Intersection Rigidity

Reed Campbell Meyerson

A dissertation
submitted in partial fulfillment of the
requirements for the degree of

Doctor of Philosophy

University of Washington
2021

Reading Committee:
Gunther Uhlmann, Chair
Hart Smith
Tatiana Toro

Program Authorized to Offer Degree:
Mathematics

University of Washington

Abstract

Intersection Rigidity

Reed Campbell Meyerson

Chair of the Supervisory Committee:
Gunther Uhlmann
Department of Mathematics

We consider three inverse problems related to geodesic intersections. First, we consider the problem of recovering the geometry of a Riemannian manifold with boundary from the knowledge of all pairs of inward pointing directions at the boundary that correspond to intersecting geodesics. We call this the intersection data and show that it determines the geometry when the manifold is simple and at least three-dimensional. Next, we consider the problem of recovering the geometry of a Riemannian manifold with boundary from the knowledge of how far along each geodesic you must travel to reach the intersection points of any other geodesic. We call this information a stitching data and show that it determines the geometry of the manifold, without any restrictions on the geometry. Finally, we consider the problem of recovering the geometry of a Riemannian manifold with boundary from knowledge of how to time particles shot along geodesics from the boundary so that they collide on the interior. We call this information the delayed collision data and show that it determines the geometry of the manifold with natural geometric restrictions. In particular, the stitching data and delayed collision data apply to non-compact and unbounded manifolds.

Acknowledgements

First, I would like to thank my PhD advisor Professor Gunther Uhlmann. His love for inverse problems is infectious, and his encouragement has been invaluable to me. In particular, I have benefited greatly from his advice to find balance between family life, teaching, and research.

Next, I would like to thank my undergraduate advisor Professor Patrick McDonald. Were it not for his Calculus with Theory class, I would certainly not be getting a PhD in mathematics.

I would also like to thank my committee members Professor Tatiana Toro, Professor Hart Smith and Professor Anne Greenbaum. Their time and feedback has been greatly appreciated in this process.

I would like to thank my parents. Their support in my educational process has been a constant force in my life, starting before kindergarten and continuing even through this dissertation.

Finally, I would like to thank my wife, Cheyenne, for all of the encouragement and sacrifices she has made to make this a possibility. Without her, I would have no direction.

Dedication

For Cheyenne.

Contents

1 Introduction

Let (M, g) be a Riemannian manifold with boundary ∂M. Then for every inward pointing direction (i.e. unit vector) at the boundary, $v \in TM\big|_{\partial M}$, there is a unique geodesic γ_v with initial velocity v. For two inward pointing directions v, w their geodesics either intersect or they do not intersect (that is, the intersection of their images is either empty or non-empty). We call knowledge of all pairs of inward pointing directions with intersecting geodesics, together with the Riemannian metric for the boundary (i.e. $(\partial M, g\big|_{T\partial M \times T\partial M})$) the *intersection data*. Our first major new contribution is the following.

Theorem 1.1. Let (M, g) be a simple Riemannian manifold with boundary such that $\dim(M) \geq 3$. Then, the interior geometry of (M, g) is determined by its intersection data.

For a more formal definition of the intersection data, as well as a more formal statement of Theorem 1.1, and of course a proof, see Chapter 8.

In some sense, the intersection data is the coarsest form of geodesic intersection information one could obtain. For each pair of directions, it simply answers the yes-or-no question "do the corresponding goedesics intersect?". On the other extreme, perhaps the finest form of geodesic intersection information is what we will call a *stitching data*. Informally, for any pair of geodesics, a stitching data describes how far along one geodesic you have to travel to reach each point of intersection with the other geodesic. We note that we speak of "a" stitching data rather than "the" stitching data because the representation will only be unique up to isometries of all of the geodesics. Our second major new contribution is the following theorem.

Theorem 1.2. Let (M, g) be a Riemannian manifold with (possibly empty) boundary. Then a stitching data on (M, g) determines the geometry of (M, g).

For a formal definition of stitching data, as well as a more formal statement of Theorem 1.2, and of course a proof, see Chapther 9. We note that there are no geometric assumptions on the manifold for Theorem 1.2, and so a stitching data is a generic way to encode the geometry of any Riemannian manifold with or without boundary relative to its geodesics.

As an application of the stitching data, we formulate the *delayed intersection data*. For each inward pointing direction, imagine shooting a particle in that initial direction. Then, wait a chosen delay, and pick a different inward pointing direction and shoot off a new particle in the new direction. Finally, if the two particles collide, record that they did collide and how long it took for the collision to occur. The data that you would collect by performing this experiment with all possible pairs of directions and all possible delays is what is contained in the *delayed intersection data*. Our third major contribution is the following.

Theorem 1.3. Let (M, g) be a semi-nontrapping and intersection-confirming Riemannian manifold with boundary. Then the delayed intersection data for (M, g) determines the geometry of (M, g).

For a formal definition of the delayed intersection data, semi-nontraping and intersection confirming, as well as a more formal statement of Theorem 1.3 and of course a proof, see Section 9.4.

1.1 Pedagogy

We would like this document to be as self contained as possible. However, for practical reasons, we cannot take basic set theory as a starting point. In particular, we will assume familiarity with the three books in Lee's series on manifolds [17, 18, 19]. We will still reproduce the statements of relevant lemmas and theorems, and reprove them when it is reasonable.

In addition to Lee's series on manifolds, we will draw heavily on Burago, Burago and Ivanov's book *A Course in Metric Geometry* [3]. Again, we aim to reproduce the statements of relevant lemmas and theorems, and reprove them when it is reasonable.

In addition to a relatively self-contained presentation of new results, we aim to provide a thorough review of the related literature on geodesic boundary measurement inverse problems. However, we will not attempt to reprove these results. This review can be found in Chapter 7.

In writing this document, the editorial choice was made to attempt to keep things as tensor-and-connection-free as possible. This is so that any graduate students or advanced undergraduate students interested in understanding or extending these results would have a reasonable possibility of understanding the statements and proofs of new results within their first couple years of graduate school. We emphasize that the ability to cite previous results without proof makes this possible, and that any reader wishing to fully understand all background will still need a basic understanding of tensors and connections. However, if the reader is willing to accept certain details about geodesics as facts, then they will be able to understand all statements and proofs of new results without any knowledge of tensors or connections.

Readers only interested in the new contributions, who are already experts in Riemannian and metric geometry, may restrict their attention to Section 7.5, and Chapters 8 and 9.

2 Topology

Our goals in this chapter are as follows.

- Develop the minimal amount of topology to define the *metric topology* and *topological manifold with boundary*
- Record a variety of small topological facts that will be referenced later

For a more complete overview of topology, metric spaces, and topological manifolds, see [23, 17]

2.1 Basic Point Set Topology

Definition 2.1. Let X be a set. A set $\mathcal{T} \subset 2^X$ of subsets of X is called a **topology** for X if

1. $\varnothing, X \in \mathcal{T}$.
2. If $T \subset \mathcal{T}$, then $\bigcup_{U \in T} U \in \mathcal{T}$.
3. If $U_1, U_2 \in \mathcal{T}$, then $U_1 \cap U_2 \in \mathcal{T}$.

A **topological space** is a pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T}$ is a topology on X. For a fixed topological space, we call elements of $\mathcal{T}$ **open** sets. If $x \in X$, $U \in \mathcal{T}$ and $x \in U$ we say that U is a **neighborhood** of x.

If we simply refer to a topological space X, we mean that there is an implied fixed topology $\mathcal{T}$ for X.

If there are two topologies $\mathcal{T}_1$, $\mathcal{T}_2$ on X such that $\mathcal{T}_1 \subset \mathcal{T}_2$, we say that $\mathcal{T}_1$ is **coarser** than $\mathcal{T}_2$ and $\mathcal{T}_2$ is **finer** than $\mathcal{T}_1$.

Definition 2.2. Let X, Y be topological spaces. A function $f : X \to Y$ is said to be **continuous** if for all open sets $U \subset Y$, the inverse image $f^{-1}(U) \subset X$ is open in X. If, in addition to being continuous, f is a bijection with continuous inverse $f^{-1} : Y \to X$, we say that f is a **homeomorphism**.

We recall the basic fact that the composition of continuous functions is continuous. We denote the set of continuous functions from X to Y by $C(X; Y)$. We recall that $C(X; \mathbb{R})$ comes equipped with a norm

$$||f||_\infty = \sup_{x \in X} |f(x)|$$

called the **sup norm**. We note that if f is not bounded, then its sup norm will fail to be finite. However, when X is compact, all continuous functions to $\mathbb{R}$ must be bounded, as we will see shortly.

Definition 2.3. Let X be a topological space. We say that $C \subset X$ is **compact** if, for every collection of open sets $\{U_\alpha\}_{\alpha \in A}$ such that $\bigcup_{\alpha \in A} U_\alpha \supset C$, there exists a finite subcollection $\{U_i\}_{i=1}^N \subset \{U_\alpha\}_{\alpha \in A}$ such that $\bigcup_{i=1}^N U_i \supset C$.

If $C = X$ in the above defintion, we say that the space X is compact. Perhaps the most important property of compactness is that it is preserved under continuous images.

Lemma 2.4. Let X be a topological space and $C \subset X$ be compact. Let $f : X \to Y$ be continuous. Then $f(C)$ is compact in Y.

In particular, the compact subsets of $\mathbb{R}$ are exactly the closed and bounded subsets (the Heine-Borel Theorem). Thus, it follows that every real-valued continuous function attains a maximum and minimum value on every compact set (this is the extreme value theorem).

General topological spaces can be quite exotic. Our next two definitions will be used to place restrictions on the spaces we consider.

Definition 2.5. Let X be a topological space. If for all $x, y \in X$ such that $x \neq y$ there exists an open set U such that either $x \in U$ but $y \notin U$ or $y \in U$ but $x \notin U$, we say X is T_0.

Definition 2.6. Let X be a topological space. If for all $x, y \in X$ such that $x \neq y$ there exists neighborhoods U of x and V of y such that $U \cap V = \varnothing$, we say that X is **Hausdorff**.

One of our primary objects of study will be metric spaces. As a metric is a function on the product of a space with itself, it will be useful to have a topology on this product.

Definition 2.7. Let X, Y be topological spaces. We define a topology on $X \times Y$ called the **product topology** such that $U \subset X \times Y$ is open if and only if for all points $(x, y) \in U$ there exists open neighborhoods A of x and B of y such that $A \times B \subset U$.

The following result follows straight from the definitions of *product toplogy* and *finer*.

Lemma 2.8. Let X be a set and $\mathcal{T}_1$, $\mathcal{T}_2$ be topologies on X such that $\mathcal{T}_1$ is finer than $\mathcal{T}_2$. Then the product topology on $X \times X$ with respect to $\mathcal{T}_1$ is finer than the product topology on $X \times X$ with respect to $\mathcal{T}_2$.

We recall that a set $S \subset X$ is **dense** if, for all $x \in X$ every open neighborhood U of x has non-empty intersection with S.

Lemma 2.9. Let X be a topological space. Suppose $U \subset X$ is dense in X. Then $U \times U$ is dense in $X \times X$.

Proof. Let $(x, y) \in X \times X$. Let $O \subset X \times X$ be a neighborhood of (x, y). Then there exists A, B neighborhoods of x and y respectively such that $A \times B \subset O$. Since U is dense in X, both A and B have non-empty intersection with U. Thus, it follows that $A \times B$ has non-empty intersection with $U \times U$. Thus, O has non-empty intersection with $U \times U$ as required. $\square$

The above lemma about density will be used in conjunction with the following lemma, which deals with extensions of functions that agree on dense sets.

Lemma 2.10. Let X and Y be topological space such that Y is Hausdorff. Let $U \subset X$ be dense. Suppose $f, g : X \to Y$ are continuous and $f\big|_U = g\big|_U$. Then $f = g$.

Proof. Let $x \in X$. For contradiction, suppose $f(x) \neq g(x)$. Since Y is Hausdorff, there exists A, B open in Y such that $x \in A, y \in B$ and $A \cap B = \varnothing$. Thus, $x \in f^{-1}(A) \cap f^{-1}(B)$ is open in X. It follows that there exists $z \in f^{-1}(A) \cap f^{-1}(B) \cap U$. Thus, $f(z) = g(z)$, by the definition of U. But then $g(z) \in A \cap B$, which is a contradiction. ☐

As the final matter of this section, we discuss paths and path-connected components. We recall that a subset S of a topological space X is **connected** if, for all pairs of open sets $A, B \subset X$ such that $A \cup B \supset S$, we have $A \cap B \neq \varnothing$. We define an **interval** to be any non-empty, connected subset of $\mathbb{R}$. In particular, this allows for open, closed, half-open, infinite intervals and singleton sets. It follows that a compact interval is either a closed interval $[a, b]$ for some $a < b$ or a singleton set $\{a\}$ for some $a \in \mathbb{R}$. However, we adopt the notation $[a, a] = \{a\}$ for consistency. If we write $[a, b]$ we mean that $a \leq b$. A **path** or **curve** in X is an element of $C(I; X)$ for some interval I.

For two points $x, y \in X$, write $x \sim_{\text{path}} y$ if there exists a path $\eta : [a, b] \to X$ such that $\eta(a) = x$ and $\eta(b) = y$. It is a standard exercise in topology to verify that $\sim_{\text{path}}$ is an equivalence relation. Thus, it partitions X into equivalence classes. We call equivalence classes of $\sim_{\text{path}}$ **path connected components** of X.

2.2 Metric Spaces

In this section, we discuss *metric spaces*. Intuitively, a metric is a function which assigns a "distance" between any two points in a set. Many of the topological results in this section are valid in a more general setting, however we will only need them in the setting of a metric space or a metrizable space. In the following definition, we adopt the convention from [3] that a metric may assign an infinite distance.

Definition 2.11. Let X be a set. A **metric** is a function $d : X \times X \to [0, \infty]$ satisfying

1. $d(x, y) = d(y, x)$ for all $x, y \in X$.
2. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.
3. $d(x, y) = 0$ if and only if $x = y$.

In such a case, we say the pair (X, d) is a **metric space**. If the range of d can be taken to be $[0, \infty)$, we say d is a **finite metric**.

We topologize the set $[0, \infty]$ by adding ∞ to $[0, \infty)$ as a discrete point. Specifically, this means that $O \in \mathcal{T}_{[0,\infty]}$ if and only if O is an open set in $[0, \infty)$ or $O = U \cup \{\infty\}$ for an open set $U \subset [0, \infty)$.

The vast majority of topological results about finite metrics can be recovered, and applied to our notion of metric by using the following lemma, and then applying the finite result on the equivalence class.

Lemma 2.12. Let (X, d) be a metric space. For $x, y \in X$, write xR_dy if $d(x, y) < \infty$. Then R_d is an equivalence relation, and d restricted to any equivalence class of R_d is finite.

Proof. An equivalence relation is reflexive, symmetric and transitive. Reflexivity follows from the fact that $d(x, x) = 0 < \infty$. Symmetry follows from the fact that $d(x, y) = d(y, x)$ and transitivity follows from the triangle inequality.

The fact that d restricted to equivalence classes of R_d is finite follows directly from the definition of R_d. $\square$

If (X, d) is a metric space, $x \in X$, and $0 < \varepsilon < \infty$, we denote the **open metric ball of radius** ε **centered at** x by $B_\varepsilon(x) = \{y \in X | d(x, y) < \varepsilon\}$. We denote the corresponding **closed ball** by $\overline{B}_\varepsilon(x) = \{y \in X | d(x, y) \le \varepsilon\}$. We use the metric balls to generate a topology for X.

Definition 2.13. Let (X, d) be a metric space. Define a $\mathcal{T}_d \subset 2^X$ by writing $U \in \mathcal{T}_d$ if and only if every point in U is contained in an open metric ball that is also contained in U. We call $\mathcal{T}_d$ the **metric topology generated by** d.

For finite metrics, the following is a standard result in any point set topology course (i.e. [23]). The proof for infinite metrics is the same, since the topology is determined in terms of finite balls.

Lemma 2.14. Let (X, d) be a metric space. Then $\mathcal{T}_d$ is a topology.

Of course, it follows from the triangle inequality that all open balls are themselves open in the metric topology (i.e. $B_\varepsilon(x) \in \mathcal{T}_d$).

Given a metric, we may generate a topology. However, one may wonder when the reverse occurs.

Definition 2.15. Let $(X, \mathcal{T})$ be a topological space. If there exists a finite metric d on X such that $\mathcal{T} = \mathcal{T}_d$, we say that $(X, \mathcal{T})$ is **metrizable**.

We note that we chose to define *metrizable* in terms of finite metrics. This is to avoid spaces with uncountably many connected components. However, an infinite metric space is metrizable in terms of a finite metric if R_d has countably many equivalence classes. Thus, everything we will discuss still applies to such infinite metric spaces.

For $x \neq y$ in a metric space, let $r < d(x, y)/2$. It follows from the triangle inequality that $B_r(x) \cap B_r(y) = \varnothing$. Thus, metric spaces (and hence metrizable spaces) are Hausdorff.

Lemma 2.16. Let (X, d) be a metric space. Then $d : X \times X \to [\mathbb{R}, \infty]$ is continuous with respect to the product of the metric topologies.

Proof. The proof may be separated into two parts. First, show that $d^{-1}(O)$ is open for any open $O \subset [0, \infty)$. Then, show that $d^{-1}(\infty)$ is open. The proof of the first part is the same as in the finite metric situation. Thus, we refer readers to [23].

Thus, it remains to show that $d^{-1}(\infty)$ is open. Let $x, y \in X$ be such that $d(x, y) = \infty$. For any $r > 0$ consider $U = B_r(x) \times B_r(y)$. If $(w, z) \in U$, then wR_dx and zR_dy. It follows that $d(w, z) = \infty$. Thus $U \subset d^{-1}(\infty)$. □

It is a simple exercise that if $f : X \times Y \to Z$ is a continuous function, then $y \mapsto f(x, y)$ is continuous for any fixed $x \in X$. Thus, when (X, d) is a finite metric space, we obtain a map from X to $C(X; \mathbb{R})$ by $x \mapsto d(x, \bullet)$. Where $d(x, \bullet)$ is the function $y \mapsto d(x, y)$. The following lemma demonstrates that the map is continuous from X to $C(X; \mathbb{R})$. (In fact, it is distance-preserving).

Lemma 2.17. Let (X, d) be a compact, finite metric space. Then the map $x \mapsto d(x, \bullet)$ from X to $C(X; \mathbb{R})$ is continuous with respect to the sup norm on $C(X; \mathbb{R})$. In particular, $||d(x, \bullet) - d(y, \bullet)||_\infty = d(x, y)$.

Proof. Let $x, y \in X$. Let $z \in X$. Without loss of generality, assume $d(x, z) > d(y, z)$. Then by the triangle inequality

$$\begin{aligned} |d(x,z) - d(y,z)| &= d(x,z) - d(y,z) \\ &\leq d(x,y) + d(y,z) - d(y,z) \\ &= d(x,y) \end{aligned}$$

It follows that $||d(x, \bullet) - d(y, \bullet)||_\infty \leq d(x, y)$

Next, observe that $(d(x, \bullet) - d(y, \bullet))\big|_y = d(x, y) - d(y, y) = d(x, y)$. Thus, it follows that $||d(x, \bullet) - d(y, \bullet)||_\infty \geq d(x, y)$. The equality follows. □

In a metric space, it is often easier to measure continuity by inspecting sequences of points, rather than inverse images of open sets.

Definition 2.18. Let X be a topological space. Let $\{x_n\}_{n=1}^\infty$ be a sequence in X and $x \in X$. We say that $\{x_n\}_{n=1}^\infty$ **converges** to x and we write $x_n \to x$ if, for all open neighborhoods U of x, there exists $N > 0$ such that $\{x_n\}_{n>N} \subset U$.

If Y is a topological space and $f : X \to Y$, we say that f is **sequentially continuous** if for all converging sequences $x_n \to x \in X$, the corresponding sequence $f(x_n)$ converges to $f(x)$ in Y.

It follows straightfowardly from Definition 2.2 that all continuous functions are sequentially continuous. However, depending on the topology of X, not all sequentially continuous functions are continuous. For metric spaces, this distinction need not be made. We refer readers to [23] for the following lemma. It is not difficult to adapt this to infinite metric spaces.

Lemma 2.19. Let X be a metric space, and Y be a topological space. Then $f : X \to Y$ is continuous if and only if it is sequentially continuous.

To prove sequential continuity, the following result is often useful.

Lemma 2.20. Let X be a topological space. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X and $x \in X$. Then $x_n \to x$ if and only if, every subsequence of $\{x_n\}_{n=1}^{\infty}$ in turn has its own subsequence which converges to x.

Proof. First, observe that if $x_n \to x$, then all subsequences of $\{x_n\}_{n=1}^{\infty}$ converge to x. Thus, one direction is trivial.

For the converse, suppose that every subsequence $\{x_n\}_{n=1}^{\infty}$ has its own subsequence which converges to x. For contradiction, suppose $x_n \not\to x$. Then there exists an open neighborhood U of x such that, for all $N > 0$ there exists $j > N$ where $x_j \notin U$. In particular, this implies that there exists a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $x_{n_j} \notin U$ for all j. In particular, this implies that no subsequence of $\{x_{n_j}\}_{j=1}^{\infty}$ converges to x. This is a contradiction. $\square$

To use the above result, one often needs to show that there is a convergent subsequence in the first place. Thus, we have the following standard result [23].

Lemma 2.21. Let X be a topological space. Let $C \subset X$ be compact. Then every sequence $\{x_n\}_{n=1}^{\infty}$ in C has a subsequence which converges to point in C.

2.3 Topological Manifolds with Boundary

In this section, we discuss topological manifolds with boundary. Intuitively, a topological manifold with boundary is a space that locally looks like an open subset of $\mathbb{H}^n = \{(x_1, ..., x_n) \in \mathbb{R}^n | x_1 \geq 0\}$.

Definition 2.22. Let X, Y be topological spaces. We say that X is **locally modelled by** Y if, for all $x \in X$ there is an open neighborhood U of X, an open set $V \subset Y$ and a homeomorphism $\varphi : U \to V$.

We call pairs (U, φ) as in the above definition **coordinate charts**. The open set U is referred to as the **coordinate domain** and the homeomorphism is referred to as the **coordinate map**.

Definition 2.23. A **topological manifold with boundary** is a metrizable space M that is locally modeled by $\mathbb{H}^n$ for some fixed n.

For the space $\mathbb{H}^n$, we write $\mathbb{H}^{n,\circ} = \{(x_1, ..., x_n) \in \mathbb{R}^n | x_1 > 0\}$ and $\partial\mathbb{H}^n = \{(x_1, ..., x_n) \in \mathbb{R}^n | x_1 = 0\}$.

For a topological manifold M and a point $x \in M$, we write $x \in \partial M$ if there exists a coordinate chart (U, φ) such that $x \in U$, and $\varphi(x) \in \partial\mathbb{H}^n$. We write $x \in M^\circ$ if there exists a coordinate chart (U, φ) such that $x \in U$ and $\varphi(x) \in \mathbb{H}^{n,\circ}$.

It follows trivially from the fact that $\mathbb{H}^n = \partial\mathbb{H}^n \cup \mathbb{H}^{n,\circ}$ that $M = \partial M \cup M^\circ$. However, the following result is deceptively difficult, and thus we refer readers to [17] for the full details.

Theorem 2.24. Let M be a topological manifold with boundary. Then $\partial M \cap M^\circ = \varnothing$.

From Theorem 2.24, it follows that x gets sent to $\partial \mathbb{H}^n$ by one coordinate map if and only if it gets sent to $\partial \mathbb{H}^n$ by all coordinate maps.

3 Length Spaces

In this chapter we discuss length spaces. Our development of length spaces closely follows that of Burago, Burago and Ivanov's *A Course in Metric Geometry* [3], and thus for a more complete development of length spaces, we refer readers to that book.

Intuitively, a length space is a metric space for which the metric is generated by minimizing the length of paths between points. However, the class of paths must be suitable, and the notion of length must be suitable.

Definition 3.1. A **length space** is a triple $(X, \mathcal{A}, L)$ where X is a T_0 topological space, $\mathcal{A}$ is a set of continuous paths on X whose domains are compact intervals, and $L : \mathcal{A} \to [0, \infty]$. We call $\mathcal{A}$ the set of **admissible paths**, and L the **length function**. We require $\mathcal{A}$ to satisfy the following properties.

1. If $\eta : [a, b] \to X$ is a path in $\mathcal{A}$, and $[c, d] \subset [a, b]$, then $\eta\big|_{[c,d]} \in X$.
2. If $\eta : [a, b] \to X$, and $c \in [a, b]$ is such that $\eta\big|_{[a,c]}, \eta\big|_{[c,b]} \in \mathcal{A}$, then $\eta \in \mathcal{A}$.
3. If $\eta : [a, b] \to X$ is a path in $\mathcal{A}$ and $\varphi : [c, d] \to [a, b]$ is an affine homeomorphism then $(\eta \circ \varphi) : [c, d] \to X$ is a path in $\mathcal{A}$.
4. If $\eta : [a, a] \to X$, then $\eta \in \mathcal{A}$.

If $x, y \in X$, we write $\mathcal{A}_{x,y}$ to denote the set of admissible paths from x to y. We require the length function, L, to satisfy

1. $L(\eta\big|_{[a,b]}) = L(\eta\big|_{[a,c]}) + L(\eta\big|_{[c,b]})$ for all $c \in [a, b]$.
2. The function $t \mapsto L(\eta\big|_{[a,t]})$ is continuous from $[a, b] \to \mathbb{R}$ whenever $\eta : [a, b] \to X$ is a path in $\mathcal{A}$.
3. If $\eta : [a, b] \to X$ is a path in $\mathcal{A}$, and $\varphi : [c, d] \to [a, b]$ is an affine homeomorphism then $L(\eta) = L(\eta \circ \varphi)$.
4. If $U \subset X$ is open and $x \in U$, then

$$\inf_{\substack{y \in X \setminus U \\ \eta \in \mathcal{A}_{x,y}}} L(\eta) > 0$$

If $(X, \mathcal{A}, L)$ is a length space, we define the **induced metric** d_L by

$$d_L(x,y) = \inf_{\eta \in \mathcal{A}_{x,y}} L(\eta)$$

For this entire document, we use the convention that the infimum over an empty set is ∞. Thus, if there are no admissible paths from x to y, then $d_L(x,y) = \infty$.

We note here that d_L also depends on the set of admissible curves. For instance, if $\mathcal{B} \subset \mathcal{A}$ then d_L may differ from $d_{(L|_{\mathcal{B}})}$. In particular, $d_{(L|_{\mathcal{B}})} \geq d_L$, from the properties of infiumum.

Next, we show that d_L deserves its name.

Lemma 3.2. Let $(X, \mathcal{A}, L)$ be a length space, then (X, d_L) is a metric space.

Proof. First we show that d_L is symmetric. Let $x, y \in X$. By property (3.) of $\mathcal{A}$, and property (3.) of L, for each path in $\mathcal{A}_{x,y}$ there is a corresponding path in $\mathcal{A}_{y,x}$ with the same length (by reversing the direction of the path). Thus, the infimums corresponding to $d_L(x,y)$ and $d_L(y,x)$ are equal.

The fact that $d_L(x,y) \geq 0$ follows from the fact that we required the range of L to be $[0, \infty]$.

Next, show $d_L(x,x) = 0$ for any $x \in X$. Let $\eta : [0,0] \to X$ be defined by $\eta(0) = x$. Then, by property (4.) of $\mathcal{A}$, $\eta \in \mathcal{A}_{x,x}$. By property (2.) of L, we obtain $L(\eta|_{[0,0]}) + L(\eta|_{[0,0]}) = L(\eta|_{[0,0]})$, which can only happen if $L(\eta|_{[0,0]}) = L(\eta) = 0$. Thus, it follows that $d_L(x,x) = 0$.

Continuing, we show that $d_L(x,y) > 0$ if $x \neq y$. Since X is T_0, it follows that there exists an open set U containg x but not y or containing y but not X. Without loss of generality, assume U contains x but not y. Then it follows from property (4.) of L that $d_L(x,y) > 0$.

Finally, we show the triangle inequality. Let $x, y, z \in X$. For every pair of paths $\eta_{x,z}, \eta_{z,y}$ such that $\eta_{x,z} \in \mathcal{A}_{x,z}$ and $\eta_{z,y} \in \mathcal{A}_{y,z}$. There is a path in $\eta_{x,y} \in \mathcal{A}_{x,y}$ formed by concatenating the paths (property (1.) of $\mathcal{A}$). Thus, from property (1.) of L, it follows that $\inf_{\eta \in \mathcal{A}_{x,y}} L_g(\eta) \leq \inf_{\eta \in \mathcal{A}_{x,z}} L_g(\eta) + \inf_{\eta \in \mathcal{A}_{z,y}} L_g(\eta)$. Thus, the triangle inequality follows immediately. $\square$

We note that if we did not require X to be T_0, then d_L may fail to be positive definite. For instance, consider the situation $X = \{1, 2\}$ with the coarsest topology $\mathcal{T} = \{\varnothing, X\}$. Let $\mathcal{A}$ be the set of *all* continuous paths into X. Since X has the coarsest topology, then any function from an interval to X is continuous. If we simply define $L(\eta) = 0$ for all admissible paths η, then this defines a length space (property (4.) of L is trivially satisfied). However, clearly $d_L(1,2) = 0$.

The topology generated by d_L may differ from the original topology on X. However, it must always be finer than the original topology.

Lemma 3.3. Let $(X, \mathcal{A}, L)$ be a length space. Then the topology generated by d_L is finer than the original topology on X.

Proof. Let $U \subset X$ be open in the original topology. Let $x \in U$. We must show that there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subset U$. By property (4.) of L, let $\varepsilon > 0$ be such that $\inf_{\substack{y \in X \setminus U \\ \eta \in \mathcal{A}_{x,y}}} L(\eta) > \varepsilon$. It follows that $B_\varepsilon(x) \subset U$ as required. $\square$

It is useful to consider curves which are defined on non-compact intervals. Thus, we have the following definition.

Definition 3.4. Let $(X, \mathcal{A}, L)$ be a length space. Let $I \subset \mathbb{R}$ be any interval (compact or otherwise), and $\eta : I \to X$ be a continuous curve. We say η is locally admissable it for any compact interval $[a, b] \subset I$, we have $\eta\big|_{[a,b]} \in \mathcal{A}$.

Consider the following path $\eta_1 : t \mapsto (\cos t, \sin t, 0)$ in the unit two-sphere. Restricted to $[0, 3\pi/2]$ this is a path from $(1, 0, 0)$ to $(0, -1, 0)$. There is a faster path between these points by going the other way around the equator (i.e. $\eta_2 : t \mapsto (\cos(-t), \sin(-t), 0)$). However, if one "zooms in" far enough on both of these paths, they are locally the most efficient efficient way to get from one point to the next. Specifically, for some $\varepsilon > 0$, we have that η_1 is the shortes path from $\eta_1(t)$ to $\eta_1(t + \varepsilon)$ for all t, and similarly for η_2. The next definition formalizes what is happening in this example.

Definition 3.5. Let $(X, \mathcal{A}, L)$ be a length space. Let $\eta : I \to X$ be a locally admissable curve. If for all $t \in I$ there exists $\varepsilon > 0$ such that, for all compact intervals $[a, b] \subset (t - \varepsilon, t + \varepsilon) \cap I$, we have $L(\eta\big|_{[a,b]}) = d_L(\eta(a), \eta(b))$, we say that η is **locally minimizing**.

As the example above demonstrates, a path $\eta : [a, b] \to X$ can be locally minimizing, and still have $d_L(\eta(a), \eta(b)) \neq L(\eta)$. Thus, we have the following definition.

Definition 3.6. Let $(X, \mathcal{A}, L)$ be a length space. Let $\eta : I \to X$ be a locally admissable curve. If for all $s, t \in I$ such that $s \leq t$, we have $L(\eta\big|_{[s,t]}) = d_L(\eta(s), \eta(t))$, we say that η is **minimizing**.

Of course, any curve that is minimizing is also locally minimizing. Additionally, any restriction of a minimizing curve is minimizing.

As property (3.) of the length function indicates, many reparameterizations of a path may have the same length. It is often nice to be able to pick out a particular parameteriztion of a curve.

Definition 3.7. Let $(X, \mathcal{A}, L)$ be a length space. Let $\eta : I \to X$ be a locally admissible curve. If, for all $[t, x] \subset I$ we have $L(\eta\big|_{[t,s]}) = C(s - t)$ for a fixed $C > 0$, we say that *eta* is a **constant speed curve**. If $C = 1$, then we say η is a **unit speed curve**.

Definition 3.8. Let $(X, \mathcal{A}, L)$ be a length space and $\eta : I \to X$ be a constant-speed, locally minimizing curve. If, for all constant-speed, locally minimizing curves $\tilde{\eta} : \tilde{I} \to X$ such that $I \subset \tilde{I}$ and $\tilde{\eta}\big|_I = \eta$ we have $I = \tilde{I}$, then we say that η is a **maximally extended locally minimizing curve**.

Both η_1 and η_2 in the example above are maximally extended, locally minimizing curves.

Definition 3.9. Let $(Y, \mathcal{A}, L)$ be a length space. Let X be a set and $\Psi : X \to Y$ be a bijection. We define the **pullback length space** $(X, \Psi^*\mathcal{A}, \Psi^*L)$ by

1. $\eta \in \Psi^*\mathcal{A}$ if and only if $\Psi \circ \eta \in \mathcal{A}$.
2. $(\Psi^*L)(\eta) = L(\Psi \circ \eta)$.

A pullback length space is essentially just a relabeling of the points in another length space. It follows that pullbacks preserve distances.

Lemma 3.10. Let $(Y, \mathcal{A}, L)$ be a length space. Let $\Psi : X \to Y$ be a bijection. Then the metric space generated by $(X, \Psi^*\mathcal{A}, \Psi^*L)$ is isometric to the metric space generated by $(Y, \mathcal{A}, L)$.

4 Smooth Manifolds

In this chapter we discuss smooth manifolds. A smooth manifold will be a topological manifold with a smooth structure on it that allows us to do calculus in each coordinate patch, in such a way that we can be consistent on overlapping patches.

The main things we will need from this chapter are as follows

- A corollary of Sard's theorem, which tells us that the image of a smooth map is "small" when the dimension of the codomain is greater than the dimension of the domain
- The basics of tangent vectors, the tangent bundle and the differential of a smooth map

For an in depth course on smooth manifolds, we refer readers to [18]. Any result in this chapter which lacks a proof can be found directly in [18] or is a direct corollary of the material in [18], unless otherwise noted.

4.1 Sard's Theorem

Let M be a topological manifold with boundary. For two coordinate charts, (U, φ) and (V, ψ) such that $U \cap V \neq \varnothing$, define the **transition map from φ to ψ** to be the composition $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$. If the transition map is smooth (as a function from a subset of $\mathbb{H}^n$ to $\mathbb{H}^n$), we say that the coordinate charts are **compatible**. We also define two coordinate charts with non-intersecting coordinate domains to be compatible.

A **smooth atlas** for a topological manifold with boundary M is a set of coordinate charts, $\mathscr{A}$, such that

- the coordinate domains in $\mathscr{A}$ cover M
- the coordinate charts in $\mathscr{A}$ are pairwise compatible

We may put a partial order on the set of all coordinate charts for M, by declaring $\mathscr{A}_1$ less than $\mathscr{A}_2$ if $\mathscr{A}_1 \subset \mathscr{A}_2$. Maximal elements of this poset are called **smooth structures** for M. It follows from Zorn's lemma that, if a smooth atlas exists, it is contained in a unique smooth structure.

Definition 4.1. A **smooth manifold with boundary** is a pair $(M, \mathscr{A})$ such that M is a topological manifold with boundary and $\mathscr{A}$ is a smooth structure on M. If $\partial M = \varnothing$, then we simply say $(M, \mathscr{A})$ is a **smooth manifold**.

Since we will only consider smooth manifolds with a fixed smooth structure, we will simply write M, and the smooth structure $\mathscr{A}$ will be implied. If M is a smooth manifold with boundary, and $\partial M \neq \varnothing$, then ∂M is a smooth manifold without boundary such that $\dim(\partial M) = \dim(M) - 1$. We refer readers to [18] for the details on this fact.

Definition 4.2. Let M, N be smooth manifolds and $f : M \to N$. We say that f is **smooth** if, for all coordinate patches (U, φ) of M and (V, ψ) of N, the composition $\hat{f} = \psi \circ f \circ \varphi^{-1}$ is smooth from $\varphi(U)$ to $\psi(U)$.

We denote the set of smooth functions from M to $\mathbb{R}$ by $C^\infty(M)$. In the context of fixed coordinate charts on a pair of manifolds, we let $\hat{f} = \psi \circ f \circ \varphi^{-1}$ as in the above definition.

When we are considering maps $f : M \to \mathbb{R}$, we let $\hat{f} = f \circ \varphi^{-1}$, for a fixed coordinate chart on M. In otherwords, unless specifically noted, we always take the global identity coordinate chart on $\mathbb{R}$ or $\mathbb{R}^n$.

As stated, we would like to be able to do calculus on manifolds. To do this, we must first define tangent vectors.

Definition 4.3. Let M be a smooth manifold with boundary, and $p \in M$. A **tangent vector at** p is a linear map $X : C^\infty(M) \to \mathbb{R}$ satisfying the following product rule

$$X(fg) = f(p)X(g) + X(f)g(p)$$

We denote the set of tangent vectors at p by T_pM.

Let $X, Y \in T_pM$ and $\alpha \in \mathbb{R}$. We compute

$$\begin{aligned}(\alpha X + Y)(fg) &= \alpha X(fg) + Y(fg)\\ &= \alpha f(p)X(g) + \alpha X(f)g(p) + f(p)Y(g) + Y(f)g(p)\\ &= f(p)(\alpha X(g) + Y(g)) + (\alpha X(f) + Y(f))g(p)\\ &= f(p)(\alpha X + Y)(g) + (\alpha X + Y)(f)g(p)\end{aligned}$$

Thus, T_pM is a vector space. We will see in a moment, that tangent vectors at p are essentially just directional derivatives, defined in a coordinate-invariant way. We refer readers to [18] for the details of the following result.

Theorem 4.4. Let M be a smooth, n-dimensional manifold with boundary, and $p \in M$. Let (U, φ) be a coordinate chart around p. Then for each $i = 1, ..., n$ there exists a unique tangent vector $X \in T_pM$ such that $X(f) = \frac{\partial}{\partial x^i}\big|_{\varphi(p)} \hat{f}$ for all $f \in C^\infty(M)$.

For a fixed coordinate chart (U, φ), we denote the tangent vector guaranteed by the above theorem by $\frac{\partial}{\partial x^i}\big|_p$.

As a direct corollary of Theorem 4.4, we get that $\{\frac{\partial}{\partial x^1}\big|_p, ..., \frac{\partial}{\partial x^n}\big|_p\}$ is a basis for T_pM, for any fixed coordinate chart (U, φ) containing p. Thus, T_pM is a finite dimensional vector space and $\dim(T_pM) = \dim(M)$.

When we have a smooth function $f : M \to N$, we induce a linear transformation between the tangent spaces T_pM and $T_{f(p)}N$ in the following way.

Definition 4.5. Let M, N be smooth manifolds and $f : M \to N$ be smooth. Then for $p \in M$, define $df_p : T_pM \to T_{f(p)}N$ by

$$df_p(X)(g) = X(g \circ f)$$

for all $g \in C^\infty(N)$. We call the map df_p the **differential of f at p**.

In the case where we have a smooth curve $\eta : I \to M$, we write $\eta'(s) = \dot{\eta}(s) = d\eta\big|_s(\frac{d}{dt}\big|_s)$.

It is relatively straightforward to show that df_p is a linear transformation. Thus, as a linear transformation between finite dimensional vector spaces, we may inquire about its rank.

Definition 4.6. Let $f : M \to N$ be smooth. We say that $p \in M$ is a **critical point** of f if $df : T_pM \to T_{f(p)}N$ is not surjective. A point $q \in N$ is called a **critical value** if it is the image of a critical point.

We are nearly in a position to state Sard's theorem, which says that the set of critical values (a subset of the codomain) is small in a technical sense. The specific technical sense, is that the set has "measure zero". We choose to omit the definition of "measure zero set", because we will only need the following property

- Let $\dim(N) > 0$. If $S \subset N$ has measure zero, then $S \neq N$.

We refer readers to [18] for a proper treatment of measure zero sets on manifolds, and for a proof of the following theorem.

Theorem 4.7 (Sard's Theorem)**.** Let M, N be smooth manifolds with boundary. Let $f : M \to N$ be a smooth map. Then the set of critical values has measure zero in N.

It follows directly from basic linear algebra, that if $\dim(M) < \dim(N)$, then the set of critical values of a smooth function $f : M \to N$ is just its image. Thus, we have the following corollary.

Corollary 4.8. Let M, N be smooth manifolds such that $\dim(M) < \dim(N)$. Let $f : M \to N$ be smooth. Then $f(M)$ has measure zero in N.

As stated before, we do not need to know any of the intricacies of measure to use the above result. We will use it only in the following way.

- If $\dim(M) < \dim(N)$ then there is no surjective smooth function $f : M \to N$.

4.2 The Tangent Bundle and Vector Fields

Consider a first order autonomous ODE in $\mathbb{R}^n$. It will be of the form $x'(t) = F(x(t))$, where $x : I \to \mathbb{R}^n$ for some interval I and $F : \mathbb{R}^n \to \mathbb{R}^n$ assigns to each point in $\mathbb{R}^n$ a vector in sufficiently regular way. To extend this idea to manifolds with boundary, we need to make sense of the idea of "assigning a vector to each point in M in a sufficiently regular way". To do this, we will describe the tangent bundle. The tangent bundle, as a set, will just be the union of all of the different tangent spaces. However, we will need to topologize it and give it a smooth structure.

For a smooth manifold with boundary, we define the **tangent bundle** $TM = \bigsqcup_{p \in M} T_pM$. An element of TM is thus an ordered pair, (p, v) where $p \in M$ and $v \in T_pM$. There is a surjective map $\pi : TM \to M$ defined by $\pi(p, v) = p$. We will typically abuse notation and write $v \in TM$ and if we wish to identify its base point, we will explicitly write $p = \pi(v)$. We use π to define an atlas for TM. First, for a chart (U, φ) in M, and a tangent vector $\frac{\partial}{\partial x^i}\big|_p \in T_pM$, define

$$\tilde{\varphi}(\frac{\partial}{\partial x^i}\big|_p) = (\varphi(p), e_i)$$

where $\{e_1, ..., e_n\}$ is the standard basis for $\mathbb{R}^n$. Then, we extend $\tilde{\varphi} : \pi^{-1}(U) \to \pi(U) \times \mathbb{R}^n$ to by requiring it to be linear on each tangent space. Thus, we take $(\pi^{-1}(U), \tilde{\varphi})$ to be elements of the atlas on TM. We refer readers to [18] for the details, that this does infact define an atlas. Thus, the smooth structure on TM is defined by extending this atlas maximally.

If $f : M \to N$ smooth then we define $df : TM \to TN$ by $df(v) = df\big|_{\pi(v)}(v)$. We refer readers to [18] for details of the following important fact.

Theorem 4.9. Let M, N be smooth manifolds and $f : M \to N$ be smooth. Then $df : TM \to TN$ is smooth.

We would like to be able to assign a vector to each point on a manifold in a smooth way. The following definition formalizes this.

Definition 4.10. Let $q : N \to M$ be a surjective smooth map. A smooth map $\sigma : M \to N$ satisfying $q \circ \sigma$ is the identity on M is called a section of q.

If there is a fixed surjective map, (such as in the case of $\pi : TM \to M$), we simply say that σ is a section of N, and we write $\sigma \in \Gamma(N)$.

An element of $\Gamma(TM)$ is called a **vector field** on M. If $X \in \Gamma(TM)$, then $X_p \in T_pM$ for all $p \in M$, and we get a map $X_p : C^\infty(M) \to \mathbb{R}$. Using this, we define a map $X : C^\infty(M) \to C^\infty(M)$ by $X(f)\big|_p = X_p(f)$.

5 Riemannian Manifolds

Euclidean geometry can be recovered entirely by considering the vector space structure on $\mathbb{R}^n$, together with the dot product $v, w \mapsto v \bullet w$. Specifically, lengths can be defined as $|v| = \sqrt{v \bullet v}$, distances can then be defined in terms of lengths $|v - w|$, and angles can be defined using the cosine formula $\angle(v, w) = \cos^{-1}(\frac{v \bullet w}{|v||w|})$. Thus, to put a geometry on a manifold with boundary, it seems natural to consider inner products on the tangent spaces T_pM.

Let M be a smooth manifold with boundary. A **Riemannian metric** on M is a smoothly varying inner product g on TM. For two vectors $v, w \in T_pM$ based at the same point, we denote their inner product according to g by $g(v, w)$. By "smoothly varying", one can take many equivalent definitions, but we specify the following. For any two smooth vector fields $X, Y \in \Gamma(TM)$, then $p \mapsto g(X\big|_p, Y\big|_p)$ defines a smooth function on M.

For a more complete review of Riemannian geometry we refer readers to [19]. As with the previous section, any result in this section which does not have a proof is either directly in [19], or a direct corollary of the material in [19].

A **Riemannian manifold with boundary** is a pair (M, g) where M is a smooth manifold with boundary and and g is a Riemannian metric on M.

For this document "Riemannian metric" will always refer to a smoothly varying inner product, while "metric" on its own will always refer to a distance function. A Riemannian metric induces a length structure on M.

Definition 5.1. Let (M, g) be a Riemannian manifold. Let $\mathcal{A}$ be the set of piecewise smooth paths on M (with compact intervals as domains). For $\eta\big|_{[a,b]} \in \mathcal{A}$, we define its length by

$$L_g(\eta) = \int_a^b |\dot{\eta}(t)|_g dt$$

We call $(M, \mathcal{A}, L_g)$ the **Riemannian length structure**. We denote the induced metric by $d_g = d_{L_g}$. We call d_g the **Riemannian distance function**.

It remains to be checked that the Riemannian length structure is in fact a length structure. We refer readers to [3] for the details of the following.

Lemma 5.2. Let (M, g) be a Riemannian manifold with boundary. Then $(M, \mathcal{A}, L_g)$ is a length space.

For a general length space, the topology generated by the induced metric may be strictly finer than the original topology. However, for a Riemannian manifold with boundary, this is not the case.

Lemma 5.3. Let (M, g) be a Riemannian manifold with boundary. Then the topology generated by d_g is the same as the topology on M.

The interior of a topological manifold with boundary is dense in the entire manifold. Thus, the following lemma follows from Lemma 5.3.

Lemma 5.4. Let (M, g) be a complete Riemannian manifold with boundary. Then the completion of (M°, d_g) is isometric to (M, d_g).

Given a subset of a general length space, there are two natural metrics to consider on the subset. First, one can simply restrict the original metric to the subset. Alternatively, one can consider only paths which are contained in the subset and then re-induce a different metric. In general, these may give different results. For instance, consider the unit circle in the plane. The restricted metric yields the distance between $(1, 0)$ and $(-1, 0)$ is 2. However, the shortest path through the subset from one point to another has length π.

For a Riemannian manifold with boundary, there is no difference between the restriction of the metric to the interior, and the re-induced metric on the interior of a manifold.

Lemma 5.5. Let (M, g) be a Riemannian manifold with boundary. Let g° be g restricted to TM°. Then $d_{g^\circ} = d_g$ on $M^\circ \times M^\circ$.

5.1 Submanifolds of TM

The Riemannian metric allows us to define a number of submanifolds of TM. We summarize these here. It is a good exercise to write all of these as regular level sets or sublevel sets of some function from TM to $\mathbb{R}^m$ (for some m depending on their codimension in TM), thus proving that they are infact submanifolds. We note that all of the following inherit the map $\pi : TM \to M$, and will either have the property that π is surjective onto M or that it is surjective onto ∂M.

We define the **tangent bundle restricted to the boundary** $TM\big|_{\partial M} = \{v \in TM | \pi(v) \in \partial M\}$. We contrast this with the **tangent bundle to the boundary** $T\partial M$, which can also be viewed as a submanifold of TM. In particular, note that $\dim(TM\big|_{\partial M}) = 2\dim(M) - 1$, while $\dim(T\partial M) = 2\dim(M) - 2$. We let g^∂ denote $g\big|_{T\partial M \times T\partial M}$. Thus, $(\partial M, g^\partial)$ is a Riemannian manifold (without boundary), and in general $d_{g^\partial} \neq d_g\big|_{\partial M \times \partial M}$.

For $v \in TM\big|_{\partial M}$, we may use the Riemannian metric to define its orthogonal projection onto $TM\big|_{\partial M}$. We denote this orthogonal projection by $P^\partial(v)$. We note that $P^\partial : TM\big|_{\partial M} \to T\partial M$ is smooth.

We define the **unit sphere bundle** $SM = \{v \in TM | g(v, v) = 1\}$. We note that SM is a manifold with boundary. Its boundary is of course denoted ∂SM. We would like to be able to talk about "inward pointing directions", which will correspond to a subset of ∂SM.

For $x \in \partial M$, there is exactly one vector $\nu_x \in T_xM$ satisfying the following constraints

- $g(\nu_x, \nu_x) = 1$

- $P^{\partial}(\nu_x) = 0$
- $df_x(\nu_x) \geq 0$ whenever $f : M \to [0, \infty)$ satisfies $f^{-1}(\{0\}) = \partial M$.

We call ν_x the **inward pointing unit normal at** x. We note that $x \mapsto \nu_x$ is a smooth function from ∂M to $TM\big|_{\partial M}$.

We use ν_x to define the **inward pointing unit vectors** $\partial_+ SM = \{v \in \partial SM | g(v, \nu_x) > 0\}$. Its closure is denoted $\overline{\partial}_+ SM = \{v \in \partial SM | g(v, \nu_x) \geq 0\}$. We define the **outward pointing unit vectors** by $\partial_- SM = \{v \in \partial SM | g(v, \nu_x) < 0\}$. It closure $\overline{\partial}_- SM$ is defined in the corresponding way.

The image of $\partial_+ SM$ under the map P^{∂} is the **open unit disk bundle for the boundary** $U\partial M = \{v \in T\partial M | g(v, v) < 1\}$. In particular, $P^{\partial} : \partial_+ SM \to U\partial M$ is a diffeomorphism. For $v \in U\partial M$, we let $\hat{v}$ denote the unique vector in $\partial_+ SM$ satisfying $P^{\partial}(\hat{v}) = v$.

We denote the **closed unit disk bundle for the boundary** by $\overline{U}\partial M = \{v \in T\partial M | g(v, v) \leq 1\}$. Then $P^{\partial} : \overline{\partial}_+ SM \to \overline{U}\partial M$ is a diffeomorphism. We use the same notation, $\hat{v}$, to denote the unique vector in $\overline{\partial}_+ SM$ satisfying $P^{\partial}(\hat{v}) = v$. When $v \in U\partial M$, these notions coincide. When $|v|_g = 1$, then $\hat{v} = v$.

5.2 Geodesics on Manifolds without Boundary

Definition 5.6. Let (M, g) be a Riemannian manifold (without boundary). A maximal, constant-speed, locally-minimizing curve on (M, g) is called a **geodesic**.

Perhaps the most important property of geodesics is that they are smooth. For $p, q \in M$, if $\gamma : I \to M$ is a geodesic and $[a, b] \subset I$ is such that $\gamma(a) = p$, and $\gamma(b) = q$, we say that $\gamma|_{[a,b]}$ is a **geodesic segment** from p to q.

Geodesics have the following very useful property.

Theorem 5.7. Let (M, g) be a Riemannian manifold (without boundary). Let $v \in TM$. Then there is a unique geodesic $\gamma : I \to M$ such that $\gamma(0) = \pi(v)$ and $\dot{\gamma}(0) = v$.

For such a geodesic, we write $I_v = I$ to denote the domain and $\gamma_v = \gamma$ to denote the curve. We will often abuse notation, by using γ_v to denote the set $\gamma_v(I_v) \subset M$. When we do this, it will either be clear from context, or we will explicitly state that we are referring to the set.

For each $p \in M$, write $\mathcal{E}_p = \{v \in T_p M | 1 \in I_v\}$. Define the **exponential map at** p, denoted $\exp_p : \mathcal{E}_p \to M$, by $\exp_p(v) = \gamma_v(1)$. We summarize the properties of the exponential map below.

Theorem 5.8. Let (M, g) be a Riemannian manifold (without boundary). Then for all $p \in M$ there exists $\varepsilon > 0$ such that

1. $B_\varepsilon(0) \subset \mathcal{E}_p$

2. $\exp_p : B_\varepsilon(0) \to M$ is a diffeomorphism onto its image.

For $p \in M$, we define the **injectivity radius** at p, denote $\mathrm{inj}(p)$ to be the supremum over all such ε in the theorem above. The above theorem guarantees that the injectivity radius at each point is always strictly positive. We will need the following fact, which can be found in [2].

Lemma 5.9. Let (M, g) be a Riemannian manifold (without boundary). Then the injectivity radius $\mathrm{inj} : M \to (0, \infty]$ is continuous.

It follows straight from the definition of *injectivity radius* that, if $p, q \in M$ and $d_g(p, q) < \mathrm{inj}(p)$, then there exists a unique vector $v \in T_pM$ such that $|v|_g < \mathrm{inj}(p)$ and $\exp_p(v) = q$. In such a situation, we write $v = \exp_p^{-1}(q)$.

Lemma 5.10. Let (M, g) be a Riemannian manifold (without boundary). Let $p, q \in M$ and $0 < d_g(p, q) < \mathrm{inj}(p)$. Let $v = \exp_p^{-1}(q)$, and $\hat{v} = \frac{v}{|v|_g} \in S_pM$. Then $\gamma_{\hat{v}}\big|_{[0,d_g(p,q)]}$ is a minimizing geodesic segment from p to q.

5.3 Geodesics on Manifolds with Boundary

So far, we have mostly focused on geodesics for Riemannian manifolds without boundary. We would like to define geodesics on manifolds with boundary by taking a geodesic on a boundariless extension of (M, g) and restricting it to M. However, if done naively, the domain of the restriction might not be an interval. As an example, consider the closed annulus $\overline{B}_2(0) \setminus B_1(0) \subset \mathbb{R}^2$. The curve $(t, 0)$ is a geodesic in $\mathbb{R}^2$ with domain $\mathbb{R}$. However, its naive restriction to the annulus would have a domain of $[-2, -1] \cup [1, 2]$. What we would actully like is for each component to correspond to a *separate* geodesic. We formalize this below.

Definition 5.11. Let (M, g) be a Riemannian manifold with boundary. A **boundariless extension** of (M, g) is a triple $(\tilde{M}, \tilde{g}, \iota)$ where $(\tilde{M}, \tilde{g})$ is a Riemannian manifold (without boundary), and $\iota : M \to \tilde{M}$ is a Riemannian isometry onto its image.

For any Riemannian manifold with boundary, a boundariless extension always exists. To do this somewhat constructively, one can take $\tilde{M}$ to be the double of M (see [18]), and then extend the Riemannian metric to $\tilde{M}$.

Let $(\tilde{M}, \tilde{g}, \iota)$ be a boundariless extension of (M, g). Let $\tilde{\eta} : \tilde{I} \to \tilde{M}$ be a continuous curve. Let $t \in \tilde{I}$ be such that $\tilde{\eta}(t) \in \iota(M)$, and I be the connected component of $\tilde{\eta}^{-1}[\iota(M)]$ containing t. Then, there is a unique continuous curve $\eta : I \to M$ such that $\iota \circ \eta = \tilde{\eta}\big|_I$. We call η a **connected restriction of** $\tilde{\eta}$ to M.

Definition 5.12. Let (M, g) be a Riemannian manifold with boundary. Let $\eta : I \to M$ be such that there exists an extension $(\tilde{M}, \tilde{g}, \iota)$ of (M, g) and a geodesic $\tilde{\eta} : \tilde{I} \to \tilde{M}$ such that η is a connected restrction of $\tilde{\eta}$ to M. Then, we say that η is a **geodesic** on M.

For $v \in TM$, we define $\gamma_v : I_v \to M$ to be the connected restriction of $\iota \circ \gamma_{d\iota(v)}$ to subset of $\gamma_{d\iota(v)}^{-1}(M)$ containing 0.

We remark that the definition above does not depend on the specific extension of (M, g) chosen. Specifically, if there is one extension for which η is the restriction of a geodesic, then it will be the restriction of a geodesic for *any* extension. Additionally, γ_v does not depend on the specific extension chosen.

If $p, q \in M$ are such that there is a unique minimizing geodesic containing both p and q, we let $\overline{pq} : [0, d_g(p, q)] \to M$ denote the geodesic segment from p to q.

We define the exponential map in the same way as for manifolds without boundary: $\mathcal{E}_p = \{v \in T_pM | 1 \in I_v\}$ and $\exp_p : \mathcal{E}_p \to M$ by $\exp_p(v) = \gamma_v(1)$. For points $p \in M^\circ$, there will be an open ball $B_\varepsilon(0) \subset E_p$, as in Theorem 5.8. However, for $p \in \partial M$, then $\mathcal{E}_p$ is contained in the halfspace of T_pM which corresponds to the "inward pointing vectors" (i.e. vectors such that $g(v, \nu_x) \geq 0$). In particular, when $p \in \partial M$, no open neighborhood of 0 is contained in $\mathcal{E}_p$.

For a manifold with boundary, we choose to only define the injectivity radius for interior points. If $x \in M^\circ$, we define $\mathrm{inj}(x)$ to be the injectivity radius of x thought of as a point in (M°, g). For $x \in \partial M$, we leave $\mathrm{inj}(x)$ undefined.

In general, the geodesics on a manifold with boundary inherit many properties from the geodesics on the boundariless extensions that they are restrictions of. In particular, they are still locally minimizing, constant-speed curves. However, the converse is not true. Specifically, there may be locally minimizing curves, which are not geodesics.

For an example, again consider the closed annulus $A = \overline{B}_2(0) \setminus B_1(0) \subset \mathbb{R}^2$. Define the curve $\eta : \mathbb{R} \to A$ by $\eta(t) = (\cos(t), \sin(t))$. Then η is a maximal, unit-speed, locally-minimizing curve in A, but not a geodesic.

The other main subtletly with geodesics on manifolds with boundary is that some directions tangent to the boundary have "no where to go". For instance, consider the annulus A, and the vertical vector $\frac{\partial}{\partial y}$ based at $(2, 0)$. If we take all of $\mathbb{R}^2$ to be the extension of A, then the extended geodesic is just the curve $t \mapsto (2, t)$. The restriction of this curve is just a singleton curve, $\{0\} \ni t \mapsto \{(2, 0)\}$. However, this is not necessarily the case for all tangent directions. For instance, consider the vertical direction at $(1, 0)$ which is is tangent to the interior circle of the annulus. The corresponding geodesic is $[-\sqrt{3}, \sqrt{3}] \ni t \mapsto (1, t)$. The distinction between the behavior of geodesics tangent to the inner circle and outer circle is that the inner circle is concave relative to the interior of the annulus and the outer circle is convex.

Of course, there is nothing inherently wrong with having a curve whose domain is $\{0\}$, except that we can not differentiate it. We would like to differentiate all geodesics to be able to discuss their entry and exit directions. Thus, in the situation that $I_v = \{0\}$, we *define* $\dot{\gamma}_v(0) = v$. In practical situations, this edge case isn't particularly important, however this definition helps our notation stay consistent.

This sublety is not necessary when the geodesic is not tangent to the boundary. In particular, Geodesics that start at the boundary and point strictly inwards must go through the interior for a non-zero amount of time.

Lemma 5.13. Let (M, g) be a Riemannian manifold with boundary. Let $v \in \partial_+ SM$. Then I_v is one of the following

1. $[0, b]$ for $b > 0$
2. $[0, b)$ for $b > 0$
3. $[0, \infty)$

Furthermore, there exists $\varepsilon > 0$ such that $\varepsilon \in I_v$, and $\gamma_v(t) \in M^\circ$ for all $t \in (0, \varepsilon)$.

Let $v \in SM$. If I_v has a closed right endpoint (i.e. $I_v = (a, b]$ or $[a, b]$ where a is allowed to be $-\infty$), then we write $\tilde{\tau}_g(v) = b$. Otherwise we let $\tilde{\tau}_g(v) = \infty$. We call $\tilde{\tau}_g : SM \to [0, \infty]$ the exit-time function. In the event that $\tilde{\tau}_g(v) < \infty$, we define the exit direction to be $\tilde{\alpha}_g(v) = \dot{\gamma}_v(\tilde{\tau}_g(v))$. In the case that $\tilde{\tau}_g(v) = \infty$, we leave $\tilde{\alpha}_g(v)$ undefined. We define corresponding operators for the boundary. For $v \in U\partial M$, let $\tau_g(v) = \tilde{\tau}_g(\hat{v})$, and $\alpha_g(v) = P^\partial(\tilde{\alpha}_g(\hat{v}))$. Then we have $\tau_g : U\partial M \to [0, \infty]$ and $\alpha_g : U\partial M \to \overline{U}\partial M$. Of course, if $\tilde{\alpha}_g(\hat{v})$ is undefined, then so is $\alpha_g(v)$.

5.4 Geodesic Triangles

Let (M, g) be a Riemannian manifold with boundary. Let $p, q, r \in M$ be distinct points such that minimizing geodesic segments $\overline{pq}$, $\overline{qr}$ and $\overline{rs}$ exist and are unique. Then, we write $\Delta pqr = (\overline{pq}, \overline{qr}, \overline{rs})$, and we say Δpqr is a **geodesic triangle**. We note that in general, the three geodesics might not necessarily exist or be unique. However, if $p, q, r \in M^\circ$ and $d_g(p, q), d_g(q, r), d_g(r, p) < \max\{\mathrm{inj}(p), \mathrm{inj}(q), \mathrm{inj}(r)\}$, then each of the geodesic segments exist and are unique. For us, we will only be interested in limits of triangles as all of the vertices approach a fixed interior point. Thus, we will not worry ourselves with such subtleties.

Let Δpqr be a geodesic triangle. Then, we define

$$\angle pqr = \angle(\overline{qp}'(0), \overline{qr}'(0))$$

Where, for two non-zero vectors $v, w \in T_q M$, their angle is given using the cosine formula $\angle(v, w) = \cos^{-1}(\frac{g(v,w)}{|v|_g |v|_w})$.

Recall that, for a Euclidean triangle, the sum of the interior angles is always π. While this is not true for a general geodesic triangle, the deviation from π goes to zero as the vertices get closer together.

Lemma 5.14. Let $p \in M^\circ$ be fixed. Let $\{q_k\}_{k=1}^\infty, \{r_k\}_{k=1}^\infty$ be sequences of points such that $(q_k, r_k) \to (p, p)$ and p, q_k, r_k are distinct for all k. Then

$$\lim_{k\to\infty} [\angle p q_k r_k + \angle q_k r_k p + \angle r_k p q_k]$$

Lemma 5.14 will be used in conjunction with the following fact about Euclidean geometry.

Lemma 5.15. Let e_1, e_2 be orthonormal vectors in a 3-dimensional inner product space $(V, \langle , \rangle)$. Let $v \in V$ be a unit vector. Suppose $\angle(e_1, v) + \angle(v, e_2) = \angle(e_1, e_2)$. Then $v \in \operatorname{span}\{e_1, e_2\}$.

Proof. Let e_1, e_2, v be as stated. Then, using the fact that they are all unit vectors, the sum-of-angles formula above becomes

$$\cos^{-1}\langle e_1, v\rangle + \cos^{-1}\langle e_2, v\rangle = \cos^{-1}\langle e_1, e_2\rangle$$

Applying "cosine" to both sides, using the angle addition formula and the fact that $\sin(\cos^{-1}(x)) = \sqrt{1-x^2}$, we obtain

$$\langle e_1, e_2\rangle = \langle e_1, v\rangle\langle e_2, v\rangle - \sqrt{1-\langle e_1, v\rangle^2}\sqrt{1-\langle e_2, v\rangle^2}$$

But $\langle e_1, e_2\rangle = 0$, so with a little rearranging, we obtain

$$\begin{aligned}\langle e_1, v\rangle^2\langle e_2, v\rangle^2 &= (1-\langle e_1, v\rangle^2)(1-\langle e_2, v\rangle^2)\\ &= 1-\langle e_1, v\rangle^2-\langle e_2, v\rangle^2+\langle e_1, v\rangle^2\langle e_2, v\rangle^2\end{aligned}$$

Thus,

$$\langle e_1, v\rangle^2 + \langle e_2, v\rangle^2 = 1$$

However, we may choose e_3 so that $\{e_1, e_2, e_3\}$ forms an orthonormal basis for V. Then $v = \langle e_1, v\rangle e_1 + \langle e_2, v\rangle e_2 + \langle e_3, v\rangle e_3$. This implies that $|v|^2 = \langle e_1, v\rangle^2 + \langle e_2, v\rangle^2 + \langle e_3, v\rangle^2$. But v is a unit vector, so it follows that $\langle e_3, v\rangle = 0$. Thus, $v \in \operatorname{span}\{e_1, e_2\}$ as required. $\square$

5.5 The Gradient

Let $f : M \to \mathbb{R}$, be smooth near $p \in M$. It follows from the Riesz representation theorem for finite dimensional vector spaces that there is a unique vector $w \in T_pM$ such that $df_p(v) = g(v, w)$ for all $v \in T_pM$. In such a situation, we define the **gradient of f at p** to be $\operatorname{grad}_g(f)\big|_p = w$. The gradient is smooth wherever f is smooth.

Just as in Euclidean space, where the gradient and dot product allow us to form a "fundamental theorem of calculus" for integration on curves, we can use the Riemannian metric and gradient to formulate a similar statement in the Riemannian setting.

Lemma 5.16. Let (M, g) be a Riemannian manifold with boundary. Let $f : M \to \mathbb{R}$ be smooth and $x, y \in M$. Let $r : [a, b] \to M$ be a smooth path from x to y. Then $f(y) - f(x) = \int_a^b g(\dot{r}(t), \operatorname{grad}_g(f)\big|_{r(t)})dt$.

Within the injectivity radius, the gradient of the distance to a fixed point behaves similarly to the Euclidean case. In particular, the gradient "points away" from the fixed point, along geodesics eminating from the fixed point.

Lemma 5.17. Let (M, g) be a Riemannian manifold. Let $x, y \in M$ be such that $d_g(x, y) < \operatorname{inj}(x)$. Then $d_g(x, \bullet)$ is smooth at y and $\operatorname{grad}_g(d_g(x, \bullet))\big|_y = \overline{xy}'(d_g(x, y))$.

We note that, as stated, the lemma above only applies to manifolds without boundary. In particular, we have not defined the injectivity radius for all points on a manifold with boundary. However, we will see that we can adapt the lemma to apply to *simple manifolds* (with boundary), which will be defined in the next chapter.

Finally, we record the relationship between taking the gradient of a function before or after restricting it to the boundary.

Lemma 5.18. Let (M, g) be a Riemannian manifold with boundary. Let $f : M \to \mathbb{R}$ be smooth. Then for $x \in \partial M$, we have $P^{\partial}(\operatorname{grad}_g(f)) = \operatorname{grad}_{g^{\partial}}(f\big|_{\partial M})$.

6 Simple Manifolds

In this chapter, we describe *simple manifolds*. Simple manifolds will be our primary object of study in Chapter 8.

Definition 6.1. Let (M, g) be a Riemannian manifold with boundary. We say that (M, g) has **strictly convex boundary** if the second fundamental form for the boundary is positive definite.

For a definition and discussion of the second fundamental form, see [19]. The only property of strict convexity we will need is the following lemma.

Lemma 6.2. Let (M, g) be Riemannian manifold with strictly convex boundary. Then, for all nonzero $v \in T\partial M$, we have $I_v = \{0\}$.

Intuitively, this implies that if you try and shoot off a geodesic tangent to the boundary, the boundary immediately curves to get in your way. We can *almost* take Lemma 6.2 to be the definition of strict convexity. However, it fails if the boundary doesn't curve fast enough. For instance, the superlevel set $\{(x, y) | y \geq x^4\}$ fails to have strictly convex boundary at the origin, however it does satisfy the conclusion of Lemma 6.2.

Definition 6.3. Let (M, g) be a compact Riemannian manifold with strictly convex boundary. If, for every point $p \in M$, the exponential map $\exp_p : \mathcal{E}_p \to M$ is a diffeomorphism onto its image, we say that (M, g) is **simple**.

In particular, if we say (M, g) is a simple manifold, it is implied that it is a compact manifold with boundary. We note that all simple manifolds are diffeomorphic to a closed disk (of the appropriate dimension).

The next two lemmas follows directly from Definition 6.3.

Lemma 6.4. Let (M, g) be a simple manifold. Then, for any two distinct points $x, y \in M$ there is a unique unit-speed geodesic from x to y, and that geodesic is minimizing.

Lemma 6.5. Let (M, g) be a simple manifold. Then distinct geodesics either have non-empty intersection or intersect at a single point.

For simple manifolds, the distance functions, exit time functions and exit direction functions have nice smoothness properties. For details of the next two lemmas see [27].

Lemma 6.6. Let (M, g) be a simple manifold and $x \in M$. Then $d_g(x, \bullet)$ is smooth on $M \setminus \{x\}$. Furthermore, if $y \in M$ is distinct from x, then $\operatorname{grad}_g(d_g(x, \bullet))\big|_y = \overline{xy}'(d_g(x, y))$.

Lemma 6.7. Let (M, g) be a simple manifold. Then $\tilde{\tau}_g$, τ_g, $\tilde{\alpha}_g$, and α_g are everywhere-defined and continuous functions. τ_g, α_g are smooth on their whole domain and the range of α_g can be taken to be $U\partial M$. Finally, $\tilde{\alpha}_g$ and $\tilde{\tau}_g$ are smooth on $SM \setminus T\partial M$.

The next lemma follows straight forwardly from the previous two lemmas.

Lemma 6.8. Let (M, g) be a simple manifold. Let $v, w \in U\partial M$. Then $\alpha_g(v) = w$ if and only if the following two equations hold

$$\operatorname{grad}_{g^\partial}(d_g(\pi(v), \bullet)\big|_{\partial M})\big|_{\pi(w)} = w$$
$$\operatorname{grad}_{g^\partial}(d_g(\pi(w), \bullet)\big|_{\partial M})\big|_{\pi(v)} = -v$$

and $\tau_g(v) = d_g(\pi(v), \pi(\alpha_g(v)))$.

Let $\operatorname{diag}(M) = \{(x, x) | x \in M\} \subset M^2$. The fact that there is always a single geodesic connecting any two distinct points in a simple manifold allows us to define the following functions.

- For $(p, q) \in M^2 \setminus \operatorname{diag}(M)$, let $\operatorname{dir}(p, q) = \frac{\exp_p^{-1}(q)}{|\exp_p^{-1}(q)|_g}$.
- For $(p, q) \in M^2 \setminus \operatorname{diag}(M)$, let $\operatorname{aim}(p, q) = -P^\partial(\tilde{\alpha}_g(-\operatorname{dir}(p, q)))$.

Then $\operatorname{dir} : M^2 \setminus \operatorname{diag}(M) \to SM$ is smooth and $\operatorname{aim} : M^2 \setminus \operatorname{diag}(M) \to U\partial M$ is smooth. Unpacking their definitions, we get the following more intuitive descriptions of aim and dir.

- $\operatorname{dir}(p, q)$ is the direction you need to shoot a geodesic starting at p to hit q.
- If $v = \operatorname{aim}(p, q)$, then $\hat{v}$ is the unique inward pointing direction at the boundary whose geodesic hits p first and then q.

7 Geodesic Inverse Problems

In this chapter we discuss the problems of *boundary rigidity*, *lens rigidity*, *source distance rigidity*, and *broken scattering rigidity*. While we will only use the results of source distance rigidity directly, we would not even be thinking about intersection rigidity or stitching data were it not for the work done on boundary rigidity or lens rigidity. For a more complete review of geodesic measurement problems, we refer readers to [29].

All four of the problems we will discuss can be thought of as different ways to measure geodesics from the boundary of a manifold. Thus, we classify such problems as *geodesic inverse problems.* Geodesic inverse problems are a subset of all *geometric inverse problems*, where the measurements made have something to do with the geometry of a space, but may not be directly phrased in terms of the geodesics. For instance, the classic *can you hear the shape of a drum?* problem posed by Kac in [13] of determining a Riemannian manifold from the Dirichlet eigenvalues of its Laplace-Beltrami operator is a geometric inverse problem which is not a geodesic inverse problem. Another classic geometric inverse problem is the Calderón problem of determining the pointwise interior conductivity of a material from voltage and current measurements at its boundary [5].

We emphasize that different types of geometric inverse problems often have connections to eachother. For instance, in [25], Pestov and Uhlmann relate the boundary rigidity problem to the problem of recovering a Riemannian metric from knowledge of its Dirichlet-to-Neumann map, which is the natural extension of the Calderón problem to Riemannian manifolds.

Finally, we note that anyone reading to simply understand the new results in this thesis can skip to Section 7.5, and then proceed to the new results in Chapters 8 and 9.

7.1 Boundary Rigidity

The *boundary rigidity problem* is to determine the interior geometry of a Riemannian manifold from the knowledge of the distance between boundary points. More specifically, given two Riemannian manifolds with boundary, (M_i, g_i) for $i = 1, 2$, suppose there exists a diffeomorphism of the boundaries $\varphi^\partial : \partial M_1 \to \partial M_2$ such that

$$\varphi^\partial : (M_1, d_{g_1}|_{\partial M_1 \times \partial M_1}) \to (M_2, d_{g_2}|_{\partial M_2 \times \partial M_2})$$

is an isometry. Does this imply that φ^∂ extends to an isometry $\varphi : (M_1, g_1) \to (M_2, g_2)$?

If (M_1, g_1) is such that any other manifold (M_2, g_2) with isometric boundary as above has isometric interiors as above, we say that (M_1, g_1) is **boundary rigid**. If we have an entire collection of Riemannian manifolds with boundary that are each boundary rigid, we label the whole collection as boundary rigid.

We note that if $\varphi^\partial : (\partial M_1, d_{g_1}) \to (\partial M_2, d_{g_2})$ is an isometry, then it is also a Riemannian isometry from $(\partial M_1, g_1^\partial)$ to $(\partial M_2, g_2^\partial)$. In particular, the Riemannian structure on the boundary can be recovered by considering the intrinsic metric generated by the restricted metric (see for [3] for details on generating an intrinsic metric from a non-intrinsic metric). Thus, the tuple $(\partial M, g^{\partial}, d_g|_{\partial M\times\partial M})$ is equivalent to $(\partial M, d_g|_{\partial M\times\partial M})$.

7.1.1 Physical Motivation

Geophysicists were among the first people interested in such problems. Let $E \subset \mathbb{R}^3$ be the (approximately ball-shaped) subset of Euclidean space corresponding to the Earth (not just the surface of the earth, but also its interior). For a point $x \in E$, let $c^2(x)$ denote the speed of sound at x. This will vary depending on the pointwise material composition of the interior of the earth. Let $\overline{g}$ denote the Euclidean inner product and $\tilde{g} = \frac{1}{c^2(x)}g$ denote the conformally equivalent "sound-speed" Riemannian metric. If $p, q \in \partial E$, then $d_{\tilde{g}}(p, q)$ corresponds to how much time it would take an earthquake with epicenter at p to be detected at q. Thus, with enough seismometers (earthquake-measuring devices), and enough earthquakes, one could hope to construct some reasonable approximation of $d_{\tilde{g}}|_{\partial E\times\partial E}$.

The recovery of $c^2(x)$ from knowledge of $d_{\tilde{g}}|_{\partial E\times\partial E}$ is the *isotropic geophysical boundary rigidity problem*. Herglotz, Wiechert and Zoeppritz showed in [12, 35] that the boundary distances do determine $c^2(x)$, provided the sound-speed function depends only on the radius and satisfies the Herglotz condition:

$$\frac{d}{dr}\left(\frac{r}{c^2(r)}\right) > 0$$

where r is the radial distance from the center of the earth.

Of course, the speed of sound may depend on position in a more complicated way than $c^2(x) = c^2(r)$. Additionally, the speed of sound may change based on the direction the soundwave is travelling in. For instance, in wood the speed of sound "with the grain" is typically about three to five times faster then the speed of sound "across the grain" [31]. For the earth, seismic measurements have demonstrated that the inner core of the Earth has an anisotropic sound speed [6]. Rather than allow for complete generality in the anisotropic sound speed, where we would have the speed of sound is $c^2(x, v)$ is a general function of position x and direction v, it is common to assume that the speed of sound is $\frac{1}{\sqrt{g(v,v)}}$ for some Riemannian metric g on E, whenever v is a Euclidean unit vector indicating direction. In this case, the sound speed boundary distances will be $d_g|_{\partial E\times\partial E}$.

7.1.2 Boundary Rigidity Counterexamples and Full Data Results

Let (M, g) be a compact Riemannian manifold with boundary. Suppose there exists $p \in M^\circ$ and an open neighborhood U of p such that no minimizing curve between boundary points passes through U. Then one can perturb the metric on U without changing the boundary distances $d_g|_{\partial M\times\partial M}$. Specifically, this happens when $d_g(p, \partial M) > \sup_{x,y\in\partial M} d_g(x, y)$. For a concrete example, consider the unit 2-sphere with a small open neighborhood of the southpole removed. Then no minimizing curve will enter the northern hemisphere.

Thus, if one hopes to recover the geometry of (M, g) from its boundary distances, we must place restrictions on the geometry. Perhaps the most common restriction is to assume that (M, g) is simple. In [20], Michel conjectures that simple manifolds are boundary rigid. The two-dimensional case of Michel's conjecture was shown in [25], however it is still an open problem in three and higher dimensions.

Before the two dimensional case was shown in general, a number of specific classes of simple manifolds were shown to be boundary rigid.

- simple subspaces of Euclidean space [10]
- simple subspaces of the open two-dimensional hemisphere [21]
- simple subspaces of symmetric spaces with contstant negative curvature [1]
- simple two dimensial spaces with negative curvature [9, 24]
- simple metrics sufficiently close to the Euclidean metric [16, 4]
- simple metrics that are conformally equivalent to the Euclidean metric [22]

It has been observed that Mukhometov's proof in [22] extends to simple metrics in any fixed conformal class. In otherwords, if you know that (M, g) is simple, and you are given the conformal class of g, and the boundary distances, then you can recover the interior geometry.

Additionally, it is known that there is a generic class of simple metrics which can be distinguished from eachother by their boundary distances [28].

It should be noted that the proof of two-dimensional simple boundary rigidity reduces the problem to determining the Dirichlet-to-Neumann map for the manifold [32, 25]. While the connection between a problem which can be stated in terms of pure geometry (boundary rigidity), and a more analytic/PDE type problem (the Dirichlet-to-Neumann map) is beautiful, one would also like to have a purely geometric proof of boundary rigidity. The intersection rigidity problem was initiated to try to find a more geometric proof of two-dimensional boundary rigidity. This is because, for simple manifolds, the boundary distances very easily determine the lens data (defined in the next section), and we will see that the lens data easily determines the cross relation (again, only for two dimensional simple manifolds). Somewhat ironically, the proof of intersection rigidity only works in three and higher dimensions. Thus, we are still left searching for a geometric proof of two-dimensional simple boundary rigidity.

7.2 Lens Rigidity

In the last section, we ran into an issue when there was a region of the manifold though which no minimizing geodesics pass. In essence, there is no geometric information about such a region contained in the boundary distance data. We avoided this issue by restricting to a class of manifolds for which this does not occur. However, we make the observation that, in any compact, connected manifold with non-empty boundary, there are always *some* geodesics from the boundary through every interior point. They just might not be minimizing geodesics. Thus, we might hope to gain more information about the interior of the manifold by considering non-minimizing geodesics.

The **lens data** is the triple $(\partial M, \tau_g, \alpha_g)$. Intuitively, the lens data corresponds to the information you would collect by performing the following experiments: for each point $x \in \partial M$ and dirction $v \in \partial_+ SM$, shoot off a geodesic in the direction of v. Then, record how long it takes to exit the manifold through the boundary (if this even occurs at all), and record what direction it exits the manifold in. In the seismic imaging setting, this corresponds to recording *all* incoming seismic waves (not just the first one, as we do to obtain the boundary distance), and recording the direction the incoming wave is traveling.

When (M, g) is simple, Lemma 6.8 demonstrates that the boundary distance data is equivalent to the lens data. However, as discussed above, when there is a region which contains no minimizing geodesics, the lens data contains strictly more information about the manifold.

We formulate the **lens rigidity problem** as follows. Let (M_i, g_i) for $i = 1, 2$ be connected Riemannian manifolds with boundary. If there exists a diffeomorphism $\varphi^\partial : \partial M_1 \to \partial M_2$ such that

- $d\varphi^\partial : U\partial M_1 \to U\partial M_2$ is a diffeomorphism
- $\tau_{g_2}(d\varphi^\partial(v)) = \tau_{g_1}(v)$ for all $v \in U\partial M_1$
- $\alpha_{g_2}(d\varphi^\partial(v)) = d\varphi^\partial(\alpha_{g_1}(v))$ for all $v \in U\partial M_1$

does this imply that φ^∂ extends to an isometry $\varphi : M_1 \to M_2$? We note here that the hypothesis that $d\varphi^\partial : U\partial M_1 \to U\partial M_2$ is a diffomorphism implies that φ^∂ is an isometry from $(\partial M_1, g_1^\partial)$ to $(\partial M_2, g_2^\partial)$.

Similarly to boundary rigidity, we say that a manifold (M_1, g_1) is **lens rigid** if for all (M_2, g_2) and φ^∂ as above, then φ^∂ extends to an isometry of the interiors.

7.2.1 Lens Rigidity Results

The following classes of manifolds are known to be lens rigid.

- real analytic manifolds (with a small assumption about conjugate points) [34]
- finite quotients of other lens-rigid manifolds [7]
- the flat solid torus $\mathbb{D}^n \times \mathbb{S}^1$ for $n \geq 2$ [8]
- two-dimensional, oriented, non-trapping compact Riemannian manifolds without conjugate points [11]

- three-and-higher dimensional compact manifolds with boundary that admit a foliation by strictly convex hypersurfaces [30]

In the situation where there is a foliation by strictly convex hypersurfaces, the proof proceeds by analyzing the geodesic x-ray transform (see [33] for details on the geodesic x-ray transform). We emphasize that there are no known examples of simple manifolds which do not admit a convex foliation, and infact it seems plausible that all simple manifolds *may* admit such a folation. Thus, anyone wishing to prove Michel's original conjecture [20] should at least consider trying to show that all simple manifolds admit such a foliation.

7.3 Broken Scattering Rigidity

Consider a diffuse medium of particles, such that each particle can scatter an incoming ray of light in a random direction. If you could shoot light beams along geodesics, and assume that a typical beam will be scattered one time, you could measure the incoming and outgoing direction and the total travel time.

A **broken geodesic** is a continuous curve $\eta : [0, L_g(\eta)] \to M$ such that

$$\eta(t) = \begin{cases} \gamma_{\hat{v}}(t), & t < t^* \\ \gamma_z(t - t^*), & t \geq t^* \end{cases}$$

For some $\hat{v} \in \partial SM$, $z \in SM$, and $t^* \in [0, L_g(\eta)]$. For such a curve, we define its incoming direction to be $v \in \overline{U}\partial M$ and its outgoing direction to be $P^{\partial}(\tilde{\alpha}_g(z))$.

Having defined broken geodesics, we define the **broken scattering relation**, R_g, by

$$\begin{multline} R_g = \{(v, w, l) \in U\partial M \times U\partial M \times [0, \infty) | \exists \text{ a broken geodesic } \eta, \text{ such that } v, w, l \\ \text{are the incoming direction, outgoing direction} \\ \text{and length of } \eta \text{ respectively}\} \end{multline}$$

We call the pair $(\partial M, R_g)$ the **broken scattering data**.

The **broken scattering rigidity problem** is the following. Let (M_i, g_i), $i = 1, 2$ Riemannian manifolds with boundary. If there exists such that there exists $\varphi^{\partial} : \partial M_1 \to \partial M_2$ satisfying

- $d\varphi^{\partial} : U\partial M_1 \to U\partial M_2$ is a diffeomorphism
- The map $(v, w, l) \mapsto (d\varphi^{\partial}(v), d\varphi^{\partial}(w), l)$ is a bijection from R_{g_1} to R_{g_2}.

does this imply that φ^{∂} extends to an isometry from M_1 to M_2?

If (M_1, g_1) is such that for all (M_2, g_2) and φ^{∂} satisfying the above constraints, then φ^{∂} extends to an isometry, we say that (M_1, g_1) is **broken scattering rigid**. In [14], Kurylev, Lassas and Uhlmann show that compact connected Riemannian manifolds with non-empty boundary are broken scattering rigid if they are at least three-dimensional.

7.4 Interior Source Rigidity

Imagine you have a large (approximately dense) collection of objects on the interior of a Riemannian manifold emitting particles in all directions along geodesics. In this section, we describe two inverse problems that can be thought of as measurements that one could make in this situation.

7.4.1 Source Distance Rigidity

The first measurement is the geodesic distance to each interior point. Let $SRC_g = \{d(x, \bullet)|_{\partial M}\}_{x \in M}$. Then we call the pair $(\partial M, \mathrm{SRC}_g)$ the **source distance data**. The **source distance rigidity problem** is the following. Let (M_i, g_i) be Riemannian manifolds with boundary for $i = 1, 2$. Suppose there exists a diffeomorphsim $\varphi^\partial : \partial M_1 \to \partial M_2$ such that the correspondence $\mathrm{SRC}_{g_2} \ni f \mapsto f \circ \varphi^\partial \in C^\infty(\partial M_1)$ is a bijection onto SRC_{g_1}. Does this imply that φ^∂ extends to an isometry from M_1 to M_2?

As before, we say that a manifold (M_1, g_1) is **source distance rigid** if for all (M_2, g_2) and φ^∂ satisfying the above, we know that φ^∂ extends to an isometry.

In [26], Pestov and Uhlmann consider the partial data situation in the case that the metric is conformally equivalent to a Euclidean metric. They provide a reconstruction procedure for the conformal factor when the source distances are known in a neighborhood of a point on the interior, provided that neighborhood is reachable via boundary normal coordinates.

We will prove source distance rigidity for simple manifolds. To do this, we take advantage of the differentiability of the source distance functions at the boundary to generate a relatively straightforward proof in section 8.7.

7.4.2 Source Direction Rigidity

Rather than knowing the distances to each source point, we could know the exit directions of particles eminating from source points. This was the convention used in [15]. In this document, we take the opposite convention and consider the inward pointing directions that correspond to geodesics that will contain a particular point. Of course these objects are equivalent, as one can convert between the two conventions simply by multiplying all vectors by -1.

Specifically, for $x \in M$, define $\mathrm{star}_g(x) = \{v \in U\partial M | x \in \gamma_{\hat{v}}\}$. Then, we let $\mathrm{STAR}_g = \{\mathrm{star}_g(x)\}_{x \in M} \subset 2^{U\partial M}$. We call the triple $(\partial M, g^\partial, \mathrm{STAR}_g)$ the **source direction data**. The **source direction rigidity problem** is as follows. Let (M_i, g_i) be Riemannian manifolds with boundary for $i = 1, 2$. Suppose there exists a diffeomorphism $\varphi^\partial : \partial M_1 \to \partial M_2$ such that the map $\mathrm{STAR}_{g_1} \ni S \mapsto d\varphi^\partial(S) \in 2^{U\partial M}$ is a bijection onto STAR_{g_2}. Does this imply that φ^∂ extends to an isometry from M_1 to M_2?

If (M_1, g_1) is such that, for all (M_2, g_2) and φ^∂ as above, we have that φ^∂ extends to an isometry from M_1, to M_2 we say that (M_1, g_1) is **source direction rigid**.

In [15], Lassas, Saksala and Zhou show that non-trapping manifolds with strictly convex boundary are source direction rigid. We will need to show that simple manifolds are source direction rigid. We emphasize that this case is completely contained within the theorem proven in [15]. However, just as in the source distance situation, the differentiability of the distance functions will allow for a more direct proof of source direction rigidity for simple manifolds. This is shown in section 8.6.

7.5 Quantification

We observe that all of the above rigidity problems are stated in a way that involves the existence of a map such that the data on any two manifolds will conjugate in a natural way with respect to the map. This definition encourages a method of arguing in parallel, where you constantly have two Riemannian manifolds (or one manifold with two metrics), and you first describe some object on one manifold, and then show that the construction "conjugates", and so the information you have identified is the "same" on the other manifold.

However, one can also argue in a more direct manner. For instance, consider the fact that the lens data is "determined" by the boundary distance data when (M, g) is simple. To show this in the "parallel" way, one would make the following argument

- If (M_i, g_i) are simple Riemannian manifolds for $i = 1, 2$ such that there exists $\varphi^\partial : \partial M_1 \to \partial M_2$ which conjugates the boundary distance data, then φ^∂ conjugates the lens data.

Alternatively, one can just write the lens data *directly* in terms of the boundary distance data. This is what is done in Lemma 6.8, however there are some sublteties for this. We claim that we can write all objects in the tuple $(\partial M, \tau_g, \alpha_g)$ in terms of the objects in $(\partial M, d_g|_{\partial M\times\partial M})$ (provided that (M, g) is simple). At first glance, Lemma 6.8 seems to do this quite transparently. However, we invoked the gradient $\mathrm{grad}_{g^\partial}$ in our description of the lens data in terms of the boundary distance data. This is still valid, since the Riemannian structure of the boundary may be written in terms of the boundary distances, as indicated in the discussion in Section 7.1.

Let us be more precise in describing the argument in the preceding paragraph. Let $(A_1, ..., A_n)$ be a tuple of mathematical objects. Let B be a mathematical object. Then we say that the tuple **quantifies** B, and we write $(A_1, ..., A_n) \rightsquigarrow B$, if it can be written directly in terms of the objects in the tuple. If $(B_1, ..., B_m)$ is such that $(A_1, ..., A_n) \rightsquigarrow B_k$ for all $k = 1, ..., m$, we write $(A_1, ..., A_n) \rightsquigarrow (B_1, ..., B_m)$.

Quantification is transitive, in the sense that if $(A_1, ..., A_n) \rightsquigarrow B$, and $(A_1, ..., A_n, B) \rightsquigarrow C$, then $(A_1, ..., A_n) \rightsquigarrow C$. Intuitively, this means that if one wishes to show that $(A_1, ..., A_n)$ quantifies C, then we may use any other object that is quantified by $(A_1, ..., A_n)$ in the quantification of C. For instance our argument that the lens data is quantified by the boundary distance data takes the following form.

- First, $(\partial M, d_g|_{\partial M\times\partial M}) \rightsquigarrow g^\partial$.

- Then $(\partial M, g^{\partial}) \rightsquigarrow \mathrm{grad}_{g^{\partial}}$.
- Finally $(\partial M, g^{\partial}, \mathrm{grad}_{g^{\partial}}, d_g\big|_{\partial M \times \partial M}) \rightsquigarrow (\partial M, \tau_g, \alpha_g)$.

When reading a claim of quantification, one should have two questions in mind. First, if the relationship decribed is a true statement, does it actually write one object in terms of the other? Second, is the relationship described true? For instance, in Lemma 6.8, we must ask if we have actually described the lens data in terms of the boundary distance data (without accidentally invoking some hidden information about the interior of the manifold). Then, we may evaluate the truthfulness of the claim. In other words, is the stated relationship between the gradients of the distance functions and the lens data accurate. In proving that the relationship is valid, one will necessarily have to invoke the interior of the manifold.

In the remainder of this document, we follow this forward and constructive style of argument, rather than the parallel "conjugation" style of argument. We will attempt to make our quantifications as transparent as possible by using one of the following styles.

- state that A quantifies B in a lemma or theorem, and then have a separate lemma which describes the relationship between A and B such that it is obvious it is a quantification.
- state that A quantifies B in a lemma or theorem, and then describe the quantification in the proof of the lemma or theorem as quickly as possible (typically within the first paragraph). Then, the remainder of the proof will be used to show that the relationship described in the quantification is valid.

8 Intersection Rigidity

In this chapter, we prove our first major new contribution: the intersection data determines the geometry of a simple manifold when the manifold is at least three dimensional. For an intuitive description of the intersection data, see Chapter 1. A formal definition of the intersection data is given in the next section.

The proof is outlined as follows.

- Show that the intersection data quantifies the source direction data
- Show that the source direction data quantifies the lens data
- Show that the source direction data quantifies the source distance data
- Show that the source distance data quantifies a metric space which is isometric to (M, d_g).

8.1 The Intersection Data

Let (M, g) be a Riemannian manifold with boundary. For vectors $v, w \in U\partial M$ write $v\mathrm{C}_g w$ if $\gamma_{\hat{v}} \cap \gamma_{\hat{w}} \neq \varnothing$. This defines a reflexive relation on $U\partial M$. We call C_g the **cross relation**.

It will be easier to deal with interior intersections first. When $\gamma_{\hat{v}} \cap \gamma_{\hat{w}} \cap M^\circ \neq \varnothing$, we write $v\mathrm{C}^\circ_g w$. Again, this defines a reflexive relation on $U\partial M$. We call C°_g the **interior cross relation**. We call the tuples $(\partial M, g^\partial, \mathrm{C}_g)$ and $(\partial M, g^\partial, \mathrm{C}^\circ_g)$ the **intersection data** and the **interior intersection data** resprectively. At the end of this chapter, we show that the interior intersection data is equivalent to the intersection data whenever (M, g) is simple.

For $v_1, ..., v_n \in U\partial M$, we define $(v_1\mathrm{C}^\circ_g\bullet) = \{w \in U\partial M | v_1\mathrm{C}^\circ_g w\}$ and $[(v_1, ..., v_n)\mathrm{C}^\circ_g\bullet] = (v_1\mathrm{C}^\circ_g\bullet)\cap(v_2\mathrm{C}^\circ_g\bullet)\cap...\cap(v_n\mathrm{C}^\circ_g\bullet)$. We define $(v_1\mathrm{C}_g\bullet)$ and $[(v_1, ..., v_n)\mathrm{C}_g\bullet]$ in the corresponding way. We call sets of the form $(v_1\mathrm{C}^\circ_g\bullet)$ and $[(v_1, ..., v_n)\mathrm{C}^\circ_g\bullet]$ **cross sets** and **multicross sets** respectively.

8.2 The Source Direction Data

For $x \in M$ define the **star set for** x to be $\mathrm{star}_g(x) = \{v \in U\partial M | x \in \gamma_{\hat{v}}\}$. We define the following collections $\mathrm{STAR}_g = \{\mathrm{star}_g(x)\}_{x\in M}$ and $\mathrm{STAR}^\circ_g = \{\mathrm{star}_g(x)\}$. We call the tuples $(\partial M, g^\partial, \mathrm{STAR}_g)$ and $(\partial M, g^\partial, \mathrm{STAR}^\circ_g)$ the **source direction data**, and the **interior source direction data** respectively. At the end of this chapter we show that the interior source direction data is equivalent to the source direction data whenever (M, g) is simple.

For general (M, g) and x, the structure of $\mathrm{star}_g(x)$ can be quite complicated. However, when (M, g) is simple and $x \in M^\circ$, for any $y \in \partial M$, there is a unique vector $\mathrm{aim}(y, x) \in U_y\partial M$ such that the corresponding geodesic, $\gamma_{\widehat{\mathrm{aim}(y,x)}}$ passes through x. In such a situation, define $\sigma_x(y) = \mathrm{aim}(y, x)$. Since aim is smooth on its domain, this defines a smooth section $\sigma_x \in \Gamma(U\partial M)$. When (M, g) is simple, $S \in \mathrm{STAR}^\circ_g$, $\sigma \in \Gamma(U\partial M)$ and $S = \sigma(U\partial M)$, we say that σ **represents** S. Of course, there is a one-to-one correspondence between sets in STAR°_g and the sections in $\Gamma(U\partial M)$ that represent them, whenever (M, g) is simple.

8.3 The Source Distance Data

For $x \in M^\circ$, let $d_g(x, \bullet)\big|_{\partial M}$ be the function from ∂M to $\mathbb{R}$ defined by $y \mapsto d_g(x, y)$. We define the **source distances** to be the set $\mathrm{SRC}_g = \{d_g(x, \bullet)\big|_{\partial M}\}_{x\in M}$, and the **interior source distances** to be the set $\mathrm{SRC}^\circ_g = \{d_g(x, \bullet)\big|_{\partial M}\}_{x\in M^\circ}$. We call the tuples $(\partial M, g^\partial, \mathrm{SRC}_g)$ and $(\partial M, g^\partial, \mathrm{SRC}^\circ_g)$ the **source distance data** and the **interior source distance data** respectively. At the end of this chapter, we show that the interior source distance data is equivalent to the source distance data whenever (M, g) is simple.

8.4 The Intersection Data Determines the Source Direction Data

In this section, we show the following.

Theorem 8.1. Let (M, g) be a simple manifold such that $\dim(M) \geq 3$. Then $(\partial M, g^{\partial}, \mathrm{C}_g^{\circ}) \rightsquigarrow \mathrm{STAR}_g^{\circ}$.

The quantification will be different when $\dim(M) = 3$ and when $\dim(M) \geq 4$. In both situations, we will be interested in the structure of the multicross set $[(v, w, z)\mathrm{C}_g^{\circ}\bullet]$ for three vectors $v, w, z \in U\partial M$.

In the four-or-higher dimensional case, we will see that the set $[(v, w, z)\mathrm{C}_g^{\circ}\bullet]$ is at least 3 dimensional whenever $\gamma_{\hat{v}}, \gamma_{\hat{w}}, \gamma_{\hat{z}}$ are distinct geodesics intersecting at a single point in M°. However, $[(v, w, z)\mathrm{C}_g^{\circ}\bullet]$ is at most 2 dimensional when $\gamma_{\hat{v}}, \gamma_{\hat{w}}, \gamma_{\hat{z}}$ intersect at 3 distinct points. Thus, we can use the interior intersection data to distinguish between these two situations. Once we are able to identify when three geodesics intersect at a single point in the interior, we will swap out one direction at a time to find all geodics that pass through a single point.

The three dimensional case will be more subtle. We will analyze the path connected component of $[(v, w, z)\mathrm{C}_g^{\circ}\bullet]$ which contains v. In the situation where $\gamma_{\hat{v}}, \gamma_{\hat{w}}, \gamma_{\hat{z}}$ intersect at 3 distinct points, we will see that this cannot be the image of a smooth section in $\Gamma(U\partial M)$, where as in the situation where $\gamma_{\hat{v}}, \gamma_{\hat{w}}, \gamma_{\hat{z}}$ pass through a single point and the tangent vectors form an orthonormal basis at the intersection point, then the path connected component is exactly the star set for the point of intersection.

8.4.1 Quantification when $\dim(M) \geq 4$

We offer the following quantification of STAR_g° in terms of $(\partial M, g^{\partial}, \mathrm{C}_g^{\circ})$ in the four-or-higher dimensional setting.

Lemma 8.2. Let (M, g) be a simple manifold such that $\dim(M) \geq 4$. Then $S \in \mathrm{STAR}_g^{\circ}$ if and only if

1. $\exists \sigma \in \Gamma(U\partial M)$ such that $S = \sigma(\partial M)$.
2. $S \subset [(v, w, z)\mathrm{C}_g^{\circ}\bullet]\ \forall v, w, z \in S$.

We will prove Lemma 8.2 at the end of this subsection. Consider two non-intersecting geodesics $\gamma_{\hat{v}}$ and $\gamma_{\hat{w}}$. Then the set of vectors in $U\partial M$ which correspond to geodesics that pass through both $\gamma_{\hat{v}}$ and $\gamma_{\hat{w}}$ on the interior of M is $[(v, w)\mathrm{C}_g^{\circ}\bullet]$. Observe that a vector $u \in [(v, w)\mathrm{C}_g^{\circ}\bullet]$ can be uniquely described by its intersection point with $\gamma_{\hat{v}}$, its intersection point with $\gamma_{\hat{w}}$, and the order in which it intersects $\gamma_{\hat{v}}$ and $\gamma_{\hat{w}}$. Thus, $[(v, w)\mathrm{C}_g^{\circ}\bullet]$ is the union of the images of the following maps

$$I_{\hat{v}} \times I_{\hat{w}} \ni (s, t) \mapsto \operatorname{aim}(\gamma_{\hat{v}}(s), \gamma_{\hat{w}}(t))$$
$$I_{\hat{w}} \times I_{\hat{v}} \ni (s, t) \mapsto \operatorname{aim}(\gamma_{\hat{w}}(s), \gamma_{\hat{v}}(t))$$

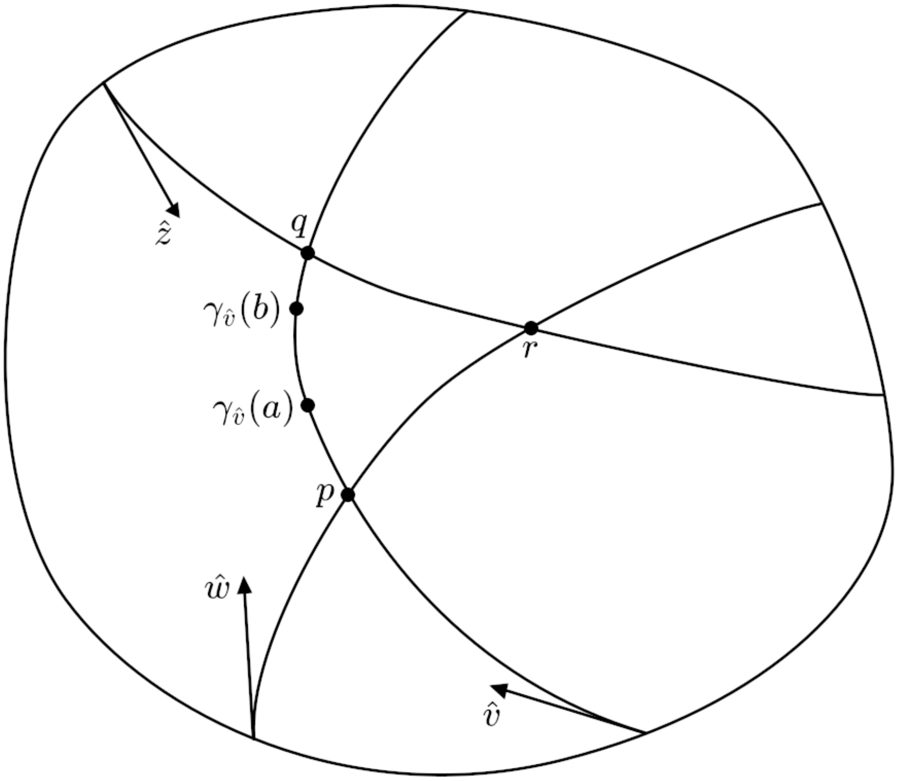


Figure 1: three distinct geodesics intersecting at three distinct points

In the following lemma, we provide a similar analysis for $[(v,w,z)\mathrm{C}^\circ_g\bullet]$ in the case where $\gamma_{\hat{v}}, \gamma_{\hat{w}}, \gamma_{\hat{z}}$ are distinct geodesics with pairwise intersections at three distinct points. We break up the geodesics into segments which do not intersect, and then apply a similar argument as above.

Lemma 8.3. Let (M,g) be simple. Suppose $v,w,z \in U\partial M$ satisfies the following

1. $\gamma_{\hat{v}}, \gamma_{\hat{w}}, \gamma_{\hat{z}}$ are distinct.
2. $v\mathrm{C}^\circ_g w$, $v\mathrm{C}^\circ_g z$, $w\mathrm{C}^\circ_g z$.
3. $\gamma_{\hat{v}} \cap \gamma_{\hat{w}} \cap \gamma_{\hat{z}} = \varnothing$..

Then there exists finitely many smooth functions $\{f_i : (a_i,b_i) \times (c_i,d_i) \to U\partial M\}_{i=1}^N$ such that $[(v,w,z)\mathrm{C}^\circ_g\bullet] \subset \bigcup\limits_{i=1}^N f_i\Big((a_i,b_i)\times(c_i,d_i)\Big)$.

Proof. It follows from Lemma 6.5 and properties (1.)-(3.) that there exists distinct points $p,q,r \in M^\circ$ such that $\gamma_{\hat{v}} \cap \gamma_{\hat{w}} = \{p\}, \gamma_{\hat{v}} \cap \gamma_{\hat{z}} = \{q\}$, and $\gamma_{\hat{w}} \cap \gamma_{\hat{z}} = \{r\}$. Without loss of generality, we assume that the intersections of the three geodesics are oriented as in Figure 1 (i.e. $\gamma_{\hat{v}}$ intersects $\gamma_{\hat{w}}$ before $\gamma_{\hat{z}}$; $\gamma_{\hat{w}}$ intersects $\gamma_{\hat{v}}$ before $\gamma_{\hat{z}}$; $\gamma_{\hat{z}}$ intersects $\gamma_{\hat{v}}$ before $\gamma_{\hat{w}}$).

Let $t_v = \tau_g(v), t_w = \tau_g(w)$, and $t_z = \tau_g(z)$. Let t_1 and t_2 be such that $\gamma_{\hat{v}}(t_1) = p$ and $\gamma_{\hat{v}}(t_2) = q$. Choose $a,b \in (t_1,t_2)$ such that $a < b$, as in Figure 1. Then the following four products are in the domain of aim (i.e. $M^2 \setminus \mathrm{diag}(M)$).

1. $\gamma_{\hat{v}}\big((0,b)\big) \times \gamma_{\hat{z}}\big((0,t_z)\big)$
2. $\gamma_{\hat{z}}\big((0,t_z)\big) \times \gamma_{\hat{v}}\big((0,b)\big)$
3. $\gamma_{\hat{v}}\big((a,t_v)\big) \times \gamma_{\hat{w}}\big((0,t_w)\big)$
4. $\gamma_{\hat{w}}\big((0,t_w)\big) \times \gamma_{\hat{v}}\big((a,t_v)\big)$

Thus, the following defines four smooth functions.

1. $f_1 : (0,b) \times (0,t_z) \to U\partial M$ by $f_1(s,t) = \mathrm{aim}(\gamma_{\hat{v}}(s), \gamma_{\hat{z}}(t))$
2. $f_2 : (0,t_z) \times (0,b) \to U\partial M$ by $f_2(s,t) = f_1(t,s)$
3. $f_3 : (a,t_v) \times (0,t_w) \to U\partial M$ by $f_3(s,t) = \mathrm{aim}(\gamma_{\hat{v}}(s), \gamma_{\hat{w}}(t)) = f_3(t,s)$
4. $f_4 : (0,t_w) \times (a,t_v) \to U\partial M$ by $f_4(s,t) = f_3(t,s)$

We claim that the union of the images of the above four smooth functions contains $[(v,w,z)\mathrm{C}^\circ_g\bullet]$. Let $u \in [(v,w,z)\mathrm{C}^\circ_g\bullet]$. Then $v\mathrm{C}^\circ_g u$, so there exists $c \in (0,t_v)$ such that $\gamma_{\hat{v}}(c) \in \gamma_{\hat{u}}$. Then at least one of the following statements is true.

A. $c \in (0, b)$

B. $c \in (a, t_v)$

In case (A), if $\gamma_{\hat{u}}$ passes through $\gamma_{\hat{v}}$ before $\gamma_{\hat{z}}$, then u is in the image of f_1, otherwise it is in the image of f_2. In case (B), if $\gamma_{\hat{u}}$ passes through $\gamma_{\hat{v}}$ before $\gamma_{\hat{w}}$, then u is in the image of f_3 otherwise it is in the image of f_4. $\square$

Thus, if $\gamma_{\hat{v}}, \gamma_{\hat{w}}, \gamma_{\hat{z}}$ intersect at three distinct points, as in Figure 1, then $[(v, w, z)\mathrm{C}_g^{\circ}\bullet]$ is contained in the smooth image of a two-dimensional set. The contrapositive of the preceding statement, and the corollary of Sard's theorem (Corollary 4.8) yield the following lemma.

Lemma 8.4. Let (M, g) be a simple manifold such that $\dim(M) \geq 4$. Suppose $v, w, z \in U\partial M$ satisfy hypotheses (1.) and (2.) from Lemma 8.3. If there exists $\sigma \in \Gamma(U\partial M)$ such that $\sigma(\partial M) \subset [(v, w, z)\mathrm{C}_g^{\circ}\bullet]$. Then $\gamma_{\hat{v}} \cap \gamma_{\hat{w}} \cap \gamma_{\hat{z}} = \{x\}$ for some $x \in M^{\circ}$.

Proof. Let $v, w, z \in U\partial M$ satisfy hypotheses (1.) and (2.) from Lemma 8.3. Additionally, suppose $\sigma(U\partial M) \subset [(v, w, z)\mathrm{C}_g^{\circ}\bullet]$. For contradiction, suppose $\gamma_{\hat{v}} \cap \gamma_{\hat{w}} \cap \gamma_{\hat{z}}$ is not a singleton. Then $\gamma_{\hat{v}} \cap \gamma_{\hat{w}} \cap \gamma_{\hat{z}} = \varnothing$ follows from the fact that distinct geodesics in simple manifolds intersect at a single point if they intersect at all. Thus, we know that v, w, z satisfies the hypotheses of Lemma 8.3. Thus, let $\{f_i : (a_i, b_i) \times (c_i, d_i) \to U\partial M\}_{i=1}^{N}$, be the functions guaranteed by the lemma.

Then $\pi \circ f_i : (a_i, b_i) \times (c_i, d_i) \to \partial M$ defines a smooth function. But $\dim(M) \geq 4$, so $\dim(\partial M) > 2$. Thus, by Corollary 4.8, the union of the images of $\pi \circ f_i$ has measure zero in ∂M. It follows from Lemma 8.3 that $\pi([(v, w, z)\mathrm{C}_g^{\circ}\bullet])$ has measure zero in ∂M. However, if $[(v, w, z)\mathrm{C}_g^{\circ}\bullet]$ is the image of a section $\sigma \in U\partial M$, then $\pi([(v, w, z)\mathrm{C}_g^{\circ}\bullet]) = \partial M$. This is a contradiction. Thus, no such section can exist. $\square$

Let $v, w, z \in U\partial M$. We say that the triple (v, w, z) corresponds to distinct geodesics if $\gamma_{\hat{v}}, \gamma_{\hat{w}}, \gamma_{\hat{z}}$ are all distinct. The next lemma shows that we can convert between triples that correspond to distinct geodesics by swapping one entry at a time with a third triple, whenever the vectors all come from a single section $\sigma \in U\partial M$.

Lemma 8.5. Let (M, g) be a simple manifold. Let $\sigma \in \Gamma(U\partial M)$. Let $p_i, q_i, r_i \in \partial M$ be such that $(\sigma(p_i), \sigma(q_i), \sigma(r_i))$ corresponds to distinct geodesics for $i = 0, 1$. Then there exists $p_2, q_2, r_2 \in \partial M$ such that

- $(\sigma(p_0), \sigma(q_0), \sigma(r_2))$
- $(\sigma(p_0), \sigma(q_2), \sigma(r_2))$
- $(\sigma(p_2), \sigma(q_2), \sigma(r_2))$
- $(\sigma(p_2), \sigma(q_2), \sigma(r_1))$

- $(\sigma(p_2), \sigma(q_1), \sigma(r_1))$

all correspond to distinct geodesics.

Proof. First, observe that a 1-dimensional manifold can never satisfy the hypotheses of this lemma (if (M, g) is 1-dimensional and simple, then there are only two points in $U\partial M$). Thus, $\dim(M) \geq 2$, so ∂M is an infinite set. It follows that the set

$$\partial M \setminus \{\gamma_{\widehat{\sigma(p_0)}}, \gamma_{\widehat{\sigma(q_0)}}, \gamma_{\widehat{\sigma(r_0)}}, \gamma_{\widehat{\sigma(p_1)}}, \gamma_{\widehat{\sigma(q_1)}}, \gamma_{\widehat{\sigma(r_1)}}\}$$

is nonempty (the six geodesics in question can intersect the boundary in at most 12 points). Thus, choose p_2 arbitrarily from the above set. Similarly, choose q_2 from the set

$$\partial M \setminus \{\gamma_{\widehat{\sigma(p_0)}}, \gamma_{\widehat{\sigma(q_0)}}, \gamma_{\widehat{\sigma(r_0)}}, \gamma_{\widehat{\sigma(p_1)}}, \gamma_{\widehat{\sigma(q_1)}}, \gamma_{\widehat{\sigma(r_1)}}, \gamma_{\widehat{\sigma(p_2)}}\}$$

and r_2 from the set

$$\partial M \setminus \{\gamma_{\widehat{\sigma(p_0)}}, \gamma_{\widehat{\sigma(q_0)}}, \gamma_{\widehat{\sigma(r_0)}}, \gamma_{\widehat{\sigma(p_1)}}, \gamma_{\widehat{\sigma(q_1)}}, \gamma_{\widehat{\sigma(r_1)}}, \gamma_{\widehat{\sigma(p_2)}}, \gamma_{\widehat{\sigma(q_2)}}\}$$

It follows that at least one endpoint of each of the geodesics corresponding to $\sigma(p_2), \sigma(q_2), \sigma(r_2)$ is different from the endpoints of the geodesics corresponding to $\sigma(p_0), \sigma(q_0), \sigma(r_0)$,$\sigma(p_1), \sigma(q_1), \sigma(r_1)$. It follows that all of the triples stated in the conclusion of this lemma correspond to distinct geodesics. □

In the following lemma, we characterize when a section σ represents some element in STAR_g°.

Lemma 8.6. Let (M, g) be simple and $\dim(M) \geq 2$. Let $\sigma \in \Gamma(U\partial M)$ be such that $\gamma_{\widehat{\sigma(p)}} \cap \gamma_{\widehat{\sigma(q)}} \cap \gamma_{\widehat{\sigma(r)}}$ is a singleton subset of M° whenever $(\sigma(p), \sigma(q), \sigma(r))$ corresponds to distinct geodesics. Then there exists $x \in M^\circ$ such that σ represents $\mathrm{star}_g(x)$.

Proof. First, we define

$$\mathscr{D} = \{(p, q, r) \in [\partial M]^3 | ((\sigma(p), \sigma(q), \sigma(r)) \text{ corresponds to distinct geodesics}\}$$

For $(p, q, r) \in \mathscr{D}$, define $\iota(p, q, r) \in M^\circ$ to be the point that the corresponding geodesics intersect at. Specifically, we mean that $\gamma_{\widehat{\sigma(p)}} \cap \gamma_{\widehat{\sigma(q)}} \cap \gamma_{\widehat{\sigma(r)}} = \{\iota(p, q, r)\}$. For two triples $(p_0, q_0, r_0), (p_1, q_1, r_1) \in \mathscr{D}$, write $(p_0, q_0, r_0) \approx (p_1, q_1, r_1)$ if they differ by at most one point (i.e. $(s, q, r) \approx (p, q, r)$, $(p, s, r) \approx (p, q, r)$ etc. . .). Let $\sim$ be the equivalence relation generated by $\approx$. Specifically, this means that $(p_0, q_0, r_0) \sim (p_n, q_n, r_n)$ if there exists $(p_1, q_1, r_1), ..., (p_{n-1}, q_{n-1}, r_{n-1}) \in \mathscr{D}$ such that $(p_{k-1}, q_{k-1}, r_{k-1}) \approx (p_k, q_k, r_k)$ for all $k = 1, 2, ..., n$.

First, observe that if $(p, q, r), (\tilde{p}, \tilde{q}, \tilde{r}) \in \mathscr{D}$ and $(p, q, r) \sim (\tilde{p}, \tilde{q}, \tilde{r})$, then $\iota(p, q, r) \sim \iota(\tilde{p}, \tilde{q}, \tilde{r})$. Thus, ι is constant on equivalence classes of $\sim$. However Lemma 8.5 shows that all elements of $\mathscr{D}$ are equivalent. Thus, there is a single point $x \in M^\circ$ such that $\iota(p,q,r) = x$ for all $(p,q,r) \in \mathscr{D}$.

Finally, we observe that every point $p \in \partial M$ is part of some triple $(p,q,r) \in \mathscr{D}$ (this follows from the fact that ∂M is an infinite set, similar to the argument in Lemma 8.5). Thus, there exists $x \in M^\circ$ such that $\gamma_{\widehat{\sigma(p)}}$ passes through x for all $p \in \partial M$. Thus, σ represents $\mathrm{star}_g(x)$ as required. $\square$

We are now ready to prove Lemma 8.2 which quantifies STAR°_g in terms of $(\partial M, g^\partial, \mathrm{C}^\circ_g)$. In the following proof, we reference many of the lemmas that we have proven in this section. However, "(1.)" and "(2.)" will always refer to the hypotheses of the original lemma (Lemma 8.2) in the proof.

Proof of Lemma 8.2. First, let $S \in \mathrm{STAR}^\circ_g$. This means that there exists $x \in M^\circ$ such that $S = \mathrm{star}_g(x)$. Then $\sigma_x \in \Gamma(U\partial M)$ represents S, so $\sigma_x(\partial M) = S$. Thus, (1.) holds.

Now, let $u, v, w, z \in S$. Then $\gamma_{\hat{u}}, \gamma_{\hat{v}}, \gamma_{\hat{w}}, \gamma_{\hat{z}}$ all pass through x, so $u \in [(v,w,z)\mathrm{C}^\circ_g\bullet]$. This is true for all such u, so $S \subset [(v,w,z)\mathrm{C}^\circ_g\bullet]$. Thus, (2.) also holds.

Conversely, suppose $S \subset U\partial M$ satisfies (1.) and (2.). We must show that there exists $x \in M^\circ$ such that $S = \mathrm{star}_g(x)$. Let $\sigma \in \Gamma(U\partial M)$ be a smooth section guarunteed by property (1.). We claim that σ satisfies the hypotheses of Lemma 8.6.

Let $p, q, r \in \partial M$ be such that $(\sigma(p), \sigma(q), \sigma(r))$ corresponds to distinct geodesics. Using (1.) and (2.) together, we know that $\sigma(p), \sigma(q), \sigma(r) \in S \subset [(\sigma(p), \sigma(q), \sigma(r))\mathrm{C}^\circ_g\bullet]$. Thus, $\sigma(p)\mathrm{C}^\circ_g\sigma(q)$, $\sigma(p)\mathrm{C}^\circ_g\sigma(r)$, and $\sigma(q)\mathrm{C}^\circ_g\sigma(r)$ are all valid relations. Additionally, $\sigma(\partial M) = S \subset [(\sigma(p), \sigma(q), \sigma(r))\mathrm{C}^\circ_g\bullet]$. Thus $\sigma(p), \sigma(q), \sigma(r)$ satisfy the hypotheses of Lemma 8.4, so $\gamma_{\widehat{\sigma(p)}} \cap \gamma_{\widehat{\sigma(q)}}\gamma_{\widehat{\sigma(r)}}$ is a singleton subset of M°. Since this is true for all such p, q, r, we know that σ satisfies the hypotheses of Lemma 8.6. Thus, σ represents $\mathrm{star}_g(x)$ for some $x \in M^\circ$ as required. $\square$

8.4.2 Quantification when $\dim(M) = 3$

In the 3-dimensional setting, Lemma 8.4 no longer applies. Thus, we must modify our approach. We will still be interested in the properties of $[(v,w,z)\mathrm{C}^\circ_g\bullet]$ but we can no long make a simple argument based on the dimension of the set.

For $v, w, z \in U\partial M$ define $\mathscr{P}_{(v,w,z)} \subset [(v,w,z)\mathrm{C}^\circ_g\bullet]$ to be the path connected component of $[(v,w,z)\mathrm{C}^\circ_g\bullet]$ which contains v. In other words, $u \in \mathscr{P}_{(v,w,z)}$ if and only if there exists a continuous path $r : [a,b] \to U\partial M$ such that $r(a) = v$, $r(b) = u$ and $r(t) \in [(v,w,z)\mathrm{C}^\circ_g\bullet]$ for all $t \in [a,b]$. Observe, that since the topology of ∂M is determined by the smooth structure on M, $\mathscr{P}_{(v,w,z)}$ is quantifiable in terms of $(\partial M, g^\partial, \mathrm{C}^\circ_g)$.

Having defined $\mathscr{P}_{(v,w,z)}$, we offer the following quantification of SRC°_g in terms of $(\partial M, g^\partial, \mathrm{C}^\circ_g)$.

Lemma 8.7. Let (M,g) be simple and $\dim(M) = 3$. Let $S \subset U\partial M$. Then $S \in \mathrm{STAR}^\circ_g$ if and only if there exists $v, w, z \in U\partial M$ such that

1. (v, w, z) corresponds to distinct geodesics.
2. $v, w, z \in [(v, w, z)\mathrm{C}^{\circ}_{g}\bullet]$
3. $\mathscr{P}_{(v,w,z)}$ is the image of a smooth section $\sigma \in \Gamma(U\partial M)$.
4. $S = \mathscr{P}_{(v,w,z)}$.

As before, we will prove Lemma 8.7 last. For Lemma 8.7 to be a valid quantification, we must know that the notion of "distinct" geodesics is quantifiable in terms of the interior intersection data. The following lemma demonstrates this fact.

Lemma 8.8. Let (M, g) be simple. Let $v, w \in U\partial M$. Then $\gamma_{\hat{v}} = \gamma_{\hat{w}}$ (as sets) if and only if $(v\mathrm{C}^{\circ}_{g}\bullet) = (w\mathrm{C}^{\circ}_{g}\bullet)$.

Proof. Let $v, w \in U\partial M$. If $\gamma_{\hat{v}} = \gamma_{\hat{w}}$, then clearly any geodesic intersects with $\gamma_{\hat{v}} \cap M^{\circ}$ if and only if it intersects with $\gamma_{\hat{w}} \cap M^{\circ}$. Thus, $(v\mathrm{C}^{\circ}_{g}\bullet) = (w\mathrm{C}^{\circ}_{g}\bullet)$.

Conversely, suppose that $\gamma_{\hat{v}} \neq \gamma_{\hat{w}}$. Since distinct geodesics in a simple manifold intersect at a single point if they intersect at all, there are three possibilities

1. $\gamma_{\hat{v}} \cap \gamma_{\hat{w}} = \varnothing$
2. $\gamma_{\hat{v}} \cap \gamma_{\hat{w}} = \{p\} \subset M^{\circ}$
3. $\gamma_{\hat{v}} \cap \gamma_{\hat{w}} = \{p\} \subset \partial M$

In case (1.), it follows that $v \in (v\mathrm{C}^{\circ}_{g}\bullet)$ but $v \notin (w\mathrm{C}^{\circ}_{g}\bullet)$. Thus, $(v\mathrm{C}^{\circ}_{g}\bullet) \neq (w\mathrm{C}^{\circ}_{g}\bullet)$.

In case (2.), let $q \in \gamma_{\hat{w}} \cap M^{\circ} \setminus \{p\}$. Let $z = \mathrm{aim}(\pi(v), q)$. Then $\gamma_{\hat{z}} \cap \gamma_{\hat{v}} = \{\pi(v)\}$. But $\pi(v) \notin M^{\circ}$, so $z \notin (v\mathrm{C}^{\circ}_{g}\bullet)$. However $\gamma_{\hat{z}}$ passes through $q \in \gamma_{\hat{v}}$, so $z \in (w\mathrm{C}^{\circ}_{g}\bullet)$. Thus, $(v\mathrm{C}^{\circ}_{g}\bullet) \neq (w\mathrm{C}^{\circ}_{g}\bullet)$.

Finally, in case (3.), we know that the only intersection off $\gamma_{\hat{w}}$ with $\gamma_{\hat{w}}$ occurs in ∂M. Thus, $v \notin (w\mathrm{C}^{\circ}_{g}\bullet)$, but $v \in (v\mathrm{C}^{\circ}_{g}\bullet)$. We conclude that $(v\mathrm{C}^{\circ}_{g}\bullet) \neq (w\mathrm{C}^{\circ}_{g}\bullet)$ as required. $\square$

For $v \in U\partial M$, let $\gamma^{\circ}_{\hat{v}} = \gamma_{\hat{v}}((0, \tau_g(v))) = \gamma_{\hat{v}} \setminus \partial M$. For $w \in U\partial M$, define $\mathrm{sing}(w) \subset U\partial M$ to be the vectors $v \in U\partial M$ such that $\gamma_{\hat{v}} \cap \gamma^{\circ}_{\hat{w}}$ is a singleton set. Specifically, we can write $\mathrm{sing}(w) = (w\mathrm{C}^{\circ}_{g}\bullet) \setminus \{w, -\alpha_g(w)\}$. We may define $\mathcal{P}_w : \mathrm{sing}(w) \to M$ by $\gamma_{\hat{v}} \cap \gamma^{\circ}_{\hat{w}} = \{\mathcal{P}_w(v)\}$. The next lemma shows that $\mathcal{P}_w$ is continuous.

Lemma 8.9. Let (M, g) be simple. Suppose $w \in U\partial M$. Then $\mathcal{P}_w : \mathrm{sing}(w) \to M$ is continuous.

Proof. Let $w \in U\partial M$. We will show that $\mathcal{P}_w$ is sequentially continuous, which is equivalent by Lemma 2.19 Let $\{v_k\}_{k=1}^{\infty}$ be a sequence in $\mathrm{sing}(w)$ such that $v_k \to v \in \mathrm{sing}(w)$. We claim that $\mathcal{P}_w(v_k) \to \mathcal{P}_w(v)$. We will show that every subsequence of $\{v_k\}_{k=1}^{\infty}$ has a subsequence

$\{v_{i(k)}\}_{k=1}^{\infty}$ such that $\mathcal{P}_w(v_{i(k)}) \to \mathcal{P}_w(v)$. From which, sequential continuity follows from Lemma 2.20.

Let $\{v_{j(k)}\}_{k=1}^{\infty}$ be a subsequence of $\{v_k\}_{k=1}^{\infty}$. Since $[0, \tau_g(w)]$ is compact, by Lemma 2.4, we know that $\gamma_{\hat{w}}$ is compact in M. Thus, by Lemma 2.21, we know that there is a subsequence $\{v_{i(k)}\}_{k=1}^{\infty}$ of $\{v_{j(k)}\}_{k=1}^{\infty}$ such that $\mathcal{P}_w(v_{i(k)}) \to p \in \gamma_{\hat{w}}$. We claim that $p = \mathcal{P}_w(v)$.

Observe that $\operatorname{aim}(\pi(v_k), \mathcal{P}_w(v_k)) = v_k$. Additionally, since aim and π are continuous, we know that $(z, q) \mapsto \operatorname{aim}(\pi(z), q)$ is continuous. Thus, it follows that $\operatorname{aim}(\pi(v_{i(k)}), \mathcal{P}_w(v_{i(k)})) \to \operatorname{aim}(\pi(v), p)$. But $\operatorname{aim}(\pi(v_{i(k)}), \mathcal{P}_w(v_{i(k)})) = v_{i(k)} \to v$. Thus, $\operatorname{aim}(\pi(v), p) = v$, since limits are unique. This implies that $p \in \gamma_{\hat{v}}$. Thus, it follows from the fact that $v \in \operatorname{sing}(w)$ that $\gamma_{\hat{v}} \cap \gamma_{\hat{w}}^{\circ} = \{p\}$, so $\mathcal{P}_w(v) = p$ as required. □

Lemma 8.10. Let $w, z \in U\partial M$ be such that

1. $\gamma_{\hat{w}} \cap \gamma_{\hat{z}} = \{x\} \subset M^{\circ}$.
2. The tangent vectors e_w, e_z to $\gamma_{\hat{w}}, \gamma_{\hat{z}}$ at x are orthogonal.

Let $\{(p_k, q_k)\}_{k=1}^{\infty}$ be a sequence in $\gamma_{\hat{w}} \times \gamma_{\hat{z}} \setminus \{(x, x)\}$ such that $(p_k, q_k) \to (x, x)$. Let $v_k = \operatorname{dir}(p_k, q_k)$. If there exists $v \in T_xM$ such that $v_k \to v$, then $v \in \operatorname{span}\{e_w, e_z\}$.

Proof. Let t_w and t_z be defined such that $\gamma_{\hat{w}}(t_w) = \gamma_{\hat{z}}(t_z) = x$. Let $u_k = \operatorname{aim}(p_k, q_k)$. Then $\gamma_{\widehat{u_k}}$ contains p_k and q_k. By passing to a subsequence, we may guaruntee that p_k and q_k are on fixed sides of their respective geodesics relative to x. Thus, without loss of generality, let $\gamma_{\hat{w}}(t_{w,k}) = p_k$ and $\gamma_{\hat{z}}(t_{z,k}) = q_k$ be such that $t_{w,k} < t_w$ and $t_{z,k} > t_z$. Consider triangle $\Delta p_k q_k x$. Define the following vectors as in Figure 2.

- Let $\tilde{v}_k$ be the tangent vector to $\gamma_{\widehat{u_k}}$ at p_k.
- Let $E_{w,k}$ be the tangent vector to $\gamma_{\hat{w}}$ at p_k.
- Let $E_{z,k}$ be the tangent vector to $\gamma_{\hat{z}}$ at q_k.

Then, by the continuity of the geodesic flow, we obtain the following limits

- $\tilde{v}_k \to v$
- $E_{w,k} \to e_w$
- $E_{z,k} \to e_z$.

The sum of the angles in $\Delta p_k q_k x$ is $\angle(E_{w,k}, v_k) + \angle(\tilde{v}_k, E_{z,k}) + \frac{\pi}{2}$. Thus, taking limit, using the continuity of $\angle(\bullet, \bullet)$, and Lemma 5.14 we obtain

$$\begin{aligned}\lim_{k\to\infty} \angle(E_{w,k}, v_k) + \angle(\tilde{v}_k, E_{z,k}) + \frac{\pi}{2} &= \angle(e_w, v) + \angle(v, e_z) + \frac{\pi}{2} \\ &= \pi\end{aligned}$$

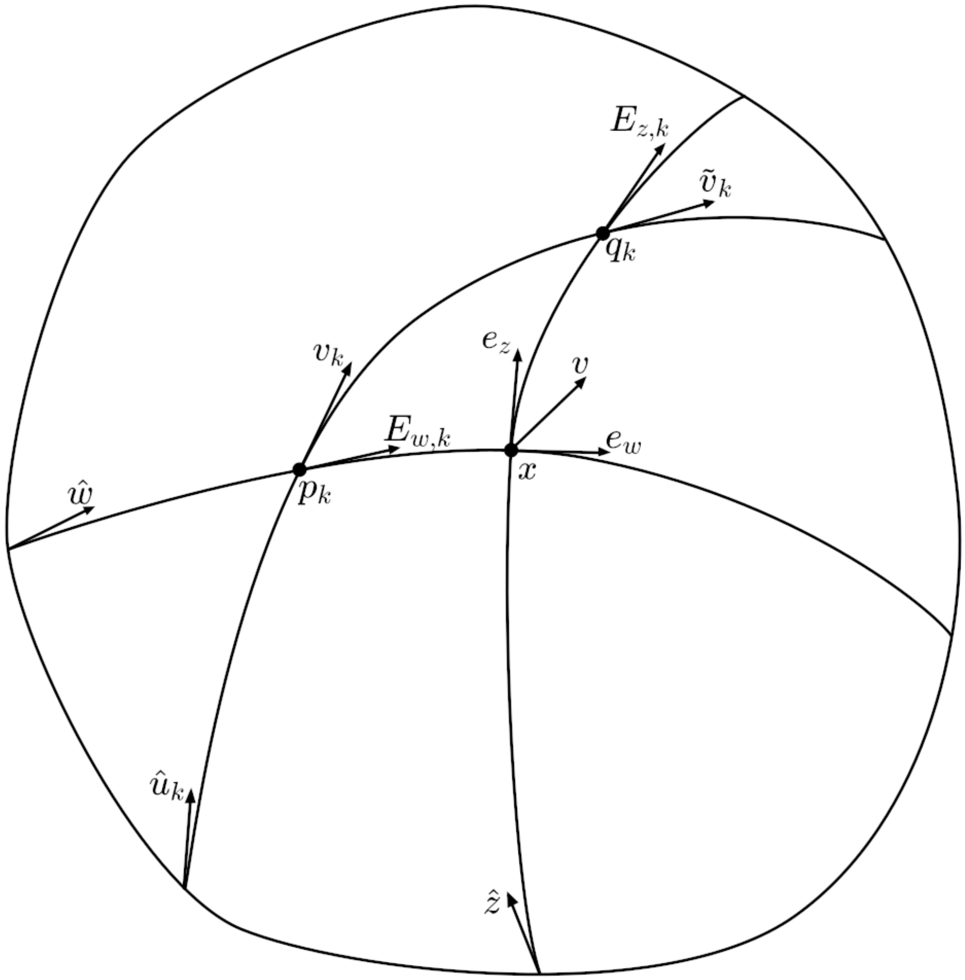


Figure 2: Diagram for proof of Lemma 8.10

Thus, $\angle(e_w, v) + \angle(v, e_z) = \frac{\pi}{2}$. So, by Lemma 5.15, we conclude that $v \in \operatorname{span}\{e_w, e_z\}$ as required. $\square$

Most directly, the above lemma constrains how a sequence of vectors can *approach* the orthogonal intersection of two geodesics. If we think about a continuous path of vectors in $U\partial M$ such that all of the corresponding geodesics pass through a collection of orthogonal vectors, we can use Lemma 8.10 to constrain how the geodesics can *leave* the intersection point. This is what we do in the following lemma. In particular, we show that if a continuous path of vectors has corresponding geodesics that start by passing through the intersection of three perpindicular geodesics, then they must always pass through that intersection point.

Lemma 8.11. Let (M, g) be a simple Riemannian manifold with boundary such that $\dim(M) = 3$. Let $x \in M^\circ$ and $v, w, z \in U\partial M$ be such that $\gamma_{\hat{v}}, \gamma_{\hat{w}}, \gamma_{\hat{z}}$ pass through x and their tangent vectors at x form an orthonormal basis for T_xM. Then $\mathscr{P}_{(v,w,z)} = \operatorname{star}_g(x)$.

Proof. First, we show that $\operatorname{star}_g(x) \subset \mathscr{P}_{(v,w,z)}$. To see this, let $\sigma \in \Gamma(U\partial M)$ represent $\operatorname{star}_g(x)$. Then, for any $w \in \operatorname{star}_g(x)$, let $r : [0,1] \to M$ be a continuous path from $\pi(v)$ to $\pi(w)$. Then $\sigma \circ r$ is a continuous path from v to w through $[(v, w, z)\mathrm{C}_g^\circ\bullet]$. So $w \in \mathscr{P}_{(v,w,z)}$.

Thus, it remains to show that $\mathscr{P}_{(v,w,z)} \subset \operatorname{star}_g(x)$. For the sake of contradiction, suppose $\tilde{v} \in \mathscr{P}_{(v,w,z)}$ but $\tilde{v} \notin \operatorname{star}_g(x)$. Then, there exists $a : [0,1] \to [(v, w, z)\mathrm{C}_g^\circ\bullet]$ continuous such that $a(0) = v$ and $a(1) = \tilde{v}$. Let $t^* = \sup\{t \in [0,1] | a(t) \in \operatorname{star}_g(x)\}$. Since $\operatorname{star}_g(x)$ is the image of the section that represints it, and ∂M, it follows that $\operatorname{star}_g(x)$ is compact. Thus, $a^{-1}(\operatorname{star}_g(x))$ is compact, so $t^* = \sup\limits_{t \in a^{-1}(\operatorname{star}_g(x))} t$ is attained. Specifically, this means that $a(t^*) \in \operatorname{star}_g(x)$. It follows that $t^* < 1$ since $a(1) = \tilde{v} \notin \operatorname{star}_g(x)$ by assumption. There are two possibilities for $a(t^*)$.

1. $\gamma_{\widehat{a(t^*)}}$ is distinct from $\gamma_{\hat{v}}, \gamma_{\hat{w}}$, and $\gamma_{\hat{z}}$.
2. $\gamma_{\widehat{a(t^*)}}$ is identical to exactly one of $\gamma_{\hat{v}}, \gamma_{\hat{w}}$, or $\gamma_{\hat{z}}$.

First, suppose that $a(t^*)$ is as in case (1.). Then $a(t) \in \operatorname{sing}(v) \cap \operatorname{sing}(w) \cap \operatorname{sing}(z)$ for all $t \in [t^*, 1]$ (this is because the only way for $a(t)$ to intersect one of the three geodesics in multiple points is if $\gamma_{\widehat{a(t)}}$ is identical to the geodesic. But this would imply that $a(t) \in \operatorname{star}_g(x)$, which contradicts the definition of t^*). By Lemma 8.9, the function $t \mapsto (\mathcal{P}_v(a(t)), \mathcal{P}_w(a(t)), \mathcal{P}_z(a(t)))$ is continuous, and has value (x, x, x) when $t = t^*$. Let $\{t_k\}_{k=1}^\infty$ be a sequence in $(t^*, 1]$ such that $t_k \to t^*$. Let $p_k = \mathcal{P}_v(a(t_k))$, $q_k = \mathcal{P}_w(a(t_k))$, and $r_k = \mathcal{P}_z(a(t_k))$. We may restrict to a subsequence such that $\gamma_{\widehat{a(t_k)}}$ intersects $\gamma_{\hat{v}}, \gamma_{\hat{w}}, \gamma_{\hat{z}}$ in the same order for all k. Without loss of generality, assume $\gamma_{\widehat{a(t_k)}}$ intersects $\gamma_{\hat{v}}$, then $\gamma_{\hat{w}}$, then $\gamma_{\hat{z}}$.

Observe that $a(t_k) = \operatorname{aim}(p_k, q_k) = \operatorname{aim}(p_k, r_k) = \operatorname{aim}(q_k, r_k)$ for all k. Let $u \in S_xM$ be the tangent vector to $\gamma_{\widehat{a(t^*)}}$ at x. Consider the following sequences of vectors in SM

$\{\mathrm{dir}(p_k,q_k)\}_{k=1}^{\infty}$, $\{\mathrm{dir}(p_k,r_k)\}_{k=1}^{\infty}$ and $\{\mathrm{dir}(q_k,r_k)\}_{k=1}^{\infty}$. It follows from the continuity of the geodesic flow that all three sequences converge to the tangent vector to $\gamma_{\widehat{a(t^*)}}$ at x.

Let e_v be the tangent vector to $\gamma_{\hat{v}}$ at x. Similarly, for e_w and e_z. Then it follows from Lemma 8.10 that $u \in \mathrm{span}\{e_v, e_w\} \cap \mathrm{span}\{e_v, e_z\} \cap \mathrm{span}\{e_w, e_z\}$. But the intersection of three mutually perpindicular two-dimensional subspaces of a three-dimensional vector space is just the set containing the zero vector. Thus $u = 0 \in T_xM$. This contradicts the fact that u is a unit vector. Thus we have eliminated case (1.).

Continuing, suppose that $a(t^*)$. is as in case (2.). Then $a(t^*)$ is based at one of the endpoints of $\gamma_{\hat{v}}, \gamma_{\hat{w}}$ or $\gamma_{\hat{z}}$, and is such that $\widehat{a(t^*)}$ is tangent to the corresponding geodesic. Without loss of generality, assume $a(t^*) = v$. The following argument can be modified slightly to match any of the other 5 possibilities.

Similar to case (1.), $a(t) \in \mathrm{sing}(w) \cap \mathrm{sing}(z)$ for all $t \in [t^*, 1]$. Thus, the function $t \mapsto (\mathcal{P}_w(a(t)), \mathcal{P}_z(a(t)))$ is continuous on $[t^*, 1]$, and its value at t^* is (x, x). Let $\{t_k\}_{k=1}^{\infty}$ be a sequence in $(t^*, 1]$ such that $t_k \to t^*$. Again, we may restrict to a subsequence such that $\gamma_{\widehat{a(t_k)}}$ intersects $\gamma_{\hat{w}}$ and $\gamma_{\hat{z}}$ in a particular order. Without loss of generality, assume $\gamma_{\widehat{a(t_k)}}$ intersects $\gamma_{\hat{w}}$ and then $\gamma_{\hat{z}}$.

As before, let $e_v, e_w, e_z \in S_xM$ correspond to the tangent vectors to $\gamma_{\hat{v}}, \gamma_{\hat{w}}, \gamma_{\hat{z}}$ at x. Let $q_k = \mathcal{P}_w(a(t_k))$ and $r_k = \mathcal{P}_z(a(t_k))$. Then by a similar argument as in case (1.), we know that $\mathrm{dir}(q_k, r_k)$ converges to a vector $u \in \mathrm{span}\{e_w, e_z\}$. But by the continuity of the geodesic flow, it follows that $\mathrm{dir}(q_k, r_k)$ converges to e_v.But $e_v \notin \mathrm{span}\{e_w, e_z\}$, so this is a contradiction. $\square$

In the next lemma, we show that when three geodesics intersect at three distinct points, then $\mathscr{P}_{(v,w,z)}$ cannot be the image of a section of $U\partial M$. Thus, by contrapositive, if $\mathscr{P}_{(v,w,z)}$ is the image of a section, and the geodesics are distinct, we know that all three geodesics must intersect at a single point.

Lemma 8.12. Let (M, g) be simple. Let $v, w, z \in U\partial M$ be such that

1. (v, w, z) corresponds to distinct geodesics.
2. $v, w, z \in [(v, w, z)\mathrm{C}^{\circ}_{g}\bullet]$
3. $\gamma_{\hat{v}} \cap \gamma_{\hat{w}} \cap \gamma_{\hat{z}} = \varnothing$.

Then $\mathscr{P}_{(v,w,z)}$ is not the image of a smooth section of $U\partial M$.

Proof. Let v, w, z be as stated. We will show that $\mathscr{P}_{(v,w,z)}$ is not closed. This will be sufficient, because the continuous image of a compact set is compact, and thus closed. Thus, if $\mathscr{P}_{(v,w,z)}$ were the image of a smoot section of $U\partial M$, it must be closed.

Form properties (1.)-(3.) it follows that the pairwise intersections of $\gamma_{\hat{v}}$,$\gamma_{\hat{w}}$, and $\gamma_{\hat{z}}$ are three distinct points. Let $\{p\} = \gamma_{\hat{v}} \cap \gamma_{\hat{w}}$,$\{q\} = \gamma_{\hat{v}} \cap \gamma_{\hat{z}}$ and $\{r\} = \gamma_{\hat{w}} \cap \gamma_{\hat{z}}$. Without loss of generality, assume that $\gamma_{\hat{v}}, \gamma_{\hat{w}}, \gamma_{\hat{z}}$ are oriented as in Figure 1. Let $u = \mathrm{aim}(\pi(v), r)$.

Observe that $\gamma_{\hat{u}}$ only intersects v at \$π(v)∉ M^^. Thus $u \notin (v\mathrm{C}_g^\circ\bullet)$, so $u \notin [(v,w,z)\mathrm{C}_g^\circ\bullet]$. We claim that there is a continuous path $a : [0,1] \to U\partial M$ such that $a(0) = v$, $a(1) = u$ and $a(t) \in [(v,w,z)\mathrm{C}_g^\circ\bullet]$ for all $t \in [0,1)$. From this, it follows directly that $u = a(1)$ is in the closuer of $\mathscr{P}_{(v,w,z)}$.

Let $t_1, t_2 \in I_z$ be such that $\gamma_{\hat{z}}(t_1) = q$ and $\gamma_{\hat{z}}(t_2) = r$. Then, let $a_1 : [t_1, t_2] \to U\partial M$ be defined by $a_1(t) = \mathrm{aim}(p, \gamma_{\hat{z}}(t))$. Then a_1 is continuous, because aim is continuous. Observe that $a_1(t_1) = v$ and $a_1(t_2) = w$. Additionally, $a_1(t)$ passes through p which is in both $\gamma_{\hat{v}}$ and $\gamma_{\hat{w}}$, and passes through $\gamma_{\hat{z}}$ by construction. Thus, $a_1(t) \in [(v,w,z)\mathrm{C}_g^\circ\bullet]$ for all $t \in [t_1, t_2]$.

Let $t_3 \in I_v$ be such that $\gamma_{\hat{v}}(t_3) = p$. Then define $a_2 : [0, t_3] \to U\partial M$ by $a_2(t) = \mathrm{aim}(\gamma_{\hat{v}}(t), r)$. Again, a_2 is continuous, $a_2(0) = u$ and $a_2(t_3) = w$. Additionally $a_2(t)$ passes through r which is in both $\gamma_{\hat{w}}$ and $\gamma_{\hat{z}}$, and passes through $\gamma_{\hat{v}} \cap M^\circ$ for all $t \in (0, t_3]$. Thus, $a_2(t) \in [(v,w,z)\mathrm{C}_g^\circ\bullet]$ for all $t \in (0, t_3]$

Finally, we may form a continuous path from v to u by concatenating a_1 and the reverse of a_2. This path is in $[(v,w,z)\mathrm{C}_g^\circ\bullet]$ for all t except at the endpoint corresponding to u. Thus, u is in the closure of $\mathscr{P}_{(v,w,z)}$, but $u \notin \mathscr{P}_{(v,w,z)}$ as required. □

We are finally able to prove Lemma 8.7, our original quantification of STAR_g° in terms of $(\partial M, g^\partial, \mathrm{C}_g^\circ)$ in the three-dimensional setting. In the following proof (1.)-(4.) will exclusively refer to statements (1.) – (4.) in Lemma 8.7.

Proof of Lemma 8.7. For the forward direction, let $S \in \mathrm{STAR}_g^\circ$. Then there exists $x \in M^c irc$ such that $S = \mathrm{star}_g(x)$. There exists $v, w, z \in U\partial M$ such that $\gamma_{\hat{v}}, \gamma_{\hat{w}}, \gamma_{\hat{z}}$ pass through x in such a way that their tangent vectors form an orthonormal basis for T_xM. Clearly (v,w,z) corresponds to distinct geodesics. Additionally, since all three geodesics contain x, we know that $v, w, z \in [(v,w,z)\mathrm{C}_g^\circ\bullet]$. Then, by Lemma 8.11, we obtain $\mathscr{P}_{(v,w,z)} = \mathrm{star}_g(x)$. Thus, let $\mathrm{star}_g(x)$ be represented by $\sigma \in U\partial M$. Then, we know that $\mathscr{P}_{(v,w,z)}$ is the image of σ.

Conversely, suppose $S \subset U\partial M$ and v, w, z satisfy properties (1.)-(4.). Then it follows from Lemma 8.12 that $\gamma_{\hat{v}} \cap \gamma_{\hat{w}} \cap \gamma_{\hat{z}} = \{x\}$ for some $x \in M^\circ$. Let $\sigma_x \in \Gamma(U\partial M)$ be the section that represents $\mathrm{star}_g(x)$. We claim that $S = \sigma_x(\partial M)$. To do this, we will show both inclusions ($\sigma_x(\partial M) \subset S$ and $S \subset \sigma_x(\partial M)$).

First, we show that $\sigma_x(\partial M) \subset S$. Let $y \in \partial M$. We wish to show that $\sigma_x(y) \in S$. Since ∂M is path connected, let $a : [0,1] \to \partial M$ be a continuous path between $\pi(v)$ and y. WE know that $x \in \gamma_{\hat{v}}$ so it follows that $\sigma_x(\pi(v)) = v$. Thus $\sigma_x \circ a$ is a continuous path through $[(v,w,z)\mathrm{C}_g^\circ\bullet]$. This implies that $\sigma_x(y) \in \mathscr{P}_{(v,w,z)}$ as required.

Finally, we show that $S \subset \sigma_x(\partial M)$. Let $u \in S$. If $\sigma_x(\pi(u)) \neq u$, then there would be two distinct vectors (u and $\sigma_x(\pi(u))$) in S which are based at the same point in ∂M. This would contradict the fact that S is the image of a section of $U\partial M$. Thus, we know that $\sigma_x(\pi(u)) = u$, so $u \in \sigma_x(\partial M)$ as required. □

8.5 The Source Direction Data Determines the Lens Data

In this section, we show that the interior source directions determine the lens data (and hence the boundary distance data) whenever (M, g) is a simple Riemannian manifold with boundary and $\dim(M) \geq 2$. Specifically, we have

Theorem 8.13. Let (M, g) be simple and $\dim(M) \geq 2$. Then $(\partial M, g^{\partial}, \mathrm{STAR}_g^{\circ}) \rightsquigarrow (\partial M, \tau_g, \alpha_g)$.

The following two lemmas follow directly from the properties of simple manifolds and Lemma 5.17

Lemma 8.14. Let (M, g) be simple. Let $x \in M^{\circ}$ and $f : \partial M \to \mathbb{R}$ be defined by $f(y) = d_g(x, y)$. Let $v \in U\partial M$. Then $\gamma_{\hat{v}}$ passes through x if and only if $\mathrm{grad}_{g^{\partial}}(f)\big|_{\pi(v)} = -v$.

Lemma 8.15. Let (M, g) be simple. Let $x \in M^{\circ}$ and $f : \partial M \to \mathbb{R}$ be defined by $f(y) = d_g(x, y)$. Let $\sigma \in \Gamma(U\partial M)$. Then σ represents $\mathrm{star}_g(x)$ (i.e. $\sigma = \sigma_x$) if and only if $\mathrm{grad}_{g^{\partial}}(f) = -\sigma$.

Thus, by Lemma 5.16 we may recover the source distance functions up to an additive constant by integrating along paths. For $p, q \in \partial M$, let $r_{pq} : [0, 1] \to \partial M$ be an arbitrary smooth path from p to q. Clearly the set of such smooth paths from p to q is quantified in terms of ∂M. The following lemma follows directly from Lemma 5.16, Lemma 5.18 and Lemma 8.15

Lemma 8.16. Let (M, g) be simple. Let $x \in M^{\circ}$. then

$$d_g(x, q) - d_g(x, p) = -\int_0^1 g^{\partial}(\dot{r}_{pq}(t), \sigma_x(r_{pq}(t)))dt.$$

For $S = \mathrm{star}_g(x) \in \mathrm{STAR}_g^{\circ}$ define $D_S : \partial M \times \partial M \to \mathbb{R}$ by

$$D_{\mathrm{S}}(p, q) = d_{\mathrm{g}}(x, q) - d_{\mathrm{g}}(x, p)$$

Then clearly from Lemma 8.16 we get the following

Lemma 8.17. Let (M, g) be simple, and $\dim(M) \geq 2$. Then

$$(\partial M, g^{\partial}, \mathrm{STAR}_g^{\circ}) \rightsquigarrow D_{\bullet}$$

The two-dimensional hypothesis is necessary so that r_{pq} exists (we need ∂M to be connected). Now, we are able to provide the following quantification of the lens data in terms of the interior source direction data.

Lemma 8.18. Let (M, g) be simple. Let $v, w \in U\partial M$. Then $\alpha_g(v) = w$ if and only if

1. $\pi(v) \neq \pi(w)$
2. For all $S \in \mathrm{STAR}_g^\circ$, then $v \in S$ if and only if $(-w) \in S$.

Furthermore, we may write τ_g in terms of $D_\bullet$ and α_g in the following way:

$$\tau_g(v) = \sup\{D_S(\pi(v), \pi(\alpha_g(v)))|(S \in \mathrm{STAR}_g^\circ) \wedge (v \in S)\}$$

Proof. First, we show that the characterization of α_g is valid. For the forward direction, let $v, w \in U\partial M$ be such that $\alpha_g(v) = w$. Property (1.) follows immediately from the fact that geodesics in simple manifolds never self intersect, and thus must enter and exit the manifold at distinct points.

Additionally, it follows from the fact that $-\alpha_g(-\alpha_g(v)) = v$ that $\gamma_{\widehat{-w}}$ is the reverse of $\gamma_{\hat{v}}$, thus $\gamma_{\hat{v}} = \gamma_{\widehat{-w}}$ as a set. This implies that $v \in \mathrm{star}_g(x)$ if and only if $(-w) \in \mathrm{star}_g(x)$, since $\gamma_{\hat{v}}$ and $\gamma_{\widehat{-w}}$ intersect the same interior points. Thus, we have property (2.) and have completed the forward direction.

Conversely, let $v, w \in U\partial M$ satisfy properties (1.) and (2.). We must show that $\alpha_g(v) = w$. For the sake of contradiction, suppose $\alpha_g(v) \neq w$. This occurs if and only if $\gamma_{\hat{v}}$ and $\gamma_{\widehat{-w}}$ are distinct geodesics. Thus, there exists a point $x \in \gamma_{\hat{v}}$ such that $x \notin \gamma_{\widehat{-w}}$. In particular, this implies that $v \in \mathrm{star}_g(x)$ but $(-w) \notin \mathrm{star}_g(x)$. But this contradicts property (2.). Thus, $\alpha_g(v) = w$ as required.

Finally, we show that the characterization of τ_g is accurate. To do this, we will show both inequalities (i.e. $\tau_g(v) \geq \sup\limits_{(S \in \mathrm{STAR}_g^\circ) \wedge (v \in S)} D_S(\pi(v), \pi(\alpha_g(v)))$ and $\tau_g(v) \leq \sup\limits_{(S \in \mathrm{STAR}_g^\circ) \wedge (v \in S)} D_S(\pi(v), \pi(\alpha_g(v))))$.

For simple manifolds, the exit time is just the distance between the entry and exit point of the geodesic, so $\tau_g(v) = d_g(\pi(v), \pi(\alpha_g(v)))$. Let $S \in \mathrm{STAR}_g^\circ$ be such that $v \in S$. Then $D_S(p, q) = d_g(x, q) - d_g(x, p)$ for some $x \in \gamma_{\hat{v}} \cap M^\circ$. Since x is on the geodesic joining $\pi(v)$ and $\pi(\alpha_g(v))$, and all geodesics are minimizing, it follows that $d_g(\pi(v), \pi(\alpha_g(v))) = d_g(x, \pi(\alpha_g(v))) + d_g(x, \pi(v))$. But this sum is greater than $d_g(x, \pi(\alpha_g(v))) - d_g(x, \pi(v)) = D_S(\pi(v), \pi(\alpha_g(v)))$. Thus, $\tau_g(v) \geq \sup\limits_{(S \in \mathrm{STAR}_g^\circ) \wedge (v \in S)} D_S(\pi(v), \pi(\alpha_g(v)))$.

Now, let $\{x_n\}_{n=1}^\infty$ be a sequence of points in $\gamma_{\hat{v}} \cap M^\circ$ such that $x_n \to \pi(v)$. Let $S_n = \mathrm{star}_g(x_n)$. Then $v \in S_n$ for all n, and $D_{S_n}(\pi(v), \pi(\alpha_g(v))) = d_g(x_n, \pi(\alpha_g(v))) - d_g(x_n, \pi(v))$. Thus, we compute $\lim\limits_{n\to\infty} D_{S_n}(\pi(v), \pi(\alpha_g(v))) = d_g(\pi(v), \pi(\alpha_g(v))) = \tau_g(v)$. Thus $\tau_g(v) \leq \sup\limits_{(S \in \mathrm{STAR}_g^\circ) \wedge (v \in S)} D_S(\pi(v), \pi(\alpha_g(v)))$, so we have both inequalities. The desired equality follows. $\square$

8.6 The Source Direction Data Determines the Source Distance Data

Equipped with the lens data and $D_\bullet$, it is relatively straightforward to recover the source distance data from the source direction data. In this section, we show the following.

Theorem 8.19. Let (M, g) be simple and $\dim(M) \geq 2$. Then

$$(\partial M, g^\partial, \mathrm{STAR}_g^\circ) \rightsquigarrow (\partial M, g^\partial, \mathrm{SRC}_g^\circ)$$

We offer the following quantification of SRC_g° in terms of $(\partial M, g^\partial, \mathrm{STAR}_g^\circ)$.

Lemma 8.20. Let (M, g) be simple. Let $f : \partial M \to \mathbb{R}$ be smooth. Then $f \in \mathrm{SRC}_g^\circ$ if and only if there exists $S \in \mathrm{STAR}_g^\circ$ such that

$$f(p) = \frac{1}{2}\Big[D_S\big(\pi(\alpha_g(\sigma(p))), p\big) + \tau_g(\sigma(p))\Big]$$

for all $p \in \partial M$, where $\sigma \in \Gamma(U\partial M)$ represents S.

Proof. Let $q = \pi(\alpha_g(\sigma(p)))$. Then we compute

$$\begin{aligned}
\frac{1}{2}\big[D_S(q,p) + \tau_g(\sigma(p))\big] &= \frac{1}{2}\big[d_g(x,p) - d_g(x,q) + d_g(p,q)\big] \\
&= \frac{1}{2}\big[d_g(x,p) - d_g(x,q) + d_g(x,p) + d_g(x,q)\big] \\
&= d_g(x,p)
\end{aligned}$$

Where the second equality above follows from the fact that x lies on the minimizing geodesic between p and q. The lemma follows from the above computation and the fact that there is a bijective correspondence between points in M° and sets in STAR_g°. $\square$

8.7 The Source Distance Data Determines the Geometry

In this section, we define a distance function $d_{\mathrm{SRC}_g^\circ}$ on SRC_g° such that $(\mathrm{SRC}_g^\circ, d_{\mathrm{SRC}_g^\circ})$ is isometric to (M°, d_g). Then we show that $(\partial M, g^\partial, \mathrm{SRC}_g^\circ)$ quantifies $d_{\mathrm{SRC}_g^\circ}$.

Lemma 8.21. Let (M, g) be a simple. Then the map $x \mapsto d_g(x, \bullet)\big|_{\partial M}$ is a bijection from M° to SRC_g°.

Proof. The map is clearly surjective, since SRC_g° is defined as the image of the map. Thus, we must show that it is injective. Let $x, y \in M^\circ$ be such that $x \neq y$. We must show that there exists $p \in \partial M$ such that $d_g(x,p) \neq d_g(y,p)$. To see this, let $v = \mathrm{aim}(x,y)$ and $p = \pi(v)$. Then, $\gamma_{\hat{v}}$ is minimizing from p to x and p to y. However, $\gamma_{\hat{v}}$ intersects x before y. Thus, the distance $d_g(x,p)$ is strictly greater than the distance $d_g(y,p)$, so $d_g(x,p) \neq d_g(y,p)$ as required. $\square$

Equipped with Lemma 8.21, we define $d_{\mathrm{SRC}_g^\circ}$ by

$$d_{\mathrm{SRC}_g^\circ}(d_g(x,\bullet)\big|_{\partial M}, d_g(y,\bullet)\big|_{\partial M}) = d_g(x,y)$$

It follows from Lemma 8.21 that $d_{\mathrm{SRC}_g^\circ}$ is well-defined and that (M°, d_g) is isometric to $(\mathrm{SRC}_g^\circ, d_{\mathrm{SRC}_g^\circ})$. We claim the following.

Theorem 8.22. Let (M,g) be a simple Riemannian manifold such that $\dim(M) \geq 2$, then

$$(\partial M, g^\partial, \mathrm{SRC}_g^\circ) \rightsquigarrow d_{\mathrm{SRC}_g^\circ}$$

The proof of Theorem 8.22 follows from the next two lemmas. This is detailed in the following two lemmas.

Lemma 8.23. Let (M,g) be simple and $\dim(M) \geq 2$. Let $f_1, f_2 \in \mathrm{SRC}_g^\circ$ be distinct functions. Then there exists $p \in \partial M$ such that $\mathrm{grad}_{g^\partial}(f_1)\big|_p = \mathrm{grad}_{g^\partial}(f_2)\big|_p$.

Lemma 8.24. Let (M,g) be simple and $\dim(M) \geq 2$. Let $f_1, f_2 \in \mathrm{SRC}_g^\circ$ be such that $f_1 = d_g(x,\bullet)\big|_{\partial M}$ and $f_2 = d_g(y,\bullet)\big|_{\partial M}$ for $x, y \in M^\circ$. Let $p \in \partial M$ be such that $\mathrm{grad}_{g^\partial}(f_1)\big|_p = \mathrm{grad}_{g^\partial}(f_2)\big|_p$. Then $d_g(x,y) = |f_1(p) - f_2(p)|$.

Proof of Lemma 8.23. We note that the two-dimensional hypothesis is necessary for the gradient to be defined on the boundary. Let $f_1, f_2 \in \mathrm{SRC}_g^\circ$ be distinct. Then, there exists $x, y \in M^\circ$ such that $f_1 = d_g(x,\bullet)\big|_{\partial M}$ and $f_2 = d_g(y,\bullet)\big|_{\partial M}$. By Lemma 8.21, we know that $x \neq y$. Thus, (x,y) is in the domain of aim. Let $v = \mathrm{aim}(x,y)$ and $p = \pi(v)$. Since $\gamma_{\hat{v}}$ passes through both x and y it follows from Lemma 8.14 that $\mathrm{grad}_{g^\partial}(f_1)\big|p = \mathrm{grad}_{g^\partial}(f_2)\big|p = v$ as required. ☐

Proof of Lemma 8.24. Let f_1, f_2, x, y, p be as stated, and $v = \mathrm{grad}_{g^\partial}(f_1)\big|_p = \mathrm{grad}_{g^\partial}(f_2)\big|_p$. Then, from Lemma 8.14 we know that $\gamma_{\hat{v}}$ passes through both x and y.

All geodesics are minimizing, since (M,g) is simple. Thus, dg(x,p)l is the length of the portion of $\gamma_{\hat{v}}$ between p and x, and $d_g(y,p)$ is the length of the portion of $\gamma_{\hat{v}}$ between p and y. It follows that $|d_g(x,p) - d_g(y,p)|$ is the length of the geodesic segment between x and y. Thus, $d_g(x,y) = |d_g(x,p) - d_g(y,p)| = f_1(p) - f_2(p)|$ as required. ☐

Thus, the following quantification of $d_{\mathrm{SRC}_g^\circ}$ in terms of SRC_g° follows immediately from the above two lemmas.

Lemma 8.25. Let (M,g) be simple. Let $f_1, f_2 \in \mathrm{SRC}_g^\circ$. Then $d_{\mathrm{SRC}_g^\circ}(f_1, f_2) = |f_1(p) - f_2(p)|$ for any value of p such that $\mathrm{grad}_{g^\partial}(f_1)\big|_p = \mathrm{grad}_{g^\partial}(f_2)\big|_p$.

Thus, we have recovered a metric space, which is isometric to the interior (M°, d_g). However, our goal is to obtain a metric space which is isometric to the whole manifold (M, d_g). The next lemma follows directly from Lemma 5.4.

Lemma 8.26. Let (M, g) be simple. Then the completion of $(\mathrm{SRC}_g^\circ, d_{\mathrm{SRC}_g^\circ})$ is isometric to (M, d_g).

Finally, we remark that the completion of a space is quantifiable in terms of the space and its distance function. Thus, we have quantified a metric space which is isometric to (M, d_g).

8.8 Equivalence of Standard and Interior Data

In this section we show that the gometric objects that were modified to avoid the boundary (i.e. the objects with "$\circ$" supserscripts) are equivalent to their unmodified counterparts. In one direction, (modified $\rightsquigarrow$ unmodified) we could use the fact that the modified data sets determine the interior geometry and then reconstruct the unmodified sets from the "copy" of the original manifold. However, we opt for a more direct approach.

Lemma 8.27. Let (M, g) be simple. Then $(\partial M, g^\partial, \mathrm{C}_g^\circ) \leftrightsquigarrow (\partial M, g^\partial, \mathrm{C}_g)$.

Proof. First, observe that if $\dim(M) = 1$, then there is only one geodesic (up to direction), so $\mathrm{C}_g = \mathrm{C}_g^\circ$. Thus, assume $\dim(M) \geq 2$. First, we show that we can recover C_g° from C_g.

Consider $v \in U\partial M$ and $x \in \partial M$. We will show that we can determine whether a vector in $w \in \pi^{-1}(\{x\}) \cap (v\mathrm{C}_g\bullet)$ satisfies $v\mathrm{C}_g^\circ w$ or not. First, observe that $x \in \gamma_{\hat{v}}$ if and only if $\pi^{-1}(\{x\}) \cap (v\mathrm{C}_g\bullet) = U_x\partial M$. Thus, we can identify which of the following two cases we are in.

1. $x \in \gamma_{\hat{v}}$
2. $x \notin \gamma_{\hat{v}}$

If we are in case (1.), then we know that $w \in (v\mathrm{C}_g^\circ\bullet)$ if and only if $\gamma_{\hat{v}} = \gamma_{\hat{w}}$. This occurs if and only if $\pi(v) \in \gamma_{\hat{w}}$, which we can determine by the same argument as above.

If we are in case (2.), then observe that $\pi^{-1}(\{x\}) \cap (v\mathrm{C}_g\bullet)$ is the immersed image of a closed interval (the immersion is $I_{\hat{v}} \ni t \mapsto \mathrm{aim}(x, \gamma_{\hat{v}}(t))$). The endpoints of this image can be identified, and they are exactly the vectors that correspond to geodesics that intersect $\gamma_{\hat{v}}$ only in ∂M. Thus, we have shown that $(\partial M, g^\partial, \mathrm{C}_g) \rightsquigarrow (\partial M, g^\partial, \mathrm{C}_g^\circ)$.

Now, we show the reverse quantification. Again, let $v \in U\partial M$ and $x \in \partial M$. Then observe that $x \in \gamma_{\hat{v}}$ if and only if $\pi^{-1}(\{x\}) \cap (v\mathrm{C}_g^\circ\bullet)$ is a singleton. Thus, we may identify which of the following two cases we are in.

1. $x \in \gamma_{\hat{v}}$
2. $x \notin \gamma_{\hat{v}}$

If we are in case (1.), then $\pi^{-1}(\{x\}) \cap (v\mathrm{C}_g\bullet)$ is the whole unit disk based at x (i.e. $U_x\partial M$), because all geodesics starting from x intersect $\gamma_{\hat{v}}$.

If we are in case (2.), then $\pi^{-1}(\{x\}) \cap (v\mathrm{C}_g\bullet)$ is the closure of $\pi^{-1}(\{x\}) \cap (v\mathrm{C}_g^\circ\bullet)$ in $U\partial M$. $\square$

Lemma 8.28. Let (M, g) be simple. Then $(\partial M, g^\partial, \mathrm{STAR}_g^\circ) \leftrightsquigarrow (\partial M, g^\partial, \mathrm{STAR}_g)$.

Proof. Let $S \in \mathrm{STAR}_g$. We must show that we can identify whether or not S corresponds to $\mathrm{star}_g(x)$ for a point in the boundary or the interior. Observe that $x \in \partial M$ if and only if $\pi : S \to \partial M$ is injective. This is because if $x \in \partial M$, then $\{v \in U\partial M | \pi(v) = x\} \subset \mathrm{star}_g(x)$. Thus, we have shown that STAR_g quantifies STAR_g°.

For the other direction, recall that the inerior source direction data quantifies the lens data. Additionally, observe that if $x \in \partial M$, and $\gamma_{\hat{v}} \neq x$, then x is one of the endpoints of $\gamma_{\hat{v}}$. Thus, we obtain the follow description of $\mathrm{star}_g(x)$: If $x \in \partial M$, then $\mathrm{star}_g(x) = \{v \in U\partial M | \pi(\alpha_g(v)) = x\} \cup \{v \in U\partial M | \pi(v) = x\}$. Thus, STAR_g is all of the sets in STAR_g° together with all of the sets of the form described in the previous sentence. ☐

Lemma 8.29. Let (M, g) be simple. Then $(\partial M, g^\partial, \mathrm{SRC}_g^\circ) \leftrightsquigarrow (\partial M, g^\partial, \mathrm{SRC}_g)$.

Proof. In one direction, we claim that SRC_g is the closure of SRC_g° in $C(\partial M; \mathbb{R})$ with respect to the sup norm. To see this, simply observe that if $f = d_g(x, \bullet)$ for $x \in \partial M$, and $\{x_n\}_{n=1}^\infty$ is a sequence in M° such that $x_n \to x$, then by Lemma 2.17 $d_g(x_n, \bullet) \to f$. Thus, SRC_g is a subset of the closure of SRC_g°. It remains to show that SRC_g is closed. This fact follows from the fact that $x \mapsto d_g(x, \bullet)$ is continuous, so SRC_g is the continuous image of a compact set, and is thus closed.

In the other direction, let $f \in \mathrm{SRC}_g$. Observe that $f \in \mathrm{SRC}_g^\circ$ if and only if $f(x) > 0$ for all $x \in \partial M$. ☐

9 The Stitching Data

The intersection data is a very coarse description of geodesics intersections. Given two directions on the boundary, do the corresponding geodesics intersect at all or not? With this information, we were able to recover the interior geomery with the restriction that the manifolds must be simple.

If we have more fine-grained information about geodesic intersections, perhaps we can lift some of the restrictions. In this section, we formulate a geodesic intersection data object called the *stitching data*. If the intersection data is the coarsest description of geodesics intersections, then the stitching data is the finest (short of just being given the whole manifold and all of its geodesics).

Intuitively, the stitching data is like being given all of your geodesics as strings in a basket, where each string is the length of the corresponding geodesic. And, for each pair of strings you pull out, you know exactly which points on one string correspond to which points on the other. In other words, you know which points correspond to intersections in the manifold, and how far along one geodesic you need to travel to reach each intersection point of the other. We call it the stitching data, because you can imagine attaching the strings together according to the given information and recovering the geometry of the manifold like *stitching* a piece of fabric.

A major difference between the stitching data and the intersection data is that the stitching data is applicable to manifolds without boundary. In fact, we will see that we can lift all geometric assumptions about the manifold with boundary. Thus, the stitching data is a generic way to encode the geometry of any manifold.

As an application of the stitching data, we formulate a geometric boundary measurement object called the *delayed intersection data*. Intuitively, the delayed intersection data corresponds to the following pseudo-physical situation. Suppose we have a Riemannian manifold with boundary (M, g) and we are allowed to shoot off two particles from any two points on the boundary along geodesics in any direction. The delayed intersection data tells you how to time your releases so that the two particles collide (if they collide at all), and how long you have to wait for the collision to occur.

We formulate geometric conditions which we call *intersection-confirming* and *semi-nontrapping*, and show that the delayed intersection data determines a stitching data if that condition is met by the manifold. The class of intersection-confirming and semi-nontrapping manifolds strictly contains the class of simple manifolds.

9.1 Definition of Stitching Data

Let (M, g) be a Riemannian manifold with (possibly empty) boundary. For $v, w \in SM$, write $v \sim_{SM} w$ if there exists $t \in I_v$ such that $w = \dot{\gamma}_v(t)$ or $w = -\dot{\gamma}_v(t)$. Then it is easy to verify that $\sim_{SM}$ is an equivalence relation on SM. Thus, $\sim_{SM}$ partitions SM into equivalence classes. Let $[v] \subset SM$ denote the equivalence class of $\sim_{SM}$ containing v. We let $\mathcal{G} = {SM}/{\sim_{SM}}$. Then $\mathcal{G}$ represents the space of geodesics.

In the following definition, we allow *interval* to mean any non-empty connected subset of $\mathbb{R}$. Specifically, this allows for open intervals, half-open intervals, infinite intervals, and even singleton sets.

Definition 9.1. Let (M, g) be a Riemannian manifold with boundary. Let $(\mathscr{G}, m_\bullet, \mathscr{C}_{\bullet,\bullet})$ be a triple where $\mathscr{G}$ is a set, m_α is an interval for each $\alpha \in \mathscr{G}$, and $\mathscr{C}_{\alpha,\beta} : m_\alpha \to 2^{m_\beta}$ is a function for each $\alpha, \beta \in \mathscr{G}$. We say $(\mathscr{G}, m, \mathscr{C})$ is a **stitching data for** (M, g) if there exists a function $f : \mathscr{G} \to SM$ satisfying

1. $\alpha \mapsto [f(\alpha)]$ is surjective from $\mathscr{G}$ to $\mathcal{G}$.
2. $m_\alpha = I_{f(\alpha)}$ for each $\alpha \in \mathscr{G}$.
3. For α, β, then $t \in \mathscr{C}_{\alpha,\beta}(s)$ if and only if $\gamma_{f(\alpha)}(s) = \gamma_{f(\beta)}(t)$.

Let $(\mathscr{G}, m, \mathscr{C})$ be a stitching data for (M, g). Intuitively, elements of $\mathscr{G}$ correspond to geodesics of (M, g), $m_\bullet$ tells us how to parametrize the corresponding geodesics, and $\mathscr{C}_{\bullet,\bullet}$ tells us which points on one geodesic correspond to which points on another geodesic. In particular, observe that $\mathscr{C}_{\alpha,\beta}(t) = \varnothing$ for all $t \in m_\alpha$ if and only if the corresponding geodesics do not intersect.

We observe that once the function f is specified, there are no choices to make in properties (2.) and (3.) above. m and $\mathscr{C}$ are completely determined by the geometry, and f. Specifically, we have the follwing lemma.

Lemma 9.2. Let (M, g) be a Riemannian manifold with boundary. If $\mathscr{G}$ is a set and $f : \mathscr{G} \to SM$ is a function such that $\alpha \mapsto [f(\alpha)]$ is surjective from $\mathscr{G}$ to $\mathcal{G}$. Then there exists a unique stitching data $(\mathscr{G}, m^f, \mathscr{C}^f)$ for (M, g) such that $f, m, \mathscr{C}$ satisfy (1.)-(3.) in Definition 9.1.

Proof. Property (2.) implies that we *must* define $m_\alpha = I_{f(\alpha)}$. From there, for $s \in m_\alpha$, we have $\mathscr{C}_{\alpha,\beta}(s) = \gamma^{-1}_{f(\beta)}(\gamma_{f(\alpha)}(s))$. □

For such an $f : \mathscr{G} \to SM$, we call $(\mathscr{G}, m^f, \mathscr{C}^f)$ the **stitching data generated by** f.

To recover the geometry of (M, g) from a stitching data, we will show that standard Riemannian length function restricted to piecewise geodesic paths generates the same distance function d_g. Then, we will show that the stitching data quatifies a particular length space which is isomorphic to the Riemannian length function, together with the piecewise geodesic paths.

9.2 Piecewise Geodesic Length Structure

In this section, we develop the piecewise geodesic length structure, which will be the standard Riemannian length structure restricted to piecewise geodesic curves. We will conclude this section by showing that the metric generated by the piecewise geodesic length structure is equal to d_g.

Definition 9.3. Let (M, g) be a Riemannian manifold with boundary. We say a continuous curve $\eta : [a, b] \to M$ is **piecewise geodesic** if there exists a partition $\{x_1, ..., x_n\}$ of $[a, b]$, vectors $\{v_1, ..., v_{n-1}\} \subset SM$, and smooth functions $\{s_1, ..., s_{n-1}\}$ such that

1. $v_k \in T_{\eta(x_k)}M$
2. $s_k : [x_k, x_{k+1}] \to I_{v_k}$
3. $\eta\big|_{[x_k,x_{k+1}]}(t) = \gamma_{v_k}(s_k(t))$ for all $t \in [a, b]$.

For such a curve, we write $\eta \in \mathcal{A}^{p.g.}$.

Lemma 9.4. $(M, \mathcal{A}^{p.g.}, L_g)$ is a length space.

Proof. Properties (1.)-(3.) of L_g in Definition 3.1 follow directly from the fact that $(M, \mathcal{A}, L_g)$ is a length space and $\mathcal{A}^{p.g.} \subset \mathcal{A}$. Property (4.) of $L_g\big|_{\mathcal{A}^{p.g.}}$ follows from the fact that the infimum over a smaller set is at least as large as the infimum over a larger set. Properties (1.)-(3.) of $\mathcal{A}^{p.g.}$ follow in a straightforward (if tedious) way from the definition of $\mathcal{A}^{p.g.}$. □

Let $d_{p.g.}$ be the distance function generated by $(M, \mathcal{A}^{p.g.}, L_g)$. Specifically, we mean

$$d_{p.g.}(x, y) = \inf_{\eta \in \mathcal{A}^{p.g.}_{x,y}} L_g(\eta)$$

Our main goal for this section is to prove the following.

Theorem 9.5. Let (M, g) be a Riemannian manifold with boundary. Then $d_g = d_{p.g.}$.

For the duration of this chapter, we let $\mathcal{A}$ denote the standard set of admissable curves for M (i.e. piecewise smooth curves). Since $\mathcal{A}^{p.g.} \subset \mathcal{A}$, and the distance functions are defined by taking the infimum over the corresponding sets of admissable curves, we easily obtain $d_g \leq d_{p.g.}$. Thus, we wish to show the oppositie inequality $d_{p.g.} \leq d_g$.

Our proof will proceed in two parts. First, we will show that $d_{p.g.}(x, y) \leq d_g(x, y)$ when $x, y \in M^\circ$. Thus, we will obtain $d_{p.g.}\big|_{M^\circ \times M^\circ} = d_g\big|_{M^\circ \times M^\circ}$. Then we will use the fact that $M^\circ \times M^\circ$ is dense in $M \times M$ to extend $d_{p.g.}\big|_{M^\circ \times M^\circ}$ uniquely to all of $M \times M$. Its unique continuous extension will be d_g.

First, we show that curves in $\mathcal{A}$ are Lipschitz.

Lemma 9.6. Let $\eta \in \mathcal{A}$. Then η is Lipschitz with respect to d_g.

Proof. Let $\eta : [a, b] \to M$ be in $\mathcal{A}$. We must show that there exists $N > 0$ such that $d_g(\eta(s), \eta(t)) < N|s - t|$ for all $s, t \in [a, b]$.

Let $\{x_1, ..., x_n\}$ be a partition of $[a, b]$ such that $\eta\big|_{[x_k, x_{k+1}]}$ is smooth. Let $\eta_k = \eta|_{[x_k, x_{k+1}]}$. Then $|\dot{\eta}_k|_g$ is continuous for each k. Thus, by the extreme value theorem there exists $N_k > 0$ usch that $|\dot{\eta}_k|_g \leq N_k$ on its domain. Let $N = \max\{N_1, ..., N_{n-1}\}$. Then

$$\begin{aligned} d_g(\eta(s), \eta(t)) &\leq L_g(\eta\big|_{[}s, t]) \\ &\leq \int_s^t |\dot{\eta}(r)|_g dr \\ &\leq \int_s^t N dr \\ &\leq N|s - t| \end{aligned}$$

as required. □

Let us define the set $\mathcal{A}^{p.g.\circ} = \{\eta\big|_{[a,b]} \in \mathcal{A}^{p.g.} | \eta([a, b]) \subset M^\circ\}$ of piecewise geodesic paths contained in M°. Additionally, let us recall the definition of $\mathcal{A}^\circ = \{\eta\big|_{[a,b]} \in \mathcal{A} | \eta([a, b]) \subset M^\circ\}$, the set of piecewise smooth paths contained in M°.

Lemma 9.7. Let (M, g) be a Riemannian manifold with boundary. Let $x, y \in M^\circ$ and $\eta \in \mathcal{A}^\circ_{x,y}$. Then there exists $\tilde{\eta} \in \mathcal{A}^{p.g.\circ}_{x,y}$ such that $L_g(\tilde{\eta}) \leq L_g(\eta)$.

Proof. Let $(M, g), x, y, \eta$ be as stated. Our strategy will be to find a partition $\{t_k\}_{k=1}^{n}$ of $[a, b]$ such that there is a minimizing geodesic between $\eta(t_k)$ and $\eta(t_{k+1})$. Then, we will form $\tilde{\eta}$ by concatenating the minimizing geodesic segments.

By Lemma 5.9, we know the injectivity radius is continuous on M°. Thux, by the extreme value theorem the function $\mathrm{inj}_{\eta(t)}$ achieves a positive minimum on $[a, b]$. Let $0 < r < \inf_{t\in[a,b]} \mathrm{inj}_{\eta(t)}$. This implies that there is a unique unit speed, minimizing geodesic from $\eta(t)$ to $\eta(s)$ whenever $d_g(\eta(t), \eta(s)) \leq r$.

By Lemma 9.6, there exists $N > 0$ such that $d_g(\eta(s), \eta(t)) \leq N|s - t|$. In particular, if $|s - t| \leq \frac{r}{N}$ then there is a minimizing geodesic segment from $\eta(s)$ to $\eta(t)$.

Thus, let $\{t_1, ..., t_n\}$ be a partition of $[a, b]$ such that $|t_k - t_{k+1}| < \frac{r}{N}$ for $k = 1, ..., n - 1$. For each such k, let $\eta_k : [0, d_g(\eta(t_k), \eta(t_{k+1}))] \to M$ be the minimizing geodesic segment connecting $\eta(t_k)$ to $\eta(t_{k+1})$. We form $\tilde{e}ta$ by concatenating all of the η_k.

It is clear that $\tilde{\eta} \in \mathcal{A}^{p.g.}_{x,y}$ by construction. Additionally, since all the η_k are minimizing, $L_g(\eta_k) \leq L_g(\eta|_{[t_k,t_k+1]})$ for all $k = 1, 2, ..., n - 1$. Thus, $L_g(\tilde{\eta}) \leq L_g(\eta)$ as required. ☐

Lemma 9.8. Let (M, g) be a Riemannian manifold with boundary. Let $x, y \in M^\circ$. Then $d_{p.g.}(x, y) = d_g(x, y)$.

Proof. Let $x, y \in M^\circ$. As we have already established, we only need to show the inequality $d_{p.g.}(x, y) \leq d_g(x, y)$. By Lemma 9.7, Lemma 5.5, and the fact that $\mathcal{A}^{p.g.} \supset \mathcal{A}^{p.g.\circ}$, we have the following inequalities and equalities

$$\begin{aligned} d_{p.g.}(x, y) &= \inf_{\eta\in\mathcal{A}^{p.g.}_{x,y}} L_g(\eta) \\ &\leq \inf_{\eta\in\mathcal{A}^{p.g.\circ}_{x,y}} L_g(\eta) \\ &\leq \inf_{\eta\in\mathcal{A}^{\circ}_{x,y}} L_g(\eta) \\ &= d_g(x, y) \end{aligned}$$

as required. ☐

We would like to use a "unique continuous extension" argument to show that $d_{p.g.} = d_g$. There are some subtleties with this, because we don't know *a priori* that the topologies generated by $d_{p.g.}$ and d_g are the same. We address these subtleties in the following lemmas.

Lemma 9.9. Let (M, g) be a Riemannian manifold with boundary. Then d_g is continuous with respect to $d_{p.g.}$.

Proof. Let $U \subset \mathbb{R}$ be open. We must show that $d_g^{-1}(U)$ is open in the topology generated by $d_{p.g.}$. It follows from Lemma 5.3 and Lemma 2.16 that $d_g^{-1}(U)$ is open in the original topology of $M \times M$. Additionally, by Lemma 3.3 and Lemma 2.8, we know that the topology on $M \times M$ generated by $d_{p.g.}$ is finer than the original product topology on $M \times M$. Thus $d_g^{-1}(U)$ is open in the product topology generated by $d_{p.g.}$ as required. ☐

Lemma 9.10. Let (M, g) be a Riemannian manifold with boundary. Then M° is dense in M with respect to the toplogy generated by $d_{p.g.}$.

Proof. Let $x \in M$ and $\varepsilon > 0$. We will show that there exists $y \in M^\circ$ such that $d_{p.g.}(x, y) < \varepsilon$. If $x \in M^\circ$, then choose $y = x$ and we are done. Thus, assume $x \in \partial M$. Let $\nu_x \in T_x M$ be the inward pointing unit normal at x. Then by Lemma 5.13, there exists $0 < \delta < \varepsilon$ such that γ_{ν_x} is defined on $[0, \delta]$, and $\gamma_{\nu_x}(\delta) \in M^\circ$. Additionally, γ_{ν_x} is a unit-speed curve with respect to L_g. Thus $L_g(\nu_x\big|_{[0,\delta]}) = \delta$. It follows from the definition of $d_{p.g.}$ that $d_{p.g.}(x, \nu_x(\delta)) \leq \delta < \varepsilon$ as required. ☐

We are now able to prove Theorem 9.5.

Proof of Theorem 9.5. By Lemma 2.16, we know that $d_{p.g.}$ is continuous with respect to the topology it generates. By Lemma 2.9 and Lemma 9.10, we know that $M^\circ \times M^\circ$ is dense in $M \times M$ with respect to the product topology generated by $d_{p.g.}$. By Lemma 9.8, we know that $d_{p.g.} = d_g$ on a $d_{p.g.}$ -dense subset of $M \times M$. Thus, by Lemma 9.9 and Lemma 2.10 it follows that $d_{p.g.} = d_g$ on all of $M \times M$. ☐

9.3 Quantifying an Isomorphic Length Space

In the previous section, we showed that $d_{p.g.} = d_g$. In this section, we show that we can quantify a length space that is isomorphic to $(M, L_g, \mathcal{A}^{p.g.})$ in terms of a stitching data.

As before, let (M, g) be a Riemannian manifold with boundary, and let $(\mathscr{G}, m, \mathscr{C})$ be a stitching data for M. We form the **stitching space** by taking the disjoint union $\mathcal{S} = \bigsqcup_{\alpha \in \mathscr{G}} m_\alpha$. For points in $\mathcal{S}$, we use subscripts to make it explicit which m_α they come from. For instance, we would write $a_\alpha \in m_\alpha \subset \mathcal{S}$.

Clearly a stitching data quantifies the corresponding stitching space. In other words

Lemma 9.11. Let (M, g) be a Riemannian manifold with boundary and $(\mathscr{G}, m, \mathscr{C})$ be a stitching data for (M, g). Then $(\mathscr{G}, m, \mathscr{C}) \rightsquigarrow \mathcal{S}$.

Fix $f : \mathscr{G} \to SM$ as in Definition 9.1. We use f to define a function $\tilde{\Psi}_f : \mathcal{S} \to M$, by

$$\tilde{\Psi}_f(a_\alpha) = \gamma_{f(\alpha)}(a_\alpha)$$

Since every point is an element of some geodesic, $\tilde{\Psi}_f$ is surjective. Thus, we form an equivalence relation $\sim_f$ on $\mathcal{S}$ by writing $a_\alpha \sim_f b_\beta$ if $\tilde{\Psi}_f(a_\alpha) = \tilde{\Psi}_f(b_\beta)$.

Write $\langle a_\alpha \rangle \subset \mathcal{S}$ to denote the equivalence class of $\sim_f$ containing a_α. Since $\tilde{\Psi}_f$ is surjective, it follows that there is a unique bijection $\Psi_f : \mathcal{S}/\sim_f \to M$ satisfying $\Psi_f(\langle a_\alpha \rangle) = \tilde{\Psi}_f(a_\alpha)$ for all $a_\alpha \in \mathcal{S}$. We use this bijection to form the pullback length space $(\mathcal{S}/\sim_f, \Psi_f^* \mathcal{A}^{p.g.}, \Psi_f^* L_{p.g.})$. Where $L_{p.g.} = L_g\big|_{\mathcal{A}^{p.g.}}$. By Lemma 3.10, and Theorem 9.5, the metric space generated by $(\mathcal{S}/\sim_f, \Psi_f^* \mathcal{A}^{p.g.}, \Psi_f^* L_{p.g.})$ is isometric to (M, d_g). Thus, to show that a stitching data determines the geometry of the manifold it comes from, we wish to show the following theorem.

Theorem 9.12. Let (M, g) be a Riemannian manifold with boundary. Let $(\mathscr{G}, m, \mathscr{C})$ be a stitching data for (M, g), and $f : \mathscr{G} \to SM$ be a function satisfying the hypotheses of Definition 9.1. Then $(\mathscr{G}, m, \mathscr{C}) \rightsquigarrow (\mathcal{S}/\sim_f, \Psi_f^* \mathcal{A}^{p.g.}, \Psi_f^* L_{p.g.})$.

We will prove Theorem 9.12 in three parts (one part for each element of the tuple $(\mathcal{S}/\sim_f, \Psi_f^* \mathcal{A}^{p.g.}, \Psi_f^* L_{p.g.})$).

For the remainder of the section, we fix a stitching data $(\mathscr{G}, m, \mathscr{C})$ and we fix an $f : \mathscr{G} \to SM$ as in Definition 9.1. Thus, we will refer to *the* stitching data.

First, we show that the equivalence relation can be quantified in terms of the stitching data. This is equivalent to saying that the equivalence relation does not depend on the particular f.

Lemma 9.13. *Let $a_\alpha, b_\beta \in \mathcal{S}$. Then $a_\alpha \sim_f b_\beta$ if and only if $b_\beta \in \mathscr{C}_{\alpha,\beta}(a_\alpha)$.*

Proof. $a_\alpha \sim_f b_\beta$ if and only if $\tilde{\Phi}_f(a_\alpha) = \tilde{\Phi}_f(b_\beta)$, which happens if and only if $\gamma_{f(\alpha)}(a_\alpha) = \gamma_{f(\beta)}(b_\beta)$. This is just a restatement of property (3.) of Definition 9.1. ☐

Next, we show that the set of admissible curves, $\Psi_f^* \mathcal{A}^{p.g.}$ can be quantified in terms of the stitching data. Note that the following quantification uses the equivalence classes of $\sim_f$. This is valid, since we just showed that the relation can be quantified in terms of the stitching data.

Lemma 9.14. *Let $\eta : [c, d] \to \mathcal{S}/\sim_f$. Then $\eta \in \Psi_f^* \mathcal{A}^{p.g.}$ if and only if there exists a partition $\{x_1, ..., x_n\}$ of $[c, d]$, elements $\alpha_1, ..., \alpha_{n-1} \in \mathscr{G}$, and smooth functions $\{\eta_k : [x_k, x_{k+1}] \to m_{\alpha_k}\}_{k=1}^{n-1}$ such that*

1. $\eta|_{[x_k, x_{k+1}]}(t) = \langle \eta_k(t) \rangle$, *for* $k = 1, 2, ..., n-1$.
2. $\langle \eta_k(x_{k+1}) \rangle = \langle \eta_{k+1}(x_{k+1}) \rangle$ *for* $k = 1, 2, ..., n-2$.

Proof. First, let $\eta : [c, d] \to \mathcal{S}/\sim_f$. For the forward direction, assume $\eta \in \Psi_f^* \mathcal{A}^{p.g.}$. From the definition of pullback length space, this occurs if and only if $(\Psi_f \circ \eta) : [c, d] \to M$ is piecewise geodesic. Thus, there exists a partition $\{x_k\}_{k=1}^n$ of $[c, d]$, vectors $\{v_k\}_{k=1}^{n-1} \subset SM$, and smooth functions $\{s_k : [x_k, x_{k+1}] \to I_{v_k}\}_{k=1}^n$ such that $(\Psi_f \circ \eta)(t) = \gamma_{v_k}(s_k(t))$ for $t \in [x_k, x_{k+1}]$.

Recall that $f : \mathscr{G} \to SM$ satisfies $\alpha \mapsto [f(\alpha)]$ is surjective onto equivalence classes of $\sim_{SM}$. Thus, for each v_k, let $\alpha_k \in \mathscr{G}$ be such that $f(\alpha_k) \sim_{SM} v_k$. This implies that $\gamma_{f(\alpha_k)}$ and γ_{v_k} are just reparameterizations of the same geodesic. Thus, there exists $\tilde{s}_k : I_{v_k} \to m_{\alpha_k}$ smooth, such that $\gamma_{f(\alpha_k)} \circ \tilde{s}_k = \gamma_{v_k}$. It follows that $(\Psi_f \circ \eta)(t) = \gamma_{f(\alpha_k)}(\tilde{s}_k(s_k(t)))$ for $t \in [x_k, x_{k+1}]$. Thus, define $\eta_k : [x_k, x_{k+1}] \to m_{\alpha_k}$ by $\eta_k(t) = \tilde{s}_k(s_k(t))$. This implies that

$\Psi_f(\eta(t)) = \gamma_{f(\alpha_k(t))}(\eta_k(t))$ for $t \in [x_k, x_{k+1}]$. By the defining property of Ψ_f, we obtain that $\eta(t) = \langle \eta_k(t) \rangle$ for $t \in [x_k, x_{k+1}]$. Thus, property (1.) is satisfied.

To show that $\langle \eta_k(x_{k+1}) \rangle = \langle \eta_{k+1}(x_{k+1}) \rangle$, we will use the fact that $\Psi_f \circ \eta : [c, d] \to M$ must be continuous. Thus, $\gamma_{f(\alpha_k)}(\eta_k(x_{k+1})) = \gamma_{f(\alpha_{k+1})}(\eta_{k+1}(x_{k+1}))$. From Lemma 9.13, we get that this occurs if and only if $\langle \eta_k(x_{k+1}) \rangle = \langle \eta_{k+1}(x_{k+1}) \rangle$. Thus, we have shown property (2.) and have completed the forward direction.

For the reverse direction, suppose that there exists a partition $\{x_k\}_{k=1}^{n}$ of $[c, d]$, elements $\{\alpha_k\}_{k=1}^{n-1}$ of $\mathscr{G}$, and smooth functions $\{\eta_k : [x_k, x_{k+1}]\}_{k=1}^{n-1}$ satisfying (1.) and (2.). We must show that $(\Psi_f \circ \eta) : [x_k, x_{k+1}] \to M$ is piecewise geodesic.

From property (1.), $(\Psi_f \circ \eta)(t) = \Psi_f(\langle \eta_k(t) \rangle)$ for $t \in [x_k, x_{k+1}]$. By the defining property of Ψ_f, we have that $(\Psi_f \circ \eta)(t) = \tilde{\Psi}_f(\eta_k(t)) = \gamma_{f(\alpha_k)}(\eta_k(t))$. Thus, $\Psi_f \circ \eta$ is a smooth reparameterization of a geodesic on each $[x_k, x_{k+1}]$.

It remains to show that $\Psi_f \circ \eta$ is continuous. This follows directly from property (2.). ☐

Finally, we show that the pullback length structure $\Psi_f^* L_{p.g.}$ can be quantified in terms of the stitching data.

Lemma 9.15. Let $\eta \in \Psi_f^* \mathcal{A}^{p.g.}$, be such that $\eta : [c, d] \to {}^{\mathcal{S}}\!/\!_{\sim_f}$. Let $\{x_1, ..., x_n\}$, $\alpha_1, ..., \alpha_{n-1} \in \mathscr{G}$, $\{\eta_1, ..., \eta_{n-1}\}$ be as in Lemma 9.14. Then

$$\left(\Psi_f^* L_{p.g.}\right)(\eta) = \sum_{k=1}^{n-1} \int_{x_k}^{x_{k+1}} |\eta_k'(s)| ds$$

Proof. We compute

$$\begin{aligned}
(\Psi_f^* L_{p.g.}(\eta)) &= L_{p.g.}(\Psi_f \circ \eta) \\
&= \sum_{k=1}^{n-1} L_g\big((\Psi_f \circ \eta)\big|_{[x_k, x_{k+1}]}\big) \\
&= \sum_{k=1}^{n-1} L_g(\gamma_{f(\alpha_k)} \circ \eta_k) \\
&= \sum_{k=1}^{n-1} \int_{x_k}^{x_{k+1}} \big|\frac{d}{ds}[\gamma_{f(\alpha_k)}(\eta_k(s))]\big|_g ds \\
&= \sum_{k=1}^{n-1} \int_{x_k}^{x_{k+1}} |\eta_k'(s)| ds
\end{aligned}$$

☐

9.4 The Delayed Collision Data

Imagine you have a Riemannian manifold with boundary, and for each point on the boundary you can shoot off particles in any inward pointing direction, and that the particles travel along geodesics. Further imagine that when you do this, you can

- introduce a delay between firing the two particles
- for a given delay, and given directions, measure if a collision between the particles occurs
- if a collision occurs, measure how long it took for the collision to occur

We call the data that you would collect in the hypothetical experiments described above the *delayed collision data*. We provide a formal definition below.

Definition 9.16. Let (M, g) be a Riemannian manifold with boundary. Define the **delayed collision operator** $\mathbb{D}_g : \overline{U}\partial M \times \overline{U}\partial M \times [0, \infty) \to [0, \infty]$ by

$$\mathbb{D}_g(v, w, D) = \inf \Bigg(\{t \in I_{\hat{v}} \cap [0, \infty) | (t + D) \in I_{\hat{w}} \cap [0, \infty) \\ \text{and } \gamma_{\hat{v}}(t) = \gamma_{\hat{w}}(t + D)\} \cup \{\infty\} \Bigg)$$

We call the triple $(\partial M, g^{\partial}, \mathbb{D}_g)$ the **delayed collision data**.

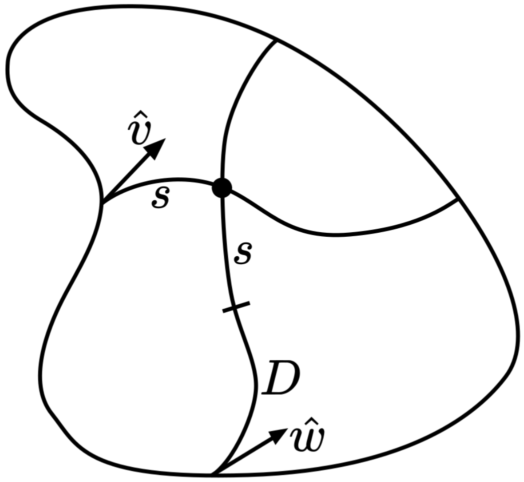
Where we have chosen to work with the **closed unit disk bundle for the boundary** $\overline{U}\partial M = \{v \in T\partial M | g(v, v) \leq 1\}$. The main considerations when dealing with $\overline{U}\partial M$ rather than $U\partial M$ are the following

- When $|v|_g = 1$, then $v = \hat{v}$ is tangent to the boundary.
- Depending on the convexity/concavity of ∂M, $I_{\hat{v}}$ might be the singleton set $\{0\}$ or might extend in the negative direction. Both of these possibilities can only occur if $\hat{v}$ is tangent to the boundary.

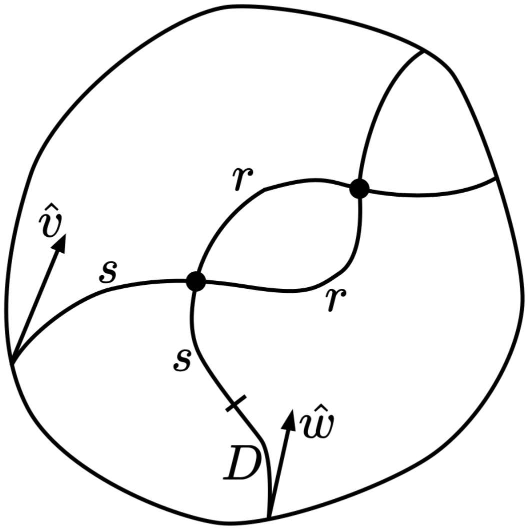
Intuitively, if you fire particles in direction $\hat{w}$ at time 0, in direction $\hat{v}$ at time D, and they first collide s seconds later (at time $D + s$), then $\mathbb{D}_g(v, w, D) = s$. This is shown in Figure 3a. If $\mathbb{D}_g(v, w, D) = \infty$, then the particles never collided.

We would like to use the delayed collision data to quantify a stitching data. However, a stitching data requires information about *all* geodesics, while the delayed collision data only directly has information about goedesics that intersect the boundary. Thus, the following definition is natural.

Definition 9.17. Let (M, g) be a Riemannian manifold with boundary. If the map $v \mapsto [\hat{v}]$ is surjective from $\overline{U}\partial M$ to $\mathcal{G}$, then we say (M, g) is **semi-nontrapping**.



(a) $\mathbb{D}_g(v, w, D) = s$



(b) One intersection point is "hiding" another, because both geodesic segments have length r.

We remark that, while there is no *direct* information in the delayed collision data for strictly trapped geodesics (i.e. geodesics which never reach the boundary), one may be able to recover some information about them by investigating limits of geodesics that do intersect the boundary. However, we will not attempt to analyze this situation in this document. We will restrict our attention to semi-nontrapping manifolds.

Let $\iota_g : \overline{U}\partial M \to SM$ be the injection defined by $\iota_g : v \mapsto \hat{v}$. For a semi-nontrapping manifold, we would like to show that the delayed collision data quantifies $(\overline{U}\partial M, m^{\iota_g}, \mathscr{C}^{\iota_g})$, the stitching data generated by ι_g. Thus, for each pair of vectors $v, w \in U\partial M$, we would like to know about their intersections.

Naïvely, one might hope to pick out all intersections between two geodesics by firing pairs with all possible delays and recording whether value of the collision operator is ∞. However, since $\mathbb{D}_g$ only selects the first collision, then one collision can "hide" behind another in the rare situation that the geodesic segments between the two collision points are exactly the same length. This is shown in Figure 3b. In particular, for no value of D will the value of $\mathbb{D}_g(v, w, D)$ or $\mathbb{D}_g(w, v, D)$ correspond to the second intersection point. However, we may use a third geodesic which passes through the second intersection point to confirm the existence of the intersection, as long as that third geodesic does not have the original problem with repsect to the first two geodesics. We formalize this situation below.

Definition 9.18. Let (M, g) be a Riemannian manifold with boundary. Let $v, w \in U\partial M$. We say the pair (v, w) is **generically delayed** if $\gamma_{\hat{v}}(s) = \gamma_{\hat{w}}(t)$ implies $\mathbb{D}_g(v, w, t - s) = s$ or $\mathbb{D}_g(v, w, s - t) = t$ (depending on whether $t - s$ or $s - t$ is non-negative) whenever $t, s \geq 0$.

When (v, w) is generically delayed, then there are no hidden intersections. In particular, for a simple manifold, all pairs (v, w) such that $\gamma_{\hat{v}}, \gamma_{\hat{w}}$ are distinct are generically delayed. However, for connected one-dimensional manifolds with boundary, there is only one geodesic (up to a change of direction), and thus any two geodesics are *not* generically delayed. Thus, while we will not explicitly mention it in the statement of our results, they will implicitly apply only when $\dim(M) \geq 2$, as there will be no one-dimensional manifolds that meet our hypotheses.

Definition 9.19. Let (M, g) be a Riemannian manifold with boundary. Let $v, w \in \overline{U}\partial M$. Let $x \in \gamma_{\hat{v}} \cap \gamma_{\hat{w}}$. Let $z \in U\partial M$ be such that

1. $x = \gamma_{\hat{z}}(t)$ for some $t \geq 0$
2. (v, z) and (w, z) are generically delayed

Then, we say that z **confirms the intersection** of $\gamma_{\hat{v}}$ and $\gamma_{\hat{w}}$ at x.

We will be interested in manifolds for which all intersections can be confirmed.

Definition 9.20. Let (M, g) be a Riemannian manifold with boundary. Suppose that for all v, w, x such that $v, w \in U\partial M$, and $x \in \gamma_{\hat{v}} \cap \gamma_{\hat{w}}$ there exists $z \in U\partial M$ that confirms the intersection of $\gamma_{\hat{v}}$ and $\gamma_{\hat{w}}$ at x. Then, we say that (M, g) is **intersection-confirming**.

Our goal is to show that the delayed collision data determines the geometry of a semi-nontrapping, intersection-comfirming Riemannian manifold. Specifically, we will show the following.

Theorem 9.21. Let (M, g) be semi-nontrapping and intersection-confirming. Then $(\partial M, g^{\partial}, \mathbb{D}_g) \rightsquigarrow (\overline{U}\partial M, m^{\iota_g}, \mathscr{C}^{\iota_g})$.

We remark that it is not obvious (at least to the author), that there are any semi-nontrapping manifolds which are *not* intersection-confirming. And certainly, if they do exist, it seems that it should be rare. However, the following conjecture proved too challenging to defeat.

Conjecture 9.22. The set of Riemannian metrics on a fixed smooth manifold which are semi-nontrapping and intersection-confirming is generic in the set of Riemannian metrics which are semi-nontrapping.

Clearly $\overline{U}\partial M$ can be written in terms of ∂M and g^{∂}. Thus, we must show that m^{ι_g} and $\mathscr{C}^{\iota_g}$ can be quantified by the delayed collision data. We break this up into two lemmas; one for each quantification.

Lemma 9.23. Let (M, g) be semi-nontrapping and intersection-confirming. Let $v \in \overline{U}\partial M$. Then

1. $s \in m^{\iota g}(v) \cap [0,\infty)$ if and only if there exists $w \in \overline{U}\partial M$ and $D \in [0,\infty)$ such that $\mathbb{D}_g(v,w,D) = s$ or $\mathbb{D}_g(w,v,s-D) = D$.

2. If $|v|_g < 1$ then $m^{\iota g}(v) = m^{\iota g}(v) \cap [0,\infty)$

3. If $|v|_g = 1$ then $m^{\iota g}(v) = \big[m^{\iota g}(v) \cap [0,\infty)\big] \cup \Big[-\big(m^{\iota g}(-v) \cap [0,\infty)\big)\Big]$

Where $-A = \{-a | a \in A\}$ for any subset $A \subset \mathbb{R}$.

Proof. From the definition of *induced stitching data*, we have that $m^{\iota g}(v) = I_{\hat{v}}$. Thus, part (2.) follows directly from Lemma 5.13. Additionally, part (3.) follows from the fact that $I_{-\hat{v}} = -I_{\hat{v}}$. Thus, it remains to show part (1.).

In one direction, let $s \in I_{\hat{v}} \cap [0,\infty)$. Then, by Definition 9.20, there exists $w \in \overline{U}\partial M$ that confirms the intersection of $\gamma_{\hat{v}}$ with itself at $\gamma_{\hat{v}}(s)$. Thus, (v,w) are generically delayed and $\gamma_{\hat{v}}(s) = \gamma_{\hat{w}}(t)$ for some $t \geq 0$. If $t - s \geq 0$ then $\mathbb{D}_g(v,w,t-s) = s$ by Definition 9.18. Thus, let $D = t - s$. If $s - t \geq 0$, then $\mathbb{D}_g(w,v,s-t) = t$ by Definition 9.18. In this case, let $D = t$. Thus, we have shown that (1.).

Conversely, suppose there exists $D \in [0,\infty)$ such that $\mathbb{D}_g(v,w,D) = s$ or $\mathbb{D}_g(w,v,s-D) = D$. First, suppose $\mathbb{D}_g(v,w,D) = s$. Then by the defintion of $\mathbb{D}_g$ there exists $s' \geq s$ such that $s' \in I_{\hat{v}}$. Thus, $s \in I_{\hat{v}}$. If we are in the other case (i.e. that $\mathbb{D}_g(w,v,s-D) = D$. Then by the definition of $\mathbb{D}_g$ there exists $D' \geq D$ such that $(D' + s - D) \in I_{\hat{v}}$. Thus, $s \in I_{\hat{v}}$ as required. ☐

Finally, we provide a quantification of $\mathscr{C}^{\iota g}$.

Lemma 9.24. Let (M,g) be semi-nontrapping and intersection-confirming. Let $v, w \in \overline{U}\partial M$. Then $t \in \mathscr{C}^{\iota g}_{v,w}(s)$ if and only if there exists $z \in \overline{U}\partial M$ and $r \in m^{\iota g}(z)$ such that

1. $\mathbb{D}_g(v,z,r-s) = s$ or $\mathbb{D}_g(z,v,s-r) = r$ depending on whether $r - s$ or $s - r$ is non-negative.

2. $\mathbb{D}_g(w,z,r-t) = t$ or $\mathbb{D}_g(z,w,t-r) = r$ depending on whether $r - t$ or $t - r$ is non-negative.

Proof. Let $v, w \in \overline{U}\partial M$, $s \in m^{\iota g}(v)$, $t \in m^{\iota g}(w)$. First, suppose that $t \in \mathscr{C}^{\iota g}_{v,w}(s)$. This occurs if and only if $\gamma_{\hat{v}}(s) = \gamma_{\hat{w}}(t)$. Then, there exists $z \in \overline{U}\partial M$ that confirms the existence of the intersection of $\gamma_{\hat{v}}$ and $\gamma_{\hat{w}}$ at $\gamma_{\hat{v}}(s) = \gamma_{\hat{w}}(t)$. Thus, (v,z) and (w,z) are generically delayed and $\gamma_{\hat{z}}(r) = \gamma_{\hat{v}}(s) = \gamma_{\hat{v}}(t)$ for some $r \geq 0$.

Thus, by Definintion 9.18, if $r - s \geq 0$ then we know that $\mathbb{D}_g(v,z,r-s) = s$, and if $s - r \geq 0$, then we know that $\mathbb{D}_g(z,v,s-r) = r$. This yields (1.). A very similar argument yields (2.).

Conversely, suppose that there exists $z \in \overline{U}\partial M$ and $r \in m^{\iota_g}(z)$ satisfying (1.) and (2.). We will show that $\gamma_{\hat{z}}(r) = \gamma_{\hat{v}}(s)$. The same argument will yield that $\gamma_{\hat{z}}(r) = \gamma_{\hat{w}}(t)$. Thus, by transitivity, we will have $\gamma_{\hat{v}}(s) = \gamma_{\hat{w}}(t)$, which is equivalent to $t \in \mathscr{C}^{\iota_g}_{v,w}(s)$.

First, suppose that $\mathbb{D}_g(v, z, r - s) = s$. Then $\inf\{s' \in I_{\hat{v}} | \gamma_{\hat{v}}(s') = \gamma_{\hat{z}}(s' + r - s)\} = s$. Thus, there exists a sequence $\{s_n\}_{n=1}^{\infty}$ such that $s_n \to s$ and $\gamma_{\hat{v}}(s_n) = \gamma_{\hat{z}}(s_n + r - s)$. Taking limits, we obtain $\gamma_{\hat{v}}(s) = \gamma_{\hat{z}}(r)$.

In the other case, suppose $\mathbb{D}_g(z, v, s - r) = r$. Then again, we have $r = \inf\{r' \in I_{\hat{z}} | \gamma_{\hat{z}}(r') = \gamma_{\hat{v}}(r' + s - r)\}$. By the same argument as above, we obtain $\gamma_{\hat{v}}(s) = \gamma_{\hat{z}}(r)$ as required.

As stated, the same argument using (2.) yields $\gamma_{\hat{z}}(r) = \gamma_{\hat{w}}(t)$. Thus, we have $\gamma_{\hat{w}}(t) = \gamma_{\hat{v}}(s)$, so $t \in \mathscr{C}^{\iota_g}_{v,w}(s)$ as required. □

10 Future Work

In this thesis, we considered three inverse problems: intersection rigidity, stitching rigidity and delayed collision rigidity. We will first discuss future work that applies to all three problems, and then we will discuss other problems individually.

With all geometric inverse problems there are the following questions.

- Injectivity
    - Does the data uniquely determine the object (up to natural isomorphism)?
- Partial Data
    - If you only have a subset of the original data, does this still determine the object (up to natural isomorphism)?
- Continuity of Inverse
    - If two data sets are approximately equal, does this imply that the objects they came from are approximately equal?
- Reconstruction Procedure
    - Is there a formula or an algorithm to reconstruct the original object from the data?
- Finite Data
    - If you can only take a finite number of measurements, can you approximately reconstruct the original object?

For each of our three problems, we only considered injectivity. Thus, the other four questions are open.

To discuss questions of continuity, one would have to topologize the three spaces. Thus, we have our first question.

Question 10.1. Is there a natural way to topologize the intersection data, stitching data and delayed collision data?

Since $\mathbb{D}_g$ is a function, one may consider the myriad of ways to topologize function spaces. For the intersection data, one may consider C_g as a subset of $M \times M$ and use the Hausdorff distance? Both of these suggestions only work directly if considering a fixed smooth manifold M. It seems less obvious where to begin thinking about topologizing the stitching data.

10.1 Future Work for Intersection Rigidity

For the intersection data, there is no obvious reason to restrict to simple manifolds as there was with the boundary distance data. Thus, we have the following question.

Question 10.2. Is there a larger class of manifolds that are intersection rigid?

A first step would be to remove the convex boundary assumption, and just consider non-trapping manifolds without conjugate points. It seems very likely that this class of manifolds should be intersection rigid. In particular, it seems that the four-and-higher dimensional argument should readily generalize to such a situation. To remove the conjugate point and non-trapping assumption would require significant more thought, although there is no obvious reason why they would not be intersection rigid.

Staying in the class of simple manifolds, if one could show that the boundary distance data determines the intersection data, then of course Michel's conjecture would immediately follow.

Question 10.3. For a simple manifold, can the intersection data be quantified in terms of the boundary distance data?

This is true for two dimensional manifolds. To see this, observe that in two dimensions, for simple manifolds, $v\mathrm{C}_g^\circ w$ if and only if the end points of $\gamma_{\hat{v}}$ and $\gamma_{\hat{w}}$ alternate as you go around the boundary of M, and of course, the boundary distance data determines the lens data, which will determine the endpoints. This proof does not immediately generalize to higher dimensions.

10.2 Future Work for Delayed Collision Rigidity

For the delayed collision data, we had two constraints.

- semi-nontrapping
- intersection-confirming

We would like to understand these constraints better, and understand if they can be lifted.

Question 10.4. Are manifolds with strictly trapped geodesics delayed collision rigid?

Question 10.5. Are all manifolds intersection-confirming? Does a generic Riemannian metric yield an intersection-confirming manifold?

Finally, for applications, one would like to find PDEs, for which initial conditions can be set up for which recording the solutions corresponds to measuring the delayed condition data.

Question 10.6. Is there a PDE that "geometrizes" to yield the delayed collision data?

For such a PDE, one should be able to set up initial conditions such that wave-packets can propogate along geodesics. Additionally, the PDE should have some nonlinearity, so that there will be an interaction of the two propogating wave packets.

References

[1] Gérard Besson, Gilles Courtois, and Sylvestre Gallot. Entropies et rigidités des espaces localement symétriques de courbure strictement négative. *Geometric & Functional Analysis GAFA*, 5(5):731–799, 1995.

[2] Nicolas Boumal. An introduction to optimization on smooth manifolds. Available online, Nov 2020.

[3] Dmitri Burago, Yuri Burago, and Sergei Ivanov. *A Course in Metric Geometry.* American Mathematical Society, 2001.

[4] Dmitri Burago and Sergei Ivanov. Boundary rigidity and filling volume minimality of metrics close to a flat one. *Annals of mathematics*, pages 1183–1211, 2010.

[5] Alberto Calderón. On inverse boundary value problem. *Computational and Applied Mathematics*, 25, 01 2006.

[6] Kenneth C Creager. Anisotropy of the inner core from differential travel times of the phases pkp and pkikp. *Nature*, 356(6367):309–314, 1992.

[7] Chrisopher B Croke, Gunther Uhlmann, Irena Lasiecka, and Michael Vogelius. *Geometric methods in inverse problems and PDE control*, volume 137. Springer Science & Business Media, 2004.

[8] Christopher Croke. Scattering rigidity with trapped geodesics. *Ergodic Theory and Dynamical Systems*, 34(3):826–836, 2014.

[9] Christopher B Croke. Rigidity for surfaces of non-positive curvature. *Commentarii Mathematici Helvetici*, 65(1):150–169, 1990.

[10] Mikhael Gromov et al. Filling riemannian manifolds. *Journal of Differential Geometry*, 18(1):1–147, 1983.

[11] Colin Guillarmou, Marco Mazzucchelli, and Leo Tzou. Boundary and lens rigidity for non-convex manifolds. *arXiv preprint arXiv:1711.10059*, 2017.

[12] Gustav Herglotz. Über die elastizitaet der erde bei beruecksichtigung ihrer variablen dichte. *Zeitschr. für Math. Phys*, 52:275–299, 1905.

[13] Mark Kac. Can one hear the shape of a drum? *The American Mathematical Monthly*, 73(4):1–23, 1966.

[14] Yaroslav Kurylev, Matti Lassas, and Gunther Uhlmann. Rigidity of broken geodesic flow and inverse problems. *American Journal of Mathematics*, 132(2):529–562, 2010.

[15] Matti Lassas, Teemu Saksala, and Hanming Zhou. Reconstruction of a compact manifold from the scattering data of internal sources. *Inverse Problems & Imaging*, 12(4):993–1031, 2018.

[16] Matti Lassas, Vladimir Sharafutdinov, and Gunther Uhlmann. Semiglobal boundary rigidity for riemannian metrics. *Mathematische Annalen*, 325(4):767–793, 2003.

[17] John M. Lee. *Introduction to Topological Manifolds.* Springer, 2nd edition, 2012.

[18] John M. Lee. *Introductions to Smooth Manifolds.* Springer, 2nd edition, 2013.

[19] John M. Lee. *Introduction to Riemannian Manifolds.* Springer, 2nd edition, 2018.

[20] René Michel. Sur la rigidité imposée par la longueur des géodeésiques. *Inventiones mathematicae*, 65:71–86, 1981.

[21] René Michel. Restriction de la distance géodésique a un arc et rigidité. *Bulletin de la Société Mathématique de France*, 122(3):435–442, 1994.

[22] RG Mukhometov. The problem of reconstructing a riemann metric and integral geometry. In *Dokl. Akad. Nauk SSSR*, volume 232, pages 32–35, 1977.

[23] James Munkres. *Topology.* Pearson, 2nd edition, 2000.

[24] Jean-Pierre Otal. Sur les longueurs des géodésiques d'une métrique à courbure négative dans le disque. *Commentarii Mathematici Helvetici*, 65(1):334–347, 1990.

[25] L Pestov and G Uhlmann. Two dimensional simple compact manifolds with boundary are boundary rigid. *Annals of Math*, 161:1089–1106, 2005.

[26] Leonid Pestov, Gunther Uhlmann, and Hanming Zhou. An inverse kinematic problem with internal sources. *Inverse problems*, 31(5):055006, 2015.

[27] Vladimir Altafovich Sharafutdinov. *Integral geometry of tensor fields*, volume 1. Walter de Gruyter, 2012.

[28] Plamen Stefanov and Gunther Uhlmann. Boundary rigidity and stability for generic simple metrics. *Journal of the American Mathematical Society*, 18(4):975–1003, 2005.

[29] Plamen Stefanov, Gunther Uhlmann, Andars Vasy, and Hanming Zhou. Travel time tomography. *Acta Mathematica Sinica*, 35:1085–1114, 2019.

[30] Plamen Stefanov, Gunther Uhlmann, and Andras Vasy. Local and global boundary rigidity and the geodesic x-ray transform in the normal gauge. *arXiv preprint arXiv:1702.03638*, 2017.

[31] Engineering ToolBox. Velocity of sound in solids and metals, 2004.

[32] Gunther Uhlmann. Inverse problems: seeing the unseen. *Bulletin of Mathematical Sciences*, 4(2):209–279, 2014.

[33] Gunther Uhlmann and András Vasy. The inverse problem for the local geodesic ray transform. *Inventiones mathematicae*, 205(1):83–120, 2016.

[34] James Vargo. A proof of lens rigidity in the category of analytic metrics. *Mathematical Research Letters*, 16, 01 2009.

[35] Emil Wiechert and Karl Bernhard Zoeppritz. *Uber Erdbebenwellen.* Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, 1907.