^3 pp' \cos pq \\
& - 8p^3 p'^4 q \cos^3 pp' \cos p'q + 4p^3 p'^3 q^2 \cos^3 pp' - 4p^4 p'^4 \cos^2 pp' + 4p^4 p'^3 q \cos^2 pp' \cos p'q \\
& + 4p^3 p'^4 q \cos^2 pp' \cos pq + 4p^4 p'^2 q^2 \cos^2 pp' \cos^2 pq + 8p^3 p'^3 q^2 \cos pq \cos^2 pp' \cos p'q \\
& - 4p^3 p'^2 q^3 \cos^2 pp' \cos pq + 4p^4 p'^3 q \cos pq \cos pp' - 4p^4 p'^2 q^2 \cos pq \cos pp' \cos p'q \\
& - 4p^3 p'^3 q^2 \cos^2 pq \cos pp' + 4p'^4 q^2 p^2 \cos^2 p'q \cos^2 pp' - 4p'^3 q^3 p^2 \cos p'q \cos^2 pp' \\
& + 4p'^4 p^3 q \cos p'q \cos pp' - 4p'^3 q^2 p^3 \cos^2 p'q \cos pp' - 4p'^4 q^2 p^2 \cos p'q \cos pp' \cos pq \\
& + q^4 p^2 p'^2 \cos^2 pp' - 2q^2 p^3 p'^3 \cos pp' + 2p^3 p'^2 q^3 \cos p'q \cos pp' \\
& + 2q^3 p'^3 p^2 \cos pp' \cos pq + p^4 p'^4 - 2p^4 p'^3 q \cos p'q \\
& - 2p^3 p'^4 q \cos pq + p^4 p'^2 q^2 \cos^2 p'q + 2p^3 p'^3 q^2 \cos p'q \cos pq \\
& + p'^4 p^2 q^2 \cos^2 pq \Big] . \qquad (306)
\end{aligned}
$$

We use Appendix 2 to expand Eq.(306) in Chebyshev polynomials, hence:

$$
\begin{aligned}
T_7^2 = {} & \frac{128}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \sum_{n=0}^{\infty} \frac{1}{n+1} \int_0^\infty dq \int_0^\infty dp' \Big[4p^4p'^3q \binom{41}{pp'}_n \binom{02}{p'q}_n \binom{02}{pq}_n - 8p^4p'^2q^2 \binom{31}{pp'}_n \binom{02}{p'q}_n \binom{12}{pq}_n \\
& - 8p^3p'^3q^2 \binom{31}{pp'}_n \binom{12}{p'q}_n \binom{02}{pq}_n + 4p^3p'^2q^3 \binom{31}{pp'}_n \binom{02}{p'q}_n \binom{02}{pq}_n \\
& - 4p^4p'^3q \binom{21}{pp'}_n \binom{02}{p'q}_n \binom{02}{pq}_n + 4p^4p'^2q^2 \binom{02}{pp'}_n \binom{12}{p'q}_n \binom{02}{pq}_n + 4p^3p'^3q^2 \binom{21}{pp'}_n \binom{02}{p'q}_n \binom{12}{pq}_n \\
& + 4p^4p'q^3 \binom{21}{pp'}_n \binom{02}{p'q}_n \binom{22}{pq}_n + 8p^3p'^2q^3 \binom{21}{pp'}_n \binom{12}{p'q}_n \binom{12}{pq}_n - 4p^3p'q^4 \binom{21}{pp'}_n \binom{02}{p'q}_n \binom{12}{pq}_n \\
& + 4p^4p'^2q^2 \binom{11}{pp'}_n \binom{02}{p'q}_n \binom{12}{pq}_n - 4p^4p'q^3 \binom{11}{pp'}_n \binom{12}{p'q}_n \binom{12}{pq}_n - 4p^3p'^2q^3 \binom{11}{pp'}_n \binom{02}{p'q}_n \binom{22}{pq}_n \\
& + 4p^2p'^3q^3 \binom{21}{pp'}_n \binom{22}{p'q}_n \binom{02}{pq}_n - 4p^2p'^2q^4 \binom{21}{pp'}_n \binom{12}{p'q}_n \binom{02}{pq}_n + 4p^3p'^3q^2 \binom{11}{pp'}_n \binom{12}{p'q}_n \binom{02}{pq}_n \\
& - 4p^3p'^2q^3 \binom{11}{pp'}_n \binom{22}{p'q}_n \binom{02}{pq}_n - 4p^2p'^3q^3 \binom{11}{pp'}_n \binom{12}{p'q}_n \binom{12}{pq}_n + p^2p'q^5 \binom{21}{pp'}_n \binom{02}{p'q}_n \binom{02}{pq}_n \\
& - 2p^3p'^2q^3 \binom{11}{pp'}_n \binom{02}{p'q}_n \binom{02}{pq}_n + 2p^3p'q^4 \binom{11}{pp'}_n \binom{12}{p'q}_n \binom{02}{pq}_n + 2p^2p'^2q^4 \binom{11}{pp'}_n \binom{02}{p'q}_n \binom{12}{pq}_n \\
& + p^4p'^3q \binom{01}{pp'}_n \binom{02}{p'q}_n \binom{02}{pq}_n - 2p^4p'^2q^2 \binom{01}{pp'}_n \binom{12}{p'q}_n \binom{02}{pq}_n - 2p^3p'^3q^2 \binom{01}{pp'}_n \binom{02}{p'q}_n \binom{12}{pq}_n \\
& + p^4p'q^3 \binom{01}{pp'}_n \binom{22}{p'q}_n \binom{02}{pq}_n + 2p^3p'^2q^3 \binom{01}{pp'}_n \binom{12}{p'q}_n \binom{12}{pq}_n + p^2p'^3q^3 \binom{01}{pp'}_n \binom{02}{p'q}_n \binom{22}{pq}_n \Big] \\
\equiv {} & \sum_{n=0}^{\infty} \frac{1}{n+1} \int_{p'<q} dp'\,dq\; g_n(\underline{p'q}) \\
\equiv {} & \sum_{n=0}^{\infty} T_{7n}^2 \,. \qquad (307)
\end{aligned}
$$

Again using the same breakup of g_n as in Eq.(285),

$$
\begin{aligned}
g_0(p'q) = \frac{1}{|p'^2-q^2||p^2-q^2|} \Big[& \frac{1}{4}\frac{p^2p'}{q}\left\langle\frac{p}{p'}\right\rangle^5\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle + \frac{3}{4}\frac{p^2p'}{q}\left\langle\frac{p}{p'}\right\rangle^3\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle \\
& + \frac{1}{2}\frac{p^2p'}{q}\left\langle\frac{p}{p'}\right\rangle\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle - p^2\left\langle\frac{p}{p'}\right\rangle^4\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle^2 - 2p^2\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle^2 - pp'\left\langle\frac{p}{p'}\right\rangle^4\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle \\
& - 2pp'\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle + \frac{1}{2}pq\left\langle\frac{p}{p'}\right\rangle^4\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle + pq\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle - \frac{p^3}{q}\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle \\
& - \frac{p'^2p}{q}\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle + \frac{p^3}{p'}\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle + pp'\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle + p^2\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle^2 \\
& + p'^2\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle^2 + \frac{3}{4}\frac{p^3q}{p'^2}\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle^3 + \frac{3}{4}pq\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle^3 \\
& + \frac{1}{4}\frac{p^3q}{p'^2}\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle + \frac{1}{4}pq\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle + \frac{2p^2q}{p'}\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 \\
& + 2p'q\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 - \frac{p^2q^2}{p'^2}\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle^2 - q^2\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle^2 \\
& + 2p^2\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle^2 - \frac{2p^2q}{p'}\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 - \frac{3}{2}pq\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle^3 \\
& - \frac{1}{2}pq\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle + \frac{3}{4}pq\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle^3\left\langle\frac{p}{q}\right\rangle - \frac{3}{4}\frac{p'^2q}{p}\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle^3\left\langle\frac{p}{q}\right\rangle + \frac{1}{4}pq\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle \\
& + \frac{1}{4}\frac{p'^2q}{p}\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle - \frac{pq^2}{p'}\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle - \frac{p'q^2}{p}\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle + 2pp'\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle \\
& - \frac{3}{2}pq\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle^3\left\langle\frac{p}{q}\right\rangle - \frac{1}{2}pq\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle - 2p'q\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 + \frac{1}{4}\frac{pq^3}{p'^2}\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle \\
& + \frac{1}{4}\frac{q^3}{p}\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle - pq\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle + \frac{pq^2}{p'}\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle + q^2\left\langle\frac{p}{p'}\right\rangle^2\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle^2 \\
& + \frac{p^2p'}{q}\left\langle\frac{p}{p'}\right\rangle\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle - 2p^2\left\langle\frac{p}{p'}\right\rangle\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle - 2pp'\left\langle\frac{p}{p'}\right\rangle\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle^2 + \frac{3}{4}\frac{p^2q}{p'}\left\langle\frac{p}{p'}\right\rangle\left\langle\frac{p'}{q}\right\rangle^3\left\langle\frac{p}{q}\right\rangle \\
& + \frac{1}{4}\frac{p^2q}{p'}\left\langle\frac{p}{p'}\right\rangle\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle + 2pq\left\langle\frac{p}{p'}\right\rangle\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 + \frac{3}{4}p'q\left\langle\frac{p}{p'}\right\rangle\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle^3 + \frac{1}{4}p'q\left\langle\frac{p}{p'}\right\rangle\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle \Big],
\end{aligned}
\tag{308}
$$

and

$$
\begin{aligned}
\delta^{1}(p'q) = \frac{1}{|p'^2-q^2||p^2-q^2|}\Bigg[& -\frac{p^4}{p'q}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 - \frac{4p^2p'}{q}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 - \frac{p'^3}{q}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 + 2\frac{p^4}{p'q}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 \\
& + \frac{2p^2p'}{q}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 + 2\frac{p^2q}{p'}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 + 2p'q\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 + \frac{p^3}{p'}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^3 + pp'\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^3 \\
& - \frac{p^3}{p'}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle - pp'\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle + 2\frac{p^2p'}{q}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 + 2\frac{p'^3}{q}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 \\
& + \frac{2qp^2}{p'}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 + 2qp'\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 + p^2\left\langle\frac{p'}{q}\right\rangle^3\left\langle\frac{p}{q}\right\rangle^2 + p'^2\left\langle\frac{p'}{q}\right\rangle^3\left\langle\frac{p}{q}\right\rangle^2 - p^2\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle^2 \\
& - p'^2\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle^2 - 2\frac{p^2q}{p'}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 - 2p'q\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 + 4\frac{p^2p'}{q}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 \\
& - 2\frac{p^2p'}{q}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 - \frac{2q}{p'}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 - p^2\left\langle\frac{p'}{q}\right\rangle^3\left\langle\frac{p}{q}\right\rangle^2 + p^2\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle^2 - \frac{2p^2p'}{q}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 \\
& - 2qp'\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 - pp'\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^3 + pp'\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle - \frac{p^4}{qp'}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 - \frac{2p^2q}{p'}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 \\
& - \frac{q^3}{p'}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 - \frac{qp^2}{p'}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^4 + \frac{qp^2}{p'}\left\langle\frac{p'}{q}\right\rangle^2 - \frac{2p^2p'}{q}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 - 2\frac{qp^2}{p'}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 \\
& - 2qp'\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 - 2\frac{q^3}{p'}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 - pp'\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^3 - \frac{q^2p}{p'}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^3 \\
& + pp'\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle + \frac{q^2p}{p'}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle - p^2\left\langle\frac{p'}{q}\right\rangle^3\left\langle\frac{p}{q}\right\rangle^2 - q^2\left\langle\frac{p'}{q}\right\rangle^3\left\langle\frac{p}{q}\right\rangle^2 + p^2\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle^2 \\
& + q^2\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle^2 - \frac{1}{2}pq\left\langle\frac{p'}{q}\right\rangle^3\left\langle\frac{p}{q}\right\rangle^3 + \frac{1}{2}pq\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle^3 + \frac{1}{2}pq\left\langle\frac{p'}{q}\right\rangle^3\left\langle\frac{p}{q}\right\rangle - \frac{1}{2}pq\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle \\
& + 2\frac{p^2q}{p'}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 + 2\frac{q^3}{p'}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 + \frac{q^2p}{p'}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^3 - \frac{q^2p}{p'}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle \\
& - \frac{p'^3}{q}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 - 2qp'\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 - \frac{q^3}{p'}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 + qp'\left\langle\frac{p'}{q}\right\rangle^4\left\langle\frac{p}{q}\right\rangle^2 + qp'\left\langle\frac{p}{q}\right\rangle^2 \\
& + 2p'q\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 + \frac{2q^3}{p'}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 + q^2\left\langle\frac{p'}{q}\right\rangle^3\left\langle\frac{p}{q}\right\rangle^2 - q^2\left\langle\frac{p'}{q}\right\rangle\left\langle\frac{p}{q}\right\rangle^2 \\
& - \frac{q^3}{p'}\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^2 \Bigg],
\end{aligned}
\tag{309}
$$

and

$$\delta^2(p'q) = \frac{1}{|p'^2-q^2||p^2-q^2|}\Big[-\frac{9}{4}\frac{p^3}{q}\left\langle\frac{p'}{q}\right\rangle^3\left\langle\frac{p}{q}\right\rangle^3 - \frac{9}{4}\frac{p'^2p}{q}\left\langle\frac{p'}{q}\right\rangle^3\left\langle\frac{p}{q}\right\rangle^3 + \frac{9}{2}\frac{p^3}{q}\left\langle\frac{p'}{q}\right\rangle^3\left\langle\frac{p}{q}\right\rangle^3$$

$$+\frac{9}{2}pq\left\langle\frac{p'}{q}\right\rangle^3\left\langle\frac{p}{q}\right\rangle^3 + \frac{3}{2}p^2\left\langle\frac{p'}{q}\right\rangle^3\left\langle\frac{p}{q}\right\rangle^4 - \frac{3}{2}p^2\left\langle\frac{p'}{q}\right\rangle^3\left\langle\frac{p}{q}\right\rangle^2 + \frac{9}{2}\frac{pp'^2}{q}\left\langle\frac{p'}{q}\right\rangle^3\left\langle\frac{p}{q}\right\rangle^3$$

$$+\frac{9}{2}pq\left\langle\frac{p'}{q}\right\rangle^3\left\langle\frac{p}{q}\right\rangle^3 + \frac{3}{2}pp'\left\langle\frac{p'}{q}\right\rangle^4\left\langle\frac{p}{q}\right\rangle^3 - \frac{3}{2}pp'\left\langle\frac{p'}{q}\right\rangle^2\left\langle\frac{p}{q}\right\rangle^3 - \frac{9}{2}pq\left\langle\frac{p'}{q}\right\rangle^3\left\langle\frac{p}{q}\right\rangle^3\Big], \quad (310)$$

and

$$\delta^3(p'q) = -\frac{1}{|p'^2-q^2||p^2-q^2|}\,\frac{4p^2p'}{q}\left\langle\frac{p'}{q}\right\rangle^4\left\langle\frac{p}{q}\right\rangle^4 . \qquad (311)$$

Symmetrizing in p' and q, then

$$T_{70}^2(R_1) = \frac{128}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2\Big[\int_0^p dq\int_0^q dp'\left(-\frac{1}{4}\frac{p'^3}{p^2q} + \frac{1}{4}\frac{p'}{qp^2} + \frac{1}{2}\frac{p'}{p^2q}\right)$$

$$+\int_0^p dp'\int_{p'}^p dq\left(-\frac{1}{4}\frac{p'^3}{p^2q^3} + \frac{1}{(p^2-p'^2)}\left(\frac{1}{4}\frac{p'q^3}{p^4} - \frac{1}{4}\frac{qp'}{p^2}\right)\right)\Big]$$

$$= \frac{128}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2\frac{1}{8}, \qquad (312)$$

$$T_{70}^2(R_2) = \frac{128}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2\Big[\int_p^\infty dq\int_0^p dp'\left(-\frac{p'^3p^4}{q^9} + \frac{1}{4}\frac{p'p^4}{q^7} + \frac{1}{4}\frac{p'^3}{q^5} + \frac{3}{4}\frac{p'}{q^3} + \frac{1}{4}\frac{p'^3p^2}{q^7} - \frac{1}{4}\frac{p'p^2}{q^5}\right)\Big]$$

$$= \frac{128}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2\frac{11}{64}, \qquad (313)$$

and

$$T^2_{70}(R_3) = \frac{128}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2\Big[\int_p^\infty dp'\int_{p'}^\infty dq\left(\frac{1}{4}\frac{p^4}{p'^3q^3}+\frac{1}{2}\frac{p^2}{p'q^3}+\frac{1}{4}\frac{p'p^4}{q^7}-\frac{p'^3p^4}{q^9}\right)$$

$$+\int_p^\infty dq\int_p^q dp'\frac{1}{(q^2-p^2)}\left(\frac{1}{4}\frac{p'p^4}{q^5}-\frac{1}{4}\frac{p'^3p^4}{q^7}\right)\Big]$$

$$= \frac{128}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2\frac{9}{64} \quad . \tag{314}$$

Adding Eqs.(312-314), then

$$T^2_{70} = \frac{25}{32}\left(\frac{128}{p^4}\right)\left(\frac{\alpha_0}{2\pi}\right)^2 \quad . \tag{315}$$

Further from Eq.(307) and Appendix 2,

$$\bar{g}^n(p'q) = \frac{\left\langle\frac{p}{p'}\right\rangle^{n+1}\left\langle\frac{p'}{q}\right\rangle^{n+1}\left\langle\frac{p}{q}\right\rangle^{n+1}}{|p'^2-q^2||p^2-q^2|}\frac{1}{2}pq\left(\left\langle\frac{p'}{q}\right\rangle-\left\langle\frac{p'}{q}\right\rangle^{-1}\right)\left(\left\langle\frac{p}{q}\right\rangle-\left\langle\frac{p}{q}\right\rangle^{-1}\right). \tag{316}$$

Then

$$\sum_{n=1}^{\infty} T^2_{7n}(R_1) = \frac{128}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2\sum_{n=1}^{\infty}\frac{1}{n+1}\int_0^p dq\int_0^q dp'\frac{p'^{2n+1}}{q\,p^{2n+2}}$$

$$= \frac{128}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2\left(\frac{1}{4}\zeta(3)-\frac{1}{4}\right) , \tag{317}$$

$$\sum_{n=1}^{\infty} T_{7n}^{2}(R_2) = \frac{128}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \sum_{n=1}^{\infty} \frac{1}{n+1} \int_p^{\infty} dq \int_0^p dp' \, \frac{p'^{2n+1}}{q^{2n+3}}$$

$$= \frac{128}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \left(\frac{1}{4}\zeta(3) - \frac{1}{4}\right) , \tag{318}$$

and

$$\sum_{n=1}^{\infty} T_{7n}^{2}(R_3) = \frac{128}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \sum_{n=1}^{\infty} \frac{1}{n+1} \int_p^{\infty} dp' \int_{p'}^{\infty} dq \, \frac{p^{2n+2}}{p' q^{2n+3}}$$

$$= \frac{128}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \left(\frac{1}{4}\zeta(3) - \frac{1}{4}\right) . \tag{319}$$

Adding Eqs.(315, 317-319), then

$$T_7^2 = \frac{96}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 \left(\zeta(3) - \frac{5}{12}\right) . \tag{320}$$

Putting Eqs.(301, 320) in Eq.(280), then

$$T_7 = \frac{8}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2 . \tag{321}$$

Observe how the zeta functions cancelled between T_7^1 and T_7^2.

Chapter 12

CALCULATION OF T_8

By definition, Eq.(70),

$$T_8 = 4 Tr\, \gamma G'^{(0)} \Gamma^{(2)}_\mu (pp) G'^{(0)} \gamma \gamma^\mu / q^2 . \tag{322}$$

Using Eqs.(58, 74, 75) and the Feynman rules in Eq.(322), then

$$T_8 = 4 Tr \int \frac{d^4 q}{(2\pi)^4} \, i e_0^2 \frac{(\gamma_\alpha \gamma p \gamma_\mu - \gamma_\mu \gamma p \gamma_\alpha)(B'^{(2)}(p^2)\gamma_\beta + 2 p_\beta \gamma p B'^{(2)}(p^2))}{i^2 2 p^4 \qquad q^2} \frac{(\gamma_\mu \gamma (p-q) \gamma_\alpha - \gamma_\alpha \gamma (p-q) \gamma_\mu)\gamma_\beta}{i^2 2 (p-q)^4} . \tag{323}$$

Let k = p − q, then

$$T_8 = 4 Tr \int \frac{d^4 k}{(2\pi)^4} \frac{i e_0^2 (\gamma_\alpha \gamma p \gamma_\mu - \gamma_\mu \gamma p \gamma_\alpha)(B'^{(2)}(p^2) + 2 p_\beta \gamma p B'^{(2)}(p^2))(\gamma_\mu \gamma k \gamma_\alpha - \gamma_\alpha \gamma k \gamma_\mu)\gamma_\beta}{2 p^4 \qquad (p-k)^2 \qquad 2 k^4} , \tag{324}$$

or

$$T_6 = \frac{i e_0^2 B'^{(2)}(p^2)}{p^4} \int \frac{d^4 k}{(2\pi)^4} \frac{Tr (\gamma_\alpha \gamma p \gamma_\mu - \gamma_\mu \gamma p \gamma_\alpha)\gamma_\beta (\gamma_\mu \gamma k \gamma_\alpha - \gamma_\alpha \gamma k \gamma_\mu)\gamma_\beta}{(p-k)^2 k^4}$$

$$+ \frac{2 i e_0^2 B'^{(2)}(p^2)}{p^4} \int \frac{d^4 k}{(2\pi)^4} \frac{Tr (\gamma_\alpha \gamma p \gamma_\mu - \gamma_\mu \gamma p \gamma_\alpha)\gamma p (\gamma_\mu \gamma k \gamma_\alpha - \gamma_\alpha \gamma k \gamma_\mu)\gamma p}{(p-k)^2 k^4} . \tag{325}$$

Using Appendix 1

$$Tr (\gamma_\alpha \gamma p \gamma_\mu - \gamma_\mu \gamma p \gamma_\alpha)\gamma_\beta (\gamma_\mu \gamma k \gamma_\alpha - \gamma_\alpha \gamma k \gamma_\mu)\gamma_\beta = 192\, p \cdot k , \tag{326}$$

and

$$Tr (\gamma_\alpha \gamma p \gamma_\mu - \gamma_\mu \gamma p \gamma_\alpha)\gamma p (\gamma_\mu \gamma k \gamma_\alpha - \gamma_\alpha \gamma k \gamma_\mu)\gamma p = 96\, p^2\, p \cdot k . \tag{327}$$

Putting Eqs.(326, 327) and using Eq.(80) in Eq.(325), then

$$T_8 = \frac{-192\, e_0^2}{\rho^4}\left(B^{(2)}_{(\rho^2)} + \rho^2 B'^{(2)}_{(\rho^2)}\right) I_1 \quad . \tag{328}$$

Finally putting Eqs.(82-83) in Eq.(328), then

$$T_8 = \frac{-128\, B^{(1)}}{\rho^4}\left(B^{(2)}_{(\rho^2)} + \rho^2 B'^{(2)}_{(\rho^2)}\right) \quad . \tag{329}$$

Chapter 13

CALCULATION OF T_{10}

By definition, Eq.(70),

$$T_{10} = 4B'^{(2)}_{(p^2)} \, Tr \, \gamma p G'^{(0)} K^{(2)} G'^{(0)} \gamma p \; . \tag{330}$$

Using Eq.(74),

$$T_{10} \propto Tr\,(\gamma_\alpha \gamma p \gamma_\mu - \gamma_\mu \gamma p \gamma_\alpha)\, p^\mu K^{(2)} G'^{(0)} \gamma p = 0 \, ,$$

or

$$T_{10} = 0 \; . \tag{331}$$

Chapter 14

CALCULATION OF T_{11}

By definition, Eq.(70),

$$T_{11} = Tr\, \gamma G'^{(0)} K^{(2)} G'^{(2)} \gamma \ , \tag{332}$$

or using Eqs.(74, 75),

$$T_{11} = -2Tr\, \gamma G'^{(0)} K^{(2)} B^{(2)} G'^{(0)} \gamma \ . \tag{333}$$

Using Eqs.(74, 75) and the Feynman rules, then

$$T_{11} = -2Tr \int \frac{d^4q}{(2\pi)^4} \frac{ie_0^2 (\gamma_\alpha \gamma p \gamma_\mu - \gamma_\mu \gamma p \gamma_\alpha)\gamma_\beta B^{(2)}(p-q)(\gamma_\mu \gamma(p-q)\gamma_\alpha - \gamma_\alpha \gamma(p-q)\gamma_\mu)\gamma_\beta}{2p^4 i^2 \qquad q^2 \qquad 2(p-q)^4 i^2} \ . \tag{334}$$

Let k = p − q, then

$$T_{11} = -2Tr \int \frac{d^4k}{(2\pi)^4} \frac{ie_0^2 (\gamma_\alpha \gamma p \gamma_\mu - \gamma_\mu \gamma p \gamma_\alpha)\gamma_\beta B^{(2)}(k)(\gamma_\mu \gamma k \gamma_\alpha - \gamma_\alpha \gamma k \gamma_\mu)\gamma_\beta}{4p^4 (p-k)^2 k^4} \ , \tag{335}$$

or using Eq.(326),

$$T_{11} = \frac{96 e_0^2}{p^4} I_2 \ , \tag{336}$$

where

$$I_2 = \int \frac{d^4k}{(2\pi)^4} \frac{B^{(2)}(k)\, p\cdot k}{i(p-k)^2 k^4} \ . \tag{337}$$

Using Appendix 2, Eq.(337) becomes

$$I_2 = \frac{1}{16\pi^2}\int \frac{dK}{K} B^{(2)}(K)\left[\left(\frac{P}{K}+\frac{K}{P}\right)\left\langle\frac{P}{K}\right\rangle - 1\right]$$

$$= \frac{1}{16\pi^2}\int_0^P dK \frac{K}{P^2} B^{(2)}(K) + \frac{1}{16\pi^2}\int_P^\infty dK \frac{P^2}{K^3} B^{(2)}(K) \ . \qquad (338)$$

As we shall show, $B^{(2)}(p^2)$ is of the following form

$$B^{(2)}(K) = A \ln K + B \ , \qquad (339)$$

where A and B are constants. Putting Eq.(339) in Eq.(338),

$$I_2 = \frac{1}{16\pi^2}\left(\int_0^P dK \frac{K}{P^2}(A\ln K + B) + \int_P^\infty dK \frac{P^2}{K^3}(A \ln K + B)\right)$$

$$= (A \ln P + B)\frac{1}{16\pi^2} = \frac{B^{(2)}(P^2)}{16\pi^2} . \qquad (340)$$

Putting Eq.(340) in Eq.(336),

$$T_{11} = \frac{64 B^{(2)} B^{(2)}(P^2)}{P^4} \ . \qquad (341)$$

Chapter 15

CALCULATION OF T_{12}

By definition, Eq.(70),

$$T_{12} = \left(4B'^{(4)}(p^2) + 4p^2 B'^{(2)2}(p^2) - 4B^{(2)}(p^2)\, B'^{(2)}(p^2)\right) Tr\, \gamma P G''^{(0)} \gamma P \; . \tag{342}$$

Using the Feynman rules,

$$Tr\, \gamma P G''^{(0)} \gamma P = \frac{4}{p^2} \; . \tag{343}$$

Putting Eq.(343) in Eq.(342), then

$$T_{12} = \frac{16}{p^4} \left(B'^{(4)}(p^2) + p^2 B'^{(2)2}(p^2) - B^{(2)}(p^2)\, B'^{(2)}(p^2)\right) . \tag{344}$$

Chapter 16

CALCULATION OF T_{13}

By definition, Eq.(70),

$$T_{13} = \left(-2\, b''^{(2)}(p^2)\, B'^{(2)}(p^2) - b''^{(4)}(p^2)\right) Tr\ \gamma G^{(0)} \gamma\ . \tag{345}$$

Using the Feynman rules,

$$Tr\ \gamma G^{(0)} \gamma = \frac{8}{p^2}\ . \tag{346}$$

Putting Eq.(346) in Eq.(345), then

$$T_{13} = \frac{8}{p^2}\left(-2\, b''^{(2)}(p^2)\, B'^{(2)}(p^2) - b''^{(4)}(p^2)\right)\ . \tag{347}$$

Chapter 17

CALCULATION OF T_{14}

By definition, Eq.(70),

$$T_{14} = -4 b''^{(2)}(p^2) B'^{(2)}(p^2) \, Tr \, \gamma P G^{(0)} \gamma P \, . \tag{348}$$

Using the Feynman rules,

$$Tr \, \gamma P G^{(0)} \gamma P = -4 \, . \tag{349}$$

Putting Eq.(349) in Eq.(348), then

$$T_{14} = 16 \, b''^{(2)}(p^2) B'^{(2)}(p^2) \, . \tag{350}$$

Chapter 18

$\sum T_i$ in TERMS OF $B^{(j)}$

Using Eqs.(52, 53)

$$b'^{(2)} = \left(B_\uparrow^{(2)} + B_\downarrow^{(2)}\right)'_{/K=0} = B'^{(2)}(p^2)\, 2p_\mu \frac{1}{2} + B'^{(2)}(p^2)\, 2p_\mu\left(-\frac{1}{2}\right) = 0 \;, \tag{351}$$

$$b''^{(2)} = 2B''^{(2)}(p^2)\, p^2 + 4B'^{(2)}(p^2) \;, \tag{352}$$

$$b'^{(4)} = 0 \;, \tag{353}$$

and

$$b''^{(4)} = 2p^2 B''^{(4)}(p^2) + 4B'^{(4)}(p^2) - 6p^2 B^{(2)}(p^2) B''^{(2)}(p^2) - 12 B^{(2)}(p^2) B'^{(2)}(p^2) - 2p^2 B'^{(2)}(p^2) \;. \tag{354}$$

Adding Eqs.(329, 84, 331, 341, 344, 347, 350) and using Eqs.(351-354), then

$$\sum_{j=8}^{14} T_j = 16B^{(2)}(p^2)\left(B''^{(2)}(p^2) + \frac{B'^{(2)}(p^2)}{p^2}\right) - 16\left(B''^{(4)}(p^2) + \frac{B'^{(4)}(p^2)}{p^2}\right)$$
$$- \frac{128}{p^2} B^{(2)} B'^{(2)}(p^2) + 96 B'^{(2)\,2}(p^2) + 32 B'^{(2)}(p^2) B''^{(2)}(p^2)\, p^2 \;. \tag{355}$$

As we shall see $B^{(2)}(p^2)$ and $B^{(4)}(p^2)$ are of the following form

$$B^{(2)}(p^2) = B^{(2)}\left[\sigma_{21} \ln\frac{p^2}{M^2} + \sigma_{20}\right] \;, \tag{356}$$

$$B^{(4)}(p^2) = B^{(2)\,2}\left[\sigma_{42}\left(\ln\frac{p^2}{M^2}\right)^2 + \sigma_{41}\ln\frac{p^2}{M^2} + \sigma_{40}\right] \;. \tag{357}$$

Then

$$B'^{(2)}(p^2) = B^{(2)} \frac{\sigma_{21}}{p^2} \;, \tag{358}$$

$$B''^{(2)}(p^2) = -B^{(2)}\frac{\sigma_{21}}{p^4}\,, \tag{359}$$

$$B'^{(4)}(p^2) = B^{(2)2}\left[\frac{2\sigma_{42}}{p^2}\ln\frac{p^2}{M^2} + \frac{\sigma_{41}}{p^2}\right], \tag{360}$$

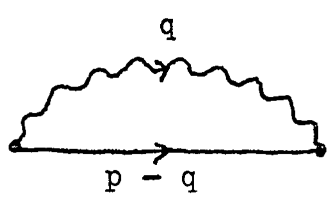
and

$$B''^{(4)}(p^2) = B^{(2)2}\left[-\frac{2\sigma_{42}}{p^4}\ln\frac{p^2}{M^2} + \frac{2\sigma_{42}}{p^4} - \frac{\sigma_{41}}{p^4}\right]. \tag{361}$$

Putting Eqs.(356-361) in Eq.(355), then

$$\sum_{j=1}^{14} T_j = -\frac{18}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^2\left(\sigma_{42} + 4\sigma_{21} - 2\sigma_{21}^{\,2}\right). \tag{362}$$

Note again that the logarithmic divergences ln(p/M) and $(\ln(p/M))^2$ conspired to cancel as they must. It is noteworthy also that this result, Eq.(362), depends only on the coefficients of the leading logarithm of $\Sigma^{(2)}$ and $\Sigma^{(4)}$. This is important as we shall not have to worry about ambiguities in σ_{20} and σ_{40}. For example in calculating $\Sigma^{(2)}$ one obtains different values for σ_{20} with the following two momentum assignments:



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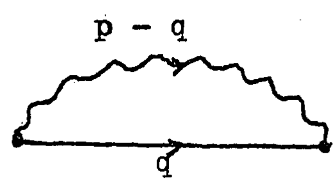

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and therefore these must be reconciled with a calculation of the vertex à la Ward's identity. Fortunately σ_{21} and σ_{42} are invariant under such momentum translations.

Chapter 19

SECOND ORDER ELECTRON SELF ENERGY AND σ_{21}

By definition, Eq.(47),

$$B^{(2)}(p^2) = -\frac{1}{i\gamma p}\Sigma^{(2)}(p^2), \tag{363}$$

where $\Sigma^{(2)}$ is the second order self energy of the electron. Since γ and p are the only independent vectors in the calculation of Eq.(363), and since $B^{(2)}(p^2)$ is a scalar, then $B^{(2)}(p^2)$ is of the following form:

$$B^{(2)}(p^2) = a(p^2) + b(p^2)\gamma p . \tag{364}$$

Taking the trace of both sides, then

$$a(p^2) = \frac{1}{4}\, Tr\, B^{(2)}(p^2) , \tag{365}$$

and

$$b(p^2) = 0 . \tag{366}$$

Therefore, using Eq.(363),

$$B^{(2)}(p^2) = -\frac{1}{4}\, Tr\, \frac{1}{i\gamma p}\Sigma^{(2)}(p) . \tag{367}$$

Using the Feynman rules, then

$$B^{(2)}(p^2) = -\frac{1}{4}\, Tr\, \frac{1}{i\gamma p}\int \frac{d^4 k}{(2\pi)^4}\; \frac{i e_0^2\, \gamma_\alpha \gamma(p-q)\gamma_\alpha}{i\,(p-k)^2 k^2} . \tag{368}$$

Using Appendix 1,

$$Tr\; \gamma_\alpha \gamma(p-k)\gamma_\alpha \gamma p = -8p\cdot(p-k) . \tag{369}$$

Putting Eq.(369) in Eq.(368), rotating contours, and using Eq.(83), then

$$B^{(2)}(p^2) = \frac{8}{3}\frac{B^{(2)}}{p^2}\left(I_3 - I_4\right) , \tag{370}$$

where

$$I_3 = \int dK \frac{d\Omega_K}{2\pi^2} \frac{p^2 K}{(p-K)^2} , \tag{371}$$

and

$$I_4 = \int dK \frac{d\Omega_K}{2\pi^2} \frac{K^2 p \cos pK}{(p-K)^2} . \tag{372}$$

Using Appendix 2 to expand Eq.(371) in Chebyshev polynomials,

$$\begin{aligned} I_3 &= \int dK \frac{d\Omega_K}{2\pi^2} p^2 K \sum_{n=0}^{\infty} \left(\begin{smallmatrix} 0 & 1 \\ p & p' \end{smallmatrix}\right)_n C_n(pK) \\ &= \int dK \frac{d\Omega_K}{2\pi^2} p \sum_{n=0}^{\infty} \left\langle \frac{p}{K} \right\rangle^{n+1} C_n(pK) C_0(pK) \\ &= \int dK\, p \left\langle \frac{p}{K} \right\rangle \\ &= p\left[\int_0^p dK \frac{K}{p} + \int_p^{\infty} dK \frac{p}{K} \right] \\ &= \frac{1}{2} p^2 + p^2 \ln \frac{M}{p} , \end{aligned} \tag{373}$$

where M is an infinite cutoff. Similarly Eq.(372) becomes:

$$I_4 = \frac{1}{8} p^2 + \frac{1}{2} p^2 \ln \frac{M}{p} . \tag{374}$$

Putting Eqs.(373, 374) in Eq.(370), then

$$B^{(2)}(p^2) = B^{(2)}\left(-\frac{2}{3} \ln\left(\frac{p}{M}\right)^2 + 1\right) . \tag{375}$$

Inserting Eq.(375) in Eq.(363) and using Eq.(83), we observe that

$$\Sigma^{(2)} = \frac{\gamma p}{i} \frac{3}{32} \frac{e_0^2}{\pi^2} \left(1 - \frac{2}{3} \ln \left(\frac{p}{M} \right)^2 \right) ,$$

which agrees with the $m_0 = 0$ limit of the well known result.[48] Finally comparing Eq.(375) with Eq.(356),

$$\sigma_{21} = -\frac{2}{3} \cdot \tag{376}$$

Chapter 20

FOURTH ORDER ELECTRON SELF ENERGY AND σ_{42}

By definition, Eq.(48),

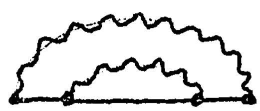
$$B^{(4)}(p^2) = -\frac{1}{i\gamma p}\Sigma^{(4)}(p)\,, \tag{377}$$

where $\Sigma^{(4)}$ is the fourth order self energy of the electron, excluding the photon self energy part. Let

$$\Sigma^{(4)} = \Sigma_1^{(4)} + \Sigma_2^{(4)} + \Sigma_3^{(4)}\,, \tag{378}$$

where

$$\Sigma_1^{(4)} = \tag{379}$$



,

$$\Sigma_2^{(4)} =$$

,

(380)

and

$$\Sigma_3^{(4)} =$$

.

(381)

Accordingly we define

$$B_j^{(4)}(p^2) = -\frac{1}{i\gamma p}\Sigma_j^{(4)}(p^2)\,, \tag{382}$$

and

$$B^{(4)}(p^2) = \sum_{j=1}^{3} B_j^{(4)}(p^2)\,. \tag{383}$$

As we argued for $B^{(2)}(p^2)$, also

$$B_i^{(4)}(p^2) = \frac{1}{4} Tr\, B_i^{(4)}(p^2) . \tag{384}$$

Using Eqs.(379, 384) and the Feynman rules, then

$$B_1^{(4)}(p^2) = -\frac{1}{4} Tr \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} (ie_0^2)^2 \gamma_\alpha \frac{1}{i\gamma k} \gamma_\beta \frac{1}{i\gamma(k-q)} \gamma_\beta \frac{1}{i\gamma k} \gamma_\alpha \frac{1}{(p-k)^2 q^2} \frac{1}{i\gamma p}$$

$$= e_0^4\, Tr \frac{1}{\gamma p} \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} \frac{\gamma k \gamma(k-q) \gamma k}{k^2 (k-q)^2 k^2 (p-k)^2 q^2} . \tag{385}$$

Using Appendix 1 to take the trace and Appendix 2 to expand in Chebyshev polynomials,

$$B_1^{(4)}(p^2) = -\frac{4e_0^4}{64\pi^4 p^2} \sum_{n=0}^{\infty} \frac{1}{n+1} \int dk\, dq \left\{ k^2 pq \begin{pmatrix} 1 & 1 \\ p & k \end{pmatrix}_n \begin{pmatrix} 0 & 1 \\ q & k \end{pmatrix}_n \begin{pmatrix} 0 & 0 \\ p & q \end{pmatrix}_n \right.$$

$$\left. + kpq^2 \begin{pmatrix} 0 & 1 \\ p & k \end{pmatrix}_n \begin{pmatrix} 0 & 1 \\ q & k \end{pmatrix}_n \begin{pmatrix} 1 & 0 \\ p & q \end{pmatrix}_n - 2kpq^2 \begin{pmatrix} 1 & 1 \\ p & k \end{pmatrix}_n \begin{pmatrix} 1 & 1 \\ q & k \end{pmatrix}_n \begin{pmatrix} 0 & 0 \\ p & q \end{pmatrix}_n \right\}$$

$$\equiv -\frac{4e_0^4}{64\pi^4 p^2} \sum_{n=0}^{\infty} b_{1n}^{(4)} . \tag{386}$$

Then using Appendix 2

$$b_{10}^{(4)} = \int dk\, dq \left\{ k^2 pq \frac{1}{2pk} \left\langle \frac{p}{k} \right\rangle^2 \frac{1}{qk} \left\langle \frac{q}{k} \right\rangle - 2kq^2 p \frac{1}{2pk} \left\langle \frac{p}{k} \right\rangle^2 \frac{1}{2qk} \left\langle \frac{q}{k} \right\rangle^2 \right\} . \tag{387}$$

Symmetrizing in k and q,

$$b_{10}^{(4)} = \int_{k<q} dk\, dq \left(\frac{1}{2} \left\langle \frac{p}{k} \right\rangle^2 \left\langle \frac{q}{k} \right\rangle - \frac{1}{2} \frac{q}{k} \left\langle \frac{p}{k} \right\rangle^2 \left\langle \frac{q}{k} \right\rangle^2 \right.$$

$$\left. + \frac{1}{2} \left\langle \frac{p}{q} \right\rangle^2 \left\langle \frac{q}{k} \right\rangle - \frac{1}{2} \frac{k}{q} \left\langle \frac{p}{q} \right\rangle^2 \left\langle \frac{q}{k} \right\rangle^2 \right) . \tag{388}$$

Define

$$b_{10}^{(4)} = \sum_{j=1}^{3} b_{10}^{(4)}(R_j), \tag{389}$$

where $b_{10}^{(4)}(R_i)$ are the integrals evaluated in regions R_1, R_2, and R_3:

$$\begin{aligned} R_1&: \; k < q < p \\ R_2&: \; k < p < q \\ R_3&: \; p < k < q \end{aligned} \quad . \tag{390}$$

Proceeding then:

$$b_{10}^{(4)}(R_1) = \int_0^p dK \int_0^K dq \left[\frac{1}{2}\left(\frac{K}{p}\right)^2\left(\frac{q}{K}\right) - \frac{1}{2}\frac{q}{K}\left(\frac{K}{p}\right)^2\left(\frac{q}{K}\right)^2 + \frac{1}{2}\left(\frac{q}{p}\right)^2\left(\frac{q}{K}\right) - \frac{1}{2}\frac{K}{q}\left(\frac{q}{p}\right)^2\left(\frac{q}{K}\right)^2\right]$$

$$= \frac{1}{32} p^2 \quad , \tag{391}$$

$$b_{10}^{(4)}(R_2) = \int_0^\infty dK \int_0^p dq \left[\frac{1}{2}\left(\frac{p}{K}\right)^2\left(\frac{q}{K}\right) - \frac{1}{2}\frac{q}{K}\left(\frac{p}{K}\right)^2\left(\frac{q}{K}\right)^2 + \frac{1}{2}\left(\frac{q}{p}\right)^2\left(\frac{q}{K}\right) - \frac{1}{2}\frac{K}{q}\left(\frac{q}{p}\right)^2\left(\frac{q}{K}\right)^2\right]$$

$$= \frac{3}{32} p^2 , \tag{392}$$

and

$$b_{10}^{(4)}(R_3) = \int_p^\infty dq \int_q^\infty dK \left[\frac{1}{2}\left(\frac{p}{K}\right)^2\left(\frac{q}{K}\right) - \frac{1}{2}\frac{q}{K}\left(\frac{p}{K}\right)^2\left(\frac{q}{K}\right)^2 + \frac{1}{2}\left(\frac{p}{q}\right)^2\left(\frac{q}{K}\right) - \frac{1}{2}\frac{K}{q}\left(\frac{p}{q}\right)^2\left(\frac{q}{K}\right)^2\right]$$

$$= \frac{1}{8} p^2 \ln\frac{M}{p} , \tag{393}$$

where M is an infinite cutoff. Further

$$b_{11}^{(4)} = \frac{1}{2}\int dk \int dq\, kpq^2 \frac{1}{pk}\left\langle\frac{p}{k}\right\rangle^2 \frac{1}{qk}\left\langle\frac{q}{k}\right\rangle^2 \frac{1}{2} . \tag{394}$$

Again symmetrizing in k and q, then

$$b_{11}^{(4)} = \frac{1}{4}\int dk \int dq \left(\frac{q}{k}\left\langle\frac{p}{k}\right\rangle^2\left\langle\frac{q}{k}\right\rangle^2 + \frac{k}{q}\left\langle\frac{p}{q}\right\rangle^2\left\langle\frac{q}{k}\right\rangle^2\right) . \tag{395}$$

Again let

$$b_{11}^{(4)} = \sum_{j=1}^{3} b_{11}^{(4)}(R_j) . \tag{396}$$

We obtain

$$b_{11}^{(4)}(R_1) = \frac{1}{32}p^2 , \tag{397}$$

$$b_{11}^{(4)}(R_2) = \frac{1}{64}p^2 + \frac{1}{16}p^2 \ln\frac{M}{p} , \tag{398}$$

and

$$b_{11}^{(4)}(R_3) = \frac{1}{16}p^2 \ln\frac{M}{p} + \frac{1}{8}p^2\left(\ln\frac{M}{p}\right)^2 . \tag{399}$$

There are no further contributions to $B_1^{(4)}$, i.e. $b_{1i}^{(4)} = 0$ for $i > 1$. Combining Eqs.(391-393) and Eqs.(397-399), then

$$B_1^{(4)}(p^2) = B^{2(2)}\left(-\frac{11}{3} + \frac{8}{3}\ln\left(\frac{p}{M}\right)^2 - \frac{2}{3}\left(\ln\left(\frac{p}{M}\right)^2\right)^2\right) . \tag{400}$$

Similarly calculating $B_2^{(4)}$ from Eq.(380), we have

$$B_2^{(4)}(p^2) = -\frac{1}{i\gamma p}\int\frac{d^4k}{(2\pi)^4}\int\frac{d^4q}{(2\pi)^4}(ie_0^2)^2\gamma_\alpha\frac{1}{i\gamma k}\gamma_\beta\frac{1}{i\gamma(k-q)}\gamma_\alpha\frac{1}{i\gamma(p-q)}\gamma_\beta\frac{1}{(p-k)^2q^2} , \tag{401}$$

or

$$B_2^{(4)}(p^2) = -\frac{2e_0^4}{p^4}\int\frac{d^4k}{(2\pi)^4}\int\frac{d^4q}{(2\pi)^4}\frac{Tr\,\gamma(k-q)\gamma p\, k\cdot(p-q)}{k^2(k-q)^2(p-q)^2(p-k)^2q^2} . \tag{402}$$

Using Appendix 1 to expand the trace, and rotating contours,

$$B_2^{(4)}(p^2) = \frac{8e_0^4}{64\pi^4 p^2} \int K^3 dK q^3 dq \frac{d\Omega_q}{2\pi^2} \frac{d\Omega_K}{2\pi^2} \frac{(p\cdot K)^2 - p\cdot K p\cdot q - K\cdot q p\cdot K + K\cdot q p\cdot q}{K^2 (K-q)^2 (p-q)^2 (p-K)^2 q^2} \quad . \tag{403}$$

Using Appendix 2 to expand in Chebyshev polynomials, then

$$B_2^{(4)}(p^2) = \frac{8e_0^4}{64\pi^4 p^2} \sum_{n=0}^{\infty} \frac{1}{n+1} \int dK dq \Big[p^2 K^3 q \begin{pmatrix} 21 \\ pK \end{pmatrix}_n \begin{pmatrix} 01 \\ qK \end{pmatrix}_n \begin{pmatrix} 01 \\ pq \end{pmatrix}_n$$

$$- p^2 K^2 q^2 \begin{pmatrix} 11 \\ qK \end{pmatrix}_n \begin{pmatrix} 01 \\ qK \end{pmatrix}_n \begin{pmatrix} 11 \\ pq \end{pmatrix}_n - K^3 q^2 p \begin{pmatrix} 11 \\ pK \end{pmatrix}_n \begin{pmatrix} 11 \\ qK \end{pmatrix}_n \begin{pmatrix} 01 \\ pq \end{pmatrix}_n + K^3 q^3 p \begin{pmatrix} 01 \\ pK \end{pmatrix}_n \begin{pmatrix} 11 \\ qK \end{pmatrix}_n \begin{pmatrix} 11 \\ pq \end{pmatrix}_n \Big]$$

$$= \frac{8e_0^4}{64\pi^4 p^2} \sum_{n=0}^{\infty} b_{2n}^{(4)} \quad . \tag{404}$$

Using Eq.(404) and Appendix 2, then

$$b_{20}^{(4)} = \int dK dq \Big[\Big\{ \frac{1}{4} \frac{p}{q} \left\langle \frac{p}{K} \right\rangle^2 \left\langle \frac{q}{K} \right\rangle \left\langle \frac{p}{K} \right\rangle + \frac{1}{4} \frac{K^2}{pq} \left\langle \frac{p}{K} \right\rangle^2 \left\langle \frac{q}{K} \right\rangle \left\langle \frac{p}{q} \right\rangle$$

$$- \frac{1}{4} \left\langle \frac{p}{K} \right\rangle^2 \left\langle \frac{q}{K} \right\rangle \left\langle \frac{p}{q} \right\rangle^2 - \frac{1}{4} \frac{K}{p} \left\langle \frac{p}{K} \right\rangle^2 \left\langle \frac{q}{K} \right\rangle^2 \left\langle \frac{p}{q} \right\rangle$$

$$+ \frac{1}{4} \frac{q}{p} \left\langle \frac{p}{K} \right\rangle \left\langle \frac{q}{K} \right\rangle^2 \left\langle \frac{p}{q} \right\rangle^2 \Big\} + \Big\{ q \leftrightarrow K \Big\} \Big] \quad . \tag{405}$$

Evaluating Eq.(405) in regions R_1, R_2, and R_3 then

$$b_{20}^{(4)}(R_1) = \left(\frac{3}{64} + \frac{1}{144} \right) p^2 , \tag{406}$$

$$b_{20}^{(4)}(R_2) = \left(\frac{1}{48} + \frac{1}{32} \right) p^2 + \frac{1}{8} p^2 \ln \frac{M}{p} \ , \tag{407}$$

and

$$b_{20}^{(4)}(R_3) = \frac{1}{8}p^2\left(\ln\frac{M}{p}\right)^2 + \frac{1}{8}p^2\ln\frac{M}{p}\,. \tag{408}$$

For n > 0 we separate the Kronecker delta terms from the rest, hence

$$b_{2n}^{(4)}(R_1) = -\frac{1}{64}p^2\delta_{n1} + p^2\Big[-\frac{1}{16}\left(\frac{1}{n+1}-\frac{1}{n+3}\right)+\frac{1}{8}\left(\frac{1}{n+2}-\frac{1}{n+3}\right)$$

$$+\frac{1}{64}\left(\frac{1}{n+1}-\frac{1}{n+3}\right)-\frac{1}{32}\frac{1}{(n+3)^2}+\frac{1}{32}\frac{1}{(n+1)^2}-\frac{1}{64}\left(\frac{1}{n+1}-\frac{1}{n+3}\right)\Big], \tag{409}$$

$$b_{2n}^{(4)}(R_2) = \left(-\frac{1}{32}p^2\ln\frac{M}{p}-\frac{1}{128}p^2\right)\delta_{n1} + p^2\Big[-\frac{1}{16}\left(\frac{1}{n}-\frac{1}{n+2}\right)+\frac{1}{8}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)$$

$$+\frac{1}{32}\frac{1}{(n+1)^2}-\frac{1}{64}\left(\frac{1}{n+1}-\frac{1}{n+3}\right)+\frac{1}{64}\left(\frac{1}{n-1}-\frac{1}{n+1}\right)(1-\delta_{n1})-\frac{1}{32}\frac{1}{(n+1)^2}(1-\delta_{n1})+\frac{1}{32}\ln\frac{M}{p}\Big], \tag{410}$$

and

$$b_{2n}^{(4)}(R_3) = -\frac{1}{16}p^2\left(\ln\frac{M}{p}\right)^2-\frac{1}{32}p^2\ln\frac{M}{p}+p^2\Big[\frac{1}{8}\left(\frac{1}{n-1}-\frac{1}{n}\right)+\frac{1}{16}\left(\frac{1}{n-1}-\frac{1}{n+1}\right)$$

$$+\frac{1}{64}\left(\frac{1}{n-1}-\frac{1}{n+1}\right)-\frac{1}{32(n+1)^2}+\frac{1}{32(n-1)^2}-\frac{1}{64}\left(\frac{1}{n-1}-\frac{1}{n+1}\right)-\frac{1}{8}\ln\frac{M}{p}+\frac{1}{32}\ln\frac{M}{p}+\frac{1}{16}\left(\ln\frac{M}{p}\right)^2\Big]. \tag{411}$$

Putting Eqs.(406–408, 409–411) in Eq.(404), then

$$B_2^{(4)}(p^2) = B^{(2)\,2}\left[\frac{4}{3}\left(\ln\left(\frac{p}{M}\right)^2\right)^2-\frac{8}{3}\ln\left(\frac{p}{M}\right)^2+\frac{10}{3}\right]. \tag{412}$$

We obtain $B_3^{(4)}$ from Eq.(381) by using our result for $B^{(2)}(p^2)$, Eq.(375); hence:

$$B_3^{(4)} = -\frac{1}{i\gamma p}\left[-\frac{2}{3}i\gamma p B^{(2)}\left(\ln\frac{M^2}{p^2}+\frac{3}{2}\right)\right]\frac{1}{i\gamma p}\left[-\frac{2}{3}i\gamma p B^{(2)}\left(\ln\frac{M^2}{p^2}+\frac{3}{2}\right)\right], \tag{413}$$

or

$$B_3^{(4)} = B^{(2)\,2}\left[-\frac{4}{9}\left(\ln\left(\frac{p}{M}\right)^2\right)^2+\frac{4}{3}\ln\left(\frac{p}{M}\right)^2-1\right]. \tag{414}$$

Inserting Eqs.(400, 412, 414) in Eq.(383), then

$$B^{(4)}(p^2) = B^{2}_{(2)}\left[\frac{2}{9}\left(\ln\left(\frac{p}{M}\right)^2\right)^2 + \frac{4}{3}\ln\left(\frac{p}{M}\right)^2 - \frac{4}{3}\right] . \tag{415}$$

Comparing Eqs.(357, 415), then

$$\sigma_{42} = \frac{2}{9} . \tag{416}$$

Chapter 21

$f^{(6)}(\alpha_0)$ RESULT

Putting Eqs.(376, 416) in Eq.(362), we obtain:

$$\sum_{j=8}^{14} T_j = \frac{60}{\rho^4}\left(\frac{\alpha_0}{2\pi}\right)^2. \tag{417}$$

Adding Eqs.(113, 145, 191, 206, 215, 276, 321), then

$$\sum_{j=1}^{7} T_j = -\frac{48}{\rho^4}\left(\frac{\alpha_0}{2\pi}\right)^2. \tag{418}$$

Adding Eqs.(417, 418), then

$$\sum_{j=1}^{14} T_j = \frac{12}{\rho^4}\left(\frac{\alpha_0}{2\pi}\right)^2. \tag{419}$$

Comparing Eqs.(37, 70), we see that

$$f^{(6)}(\alpha_0) = -\frac{\rho^4}{48}\left(\frac{\alpha_0}{2\pi}\right)\sum_{j=1}^{14} T_j. \tag{420}$$

Putting Eq.(419) in Eq.(420), then finally

$$f^{(6)}(\alpha_0) = -\frac{1}{4}\left(\frac{\alpha_0}{2\pi}\right)^3, \tag{421}$$

which agrees with Rosner.

Chapter 22

COMPARISON WITH ROSNER'S CALCULATION

As stated previously, Rosner's calculation was performed in the generalized Landau gauge

$$\mathcal{D}_{\mu\nu} = \left(g_{\mu\nu} - \lambda \frac{k_\mu k_\nu}{k^2}\right) D(k^2) \quad ,$$

with a particular choice of λ such that vertex and self mass insertions are finite. In this gauge $f^{(6)}(\alpha_0)$ is given by[30]

$$f^{(6)}(\alpha_0) = -\frac{p^4}{48}\left(\frac{\alpha_0}{2\pi}\right)\sum_{i=1}^{7} T_i \quad , \tag{422}$$

i.e. the T_i for $i > 7$ are vanishing. The T_i will have both pieces indepependent and dependent on λ :

$$T_i \equiv T_i(0) + T_i(\lambda) \ . \tag{423}$$

Comparing Rosner's expression, Eq.(422), with Eq.(420), gauge invariance requires

$$\sum_{i=1}^{7} T_i^R = \sum_{i=1}^{14} T_i^F \ , \tag{424}$$

where the superscripts R and F denote Rosner's and Feynman gauge respectively. Inserting Eq.(423) in Eq.(424) then

$$\sum_{i=1}^{7} T_i^R(0) + \sum_{i=1}^{7} T_i^R(\lambda) = \sum_{i=1}^{14} T_i^F \ . \tag{425}$$

Further, by definition we note that

$$T_i^R(0) = T_i^F \ . \tag{426}$$

Putting Eq.(426) in Eq.(425), we conclude that

$$\sum_{j=1}^{7} T_j^R(\lambda) = \sum_{j=8}^{14} T_j^F . \tag{427}$$

From Rosner's paper[30] we note that

$$-\frac{p^4}{48}\left(\frac{\alpha_0}{2\pi}\right)\sum_{j=1}^{7} T_j^R(\lambda) = -\frac{5}{4}\left(\frac{\alpha_0}{2\pi}\right)^3 , \tag{428}$$

or

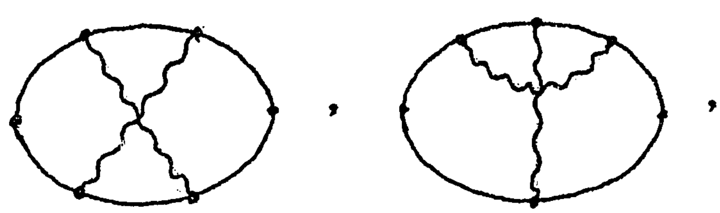
$$\sum_{j=1}^{7} T_j^R(\lambda) = \frac{60}{p^4}\left(\frac{\alpha_0}{2\pi}\right)^3 , \tag{429}$$

which is the same as Eq.(417), thus confirming Eq.(427).

Returning to our calculation it is further interesting to note the following. Comparing Eq.(418) with T_1, Eq.(113), we see that

$$\sum_{j=2}^{7} T_j = 0 \quad . \tag{430}$$

These terms are all graphs containing crossed photon lines, viz.



We conclude that in sixth order in Feynman gauge, graphs with crossed photon lines (excluding the fourth order self energy graph) make no contribution. It would be of interest to see if this behavior persists in higher order.

Let us examine the cancellations in Eq.(430) more closely. We first recall from Eqs.(206, 215) that T_4 and T_5

are identically zero:

$$T_4 = 4\,Tr\,\gamma G'^{(0)} K_b'^{(4)} G'^{(0)}\gamma = 0\ , \tag{431}$$

$$T_5 = 8\,Tr\,\gamma G'^{(0)} K_c'^{(4)} G'^{(0)}\gamma = 0\ . \tag{432}$$

Adding Eqs.(145, 191), we see that

$$\begin{aligned} T_2 + T_3 &= 2\,Tr\,\gamma G'^{(0)} K_b^{(4)} G'^{(0)}\gamma \\ &\quad + 4\,Tr\,\gamma G'^{(0)} \left[K_c^{(4)} - \bar{K}^{(4)}\right] G'^{(0)}\gamma \\ &= 96\left(\zeta(3) - \frac{5}{6}\right)\ . \end{aligned} \tag{433}$$

Similarly, adding Eqs.(276, 321),

$$\begin{aligned} T_6 + T_7 &= Tr\,\gamma G'^{(0)} K_b''^{(4)} G'^{(0)}\gamma + 2\,Tr\,\gamma G'^{(0)} K_c''^{(4)} G'^{(0)}\gamma \\ &= -96\left(\zeta(3) - \frac{5}{6}\right) = -\left(T_2 + T_3\right)\ . \end{aligned} \tag{434}$$

Equations (431-434) give us some understanding of the cancellations implicit in Eq.(430).

Chapter 23

CONCLUSION

We have calculated the sixth order contribution to the divergent part of Z_3^{-1} from the vacuum polarization tensor. The result is

$$\left(Z_3^{-1}\right)^{(6)}_{\text{div.}} = f^{(6)}(\alpha_0)\ln\frac{M^2}{m^2} = -\frac{1}{4}\left(\frac{\alpha_0}{2\pi}\right)^3 \ln\frac{M^2}{m^2} ,$$

which agrees with Rosner. The calculation was performed in the Feynman gauge, giving rise to simplifications not present in the generalized Landau gauge employed by Rosner. To be sure, logarithmic divergences arising from vertex and electron self energy insertions, and Riemann zeta functions arising from crossed photon graphs, appeared at intermediate stages of the calculation, but cancelled in the final result. The integrals are both fewer and simpler in Feynman gauge.

It is of interest to see where the root of Eq.(10) lies: Eq.(10) vanishes for $\alpha_0 = 4\pi(1+\sqrt{5/3})$. For this value of α_0 , $f'(\alpha_0)$ is in fact negative; however it is clear from this value of α_0 that the series has not begun to converge sufficiently in sixth order. Therefore, still higher order calculations of the fundamental function $f(\alpha_0)$ are essential.

The observations made in Chapter 22 might lead one to expect that the only graphs which will contribute to $f^{(8)}(\alpha_0)$ are the simple three-photon-exchange ladder graph along with still simpler graphs arising from the momentum dependence of

$B^{(i)}(p^2)$ in Feynman gauge. This problem is under investigation.

It is desirable to explore nonperturbative methods in calculating $f(\alpha_0)$. The work of V. Blank and V. Sudakov[49–50] is suggestive, except for the fact that there appears to be no most important region of integration in the problem.

The recent work of A. Abdellatif[31] attempts to develope a scheme for calculating $f(\alpha_0)$ to all orders, by using certain invariance properties under the conformal group. The Jost-Luttinger result is reproduced, but ambiguities are encountered in sixth order. There is still some hope that this approach will yield a closed form for $f(\alpha_0)$.

The possibility of a finite canonical quantum electrodynamics rests on a calculation of the fundamental function $f(\alpha_0)$ to all orders, and the determination of its roots.

" Hier sitzt er, unser lieber Planck, und lächelt innerlich über dies mein kindliches Hantieren mit der Laterne des Diogenes. Unsere Sympathie für ihn bedarf keiner fadenscheinigen Begründung. Möge die Liebe zur Wissenschaft auch in Zukunft seinen Lebensweg verschönern und ihn zu der Lösung des von ihm selbst gestellten, mächtig geförderten wichtigsten physikalischen Problems der Gegenwart fuhren! Möge es ihm gelingen, die Quantentheorie mit der Elektrodynamik und Mechanik zu einem logish einheitlichen System zu vereinigen." ....
.......Albert Einstein Rede zum 60. Geburtstag von Max Planck.[51]

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Appendix 1

DIRAC ALGEBRA

From the Dirac commutation relations,

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$$

we obtain the following relations:

$$\gamma_\mu \gamma^\mu = 4$$

$$\gamma_\mu \gamma a \gamma^\mu = -2\gamma a$$

$$\gamma_\mu \gamma a \gamma b \gamma^\mu = 4a\cdot b$$

$$\gamma_\mu \gamma a \gamma b \gamma c \gamma^\mu = -2\gamma c \gamma b \gamma a$$

$$\gamma_\mu \gamma a \gamma b \gamma c \gamma d \gamma^\mu = 2(\gamma d \gamma a \gamma b \gamma c + \gamma c \gamma b \gamma a \gamma d)$$

$$Tr\, \gamma a \gamma b = 4a\cdot b$$

$$Tr\, \gamma a_1 \gamma a_2 \ldots \gamma a_{2n+1} = 0$$

$$Tr\, \gamma a_1 \gamma a_2 \gamma a_3 \gamma a_4 = 4(a_1\cdot a_2\, a_3\cdot a_4 + a_1\cdot a_4\, a_2\cdot a_3 - a_1\cdot a_3\, a_2\cdot a_4)$$

Appendix 2

CHEBYSHEV POLYNOMIALS

The Chebyshev coefficients $\binom{m\,K}{p\,p'}_n$ are defined by

$$\frac{\cos^m pp'}{(p-p')^{2K}} = \sum_{n=0}^{\infty} \binom{m\,K}{p\,p'}_n C_n(pp') , \qquad \text{A2-1}$$

where C_n is the nth order Chebyshev polynomial defined by

$$C_n(\theta) = \frac{\sin(n+1)\theta}{\sin\theta} . \qquad \text{A2-2}$$

The C_n obey the following orthogonality relation:

$$\int \frac{d\Omega_q}{2\pi^2} C_n(pq) C_m(p'q) = \frac{\delta_{nm}}{n+1} C_n(pp') . \qquad \text{A2-3}$$

We define

$$\begin{bmatrix} n_1 l_1 , n_2 l_2 , n_3 l_3 \\ pp' , p'q , pq \end{bmatrix} \equiv \int \frac{d\Omega_{p'}}{2\pi^2} \int \frac{d\Omega_q}{2\pi^2} \frac{\cos^{n_1} pp' \cos^{n_2} p'q \cos^{n_3} pq}{(p-p')^{2l_1} (p'-q)^{2l_2} (p-q)^{2l_3}}$$

$$= \sum_{rst=0}^{\infty} \int \frac{d\Omega_{p'}}{2\pi^2} \int \frac{d\Omega_q}{2\pi^2} \binom{n_1 l_1}{pp'}_r \binom{n_2 l_2}{p'q}_s \binom{n_3 l_3}{pq}_t C_r(pp') C_s(p'q) C_t(pq)$$

$$= \sum_{rst=0}^{\infty} \binom{n_1 l_1}{pp'}_r \binom{n_2 l_2}{p'q}_s \binom{n_3 l_3}{pq}_t \frac{\delta_{rs} \, C_r(pp) \, \delta_{st}}{r+1 \qquad s+1}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{n_1 l_1}{pp'}_n \binom{n_2 l_2}{p'q}_n \binom{n_3 l_3}{pq}_n , \qquad \text{A2-4}$$

where we have used A2-1, A2-3 respectively, and used A2-2 to note that

$$C_n(0) = \lim_{\theta\to 0} \frac{\sin(n+1)\theta}{\sin\theta} = \lim_{\theta\to 0} (n+1)\frac{\cos(n+1)\theta}{\cos\theta} = n+1 \quad . \qquad \text{A2-5}$$

The Chebyshev polynomials are also given by

$$(1-2tz+t^2)^{-1} \equiv \sum_{n=0}^{\infty} t^n C_n(z) , \qquad \text{A2-6}$$

and satisfy the following recurrence relation:

$$2z\,C_n(z) = C_{n+1}(z) + C_{n-1}(z) \ , \quad n \geq 1 \quad . \qquad \text{A2-7}$$

Denoting

$$\left\langle \frac{\rho}{\rho'} \right\rangle = \begin{cases} \rho/\rho' \ , & \rho<\rho' \\ \rho'/\rho \ , & \rho'<\rho \end{cases}$$

and using A2-6 and A2-7, we obtain the following Chebyshev coefficients:

$$\begin{pmatrix} 0 & 1 \\ \rho & \rho' \end{pmatrix}_0 = \frac{1}{\rho\rho'}\left\langle\frac{\rho}{\rho'}\right\rangle ,$$

$$\begin{pmatrix} 1 & 1 \\ \rho & \rho' \end{pmatrix}_0 = \frac{1}{2\rho\rho'}\left\langle\frac{\rho}{\rho'}\right\rangle^2 ,$$

$$\begin{pmatrix} 2 & 1 \\ \rho & \rho' \end{pmatrix}_0 = \frac{1}{4\rho\rho'}\left(\frac{\rho}{\rho'}+\frac{\rho'}{\rho}\right)\left\langle\frac{\rho}{\rho'}\right\rangle^2 ,$$

$$\begin{pmatrix} 3 & 1 \\ \rho & \rho' \end{pmatrix}_0 = \frac{1}{8\rho\rho'}\left(\left\langle\frac{\rho}{\rho'}\right\rangle^4 + 2\left\langle\frac{\rho}{\rho'}\right\rangle^2\right),$$

$$\binom{4\ 1}{p\ p'}_0 = \frac{1}{16pp'}\left(\left\langle\frac{p}{p'}\right\rangle^5 + 3\left\langle\frac{p}{p'}\right\rangle^3 + 2\left\langle\frac{p}{p'}\right\rangle\right),$$

$$\binom{0\ 2}{p\ p'}_0 = \frac{1}{pp'|p^2-p'^2|}\left\langle\frac{p}{p'}\right\rangle,$$

$$\binom{1\ 2}{p\ p'}_0 = \frac{1}{pp'|p^2-p'^2|}\left\langle\frac{p}{p'}\right\rangle^2,$$

$$\binom{2\ 2}{p\ p'}_0 = \frac{1}{4pp'|p^2-p'^2|}\left(3\left\langle\frac{p}{p'}\right\rangle^3 + \left\langle\frac{p}{p'}\right\rangle\right),$$

$$\binom{3\ 2}{p\ p'}_0 = \frac{1}{2pp'|p^2-p'^2|}\left(\left\langle\frac{p}{p'}\right\rangle^4 + \left\langle\frac{p}{p'}\right\rangle^2\right),$$

$$\binom{0\ 0}{p\ p'}_n = \delta_{n0},$$

$$\binom{1\ 0}{p\ p'}_n = \frac{1}{2}\delta_{n1},$$

$$\binom{0\ 1}{p\ p'}_n = \frac{1}{pp'}\left\langle\frac{p}{p'}\right\rangle^{n+1},$$

$$\binom{1\ 1}{p\ p'}_n = \frac{1}{2pp'}\left\{\left(\frac{p}{p'}+\frac{p'}{p}\right)\left\langle\frac{p}{p'}\right\rangle^{n+1} - \delta_{n0}\right\},$$

$$\binom{2\ 1}{p\ p'}_n = \frac{1}{4pp'}\left\{\left(\frac{p}{p'}+\frac{p'}{p}\right)^2\left\langle\frac{p}{p'}\right\rangle^{n+1} - \delta_{n1} - \delta_{n0}\left(\frac{p}{p'}+\frac{p'}{p}\right)\right\},$$

$$\binom{3\ 1}{p\ p'}_n = \frac{1}{8pp'}\left\{\left(\frac{p}{p'}+\frac{p'}{p}\right)^3\left\langle\frac{p}{p'}\right\rangle^{n+1} - \delta_{n2} - \delta_{n1}\left(\frac{p}{p'}+\frac{p'}{p}\right) - \delta_{n0}\left(3+\left\langle\frac{p}{p'}\right\rangle^2+\left\langle\frac{p}{p'}\right\rangle^{-2}\right)\right\},$$

$$\binom{4\,1}{\rho\rho'}_n = \frac{1}{16\rho\rho'}\left\{\left(\frac{\rho}{\rho'}+\frac{\rho'}{\rho}\right)^4\left\langle\frac{\rho}{\rho'}\right\rangle^{n+1} - \delta_{n3} - \delta_{n2}\left(\frac{\rho}{\rho'}+\frac{\rho'}{\rho}\right) - \delta_{n1}\left(\left\langle\frac{\rho}{\rho'}\right\rangle^2 + 4 + \left\langle\frac{\rho'}{\rho}\right\rangle^{-2}\right)\right.$$

$$\left. - \delta_{n0}\left(\left\langle\frac{\rho}{\rho'}\right\rangle^3 + \left\langle\frac{\rho'}{\rho}\right\rangle^3 + 4\left\langle\frac{\rho}{\rho'}\right\rangle + 4\left\langle\frac{\rho}{\rho'}\right\rangle^{-1}\right)\right\},$$

$$\binom{0\,2}{\rho\rho'}_n = \frac{n+1}{\rho\rho'|\rho^2-\rho'^2|}\left\langle\frac{\rho}{\rho'}\right\rangle^{n+1},$$

$$\binom{1\,2}{\rho\rho'}_n = \frac{1}{2\rho\rho'|\rho^2-\rho'^2|}\left\{(n+1)\left(\frac{\rho}{\rho'}+\frac{\rho'}{\rho}\right) + \left\langle\frac{\rho}{\rho'}\right\rangle - \left\langle\frac{\rho}{\rho'}\right\rangle^{-1}\right\}\left\langle\frac{\rho}{\rho'}\right\rangle^{n+1},$$

$$\binom{2\,2}{\rho\rho'}_n = \frac{1}{4\rho\rho'}\left\{(n+1)\left(\frac{\rho}{\rho'}+\frac{\rho'}{\rho}\right)^2\left\langle\frac{\rho}{\rho'}\right\rangle^{n+1} + 2\left(\left\langle\frac{\rho}{\rho'}\right\rangle^2 - \left\langle\frac{\rho}{\rho'}\right\rangle^{-2}\right)\left\langle\frac{\rho}{\rho'}\right\rangle^{n+1}\right.$$

$$\left. + \delta_{n0}\left(\left\langle\frac{\rho}{\rho'}\right\rangle^{-1} - \left\langle\frac{\rho}{\rho'}\right\rangle\right)\right\}\frac{1}{|\rho^2-\rho'^2|},$$

$$\binom{3\,2}{\rho\rho'}_n = \frac{1}{8\rho\rho'|\rho^2-\rho'^2|}\left\{(n+1)\left(\frac{\rho}{\rho'}+\frac{\rho'}{\rho}\right)^3\left\langle\frac{\rho}{\rho'}\right\rangle^{n+1} + 3\left(\left\langle\frac{\rho}{\rho'}\right\rangle - \left\langle\frac{\rho}{\rho'}\right\rangle^{-1}\right)\left(\frac{\rho}{\rho'}+\frac{\rho'}{\rho}\right)^2\left\langle\frac{\rho}{\rho'}\right\rangle^{n+1}\right.$$

$$\left. - \delta_{n1}\left(\left\langle\frac{\rho}{\rho'}\right\rangle - \left\langle\frac{\rho}{\rho'}\right\rangle^{-1}\right) - 2\delta_{n0}\left(\left\langle\frac{\rho}{\rho'}\right\rangle^2 - \left\langle\frac{\rho}{\rho'}\right\rangle^{-2}\right)\right\},$$

$$\binom{4\,2}{\rho\rho'}_n = \frac{1}{16\rho\rho'|\rho^2-\rho'^2|}\left\{(n+1)\left(\frac{\rho}{\rho'}+\frac{\rho'}{\rho}\right)^4\left\langle\frac{\rho}{\rho'}\right\rangle^{n+1} + 4\left(\left\langle\frac{\rho}{\rho'}\right\rangle - \left\langle\frac{\rho}{\rho'}\right\rangle^{-1}\right)\left(\frac{\rho}{\rho'}+\frac{\rho'}{\rho}\right)^3\left\langle\frac{\rho}{\rho'}\right\rangle^{n+1}\right.$$

$$- \delta_{n2}\left(\left\langle\frac{\rho}{\rho'}\right\rangle - \left\langle\frac{\rho}{\rho'}\right\rangle^{-1}\right) - 2\delta_{n1}\left(\left\langle\frac{\rho}{\rho'}\right\rangle^2 - \left\langle\frac{\rho}{\rho'}\right\rangle^{-2}\right)$$

$$\left. - \delta_{n0}\left(3\left\langle\frac{\rho}{\rho'}\right\rangle^3 + 4\left\langle\frac{\rho}{\rho'}\right\rangle - 4\left\langle\frac{\rho}{\rho'}\right\rangle^{-1} - 3\left\langle\frac{\rho}{\rho'}\right\rangle^{-3}\right)\right\}.$$