Rectifiability via curvature and regularity in anisotropic problems

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Abstract

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Understanding the geometry of rectifiable sets and measures has led to a fascinating interplay of geometry, harmonic analysis, and PDEs. Since Jones' work on the Analysts' Traveling Salesman Problem, tools to quantify the flatness of sets and measures have played a large part in this development. In 1995, Melnikov discovered and algebraic identity relating the Menger curvature to the Cauchy transform in the plane allowing for a substantially streamlined story in $\mathbb{R}^2$. It was not until the work of Lerman and Whitehouse in 2009 that any real progress had been made to generalize these discrete curvatures in order to study higher-dimensional uniformly rectifiable sets and measures.

Since 2015, Meurer and Kolasinski began developing the framework necessary to use discrete curvatures to study sets that are countably rectifiable. In Chapter 2 we bring this part of the story of discrete curvatures and rectifiability to its natural conclusion by producing multiple classifications of countably rectifiable measures in arbitrary dimension and codimension in terms of discrete measures. Chapter 3 proceeds to study higher-order rectifiability, and in Chapter 4 we produce examples of 1-dimensional sets in $\mathbb{R}^2$ that demonstrate the necessity of using the so-called "pointwise" discrete curvatures to study countable rectifiability.

Since at least the 1930s with the work of Douglas and Rad'o, Plateau's problem has been at the heart of many developments in geometric measure theory and PDEs. After the pioneering work by De Giorgi on the regularity of area minimizing surfaces, studying the

regularity of anisotropic energy minimizing surfaces has been an extremely active area of research leading to new interactions between geometry and the fields of PDEs, Calculus of Variations, and Optimal Transport.

For anisotropic minimal surfaces all known higher-order regularity results, that is, regularity beyond countable rectifiability, require an assumption on the energy that is similar in effect to the "ellipticity" condition of Almgren in so far as the condition forces the PDE that eventually arises to be a linear elliptic PDE. The second part of this thesis discusses regularity results for anisotropic problems in geometry and PDEs which do not benefit from a naturally arising elliptic PDE. Chapter 5 begins with results in low-dimensions, including a quick proof of regularity in dimension 2 for all anisotropic minimal surfaces, and a Bernstein-type theorem. Chapter 6 studies $\| \cdot \|_p$-minimal surfaces by a monotonicity formula, allowing one to consider types of surfaces not covered in Chapter 5. The monotonicity formula ensures that blow-ups of $\| \cdot \|_p$-stationary surfaces are stationary cones, motivating an exploration of what anisotropic stationary cones in $\mathbb{R}^2$ can look like. Chapter 7 produces regularity results for a general family of PDEs, including a maximum principle, a Harnack inequality, and a Liouville theorem.

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DEDICATION

To the one with whom I laugh the most.

Thanks for all the adventures Sam.

Chapter 1

INTRODUCTION

This thesis has two major parts. The first part, Chapters 2-4, analyzes the relationship between rectifiability and discrete curvatures. Chapters 5-7 form the second part, where we turn our focus to the regularity of anisotropic problems. The regularity of two distinct, but closely related, types of anisotropic problems are presented: anisotropic minimal surface problems and anisotropic PDEs.

1.1 Rectifiability via curvature

In geometry, manifolds are of particular interest because of their nice, rigid, tangent structure. In the calculus of variations, one can study “surfaces” (i.e., manifolds, currents, varifolds, etc.) that minimize some energy. Unfortunately, the same rigidity of the tangent structure of manifolds that makes studying them so interesting, also means that the limit of manifolds in some ambient space is unlikely to remain a manifold.

This lack of “closedness” of the space of submanifolds leads to the desire to find more general notions of manifolds that are closed under some appropriate limit. The tangent structure, which depends upon a manifold becoming flat as you zoom in on it, is an important geometric property for any suitable generalization of manifolds.

Countably (Lipschitz, n)-rectifiable measures on $\mathbb{R}^m$ are precisely the measure-theoretic objects that have a tangent structure. Intuitively, that is they are flat after zooming in far enough and approach a unique tangent. More precisely, for some class of functions $\mathcal{C}$ from $\mathbb{R}^n \to \mathbb{R}^m$, we say that a Radon measure μ on $\mathbb{R}^m$ is countably $(\mathcal{C}, n)$-rectifiable if there exists $\{f_i\}_{i\in\mathbb{N}} \subset \mathcal{C}$ so that $\mu(\mathbb{R}^m \setminus \cup_i f_i(\mathbb{R}^n)) = 0$. A set $E \subset \mathbb{R}^m$ is said to be countably $(\mathcal{C}, n)$-rectifiable if the restriction of the n-dimensional Hausdorff measure, $\mathcal{H}^n|_E$ is countably $(\mathcal{C}, n)$-rectifiable.

Due to Rademacher's theorem, that is that Lipschitz functions are differentiable almost everywhere, one could suspect that geometrically this means a measure μ is (Lipschitz, n)-rectifiable precisely when μ-a.e. point has an n-dimensional measure theoretic tangent space. While not straightforward, this indeed turns out to be the case, see for instance [51, Theorem 15.19] when considering sets. This demonstrates that reasonable generalizations of manifolds should at least be rectifiable, and in turn shows the need for necessary and/or sufficient conditions to test rectifiability.

Since the Analysts' Traveling Salesman Theorem, see [42] in $\mathbb{R}^2$ and [57] in $\mathbb{R}^n$, tools to quantify how flat a set or measure is at all locations and scales have become a ubiquitious when characterizeing the rectifiability of sets and measures. In fact, these quantitative tools have helped identify a subclass of rectifiable measures, called uniformly rectifiable measures which have interesting roles in geometry and PDEs, see [19, 20].

A measure μ on $\mathbb{R}^m$ is said to be uniformly n-rectifiiable if it is Ahlfors n-regular, that is if there exists $0 < c < C < \infty$ so that $cr^n \leq \mu(B(x,r)) \leq Cr^n$ for all $x \in \text{spt}\mu$ and all $0 < r < \text{diam}(\text{spt}\mu)$, and if it also has the property that there exists some $\theta > 0$, so that for all $x \in \text{spt}\mu$ and all $0 < r < \text{diam}(\text{spt}\mu)$, there exists one Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}^m$ with the property that $\mu(f(B(0,r))) > \theta\mu(B(x,r))$ where the dimension of each ball can be understood from context.

The most popular tool to quantify flatness of sets and measures are the β-numbers. For a measure μ, we define

$$\beta^n_{\mu;p}(x,r) := \left(\inf_{n\text{-planes } L} \frac{1}{r^n} \int_{B(x,r)} \left(\frac{\text{dist}(y,L)}{r} \right)^p d\mu(y) \right)^{\frac{1}{p}}.$$

Geometrically, one can interpret $\beta^n_{\mu;p}(x,r)$ as telling you in an L^p-sense how far $\mu|_{B(x,r)}$ is from being supported on an n-dimensional plane. Therefore, assumptions like

$$\int_0^R \beta^n_{\mu;2}(x,r)^2 \frac{dr}{r} < \infty \tag{1.1}$$

imply that as r approaches zero, the support of μ looks more and more like it is contained within an n-dimensional plane, and hence like the measure μ is becoming "flat".

There are many very useful "parametrization results" which say under sufficient flatness assumptions, like (1.1), one can produce a function f parametrizing a set (or measure) and that f is regular. A woefully incomplete list of some of these parametrization results with different levels of regularity includes: continuous [61], Lipschitz [65, 74, 21, 30], $C^{1,\alpha}$ [38].

Quantitative notions of flatness are not the only method to characterize rectifiability. In [19], one of the many characterizations of uniform rectifiability is by the L^2-boundedness of singular integral operators. When $n = 2$, in [54], an algebraic identity was found that directly relates the Menger curvature of triples of points to the Cauchy transform in $\mathbb{C}$. The Menger curvature of three points x_1, x_2, x_3 is defined as the inverse of the radius of the unique circle passing through the three points. In the special case of $\mathbb{R}^2$, this identity lead to a dramatic simplification of the equivalences in [19] by passing through the Menger curvature, [53, 52].

Unfortunately, in [32] Farag demonstrates that no algebraic identity of the same form could exist in higher-dimensions to relate generalizations of Menger curvature to the Riesz transforms. Nonetheless, in [45] Leger used geometric methods to show that the Menger curvature can, without Melnikov's identity, still characterize countably (Lipschitz,1)-rectifiable sets and measures in $\mathbb{R}^m$ thanks to the flatness it imposes. Beginning in 2009, [46, 47] classify uniformly n-rectifiable sets by many examples of geometrically motivated generalizations of Menger curvature, which we call discrete curvatures. Later [56] Meurer provides a general definition which encompasses these successful discrete curvatures, and produced a sufficient condition analogous to the work of Leger which guarantees rectifiability of sets. Around the same time, in [43], Kolasinski produced a sufficient condition in terms of some specific discrete curvatures to guarantee countably $(C^{1,\alpha}, n)$-rectifiability of measures.

Chapter 2 is work done solely by the author. It provides two new characterizations of countably (Lipschitz,n)-rectifiable Radon measures on $\mathbb{R}^m$ under relatively weak density assumptions, see Theorem 16. We also show a converse to work of Kolasinski, providing a direct quantitative comparison between (centered) β-numbers and Menger curvatures, see Theorem 19.

Chapter 3 is joint work with Silvia Ghinassi (a graduate student at the time). We further develop the ideas behind Theorem 19 demonstrating a sufficient condition for a Radon measure to be a countably ($C^{1,\alpha}$, n)-rectifiable measure, see Theorem I. This is done by demonstrating that finiteness of appropriately modified discrete curvatures imply a set satisfies the hypotheses of a parametrization result [38] by Ghinassi, Theorem 51.

Chapter 4 is joint work with Sean McCurdy (a graduate student at the time). We produce two examples of sets in $\mathbb{R}^2$ demonstrating that the so-called "integral" Menger curvatures cannot be used to detect countable rectifiability by blowing-up– even for connected sets or Ahlfors regular sets. Due to the fact that β-numbers are more popular than Menger curvatures, this chapter is stated in terms of β-numbers. Theorems 5 and 19 provide a direct translation of these results to results about discrete curvatures.

In fact, the original motivation to construct these examples arose from a desire to show the failure of a third attempted characterization of rectifiability by discrete curvatures. That is, we wanted to know if countably (Lipschitz, 1)-rectifiable measures could be characterized by the limiting behavior of integral Menger curvature, i.e., by studying what happens to the expression in (2.33) as $r \downarrow 0$.

A major hurdle immediately presents itself: What if for no $r > 0$ the quantity in (2.33) is finite? Originally, this seemed like just a technicality. However, Theorem 61 (resp. 62) shows that even for countably (Lipschitz,1)-rectifiable sets in $\mathbb{R}^2$, even with the restrictive topological (resp. geometric) constraint of connectedness (resp. Ahlfors regularity), one cannot expect the quantity in (2.33) can to be finite for any x in the set or any $r > 0$.

1.2 Regularity in anisotropic problems

The anisotropic Plateau's problem asks, "If W is an $(n-1)$-dimensional *boundary*, how nice must the n-dimensional *surface* S that has the least energy out of all surfaces that *span* the boundary W be?" There are many ways to interpret this question depending upon how one chooses to define "boundary", "surface", and "spans". We emphasize some of these interpretations during a brief and incomplete survey of the topic.

In the 1930s, Douglas [29] and Radó [60] solved the isotropic Plateau problem (area minimizing surfaces) in the plane by studying surfaces parametrized by continuous functions on the disc, with fixed values on the boundary.

In the 1950s, De Giorgi [22, 23, 24] introduced another notion of surface, which we now call sets of locally finite perimeter. In 1961 De Giorgi [25] demonstrated regularity of the boundary for area minimizing sets of locally finite perimeter near flat points. To do so, he introduced the “tilt-excess improvement” method.

In 1960, Reifenberg [61] used a clever local parametrization argument to produce solutions to the Plateau problem. At the same time, Federer and Fleming generalized De Giorgi’s set of finite perimeter to normal currents, which allow one to study the regularity of n-dimensional energy minimizing surfaces in $\mathbb{R}^m$, ultimately leading to a very satisfying collection of regularity results for anisotropic energy minimizing surfaces of Almgren’s [5] so-called elliptic energies. The epic book by Federer [35, Chapter 5] provides a thorough description of regularity theory for elliptic energy minimizing currents.

The general story of regularity for energy minimizing currents, more-or-less originating from [25] is as follows:

(1) Start at a point on a surface where you have zoomed in far enough that the current looks “flat”.

(2) show that this flatness implies the current can largely be written as the graph of a function u. In turn,

(3) setting the first variation equal to zero, one discovers this function almost solves some PDE.

(4) an ϵ-regularity theorem says that u is close to some v which truly solves the PDE, and

(5) since v solves the PDE, *if the PDE is nice*, then v is regular, so in fact u is even flatter than you expected.

The way that one defines “flatness” turns out to be quantified by a notion of “tilt”. The more tilt a surface has (i.e., the excess tilt) the less flat it is. Hence, the conclusion in step (5) is often called “tilt-excess decay”.

The desire to have a nice PDE in (4) is precisely where Almgren's ellipticity condition comes from: it is a constraint on the energy that forces the PDE to be a linear uniformly elliptic divergence form PDE.

There has been some work attempting to weaken the ellipticity condition, for example [63, 36], for currents. Nonetheless, the weaker assumption still boils down to some sufficient condition to ensure the PDE from (4) is elliptic. The more recent work of Figalli is the only known higher-regularity result for anisotropic energy minimizing currents where it is not assumed that regardless of the orientation, the first variation always produces an elliptic PDE. Still, Figalli assumes that in all "nearby" directions, the resulting PDE is elliptic, and in particular the theory of elliptic PDEs is still the workhorse that yields $C^{1,\alpha}$-regularity near flat points.

Since the PDE, which arises from the first variation of the anisotropic energy, is so important to this technique, it is common to study regularity of surfaces whose first variation is zero. These surfaces are called anisotropic minimal surfaces. Energy minimizing surfaces are always minimal surfaces, but not vice-versa.

In the 1970s, Allard [2, 3] introduced the notion of a varifold as a new way to interpret "surface". The primary tool to study the regularity of area minimal varifolds is through Allard's monotonicity formula. A nice exposition can be found in [67].

Compared to the regularity theory for energy minimizing currents, the regularity theory for minimal varifolds has lagged behind. The regularity theory for area minimal varifolds is based on a monotonicity formula. When considering $n \geq 2$-dimensional varifolds this monotonicity formula holds exclusively for energies that are, up to a constant-coefficient linear change of variables, the area integrand [4]. When the varifold is of dimension $n = 1$, a monotonic quantity is still known (also [4]) but to the best of the author's knowledge there has been no work demonstrating regularity for 1-dimensional anisotropic minimal varifolds using this method because the monotonic quantity has no specific formula that has been shown to be beneficial.

There has been recent progress moving regularity theory beyond area minimal varifolds to anisotropic energy minimal varifolds [27, 26, 28]. In the absence of a monotonicity formula, attempts at showing the varifold is locally $C^{1,\alpha}$ near flat points, depend once again upon some condition forcing the first variation to produce an elliptic PDE.

In chapters 5-6, we study anisotropic minimal surfaces which do not have a first variation that produces an elliptic PDE. The main focus is on surfaces minimizing the $\|\cdot\|_p$-energy. Specifically, we focus on the local regularity of the boundary of sets of locally finite perimeter which locally minimize the $\|\cdot\|_{\ell^p}$-energy. To this end, we need an open set $A \subset \mathbb{R}^n$ and define the functional Φ_ρ on sets of locally finite perimeter by

$$\Phi_p(E;A) := \int_{\partial^* E \cap A} \rho(\nu_E) d\mathcal{H}^{n-1},$$

where $\rho : \mathbb{R}^n \to \mathbb{R}^n$ is some strictly-convex, positively 1-homogeneous function that is C^1 on $\mathbb{R}^n \setminus \{0\}$. We then study the regularity $\partial^* E$ whenever E satisfies

$$\Phi_p(E;A) \leq \Phi_p(F;A) \qquad \forall E \Delta F \subset\subset A.$$

The main focus is when $\rho = \|\cdot\|_{\ell^p}$. Any more general regularity statements are most interesting when considering some ρ where the first variation does not provide us with an elliptic PDE.

Chapter 5 includes a short, but complete, proof that in $\mathbb{R}^2$, all Φ_p-energy minimizing sets of locally finite perimeter have boundaries that are straight edges. Theorem 94 also includes that global minimizers must be halfplanes.

Chapter 6 provides a general sufficient condition for a monotonicity formula for Φ_ρ-minimal surfaces in $\mathbb{R}^n$, namely the existence of a positively 1-homogeneous function f, so that

$$\langle x, \xi \rangle \langle \nabla f(x), \nabla \rho(\xi) \rangle \geq 0 \qquad \forall x, \xi. \tag{1.2}$$

We then demonstrate that $\|\cdot\|_p$-energy minimizing currents in $\mathbb{R}^2$ satisfy a monotonicity formula, which differs from the one in [4]. This monotonicity formula is used to show that blow-ups of energy minimizing currents are global energy minimizing cones. The only specific property of the monotonicity formula used, is that when $n = 2$ and $f = \|\cdot\|_p = \rho$ the expression in (1.2) is zero if and only if $x \cdot \xi$ is zero.

All of the work done in Chapter 6 could be carried over for $\|\cdot\|_p$-minimal varifolds in $\mathbb{R}^2$, ultimately demonstrating that blow-ups of $\|\cdot\|_p$-minimal varifolds are $\|\cdot\|_p$-minimal cones. It is known that the only area-minimal cones in the plane are (1) lines and (2) the so-called triple junction, that is where three half-lines join at a point with 120° angles. Naturally, this rigidity is lost in the anisotropic problem, but we provide a constructive method to produce all anisotropic stationary triple junctions in the plane. This analysis of triple junctions in the plane can also be extended to triple junctions in higher-dimensions, as long as you know what 2-dimensional plane your triple junction is contained in.

Chapter 7 studies the regularity of a wide class of PDEs, which include those that arise as (partially linearized) versions of the PDEs one would get from the first variation in the anisotropic minimal surface problem. In particular, this class of PDEs contains lots of non-elliptic PDEs, including the p-Laplacian and the pseudo p-Laplacian. We show that a De Giorgi-Nash-Moser theory holds for these PDEs. In particular, positive subsolutions are locally bounded (Theorem 136), solutions satisfy a maximum principle (Theorem 138), positive solutions satisfy a Harnack inequality (Theorem 139), and a Liouville theorem holds (Theorem 142).

Chapter 2

CHARACTERIZATION OF COUNTABLY (LIPSCHITZ,n)-RECTIFIABLE MEASURES ON $\mathbb{R}^m$

2.1 An introduction to discrete curvatures and rectifiability

In the late 1990s there was a flurry of activity relating 1-rectifiable sets, boundedness of singular integral operators, the analytic capacity of a set, and the integral Menger curvature in the plane. In 1999 Léger extended the results for Menger curvature to 1-rectifiable sets in higher dimension, as well as to the codimension one case.

A decade later, Lerman and Whitehouse, and later Meurer, found higher-dimensional geometrically motivated generalizations of Menger curvature that yield results about the uniform rectifiability of measures and the rectifiability of sets respectively.

Primarily, these higher-dimensional Menger curvatures have been used to study the regularity of surfaces and knots, for instance, self-avoidance and smoothness of the normal. Herein, we use these tools to find new characterizations of rectifiable Radon measures in arbitrary dimension and codimension (see Theorems 16 and 18). Work in progress indicates the characterization in Theorem 18 (4) and (5) is more likely more useful in practice than the characterization in Theorem 16. We also relate the pointwise Menger curvature to the sum of the β numbers over scales (see Theorem 19). As a consequence of taking tools from knot theory to answer geometric measure theory questions, we include extra details to ensure this paper is sufficiently self-contained for readers from either discipline. Despite this, when classic arguments make repeat appearances, we attempt to avoid repetition by referring the reader to the analogous argument earlier in the paper.

2.1.1 (Uniform) Rectifiability and β_p-coefficients

Studying rectifiable sets and measures (see Definition 21) is a central topic in geometric measure theory. In his 1990 work on the Analysts' traveling salesman problem in the plane [42] (later generalized to 1-dimensional sets in $\mathbb{R}^m$ by Okikiolu in [57] and to n-dimensional sets in $\mathbb{R}^m$ by Pajot in [58]) Peter Jones introduced what are now called the Jones' β-numbers, which have dominated the landscape in quantitative techniques relating to rectifiability, analytic capacity, and singular integrals.

In the joint monograph on the topic [20] David and Semmes laid the framework to understanding the quantitative structures Jones introduced, as well as how to properly generalize these ideas to Ahlfors regular sets and measures higher dimensions. In doing so, they introduced the notion of uniform rectifiability (see Definition 22).

David and Semmes gave many equivalent characterization of uniform n-rectifiability. One characterization is related to Jones' β-numbers the definition of which is included for completeness. For $1 \leq p < \infty$ one defines

$$\beta^n_{\mu;p}(x,r) = \inf_L \left(\frac{1}{r^n} \int_{B(x,r)} \left(\frac{\operatorname{dist}(y,L)}{r} \right)^p \, d\mu(y) \right)^{\frac{1}{p}}, \tag{2.1}$$

where the infimum is taken over all n-dimensional affine subspaces $L \subset \mathbb{R}^m$. When the dimension n is understood from context, the superscript is typically forgotten.

The relevant characterization of uniformly n-rectifiable sets was discovered in [19].

Theorem 1 ([19])**.** *If μ is an n-Ahlfors regular measure on $\mathbb{R}^m$ then the following are equivalent.*
(1) μ is uniformly rectifiable.
(2) For $1 \leq p < \frac{2n}{n-2}$ there exists some $c > 0$ depending on p such that the β_p-numbers satisfy the following so called Carleson-condition.

$$\int_{B(x,R)} \int_0^R \beta^n_{\mu;p}(y,r)^2 \frac{dr}{r} \, d\mu(y) \leq cR^n \quad \textit{for all } x \in \operatorname{spt}\mu, \ \mathrm{R} > 0. \tag{2.2}$$

More recently, a much desired characterization of countably n-rectifiable measures (a characterization for measures, similar to the characterization for sets from [58]) was discovered by Azzam and Tolsa in a pair of papers [72] and [8]. In particular, the following theorem is from [72].

Theorem 2 ([72])**.** *Let $1 \leq p \leq 2$. If μ is a finite Borel measure on $\mathbb{R}^m$ which is countably n-rectifiable, then*

$$\int_0^\infty \beta^n_{\mu;p}(x,r)^2 \frac{dr}{r} < \infty \quad \textit{for } \mu \textit{ a.e. } x \in \mathbb{R}^m. \tag{2.3}$$

The converse has hypothesis on the densities of the measure (see Definition 24).

Theorem 3 ([8]). *Let μ be a finite Borel measure in $\mathbb{R}^m$ such that $0 < \Theta^{n,*}(\mu, x) < \infty$ for μ-a.e. $x \in \mathbb{R}^m$. If*

$$\int_0^\infty \beta^n_{\mu;2}(x,r)^2 \frac{dr}{r} < \infty \quad \mu \text{ a.e. } x \in \mathbb{R}^m, \tag{2.4}$$

then μ is countably n-rectifiable.

Later, [30] (with an alternative proof in [73]) removed the *a priori* assumption of absolute continuity with respect to the Hausdorff measure. Together, these results can be summarized in the following theorem.

Theorem 4 ([30]). *Let μ be a Radon measure in $\mathbb{R}^m$ with $0 < \Theta^{n,*}(\mu, x)$ and $\Theta^n_*(\mu, x) < \infty$. Then μ is countably n-rectifiable if and only if*

$$\int_0^1 \beta^n_{\mu,2}(x,r)^2 \frac{dr}{r} < \infty \quad \text{for } \mu \text{ a.e. } x \in \mathbb{R}^m. \tag{2.5}$$

We note that prior to [30] some partial results on rectifiability of measures without the *a priori* assumption of absolute continuity with respect to the Hausdorff measure were also demonstrated in [9], [10], [11].

In light of the characterization for uniformly rectifiable measures, one might expect that the restriction to $p = 2$ in Theorem 3 is a consequence of the proof, and potentially a range of values for p could be found. However, in [73] a very flexible construction of 1-rectifiable sets with positive and finite measure was laid out, which, by varying the parameters of this construction demonstrate that for $1 \leq p < 2$ no conclusion can be drawn from μ almost everywhere finiteness of the quantity in (2.3). Under the condition that $\Theta^{n,*}(\mu, x) < \infty$, Hölder's inequality guarantees that if $p > 2$ then (2.3) implies (2.4), so only the range $1 \leq p < 2$ was of particular interest.

2.1.2 *Uniform Rectifiability and integral Menger curvatures*

In 1995 Melnikov discovered a beautiful identity connecting Menger curvature to the analytic capacity in the plane, which kicked off a great deal of research to find curvature-based techniques to classify rectifiable and uniformly rectifiable sets, and further to find relations to singular integrals ([54], [53], [52], [50], [45]).

Unfortunately, all of the progress made applied exclusively to dimension 1 and/or codimension 1 sets. In fact, in [32], Farag showed that in the higher-(co)dimension case there does not exist an algebraic generalization of Melnikov's identity that could directly relate to Riesz kernels. In particular, this result does not preclude the usefulness of Menger-type curvatures whose integrand take a different form.

After a hiatus in progress on this subject, Lerman and Whitehouse showed that a geometrically motivated generalization of Menger curvature was sufficient to characterize uniformly n-rectifiable measures in higher codimension (in fact, their results hold for real separable Hilbert spaces). The work, completed in [46] and [47] was motivated by the original work of David and Semmes [19] and [20], but required a nontrivial and creative idea of "geometric multipoles". They defined several versions of Menger curvatures of $(n+2)$-points in $\mathbb{R}^m$ (see [46], [47], and [48]). Among these curvatures, one that is well suited for our setting is

$$\mathcal{K}_1(x_0, \ldots, x_{n+1}) = \left(\frac{\mathcal{H}^{n+1}(\Delta(x_0, \ldots, x_{n+1}))^2}{\operatorname{diam}\{x_0, \ldots, x_{n+1}\}^{(n+1)(n+2)}} \right)^{\frac{1}{2}} \tag{2.6}$$

where $\Delta(x_0, \ldots, x_{n+1})$ is the simplex with corners $\{x_0, \ldots, x_{n+1}\}$. Lerman and Whitehouse showed that control of integrals of the integrand $\mathcal{K}_1$ has interesting geometric consequence, this is explained in more detail below.

One necessary object to understand and state the results of Lerman and Whitehouse, is a notion of the "space of well-scaled simplices". More precisely, for some $0 < \lambda < 1$ and letting $X = (x_0, \ldots, x_{n+1})$ denote an $(n+2)$-tuple in $\mathbb{R}^m$, define

$$W_\lambda(B(x,r)) = \{X \in B(x,r)^{n+2} : \frac{\min(X)}{\mathrm{diam}(X)} \geq \lambda > 0\}. \tag{2.7}$$

Where

$$\min(X) = \min_{0 \leq i < j \leq n+1} |x_i - x_j|.$$

Then, one can think of $W_\lambda(B(x,r))$ as the space of well-scaled simplices in $B(x,r)$. We also define the quantities

$$c_1^2(\mu|_{B(x,r)}) = \int_{B(x,r)^{n+2}} \mathcal{K}_1^2(X)\ d\mu^{n+2}(X). \tag{2.8}$$

and

$$c_1^2(\mu|_{B(x,r)}, \lambda) = \int_{W_\lambda(B(x,r))} \mathcal{K}_1^2(X)\ d\mu^{n+2}(X) \tag{2.9}$$

The measure μ^{n+2} is the measure on $(\mathbb{R}^m)^{n+2}$ resulting from taking $(n+2)$-products of μ with itself.

The quantity in (2.8) can be thought of as the integral Menger curvature of the measure $\mu|_{B(x,r)}$. The quantity in (2.9) can be thought of as the amount of integral Menger curvature of the measure $\mu|_{B(x,r)}$ which arises from "well-scaled simplices". The main results of [46] and [47] are combined to show the following theorem, which notably holds in the more difficult setting of real-seperable Hilbert spaces.

Theorem 5 ([46], [47])**.** *If μ is an n-Ahlfors regular measure on a possibly infinite dimensional, real, separable Hilbert space, then the following are true:*

$$c_1^2(\mu|_{B(x,R)}) \leq C_2 \int_{B(x,R)} \int_0^{12R} \beta^n_{\mu;2}(y,r)^2 \frac{dr}{r}\ d\mu(y)$$

and there exists a $\lambda \in (0,1)$ such that for all $B(x,R)$ with $2R \leq \mathrm{diam}(\mathrm{spt}\mu)$,

$$\int_{B(x,R)} \int_0^{2R} \beta^n_{\mu;2}(y,r)^2 \frac{dr}{r}\ d\mu(y) \leq C_3 \cdot c_1^2(\mu|_{3 \cdot B(x,R)}, \lambda/2)$$

Here, both C_2 and C_3 depend only on n and the Ahlfors regularity constants of μ.

In particular, combining the two parts of Theorem 5 with the observation that $c_1^2(\mu|_{B(x,R)}, \lambda) \leq c_1^2(\mu|_{B(x,R)})$, it follows that

$$c_1^2(\mu|_{B(x,R)}) \leq CR^n \iff \int_{B(x,R)} \int_0^R \beta_{\mu;2}^n(x,r)^2 \frac{dr}{r} \leq \tilde{C}R^n,$$

which by passing through Theorem 1 leads to the following characterization.

Theorem 6 ([19], [46], [47]). *If $n \geq 2$ and μ is an n-Ahlfors regular measure on $\mathbb{R}^m$ then the following are equivalent:*
1) There exists a constant C independent of x and R so that $c_1^2(\mu|_{B(x,R)}) \leq CR^n$ for all $x \in \operatorname{spt}\mu$ and $R > 0$.
2) μ is uniformly n-rectifiable.

2.1.3 Rectifiability and pointwise Menger curvature

In 2015, Meurer found a general class of Menger-type curvatures which satisfy a one-sided comparison to β_p-coefficients and proved a sufficient condition for rectifiability of a set based off these integral menger curvatures. The integrand $\mathcal{K}_1$ of Lerman and Whitehouse is an example of the general class of Menger-type integrands laid out in [55] (the shortened published version is [56]). So, throughout the remainder this section, on a first read one should interpret the phrase (μ, p)-proper integrand, to mean the integrand $\mathcal{K}_1$. The formal definition of a "(μ, p)-proper integrand" is given in Definition 28. A symmetric (μ, p)-proper integrand is defined in Definition 31. A review of notation is also included in the preliminaries in section 2.2.2.

One main result of [55] is as follows,

Theorem 7 ([55]). *Let $E \subset \mathbb{R}^m$ be a Borel set and $\mu = \mathcal{H}^n|_E$. If $\mathcal{K}$ is a $(\mu, 2)$-proper* *integrand such that*

$$\mathcal{M}_{\mathcal{K}^2}(\mu) := \int_{\mathbb{R}^m} \cdots \int_{\mathbb{R}^m} \mathcal{K}^2(x_0, \ldots, x_{n+1})\ d\mu^{n+2}(x_0, \ldots, x_{n+1}) < \infty,$$

then E is countably n-rectifiable.

Remark 8. *Meurer also showed that $\mathcal{M}_{\mathcal{K}^2}(\mathcal{H}^n|_E) < \infty$ implies $\mathcal{H}^n|_E$ is locally finite. A similar result does not hold for Borel measures μ who are absolutely continuous with respect to the Hausdorff measure. For instance, consider μ on $\mathbb{R}^2$ defined by*

$$d\mu = \frac{1}{|x_1|} d\mathcal{H}^1|_{\{x_2=0\}}$$

where $x \in \mathbb{R}^2$ is written as $x = (x_1, x_2)$. Then $\mathrm{spt}\mu = \{\mathrm{x}_2 = 0\}$ implies $\mathcal{K}_1(y_0, y_1, y_2) = 0$ whenever $y_i \in \mathrm{spt}(\mu)$ for all $i = 1, 2, 3$. Consequently, $\mathcal{M}_{\mathcal{K}_1}(\mu) = 0$. Nonetheless, $\mu(B(0,\delta)) = \displaystyle\int_{-\delta}^{\delta} \frac{dx_1}{|x_1|} = \infty$ for all $\delta > 0$.

The next theorem is an interesting intermediate result of [55] which is of similar style to the work of Lerman and Whitehouse. As such, one must again have the correct interpretation of the "space of well-scaled simplices", namely

$$\mathcal{O}_k(x,t) := \{(x_0, \ldots, x_{n+1}) \in B(x, kt)^{n+2} \mid |x_i - x_j| \geq \frac{t}{k}\ \forall i \neq j\}. \tag{2.10}$$

Given some (μ, p)-proper integrand $\mathcal{K}$, define the notation

$$\mathcal{M}_{\mathcal{K}^p;k}(x,t) := \int_{\mathcal{O}_k(x,t)} \mathcal{K}^p(x_0, \ldots, x_{n+1})\ d\mu^{n+2}(x_0, \ldots, x_{n+1})$$

so that one can succinctly state:

Theorem 9 ([55])**.** *Let μ be an n-upper Ahlfors regular Borel measure, with upper-regularity constant C_0 (see Definition 20). Let $0 < \lambda < 2^n$ and $k > 2, k_0 \geq 1$. Then there exist* *constants*

$$k_1 = k_1(m, n, C_0, k, k_0, \lambda) > 1 \text{ and } C = C(m, n, \mathcal{K}, p, C_0, k, k_0, \lambda) \geq 1$$

such that if $\mu(B(x,t)) \geq \lambda t^n$*, then for every* $y \in B(x, k_0 t)$ *we have*

$$\beta^n_{\mu;p}(y, kt)^p \leq C \frac{\mathcal{M}_{\mathcal{K}^p;k_1}(x,t)}{t^n} \leq C \frac{\mathcal{M}_{\mathcal{K}^p,k_1+k_0}(y,t)}{t^n}.$$

A corollary of the previous result, whose relevance to the results of Lerman and Whitehouse is more immediate can be stated as

Corollary 10 ([55])**.** *Let* μ *be an* n*-upper Ahlfors regular Borel measure on* $\mathbb{R}^m$ *with upper-regularity constant* C_0 *. Fix* $0 < \lambda < 2^n, k > 2, k_0 \geq 1$ *and* $\mathcal{K}^p$ *any symmetric* (μ, p)*-proper integrand. Then there exists a constant* $C = C(m, n, \mathcal{K}, p, C_0, k, k_0, \lambda)$ *such that*

$$\int_{\mathbb{R}^m} \int_0^\infty \beta^n_{\mu;p}(x,t)^p \mathbb{1}_{\{\tilde{\delta}_k(B(x,t)) \geq \lambda\}} \frac{dt}{t}\, d\mu(x) \leq C \mathcal{M}_{\mathcal{K}^p}(\mu),$$

where

$$\tilde{\delta}_k(x, kt) = \sup_{y \in B(x,kt)} \frac{\mu(B(y,t))}{t^n}.$$

Remark 11. *Although it seemingly goes unmentioned in [55], we note that this corollary implies the following statement:*

If μ *is an* n*-Ahlfors regular measure on* $\mathbb{R}^m$*, and* $\mathcal{M}_{\mathcal{K}^p}(\mu|_{B(x,r)}) \lesssim r^n$ *with suppressed constant independent of* x*, for all* $x \in \operatorname{spt}\mu$ *and all* $0 < r < \operatorname{diam}\operatorname{spt}(\mu)$ *for some symmetric* (μ, p)*-proper integrand, and* $p \in [2, \frac{2n}{n-2})$*, then* μ *is uniformly* n*-rectifiable.*

This follows directly from Corollary 10 and Theorem 1.

In [47, Equation 10.1] Lerman and Whitehouse also introduce the curvature

$$\mathcal{K}_2(x_0, \ldots, x_{n+1}) = \left(\frac{h_{\min}(x_0, \ldots, x_{n+1})^2}{\operatorname{diam}(\{x_0, \ldots, x_{n+1}\})^{n(n+1)+2}} \right)^{\frac{1}{2}},$$

where

$$h_{\min}(x_0,\ldots,x_{n+1}) = \min_i \operatorname{dist}(x_i, \operatorname{aff}\{x_0,\ldots,x_{i-1},x_{i+1},\ldots,x_{n+1}\}) \tag{2.11}$$

and $\operatorname{aff}\{y_0,\ldots,y_k\}$ denotes the smallest affine plane containing $\{y_0,\ldots,y_k\}$.

Notably, $\mathcal{K}_2$ satisfies

$$\mathcal{K}_1^2(x_0,\ldots,x_{n+1}) \leq \mathcal{K}_2^2(x_0,\ldots,x_{n+1}) \tag{2.12}$$

for all $(x_0,\ldots,x_{n+1}) \in (\mathbb{R}^m)^{n+2}$ and is also a $(\mu,2)$-proper integrand. We define the pointwise Menger curvature of μ with respect to a (μ,p)-proper integrand $\mathcal{K}$ at x and scale r by

$$\operatorname{curv}^n_{\mathcal{K}^p;\mu}(x,r) = \int_{(B(x,r))^{n+1}} \mathcal{K}^p(x,x_1,\ldots,x_{n+1})\; d\mu^{n+1}(x_1,\ldots,x_{n+1}). \tag{2.13}$$

Then a simplified (and strictly weaker) version of the main result of [44] is:

Lemma 12. *[44] Let μ be a Borel measure on $\mathbb{R}^m$. Then, there exists $\Gamma = \Gamma(n,m)$ such that*

$$\operatorname{curv}^n_{\mathcal{K}_2^2;\mu}(x,R) \leq \Gamma \int_0^{2R} \Theta^n(\mu,x,r)^n \hat{\beta}^n_{\mu,2}(x,r)^2 \frac{dr}{r},$$

where

$$\Theta^n(\mu,x,r) = \frac{\mu(B(x,r))}{r^n}$$

and

$$\hat{\beta}^n_{\mu,p}(x,r)^p := \inf_{L\ni x} \frac{1}{r^n}\int_{B(x,r)} \left(\frac{\operatorname{dist}(y,L)}{r}\right)^p \; d\mu(y) \tag{2.14}$$

denotes the "centered" β_p-numbers. Notably, the infimum is taken over all n-planes passing through the center of $B(x,r)$.

Another useful result which can be found in a more general setting in [44] is

Lemma 13. *[44]*

Let μ be a Radon measure on $\mathbb{R}^m$ and x be such that $0 < \Theta^n_*(\mu,x) \leq \Theta^{n,*}(\mu,x) < \infty$.

Then, for $p \in [1,\infty]$ and $0 < \rho < \infty$,

$$\int_0^\rho \hat{\beta}^n_{\mu;p}(x,r)^p \frac{dr}{r} < \infty \quad \mu - a.e.\ x \in \mathbb{R}^m \tag{2.15}$$

if and only if

$$\int_0^\rho \beta^n_{\mu;p}(x,r)^p \frac{dr}{r} < \infty \quad \mu - a.e.\ x \in \mathbb{R}^m. \tag{2.16}$$

Moreover, if μ is n-Ahlfors regular, and $q \le p$ then there exists C depending on m, n, p, q and the Ahlfors regularity constants such that

$$\begin{aligned} &\int_{B(x,r)} \int_0^r \beta^n_{\mu;p}(y,s)^q \frac{ds}{s} d\mu(y) \\ &\quad \le \int_{B(x,r)} \int_0^r \hat{\beta}^n_{\mu;p}(y,s)^q \frac{ds}{s} d\mu(y) \\ &\quad \le C \int_{B(x,Cr)} \int_0^{Cr} \beta^n_{\mu;p}(y,s)^q \frac{ds}{s} d\mu(y). \end{aligned} \tag{2.17}$$

A consequence of Lemma 12 and [72] is:

Theorem 14 ([72], [44])**.** *Let μ be a Radon measure on $\mathbb{R}^m$ with $0 < \Theta^n_*(\mu,x) \le \Theta^{n,*}(\mu,x) < \infty$ for μ almost every $x \in \mathbb{R}^m$. If μ is countably n-rectifiable, then $\operatorname{curv}^n_{\mathcal{K}^2_2;\mu}(x,1) < \infty$ for μ almost every $x \in \mathbb{R}^m$.*

Indeed, the equivalence of (i) and (ii) in Lemma 25 ensures that Theorem 14 follows from Lemmata 12, 13, and Theorem 2.

2.1.4 A new look at rectifiability via Menger curvature

The first result of this paper is a generalization of Theorem 7 to the case of Radon measures with upper-density bounded above and below.

Theorem 15. *If μ is a Radon measure on $\mathbb{R}^m$ with $0 < \Theta^{n,*}(\mu,x) < \infty$ for μ almost every $x \in \mathbb{R}^m$ and $\mathcal{M}_{\mathcal{K}^2}(\mu) < \infty$ for some $(\mu,2)$-proper integrand $\mathcal{K}$, then μ is countably* *n-rectifiable.*

In light of Theorem 15, with some additional work we can now answer an open question posed in [47, Section 6] and characterize n-rectifiability with respect to Menger-type curvatures.

Theorem 16. *Let μ be a Radon measure on $\mathbb{R}^m$ with $0 < \Theta^n_*(\mu, x) \leq \Theta^{n,*}(\mu, x) < \infty$ for μ almost every $x \in \mathbb{R}^m$. Then the following are equivalent:*
1) μ is countably n-rectifiable.
2) For μ almost every $x \in \mathbb{R}^m$, $\mathrm{curv}^n_{\mathcal{K}^2;\mu}(x, 1) < \infty$, where $\mathcal{K} \in \{\mathcal{K}_1, \mathcal{K}_2\}$.
3) μ has σ-finite integral Menger curvature in the sense that μ can be written as $\mu = \sum_{j=1}^{\infty} \mu_j$ where each μ_j satisfies $\mathcal{M}_{\mathcal{K}^2}(\mu_j) < \infty$ where $\mathcal{K} \in \{\mathcal{K}_1, \mathcal{K}_2\}$.

Remark 17. *Note that Theorem 16 is to Theorem 4 as 5 is to 1, except that unfortunately, the hypothesis in Theorem 16 are strictly stronger than the hypothesis in Theorem 4. The stronger hypothesis, which may also be an artifact of the proof, does suggest that there may be a better way to define the Menger curvature integrands.*

In particular, with some additional work, combining Theorem 4, the equivalence of (2.15) with (2.16) in Lemma 13, and Theorem 16 yields another new characterization of rectifiable Radon measures in Theorem 18. Moreover, combining Theorem 1, Theorem 5, and (2.17) yields the characterization of uniformly rectifiable measures in Theorem 18.

Theorem 18 ([19], [47], [46], [55], [44], Theorem 16)**.** *If μ is a Radon measure on $\mathbb{R}^m$ with $0 < \Theta^n_*(\mu, x) \leq \Theta^{n,*}(\mu, x) < \infty$ for μ almost every $x \in \mathbb{R}^m$, then the following are equivalent*

1. *μ is countably n-rectifiable.*

2. $\displaystyle\int_0^1 \beta^n_{\mu;2}(x, r)^2 \frac{dr}{r} < \infty$ *for μ almost every $x \in \mathbb{R}^m$.*

3. $\displaystyle\int_0^1 \hat{\beta}^n_{\mu;2}(x, r)^2 \frac{dr}{r} < \infty$ *for μ almost every $x \in \mathbb{R}^m$.*

(4) $\operatorname{curv}^n_{\mathcal{K}_2^2;\mu}(x,1) < \infty$ *for* μ *almost every* $x \in \mathbb{R}^m$.

(5) $\operatorname{curv}^n_{\mathcal{K}_1^2;\mu}(x,1) < \infty$ *for* μ *almost every* $x \in \mathbb{R}^m$.

Moreover, if μ *is an* n*-Ahlfors regular Borel measure on* $\mathbb{R}^m$ *and* $p \in [2, \frac{2n}{n-2})$ *then the following are equivalent*

(a) μ *is* n *uniformly-rectifiable*

(b) $\displaystyle\int_{B(x,R)} \int_0^R \beta^n_{\mu;p}(y,r)^2 \frac{dr}{r}\, d\mu(y) \leq CR^n$ *for all* $R > 0$.

(c) $\displaystyle\int_{B(x,R)} \int_0^R \hat{\beta}^n_{\mu;p}(y,r)^2 \frac{dr}{r}\, d\mu(y) \leq \tilde{C}R^n$ *for all* $R > 0$.

(d) $\mathcal{M}_{\mathcal{K}_1^2}(\mu|_{B(x,R)}) \leq C'R^n$ *for all* $R > 0$.

(e) $\mathcal{M}_{\mathcal{K}_2^2}(\mu|_{B(x,R)}) \leq C''R^n$ *for all* $R > 0$.

The final main result more directly shows a comparability of $\operatorname{curv}^n_{\mathcal{K}^2;\mu}(x,1)$ and $\int_0^1 \hat{\beta}^n_{\mu;2}(x,r)^2 \frac{dr}{r}$, (see 2.13 and 2.14 respectively) but in the present state requires stronger hypothesis on the density of μ. This is done by proving a converse to Lemma 12

Theorem 19. *If* μ *is an* n*-Ahlfors upper-regular Radon measure on* $\mathbb{R}^m$*,* $\mathcal{K}$ *is a* $(\mu,2)$*-proper integrand, and* x *is such that* $\Theta^n_*(\mu,x) > 0$ *then*

$$\int_0^R \hat{\beta}^n_{\mu;2}(x,r)^2 \frac{dr}{r} \lesssim \operatorname{curv}^n_{\mathcal{K}_1^2;\mu}(x,R). \tag{2.18}$$

In particular, in conjunction with Lemma 12

$$\int_0^R \hat{\beta}^n_{\mu;2}(x,r)^2 \frac{dr}{r} \lesssim \operatorname{curv}^n_{\mathcal{K}_1^2;\mu}(x,R) \leq \operatorname{curv}^n_{\mathcal{K}_2^2;\mu}(x,R) \lesssim \int_0^{C_1 R} \hat{\beta}^n_{\mu;2}(x,r)^2 \frac{dr}{r}. \tag{2.19}$$

In both (2.18) *and* (2.19) *the suppressed constants*[1] *depend on* x*,* R*, the upper–regularity constant of* μ*,* m *and* n*.*

Since the (suppressed) constant in (2.18) depends on x, it would be interesting to see if μ being n-Ahlfors upper-regular can be weakened to say $\Theta^{n,*}(\mu, x) < \infty$.

2.2 Preliminaries

2.2.1 Sets and measures

When comparing two quantities and the precise constant is unimportant, we adopt the notation that

$$A \lesssim_{x,r,m} B$$

means $A \leq CB$ for some constant C depending on x, r, m. Fewer or more dependencies may be attached to the symbol $\lesssim$. If no dependencies are appended to the symbol, they are explained shortly after the equation appears.

Definition 20 (Ahlfors-regularity of measures)**.** A measure μ on $\mathbb{R}^m$ is said to be n-Ahlfors regular if there exist constants $0 < c, C < \infty$ such that

$$\mu(B(x,r)) \leq Cr^n \qquad \forall x \in \operatorname{spt}(\mu) \tag{2.20}$$

and

$$\mu(B(x,r)) \geq cr^n \qquad \forall 0 < r < \operatorname{diam}\{\operatorname{spt}(\mu)\}, \qquad \forall \mathrm{x} \in \operatorname{spt}(\mu). \tag{2.21}$$

A measure μ is said to be n-upper Ahlfors regular if (2.20) holds, and n-lower Ahlfors regular if (2.21) holds. The smallest constant C such that (2.20) holds is called the upper regularity constant for μ, and the largest c such that (2.21) holds is called the lower regularity constant for μ.

---

[1]The dependence on x comes from $\lambda = \lambda_{x,R} > 0$ such that $\lambda r^n \leq \mu(B(x,r))$ for all $0 < r < R$, see Lemma 25.

A measure μ on $\mathbb{R}^m$ is said to be absolutely continuous with respect to a measure ν, denoted $\mu \ll \nu$ if $\nu(E) = 0 \implies \mu(E) = 0$.

Definition 21 (Countably Rectifiable)**.** In this paper, we follow the convention that a Borel measure μ on $\mathbb{R}^m$ is said to be countably n-rectifiable if there exist Lipschitz maps $f_i : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\mu\left(\mathbb{R}^m \setminus \bigcup_{i=1}^{\infty} f_i(\mathbb{R}^n)\right) = 0 \tag{2.22}$$

and $\mu \ll \mathcal{H}^n$. A Borel set E is countably n-rectifiable if $\mathcal{H}^n|_E$ is countably n-rectifiable.

Definition 22 (Uniformly rectifiable)**.** A Radon measure μ on $\mathbb{R}^m$ is said to be uniformly n-rectifiable if it is n-Ahlfors regular and there exist constants $\Lambda > 0$ and $0 < \theta < 1$ such that for all $x \in \text{spt}\mu$ and all $r \geq 0$ there exist a Lipschitz map $f_{x,r} : B^n(0,r) \to \mathbb{R}^m$ such that

$$\mu\left(B(x,r) \setminus f_{x,r}(B^n(0,r))\right) \leq \theta\mu(B(x,r)).^2 \tag{2.23}$$

A Borel set $E \subset \mathbb{R}^m$ is said to be uniformly n-rectifiable if $\mathcal{H}^n|_E$ is uniformly n-rectifiable.

Definition 23 (Purely unrectifiable)**.** A Borel measure μ on $\mathbb{R}^m$ is said to be n-purely unrectifiable if every Lipschitz map $f : \mathbb{R}^n \to \mathbb{R}^m$ has the property that

$$\mu(f(\mathbb{R}^n)) = 0. \tag{2.24}$$

A Borel set E is said to be countably n-rectifiable (uniformly n-rectifiable/n-purely unrectifiable respectively) if $\mathcal{H}^n|_E$ is countably n-rectifiable (uniformly n-rectifiable/n-purely unrectifiable respectively).

Definition 24 (Density ratios)**.** Given a Borel measure μ on $\mathbb{R}^m$, we define the function

$$\Theta^n(\mu, x, r) = \frac{\mu(B(x,r))}{r^n}. \tag{2.25}$$

[2]Here and after, $B^n(0,r)$ denotes an n-dimensional ball of radius r centered at 0. Similarly $B^m(x,r)$ is an m-dimensional ball centered at x with radius r. Typically, the dimension is clear from context and hence neglected.

Moreover, the n-dimensional upper-density of μ at x, denoted $\Theta^{n,*}(\mu,x)$ is defined as

$$\Theta^{n,*}(\mu,x) = \limsup_{r\to 0} \frac{\mu(B(x,r))}{r^n} \tag{2.26}$$

and the n-dimensional lower-density of μ at x, denoted $\Theta^n_*(\mu,x)$ is defined by

$$\Theta^n_*(\mu,x) = \liminf_{r\to 0} \frac{\mu(B(x,r))}{r^n} \tag{2.27}$$

If $\Theta^{n,*}(\mu,x) = \Theta^n_*(\mu,x)$ their common value is called the density of μ at x and is denoted by $\Theta^n(\mu,x)$. Notably, $\mu \ll \mathcal{H}^n$ if and only if $\Theta^{n,*}(\mu,x) < \infty$ for μ a.e. $x \in \mathbb{R}^m$.

The following lemma is a useful characterization of density properties of measures.

Lemma 25. *If μ is a Borel measure on $\mathbb{R}^m$ and $x \in \mathbb{R}^m$, then for any $R > 0$, the following are equivalent:*

1. $\Theta^n_*(\mu,x) > 0$

2. *There exists $\lambda > 0$ such that $\mu(B(x,r)) \geq \lambda r^n$ for all $0 < r \leq R$.*

Similarly, the following are equivalent:

(i) $\Theta^{n,*}(\mu,x) < \infty$

(ii) There exists a $\Lambda > 0$ such that $\mu(B(x,r)) \leq \lambda r^n$ for all $0 < r \leq R$.

Proof. We only discuss the proof that (1) and (2) are equivalent. The proof that (i) and (ii) are equivalent follows the same structure.

First, note that (2) implies $\Theta^n_*(\mu,x) \geq \lambda$. So, we assume (1) and show (2). Since $\Theta^n_*(\mu,x) > 0$ it follows that there exists $\delta = \delta(x)$ such that for all $r \leq \delta, \mu(B(x,r)) \geq \frac{\Theta^n_*(\mu,x)}{2} r^n$. In particular,

$$\mu(B(x,\delta)) \geq \frac{\Theta^n_*(\mu,x)}{2}\delta^n,$$

so for $\delta \leq r \leq R$ it follows

$$\mu(B(x,r)) \geq \mu(B(x,\delta)) \geq \frac{\Theta_*^n(\mu,x)}{2}\delta^n \geq \frac{\Theta_*^n(\mu,x)}{2}\frac{\delta^n}{R^n}r^n,$$

so $\lambda = \frac{\Theta_*^n(\mu,x)}{2}\left(\frac{\delta}{R}\right)^n \leq \Theta_*^n(\mu,x)/2$ suffices. □

Finally, given a measure μ on $\mathbb{R}^m$ we let μ^k denote the measure on $(\mathbb{R}^m)^k$ defined as the k-fold product of μ with itself. Similarly, given a set $E \subset \mathbb{R}^m$ we let E^k denote the k-fold product of E as a set in $(\mathbb{R}^m)^k$.

2.2.2 Menger-type curvature, a formal review

Simplices and Notation

Given points $\{x_0, \ldots, x_n\} \subset \mathbb{R}^m$ then $\Delta(x_0, \ldots, x_n)$ will denote the convex hull of $\{x_0, \ldots, x_n\}$. In particular, if $\{x_0, \ldots, x_n\}$ are not contained in any $(n-1)$-dimensional plane, then $\Delta(x_0, \ldots, x_n)$ is an n-dimensional simplex. Moreover, $\text{aff}\{x_0, \ldots, x_n\}$ denotes the smallest affine subspace containing $\{x_0, \ldots, x_n\}$. That is $\text{aff}\{x_0, \ldots, x_n\} = x_0 + \text{span}\{x_1 - x_0, \ldots, x_n - x_0\}$.

If Δ is an n-simplex, it is additionally called an (n, ρ)-simplex if

$$h_{\min}(x_0, \ldots, x_n) = \min_i \text{dist}(x_i, \text{aff}\{x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\}) \geq \rho.$$

The next lemma can be found in [56, Lemma 3.7] and quantifies some geometric properties of simplices, which is of particular interest when showing a certain integrand is (μ, p)-proper.

Lemma 26. *Let $C \geq 1, t > 0, x \in \mathbb{R}^m, w \in B(x, Ct)$ and $S = \Delta(x_0, \ldots, x_n) \subset B(x, Ct)$ be some $(n, \frac{t}{C})$-simplex. Define $S_w = \Delta(x_0, \ldots, x_n, w)$ and choose distinct $i, j \in \{0, \ldots, n\}$. Then*

1. $\frac{t}{C} \leq |x_i - x_j| \leq \text{diam}(S_w) \leq 2Ct$,

2. $|x_i - w| \leq 2Ct$,

3. $\frac{t^n}{C^n n!} \leq \mathcal{H}^n(S) \leq \frac{(2C)^n}{n!} t^n$,

4. $\operatorname{dist}(w, \operatorname{aff}(x_0, \ldots, x_n)) = n \frac{\mathcal{H}^{n+1}(S_w)}{\mathcal{H}^n(S)}$

The next lemma is an immediate consequence of repeated applications of [56, Lemma 2.17].

Lemma 27. *Let $0 < k \leq n$. If $T_x = \Delta(x_0, \ldots, x_n)$ is an (n, ρ)-simplex, and $\{y_0, \ldots, y_k\}$ are such that for all $i \in \{1, \ldots, k\}$ $|x_i - y_i| < \delta$ for some small $\delta > 0$ satisfying $(k+1)\delta < \rho$, then $\Delta(y_0, \ldots, y_k, x_{k+1}, \ldots, x_n)$ is an $(n, \rho - (k+1)\delta)$-simplex.*

Meurer's proper integrands

The ability to use a general class of integrands $\mathcal{K} : (\mathbb{R}^m)^{n+2} \to [0, \infty)$ as a tool to study countably n-rectifiable sets and measures in $\mathbb{R}^m$ was demonstrated by Meurer in [56]. Below is the definition of the general class of integrands as laid out by Meurer.

Definition 28 ((μ, p)-proper integrand)**.** Let $n, m \in \mathbb{N}$ with $1 \leq n < m$. Let $\mathcal{K} : (\mathbb{R}^m)^{n+2} \to [0, \infty)$ and $1 < p < \infty$. One says that $\mathcal{K}$ is a (μ, p)-proper integrand if it fulfills the following four conditions:

1. $\mathcal{K}$ is μ^{n+2}-measurable, where μ^{n+2} denotes the $(n+2)$-product measure of μ.

2. There exists some constants $c = c(n, \mathcal{K}, p) \geq 1$ and $\ell = \ell(n, \mathcal{K}, p) \geq 1$ so that, for all $t > 0$, $C \geq 1$, $x \in \mathbb{R}^m$ and all $(n, \frac{t}{C})$-simplices $\Delta(x_0, \ldots, x_n) \subset B(x, Ct)$, it follows

$$\left(\frac{d(w, \operatorname{aff}(x_0, \ldots, x_n))}{t} \right)^p \leq c C^\ell t^{n(n+1)} \mathcal{K}^p(x_0, \ldots, x_n, w) \tag{2.28}$$

   for all $w \in B(x, Ct)$.

3. For all $\lambda > 0$,

$$\lambda^{n(n+1)}\mathcal{K}^p(\lambda x_0, \ldots, \lambda x_{n+1}) = \mathcal{K}^p(x_0, \ldots, x_{n+1}) \tag{2.29}$$

4. $\mathcal{K}$ is translation invariant in the sense that for every $b \in \mathbb{R}^m$,

$$\mathcal{K}(x_0 + b, \ldots, x_{n+1} + b) = \mathcal{K}(x_0, \ldots, x_{n+1}) \tag{2.30}$$

Remark 29. *The preceding definition is rather long and written so that expressions show-up in the same form that they do in the proof of [55, Theorem 5.6], a theorem which roughly provides a bound on β_p-numbers by Menger curvature. As written above, one may notice that part (2) looks vaguely like one is bounding β_p-numbers. However, the relationship becomes more obvious after applying part 3 to re-write part 2 of the definition of a (μ, p)-proper integrand in the following way:*

There exists some constant $c = c(n, \mathcal{K}, p) \geq 1$ and $\ell = \ell(n, \mathcal{K}, p) \geq 1$ so that, for all $t > 0$, $C \geq 1, x \in \mathbb{R}^m$ and all $(n, \frac{t}{C})$-simplices $\Delta(x_0, \ldots, x_{n+1}) \subset B(x, Ct)$, it follows

$$\left(\frac{d(w, \operatorname{aff}(x_0, \ldots, x_n))}{t}\right)^p \leq cC^\ell \mathcal{K}^p\left(\frac{x_0}{t}, \ldots, \frac{x_{n+1}}{t}, \frac{w}{t}\right)$$

for all $w \in B(x, Ct)$. In particular, ignoring all details and technicalities it looks like integrating the left-hand side over w yields the L^p-distance to a specific plane at scale t is bounded by the Menger curvature integrand "at scale t" when integrated over just one input, while the other inputs span the given affine plane.

Given a Borel measure μ and a (μ, p)-proper integrand $\mathcal{K}$, the integral Menger curvature of μ with respect to $\mathcal{K}$ (or simply integral Menger curvature) is

$$\mathcal{M}_{\mathcal{K}^p}(\mu) = \int_{(\mathbb{R}^m)^{n+2}} \mathcal{K}^p(x_0, \ldots, x_{n+1})\ d\mu^{n+2}(x_0, \ldots, x_{n+1}).$$

The pointwise Menger curvature of x in μ with respect to $\mathcal{K}$ at scale r is

$$\operatorname{curv}^n_{\mathcal{K}^p;\mu}(x,r) = \int_{B(x,r)^{n+1}} \mathcal{K}^p(x,x_1,\dots,x_{n+1})\ d\mu^{n+1}(x_1,\dots,x_{n+1}).$$

We next show why one of the two integrands emphasized throughout this paper does indeed satisfy the definition of a $(\mu,2)$-proper integrand. As noted in [55, Lemma 3.9], the computations here are analogous to [55, Lemmata 3.7 and 3.8]. Nevertheless, we include them for the reader's convenience.

To precisely express the two integrands we define

$$X = \{(x_0,\dots,x_{n+1}) : \mathcal{H}^{n+1}(\Delta(x_0,\dots,x_{n+1})) > 0\}.$$

Example 30. The following integrands are $(\mu,2)$-proper although they first appear in the works of Lerman and Whitehouse (see [46], [47], [48]).

$$\mathcal{K}_1(x_0,\dots,x_{n+1}) := \mathbb{1}_X(x_0,\dots,x_{n+1}) \frac{\mathcal{H}^{n+1}(\Delta(x_0,\dots,x_{n+1}))}{(\operatorname{diam}\{x_0,\dots,x_{n+1}\})^{\frac{(n+1)(n+2)}{2}}}$$

and

$$\mathcal{K}_2(x_0,\dots,x_{n+1}) := \mathbb{1}_X(x_0,\dots,x_{n+1}) \frac{h_{\min}(x_0,\dots,x_{n+1})}{\operatorname{diam}(\{x_0,\dots,x_{n+1}\})^{\frac{n(n+1)+2}{2}}}$$

where $h_{\min}(x_0,\dots,x_{n+1})) = \min_i \operatorname{dist}\{x_i, \operatorname{aff}\{x_0,\dots,x_{i-1},x_{i+1},\dots,x_{n+1}\}$.

Since the expressions for the Menger-type curvatures look like they tend to zero as $(x_0,\dots,x_{n+1})$ tend to X^c, the indicator function is typically neglected. Below is an outline of why (by considering $\mathcal{K}_1$) these examples are proper integrands. The proof of $\mathcal{K}_2$ is more straightforward.

Measurability follows due to the fact that X and $(\mathbb{R}^m)^{n+2} \setminus X$ are open and closed respectively in $(\mathbb{R}^m)^{n+2}$. So μ^{n+2}-measurability follows since μ is Borel and $\mathcal{K}_1$ is continuous on X and $(\mathbb{R}^m)^{n+2} \setminus X$.

For the second condition in the definition of $(\mu,2)$-proper, consider $t > 0, C \geq 1, x \in$

$\mathbb{R}^m, \Delta = \Delta(x_0, \ldots, x_n) \subset B(x, Ct)$ is an $(n, \frac{t}{C})$-simplex, fix $w \in B(x, Ct)$ and let $\Delta_w = \Delta(x_0, \ldots, x_n, w)$. Then,

$$\begin{aligned}
\left(\frac{d(w, \operatorname{aff}(x_0, \ldots, x_n))}{t}\right)^2 &= \left(n\frac{\mathcal{H}^{n+1}(\Delta_w)}{t\ \mathcal{H}^n(\Delta)}\right)^2 && \text{(26 part (4))} \\
&\leq (n \cdot n! \cdot C^n)^2 \left(\frac{\mathcal{H}^{n+1}(\Delta_w)}{t \cdot t^n}\right)^2 && \text{(26 part (3))} \\
&= (n \cdot n! \cdot C^n)^2 t^{n(n+1)} \left(\frac{\mathcal{H}^{n+1}(\Delta_w)}{t^{(n+1)+\frac{n(n+1)}{2}}}\right)^2 \\
&= (n \cdot n! \cdot C^n)^2 t^{n(n+1)} \left(\frac{\mathcal{H}^{n+1}(\Delta_w)}{t^{\frac{(n+1)(n+2)}{2}}}\right)^2 \\
&\leq (n \cdot n! \cdot C^n)^2 C^{\frac{(n+1)(n+2)}{2}} t^{n(n+1)} \mathcal{K}_1^2(\Delta) && \text{(26 parts (1,2))}
\end{aligned}$$

hence, the second property holds with $\ell = \frac{(n+2)(n+1)}{2} + 2n$ and $c = (n \cdot n!)^2$.

For homogeneity, note that if $\lambda > 0$, then $(x_0, \ldots, x_{n+1}) \in X \iff (\lambda x_0, \ldots, \lambda x_{n+1}) \in X$. Moreover, for $(x_0, \ldots, x_{n+1}) \in X$ it follows that

$$\mathcal{H}^{n+1}(\lambda x_0, \ldots, \lambda x_{n+1}) = \lambda^{n+1}\mathcal{H}^{n+1}(x_0, \ldots, x_{n+1}).$$

Consequently

$$\begin{aligned}
\mathcal{K}_1^2(\lambda x_0 \ldots, \lambda x_{n+1}) &= \frac{\lambda^{2(n+1)}}{\lambda^{(n+2)(n+1)}}\mathcal{K}_1^2(x_0, \ldots, x_{n+1}) \\
&= \lambda^{-n(n+1)}\mathcal{K}_1^2(x_0, \ldots, x_{n+1}).
\end{aligned}$$

Translation invariance follows from the geometric nature of the definition.

Definition 31 (Symmetric (μ, p)-proper integrand)**.** A (μ, p)-proper integrand is said to be symmetric if for all permutations $\sigma \in S_{n+2}$

$$\mathcal{K}^p(x_0, \ldots, x_{n+1}) = \mathcal{K}^p(x_{\sigma(0)}, \ldots, x_{\sigma(n+1)})$$

The next lemma, due to [56, Lemma 5.1], demonstrates that the restriction to symmetric proper integrands is a non-issue.

Lemma 32. *Let μ be a Radon measure on $\mathbb{R}^m$ and fix $\mathcal{K}^p$ some (μ, p)-proper integrand. Then, there exists $\tilde{\mathcal{K}}^p$ a symmetric (μ, p)-proper integrand which satisfies $\mathcal{M}_{\mathcal{K}^p}(\mu \cap E) = \mathcal{M}_{\tilde{\mathcal{K}}^p}(\mu \cap E)$ for all Borel sets E.*

The proof is to use Fubini's theorem to check that the integrand

$$\tilde{\mathcal{K}}^p(x_0, \ldots, x_{n+1}) = \frac{1}{\#|S_{n+2}|} \sum_{\sigma \in S_{n+2}} \mathcal{K}^p(x_{\sigma(0)}, \ldots, x_{\sigma(n+1)})$$

satisfies $\mathcal{M}_{\mathcal{K}^p}(\mu \cap E) = \mathcal{M}_{\tilde{\mathcal{K}}^p}(\mu \cap E)$ for all Borel E. Moreover, it clearly satisfies conditions (1), (3), and (4) in Definition 28. So, it only remains to check that condition (2) holds. But, $\mathcal{K}^p \leq \#|S_{n+2}|\tilde{\mathcal{K}}^p$ validates condition (2) of Definition (28).

2.3 Proofs of main results

One main tool of Meurer's work (see [55, Theorem 4.1]), a non-trivial generalization of an analogous result from [45], is the following:

Theorem 33. *Let $\mathcal{K} : (\mathbb{R}^m)^{n+2} \to [0, \infty)$ be a $(\mu, 2)$-proper integrand. If μ is a Borel measure on $\mathbb{R}^m$, then there exists some small $\eta = \eta(\mathcal{K}, n, m, C_0) > 0$, so that if μ satisfies*

(A) $\mu(B(0, 2)) \geq 1$ *and* $\mu(\mathbb{R}^m \setminus B(0, 2)) = 0$

(B) $\mu(B) \leq C_0(\operatorname{diam} B)^n$ *for every ball B.*

(C) $\mathcal{M}_{\mathcal{K}^2}(\mu) \leq \eta$

then there exists some Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}^{m-n}$ with Lipschitz constant bounded above by some $\Lambda = \Lambda(\mathcal{K}, n, m, C_0)$ such that a rotation of the graph of f, named Γ, satisfies

$$\mu(\mathbb{R}^m \setminus \Gamma) < \frac{1}{100}\mu(\mathbb{R}^m). \tag{2.31}$$

Moreover, for given $\mathcal{K}$ and C_0 the Lipschitz constant of f goes to zero as $\mathcal{M}_{\mathcal{K}^2}(\mu)$ approaches zero.

2.3.1 Scaling Menger Curvature

Theorem 33 is one of the main tools for proving Theorem 15. It will also be useful to know how integral Menger curvature scales, and how this impacts Theorem 33.

Proposition 34. *Let μ be a Radon measure and $\mathcal{K}$ a (μ, p)-proper integrand. Let ν be the Radon measure defined by $\nu(A) = \lambda\mu(aA + x)$ for some $a, \lambda > 0$ and $x \in \mathbb{R}^m$ then*

$$\mathcal{M}_{\mathcal{K}^2}(\nu) = \lambda^{n+2} a^{-n(n+1)} \mathcal{M}_{\mathcal{K}^2}(\mu). \tag{2.32}$$

In particular, if $\mu_{x,r}$ is defined so that $\mu_{x,r}(E) = \frac{\mu(rE+x)}{r^n}$ for all $E \subset \mathbb{R}^m$, then

$$\mathcal{M}_{\mathcal{K}^p}(\mu_{x,r}|_{B(0,1)}) = \frac{\mathcal{M}_{\mathcal{K}^p}(\mu|_{B(x,r)})}{r^n} \tag{2.33}$$

Proof. Fix a Borel measure μ on $\mathbb{R}^m$ a point, $x \in \mathbb{R}^m$ and $a, \lambda > 0$. Define $f : \mathbb{R}^m \to \mathbb{R}^m$ by $f(y) = \frac{y-x}{a}$, then $\nu = \lambda f_{\#}\mu$ (See [51, 1.17-1.19] for definition and use of image measures). Consequently,

$$\mathcal{M}_{\mathcal{K}^p}(\nu) = \int_{(\mathbb{R}^m)^{n+2}} \mathcal{K}^p(y_0, \ldots, y_{n+1})\, d(\lambda f_{\#}\mu(y_0)) \cdots\, d(\lambda f_{\#}\mu(y_{n+1}))$$
$$= \lambda^{n+2} \int_{(\mathbb{R}^m)^{n+2}} \mathcal{K}^p\left(ax_0 + x, ax_1 + x, \ldots, ax_{n+1} + x\right)\, d\mu(x_0) \cdots\, d\mu(x_{n+1})$$
$$= \lambda^{n+2} a^{-n(n+1)} \int_{(\mathbb{R}^m)^{n+2}} \mathcal{K}^p(x_0, \ldots, x_{n+1})\, d\mu(x_0) \cdots\, d\mu(x_{n+1})$$

where the final line follows by first applying translation invariance and then the homogeneity of $\mathcal{K}^p$ (see conditions (2.29) and (2.30) in the definition of a (μ, p)-proper integrand). This proves (2.32).

We see (2.33) follows from choosing $\lambda = r^{-n}$ and $a = r^{-1}$ combined with the observation that $f^{-1}(B(0,1)) = B(x,r)$ when $f(y) = \frac{y-x}{r}$. □

We note that (2.33) is of interest due to the following modification of Theorem 33

Theorem 35. *Let μ be an n-Ahlfors upper-regular Radon measure with upper-regularity constant C. Let $\mathcal{K}$ be a $(\mu,2)$-proper integrand. Then, there exists a constant $\eta_1 = \eta_1(\mathcal{K}, n, m, \Theta^n(\mu,x,r), C) > 0$ such that*

$$\frac{\mathcal{M}_{\mathcal{K}^2}\left(\mu|_{B(x,r)}\right)}{r^n} \leq \eta_1 \tag{2.34}$$

implies there exists some Lipschitz graph Γ with Lipschitz constant bounded above by some $\Lambda = \Lambda(\eta_1)$ such that

$$\mu\left(B(x,r) \setminus \Gamma\right) < \frac{1}{100}\mu(B(x,r)). \tag{2.35}$$

Moreover, given $\mathcal{K}, \Theta^n(\mu,x,r), C$ the Lipschitz constant of Γ tends to zero as $\dfrac{\mathcal{M}_{\mathcal{K}^2}(\mu|_{B(x,r)})}{r^n}$ approaches zero.

An immediate corollary to Theorem 35 is

Corollary 36. *Let μ be an n-Ahlfors regular Radon measure with lower-regularity constant c, and upper-regularity constant C. If $\mathcal{K}$ is a $(\mu,2)$-proper integrand and (2.34) is satisfied with $\eta_1 = \eta_1(\mathcal{K}, n, m, c, C)$ for all $x \in \operatorname{spt}(\mu)$ and all $0 < r < \operatorname{diam}(\operatorname{spt}(\mu))$ then μ is uniformly n-rectifiable.*

Proof. (of Theorem 35). We claim that $\nu = \dfrac{\mu_{x,r}|_{B(0,1)}}{\Theta^n(\mu,x,r)}$ satisfies

1. $\nu(B(0,1)) \geq 1$ and $\nu(\mathbb{R}^m \setminus B(0,1)) = 0$.

2. $\nu(B(y,t)) \leq \dfrac{C}{c} t^n$

3.
$$\mathcal{M}_{\mathcal{K}^2}(\nu) = \frac{\mathcal{M}_{\mathcal{K}^2}\left(\mu|_{B(x,r)}\right)}{r^n c^{n+2}}.$$

Indeed, (1) follows since $\nu(B((0,1))) = \frac{1}{\Theta^n(\mu,x,r)}\frac{\mu(B(x,r))}{r^n} = \frac{\Theta^n(\mu,x,r)}{\Theta^n(\mu,x,r)} = 1$ and ν is the restriction of a measure to $B(0,1)$.

To see (2) follows, consider $y \in B(0,1)$ and $t > 0$. Then,

$$\begin{aligned}\nu(B(y,t)) &= \frac{1}{\Theta^n(\mu,x,r)}\frac{\mu\left(B(x+ry,rt)\cap B(x,r)\right)}{r^n} \\ &\leq \frac{1}{\Theta^n(\mu,x,r)}\frac{\mu(B(x+ry,rt))}{r^n} \leq \frac{1}{\Theta^n(\mu,x,r)}\frac{C(rt)^n}{r^n} \\ &\leq \frac{C}{\Theta^n(\mu,x,r)}t^n.\end{aligned}$$

Finally, (3) follows since (2.33) in Proposition 34 guarantees

$$\mathcal{M}_{\mathcal{K}^2}\left(\mu_{x,r}|_{B(0,1)}\right) = \frac{\mathcal{M}_{\mathcal{K}^2}(\mu|_{B(x,r)})}{r^n}$$

then applying (2.32) with $\lambda = \Theta^n(\mu,x,r)^{-1}$, $a = 1$ yields

$$\mathcal{M}_{\mathcal{K}^2}(\nu) = \mathcal{M}_{\mathcal{K}^2}(\Theta^n(\mu,x,r)^{-1}\mu_{x,r}|_{B(0,1)})$$
$$= \Theta^n(\mu,x,r)^{-(n+2)}\mathcal{M}_{\mathcal{K}^2}\left(\mu_{x,r}|_{B(0,1)}\right) = \Theta^n(\mu,x,r)^{-(n+2)}\frac{\mathcal{M}_{\mathcal{K}^2}(\mu|_{B(x,r)})}{r^n}$$

Consequently, if $\eta_1\left(\mathcal{K},m,n,\Theta^n(\mu,x,r),C\right)$ is chosen so that it is at most as large as $\Theta^n(\mu,x,r)^{n+2}\eta(\mathcal{K},m,n,\frac{C}{c})$ where $\eta(\mathcal{K},m,n\frac{C}{c})$ is as in Theorem 33, then ν satisfies the hypothesis of Theorem 33. Hence, there exists a Lipschitz graph $\tilde{\Gamma}$ with $\nu(\mathbb{R}^m\setminus\tilde{\Gamma}) < \frac{1}{100}\nu(\mathbb{R}^m)$. But $\frac{\nu(\mathbb{R}^m\setminus\tilde{\Gamma})}{\nu(\mathbb{R}^m)} = \frac{\mu(B(x,r)\setminus\Gamma)}{\mu(B(x,r))}$ where $\Gamma = r\tilde{\Gamma}+x$. In particular, $\mu(B(x,r)\setminus\Gamma) < \frac{1}{100}\mu(B(x,r))$ as desired. □

Remark 37. *Note that in the case $\mathcal{K} = \mathcal{K}_1$ (along with several other specific integrands studied in [46], [47]) Corollary 36 is already known from the work of Lerman and Whitehouse. In fact, the hypothesis in Theorem 35 is stronger than theirs, because Theorem 35 requires not only a Carleson-type bound on the local integral Menger curvature, but that the bound be by a small constant.*

To create an effective theory of quantitative, albeit non-uniform rectifiability, it would be interesting to address this seemingly unnecessary "smallness" condition being imposed by η_1 *in* (2.34). *It would also likely be useful to allow* Γ *to only satisfy* $\mu(B(x,r) \setminus \Gamma) \leq (1-\epsilon)\mu(B(x,r))$ *for some* $\epsilon = \epsilon(\eta, m, n, \mathcal{K}, C, \Theta^n(\mu, x, r))$.

2.3.2 Rectifiability from integral Menger curvature

The goal of this section is to prove Theorem 15 included below for completeness.

Theorem 38. *If* μ *is a Radon measure on* $\mathbb{R}^m$ *with* $0 < \Theta^{n,*}(\mu, x) < \infty$ *for* μ *almost every* $x \in \mathbb{R}^m$ *and* $\mathcal{M}_{\mathcal{K}^2}(\mu) < \infty$ *for some* $(\mu, 2)$*-proper integrand* $\mathcal{K}$*, then* μ *is countably* n*-rectifiable.*

The proof could be separated into two parts. The first part is a sequence of arguments to show that Radon measures which are mutually absolutely continuous with respect to the Hausdorff measure behave sufficiently similar the Hausdorff measure restricted to sets. The second part follows an argument from Sections 1 and 2 of [45] and is also similar to the case for sets as presented in [56]. Several preparatory lemmas are required.

Lemma 39. *Let* μ *be a Radon measure on* $\mathbb{R}^m$ *with* $0 < \Theta^{n,*}(\mu, x) < \infty$ *for* μ *almost every* $x \in \mathbb{R}^m$*. Then there exists some Borel set* E *such that* $\nu = \mu|_E$ *satisfies* $\nu(\mathbb{R}^m) \geq \frac{1}{2}\mu(\mathbb{R}^m)$ *and*

$$0 < c \leq \Theta^{n,*}(\nu, x) \leq C < \infty \text{ for } \nu \text{ almost every } x \in \mathbb{R}^m.$$

Proof. Without loss of generality, suppose $\mu(\mathbb{R}^m) < \infty$. Otherwise, apply the finite case to a family of disjoint annulli that exhaust $\mathbb{R}^m$. Moreover, suppose $\mu(\mathbb{R}^m) > 0$ to avoid trivialities.

Define $E_j = \{x \in \mathbb{R}^m : 2^{-j} \leq \Theta^{n,*}(\mu, x) \leq 2^j\}$. Then $E_j^c \supset E_{j+1}^c$ for all j and moreover

$$\bigcap_{j=1}^{\infty} E_j^c = \{x \in \mathbb{R}^m : 0 = \Theta^{n,*}(\mu, x) \text{ or } \Theta^{n,*}(\mu, x) = +\infty\}.$$

Since $\mu(E_1^c) \leq \mu(\mathbb{R}^m) < \infty$ it follows that

$$\lim_{j\to\infty} \mu(E_j^c) = \mu\left(\bigcap_{j=1}^{\infty} E_j^c\right) = 0$$

so there exists k with $\mu(E_k) \geq \frac{1}{2}\mu(\mathbb{R}^m)$. Fix such k and let $c = 2^{-k}$ and $C = 2^k$. Then, define $\nu = \mu_{E_k}$. Since E_k is μ-measurable, it follows ν is Radon and $\nu(\mathbb{R}^m) \geq \frac{1}{2}\mu(\mathbb{R}^m)$. On the other hand the Lebesgue-Besicovitch differentiation theorem ensures $\Theta^{n,*}(\nu, x) = \Theta^{n,*}(\mu, x)$ for ν a.e. $x \in \mathbb{R}^m$ since $\frac{\nu(B(x,r))}{\mu(B(x,r))} \xrightarrow{r\downarrow 0} 1$ for ν a.e. $x \in \mathbb{R}^m$. Consequently $2^{-k} \leq \Theta^{n,*}(\nu, x) \leq 2^k$ for ν a.e. $x \in \mathbb{R}^m$ as desired. $\square$

Lemma 40. *Let μ be a Radon measure with $0 < c \leq \Theta^{n,*}(\mu, x) \leq C < \infty$ for μ a.e. $x \in \mathbb{R}^m$ such that $\operatorname{spt}(\mu)$ is bounded. If $\mathcal{K}$ is a (μ, p)-proper integrand for some $1 < p < \infty$ and $\mathcal{M}_{\mathcal{K}^p}(\mu) < \infty$, then for all $\zeta_0 > 0$ there exists a compact set $E^* \subset \operatorname{spt}(\mu)$ with*

(i) $\mu(E^*) \geq \frac{c}{2^{n+2}} \left(\operatorname{diam} E^*\right)^n$

(ii) For all $x \in E^*$ and all $t > 0$, $\mu\left(E^* \cap B(x,t)\right) \leq 2Ct^n$.

(iii) $\mathcal{M}_{\mathcal{K}^p}(\mu|_{E^*}) \leq \zeta_0 \left(\operatorname{diam} E^*\right)^n$

Proof. Since μ is Radon, and $\operatorname{spt}(\mu)$ is bounded, it follows that $\mu(\mathbb{R}^m) < \infty$. Define

$$E_\ell = \{x \in \mathbb{R}^m : \forall t \in (0, 2^{-\ell}),\ \mu(B(x,t)) \leq 2Ct^n\}. \tag{2.36}$$

Evidently, $E_\ell \subset E_{\ell+1}$. Moreover, the assumption $c \leq \Theta^{n,*}(\mu, x) \leq C$ for μ almost every $x \in \mathbb{R}^m$ ensures

$$\mu(E_\ell) \xrightarrow{\ell\to\infty} \mu(\mathbb{R}^m).$$

Hence, by the finiteness of $\mu(\mathbb{R}^m)$ there exists some ℓ such that $\mu(E_\ell) \geq \frac{1}{2}\mu(\mathbb{R}^m)$. Define $\nu = \mu|_{E_\ell}$. Then, notably

$$\nu(B(x,r)) \leq 2Cr^n \qquad \forall x \in E_\ell \qquad \forall 0 < r \leq 2^{-\ell} \tag{2.37}$$

and by Lebesgue-Besicovitch differentiation theorem [37, Theorem 1.7.1],

$$\frac{\mu(B(x,r)\cap E_\ell)}{\mu(B(x,r))} \xrightarrow{r\to 0} 1 \quad \mu \text{ } a.e. \text{ } x \in E_\ell$$

which implies

$$c \le \Theta^{n,*}(\nu, x) \qquad \nu \text{ } a.e. \text{ } x \in E_\ell \tag{2.38}$$

since, by assumption $\limsup_{r\to 0} \frac{\mu(B(x,r)}{r^n} \ge c$ for μ almost every $x \in \mathbb{R}^m$.

Define

$$A(\tau) = \left\{(x_0, \ldots, x_{n+1}) \in (E_\ell)^{n+2} : |x_0 - x_i| < \tau \;\forall i \in \{1, \ldots, n+1\}\right\}. \tag{2.39}$$

Claim 1: $\nu^{n+2}(A(\tau)) \xrightarrow{\tau\to 0} 0$.

Indeed, note that $A(\tau) = \bigcup_{x\in E_\ell}\{x\} \times (B(x,\tau)\cap E_\ell)^{n+1}$. In particular (2.37) ensures that for all $\tau \le 2^{-\ell}$,

$$\begin{aligned}\nu^{n+2}(A(\tau)) &= \int_{E_\ell} \nu^{n+1}(B(x,\tau)\cap E_\ell)\; d\nu(x)\\ &\le \int_{E_\ell} (2C\tau^n)^{(n+1)}\; d\nu(x) = (2C\tau^n)^{(n+1)}\nu(E_\ell).\end{aligned}$$

Since ν is finite, the claim follows.

Claim 2: $I(\tau)$ defined in (2.40) satisfies $I(\tau) \xrightarrow{\tau\to 0} 0$.

$$I(\tau) = \int_{A(\tau)} \mathcal{K}^p(x_0, \ldots, x_{n+1})\; d\nu^{n+2}(x_0, \ldots, x_{n+1}) \tag{2.40}$$

Indeed, $\mathcal{M}_{\mathcal{K}^p}(\nu) \le \mathcal{M}_{\mathcal{K}^p}(\mu) < \infty$ implies $\mathcal{K}^p \in L^1((\mathbb{R}^m)^{n+2}, \nu^{n+2})$. Then define

$$I(\tau) = \int_{(\mathbb{R}^m)^{n+2}} \mathbb{1}_{A(\tau)}\mathcal{K}^p\; d\nu^{n+2} \le \int_{(\mathbb{R}^m)^{n+2}} \mathcal{K}^p\; d\nu^{n+2}.$$

Consequently, for any sequence of τ_k converging to zero, Claim 1 ensures that the corresponding sequence of functions $\{\mathbb{1}_{A(\tau_k)}\mathcal{K}^p\}$ converges to zero ν^{n+2} a.e., and is bounded by the L^1

function $\mathcal{K}^p$. So, the dominated convergence theorem validates Claim 2. Consequently given $\zeta > 0$ depending only on c, C and ζ_0 to be chosen later, there exists τ_0 such that $0 < 2\tau_0 < 2^{-\ell}$ and

$$I(2\tau_0) \leq \frac{\zeta}{16C}\nu(\mathbb{R}^m). \tag{2.41}$$

Next, define a cover of E_ℓ by

$$\mathcal{G} = \left\{ B(x,\tau) : x \in E_\ell,\ 0 < \tau < \tau_0,\ \text{and } \Theta^n(\nu, x, \tau) \geq \frac{c}{2} \right\}. \tag{2.42}$$

In particular, (2.38) guarantees $\mathcal{G}$ is a fine cover of some ν-measurable set $E \subset E_\ell$ with ν-full measure, i.e., $\nu(E) = \nu(\mathbb{R}^m)$. So, the corollary to Besicovitch's covering theorem, [37, Corollary 1.5.2] ensures that there exists a countable, disjoint subfamily $\{B_i\}$ of $\mathcal{G}$ with

$$\nu(\mathbb{R}^m \setminus \bigcup_{i=1}^{\infty} B_i) = 0. \tag{2.43}$$

In light of (2.36) and (2.42), $\tau < \tau_0 < 2^{-\ell}$ ensures

$$\nu(\mathbb{R}^m) = \sum_{i=1}^{\infty} \nu(B_i) \leq \sum_{i=1}^{\infty} (2C) \left(\frac{\operatorname{diam} B_i}{2} \right)^n$$

so that

$$\frac{\nu(\mathbb{R}^m)}{2C} \leq \sum_{i=1}^{\infty} \left(\frac{\operatorname{diam} B_i}{2} \right)^n. \tag{2.44}$$

Moreover, $(B_i \cap E_\ell)^{n+2} \subset A(2\tau_0) \cap B_i$ so, (2.41) and (2.40) yields

$$\sum_{i=1}^{\infty} \mathcal{M}_{\mathcal{K}^p}(\nu|_{B_i}) \leq I(2\tau_0) \leq \frac{\zeta}{16C}\nu(\mathbb{R}^m). \tag{2.45}$$

Define the index set of "bad" balls, or balls with too much Menger curvature by

$$I_b = \left\{ i \in \mathbb{N} : \mathcal{M}_{\mathcal{K}^p}(\nu|_{B_i}) \geq \zeta \frac{\left(\frac{\operatorname{diam} B_i}{2}\right)^n}{4} \right\} \tag{2.46}$$

Then,

$$\sum_{i\in I_b}\mathcal{M}_{\mathcal{K}^p}(\nu|_{B_i}) \geq \frac{\zeta}{4}\sum_{i\in I_b}\left(\frac{\operatorname{diam} B_i}{2}\right)^n \tag{2.47}$$

Notice that if $\sum_{i\in I_b}\left(\frac{\operatorname{diam} B_i}{2}\right)^n > \frac{\nu(\mathbb{R}^m)}{4C}$ then additionally considering (2.45) and (2.47) implies

$$\sum_{i\in\mathbb{N}}\mathcal{M}_{\mathcal{K}^p}(\nu|_{B_i}) \leq \frac{\zeta}{16C}\nu(\mathbb{R}^m) < \frac{\zeta}{4}\sum_{i\in I_b}\left(\frac{\operatorname{diam} B_i}{2}\right)^n \leq \sum_{i\in I_b}\mathcal{M}_{\mathcal{K}^p}(\nu|_{B_i})$$

which is a contradiction. It follows

$$\sum_{i\in I_b}\left(\frac{\operatorname{diam} B_i}{2}\right)^n \leq \frac{\nu(\mathbb{R}^m)}{4C}. \tag{2.48}$$

Now, (2.44) and (2.48) together ensure that $I_b \neq \mathbb{N}$.

From now on, fix $i \in \mathbb{N}\setminus I_b$. The inner-regularity of Radon measures ensures that there exists some compact E^* with

$$E^* \subset B_i \cap E_\ell \quad\text{and}\quad \nu(E^*) \geq \frac{1}{2}\nu(B_i). \tag{2.49}$$

Then evidently, E^* satisfies:

1. $\nu(E^*) \geq \frac{1}{2}\nu(B_i) \geq \frac{1}{2}\frac{c}{2}\left(\frac{\operatorname{diam} B_i}{2}\right)^n \geq \frac{c}{2^{n+2}}\operatorname{diam}(E^*)^n$, where the second inequality is because $B_i \in \mathcal{G}$, see (2.42).

2. For all $x \in E^*$ and for all $0 < t < 2^{-\ell}$ it follows from (2.36) and $E^* \subseteq E_\ell$ that

$$\nu\left(E^* \cap B(x,t)\right) \leq 2C\left(\frac{\operatorname{diam}\left(E^* \cap B(x,t)\right)}{2}\right)^n. \tag{2.50}$$

On the other hand, $E^* \subset B_i$ and $\operatorname{diam}(B_i) \leq 2\tau_0 < 2^{-\ell}$. So, for $t \geq 2^{-\ell}$ it follows,

$$\nu\left(E^* \cap B(x,t)\right) \leq \nu(B_i) \leq 2C\left(\frac{\operatorname{diam} B_i}{2}\right)^n < 2C\left(2^{-\ell}\right)^n \leq 2Ct^n$$

so that in fact $\nu|_{E^*}$ is n-Ahlfors upper regular with regularity constant $2C$, that is (2.50) holds for all $t > 0$.

3. It follows
$$\frac{1}{4}\left(\frac{\operatorname{diam}(B_i)}{2}\right)^n \leq \frac{2C}{c}(\operatorname{diam} E^*)^n. \tag{2.51}$$
Indeed, choose a ball B with
$$\operatorname{diam} B \leq 2 \operatorname{diam} E^* \quad \text{and} \quad E^* \subset B. \tag{2.52}$$
Combining (2.42), (2.49), (2.50), and (2.52) yields
$$\begin{aligned}\frac{c}{4}\left(\frac{\operatorname{diam} B_i}{2}\right)^n \leq \frac{\nu(B_i)}{2} &\leq \nu(E^*) = \nu(E^* \cap B) \\ \leq 2C\left(\frac{\operatorname{diam} B}{2}\right)^n &\leq 2C \operatorname{diam}(E^*)^n.\end{aligned}$$
Hence, the lower bound on $\operatorname{diam} E^*$ follows.

4. Finally, since $i \in \mathbb{N} \setminus I_b$, (2.46) and (2.51) yields
$$\mathcal{M}_{\mathcal{K}^p}(\nu|_{E^*}) < \frac{\zeta}{4}\left(\frac{\operatorname{diam} B_i}{2}\right)^n \leq \frac{2C}{c}\zeta\,(\operatorname{diam} E^*)^n$$

Choosing $\zeta = \frac{c\zeta_0}{2C}$ completes the proof. □

The next technical lemma is a standard "structure theorem" and is contained in for instance [35, 3.3.12 - 3.3.15]

Lemma 41. *If μ is a Radon measure with $0 < \Theta^{n,*}(\mu, x)$ for μ almost every x, then one can write $\mu = \mu_r + \mu_u$ where $\mu_r = \mu|_E$ for some μ-measurable E, and μ_r is countably n-rectifiable. On the other hand, μ_u is purely unrectifiable.*

Now, we are prepared to show

Lemma 42. *Let μ be a Radon measure on $\mathbb{R}^m$ with $0 < \Theta^{n,*}(\mu, x) < \infty$ for μ almost every $x \in \mathbb{R}^m$. Let $\mathcal{K}^2$ a $(\mu, 2)$-proper integrand. After writing $\mu = \mu_u + \mu_r$ as in Lemma 41, if $\mu_u(\mathbb{R}^m) > 0$ then $\mathcal{M}_{\mathcal{K}^2}(\mu) = +\infty$.*

Notably, Lemma 42 is the contrapositive of Theorem 38.

Proof. Let $\mathcal{K}^2$ and μ, μ_u, μ_r be as in the statement of Lemma 42. Without loss of generality, suppose $0 < \mu(\mathbb{R}^m) < \infty$. It follows $0 < \Theta^{n,*}(\mu_u, x) < \infty$ for μ_u almost every $x \in \mathbb{R}^m$, since the Lebesgue-Besicovitch differentiation theorem guarantees

$$\Theta^{n,*}(\mu_u, x) = \limsup_{r \to 0} \frac{\mu_u(B(x,r))}{\mu(B(x,r))} \frac{\mu(B(x,r))}{r^n} = \Theta^{n,*}(\mu, x)$$

for μ_u a.e. $x \in \mathbb{R}^m$.

Moreover, by assumption $\mu_u(\mathbb{R}^m) > 0$. By Lemma 39 it follows that there exists ν_0 a restriction of μ_u and some c, C such that $0 < c \leq \Theta^{n,*}(\nu_0, x) \leq C < \infty$ for ν_0 almost every $x \in \mathbb{R}^m$. Lemma 39 also guarantees that, $0 < \frac{1}{2}\mu_u(\mathbb{R}^m) < \nu_0(\mathbb{R}^m) \leq \mu(\mathbb{R}^m) < \infty$. Since ν_0 is a restriction of μ_u to some Borel set, it follows that ν_0 is a Radon measure satisfying $\mathcal{M}_{\mathcal{K}^2}(\nu_0) \leq \mathcal{M}_{\mathcal{K}^2}(\mu_u) \leq \mathcal{M}_{\mathcal{K}^2}(\mu)$. In the spirit of contradiction, suppose $\mathcal{M}_{\mathcal{K}^2}(\mu) < \infty$.

Since $\nu_0(\mathbb{R}^m) < \infty$, without loss of generality, suppose $\operatorname{spt}(\nu_0)$ is bounded. Then, ν_0 satisfies the hypothesis of Lemma 40. In particular, for

$$\zeta_0 < \eta \cdot \left(\frac{2^{n+2}}{c}\right)^{-(n+2)} \text{ where } \eta = \eta\left(\mathcal{K}, n, m, 2^{n+3}Cc^{-1}\right) \text{ is from Theorem 33,} \tag{2.53}$$

there exists some compact $E^* \subset \operatorname{spt}(\nu_0)$ such that

(i)

$$\nu_0(E^*) \geq \frac{c}{2^{n+2}} \left(\operatorname{diam} E^*\right)^n \tag{2.54}$$

(ii) For all $x \in E^*$ and all $t > 0$,

$$\nu_0 \left(E^* \cap B(x,t)\right) \leq 2Ct^n. \tag{2.55}$$

(iii)

$$\mathcal{M}_{\mathcal{K}^p}(\nu_0|_{E^*}) \leq \zeta_0 \left(\operatorname{diam} E^*\right)^n \tag{2.56}$$

Our next goal is to scale and translate ν_0 to find a measure ν_1 which satisfies the hypothesis of Theorem 33. To this end, choose $x_0 \in E^*$. Then,

$$E^* \subset B(x_0, \operatorname{diam}(E^*)). \tag{2.57}$$

Let $f(y) = \frac{y-x_0}{\operatorname{diam}(E^*)}$, so that

$$f^{-1}(B(x_0, \operatorname{diam}(E^*))) = B(0,1). \tag{2.58}$$

Define

$$\nu_1 = \left(\operatorname{diam} E^*\right)^{-n} \left(\frac{2^{n+2}}{c}\right) f_\sharp(\nu_0|_{E^*}). \tag{2.59}$$

It follows by computations similar to those at the beginning of the proof of Corollary 36 that ν_1 satisfies the hypotheses of Theorem 33.

Consequently, Theorem 33 ensures there exists a Lipschitz graph Γ such that

$$\frac{\nu_1(\mathbb{R}^m \setminus \Gamma)}{\nu_1(\mathbb{R}^m)} < \frac{1}{100}.$$

But, recalling (2.59), it is clear this implies

$$\frac{\left(\nu_0|_{E^*}\right)\left(\mathbb{R}^m \setminus f(\Gamma)\right)}{\nu_0(\mathbb{R}^m)} < \frac{1}{100}.$$

Since f is a translation and scaling, $f(\Gamma)$ is still a Lipschitz graph, contradicting the fact that ν_0 is the restriction of an n-purely unrectifiable measure. □

2.3.3 Pointwise Menger curvature and β-numbers.

The first goal of this section is to prove Theorem 16, included below for convenience.

Theorem 43. *Let μ be a Radon measure on $\mathbb{R}^m$ with $0 < \Theta^n_*(\mu, x) \leq \Theta^{n,*}(\mu, x) < \infty$ for μ almost every $x \in \mathbb{R}^m$. Then the following are equivalent:*

1. *μ is countably n-rectifiable.*

2. *For μ almost every $x \in \mathbb{R}^m$, $\mathrm{curv}^n_{\mathcal{K}_1^2;\mu}(x, 1) < +\infty$.*

3. *For μ almost every $x \in \mathbb{R}^m$, $\mathrm{curv}^n_{\mathcal{K}_2^2;\mu}(x, 1) < +\infty$.*

4. *μ has σ-finite integral Menger curvature in the sense that μ can be written as $\mu = \sum_{j=1}^{\infty} \mu_j$ where each μ_j satisfies $\mathcal{M}_{\mathcal{K}_1^2}(\mu_j) < \infty$.*

5. *μ has σ-finite integral Menger curvature in the sense that μ can be written as $\mu = \sum_{j=1}^{\infty} \mu_j$ where each μ_j satisfies $\mathcal{M}_{\mathcal{K}_2^2}(\mu_j) < \infty$.*

Proof. The fact that (1) $\implies$ (2) is the content of [44, Lemma 1.1] combined with the characterization by Azzam and Tolsa in Theorem 3. Since $\mathcal{K}_1 \leq \mathcal{K}_2$ pointwise, it also follows that (2) $\implies$ (3). So, it suffices to show (3) $\implies$ (4) $\implies$ (1) and (2) $\implies$ (5) $\implies$ (1).

To this end, fix μ as in the statement of the theorem. Moreover, without loss of generality suppose that

$$\mu(\mathbb{R}^m) < \infty \tag{2.60}$$

Then, for $j \in \mathbb{N}_0$ define

$$E_j = \{x \in \mathbb{R}^m : \mathrm{curv}^n_{\mathcal{K}_1^2;\mu}(x, 1) \in [j, j+1)\} \quad \text{and} \quad \mu_j = \mu|_{E_j}. \tag{2.61}$$

Fubini's theorem [37, Theorem 1.4.1] ensures that the map

$$x \mapsto \int_{(\mathbb{R}^m)^{n+1}} \mathcal{K}_1^2(x, x_1, \ldots, x_{n+1})\ d\mu^{n+1}(x_1, \ldots, x_{n+1}) = \operatorname{curv}^n_{\mathcal{K}_1^2;\mu}(x, \infty)$$

is measurable. In particular the sets E_j are measurable, as each E_j is the preimage of a Borel set by a measurable function. For each $j \in \mathbb{N}_0$ it follows from (2.60) that $\mu_j = \mu|_{E_j}$ is a finite, Borel measure. The Lebesgue-Besicovitch differentiation theorem ensures that $0 < \Theta^{n,*}(\mu_j, x) < \infty$ for μ_j a.e. $x \in \mathbb{R}^m$ since $\lim_{r\to 0} \dfrac{\mu_j(B(x,r))}{\mu(B(x,r))} = 1$ for μ_j a.e. $x \in \mathbb{R}^m$. Since $\mathcal{K}_1^2$ is a non-negative function, and μ_j satisfies $\mu_j(E) \leq \mu(E)$ for all μ-measurable sets E it follows

$$\begin{aligned}
\operatorname{curv}^n_{\mathcal{K}_1^2;\mu_j}(x, \infty) &= \int_{(\mathbb{R}^m)^{n+1}} \mathcal{K}_1^2(x, x_1, \ldots, x_{n+1})\ d\mu_j^{n+1}(x_1, \ldots, x_{n+1}) \\
&\leq \int_{(\mathbb{R}^m)^{n+1}} \mathcal{K}_1^2(x, x_1, \ldots, x_{n+1})\ d\mu^{n+1}(x_1, \ldots, x_{n+1}) \\
&\leq (j+1) \qquad \forall x \in E_j,
\end{aligned}$$

where the final line follows from the definition of E_j, (2.61). Combining the above computation with (2.60) yields,

$$\mathcal{M}_{\mathcal{K}_1^2}(\mu_j) = \int_{\mathbb{R}^m} \operatorname{curv}^n_{\mathcal{K}_1^2;\mu_j}(x, \infty)\ d\mu_j(x) < (j+1)\mu_j(\mathbb{R}^m) < \infty.$$

Therefore, (3) $\implies$ (4). To see (4) $\implies$ (1) , note that Theorem 38 ensures each μ_j is n-countably rectifiable. Since $\mu = \sum_j \mu_j$ satisfies $\mu \ll \mathcal{H}^n$ has been decomposed into countably many n-countably rectifiable pieces, it follows μ is countably n-rectifiable.

The fact that (2) $\implies$ (5) $\implies$ (1) follows identically with $\mathcal{K}_2$ in place of $\mathcal{K}_1$. □

Given the history of the subject, this method of proof is not very satisfying, namely due to the fact that Lerman and Whitehouse's characterization of uniform rectifiability in terms of integral Menger curvature demonstrated an equivalence of a Carleson-type condition for integral Menger curvature and the Carleson condition for the β-numbers. Moreover, the first direction in this characterization follows from a direct comparison with β-numbers due to Kolasiński.

Our next goal is to show Theorem 19, which is equivalent to Theorem 44. We simply use Lemma 25 to restate the theorem for the sake of illuminating the dependencies on x.

Theorem 44. *If μ is an n-Ahlfors upper-regular Radon measure on $\mathbb{R}^m$ with upper-regularity constant C_0, and there exists λ such that $\mu(B(x,r)) \geq \lambda r^n$ for all $0 < r \leq R$, then*

$$\int_0^R \hat{\beta}^n_{\mu;2}(x,r)^2 \frac{dr}{r} \leq C_3 \operatorname{curv}^n_{\mathcal{K}^2;\mu}(x,R), \tag{2.62}$$

where $C_3 = C_3(\lambda, m, n, C_0, \mathcal{K})$, and $\mathcal{K}$ is any $(\mu,2)$-proper integrand.

In fact, if $\mu, \lambda, x, r,$ and R are as above, then for $\mathcal{K} \in \{\mathcal{K}_1, \mathcal{K}_2\}$

$$\begin{aligned}\int_0^R \hat{\beta}^n_{\mu;2}(x,r)^2 \frac{dr}{r} \leq C \operatorname{curv}^n_{\mu;\mathcal{K}^2}(x,R) \leq C \cdot \Gamma \int_0^{2R} \Theta^n(\mu,x,r) \hat{\beta}^n_{\mu;2}(x,r)^2 \frac{dr}{r} \\ \leq \tilde{C} \int_0^{2R} \hat{\beta}^n_{\mu;2}(x,r)^2 \frac{dr}{r}\end{aligned} \tag{2.63}$$

with constants $C, \Gamma, \tilde{C}$ depending on m, n, λ the upper-regularity constant of μ, and $\mathcal{K}$.

This theorem demonstrates a more direct converse to Kolasiński's bound on β-numbers. Alas, notice that in its present form, Theorem 44 requires stronger density conditions than Theorem 43. It would be interesting to try to weaken the density conditions at least as far as they are in Theorem 43.

The following technical lemma plays a central role in the proof of Theorem 44. For a review of the notation used in the proof, see Section 2.2.2.

Lemma 45. *Let μ be an n-Ahlfors upper-regular Radon measure on $\mathbb{R}^m$ with upper-regularity constant C_0. Suppose $x \in \mathbb{R}^m$ and $\lambda, R > 0$ such that*

$$\mu(B(x,r)) \geq \lambda r^n \tag{2.64}$$

holds for all $0 < r \leq R$.

Then, for

$$\delta = \delta(n, \lambda, C_0) = \frac{\lambda}{2^{k+2}5^{n-1}C_0} \tag{2.65}$$

and

$$\eta = \eta(n, \lambda, C_0) = \frac{\delta}{10n} = \frac{\lambda}{2^{k+3}5^n n C_0} \tag{2.66}$$

and all $0 < r \leq R$ *there exist points* $\{x_{i,r}\}_{i=1}^n \subset B(x, r)$ *such that*

$$h_{\min}(x, x_{1,r}, \ldots, x_{n,r}) \geq \delta r \tag{2.67}$$

and

$$(\mu|_{B(x,r)})(B(x_{i,r}, 5\eta r)) \geq \left(\frac{\lambda\eta^m}{2^{m+1}}\right) r^n = C_2(m, n, \lambda, C_0) r^n. \tag{2.68}$$

In particular, if for each $i \in \{1, \ldots, n\}$, $B_{i,r} := B(x_{i,r}, 5\eta r)$ *for any choices of* $y_i \in B_{i,r}$ *it follows that*

$$h_{\min}(x, y_1, \ldots, y_n) \geq \delta r - 5n\eta r = \frac{\delta r}{2}. \tag{2.69}$$

Finally, if $\mathbb{B}_r := B_{1,r} \times \cdots \times B_{n,r}$ *then*

$$\mathbb{B}_{\delta r/3} \cap \mathbb{B}_r = \emptyset. \tag{2.70}$$

Proof. (of Lemma 56). Let m, n, μ, C_0, λ and R be as in the lemma statement. In particular, $\lambda \leq C_0$. Define δ, η as in (3.9) and (3.10).

Fix $0 < r < R$, and suppose there exist $\{x_1, \ldots, x_k\}$ satisfying

$$h_{\min}(x, x_1, \ldots, x_k) \geq \delta r \text{ and } \mu(B(x_i, 5\eta r)) \geq \left(\frac{\lambda\eta^m}{2^{m+1}}\right) r^n$$

for all $i = 1, \ldots, k$ and assume that $k < n$. Then, we will find a point x_{k+1} such that $h_{\min}(x, x_1, \ldots, x_{k+1}) \geq \delta r$ and $\mu(B(x_{k+1}, 5\eta r)) \geq \left(\frac{\lambda\eta^m}{2^{m+1}}\right) r^n$. Hence, induction will guarantee the theorem.[3]

Let $V_k = \operatorname{aff}\{x, x_1, \ldots, x_k\} = x + \operatorname{span}\{x_1 - x, \ldots, x_k - x\}$. Define

$$(V_k)_{\delta r} = \{y \in B(x,r) : \operatorname{dist}(y, V_k) < \delta r\}. \tag{2.71}$$

Define $\rho = \rho(\lambda, C_0, n, k, r)$ by

$$\rho = sr \qquad \text{where } s = \left(\frac{\lambda}{C_0}\frac{1}{2^{k+1}\cdot 5^n}\right) < 1. \tag{2.72}$$

Note, $s < 1$ since $\lambda \leq C_0$. Let

$$\mathcal{G}_1 = \{B(y, 5\rho) \mid y \in V_k \cap B(x,r)\}. \tag{2.73}$$

We first note that $\mathcal{G}_1$ is a cover of $B(x,r) \cap (V_k)_{\delta r}$ since $\delta = \frac{5}{2}s$ implies $\delta r = \frac{5\rho}{2}$. So, by Vitali we can find a subfamily of sets $\{B(x_i, 5\rho)\}_{i=1}^{N'}$ such that $B(x,r) \cap (V_k)_{\delta r} \subset \bigcup_i B(x_i, 5\rho)$ and $\{B(x_i,\rho)\}_{i=1}^{N'}$ is disjoint.

A priori, N' could be infinite, but we will see that $N' \leq N_{k,s} = N_{n,k,\lambda,C_0}$ where

$$N_{n,k,\lambda,C_0} = \left(\frac{2}{s}\right)^k. \tag{2.74}$$

For α a positive integer, letting ω_α denote the volume of a k-dimensional unit ball, since $B \in \mathcal{G}_1$ implies $B \cap V_k$ is a k-dimensional ball of radius 5ρ,

$$\begin{aligned} N'\omega_k\rho^k &= \sum_{i=1}^{N'} \mathcal{H}^k\left(V_k \cap B\left(x_i,\rho\right)\right) = \left(\mathcal{H}^k|_{V_k}\right)\left(\bigcup_{i=1}^{N'} B\left(x_i,\rho\right)\right) \\ &\leq \left(\mathcal{H}^k|_{V_k}\right)\left(B(x,2r)\right) = \omega_k(2r)^k \end{aligned}$$

---

[3]The proof of the inductive step clearly shows that we can also find a point x_1 with $|x_1 - x| \geq \delta r$ and $\mu(B(x_1, 5\eta r)) \geq \left(\frac{\lambda\eta^m}{2^{m+1}}\right) r^n$.

so that $N' \leq 2^k (r\rho^{-1})^k = (2s^{-1})^k = N_{n,k,\lambda,C_0}$.

We wish to show that our choice of δ forces $\mu(B(x,r) \cap (V_k)_{\delta r}) \leq \dfrac{\lambda r^n}{2}$ so that

$$\mu(B(x,r) \setminus (V_k)_{\delta r}) \geq \frac{\lambda r^n}{2}. \tag{2.75}$$

Indeed,

$$\begin{aligned}\mu\left((V_k)_{\delta r} \cap B(x,r)\right) \leq \sum_{i=1}^{N'} \mu(B(x_i, 5\rho)) \leq \sum_{i=1}^{N'} C_0 (5\rho)^n \leq N_{n,k,\lambda,C_0} C_0 5^n s^n r^n \\ = \left(\frac{2}{s}\right)^k C_0 5^n s^n r^n.\end{aligned}$$

So, it suffices to show

$$2^k C_0 5^n s^{n-k} r^n \leq \frac{\lambda r^n}{2}$$

which holds if and only if

$$s^{n-k} \leq \frac{\lambda}{2^{k+1} 5^n C_0}.$$

Since $s < 1$ and $k < n$ this implies that our choice of s in (2.72) suffices to ensures (2.75).

Now we claim that (2.75) guarantees the existence of some $x_{k+1} \in B(x,r) \setminus (V_k)_{\delta r}$ such that (3.12) holds. The fact that $x_{k+1} \notin (V_k)_{\delta r}$ will guarantee (3.11).

To this end, let us consider the family of balls

$$\mathcal{G}_2 = \{B(y, 5\eta r) \mid y \in B(x,r) \setminus (V_k)_{\delta r}\}.$$

Then $B \in \mathcal{G}_2$ implies $B \subset B(x, 2r)$ since $\lambda \leq C_0$, (3.9), and (3.10) guarantee $5\eta < \frac{\delta}{2n} < 1$. Moreover, $\mathcal{G}_2$ covers $B(x,r) \setminus (V_k)_{\delta r}$. In particular, Vitali ensures there exists a subfamily $\{B(x_i, 5\eta r)\}_{i=1}^{M'}$ that covers $B(x,r) \setminus (V_k)_{\delta r}$ and $\{B(x_i, \eta r)\}_{i=1}^{M'}$ is a disjoint family. Again, we have no apriori estimate on M', but disjointness and containment in $B(x, 2r)$ yields

$$\omega_m(r\eta)^m M' = \sum_{i=1}^{M'} \mathcal{H}^m\left(B(x_i, r\eta)\right) \leq \mathcal{H}^m(B(x, 2r)) = \omega_m(2r)^m.$$

Consequently, we define M_{m,n,λ,C_0} so that

$$M' \leq (2\eta^{-1})^m = M_{\eta,m} = M_{m,n,\lambda,C_0}. \tag{2.76}$$

Combining (2.75) and (2.76), we deduce

$$\begin{aligned}
\frac{\lambda r^n}{2} &\leq \mu\left(B(x,r) \setminus (V_k)_{\delta r}\right) \\
&\leq \left(\mu|_{B(x,r)}\right)\left(\bigcup_{i=1}^{M'} B(x_i, 5\eta r)\right) \\
&\leq \sum_{i=1}^{M'} \left(\mu|_{B(x,r)}\right)\left(B(x_i, 5\eta r)\right) \\
&\leq M_{m,n,\lambda,C_0} \max\left\{(\mu|_{B(x,r)})(B(x_i, 5\eta r)) \mid i \in \{1, \ldots, M'\}\right\}. \qquad (2.77)
\end{aligned}$$

Choosing $k + 1 = j$ such that

$$(\mu|_{B(x,r)})(B(x_j, 5\eta r)) = \max\left\{(\mu|_{B(x,r)})(B(x_i, 5\eta r)) \mid i \in \{1, \ldots, M'\}\right\},$$

we have from (2.77) that

$$(\mu|_{B(x,r)})(B(x_{k+1}, 5\eta r)) \geq \frac{\lambda r^n}{2M_{m,n,\lambda,C_0}} \geq \frac{\lambda r^n \eta^m}{2^{m+1}}$$

verifying that $x_{k+1} \in B(x,r) \setminus (V_k)_{\delta r}$ satisfies (3.12).

It only remains to show (3.14) and (3.13), which follow quickly from the work already done. Indeed, Lemma 27 and (3.11) verify (3.14). On the other hand, (3.13) follows from $\mathbb{B}_{\delta r/3} \subset B(x, \delta r/3)$ and (3.11). □

Proof. (Of Theorem 44) First, observe that (2.63) follows from (2.62) combined with (2.12) and Lemma 12 and the Ahlfors upper-regularity assumption on μ. Hence, the goal is to verify (2.62).

Fix μ, R, λ as in the theorem statement. Let $\mathcal{K}$ be some $(\mu, 2)$-proper integrand, and $0 < r \leq R$. Let $\{x_{i,r}\}$, $B_{i,r}$ and $\mathbb{B}_r$ be as in Lemma 56. Then, first replacing the infimum with an average over fixed planes, and then applying (3.12), yields

$$\begin{aligned}\hat{\beta}^n_{\mu;2}(x,r)^2 &= \inf_{L \ni x} \frac{1}{r^n} \int_{B(x,r)} \left(\frac{\operatorname{dist}(z,L)}{r}\right)^2 d\mu(z) \\ &\leq \int_{\mathbb{B}_r} \int_{B(x,r)} \left(\frac{\operatorname{dist}(z, \operatorname{aff}\{x, y_1, \ldots, y_n\})^2}{r}\right)^2 \frac{d\mu(z) d\mu^n(y_1, \ldots, y_n)}{\mu^n(\mathbb{B}_r) r^n} \\ &\leq C \int_{\mathbb{B}_r} \int_{B(x,r)} \left(\frac{\operatorname{dist}(z, \operatorname{aff}\{x, y_1, \ldots, y_n\})}{r}\right)^2 \frac{d\mu(z) d\mu^n(y_1, \ldots, y_n)}{r^{n^2+n}},\end{aligned}$$

where $C = C(m, n, \lambda, C_0)$. Since $\{x, z\} \cup B_{i,r} \subset B(x,r)$ for all $i = 1, \ldots, n$, (3.14) ensures we can apply (2.28) in the final integral above, so that

$$\hat{\beta}^n_{\mu;2}(x,r)^2 \leq C \int_{\mathbb{B}_r} \int_{B(x,r)} \mathcal{K}(x, z, y_1, \ldots, y_n)^2 d\mu(z) d\mu^n(y_1, \ldots, y_n). \tag{2.78}$$

Finally, using the fact that for any $0 < \sigma < 1$,

$$\int_0^R \hat{\beta}^n_{\mu;2}(x,r)^2 \frac{dr}{r} \leq C_\sigma \sum_{j \geq 0} \hat{\beta}^n_{\mu;2}(x, \sigma^j R)^2$$

when $\sigma = \delta/3$ and writing $r_j := \left(\frac{\delta}{3}\right)^j R$, (3.13) and (2.78) yield

$$\int_0^R \hat{\beta}^n_{\mu;2}(x,r)^2 \frac{dr}{r} \leq C \int_{\cup_{j \geq 0} \mathbb{B}_{r_j}} \int_{B(x,r)} \mathcal{K}(x, z, y_1, \ldots, y_n)^2 d\mu^{n+1}(z, y_1, \ldots, y_n), \tag{2.79}$$

where $C = C(m, n, \lambda, C_0)$. Since for all j, $\mathbb{B}_{r_j} \times B(x,r) \subset B(x,r)^{n+1}$, (2.62) follows from non-negativity of the integrand after replacing $\cup_{j \geq 0} \mathbb{B}_{r_j} \times B(x,r)$ in (2.79) with $B(x,r)^{n+1}$. $\square$

Chapter 3

SUFFICIENT CONDITION FOR ($C^{1,\alpha}$, n)-RECTIFIABLE SETS ON $\mathbb{R}^m$

3.1 An introduction to higher-order rectifiability

In 1990 Peter Jones introduced the β-numbers as a quantitative tool to provide control of the length of a rectifiable curve and to prove the Analyst's Traveling Salesman Theorem [42] in the plane. Kate Okikiolu extended the result to one-dimensional objects in $\mathbb{R}^n$ [57]. In order to study the regularity of Ahlfors regular sets and measures of higher dimensions [19, 20], David and Semmes generalized the notion of β-numbers, see (3.4). This was the beginning of quantitative geometric measure theory and has led to lots of activity around characterizing uniformly rectifiable measures and their connections to the boundedness of a certain class of singular integral operators.

More recently, rectifiable sets and measures have been studied using the quantitative techniques previously used for uniformly rectifiable measures. For instance, β-numbers can characterize rectifiability of measures, amongst the class of all measures with various density and mass bounds. See for instance [58, 8, 72, 30, 11].

Several other geometric quantities have also proven to be useful in quantifying the regularity of sets and measures. In this paper, we wish to explore how Menger-type curvatures, see Definitions 53 and 55, yield information about $C^{1,\alpha}$ n-rectifiability, see Definition 50, of measures. In 1995, Melnikov discovered an identity for the (1-dimensional or classical) Menger curvature [54] which, in the complex plane, greatly simplified the existing proofs relating rectifiability to the L^2-boundedness of the Cauchy integral operator [53, 52]. The dream was that a notion of curvature, and similar identity, could be found in higher dimensions to produce simpler proofs demonstrating the equivalence of uniform rectifiability to the L^2-boundedness of singular integral operators. Alas, in 1999 Farag showed that in higher-dimensions no such identity could exist [32]. Nonetheless, geometric arguments made with non-trivial adaptations from [45] have since been used to characterize uniform rectifiability in all dimensions and codimensions in terms of Menger-type curvatures, [46, 47]. A sufficient condition for rectifiability of sets in terms of higher dimensional Menger-type curvatures appears in [56] and was extended to several characterizations of rectifiable measures under suitable density conditions [39].

Menger curvatures have also been used to quantify higher regularity (in a topological sense) of surfaces, see, for instance [68, 14]. Of particular relevance in our context, [43] showed that finiteness of $\mathrm{curv}^{\alpha}_{\mu;p}$, see Definition 55, is a sufficient condition for $C^{1,\alpha}$ n-rectifiability of measures. A formulation of the theorem is[1]

Theorem 46. *Let μ be a Radon measure on $\mathbb{R}^m$, with $0 < \Theta^n_*(\mu, x) \leq \Theta^{n,*}(\mu, x) < \infty$, for μ-a.e. $x \in \mathbb{R}^m$ and let $1 \leq p < \infty$, $0 < \alpha \leq 1$. If for μ- a.e. $x \in \mathbb{R}^m$*

$$\mathrm{curv}^{\alpha}_{\mu;p}(x, 1) < \infty, \tag{3.1}$$

then μ is $C^{1,\alpha}$ n-rectifiable.

The goal of this article is to prove that for a Radon measure μ satisfying relaxed density assumptions and for any $\alpha \in [0, 1)$ if the pointwise Menger curvature with $p = 2$, see Definition 55, is finite at μ a.e. x then μ is $C^{1,\alpha}$ n-rectifiable. More precisely,

Theorem I. Let μ be a Radon measure on $\mathbb{R}^m$, with $0 < \Theta^{n,*}(\mu, x) < \infty$ for μ-a.e. $x \in \mathbb{R}^m$, and let $\alpha \in [0, 1)$. If for μ- a.e. $x \in \mathbb{R}^m$

$$\mathrm{curv}^{\alpha}_{\mu;2}(x, 1) < \infty, \tag{3.2}$$

then μ is $C^{1,\alpha}$ n-rectifiable.

The $\alpha = 0$ case in Theorem I appears in [39]. The case $\alpha > 0$ is an improvement of a special case of [43] where the lower density assumption is relaxed.

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[1]The familiar reader may be aware that there are two additional parameters in the theorem of [43]. One such parameter is the ability to choose from a small family functions to replace the $h_{\min}$ in the integrand that defines $\mathrm{curv}^{\alpha}_{\mu;p}$. Such choices have previously been shown to be comparable to one another, see [46, 47, 48] or [43, 8.6-8.8]. The second such parameter was originally denoted by l. The $l = n + 2$ case is written here. Since any other choice of l is a stronger assumption, (changing the parameter ℓ is equivalent to replacing an L^p bound on some number of components of the integrand with an L^∞ bound) we chose to remove this for readability.

A rough sketch of the proof is as follows: the condition (3.2) is shown to imply "flatness" of the support of μ in terms of Jones' square function, and consequently (pieces of) the support of μ can be parametrized by Lipschitz graphs (see [56, Theorem 5.4]) when $\alpha = 0$ or $C^{1,\alpha}$ images (see [38, Theorem II]) when $\alpha > 0$.

On the other hand, the original proof provided by Kolasiński had two major parts. First, under the additional assumption that μ is Lipschitz n-rectifiable Kolasiński showed that the condition (3.1) forced additional flatness on the Lipschitz functions that cover the support of μ, which consequently implied better regularity on each such function (see [62, Lemma A.1]). Second, it remained to show that (3.1) implies Lipschitz n-rectifiability of μ. This was done by appealing to a characterization of rectifiability from [3, §2.8 Theorem 5] which roughly says that if the approximate tangent cone (in the sense of Federer) of μ is contained in an n-plane at almost every point, then μ is Lipschitz n-rectifiable.

Remark 47. *Note that Theorem I requires* $0 < \Theta^{n,*}(\mu, x)$ *for* μ *a.e.* x*, whereas Theeorem 46 requires the stronger assumption that* $0 < \Theta^{n}_{*}(\mu, x)$ *for* μ *a.e.* x*. The stronger density assumption in Theorem 46 allows one to apply Theorem 57. Then Remark 58 and Proposition 59 provide a direct proof that*

$$\int_0^1 \beta_2^{\mu}(x,r)^2 \frac{dr}{r} < \infty \quad \mu \text{ a.e. } x \in \mathbb{R}^m$$

when working under the hypotheses of Theorem 46. This provides an alternative proof to what we previously called the second major part of the original proof of Theorem 46.

The question of whether the proof of Theorem I when $\alpha = 0$ *can be completed by appealing to [8] after controlling Jones' function as above is an interesting one. Presently, the authors do not know how to do this without additionally assuming* $0 < \Theta^{n}_{*}(\mu, x)$ *for* μ *almost every* x*.*

Another difference between the two theorems is that Theorem I is stated only when $p = 2$*. Since increasing* p *only makes condition* (3.1) *harder to satisfy, the results in Theorem 46 are not sharp in terms of the parameter* p*. We expect that varying the parameter* p *would* *lead to results about rectifiability in the sense of Besov spaces, which is beyond the scope of this article.*

The proof of Theorem I is divided into two parts. First, we prove the claim for a measure μ which is n-Ahlfors upper regular on $\mathbb{R}^m$ and with positive lower density. Then, we use standard techniques to reduce the general case to the previous one.

3.2 Notation and Background

We begin by stating some definitions and notation.

Definition 48. Let $0 \leq s < \infty$ and let μ be a measure on $\mathbb{R}^m$. The upper and lower s-densities of μ at x are defined by

$$\Theta^{s,*}(\mu, x) = \limsup_{r \to 0} \frac{\mu(B(x,r))}{r^s}$$
$$\Theta^s_*(\mu, x) = \liminf_{r \to 0} \frac{\mu(B(x,r))}{r^s}.$$

If they agree, their common value is called the s-density of μ at x and denoted by

$$\Theta^s(\mu, x) = \Theta^{s,*}(\mu, x) = \Theta^s_*(\mu, x). \tag{3.3}$$

Definition 49 (*β_p-numbers*)**.** Given $x \in \mathbb{R}^m$, $r > 0$, $p \in (1, \infty)$, an integer $0 \leq n \leq m$, and a Borel measure μ on $\mathbb{R}^m$ define

$$\beta_p^\mu(x, r) = \left(\inf_L \frac{1}{r^n} \int_{B(x,r)} \left(\frac{\operatorname{dist}(y, L)}{r} \right)^p d\,\mu(y) \right)^{\frac{1}{p}}, \tag{3.4}$$

where the infimum is taken over all n-planes L.

When we talk about rectifiability and higher-order rectifiability, we mean what Federer would call "countably n-rectifiable".

Definition 50. A measure μ on $\mathbb{R}^n$ is said to be (Lipschitz) n-rectifiable if there exist countably many Lipschitz maps $f_i\colon \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\mu\left(\mathbb{R}^m \setminus \bigcup_i f_i(\mathbb{R}^n)\right) = 0. \tag{3.5}$$

A measure μ on $\mathbb{R}^n$ is $C^{1,\alpha}$ n-rectifiable if there exist countably many $C^{1,\alpha}$ maps $f_i\colon \mathbb{R}^n \to \mathbb{R}^m$ such that (3.5) holds.

In [38], a sufficient condition for $C^{1,\alpha}$ n-rectifiability in terms of β-numbers is provided.

Theorem 51 ([38])**.** *Let μ be a Radon measure on $\mathbb{R}^m$ such that $\Theta^n_*(\mu, x) < \infty$ and $\Theta^{n,*}(\mu, x) > 0$ for μ-almost every $x \in \mathbb{R}^m$, and $\alpha \in (0,1)$. Moreover, suppose, for μ-almost every $x \in \mathbb{R}^m$,*

$$J^{\mu}_{2,\alpha}(x) := \int_0^1 \frac{\beta^{\mu}_2(x,r)^2}{r^{2\alpha}}\,\frac{dr}{r} < \infty \tag{3.6}$$

Then, μ is $C^{1,\alpha}$ n-rectifiable.

When $\alpha = 1$, if we replace r in the left hand side of (3.6) by $r\eta(r)$, where $\eta(r)^2$ satisfies the Dini condition, then we obtain that μ is C^2 n-rectifiable.

Remark 52. *We say that a function ω satisfies the Dini condition if $\int_0^1 \frac{\omega(r)}{r} dr < \infty$. A possible choice for η in Theorem 51 is $\eta(r) = \frac{1}{\log(1/r)^{\gamma}}$, for $\gamma > \frac{1}{2}$.*

Definition 53 (*Classical Menger curvature*)**.** Given three points $x, y, z \in \mathbb{R}^m$, the (classical) Menger curvature is defined to be the reciprocal of the circumradius of x, y, z. That is,

$$c(x,y,z) = \frac{1}{R(x,y,z)},$$

where $R(x,y,z)$ is the radius of the unique circle passing through x, y, z.

In order to work with higher dimensional Menger curvatures, we introduce some notation for simplices in $\mathbb{R}^m$.

Definition 54 (*Simplices*)**.** Given points $\{x_0, \ldots, x_n\} \subset \mathbb{R}^m$, $\Delta(x_0, \ldots, x_n)$ denotes the convex hull of $\{x_0, \ldots, x_n\}$. In particular, if $\{x_0, \ldots, x_n\}$ are not contained in any $(n-1)$-dimensional plane, then $\Delta(x_0, \ldots, x_n)$ is an n-dimensional simplex with corners $\{x_0, \ldots, x_n\}$. Moreover, we denote by $\operatorname{aff}\{x_0, \ldots, x_n\}$ the smallest affine subspace containing $\{x_0, \ldots, x_n\}$. That is $\operatorname{aff}\{x_0, \ldots, x_n\} = x_0 + \operatorname{span}\{x_1 - x_0, \ldots, x_n - x_0\}$. Then, we define

$$h_{\min}(x_0, \ldots, x_n) = \min_i \operatorname{dist}(x_i, \operatorname{aff}\{x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\})$$

to be the minimum height of a vertex over the plane spanned by the opposing face. If $\Delta = \Delta(x_0, \ldots, x_n)$ we occasionally abuse notation and write $h_{\min}(\Delta)$ in place of $h_{\min}(x_0, \ldots, x_n)$. If Δ as before is an n-simplex, it is additionally called an (n, ρ)-simplex if

$$h_{\min}(x_0, \ldots, x_n) \geq \rho.$$

Definition 55 (*Menger curvatures*)**.** For $x \in \mathbb{R}^m$, $r > 0$, $\alpha \in [0, 1)$, an integer $0 \leq n \leq m$, and $p \in [1, \infty]$, we define the curvature of μ at x of scale r to be

$$\operatorname{curv}^{\alpha}_{\mu;p}(x, r) = \int_{B(x,r)^{n+1}} \frac{h_{\min}(x, x_1, \ldots, x_{n+1})^p}{\operatorname{diam}\left(\{x, x_1, \ldots, x_{n+1}\}\right)^{p(1+\alpha)+n(n+1)}} d\mu^{n+1}, \tag{3.7}$$

where μ^{n+1} is the product measure defined by taking $(n+1)$-products of μ with itself.

3.3 Proof of Theorem I

We now proceed to prove the theorem in the case where μ is n-Ahlfors upper regular on $\mathbb{R}^m$ and has positive lower density μ-almost everywhere.

We recall the following Lemma from [39, Lemma 3.13], which says that, under the appropriate density assumptions on a measure μ, given a point x and radius r, the ball $B(x, r)$ contains a large number of effective n-dimensional secant planes through x.

Lemma 56. *Let μ be an n-Ahlfors upper-regular Radon measure on $\mathbb{R}^m$ with upper-regularity* *constant* C_0*. Suppose* $x \in \mathbb{R}^m$ *and* $\lambda, R > 0$ *such that*

$$\mu(B(x,r)) \geq \lambda r^n \tag{3.8}$$

holds for all $0 < r \leq R$*.*

Then, for

$$\delta = \delta(n, \lambda, C_0) = \frac{\lambda}{2^{k+2}5^{n-1}C_0} \tag{3.9}$$

and

$$\eta = \eta(n, \lambda, C_0) = \frac{\delta}{10n} = \frac{\lambda}{2^{k+3}5^n n C_0} \tag{3.10}$$

and all $0 < r \leq R$ *there exist points* $\{x_{i,r}\}_{i=1}^n \subset B(x,r)$ *such that*

$$h_{\min}(x, x_{1,r}, \ldots, x_{n,r}) \geq \delta r \tag{3.11}$$

and

$$(\mu \llcorner B(x,r))(B(x_{i,r}, 5\eta r)) \geq \left(\frac{\lambda \eta^m}{2^{m+1}}\right) r^n = C_2(m,n,\lambda,C_0) r^n. \tag{3.12}$$

For any $0 < r \leq R$ *and* $i \in \{1, \ldots, n\}$*, let* $B_{i,r} := B(x_{i,r}, 5\eta r) \cap B(x,r)$ *and* $\mathbb{B}_r := B_{1,r} \times \cdots \times B_{n,r}$*. Then*

$$\mathbb{B}_{\frac{\delta r}{3}} \cap \mathbb{B}_r = \varnothing \tag{3.13}$$

and, whenever $(y_1, \ldots, y_n) \in \mathbb{B}_r$*,*

$$h_{\min}(x, y_1, \ldots, y_n) \geq \frac{\delta r}{2}. \tag{3.14}$$

Theorem 57. *If* μ *is an* n*-Ahlfors upper-regular measure on* $\mathbb{R}^m$ *such that* $\Theta_*^n(\mu, x) > 0$ *for all* x*, and*

$$\mathrm{curv}_{\mu;2}^{\alpha}(x,1) < \infty$$

for μ*-almost every* x*, then* μ *is* $C^{1,\alpha}$ n*-rectifiable.*

We prove Theorem 57 by showing that for μ as in the statement of the theorem, we can in fact show that

$$\int_0^1 \frac{\beta_2^\mu(x,r)^2}{r^{2\alpha}}\,\frac{dr}{r} < \infty$$

for almost every $x \in \mathbb{R}^m$, and then appeal to Theorem 51. In the proof we use a slight modification of the usual β-numbers introduced above, the so-called "centered β-numbers", that we denote by $\hat{\beta}_2^\mu$. These numbers are defined exactly as the β-numbers, except that the infimum in the definition of $\hat{\beta}_2^\mu(x,r)$ is taken over n-planes passing through x. That is, for $x \in \mathbb{R}^m$, $r > 0$ define

$$\hat{\beta}_2^\mu(x,r)^2 := \inf_{L \ni x} \int_{B(x,r)} \left(\frac{\mathrm{dist}(z,L)}{r}\right)^2 \frac{d\mu(z)}{r^n}.$$

In particular, because the infimum in the definition of $\beta_2^\mu(x,r)$ is taken over a larger class than the one in $\hat{\beta}_2^\mu(x,r)$, we have $\beta_2^\mu(x,r) \le \hat{\beta}_2^\mu(x,r)$ for all $x \in \mathbb{R}^m, r > 0$.

Proof of Theorem 57. Let μ be as in the theorem statement, and x a point so that $\Theta_*^n(\mu,x) > 0$. Then, there exists some $\lambda > 0$ so that for all $0 < r \le 1$, $\mu(B(x,r)) \ge \lambda r^n$. Now, fix $0 < r \le 1$. By the definition of infimum, for any $(y_1,\ldots,y_n) \in (\mathbb{R}^n)^n$,

$$\begin{aligned}\hat{\beta}_{\mu;2}^n(x,r)^2 &= \inf_{L \ni x} \frac{1}{r^n} \int_{B(x,r)} \left(\frac{\mathrm{dist}(z,L)}{r}\right)^2 d\mu(z)\\ &\le \frac{1}{r^n} \int_{B(x,r)} \left(\frac{\mathrm{dist}(z,\mathrm{aff}\{x,y_1,\ldots,y_n\})}{r}\right)^2 d\mu(z). \qquad (3.15)\end{aligned}$$

Choose $x_{i,r}, B_{i,r}$, and $\mathbb{B}_r$ as in Lemma 56. Averaging (3.15) over all $(y_1,\ldots,y_n) \in \mathbb{B}_r$, and then applying (3.12) yields

$$\begin{aligned}\hat{\beta}_{\mu;2}^n(x,r)^2 &\le \int_{\mathbb{B}_r}\int_{B(x,r)} \left(\frac{\mathrm{dist}(z,\mathrm{aff}\{x,y_1,\ldots,y_n\})}{r}\right)^2 \frac{d\mu(z)d\mu^n(y_1,\ldots,y_n)}{\mu^n(\mathbb{B}_r)r^n}\\ &\le C\int_{\mathbb{B}_r}\int_{B(x,r)} \left(\frac{\mathrm{dist}(z,\mathrm{aff}\{x,y_1,\ldots,y_n\})}{r}\right)^2 \frac{d\mu(z)d\mu^n(y_1,\ldots,y_n)}{r^{n^2+n}}, \qquad (3.16)\end{aligned}$$

where $C = C(m, n, \lambda, C_0)$. We now claim that,

$$\operatorname{dist}(z, \operatorname{aff}\{x, y_1, \ldots, y_n\}) \leq \left(\frac{2}{\delta}\right)^n h_{\min}(x, z, y_1, \ldots, y_n). \tag{3.17}$$

Indeed, let $\Delta = \Delta(x, z, y_1, \ldots, y_n)$ and $\Delta_w = \Delta(\{x, z, y_1, \ldots, y_n\} \setminus \{w\})$, for each $w \in \{x, z, y_1, \ldots, y_n\}$. Basic Euclidean geometry ensures that

$$\begin{aligned} \operatorname{dist}(z, \operatorname{aff}\{\Delta_z\})\mathcal{H}^n(\Delta_z) &= (n+1)\mathcal{H}^{n+1}(\Delta) \\ &= h_{\min}(x, z, y_1, \ldots, y_n)\mathcal{H}^n(\Delta_{w_0}), \end{aligned} \tag{3.18}$$

where w_0 is any vertex such that

$$\operatorname{dist}(w_0, \operatorname{aff}\{\{x, z, y_1, \ldots, y_n\} \setminus \{w_0\}\}) = h_{\min}(x, z, y_1, \ldots, y_n).$$

On the other hand, since $\{x, z\} \cup B_{i,r} \subset B(x, r)$ for all $i = 1, \ldots, n$, Equation (3.14) ensures

$$\frac{\mathcal{H}^n(\Delta_{w_0})}{\mathcal{H}^n(\Delta_z)} \leq \left(\frac{2r}{\delta r}\right)^n. \tag{3.19}$$

The claim (3.17) now follows from (3.18) and (3.19).

Evidently, $\operatorname{diam}\{x, z, y_1, \ldots, y_n\} \leq 2r$. Using this diameter bound, (3.17) and (3.16), we conclude

$$\hat{\beta}_2^\mu(x, r)^2 \leq C \int_{\mathbb{B}_r} \int_{B(x,r)} \frac{h_{\min}(x, z, y_1, \ldots, y_n)^2}{\operatorname{diam}\{x, z, y_1, \ldots, y_n\}^{n^2+n+2}} d\mu(z) d\mu^n(y_1, \ldots, y_n). \tag{3.20}$$

Setting $r_j = \left(\frac{\delta}{3}\right)^j$ and using the fact that $0 < \frac{\delta}{3} < 1$, it follows that

$$\int_0^1 \frac{\hat{\beta}_2^\mu(x, r)^2}{r^{2\alpha}} \frac{dr}{r} \leq C_\delta \sum_{j \geq 0} \frac{\hat{\beta}_2^\mu(x, r_j)^2}{(r_j)^{2\alpha}}. \tag{3.21}$$

It now follows from (3.21), (3.13), and (3.20) that

$$
\begin{aligned}
&\int_0^1 \hat{\beta}_2^{\mu}(x,r)^2 \frac{dr}{r^{1+2\alpha}} \\
&\leq C \int_{\cup_j \mathbb{B}_{r_j} \times B(x,r_j)} \frac{h_{\min}(x,z,y_1,\dots,y_n)^2}{\operatorname{diam}\{x,z,y_1,\dots,y_n\}^{n^2+n+2} r_j^{2\alpha}} d\mu^{n+1}(y_1,\dots,y_n,z) \\
&\leq C \int_{\cup_j \mathbb{B}_{r_j} \times B(x,r_j)} \frac{h_{\min}(x,z,y_1,\dots,y_n)^2}{\operatorname{diam}\{x,z,y_1,\dots,y_n\}^{n^2+n+2+2\alpha}} d\mu^{n+1}(y_1,\dots,y_n,z) \\
&\leq C \int_{B(x,1)^{n+1}} \frac{h_{\min}(x,x_1,\dots,x_{n+1})^2}{\operatorname{diam}\left(\{x,x_1,\dots,x_{n+1}\}\right)^{2(1+\alpha)+n(n+1)}} d\mu^{n+1}(x_1,\dots,x_{n+1}) \\
&= C \operatorname{curv}^{\alpha}_{\mu;2}(x,1),
\end{aligned}
$$

where in the penultimate step we used that $\mathbb{B}_{r_j} \times B(x,r_j) \subset B(x,1)^{n+1}$ for all j, and the non-negativity of the integrand. □

Remark 58. *At this point, we briefly focus on the difference between conditions* (3.2) *and* (3.1)*. The proof of Theorem 57 could be followed out identically, using* $\operatorname{curv}^{\alpha}_{\mu;p}(x,1) < \infty$ *in place of* $\operatorname{curv}^{\alpha}_{\mu;2}(x,1)$ *and by replacing appropriate* 2*'s with* p*'s, to obtain*

$$\int_0^R \left(\frac{\hat{\beta}_p^{\mu}(x,r)}{r^{\alpha}}\right)^p \frac{dr}{r} \leq C \operatorname{curv}^{\alpha}_{\mu;p}(x,R) < \infty.$$

Consequently, the following proposition is of interest, see Remark 47.

Proposition 59. *Let* μ *be a Radon measure on* $\mathbb{R}^m$ *which is is Ahlfors upper-regular with constant* C_0*, and such that* $0 < \Theta^{n,*}(\mu,x)$ *for* μ*-almost every* $x \in \mathbb{R}^m$*. Let* $p \in [1,\infty)$*, and* $\alpha \in (0,1]$*. If for* μ*-a.e.* $x \in \mathbb{R}^m$*,*

$$\int_0^1 \left(\frac{\beta_p^{\mu}(x,r)}{r^{\alpha}}\right)^p \frac{dr}{r} < \infty,$$

then μ *is* n*-rectifiable.*

Proof. We show that the hypotheses imply that for μ- a.e. $x \in \mathbb{R}^m$,

$$\int_0^1 \beta_2^\mu(x,r)^2 \frac{dr}{r} < \infty$$

and then employ [30] to obtain rectifiability.

For $p \leq 2$, it is enough to observe, from the definition of $\beta_p^\mu(x,r)^p$, that

$$\left(\frac{\operatorname{dist}(y,P)}{r}\right)^2 \leq 2^{\frac{2}{p}} \left(\frac{\operatorname{dist}(y,P)}{r}\right)^p$$

as $\frac{\operatorname{dist}(y,P)}{r} \leq 2$. This immediately implies that $\beta_2^\mu(x,r)^2 \leq \beta_p^\mu(x,r)^p$, and hence we are done. Note that in this case we did not use that $\alpha > 0$.

For $p > 2$, we use Hölder inquality:

$$\begin{aligned}
\int_0^1 \beta_2^\mu(x,r)^2 \frac{dr}{r} &= \int_0^1 \frac{\beta_2^\mu(x,r)^2}{r^{2\alpha}} \cdot r^{2\alpha} \frac{dr}{r} \\
&\leq \left(\int_0^1 \left(\frac{\beta_2^\mu(x,r)^2}{r^{2\alpha}}\right)^{\frac{p}{2}} \frac{dr}{r}\right)^{\frac{2}{p}} \left(\int_0^1 (r^{2\alpha})^{\frac{p}{p-2}} \frac{dr}{r}\right)^{\frac{p-2}{p}} \\
&\leq \left(\int_0^1 \left(\frac{\beta_2^\mu(x,r)}{r^{\alpha}}\right)^{p} \frac{dr}{r}\right)^{\frac{2}{p}} \left(\int_0^1 r^{\frac{2p\alpha}{p-2}-1}\, dr\right)^{\frac{p-2}{p}} \\
&\leq C_{p,\alpha,C_0} \left(\int_0^1 \left(\frac{\beta_p^\mu(x,r)}{r^{\alpha}}\right)^{p} \frac{dr}{r}\right)^{\frac{2}{p}},
\end{aligned}$$

where in the last inequality we used the fact that, for $p > 2$, $\beta_2^\mu(x,r) \leq \left(\frac{\mu(B(x,r))}{r^n}\right)^{\frac{1}{2}-\frac{1}{p}} \beta_p^\mu(x,r)$. □

Finally, we reduce Theorem I to Theorem 57.

Lemma 60. *Let μ be a Radon measure with compact support on $\mathbb{R}^m$ such that $0 < \Theta_*^n(\mu,x) \leq \Theta^{n,*}(\mu,x) < \infty$ for μ-a.e. $x \in \mathbb{R}^m$. Then there exist an increasing sequence of sets $\{E_k\}$ such that*

$$\mu\left(\mathbb{R}^m \setminus \bigcup_{k=1}^\infty E_k\right) = 0,$$

$\mu_k := \mu \llcorner E_k$ is n-Ahlfors upper-regular and $\Theta^n_*(\mu_k, x) > 0$ for μ_k-a.e. $x \in \mathbb{R}^m$.

Proof of Lemma 60. For any positive integer k, let E_k be given by

$$E_k = \{x \in \mathbb{R}^m \mid \mu(B(x,r)) \leq kr^n \text{ for every } r < 2^{-k}\}.$$

Evidently, $E_k \subseteq E_{k+1}$ and $\mu\left(\mathbb{R}^m \setminus \cup_{k=1}^{\infty} E_k\right) = 0$.

Notice that $\mu_k(B(x,r)) \leq \mu(B(x,r)) \leq kr^n$, for $r < 2^{-k}$. Since $\mathrm{spt}(\mu_\mathrm{k}) \subseteq \mathrm{spt}(\mu)$ has finite diameter, it follows that μ_k is upper n-Ahlfors regular. Moreover $\Theta^n_*(\mu_k, x) > 0$ for almost every $x \in E_k$, by Theorem 2.12(2) in [51], since $\mu_k \ll \mu$ and $\Theta^n_*(\mu, x) > 0$. □

Proof of Theorem I. Let μ and α be as in Theorem I. Without loss of generality, by considering $\mu \llcorner B(0, R_k)$ for some sequence of $R_k \uparrow \infty$, we can assume that μ has compact support.

To apply Lemma 60 we first need to show that $\Theta^n_*(\mu, x) > 0$ for μ-a.e. $x \in \mathbb{R}^m$. Indeed, when $\alpha = 0$, μ is n-rectifiable (see [39, Theorem 1.19]). Moreover, for $\alpha \in (0,1)$, $\mathrm{curv}^{\alpha}_{\mu;2}(x,1) < \infty$ implies $\mathrm{curv}^{0}_{\mu;2}(x,1) < \infty$. Therefore μ is n-rectifiable and the n-rectifiability of μ implies $\Theta^n_*(\mu, x) > 0$, for μ-a.e. $x \in \mathbb{R}^m$.

Now, define μ_k as in Lemma 60 and apply Theorem 57 to each μ_k. Then each μ_k is $C^{1,\alpha}$ n-rectifiable which implies μ is $C^{1,\alpha}$ n-rectifiable as desired. □

Chapter 4

EXAMPLES OF VERY NON-UNIFORMLY RECTIFIABLE, RECTIFIABLE SETS

4.1 An introduction to local and non-local quantitative flatness

In his solution to the Analyst's Traveling Salesman Problem [42], Peter Jones introduced a local gauge of flatness which has been generalized by David and Semmes [19] to higher dimensions. These families of local gauges of flatness are called the Jones β-numbers, and they have come to dominate the landscape in quantitative techniques relating rectifiability, potential theory, and boundedness of singular integrals. See, for example the landmark book [20].

For a set $E \subset \mathbb{R}^d$, $1 \leq p < \infty$, and an integer $1 \leq n \leq d-1$, we write $\mu = \mathcal{H}^n \llcorner E$ and define the Jones β-numbers as follows,

$$\beta^n_{E;p}(x,r) = \left(\inf_{L \subset \mathbb{R}^d \text{ an } n\text{-plane}} \int_{B(x,r)} \left(\frac{\mathrm{dist}(y,L)}{r} \right)^p \frac{d\mu(y)}{r^n} \right)^{\frac{1}{p}}. \tag{4.1}$$

We also write $\beta^n_{\mu;p}(x,r)$ for $\beta^n_{E;p}(x,r)$, when $\mu = \mathcal{H}^n \llcorner E$ is understood. If $p = \infty$, the β-numbers are defined in terms of the sup-norm instead of the L^∞-norm.

Various notions of "rectifiability" have been studied over the years and are frequently characterized by β-numbers. We introduce them from most to least regular. The original notion is for 1-dimensional sets in $E \subset \mathbb{R}^d$. It is said that E is rectifiable if E can be contained in a curve of finite length. Thanks to [42] in dimension $d = 2$ and [57] for dimension $d \geq 3$, the following characterization is known:

$$E \subset \mathbb{R}^d \text{ is (finitely) rectifiable} \iff \int_E \int_0^\infty \beta^1_{E;\infty}(x,r)^2 \frac{dr}{r}\, d\mathcal{H}^1(x) < \infty. \tag{4.2}$$

In addition to generalizing the Jones β-numbers, [19] also introduced the notion of uniform rectifiability. A set $E \subset \mathbb{R}^d$ is said to be Ahlfors n-regular if there exists $0 < c < C < \infty$ such that $cr^n \leq \mathcal{H}^n(E \cap B(x,r)) \leq Cr^n$ for all $x \in E$ and all $0 < r < \mathrm{diam}(E)$. An n-Ahlfors regular $E \subset \mathbb{R}^d$ is said to be uniformly n-rectifiable if there exist finite constants $\theta, \Lambda > 0$ such that for all $x \in E$ and all $0 < r < \mathrm{diam}(E)$ there is a Lipschitz mapping $g : B(0,r) \subset \mathbb{R}^n \to \mathbb{R}^d$ with $\mathrm{Lip}(g) \leq \Lambda$ such that $\mathcal{H}^n(E \cap B(x,r) \cap g(B(0,r))) \geq \theta r^n$.

In [19] the authors show that an n-Ahlfors regular set $E \subset \mathbb{R}^d$, is n-uniformly rectifiable if and only if the Jones β-numbers satisfy the following Carleson condition for some $1 \leq p < \frac{2n}{n-2}$,

$$C^n_{E;p}(x,R) := \int_{B(x,R)} \int_0^R \beta^n_{E;p}(y,r)^2 \frac{dr}{r}\, d\mu(y) \leq cR^n \quad \text{for all } x \in E,\ R > 0. \tag{4.3}$$

A set $E \subset \mathbb{R}^d$ is said to be countably n-rectifiable if there are Lipschitz maps $f_i : \mathbb{R}^n \to \mathbb{R}^d$ with $i = 1, 2, \ldots$, such that

$$\mathcal{H}^n(\mathbb{R}^d \setminus \cup_i f_i(\mathbb{R}^n)) = 0.$$

Recently, Tolsa [72] and Azzam and Tolsa [8] show, as a special case, that E is countably n-rectifiable if and only if

$$J^n_E(x,1) = \int_0^1 \beta^n_{E;2}(x,r)^2 \frac{dr}{r} < \infty \quad \text{for } \mathcal{H}^n - a.e.\ x \in E \tag{4.4}$$

where $J^n_E(x,1)$ is the Jones function at x and scale 1. See [51] for more about countably n-rectifiable sets and also [58] and [10] for more about identifying countably n-rectifiable sets and measures via β-numbers.

In this paper, we show that sets which satisfy (4.4) can fail to satisfy (4.3) as dramatically as possible. We show this through two examples in the plane. The first, Theorem 61, is connected and the second, Theorem 62, is Ahlfors regular.

Theorem 61. *There exists a rectifiable curve (of finite length), $K_0 \subset \mathbb{R}^2$, such that for $\mu = \mathcal{H}^1 \llcorner K_0$, for any $x \in K_0$, and any $\delta > 0$*

$$\int_{B_\delta(x)} \int_0^\delta \beta^1_{K_0;2}(y,r)^2 \frac{dr}{r}\, d\mu(y) = \infty.$$

The set K_0 arises from unions of modifications of approximations to snowflake-like sets. Since K_0 is a rectifiable curve, by the Analyst's Traveling Salesman theorem, i.e., (4.2), it follows

$$\int_{\mathbb{R}^2}\int_0^\infty \beta^1_{K_0,\infty}(y,r)^2\frac{dr}{r}d\mu(y)<\infty,$$

which indicates that K_0 fails to be Ahlfors upper-regular at generic points.

We note that this example also gives rise to a curve of finite length (see Remark 73) which has classical tangents nowhere. This is in contrast to the well-known theorem that simple rectifiable curves have tangents almost everywhere, see [31], and also demonstrates the necessity of the "simple" assumption in the main theorem of [17], which states that σ-finite simple curves have classical tangents on a set of positive measure.

Theorem 62. *There is a* 1*-Ahlfors regular, countably* 1*-rectifiable set* A_0 *contained in the unit cube in* $\mathbb{R}^2$ *such that for* $\mu=\mathcal{H}^1 \llcorner A_0$*, for every* $x\in A_0$*, and for every* $\delta>0$*,*

$$\int_{B_\delta(x)}\int_0^\delta \beta^1_{A_0;2}(y,r)^2\frac{dr}{r}\;d\mu(y)=\infty.$$

The set A_0, whose construction was initially motivated by the machinery introduced in [73], is created from scaled unions of approximations to the 4-corner Cantor set. Ultimately the presentation was simpler using the framework of self-similar sets.

Remark 63. *These examples can be used to create higher-dimensional ones by taking Cartesian products with finite intervals. That is, if* $A\in\{K_0,A_0\}$ *for any positive integer* $n<d$*, define* $E'=A\times[0,1]^{n-1}\subset\mathbb{R}^{n+1}$*. Embedding* E' *into the first* $(n+1)$*-dimensions of* $\mathbb{R}^d$ *preserves the properties of* A*. In particular, it is standard that defining* β*-numbers over cubes (with sides parallel to the axes in* $\mathbb{R}^d$*) instead of balls leads to an equivalent definition of the* β*-numbers. Consequently finiteness of* $C^n_{E;2}(x,R)$ *is equivalent to the finiteness of* $C^1_{A;2}(x',R)$ *where* x' *is the orthogonal projection of* x *into* $\mathbb{R}^2$*.*

4.2 Construction of K_0

To construct a 1-rectifiable set, K_0, that is connected (hence Ahlfors lower-regular) for which the Jones function is locally non-integrable, we modify approximations to the Koch snowflake. This set will not be Ahlfors upper regular, i.e., $\mathcal{H}^1(B(x,r) \cap K_0) \leq Cr$ fails for all C and some (x,r). We begin with an informal description of the technical construction which follows.

The construction splits into two parts. First, build a "base set" E_∞ which satisfies $C_{E_\infty}(0,\delta) = +\infty$. The base set E_∞ is designed from modified approximations of the Koch snowflake, see Definition 65 and subsequent discussion. The goal is to build the connected base set E_∞ so that within the triadic strips $[3^{-i}, 3^{-(i-1)}] \times \mathbb{R}$ the set E_∞ looks like successive approximations to the Koch snowflake which arise from more iterations of the "bumping process". See Figures 5.3a - 4.3 for example sets that could be scaled and set on the triadic intervals $[3^{-1}, 2 \cdot 3^{-1}], [3^{-2}, 2 \cdot 3^{-2}], [3^{-3}, 2 \cdot 3^{-3}]$, as in Figure 4.4, to begin creating the base set E_∞. After doing this infinitely many times, and taking care to balance the number of corners with the shallowness of the corners, we create a connected set with finite length such that the infinitely many "bumps" in any neighborhood of the origin give $C_{E_\infty}(0,\delta) = +\infty$ for all $0 < \delta$.

After we have constructed the base set E_∞, we build the desired final set K_0. Roughly speaking, this happens by iteratively adding scaled copies of E_∞ in a dense way along E_∞ itself.

For the remainder of this paper, we only consider $E \subset \mathbb{R}^2$ and the β-numbers when $p = 2$. As such, we write β_E, β_μ, C_E, and C_μ in place of $\beta^1_{E;2}, \beta^1_{\mu;2}$, $C^1_{E;2}$, and $C^1_{\mu;2}$, see (4.1), (4.3). Moreover, for any set $L \subset \mathbb{R}^2$ we write $B_r(L) = \{x : \text{dist}(x, L) < r\}$ and $B_r = B_r(\{0\})$.

We begin by stating two basic properties of the Jones β-numbers. The first, often called "doubling" despite our choice to scale by a factor of 3, controls how fast the β-numbers can shrink by relating the β-numbers at comparable scales. The second shows how the β-numbers behave under rescaling.

Proposition 64. *Let $E \subset \mathbb{R}^2$ have $dim_{\mathcal{H}}(E) = 1$.*

1. *For any ball $B_r(y) \subset B_{3r}(x)$,*

$$\beta_E(y,r)^2 \leq 3^3 \beta_E(x,3r)^2$$

2. *The β-numbers have the following scaling property. If $E^{z,t} = tE + z$ then $\beta_{E^{z,t}}(x,r)^2 = \beta_E\left(\frac{x-z}{t}, \frac{r}{t}\right)^2$. Consequently, $C_{E^{z,t}}(z,r) = tC_E(0, t^{-1}r)$.*

Next, for the reader's convenience we record some facts about and give a construction of the standard approximations to the Koch snowflake.

Definition 65. Let $I \subset \mathbb{R}^2$ be a line segment, and fix $0 < \alpha < \pi/2$. Define $P(I)$ as the set which results from the following operation:

1. Divide I into three equal subintervals, $I_{\text{left}} \cup I_{\text{center}} \cup I_{\text{right}}$.

2. Over the middle interval, I_{center}, construct an isosceles triangle with angles α and base I_{center}.

3. Delete I_{center}, the base of the isosceles triangle.

We define

$$S(I) = \overline{P(I) \setminus I}, \tag{4.5}$$

and call $S(I)$ the *bump*. If q_I is the orthogonal projection onto the line containing I and $q_I^\perp$ is the orthogonal projection onto $I^\perp$, then $\text{height}(S(I)) = \text{diam}\{q_I^\perp(S(I))\}$ and $\text{width}(S(I)) = \text{diam}\{\pi_I(S(I))\} = \frac{1}{3}\mathcal{H}^1(I)$. We shall abuse our notation slightly by saying that for a collection of line segments, E, the set $P(E)$ is obtained by applying P to each maximal line segment contained in E.

If $I = [0,1] \times \{0\}$ and $\alpha = \frac{\pi}{3}$, the standard approximations to the Koch snowflake are given by $\{P^k(I)\}_{k=1}^{\infty}$, where P^k denotes applying P iteratively k times. We emphasize a few properties about deformations under the operation P.

Proposition 66. *For any finite line segment $I \subset \mathbb{R}^2$ and positive integer n,*

$$\text{height}(S(I)) = \frac{\tan(\alpha)}{6}|I| \tag{4.6}$$

$$\mathcal{H}^1(S(I)) = \frac{\sec(\alpha)}{3}|I| \tag{4.7}$$

$$\mathcal{H}^1(P^n(E)) = \left(\frac{\sec(\alpha)+2}{3}\right)^n \mathcal{H}^1(E) \tag{4.8}$$

When $\tau = \frac{1}{20}\min\left\{\frac{\tan(\alpha)}{6}, \frac{1}{3}\right\}$, *there exists* $c_0 = c(\alpha)$ *such that for all lines* L

$$\mathcal{H}^1\left(S(I) \setminus B_\tau(L)\right) \geq c_0 \mathcal{H}^1(S(I)). \tag{4.9}$$

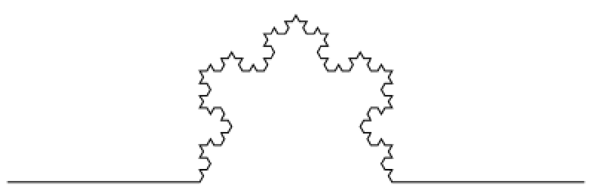

Figure 4.1: The set $\mathcal{P}_3(I)$ when $\alpha = \pi/3$. Note: despite the least iterations, this has more length than the following two images.

Figure 4.2: The set $\mathcal{P}_4(I)$ when $\alpha = \pi/9$.

Figure 4.3: The set $\mathcal{P}_5(I)$ when $\alpha = \pi/27$.

Proof. (4.6) and (4.7) follow from planar geometry. The $n = 1$ case for (4.8) follows by

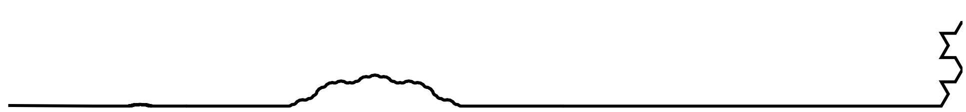

Figure 4.4: Zoomed in and truncated picture of the 3rd approximation to a base set, made by placing the sets from Figures 5.3a - 4.3 in their appropriate triadic intervals.

adding back in the unchanged intervals I_{left} and I_{right}, which have total length $\frac{2}{3}|I|$. The geometric nature of the definition of P allows us to then iterate this to achieve (4.8).

To verify (4.9) we proceed by contradiction. Suppose no such constant c_0 exists. Then, there exists a sequence of lines intersecting $S(I)$ such that

$$\mathcal{H}^1\left(S(I) \setminus B_\tau(L_i)\right) < 2^{-i}\mathcal{H}^1(S(I)).$$

After passing to a subsequence, L_i converge to some line L with the property that $\mathcal{H}^1\left(S(I) \setminus B_\tau(L)\right) = 0$. Since $S(I)$ is connected, this implies $S(I) \subset B_{2\tau}(L)$. However, this contradicts the fact that $2\tau \leq \frac{1}{10}\min\{\text{height}(S(I)), \text{width}(S(I))\}$. □

Definition 67. Define $\mathcal{P}_j$ to be the set operation defined on line-segments by

$$\mathcal{P}_j(I) = P^{j-1}(S(I)) \bigcup \left(I \setminus I_{\text{center}}\right),$$

recalling the definition of $S(I)$ can be found in (4.5). Loosely speaking, for any line segment, I, $\mathcal{P}_j(I)$ is the set that replaces the center of I with a jth approximation of the Koch curve.

Remark 68. *The Hausdorff distance between two compact sets $A, F \subset \mathbb{R}^2$ is defined by*

$$\text{dist}_{\mathcal{H}}(A, F) := \max\left\{\sup_{y\in A}\inf_{x\in F}|x-y|, \sup_{y\in F}\inf_{x\in A}|x-y|\right\}.$$

In addition to metrizing the collection of all non-empty compact sets, the Hausdorff distance generates a topology on the collection of non-empty compact sets that is complete, since $\mathbb{R}^2$ is complete.

Corollary 69. *For any line segment $I \subset \mathbb{R}^2$ and positive integer n*

$$\mathcal{H}^1\left(\mathcal{P}_n(I)\right) = \frac{2}{3}|I| + \left(\frac{\sec(\alpha)+2}{3}\right)^{n-1}\frac{\sec(\alpha)}{3}|I|. \tag{4.10}$$

Moreover, if $\alpha \leq \pi/3$*,*

$$\mathrm{dist}_{\mathcal{H}}(I, P^n(I)) \leq \frac{\tan(\alpha)}{12}|I|. \tag{4.11}$$

Proof. Equations (4.7) and (4.8) verify (4.10). Indeed,

$$\mathcal{H}^1\left(P^{n-1}(S(I))\right) = \left(\frac{\sec(\alpha)+2}{3}\right)^{n-1}\mathcal{H}^1(S(I)) = \left(\frac{\sec(\alpha)+2}{3}\right)^{n-1}\frac{\sec(\alpha)}{3}|I|.$$

The restriction to $\alpha \leq \pi/3$ ensures the longest line segment of $P^i(I)$ has length at most 3^{-i}. Consequently, (4.6) guarantees

$$\mathrm{dist}_{\mathcal{H}}(P^n(I), I) \leq \sum_{i=1}^{n}\mathrm{dist}_{\mathcal{H}}(P^i(I), P^{i-1}(I)) \leq \sum_{i=1}^{n} 3^{-i}\mathrm{height}(S(I)) \leq \frac{\tan(\alpha)}{12}|I|.$$

□

We now define a sequence of sets E_k which will be instrumental in defining the “base set” E_∞ in our construction.

Definition 70. Now, we let n be a natural number to be chosen later and $E_0 = I = [0,1]\times\{0\}$. We define $E_1 = \mathcal{P}_n(I)$. For $k \geq 2$ inductively define

$$E_k = \mathcal{P}_{kn}\left(\left[0, 3^{-(k-1)}\right]\times\{0\}\right)\bigcup\left(\left\{\left[3^{-(k-1)}, 1\right]\times\mathbb{R}\right\}\cap E_{k-1}\right). \tag{4.12}$$

Notably, for all integers j the operation $\mathcal{P}_j$ applied to $[0, 3^{-(k-1)}]\times\{0\}$ leaves the segment $[0, 3^{-k}]\times\{0\}$ untouched. Consequently, the sequence of sets $\{E_k\}$ are defined by replacing the “next” triadic interval with a scaled approximation of the Koch snowflake. The fact that each triadic strip $[3^{-k}, 3^{-(k-1)}]\times\mathbb{R}$ is only modified once in the sequence of sets E_k is ensures the Hausdorff dimension of the final set remains 1.

Lemma 71 (Base Set)**.** *Fix $\alpha \leq \pi/3$ and any integer n satisfying*[1]

$$3^{-1}\left(\frac{\sec(\alpha)+2}{3}\right)^{n} < 1 < 3^{-1}\left(\frac{\sec(\alpha)+2}{3}\right)^{2n}. \tag{4.13}$$

Then the sequence of sets E_k from (4.12) *converge to a compact and connected Borel set E_∞ in the Hausdorff topology on compact subsets. Furthermore, E_∞ satisfies:*

1. *$\mathcal{H}^1(E_\infty) < \infty$*

2. *For all $\delta > 0$, $C_{E_\infty}(0,\delta) = +\infty$.*

Proof. Note that (4.11) ensure that $\operatorname{dist}_{\mathcal{H}}(E_{k+1}, E_k) \sim 3^{-k}$. In particular, $\{E_k\}$ is a Cauchy sequence in the Hausdorff topology. Hence, the existence of the limiting compact set E_∞ follows from completeness of the Hausdorff topology on compact sets, see Remark 68.

By construction each E_k is connected. In fact, since $E_k \cap \overline{B_{3^{-k}}} = [0, 3^{-k}] \times \{0\}$ it follows that $E_k \setminus B_{3^{-k}}(0)$ is connected for each k. Connectedness of E_∞ now follows since $E_\infty \setminus B_{3^{-k}} = E_k \setminus B_{3^{-k}}$. This demonstrates that outside every neighborhood of the origin E_∞ is connected. Consequently, E_∞ is connected.

To see that E_∞ has finite length we write the $\mathcal{H}^1$-measure of E_k as the measure of E_k outside $B_{3^{1-k}}$ plus the measure of E_k inside the ball $B_{3^{1-k}}$. The two key observations being $E_k \setminus B_{3^{1-k}} = E_{k-1} \setminus B_{3^{1-k}}$ and $\mathcal{H}^1(E_{k-1} \setminus B_{3^{1-k}}) = \mathcal{H}^1(E_{k-1}) - 3^{1-k}$. Indeed, (4.10) and these observations imply,

$$\begin{aligned}\mathcal{H}^1(E_k) &= \mathcal{H}^1(E_k \cap B_{3^{1-k}}) + \mathcal{H}^1(E_{k-1} \setminus B_{3^{1-k}}(0)) \\ &= \frac{2}{3}|[0, 3^{1-k}] \times \{0\}| + 3^{1-k}\left(\frac{\sec(\alpha)+2}{3}\right)^{nk-1}\frac{\sec(\alpha)}{3} + \left(\mathcal{H}^1(E_{k-1}) - 3^{-k}\right),\end{aligned}$$

or, equivalently

[1]Note that for instance, $\alpha = \pi/3$ and $n \in \{2, 3\}$ satisfies (4.13).

$$\mathcal{H}^1(E_k) - \mathcal{H}^1(E_{k-1}) = 3^{-k}\left[\left(\frac{\sec(\alpha)+2}{3}\right)^{nk}\sec(\alpha) - 1\right].$$

Since $\mathcal{H}^1(E_0) = 1$, iteration yields

$$\mathcal{H}^1(E_k) = 1 + \sum_{i=1}^{k} 3^{-i}\left[\left(\frac{\sec(\alpha)+2}{3}\right)^{ni}\sec(\alpha) - 1\right]. \tag{4.14}$$

In particular, $\lim_{k\to\infty}\mathcal{H}^1(E_k) < \infty$ whenever n satisfies the lower bound from (4.13). Moreover $\mathcal{H}^1(E_\infty) = \lim_{k\to\infty}\mathcal{H}^1(E_k)$ since for all $j \geq k$,

$$\mathcal{H}^1\left(E_j \Delta E_k\right) \leq 2\sum_{i=k+1}^{\infty} 3^{-i}\left(\frac{\sec(\alpha)+2}{3}\right)^{ni}\sec(\alpha),$$

which decays to zero as $k \to \infty$. Hence, (4.14) holds for E_∞ and $0 < \mathcal{H}^1(E_\infty) < \infty$.

It only remains to show $C_{E_\infty}(0,\delta) = +\infty$ for all $\delta > 0$. To this end, we first note that when $r = r(n,\alpha) = 3^{-1}\left(\frac{\sec(\alpha)+2}{3}\right)^n$,

$$\mathcal{H}^1(E_\infty \cap B_{3^{-k}}(0)) = 3^{-k} + \sec(\alpha)\frac{r^{k+1}}{1-r} - \frac{3^{-(k+1)}}{1-3^{-1}}. \tag{4.15}$$

Indeed, by (4.14) and the trick used to prove (4.14)

$$\mathcal{H}^1(E_\infty \cap B_{3^{-k}}(0)) = 3^{-k} + \sum_{i=k+1}^{\infty} 3^{-i}\left[\sec(\alpha)\left(\frac{\sec(\alpha)+2}{3}\right)^{ni} - 1\right].$$

Claim: With τ as in Proposition 66 and $\alpha \leq \pi/3$, there exists a constant c_1 and integer j_0 independent of k such that for any line L, and all k such that $nk - 1 - j_0 \geq 0$,

$$\mathcal{H}^1\left(\left(E_\infty \setminus B_{\frac{\tau}{2\cdot 3^k}}(L)\right) \cap B_{3^{-k}}\right) \geq c_1 3^{-k}\left(\frac{\sec(\alpha)+2}{3}\right)^{nk-1-j_0}. \tag{4.16}$$

Proof of Claim. Writing $I' = [0,1]\times\{0\}$, we will in fact scale by 3^k and show the stronger result that

$$\mathcal{H}^1\left(\left(\mathcal{P}_{nk}(I')\setminus B_{\frac{\tau}{2\cdot 3^0}}(L)\right)\cap B_{3^0}\right)\geq c_1 3^0\left(\frac{\sec(\alpha)+2}{3}\right)^{nk-1-j_0}|I'|.$$

To do so, we find a line segment $J\subset S(I')\setminus B_\tau(L)$ such that J has an endpoint in common with one of the two line segments of $S(I')$ and $|J|=3^{-j_0}\mathcal{H}^1(S(I'))/2$, where j_0 to be chosen later is independent of L. This specific choice of length and endpoint ensure that $P^{nk-1-j_0}(J)\subset\mathcal{P}_{nk}(I')$. Moreover, the choice of j_0 will both guarantee that $|J|$ is large enough and that $P^{nk-1-j_0}(J)$ remains outside of $B_{\tau/2}(L)$, hence verifying the claim.

To find J, we note that the simple shape of $S(I')$ guarantees that $S(I')\setminus B_\tau(L)$ has at most 4 maximal line segments. Hence, there exists a maximal line segment $K_L\subset S(I')\setminus B_\tau(L)$ with $\mathcal{H}^1(K_L)\geq\frac{1}{4}\mathcal{H}^1\left(S(I')\setminus B_\tau(L)\right)$. If K_L is parallel to L let x_L denote either endpoint of K_L. Otherwise, let x_L denote the unique endpoint of K_L that is not contained in $\overline{B_\tau(L)}$. Define J to be the unique subset of K_L of length $3^{-j_0}\frac{\sec(\alpha)}{6}|I'|$ with endpoint x_L. Now, define j_0 as the smallest integer such that

$$3^{-j_0}<\min\left\{\frac{c_0}{4},\left(\frac{\tan(\alpha)}{12}\cdot\frac{\sec(\alpha)}{6}|I'|\right)^{-1}\frac{\tau}{2}\right\},$$

where c_0 is as in Proposition 66. The first condition ensures that $J\subset K_L$ and (4.9) guarantees that the first constraint on j_0 is independent of L and k. The second constraint combined with (4.7) and (4.11) ensure that $\mathrm{dist}_{\mathcal{H}}(P^{nk-1-j_0}(J),J)\leq\frac{\tau}{2}$. Moreover, choosing j_0 to be the smallest admissible integer guarantees that $|J|=3^{-j_0}\frac{\sec(\alpha)}{6}|I'|\geq c'|I'|$ where c' is independent of L and k. Finally, (4.8) completes the proof of the Claim since

$$\mathcal{H}^1\left(\mathcal{P}_{nk}(I')\setminus B_{\tau/2}(L)\right)\geq\mathcal{H}^1(P^{nk-1-j_0}(J))\geq c_1\left(\frac{\sec(\alpha)+2}{3}\right)^{nk-1-j_0}|I'|,$$

where c_1 depends only on α.

Whenever $nk-1-j_0 \geq 0$, (6.18) implies

$$\beta_{E_\infty}(0,3^{-k})^2 \geq \frac{1}{3^{-k}}\left(\frac{\frac{\tau}{2\cdot 3^k}}{3^{-k}}\right)^2\left(c_1 3^{-k}\left(\frac{\sec(\alpha)+2}{3}\right)^{nk-1-j_0}\right) = c_2\left(\frac{\sec(\alpha)+2}{3}\right)^{nk} \tag{4.17}$$

Fix $\delta > 0$ and any integer k_δ such that $3^{-k_\delta} < \delta$ and $nk_\delta - 1 - j_0 \geq 0$. Then, with $\mu = \mathcal{H}^1 \llcorner E_\infty$, repeated applications of Proposition 64, (4.17), and (4.15) yield

$$\begin{aligned}\int_{B_\delta(0)}\int_0^\delta \beta_\mu(x,r)^2\frac{dr}{r}d\mu(x) &\geq \ln(3)3^{-2}\sum_{k=k_\delta}^{\infty}\mu(B_{3^{-(k+2)}})\beta_\mu(0,3^{-(k+2)})^2\\ &\geq \ln(3)3^{-2}\sum_{k=k_\delta}^{\infty}\left(3^{-k}+\sec(\alpha)\frac{r^{k+1}}{1-r}-\frac{3^{-(k+1)}}{1-3^{-1}}\right)\left(c_2\left(\frac{2+\sec(\alpha)}{3}\right)^{nk}\right).\end{aligned}$$

Due to the lower bound in (4.13), this sum diverges if and only if

$$\sum_{k=k_\delta}^{\infty}\left[\sec(\alpha)\frac{r^{k+1}}{1-r}-\frac{1}{3^{k+1}-3^k}\right]\left(\frac{2+\sec(\alpha)}{3}\right)^{nk} = \sum_{k=k_\delta}^{\infty}\left[\sec(\alpha)\frac{3^k r^{2k+1}}{1-r}-\frac{r^k}{3-1}\right]$$

diverges. Since the lower bound in (4.13) ensures $r < 1$, this diverges if and only if $\sum_{k=k_\delta}^{\infty}(3r^2)^k$ diverges which is equivalent to the upper bound in (4.13). □

Theorem 72. *There exists a connected set, $K_0 \subset \mathbb{R}^2$ of finite $\mathcal{H}^1$-measure such that for any $x \in K_0$ and $\delta > 0$*

$$C_{K_0}(x,\delta) = \infty.$$

Proof of Theorem 61. . Let $\{r_i\}_{i=1}^{\infty}$ be a sequence of positive numbers such that $\sum_i r_i \leq 1$. Let $E^{x,r} \subset \mathbb{R}^2$ be the set $E^{x,r} = rE_\infty + x$. We construct K_0 as the union of a countable collection of nested sets $\{\Gamma_i\}$.

Let $\Gamma_0 = E_\infty$. Now, let $\{x_{1,j}\}_{j=1}^{N_1}$ be a maximal 2^{-1-1}-separated net in Γ_0. Let

$$\Gamma_1 = \Gamma_0 \cup \bigcup_{j=1}^{N_1} E^{x_{1,j},\frac{r_1}{N_1}}.$$

Suppose that we have defined Γ_{i-1}, some positive integers $\{N_\ell\}_{\ell=1}^{i-1}$ and a collection of points $\{x_{\ell,j} \in \Gamma_{i-2} \mid 1 \le \ell \le i-1, 1 \le j \le N_\ell\}$ that form a maximal $2^{-(i-1)-1}$-separated net for Γ_{i-2}. Then choose $N_i \in \mathbb{N}$ and points $\{x_{i,j}\}_{1 \le j \le N_i} \subset \Gamma_{i-1}$ so that $\{x_{\ell,j} \in \Gamma_{i-1} \mid 1 \le \ell \le i, 1 \le j \le N_\ell\}$ is a maximal 2^{-i-1}-separated net in Γ_{i-1}. Then define Γ_i by

$$\Gamma_i = \Gamma_{i-1} \cup \left(\bigcup_{j=1}^{N_j} E^{x_{i,j} \frac{r_i}{N_i}} \right).$$

We claim that $K_0 = \cup_{i=0}^{\infty} \Gamma_i$ is the desired set. First note that since each Γ_i is countably rectifiable, then K_0 is countably rectifiable. Moreover, $\{x_{i,j}\}_{j=1}^{N_i} \subset \Gamma_{i-1}$ for all i ensures K_0 inherits connectivity from E_∞. Furthermore, since $\{\Gamma_i\}$ is a nested sequence increasing to K_0 and $\sum_i r_i \le 1$,

$$\mathcal{H}^1(K_0) = \mathcal{H}^1\left(E_\infty \cup \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{N_i} E^{x_{i,j}, \frac{r_i}{N_i}} \right) \le \mathcal{H}^1(E_\infty) \left(1 + \sum_{i=1}^{\infty} r_i \right) \le 2\mathcal{H}^1(E_\infty).$$

It only remains to show that for $x \in K_0$ and $\delta > 0$ that $C_{K_0}(x, \delta) = \infty$. To this end, fix $x \in K_0$, and $\delta > 0$. By definition of K_0, there exists ℓ_0 such that $x \in \Gamma_{\ell_0}$. Then, by the net property of the points $\{x_{i,j}\}$, it follows that for $\ell - 1 \ge \ell_0$ large enough that $2^{-\ell-1} < \delta/4$, there exists $i \le \ell$ with $x_{i,j} \in \Gamma_{\ell-1} \cap B(x, \delta/2) \subset K_0 \cap B(x, \delta/2)$. Writing $\mu = \mathcal{H}^1 \llcorner K_0$ and $\mu_{i,j} = \mathcal{H}^1 \llcorner E^{x_{i,j}, \frac{r_i}{N_i}}$ it follows from monotonicity of the integral that

$$\int_{B_\delta(x)} \int_0^\delta \beta_{K_0;2}(y,r)^2 \frac{dr}{r} d\mu(y) \ge \int_{B_{\delta/2}(x_{i,j})} \int_0^{\delta/2} \beta_{\mu_{i,j};2}(y,r) \frac{dr}{r} d\mu_{i,j}(y), \tag{4.18}$$

or equivalently $C_{K_0}(x, \delta) \ge C_{\mu_{i,j}}(x_{i,j}, \delta/2)$. Recalling that $E^{z,t} = tE_\infty + z$, we use (4.18), Proposition 64(2), and Lemma 71 to conclude

$$C_{K_0}(x, \delta) \ge C_{E^{x_{i,j}, \frac{r_i}{N_i}}}\left(x_{i,j}, \frac{\delta}{2} \right) = \frac{r_i}{N_i} C_{E_\infty}\left(0, \frac{\delta N_i}{2r_i} \right) = \infty.$$

Since $x \in K_0$ and $\delta > 0$ are arbitrary this finishes the proof. $\square$

Remark 73. *Since K_0 from Theorem 61 is connected, $\mathcal{H}^1(\overline{K_0}) = \mathcal{H}^1(K_0) < \infty$ and $\overline{K_0}$ is compact, see [64, Lemma 3.4, 3.5]. Thus $\overline{K_0}$ is a rectifiable curve by Ważewski's theorem, see [64, Lemma 3.7] or [1, Theorem 4.4].*

The authors thank Matthew Badger for pointing out that $\overline{K_0}$ coincides with the Hausdorff-limit of $\{\Gamma_i\}$. So, Gołab's semi-continuity theorem and Ważewski's theorem suffice to ensure $\overline{K_0}$ is a rectifiable curve. See, for instance, [1] or [31] for relevant theorem statements.

4.3 Construction of A_0

The unique compact set fixed by the iterated function system (IFS),

$$\{F_{i,j} : \mathbb{R}^2 \to \mathbb{R}^2 | F_{i,j}(E) = 2^{-2}(E + (i,j)), \quad i,j \in \{0,3\}\}$$

is called the 4-corner Cantor set, $\mathcal{C}$. The 4-corner Cantor set is an Ahlfors regular set with positive and finite $\mathcal{H}^1$-measure and is purely unrectificable. That is, $\mathcal{H}^1(\mathcal{C} \cap f(\mathbb{R})) = 0$ for all Lipschitz functions $f : \mathbb{R} \to \mathbb{R}^2$.

Typically, one approximates the 4-corner Cantor set by beginning with the "initial set" $[0,1]^2$ in their iteration scheme. However, our motivation for the construction of A_0 arises from considering the initial set $[0,1) \times \{0\}$. Beginning with a 1-dimensional set allows every approximating set to have positive and finite $\mathcal{H}^1$-measure. This is also critical to produce estimates on the β-numbers of each approximation.

The general strategy for producing the desired set A_0 in Theorem 62 is as follows. We produce a base set Σ_0 such that within successive tetradic strips $[2^{-2i}, 2^{-2i+2}] \times \mathbb{R}$ the set $\Sigma_0 \cap ([2^{-2i}, 2^{-2i+2}] \times \mathbb{R})$ is a scaled version of a higher-iteration approximation to the 4-corner Cantor set. This allows for precise control on the β-numbers in every neighborhood of the origin. Then, following the strategy for Theorem 61 we carefully iterate this set "on itself" in a dense way, taking care to preserve Ahlfors regularity.

4.3.1 Approximations to the 4-corner Cantor set

Consider the following sequence of approximations to the 4-corner cantor set, by sets of positive and finite $\mathcal{H}^1$-measure.

Let $E_0 = [0,1) \times \{0\}$ and inductively define

$$E_k = \sum_{(i,j)\in\{0,3\}^2} p_{ij} + 2^{-2}E_{k-1} \quad \text{where} \quad p_{ij} = \left(\frac{i}{2^2}, \frac{j}{2^2}\right). \tag{4.19}$$

The word similarity is used to refer to any mapping that can be written as a composition of scalings, rotations, reflections, and translations. Throughout the rest of the paper, we say that two sets are similar if one is the image of the other by a similarity. In reality the similarities we discuss can always be written as a scaling and translation, as in (4.19).

We let Δ denote the collection of tetradic half-open cubes in $\mathbb{R}^2$, that is

$$\Delta = \{[a2^{-2k}, (a+1)2^{-2k}) \times [b2^{-2k}, (b+1)2^{-2k}) \mid a, b, k \in \mathbb{Z}\}.$$

For some $Q \in \Delta$, we let $\ell(Q)$ denote the sidelength of Q. We partition the tetradic cubes into cubes of fixed sidelength by defining $\Delta^i = \{Q \in \Delta \mid \ell(Q) = 2^{-2i}\}$.

In general, for a set $E \subset \mathbb{R}^2$ we respectively denote the *length of* E and the *height of* E by

$$\ell(E) = \text{diam}\{\pi_x(E)\} \quad \text{and} \quad h(E) = \text{diam}\{\pi_y(E)\}$$

where π_x and π_y denote the orthogonal projection onto the horizontal and vertical axes. In particular, for a cube Q with axis-parallel sides, this notion of length coincides with the cubes sidelength. Hence no confusion with the earlier convention that $\ell(Q)$ is the sidelength of Q will arise.

Definition 74 (Clusters and sub-clusters)**.** Any set which is similar to any E_k or $E_k \cup [0,1) \times \{0\}$ for $k \in \mathbb{N}$ will be called a *cluster*.

Moreover, for fixed $k \in \mathbb{N}$, we will call E_k the 0*th sub-cluster of* E_k and the 2^{2k} line segments that make up E_k are called the *kth -subclusters of E_k*. For $\ell \in \{1, \ldots, k-1\}$, the $2^{2\ell}$-sets contained in E_k which are similar to $E_{k-\ell}$ are called the *ℓth sub-clusters of E_k.*

Definition 75 (Root points)**.** We associate to each cluster and each cube a root point. The *root point* of a cluster E is the lower-most and left-most point in the cluster. Since a sub-cluster is itself a cluster, the notion of a root point extends to sub-clusters. For a cluster E, we let x_E denote its root point. For a tetradic cube $Q \in \Delta$ we let x_Q denote the lower-most and left-most point of Q and call x_Q *the root point of Q.*

Proposition 76. *For fixed non-negative integer k, the set E_k has the following properties.*

1. *Each E_k is a finite union of 2^{2k} intervals each of length 2^{-2k}. In particular, $\mathcal{H}^1(E_k) = 1$ and E_k is countably 1-rectifiable. Moreover, each connected component I of E_k has $\partial I \subset \ell(I)\mathbb{Z}^2 = 2^{-2k}\mathbb{Z}^2$ and consequently is contained in a line $\mathbb{R} \times \{a2^{-2k}\}$ for some $a \in \mathbb{N}_0$.*

2. *If $j \geq 0$ is an integer and if $Q \in \Delta^j$ is such that $Q \cap E_k$ is non-empty, then*

$$Q \cap E_k = \begin{cases} x_Q + [0, \ell(Q)) \times \{0\} & j \geq k \\ x_Q + 2^{-2j} E_{k-j} & j \leq k \end{cases} \tag{4.20}$$

3. *Each E_k is Ahlfors regular with regularity constant independent of k.*

4. *For $0 \leq j \leq k$ an integer, each jth subcluster of E_k has $\mathcal{H}^1$-measure 2^{-2j}.*

5. *For $1 \leq j \leq k$ an integer, the jth subclusters of E_k are $2 \cdot 2^{-2j}$-separated horizontally and at least $2 \cdot 2^{-2j}$-separated vertically. In fact, they are $\left(3 - \frac{3}{4}\sum_{i=1}^{k-j} 2^{-2i}\right) \cdot 2^{-2j}$-separated vertically.*

6. *If $J \subset E_k$ is a connected component, then J is a vertical distance of $3 \cdot 2^{-2k}$ from the nearest connected component J' of E_k.*

7. *There exists a universal constant $c > 0$ such that if $k \geq 2$ and $\mu_k = \mathcal{H}^1 \llcorner E_k$, then for all $x \in E_k$,*

$$\int_{6\cdot 2^{-2k}}^{1} \beta_{\mu_k}(x,r)^2 \frac{dr}{r} \geq c(k-2)$$

Proof. (1) follows immediately from (4.19) since each $p_{ij} \in 2^{-2}\mathbb{Z}^2$.

To see (2), we first note that the case $j = 0$ is clear for any $k \in \mathbb{N}$. Further, the case $k = 0$ is clear for all $j \in \mathbb{N}$. To procede inductively suppose that (4.20) holds for all $k \in \mathbb{N}$ when $j = \ell - 1$. We will show it holds for all $k \in \mathbb{N}$ when $j = \ell$. Indeed, suppose that $Q \in \Delta^\ell$ has non-empty intersection with E_ℓ. Let x_Q be the root of Q. Choose $p \in \{p_{ij}\}_{(i,j)\in\{0,3\}^2}$ such that $Q \subset p + [0, 2^{-2})^2$. Then, $4(Q \cap E_k - p) = (4Q - 4p) \cap (4E_k - 4p) = \tilde{Q} \cap E_{k-1}$ where $\tilde{Q} := 4Q - 4p \in \Delta^{\ell-1}$. By the inductive assumption,

$$\tilde{Q} \cap E_{k-1} = \begin{cases} x_{\tilde{Q}} + [0, \ell(\tilde{Q})) \times \{0\} & \ell - 1 \geq k - 1 \\ x_{\tilde{Q}} + 2^{-2(i-1)} E_{(\ell-1)-(i-1)} & \ell - 1 \leq k - 1. \end{cases}$$

Translating and scaling this back to what this means about $Q \cap E_k$ verifies the induction.

(3) follows from (1) and (2) since these imply that $\frac{\mathcal{H}^1(Q\cap E_k)}{\ell(Q)} = 1$ for tetradic cubes Q with $\ell(Q) \leq 1$ that intersect E_k. This suffices since any ball contains a tetradic cube of comparable sidelength and is contained in 4^2 tetradic cubes of comparable sidelength.

(4) is equivalent to showing that E_k is made of 2^{2k} intervals, each of length 2^{-2k}.

(5) The horizontal separation is verified by an argument similar to the vertical separation. For the vertical separation, we only verify that the vertical separation is at least $2 \cdot 2^{-2j}$. Indeed, this follows since E_ℓ is contained in the horizontal strips $\mathbb{R} \times [0, 1/4] \cup [3/4, 1]$ for all ℓ. Then, the scaling from (4.19) ensures that the jth subclusters, which arise by applying (4.19) j times to the sets E_{k-j} are vertically $2 \cdot 2^{-2j} = \frac{1}{2} 2^{-2(j-1)}$-separated. The reason the height-bound can be improved, is because the jth subclusters are actually contained in smaller strips. See for instance, E_1, where the first subclusters are contained in lines, and E_2 where the first subclusters are contained in the strips $\mathbb{R} \times [0, \frac{3}{16}] \cup [\frac{12}{16}, \frac{15}{16}]$.

(6) follows from the fact that vertically-closest connected components in E_k come from the connected components of E_1 which are $3 \cdot 2^{-2}$ separated. After being scaled by 2^{-2} in (4.19) another $(k-1)$ times the separation is reduced to a distance of $3 \cdot 2^{-2k}$ as claimed. This coincides with the precise formula in (5) and could be considered as a base case for induction on j for the interested reader.

(7) Throughout the proof of (7), we fix integers $1 \leq j < k$ and $k \geq 2$.

Claim 1: For all $x \in E_k$ there exists some $x' \in E_j$ with

$$\operatorname{dist}(x, x') \leq 2^{-2j} \tag{4.21}$$

Proof of Claim 1. Note that the scaling in (4.19) ensures that for some ℓ, we know that every $x \in E_{\ell+1}$ is within a distance $3 \cdot 2^{-2(\ell+1)}$ of a point in E_ℓ. Iterating verifies the claim by showing for $x \in E_k$ there exists $x' \in E_j$ such that

$$\operatorname{dist}(x, x') \leq \sum_{\ell=j+1}^{k} 3 \cdot 2^{-2\ell} \leq 3 \sum_{\ell=j+1}^{\infty} 2^{-2\ell} = 4 \cdot 2^{-2(j+1)}.$$

Claim 2: There exists c independent of j such that for all $5 \cdot 2^{-2j} \leq r \leq 11 \cdot 2^{-2j}$ and all $x' \in E_j$,

$$\beta^1_{\mu_j;2}(x', r)^2 \geq c$$

Proof of Claim 2. Let $J \subset E_j$ be the connected component containing x'. By (4)-(6) of this proposition, it follows that for $r \geq 5 \cdot 2^{-2j} = \sqrt{(3 \cdot 2^{-2j})^2 + (4 \cdot 2^{-2j})^2}$, the ball $B_r(x')$ contains J and 3 other connected components of E_j. Consequently, there are two horizontal lines, L^u and L^d, such that $B_r(x') \cap \left(L^u \cup L^d\right)$ contains at least 4 connected components of E_j. Part (1) of this proposition ensures,

$$\min\{\mu_j\left(L^u \cap B_r(x')\right), \mu_j\left(L^d \cap B_r(x')\right)\} \geq 2 \cdot 2^{-2j}. \tag{4.22}$$

Moreover, part (6) ensures that the distance between L^u and L^d is $3 \cdot 2^{-2j}$, which combined with (4.22) forces that any line L satisfies,

$$\mu_j\left(\left\{y \in B_r(x') \middle| \operatorname{dist}(y,L) \geq 3 \cdot 2^{-2j-1}\right\}\right) \geq 2 \cdot 2^{-2j}. \tag{4.23}$$

Finally, recalling $5 \cdot 2^{-2j} \leq r \leq 11 \cdot 2^{-2j}$, (4.23) implies

$$\inf_L \int_{B_r(x')} \left(\frac{\operatorname{dist}(y,L)}{r}\right)^2 \frac{d\mu_j(y)}{r} \geq \left(\frac{\frac{3 \cdot 2^{-2j}}{2}}{r}\right)^2 \left(\frac{2 \cdot 2^{-2j}}{r}\right) \geq c$$

which verifies Claim 2.

Claim 3: There exists c' such that for all $x \in E_k$ and all integers $1 \leq j < k$ and ρ such that $6 \cdot 2^{-2j} \leq \rho \leq 12 \cdot 2^{-2j}$,

$$\beta^1_{\mu_k;2}(x,\rho)^2 \geq c'. \tag{4.24}$$

Proof of Claim 3. Claim 1 ensures that for all $5 \cdot 2^{-2j} \leq r \leq 11 \cdot 2^{-2j}$ there exists $x' \in E_j$ such that $B_r(x') \subset B_\rho(x)$. As in Claim 2, fix lines L^d and L^u such that $B_r(x) \cap \left(L^u \cup L^d\right)$ contains at least 4 connected components of E_j. Choose a so that $L^d = \mathbb{R} \times \{a\}$ and $L^u = \{a + (0, 3 \cdot 2^{-2j})\} + \mathbb{R} \times \{0\}$. Moreover, suppose the left-most connected component of L^u has right-most endpoint with x-value equal to c_1. Define $L_v = \{c_1 + 2^{-2j}\} \times \mathbb{R}$ and $L_h = a + 2^{-2j}$. By Proposition 76(5,6), the neighborhoods $N_v = B_{2^{-2j}}(L_v)$ and $N_h = B_{2^{-2j}}(L_h)$ are disjoint from E_ℓ for all $\ell \geq j$. See Figure 4.5.

Consequently, for any line L the nieghborhood $B_{2^{-2j-1}}(L)$ can intersect at most 4 of the "quadrants" made by the neighborhoods of N_v and N_L. Making a generous estimate since the ball may cut-off part of one of the quadrants in Figure 4.5, we conclude

$$\mu_k\left(\{y \in B_r(x') \mid \operatorname{dist}(y,L) \geq 2^{-2j-2}\}\right) \geq 2^{-2j-2} \tag{4.25}$$

where the measure-bound comes Proposition 76(1). Since $B_r(x') \subset B_\rho(x)$ and $1 \leq \frac{\rho}{r} \leq C < \infty$, Claim 3 follows from (4.25) analogously to how Claim 2 followed from (4.23).

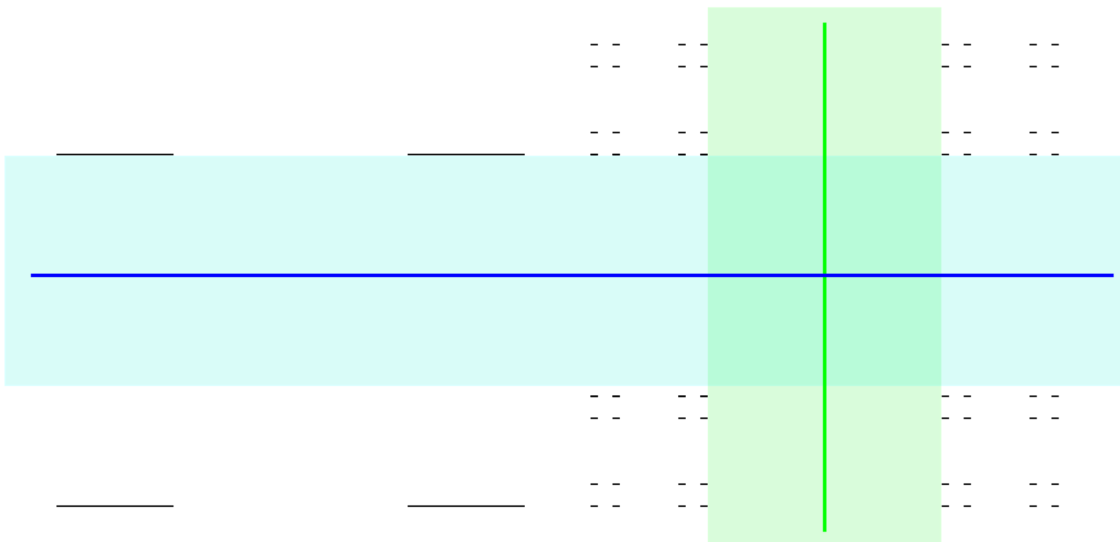

Figure 4.5: When $j = k-2$, the picture displays a subclusters of equal length for E_j and E_k on the left and right respectively. In E_k, the line L_v and its neighborhood N_v are in green, whereas the line L_h and its neighborhood N_h are drawn where it would pass through both E_j and E_k

Finally, we verify (7) because

$$\int_{6\cdot 2^{-2j}}^{1} \beta_{\mu_k}(x,\rho)^2 \frac{d\rho}{\rho} \geq \sum_{j=2}^{k} \int_{6\cdot 2^{-2j}}^{11\cdot 2^{-2j}} c' \frac{d\rho}{\rho} = c(k-2).$$

□

We construct the base set Σ_0 from approximations to the 4-corner Cantor set by first defining

$$E(n) := (2^{-2n}, 0) + 2^{-2n} E_{2^{2n}} \quad \text{and} \quad \Sigma_0 := \bigcup_n E(n) \cup ([0,1) \times \{0\}) . \tag{4.26}$$

Proposition 77. Σ_0 *has the following properties.*

1. $0 < \mathcal{H}^1(\Sigma_0) < \infty$ *and* Σ_0 *is countably* 1*-rectifiable.*

2. *If* $j \geq 0$ *is an integer and* $Q \in \Delta^j$ *is such that* $Q \cap \Sigma_0 \neq \emptyset$*, then*

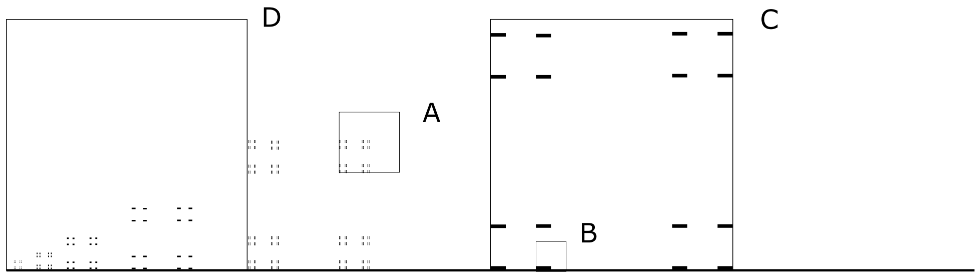

Figure 4.6: Here we see Σ_0 and several examples of $Q \in \Delta$. The cube D illustrates the first case in Equation (4.27). The cube A illustrates an example of the second case in Equation (4.27). The cubes B and C illustrate examples of the last case in Equation (4.27).

$$Q \cap \Sigma_0 = \begin{cases} \Sigma_0 \cap [0, \ell(Q))^2 & x_Q = (0,0) \\ x_Q + 2^{-2j} E_k \textit{ for some } k & x_Q \neq (0,0) \textit{ and } \pi_y(x_Q) \neq 0 \\ x_Q + 2^{-2j} E_k \cup [0, \ell(E_k)) \times \{0\} & x_Q \neq (0,0) \textit{ and } \pi_y(x_Q) = 0. \end{cases} \tag{4.27}$$

3. $C_{\Sigma_0}(0, \delta) = +\infty$ *for all* $\delta > 0$.

Proof. (1) Σ_0 has positive and finite mass due to Proposition 76(1) and the geometric scaling in (4.26). It is also the countable union of countably 1-rectifiable sets by Proposition 76(1).

(2) The case when $x_Q = (0,0)$ is clear. Suppose $x_Q \neq (0,0)$. There exists unique a, b such that

$$x_Q = \left(a2^{-2j}, b2^{-2j}\right). \tag{4.28}$$

If $j = 0$, $Q \cap \Sigma_0 \neq \emptyset$, and $\Sigma_0 \subset [0,1)^2$ forces $a = b = 0$. Therefore, $j \geq 1$. Since $h(E_{2^{2n}}) < \ell(E_{2^{2n}})$ and the $E(n)$ only use a translation in the positive horizontal direction of $E_{2^{2n}}$ and a homogeneous scaling, it follows that $\Sigma_0 \cap Q \neq \emptyset$ implies $0 \leq b < a$ so that $a \geq 1$. Since, $\ell(Q) = 2^{-2j}$ it follows that $a2^{-2j} \geq \ell(Q)$. Comparing the translation and scaling sizes in (4.19), $a \geq 2^{2j}\ell(Q)$ implies

$$\Sigma_0 \cap Q = \begin{cases} Q \cap E(n) & b \geq 1 \\ Q \cap (E(n) \cup [0, \ell(E(n))) \times \{b\}) & b = 0 \end{cases} \tag{4.29}$$

for some specific $n \leq j$. For simplicity of writing, assume we are in the first case. Then, $2^{2n}\left(Q \cap E(n) - (2^{-2n}, 0)\right) = \left(2^{2n}\left(Q - (2^{-2n}, 0)\right)\right) \cap E_{2^{2n}}$ or equivalently

$$Q \cap E(n) = (2^{-2n}, 0) + 2^{-2n}\left(2^{2n}\left(Q - \left(2^{-2n}, 0\right)\right) \cap E_{2^{2n}}\right). \tag{4.30}$$

In light of (4.30), it follows that (4.20) implies the 2nd case of (4.27) since $2^{2n}(Q-(2^{-2n}, 0)) \in \Delta^{j-n}$ and $n \leq j$. Analogously the $b = 0$ case corresponds to the 3rd case of (4.27).

(3) Fix $\delta > 0$. Choose N large enough that $11 \cdot 2^{-2N} < \delta/2$. In particular, for all $n \geq N$, $E(n) \subset B_\delta(0)$. Then, with $\mu = \mathcal{H}^1 \llcorner \Sigma_0$ and $\mu_n = \mathcal{H}^1 \llcorner E(n)$, it follows from Proposition 76 (1,7), Proposition 64 (2), and the scaling in (4.26) that

$$C_{\Sigma_0}(0, \delta) \geq \sum_{n \geq N} \int_{E(n)} \int_0^{2^{-2n}} \beta^1_{\mu_n;2}(x, r)^2 \frac{dr}{r} d\mu_n(x) \geq \sum_{n \geq N} c(2^{2n} - 2)\mathcal{H}^1(E(n)),$$

which diverges and completes the proof. □

We wish to iterate Σ_0 densely along itself while being careful to maintain Ahlfors upper- and lower-regularity. This is attained by scaling, and being careful where we iterate.

Definition 78 (Tail points)**.** We say a point y is a *tail point of* E if $0 < \mathcal{H}^1(E) < \infty$ and there exists a tetradic number r and $\delta > 0$ such that

$$y + r\Sigma_0 \cap B_\delta \subseteq E.$$

Note, if $y \in B_\delta(x)$ is a tail point of a set E, then $C_E(x, \delta) \equiv \infty$. See Claim 1 from the proof of Theorem 62.

Definition 79 (Iterative construction)**.** Let Σ_0 be as above. Supposing that Σ_{i-1} has been defined, we define a (possibly empty) special collection of tetradic points,

$$D^i = \left\{ x \in 2^{-2i}\mathbb{Z}^2 \middle| \left(x + [0, 2^{-2i})^2\right) \cap \Sigma_{i-1} = x + [0, 2^{-2i}) \times \{0\} \right\}, \tag{4.31}$$

and define Σ_i by

$$\Sigma_i = \Sigma_{i-1} \cup \left\{ \bigcup_{x \in D^i} x + 2^{-8i}\Sigma_0 \right\}. \tag{4.32}$$

Define,

$$A_0 = \bigcup_{j \in \mathbb{N}} \Sigma_j. \tag{4.33}$$

Proposition 80. *The sets* $\{\Sigma_j\}_{j=0}^{\infty}$ *and* $\{D^j\}_{j=1}^{\infty}$ *as in Definition 79 have the following properties:*

(1) $\Sigma_{j-1} \subset \Sigma_j$ *for all* $j \geq 1$*,*

(2) Σ_j *is contained in countably many horizontal line segments with tetradic heights.*

(3) There are infinitely many j *so that* D^j *is non-empty.*

(4) If I *is a connected component of* Σ_j *then* $\partial I \subset \ell(I)\mathbb{Z}^2$*.*

(5) Σ_j *contains no connected component of length at least* 2^{-2j} *that contain no tail point.*

Proof. Indeed, (1) follows from (4.32).

(2) Follows by induction. For Σ_0 it follows from Proposition 76 (1) combined with the scaling in (4.26). For general Σ_j induction holds due to the fact that each scaled copy of Σ_0 in (4.32) has a tail point on the dyadic lattice D^i which is coarser than the tetradic scaling factor of Σ_0.

(3) folows from (2). (5) follows from (4) and the definition of D^j in (4.31).

(4) If I is a connected component of Σ_j then there exists $y \in D^i$ some $i \leq j$ such that I is a connected component of $y + 2^{-8i}\Sigma_0$. But then, $2^{8i}(I - y)$ is a connected component of Σ_0. Since $y \in 2^{-2i}\mathbb{Z}^2$, Propositions 76(1) and 77(2) ensure $\partial\left(2^{8i}(I - y)\right) \in 2^{8i}\ell(I)\mathbb{Z}^2$ which verifies (4).

□

Definition 81 (Associated cubes)**.** Any cluster (or subcluster) E has associated to it the dyadic cube $Q_E = x_E + [0, \ell(E))^2$. In particular, by Proposition 76 (5) it follows that if clusters E, E' are disjoint with $\ell(E) = \ell(E')$, then $Q_E, Q_{E'}$ are disjoint cubes. Moreover, for some cluster E, the root point of Q_E and the root point of E coincide.

Definition 82. We associate to the base set Σ_0 the following family of cubes

$$\mathcal{Q}_{\Sigma_0} = \left\{[0, 2^{-2i})^2 : i \geq 0\right\} \cup \{Q_E : E \text{ is a subcluster of } E(n) \subset \Sigma_0, n \geq 1 \ \} \tag{4.34}$$

By similarity, for any $y \in D^i$ we associate to $y + 2^{-8i}\Sigma_0$ the family of cubes

$$\mathcal{Q}_y = \left(y + 2^{-8i}\mathcal{Q}_{\Sigma_0}\right) \bigcup \left(y + \{[0, 2^{-2k})^2 : i \leq k\}\right). \tag{4.35}$$

We will let

$$\mathcal{Q} = \cup_{i \geq 0} \cup_{y \in D^i} \mathcal{Q}_y \tag{4.36}$$

which we stratify by scale in the following sense

$$\mathcal{Q}^i = \{Q \in \mathcal{Q} \mid \ell(Q) = 2^{-2i}\} \tag{4.37}$$

and we enumerate the elements $\mathcal{Q}^i$ so that

$$\mathcal{Q}^i = \{Q^i_j\}_{j=1}^{N(i)}. \tag{4.38}$$

Finally, for $Q \in \mathcal{Q}$ and any positive integer ℓ we let $\mathcal{C}_\ell(Q) = \{Q' \in \mathcal{Q} \mid Q' \subset Q, \ \ell(Q') = 2^{-2\ell}\ell(Q)\}$, and call $\mathcal{C}_\ell(Q)$ the *ℓth descendent cubes of Q*.

Lemma 83. *For all $i \geq 0$ and all cubes, $Q^i_j \in \mathcal{Q}^i$, $\Sigma_i \cap Q^i_j$ is similar to one of the following:*

1. $\left(2^{-2k}\Sigma_0 \cup [0, 1) \times \{0\}\right) \cap [0, 1)^2$ *for some integer k.*

2. $E \cap Q_E$ *for some sub-cluster $E \subset E(n)$ for some integer $n \geq 1$*

This follows immediately from the explicit definition of cubes.

Lemma 84. *$\mathcal{Q}^j \subset \Delta^j$ and for all $Q \in \Delta^j$, then either $Q \cap \Sigma_j = \emptyset$ or $Q \in \mathcal{Q}_j$.*

This follows from an induction argument similar to the proofs of Propositions 76 (1) and 77 (2). The key observation in the induction is that the scaling in (4.32) ensures that all tail points added in the jth stage have root points in tetradic lattices that are coarser than the length of the scaled copy of Σ_0 being added.

Corollary 85. *The cubes $\mathcal{Q}$ have the following nice properties:*

1. *Each collection $\mathcal{Q}_i$ is a disjoint collection of cubes, and for any $Q \in \mathcal{Q}$ and any integer $\ell \geq 0$, $\mathcal{C}_\ell(Q)$ is a disjoint collection of subcubes of Q.*

2. *For all non-negative integers i and j,*

$$\Sigma_i \subseteq \cup_{Q \in \mathcal{Q}_j} Q \tag{4.39}$$

3. *In particular, for any $Q_0 \in \mathcal{Q}_i$*

$$\Sigma_i \cap Q_0 = \Sigma_i \bigcap \left(\cup_{Q \in \mathcal{C}_1(Q)} Q \right) \tag{4.40}$$

Proof of Theorem 62. By Lemma 77 (1), Σ_0 is 1-rectifiable, and A_0 is a countable union of scaled translations of Σ_0 so A_0 is 1-rectifiable.

Next, we show that A_0 is 1-Ahlfors regular. Indeed, it suffices to show that there exists $0 < c \leq C < \infty$ independent of i such that for for any $j \geq 0$, $Q \in \Delta^j$, and $Q \cap A_0 \neq \emptyset$,

$$c\ell(Q) \leq \mathcal{H}^1(Q \cap A_0) \leq C\ell(Q). \tag{4.41}$$

We do this by showing similar bounds for $\frac{\mathcal{H}^1(Q \cap \Sigma_j)}{\ell(Q)}$ for cubes $Q \in \Delta^j$ that intersect Σ_j, and then proving that not too much additional mass is added to the cube Q.

Due to Lemma 84 the condition that $Q \in \Delta^j$ and $Q \cap A_j \neq \emptyset$ is equivalent to $Q \in \mathcal{Q}_j$. Since $Q \in \mathcal{Q}_j$ Lemma 83 characterizes what $Q \cap \Sigma_j$ looks like and we conclude

$$\ell(Q) \leq \mathcal{H}^1(Q \cap \Sigma_j) \leq 3\ell(Q), \tag{4.42}$$

by considering each of the three cases in Lemma 83. Indeed, each cube either contains its entire bottom portion, or contains a cluster E with $\ell(E) = \ell(Q)$. In either case this implies the lower bound in (4.42). On the other hand, we know that a rough upper-bound is to assume that $Q \cap \Sigma_j$ contains a cluster with a line segment at the bottom, and contains Σ_0 scaled by 2^{-2k}, then by Proposition 76, the upper bound in (4.42) follows.

It remains to show that (4.42) implies (4.41). Due to Proposition 80 (1), the lower-bound in (4.41) is inherited directly from (4.42). The upper-bound follows with the additional observation that for $\ell \geq j$,

$$\mathcal{H}^1\left(Q \cap \Sigma_{\ell+1} \setminus \Sigma_\ell\right) \leq \#|D_{\ell+1}| 2^{-8(\ell+1)} \mathcal{H}^1\left(\Sigma_0\right) \leq 2^{-4(\ell+1)} \mathcal{H}^1(\Sigma_0).$$

Summing over $\ell \geq j$ verifies (4.41). It is a standard argument to go from Ahlfors regularity in tetradic/dyadic cubes to in balls, see for instance the brief description in the proof of Proposition 76(3). Since the cubes in $\mathcal{Q}$ are all the tetradic cubes with non-empty intersection with A_0, we have regularity in tetradic cubes.

Finally, to see that $C_{A_0}(x, \delta) = \infty$ it suffices to show the following claim.

Claim 1- If $x \in A_0$ and $\delta > 0$, then there is a tail point in $A_0 \cap B_{\delta/2}(x)$.

Briefly assuming that Claim 1 holds, the fact that $C_{A_0}(x, \delta) = \infty$ for all $x \in A_0$ and $\delta > 0$ follows since if y is the tail point in $B_{\delta/2}(x)$ then, by Proposition 77 (3) and monotonicity of integrals of non-negative functions:

$$C_{A_0}(x, \delta) \geq C_{A_0}(y, \delta/2) \geq C_{\Sigma_0}(0, \epsilon_y) = \infty,$$

where $\epsilon_y > 0$ is some scale dependent on which D^i the tail point y is in.

To verify Claim 1, fix x and δ as in the claim. Adopting the convention that $\Sigma_{-1} = \emptyset$ fix i_0 such that $x \in \Sigma_{i_0} \setminus \Sigma_{i_0-1}$. Choose k to be the smallest natural number such that $\operatorname{diam}\left(2^{-8k}\Sigma_0\right) \leq \delta/4$.

Case 1- $B_{\delta/4}(x) \cap \Sigma_k$ contains a tail. Since $\Sigma_k \subset A_0$ in this case the claim holds.

Case 2- Otherwise, choose $k_0 \geq k$ such that

$$\begin{cases} (\Sigma_{k_0-1} \setminus \Sigma_k) \cap B_{\delta/4}(x) = \emptyset \\ (\Sigma_{k_0} \setminus \Sigma_k) \cap B_{\delta/4}(x) \neq \emptyset, \end{cases}$$

that is k_0 is the first stage after k where something new is added to the ball $B_{\delta/4}(x)$. The way something new is added to the ball $B_{\delta/4}(x)$ in the k_0th stage is if there exists y such that,

$$\{y + 2^{-8k_0}\Sigma_0\} \cap \{\Sigma_{k_0} \cap B_{\delta/4}(x)\} \neq \emptyset.$$

But then, y is a tail point of Σ_{k_0} and consequently of A_0. By our choice of k, we conclude

$$|x - y| < \operatorname{diam}(2^{-4k_0}\Sigma_0) + \delta/4 \leq \delta/2.$$

Hence the tail point y is indeed in $B_{\delta/2}(x)$. So, by Proposition 64(2)

$$C_{A_0}(x, \delta) \geq C_{A_0}(y, \delta/2) \geq cC_{\Sigma_0}(0, \delta') = \infty.$$

This completes the theorem. □

Chapter 5

REGULARITY IN LOW DIMENSIONS

5.1 Motivation and history

Plateau's original conjecture was an experimentally determined "classification" of the structure of soap bubbles [59]. The classification problem for 2-dimensional area minimal varifolds in $\mathbb{R}^3$ was ultimately solved by Taylor [70] confirming Plateau's experimental results. Area minimizing surfaces are particularly interesting in $\mathbb{R}^3$ due to their relationship to so surface tension phenomena. According to Taylor, understanding how thermodynamic forces cause constraints on the shape of surfaces was studied by, amongst others, Maxwell, Laplace, Gauss, Poisson, and Gibbs, before Plateau's original "laws".

The techniques used by Taylor were dependent upon a deep understanding of how 1-dimensional minimizers on the sphere behaved, [69]. This type of "dimensionality reduction" is ubiquitous, arising in Federer's dimension reduction [34], structure theorems for sets [75], and free boundary problems [16], demonstrating the necessity for a deep understanding in the low-dimensional case.

In this chapter, we begin cataloging properties of low-dimensional anisotropic minimizers. First we study general anisotropic energy minimizing sets of locally finite perimeter in $\mathbb{R}^2$ demonstrating that all minimizers are locally straight line segments and producing a Bernstein theorem in the plane, Theorem 91. This proof cannot be reproduced in higher-dimensions because of the existence of saddle points. At a saddle point, one cannot create a competitor by the type of localization argument presented. This observation can be thought of as a qualitative version of the statement that anisotropic minimal surfaces have anisotropic mean curvature zero.

We then turn our attention to $\|\cdot\|_p$ energy minimizing sets of locally finite perimeter. Given the more general results for anisotropic energy minimizing sets in the plane from the previous section, the main interest of this section is the technique, namely the development and exploitation of a new monotonicity formula, Theorem 101 arising from the first-variation of the $\|\cdot\|_p$-energy. As the only tool used comes fromthe first-variation, the analysis within this section can be repeated for $\|\cdot\|_p$ minimal sets of locally finite perimeter and for minimal varifolds in $\mathbb{R}^2$

Since monotonicity formulas demonstrate that blow-ups of minimal sets are minimal cones, Corollary 109, this provides another reason to understand what minimal cones look like in $\mathbb{R}^2$. We provide an "algorithm" to construct and classify stationary triple junctions in $\mathbb{R}^2$, see Theorem 111 and Figures 5.5a, 5.5b.

5.2 *Preliminaries*

Suppose $A \subset \mathbb{R}^n$ is open and $\rho \in C^1(\mathbb{R}^n \setminus \{0\})$ satisfies

$$\rho(\lambda x) = \lambda \rho(x) \quad \forall\ \lambda > 0 \quad \text{and} \quad \rho(x+y) < \rho(x) + \rho(y) \quad \forall\ |x| = 1 = |y|. \tag{5.1}$$

Given an open set A, we wish to study the regularity of the local minimizers of the functional

$$\Phi_\rho(E; A) := \int_{A \cap \partial^* E} \rho(\nu_E) d\mathcal{H}^n \tag{5.2}$$

when the functional is defined on all sets of locally finite perimeter $E \subset \mathbb{R}^n$ such that $E \setminus A = Z \setminus A$ for some fixed $Z \subset \mathbb{R}^n$ a fixed set of locally finite perimeter. More precisely,

Definition 86. For an open set A_0 and a mapping $\|\cdot\| : \mathbb{S}^1 \to (0, \infty)$, we say that a set of locally finite perimeter E minimizes $\Phi(\cdot\,; A_0)$ if $\partial E = \text{spt}\mu_{\text{E}}$ and for all sets of locally finite perimeter F such that $\overline{E \Delta F} \subset\subset A_0$ it holds that

$$\Phi_\rho(E; U) \leq \Phi_\rho(F; U),$$

where $U \supset \overline{E \Delta F}$ is a pre-compact, open subset of A_0.

We say that E is a (Λ, R_0) almost-minimizer of $\Phi_\rho(\cdot; A_0)$ if $\partial E = \text{spt}\mu_{\text{E}}$ and

$$\Phi_\rho(E; U) \leq \Phi(F; U) + \Lambda r^{n-1+\alpha} \quad \forall E \Delta F \subset\subset U \subset\subset A_0, \quad \text{diam}\{E \Delta F\} = r \leq R_0,$$

for any absolute constant $\alpha > 0$.

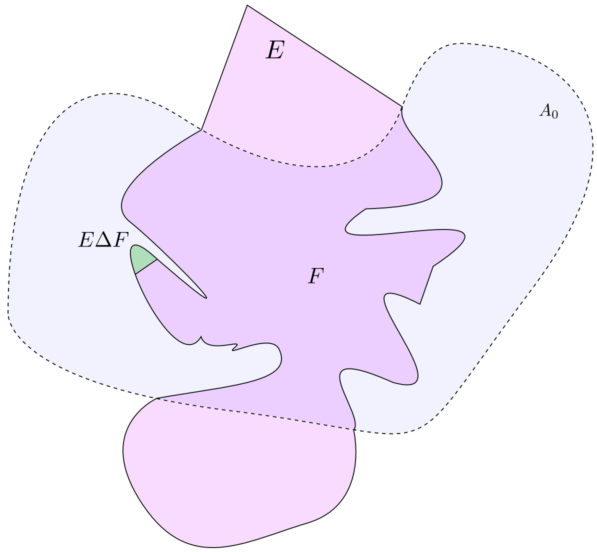

Figure 5.1: A valid competitor F relative to the set A_0.

Remark 87. *The purpose of the set A_0 in Definition 86 is to define boundary conditions. See fig. 5.1.*

The requirement that $\partial E = \mathrm{spt}\mu_{\mathrm{E}}$ in Definition 86 is necessary in order to be able to make topological claims about the boundary of an anisotropic minimizer. Fortunately, given any set of locally finite perimeter E, there exists some Borel set E' so that $\mathrm{spt}\mu_{\mathrm{E'}} = \partial \mathrm{E'}$. See, for instance, [49, Remark 16.11]. Therefore, this requirement boils down to choosing the "correct representative" of E among all equivalent sets of locally finite perimeter.

The only specific energies we consider are

$$\Phi_p(E; A) := \int_{\partial^* E \cap A} \|\nu_E\|_{\ell^p} d\mathcal{H}^{n-1} \qquad 2 \leq p < \infty. \tag{5.3}$$

Note Φ_2 recovers precisely the perimeter of E inside A.

Definition 88. An energy is called semi-elliptic if, whenever F is a set of locally finite perimeter with boundary $\partial F \cap A \subset P$ for some $n-1$ dimensional plane P, then for all

$E \neq F$ with $E \Delta F \subset\subset A$,

$$\Phi_\rho(E;A) - \Phi_\rho(F;A) \geq c\,[\Phi_2(E;A) - \Phi_2(F;A)] \geq 0.$$

The energy Φ_ρ is called elliptic if $c > 0$.

$$P(E;A) = \int_{\partial^* E \cap A} \|\nu_E\|_2 d\mathcal{H}^{n-1} = \mathcal{H}^{n-1}(\partial^* E \cap A).$$

It is well-known that strict convexity of ρ implies semi-ellipticity, [35, 5.1.2].

Proposition 89. *For $p > 2$, the energy Φ_p is not elliptic.*

Proof. Consider $A = \{(x,t) \in \mathbb{R}^n : |x| \leq 1, t \in \mathbb{R}\}$ and $F = \{(x,t) \in \mathbb{R}^n : t \leq 0\}$. For $\alpha > 0$, define the surface $C_\alpha = \{\{x,t\} : |x| \leq 1, t \leq \alpha(1 - |x|^2)\}$. Since ∂C_α is the graph of the function $u(x) = \alpha(1 - |x|^2)$ it follows that

$$\Phi_p(C_\alpha, A) = \int_{|x| \leq 1} \|(-Du, 1)\|_p d\mathcal{H}^{n-1} = \int_{|x| \leq 1} \left(1 + (2\alpha)^p \|x\|_p^p\right)^{\frac{1}{p}} d\mathcal{H}^{n-1}.$$

Hence, for small $\alpha > 0$,

$$\Phi_p(C_\alpha, A) = \mathcal{H}^{n-1}(B(0,1)) + \frac{(2\alpha)^p}{p} \int_{|x|<1} \|x\|_p^p + o(\alpha^{p+1}).$$

In particular, for all $p \geq 2$

$$\Phi_p(C_\alpha, A) - \Phi_p(F, A) = c_{n,p} \frac{\alpha^p}{p} + o(\alpha^{p+1}).$$

Consequently, if $p > 2$,

$$\lim_{\alpha \downarrow 0} \frac{\Phi_p(C_\alpha, A) - \Phi_p(F, A)}{\Phi_2(C_\alpha, A) - \Phi_2(F, A)} = \lim_{\alpha \downarrow 0} \frac{c_{n,p}\alpha^p + o(\alpha^{p+1})}{c_n \alpha^2 + o(\alpha^3)} = 0,$$

verifying the Proposition. □

Remark 90. *If $E \subset \mathbb{R}^2$ is $\Phi(\cdot\,; A_0)$ minimizing, then $\partial E \cap A_0$ contains no self-crossings, or else one could reduce the energy Φ by removing the loop formed by ∂E crossing itself.*

We follow the convention that if $A, B \subset \mathbb{R}^2$ then $A \approx B$ means $\mathcal{H}^1(A \Delta B) = 0$, and $A \precapprox B$ means $\mathcal{H}^1(A \setminus B) = 0$. Moreover, when considering a set of locally finite perimeter A we will always work with a representation of A so that $\partial A = \mathrm{spt}\mu_{\mathrm{A}}$.

For a set of locally finite perimeter A, let μ_A denote the Gauss-Green measure associated to A, ν_A denote the outward pointing measure theoretic normal, and $\partial^* A$ denote the reduced boundary of A.

For any measurable set $A \subset \mathbb{R}^2$ and a number $s \in [0,1]$ define

$$A^{(s)} = \left\{ x \in \mathbb{R}^2 : \lim_{r \downarrow 0} \frac{\mathcal{H}^2\left(A \cap B(x,r)\right)}{\mathcal{H}^2(B(x,r))} = s \right\}.$$

For a set of locally finite perimeter $A \subset \mathbb{R}^2$ the essential boundary of A, denoted $\partial^e A$ is defined to be the set $\mathbb{R}^2 \setminus \left(E^{(0)} \cup E^{(1)}\right)$.

5.3 Regularity of 1-dimensional minimizing sets of locally finite perimeter

Throughout this section, we use Φ in place of Φ_ρ.

It is well-known that strict convexity of the integrand ρ is necessary for there to be a robust regularity theory, see for instance [49, Remark 20.4]. It is also known in $\mathbb{R}^n$ that creating competitors by intersecting with half-spaces can only reduce the energy, see for instance [49, Remark 20.3]. It turns out that for 1-dimensional energy minimizers boundaries in $\mathbb{R}^2$ strict convexity is not only necessary, but also sufficient for a robust regularity result The heart of the proof boils down to a localized version of the fact that intersections with half-spaces reduce energy.

Theorem 91. *Suppose $\rho : \mathbb{S}^1 \to (0, \infty)$ is a lower semicontinuous, bounded, strictly convex function and $A_0 \subset \mathbb{R}^2$ is an open set of locally finite perimeter. Then there exists a $\Phi(\cdot\,; A_0)$ minimizer which we denote by E.*

Moreover, if E minimizes $\Phi(\cdot\,; A_0)$ then there exists a set equivalent to the minimizer, which we also call E,so that whenever $\partial E \cap A_0 \neq \emptyset$ it follows $\partial E \cap A_0$ is a non-intersecting collection of line segments. In the case that $A_0 = \mathbb{R}^2$, E must be a half-space.

Remark 92. *The existence portion of Theorem 91 is well-known. See, for instance [49, Remark 20.5] and the historical notes and citations therein.*

The geometric idea behind the of proof of Theorem 91 can be seen even in Almgren's definition of an elliptic integrand. The technicalities that arise are primarily due to showing that a point where the boundary is not flat allows one to reproduce a localized version of the half-plane argument from, for instance, [49, Remark 20.3] and create a valid competitor.

We first make use of the semicontinuity and boundedness of ρ to make a substantial simplification.

Remark 93 (∂E is locally Lipschitz for anisotropic minimal surfaces)**.** *Let ρ be as in the statement of Theorem 91. Since ρ is a positive lower semicontinuous function on $\mathbb{S}^1$, it achieves a minimum. Since it is also bounded this means there exist $c, C > 0$ such that $c|\nu| \leq \rho(\nu) \leq C|\nu|$ for all $\nu \in \mathbb{R}^2 \setminus \{0\}$. By a standard competitor argument, see Lemma 118, and the differential inequality afforded by the isoperimetric inequality this implies that if E minimizes $\Phi(\cdot, A_0)$ and $x \in \partial^* E \cap A_0$ then there exists $C_A = C_A(c, C)$ independent of x such that for all $r \in (0, \operatorname{dist}(x, \partial A_0))$,*

$$C_A^{-1} \leq \frac{\mathcal{H}^1(\partial^* E \cap B(x,r))}{r} \leq C_A. \tag{5.4}$$

That is, $|\mu_E|$ is Ahlfors regular at small, but locally uniform, scales for points $x \in \partial^* E$. This has two immediate consequences: (1) the lower bound ensures that there are no isolated points in ∂E, and (2) since by our choice of representation $\partial E = \mathrm{spt}\mu_{\mathrm{E}} = \overline{\partial^* \mathrm{E}}$, (5.4) ensures

$$\mathcal{H}^1\left((\partial E \setminus \partial^* E) \cap A_0\right) = 0. \tag{5.5}$$

Indeed, if (5.5) were false one can readily contradict that $\lim_{r \downarrow 0} r^{-1}\mathcal{H}^1(B(x,r) \cap \partial^* E) = 0$

for $\mathcal{H}^1$*-a.e.* $x \notin \partial^* E$*.*

If K *is a compact subset of* A_0*, Ważewski's theorem (See, for instance, [1] for a formal theorem statement) therefore ensures that each connected component of* $\partial E \cap K$ *is a Lipschitz curve since* $\mathcal{H}^1(K \cap \partial E) < \infty$ *and* $K \cap \partial E$ *is compact. In particular, connected components of* $\partial E \cap A_0$ *are locally Lipschitz curves.*

Theorem 94. *If* $\rho : \mathbb{S}^1 \to (0, \infty)$ *is a lower semicontinuous, bounded, strictly convex function,* $A_0 \subset \mathbb{R}^2$ *is an open set, and* $E \subset \mathbb{R}^2$ *minimizes* $\Phi(\cdot\,; A_0)$ *then there exists an equivalent set of locally finite perimeter which we also call* E*, so that* $\partial E \cap A_0 \neq \emptyset$ *implies* $\partial E \cap A_0$ *is a collection of non-intersecting line segments. In the case that* $A_0 = \mathbb{R}^2$*,* E *must be a half-space.*

Proof. Without loss of generality, assume $E = E^{(1)}$. Suppose for the sake of contradiction that $\partial E \cap A_0 \neq \emptyset$ is not made up of exclusively straight, non-intersecting line segments.

Then, there exists a non-flat curve $\gamma \subset \partial E$ such that the endpoints of γ, denoted by $\{x_1, x_2\}$, satisfy

$$|x_1 - x_2| < \operatorname{dist}(\gamma, \partial A_0). \tag{5.6}$$

By Remark 90, γ has no self-crossings nor does it cross $\partial E \setminus \gamma$.

Let ℓ be the line segment between x_1 and x_2. If $x \in \ell$ then in light of (5.6)

$$\operatorname{dist}(x, \gamma) \leq \frac{1}{2} \operatorname{dist}(x, \{x_1, x_2\}) < \operatorname{dist}(\gamma, \partial A_0)$$

Which verifies $\ell \subset\subset A_0$ and consequently, $\ell \cup \gamma \subset\subset A_0$. If necessary, shorten ℓ (and then γ accordingly) so that $\ell \cap \partial E = \gamma \cap \ell = \{x_1, x_2\}$. The fact that "the next crossing" of ℓ with ∂E exists follows from Remark 93.

In particular, $\gamma \cup \ell$ is a Jordan curve. Since $\ell \cup \gamma \subset\subset A_0$, this ensures there exists a unique connected component G of $A_0 \setminus (\gamma \cup \ell)$ whose closure does not meet ∂A_0. See Figure 5.3b.

At this point there are two cases to consider: when $G \subset E$ and when $G \subset E^c$. [1]

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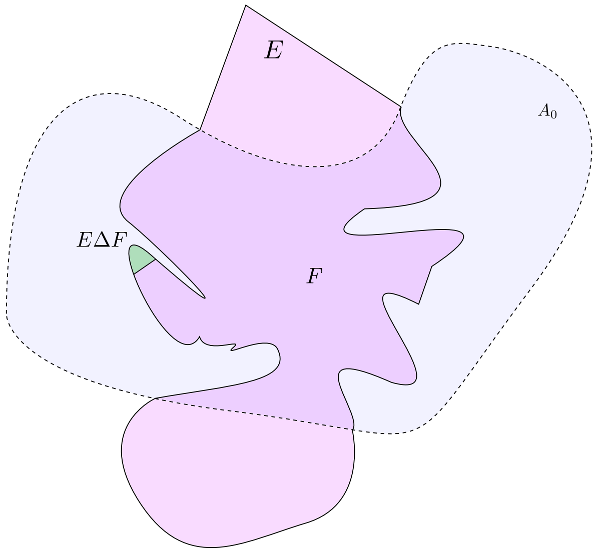
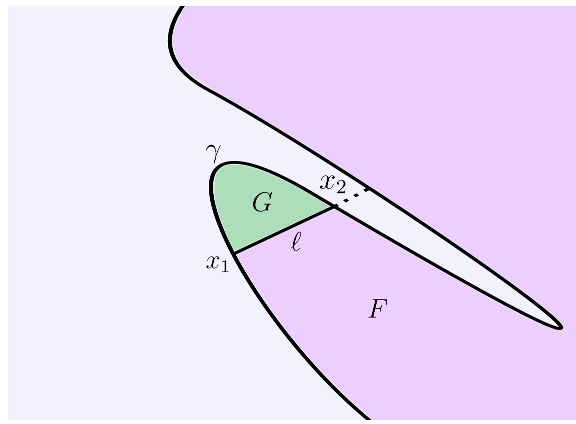
[1]If $\|\cdot\|$ were such that $\|x\| = \|-x\|$ for all $x \in \mathbb{R}^2 \setminus \{0\}$ one could just replace E with E^c to cover both



(a) A valid competitor F relative to the set A_0.



(b) If $\gamma \cap \ell \neq \{x_1, x_2\}$ shorten ℓ by removing the dashed line segment and redefining γ accordingly.

Figure 5.2

First consider the case where $G \subset E$. Define the competitor $F = E \setminus G$. By choice of G, $E\Delta F \subset\subset A_0$. So that F is a valid competitor for E in A_0.

Moreover, $F \subset E$ ensures $\{\nu_E = -\nu_F\} = \emptyset$. Hence, (5.56) implies that G satisfies

$$\mu_G = \mu_{E\setminus F} = \mu_E \llcorner F^{(0)} - \mu_F \llcorner E^{(1)}. \tag{5.7}$$

Since $F^{(1)} \subset E^{(1)}$ is disjoint from $E^{(1/2)} \supset \partial^* E$ we have $\mu_E \llcorner F^{(0)} = \mu_E \llcorner \left(F^{(0)} \cup F^{(1)}\right)$. Since $\mathcal{H}^1\left(\mathbb{R}^2 \setminus \left(F^{(0)} \cup F^{(1)} \cup \partial^* F\right)\right) = 0$ and $|\mu_E| = \mathcal{H}^1 \llcorner \partial^* E$ is in particular absolutely continuous with respect to $\mathcal{H}^1$, this in turn implies

$$\mu_E \llcorner F^{(0)} = \mu_E \llcorner \left(F^{(0)} \cup F^{(1)}\right) = \mu_E \llcorner \left(\partial^* E \setminus \partial^* F\right). \tag{5.8}$$

Similarly

$$\mu_F \llcorner E^{(1)} = \mu_F \llcorner \left(\partial^* F \setminus \partial^* E\right). \tag{5.9}$$

---

cases simultaneously. However, this additional assumption on $\|\cdot\|$ is not necessary.

Combining (5.7), (5.8), and (5.9) yields

$$\mu_G = \mu_E \llcorner (\partial^* E \setminus \partial^* F) - \mu_F \llcorner (\partial^* F \setminus \partial^* E). \tag{5.10}$$

Next, we aim to show geometrically evident fact (see Figure 5.3b) that

$$\mu_G = \mu_E \llcorner \gamma - \mu_F \llcorner \ell. \tag{5.11}$$

To this end, first note that Remark 93 ensures $\partial E \approx \partial^* E$[2]. But, since $\partial F \subset (\ell \cup \partial E)$ and $\mathcal{H}^1(\ell \cap \partial E) = 0$, it follows from the flatness of ℓ that $\partial F \approx \partial^* F$. Similarly, $\partial^* G \approx \partial G$. By Federer's theorem and (5.5) this also implies, $G^{(0)} \approx \mathbb{R}^2 \setminus \overline{G}$.

Therefore, since $G = E \Delta F$ implies $\partial E \setminus \overline{G} = \partial F \setminus \overline{G}$, it follows

$$G^{(0)} \cap \partial^* F \approx (\mathbb{R}^2 \setminus \overline{G}) \cap \partial F = (\mathbb{R}^2 \setminus \overline{G}) \cap \partial E \approx G^{(0)} \cap \partial^* E. \tag{5.12}$$

Moreover, $F \cap G = \emptyset$ implies $\{\nu_F = \nu_G\} = \emptyset$ so that (5.57) implies

$$\partial^* E = \partial^* (F \cup G) \approx \left(F^{(0)} \cap \partial^* G\right) \cup \left(G^{(0)} \cap \partial^* F\right). \tag{5.13}$$

Similarly,

$$\partial^* F = \partial^* (E \setminus G) \approx \left(E^{(1)} \cap \partial^* G\right) \cup \left(G^{(0)} \cap \partial^* E\right). \tag{5.14}$$

However, since $\partial^* E \cap E^{(1)} = \emptyset$ and $\partial^* F \cap F^{(0)} = \emptyset$, (5.12) (5.13) and (5.14) imply

$$\begin{cases} \partial^* E \setminus \partial^* F \approx \left(F^{(0)} \cap \partial^* G\right) \\ \partial^* F \setminus \partial^* E \approx \left(E^{(1)} \cap \partial^* G\right). \end{cases}$$

Since $\partial G = \gamma \cup \ell$ with $\gamma \subsetneq F^{(0)}$, $\ell \subsetneq E^{(1)}$ and $\ell \cap \gamma \approx \emptyset$, this verifies (5.11).

Since $G \subset\subset A_0$, it follows $\mu_G(A_0) = 0$. Indeed, recalling that E is an energy minimizer

[2] Recall for $M_1, M_2 \subset \mathbb{R}^2$, $M_1 \approx M_2$ means $\mathcal{H}^1(M_1 \Delta M_2) = 0$.

inside $A_0 \subset \mathbb{R}^2$, choose $\varphi \in C_c^1(A_0)$ such that $\varphi \equiv 1$ on $\overline{G} \supset \mathrm{spt}\mu_G$ and observe

$$\mu_G(A_0) = \int_{A_0} \varphi d\mu_G = \int_{A_0} \nabla\varphi dx = 0. \tag{5.15}$$

Combining (5.11), (5.15), and the fact that $\nu_F \llcorner \ell$ is constant yields

$$\int_\ell \rho(\nu_F) d\mathcal{H}^1 = \rho\left(\int_\ell \nu_F d\mathcal{H}^1\right) = \rho\left(\int_\gamma \nu_E d\mathcal{H}^1\right). \tag{5.16}$$

Since ρ is strictly convex and γ is not flat (so $\nu_E \llcorner \gamma$ is not constant) we further have

$$\rho\left(\int_\gamma \nu_E d\mathcal{H}^1\right) < \int_\gamma \rho(\nu_E) d\mathcal{H}^1. \tag{5.17}$$

It now follows from (5.10), (5.11), (5.16), and (5.17) that $\Phi(E; A_0) > \Phi(F; A_0)$. Since F is a valid competitor, this contradicts the $\Phi(\cdot\,; A_0)$ minimality of E, completing Case 1.

In case $G \subset E^c$ define $F = E \cup G$. Since $G = E\Delta F$, is compactly contained in A_0, this case follows analogously to previous one.

It remains to show that if $A_0 = \mathbb{R}^2$ then E is a half-space. Indeed, we just verified that ∂E must be a collection of non-intersecting lines. Therefore, if ∂E contains more than one line, they must be parallel. Since $\mathcal{H}^1 \llcorner \partial E$ is locally finite, we can select two consecutive lines in ∂E which we call L_1 and L_2. Let $\vec{s}$ be a unit vector parallel to L_1 and $\vec{t}$ be orthonormal to $\vec{s}$.

The idea is is to build a competitor F, see Figure 5.4, whose boundary is identical to ∂E, except on some rectangle, where on this rectangle, the $\vec{s}$-directional sides will be in $\partial E \setminus \partial F$ whereas the $\vec{t}$-directional sides are in $\partial F \setminus \partial E$. By making the $\vec{s}$-directional sides sufficiently long it will follow that F will have less Φ-energy than E, contradicting that a $\Phi(\cdot\,; \mathbb{R}^2)$-energy minimizing set E can have ∂E containing more than one line. One difficulty that makes the proof more technical, is we need some bounded open set A_0 so that making this change on the rectangle above ensures that $E\Delta F$ is compactly supported in A_0. We do this by slightly fattening the rectangle we modify.

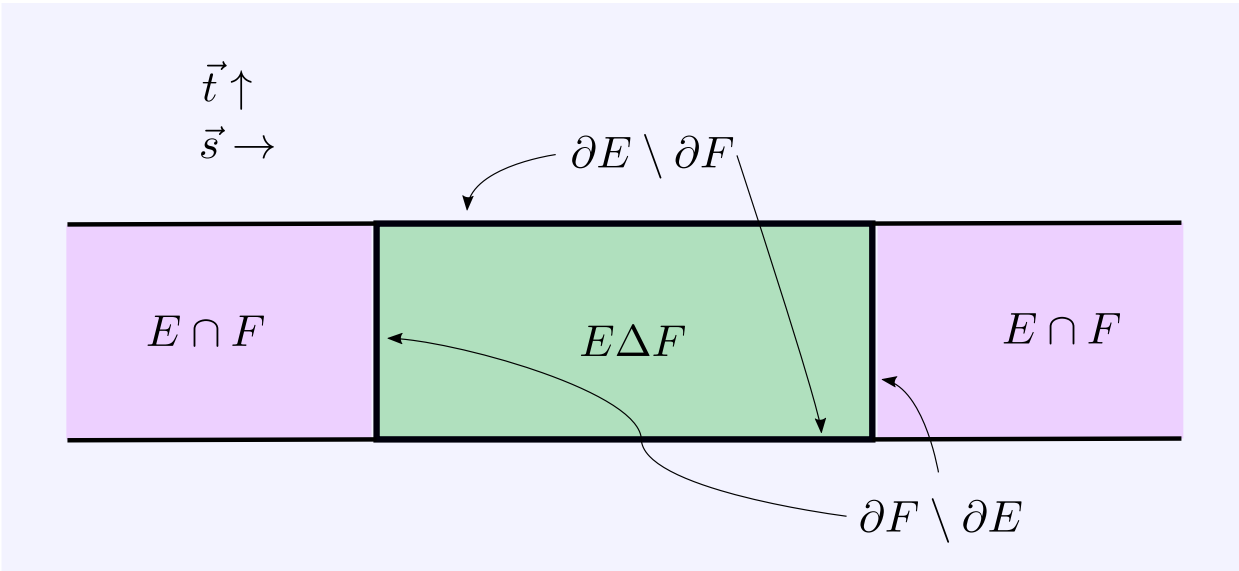


Figure 5.4: The desired competitor in the case the strip between the consecutive lines is contained in E, in which point $E\Delta F = E \setminus F$. In the case where the strip between the consecutive lines is contained in E^c, the boundaries for E and F remain unchanged, however $E\Delta F = F \setminus E$ and similarly $E \cap F$ should be replaced with $E^c \cap F^c$.

More precisely, rescale and choose your origin so that L_i is the line $\{x \in \mathbb{R}^2 : x \cdot \vec{t} = (-1)^i\}$ for $i \in \{1, 2\}$.

For each $\sigma, \tau > 0$ define the rectangle

$$R_{\sigma,\tau} = \{x \in \mathbb{R}^2 : -\sigma \leq x \cdot \vec{s} \leq \sigma, -\tau \leq x \cdot \vec{t} \leq \tau\}$$

Define $a, b > 0$ so that $\max\{\rho(\vec{s}), \rho(-\vec{s})\} = a$ and $\min\{\rho(\vec{t}), \rho(-\vec{t})\} = b$. Choose $\delta > 0$ so that $R_{1,1+\delta} \cap \partial E = R_{1,1+\delta} \cap (L_1 \cup L_2)$. That is, choose δ so that "fattening" R vertically by a distance of δ does not meet any new pieces of ∂E. Fix $c > \frac{b}{a}$ and observe that

$$\int_{(L_1\cup L_2)\cap\partial R_{c,1}} \rho(\nu_E) \geq 4bc > 4a \geq \int_{\partial R_{c,1}\setminus(L_1\cup L_2)} \rho(\nu_R).$$

Then, defining $F = E \setminus R_{c,1}$ or $F = E \cup R_{c,1}$ depending on whether or not $R_{c,1} \subset E$ it follows that $\Phi(F; R_{c+\delta,1+\delta}) < \Phi(E; R_{c+\delta;1+\delta})$ contradicting the minimality of E and hence verifying ∂E is a single line, so that E is a half-space.

$\square$

5.4 The Anisotropic First and Second Variations of Φ_ρ

In this section, we derive the first and second variations for the anisotropic energies Φ_ρ defined in (5.18). To this end, we consider a vectorfield $T \in C_c^1(A;\mathbb{R}^n)$ and we compute the Taylor expansion of the functional

$$\Phi_\rho(f_t(E);A) = \int_{f_t^{-1}(A)\cap\partial^* E} \rho\left(\left((\nabla f_t(x))^{-1}\right)^*[\nu_E]\right) Jf_t(x)d\mathcal{H}^{n-1}, \tag{5.18}$$

$f_t(x) = x + tT(x)$. In particular, $Jf_t(x) = \sqrt{\det((\nabla f_t(x))^*(\nabla f_t(x)))}$. We assume that t is small enough that $f_t^{-1}(A) = A$ so that $E_t = f_t(E)$ is a valid competitor for E inside A.

In what follows, we may write $\nabla T^*(x)$ in place of $(\nabla T(x))^*$.

Theorem 95. *If E is a set of locally finite perimeter, $\rho \in C^1(\mathbb{R}^n\setminus\{0\};(0,\infty))$ is a positively 1-homogeneous function, and $f_t(x) = x + tT(x)$ for some $T(x) \in C_c^1(A;\mathbb{R}^n)$ then*

$$\left.\frac{d}{dt}\right|_{t=0} \Phi_\rho(f_t(E);A) = \int_{\partial^* E\cap A} \rho(\nu_E)\operatorname{tr}(\nabla T(x)) - \langle d\rho(\nu_E), (\nabla T(x))^*[\nu_E]\rangle d\mathcal{H}^{n-1}, \tag{5.19}$$

where $d\rho$ denotes the differential of ρ. If additionally $\rho, T \in C^2$, then also

$$\left.\frac{d^2}{dt^2}\right|_{t=0} \Phi_\rho(f_t(E);A) = \operatorname{tr}\left(D^2\rho(\nu_E)\left[(\nabla T(x))^*[\nu_E]\otimes(\nabla T(x))^*[\nu_E]\right]\right) + 2\langle d\rho(\nu_E), (\nabla T(x))^2\rangle \tag{5.20}$$

$$- \langle d\rho(\nu_E), (\nabla T(x))^*[\nu_E]\rangle\operatorname{tr}(\nabla T(x)) + 2\left((\operatorname{tr}(\nabla T(x)))^2 - \operatorname{tr}\left((\nabla T(x))^2\right)\right).$$

Proof. We begin with some preliminary observations. First, if $a, b, c \in \mathbb{R}^n$ and $\rho \in C^2$ then

$$\left.\frac{d}{dt}\right|_{t=0}\left[\rho(a + bt + ct^2 + O(t^3))\right] = \langle d\rho(a), b\rangle \tag{5.21}$$

and

$$\left.\frac{d^2}{dt^2}\right|_{t=0}\left[\rho(a+bt+ct^2+O(t^3))\right]=\operatorname{tr}\left(D^2\rho(a)\left[b\otimes b\right]\right)+2\langle d\rho(a),c\rangle. \tag{5.22}$$

We also recall the Taylor series expansion:

$$(1+at+bt^2)^{1/2}=1+\frac{a}{2}t+\frac{t^2}{2}\left(b-\frac{a^2}{4}\right)+O(t^3) \tag{5.23}$$

Next we recall

$$\det(Id+tZ)=1+t\operatorname{tr}(Z)+\frac{t^2}{2}\left(\operatorname{tr}(Z)^2-\operatorname{tr}(Z^2)\right)+O(\|tZ\|^3) \tag{5.24}$$

where $\|A\|^2=\operatorname{tr}(A^*A)$ and that

$$(Id+tZ)^{-1}=Id-tZ+t^2Z^2+O(\|tZ\|^3). \tag{5.25}$$

We begin by expanding the Jacobian term. Since $f_t(x)=x+tT(x)$ then $\nabla f_t(x)=Id+t\nabla T(x)$. Hence,

$$\begin{aligned}(\nabla f_t(x))^*(\nabla f_t(x))&=Id+t\left(\nabla T(x)+\nabla T^*(x)\right)+t^2\left(\nabla T^*(x)\right)\left(\nabla T(x)\right)\\&=Id+t\left[\nabla T^*(x)+\nabla T(x)+t\nabla T^*(x)\nabla T(x)\right]\end{aligned} \tag{5.26}$$

Then, (5.24) and (5.26) imply

$$\begin{aligned}
\det\left(\nabla f_t(x)^*(\nabla f_t(x))\right) &= 1 + t\left(\operatorname{tr}(\nabla T^*(x) + \nabla T(x)) + t\operatorname{tr}\left(\nabla T^*(x)\nabla T(x)\right)\right) \\
&+ \frac{t^2}{2}\left[\left(\operatorname{tr}\left(\nabla T^*(x) + \nabla T(x)\right)\right)^2 - \operatorname{tr}\left(\left(\nabla T^*(x) + \nabla T(x)\right)^2\right)\right] \\
&+ O(t^3) \\
&= 1 + 2t\operatorname{tr}(\nabla T(x)) \\
&+ \frac{t^2}{2}\Bigg[\left(2\operatorname{tr}\left(\nabla T(x)\right)\right)^2 + 2\operatorname{tr}\left(\nabla T^*(x)\nabla T(x)\right) \\
&- \operatorname{tr}\left((\nabla T^*(x))^2 + 2\nabla T^*(x)\nabla T(x) + (\nabla T(x))^2\right)\Bigg] + O(t^3) \\
&= 1 + t\left[2\operatorname{tr}(\nabla T(x))\right] \qquad (5.27) \\
&+ t^2\left[2(\operatorname{tr}(\nabla T(x)))^2 - \operatorname{tr}((\nabla T(x))^2)\right] + O(t^3)
\end{aligned}$$

Applying the expansion (5.23) to (5.27) yields

$$\begin{aligned}
Jf_t(x) &= 1 + t\operatorname{tr}\left(\nabla T(x)\right) \\
&+ \frac{t^2}{2}\left[2\left(\operatorname{tr}\left(\nabla T(x)\right)\right)^2 - \operatorname{tr}\left(\left(\nabla T(x)\right)^2\right) - \frac{\left(2\operatorname{tr}\nabla T\left(x\right)\right)^2}{4}\right] + O(t^3) \\
&= 1 + t\operatorname{tr}(\nabla T(x)) + t^2\left[\left(\operatorname{tr}(\nabla T(x))\right)^2 - \operatorname{tr}\left(\left(\nabla T(x)\right)^2\right)\right] + O(t^3) \qquad (5.28)
\end{aligned}$$

Therefore,

$$\left.\frac{d}{dt}\right|_{t=0} Jf_t(x) = \operatorname{tr}(\nabla T(x)) \quad \text{and} \quad \left.\frac{d^2}{dt^2}\right|_{t=0} Jf_t(x) = 2\left(\left(\operatorname{tr}(\nabla T(x))\right)^2 - \operatorname{tr}\left((\nabla T(x))^2\right)\right). \qquad (5.29)$$

It remains to find the expansion for $\rho\left((\nabla f_t(x))^{-1}[\nu_E]\right)$. Following the convention that

$A^0 = Id$ for any matrix A, (5.25) implies

$$(\nabla f_t(x))^{-1} = \sum_{i=0}^{\infty}(-t\nabla T(x))^i = Id - t\nabla T(x) + t^2(\nabla T(x))^2 + O(\|t\nabla T(x)\|^3), \tag{5.30}$$

so that

$$\rho\left(\left((\nabla f_t(x))^{-1}\right)^*[\nu_E]\right) = \rho\left(\nu_E - t\,(\nabla T(x))^*[\nu_E] + t^2\left((\nabla T(x))^2\right)^*[\nu_E] + O(t^3)\right) \tag{5.31}$$

Applying (5.21) to (5.31) verifies

$$\left.\frac{d}{dt}\right|_{t=0}\left[\rho\left(\left((\nabla f_t(x))^{-1}\right)^*[\nu_E]\right)\right] = -\langle d\rho(\nu_E), (\nabla T(x))^*[\nu_E]\rangle \tag{5.32}$$

and applying (5.22) to (5.31) yields

$$\left.\frac{d^2}{dt^2}\right|_{t=0}\left[\rho\left(\left((\nabla f_t(x))^{-1}\right)^*[\nu_E]\right)\right] = \operatorname{tr}\left(D^2\rho(\nu_E)\left[(\nabla T(x))^*[\nu_E]\otimes(\nabla T(x))^*[\nu_E]\right]\right) + 2\langle d\rho(\nu_E), c\rangle. \tag{5.33}$$

Finally, combining (5.28) - (5.33) with the product rule we can do the Taylor expansion of the integrand in (5.18). Namely,

$$\rho\left(\left((\nabla f_t(x))^{-1}\right)^*[\nu_E]\right) Jf_t(x) = \rho(\nu_E) + t\left[\rho(\nu_E)\operatorname{tr}(\nabla T(x)) - \langle d\rho(\nu_E), (\nabla T(x))^*[\nu_E]\rangle\right] \tag{5.34}$$

$$+\frac{t^2}{2}\Big[\operatorname{tr}\left(D^2\rho(a)\left[(\nabla T(x))^*[\nu_E]\otimes(\nabla T(x))^*[\nu_E]\right]\right) + 2\langle d\rho(a), c\rangle$$

$$-\langle d\rho(\nu_E), (\nabla T(x))^*[\nu_E]\rangle\operatorname{tr}(\nabla T(x)) + 2\left((\operatorname{tr}(\nabla T(x)))^2 - \operatorname{tr}\left((\nabla T(x))^2\right)\right)\Big]$$

$$+O(t^3).$$

□

Consequently, for $k = 1, 2$ and any $T \in C_c^1(A; \mathbb{R}^n)$ we can define the mapping

$$\delta^k \Phi_\rho(E; A)[T] := \frac{d^k}{dt^k}\bigg|_{t=0} \Phi_\rho(f_t(E); A),$$

where f_t is as before. $\delta^k \Phi_\rho(E; A)$ is called the kth variation of $\Phi_\rho(\cdot; A)$ at E. For future convenience, we observe

$$\begin{aligned}\delta \Phi_\rho(E; A)[T] &= \int_{\partial^* E \cap A} \rho(\nu_E) \operatorname{tr}(\nabla T(x)) - \langle \nu_E, \nabla T(x)[d\rho(\nu_E)] \rangle d\mathcal{H}^{n-1} \\ &= \int_{\partial^* E \cap A} \operatorname{tr}\left(\left(\rho(\nu_E) Id - (D\rho)(\nu_E) \otimes \nu_E \right) \nabla T \right). \end{aligned} \tag{5.35}$$

A set E is called stationary Φ_ρ-stationary if it is Φ_ρ-minimal. That is,

$$\delta \Phi_\rho(E; A) \equiv 0.$$

5.5 Monotonicity formula and basic consequences

In this section we will produce a sufficient condition for the existence of a specific monotonic quantity on Φ_ρ-minimal sets. We then consider $n = 2, \rho = \| \cdot \|_{\ell^p}, p > 2$ to provide a new monotonicity formula, Theorem 101.

Monotonicity will follow from plugging an appropriate test function into the first variation. These computations will be done for all n and any ρ, in order to produce the sufficient condition (5.42).

We will then turn our immediate focus to the special case when $n = 2$ and $\rho = \| \cdot \|_p$ and demonstrate that this special case satisfies (5.42). We proceed to record some consequences of this monotonicity formula.

Definition 96 (f- and p-balls)**.** Given any positively 1-homogeneous function f, we define

$$B_f(x, r) := \{ y : f(y - x) < r \}. \tag{5.36}$$

We call $B_f(x,r)$ an f-ball. In the case $f = \|\cdot\|_p$, we will abuse both notation and language to write $B_p(x,r)$ in place of $B_{\|\cdot\|_p}(x,r)$ and we then call the set a p-ball.

Theorem 97. *If E is Φ_ρ-stationary, then for any positively 1-homogeneous function $f \in C^1(\mathbb{R}^n \setminus \{0\}; \mathbb{R}^n)$ it follows that for almost every r and any $x_0 \in \partial E$,*

$$\begin{aligned} &\frac{d}{dt}\bigg|_{t=r} \left[t^{-(n-1)}\Phi_\rho(E; B_f(x_0,r))\right] \\ &\quad = \frac{d}{dt}\bigg|_{t=r} \left[\int_{\partial^* E\cap B_f(x_0,r)} \frac{\langle \nabla f(x-x_0), \nabla\rho(\nu_E)\rangle\langle \nu_E, \frac{x-x_0}{f(x-x_0)}\rangle}{f(x-x_0)^{n-1}} d\mathcal{H}^{n-1}\right]. \end{aligned} \tag{5.37}$$

Proof. We will prove the case when $x_0 = 0$. Then (5.37) follows by translating. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a smooth-cutoff, chosen later, and $f \in C^1(\mathbb{R}^n \setminus \{0\}; \mathbb{R}^n)$ a positively 1-homogeneous function. Consider $T(x) = \varphi\left(t^{-1}f(x)\right)x$. Then,

$$\nabla T(x) = \varphi\left(t^{-1}f(x)\right) Id + t^{-1}\varphi'(t^{-1}f(x))\left(x \otimes \nabla f(x)\right).$$

Hence,

$$\begin{aligned} \operatorname{tr}\left(\left(\rho(\nu_E)Id - (D\rho)(\nu_E)\otimes\nu_E\right)\nabla T\right) &= n\rho(\nu_E)\varphi(t^{-1}f(x)) - \rho(\nu_E)\varphi(t^{-1}f(x)) \\ &+ t^{-1}\varphi'(t^{-1}f(x))\left[\rho(\nu_E)f(x) - \langle (D\rho)(\nu_E), \nabla f(x)\rangle\langle x, \nu_E\rangle\right]. \end{aligned} \tag{5.38}$$

Plugging (5.38) into (5.35) implies,

$$\begin{aligned} &\int \rho(\nu_E)(n-1)\varphi(t^{-1}f(x)) + t^{-1}\varphi'\left(t^{-1}f(x)\right)f(x)\rho(\nu_E) \\ &\quad = \int \frac{\varphi'(t^{-1}f(x))}{t}\langle \nabla f(x), \nabla\rho(\nu_E)\rangle\langle \nu_E, x\rangle. \end{aligned} \tag{5.39}$$

Since

$$\frac{d}{dt}\left[\varphi\left(t^{-1}f(x)\right)\right] = -t^{-2}f(x)\varphi'(t^{-1}f(x))$$

implies

$$t^{-1}\varphi'(t^{-1}f(x)) = \frac{-t}{f(x)}\frac{d}{dt}\left[\varphi\left(t^{-1}f(x)\right)\right],$$

it follows (5.39) is equivalent to

$$\begin{aligned}\int \rho(\nu_E)(n-1)\varphi(t^{-1}f(x)) - t\frac{d}{dt}\left[\varphi\left(t^{-1}f(x)\right)\right]\rho(\nu_E) \\ = -\int \frac{t}{f(x)}\frac{d}{dt}\left[\varphi\left(t^{-1}f(x)\right)\right]\langle\nabla f(x), \nabla\rho(\nu_E)\rangle\langle\nu_E, x\rangle.\end{aligned} \tag{5.40}$$

The LHS of (5.40) is precisely

$$-t^n\frac{d}{dt}\left[t^{-(n-1)}\int_{\partial^* E}\rho(\nu_E(x))\varphi(t^{-1}f(x))d\mathcal{H}^{n-1}(x)\right]$$

so that (5.40) ensures

$$\begin{aligned}\frac{d}{dt}\left[t^{-(n-1)}\int_{\partial^* E}\rho(\nu_E(x))\varphi(t^{-1}f(x))d\mathcal{H}^{n-1}(x)\right] \\ = t^{-(n-1)}\int \varphi'(t^{-1}f(x))\langle\nabla f(x), \nabla\rho(\nu_E)\rangle\langle\nu_E, \frac{x}{f(x)}\rangle.\end{aligned} \tag{5.41}$$

Replacing φ with $\varphi_k \in C^\infty(\mathbb{R};\mathbb{R})$ chosen so that

$$-C2^k \leq \varphi_k' \leq 0 \quad \text{and} \quad \varphi_k(s) = \begin{cases} 1 & s \leq 1 - 2^{-k} \\ 0 & s \geq 1 \end{cases}$$

for some constant C that depends on n, we pass to the limit taking $k \to \infty$ to discover that for almost every r,

$$\left.\frac{d}{dt}\right|_{t=r}\left[t^{-(n-1)}\Phi_\rho(E; B_f(0,t))\right] = \left.\frac{d}{dt}\right|_{t=r}\left[\int_{\partial^* E\cap B_f(0,t)}\frac{\langle\nabla f(x), \nabla\rho(\nu_E)\rangle\langle\nu_E, \frac{x}{f(x)}\rangle}{f(x)^{n-1}}d\mathcal{H}^{n-1}\right].$$

□

Corollary 98. *If $f \in C^1(\mathbb{R}^n \setminus \{0\}; \mathbb{R}^n)$ is positively* 1*-homogeneous and satisfies*

$$\langle \nabla f(x), \nabla \rho(\nu) \rangle \langle x, \nu \rangle \geq 0 \qquad \forall x, \nu \in \mathbb{S}^{n-1} \tag{5.42}$$

then for all Φ_ρ-stationary sets, any $x_0 \in \partial E$,

$$r \mapsto \frac{\Phi_\rho(E; B_f(x_0, r))}{r^{n-1}}$$

is monotonically increasing. More specifically, for almost every $0 < \sigma < \tau$

$$\frac{\Phi_\rho(E; B_f(x_0, \tau))}{\tau^{n-1}} - \frac{\Phi_\rho(E; B_f(x_0, \sigma))}{\sigma^{n-1}}$$
$$= \int_{\partial^* E \cap (B_f(x_0,\tau) \setminus B_f(x_0,\sigma))} \frac{\langle \nabla f(x - x_0), \nabla \rho(\nu_E) \rangle \langle \nu_E, \frac{x - x_0}{f(x - x_0)} \rangle}{f(x - x_0)^{n-1}} d\mathcal{H}^{n-1}.$$

The corollary is immediate from the Theorem 101, (5.37). It seems unlike that a statement like that in (5.42) will hold for every pair of points $\{(x, \nu) \subset \mathbb{R}^{2n}\}$. This is a similar condition to that studied by [4], where Allard uses Morse-theoretic techniques to show that the only ρ where such an f exists are $\rho(\cdot) = |A \cdot|$ for some constant elliptic matrix A. Nonetheless, assuming all

Lemma 99. *If $n = 2$,*

$$\operatorname{sgn}(x \cdot y) = \operatorname{sgn}(\nabla \|x\|_p \cdot \nabla \|y\|_p).$$

In particular

$$\nabla \|x\|_p \cdot \nabla \|y\|_p = 0 \iff x \cdot y = 0. \tag{5.43}$$

Proof. First compute

$$\langle x, \nu \rangle = x_1 y_1 + x_2 y_2 \quad \text{and} \quad \langle \nabla \|x\|_p, \nabla \|y\|_p \rangle = x_1 |x_1|^{p-2} y_1 |y_1|^{p-2} + x_2 |x_2|^{p-2} y_2 |y_2|^{p-2}.$$

Then

$$x \cdot y > 0 \iff x_1 y_1 > -x_2 y_2 \iff \begin{cases} \frac{x_1 y_1}{x_2 y_2} > -1 & x_2 y_2 > 0 \\ \frac{x_1 y_1}{x_2 y_2} < -1 & x_2 y_2 < 0, \end{cases} \tag{5.44}$$

and

$$\begin{aligned} \langle x, y \rangle \langle \nabla \|x\|_p, \nabla \|y\|_p \rangle &= (x_1 y_1 + x_2 y_2) \left(x_1 |x_1|^{p-2} y_1 |y_1|^{p-2} + x_2 |x_2|^{p-2} y_2 |y_2|^{p-2} \right) \\ &= |x_1|^p |y_1|^p + |x_2|^p |y_2|^p + x_1 x_2 y_1 y_2 \left(|x_1|^{p-2} |y_1|^{p-2} + |x_2|^{p-2} |y_2|^{p-2} \right). \end{aligned}$$

We see immediately that if $x_1 x_2 y_1 y_2 \geq 0$ then $\langle x, y \rangle \langle \nabla \|x\|_p, \nabla \|\nu\|_p \rangle \geq 0$. So, we suppose $x_1 x_2 y_1 y_2 < 0$. This in turn implies $\frac{x_1 y_1}{x_2 y_2} < 0$. So, within our case, (5.44) implies

$$x \cdot y > 0 \iff \begin{cases} \left| \frac{x_1 y_1}{x_2 y_2} \right| < 1 & x_2 y_2 > 0 \\ \left| \frac{x_1 y_1}{x_2 y_2} \right| > 1 & x_2 y_2 < 0. \end{cases} \tag{5.45}$$

On the other hand, still assuming $x_1 x_2 y_1 y_2 < 0$,

$$\begin{aligned} \langle \nabla \|x\|_p, \nabla \|y\|_p \rangle > 0 &\iff \begin{cases} \frac{x_1 |x_1|^{p-2} y_1 |y_1|^{p-2}}{x_2 |x_2|^{p-2} y_2 |y_2|^{p-2}} > -1 & x_2 y_2 > 0 \\ \frac{x_1 |x_1|^{p-2} y_1 |y_1|^{p-2}}{x_2 |x_2|^{p-2} y_2 |y_2|^{p-2}} < -1 & x_2 y_2 < 0 \end{cases} \\ &\iff \begin{cases} \left| \frac{x_1 y_1}{x_2 y_2} \right|^{p-1} < 1 & x_2 y_2 > 0 \\ \left| \frac{x_1 y_1}{x_2 y_2} \right|^{p-1} > 1 & x_2 y_2 < 0 \end{cases} \iff x \cdot y > 0. \end{aligned}$$

It remains to show $x \cdot y = 0 \iff \langle \nabla \|x\|_p, \langle \nabla \|y\|_p \rangle = 0$.

If $x \in \{e_1, e_2\}$ then $x \cdot y = 0$ if and only if y is the other basis direction. In any case, $\nabla \|e_i\|_p = e_i$, so if either vector is a basis direction both quantities are zero.

So now suppose $\langle x, y \rangle = 0$ and $x \neq e_i$. Then

$$x_1 y_1 \neq 0 \neq x_2 y_2 \quad \text{and} \quad \frac{x_1 y_1}{x_2 y_2} = -1.$$

But then

$$\langle \nabla \|x\|_p, \nabla \|y\|_p \rangle = 0 \iff \frac{x_1 y_1}{x_2 y_2} \left| \frac{x_1 y_1}{x_2 y_2} \right|^{p-2} = -1 \iff x \cdot y = 0.$$

□

Remark 100. *For geometric reasons, it would make more sense to study*

$$r \mapsto \frac{\Phi_p(E; K_q(x, r))}{r}$$

where $q^{-1} + p^{-1} = 1$*. However, if one tries to show* $\langle \nabla \|x\|_p, \nabla \|y\|_q \rangle \langle x, y \rangle \geq 0$ *for any* $q \neq p$*, then the proof proceeds identically by replacing powers of* y_i^p *with* y_i^q *up until* (5.45)*. But then, in the case* $x_1 y_1 x_2 y_2 < 0$*,*

$$\langle \nabla \|x\|_p, \nabla \|y\|_q \rangle > 0 \iff \begin{cases} \left|\frac{x_1}{x_2}\right|^{p-1} \left|\frac{y_1}{y_2}\right|^{q-1} < 1 & x_2 y_2 > 0 \\ \left|\frac{x_1}{x_2}\right|^{p-1} \left|\frac{y_1}{y_2}\right|^{q-1} > 1 & x_2 y_2 < 0 \end{cases}$$

$$\not\Longleftrightarrow \begin{cases} \left|\frac{x_1 y_1}{x_2 y_2}\right| < 1 & x_2 y_2 > 0 \\ \left|\frac{x_1 y_1}{x_2 y_2}\right| > 1 & x_2 y_2 < 0 \end{cases} \iff x \cdot y > 0.$$

So, despite the the fact that studying the $\| \cdot \|_p$*-energy ratios over* $\| \cdot \|_q$*-balls would be more natural, the desired algebraic inequality does not hold.*

Theorem 101. *If* $n = 2, \rho = \| \cdot \|_{\ell^p}$ *and* $x_0 \in \partial E$*, then*

$$r \mapsto \frac{\Phi_p(E; B_p(x_0, r))}{r}$$

is monotonically increasing and if $0 < \sigma < \tau$ *satisfy* $\mathcal{H}^1(\partial B_f(x_0, \sigma) \cap \partial^* E) = 0 = \mathcal{H}^1(\partial B_f(x_0, \tau) \cap$

$\partial^* E$) then

$$\frac{\Phi_p(E;B_f(x_0,\tau))}{\tau}-\frac{\Phi_p(E;B_f(x_0,\sigma))}{\sigma} = \int_{\partial^* E\cap\{\sigma\le\|x-x_0\|_p\le\tau\}} \frac{\langle\nabla\|x\|_p,\nabla\|\nu_E\|\rangle\langle\nu_E,\frac{x}{\|x\|_p}\rangle}{\|x\|_p}d\mathcal{H}^1.$$

Consequently, if $\sigma<\tau$ and $\mathcal{H}^1(\partial B_f(x_0,\sigma)\cap\partial^* E)=0=\mathcal{H}^1(\partial B_f(x_0,\tau)\cap\partial^* E)$, which in particular means for almost every $0<\sigma<\tau$ and

$$\frac{\Phi_p(E;B_f(x_0,\tau))}{\tau}=\frac{\Phi_p(E;B_f(x_0,\sigma))}{\sigma}$$

then

$$\nu_E(x)\cdot(x-x_0)=0 \quad \mathcal{H}^1-a.e.\ x\in(B_f(x_0,\tau)\setminus B_f(x_0,\sigma))\cap\partial^* E.$$

Proof. The monotonicity formula follows immediately from Theorem 101 and Lemma 99. The fact that $\nu_E(x)\cdot(x-x_0)=0$ a.e. $B_f(x_0,\tau)\setminus B_f(x_0,\sigma)$ is due to (5.43), as the integrand must be zero almost everywhere. □

Definition 102 (Energy densities)**.** Given $x\in\partial E\subset\mathbb{R}^n$, and a continuous function $f:\mathbb{R}^n\to\mathbb{R}$, we define the f upper- and f lower- energy density of E at x as

$$\Theta^n_{f,*}(E,x)=\liminf_{r\downarrow 0}\frac{\Phi_p(E;B_f(x,r))}{r^{n-1}}$$

and

$$\Theta^{n,*}_f(E,x)=\limsup_{r\downarrow 0}\frac{\Phi_p(E;B_f(x,r))}{r^{n-1}}.$$

Whenever $\Theta^n_{f,*}(E,x)=\Theta^{n,*}_f(E,x)$ we call the common value the f energy density of E at x, and denote it $\Theta^n_f(E,x)$.

Whenever f is clear from context, the dependency on f may not be explicitly stated. If $f=\|\cdot\|_p$, we will denote the energy densities by $\Theta^n_{p,*},\Theta^{n,*}_p,\Theta^n_p$ and call them the p densities instead of $\|\cdot\|_p$-densities.

The easiest corollary of the monotonicity formula, Theorem 101, is that the energy density $\Theta_p^1(E,x)$ exists for every $x \in \partial E$.

Corollary 103. *If $1 < p < \infty$, then $\Theta_p^1(E,x_0)$ exists for every $x_0 \in \partial E$.*

The corollary holds because bounded monotonic sequences always have limits. We next observe the upper semicontinuity of the density:

Lemma 104. *If $\{E_j\}$ are Φ_p-minimizing sets of locally finite perimeter in $\mathbb{R}^2$, such that $E_j \to E$ in L^1, and $x_j \in E_j$ for all j are such that $x_j \to x_0$, then $\limsup_j \Theta_p^1(E_j,x_j) \leq \Theta_p^1(E_\infty,x_0)$.*

Proof. First note that by Theorem 121, E is a Φ_p-minimizing set of locally finite perimeter. For almost every $r \in (0,\infty)$ it follows that

$$\mathcal{H}^1\left(\partial B_p(x_j,r) \cap \partial E_j\right) = 0 = \mathcal{H}^1\left(\partial B_p(x_0,r) \cap \partial E_\infty\right) \tag{5.46}$$

Let $\epsilon > 0$ and $0 < r_0$ be so that (5.46) holds with $r \in \{r_0, r_0+\epsilon\}$. Fix J large enough that $|x_j - x_0| < \epsilon$ for all $j \geq J$. If $\alpha = \limsup_j \Theta_p^1(E_j,x_j)$ then by monotonicity, if $j \geq J$

$$\begin{aligned}\alpha &\leq \limsup_j \frac{\Phi_p(E_j; B_p(x_j,r_0))}{r_0} \\ &\leq \limsup_j \frac{\Phi_p(E_j; B_p(x_0,r_0+\epsilon))}{r_0+\epsilon}\frac{r_0+\epsilon}{r_0}.\end{aligned}$$

By continuity of Φ_p on sequences of minimizing sets, Theorem 121, implies

$$\alpha \leq \frac{\Phi_p(E; B_p(x_0,r_0+\epsilon))}{r_0+\epsilon}\frac{r_0+\epsilon}{r_0}.$$

Taking $\epsilon \downarrow 0$ then $r_0 \downarrow 0$ completes the proof. □

Ultimately, we want to show that the minimizers are flat. The first big step to achieve this is to show that the tangents are minimizing cones. The next result shows that constant energy density implies the surface is a cone. A quantitative version of Lemma 99, i.e., something that says

$$\nabla\|x\|_p \cdot \nabla\|y\|_p \text{ is small } \iff x \cdot y \text{ is small}$$

would lead to a much stronger result that small drops in energy imply the surface is nearly a cone. For a rather recent example, see [30] for similarly quantitative results in different settings.

Lemma 105. *Suppose $E \subset \mathbb{R}^2$ is Φ_p-stationary, $x_0 \in \partial E$, $\tau > 0$, and*

$$\frac{\Phi_p(E; B_p(x_0, \tau))}{\tau} - \Theta^1_p(E, x_0) = 0. \tag{5.47}$$

Then for all $\tau' < \tau$,

$$\partial^* E \cap B_p(x_0, \tau') = \{(1-\lambda)x_0 + \lambda y : y \in \partial^* E \cap \partial B_p(x_0, \tau'), \lambda \in [0, 1]\}.$$

In particular, if (5.47) holds for all $\tau > 0$, then $\partial^* E$ must be a cone.

Proof. First note (5.47) allows us to apply Theorem 101 to conclude $x \cdot \nu_E(x) = 0$ for $\mathcal{H}^1$-a.e. $x \in B_p(x_0, \tau) \cap \partial^* E$.

Without loss of generality, let $x_0 = 0$. We next wish to consider $\widetilde{T}(x) = h(x)T(x)$, where $T(x)$ is as in Theorem 97, and $h(x)$ is an arbitrary positively 0-homogeneous function, continuous on $\mathbb{S}^1$. However, we may only consider test vector fields $\widetilde{T} \in C^1$, so we currently consider $h \in C^1$ and later make an approximation argument. Given $h \in C^1$, observe

$$\nabla\widetilde{T}(x) = x \otimes \nabla h(x)\varphi(t^{-1}f(x)) + h(x)\nabla T(x).$$

Therefore,

$$\begin{aligned}
&\operatorname{tr}\Big((\|\nu_E\|_p Id - (\nabla\|\nu_E\|_p)(\nu_E)\otimes\nu_E)\,\nabla\widetilde{T}(x)\Big)\\
&\quad= h(x)\operatorname{tr}\left((\|\nu_E\|_p Id - (\nabla\|\nu_E\|_p\otimes\nu_E)\,\nabla T(x)\right)\\
&\quad+\varphi(t^{-1}f(x))\left[\|\nu_E\|_p x\cdot\nabla h(x) - \langle\nabla\|\nu_E\|_p,\nabla h(x)\rangle\langle\nu_E,x\rangle\right].
\end{aligned}$$

Hence, one can proceeding identically as in the proof of Theorem 97 (up to multiplying everything that occurs in Theorem 97 by the function h and carrying along the new terms involving ∇h) to compute that for almost every $t<\tau$,

$$\begin{aligned}
\frac{d}{dt}\left[t^{-1}\int_{\partial^* E\cap B_p(0,t)}\|\nu_E\|_p h d\mathcal{H}^1\right] &= t^{-1}\int_{\partial^* E\cap\partial B_p(0,t)} h(x)\langle\nu_E,\frac{x}{f(x)}\rangle\langle\nabla\|\nu_E\|_p,\nabla\|x\|_p\rangle h(x)d\mathcal{H}^0\\
&+t^{-1}\int_{\partial^* E\cap B_p(0,t)}\left[\|\nu_E\|_p\left\langle\frac{x}{t},\nabla h(x)\right\rangle - \langle\nabla\|\nu_E\|_p,\nabla h(x)\rangle\left\langle\frac{x}{t},\nu_E\right\rangle\right].
\end{aligned}$$

In fact, because $x\cdot\nu_E=0$ for $\mathcal{H}^1$ almost every $x\in\partial^* E\cap B_\tau$, it follows that for almost every $t<\tau$, the preceding reduces to

$$\frac{d}{dt}\left[t^{-1}\int_{\partial^* E\cap B_p(0,t)}\|\nu_E\|_p h d\mathcal{H}^1\right] = t^{-1}\int_{\partial^* E\cap B_p(0,t)}\|\nu_E\|_p\left\langle\frac{x}{t},\nabla h(x)\right\rangle. \tag{5.48}$$

In (5.48) we replace h with a sequence of $h_\epsilon\in C^1$ which we let approach an arbitrary 0-homogeneous $g\in C^1(\mathbb{R}^2\setminus\{0\})$. More precisely, choose $h_\epsilon\in C^1(\mathbb{R}^n)$ so that

$$h_\epsilon(x)=\begin{cases} g(x) & \|x\|_p\geq\epsilon\\ 0 & x=0\end{cases}\quad\text{and}\quad |x\cdot\nabla h_\epsilon(x)|\leq C\|g\|_\infty\epsilon^{-1}\quad\forall\|x\|_p\leq\epsilon.$$

Then, 0-homogeneity of g ensures

$$\left| t^{-1} \int_{\partial^* E \cap B_p(0,t)} \|\nu_E\|_p \left\langle \frac{x}{t}, \nabla h_\epsilon(x) \right\rangle \right| = \left| t^{-1} \int_{\partial^* E \cap B_p(0,\epsilon)} \|\nu_E\|_p \left\langle \frac{x}{t}, \nabla h(x) \right\rangle \right| \leq C \frac{\Phi_p(E; B_p(0,\epsilon))}{t^2}$$

which, for instance due to (5.47), approaches zero as $\epsilon \downarrow 0$. On the other hand, whenever $\epsilon < t$, the remaining two terms in (5.48) are independent of ϵ. Consequently, it follows from (5.48) that for almost every $0 < t < \tau$

$$\begin{aligned} &\frac{d}{dt}\left[t^{-1} \int_{\partial^* E \cap B_p(0,t)} \|\nu_E(x)\|_p g(x) d\mathcal{H}^1 \right] \\ &\quad = \lim_{\epsilon \downarrow 0} \frac{d}{dt} \left[t^{-1} \int_{\partial^* E \cap B_p(0,t)} \|\nu_E(x)\|_p h_\epsilon(x) d\mathcal{H}^1 \right] \\ &\quad = \lim_{\epsilon \downarrow 0} t^{-1} \int_{\partial^* E \cap B_p(0,t)} \|\nu_E(x)\|_p \left\langle \frac{x}{t}, \nabla h_\epsilon(x) \right\rangle = 0. \end{aligned} \tag{5.49}$$

In other words,

$$\frac{1}{\tau'} \int_{B_p(0,\tau')} g(x)\rho(\nu_E) d\mathcal{H}^1 = \text{constant} \qquad a.e.\ 0 < \tau' < \tau.$$

Since $g \in C^1(\mathbb{R}^n \setminus \{0\}; \mathbb{R})$ is an arbitrary 0-homogeneous function, this implies the result, see for [67, Theorem 19.3]. □

To relate Lemma 105 to information about tangents, we must better understand how tangents relate to the energy density.

Definition 106 (Enlargements)**.** If $E \subset \mathbb{R}^n$ then $E_{x,r}$ is the set defined by $E_{x,r} = \frac{E-x}{r}$.

Lemma 107. *If E is a set of locally finite perimeter, then*

$$\frac{\Phi_p(E; K_{\sigma r})}{\sigma r} = \frac{\Phi_p(E_{x,r}; K_\sigma)}{\sigma}.$$

Moreover,

$$\int_{\partial^* E\cap\{\rho r<\|y-x\|_p<\sigma r\}} \frac{\langle \frac{y-x}{f(y-x)}, \nu_E\rangle \langle \nabla \|y-x\|_p, \nabla \|\nu_E\|_p\rangle}{f(y-x)} d\mathcal{H}^{n-1}$$
$$= \int_{\partial^* E_{x,r}\cap\{\rho<\|y\|_p<\sigma\}} \frac{\langle \frac{y}{f(y)}, \nu_{E_{x,r}}\rangle \langle \nabla \|y\|_p, \nabla \|\nu_{E_{x,r}}\|_p\rangle}{f(y)} d\mathcal{H}^{n-1}.$$

Proof. Indeed, since $\nu_{E_{x,r}}(y) = \nu_E(rx+y)$ it follows that

$$\begin{aligned}\frac{\Phi_p(E_{x,r}; K_\sigma)}{\sigma} &= \frac{1}{\sigma}\int_{\partial^* E_{x,r}\cap B_p(0,\sigma)} \|\nu_{E_{x,r}}(y)\|_p d\mathcal{H}^1 \\ &= \frac{1}{\sigma}\int_{\partial^* E\cap B_p(x,r\sigma)} \|\nu_E(y)\|_p \frac{d\mathcal{H}^1}{r} \\ &= \frac{\Phi_p(E; B_p(x,r\sigma))}{r\sigma}.\end{aligned}$$

The second claim follows since the integrand is -1-homogeneous, and the measure is 1-homogeneous. ☐

The next corollary follows from the scaling from Lemma 107 and the continuity of the Φ_p-energy when restricted to energy minimizing sets, see Theorem 121.

Corollary 108. *If E is minimizing and $x \in \partial E$, $r_j \to 0$ as $j \to \infty$, and $E_{r_j} \to E_\infty$ then*

$$\Theta^1_p(E;x) = \frac{\Phi_p(E_\infty; B_p(0,\sigma))}{2\sigma} \qquad \forall \sigma > 0. \tag{5.50}$$

Finally, we conclude that blow-up limits of energy minimizers are energy minimizing cones. That is,

Corollary 109. *If $E \subset \mathbb{R}^2$ is Φ_p-energy minimizing, $x \in \partial E, r_j \to 0$, and $E_{x,r_j} \to E_\infty$ then $E = \lambda E$ for all $\lambda > 0$.*

Proof. The fact that E_∞ is energy-minimizing follows from Theorem 121. The fact that E_∞ is a cone, follows from Corollary 108 and Lemma 105. ☐

Remark 110. *We reiterate that the above theory can be implemented for sets and varifolds. Indeed, Theorem 101 only requires the first variation being zero and does not directly use minimizing. Lemmas 104, 105, and Corollary 108 only uses that the monotonicity formula applies to each set. Here, we merely used the results of Section 5.7 in order to gain monotonicity from the minimizing property, including in limiting sets.*

5.6 Anisotropic minimal cones

In this section, we consider global minimal cones in $\mathbb{R}^2$. That is, sets of the form

$$C(E) = \{\lambda x : \lambda > 0, x \in E \subset \partial B_\rho(0,1)\}. \tag{5.51}$$

First note that $E \subset \mathbb{S}^1$ must be a finite collection of points, or else the energy of the cone would be infinite.

For area minimal cones in the plane, it's known the only cones can be: (1) straight lines, and (2) a triple junction, where all three half-lines meet at $120°$ angles. We produce a new anisotropic characterization of triple junctions for anisotropic minimal cones.

The key tool to understand anisotropic minimal cones is convex duality. Given ρ as in (5.1), the convex dual to ρ, denoted ρ_* is defined by

$$\rho_*(\xi) = \sup\{x \cdot \xi : \rho(x) < 1\}.$$

If ρ is C^1 and strictly convex, then so is ρ_* and $(\rho_*)_* = \rho$. Moreover,

$$\begin{cases} \rho_*\left((D\rho)(x)\right) \equiv 1 & \text{and} \quad \rho\left((D\rho_*)(\xi)\right) \equiv 1 \\ (D\rho_*)\left[(D\rho)(x)\right] = \frac{x}{\rho(x)} & \text{and} \quad (D\rho)\left[(D\rho_*)(\xi)\right] = \frac{\xi}{\rho_*(\xi)}. \end{cases} \tag{5.52}$$

The first observation in (5.52) is why it is convenient to consider $E \subset \partial B_\rho(0,1)$ in (5.51).

Next, we note a small departure from the typical assumptions in this document and proceed using an abuse of notation. We consider the energy

$$F_\rho(V; A_0) := \int \rho(T_x V) d\|V\|,$$

for a strictly convex norm ρ, defined for either a current, varifold, or closed rectifiable set V, where $T_x V$ is the tangent space to V. In the case of currents, it makes sense to consider F_ρ for positively one-homogeneous ρ, due to the orientability of currents. F_ρ is only well-defined for norms ρ. The considerations of minimal closed rectifiable sets arise, for instance, in the setting of sliding minimizers studied by Almgren and David [6, 18].

Theorem 111. *Fix $x_1 \in \partial B_\rho(0,1)$ and $E = \{x_1, x_2, x_3\}$. If ρ is a norm, the following are equivalent.*

1. *The cone $C = C(E)$ in (5.51) is minimal with respect to the anisotropic energy F_ρ*

2. *For $j = 2, 3$, $x_j = (D\rho_*)(y_j)$ where $\{y_2, y_3\} = \partial B_{\rho_*}(-(D\rho)(x_1), 1) \cap \partial B_{\rho_*}(0, 1)$.*

Remark 112. *If ρ is strictly convex and C^1, but only positively 1-homogeneous, the proof of Theorem 111 will still show that if E is stationary, then for $j = 2, 3$, x_j take the form $x_j = (D\rho_*)(y_j)$ for some y_j satisfying*

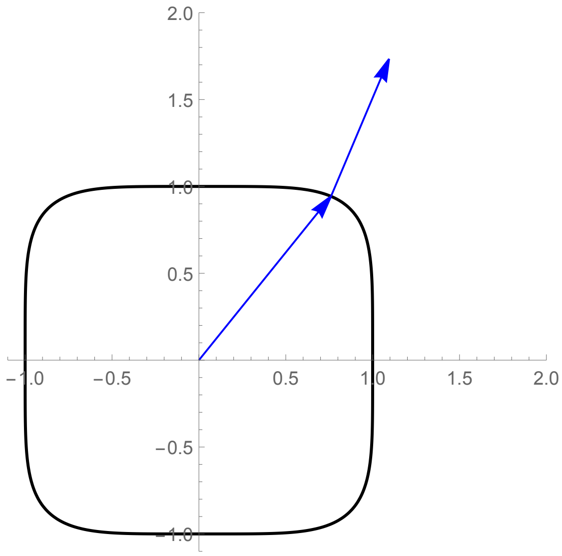
$$\rho_*(y_j) = 1 = \rho_*(-y_j - y_1) \iff y_j \in \partial B_{\rho_*}(0,1) \cap \partial \{-y_1 - B_{\rho_*}(0,1)\}.$$

Example 113. Let e_2 denote the second standard basis direction in $\mathbb{R}^2$ and define

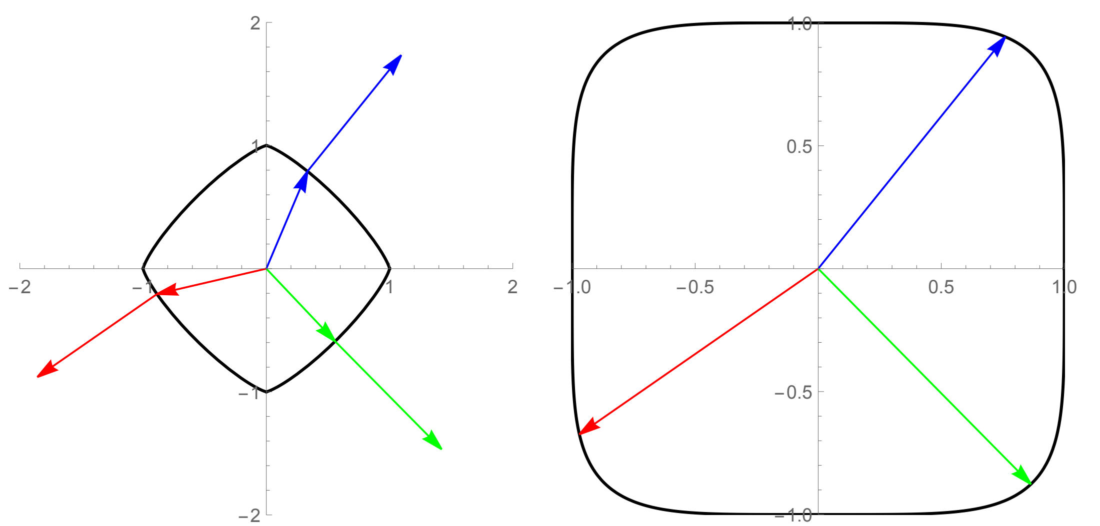
$$\rho_*(x_1, x_2) = \begin{cases} \sqrt{x_1^2 + x_2^2} & x_2 \geq 0 \\ \sqrt{x_1^2 + (10x_2)^2} & x_2 < 0. \end{cases}$$

Then $\rho_* \in C^1$ is strictly convex and positively 1-homogeneous, and therefore so is $\rho = (\rho_*)_*$.

Since $(D\rho_*)(e_2) = e_2$ it follows $(D\rho)(e_2) = e_2$. Note, if $\rho_*(y) = 1$ then $y \cdot e_2 \geq -1/10$. On the other hand, if $y \in \{\rho(-e_2 - \ \cdot\ ) = 1\}$, then $y \cdot e_2 \leq -9/10$. In particular, by Remark



(a) On the left, the set of points with $\|x\|_5 = 1$ is drawn, with a specific vector x_1 chosen. Its dual vector, y_1, is also drawn and is normal to $B_5(0,1)$ at x_1. On the right, y_1 is translated to the origin. The "dual sphere" $\partial B_{5/4}(0,1)$ is drawn with a solid line, while the "dual sphere" $\partial B_{5/4}(-y_1,1)$ is drawn with a dashed line. The vectors in green and red point to the unique points in the intersection of both dual spheres. The red and green vectors are also shown to add to the dashed blue vector, representing $-y_1$.



(b) On the left, the solid red, blue, and green vectors are translated to the origin. Their respective duals are drawn tangent to $B_{5/4}(0,1)$. On the right, the respective duals are translated to the origin, drawn inside the ball $B_5(0,1)$. The resulting three vectors create a $\|\cdot\|_5$-stationary cone.

112 it follows there is no minimal triple junction for F_ρ containing the point e_2.

Of Theorem 111. Observe that

$$F_\rho(C(E); B_\rho(0,1)) = \sum_{x\in E} \rho(x).$$

Since we only consider compactly supported variations, and straight line-segments are minimal, the only non-trivial variations one can consider are those which perturb the point of intersection. If said perturbation is in the direction of e, then letting $f_t(C)$ denote the set constructed by sliding the intersection point of $C(E)$ a distance t in the direction e, we find

$$F_\rho(f_t(C); B_\rho(0,1)) = \sum_{x\in E} \rho(x+te).$$

Hence,

$$\left.\frac{d}{dt}\right|_{t=0} F_\rho(f_t(C); B_\rho(0,1)) = \sum_{x\in E} e\cdot (D\rho)(x).$$

Since e is arbitrary, this means $C(E)$ is stationary if and only if

$$\delta F_\rho(C; B_\rho(0,1)) := \sum_{x\in E} (D\rho)(x) \equiv 0. \tag{5.53}$$

We will now show $1 \implies 2$. Indeed, assume $E = \{x_1, \ldots, x_3\}$ and $C(E)$ is stationary. Denote $y_i = (D\rho)(x_i)$ for $i = 1,2,3$. Then, (5.52) implies $\rho_*(y_i) = 1$ for all i. On the other hand, (5.53) implies $-y_1 - y_3 = y_2$ and $-y_1 - y_2 = y_3$. That is, $\rho_*((-y_1) - y_2) = \rho_*((-y_1) - y_2) = 1$. Therefore, $y_2, y_3 \in \partial B_{\rho_*}(-y_1, 1) \cap \partial B_{\rho_*}(0,1)$. It now follows from (5.52) and $\rho(x_j) = 1$ that

$$(D\rho_*)(y_j) = (D\rho_*)\left[(D\rho)(x_j)\right] = \frac{x_j}{\rho(x_j)} = x_j,$$

as desired.

It remains to show $2 \implies 1$. First note, by strict convexity of ρ and $(-D\rho)(x_1) \in \partial B_\rho(0,1)$, it follows that the intersection $\partial B_{\rho_*}(0,1) \cap \partial B_{\rho_*}(-(D\rho)(x_1), 1)$ has precisely two points. Let $y_1 = (D\rho)(x_1)$. For $j = 2, 3$, translation invariance and symmetry about the origin of ρ_* implies,

$$-(y_1+y_j) \in \partial B_{\rho_*}(0,1) \iff y_1+y_j \in \partial B_{\rho_*}(0,1) \iff y_j \in \partial B_{\rho_*}(-y_1,1) \iff -y_j \in \partial B_{\rho_*}(y_1,1).$$

In particular, $y_j \in \partial B_{\rho_*}(0,1) \cap \partial B_{\rho_*}(-y_1,1)$ is equivalent to $-y_1 - y_j \in \partial B_{\rho_*}(0,1) \cap \partial B_{\rho_*}(-y_1,1)$.

Strict convexity of ρ_* and $\rho_*(y_j) = 1$ for all $j = 1, 2, 3$ ensures that for $j = 2, 3$, $y_j \neq -y_j - y_1$. Since the intersection only has two points, it must be that $y_2 = -y_3 - y_1$, or equivalently, $y_2 + y_3 = -y_1$. But then, if for $j = 1, 2, 3$, $y_j = (D\rho)(x_j)$ it follows from (5.53) that $C(E)$ is minimal as desired.

We note, that this is a sum of points in $\partial B_{\rho_*}(0,1)$ by (5.52). Moreover, if $E_1 = E \setminus \{x_1\}$, then

$$-(D\rho)(x_1) = \sum_{x \in E_1} (D\rho)(x).$$

So, in the case that $E = \{x_1, x_2, x_3\}$, we have not only that $\rho_*\left((D\rho)(x_i)\right) = 1$, but also that $(D\rho)(x_1) + (D\rho)(x_2)$

□

5.7 Appendix: Compactness of energy minimizers and related tools

First we'll recall the relative isoperimetric inequality

Theorem 114. *If $n \geq 2$ and $t \in (0,1)$, $x \in \mathbb{R}^n$ and $r > 0$, then there exists a positive constant $c(n,t)$ such that*

$$P(E; B(x,r)) \geq c(n,t)|E \cap B(x,r)|^{\frac{n-1}{n}} \tag{5.54}$$

for every set of locally finite perimeter E such that $|E \cap B(x,r)| \leq t|B(x,r)|$. In particular, *choosing* $t = \frac{1}{2}$ *one learns,*

$$P(E; B(x,r)) \geq c(n) \min \{|E \cap B(x,r)|, |B(x,r) \setminus E|\}^{\frac{n-1}{n}} . \tag{5.55}$$

The isoperimetric inequalities above both are extremely useful for relating information about the smallness of the perimeter of a set (or largeness of a volume of a set, respectively) to the smallness of the volume of the set (or largeness of the perimeter of the set, respectively).

It is frequently useful to consider certain representations of sets of locally finite perimeter. One convenient representative is the one defined by the measure-theoretically relevant points in a set. More precisely, for a set of locally finite perimeter E, we define

$$E^{(s)} := \left\{ x \in \mathbb{R}^n : \lim_{r \downarrow 0} \frac{|E \cap B(x,r)|}{|B(x,r)|} = s \right\} .$$

It is well know that $\nu_E = \nu_{E^{(1)}}$ and $\nu_{E^c} = \nu_{E^{(0)}} = -\nu_E$. That is the set $E^{(1)}$ is a choice of representation of E. Federer defined the essential boundary

$$\partial^* E = \mathbb{R}^n \setminus \left(E^{(0)} \cup E^{(1)} \right) .$$

The following theorem can be found, for instance, in [49, Theorem 16.2]

Theorem 115 (Federer's theorem)**.** *If E is a set of locally finite perimeter in $\mathbb{R}^n$, then $\partial^* E \subset E^{(1/2)} \subset \partial^e E$. In fact,*

$$\mathcal{H}^{n-1}(\partial^e E \setminus \partial^* E) = 0.$$

In particular, for any Borel set $M \subset \mathbb{R}^n$,

$$M \approx \left(M \cap E^{(1)} \right) \cup \left(M \cap E^{(0)} \right) \cup \left(M \cap E^{(1/2)} \right) ,$$

where $M_1 \approx M_2$ means M_1 and M_2 are $\mathcal{H}^{n-1}$-equivalent, that is, $\mathcal{H}^{n-1}(M_1 \Delta M_2) = 0$.

Furthermore, it is known that $\mathrm{spt}\mu_{\mathrm{E}} = \mathrm{\mathbb{R}^n} \setminus \left(\mathrm{E^{(0)}} \cup \mathrm{E^{(1)}} \right)$, and $\mathcal{H}^{n-1} \left(E^{(1/2)} \Delta \partial^* E \right) = 0$.

In fact,

$$\mathcal{H}^{n-1}\left(\mathbb{R}^n \setminus \left(E^{(0)} \cup E^{(1/2)} \cup E^{(1)}\right)\right) = 0.$$

When considering two sets of locally finite perimeter E and F, define the sets

$$\{\nu_E = \nu_F\} := \{x \in \partial^* E \cap \partial^* F : \nu_E(x) = \nu_F(x)\}$$
$$\{\nu_E = -\nu_F\} := \{x \in \partial^* E \cap \partial^* F : \nu_E(x) = -\nu_F(x)\}.$$

The next technical theorem, is geometrically evident, but the technical details of the proof can be found, for instance, in [49, Theorem 16.3].

Theorem 116 (Set operations on Gauss-Green measures)**.** *If E and F are sets of locally finite perimeter, then*

$$\begin{aligned}
\mu_{E\cap F} &= \mu_E \llcorner F^{(1)} + \mu_F \llcorner E^{(1)} + \nu_E \mathcal{H}^{n-1} \llcorner \{\nu_E = \nu_F\}, \\
\mu_{E\setminus F} &= \mu_E \llcorner F^{(0)} - \mu_F \llcorner E^{(1)} + \nu_E \mathcal{H}^{n-1} \llcorner \{\nu_E = -\nu_F\}, \\
\mu_{E\cup F} &= \mu_E \llcorner F^{(0)} + \mu_F \llcorner E^{(0)} + \nu_E \mathcal{H}^{n-1} \llcorner \{\nu_E = \nu_F\}.
\end{aligned} \tag{5.56}$$

Equivalent statements to the above can be said about the reduced boundary. The only one we use here is,

$$\partial^* (E \cup F) \approx \left(F^{(0)} \cap \partial^* E\right) \cup \left(E^{(0)} \cap \partial^* F\right) \cup \{\nu_E = \nu_F\}\,. \tag{5.57}$$

Theorem 116 is a technical tool that is useful to compute the energy of modified sets. For example, in proving Lemma 117 and 118. Lemma 117 will be used anytime we build a competitor. However, it is particularly useful when considering convergent sequences of energy minimizing sets. In those applications, the set F will play the role of a competitor to the limiting set E and the set $\tilde{E}$ will eventually be replaced with each set E_h in the sequence of minimizers E_h which converge to E. Finally $\tilde{F}$ will be a desirable competitor for the set $\tilde{E}$. Its desirability will follow from (5.61).

The proof of Lemma 117 follows from modifications of the proof [49, Theorem 16.16]. See [66, Proposition 4.3] for an example of necessary modifications, but for a different energy.

Lemma 117 (Φ_p-energy of modified competitors)**.** *Suppose $\tilde{A}_0 \subset\subset A \subset \mathbb{R}^n$ are both open sets, $\tilde{A}_0$ has finite perimeter. If $\tilde{E}$ and F, G are sets of locally finite perimeter satisfying*

$$\mathcal{H}^{n-1}\left(\partial^* G \cap \partial^* \tilde{E}\right) = 0 = \mathcal{H}^{n-1}\left(\partial^* G \cap \partial^* F\right). \tag{5.58}$$

Define

$$\tilde{F} := (F \cap G) \cup \left(\tilde{E} \setminus G\right) \tag{5.59}$$

then $\tilde{F}$ is a set of locally finite perimeter, and

$$\Phi_p\left(\tilde{F}; \tilde{A}_0\right) = \Phi_p\left(\tilde{E}; \tilde{A}_0 \setminus \overline{G}\right) + \Phi_p(F; G) + \Phi_p(G; E\Delta F). \tag{5.60}$$

In particular, if $\tilde{E}\Delta F \subset\subset G \subset\subset \tilde{A}_0$ then

$$\Phi_p(\tilde{F}; \tilde{A}_0) = \Phi_p(F; \tilde{A}_0 \setminus \partial G) = \Phi_p(F; \tilde{A}_0). \tag{5.61}$$

Lemma 118 will be used to demonstrate that sequences of minimizers satisfy the properties of a pre-compactness theorem. The proof of Lemma 118 arises as a consequence of the comparability of norms on finite vector spaces and slight modifications of the proof in [49, Theorem]. To see how this comparison and modifications are done, but for a different energy, see [66, Proposition 4.10].

Lemma 118 (Volume density bounds)**.** *Let A be an open set and E be a (Φ, r_0)-minimizer in A. For each $x \in \partial E \cap A$, and all $r \in (0, d_x)$ where*

$$d_x = \min\{\text{dist}(x, \partial A), r_0\} \tag{5.62}$$

E satisfies the following density bounds:

$$\frac{1}{\left(n^{\frac{p-2}{2p}}+1\right)^n} \leq \frac{\mathcal{H}^n(E\cap B(x,r))}{\omega_n r^n} \leq 1-\frac{1}{\left(n^{\frac{p-2}{2p}}+1\right)^n} \tag{5.63}$$

and

$$c(n,p)\frac{\omega_n^{n-1}}{\left(n^{\frac{p-2}{2p}}+1\right)^{n-1}} \leq \frac{\mathcal{H}^{n-1}(\partial^* E\cap B(x,r))}{r^{n-1}} \leq n\cdot n^{\frac{p-2}{2p}}\omega_n, \tag{5.64}$$

where $c(n,p)$ is the constant $c(n,t)$ from the relative isoperimetric inequality with t satisfies

$$t=\frac{1}{\left(n^{\frac{p-2}{2p}}+1\right)^n}$$

In particular,

$$\mathcal{H}^{n-1}(A\cap(\partial E\setminus\partial^* E))=0.$$

By Lemma 118, it follows that if E is a Φ_p-energy minimizer up to scale r_0 in $\mathbb{R}^n$ then E has a representative in the class $\mathcal{A}(C_A(n,p),r_0)$ where $C_A(n,p)$ is the Ahlfors $(n-1)$-regularity constant implicit in Lemma 118, and

$$\mathcal{A}(C_A,r_0)=\left\{E\subset\mathbb{R}^n \,\middle|\, \begin{array}{c}E \text{ is a set of locally finite perimiter satisfying } \partial E=\mathrm{spt}\mu_{\mathrm{E}} \text{ and its perimeter}\\ \text{measure } |\mu_E| \text{ is } (n-1)\text{-Ahlfors regular up to scale } r_0 \text{ with constant } C_A\end{array}\right\}. \tag{5.65}$$

The following compactness theorem can be found in [15, Theorem 3.6].

Theorem 119. *If $\{E_k\}\subset\mathcal{A}(C_A,r_0)$ with $0\in\partial E_k$ for all $k\geq 1$, there exists a subsequence $\{E_{k_j}\}$, a set E of locally finite perimeter, and a non-negative Radon measure, μ, such that*

$$E_{k_j}\xrightarrow{L^1_{loc}(\mathbb{R}^n)} E,\quad \mu_{E_{k_j}}\rightharpoonup\mu_E \quad \textit{and} \quad |\mu_{E_{k_j}}|\rightharpoonup\mu. \tag{5.66}$$

Additionally, $\partial E=\mathrm{spt}\mu_{\mathrm{E}}$ and μ is $(n-1)$-Ahlfors regular up to scale r_0 with constant C_A. Furthermore, $|\mu_E|\leq\mu$ and

1. *If $x\in\partial E$, then there exists $x_{k_j}\in\partial E_{k_j}$ such that $x_{k_j}\to x$.*

2. If $x \in \mathrm{spt}\mu$, then there exists $x_{k_j} \in \partial E_{k_j}$ such that $x_{k_j} \to x$.

3. If $x_{k_j} \in \partial E_{k_j}$ and $x_{k_j} \to x$ then $x \in \mathrm{spt}\mu$.

Combining Lemma 118 and Theorem 119 we achieve our pre-compactness theorem for Φ_p-minimizers up to scale r_0.

Theorem 120 (Pre-compactness of Φ_p-minimizers up to scale r_0)**.** *If $\{E_h\}_{h\in\mathbb{N}}$ is a sequence of Φ_p-minimizers up to scale r_0 in the open set A, then for every $A_0 \subset\subset A$ with finite perimeter, there exists a subsequence E_{h_k} and a set of finite perimeter $E \subset A_0$ and a Radon measure μ supported on A_0 that is Ahlfors regular up to scale r_0 with constant C_A, such that*

$$A_0 \cap E_{h_k} \to E, \qquad \mu_{A_0 \cap E_{h_k}} \rightharpoonup \mu_E, \qquad |\mu_{E_{h_k}}| \rightharpoonup \mu.$$

Additionally, $\partial E = \mathrm{spt}\mu_{\mathrm{E}}, |\mu_{\mathrm{E}}| \leq \mu$, and

1. If $x \in \partial E$, then there exists $x_{h_k} \in \partial E_{h_k}$ such that $x_{h_k} \to x$.

2. If $x \in \mathrm{spt}\mu$, then there exists $x_{h_k} \in \partial E_{h_k}$ such that $x_{h_k} \to x$.

3. If $x_{h_k} \in \partial E_{h_k}$ and $x_{h_k} \to x$ then $x \in \mathrm{spt}\mu$.

There are two lingering questions after Theorem 120. First, is the limiting set E a Φ_p-minimizer up to scale r_0? Second, is $\mu = |\mu_E|$? The Theorem 121 provides a positive answer to both questions. The proof of Theorem 121 follows from small modifications of [49, Theorem 21.14]. See [66, Proposition 4.13] for an example of the necessary modifications, but for a different anisotropic energy.

Theorem 121 (Anisotropic closure theorem)**.** *If $\{E_h\}_{h\in\mathbb{N}}$ is a sequence of $\Phi_p(\cdot\ ; A)$ minimizers for the open set $A \subset \mathbb{R}^n$ and if $A_0 \subset\subset A$ is an open set finite perimeter such that $A_0 \cap E_h \to E$ for some set of finite perimeter E, then E is a $\Phi_p(\cdot\ ; A_0)$-energy minimizer.*

Moreover,

$$D_{|\mu_E|}\mu \leq 1 \quad \mu_E \text{ a.e. } x \in A_0 \quad \text{and} \quad D_\mu|\mu_E| \geq c_{n,p} > 0 \quad \mu \text{ a.e. } x \in A_0. \tag{5.67}$$

In particular, if $x_h \in \partial E_h$ *and* $x_h \to x$ *then* $x \in \operatorname{spt}|\mu_{\mathrm{E}}|$. *Moreover, for almost every* $x \in \mathbb{R}^n$ *and almost every* s *small enough,* $\lim_{h\to\infty} \Phi_p(E_h; B(x,s))$ *exists and equals* $\Phi_p(E; B(x,s))$.

Lemma 122. *If* $\{E_h\}$ *are a sequence of* Φ_p*-minimizers and* $E_h \to E$*, then*

$$\Phi_p(E; B(x,s)) = \lim_{h\to\infty} \Phi_p(E_h; B(x,s)) \tag{5.68}$$

for almost every $x \in \partial E$ *and* $s > 0$. *The set of* s *depends on* x.

Proof. Choose $x \in \partial E$ and $s < r$ so that

$$\mathcal{H}^{n-1}\left(\partial^* E \cap \partial B(x,s)\right) = 0 = \mathcal{H}^{n-1}\left(\partial^* E_h \cap \partial B(x,s)\right)$$

for all $h \in \mathbb{N}$ and

$$\liminf_{h\to\infty} \mathcal{H}^{n-1}\left(\partial B(x,s) \cap \left(E_h^{(1)} \Delta E^{(1)}\right)\right) = 0.$$

As before, the set of such s is $\mathcal{H}^1$-a.e. $s \in (0,r)$ Then define

$$F_h = (E \cap B(x,s)) \cup (E_h \setminus B(x,s)).$$

We know that $\Phi_p(E; B(x,s)) \leq \liminf_h \Phi_p(E_h; B(x,s))$ from lower-seminconinuity. On the other hand, using the minimizing property of E_h and (5.60) we find,

$$\begin{aligned}\Phi_p(E_h, B(x,s)) \leq \Phi_p(F_h, B(x,s)) &= \Phi_p(E; B(x,s)) + \Phi_p(B(x,s); E\Delta E_h)\\ &\leq \Phi_p(E; B(x,s)) + P(B(x,s); E\Delta E_h).\end{aligned}$$

Taking the lim inf on both sides yields

$$\liminf_h \Phi_p(E_h; B(x,s)) \le \Phi_p(E; B(x,s)).$$

Since $E_h \to E$, it follows that we could now repeat the argument with any subsequence $E_{h(k)}$ (since all such subsequences converge to E). In particular, if we choose $E_{h(k)}$ such that $\lim_{k\to\infty} \Phi_p(E_{h(k)}; B(x,s)) = \limsup_h \Phi_p(E_{h(k)}; B(x,s))$ then (with help from lower-semicontinuity) we learn that

$$\limsup_h \Phi_p(E_h; B(x,s)) = \liminf_k \Phi_p(E_{h(k)}; B(x,s)) \le \Phi_p(E_h; B(x,s)) \le \liminf_h \Phi_p(E_h; B(x,s))$$

□

Chapter 6

A HARNACK INEQUALITY FOR ANISOTROPIC PDES

6.1 Introduction

While uniformly elliptic partial differential equations are in general well understood, degenerate elliptic PDEs arising in the case when the lower ellipticity constant is not bounded away from zero present many challenges. These PDEs are often studied from the perspective that they are *close* to a uniformly elliptic PDE. In this paper we propose a different approach for operators that fail to be uniformly elliptic, but instead are γ-homogeneous. More precisely we focus on operators which arise naturally in the study of anisotropic geometric variational problems. That is, for $1 < \gamma < \infty$,

$$\int_\Omega \mathfrak{D}(u, \varphi) = 0 \quad \forall \varphi \in W_0^{1,\gamma'}(\Omega), \tag{6.1}$$

where $\mathfrak{D} : W^{1,\gamma}(\Omega) \times W_0^{1,\gamma'}(\Omega) \to W^{1,1}(\Omega)$ is defined by

$$\mathfrak{D}(u, \varphi) = \left\langle \rho(x, Du)^{\gamma-1}(D\rho(x, \cdot))(Du), D\varphi \right\rangle, \tag{6.2}$$

and $\rho(x, \cdot)$ is 1-homogeneous and strictly convex in the second variable. In particular, PDEs of the form (6.1) (when there is no dependence of ρ on x) arise as the Euler-Lagrange equation of the functional

$$\int_\Omega \rho(Du)^\gamma.$$

For this reason we consider the γ in (6.1) as defining the homogeneity of the PDE. A Moser iteration approach allows us to exploit this γ-homogeneity to prove a Harnack inequality and a strong maximum principle, in addition to a Liouville-type theorem for global solutions. These results arise as corollaries of the theorems described below.

For any real number $\alpha > 1$ we let α' denote the Holder conjugate of α, i.e., $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$. Also, whenever $\rho(x, \cdot)$ is a 1-homogeneous function $\rho_*(x, \cdot)$ denotes its convex dual defined by $\rho_*(x, \zeta) = \sup_{\rho(x,\xi)<1} \zeta \cdot \xi$. Throughout this paper we consider the following PDE generalizing the PDE in (6.1)

$$\int_\Omega \langle \rho(x,(Du))^{\gamma-1}(D\rho(x,\cdot))(Du), D\varphi\rangle = \int_\Omega \left\langle \vec{F}, D\varphi\right\rangle + f\varphi \quad \forall\varphi\in W^{1,\gamma'}(\Omega) \tag{6.3}$$

where

$$\vec{F}\in L^q_{\text{loc}}(\Omega) \quad \text{and} \quad f\in L^{\frac{q}{\gamma'}}_{\text{loc}}(\Omega) \quad \text{for some } q>\frac{n}{\gamma-1}. \tag{6.4}$$

Theorem 123. *Suppose that for some $1<\gamma<n$, $u\in W^{1,\gamma}(\Omega)$ is a non-negative subsolution of (6.3), that $\rho:\Omega\times\mathbb{R}^n\setminus\{0\}\to(0,\infty)$ satisfies (6.7), (6.8), (6.9), and (6.10), and $\vec{F}, f$, and q satisfy (6.4). If $0<r<R<1$ and $B_R\subset\Omega$, then for all $0<p<\infty$, there exists $C=C(n,\gamma,\nu,p)$ so that*

$$\sup_{B_r} u \le C\left[(R-r)^{-\frac{n}{p}}\|u\|_{L^p(B_R)} + R^\delta\|\rho_*(x,\vec{F})\|^{\frac{1}{\gamma-1}}_{L^q(B_R)} + R^{\gamma'\delta}\|f\|^{\frac{1}{\gamma-1}}_{L^{\frac{q}{\gamma'}}}\right] \tag{6.5}$$

where $\delta = 1-\frac{n}{q(\gamma-1)}>0$.

In the case when $\gamma=2=\gamma'$, which includes the elliptic case, (6.5) recovers the local-boundedness results of De Giorgi, Nash, and Moser for elliptic divergence form equations. One can verify this includes the elliptic case with symmetric coefficients by considering $\rho(x,\xi)=|S(x)\xi|$ where $S(x)=A(x)^{1/2}$, which ensures $\langle A(x)\xi,\xi\rangle = |S(x)\xi|^2$. The next example demonstrates that the $\gamma=2$ case is not restricted to the uniformly elliptic setting.

Example 124. If $\gamma=2$ and $\rho=\|\cdot\|_{\ell^p}$ for some $p>2$, then (6.1) with integrand (6.2) becomes the PDE $\operatorname{div}(D(\|\cdot\|^2_{\ell^p})(Du))=0$. A computation yields

$$(\partial_i\partial_j\|\cdot\|^2_{\ell^p})(x) = 2(p-1)\left[\delta_{ij}\left(\frac{x_i}{\|x\|_p}\right)^{p-2} - \frac{x_i|x_i|^{p-2}x_j|x_j|^{p-2}}{\|x\|_p^{2(p-1)}}\right].$$

Notably, $(D^2\|\cdot\|^2_{\ell^p})(e)\equiv 0$ whenever e is a standard basis vector.

The fact that Example 124 is handled with the methods in this paper highlights that the homogeneity, not the uniform strong convexity, is what matters for studying the 0th order regularity of this type of PDE. The convexity conditions typically imposed when studying elliptic equations additionally requires the degree of homogeneity is fixed at 2.

Convexity has always played a strong role in the Calculus of Variations. It is known that in certain settings specific convexity conditions are necessary to guarantee higher-order regularity. A natural question is whether there exist weaker convexity conditions depending upon the homogeneity that are sufficient to imply higher-order regularity results. One could hope that teasing apart the roles of homogeneity and convexity might recover simpler conditions for higher-order regularity than presently exist.

Example 125. We revisit the previous example in more generality. Consider $\rho(\cdot) = \|A(\cdot)\|_{\ell^p}$ for $p > 2$, where A is an orthogonal matrix. This time, we consider any $\gamma > 2$. The PDE under consideration takes the form

$$\operatorname{div}(D(\|A(\cdot)\|_{\ell^p}^{\gamma})(Du)) = 0.$$

We note that when $A = I$ and $\gamma = p$ this is precisely the so-called pseudo p-Laplacian or anisotropic p-Laplacian. Our results imply a new Harnack inequality for solutions to the pseudo p-Laplacian.

By the previous example, whenever $A(Du) \in \{\pm e_1, \pm e_2, \dots, \pm e_n\}$ the Hessian, $D^2\|A(\cdot)\|_{\ell^p}^{\gamma}(Du)$, is zero. Performing a Taylor series expansion of $\|A(\cdot)\|_{\ell^p}^{\gamma}$ around an arbitrary $\xi_1 \in \mathbb{R}^n \setminus \{0\}$, you find for $\xi_2 \in \langle \xi_1 \rangle^{\perp}$ small enough,

$$\|A(\xi_1 + \xi_2)\|_p^{\gamma} - \|A(\xi_1)\|_p^{\gamma} = O\left(\sum_{e_i : A(\xi_1)\cdot e_i = 0} |\xi_2 \cdot e_i|^p \right) + O\left(\sum_{e_i : A(\xi_1)\cdot e_i \neq 0} |\xi_2 \cdot e_i|^2 \right)$$

recovering different growth conditions in different directions. This seems to demonstrate that the gains and losses in energy of functional

$$\int_{\Omega} \|A(Du)\|_{\ell^p}^{p} \tag{6.6}$$

by small perturbations behaves similarly to a "mixed growth" problem. The Euler-Lagrange equation for minimizers of (6.6) is covered as a special case of our results.

Finally, without straying too far from this same family of PDEs, we note that the techniques below even cover the case when $\rho(x,\xi) = \|A(x)[\xi]\|_{\ell^{p(x)}}$ so long as $p(x) \in (1,\infty)$ and $A(x)$ has reasonably bounded eigenvalues.

The second main result of this paper is for non-negative supersolutions.

Theorem 126. *Suppose* $1 < \gamma < n$ *and* $u \in W^{1,\gamma}(\Omega)$ *is a non-negative supersolution to* (6.3)*, for some* ρ *satisfying* (6.7) - (6.10)*. Assume* $\vec{F}$*,* f*, and* q *are as in* (6.4)*. If* $0 < r < R < 1$ *and* $B_R \subset \Omega$*, then for all* $0 < p < \frac{n(\gamma-1)}{n-\gamma}$ *and all* $0 < \theta < \tau < 1$*, there exists* $C = C(n,\gamma,\nu,\Lambda,q,p,\theta,\tau) > 0$ *so that*

$$\inf_{B_{\theta R}} u + R^{\delta}\|\rho_*(\vec{F})\|_{L^q(B_R)} + R^{\gamma'\delta}\|f\|_{L^{\frac{q}{\gamma'}}(B_R)} \geq CR^{-\frac{n}{p}}\|u\|_{L^p(B_{\tau R})},$$

where $\delta = 1 - \frac{n}{q(\gamma-1)}$.

In the case $\gamma = 2 = \gamma'$ this recovers Moser's weak Harnack inequality. Moreover, we want to emphasize that the above theorem does not require any *a priori* boundedness of u.

As in the classical case, and addressed in Section 6.3, these two main theorems can be combined to achieve a strong maximum principle and a Harnack inequality. Hence as corollaries these theorems also prove C^{α}-regularity of solutions and a Liouville-type theorem for bounded solutions on $\mathbb{R}^n$.

In the preparation of this paper the author learned about the very recent work in [12]. After discovering that paper we learned that there is also a wealth of literature about minimizers with Orlicz-type growth conditions. See, for instance, [41, 13, 7], and the citations therein. In the special case where $\vec{F}, f = 0$ the works of [13, 12] recover some of the results of this paper. Using Orlicz spaces they address additional difficulties including equations with different upper and lower (almost)-homogeneities of the integrand. In the isotropic setting, [7] shows a Harnack inequality similar to the one in Theorem 139. Meanwhile [71] proves a general Harnack inequality, while assuming that the terms $\vec{F}, f$ are in L^{∞}.

Also during preparation of this paper [33] proved a Liouville type theorem when $\gamma = 2$ and ρ has strictly positive Hessian in the sense that

$$\lambda^2|\zeta|^2 \leq (\partial_{\xi_i}\partial_{\xi_j}\rho)(\xi)\zeta_i\zeta_j \leq \Lambda|\zeta|^2 \qquad \forall \zeta \in \xi^{\perp}.$$

Their techniques made use of an interesting monotonicity formula.

6.2 Notation and Preliminaries

Throughout we will suppose $\rho : \Omega \times \mathbb{R}^n \setminus \{0\} \to (0, \infty)$ is so that

$$\begin{cases} \rho(x, \cdot) \in C^1(\mathbb{R}^n \setminus \{0\}) & \forall x \in \Omega \\ \rho(\cdot, \xi) \in L^\infty(\Omega) & \forall \xi \in \mathbb{R}^n \setminus \{0\}. \end{cases} \tag{6.7}$$

and that ρ positively 1-homogeneous function in its second variable, i.e.,

$$\rho(x, \lambda\xi) = \lambda\rho(x, \xi) \qquad \forall \lambda > 0, \quad \forall x \in \Omega, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}. \tag{6.8}$$

We further assume that $\rho(x, \cdot)$ is strictly convex in the sense that

$$\rho(x, \xi_1 + \xi_2) < \rho(x, \xi_1) + \rho(x, \xi_2) \quad \forall x \in \Omega, \quad \forall \xi_1 \neq \lambda\xi_2 \in \mathbb{R}^n \setminus \{0\}. \tag{6.9}$$

Finally, we assume there exists $0 < \nu \leq \lambda < \infty$ independent of x so that

$$\nu \leq \rho(x, \xi) \leq \Lambda \qquad \forall |\xi| = 1, \qquad \forall x \in \Omega. \tag{6.10}$$

We will let ρ_x denote $\rho(x, \cdot)$ to simplify notation. Namely, $(D\rho_x)(Du) = (D\rho(x, \cdot))(Du)$. We say that $u \in W^{1,\gamma}(\Omega)$ is a subsolution (supersolution) to (6.3) if

$$\int_\Omega \left\langle \rho_x(Du)^{\gamma-1}(D\rho_x)(Du), D\varphi \right\rangle \leq (\geq) \int_\Omega \left\langle \vec{F}, D\varphi \right\rangle + f\varphi$$

for all non-negative $\varphi \in W_0^{1,\gamma'}(\Omega)$. We say that u is a solution if it is both a subsolution and supersolution.

Given a real number, say α, in $(1,\infty)$ or $[1,n)$, we will respectively always let α' and α^* denote the Holder and Sobolev exponents. That is,

$$\frac{1}{\alpha} + \frac{1}{\alpha'} = 1 \qquad \alpha^* = \frac{n\alpha}{n-\alpha}.$$

Theorem 127 (Gagliado-Nirenberg-Sobolev)**.** *If $u \in W_0^{1,\gamma}(\Omega)$ then there exists $C = C(n,\gamma) > 0$ so that*

$$\|u\|_{L^{\gamma^*}(\Omega)} \leq C\|Du\|_{L^{\gamma}(\Omega)}$$

Theorem 128 (Poincare in a ball)**.** *For each $1 \leq \gamma < n$ there exists a $C = C(n,\gamma)$ so that*

$$\left(r^{-n}\int_{B(x,r)} |f-(f)_{x,r}|^{\gamma^*} dy\right)^{\frac{1}{\gamma^*}} \leq C_2 r^{1-\frac{n}{\gamma}} \left(\int_{B(x,r)} |Df|^{\gamma} dy\right)^{\frac{1}{\gamma}}$$

Remark 129. *Since ρ will always satisfies* (6.7), (6.8)*, and* (6.9)*, if it also solves the first inequality in* (6.10) *then the Sobolev embedding theorem can be re-written as*

$$\|u\|_{L^{\gamma^*}(\Omega)} \leq C\nu^{-1}\|\rho_x(Du)\|_{L^{\gamma}(\Omega)}.$$

Similarly Poincare in a ball can be re-written as

$$\left(r^{-n}\int |f-(f)_{x,r}|^{\gamma^*} dy\right)^{\frac{1}{\gamma^*}} \leq C_2 \nu^{-1} r^{1-\frac{n}{\gamma}} \left(\int_{B(x,r)} \rho_x(Df)^{\gamma} dy\right)^{\frac{1}{\gamma}}.$$

Given $\rho : \Omega \times \mathbb{R}^n \setminus \{0\} \to (0,\infty)$ as in (6.7), (6.8), (6.9), define $\rho_* : \Omega \times \mathbb{R}^n \setminus \{0\} \to (0,\infty)$ by

$$\rho_*(x,\xi^*) = \sup_{\rho(x,\xi)<1} \xi \cdot \xi^*.$$

That is, $\rho_*(x,\cdot)$ is the convex dual of $\rho(x,\cdot)$ for all $x \in \Omega$. We record for the reader's convenience a few facts that are frequently used:

Proposition 130. *Let* $a, b, c, \epsilon > 0$*,* $\alpha \in (1, \infty)$*,* ρ *as in* (6.7)*,* (6.8)*, and* (6.9)*. Suppose* $\xi_1, \xi_2 \in \mathbb{R}^n \setminus \{0\}$*.*

- *Young's inequality says*

$$abc \leq a\epsilon^{\alpha}\frac{b^{\alpha}}{\alpha} + a\epsilon^{-\alpha'}\frac{c^{\alpha'}}{\alpha'}.$$

- *Fenchel's inequality guarantees*

$$\xi_1 \cdot \xi_2 \leq \rho(\xi_1)\rho_*(\xi_2)$$

- *It holds,*

$$\rho_*(x, D\rho(\xi_1)) \equiv 1. \tag{6.11}$$

- *If* ρ *satisfies* (6.8) *and* (6.9) *then so does* ρ_**.*

- *If* ρ *satisfies* (6.10) *then for all* $|\xi| = 1$*,*

$$\Lambda^{-1} \leq \rho_*(\xi) \leq \nu^{-1}.$$

  In particular, $\vec{F} \in L^q(\Omega)$ *if and only if* $\rho_*(\cdot, \vec{F}) \in L^q(\Omega)$

A warm-up exercise is the following Cacciopolli type inequality.

Lemma 131. *If* u *is a subsolution to* (6.3) *with* $\vec{F}, f \equiv 0$ *and* $1 < \gamma < \infty$*, then*

$$\|\eta\rho_x(Du)\|_{L^\gamma(\Omega)} \leq C(n, \gamma)\|u\rho_x(D\eta)\|_{L^\gamma(\Omega)}.$$

Proof. Consider $\varphi = \eta^\gamma u$. Then $D\varphi = \gamma\eta^{\gamma-1}uD\eta + \eta^\gamma Du$. So, using the 1-homogeneity of ρ

and Fenchel's inequality,

$$\begin{aligned}&\big\langle \rho_x(Du)^{\gamma-1}(D\rho_x)(Du), D\varphi\big\rangle\\ &\geq -\gamma(\eta\rho_x(Du))^{\gamma-1}\rho_*(x,(D\rho_x)(Du))u\rho_x(D\eta)+\eta^\gamma\rho_x(Du)^\gamma.\end{aligned}$$

Using u is a subsolution with $\vec{F}, f\equiv 0$, and (6.11) yields

$$\begin{aligned}\int_\Omega \eta^\gamma\rho_x(Du)^\gamma &\leq \gamma\int_\Omega(\eta\rho_x(Du))^{\gamma-1}u\rho_x(D\eta)\\ &\leq \left(\int(\eta\rho(Du))^\gamma\right)^{1-\frac{1}{\gamma}}\left(\int_\Omega u^\gamma\rho(D\eta)^\gamma\right)^{\frac{1}{\gamma}}.\end{aligned}$$

Dividing completes the proof. □

Finally, we recall a technical lemma for later use.

Lemma 132. *Let* ω,σ *be non-decreasing functions in* $(0,R]$. *Suppose there exists* $0<\tau,\tilde{\delta}<1$ *so that for all* $r\leq R$,

$$\omega(\tau r)\leq\tilde{\delta}\omega(r)+\sigma(r).$$

Then for any $\mu\in(0,1)$ *and* $r\leq R$

$$\omega(r)\leq C\left\{\left(\frac{r}{R}\right)^\alpha\omega(R)+\sigma(r^\mu R^{1-\mu})\right\}$$

where $C=C(\tilde{\delta},\tau)$ *and* $\alpha=\alpha(\tilde{\delta},\tau,\mu)$.

6.3 Main results

6.3.1 Fundamental Solutions

We start with discussing fundamental solutions of the PDE

$$\operatorname{div}\left(\rho(Du)^{\gamma-1}(D\rho)(du)\right)=0.$$

Proposition 133. *If $\rho \in C^1(\mathbb{R}^n \setminus \{0\})$ satisfies*

1. $\rho(\lambda\xi) = \lambda\rho(\xi)$ *for all* $\lambda > 0$ *and* $\xi \in \mathbb{R}^n \setminus \{0\}$

2. $\rho(\xi_1 + \xi_2) = \rho(\xi_1) + \rho(\xi_2) \iff \xi_1 = \lambda\xi_2$ *for some* $\lambda > 0$

and $c_\rho = \int_{\rho_*=1} |D\rho_*(x)|^{-1} d\mathcal{H}^{n-1}(x)$*, then*

$$c_\rho v(x) = \begin{cases} \frac{\gamma-1}{\gamma-n}\rho_*(x)^{\gamma-n}\gamma - 1 & \gamma \neq n \\ \ln(\rho_*(x)) & \gamma = n \end{cases} \tag{6.12}$$

is a fundamental solution of

$$\int_{\mathbb{R}^n\setminus 0} \left\langle \rho(Dv)^{\gamma-1}(D\rho)(Dv), D\varphi \right\rangle = 0 \qquad \forall \varphi \in W_0^{1,\gamma'}(\mathbb{R}^n \setminus \{0\}), \tag{6.13}$$

in the sense that it is both a strong solution, and

$$p.v. \int_{\mathbb{R}^n} \varphi \operatorname{div}\left(\rho(Dv)^{\gamma-1}(D\rho)(Dv)\right) = \varphi(0).$$

Strict convexity is necessary for the differentiability of ρ_*. Note also, that we do not know that $v \in C^2$ even though it is known to be a strong solution to a 2nd order PDE.

Proof. To see that v is a strong solution, it suffices to show in each case that

$$\rho(Dv)^{\gamma-1}(D\rho)(Dv) = c_\rho^{1-\gamma}\rho_*(x)^{-n}x.$$

Indeed,

$$\operatorname{div}\left(\rho_*(x)^{-n}x\right) = \operatorname{tr}\left(\rho_*(x)^{-n}Id - n\rho_*(x)^{-n-1}x \cdot (D\rho_*)(x)\right) = 0.$$

When $\gamma \neq n$ we compute:

$$\begin{aligned} Dv &= c_\rho^{-1}\rho_*(x)^{\frac{1-n}{\gamma-1}}(D\rho_*)(x) \\ \rho(Dv)^{\gamma-1} &= c_\rho^{1-\gamma}\rho_*(x)^{1-n} \\ (D\rho)(Dv) &= \frac{x}{\rho_*(x)} \end{aligned}$$

which implies

$$\rho(Dv)^{\gamma-1}(D\rho)(Dv) = c_\rho^{1-\gamma}\rho_*(x)^{-n}x.$$

Now, if $\gamma = n$ we compute:

$$\begin{aligned} Dv &= \frac{D\rho_*(x)}{\rho_*(x)} \\ \rho(Dv)^{n-1} &= c_\rho^{1-n}\rho_*(x)^{-(n-1)} \\ (D\rho)(Dv) &= \frac{x}{\rho_*(x)}, \end{aligned}$$

again guaranteeing

$$\rho(Dv)^{n-1}(D\rho)(Dv) = c_\rho^{1-\gamma}\rho_*(x)^{-n}x$$

as desired.

Next, using the fact that in either case, $\rho(Dv)^{\gamma-1}(D\rho)(Dv) = x\rho_*(x)^{-n}$, we observe that for $\varphi \in C_c^1(\mathbb{R}^n)$

$$\begin{aligned} p.v.\int_{\mathbb{R}^n}\varphi\,\mathrm{div}\left(\rho(Dv)^{\gamma-1}(D\rho)(Dv)\right)d\mathcal{H}^n &= p.v.\int_{\mathbb{R}^n}\mathrm{div}\left(\varphi x\rho_*(x)^{-n}\right) + \left\langle x\rho_*(x)^{-n}, D\varphi\right\rangle d\mathcal{H}^n \\ &= \lim_{\epsilon\downarrow 0}\int_{\mathbb{R}^n\setminus\{\rho_*\leq\epsilon\}}\mathrm{div}\left(\varphi x\rho_*(x)^{-n}\right)d\mathcal{H}^n = \lim_{\epsilon\downarrow 0}\int_{\{\rho_*=\epsilon\}}\frac{\varphi}{\epsilon^{n-1}|D\rho_*(x)|}d\mathcal{H}^{n-1} \\ &= \lim_{\epsilon\downarrow 0}\int_{\{\rho_*=1\}}\frac{\varphi(\epsilon x)}{|D\rho_*(x)|}d\mathcal{H}^{n-1} = \lim_{\epsilon\downarrow 0}\fint_{\rho_*=1}\varphi(\epsilon x)d\nu, \end{aligned}$$

where $\nu = \mathcal{H}^{n-1} \llcorner \{\rho_* = 1\}$. By continuity of φ, the result follows. $\square$

6.3.2 Regularity

In this section we focus on functions u that solve

$$\int_\Omega \left\langle \rho_x(Du)^{\gamma-1}(D\rho_x)(Du), D\varphi \right\rangle dx = \int_\Omega \left\langle \vec{F}, D\varphi \right\rangle + f\varphi \qquad \forall \varphi \in W_0^{1,\gamma'}(\Omega). \tag{6.14}$$

As a corollary of these results, we can answer further questions about functions u that solve

$$\int_\Omega \left\langle \rho_x(Du)^{\gamma-1}(D\rho_x)(Du), D\varphi \right\rangle dx = 0. \tag{6.15}$$

We begin with a Caccioppoli inequality when $\vec{F}, f \not\equiv 0$.

Theorem 134. *Suppose $u \in W^{1,\gamma}(\Omega)$ is a subsolution of (6.14) with $1 < \gamma < \infty$ and $\rho : \Omega \times \mathbb{R}^n \setminus \{0\}$ satisfies (6.7), (6.8), (6.9), and (6.10). Assume $\vec{F}, f \in L^{\tilde{\gamma}}(\Omega)$ where $\tilde{\gamma} = \max\{\gamma, \gamma'\}$. If $B_{2R} \subset \Omega$ and $0 < R \leq 10$ then,*

$$\|\rho_x(Du)\|_{L^\gamma(B_R)} \leq c_{\gamma,\Lambda} \left[R^{-1}\|u\|_{L^\gamma}(B_{2R}) + \|\rho_*(x,\vec{F})\|_{L^{\gamma'}(B_{2R})}^{\frac{1}{\gamma-1}} + R^{\frac{1}{\gamma-1}}\|f\|_{L^{\gamma'}(B_{2R})}^{\frac{1}{\gamma-1}} \right]. \tag{6.16}$$

Remark 135. *Note, it is necessary for $\vec{F}, f \in L^\gamma$ (resp., $\vec{F}, f \in L^{\gamma'}$) for the equation (resp., conclusion) to make sense*[1].

Proof. Suppose without loss of generality, $R = 1$. Consider the test function $\varphi = \eta^\gamma u$ for some $\eta \in C_0^1(B_2)$ to be chosen later. Note,

$$D\varphi = \eta^\gamma Du + \gamma u \eta^{\gamma-1} D\eta.$$

Taking advantage of the 1-homogeneity of ρ_x, Fenchel's inequality, Young's inequality, and

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[1]Technically, by Sobolev embedding we only need $f \in L^{(\frac{\gamma' n}{n-\gamma'})'} \cap L^{\gamma'}$.

(6.11) we choose $\epsilon > 0$ so that $(\gamma-1)\epsilon^{\gamma'} = \frac{1}{2}\rho(\eta Du)^\gamma$ and compute

$$\begin{aligned}\langle \rho_x(Du)^{\gamma-1}(D\rho_x)(Du), D\varphi\rangle &\geq \rho_x(\eta Du)^\gamma - \gamma\rho_x(\eta Du)^{\gamma-1}\rho_*(x,(D\rho)(Du))\rho_x(uD\eta) \\ &\geq \rho_x(\eta Du)^\gamma - \gamma\left[\frac{\epsilon^{\gamma'}\rho_x(\eta Du)^\gamma}{\gamma'} + \frac{\rho_x(uD\eta)^\gamma}{\epsilon^\gamma\gamma}\right] \\ &\geq \frac{\rho_x(\eta Du)^\gamma}{2} - c_\gamma\rho_x(uD\eta)^\gamma. \end{aligned} \tag{6.17}$$

On the other hand, for $\epsilon > 0$ chosen so that $\gamma^{-1}\epsilon^\gamma = 1/4$, Fenchel's and Young's inequalities ensure

$$\begin{aligned}\langle \vec{F}, D\varphi\rangle &\leq \gamma\left(\rho_*(x,\vec{F})\eta^{\gamma-1}\right)\rho_x(uD\eta) + \eta^\gamma\left(\rho_*(\vec{F})\rho(Du)\right) \\ &\leq \gamma\left[\frac{\rho_*(x,\vec{F})^{\gamma'}\eta^\gamma}{\gamma'} + \frac{\rho_x(uD\eta)^\gamma}{\gamma}\right] + \eta^\gamma\left[\frac{\rho_*(x,\vec{F})^{\gamma'}}{\epsilon^{\gamma'}\gamma'} + \epsilon^\gamma\frac{\rho_x(Du)^\gamma}{\gamma}\right] \\ &= \frac{\rho_x(\eta Du)^\gamma}{4} + c_\gamma\rho_*(x,\vec{F})^{\gamma'}\eta^\gamma + \rho_x(uD\eta)^\gamma. \end{aligned} \tag{6.18}$$

Combining (6.17), (6.18), and (6.14) yields,

$$\begin{aligned}\int \rho_x(\eta Du)^\gamma &\leq c_\gamma\left[\int \rho_*(x,\vec{F})^{\gamma'}\eta^\gamma + \int \rho_x(uD\eta)^\gamma + \int f\eta^\gamma u\right] \\ &\leq c_\gamma\left[\int \rho_*(x,\vec{F})^{\gamma'}\eta^\gamma + \int u^\gamma(\eta^\gamma + \rho_x(D\eta)^\gamma) + \int \eta^\gamma f^{\gamma'}\right]. \end{aligned} \tag{6.19}$$

Choosing $0 \leq \eta \leq 1$, $\eta \equiv 1$ on B_1, $\eta \equiv 0$ on B_2^c and $|D\eta| \leq 2$ we find

$$\|\rho_x(Du)\|_{L^\gamma(B_R)} \leq c_{\gamma,\Lambda,}\left[\|u\|_{L^\gamma(B_{2R})} + \|\rho_*(x,\vec{F})\|^{\gamma'-1}_{L^{\gamma'}(B_{2R})} + \|f\|^{\gamma'-1}_{L^{\gamma'}(B_{2R})}\right].$$

Equation (6.16) is recovered by scaling. □

We note that in Theorem 134, the fact that ρ can depend on x never needs to be dealt with separately. This is unsurprising because conditions (6.8), (6.9), and Fenchel's inequality are used at a pointwise level while (6.10) is used at a global level, see Remark 129. Hence, to simplify notation, we only explicitly write-out the dependence of ρ on x in the statements of theorems and suppress this dependence throughout all remaining proofs.

Theorem 136. *Suppose* $\rho : \Omega \times \mathbb{R}^n \setminus \{0\}$ *satisfies* (6.7), (6.8), (6.9), *and* (6.10). *Let* u *be a subsolution to* (6.14) *and fix* $1 < \gamma < n$. *If* $\vec{F}, f$ *and* q *are as in* (6.4), $0 < r < R < 1$, *and* $\overline{B_R} \subset \Omega$ *then there exists some* $C = C(n, \nu, \Lambda, \gamma, q, p)$ *and* $\delta = 1 - \frac{n}{q(\gamma-1)} > 0$ *so that*

$$\sup_{B_r} u^+ \leq C\left[\frac{\|u^+\|_{L^p(B_R)}}{(R-r)^{\frac{n}{p}}} + R^{\delta}\|\rho_*(x,\vec{F})\|_{L^q(B_R)}^{\frac{1}{\gamma-1}} + R^{\gamma'\delta}\|f\|_{L^{\frac{q}{\gamma'}}(B_R)}^{\frac{1}{\gamma-1}}\right]$$

Proof. We consider the test function $\varphi = \eta^\gamma v^\beta \bar{u}$ for $\beta \geq 0$ where $\bar{u} = u^+ + k$ and $v = \min\{u, m\}$ for some $0 < k < m < \infty$, k to be chosen later. Notice

$$D\varphi = \gamma\eta^{\gamma-1}v^\beta \bar{u} D\eta + \beta\eta^\gamma v^\beta \bar{U} Dv + \eta^\gamma v^\beta D\bar{u}.$$

We wish to expand out (6.14) with this choice of φ. To this end, first observe 1-homogeneity, i.e., (6.8) ensures

$$\begin{aligned}\big\langle \rho(Du)^{\gamma-1}(D\rho)(Du), &\beta v^{\beta-1}\bar{u}\eta^\gamma Dv + \eta^\gamma v^\beta D\bar{u}\big\rangle \\ &= \beta\rho(Dv)^\gamma v^\beta \eta^\gamma + \rho(D\bar{u})^\gamma \eta^\gamma v^\beta. \end{aligned} \tag{6.20}$$

Next we apply Fenchel's and Young's inequalities in combination with (6.11) for some $\epsilon = \epsilon(\gamma) > 0$ to be chosen immediately after (6.21),

$$\begin{aligned}\big\langle \rho(Du)^{\gamma-1}(D\rho)(Du), &\gamma\eta^{\gamma-1}v^\beta \bar{u} D\eta\big\rangle \\ &\geq -\gamma v^\beta \left(\rho(D\bar{u})\eta)\right)^{\gamma-1} \rho_*(D\rho(Du))\left(\rho(D\eta)\bar{u}\right) \\ &\geq -\gamma v^\beta \left[\epsilon^{\gamma'}\frac{\rho(D\bar{u})^\gamma \eta^\gamma}{\gamma'} + \frac{\rho(D\eta)^\gamma \bar{u}^\gamma}{\epsilon^\gamma \gamma}\right]. \end{aligned} \tag{6.21}$$

Choose ϵ so that $(\gamma-1)\epsilon^{\gamma'}=1/2$. Since $\frac{\gamma}{\gamma'}=\gamma-1$, this choice of ϵ ensures (6.21) becomes

$$\left\langle \rho(Du)^{\gamma-1}(D\rho)(Du),\gamma\eta^{\gamma-1}v^{\beta}\bar{u}D\eta\right\rangle \geq -\frac{v^{\beta}\rho(D\bar{u})^{\gamma}\eta^{\gamma}}{2}-c_{\gamma}v^{\beta}\rho(D\eta)^{\gamma}\bar{u}^{\gamma}, \tag{6.22}$$

where c_γ may change depending on the line, but depends only on γ.

Now we look at the righthand side. We split this into two pieces and treat the first piece in much the same fashion as above.

$$\begin{aligned}
\left\langle \vec{F},\beta v^{\beta-1}\bar{u}\eta^{\gamma}Dv+\eta^{\gamma}v^{\beta}D\bar{u}\right\rangle &\leq \rho_*(\vec{F})\left[\beta v^{\beta}\eta^{\gamma}\rho(Dv)\right]+\rho_*(\vec{F})\left[\eta^{\gamma}v^{\beta}\rho(D\bar{u})\right]\\
&= (\eta^{\gamma}v^{\beta})\left[(\beta)\left(\rho_*(\vec{F})\right)(\rho(Dv))+\rho_*(\vec{F})\rho(D\bar{u})\right]\\
&\leq \eta^{\gamma}v^{\beta}\left[\left(\beta\frac{\rho_*(\vec{F})^{\gamma'}}{\epsilon_1^{\gamma'}\gamma'}+\beta\epsilon_1^{\gamma}\frac{\rho(Dv)^{\gamma}}{\gamma}\right)+\left(\frac{\rho_*(\vec{F})^{\gamma'}}{\epsilon_2^{\gamma'}\gamma'}+\epsilon_2^{\gamma}\frac{\rho(D\bar{u})^{\gamma}}{\gamma}\right)\right]\\
&= \eta^{\gamma}v^{\beta}\rho_*(\vec{F})^{\gamma'}\left(\frac{\beta}{\epsilon_1^{\gamma'}\gamma'}+\frac{1}{\epsilon_2^{\gamma'}\gamma'}\right)+\frac{\beta\epsilon_1^{\gamma}}{\gamma}\eta^{\gamma}v^{\beta}\rho(Dv)^{\gamma}+\frac{\epsilon_2^{\gamma}}{2}\eta^{\gamma}v^{\beta}\rho(D\bar{u})^{\gamma}.
\end{aligned} \tag{6.23}$$

Since we want to absorb the last two terms of (6.23) into (6.20) we choose ϵ_1,ϵ_2 so that $\gamma^{-1}\beta\epsilon_1^{\gamma}=\frac{\beta}{2}$ and $\gamma^{-1}\epsilon_2^{\gamma}=\frac{1}{4}$. The need for choosing 1/4 for the ϵ_2 coefficient is due to the fact that we'll also be absorbing the $\rho(D\bar{u})$-term from (6.22) into (6.20). We note both ϵ_1 and ϵ_2 depend solely on γ. All-in-all this allows us to re-write (6.23) as

$$\begin{aligned}
\left\langle \vec{F},\beta v^{\beta-1}\bar{u}\eta^{\gamma}Dv+\eta^{\gamma}v^{\beta}D\bar{u}\right\rangle &\leq c_{\gamma}(1+\beta)\eta^{\gamma}v^{\beta}\rho_*(\vec{F})^{\gamma'}+\frac{\beta}{2}\eta^{\gamma}v^{\beta}\rho(Dv)^{\gamma}\\
&\quad+\frac{1}{4}\eta^{\gamma}v^{\beta}\rho(D\bar{u})^{\gamma}.
\end{aligned} \tag{6.24}$$

We now deal with the final term via Fenchel and Young's inequalities

$$\begin{aligned}
\left\langle \vec{F},\gamma\eta^{\gamma-1}v^{\beta}\bar{u}D\eta\right\rangle &\leq \gamma\rho_*(\vec{F})\eta^{\gamma-1}v^{\beta}\bar{u}\rho(D\eta)\\
&\leq \gamma v^{\beta}\left[\frac{\rho_*(\vec{F})^{\gamma'}\eta^{\gamma}}{\gamma'}+\frac{\bar{u}^{\gamma}\rho(D\eta)^{\gamma}}{\gamma}\right]\\
&= (\gamma-1)\rho_*(\vec{F})^{\gamma'}\eta^{\gamma}v^{\beta}+\bar{u}^{\gamma}\rho(D\eta)^{\gamma}v^{\beta}.
\end{aligned} \tag{6.25}$$

Finally, plugging (6.20), (6.22), (6.24), and (6.25) into (6.14) yields

$$\begin{aligned}
&\frac{\beta}{2}\int_\Omega \rho(Dv)^\gamma v^\beta \eta^\gamma + \frac{1}{4}\int_\Omega \rho(D\bar{u})^\gamma \eta^\gamma v^\beta \\
&\quad \le c_\gamma \left[\int_\Omega v^\beta \rho(D\eta)^\gamma \bar{u}^\gamma + \int_\Omega \eta^\gamma v^\beta \rho_*(\vec{F})^{\gamma'} + \int_\Omega f\eta^\gamma v^\beta \bar{u}\right] \\
&\quad \le c_\gamma \left[\int_\Omega v^\beta \rho(D\eta)^\gamma \bar{u}^\gamma + (1+\beta)\int_\Omega \eta^\gamma v^\beta \frac{\bar{u}^\gamma}{k^\gamma}\rho_*(\vec{F})^{\gamma'} + \int_\Omega \frac{f}{k^{\gamma-1}}\eta^\gamma v^\beta \bar{u}^\gamma\right]. \qquad (6.26)
\end{aligned}$$

The final inequality used $\bar{u} \ge k$. Set $w = v^{\beta/\gamma}\bar{u}$, we note

$$Dw = \frac{\beta}{\gamma} v^{\frac{\beta}{\gamma}-1}\bar{u}Dv + v^{\frac{\beta}{\gamma}} D\bar{u} = \frac{\beta}{\gamma} v^{\frac{\beta}{\gamma}} Dv + v^{\frac{\beta}{\gamma}} D\bar{u}.$$

Since $\rho(\xi_1+\xi_2) \le \rho(\xi_1) + \rho(\xi_2)$ for all ξ_1, ξ_2 it follows that

$$\begin{aligned}
\eta^\gamma \rho(Dw)^\gamma &\le \eta^\gamma \left(\frac{\beta}{\gamma} v^{\frac{\beta}{\gamma}} \rho(Dv) + v^{\frac{\beta}{\gamma}} \rho(D\bar{u})\right)^\gamma \\
&\le 2^{\gamma-1}\left(\left(\frac{\beta}{\gamma}\right)^\gamma \eta^\gamma v^\beta \rho(Dv)^\gamma + \eta^\gamma v^\beta \rho(D\bar{u})^\gamma\right).
\end{aligned}$$

In particular, this guarantees that for some c_γ

$$\int_\Omega \eta^\gamma \rho(Dw)^\gamma \le c_\gamma(1+\beta^{\gamma-1})\left[\frac{\beta}{2}\int_\Omega \rho(Dv)^\gamma v^\beta \eta^\gamma + \frac{1}{4}\int_\Omega \rho(D\bar{u})^\gamma \eta^\gamma v^\beta\right]. \qquad (6.27)$$

Combining (6.26) and (6.27) yields

$$\int \eta^\gamma \rho(Dw)^\gamma \le c_\gamma(1+\beta^\gamma)\left[\int_\Omega w^\gamma \rho(D\eta)^\gamma + \int_\Omega (\eta w)^\gamma \left(\frac{\rho_*(\vec{F})^{\gamma'}}{k^\gamma} + \frac{f}{k^{\gamma-1}}\right)\right]. \qquad (6.28)$$

Due to the observation that $\rho(D(\eta w))^\gamma \le 2^{\gamma-1}(w^\gamma \rho(D\eta)^\gamma + \eta^\gamma \rho(Dw)^\gamma)$, (6.28) implies

$$\int_\Omega \rho(D(\eta w))^\gamma \le c_\gamma(1+\beta^\gamma)\left[\int_\Omega w^\gamma \rho(D\eta)^\gamma + \int_\Omega (\eta w)^\gamma \left(\frac{\rho_*(\vec{F})^{\gamma'}}{k^\gamma} + \frac{f}{k^{\gamma-1}}\right)\right]. \qquad (6.29)$$

Our next goal is to deal with the term $\int (\eta w)^{\gamma} \frac{\rho_*(\vec{F})^{\gamma'}}{k^{\gamma}}$. We roughly explain in words how we do this. We first use Holder's inequality to make the L^q norm of $\vec{F}$ appear. To this end, we will introduce the parameter $\alpha_1 = \alpha(q,\gamma) = \frac{q}{q-\gamma'}$ which is equivalent to $\alpha_1' = \frac{q}{\gamma'}$. This is precisely where the requirement $f \in L^{\frac{q}{\gamma'}}$ comes from.

Next, by choosing k appropriately, we will make the term with $\rho(\vec{F}) + f$ be absorbed into a 1. At this point, the remaining term with ηw will have a strange power. So, we use interpolation, and the fact that $\gamma < \gamma\alpha_1 < \frac{n\gamma}{n-\gamma}$, to re-write our strange power as a linear combination of the L^{γ} and L^{γ^*} norms of ηw. The necessary upper-bound on α_1 is satisfied so long as $q > \frac{n}{\gamma - 1}$.

By making the coefficient of the L^{γ} norm of ηw larger, we can choose the L^{γ^*} norm of ηw to be arbitrarily small. This is necessary, as we will apply the Gagliardo-Nirenberg-Sobolev inequality to turn this latter norm into an estimate on the L^{γ} norm of $\rho(D(\eta w))$ which we can finally absorb into the left hand side. When we apply Young's inequality in order to make this weighted-linear combination of norms appear, we will choose $\alpha_2 = \alpha(n,q,\gamma) = \theta_1^{-1}$, where θ_1 is the interpolation power. This conveniently makes all exponents outside of integrals disappear, allowing the desired simplifications to all occur.

We begin the process outlined above by applying Holder's inequality,

$$\begin{aligned} \int_{\Omega} (\eta w)^{\gamma} \left(\frac{\rho_*(\vec{F})^{\gamma'}}{k^{\gamma}} + \frac{f}{k^{\gamma-1}} \right) &\leq \|(\eta\omega)^{\gamma}\|_{L^{\alpha_1}(\Omega)} \left\| \frac{\rho_*(\vec{F})^{\gamma'}}{k^{\gamma}} + \frac{f}{k^{\gamma-1}} \right\|_{L^{\alpha_1'}(\Omega)} \\ &\leq C(\gamma,q) \|\eta w\|^{\gamma}_{L^{\gamma\alpha_1}(\Omega)} \end{aligned} \tag{6.30}$$

where $\alpha_1' = \frac{q}{\gamma'}$ is as above, and k is chosen so that $k = k_1 + k_2$ where $k_1^{\gamma} = \|\rho_*(\vec{F})\|^{\gamma'}_{L^q(\Omega)}$ and $k_2^{\gamma-1} = \|f\|_{L^{\frac{q}{\gamma'}}(\Omega)}$. If $\vec{F}, f \equiv 0$ choose $k > 0$ arbitrary, and you can later send k to zero. Next, we define $\theta_1 = \theta_1(q,n,\gamma) \in (0,1)$ so that $\frac{1}{\gamma\alpha_1} = \frac{\theta_1}{\gamma} + \frac{(1-\theta_1)(n-\gamma)}{n\gamma}$. Note that if $\alpha_2 = \alpha(q,n,\gamma) = \theta_1^{-1}$ then $\alpha_2' = (1-\theta_1)^{-1}$. Riesz-Thorin interpolation applied to (6.30), Young's inequality, and the Gagliardo-Nirenberg-Sobolev inequality consecutively ensure

$$\begin{aligned}
\int_\Omega (\eta w)^\gamma \frac{\rho_*(\vec{F})^{\gamma'}}{k^\gamma} &\leq \left(\|\eta w\|^{\theta_1}_{L^\gamma(\Omega)} \|\eta w\|^{1-\theta}_{L^{\gamma^*}(\Omega)} \right)^\gamma \\
&\leq \frac{1}{\epsilon^{\alpha_2}\alpha_2} \|\eta w\|^{\gamma\theta_1\alpha_2}_{L^\gamma(\Omega)} + \frac{\epsilon^{\alpha_2'}}{\alpha_2'} \|\eta w\|^{\gamma(1-\theta_1)\alpha_2'}_{L^{\gamma^*}(\Omega)} \\
&= \frac{1}{\epsilon^{\alpha_2}\alpha_2} \|\eta w\|^{\gamma}_{L^\gamma(\Omega)} + \frac{\epsilon^{\alpha_2'}}{\alpha_2'} \|\eta w\|^{\gamma}_{L^{\gamma^*}(\Omega)} \\
&\leq \epsilon^{-\alpha_2}\alpha_2^{-1} \|\eta w\|^{\gamma}_{L^\gamma(\Omega)} + C(q,n,\gamma,\rho)\epsilon^{\alpha_2'} \|\rho(D(\eta w))\|^{\gamma}_{L^\gamma(\Omega)}. \qquad (6.31)
\end{aligned}$$

See Remark 129 for our non-standard application of Gagliardo-Nirenberg-Sobolev inequality. Next, we want to plug (6.31) into (6.29) and subtract over the $\rho(D(\eta w))$ term, so we choose ϵ so that the coefficient of $\|\rho(D(\eta w))\|^{\gamma}_{L^\gamma(\Omega)}$ is 1/2. That is, choose $\epsilon = \epsilon(q,n,\gamma,\rho) > 0$ by

$$c_\gamma(1+\beta^\gamma)C(q,n,\gamma,\nu)\epsilon^{\alpha_2'} = \frac{1}{2}.$$

Then using our choice of ϵ and plugging (6.31) into (6.29) yields

$$\begin{aligned}
\int_\Omega \rho(D(\eta w))^\gamma &\leq c_\gamma(1+\beta^\gamma)\left[\int_\Omega w^\gamma \rho(D\eta)^\gamma + C_{q,n,\gamma,\nu}(1+\beta^\gamma)^{\alpha_2-1} \int_\Omega w^\gamma \eta^\gamma \right] \\
&\leq C_{q,n,\gamma,\nu}(1+\beta^\gamma)^{\alpha_2} \left[\int_\Omega w^\gamma (\rho(D\eta)^\gamma + \eta^\gamma) \right], \qquad (6.32)
\end{aligned}$$

where as always, $\alpha_2 = \alpha(n,q,\gamma) > 0$. Finally, the Gagliardo-Nirenberg-Sobolev inequality applied to (6.32) implies

$$\|\eta w\|_{\gamma\chi} \leq C_{q,n,\gamma,\nu}(1+\beta^\gamma)^{\frac{\alpha_2}{\gamma}} \left[\int_\Omega w^\gamma (\rho(D\eta)^\gamma + \eta^\gamma) \right]^{\frac{1}{\gamma}}, \qquad (6.33)$$

where $\chi = \frac{n}{n-\gamma}$. Choose the cut-off function η so that with $0 < r < R < 1$ and some $B_R(x) \subset\subset \Omega$, $\eta \in C_0^1(B_R)$ and

$$\eta \equiv 1 \text{ in } B_r \quad \text{and} \quad |D\eta| \leq \frac{2}{R-r}. \qquad (6.34)$$

Then (6.33) guarantees

$$\left(\int_{B_r(x)} w^{\gamma\chi}\right)^{\frac{1}{\chi}} \leq C_{n,\gamma,q,\nu,\Lambda}\frac{(1+\beta^{\gamma-1})^{\alpha_2}}{(R-r)^{\gamma}}\int_{B_R} w^{\gamma}$$

where our constant gained a dependence on Λ, which arises by applying (6.34) in the form $\rho(D\eta) \leq \Lambda|D\eta| \leq 2\Lambda(R-r)^{-1}$. Recall the definition of w and use $v \leq \bar{u}$ to discover

$$\left(\int_{B_r(x)} v^{(\beta+\gamma)\chi}\right)^{\frac{1}{\chi}} \leq C_{n,\gamma,q,\nu,\Lambda}\frac{(1+\beta^{\gamma-1})^{\alpha_2}}{(R-r)^{\gamma}}\int_{B_R} \bar{u}^{\beta+\gamma}. \tag{6.35}$$

Taking $m \uparrow \infty$ in (6.35) yields

$$\|\bar{u}\|_{L^{(\beta+\gamma)\chi}B_r(x)} \leq \left(C_{n,\gamma,q,\nu,\Lambda}\frac{(1+\beta^{\gamma-1})^{\alpha_2}}{(R-r)^{\gamma}}\right)^{\frac{1}{\beta+\gamma}}\|\bar{u}\|_{L^{\beta+\gamma}(B_R(x))}, \tag{6.36}$$

so long as the right hand side is finite. Note the constant on the righthand side is independent of β, suggesting we iterate. We begin with $\beta = \beta_0 = 0$ and for $k = 1, 2, 3\ldots$ define $\beta_k = \gamma(\chi^{k-1}-1)$. For $k = 0, 1, 2, \ldots$ we consider $r_k = r + \frac{R-r}{2^{k+1}}$. Note, $(\beta_k+\gamma)\chi = \gamma\chi^k = (\beta_{k+1}+\gamma)$ and $r_{k-1} - r_k = 2^{-(k+1)}(R-r)$.

Writing $C = C_{n,\gamma,q,\nu,\Lambda}$, for $k \geq 0$ this choice of β_k, r_k (6.36) reads

$$\|\bar{u}\|_{L^{\beta_k+\gamma}(B_{r_k}(x))} \leq \left(C\frac{(1+\beta_k^{\gamma-1})^{\alpha_2}}{(R-r)^{\gamma}}\right)^{\frac{1}{\beta_k+\gamma}} 2^{\frac{k+1}{\gamma\chi^{k-1}}}\|\bar{u}\|_{L^{\beta_{k-1}+\gamma}(B_{r_{k-1}}(x))}.$$

Recalling $\beta_k = \gamma(\chi^{k-1}-1)$ we observe that for $k \geq 2$,

$$\begin{aligned}\left(\frac{(1+\beta_k^{\gamma-1})^{\alpha_2}}{(R-r)^{\gamma}}\right)^{\frac{1}{\beta_k+\gamma}} &\leq (R-r)^{-\chi^k}(C(1+\gamma\chi))^{\frac{k(\gamma-1)\alpha_2}{\gamma\chi^{k-1}-1}}\\ &\leq (R-r)^{-\chi^k}C^{\frac{k}{\chi^k}}.\end{aligned}$$

Consequently, after iterating out the first few cases by hand, for all k,

$$\|\bar{u}\|_{L^{\beta_k+\gamma}(B_{r_k}(x))} \leq (R-r)^{-\sum_{i=0}^{\infty}\chi^i} C^{\sum_{i=0}^{\infty}\frac{i}{\chi^{i-1}}} \|\bar{u}\|_{L^{\beta_0+\gamma}(B_{r_1}(x))},$$

where still $C = C(n,\gamma,q,\nu,\Lambda)$. Since $u \in W^{1,\gamma}(\Omega)$ and $\beta_0 = 0$, the right hand side is seen to be finite, and is independent of k. Taking $k \to \infty$ on the left hand side, and recalling $\bar{u} = u^+ + k$ yields

$$\|u^+\|_{L^\infty(B_r(x))} \leq C(R-r)^{-\frac{n}{\gamma}} \left[\|u^+\|_{L^\gamma(B_R)} + k\right].$$

Recalling $k = \|\rho_*(\vec{F})\|_q^{\frac{\gamma'}{\gamma}} + \|f\|_{\frac{q}{\gamma'}}^{\frac{1}{\gamma-1}}$ and noting $\frac{1}{1-\chi^{-1}} = \frac{n}{\gamma}$ yields the result for the case $p \geq \gamma$. A classical scaling argument covers the case $0 < p < \gamma$. See, for instance, [40, p. 75]. □

Theorem 137. *Let $\rho : \Omega \times \mathbb{R}^n \setminus \{0\}$ satisfy* (6.7), (6.8), (6.9), *and* (6.10). *Suppose $u \in W^{1,\gamma}(\Omega)$ is a supersolution to* (6.14) *in Ω and $\vec{F}, f$, and q are as in* (6.4). *If $B_{2R} \subset \Omega$ and $0 < p < \frac{(\gamma-1)n}{n-\gamma}$, then for any $0 < \theta < \tau < 1$ there exists $C = C(n,\gamma,q,p,\nu,\Lambda,\theta,\tau)$ and $\delta = 1 - \frac{n}{q(\gamma-1)} > 0$ so that*

$$\inf_{B_{\theta R}} u^+ + R^\delta \|\rho_*(\vec{F})\|_{L^q(B_R)}^{\frac{1}{\gamma-1}} + R^{\gamma'\delta}\|f\|_{L^{\frac{q}{\gamma'}}(B_R)}^{\frac{1}{\gamma-1}} \geq CR^{-\frac{n}{p}}\|u^+\|_{L^p(B_{\tau R})}$$

and the dependence on ρ only appears as a dependence on $\sup_{|\xi|=1}\rho(\xi)$ and $\inf_{|\xi|=1}\rho(\xi)$.

Two corollaries of Theorem 136 and Theorem 137 are the strong maximum principle and the weak-Harnack inequality (Theorems 138 and 139), which we will state and prove before proving Theorem 139.

Theorem 138. *Let $\rho : \Omega \times \mathbb{R}^n \setminus \{0\}$ satisfy* (6.7), (6.8), (6.9), *and* (6.10). *Suppose $u \in W^{1,\gamma}(\Omega)$ is a super solution to* (6.14) *in Ω, and $\vec{F}, f \equiv 0$. Then, if for some ball $B \subset\subset \Omega$*

$$\sup_B u = \sup_\Omega u \geq 0,$$

then the function u must be constant in Ω.

Proof. Suppose $B' \subset\subset \Omega$ is so that $\sup_{B'} u = \sup_\Omega u$. If necessary there is a much smaller ball $B_R \subset\subset \Omega$ so that $\sup_{B_{R/3}} u = \sup_\Omega u$. Without loss of generality $B_{2R} \subset\subset \Omega$. Let $M = \sup_\Omega u$ and applying Theorem 139 to the non-negative supersolution $M - u$ it follows that

$$R^{-n}\|M-u\|_{L^1(B_{2R/3})} \leq C \inf_{B_{R/3}} (M-u) = 0.$$

Hence, $u \equiv M$ on $B_{2R/3}$, which implies the theorem. □

Theorem 139. *Suppose $\rho : \Omega \times \mathbb{R}^n \setminus \{0\} \to (0,\infty)$ satisfies (6.7), (6.8), (6.9), and (6.10). If $1 < \gamma < n$ and $u \in W^{1,\gamma}(\Omega)$ is a nonnegative solution to (6.14) in Ω for some $\vec{F}$, f, and q are as in (6.4) and some $\overline{B_{3R}} \subset \Omega$, then there exists some $C = C(n,\gamma,q,\nu,\Lambda)$ and $\delta = 1 - \frac{n}{q(\gamma-1)} > 0$ so that*

$$\sup_{B_R} u \leq C_{n,\gamma,\nu,\Lambda,q}\left[\inf_{B_{2R}} u + R^{\delta}\|\rho_*(\vec{F})\|^{\frac{1}{\gamma-1}}_{L^q(B_{3R})} + R^{\gamma'\delta}\|f\|^{\frac{1}{\gamma-1}}_{L^{\frac{q}{\gamma'}}(B_{3R})}\right]. \tag{6.37}$$

The preceding Theorem follows readily from Theorem 136 and Theorem 137 by choosing, for instance, $p = \frac{(\gamma-1)n}{2(n-\gamma)}$ in both theorems. In the case that $\vec{F}, f \equiv 0$ this recovers an inequality of the same form as the classical Harnack inequality.

Proof. (Of Theorem 137.)

Let $k > 0$ to be chosen later, $\bar{u} = u^+ + k$, $v = \min\{\bar{u}, m\}$. Consider the test function $\varphi = \eta^\gamma v^{-\beta}\bar{u}$ for $\beta \geq \beta_p > 1$, where $\beta_p = \gamma - \frac{p}{\chi}$, $\chi = \frac{n}{n-\gamma}$.

We now proceed in a similar fashion to the proof of Theorem 136. Observe,

$$\begin{aligned}\Big\langle \rho(Du)^{\gamma-1}(D\rho)(Du), -\beta v^{-\beta-1}\bar{u}\eta^\gamma Dv + \eta^\gamma v^{-\beta} D\bar{u}\Big\rangle \\ = -\beta\rho(Dv)^\gamma v^{-\beta}\eta^\gamma + \rho(D\bar{u})^\gamma \eta^\gamma v^{-\beta} \\ = (1-\beta)\rho(Dv)^\gamma v^{-\beta}\eta^\gamma.\end{aligned} \tag{6.38}$$

We note that $(\beta - 1) \geq 1 - \beta_p$; this observation will simplify our computations later as, when we iterate β, we may now keep constants independent of β. They will instead depend on β_p, which depends on n, γ, p.

Analogous to (6.21) we compute

$$\left\langle \rho(Du)^{\gamma-1}(D\rho)(Du), \gamma\eta^{\gamma-1}v^{-\beta}\bar{u}D\eta \right\rangle \leq \gamma v^{-\beta}\left[\epsilon^{\gamma'}\frac{\rho(D\bar{u})^{\gamma}\eta^{\gamma}}{\gamma'} + \frac{\rho(D\eta)^{\gamma}\bar{u}^{\gamma}}{\epsilon^{\gamma}\gamma}\right]. \tag{6.39}$$

Choosing ϵ so that $(\gamma-1)\epsilon^{\gamma'} = -(1-\beta)/2 > (1-\gamma)/2 > 0$ yields

$$\begin{aligned}\Big\langle \rho(Du)^{\gamma-1}(D\rho)(Du), &\gamma\eta^{\gamma-1}v^{-\beta}\bar{u}D\eta \Big\rangle \\ &\leq \frac{-(1-\beta)}{2}v^{-\beta}\eta^{\gamma}\rho(Dv)^{\gamma} + c_{n,\gamma,p}v^{-\beta}\rho(D\eta)^{\gamma}\bar{u}^{\gamma}.\end{aligned} \tag{6.40}$$

Next we treat the $\vec{F}$ term. Proceeding as in (6.23) we discover

$$\begin{aligned}\left\langle \vec{F}, -\beta v^{-\beta-1}\bar{u}\eta^{\gamma}Dv + \eta^{\gamma}v^{\beta}D\bar{u}\right\rangle &\geq -(\beta-1)\eta^{\gamma}v^{-\beta}\left[\left(\rho_*(\vec{F})\right)(\rho(Dv))\right] \\ \geq (1-\beta)&\left[\frac{\eta^{\gamma}v^{-\beta}\rho_*(\vec{F})^{\gamma'}}{\epsilon^{\gamma'}\gamma'} + \epsilon^{\gamma}\frac{\eta^{\gamma}v^{-\beta}\rho(Dv)^{\gamma}}{\gamma}\right] \\ \geq \frac{(1-\beta)}{4}&\eta^{\gamma}v^{-\beta}\rho(Dv)^{\gamma} - c_{n,\gamma,p}\eta^{\gamma}v^{-\beta}\rho_*(\vec{F})^{\gamma'}\end{aligned} \tag{6.41}$$

where we chose $\epsilon > 0$ so that $\gamma^{-1}\epsilon^{\gamma} = 1/4$. To bound the final piece, we compute

$$\begin{aligned}\left\langle \vec{F}, \gamma\eta^{\gamma-1}v^{-\beta}\bar{u}D\eta\right\rangle &\geq -\gamma\rho_*(\vec{F})\eta^{\gamma-1}v^{-\beta}\bar{u}\rho(D\eta) \\ &\geq -\gamma v^{-\beta}\left[\frac{\rho_*(\vec{F})^{\gamma'}\eta^{\gamma}}{\gamma'} + \frac{\bar{u}^{\gamma}\rho(D\eta)^{\gamma}}{\gamma}\right] \\ &= -(\gamma-1)\rho_*(\vec{F})^{\gamma'}\eta^{\gamma}v^{-\beta} - \bar{u}^{\gamma}\rho(D\eta)^{\gamma}v^{-\beta}.\end{aligned} \tag{6.42}$$

Using that u is a supersolution and plugging (6.38), (6.40), (6.41), and (6.42) into (6.14) achieves

$$\frac{-(\beta-1)}{4}\int_\Omega \rho(Dv)^\gamma v^{-\beta}\eta^\gamma$$
$$\geq -c_{n,\gamma,p}\int_\Omega v^{-\beta}\rho(D\eta)^\gamma \bar{u}^\gamma - c_{n,\gamma,p}\int_\Omega \eta^\gamma v^{-\beta}\rho_*(\vec{F})^{\gamma'} - \int_\Omega \eta^\gamma v^{-\beta+1} f$$

or after recalling $\bar{u} \geq k$, $\beta - 1 \geq \beta_p - 1 > 0$ and multiplying by -1,

$$\int_\Omega \rho(Dv)^\gamma v^{-\beta}\eta^\gamma \leq c_{n,\gamma,p}\Bigg[\int_\Omega v^{-\beta}\bar{u}^\gamma \rho(D\eta)^\gamma + + \int_\Omega \eta^\gamma v^{-\beta}\bar{u}^\gamma \frac{\rho_*(\vec{F})^{\gamma'}}{k^\gamma} + \int_\Omega \eta^\gamma v^{-\beta}\bar{u}^\gamma \frac{f}{k^{\gamma-1}}\Bigg]. \tag{6.43}$$

Note the striking similarity to (6.26). The only difference being the benefit that (6.43) has no constants depending on β. Hence, we follow the process done in the proof of Theorem 136 and choose

$$k = \|\rho_*(\vec{F})\|_{L^q}^{\frac{1}{\gamma-1}} + \|f\|_{L^{\frac{q}{\gamma'}}}^{\frac{1}{\gamma-1}}.$$

Recalling $0 < \theta < \tau < 1$, setting $w = v^{-\beta/\gamma}\bar{u}$ and proceeding as in the proof of Theorem 136 leads to

$$\left(\int_{B_{R\theta}} \bar{u}^{(\gamma-\beta)\chi}\right)^{\frac{1}{\chi}} \leq C_{n,\gamma,\nu,\Lambda,p,q,R\theta,R\tau}\int_{B_{R\tau}} \bar{u}^{\gamma-\beta}. \tag{6.44}$$

The dependence on $R\theta, R\tau$ comes from the magnitude of the derivative of the cut-off function. Moreover, in (6.44), the dependence on p that is not present in (6.33) is due to having assumed $\beta \geq \beta_p$. To improve readability, we write $C = C_{n,\gamma,\nu,\Lambda,p,q,R\theta,R\tau}$ and suppose $R = 1$ through the end of (6.50). Now, whenever $\gamma - \beta < 0$ (6.44) ensures

$$C\|\bar{u}\|_{L^{\gamma-\beta}(B_\tau)} \leq \|\bar{u}\|_{L^{(\gamma-\beta)\chi}(B_\theta)}. \tag{6.45}$$

Since $\inf v^+ = \lim_{p\to-\infty}\|v\|_{L^p}$, iterating as in Theorem 136, for any $\beta > \gamma$ (6.45) implies,

$$C\|\bar{u}\|_{L^{\gamma-\beta}(B_\tau)} \leq \inf_{B_\theta} \bar{u}. \tag{6.46}$$

On the other hand, whenever $0 < \beta < \gamma - 1$, (6.44) guarantees

$$\|\bar{u}\|_{L^{(\gamma-\beta)\chi}(B_\theta)} \leq C\|\bar{u}\|_{L^{\gamma-\beta}(B_\tau)}. \tag{6.47}$$

Recalling $-\beta < -1$, we can iterate (6.47) finitely many times depending on p, p_0 so that when $0 < p_0 < p < (\gamma-1)\chi$, choosing $\gamma - \beta = p_0$ and finitely many iterations of (6.47) ensures

$$\|\bar{u}\|_{L^p(B_\theta)} \leq C \cdot C(p_0)\|\bar{u}\|_{L^{p_0}(B_\tau)}. \tag{6.48}$$

We claim the proof is complete once we show that there exists $p_0 > 0$ so that

$$\int_{B_\tau} \bar{u}^{-p_0} \int_{B_\tau} \bar{u}^{p_0} \leq C. \tag{6.49}$$

Indeed, choosing $\beta = \gamma + p_0$ in (6.46) combined with (6.48) and (6.49) yields

$$\begin{aligned} \inf_{B_1} \bar{u} &\geq C \cdot C(p_0)\|\bar{u}\|_{L^{-p_0}} \\ &= C \cdot C(p_0)\left(\|\bar{u}\|_{L^{-p_0}}\|\bar{u}\|_{L^{p_0}}^{-1}\right)\|\bar{u}\|_{L^{p_0}} \\ &\geq C \cdot C(p_0)\|\bar{u}\|_{L^p}. \end{aligned} \tag{6.50}$$

It happens that $p_0 = p_0(n, \gamma, \nu, \Lambda)$ so that the constant $C(p_0)$ can be absorbed into our universal constant. Recalling that $\bar{u} = u^+ + k$ and using scaling, this says

$$\inf_{B_{R\theta}} u^+ + R^\delta \|\rho_*(\vec{F})\|_{L^q(B_R)}^{\frac{1}{\gamma-1}} + R^{\gamma'\delta}\|f\|_{L^{\frac{q}{\gamma'}}(B_R)}^{\frac{1}{\gamma-1}} \geq C_{n,\gamma,\nu,\Lambda,q,p,\theta,\tau}\|u^+\|_{L^p(B_{R\tau})},$$

as desired.

Hence, it only remains to show (6.49). We consider the test function $\varphi = \frac{\eta^\gamma}{\bar{u}^{\gamma-1}}$. Note,

$$D\varphi = \gamma\left(\frac{\eta}{\bar{u}}\right)^{\gamma-1} D\eta - (\gamma-1)\left(\frac{\eta}{\bar{u}}\right)^{\gamma} D\bar{u}.$$

We estimate as usual:

$$\left\langle \rho(D\bar{u})^{\gamma-1}(D\rho)(D\bar{u}), -(\gamma-1)\left(\frac{\eta}{\bar{u}}\right)^{\gamma} D\bar{u} \right\rangle = -(\gamma-1)\rho\left(\frac{\eta D\bar{u}}{\bar{u}}\right)^{\gamma}. \tag{6.51}$$

and choosing $\epsilon > 0$ so that $(\gamma-1)\epsilon^{\gamma'} = (\gamma-1)/2$.

$$\begin{aligned} \left\langle \rho(D\bar{u})^{\gamma-1}(D\rho)(D\bar{u}), \gamma\left(\frac{\eta}{\bar{u}}\right)^{\gamma-1} D\eta \right\rangle &\leq \gamma\left[\frac{\epsilon^{\gamma'}}{\gamma'}\rho\left(\frac{\eta D\bar{u}}{\bar{u}}\right)^{\gamma} + \frac{\rho(D\eta)^{\gamma}}{\epsilon^{\gamma}\gamma}\right] \\ &\leq \frac{\gamma-1}{2}\rho\left(\frac{\eta D\bar{u}}{\bar{u}}\right)^{\gamma} + c_{\gamma}\rho(D\eta)^{\gamma}. \end{aligned} \tag{6.52}$$

On the other hand, since $\bar{u} \geq k$

$$\begin{aligned} \left\langle \vec{F}, \gamma\left(\frac{\eta}{\bar{u}}\right)^{\gamma-1} D\eta \right\rangle &\geq -\gamma\rho_*(\vec{F})^{\gamma'}\left(\frac{\eta}{\bar{u}}\right)^{\gamma-1}\rho(D\eta) \\ &\geq -\gamma\left[\frac{\rho_*(\vec{F})^{\gamma'}\eta^{\gamma}}{k^{-\gamma}\gamma'} + \frac{\rho(D\eta)^{\gamma}}{\gamma}\right] \end{aligned} \tag{6.53}$$

and choosing $\epsilon > 0$ so that $(\gamma')^{-1}\epsilon^{\gamma} = 1/4$ guarantees

$$\begin{aligned} \langle \vec{F}, -(\gamma-1)\left(\frac{\eta}{\bar{u}}\right)^{\gamma} D\bar{u}\rangle &\geq -(\gamma-1)\left(\frac{\eta}{\bar{u}}\right)^{\gamma}\left[\frac{\epsilon^{\gamma}\rho(D\bar{u})^{\gamma}}{\gamma} + \frac{\rho_*(\vec{F})^{\gamma'}}{\epsilon^{\gamma'}\gamma'}\right] \\ &\geq -\frac{(\gamma-1)}{4}\rho\left(\frac{\eta D\bar{u}}{\bar{u}}\right)^{\gamma} - c_{\gamma}\eta^{\gamma}k^{-\gamma}\rho_*(\vec{F})^{\gamma'}. \end{aligned} \tag{6.54}$$

Finally, note

$$\int f\eta^{\gamma}\bar{u}^{-(\gamma-1)} \geq -k^{-(\gamma-1)}\|f\|_{\frac{n}{\gamma}}\|\eta\|^{\gamma}_{L^{\frac{n\gamma}{n-\gamma}}} \geq -c_{\gamma,n,\nu}k^{-(\gamma-1)}\|f\|_{\frac{n}{\gamma}}\|D\eta\|^{\gamma}_{L^{\gamma}} \tag{6.55}$$

Combining (6.51) - (6.55), with the fact that u is a supersolution yields

$$-\int_\Omega \eta^\gamma \rho\left(D(\log \bar{u})\right)^\gamma \geq -c_{n,\gamma,\nu}\left[\int \rho(D\eta)^\gamma\left(1+\frac{\|f\|_{\frac{n}{\gamma}}}{k^{\gamma-1}}\right)+\int \frac{\rho_*(\vec{F})^{\gamma'}}{k^\gamma}\eta^\gamma\right]$$
$$\geq -c_{n,\gamma,\nu}\left[\int \rho(D\eta)^\gamma\left(1+k^{-(\gamma-1)}\|f\|_{\frac{n}{\gamma}}+k^{-\gamma}\|\rho_*(\vec{F})\|^{\gamma'}_{L^{\frac{n}{\gamma-1}}}\right)\right], \tag{6.56}$$

where the 2nd inequality follows from the first by Holder and Sobolev embedding similar to (6.55). Choosing $k=\|f\|^{\frac{1}{\gamma-1}}_{\frac{n}{\gamma}}+\|\rho_*(\vec{F})\|^{\frac{1}{\gamma-1}}_{L^{\gamma'}}$ this can be written

$$\int_\Omega \eta^\gamma \rho(D(\log \bar{u}))^\gamma \leq c_{n,\gamma,\nu}\int \rho(D\eta)^\gamma. \tag{6.57}$$

Now, for all $B'_{2r}\subset B_{2R}$, we can choosing $\eta\equiv 1$ on B'_r and $\eta\equiv 0$ on $\Omega\setminus B'_{2r}$ with $|D\eta|\leq \frac{2}{r}$, (6.57) says

$$\int_{B'_r}|D(\log \bar{u})|^\gamma \leq c_{n,\gamma,\nu}\int_{B'_r}\rho(D(\log \bar{u}))^\gamma \leq c_{n,\gamma,\nu,\Lambda} r^{n-\gamma}$$

or taking the γ-root and multiplying by $r^{1-\frac{n}{\gamma}}$ that is

$$r^{1-\frac{n}{\gamma}}\left(\int_{B'_r}|D(\log \bar{u})|^\gamma\right)^{\frac{1}{\gamma}} \leq C_{n,\gamma,\nu,\Lambda}. \tag{6.58}$$

Noticing $(r^{-n})^{\frac{1}{\gamma^*}} = r^{1-\frac{n}{\gamma}}$, we consecutively apply Jensen's inequality, the Poincare inequality in a ball, and (6.58) to achieve

$$\frac{1}{r^n}\int_{B'_r}|\log \bar{u}-(\log \bar{u})_{B'_r}| \leq C\left(\frac{1}{r^n}\int_{B'_r}|\log \bar{u}-(\log \bar{u})_{B'_r}|^{\gamma^*}\right)^{\frac{1}{\gamma^*}} \tag{6.59}$$
$$\leq C_{n,\gamma,\nu,\Lambda}.$$

Since this holds uniformly for all $B'_{2r}\subset B_{2R}$, and hence (6.59) holds for all $B'_r\subset B_R$, this ensures $\log \bar{u}\in BMO(B_R)$ and consequently, by the John-Nirenberg lemma there exists

$0 < p_0$ depending only on n and the constant in (6.59) so that

$$\sup_{B \subset B_3} \frac{1}{|B|} \int_B e^{p_0 |\log \bar{u} - (\log \bar{u})_B|} < \infty. \tag{6.60}$$

Finally, since $e^{|a-b|} \geq 1$ we note that $e^{p_1|a-b|} \leq e^{p_0|a-b|}$ if $p_1 \leq p_0$. In particular, without loss of generality, suppose $p_0 < \frac{(\gamma-1)n}{n-\gamma}$. But then, (6.60) implies (6.49). □

Theorem 140 (Improvement of Oscillation)**.** *Suppose $u, \rho, \gamma, \vec{F}, f$, and q are as in Theorem 139. If $\overline{B_{3R}} \subset \Omega$, then for all $0 < \theta < 1$,*

$$\text{osc}_{B_{\theta R}} u \leq C_{n,\rho,\gamma,q} \theta^\alpha \left[\text{osc}_{B_R} u + \|\rho_*(\vec{F})\|_{L^q(B_{2R})}^{\frac{1}{\gamma-1}} + \|f\|_{L^{\frac{q}{\gamma'}}(B_{2R})}^{\frac{1}{\gamma-1}} \right].$$

Proof. For $0 < s < 2R$ let

$$M_s = \sup_{B_s} u \qquad \text{and} \qquad m_s = \inf_{B_s} u.$$

Then $v \in \{M_{2R} - u, u - m_{2r}\}$ is a positive solution of (6.14) on B_{2R}.

Consider $p = 1$ and $\theta = 1/2 < \tau < 1$ in Theorem 137. That is, for a positive solution v,

$$\inf_{B_{R/2}} v + G(R) \geq C R^{-n} \int v \tag{6.61}$$

where

$$G(R) = R^\delta \|\rho_*(\vec{F})\|_{L^q(B_R)}^{\frac{1}{\gamma-1}} + R^{\gamma'\delta} \|f\|_{L^{\frac{q}{\gamma'}}(B_{2R})}^{\frac{1}{\gamma-1}}.$$

Note, $\inf_{B_{R/2}} M_{2R} - u = M_{2R} - M_{R/2}$ and $\inf_{B_{R/2}} u - m_{2R} = m_{r/2} - m_{2r}$. Therefore, applying (6.61) to $M_{2R} - u$ and $u - m_{2R}$ yields

$$M_{2R} - M_{R/2} + G(R) \geq C R^{-n} \int M_{2R} - u$$
$$m_{R/2} - m_{2R} + G(R) \geq C R^{-n} \int u - m_{2R}.$$

Adding yields

$$(M_{2R} - m_{2R}) - (M_{R/2} - m_{R/2}) + 2G(R) \geq C(M_{2R} - m_{2R}),$$

or equivalently,

$$\operatorname{osc}_{B_{R/2}} u \leq (1 - C) \operatorname{osc}_{B_{2R}} u + 2 \left[R^{\delta} \|\rho_*(\vec{F})\|_{L^q(B_R)}^{\frac{1}{\gamma - 1}} + R^{\gamma' \delta} \|f\|_{L^{\frac{q}{\gamma'}}(B_{2R})}^{\frac{1}{\gamma - 1}} \right].$$

Lemma 132 verifies the result by choosing the parameters from Lemma 132 by $\tau = 1/4, \tilde{\delta} = (1 - C)$, and $\mu(1 - \frac{n}{q(\gamma - 1)}) > \alpha$. The latter can be done by making α smaller if necessary. Notably $\alpha = \alpha(1/4, \tilde{\delta})$ so it has the expected dependencies. □

Holder regularity is a classic result of the improvement of oscillation in Theorem 140.

Corollary 141. *Suppose $u, \rho, \gamma, \vec{F}, f$ and q are as in Theorem 139. Then $u \in C^{\alpha}_{\text{loc}}(\Omega)$ for some $\alpha = \alpha(n, \gamma, \nu, \Lambda, q)$.*

Last, we conclude with a Liouville-type theorem in the case that $f, \vec{F} \equiv 0$.

Theorem 142 (Liouville Theorem)**.** *Let ρ, γ, and u be as in Theorem 139. Suppose additionally that $f, \vec{F} \equiv 0$. If $\Omega = \mathbb{R}^n$ and u is bounded from above or below, then u is constant.*

Proof. It suffices to assume $u \geq 0$ by replacing u with $-u + \sup_{\mathbb{R}^n} u$ ($u - \inf_{\mathbb{R}^n} u$, resp.) when u is bounded above (below, resp.). Since $f, \vec{F} \equiv 0$, and $\Omega = \mathbb{R}^n$, Theorem 139 implies u is bounded above. Indeed, for $x \in \mathbb{R}^n$, $u(x) \leq \sup_{B_{|x|}} u \leq C_{n,\gamma,\nu,\lambda,Q} \inf_{B_{2|x|}} u \leq Cu(0)$. In particular, $\|u\|_{L^\infty(\mathbb{R}^n)} < \infty$. Now, iterating Theorem 140 says there exists $0 < \theta < 1$ so that for any $R > 0$ and integer k,

$$\operatorname{osc}_{B_R} u \leq \theta^k \operatorname{osc}_{B_{2^k R}} \leq \theta^k \left(2\|u\|_{L^\infty(\mathbb{R}^n)} \right).$$

Taking k and R to infinity consecutively completes the proof. □

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