Identifying and Addressing Student Difficulties with the Source-Field Relationship using Tutorials in Upper-Division Electromagnetism

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A dissertation

submitted in partial fulfillment of the

requirements for the degree of

Doctor of Philosophy

University of Washington

2021

Reading Committee:

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Program Authorized to Offer Degree:

Physics

University of Washington

Abstract

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This dissertation describes an investigation of student conceptual understanding of the source-field relationship in upper-division electromagnetism. The three core aspects of the source-field relationship discussed in this dissertation are: the interpretation and use of integrals in describing superposition from distributed sources, the use of symmetry arguments to infer criteria necessary for simplifying flux and contour integrals, and the interpretation and use of vector derivatives to generalize the source-field relationship. This research is providing the beginnings of a research base for the creation and development of several tutorials in *Tutorials in Physics: Electrodynamics,* a set of worksheets for use in upper-division electromagnetism courses.

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Acknowledgements

I would like to express my sincere gratitude to my supervisor, Peter Shaffer, for his guidance through my Ph.D. journey. His inquiries challenged me to explore student understanding with both breadth and depth, and his suggestions were invaluable to the completion of this dissertation.

I would also like to thank the other faculty in the Physics Education Group (PEG): Paula Heron, Suzanne Brahmia, Donna Messina, and the late Lillian McDermott. Their mentorship and expertise helped teach me about prevailing theories in physics education, and they provided additional perspectives to my research.

I would like to express my sincere gratitude to Ryan Hazelton, who served as a collaborator and peer mentor during the first couple years of my journey. The groundwork that he laid in introducing tutorials to the upper-division electromagnetism course sequence at the University of Washington (UW) was essential in enabling my research, and the discussions we had really helped me develop a depth of understanding about my research topics.

I would also like to thank the other PEG group members for lively and inclusive community they formed. My interactions and discussions with them were very insightful and memorable, as each brought their unique viewpoints, expertise, and personalities. Those who have graduated before me are Paul Emigh, Aziz Khan, Alexis Olsho, Brian Stephanik, Tong Wan, Lisa Goodhew, and Sheh-lit Chang. The current graduate students are Charlotte Zimmerman, Anne Alesandrini, Jesse Ashworth, Jared Canright, Yasmene Elhady, and Kristin Kellar.

I would like to thank the faculty and instructors at UW who have allowed me to conduct research in their courses. This includes Silas Beane, Pavel Bolokhov, David Cobden, Matthew DePies, Alejandro Garcia, Subhadeep Gupta, Christopher Laumann, Gerald Miller, the late Jim Reid, Leslie Rosenberg, Gerald Seidler, and Samu Taulu.

Lastly, I would like to thank my family for their continued support and encouragement.

Chapter 1 Introduction

This dissertation describes an investigation of student conceptual understanding of the *source-field relationship*[i] in the upper-division electromagnetism course sequence. The course sequence requires students to apply their knowledge of the source-field relationship from introductory electromagnetism at a higher level of logical complexity and mathematical rigor. In these courses, students are introduced to various mathematical techniques in the context of familiar sources and fields, and are then guided to generalize the relationships to describe arbitrary sources and fields. However, many students find the course sequence challenging in many ways, particularly in making connections between the advanced mathematics and the physical concepts[1]. The research described here investigates student ability to interpret and use integrals of distributed sources, symmetry arguments, and vector derivatives in relating sources and fields. Throughout this research, specific student difficulties regarding the source-field relationship have been identified, and this dissertation describes an attempt to address these with the development of select topics in *Tutorials in Physics: Electrodynamics*[2].

A. Physics Education Research in general

The 1970s are often said to mark the beginning of the field of Physics Education Research (PER), which includes the study of how students learn physics topics and of best practices to adopt in physics instruction. Since then, PER has produced a large volume of research ranging from the identification of common student conceptions about physics, development of analytical tools and assessments to quantify student thinking, development of curricula that promote conceptual learning in students, to diversity studies on how instruction impacts various demographic groups.

[i] In this dissertation, the phrase *source-field relationship* refers to the way in which *sources* like charge or current that "create" vector *fields* like the electric or magnetic fields, respectively.

The University of Washington (UW) Physics Education Group philosophy is based on the interplay between three main pillars: research, instruction, and curriculum. Research on student understanding of physics and the identification of conceptual and reasoning difficulties provides insights that can guide physics instruction and the development of effective curricula. Instruction that takes place in multiple classes over multiple years provides data for research and feedback on student interactions with the curricula. Changes to curricula can then spark new research questions and improve the effectiveness of physics instruction. It is these interactive processes that help us develop effective, research-validated teaching methods and curricula.

Much of PER is devoted to helping students understand concepts. One key finding[3] is that after traditional lecture-based instruction, introductory students may show proficiency in quantitative skills like solving typical end-of-chapter problems, but many of the same students can show deficiency in qualitative reasoning when answering conceptual questions[i] within the same topic. Other findings[4,5] suggest that instructional strategies that incorporate a high degree of active engagement between students and their peers, instructors, and curricula tend to do much better at promoting conceptual development and qualitative reasoning skills for students than strategies that lack active engagement. However, simply getting students to engage is insufficient, as curricula also need to be designed to promote student engagement in meaningful ways[6].

Faculty in PER and associated fields have developed many theories about student learning, and I briefly describe a small subset that have influenced my perspective in teaching, research, and curriculum development at UW. The psychological theory of *constructivism*[7] states that learners form or construct their own knowledge based on their own experiences, which suggests that everyone starts with different pre-conceptualizations[ii] when learning new or advanced topics and that learning occurs at a personal level. It is important to find common patterns in alternate mental models that compete with the model

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[i] *E.g.,* Is quantity A *greater than, less than,* or *equal to* quantity B?
[ii] This could be conceptualizations one made about the real world, or conceptualizations developed in ones past in related topics.

being taught through the identification of student difficulties[8,9]. To address student difficulties, the *Elicit-Confront-Resolve*[10] method is an instructional strategy designed to help students become aware of inconsistencies that may be present in their current conceptualization of a topic in order to promote conceptual development toward a more coherent model.

In addition to educational theories, faculty in PER have developed numerous theoretical frameworks, which can provide structure in research by framing the way data can be analyzed and interpreted. Although I did not structure my research around any specific framework, the dissertation makes some references when any are particularly relevant or insightful to the work that has been done as part of this research. The *resources*[11] framework breaks down larger conceptual ideas into much smaller pieces that can be contextually activated and combined, and suggests that misconceptions may stem from inadequate activation or synthesis of basic ideas. The *Mathematization of Physics*[12] framework categorizes the interaction between mathematics and physics in terms of interpretation and application. The *Symbolic Forms*[13] framework describes the physical interpretation of mathematical equations and expressions based on the relative positioning and links between symbols.

Studies in K-12 education and introductory-level college courses make up the bulk of PER. In the past decade, some members of the PER community began to study content taught in upper-division courses, although mainly in quantum mechanics. As such, studies in upper-division electromagnetism form a very small portion of the PER base, so much of the context for this dissertation is unexplored.

B. Context for research

Almost all my research has taken place in the context of the upper-division electromagnetism course sequence at the UW. Here, I provide an overview of the course structure and classroom environment pertaining to my research and the development of the upper-division electromagnetism tutorials.

1. UW Upper-division Electromagnetism course structure

The upper-division electromagnetism course sequence at UW uses Griffiths' *Introduction to Electrodynamics* textbook[14]. The course sequence consists of a year-long set of three classes taught on a quarter system[i]: Phys321, Phys322, and Phys323. Phys321 focuses on electrostatics, covering chapters 1-4 in Griffiths. Phys322 focuses on magnetostatics and Maxwell's Equations, typically covering chapters 5-9.2 in Griffiths. Phys323 focuses on applications of electrodynamics, time-dependence, and special relativity, covering chapters 9-12 in Griffiths. The first two classes are required by all physics majors, whereas Phys323 is an elective.

Almost all students who take the upper-division electromagnetism courses are undergraduate physics majors in their third or fourth year. Phys321 is offered in the Autumn and Spring, with about 125 and 110 students per quarter[ii], respectively. Phys322 is offered in the Winter and Summer, with about 175 and 50 students per quarter, respectively. Phys323 is typically only offered in Spring, with about 90 students per quarter. These relatively large class sizes allow research at UW to be relatively quantitative, whereas research at universities with smaller physics programs are limited to qualitative data.

Almost all instructors of the UW upper-division electromagnetism courses during my research opted to teach the course with lectures, associated lecture homework, and a tutorial component as a discussion section. The lectures tend to follow closely with Griffiths, while the lecture homework tends to be quantitative, focusing on derivation and problem solving based on Griffiths' example problems.

The tutorial component focuses on conceptual understanding, interpretation, and logical reasoning to complement the quantitative skills that are typically the focus of lecture. This course structure is modeled after the calculus-based introductory physics sequence at UW which uses the *Tutorials in Introductory Physics*[15], which has proved to be an effective strategy[10] at the introductory

[i] Autumn quarter spans from late-September to December, Winter quarter spans from January to mid-March, Spring quarter spans from late-March to mid-June, and Summer quarter spans from late-June to mid-August.

[ii] Numbers based on the 2019-2020 academic year. Class sizes are steadily growing (e.g., 100 students in Phys321 Autumn 2015 suggests about a 5% growth per year).

level. The tutorial component has three pieces that occur weekly: pretests, in-class tutorials, and associated tutorial homework. These are described below.

The pretest is an online, pre-tutorial survey designed to elicit conceptual ideas about a given topic. A typical pretest takes between 10-20 minutes to complete and pairs multiple choice or free-response questions with prompts for students to explain their reasoning. Pretests are typically given a few days after the relevant topic is covered in lecture, but before the topic is covered in tutorial. Students are given credit based on effort, which incentivizes adequate completion of the pretest. From an instructional perspective, the intent of the pretest is to help students preview a topic and articulate their current understanding of the topic, which may provide a basis upon which they may reflect upon after instruction. From a research perspective, the pretest provides a measurement of the students' understanding of a topic after traditional instruction but before tutorial intervention, which serves to provide a baseline for comparison in lieu of a control group. Pretests are also used to inform teaching assistants about difficulties that the tutorial is intended to address, which helps the teaching assistances gain pedagogical content knowledge[16] that enhances their teaching ability.

The in-class tutorials are small-group discussion sections that are guided by tutorial worksheets. A typical tutorial section contains a subset of about 20 students from the larger class, within which students form smaller groups of 3-5 students each. Each group is given a whiteboard and works through the tutorial worksheets, which are a highly scaffolded set of questions designed to elicit conceptual difficulties and promote discussion within the groups. The tutorials are 50-60 minutes long, and students are given credit for participation to incentivize attendance. Each section is typically led by two experienced graduate teaching assistants who have taken a preparatory course for teaching the introductory-level tutorials. In addition, the teaching assistants prepare weekly by discussing the concepts of the upcoming tutorial and predicting places where students need help.

After completing the tutorial, students work through the tutorial homework. These are sets of questions that focus on conceptual understanding and reasoning, reinforcing or extending concepts covered in the tutorial. Students typically turn in the homework within a week after their tutorial section.

The homework is graded such that only between one-third to one-half of the credit is awarded for getting correct answer, while the remaining one-half to two-thirds of the credit is awarded for proper reasoning.

The courses also include midterm examinations and finals. Typically, the instructor contributes 2-4 pages of questions based on the lecture component of the course, while a member of the Physics Education Group contributes one page of questions based on the tutorial component. The tutorial exam questions emphasize qualitative reasoning and are graded similarly to the tutorial homework.

2. Development of *Tutorials in Physics: Electrodynamics*

The development of the tutorials in upper-division electromagnetism at UW drew heavily on the experience gained in the development of the *Tutorials in Introductory Physics* and the subsequent development of tutorials in quantum mechanics. The UW Physics Education Group first began developing *Tutorials in Introductory Physics* in 1991 for use in calculus-based introductory physics, which is now widely known throughout the physics education community. The first upper-division quantum mechanics tutorials at UW were written in 1996, which the group continued developing in the early 2000s into what is now called *Tutorials in Physics: Quantum Mechanics*[17]. Meanwhile, the upper-division electromagnetism course lectures were paired with traditional problem-solving review sessions in which a graduate teaching assistant would work through example and past homework questions. However, many upper-division students who were exposed to the tutorials in quantum mechanics asked for tutorials to be implemented in upper-division electromagnetism as well. We first introduced tutorials to the upper-division electromagnetism course sequence in Summer 2014.

At first, we imported 16 of the E&M tutorials written at the University of Colorado Boulder (UC-Boulder). These were part of a larger initiative at CU-Boulder in transforming the entire upper-division program around 2007[18] to better incorporate active engagement throughout their courses. The tutorials served as a good starting point for determining which topics students needed the most help on conceptually. However, we quickly found that CU-Boulder's tutorials were not well suited to the course structure and student population at UW. In part, this may have been that CU-Boulder employed a flipped classroom approach, so their tutorials were not meant to be the primary tool for helping student develop conceptual understanding of the topic. This may have been why applying them in isolation from the rest of the transformed course structure did not seem to be very effective at UW. Additionally, their tutorials tended to have mathematical parts that prompted for derivation with little scaffolding, with which the students at UW seemed to have difficulty. Overall, student progression through the CU-Boulder tutorials did not expectations, as students tended to take longer than expected and tended to rely on teaching assistants to interpret questions and progress. This suggested that we needed to adapt or rewrite the tutorials for use at UW.

Since then, we have been developing the set of tutorials at UW *Tutorials in Physics: Electrodynamics*, which are listed in Table 1-1. Those in italics were originally inspired by the CU-Boulder tutorials but have since been significantly modified or rewritten. Those in bold represent the tutorials that are most relevant to my research, grouped by chapter in Table 1-2.

Table 1-1: List of tutorials in Tutorials in Physics: Electrodynamics.

Electrostatics	Magnetostatics	Waves and Time-Dependence
Coulomb's Law	**Magnetostatic Fields**	EM Plane Waves
Integrals with Charge	***Ampère's Law***	Waves at Boundaries
Vector Derivatives	***Magnetic Vector Potential***	Waveguides
Divergence as a Source	Magnetization	*Gauge Transforms*
Gauss' Law	**Auxiliary Field**	*Retarded Time*
Potential	*Electromotive Force*	Radiation
Conductors	Fields at Boundaries	Relativistic Length and Time
Separation of Variables	*Maxwell-Ampère's Law*	Intervals and Causality
Fourier's Trick	*Energy Flow in Circuits*	Energy, Mass, and Momentum
Method of Images		Relativistic Electromagnetism
Multipole Expansion		
Polarization		
Displacement Field		

Table 1-2: Relevant tutorials by chapter with three-letter codes.

Chapter 2	Chapter 3	Chapter 4
COU - Coulomb's Law INT - Integrals with Charge MPE – Multipole Expansion	GSL – Gauss' Law AML – Ampère's Law	DEL – Vector derivatives VDS – Divergence as a source DPF – Displacement Field MSF – Magnetostatic Fields MVP – Magnetic Vector Potential AUX – Auxiliary Field

C. Methods of investigation

1. Data sources

The quantitative data presented in this dissertation come from tutorial pretests and tutorial exam questions in the upper-division electromagnetism course sequence. Some tutorial homework data were collected but are not presented unless otherwise stated. Pretest data were filtered to exclude responses that were incomplete, duplicate[i], non-sensible, or appeared to have been completed in a rush by students[ii]. As pretests tend to be administered over many quarters and produce similar results, pretest data is aggregated unless there are any noticeable differences or trends to report. Exam data were not filtered unless explicitly stated. For each task, student reasoning was categorized using an emergent coding scheme[19], in which student explanations and equations were coded based on the core logic that appeared to be the basis for their response. The codes were then condensed and cross-referenced between data sets until fewer than 5 categories could describe most of the responses.

Informal observation in the tutorial classroom, office hours, and homework grading also provided insight into student conceptual difficulties and helped with the development of the tutorials.

2. Analysis and presentation of data

Percentages in the dissertation text are rounded to the nearest 5% to improve readability and to give a sense for the instructional relevance of the findings. Percentages in figures and tables are rounded

[i] If a student submitted multiple responses to a pretest, the first non-blank response of every field was used, and subsequent responses were discarded.

[ii] Students who gave non-sensible responses or purposefully left explanations blank tended to take very little time. Responses that took less than 5 minutes for the 25-minute pretest were omitted as an extra precaution to ensure that the data represented a population of students who carefully considered their responses.

to the nearest percent to provide precision. The dissertation text makes statements about statistical significance, while footnotes provide details about the statistical test used. Most of the data are categorical[i], so the primary analysis technique used in this dissertation was the chi-squared test for independence, which is suitable for large sample sizes. The Fisher exact test was used when the count of any occurrence was less than 10, as the chi-squared test is inaccurate with smaller sample sizes or heavily skewed data. The 95% confidence level was used to determine statistical significance.

The labels for the data sources are shortened to condense figure and table sizes. Tutorials have an associated three-letter code, which can be found in Table 1-2. Quarters are labeled by three numbers[ii], where the first two numbers correspond to the year and the third number corresponds to the quarter. Further information on labeling schemes can be found in the guide to the appendix <App – 1>.

D. Organization of dissertation

The overarching theme of this dissertation is on student interpretation and application different aspects of the *source-field relationship* in upper-division electromagnetism. The aspects that are covered in this dissertation are: the interpretation and use of integrals in describing superposition from distributed sources, the use of symmetry arguments to infer criteria necessary for simplifying flux and contour integrals, and the interpretation and use of vector derivatives to generalize the source-field relationship.

Each chapter begins with an introduction reviewing the theory of the content and continues with an overview of past research in PER relevant to that aspect. This is followed by the bulk of each chapter, which presents my investigation on the student interpretation and application of that aspect, the development of curricula to address student difficulties, and the assessment of the curricula.

Chapter 2 presents research on how students use integrals to describe a superposition of sources. It also describes the development of the *Integrals with Charge* tutorial. Chapter 3 presents research on how students use symmetry arguments to infer properties of the field from the source and whether

[i] Non-numerical data, *e.g.*, yes/no, color, etc.

[ii] *E.g.*, “151” and “15(1Win) correspond to Winter 2015, while “194” and “19(4Aut) correspond to Autumn 2019.

students recognize the role of symmetry in Gauss' Law and Ampère's Law simplifications. The first part of Chapter 4 presents research on how students interpret and identify divergence and curl, while the second part of Chapter 4 presents research on how students use vector derivatives to generalize the source-field relationship to unfamiliar vector fields.

Finally, Chapter 5 presents a more holistic assessment of the impact of *Tutorials in Physics: Electrodynamics* on student understanding of electromagnetism than is given in the other chapters. It also includes a summary reflection on suggested areas for future research and curriculum development.

In the dissertation, the first person is used frequently as a stylistic preference. The singular first person typically indicates my personal research interest, ideas, or speculation about inferences from data. The plural first person typically includes Ryan L.C. Hazelton, a former graduate student and post-doc at UW with whom I worked closely in co-developing the early versions of *Tutorials in Physics: Electrodynamics* from 2014-2015. Occasionally, the plural first person refers more broadly to the UW Physics Education Group.

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Chapter 2 Superposition and Integrals

One core aspect of the source-field relationship is that it obeys the principle of superposition. The principle of superposition as used in physics leverages the linearity of a quantity to decompose the quantity into a sum of individual parts. One can then interpret each individual part as an effect from an isolated cause, so that one can model the total effect as an accumulation from multiple causes. For the source-field relationship, one can interpret each source as creating its own field, which one then combines with the fields from all the other sources to find the net field.

The principle of superposition for the source-field relationship is best represented by the application of Coulomb's Law[i]. In point-charge form, Coulomb's Law is given by $\vec{E} = \frac{kQ}{d^2}\hat{d}$, where k or $\frac{1}{4\pi\epsilon_o}$ is the Coulomb constant and $\vec{d}$ is the separation vector[ii] from the point charge to the field point. One can see that the electric field produced by a point charge is linearly proportional to the amount of charge. When applying Coulomb's Law to a series of point charges, one would express the partial electric field from each point charge with its own separation vector and charge value, and then find the vector sum of the partial electric fields to find the net electric field. When applying Coulomb's Law to distributed charges though, the point-charge form of Coulomb's Law in insufficient, and one must be able to encapsulate the principle of superposition in the form of an integral.

The Riemann sum interpretation of integration perfectly embodies the principle of superposition for the source-field relationship. For example, in the expression $\phi = \int_{all\ u} f(u)du$, the infinitesimal du can be interpreted as an isolated cause, the product $f(u)du$ as the effect of any individual cause, the integral sign represents the accumulation of small effects, and the bounds represent the set of all causes.

[i] And Biot-Savart Law as the magnetostatic analogue.
[ii] Represented in Griffiths as a script r.

This leads to the interpretation that ϕ represents the total effect from all causes. Other common interpretations of integration[i] do not have as strong of a connection to the principle of superposition as the Riemann sum.

In the physical context[ii] of distributed charges, du is represented by an infinitesimal charge element dq, which results in a set of *charge-based integrals* that have the form $\phi = \int_{all\ charges} f(\vec{r}')dq$, where $\vec{r}'$ represents the position vector of the infinitesimal change element. This set not only includes determining electric field via Coulomb's Law [when $f(\vec{r}') = \frac{k}{d^2}\hat{d}$], but also includes determining net charge [when $f(\vec{r}') = 1$], electric potential [when $f(\vec{r}') = \frac{k}{d}$], and dipole moment [when $f(\vec{r}') = \vec{r}'$]. These integrals all require the same basic steps to set up. One must first identify the dimensionality of the charge distribution and express the infinitesimal charge element as $\rho d\tau'$, $\sigma dA'$, or $\lambda dl'$. Next, one must parametrize the infinitesimal spatial element in the appropriate coordinate system, which requires the understanding of infinitesimal arc lengths if using curvilinear coordinate systems. Then, one must represent the shape and location of the charge distribution with the bounds of the integral in the chosen coordinate system. Finally, one must identify and substitute elements in $f(\vec{r}')$ that would vary based on the location of the various infinitesimal charges.

This chapter discusses my research on the student understanding of charged-based integrals and how that has inspired the creation and development of the *Integrals with Charge* tutorial. Section 0 discusses relevant research on the interpretation of integration in physics. Section B describes the initial attempt to address issues with determining electric field with the *Coulomb's Law* tutorial starting Autumn 2014. Section C discusses an investigation in student difficulties with charge-based integrals. Section D describes the first version of the *Integrals with Charge* tutorial in Autumn 2015 as an attempt to address student difficulties with charge-based integrals. Section E discusses further investigation in student

[i] *e.g.*, area under a curve, inverse operation of the derivative
[ii] Although the research on this aspect of the source-field relationship is in the context of electrostatics, the concepts discussed here are generalizable to other sources, *e.g.*, current.

interpretation of density and infinitesimal elements. Section F discusses two major revisions in Autumn 2017 and Autumn 2018 designed to address student difficulties with infinitesimal elements. Lastly, section G summarizes the important findings of this research.

A. Relevant research on integration in physics

A common theme in existing literature in math and physics education research related to the charge-based integral is a failure of students to construct a fundamental understanding of the integral and the differential. An early British study by Orton[1] suggests that many students do not interpret the integral as a Riemann sum, though many are able to compute the integral when asked. In the context of physics, this leads to difficulties in solving physics questions involving distributed charge if students are unable to recognize how to apply their understanding of superposition as a sum to an integral. An interview study by Nguyen and Rebello[2] showed that most of the students interviewed had difficulties in correctly expressing the infinitesimal quantity to be summed and subsequently accumulating the infinitesimal, although most could easily identify the need for an integral and evaluate an integral. Student difficulties with interpreting the infinitesimal in physics contexts is likely tied to differences in how expert physicists and mathematicians treat the infinitesimal, both historically and in teaching[3]. For example, novice physicists likely have only seen differentials in the context of integrals and derivatives in calculus courses, where the concept of the differential itself carries little physical meaning. Much of the focus of research on the use of integrals in physics is based on the interpretation of the integral[4,5,6], the Riemann sum or "layers" framework[7], and the infinitesimal at both the introductory level[8,9,10] and the upper-division[11,12].

At UW, the *Charge* tutorial in *Tutorials in Introductory Physics*[13] focuses on other student difficulties with Coulomb's Law. It focuses on applying superposition of Coulomb's Law as a vector sum and conceptualizing distributed charges as series of point charges, which is based on research by Kanim[14] showing that many students at the introductory level have difficulty with the fundamental ideas of vector addition, superposition, and distributed charges. Thus, at the introductory level, we address several concepts necessary to form the integral, but not the interpretation and construction of the integral itself.

My research expands on the current research base by further investigating student difficulties with integrals and infinitesimals in the context of electrostatics at the upper-division level. My research also resulted in the development of an upper-division tutorial that complements the *Charge* tutorial at the introductory level in addressing student difficulties with charge-based integrals.

B. Implementation of the initial tutorial: *Coulomb's Law*

The first upper-division electromagnetism tutorials implemented at UW were borrowed from CU-Boulder. Their *Coulomb's Law* tutorial was designed to address some of the student difficulties at CU-Boulder with writing Coulomb's Law integrals. The initial adaptations of the *Coulomb's Law* tutorial <App – 5> followed a similar structure with the same goals in mind.

1. Primary goals of the *Coulomb's Law* tutorial

According to CU-Boulder's Instructors' Manual[15], this tutorial was designed to fulfill the following learning goals: familiarize students with conventional notation of the "r" vectors, represent a physical problem mathematically, understand the increased difficulty of the upper-division level while making connection to the introductory level, and check limiting cases for sense-making. In the context of charged-based integrals ($\int f(\vec{r}')\rho d\tau$), I claim that the primary focus of this tutorial was on the mathematization of the functional part of Coulomb's Law, $f(\vec{r}') = \frac{k}{d^2}\hat{d}$.

2. Description of the *Coulomb's Law* tutorial

The first half of the tutorial was focused on learning the convention of displacement and coordinates. The first section of this tutorial was dedicated to the separation vector. Students were asked to sketch out and determine the relationship between three vectors: $\vec{r}$ for the position where a field would be evaluated, $\vec{r}'$ for the position where a charge is located, and $\vec{d}$ for the separation vector from the source charge to the field charge. Students were also asked to investigate the unit vector $\hat{d}$ and use its definition to rewrite $f(\vec{r}')$ in the mathematically more convenient form, $\frac{k}{d^3}\vec{d}$. The second section of the tutorial was dedicated expressing Cartesian coordinates in terms of cylindrical and spherical coordinates, which helps students learn to apply trigonometry for coordinate transforms.

The second half of the tutorial focused on the idea of superposition and integration. In the third part, students were first asked to write the potential from two point charges and then generalize to a string of point charges. Finally, students were asked to write an integral expression for the potential due to a line charge. In the last part, students were asked to determine an integral expression for the electric field for a charged hoop. This was intended as a synthesis step for students to incorporate all the ideas of separation vector, curvilinear coordinates, and integral together with little scaffolding.

3. Informal Observations of the *Coulomb's Law* tutorial

For the first half of the tutorial, what we observed with our students were consistent with reflections provided by CU-Boulder[15]; however, we saw deviation in the second half. We noted that many students struggled to generalize to a Coulomb potential integral in the third part in Cartesian coordinates, and consequently not many students got to setting up the electric field integral for the charged hoop in curvilinear coordinates.

4. Formal Assessment of the *Coulomb's Law* tutorial

On the first exam of Phys321 in Autumn 2014 (N = 96 students) <App – 22>, students were given a cubical charge distribution of side length a and volume charge density $\rho = bz$, centered on the origin (Figure 2-1). Students were first asked to determine the net charge of the cube as a primer question, to get them to interpret the system and think about the odd symmetry across the *xy*-plane. Later, students were asked to determine the displacement vector from an arbitrary location (x', y', z') and point P, located at coordinates $(0, 7, 0)$. Finally, students were asked to determine an explicit integral expression for the

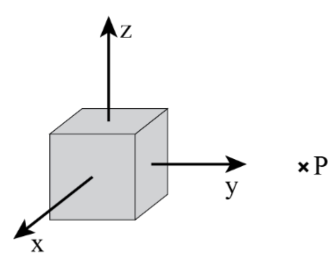

Figure 2-1: Context for the Coulomb's Law exam.

electric field at point P, a type of question that instructors typically ask without helper questions to test understanding of Coulomb's Law.

For determining the net charge of the cube, students were expected to identify that the charge distribution was positive on top and equally negative on bottom or write out an integral expression with $\int \rho dV$ and evaluate the integral to be zero. Only 45% of students were able to correctly determine that the net charge was zero. An additional 15% of students attempted to write an integral but either did not evaluate it or made mistakes in the evaluation that led to a non-zero answer. The most common incorrect response that 40% of students made was claiming that $Q_{net} = \rho V = bz * a^3$, implying that multiplication is enough to determine the net charge from a non-uniform charge density. The prevalence of this misconception suggests that many students may not be interpreting density as an infinitesimal ratio or are not internalizing the process of finding the net charge as an addition operation, especially when the density is non-uniform.

For finding the separation vector, 65% of students were able to describe it correctly in terms of the given coordinates (x', y', z') and the location of the field point, like $\vec{d} = (-x', 7 - y', -z')$. Most of the rest of the students had a sign reversal and described $\vec{r}' - \vec{r}$ instead.

For determining an integral expression for the electric field with Coulomb's Law, students were expected to write the integral $\vec{E} = \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} \frac{kbz'dx'dy'dz'}{(x'^2+(7-y')^2+z^2)^{\frac{3}{2}}} (-x', 7 - y', -z')$, which explicitly shows substitution of the volume element, the bounds of the cube, and substitutions for the separation distance and unit vector. For the volume element, about 55% of students showed the infinitesimal volume element *dxdydz* or *dx'dy'dz',* while the rest of the students either left it unspecified as $d\tau'$ or incorrectly as *dz'*. For the bounds, only 35% of student explicitly referenced bounds based on the location of the charge, while most of the students left the bounds of the integral blank, with a few instances of distance-related bounds ending at the value 7, where the point-of-interest is. About 60% of students were able to unambiguously describe[i] the separation distance and unit vector within the Coulomb's Law integral.

Overall, students performed reasonably on aspects regarding the separation vector, which is expected because that is a primary focus of the *Coulomb's Law* tutorial. The mediocre performance on the explicitness of the integral could represent a mismatch in the expectation between what students believe an explicit integral is, but it could also signal that many students do not understand the role of the bounds in a charge-based integral. The low performance on the net charge question was very unexpected and suggests that many students are not properly interpreting the process of determining net charge from a non-uniform charge density.

C. Investigation of student difficulties with charge-based integrals

To further investigate student difficulties with charge-based integrals, I developed a pretest <App – 2> to see how students were interpreting net charge, integration, and infinitesimal length elements. This pretest was scheduled after the first one or two lectures, in which instructors typically cover Coulomb's Law and curvilinear coordinates. The data from this pretest spans 8 quarters[ii] since Autumn 2015. The pretest data contains a total sample size of N = 512 students.

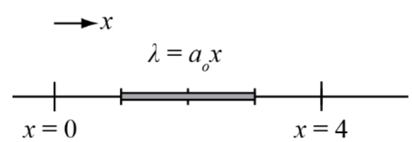

Figure 2-2: Context for the Integrals with Charge pretest.

One set of questions on the pretest shows a line charge with a density of $\lambda = a_o x$ spanning from $x = 1$ to $x = 3$ (Figure 2-2), and students were asked to determine the net charge and the magnitude of the dipole moment[iii] of the charge distribution. To aid in the analysis of these questions, they were formatted as multiple choice with distractors devised

[i] Either by explicitly substituting in their separation vector in the integral or by explicitly stating that the distance in the denominator was the separation distance.
[ii] This excludes the Spring 2016 quarter where students took the "pretest" after the tutorial.
[iii] Students were given $p = \sum(qx)$ with a point-charge example.

based on various mistakes that I hypothesized students might make, but students were asked to explain for each question.

1. Student difficulties with determining net charge

For net charge question, it was intended for students to come up with and evaluate the integral $\int_1^3 a_o x dx$, which evaluates to $4a_o$. Students averaged 50% accuracy on this question. Some students determined the net charge without explicitly mentioning integration by evaluating the charge density at the center of the line and multiplying by length[i], $\lambda(2) * L = (a_o 2) * 2 = 4a_o$. With reasoning considered, 45% of students got the correct answer and justified their answer with the use of an integral, which is consistent[ii] with the 45% of students who determined the net charge correctly on the Autumn 2014 exam.

The most common incorrect answer of $2a_o$ was chosen 40% of the time. The most common line of reasoning for this answer mentioned the length of the line as 2 or the difference between the endpoints 3 and 1. One interpretation to why students may be replacing *x* with the length is that they may be converting the *x* to *dx* and evaluating $\int_1^3 a_o dx$, as some students claimed to be doing an integral but got $2a_o$ as their answer.

2. Student performance with unfamiliar charge-based integrals

The dipole question requires taking the point-charge version of the dipole moment $p = \sum(qx)$ and converting it to the integral $\int dq * x$, where one expresses the infinitesimal charge element as $dq = \lambda dx$. Thus, the correct integral expression is

Table 2-1: Breakdown of student performance on charge and dipole tasks.

N = 512 students	% correct on dipole	% incorrect on dipole	Row total
% correct on net charge	22%	29%	51%
% incorrect on net charge	4%	45%	49%
Column total	25%	75%	

[i] This method only works for uniform and linear functions where the center represents the average of the function.

[ii] This suggests that the *Coulomb Law* tutorial did not affect student ability to use integration to determine net charge.

$\int_1^3 a_o x dx * x$, which evaluates to $(8\frac{2}{3})a_o$. Overall, only 25% of students answered correctly, most of whom answered the net charge question correctly. The breakdown of student responses on the charge and dipole questions is shown in Table 2-1. There is a significant[i] difference in accuracy on the dipole question based on how students answered the net charge question, with students who correctly determined the net charge of the system significantly outperforming the students with incorrect net charge 45% to 5%.

3. Student interpretations of the process of integration

Table 2-2: Breakdown of student responses on interpretation of integration, with performance on net charge and dipole tasks.

Interpretation of "integrate"	% of students (N = 512)	% correct on net charge	% correct on dipole
Sum	54%	62%	34%
Area under the curve	34%	37%	15%
Anti-derivative	5%	36%	12%
Miscellaneous	7%	34%	20%

Another question on the pretest asked students to define what it means to integrate in five words or less. Ideally one would like to see the students interpreting the process of integration as a Riemann sum, and an average of 55%[ii] of students included the idea of summation, accumulation, or addition in their definition. Alternatively, 35% of students defined integration as the area under a curve, and 10% of students defined it as the inverse of differentiation or miscellaneous. The breakdown of student responses on this question in relationship to the charge and dipole questions are shown in Table 2-2. There is a statistically significant difference[iii] between the students' interpretation of integration with performance on the net charge and dipole questions as well. Students who include ideas of addition answer those questions correctly 60% and 35% of the time, respectively, outperforming students with alternate interpretations scoring 35% and 15%,

[i] The chi-squared test for independence on the 2x2 contingency table (Table 2-1) gives a p-value of 4.02e-20. This allows one to reject the null hypothesis that student accuracy on the net charge and dipole questions are independent

[ii] Quarterly data varies from 45% to 65%.

[iii] The chi-squared test for independence on the two 2x2 contingency tables comparing accuracy to whether students interpreted integration as a sum showed give p-values of 9.24e-9 for the net charge question and 1.82e-6 for the dipole question. This allows one to reject the null hypothesis that student interpretation of integration does not affect accuracy on determining net charge and dipole.

respectively. This suggests that identifying addition as the primary operation involved in the process of integration is a key element in correctly setting up and evaluating charge-based integrals.

4. Student difficulties with infinitesimal arc length

Lastly, the pretest contained a set of questions asking students to express infinitesimal length and area elements, with descriptors and images given. The most notable piece is how students interpret "the length of a small step around the origin" to see if students can distinguish between arc length and angle, and correctly express arc length in curvilinear coordinates. Only an average of 10% of students gave the correct answer resembling $rd\theta$. 55% of students simply described the infinitesimal arc length as $d\theta$, suggesting that many students do not distinguish between infinitesimal arc length and infinitesimal angle. 15% of students attempted to express it in terms of Cartesian coordinates like $\sqrt{dx^2 + dy^2}$, which suggests that some students do not recognize how to use an appropriate coordinate system.

D. Creation of the *Integrals with Charge* tutorial

I hypothesized that student difficulties with expressing Coulomb's Law via an integral were more fundamental than the expression of the separation vector. The separation distance and unit vector does add layers of complication to the electric field integral and are pieces that eventually need to be addressed, but my findings suggest that many students at UW had difficulty interpreting the quantity $\rho d\tau'$ as an infinitesimal charge and the integral as describing a process of accumulation. So, in Autumn quarter of 2015, I developed a new tutorial titled *Integrals with Charge* <App – 10>, intended to replace the *Coulomb's Law* tutorial.

1. Initial goals of the *Integrals with Charge* tutorial

The overall goal of the tutorial is to help students better understand the meaning of the integral sign and the infinitesimal charge element within charge-based integrals. For the integral, the focus is teaching students that the process of integration is, at its core, a summation. To a lesser extent, the tutorial also aims to teach students that the bounds correspond to the location of the charges. For the infinitesimal charge element, the focus is teaching students how to physically interpret infinitesimal quantities.

2. Description of the *Integrals with Charge* tutorial

On the first page of this tutorial, students explore the difference between finding the net charge for a series of equal and unequal point charges (Figure 2-3). The purpose of this first page is to suggest that addition is the primary operation necessary to account for multiple charges, whereas multiplication is a simplification when things are equal. This leads to the core concept in the second page establishing that integration is best interpreted as the summation of many infinitesimal quantities. The second page then proceeds to discuss the meaning of infinitesimal quantities like "*dx*" and the product within the integral. A discussion in homework further delves into the meaning of charge density, ultimately as ratio of infinitesimal charge to an infinitesimal spatial measurement.

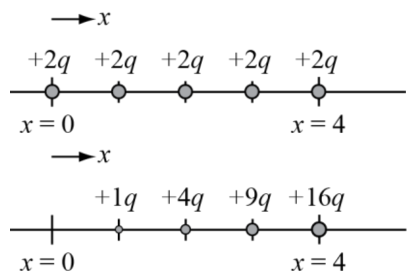

Figure 2-3: Series of point charges in Integrals with Charge tutorial.

The third and fourth parts of the tutorial complicate the infinitesimal charge element by introducing polar coordinates and surface charge densities, which target the parametrizing step of writing the charge-based integral. There are also a few questions that discuss the bounds. The core concept with curvilinear coordinates that students should understand is that the charge densities need to be paired with the correct dimensions of length to represent an infinitesimal charge, and thus curved segments must be described by arc length instead of infinitesimal angles to show the correct dimensionality. The concepts in these parts are extended in the homework, where students are asked to sketch and determine infinitesimal length and volume elements in cylindrical and spherical coordinates. In comparison to the curvilinear coordinates part of *Coulomb's Law,* the focus is on expressing infinitesimal arc length as $Rd\theta$ for use in the infinitesimal charge element rather than on applying coordinate substitutions for use in the displacement vector.

The rest of the tutorial and homework provide support for the functional part of the charge-based integral ($\int f(\vec{r}')\rho d\tau'$), because the initial core steps have all been addressed with the basic case of determining the net charge, when $f(\vec{r}') = 1$. Ultimately, students should be able to write Coulomb's Law generally, when $f(\vec{r}') = \frac{k}{d^2}\hat{d}$, but there is not enough room to fit essentially the entire *Coulomb's Law* tutorial. Rather, the concepts are broken down into pieces. In the fifth part and supplement, students investigate the dipole moment, which offers the simplest way to introduce a spatially weighted integral when $f(\vec{r}') = \vec{r}'$. This teaches the concept that the position of an infinitesimal charge needs to be included as a variable within the product of the integral rather than as a multiplication of net charge with a single position. The dipole moment also is a vector quantity, which helps tease out some conflation between components and coordinates[i], which will be elaborated further in the symmetry chapter. In the homework, students take on the next step with the Coulomb potential when $f(\vec{r}') = \frac{k}{d}$, which uses the skill of parametrizing the separation distance in terms of the position of the infinitesimal charges. Thus, with the completion of the tutorial and homework, students would have all the resources necessary to use Coulomb's Law for the electric field, except for addressing the unit vector.

3. Assessment of the *Integrals with Charge* tutorial with respect to initial goals

a. Post-tutorial "Pretest" data

In Autumn 2015, the tutorials *Vector Derivatives* and *Integrals with Charge* were both created and used as the first and second tutorial of Phys321, respectively. This was the planned sequence for Winter 2016 as well, but we decided after students had already seen the *Vector Derivatives* pretest to switch the order of these tutorials such that students saw *Integrals with Charge* first. Consequently, these students ended up taking the *Integrals with Charge* "pretest" after they had seen the tutorial, but before

[i] Students are often enlightened by the fact that the *y*-component of the dipole moment should have *dx* instead of *dy* for a horizontally distributed line charge.

they had submitted the homework. The results from this assessment serves as a posttest to the tutorial, with a sample size of N = 55 students.

The post-test students significantly[i] outperformed the pretest students on three of the four questions: dipole moment by 15%, integral interpretation by 35%, and infinitesimal arc length by 20% (Figure 2-4). The post-test students outperformed the pretest students on the net charge question by 15%, although the difference is only significant[ii] when considering correct answers with reasoning that references the use of an integral. This suggests that the early iteration of the *Integrals with Charge* tutorial was fulfilling its primary goal of teaching students to interpret integration as an addition operation and had made a positive impact on the students' understanding of charged-based integrals.

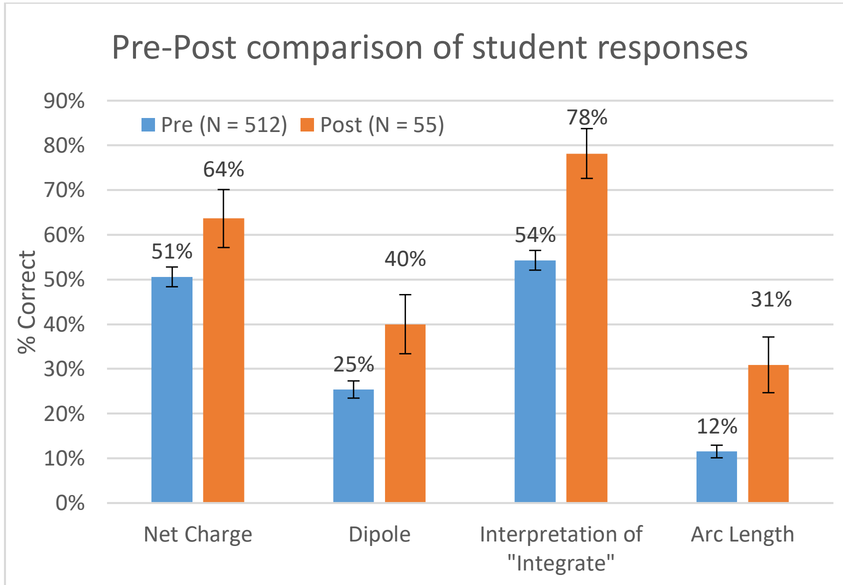

Figure 2-4: Pre-post comparison of student responses on Integrals with Charge pretest.

[i] The chi-squared test for independence on 2x2 contingency tables for a pre-post comparison on accuracy on each question give p-values of 0.0201 for the dipole question, 6.83e-4 for the integration question, and 6.07e-5 for the arc length question. This allows us to reject the null hypothesis that the tutorial does not affect student accuracy on each of these questions.

[ii] The chi-squared test for independence on the 2x2 contingency table for a pre-post comparison on accuracy alone gives a p-value of 0.0657, but a p-value of 0.0213 on correct answer justified with a reference to an integral. One cannot reject the null hypothesis that the tutorial does not affect student accuracy on the net charge question, but one can reject the null hypothesis that the tutorial does not affect students' reasoning about the net charge question.

b. Slanted-line task

On the first exam of Phys 322 in Autumn 2016 <App – 23>, I asked an exam question probing student understanding of the infinitesimal spatial element. Students were given a charged line segment with a linear charge density $\lambda(x, y) = a_o x^2 y$ that was angled between the origin and (3, 4) (Figure 2-5). As a primer question, students were first asked to identify the flaw(s) in the statement claiming that the net charge of the system was $\int_0^3 \int_0^4 a_o x^2 y \, dx \, dy$. The fundamental flaw is that the product *dx***dy* represents an infinitesimal area, so the integral is attempting to accumulate λdA, which does not represent an infinitesimal charge element. The following part then asked students to determine a correct integral expression for the net charge. The correct integral is mathematically tricky as it involves using trigonometric ratios based on the slope of the line segment to parametrize $\int \lambda dl$ in terms of a single variable, such as $\int_0^3 a_o x^2 \left(\frac{4}{3}x\right)\left(\frac{5}{3}dx\right)$. The sample size was N = 93 students.

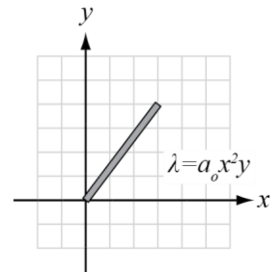

Figure 2-5: Context for the slanted-line task.

For identifying the flaw in the provided statement, 70% of students recognized the flaw of λdA, which at the time I thought was surprisingly low. The remaining students stated that the only flaw was that the order of the bounds was not written according to the convention $\int_{y_i}^{y_f} \int_{x_i}^{x_f} f(x, y) \, dx \, dy$. Of the students who identified the flaw as the infinitesimal area, 80% attempted to express the revised net charge integral as a single one-dimensional integral that included trigonometric ratios, although most did not have the correct factor associated with *dl*. Thus about 55% of the class performed reasonably on this task.

c. Rectangular-sheet task

I wanted to investigate further as to why some of the students did not recognize that the product *dx*dy* refers to an infinitesimal area element, so that same quarter I asked another question on the final exam of Phys 321 in Autumn 2016 <App – 24> to elicit how students construct a charge-based integral. Students were given a charged sheet of height h_o, width w_o, charge density $\sigma(x,y) = a_o x^2$ located in quadrant II with the origin at the sheet's bottom right corner (Figure 2-6). Students were asked to determine an integral expression for the *x*-component of the dipole moment and explain how they determined each part based on the point-charge form $\vec{p} = \sum Q_i \vec{r}_i$. Students were expected to show that $Q_i \to \sigma dA \to a_o x^2 * dxdy$ and $r_x \to x$ such that $p_x = \int_0^{h_o} \int_{-w_o}^{0} (a_o x^2 dxdy)x$, or in a simplified form $h_o \int_{-w_o}^{0} a_o x^3 dx$. The sample size was N = 89 students.

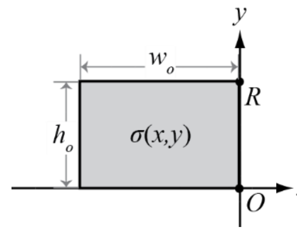

Figure 2-6: Context for the rectangular-sheet task.

For describing the infinitesimal charge element, 45% of students correctly showed the replacement $Q_i \to \sigma dA$ or with a pre-evaluated *y*-integral $Q_i \to h_o \sigma dx$. 40% of the students did not include the infinitesimal spatial element, showing their replacement as $Q_i \to \sigma$ or $Q_i \to h_o \sigma$, while the remaining 15% showed $Q_i \to \sigma A \to \sigma h_o w_o$ or something else. This suggests that many students were not interpreting the infinitesimal spatial element as a key part of the infinitesimal charge element, but instead were mapping charge to charge density. About half of the students who did not include an infinitesimal with the charge suggested the incorrect substitution $r_x \to dx$ for where the infinitesimal was derived, which resulted in an integral that closely resembled net charge; the other half suggest that *dx* was not a part of either Q_i or $\vec{r}_i$, which suggests that they were including *dx* because they recognized a need for an integral along *x*, likely cued by the *x*-dependence in the charge density. For including a vertical length in their answer, 50% of students correctly represented the vertical length as $\int_0^{h_o} dy$ or h_o. 10% of students explicitly mentioned that because the question asked for the *x*-component, they are neglecting the *y* part of their answer (Figure 2-7), indicative of a conflation between the vector components and coordinate dependence. 30% of students left out the vertical length without further explanation.

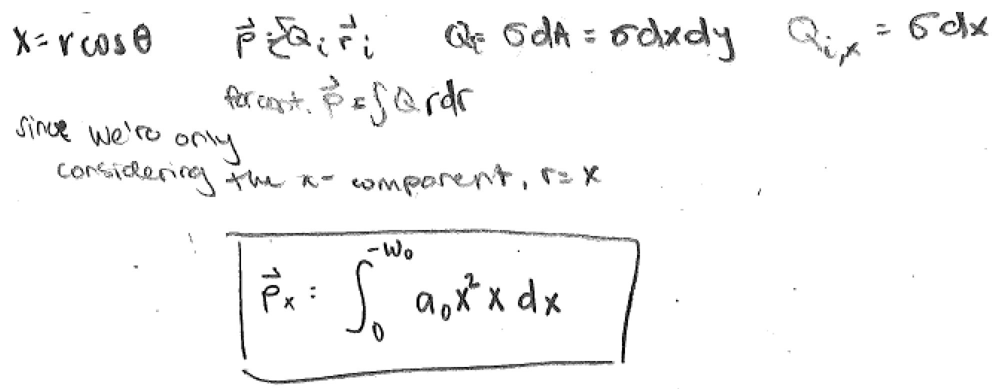

Figure 2-7: Excerpt of student response on rectangular-sheet task.

For providing bounds in their answer, 55% of students identified acceptable[i] bounds of the *x*-integral, an improvement of 20% from Coulomb's Law integral in the Autumn 2014 exam. 30% of students wrote bounds from 0 to w_o, which show that they are thinking about the width of the charge, but not the location of the charge. The remaining 15% of students left out the bounds or had something not based on the charge's width.

The inclusion of the infinitesimal spatial element in their substitution and the presence of a vertical length in their answer are not independent[ii] (Table 2-3). There is a positive correlation between the two categories, as students are more likely to answer either both correctly or both incorrectly (diagonal terms) compared to the expected value from the column and row total proportions independently. This result could be due the common skill of physically interpreting the units of quantities like σ and dx, which

Table 2-3: Breakdown of responses on the rectangular sheet task in Autumn 2016.

N = 89	Included dx or dA in Q_i	No infinitesimal	Row total
Included dy or h_o	36%	15%	51%
No height	11%	38%	49%
Column total	47%	53%	

[i]The correct bounds are from $-w_o$ to 0, but students who stated that the bounds go from w_o to 0 or from 0 to $-w_o$ suggest that they are thinking about the placement of the charge at negative values of *x*. Thus, these three sets of bounds were considered acceptable.

[ii] The chi-squared test for independence on the 2x2 contingency table (Table 2-3) gives a p-value of 4.84e-6. This allows one to reject the null hypothesis that including the infinitesimal spatial element in the charge element in student responses is independent of including height as part of the spatial element with the surface charge density.

suggests that emphasizing the dimensionality of the charge density could lead to a better physical interpretation of the infinitesimal spatial element.

d. Summary of assessment

It seems that the initial *Integrals with Charge* tutorial was fulfilling most of its original goals, but there were other student difficulties with charge-based integrals that were not addressed by the tutorial. The data suggest that students do better with interpreting the meaning of integration, determining the bounds of charge-based integrals, and distinguishing between arc length and angle. However, the exam questions uncovered student difficulties with interpreting infinitesimal elements and physical units.

E. Further investigation of student difficulties with charge-based integrals

The exam data suggest that many students are not correctly interpreting the infinitesimal charge element. Those students seem to reason that the charge density alone represents an infinitesimal charge within an integral, while being unable to correctly interpret the physical meaning of the infinitesimal spatial elements. Instead, they seem to append the infinitesimal spatial elements to the charge density based on coordinate dependence and a recognized need for an integral rather than the dimensionality of the charge density. As a follow up in Autumn 2017[i], I introduced a new pair of questions to the *Integrals with Charge* pretest <App – 4>. I wanted to investigate if student difficulty with expressing the infinitesimal charge element stemmed from either their interpretation of charge density or infinitesimal spatial elements. The following data comes from 3 quarters with a sample size of N = 289 students.

1. Student interpretation of linear charge density

The first question asked students to describe what “linear charge density” meant. The intended response was to describe a ratio between the amount of charge to a measurement of length at an

[i]The instructor for Phys 321 in Spring 2017 chose to omit tutorials.

infinitesimal scale, which is commonly summarized as "charge per unit length." About 45% of students used the words "per" and "length" in their explanation, or explicitly described a ratio of small quantities.

There are minute variations in how the students interpret "linear" and "density," and often students did not explain what both mean. For "linear," there were about 10% of students that interpreted it incorrectly, stating that it refers to a linear dependence of the function on a variable; this was likely because students saw that $\lambda(x) = a_o x$ in the context of the rest of the problem. About 15% of students stated that it meant that charge is distributed along a line, recognizing that "linear" refers to a one-dimensional object. About 20% of students opted to describe charge density as "the amount of charge" within some region, without explicitly mentioning a ratio; not all these students mentioned a length, as many defined the region as an area, volume, or section.

2. Student interpretation of the infinitesimal elements

The second question asked students to describe what *"dx"* physically means in the context of a charge integral. The intended response was to describe *dx* as an infinitesimal length, or the length of a small piece of the line.

45% of students mentioned that *dx* represented a length, distance, or displacement, showing that they understood the physical units of *dx*. 5% of students described *dx* as the charge or charge density of a small segment, while 20% of students referred to *dx* as a small piece or section of the line without labeling it with a physical quantity, which suggest that these 25% of students are thinking physically about the subdividing of the line but did not associate *dx* with a length.

The remaining 30% of the students answered more mathematically, with 15% of students interpreting *dx* as a change in position or the variable *x*, 5% as a change in the charge, 5% referring to it as a differential or rate of change, and 5% stating that it represents the axis of integration.

Table 2-4: Breakdown of student responses comparing accuracy on net charge and dipole questions based on interpretation of dx.

Interpretation of dx	% of students (N = 289)	% correct on net charge	% correct on dipole
Physical	68%	58%	30%
Mathematical	32%	37%	13%

Table 2-4 shows the breakdown of student responses based on whether they seemed to interpret *dx* physically or mathematically, and associated accuracy on the net charge and dipole tasks. The 70% of students who interpret *dx* physically significantly[i] outperform students who interpret *dx* mathematically by 20% on the net charge question and 15% on the dipole question.

F. Modifications to the *Integrals with Charge* tutorial

The assessment of the initial version of the *Integrals with Charge* tutorial showed that it was fulfilling most of its original goals of helping students better understand the Riemann sum interpretation of the integral, become more aware of the importance of the bounds of a charge-based integral, and determine infinitesimal arc lengths in curvilinear coordinates. However, the assessment and additional pretest questions revealed student difficulties with infinitesimal elements that affected many students' ability to properly construct charge-based integrals.

1. Additional goals of the *Integrals with Charge* tutorial

The original goal of teaching students how to physically interpret infinitesimal quantities needed to be refined. Additional goals to help students physical interpret infinitesimal quantities are teaching students to distinguish between infinitesimal charge and infinitesimal length, and teaching students how to relate the two infinitesimals to charge density with dimensional analysis.

2. Shortcomings of the *Integrals with Charge – initial version*

The portion of the tutorial designed to address the interpretation of the infinitesimal elements is the second half of the second page, where students are guided to discuss the infinitesimal element within the net charge integral. The discussion is prefaced by the definition of integration given as "summing many infinitesimal quantities."

[i] The chi-squared test for independence on the 2x2 contingency tables comparing interpretation of *dx* to accuracy gives a p-value of 9.32e-4 for the net charge question and 2.44e-3 for the dipole question. This allows one to reject the null hypothesis that student accuracy on determining charge-based integrals is independent of the interpretation of *dx.*

In the original version, students were asked to define the infinitesimal quantity, with a hint prompting them to describe it in terms of $\lambda(x)$. It was intended for students to answer that the infinitesimal quantity was an infinitesimal charge quantity, $\lambda(x)dx$, since charges are the quantities that need to be summed when determining net charge. However, many students seemed to be confused by this question and were tempted to only answer *dx*. The question also does not prompt for a physical interpretation of the quantity *dx* or a dimensional analysis of the three quantities, *dx*, λ, and *dq*.

The homework question <App – 17> associated with the original version has students consider a rectangular slab with dimensions *w*, *l*, and *h* (Figure 2-8) and asks students to determine an integral expression for the net charge, which students should answer with $\int_0^h \int_0^l \int_0^w \rho dxdydz$. Students were then asked to consider cases where ρ did not vary along specified axes, simplify the integral, and determine a relationship between ρ and the other charge densities σ and λ. Ultimately, students were to define *dq* in three different ways, for which the expected answer is $\rho dxdydz$, $\sigma dxdy$, and λdy. In scanning the homework responses, I found that many students did not seem to understand the question and had factors of *w* and *h* in the wrong places, including incorrect expressions for *dq* like $wh\lambda dy$. This suggests that the homework question was not effective.

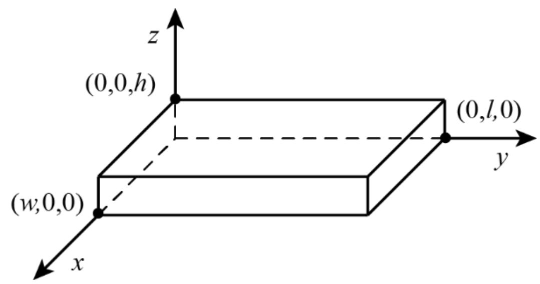

Figure 2-8: Context for the rectangular slab homework question.

3. Modifications to the *Integrals with Charge* tutorial – second version

In Autumn 2017, I modified the original version to focus on the formal definition of the infinitesimal elements. In the first modified version <App – 15>, students were asked to define and compare the quantities *dx* and *dq*, considering both the literal meaning and a physical interpretation, and then describe how these are related to the charge density λ. In tutorial, most students were confused by the wording of the first question. Many students understood the need to divide the line into many small chunks, but had difficulty articulating that *dx* represented the *length* of the small chuck, whereas *dq* represented the *charge* of that small chunk.

In the homework <App – 21>, students are asked about whether it is possible to define linear charge density from a net charge and total length of an object, both in cases where the segment is uniformly charged and non-uniformly charged. Students are led to interpret the phrase "charge per unit length" word for word. The purpose of this question is to have students understand density as a ratio of two infinitesimal quantities. In the next question, students are asked to interpret $d\tau'$ in the context of $\int f(r, r')\,\rho d\tau'$, and then define the suitable replacements of $\rho d\tau'$ if they were given σ or λ instead. The purpose of these questions is to emphasize the physical meaning of the infinitesimal spatial elements and how their use in *dq* depends on the way the charge is distributed.

4. Assessment of *Integrals with Charge* – second version

a. Charged-disk task

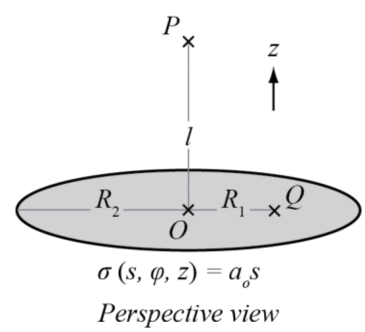

Figure 2-9: Context for the charged-disk task.

The first exam of Autumn 2017 <App – 25> included a question similar to the Coulomb's Law question in Autumn 2014 to compare the effects of the *Coulomb's Law* and *Integrals with Charge* tutorials directly. For this question, students were presented with a circular disk of radius R_2 with surface charge density $\sigma(s, \phi, z) = a_o s$ (Figure 2-9). The first two parts were primer questions: students needed to describe an infinitesimal charge element within a region located a radius R_1 away from the center as σdA, $a_o s * s ds d\phi$, or $a_o R_1 * R_1 ds d\phi$, and determine the distance from that point to another point a distance *l* above the center as $\sqrt{R_1^2 + l^2}$. Finally, students were asked to determine the integral expression for the electric potential, given that the point-charge Coulomb potential is $V = \frac{kq}{d}$. Students were expected to derive the expression $V =$

$\int_0^{2\pi}\int_0^{R_2}\frac{k*a_o s*sdsd\phi}{\sqrt{s^2+l^2}}$ by making appropriate substitutions of q and d based on their answers to the primer questions and including appropriate bounds. The sample size was N = 154 students.

For the infinitesimal charge question, 70% of students were able to correctly pair the surface charge density with an area. However, some of these students did not properly describe an infinitesimal charge. For example, some students showed the area as πr^2, some attempted the integral $\int_0^{2\pi}\int_0^{R_1} a_o s * sdsd\phi$ as dq, and some described an incorrect relationship $Q = \frac{\sigma}{A}$. 60% of the students who described an area weighted the angle properly such that their answer had correct dimensions of length-squared, while most of the other students gave the infinitesimal "area" element as $dsd\phi$. For the 30% of students who did not pair the charge density with an area, most of these responses were blank or a strange attempt at using Gauss' Law, with a few instances of σdl or σdV. For the *Coulomb's Law* exam in Autumn 2014, 55% of students included *dxdydz* in their integral, showing that volume charge density should be matched with an infinitesimal volume within the Coulomb's Law integral. Thus the *Integrals with Charge* tutorial seems to do slightly better than the *Coulomb's Law* tutorial at teaching students how to determine infinitesimal charge elements, although some students still seem to struggle with density and infinitesimal spatial elements.

For describing a distance, 95% of students showed the Pythagorean theorem. However, only 45% of students who attempted a Coulomb's potential integral on last question showed that the distance should vary depending on the position of the charge as $\sqrt{s^2+l^2}$ instead of $\sqrt{R_1^2+l^2}$ or l. This represents a 15% decline in comparison to the *Coulomb's Law* exam data, where 60% of students showed that the displacement varies as different charges in the charge distribution are considered. The decline in the percentage of students making the distance variable is likely due to a difference in how the questions were primed; on this exam, students were asked to find the distance from a fixed point with a parameter R_1, whereas on the *Coulomb's Law* exam students were asked to find the displacement from a hypothetical point with the coordinates (x', y', z'). The difference between parameters, variables, and coordinates is a nuance that many students seem to have difficulty understanding, yet it represents another hurdle in formulating correct charge-based integrals.

For expressing the potential as an integral, 70% of students attempted a Coulomb's potential integral. 10% of students showed $-\int \vec{E} \cdot d\vec{l}$, while the remaining 20% of students left it blank or gave neither type of integral. In looking at the bounds for the students who attempted a Coulomb's potential integral, 50% of those correctly determined bounds that represented the locations of the charge distribution, a 15% improvement from the *Coulomb's Law* exam. For incorrect bounds, 25% the students with Coulomb's potential integrals did not specify their bounds, 10% included infinity[i] in their bounds, and 15% showed something else for their bounds.

b. Slanted-charge task revisited

On the final of Phys321 in Autumn 2017 <App – 26>, I reused the slanted line charge context from Autumn 2016, weaving it into a multipole-expansion context. Like the previous instance, students were asked to identify a flaw in the incorrect statement about the net charge being $\int_0^3 \int_0^4 a_o x^2 y dx dy$, where the charge density was given as $\lambda(x, y) = a_o x^2 y$ and the line segment went from the origin to $(3, 4)$ (Figure 2-10). As a slight change to this question, the statement correctly identified the integral as the monopole moment, but the primary error is still that the infinitesimal charge is incorrectly represented by λdA. I wanted to see if the inclusion of the discussion about *dx* and *dq* would improve student ability in identifying this error. The following part asked students to determine an integral expression for the *x*-component of the dipole moment. The sample size was N = 82 students.

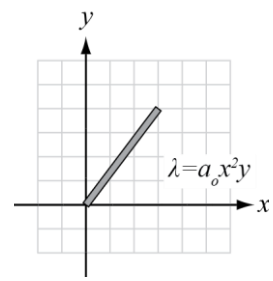

Figure 2-10: Context for the slanted-line task.

[i] The question prompt states "relative to infinity" in an effort to be precise, as a reference point needs to be defined in order to determine absolute potentials. This likely primed some students to inappropriately include infinity in their bounds, which elicits a conflation between the Coulomb's potential integral and the electric-field line integral.

For identifying the error, 45% correctly identified that the integral contained an area element, which does not match the line charge. 25% of students said that the only error was the order of the bounds, and 20% of students mentioned something about multipole expansion, such as that the integral represented the dipole moment, or that the monopole moment did not describe the net charge. The remaining 5% of students left the question blank or responded in some other way.

Table 2-5: Comparison of slanted-line task between first exam in Autumn 2016 and third exam in Autumn 2017.

Phys 321 Exam timing	Sample size	Identification of error			Ratio of dl:bounds
		$dl \neq dx * dy$	Bounds	Misc.	
Exam 1 16(4Aut)	N = 93	68%	32%		2.10
Exam 3 17(4Aut)	N = 151	46%	26%	28%	1.77

Table 2-5 shows the breakdown of student responses on this task, comparing Autumn 2016 exam 1 to Autumn 2017 final exam. If only looking at the proportion of students who correctly identified the error, there seems to be a significant[i] decline in performance; however, the timing of the exam is different and the incorporation of multipole expansion in the question add additional layers of complexity to the Autumn 2017 task. There is no significant[ii] difference in the ratio of students who identify the infinitesimal spatial element to students who identify the bounds as the error between the two exams, which suggests that if the layers of complexity were removed, student performance on the task is comparable between the versions of the *Integrals with Charge* tutorial.

For setting up the integral of the *x*-component of the dipole moment, I looked for whether students gave a single infinitesimal spatial element in their integral; either *dx, dy,* or *dr*. Overall, 70% of students had a single integral. 85% of the students who identified that the spatial element in the statement

[i] The chi-squared test for independence on the 2x2 contingency table comparing student accuracy in determining the infinitesimal spatial element as the error between the two exams gives a p-value of 7.90×10^{-4}. This allows one to reject the null hypothesis that differences between the two quarters (such as tutorial version and exam timing) do not affect student performance on this question.

[ii] The chi-squared test for independence on the 2x2 contingency table comparing students who determine either the infinitesimal spatial element as the error or the bounds between the two exams gives a p-value of 0.566. One is unable to reject the null hypothesis that differences between the two quarters (such as tutorial version and exam timing) do not affect the ratio of students who identified the infinitesimal spatial element to students who identify the bounds as the error.

had an error corrected for it in their expression, which represents a 5% increase from the Autumn 2016 exam data. 60% of the students who did not recognize *dxdy* as an area element gave their integral with one infinitesimal, which did not happen before. Most of these students implied in their reasoning that the *x*-component is linked to the *dx* in the integral[i], which is why they dropped the *dy* factor from their integral.

Overall, the difference in student performance between the Autumn 2016 first exam and the Autumn 2017 final exam seems to be due to the timing and content coverage of the exam rather than the version of the *Integrals with Charge* tutorial. The inclusion of multipole expansion leads many students to misidentify errors other than the dimensionality of the infinitesimal spatial element, while dipole moment introduces vector aspects that conflate with the interpretation of the infinitesimal spatial element.

c. Smiley-face task

The first few parts of the first exam of Phys321 in Spring 2018 <App – 27> test the same ideas in the *Coulomb's Law* exam, although in a different way than the first exam in Autumn 2017. For this context, students were given a charge distribution that resembled a smiley face: two straight rods with variable linear charge density λ_{eye} located at $x = \pm 1$ spanning from $y = 1$ to 3, and a curved rod with variable linear charge density λ_{mouth} located below the *x*-axis as a semi-circle of radius 2 (Figure 2-11).

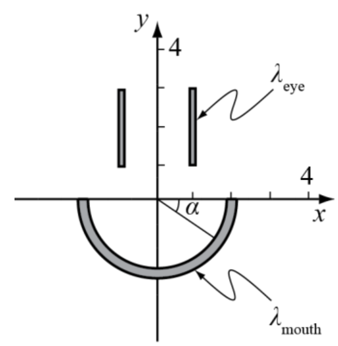

Figure 2-11: Context for the smiley-face task.

As a three-part question, students were asked to determine the net charge of one eye, the net charge of the mouth, and the potential at the origin relative to infinity. The correct answers respectively are $\int_1^3 \lambda_{eye} dy$, $\int_0^\pi \lambda_{mouth} 2d\alpha$, and $2k\int_1^3 \frac{\lambda_{eye}dy}{\sqrt{1^2+y^2}} + k\int_0^\pi \frac{\lambda_{mouth}2d\alpha}{2}$ or equivalent. The sample size was N = 95 students.

[i] This is related to the conflation of the vector nature and field nature of vector fields, which is discussed in Chapter 3 with symmetry.

For the net charge of the eye, I looked for whether students show the net charge as an integral, and whether they correctly pair the linear charge density with a length. 60% of students properly showed the net charge as an integral, whereas the other 40% of students showed multiplication without an integral. Although the question stated that λ_{eye} was variable, many students may have misinterpreted the density as uniform since it was not shown explicitly as the function $\lambda_{eye}(y)$. Regardless, this result is comparable to the *Coulomb's Law* exam where 60% of students either evaluated the net charge correctly as zero or showed their work with an integral. For dimensionality, 85% of students showed the net charge of the eye as λ_{eye} multiplied by a length, while a few others implied λ_{eye} times a volume or an area. This result is also comparable to the 80% of students on the *Coulomb's Law* exam who successfully paired the volume charge density with dV or V.

For the net charge of the mouth, I looked for whether students included the radius in their calculation, showing the distinction between arc length and angle. 75% of students included a factor for the radius, while most of the remaining 25% of students simply showed $Q_{mouth} = \int_0^{\pi} \lambda_{mouth} d\alpha$. Students who showed the correct dimensionality for the eye outperform students who showed incorrect dimensionality on including the radius for the mouth by 15%, but the difference is insignificant[i].

Table 2-6: Breakdown of student responses for determining potential on the smiley-face task.

N=95	Integral	No integral	Row total
Summation	39%	24%	63%
No Sum	7%	29%	36%
Column total	46%	54%	

Table 2-6 shows a breakdown of student responses for determining the potential on the smiley-face task. 65% of students showed a summation of multiple terms, showing that they are applying superposition to determine the potential from all three rods. However, only 45% of students attempted to describe the potential due an eye[ii] with an integral. The use

[i] The Fisher exact test on the 2x2 contingency table comparing correct dimensionality for the eye and including radius for the mouth gives a p-value of 0.511. One is unable to reject the null hypothesis that including the correct spatial element for the eye in student responses is independent of including the radius for the arc length.

[ii] I only looked at their formulation for the potential due to an eye because if students interpreted the densities as uniform, the potential for the mouth simplifies to multiplication, whereas the potential for each eye still requires an integral.

of superposition and use of integrals is not statistically independent[i], with 60% of responses that show superposition contained integrals, while only 20% of responses without addition contained integrals. Out of the responses that had an integral for the potential from the eyes, 55% showed a coordinate-dependent distance in the denominator, which is comparable to the 60% on the *Coulomb's Law* exam and a slight increase from the 45% in Autumn 2017, both of which included a primer question. Many of the incorrect integrals showed a fixed distance like $\sqrt{2}$ (closest distance) or $\sqrt{5}$ (distance to center).

d. Summary of assessment

The exam data show that the performance of students who saw the *Integrals with Charge* tutorial was not different than that of students who saw the *Coulomb's Law* tutorial on a couple of tasks related to goals of the *Coulomb's Law* tutorial. This suggests that the replacement of the *Coulomb's Law* tutorial with the *Integrals with Charge* tutorial is not detrimental to the goal of mathematizing the functional part of the charge-based integral while allowing for the focus of the tutorial to shift toward the infinitesimal charge element and Riemann sum interpretation of the integral.

The exam data also show that the performance of students who saw the second version of the *Integrals with Charge* tutorial was not different than that of students who saw the first version on the slanted-line task regarding the interpretation of infinitesimal spatial elements and density. One way to interpret the result is that the "surprisingly low" performance of the task in Autumn 2016 at 70% represented my oversight of student difficulties with infinitesimals. Another way to interpret the result is that modifications to the first version of the tutorial and homework may have been unnecessary, although there were no matched results on the rectangular-sheet task.

[i] The Fisher exact test on the 2x2 contingency table comparing the expression of superposition as a sum and the expression of the potential of an eye as an integral gives a p-value of 1.03e-4. This allows one to reject the null hypothesis that expressing superposition as a sum in student responses is independent to expressing superposition as an integral.

5. Modifications to *Integrals with Charge* – third version

In Spring 2018, two professors Dr. Corrinne Manogue (physics) and Dr. Tevian Dray (mathematics) visited UW from Oregon State University and observed a section doing the *Integrals with Charge* tutorial. Afterward, I discussed some of the difficulties I saw on the slanted line-charge exam question and charged-based integrals in general. As part of their feedback, they critiqued the use of *"dq"* in the tutorial, based on the idea that *dq* was a strictly physical idea and could conflate with their ideas of *dx* when students attempted to parametrize the infinitesimal charge element mathematically. They also noted that a group of students struggled to define the terms *dq* and *dx*, even with a TA present. They suggested that dimensional analysis may be sufficient for students to correctly describe the infinitesimal charge as $\rho dV'$, $\sigma dA'$, or $\lambda dl'$ in a charge-based integral.

As an updated version to the *Integrals with Charge* tutorial in Autumn 2018 <App – 16>, I removed the explicit mention of *dq* from the tutorial and only asked students to interpret *dx*, both mathematically and physically. The follow-up question asked students how to combine *dx* with λ to describe a charge, and to support their reasoning with dimensional analysis. The homework remained unchanged, such that it still asked students to define the meaning of charge density and asked them to determine replacements to $\rho dV'$ using σ and λ. I hypothesized that students with a stronger conceptual grasp of infinitesimal charge may come up with *dq* as a descriptor on their own, while students with a weaker conceptual grasp of infinitesimals may simply learn and apply the dimensional analysis heuristic.

6. Assessment of *Integrals with Charge* – third version

a. Curved-arc task

For the first exam of Phys321 in Spring 2019 <App – 28>, I wanted to test if students could identify flaws similar to the slanted-line task in Fall 2016 and Fall 2017. This modification of the slanted-line task is designed to test for three major concepts in the *Integrals with Charge* tutorial: variable density needs integration rather than multiplication, linear charge density needs to be matched with one dimension of length, and the appropriateness and expression of coordinate system. Students were presented with a linear charge density $\lambda(\xi)$ in the shape of a half-circle of radius 2 (Figure 2-12) with the incorrect statement states that "the net charge of this object is $\lambda(\xi) * 4 * 2$, because it is 4 wide by 2 high." In the prompt, students were notified that they were graded for three distinct ideas. The sample size for this exam was N = 108 students.

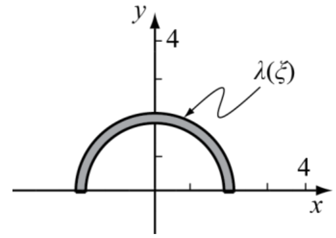

Figure 2-12: Context for the curved-arc task.

Figure 2-13 shows a breakdown of student responses on the curved-arc task. 80% of students were able to identify that multiplication was not a valid method and that they needed integration, which is an improvement from the *Coulomb's Law* exam where 40% of students attempted to use multiplication to find the net charge of the cube as $\rho(x, y, z) * V$. 80% of students were able to identify that using an area element with a linear charge density was incorrect, which is a

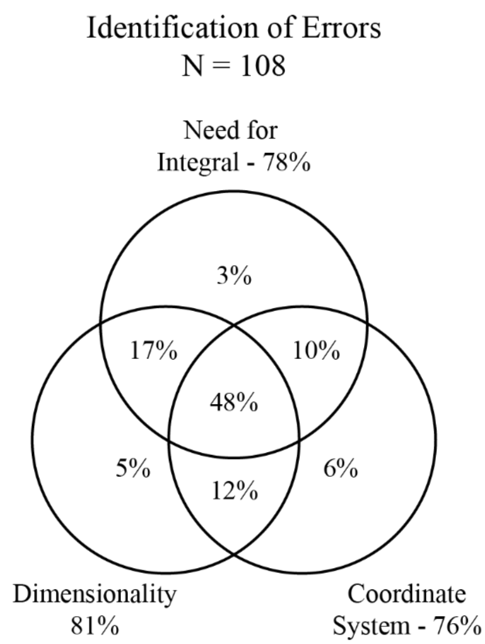

Figure 2-13: Breakdown of student responses on the curved-arc task.

significant[i] improvement from the Autumn 2016 exam where 70% of students identified that $\lambda(x,y)dxdy$ could not describe the charge of a slanted line. Lastly, 75% of students identified that Cartesian coordinates were inappropriate. Overall, 50% of students were able to identify all three ideas, 40% identified two out of the three, and the rest identified one. Many students combined the area and coordinate-system ideas into one comment by saying that the mock-student was describing a rectangle while they should be describing an arc instead, but they were credited for both.

b. Rectangular-sheet task revisited

On the final exam of Phys321 in Autumn 2019 <App – 29>, I reused the question from the Autumn 2016 final, although the presentation of the dipole integral question was modified. On the original version, students were asked to show their work as they derive an integral expression for the *x*-component of the dipole moment. On the modified version, students were first asked to determine the integral expression, and then identify the factor(s) in their expression that corresponded to Q_i within the point-charge expression of the dipole moment, $\sum Q_i \vec{r_i}$. This sample had a size of N = 121 students.

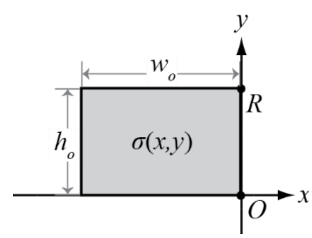

Figure 2-14: Context for the rectangular-sheet task.

Table 2-7 shows the breakdown of student responses on the rectangular-sheet task in Autumn 2019. For the identification of the factors for infinitesimal charge, 25% of students correctly identified σdA or equivalent. An additional 10% of students showed σdx, most of whom did not include a height in their integral expression. 15% of students did not describe an infinitesimal charge

Table 2-7: Breakdown of student responses on the rectangular-sheet task on third exam of Autumn 2019.

N = 121	Included dx or dA in Q_i	No infinitesimal	Row total
Included dy or h_o	27%	31%	59%
No height	11%	31%	41%
Column total	38%	62%	

[i] The chi-squared test for independence on the 2x2 contingency table comparing student ability to identify the error of dimensionality between the two exams gives a p-value of 0.0247. This allows one to reject the null hypothesis that differences between the two exams (such as tutorial version and content) do not affect student ability to identify errors in dimensionality.

in that they either implied an integral $\iint \sigma dA$ or a bulk multiplication like σA. 25% of students did not include the infinitesimal dimensions in their expression, showing either σ or σh_o. Lastly, 25% of students gave obscure answers, suggesting that they did not (or could not) interpret the question correctly.

For the *y*-dimension within integral expression, 60% of students included the height as dy or h_o, which represents a 10% increase from the Autumn 2016 exam data. For the bounds of the x-integral, 50% of students show acceptable bounds indicating both width and location, which represents a slight decrease of 5%.

Table 2-8 shows how the breakdown on the infinitesimal element and the height change from the Autumn 2016 data to the Autumn 2019 data. On each individual aspect of the task, there is no significant[i] difference in performance between the two sets of exam data. However, there is a significant[ii] difference in the distribution of responses collectively. This is likely due to the shift of focus in the tutorial away from the physical interpretation of infinitesimal charge elements to the dimensional analysis of charge density. This shows that understanding the dimensionality of the charge density and the physical meaning of infinitesimal elements in the integral are distinct skills, which suggests that teaching students dimensional analysis alone is not enough to address the difficulties in interpreting the infinitesimal charge element.

Table 2-8: Comparison of breakdown of student responses on rectangular-sheet tasks.

From 16(4Aut) to 19(4Aut)	Included dx or dA in Q_i	No infinitesimal	Row total
Included dy or h_o	−9%	+16%	+7%
No height	0%	−7%	−7%
Column total	−9%	+9%	

[i] The chi-squared test for independence on the 2x2 contingency table comparing each aspect of the task between the two exams gives a p-value of 0.242 for the height aspect and a p-value of 0.183 for the infinitesimal aspect. One is unable to reject the null hypotheses that differences between the two exams (such as tutorial version and presentation) do not affect student ability on either the infinitesimal or dimensionality aspect of the task individually.

[ii] The chi-squared test for independence on the 2x4 contingency table comparing both aspects of the task between the two exams gives a p-value of 0.0440. This allows one to reject the null hypothesis that differences between the two exams (such as tutorial version and presentation) do not affect the distribution of student responses on the infinitesimal and dimensionality aspect collectively.

c. Summary of assessment

The exam data show that there is improvement of the third version of the *Integrals with Charge* tutorial over the adapted version and the *Coulomb's Law* tutorial in addressing the dimensionality of spatial elements and charge density. However, the exam data also show that student ability to interpret dimensionality and infinitesimal quantities are distinct skills, so the improvement student ability to interpret dimensionality is offset by the decline in student ability to physically interpret infinitesimal quantities. This suggests that the second version of the tutorial, which focuses on the distinction between *dx* and *dq*, best matches the goals of the tutorial, although there is currently no direct evidence that it is superior to the first or third version.

G. Summary and reflections on superposition and integrals

Overall, we found that many of the difficulties with students have with Coulomb's Law and other charge-based integrals stem from the inability to interpret and express the infinitesimal charge element, which is consistent with previous research on students' difficulties with integration in physics in general. The creation and development of the *Integrals with Charge* tutorial has been an attempt to teach students to better understand the idea that integrals describe an accumulation of an infinitesimal quantity as well as properly interpret and express infinitesimal charge elements, which the precursor *Coulomb's Law* tutorial does minimally.

The tutorial still has room for improvement. Recent exam data show that there is still a deficiency in student understanding of the spatial infinitesimal, especially with vector quantities. This could be better addressed in the supplement where the tutorial asks for integral expressions for the components of a dipole moment, but the supplement lacks a deep discussion about the role of the spatial infinitesimal in each of those integrals and is not a required piece of the tutorial. Also, the exam question about the potential due to a complex charge distribution shows student deficiency in expressing weighted integrals. Determining potential from a distributed charge is addressed in the homework, but the question could be refined to better teach the concept in the absence of group discussion and TA support.

Although the *Integrals with Charge* tutorial was developed to be used in the upper-division electromagnetism sequence at UW, most of the core ideas[i] are relevant to a typical calculus-based introductory electromagnetism course. Much of the supporting research from other universities[2,4-10] has been done at the introductory level. California Polytechnic State University[16] also independently developed a worksheet targeting these ideas, which coincidentally follows a similar structure that focuses on the Riemann sum interpretation of integration, although with an analogy to a mechanics context with net mass and center of mass instead of net charge and dipole moment. I hypothesize that parts of the *Integrals with Charge* tutorial could be used at the introductory level as well with similar efficacy.

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[i] Except for curvilinear coordinates and Griffiths' notation.

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[16] A. Profeta, M. Moelter, S. Vokos, Identifying student … resources in setting up integral expressions for the electric field…, [Poster], presented at the Physics Education Research Conference 2019, Provo, UT, 2019.

Chapter 3 Symmetry and Symmetry Arguments

Symmetry as used in physics allows one to draw inferences about the behavior of physical quantities without having to explicitly calculate them. In upper-division electromagnetism, understanding symmetry and symmetry arguments can lead to a stronger intuition about field behavior and mastery of flux and line integrals. Although there are many categories of symmetry, we focus on even spatial symmetry in the course sequence, as that category is most relevant to applying Green's theorem integrals, notably Gauss' and Ampère's Law in integral form.

Green's theorem integrals as applied to the source-field relationship all follow a similar form: the field vector $\vec{\psi}$ at the boundary of an enclosed region of space is equated to the source S or $\vec{S}$ enclosed within that region: $\oiint \vec{\psi} \cdot d\vec{a} = \iiint S dV$ and $\oint \vec{\psi} \cdot d\vec{l} = \iint \vec{S} \cdot d\vec{a}$. They are always true, in that if one knew the field and source explicitly, one could calculate both integrals and see that they equate; however, they are not always useful, in that only in specific cases can the magnitude of the field be determined when the field is not known explicitly. To solve for $|\vec{\psi}|$, one must follow several steps: (1) construct an appropriate region, (2) separate the closed boundary integral into pieces if needed, (3) easily resolve the dot product for each piece, (4) simplify any non-zero integrals to a simple product, and (5) algebraically solve the rest of the equation. In doing these steps, two important criteria must be met: (A) the field direction must be known, and (B) the field must be independent of integration coordinates. Both criteria must be fulfilled without calculation, which is the purpose of using symmetry.

There are several types of symmetry arguments that allow one to infer information about a system without calculation:

- **Transformative arguments** involve taking a system *A* and transforming it to *A'* via translation, rotation, reflection, and/or inversion such that all physical aspects of *A* and *A'* are the same. One location in *A'* is then related to two separate locations in *A* – one by the symmetrical transformation and the other by explicit coordinates. In short, identifying a symmetry in the physical system allows one to link multiple points by that symmetry, providing a description for the (in)dependence of a field on a coordinate. Reflection and inversion are considered improper transforms in that they invert chirality, reversing any quantity resulting from cross products (*e.g.*, the magnetic field). Thus, we choose to only focus on translations and rotations, as they can provide the necessary information without introducing extra concepts of improper transforms and pseudovectors.

- **Superposition arguments** involve splitting a source in half such that one half contributes positively to a scalar or component of a vector, while the other half contributes negatively. In this way, one can argue that the net value is zero by only looking at the signed contributions instead of calculating explicitly. Because superposition arguments are based on identifying halves, reflections and 180° rotations are ideal for this type of argument.
- **Proof-by-contradiction arguments** involve assuming a non-zero value or component at a point, and then using a symmetry transformation to map a point onto itself. If symmetry dictates that the value or component must be the negative of itself, it must be zero. Reflections and 180° rotations are also ideal for this type of argument, and proof-by-contradiction and superposition arguments can be used interchangeably to get the same information.
- **Zero-contribution arguments** involve splitting a source infinitesimally and claiming that each source contributes nothing to a given value or component. This type of argument does not explicitly reference any symmetry, but it does complete the arguments necessary to verify if the two criteria (A) and (B) above are satisfied.

Section A describes research on the student understanding of Gauss' Law and Ampère's Law at both the introductory and upper-division levels. Section B describes preliminary research done in collaboration with Ryan L.C. Hazelton, which is elaborated on in his dissertation[1]. Section C and D presents research on the student understanding of symmetry and symmetry arguments. Section E describes the current curriculum addressing symmetry and symmetry arguments. Lastly, Section **Error! Reference source not found.** suggests avenues for future research related to Gauss' Law and Ampère's Law.

A. Relevant research on symmetry in electromagnetism

The topics of Gauss' Law and Ampère's Law are typically included in both introductory calculus-based and upper-division electromagnetism courses. As such, there has been research interest at both levels, and the focus and depth at each level vary from university to university. I present four areas of study at the introductory level and implications for these topics at the upper-division level.

1. Relevant research at the introductory level

At the University of Pittsburgh, Singh[2] investigated introductory-level student difficulties with discerning symmetry and applying Gauss' Law through free-response and multiple-choice questions, as well as student interviews. In the conference proceedings, she reports on a 6 question from a larger 25-item test. She notes that many students inaccurately extend results from continuous symmetry to cases with discrete symmetry, interpret inferences from conductors as general statements about symmetry or charge behavior, conflate the ideas of electric flux through a surface with electric field at a point, the idea that zero net flux can come from multiple non-zero contributions, and general application of Gauss' Law relating the charge enclosed to the net flux through a closed surface, not open surfaces. This study led to the development[3] of a 5-tutorial sequence on the fundamental electrostatics and the basic source-field reasoning needed to exploit Gauss' Law. They dedicate two tutorials on Coulomb's Law, superposition, and symmetry. The third tutorial is on helping students distinguish between electric field and flux. The fourth and fifth tutorial are on exploiting symmetry to calculate the electric field via Gauss' Law.

At the University of Maine, Traxler et. al.[4] conducted an interview study on similar ideas. They asked students to describe the electric field of a symmetric (spherical) and semi-symmetric (ellipsoid) system of charge and determine whether Gauss' Law is an appropriate strategy for determining the magnitude of the electric field. In their results, they confirm that many student difficulties from Singh's study at Pittsburgh, as well as conflation of electrostatic and magnetostatic ideas and incorrect field line diagrams. In analyzing why most students performed poorly on the semi-symmetric case, they note that many students were applying a primitive form of rote recall rather than a step-by-step problem-solving approach, suggesting a tendency for students to skip the reasoning process to arrive at their intuitive answer with little justification.

An interview study by Guisasola et. al.[5] in Europe also elicits many of these same student difficulties. They note two major student difficulties: the tendency to treat the field calculated from the Laws as only due to the sources within the pathway rather than from all sources in the system, and the tendency to claim that if a net flux or line integral is zero, the corresponding field is also zero. They claim that the latter is a confusion between the flux and circulation operators and the field, although one could attribute this tendency as an inappropriate simplification of the dot product and integral.

At UW, the introductory level tutorials[6] dedicate one and a half tutorials to electric flux and Gauss' Law, and one tutorial to Ampère's Law. These tutorials focus on many fundamental concepts such as area and path vectors, determining the sign of the dot product, and the definition of closed surfaces. However, there is not a strong focus on symmetry and its inferences, and only a brief derivation for the application of Ampère's Law. Unfortunately, due to the timing of context in the course, we are not able to address all the difficulties that students have with Gauss' and Ampère's Laws.

2. Relevant research at the upper-division level

Regarding the upper-division level, Manogue et. al.[7] posit many key points that make Ampère's Law[i] conceptually difficult for upper-division students to understand that are often left unaddressed at the

[i] and Gauss' Law by extension

introductory level. They note on the field side of the equation that: vector fields could vary in both direction and dependence, symmetry arguments are not entirely straightforward, dealing with imaginary surfaces have many nuances and may not be obvious, the exploitation of the Laws are essentially an inverse problem of the integral, and curvilinear coordinates are not too familiar. They also note on the source side of the equation that: steady current seems to be a contradiction; density[i] may not be well generalized; line, surface, and volume densities are all different; and for Ampère's Law realizing that current encircled is a flux. They posit that when it comes to using Ampère's Law, "most students just yank."

Further investigation at the University of Colorado, Boulder confirms many of the student difficulties related to the claims of Manogue et. al. Quantitative evidence[8] from the Colorado Upper Division Electrostatics (CUE) Assessment[9] after the electrostatic course shows that still about 35% student have difficulty associating a highly symmetric problem with Gauss' Law, and about 25% of student choose a Gauss' Law approach for an incompatible, semi-symmetric problem. Interview studies[8,10] suggest that difficulties stem from incorrect application and inferences about the field from the flux or line integral, difficulty with formulating symmetry arguments based only on the geometry of the system. They found that many misconceptions identified at the introductory level such as "no net flux means no electric field" and "field calculated only depends on source enclosed" persist into the upper-division level. Ultimately, much of this research contributed to the course transformation[11] at Boulder, from which the UW upper-division E&M tutorials originated.

B. Early investigation of upper-division student difficulties with Ampère's Law

Much of my early research on the student understanding of symmetry, Gauss' Law, and Ampère's Law was in collaboration with Ryan L.C. Hazelton, who was investigating the student understanding of select topics in electromagnetism at both the introductory and upper-division levels.

[i] In particular, spatially varying density

Hazelton's dissertation[1] reports on some key early findings that guided my subsequent research on this topic.

In describing upper-division student difficulties with Ampère's Law, Hazelton identifies several key areas where many students struggle. He notes that most students do not understand that additional information in necessary to solve Ampère's Law for the magnetic field. Consequently, many students are not able to justify why certain loops are appropriate in simpler cases, and many choose inappropriate loops for more complicated geometries. Additionally, Hazelton notes that understanding symmetry and using symmetry arguments seems to be challenging for most upper-division students.

We have attempted to address these difficulties in the initial rewrites of the *Gauss' Law* <App – 33> and *Ampère's Law* <App – 37> tutorials in *Tutorials in Physics: Electrodynamics*[12] as discussed in Hazelton's dissertation. In these tutorials, we model how to conceptually determine the field's direction and independence on coordinates through symmetry arguments and emphasize why the preliminary information is necessary to simplify the vector integral and solve for the magnitude of the field. Hazelton claims that the curriculum "seems somewhat promising for improving students' understanding", but "the problems seem not to be trivial, and more work in necessary to fully address them."

My research serves as a continuation of this collaboration by further investigating student difficulties with understanding symmetry and symmetry arguments.

C. Investigation of student ability to identify symmetry

One of the most basic skills students need to show for the application of Gauss' Law and Ampère's Law to determine the vector field is the ability to identify symmetry.

Here, I present only one task designed to elicit student ability to identify symmetry and its effects, although there are many more. Most of the exam questions about Gauss' Law and Ampère's Law that were designed during my research included a part that asked students to identify symmetry, and anecdotally, the results seem to be similar. The following question was chosen because it presented an unfamiliar geometry that still possessed both continuous and discrete symmetries, asked about both relevant effects from the symmetry, and the layout and wording of the question allowed for quick analysis.

1. Hourglass task

On the second exam of Phys321 in Autumn 2017 <App – 51>, students were presented with an hourglass-shaped object with uniform charge density (Figure 3-1). For the identifying symmetry task, students were asked to identify the type(s) of symmetry possessed by the object. Two additional tasks asked students about a conclusion that could be drawn from the symmetry. The first was the dependence task, which asked students to sketch or describe other locations where the magnitude of the electric field was the same as point *P*; the second was the component task, which asked students in what direction the electric field could point at point *P*.

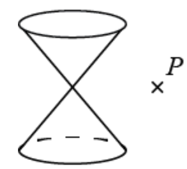

Figure 3-1: Context for the hourglass task.

For the identifying symmetry task, the intended correct response is that the object has continuous rotational symmetry about the vertical axis, and discrete rotational symmetry around any horizontal axis, which could also be described as a reflection across the horizontal plane. There is also inversional symmetry about the origin, although reflections and inversions were not covered in the tutorial.

For the dependence task, the correct response is that any point in the horizontal plane the same distance from the center of the hourglass as point *P* must have the same magnitude for the electric field. This is related to the continuous rotational symmetry about the vertical axis, which links these points without changing the apparent charge distribution. For the component task, the correct response is that the electric field could point rightward or leftward. This is related to the discrete rotational symmetry about the horizontal axis, for which one could use a proof-by-contradiction or a superposition argument to show that the vertical[i] component is zero.

2. Analysis of data for student identification of symmetry

Table 3-1: Breakdown of student responses on hourglass task. The center 2x2 cells show the breakdown of student responses on the identifying symmetry task. The other four percentages represent the accuracy on a symmetry result for the subset of students who did (not) identify the given axis of symmetry.

N = 141	Vertical axis of symmetry			Accuracy on component task
Horizontal axis of symmetry		Identified	Not identified	
	Identified	79% of N	4% of N	92%
	Not identified	13% of N	3% of N	87%
Accuracy on dependence task		82%	60%	

Most students were elaborate enough in their description to show that they could identify symmetry, although many were not very technical in their description. The identifying symmetry task was analyzed based on whether students described a vertical and/or horizontal axis for rotation or reflection. The dependence task was analyzed based on identification of the circle, with the most common incorrect response as a single point reflected across the origin. The component task was analyzed based on the lack of vertical and in/out components in their answer, so responses which showed rightward only instead of both rightward and leftward were counted as correct. Table 3-1 shows a summary of the data analysis.

Overall, on the identifying symmetry task, 95% of students identified the vertical axis of symmetry, while 85% of students identified the horizontal axis. For the symmetry-conclusion tasks, 80% of students answered correctly on the dependence task while 90% of students answered correctly on the component task. There are no significant[ii] differences in accuracy on the symmetry-conclusion tasks between students who did and did not identify the relevant axis of symmetry.

[i] The proof-by-contradiction argument also shown that the in/out component must be zero. To argue this with superposition, one would make use of the reflectional symmetry across the plane of the paper. Most students did not consider the in/out component explicitly, and it was neglected in the data analysis as well because the justification is similar to the vertical component being zero.

[ii] The Fisher exact test for the two 2x2 contingency tables comparing the accuracy of a symmetry-conclusion task students who identified the relevant symmetry gives p-values of 0.111 on the dependence task and 0.414 on the component task. One is unable to reject the null hypothesis that the accuracy on the symmetry-result tasks is independent of explicitly identifying the relevant axis of symmetry on the identifying symmetry task.

The high accuracy on these tasks suggest that most students are proficient at identifying symmetry. However, the independence of the symmetry-conclusion tasks and the relevant identifying symmetry task seems strange, as one would expect that identifying the symmetry is a prerequisite to determining the relevant conclusion. This may be because the relatively high accuracy on these tasks makes it difficult to find statistical patterns with this sample size. It may also be because students are not properly using symmetry arguments to justify how they arrived at conclusions about the electric field. Additional analysis is done on the reasoning students employ in the following section on tasks more relevant to Gauss' and Ampère's Law.

D. Investigation of student understanding of symmetry arguments

The analysis on the hourglass task suggests that students are proficient in identifying symmetry after instruction and can reach the necessary conclusions about the vector field for applying Gauss' and Ampère's Law. However, prior research suggests that students struggle with forming and understanding symmetry arguments. Additionally, I observed evidence previously discussed in Chapter 2 that some students have additional difficulty interpreting infinitesimal spatial elements in the context vector integrals. This section investigates student reasoning in tasks regarding vector components and coordinate dependence of vector fields to see to what extent students understand how symmetry arguments describe vector and field properties.

1. Description of tasks for student understanding of symmetry arguments

A typical task that elicits conflation between the vector nature and field nature of symmetry arguments asks students to determine which component(s) of a field must be zero, and/or what coordinate(s) the field cannot depend on. Each task-of-interest present was also coupled with an associated task that asked students about a Gaussian surface or Ampèrian loop.

a. Mathematical interpretation task

On a pretest for the *Ampère's Law* tutorial in Phys322 <App – 30>, students were given the mathematical expression of the magnetic field: $\vec{B} = c(x - y)\hat{z}$. They were asked to determine on which coordinate(s) the magnetic field depended, and which component(s) the magnetic field has. On a follow-up question, students were asked along which axis (or axes) the magnetic field has continuous translational symmetry. The associated task asked students whether a square Ampèrian loop around a long straight wire was valid to determine the magnitude of the magnetic field at a given point.

The correct response for this task is that the magnetic field depends on the coordinates *x* and *y*, and has a non-zero *z*-component. Since continuous translational symmetry is related to independence on coordinates, this magnetic field has symmetry along the *z*-axis. The correct response on the associated task is no; the square Ampèrian loop is not valid for determining the magnitude of the magnetic field since there are points along the loop where the magnetic field is not parallel or perpendicular and the magnitude of the magnetic field is not constant along the loop.

b. Spherical charge task

On an exam for *Gauss' Law* in Phys321 <App – 52>, students were given a uniformly charged sphere. The first task is to determine which spherical coordinates the electric field does not depend on and explain. The associated task is to determine the validity of Gaussian surface (Figure 3-2) for determining the magnitude of the electric field at a point. This exam question was given in Autumn 2016 Exam 1 and Spring 2019 Exam 1.

Figure 3-2: Context for the spherical charge task.

The correct response for the first task is that the electric field does not have dependence on the ϕ or θ coordinates. This is because the uniformly charged sphere has continuous rotational symmetry along any central axis. The correct response for the associated task is that the Gaussian surface is valid, since the flat faces are parallel to the electric field an contribute no flux, while the curved surface is perpendicular and has uniform electric field along the entire surface.

c. Non-uniformly charged slab task

On the first exam of Phys321 Autumn 2018 <App – 53>, students were given a non-uniformly charged slab with spatial dependence in *z* (Figure 3-3). For the task-of-interest, students were asked how the electric field due to the slab depends on *x*. For the associated task, students were asked to describe a Gaussian surface for which the flux integral would be easy to evaluate.

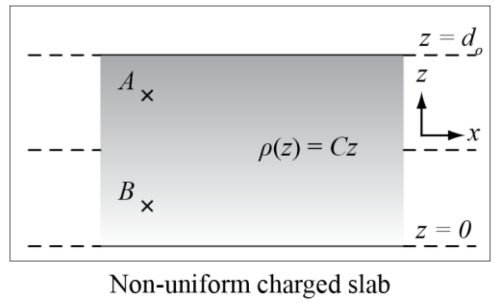

Figure 3-3: Context for the non-uniformly charged slab task.

The correct response for the task-of-interest is that the electric field does not depend on the *x*-coordinate, because the system has continuous translational symmetry along the *x*-axis. For the Gaussian surface, any right prism or “pillbox” with the longitudinal axis along *z* would work.

d. Current-carrying slab task

On the first exam of Phys322 Winter 2016 <App – 54>, students were given a current-carrying slab with uniform current density, as shown in Figure 3-4. For the first task-of-interest, students were asked which component(s) of the magnetic field must be zero at point *P*. For the second task-of-interest, students were asked how the magnetic field depends on *y*. For the associated task, students were asked to sketch an Ampèrian loop for which it was possible to determine the magnitude of the magnetic field at point *P*.

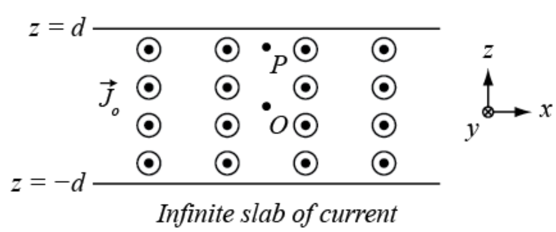

Figure 3-4: Context for the current-carrying slab task.

The correct response to the first task-of-interest is that the magnetic field does not have a *y*-component because that is the direction of the current, and it does not have a *z*-component because contributions from the left side of the slab cancel that from the right side. The correct response to the second task-of-interest is that the magnetic field does not depend on the *y*-coordinate because of continuous translational symmetry along the *y*-axis. The correct response for the associated task is a rectangular loop in the *xz*-plane with the top crossing point *P* and the bottom at either the center or symmetrically below the center of the slab.

2. Evidence of conflation between vector and field properties in student responses

There are different types of conflation that were evident in student written responses. These were categorized as descriptive, logical, notational, and overelaborate conflation. They have been divided into severe and mild degrees of conflation. For the mathematical interpretation task, conflation was determined based on multiple-selection responses.

Descriptive conflation:

Responses that exhibit this type of conflation verbally describe dependence as a vector property, or components as a field property through elaboration of their answer. This seems to be a severe form of conflation, as the student understanding of concepts and descriptions of field and vector properties seem to be reversed or melded. For example, on a task about the dependence of the electric field:

- *"Since the sphere has rotational symmetry, it's electric field can only depend on r as it will be perpendicular to the surface regardless of position."*
- *"Can't depend on ϕ and θ since there is continuous rotational symmetry about all 3 axis and the sphere is uniformly charged (vectors drawn on picture[i] to show E dependence) E is always normal to surface."*

Logical conflation:

[i] Student drew two vectors at the location, showing a "correct" vector with only an *r* component and an "incorrect" vector with *r* and θ components.

Responses that exhibit this type of conflation apply the incorrect symmetry argument for the task. This could be evidence that students do not understand symmetry arguments enough to know what property of vector fields they give information about, but this could also be an alternate form of descriptive conflation if students were trying to give a symmetry argument for the property that the task did not ask about. This also seems to be a severe form of conflation, as it affects student interpretation of symmetry arguments. For example, on a task about the dependence of the electric field:

- *"Electric field **doesn't depend on** the ϕ and θ components because sphere has continuous (rotational) symmetry in the θ and ϕ direction, **the components in those 2 directions cancel out.**"*

Notational conflation:

Responses that exhibit this type of conflation are correct in associating the correct reason for the task but provide technically imprecise language or notation. This seems to be a mild form of conflation as the general description of the property is intact, but students may be unaware of the technical precision separating descriptions of vector and field properties. For example, on a task about the dependence of the electric field:

- *"It will **not depend on $\boldsymbol{\phi}$** because the sphere has infinitely many symmetries when rotating and because of so the rotational **component** (ϕ) can be ignored."*
- *"The sphere has continuous rotational symmetry, so if the $\vec{\boldsymbol{E}}$ had $\boldsymbol{\phi}$ and $\boldsymbol{\theta}$ components, the $\vec{E}$ field could change upon rotation which would break the symmetry of the system, thus it must only have r components.*

Overelaborate conflation:

Responses that exhibit this type of conflation are technically correct, but overelaborate by describing behavior of the other property. These responses are labeled as conflation because although students are trying to show that vector and field properties are distinct properties of the vector field, they end up addressing the task with irrelevant information. This seems to be a mild form of conflation because students are not describing any incorrect behavior of the vector field or inappropriately linking symmetry arguments to vector field properties. For example, on a task about the dependence of the electric field:

- *"The electric field* ***cannot depend on $\boldsymbol{\theta}$ and $\boldsymbol{\phi}$*** *because the sphere has continuous symmetry about θ and ϕ, so the electric field* ***will be equivalent at any values $\boldsymbol{\theta}, \boldsymbol{\phi}$*** *and* ***in the $\hat{\boldsymbol{r}}$ direction****."*

3. Analysis of data on tasks investigating student understanding of symmetry arguments

The tasks-of-interest were analyzed by categorizing responses into non-conflated, mild conflation, severe conflation, and miscellaneous[i] reasoning. The associated tasks were analyzed solely based on correctness. Key statistics-of-interest are proportion of students attempting symmetry arguments, relative accuracy on the associated task for the degrees of conflation, and the *conflation rate*[ii]. Table 3-2 shows the breakdown of student responses for each task.

Table 3-2: Breakdown of student responses on symmetry argument tasks. A. T. stands for associated task.

Task	Timing	Sample size	Conflation degree	% of N	Accuracy on A.T.	Conflation rate
Spherical charge	Phys321 Exam 1 16(4Aut)	N = 93	None	76% of N	56%	11%
			Mild + Severe	10% of N	11%	
			Severe	8% of N	0%	
Non-uniformly charged slab	Phys321 Exam 1 18(4Aut)	N = 161	None	64% of N	79%	20%
			Mild + Severe	16% of N	81%	
			Severe	11% of N	89%	
Spherical charge	Phys321 Exam 1 19(2Spr)	N = 108	None	72% of N	62%	16%
			Mild + Severe	14% of N	33%	
			Severe	9% of N	20%	
Mathematical interpretation	Phys322 Pretest	N = 408	None	61% of N	58%	32%
			Present	29% of N	45%	
Current-carrying slab	Phys321 Exam 1 16(1Win)	N = 131	None	50% of N	86%	39%
			Mild + Severe	31% of N	73%	
			Severe	29% of N	71%	

[i] Responses that were not justified explicitly with symmetry arguments.
[ii] The percentage of responses attempting symmetry-based arguments that show conflation: $\frac{Conflated}{Nonconflated+Conflated} \times 100\%$.

On all the exam tasks, at least 80% of the students attempted to justify their answer with symmetry arguments. Most of the students who did not answer based on symmetry arguments attempted to calculate the functional behavior of the field without justifying the assumptions that were made.

Apart from the non-uniformly charged slab task, there is a general trend that the accuracy on the associated task decreases as the severity of conflation increases. The difference in performance between responses that showed no conflation and those that showed some conflation is statistically significant[i] for all but the non-uniformly charged slab.

The conflation rate lies between 10% and 20% for the Phys321 exam tasks, which asked only about the dependence of the electric field. However, the conflation rate lies between 30% and 40% for the Phys322 tasks, which asked about both dependence and components of the magnetic field.

4. Discussion of results

The data suggest that conflation between vector and field properties seem correlated to correct choice of surfaces for using Gauss' and Ampère's Laws. This could be a causal relationship, as the ability to distinguish vector and field properties is likely important for understanding the key steps to simplifying the flux and line integrals in Gauss' and Ampère's Laws, which is likely important for identifying appropriate surfaces.

The increase in conflation rates from Phys321 to Phys322 is alarming. At the surface level, a simple explanation is that the Phys321 tasks only asked about dependence, whereas the Phys322 tasks asked about both dependence and components, which provides more opportunity for students to show conflation. A deeper look into conflated responses on the current-carrying slab task is shown in Table 3-3, which treats the component question and dependence question as independent tasks.

[i] For statistical analysis, the chi-squared test or Fisher exact test for independence (as appropriate) was done first on 2x2 contingency tables comparing accuracy on the associated task between no conflation versus some conflation. The p-values were 0.0133, 1.00, 0.0514, 0.0244, and 0.0961, respective to the order presented in Table 3-2. If the p-value was between 0.05 and 0.10, a follow-up test was done on the 2x2 contingency table comparing accuracy on the associated task between none and mild conflation versus severe conflation. The p-values were 0.0175 for the spherical charge task in 192 and 0.0474 for the current-carrying slab task. This allows us to reject the null hypothesis that accuracy on an associated Gauss' or Ampère's Law tasks is independent of conflation between field and vector properties on symmetry-argument tasks for all but the non-uniformly charged slab task.

Table 3-3: Breakdown of student responses on current-carrying slab task for Phys322 Exam 1, Winter 2016 by individual property.

Property-of-interest	Timing	Sample size	Conflation degree	% of N	Conflation rate
Component direction	Phys322 Exam 1 16(1Win)	N = 131	None	82% of N	11%
			Mild + Severe	10% of N	
			Severe	9% of N	
Coordinate dependence	Phys322 Exam 1 16(1Win)	N = 131	None	60% of N	28%
			Mild + Severe	23% of N	
			Severe	21% of N	

The difference in conflation rates between the two individual tasks is statistically significant[i]. This suggests that for the magnetic field, students who conflate vector and field properties are more likely to impose vector properties and the associated symmetry arguments onto field properties rather than vice versa. Even after isolating the analogous question, the conflation rate on the coordinate dependence task in Phys322 is still significantly[ii] higher than the Phys321 exam tasks.

These results raise a few research questions about the dependence of conflation on context. Why does the dependence task for magnetostatics elicit conflation more than the component task? Why does the dependence task elicit conflation more in magnetostatics than electrostatics? How does conflation rate differ between the dependence and component tasks in electrostatics? I speculate that for magnetostatics, the direction of current is a salient feature in the task that may prime the null-contribution argument for the component direction, such that students who exhibit conflation use that line of reasoning over the transformative symmetry argument. However, in electrostatics, the transformative symmetry argument for dependence may be easier for students to grasp than superposition or proof-by-contradiction arguments for components, so the Phys321 dependence task alone may not be fully eliciting the conflation rate. Informal anecdotal evidence in the grading of tutorial homework suggests that the conflation rate for the component task in Phys321 is very high, but additional post-test data is necessary to investigate these hypotheses.

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[i] The chi-squared test for independence on the 2x2 contingency table comparing presence of conflation between the two tasks gives a p-value of 1.24×10^{-3}. This allows one to reject the null hypothesis that conflation rate is independent of the property-of-interest.

[ii] The chi-squared test for independence on the 2x2 contingency table comparing no conflation to some conflation between the combined Phys321 exams and the Phys322 exam gives a p-value of 0.0232. This allows one to reject the null hypothesis that conflation rate is independent of the course.

E. Description of tutorials addressing symmetry and symmetry arguments

Compared to the curriculum development of the tutorials related to integrals (discussed in Chapter 2) and vector derivatives (discussed in Chapter 4), modifications to the *Gauss' Law* and *Ampère's Law* tutorials have been minor, where only a few parts from the original version have been changed with no major rewrite or refocusing of the overall tutorial. As such, I discuss only the parts of the current version <App – 41,46> of the tutorials related to symmetry and symmetry arguments.

The first two pages of the *Gauss' Law* and *Ampère's Law* tutorials are related to symmetry and symmetry arguments. The first context is a primer where students investigate effects of observers around an Egyptian pyramid. This part was introduced to elicit student intuition about symmetry and symmetry arguments, so that the students' intuition may be developed into more formal language in the following part. The core part of the tutorials first models this formal language of symmetry by asking students to identify translational and rotational symmetry of an infinite line and specify whether each symmetry is continuous or discrete. Then, it asks students to compare various points around the wire and make inferences about the relative behavior of the field, essentially modelling a transformative argument for spatial dependence. Next, it asks students to determine what components of the vector field are zero, providing the starting point of the relevant argument. Finally, students are asked whether the statement "the field is independent of z" describes the vector nature or the field nature of the vector field, designed to elicit conflation about vector field properties.

The core part of the tutorials seems to be working effectively, as at least 80% of students are identifying symmetries and attempting to answer post-test questions conceptually with symmetry arguments. Informal observation in the classroom showed that most students have questions about the terminology but seem to show an understanding of the underlying concept.

However, the part of the *Gauss' Law* tutorial about distinguishing between vector and field properties is difficult for students. Informal observation in the classroom showed that almost all groups are confused by the wording "vector nature or field nature" and required a TA to discuss the concepts. Most students were able to give basic definitions of vectors and easily give an example of a vector that is not a field, but many students struggled to define a field without including vector terminology or with giving an example of a field that is not a vector. Almost all groups first believe that the statement "the field is independent of *z*" is a description of the vector nature of the vector field. Students showed similar difficulty on the *Ampère's Law* tutorial, despite being a repeated question in the analogous context. These parts seem effective at eliciting student conflation of vector and field properties and opening discussion with TAs, but more could be done within the tutorial to address this difficulty.

F. Summary and reflections on symmetry and symmetry arguments

Student conflation of the vector and field properties of vector fields appears to be an underlying issue that affects student understanding of symmetry arguments and the application of Gauss' and Ampère's Laws. Additional research is necessary to determine how and why conflation differs between electrostatics and magnetostatics, and between component tasks and dependence tasks.

Although conflation was the major student difficulty documented in this research on the student understanding of symmetry arguments, there was evidence of many other student difficulties about the application of Gauss' and Ampère's Laws. For example, the associated task on the spherical charge task elicited student misconceptions about how the Gaussian surface must have the same symmetries as the source and how the Gaussian surface only "solves for" the field due to the charges enclosed. Likewise, some students sketched open surfaces on the associated task of the non-uniformly charged slab task. This suggests that there are some student difficulties identified at the introductory level that persist into upper division. Additional research is necessary to determine to what extent these difficulties persist, and whether it is necessary to continue addressing them at the upper-division level.

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Chapter 4 Vector Derivatives

This chapter describes research on the ways in which students interpret and apply vector derivatives[i] in the upper-division electromagnetism course sequence. The focus of this research is on student ability to physically interpret vector derivatives as sources within the source-field relationship. This chapter contains two main parts that discuss findings related to (1) an investigation on student ability to interpret and relate vector derivatives to sources and (2) an investigation of student ability to generalize and apply the model of vector derivatives as sources in the context of unfamiliar fields.

The interpretation and application of vector derivatives are crucial to the mastery of upper-division electromagnetism because the derivatives concisely define the relationship between sources and fields. One physical interpretation of divergence and curl is that they represent the sources of a vector field, with divergence representing a source of flux[ii] and curl representing a source of rotation[iii]. As such, the modern set of Maxwell's Equations[iv] can be interpreted as enumerating the sources for the electric and magnetic fields.

A typical task for students in advanced physics courses is to determine a vector field due to a system of sources. Mathematically speaking, determining any vector field $\vec{\psi}$ from its divergence and curl is an exercise in solving differential equations, with the help of Helmholtz' Theorem. The homogeneous solution to $\vec{\nabla} \cdot \vec{\psi} = 0$ and $\vec{\nabla} \times \vec{\psi} = 0$ is $\vec{C}$, a uniform field with arbitrary magnitude pointing in any

[i] Primarily divergence and curl, although some concepts and findings are extendable to other vector derivatives.

[ii] Divergence within a volume is related to the flux entering/leaving the bounding surface through the divergence theorem, $\oiint \vec{\psi} \cdot d\vec{a} = \iiint (\vec{\nabla} \cdot \vec{\psi}) dV$. A physical interpretation suggests a causal relationship in that the divergence of the field describes faucets and sinks that are responsible for the net flux through the surface.

[iii] Likewise, curl through an area is related to a contour integral of the field around the closed loop through Stokes' Theorem, $\oint \vec{\psi} \cdot d\vec{l} = \iint (\vec{\nabla} \times \vec{\psi}) \cdot d\vec{a}$. A physical interpretation is that the curl describes pinwheels or motors that are responsible for some rotation of the field.

[iv] $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_o}, \vec{\nabla} \times \vec{E} = -\frac{d\vec{B}}{dt}, \vec{\nabla} \cdot \vec{B} = 0$, and $\vec{\nabla} \times \vec{B} = \mu_o(\vec{J} + \epsilon_o \frac{d\vec{E}}{dt})$

Cartesian direction. Helmholtz' Theorem states that the inhomogeneous solution to $\vec{\nabla} \cdot \vec{\psi} = S$ and $\vec{\nabla} \times \vec{\psi} = \vec{S}$[i] can be uniquely decomposed into two vector fields: $\vec{\psi}_{div}$, which satisfies $\vec{\nabla} \cdot \vec{\psi} = S$ but has no curl anywhere; and $\vec{\psi}_{curl}$, which satisfies $\vec{\nabla} \times \vec{\psi} = \vec{S}$ but has no divergence anywhere. Thus, the solution follows the form $\vec{\psi} = \vec{\psi}_{div} + \vec{\psi}_{curl} + \vec{C}$. With a physical source-based interpretation, the arbitrary uniform field $\vec{C}$ represents a contribution that is sourced by something external to the system, so it is often ignored as it is not "due to" the system. The field due to the system can then be found by using $\vec{\nabla} \cdot \vec{\psi} = S$ to determine $\vec{\psi}_{div}$, and $\vec{\nabla} \times \vec{\psi} = \vec{S}$ to determine $\vec{\psi}_{curl}$.

At UW[ii], the typical progression of instruction on the vector-derivative model of sources occurs throughout the undergraduate education. At the introductory level, students learn about two basic source-field relationships: the first is the charge-electric field relationship via Coulomb's Law and Gauss' Law, and the second is the current-magnetic field relationship via Biot-Savart Law and Ampère's Law. Then at the sophomore level, students typically learn vector calculus as part of a mathematical methods course. Finally, students typically[iii] encounter Maxwell's equations at the upper-division or junior level and learn to apply vector calculus in the context of electromagnetism. In this dissertation I break the conceptual development at the upper-division level into a two-step process: the first step is learning to physically interpret divergence and curl as sources, and the second step is transferring knowledge of the basic source-field relationships to other vector fields based on the divergence and curl. As most experienced instructors likely recognize, teaching students how to effectively interpret and use divergence and curl in electromagnetism can be difficult.

This chapter is organized as follows. Section A discusses relevant research at other institutions on the student understanding of vector derivatives in the context of electromagnetism. This is followed by my research, which is split in two sections, each with two parts. Section B is on how students identify

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[i] S and $\vec{S}$ represent arbitrary scalar and vector fields respectively. The character "S" was chosen to denote "source" when using a source-based interpretation to vector derivatives.

[ii] And likely many other undergraduate physics programs

[iii] Maxwell's equations are sometimes introduced in derivative form in introductory electromagnetism, although the use of divergence, curl, and associated theorems is minimal at that level.

vector derivatives as sources. Part 1 is on how students interpret cause and effect in differential relationships, which probes how students interpret sources in relationships more generally. Part 2 is on how students conceptually evaluate divergence and curl of the electric or magnetic field, which probes how students apply their interpretation of vector derivatives in a familiar physical context. Section C is on how students apply the source-field relationship more generally to newly encountered vector fields. Part 1 is on how students determine the magnetic vector potential from the magnetic field, which probes how students transfer knowledge of the familiar current-magnetic field relationship to the analogous but unfamiliar relationship. Part 2 is on how students determine the displacement field or auxiliary field from polarization or magnetization, respectively, which probes how students interpret the vector-derivative model of sources holistically. Lastly, section D summarizes the important findings of this research.

A. Relevant research on vector derivatives in physics

To date, most research on the student understanding of physics has been conducted at the introductory or pre-university levels. Since vector derivatives are outside of the scope of most introductory courses, prior research on how students interpret and apply divergence and curl in physics is sparse. Most instructors recognize that students have difficulties with these concepts, but there has been little formal research in the physics education literature.

Pepper et al.[1] at the University of Colorado at Boulder published some observations on student difficulties with vector derivatives from their work in transforming[2] their upper-division courses. They noted that students are often adept[i] at mathematically calculating vector derivatives when given an explicit mathematical expression for a field, and students perform slightly worse[ii] when asked to determine divergence and curl within visual representations of vector fields. However, Pepper et al.

[i] 90% correct on pretest for N=90 students.
[ii] 77% correct on pretest for N = 90 students.

found that students perform significantly worse[i] in a physical context.[ii] They found that students tend to base their reasoning on mathematical misinterpretations[iii] of divergence rather than interpreting and applying Gauss' Law physically or relating divergence visually to flux. Pepper et al. claimed that "many of these difficulties seem to be, in part, tied to difficulty accessing all their mathematical tools and trouble accounting for the physical situation when doing a calculation."

A few universities in Europe also have published some research on student understanding of vector derivatives in upper-division electromagnetism. Baily and Astolfi[3] confirmed the findings of Pepper et al. and noted how a few student misconceptions[iv] about vector derivatives persist despite targeted instruction. In a following paper, Baily et al.[4] investigated the prevalence of certain conceptions about vector derivatives among mathematics and physics students, noting a few differences between the groups primarily on visual representations and physical concepts. Bollen et. al.[5] conducted an exploratory study of how students use vector calculus in the intermediate electromagnetism course and noted that students initially "focused on evaluation and appeared to pay little attention to the conceptual meaning of the vector operators." Bollen et al.[6] followed up with student interviews, investigating how different conceptions blend to account for student interpretation of divergence and curl.

My research expands on the current research base by confirming student difficulties with vector derivatives in general and basic contexts, while also exploring student approaches in the applied contexts of magnetic vector potential and fields in matter. My research also resulted in the development of curricula designed to address difficulties with vector derivatives, while past research has been mainly exploratory of student difficulties.

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[i] 26% correct on a midterm for N = 51 students.

[ii] Students were asked to identify regions where the divergence of the electric field was zero for a thin spherical shell of charge.

[iii] E.g., "Outside $\vec{\nabla} \cdot \vec{E} \neq 0$ because $\vec{E}$ is not zero or uniform"

[iv] "primarily having to do with the divergence being non-zero when at least one of the components of a field is changing with distance, or whenever field lines in a diagram are becoming more or less closely spaced."

B. Investigation of student ability to identify vector derivatives as sources

I have divided the process of identifying vector derivatives as sources into two steps: (1) interpreting the source within a differential relationship, and (2) conceptually evaluating the vector derivative when sources are given. Each step is discussed in turn with the associated tasks used to probe student understanding and then the results.

1. Interpreting the source within a differential relationship

The first stage in the conceptual development of the vector-derivative model of sources is simply to recognize which quantity in the relationship represents the source. I developed a set of tasks that have proved useful in eliciting student reasoning as they try to identify the sources in differential relationships to investigate factors that could lead to difficulties in interpreting vector derivatives as sources.

a. Tasks on interpreting sources

The general theme for the following tasks is that students are given a relationship and asked what quantity, if any, best represents as the source or cause in the relationship. The first task was developed to directly probe how students interpret sources within the vector-derivative relationship. The second set of tasks was developed to probe how students interpret sources in a more general context.

i. Curl tasks

Pretest questions in Phys322 <App – 55> presented students with both a mathematical and

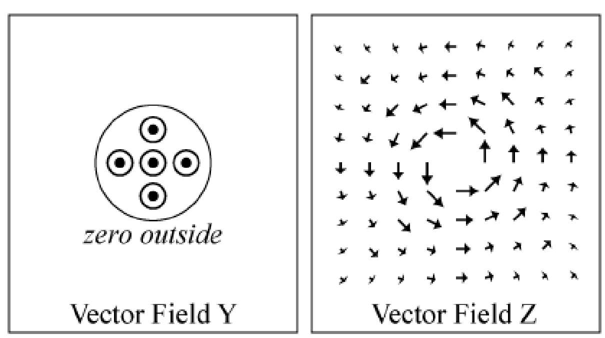

Figure 4-1: Visual context for the curl task.

visual representation of the curl[i] relationship between two vector fields. The mathematical representation was given as $\vec{\nabla} \times \vec{W} = \vec{X}$, and the visual representation was given as Figure 4-1. The task asked students to determine which quantity within each pair of vector fields represents the source.

The intended answer is that $\vec{X}$ and $\vec{Y}$ best represent the source in their respective representation. To arrive at this conclusion, one can make an analogy from the non-contextual representations to the physical relationship between current and magnetic field, where current acts as the source of the magnetic field. For the mathematical representation, Ampère's Law in differential form $\vec{\nabla} \times \vec{B} = \mu_o \vec{J}$ suggests that $\vec{X}$ best represents the source. For the visual representation, $\vec{Y}$ would be analogous to a current-carrying wire while $\vec{Z}$ would be analogous to magnetic field that the current creates.

There were two versions of how the tasks were given. In one version of the pretest, the questions were asked together such that students could see and recognize any connection between the mathematical and visual representations. On a different version <App – 57>, the two representations were given separately with the visual representation given as the first question, then an unrelated question, then the mathematical representation.

ii. Differential relationship tasks

After collecting preliminary data, I asked a more general set of questions on pretests in Phys321 <App – 59> earlier in the course sequence than the curl task. These explore to what extent students coming into the course sequence could interpret causation in other differential relationships. There was a total of three relationships given: the non-contextual relationship $\frac{df(u)}{du} = g(u)$, the contextual acceleration relationship $\frac{d\vec{v}}{dt} = \vec{a}$, and the contextual force relationship $\frac{d\vec{p}}{dt} = \vec{F}_{net}$. In all three relationships, students were asked if the relationship described a source-effect or cause-effect, and if so,

[i] The analogous divergence question would have one field as a vector but one field as a scalar, which may not elicit inconsistencies in student conception as well.

which quantity[i] best represented the 'source' or 'cause.' The *derivative quantity*[ii], or the quantity on the right-hand side of each relationship that is the derivative of another quantity, is analogous to sources in the vector-derivative model of the source-field relationship. On the other hand, the *integral quantity*[iii],or the quantity that the derivative operator acts on, is analogous to the resultant vector field of the source-field relationship. I treated derivative quantity as the intended answer due to the analogy with the vector-derivative model, but the interpretation of the differential relationships is both subjective and context dependent.

There were three versions of how these tasks were administered. On one version, students first answered the non-contextual relationship, and then answered the contextual force relationship. On the second version, students first answered the contextual acceleration relationship, and then the contextual force relationship. On the third version <App – 62>, students only answered the contextual force relationship, but the derivative operator $\left(\frac{d}{dt}\right)$ was an additional answer choice for the 'cause' of the relationship.

b. Common student reasoning for interpreting sources

Below is an overview of student reasoning in the responses to these tasks. This gives a context for the data that follows in part c. Student reasoning supporting why they identified a quantity as the source shows a consistent theme for each type of quantity.

For the equation-based[iv] tasks, students who identified the derivative quantity as the source tended express their reasoning by interpreting the relationship from a physical perspective with causal reasoning. Some of their reasoning suggested that they interpret the derivative quantity as causing

[i] Answer choices were *e.g.,* $\vec{F}$, $\vec{p}$, and "This relationship has no source or cause." On one version, the derivative operator *d/dt* was also given as an answer choice.
[ii] The name was chosen to represent the quantity that is equal to the derivative of the integral quantity.
[iii] The name was chosen to represent the quantity that is related to the integral of the derivative quantity.
[iv] *I.e.,* the mathematical representation of the curl task and the three differential relationship tasks.

changes in the integral quantity over some interval, which is related to the integral from of the relationship ($\Delta\vec{v} = \int \vec{a}dt$). For example:

- *"g(u) describes the rate at which f(u) changes and thus f(u) is dependent on g(u)."*
- *"As velocity would be affected by acceleration, so acceleration is the cause."*
- *"The key point is that you do not apply momentum to an object, but you would apply force."*
- *"The curl of W describes how W curls around X."*

On the other hand, students who identified the integral quantity as the source tended to express their reasoning by interpreting the relationship from a mathematical or operational perspective. Some implied that to use the differential relationship, one must initially know the integral quantity explicitly, such that one can then apply the derivative operator to find the derivative quantity. Similarly, students who chose the derivative operator expressed their reasoning from a mathematical or operational perspective. They suggested that the derivative operator acts as the cause or agent of action in the relationship, in that it acts on the integral quantity to give the derivative quantity. Students who did not choose the derivative quantity may be interpreting the question as asking about the cause in the equation in isolation, rather than all ways that the quantities are related. Consequently, they may be interpreting the given equations as defining[i] the derivative quantity with the integral quantity through the process of taking the derivative. For example:

- *"I see this as 'The derivative of f(u) leads to g(u)' meaning that you need to have f(u) first, and it is therefore the cause."*
- *"I think that it is velocity because, first to get a position, then we derive that to get a velocity, then we derive it once more to get an acceleration."*

[i] This tends to align with the typical Symbolic Form of a definition as $\square = \cdots$, where the right-hand-side of this form defines the quantity on the left-hand side. This could be one reason students show resistance to interpreting vector derivatives as sources, as $\vec{\nabla} \cdot \square = \cdots$ and $\vec{\nabla} \times \square = \cdots$ do not follow the format of a typical definition.

- *"So by taking the derivative of mv, we're finding the change in momentum, mdv/dt, and thus the change in force."*

- *"It results in vector X. Which is the B."*

The last student response shows a failed attempt at a math-physics connection. The student seems to be interpreting causality in $\vec{\nabla} \times \vec{W} = \vec{X}$ operationally as described above while trying to map the relationship physically to the current-magnetic field relationship. Consequently, the student seems to imply that Ampère's Law is represented by $\vec{\nabla} \times \mu_O \vec{J} = \vec{B}$. This suggests that the potential for conflation between the operational and physical interpretations of causality could lead to student difficulty with understanding the vector-derivative model of sources.

On the visual representation of the curl task, students who identified the derivative quantity (vector field $\vec{Y}$ in Figure 4-1) as the source tended to express their reasoning from a physical perspective, as was the case for the equation-based tasks. Most made analogies to current and magnetic field or used the visual cues of curl via Stokes' theorem or rotatable pinwheels. Students who identified the integral quantity (vector field $\vec{Z}$) as the source tended to express their reasoning in mathematical terms, in that they seemed to be mapping the vector fields in the figure correctly to the mathematical representation $\vec{\nabla} \times \vec{Z} = \vec{Y}$, but were then interpreting $\vec{Z}$ as a starting point in the operation, as was the case for the equation-based tasks. Student responses suggest that the reasoning for choosing which quantity corresponds to the source is consistent between the equation-based tasks and the visual task.

c. Analysis of student responses for interpreting sources

Below is a detailed discussion of the data of the student responses on the identifying sources tasks. I first present the Phys321 pretest data from the differential relationship task because the results help make sense of results from the curl task in Phys322. The data from the differential relationship task also represent a pretest of the course sequence, while the data from the curl task, as a pretest of Phys322, may be influenced by instruction in Phys321.

i. Analysis of student responses for the differential relationship tasks

Table 4-1: Breakdown of student responses for interpreting sources on the three versions of the differential relationship tasks. Highlighted in yellow is the derivative quantity, which is analogous to the source in the vector-derivative model of sources, e.g., current in $\vec{\nabla} \times \vec{B} = \mu_o \vec{J}$.

Task	Derivative quantity	Integral quantity	Derivative operator	None
Version 1: $\frac{f(u)}{du} = g(u)$, then $\frac{d\vec{p}}{dt} = \vec{F}$, N=233				
Non-contextual	22%	70%	n/a	9%
Contextual (Force)	44%	46%	n/a	10%
Version 2: $\frac{d\vec{v}}{dt} = \vec{a}$, then $\frac{d\vec{p}}{dt} = \vec{F}$, N=166				
Contextual (Acceleration)	36%	58%	n/a	6%
Contextual (Force)	52%	42%	n/a	7%
Version 3: $\frac{d\vec{p}}{dt} = \vec{F}$ only, N=84				
Contextual (Force)	48%	26%	24%	2%

Table 4-1 summarizes the data from the differential relationship tasks. The statistic-of-interest is the proportion of students who chose the derivative quantity for each task, as the derivative quantity is analogous to the source in the vector-derivative model of the source-field relationship.

Version 1 presented students with the non-contextual derivative task before the force task. A total of N = 223 students from 4 quarters[i] in Phys321 saw this version of the pretest. For the mathematical relationship, 20% of students identified the derivative quantity [$g(u)$] as the source, while 70% of students identified the integral quantity as the source. When later asked about the force relationship, 45% of students chose the derivative quantity (force) as the source, while 45% chose the integral quantity. About 60% of students chose a consistent answer, although only 20% chose the derivative quantity for both tasks as the intended ‘correct’ answer.

[i] Statistics were similar across iterations of the same version, so the data have been grouped together.

Version 2 presented students with the acceleration task before the force task. A total of N = 166 students from the same[i] 4 quarters in Phys321 saw this version of the pretest. For the acceleration relationship, 35% of students identified the derivative quantity (acceleration) as the source, while 60% identified the integral quantity as the source. When later asked about the force relationship, 50% chose the derivative quantity (force) as the source, while 40% chose the integral quantity. About 65% of students chose a consistent[ii] answer, although only 30% chose the derivative quantity for both tasks as the intended ‘correct’ answer.

Version 3 only presented students with the force task. The framing of this version differed slightly from the other two as the derivative operator $\left(\frac{d}{dt}\right)$ was also listed as an answer choice, but the patterns in the student reasoning suggest that it likely did not affect the proportion of students selecting the derivative quantity (force) as the cause. A total of N = 84 students from one quarter saw this version of the pretest. For this group of students, 50% of students identified the derivative quantity (force) as the source, 25% chose the derivative operator $\left(\frac{d}{dt}\right)$, and 25% the integral quantity (momentum).

There is no significant[iii] difference in the proportion of students choosing the derivative quantity on the force question between the three versions of the pretest. This suggests that the response rate is not suspect to priming[iv], as all three formats showed similar results for the force task. There is a significant[v] difference in the proportion of students choosing the derivative quantity in the three different contexts. The three individual tasks form a hierarchy with the force task at about 50%, the acceleration task in the

[i] The pretest randomized which questions the student saw based on their class section.
[ii] *I.e.*, they chose the derivative quantity, integral quantity, or none for *both* questions.
[iii] The chi-squared test for independence on the 2x3 contingency table comparing the proportion of students who identified the derivative quantity as the source on the force context between the pretest gives a p-value of 0.343. One is unable to reject the null hypothesis that the proportion of students identifying the derivative quantity as the source on the force context is independent of the priming context.
[iv] Priming is the psychological effect where differences in a prior task or context affects performance on a target task.
[v] The chi-squared test for independence on the 2x3 contingency table comparing the proportion of students who identified the derivative quantity as the source between the first task seen gives a p-value of 4.26e-12. This allows one to reject the null hypothesis that the proportion of students identifying the derivative quantity as the source is independent of the question context.

middle at 35%, and the non-contextual derivative task as the least at 20%, which seems to correspond to how physically interpretable a context is.

ii. Analysis of student responses for the curl tasks

As stated before, the results on the differential relationship tasks were presented first to help make sense of the results on the curl tasks, which are more relevant to the research topic. The results of the differential relationship tasks show that the context affects the rate at which students choose the derivative quantity as the cause in the relationship, suggesting that more physically interpretable contexts result in higher rates.

Table 4-2: Breakdown of student responses on curl tasks. Math refers to the mathematical representation $\vec{\nabla} \times \vec{W} = \vec{X}$, while Visual refers to the visual representation given in Figure 4-1. Highlighted in green is the derivative quantity, which is analogous to the sources in the vector-derivative model for sources, e.g., current in $\vec{\nabla} \times \vec{B} = \mu_o \vec{J}$.

Curl tasks	Math: Derivative quantity	Math: Integral quantity	Row total
Visual: Derivative quantity	17%	56%	72%
Visual: Integral quantity	5%	23%	28%
Column total	22%	78%	N = 716

A total of N = 716 students over 8 quarters took the Phys322 pretest which asked students to identify the source within the curl relationship in both the mathematical and visual representations. About 20% of students identified the derivative quantity as the source for the mathematical representation, while 70% of students identified the derivative quantity as the source for the visual representation. Table 4-2 shows a detailed breakdown of student responses for the curl task. Overall, 15% of students chose the derivative quantity for both, which is analogous to the current in the current-magnetic field relationship. The disparity in the response rates between the two representations suggests that there is a significant hurdle in student ability to physically interpret the vector derivatives when presented mathematically, even though the students seem to be relatively proficient at physically interpreting the visual representation.

Table 4-3: Breakdown of student responses on interpreting source in ***coupled*** *curl tasks. See Table 4-2 caption for details.*

Coupled questions (N = 424)		Mathematical representation: Derivative quantity	Mathematical representation: Integral quantity	Row total
Visual representation	Derivative quantity	13%	50%	63%
	Integral quantity	6%	31%	37%
Column total		19%	81%	

The results above include student responses from two different versions of the curl tasks. The first version <App – 55> presented students with the mathematical and visual representations coupled on the same page of the pretest, while the second version decoupled <App – 57> the questions by presenting students with the visual representation first, then an unrelated question, and finally the mathematical representation. Table 4-3 and Table 4-4 show a detailed breakdown of student responses on the coupled and decoupled versions of the pretest, respectively.

Table 4-4: Breakdown of student responses on interpreting source in ***uncoupled*** *curl tasks. See Table 4-2 caption for details.*

Coupled questions (N = 292)		Mathematical representation: Derivative quantity	Mathematical representation: Integral quantity	Row total
Visual representation	Derivative quantity	22%	64%	86%
	Integral quantity	3%	11%	14%
Column total		25%	75%	

For the visual task, there is a significant[i] increase from 65% when coupled to 85% when decoupled, but there is no significant difference for the mathematical task. There is a significant[ii] difference in the two versions of the pretest in terms of finding a consistent pair[iii], with 35% consistent on the decoupled tasks and 45% consistent on the coupled tasks.

<u>Possible implication of results:</u>

[i] The chi-squared test for independence on the two 2x2 contingency tables comparing the pretest version to whether students chose the derivative quantity as the source gives p-values of 2.99e-11 for the visual task, but 0.0592 for the mathematical task. This allows one to reject the null hypothesis that the proportion of students who chose the derivative quantity on the visual representation is independent of the pretest version, but one is unable to reject the null hypothesis that the proportion of students who chose the derivative quantity on the mathematical representation is independent of the pretest version.

[ii] The chi-squared test for independence on the 2x2 contingency table comparing the pretest version to consistency in sources gives a p-values of 4.18e-3. This allows one to reject the null hypothesis that consistency in student responses on the two tasks is independent of the pretest version.

[iii] *I.e.*, choosing the derivative or integral quantity for *both* tasks.

The results suggest that student may have conflicting beliefs about the “source” in the mathematical and visual representations of curl. It is possible that helping students recognize consistency could serve to help them better interpret the mathematical statement physically but could also distract students from applying a physical interpretation in the visual representation. However, the data suggest that students who favor the consistency between the tasks tend to be resistant to changing their interpretation of the mathematical representation, such that they tend to release their physical interpretation of the visual representation.

2. Determining vector derivatives when the sources are given

In part 1 above, I described the first stage in the conceptual development of the vector-derivative model of sources as the ability to identify the source in the vector-derivative relationship. Here, I discuss the second stage, which is the ability to activate the source interpretation of the model to conceptually evaluate vector derivatives. I developed a set of tasks where students determine the divergence and curl in a familiar context of electric and magnetic fields. These tasks are designed to investigate student tendencies in interpreting vector derivatives to determine what factors could lead to difficulties in student understanding of vector derivatives.

a. Tasks for determining vector derivatives

The general theme for the following tasks is that they asked students to determine the divergence and/or curl of the electric or magnetic field when a charge or current distribution is given, respectively. Key features that varied between tasks are the timing within the course sequence, the representation of the field, which vector derivative was to be determined, and the location(s)-of-interest.

i. Charged cube task

The target question of the charged cube task asked students to qualitatively describe where the divergence of the electric field is zero/non-zero for the charged cube (Figure 4-2). The context for this task was given on the first exam of Phys321 in Autumn 2014 <App – 131>. No representation of the field was given, but students were asked to generate their own sketch of the electric field prior to the target question. This question is similar to a question that is discussed by Pepper et. al.[1]

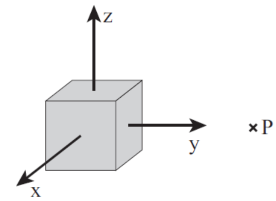

Figure 4-2: Context for the charged cube task.

ii. Dipole task

The target questions of the dipole task asked students to determine the sign of the divergence of the field at a point above a dipole, and the direction of the curl of the field at a point to the side (Figure 4-3). Many variations of this task were given during my research. The task on Phys321 pretests <App – 59> and the Autumn 2017 first exam <App – 132> showed students two charges and asked about the vector derivatives of the electric field, while the task on Phys322 pretests <App – 65> and the Winter 2017 first exam <App – 133> showed students a current loop and asked about vector derivatives of the magnetic field. A visual representation of the field was always given, but some iterations used a vector-field diagram, while others used a field-line diagram <App – 62>. This task is similar to a question that is discussed by Baily et. al.[3]

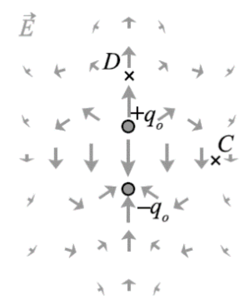

Figure 4-3: Context for the dipole task. Other variations include a diagram given as a field-line diagram and using the magnetostatic context with a current loop and magnetic field.

iii. Charged smiley task

The target question on the charged smiley task asked students to determine the sign of the divergence of the electric field at a point near the center of a charge distribution that resembled a smiley face (Figure 4-4). The task was given on the first exam of Phys321 in Spring 2018 <App – 134>. For the part of the exam relevant to vector derivatives, vector field arrows were drawn around the point-of-interest where there was no charge. This is essentially a variant of the dipole question, although with substantial modifications.

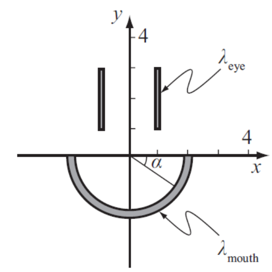

Figure 4-4: Context for the charged smiley task.

iv. Inside-outside task

The target questions on the inside-outside task asked students to determine the sign of the net divergence of the electric field within labeled regions entirely inside and entirely outside of a uniformly charged sphere (Figure 4-5). This task was given on a Phys321 pretest <App – 62>, and the vector field diagram was drawn.

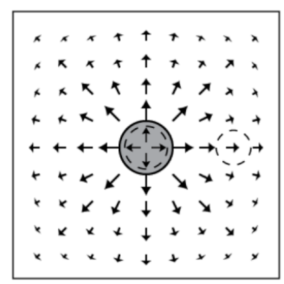
Figure 4-5: Context for the inside-outside task.

v. Comparison tasks

The target question of the comparison tasks asked students to compare the divergence of the electric field at two points. Two variations of this task were given on the first exam of Phys321, with one in Autumn 2018 <App – 135> and the other in Autumn 2019 <App – 136>.

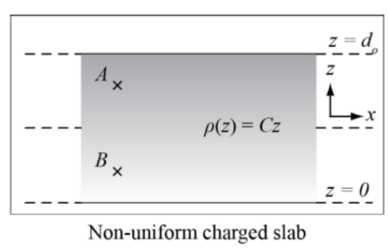

Figure 4-6: Context for the comparison task on first exam of Phys321 in Autumn 2018.

On the Autumn 2018 exam, students were asked to compare two points within a charged slab with non-uniform charge density (Figure 4-6). No visual representation of the field was given, but students were asked on a previous question to determine the functional behavior of the electric field.

On the Autumn 2019, students were asked to compare a point within a charged cylinder to a point outside (Figure 4-7). No visual representation of the field was given. Although this is essentially the inside-outside task with a different geometry, the framing of the question prompt makes it more similar to the comparison task in Autumn 2018.

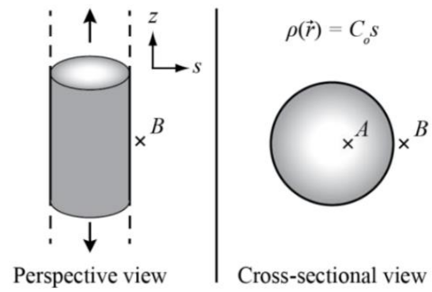

Figure 4-7: Context for the comparison task on first exam of Autumn 2019.

b. Common student responses for determining vector derivatives

Student explanations for how they determined vector derivatives varied greatly, but it was useful to categorize them into one of three primary modes of reasoning: physical, mathematical, and visual. This was done with at first through emergent coding, then further condensed with insight from the resources framework discussed below. Here I discuss common patterns in student reasoning as they fall into these three categories. I discuss the relative proportion of the modes of reasoning in part c.

i. Resources framework and vector derivatives

To help make sense of the student responses and develop the categories for the primary modes of reasoning, I drew from the resources framework[7], which posits that many micro-conceptions or *resources* can be activated and brought together to form a larger conception. With this theoretical lens, vector derivatives can be considered a very resource-rich topic due to the vast number of representations and relationships involved.

The resources involved with vector derivatives can be split into several categories: physical, mathematical, and visual.

- Physically, vector derivatives can be interpreted as sources. This also involves the activation of associated resources related to the source-field relationship, *e.g.*, superposition, symmetry, and Coulomb's Law. Student responses that use physical resources typically rely on the presence of, absence of, or proximity to charges or currents.
- Mathematically, the del operator is a vector sum of partial spatial derivatives, *e.g.*, $\vec{\nabla} \equiv \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$ in Cartesian coordinates. The mathematical complexity of vector derivatives means that there are many mathematical resources involved, *e.g.*, spatial dependence, vector components, and vector multiplication. Student responses that use mathematical resources typically rely on looking for a change in a quantity or thinking about individual components of the vector field.
- Visually, divergence can be described as a net in- or out-ness of the vector field around some closed volume, while curl can be described as a net rotation around some closed loop. These interpretations are related to the divergence and Stokes' theorems, which link the vector derivatives to flux and contour integrals, respectively. Some other visual resources that can be activated are field-line features like spreading or turning. Student responses that use visual resources typically rely on visual cues based on flux or rotation and often include keywords like "in," "out," or "around." The following parts discuss common student responses for each mode of reasoning in detail.

In categorizing student responses with the resources framework, one typically[8] notes all the resources activated in each response as isolated categories; however, I deviate from the resources framework by identifying a primary mode of reasoning based on the way in which a student seems to be tying the resources together. In some ways, this brings in elements of a misconceptions approach to analyze student reasoning and helps refine the data by condensing student responses from eight total combinations[i] of modes to three primary modes. However, this approach introduces a subjective element to interpreting which resource is considered "primary" when resources span multiple categories. For example, certain student statements[ii] are too vague, such that how each primary mode of reasoning was determined relied on the statement's conclusion. Some students also gave multiple lines of reasoning, so the primary reasoning was determined based on grammatical cues[iii] or the first[iv] line of reasoning provided.

As such, the research framework used in analyzing the data here leverages the qualitative insight gained from using resources as a theoretical lens while maintaining a categorization scheme that allows for quantitative assessment. This categorization scheme is intended to be a guide for curriculum development by looking at broad patterns in student reasoning.

ii. Physical responses for determining vector derivatives

For most of the tasks for determining vector derivatives described in part a. above, the intended correct response would be relating the vector derivative of the field to the source, and then evaluating the source density at the point(s)-of-interest. For example:

- *"By Gauss' Law, $\vec{\nabla} \cdot \vec{E} = \frac{\rho_{enc}}{\epsilon_o}$, meaning that the divergence of $\vec{E}$ is proportional to the charge enclosed. There is no charge enclosed at D, so there is no div."*

Most of the incorrect student responses that were categorized as physical rely on proximity to the source. For example:

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[i] *I.e.,* None, P, M, V, PM, PV, MV, PMV; where P, M, and V represent activation of a physical, mathematical, or visual resource, respectively.

[ii] *E.g.,* "Field lines spread" represents a visual resource, which students typically link to positive divergence, but the interpretation of decreasing field-line density which corresponds to magnitude of the vector field represents a mathematical resource, which students typically link to negative divergence.

[iii] One can sometimes determine which line of reasoning is the "proof" and which is a subsequent inference. *E.g.,* "There is no net flux because there is no charge enclosed." represents a physical primary mode of reasoning, while "There is no net flux, so there is no charge enclosed." represents a visual primary mode of reasoning.

[iv] The assumption is that the first independent line of reasoning is written based on the student's intuition, while the second line supports it. *E.g.,* "There is no net flux. Also, there is no charge enclosed." represents independent lines of reasoning and would be categorized as visual.

- *"$\nabla \cdot E = \frac{\rho}{\epsilon_o}$. At point D, $\nabla \cdot E$ is positive because the charge **around D** is positive."*

The student recognizes that the vector derivative is related to the source, but rather than evaluating the charge density at the point-of-interest, they look at the nearest charge. There are a few interpretations about how this misconception may have developed. One interpretation is that some students may not understand that vector derivatives and/or densities are inherently fields, such that they seem to treat vector derivatives as a global or regional property of the field rather than individual values at each point. This common misunderstanding has been observed by others[3]. Another interpretation is that students may be overapplying the spatial relationship in the source-field relationship to the source-derivative equality, such that they may believe that the vector derivative is closer related to a property of the field than the source.

Another type of incorrect student response that was categorized as physical was an incorrect use of symmetry, which was elicited by the charged slab comparison task. Some students suggested that the divergence at the two points were equal because the two points were equidistant from the midplane of the non-uniformly charged slab, even though the midplane did not represent a symmetry of the charge distribution.

iii. Mathematical responses for determining vector derivatives

A type of correct response that is based on mathematical reasoning would be to determine the functional form of the field and then evaluate the sum of the derivatives. However, students rarely took this approach since the tasks were framed as conceptual rather than quantitative. Some students used this line of reasoning in the charged-slab comparison task since a few leading questions first asked students to determine the (in)dependence of the field on coordinates.

The most common type of incorrect response that was categorized as mathematical is isolating a single term in the sum of partial derivatives. For example:

- *"At point D, the divergence of the electric field is negative. Magnitude E decreases as distance from the origin increases."*

This response represents a very common misconception that divergence describes how the magnitude of the field changes as one travels in the direction of the field, while curl describes how the direction of the field changes. In this response, resources related to derivative are activated, as the student notes a ratio of changes. However, resources related to the vector are improperly activated, as the student is linking divergence to the magnitude property of vectors (which both happen to be scalars) rather than a vector sum of partial derivatives. This misconception may be particularly common among students in part because the two resources that are present in the misconception make up the category name "vector derivative."

Another type of response categorized as mathematical relies on reasoning about the sum of partial derivatives in Cartesian coordinates. Some of these responses arrive at the correct answer based on compensation reasoning[i], but many either neglect or incorrectly evaluate some of the partial derivatives. For example:

- *"Positive, since $\frac{d}{dx} = 0$ but $\frac{d}{dy}$ is positive."*

This response implies that the student has activated the mathematical resources of partial derivatives and sums, but the partial derivatives are both incorrect. For context, the points to the left and right of the point-of-interest have vector components that point equally left and right respectively, which evaluates to a positive partial derivative with respect to *x*. The point below has a stronger *y*-component than the point above, which evaluates to a negative partial derivative with respect to *y*.

[i] Compensation reasoning is a tactic used by students to argue reasonableness rather than an explicit proof. Logically speaking, if some terms are positive while others are negative, the answer could be positive, negative, or zero. One cannot determine the sum without knowing the exact values.

The charged-slab comparison question elicited a type of mathematical response pertaining to the mathematical interpretation of $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_o}$. Although it is true that the charge density is directly proportional to the divergence of the electric field, many responses inappropriately extended the proportional relationship to the electric field or applied the derivative operator to the charge density. For example:

- *"The divergence as A is greater than the divergence at B, because the electric field A will be greater at A due to increased charge density."*
- *"In z direction, ρ grow with a constant ratio C, and thus E also grow in a constant ratio. $\nabla E_A = \nabla E_B$."*

These students seem to be incorporating the idea that "the electric field is proportional to charge" to interchange the two quantities, without realizing that they are not *directly* proportional as the relationship contains spatial variables.

iv. Visual responses for determining vector derivatives

A type of correct response that was categorized as visual is similar to those categorized as physical in that they identify the source of the spread or circulation of the field, but the responses do not mention the physical context. For example:

- *"Zero. 0 net E field lines appear or disappear (originate or end) at D." (Note: the diagram provided was a vector field diagram.)*

This student recognizes that the best visual cue for divergence is a source or sink of field lines but does not relate the source of the electric field lines to charges.

The most common type of student response that was categorized as visual uses compensation reasoning of the flux or contour integral. Compensation reasoning based on visual resources is similar to compensation reasoning based on mathematical resources, although the visual reasoning describes the sum of flux or rotation rather than a sum of spatial changes in the components. For example:

- *"The divergence is zero at point D. All vector field lines that enter point D also leave point D." (Note: the diagram provided was a vector field diagram.)*
- *"At point D the flux is negative because inward flux through an infinitesimal volume around D is greater than the outward flux – i.e., the field is diminishing in strength at D."*
- *"Positive ~~negative positive~~ because there is more coming in and less going out."*

These students all imply a comparison of flux, a reasoning which offers a reasonableness argument for answer. For curl, similar responses show a comparison of clockwise and counterclockwise contributions to the line integral. Many students also explain with "right hand rule" without further elaboration, which was assumed to be this type of reasoning, as it implies the use of Stokes' Theorem where a net rotation is converted to the direction of the curl.

Some other responses that were categorized as visual show a misunderstanding of the field-nature of divergence, similar to the proximity argument in the physical mode of reasoning. For example:

- *"Positive. Divergence is a measure of how much the vector field is 'spreading out' at a point."*

This idea that "the field is spreading" is likely describing divergence like a global characteristic, as students may relate positive divergence to the visual picture of the entire electric field due to a positive charge, but not recognizing that the divergence is only positive at the charge and zero elsewhere. This misconception has been observed by others[1] and is attributed to conflation of the technical definition of divergence with the English definition.

c. Analysis of student responses for determining vector derivatives

Due to the large number of variations for the task for determining vector derivatives when sources are given, there is a lot of data, as shown in Table 4-5. The different tasks elicited different types of reasoning to different extents, so the data between tasks may not be directly comparable. However, general trends can be extracted by analyzing the response rates[i] and accuracy[ii] of the primary modes of reasoning for each task. Figure 4-8 through Figure 4-13 show graphical representations of this data when relevant.

[i] Response rates refer to the distribution of student responses in the three primary modes of reasoning, ignoring accuracy.

[ii] Accuracy as in percent of students who chose a correct answer. This statistic ignores the validity of the line of reasoning.

Table 4-5: Breakdown of student responses on conceptually determining vector derivatives. For the reasoning columns, the first percent describes the accuracy of responses with that primary mode of reasoning, while the (% of N) describes the proportion of total students who used that primary mode of reasoning.

Task	Div or Curl	Timing	N	Overall correct	Physical reasoning	Math reasoning	Visual reasoning	Misc/blank
Visualization given as vector field diagram								
Charge dipole	Div	Phys321 Pretest	495	18%	44% (14% of N)	9% (33% of N)	17% (45% of N)	11% (8% of N)
Charge dipole	Curl	Phys321 Pretest	367	31%	96% (7% of N)	50% (33% of N)	13% (60% of N)	
Inside-outside	Div	Phys321 Pretest	84	69%	81% (31% of N)	13% (19% of N)	85% (46% of N)	67% (4% of N)
Inside-outside	Div	Phys321 Pretest	84	44%	88% (29% of N)	21% (17% of N)	31% (50% of N)	0% (5% of N)
Current dipole	Div	Phys322 Exam 1 17(1Win)	127	73%	100% (58% of N)	29% (11% of N)	41% (29% of N)	0% (2% of N)
Current dipole	Curl	Phys322 Exam 1 17(1Win)	127	47%	82% (40% of N)	33% (12% of N)	15% (42% of N)	50% (6% of N)
Charge dipole	Div	Phys321 Exam 1 17(4Aut)	154	53%	89% (30% of N)	3% (19% of N)	47% (44% of N)	70% (7% of N)
Charge dipole	Curl	Phys321 Exam 1 17(4Aut)	154	42%	94% (23% of N)	70% (15% of N)	16% (58% of N)	29% (5% of N)
Charged smiley	Div	Phys321 Exam 1 18(2Spr)	95	21%	87% (16% of N)	0% (34% of N)	13% (42% of N)	25% (8% of N)
Visualization given as field line diagram								
Charge dipole	Div	Phys321 Pretest	85	48%	75% (28% of N)	33% (14% of N)	42% (53% of N)	0% (5% of N)
Charge dipole	Curl	Phys321 Pretest	85	24%	100% (15% of N)	0% (8% of N)	7% (64% of N)	27% (13% of N)
Visualization not given								
Charged cube	Div	Phys321 Exam 1 14(4Aut)	96	32%[i]	85% (34% of N)	10% (22% of N)	0% (24% of N)	5% (20% of N)
Charged-slab comparison	Div	Phys321 Exam 1 18(4Aut)	161	75%	89% (52% of N)	57% (35% of N)	90% (6% of N)	50% (7% of N)
Charged-rod comparison	Div	Phys321 Exam 1 19(4Aut)	121	71%	92% (71% of N)	14% (12% of N)	31% (11% of N)	13% (7% of N)

[i] There is no statistical difference between this result and the 26% correct on the analogous question at CU-Boulder reported by Pepper et. al.[1] The chi-squared test for independence on the 2x2 contingency comparing accuracy between the two exam questions gives a p-value of 0.289. One is unable to reject the null hypothesis that student accuracy on these questions is independent of the contexts of the two exam questions.

Physical reasoning seems to be most accurate:

Except for the charge-dipole pretest questions, the accuracy of students using reasoning categorized as physical is above 80% correct, which is typically much higher than the accuracy of the other modes of reasoning. This result is not surprising as there is bias in the tasks that favor physical reasoning as the intended approach, but it does confirm that many students who use physical reasoning are able to apply it proficiently.

Student use of visual reasoning seems to be higher when visualization is given:

When visualization is given, the proportion of students who use visual reasoning is above 40% and represents the most common mode of reasoning, except for the current-dipole question about the divergence of the magnetic field. For the questions where visualization was not given, the proportion of visual reasoning is higher on the cube question than the comparison questions, likely because students were asked to produce their own sketch.

This suggests that the mode of reasoning students use is highly suspect to priming[i]. However, this is not always the case, as illustrated by the high proportion of students who answer the divergence of the magnetic field question physically. Despite the presence of the visual representation, the high proportion of physical reasoning is likely due to the reinforcement and simplicity of the concept that the divergence of the magnetic field is zero by Maxwell's Equations.

Students seem to be more accurate with field-line diagrams than vector-field diagrams:

[i] Priming is the psychological effect where differences in a prior task or context affect the response on a target task.

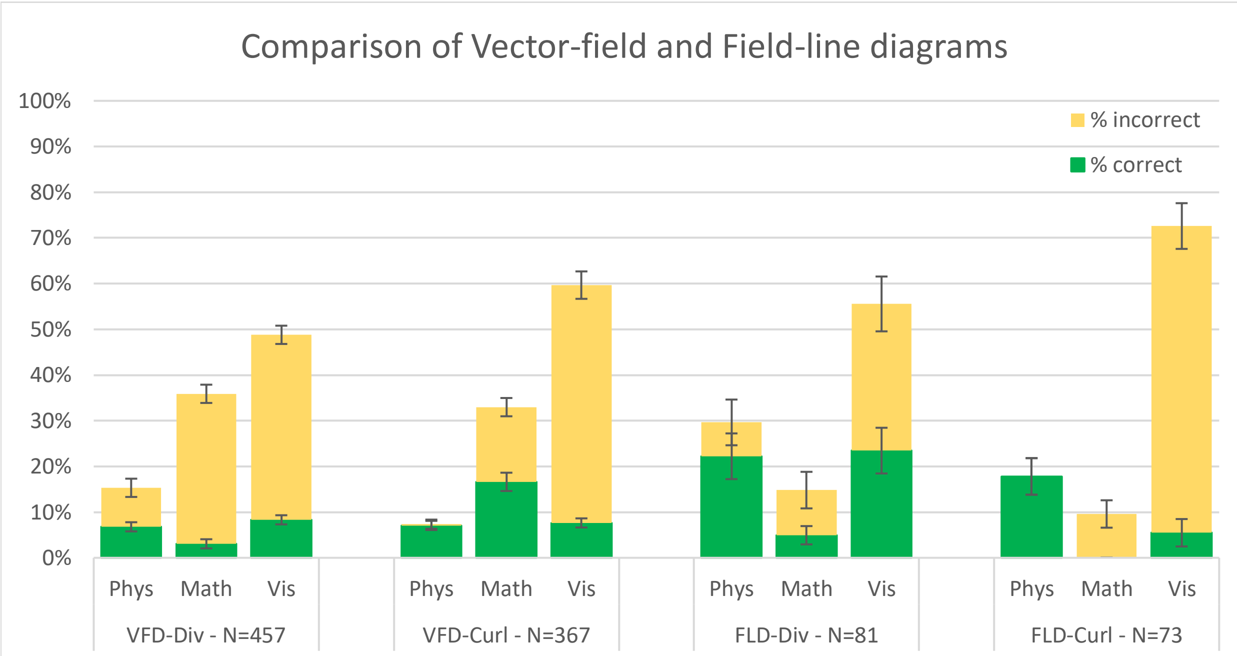

Figure 4-8: Breakdown and comparison of student responses for the charge-dipole task on a Phys321 pretest. Phys, Math, and Vis stand for the primary mode of reasoning used, with blank and miscellaneous omitted. VFD stands for vector-field diagram, and FLD stands for field-line diagram.

Although the charge-dipole pretest questions about divergence are identical other than the representation, the results differ greatly as shown in Figure 4-8. There are significant[i] differences in the response rates between the two representations. The mathematical and physical reasoning response proportions are reversed, as students are more likely to answer physically in the field-line diagram (3rd graph of Figure 4-8) while students are more likely to answer mathematically in the vector-field diagram (1st graph of Figure 4-8). The accuracy is also significantly[ii] higher in the field-line version for all modes of reasoning, with an increase from 20% overall with the vector-field diagram to 50% overall with the field-line diagram. This is additional evidence that priming has a major effect on student responses about determining vector derivatives. Field-line diagrams are inherently a visualization model based on

[i] The chi-squared test for independence on the two 2x3 contingency tables comparing response rates between the two representations gives p-values of 1.21e-4 for the divergence question and 3.71e-5 for the curl question. This allows one to reject the null hypothesis that the response rates are independent of the visual representation given.

[ii] The chi-squared test for independence on the two 2x2 contingency tables comparing accuracy within a mode of reasoning between the two representations gives p-values of 9.34e-3 for physical and 1.66e-4 for visual. The Fisher exact test for independence on the 2x2 contingency table comparing accuracy within the mathematical mode of reasoning between the two representations gives a p-value of 0.0228. This allows one to reject the null hypothesis that the accuracy of the modes of reasoning is independent on the representation given.

divergence and flux. The field lines likely draw students' attention toward the charges as the source, which improves the proportion of physical reasoning as well as increases the accuracy of visual reasoning. There was a small number of students who correctly answered the field-line diagram mathematically, so it is difficult to make conclusions as to why that mode is more accurate with the field-line diagram. On the contrary, vector-field diagrams emphasize the differences in the vectors around the point in question, which leads to a higher rate of mathematical reasoning with evaluating changes in the field, as well as compensation reasoning of flux for the visual mode.

The charge-dipole pretest questions about curl show a similar trend with the relative proportions[i] of the modes of reasoning but do not follow the same pattern about accuracy. For mathematical reasoning with the vector field representation (2nd graph of Figure 4-8), students tended to get zero curl for the wrong reason, stating that the direction of the field does not change. This line of reasoning was not present in the field-line diagram (4th graph of Figure 4-8), where the field lines are clearly curving. There is also no significant[ii] difference in the accuracy of the visual reasoning between the two versions. The discrepancy between the difference in accuracy for divergence and similarity in accuracy for curl is likely tied to the fact field-line diagrams are ultimately a divergence-based visualization model.

Within each representation, there is a significant[iii] difference in response rates for the vector-field diagram (1st two graphs of Figure 4-8), but not for the field-line diagram (last two graphs of Figure 4-8). This suggests that there may be differences between divergence and curl that affect the primary mode of reasoning that students use. On the one hand, the difference may have been found with the vector-field

[i] There was an error in the vector-field version where answer choices for curl were erroneously given as *positive, negative, zero,* or *cannot be determined*. This led to a higher rate of miscellaneous responses commenting about an undefined coordinate system. However, the relative proportion between the physical, mathematical, and visual responses are still similar to the divergence question.

[ii] The one-tailed Fisher exact test for independence on the 2x2 contingency tables comparing accuracy of the responses with visual reasoning between the two representations gives a p-value of 0.196. The two-tailed Fisher exact test would give an estimated p-value of 0.392. One is unable to reject the null hypotheses that the difference in visual representation presented does not affect accuracy for the students who use visual reasoning.

[iii] The chi-squared test for independence on the two 2x3 contingency tables comparing response rates between the divergence and curl questions gives p-values of 3.45e-4 for the vector-field diagram and 0.0891 for the field-line diagram. One can reject the null hypothesis that the response rates are independent for the divergence and curl questions with the vector-field diagram, but not the field-line diagram.

diagram due to the larger N; on the other hand, the difference may be due to a high number of filtered miscellaneous responses in the vector-field curl question that could have distracted from the three primary modes of reasoning disproportionally.

There seems to be similarities in student reasoning on determining vector derivatives between electrostatics and magnetostatics:

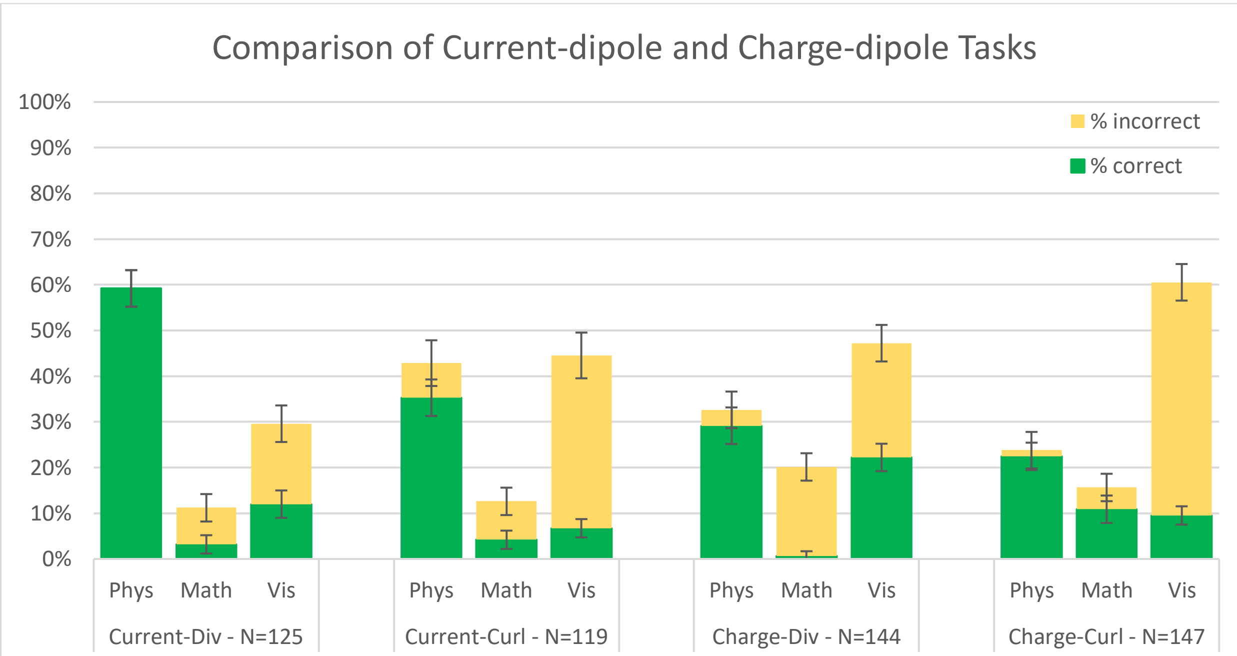

Figure 4-9: Breakdown and comparison of student responses for the dipole task on the first Phys322 Winter 2017 exam and the first Phys321 Autumn 2017 exam. Phys, Math, and Vis stand for the primary mode of reasoning used, with blank and miscellaneous omitted. Both exam questions used a vector-field diagram.

The concepts and presentation of the charge dipole and current dipole are completely analogous. However, there are a variety of factors that differ when comparing the Winter 2017 Phys322 exam to the Autumn 2017 Phys 321 exam. Between electrostatics and magnetostatics, there is a cross-analogy in sources, in that the electric field has divergence but no curl while the magnetic field has curl but no divergence. In other words, the charge-dipole divergence question (3rd graph of Figure 4-9) is physically analogous to the current-dipole curl question (2nd graph of Figure 4-9), although the two divergence questions are analogous to each other in other aspects. There is also a major difference in terms of timing, as the electrostatic context occurs early in the course sequence, whereas the magnetostatics context occurs near the middle. Lastly, there are differences in the Phys321 course for these students, namely the version of the *Vector Derivatives* tutorial.

One can compare response rates between the four questions to see where there may be similarities. A total of six combinations can be made with forming pairs within the four questions. There are two pairs that show no significant[i] difference in response rates: the divergence question on the charge-dipole task (3rd graph of Figure 4-9) with the curl question on the current-dipole task (2nd graph of Figure 4-9), and the two questions on the charge-dipole task (last two graphs of Figure 4-9). The former suggests that the physical analogy between divergence in electrostatics and curl in magnetostatics is particularly relevant, as each of these questions asks about the relevant source at the time of its focus. On the contrary, differences in response rates on the divergence in magnetostatics (1st graph of Figure 4-9) and curl in electrostatics (4th graph of Figure 4-9) are likely due to timing and preparation. The lack of a significant difference between response rates on the charge-dipole task is likely due to a smaller sample size. In comparison to the visual representation discussion above, there may be differences in how students approach divergence and curl in Phys321 that only become statistically significant with a larger sample size.

One can also compare accuracy within the modes of reasoning between the four questions to see where there may be similarities. Within reasoning categorized as physical, there are no significant[ii] differences between the current-dipole curl and the charge-dipole divergence questions (2nd and 3rd graphs of Figure 4-9), nor the current-dipole divergence and the charge-dipole curl question (1st and 4th graphs of Figure 4-9). This suggests that the student application of physical reasoning across physically analogous

[i] The chi-squared test for independence on six 2x3 contingency tables comparing response rates on each pair of questions gives p-values of 0.0307 on the current-dipole pair, 0.0733 on the charge-dipole pair, 6.90e-5 on the divergence pair, 4.17e-3 on the curl pair, 1.48e-8 on the source-less pair, and 0.126 on the physical sources pair. One is unable to reject the null hypothesis that the response rate of the physical sources pair and the charge-dipole pair is independent of the various factors that are different between the two exams.

[ii] The Fisher exact test for independence on the two 2x2 contingency tables comparing accuracy of the physical mode of reasoning between questions gives a p-value of 0.101 for the source-less pair and 0.394 for the physical sources pair. One is unable to reject the null hypothesis that accuracy of the physical mode of reasoning is independent of various factors that are different between the two exams on the physically analogous questions.

questions is consistent. Within reasoning categorized as visual, there are no significant[i] differences on accuracy between similar vector derivatives (1st and 3rd graphs, and 2nd and 4th graphs of Figure 4-9). The accuracy of reasoning categorized as mathematical appears to be inconsistent, but this may be due to small number as the mathematical mode of reasoning represents the smallest proportion of the three modes. Overall, the consistency in accuracy of the visual and physical modes of reasoning suggests that a potential strategy to improve on accuracy for these vector derivative questions may be to attempt to shift student reasoning about vector derivatives from a visual mode of reasoning to a physical one.

Visual reasoning seems to be better at determining non-zero divergence than zero divergence:

For the inside-outside task, one region was given inside of the sphere where the divergence is positive, while the second region was given outside of the sphere where the divergence is zero. The students were only given one explanation prompt to describe how they determined the divergence at the two locations. Many students used one line of reasoning for both tasks or showed how a single mode of reasoning was applied differently to the two regions, but 15% of the students explicitly showed the use of different modes of reasoning for the two regions.

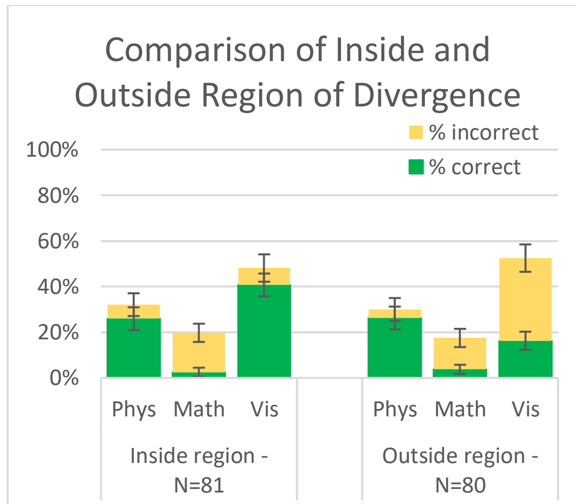

Figure 4-10: Breakdown of student responses on the inside-outside task on a Phys321 pretest. Phys, Math, and Vis stand for mode of reasoning used, with blank and miscellaneous omitted.

For accuracy on the inside-outside task, there are no significant[ii] differences between the two locations for physical and mathematical reasoning, but there is a significant difference for visual

[i] The chi-squared test for independence on two 2x2 contingency tables comparing accuracy of the visual mode of reasoning between questions gives a p-value of 0.919 for curl and 0.521 for divergence. One is unable to reject the null hypothesis that accuracy of the visual mode of reasoning is independent of various factors that are different between the two exams on the questions about the same vector derivative.

[ii] The Fisher exact test for independence on the three 2x2 contingency tables comparing accuracy for a given mode of reasoning to the location-of-interest gives p-values of 0.704 for physical, 0.642 for mathematical, and 1.36e-6 for visual. One is unable to reject the null hypothesis accuracy with the physical or mathematical mode of reasoning is independent of the location-of-interest, but one can reject it for the visual mode of reasoning.

reasoning. When the divergence is positive, 85% of students using reasoning categorized as visual were able to correctly identify positive divergence. However, when the divergence is zero, the answers using visual reasoning are more evenly split between the answer choices, with 30% correctly getting zero, 40% saying positive, and 25% saying negative. This suggests that visual reasoning is a better at determining the presence of divergence than justifying the absence of divergence, which is consistent with the finding that students using the visual mode of reasoning tend to rely on compensation reasoning.

3. Summary of analyses for student reasoning in interpreting and determining vector derivatives

On the first set of tasks where students interpret the source within differential and curl relationships, students tend to reason either physically in interpreting the derivative quantity as causing changes in the integral quantity, or mathematically in thinking about the operation of taking the derivative of the integral quantity to get the derivative quantity. The variation of the ratio of reasoning types seemed to depend on how physically interpretable the context was, with the mathematical and visual representations of curl on opposite sides of the spectrum. The conflicting interpretations about causation likely represent an obstacle in teaching students to interpret vector derivatives as sources.

For the second set of tasks where students determine vector derivatives when sources are given, it was useful to be able to categorize the student responses into three primary modes of reasoning: physical, mathematical, and visual. The data show that students who used reasoning categorized as physical tended to outperform students who used other modes of reasoning. However, many mathematical misconceptions about vector derivatives distract students from physically relating vector derivatives to sources. Additionally, student reasoning about vector derivatives seems to be highly suspect to priming, and various features of vector-field and field-line diagrams elicit many visual misconceptions that also distract from students applying physical reasoning.

4. Development of the *Vector Derivatives* and *Divergence as a Source* tutorials

The tutorials most relevant to the concept of interpreting and determining vector derivatives as sources are the *Vector Derivatives* and *Divergence as a Source* tutorials. The *Vector Derivatives* tutorial was first written in Autumn 2015 as one of the first two tutorials in Phys321 and served as a non-contextual introduction to the del operator, gradient, divergence, and curl, before the *Gauss' Law* tutorial. It was updated in Autumn 2017 to include a deeper discussion about divergence based on insight from the resources framework. The *Divergence as a Source* tutorial replaced the *Vector Derivatives* tutorial in Autumn 2018, appearing as the third tutorial in Phys321 coming right after *Gauss' Law*. It forgoes discussion of the del operator and gradient to focus on the interpretation of divergence as sources of flux, within the context of charges and electric fields. The *Magnetostatic Fields* tutorial in Phys322 is also relevant as it extends the concept of divergence-as-a-source to curl as well, but it will be discussed in further detail in section C1.

a. *Vector Derivative* tutorial – initial version

The focus of the early versions of the *Vector Derivatives* tutorial <App – 70> was on mathematically understanding the del operator and its applications. The four topics were on generalizing derivatives to three dimensions, gradient, divergence, and curl, with each topic having its own page for a brief discussion.

The first page began with single derivatives and targeted misconceptions about the derivative relationship, such as "if the derivative is zero, the field is zero" and vice versa. It then continued to generalize the derivative operator to multiple dimensions, constructing the del operator as multiple single derivatives summed as a vector. For the page on gradient, students were given a topographical graph and then asked about the direction of the gradient at a point on the graph. This targeted the misconception where rather than relying on local characteristics such as the line of equal height, student thought that the gradient to the top of the hill, which has been observed by others[9]. For the page on divergence, students were given a water-flow analogy and the definition of divergence as a source or sink. Then students were asked to determine the divergence at a point to the right of a positive point charge, with field lines drawn. Students were then asked to consider a mock-student discussion where each mock-student answered positive, negative, and zero, each justifying their answer with a common line of reasoning, designed to elicit common misconceptions about determining divergence. For the page on curl, students were given the pinwheel analogy and then asked to determine the curl at various locations in a piecewise vector field with Cartesian geometry.

In terms of modes of reasoning used, the early version started in the mathematical mode in the sense that the foundation of the tutorial was on the derivative aspect of vector derivatives. The tutorial then moved on to develop the visual and physical modes via analogies to non-electrostatic contexts. In a way, this followed the Griffiths' textbook approach[10] of summarizing vector derivatives noncontextually before moving into electrostatics.

b. *Vector Derivatives* tutorial – updated version

Informal classroom observation showed that the mock-student discussion about divergence in the early version was not fulfilling its purpose of an elicit-confront-resolve approach. Students in each group tended to agree confidently and adamantly with the mock-student aligned with their intuition, so some groups attempted to skip an in-depth discussion and moved on without correcting their intuition, while others required extensive intervention from teaching assistants to fully elaborate why certain lines of reasoning were incorrect. When the dipole task was first implemented in a Phys322 exam in Winter 2017, I was surprised that a fourth of the class did not simply state that the divergence of the magnetic field was zero as a physical law, but instead used a non-physical reasoning on the question.

In Autumn[i] 2017, I revised the divergence page of the *Vector Derivatives* tutorial <App – 75> to focus on various modes of reasoning used to evaluate divergence. The tutorial overall kept the first two

[i] The updated version was prepared for Spring 2017, but the professor of that class opted for traditional recitation sections instead of tutorials.

pages on the del operator and gradient intact, but expanded the divergence discussion to two pages, pushing the curl discussion into a supplement. This version also kept the initial setup of the divergence section with the same definition, context, and prediction.

Instead of a mock-student discussion, the remaining content iterated through mathematical, visual, and physical modes of reasoning and attempted to guide students to arrive at the correct conclusion using each line of reasoning. This was inspired by the resources framework, as each misconception previously targeted had some merits that made it intuitively appealing. For mathematical reasoning, students are presented with the expanded form of divergence in Cartesian coordinates. Students were then asked to evaluate the single derivative along each coordinate in turn. If students could realize that one derivative is positive while others are negative, they should logically arrive at the conclusion that the total divergence is undeterminable. For visual reasoning, students are asked to consider the water analogy and determine where sources, sinks, and regions where the field freely flow are. They are also guided to distinguish between the colloquial definition of “diverge” and the technical definition of divergence. Lastly, students are asked to physically interpret Gauss’ Law in differential form and describe where the charge density was zero. This was intended to be the most concrete line of reasoning that would help disperse any uncertainty students might have developed from working through the other modes of reasoning.

The intent of this structure was that if one could help students decompose the resources involved in each misconception and show consistency between multiple modes of reasoning, perhaps they could develop a better intuition about divergence and be able to consistently determine vector derivatives correctly with whichever mode of reasoning they found most appealing.

c. Assessment of the *Vector Derivatives* tutorial

*Table 4-6: Breakdown of student identification of physical source in the curl relationship across 5 quarters. * indicates decoupled version of the question.*

Quarter	Mathematical	Visual	Both	Sample size
‘16(1Win)	10%	56%	7%	N = 101
‘17(1Win)	19%	57%	10%	N = 98
‘18(1Win)	24%	73%	18%	N = 133
‘19(1Win)*	22%	83%	18%	N = 143
‘20(1Win)*	28%	89%	26%	N = 149

For the interpretation of source in vector-derivative relationships, there is a positive trend of interpreting the derivative quantity as the source on the curl tasks when looking at the on-sequence quarters[i] of Phys322, as shown Table 4-6 and Figure 4-11. For the visual representation, students in 2016 started at 55%, while students in 2020 answered at 90%. For the mathematical representation, the improvement is from 10% to 30%. Since the data stems from Phys322 pretests, the data is essentially a posttest of concepts developed[ii] in the first quarter. Thus, improvement on this task is likely related to development of the Phys321 tutorials[iii].

[i] Off-sequence Phys322 courses in the summer only have about 30 students.

[ii] The concept of divergence acting as a source for a field is developed early in Phys321 and is analogous to treating curl as a source. However, the task requires students to not only understand the divergence concept, but also recognize that it can be generalized to curl as well.

[iii] The students in the Winter Phys322 class do not perfectly correspond to the students in the preceding Autumn Phys321, as about a third of the class comes from the prior off-sequence Spring Phys321. Thus, results are not a precise measurement of changes made in the previous Phys321 tutorials.

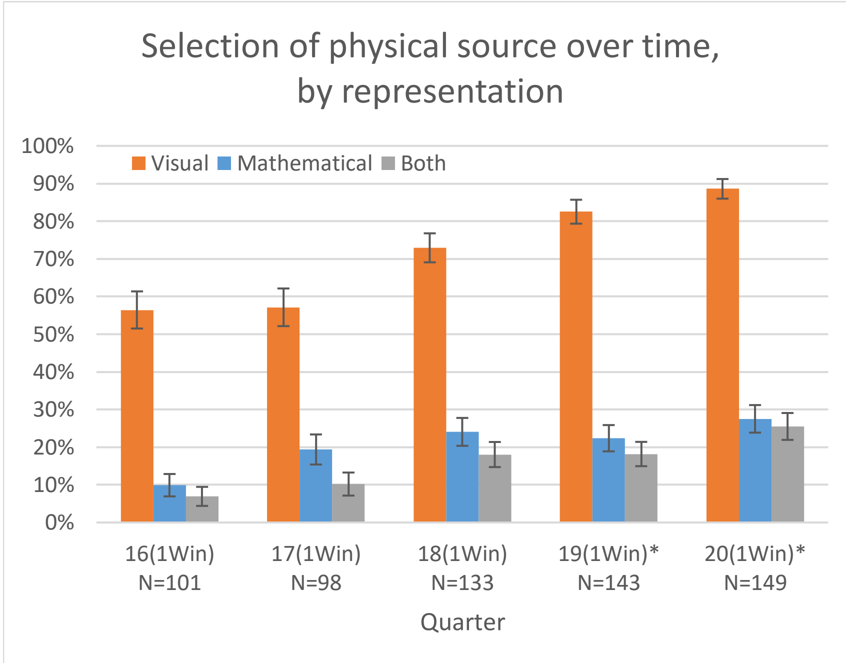

*Figure 4-11: Shows student identification of physical source in the curl relationship across 5 quarters. * indicates decoupled version of the question.*

There are no significant[i] differences in results between any sequential pair of quarters. Based on the previous discussion about pretest versions, one would expect the decoupling of the questions between Winter 2018 and Winter 2019 would have the largest effect; however, the strongest signal is between Winter 2017 and Winter 2018. When combining the results from Winter 2016 and Winter 2017 and comparing it to results from Winter 2018 to amplify the signal, there is a significant[ii] difference despite the pretest version being the same. This difference is significant[iii] on each individual task as well. This suggests that the cumulative changes in the preparation for students in Phys321 are steadily improving student ability to physically interpret the curl relationship, with the changes in Autumn 2017 being the most impactful.

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[i] The chi-squared test for independence on the four 2x4 contingency tables comparing results from sequential quarters give p-values of 0.211, 0.0651, 0.270, and 0.272 for 161-171, 171-181, 181-191, and 191-201 respectively. One is unable to reject the null hypothesis that the response rates are independent of changes between quarters.

[ii] The chi-squared test for independence on the 2x4 contingency table comparing results from 161 and 171 combined to 181 gives a p-value of 4.01e-3. This allows one to reject the null hypothesis that the response rates are independent of quarter.

[iii] The chi-squared test for independence on the two 2x2 contingency tables comparing results from 161 and 171 combined to 181 gives p-values of 2.78e-3 and 0.0287 on the visual and mathematical tasks, respectively. This allows one to reject the null hypothesis that the response rates on individual tasks are independent of quarter.

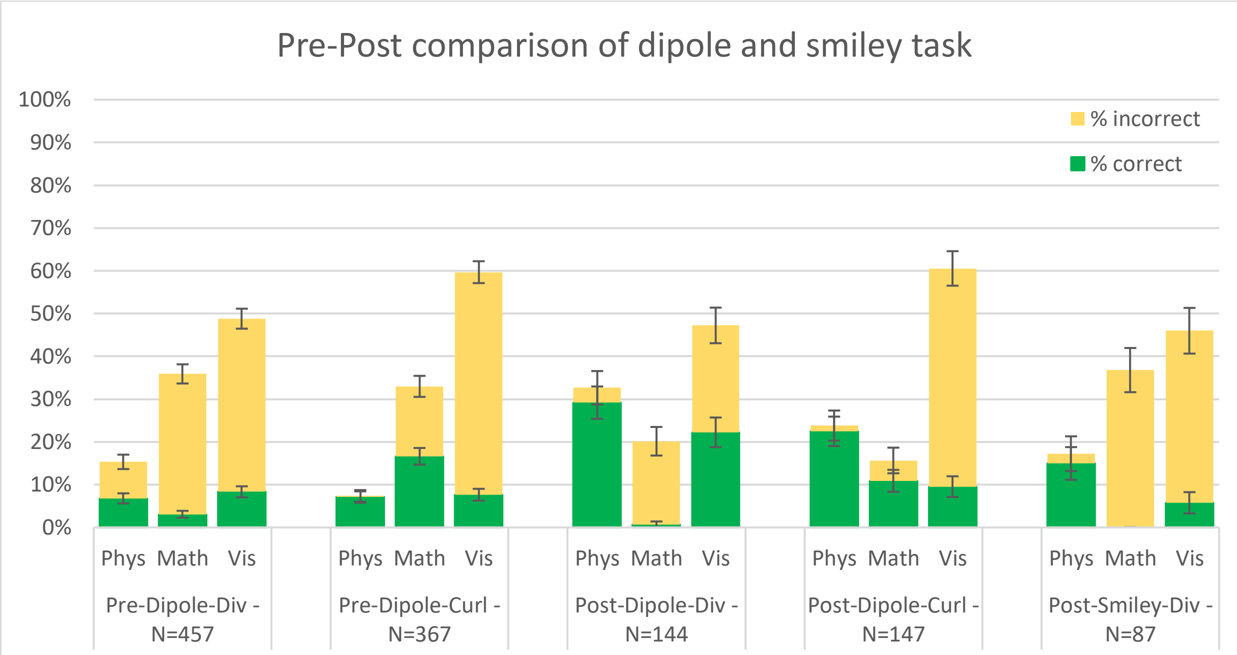

Figure 4-12: Breakdown and comparison of student responses on the charge-dipole task on a Phys321 pretest and the first Phys321 Autumn 2017 exam and the charged smiley task on the first Phys321 Spring 2018 exam. Phys, Math, and Vis stand for the primary mode of reasoning used, with blank and miscellaneous omitted. All questions used a vector-field diagram.

For the conceptual evaluation of derivatives, Figure 4-12 shows the breakdown of student responses on the dipole task with a vector-field representation on both the pretest and an exam in Autumn 2017, as well as the charged-smiley task as an exam in Spring 2018. One can see that there is a clear difference in response rates between pre- and post- tutorial instruction on the dipole task (1st four graphs of Figure 4-12). For both divergence and curl, student responses shift from the mathematical mode of reasoning to the physical, while the visual mode of reasoning remains about the same. There is also an increase in accuracy for the physical mode of reasoning for divergence. This suggests that the *Vector Derivatives* tutorial was impactful in helping some students interpret divergence physically. However, there is no significant[i] difference in response rates between the pretest dipole task and the posttest charged smiley task (1st and 5th graphs of Figure 4-12), although there is still a noticeable increase in accuracy for the physical mode of reasoning. The charged smiley task seems to have a stronger priming effect than the

[i] The chi-squared test for independence on the 2x3 contingency table comparing response rates between the two tasks gives a p-value of 0.857. One is unable to reject the null hypothesis that response rates are independent of the context of the two tasks.

dipole task, as only a few vectors near the point-of-interest were drawn and the context describing the charge distribution was disjointed from the prompt about the divergence of the electric field, whereas the description of the charge distribution and diagram was immediately followed by the prompt about the divergence for the dipole task. This suggests that the *Vector Derivatives* tutorial was ineffective at helping students overcome certain priming factors that led them to use non-physical reasoning.

The data suggests that there were major shortcomings in the approach taken in the *Vector Derivatives* tutorial. Although the intention of having students work through each mode of reasoning was to improve intuition and thus student accuracy on this type of task, students show no improvement in accuracy with both the mathematical and visual modes of reasoning. Additionally, students were not strongly drawn to the physical mode of reasoning as the most consistent approach to this type of task. This motivated the rewrite of the tutorial to focus on the physical mode of reasoning.

d. Divergence as a Source tutorial

In Autumn 2018, I composed a new tutorial <App – 77> to focus on the physical interpretation of vector derivatives. It was originally titled *Vector Derivatives as Sources* but was later renamed *Divergence as a Source* because of the strong focus on divergence in the context of electrostatics. This tutorial comes after *Integrals with Charge* and *Gauss' Law*, where the concepts of the source-field relationship for electrostatics are introduced.

In the core section of the tutorial, students are given an explicit expression for an electric field due to a slab of unknown, non-uniform charge density. Students are asked to consider the process of determining the charge density with the integral form of Gauss' Law $\oint \vec{E} \cdot d\vec{A} = \frac{\int \rho dV}{\epsilon_o}$ by finding a net electric flux and dividing by the volume. After students do this with a Gaussian surface outside the slab and an infinitesimal Gaussian surface inside the slab, they are told that this process is called the divergence. With this instructional approach, students are learning the operational definition[i,11] of

[i] The instructional method of describing an explicit procedure for determining a quantity before giving the name is inspired by *Physics by Inquiry*.

divergence as taking the infinitesimal ratio of net flux to volume to determine the source of the field. Afterwards, students are asked to apply the mathematical definition of divergence as a sum of partial derivatives, which should confirm their findings.

In terms of resources, this approach starts with in the visual mode of reasoning with net flux since many students employ visual reasoning in many contexts. Then, the reasoning develops into the physical interpretation of a source, which is established as the core concept of divergence. Finally, the mathematical reasoning is presented in hindsight, to show that the purpose behind the mathematical operation is to equate divergence to the source. This offers a more blended approach to resource development than the one used in *Vector Derivatives* and focuses on shifting students' mode of reasoning toward physical rather than improving accuracy of all three resources.

The core section was originally <App – 77> proceeded by an introductory page recapping similarities and differences between Coulomb's Law and Gauss' Law in effort to formally establish the idea of the source-field relationship, for which the vector derivative interpretation could be fit in afterward. Informal observation within the classroom showed this to be confusing and overly ambitious, as students were having difficulty abstracting the two laws into the bigger picture. This page was later <App – 82> simplified and combined in the latter discussion, which made the generalization process more tangible for students.

The core section is done in Cartesian geometry where the modes of reasoning tend to agree, but many misconceptions about divergence stem from the overgeneralization of ideas in simple contexts. Thus, the core section is followed by a mock-discussion of divergence in spherical geometry that leads to elaboration on each of the modes of reasoning. This serves to elicit and address the common misconceptions about divergence and properly generalize the concept that divergence identifies the source of a vector field.

e. Preliminary assessment of the *Divergence as a Source* tutorial

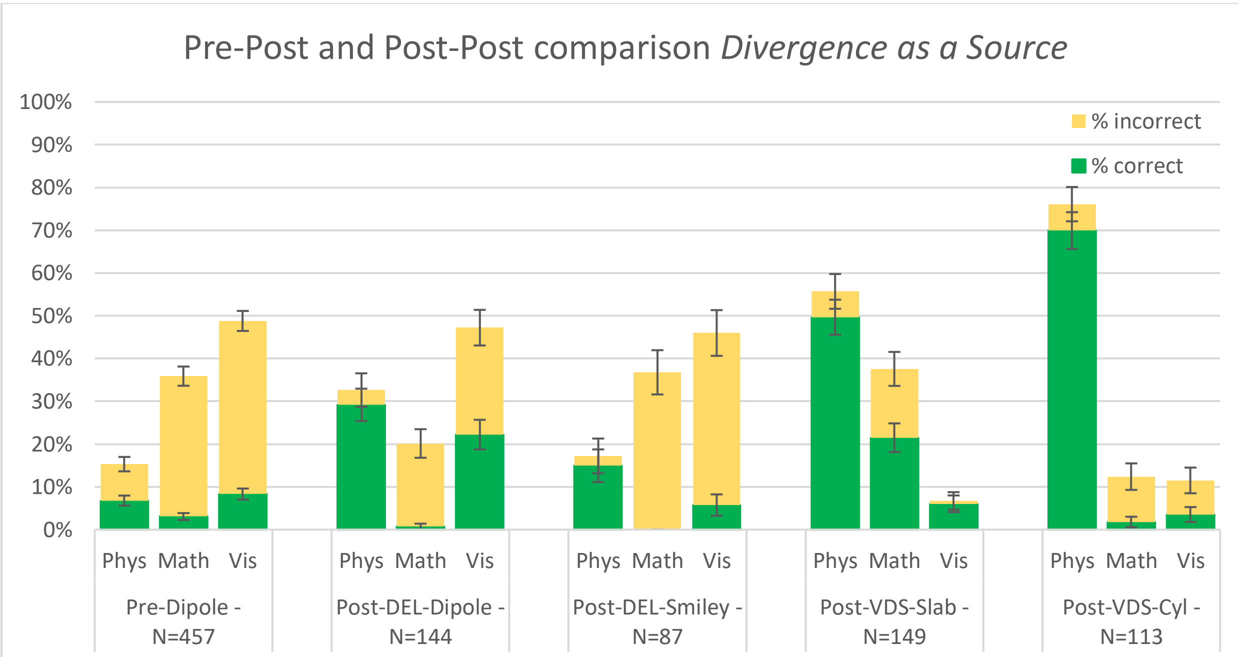

Figure 4-13: Breakdown and comparison of student responses for divergence questions on the dipole, charged-smiley, and comparison tasks. Phys, Math, and Vis stand for the primary mode of reasoning used, with blank and miscellaneous omitted. Slab and Cyl stand for the variant of the comparison task.

Figure 4-13 shows how results from exams with the *Divergence as a Source* (VDS) tutorial compare to the pretest and exams with the *Vector Derivatives* (DEL) tutorial. There is a clear increase in the proportion of students who are using the physical mode of reasoning after the *Divergence as a Source* tutorial. As the context of the comparison tasks is different than the context of the dipole and charged smiley tasks, one cannot be certain that this difference is due to the tutorial version, but the result is promising as it suggests that the *Divergence as a Source* may be effective at teaching student to interpret divergence physically.

C. Investigation of student ability to apply vector derivatives as sources

Once students can interpret vector derivatives as sources, the next step is to be able to apply the model to novel vector fields. In the upper-division electromagnetism sequence, the magnetic vector potential, displacement field, and auxiliary fields represent unfamiliar fields that students encounter for the first time. Investigating how students determine the direction of these fields can gauge to what extent students have mastered the physical interpretation of vector derivatives as sources.

1. Applying the model to magnetic vector potential

The magnetic vector potential, $\vec{A}$, is the magnetostatic analog to the electric scalar potential in that the potentials' relationships to the sources (charge and current) and the fields (electric and magnetic fields) are functionally the same. Practically speaking, the vector potential offers an alternate approach to determine the magnetic field, as doing the Coulomb-like "potential integral" $\vec{A} = \frac{\mu_o}{4\pi}\int\frac{\vec{J}d\tau}{|\vec{r}-\vec{r}\prime|}$ and then taking its curl is often mathematically easier than the rather tedious Biot-Savart integral. However, teaching students the process of using the magnetic field to find the magnetic vector potential via $\vec{\nabla}\times\vec{A} = \vec{B}$ and $\oint\vec{A}\cdot d\vec{l} = \int\vec{B}\cdot d\vec{a}$ is instructionally valuable as it offers another context for students to master curl and Stokes' Theorem applications.

a. Preliminary research on the student understanding of magnetic vector potential

My preliminary research regarding how students interpret and apply $\vec{\nabla}\times\vec{A} = \vec{B}$ to conceptualize the magnetic vector potential was presented and published at the Physics Education Research Conference in 2016[12]. To summarize, we compared the efficacy of two different teaching methods: one focused on building an analogy between the $\vec{J}\rightarrow\vec{B}$ approach and the $\vec{B}\rightarrow\vec{A}$ approach, and the second on the generalization of the $\vec{J}\rightarrow\vec{B}$ approach to a curl-based source-field relationship before learning the application of $\vec{\nabla} \times \vec{A} = \vec{B}$. The first approach relied solely on the tutorial titled *Magnetic Vector Potential* <App – 85>, while the second approach included an additional tutorial titled *Magnetostatic Fields* <App – 90> and an updated *Magnetic Vector Potential* tutorial <App – 95>. For post-tutorial assessment, students from both groups performed similarly on a task on the *Magnetic Vector Potential* homework <App –128 > where an analogy was given, but the generalization group outperformed the analogy group on similar midterm questions where analogies were not given.[i] This suggests that students in the generalization group were more capable of transferring knowledge of $\vec{J} \rightarrow \vec{B}$ concepts to the $\vec{B} \rightarrow \vec{A}$ relationship.

As an extension to this research, I looked in depth into how students are approaching the magnetic vector potential. Rather than evaluating student accuracy on exams, I investigated what lines of reasoning were common and in what proportion to see if the *Magnetostatic Fields* tutorial was impacting how students approach the magnetic vector potential. In addition, I also compared various features on exam questions to see how certain features elicit certain ideas.

b. Tasks for determining magnetic vector potential

A typical task asked on a midterm in Phys322 is for students to determine the direction of $\vec{A}$ at a point, or state explicitly if it is zero. However, the context of the question can differ widely based on geometry, starting physical quantity given, and location-of-interest. Rather than describe each individual question given, I first describe the general categories of the context. Each exam that contained this type of task is listed in the appendix <App – 137-142>.

For geometry, students were typically given systems with a high degree of symmetry such that $\vec{A}$ could be calculated via $\oint \vec{A} \cdot d\vec{l} = \int \vec{B} \cdot d\vec{a}$, even though conceptual answers were suffice. In Cartesian coordinates, this looked like infinite slabs, where the system had continuous translational symmetry along

[i] The two groups were in different quarters, so they had different exam questions. On the homework, both groups were about 70% accurate, whereas on the exam, the generalization group was at about 65% while the analogy group's accuracy dropped to about 35%.

two axes and had coordinate dependence on one. In cylindrical coordinates, this looked like thick or multiple solenoids where the magnetic field was axial but with an atypical radial dependence. There was one quarter where a long, triangular wire was given. This is a potential factor in response rates because the familiarity with a coordinate system may affect the mode of reasoning used.

For starting quantity given, this refers to whether the question prompt provides a description of the current density, magnetic field, magnetization, or some combination. Although the targeted mode of reasoning is whether students can apply source-based ideas to go directly from $\vec{B}$ to $\vec{A}$, the presence of other physical quantities may affect the relationship students opt to use. When the magnetic field was not given, exam responses were filtered to only include those that described reasonably correct magnetic fields such that the responses could be compared to other exams.

For location-of-interest, almost all the tasks contained a localized magnetic field, such that the magnetic field is zero outside of a bounded region. Some tasks asked students to determine the direction of $\vec{A}$ within the bounded region, even though the task may ask about a location where the magnetic field is zero. The other tasks asked students to determine the direction of $\vec{A}$ outside of that bounded region. The location asked could affect response rates if the type of reasoning students use is dependent on local features of the magnetic field.

For example, the exam in Summer 2016 described the following magnetic field: "The magnetic field is zero everywhere for $y < -d/2$ and $y > d$. The magnetic field from $-d/2 < y < 0$ is given by $\vec{B} = (2Cy + Cd)\,\hat{x}$ for all x and z, and from $0 < y < d$ is given by $\vec{B} = (-Cy + Cd)\,\hat{x}$ for all x and z." Visual and graphical depictions of the magnetic field were also given, as shown in Figure 4-14. The task was to determine the direction of $\vec{A}$ at a point where $y > d$. This task was categorized as Cartesian geometry, starting with magnetic field, and with a location-of-interest outside. The correct

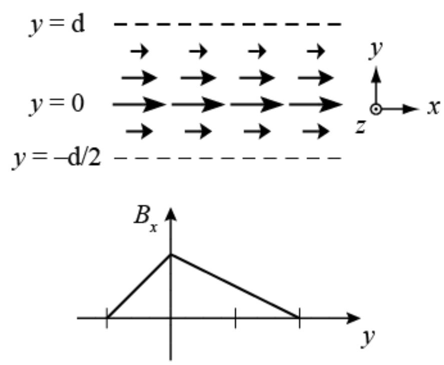

Figure 4-14: Context for determining magnetic vector potential on first exam of Phys322 in Summer 2016.

answer is out of the page or in $+\hat{z}$, which can most easily be shown by a right-hand rule where the thumb points rightwards in the direction of magnetic field, where wrapping fingers around that axis shows $\vec{A}$ to point outwards at points $y > d$.

c. Common student responses for determining magnetic vector potential

The ideal response shows mastery of the curl relationship in general. In most cases for determining the direction of the magnetic vector field, one can apply $\vec{\nabla} \times \vec{A} = \vec{B}$ via a right-hand rule. Similarly, one can interpret and apply the integral form $\oint \vec{A} \cdot d\vec{l} = \int \vec{B} \cdot d\vec{a}$ as capturing magnetic flux with a surface, such that the magnetic vector potential contributes a net contour integral around the surface. Lastly, one can opt to construct an analogy: $\vec{J} \to \vec{B}$ as $\vec{B} \to \vec{A}$. Since both are curl relationships, one can pretend the given magnetic field is a current density, solve for the "magnetic field" with the more familiar right-hand rule or Ampere's Law, then transfer their answer to the magnetic vector potential. Many students are not very detailed about the process in which they apply these lines of reasoning, but responses that list $\vec{\nabla} \times \vec{A} = \vec{B}$ or $\oint \vec{A} \cdot d\vec{l} = \int \vec{B} \cdot d\vec{a}$ and imply a right-hand rule or Ampèrian loop are categorized as source-based approaches.

An alternative line of reasoning circumvents the $\vec{B} \to \vec{A}$ direct approach above by using current-based approach via $\vec{B} \to \vec{J} \to \vec{A}$. With this approach, students use the right-hand rule in the familiar context to determine the direction of the current, if it was not already given. Then, they use the heuristic that the magnetic vector potential is typically in the direction of the current, which is based on the source-potential relationship $\nabla^2 \vec{A} = -\mu_o \vec{J}$ and $\vec{A} = \frac{\mu_o}{4\pi} \int \frac{\vec{J} d\tau}{|\vec{r} - \vec{r}\prime|}$ suggesting that each current element contributes to the component of the magnetic vector potential in the direction of that current element. Although this is a correct line of reasoning and involves the first part of the more practical $\vec{J} \to \vec{A} \to \vec{B}$ application, the need to take this approach when already given the magnetic field shows a lack of generalization of the $\vec{J} \rightarrow \vec{B}$ relationship and does not show a mastery of the concept of curl.

The most common incorrect reasoning implies a non-spatial, proportional relationship between the magnetic field and magnetic vector potential. This is most strongly elicited when asked about the magnetic vector potential in a region where the magnetic field is zero, for which many students state that the magnetic vector potential must also be zero. Another manifestation of this idea is stating that the magnetic vector potential and magnetic field point in the same direction. Both lines of reasoning improperly interpret the relationships $\vec{\nabla} \times \vec{A} = \vec{B}$ and $\oint \vec{A} \cdot d\vec{l} = \int \vec{B} \cdot d\vec{a}$ by ignoring the spatial and/or derivative nature[13,14] of the curl relationship, implying that the two fields are linearly proportional to each other by some constant. These are considered proportional-based approaches. It is interesting to note that many students applied the same misconception to the $\vec{J} \rightarrow \vec{A}$ relationship, although those responses were categorized as current-based as students attempted to circumvent the $\vec{B} \rightarrow \vec{A}$ approach.

Other types of reasoning were categorized as miscellaneous, as they were not common enough across the various tasks. However, the comparison[i] task in Winter 2019 elicited two other common lines of reasoning. Some students commented that if the divergence and/or curl of the field were zero, the field would be uniform. This shows ideas of the differential relationship of divergence and curl but was often misused to state that the magnitude of the magnetic vector potential between the two points were equal despite the presence of a magnetic field between the points that would cause a difference. The other line of reasoning relied on students' perceived proximity to a source. This could have referred to the potential integral $\vec{A} = \frac{\mu_o}{4\pi} \int \frac{\vec{J} d\tau}{|\vec{r} - \vec{r}\prime|}$ or a generalized version of the Biot-Savart integral using $\vec{B}$ as the source for $\vec{A}$. However, students were not expected to calculate these integrals, and students' intuition often led them to an incorrect answer.

[i] Rather than determining the direction of the magnetic vector field at a point, the comparison task asked students to compare the magnitude of the magnetic vector field between two points.

d. Analysis of student responses for determining magnetic vector potential

Each exam question was categorized based on the geometry, starting quantity, and point-of-interest, as shown in Table 4-7. Additionally, each quarter was categorized based on whether students worked through the *Magnetostatic Fields* tutorial in class where attendance was required[i]. Student responses were sorted into source-based, current-based, proportional-based, and miscellaneous reasoning from what could be inferred in their explanation, ignoring accuracy. For the exams where the magnetic field was not given, responses were filtered to only include students who had derived a magnetic field with a form that could be compared to the other questions.

*Table 4-7: Categorization of contexts and breakdown of student responses for the magnetic vector potential task. * denotes the number of students after filtering responses that described a reasonable magnetic field with features that allow comparison to other questions.*

Quarter/exam	15(1Win) E1	161 E2	16(3Sum) E1	183 E1	191 E1	201 E1
Geometry	Cartesian	Cylindrical	Cartesian	Triangular	Cylindrical	Cylindrical
Starting quantity(s)	J	M	B	J	J and B	B
Point(s)-of-interest	Outside	Within	Outside	Within	Both	Outside
Magnetostatic Fields status	Omitted	Required	Required	Required	Omitted	Omitted
Students	N=80*	N=96*	N=23	N=45*	N=169	N=173
Source	35% (of N)	66%	26%	44%	24%	24%
Current	21%	9%	9%	13%	17%	13%
Proportional	15%	6%	39%	11%	18%	50%
Miscellaneous	29%	19%	26%	31%	41%	13%

I made intuitive hypotheses on how each category might have affected response rates to condense the data, and then tested them against the appropriate null hypotheses. Graphical representations of these comparisons are shown in Figure 4-15. I hypothesized that the geometry of the context may affect overall response rates, as students may be more familiar with curl in cylindrical contexts than others. There is no

[i] In 191 and 201, the *Magnetostatic Fields* tutorial fell on a week of a holiday, so the tutorial was given to students as a “take-home tutorial.”

significant[i] difference of overall response rates between different geometries. I hypothesized that when current was provided in the context of the problem, there would be a higher proportion of students who used current-based reasoning. There is a significant[ii] increase from 10% to 15%, although the effect size[iii] was small[iv]. I hypothesized that when the point-of-interest was in a region where the magnetic field were zero, there would be a higher proportion of students who used proportion-based reasoning. There is a significant[v] increase from 5% to 40% and represents a medium[vi] effect size. Lastly, I hypothesized that requiring students to work through the *Magnetostatic Fields* tutorial would increase the proportion of students using source-based reasoning. There is a significant[vii] difference from 25% to 55% and represents a medium[viii] effect size.

[i] The chi-squared test for independence on the 3x4 contingency table comparing the response rates between geometries gives a p-value of 0.179. One is unable to reject the null hypothesis that the response rate is independent of the geometry of the context.

[ii] The chi-squared test for independence on the 2x2 contingency table comparing the proportion of students using current-based reasoning based on whether current was provided in the context gives a p-value of 0.0253. This allows one to reject the null hypothesis that whether current is a starting quantity does not affect the proportion of students using current-based reasoning.

[iii] The phi statistic for 2x2 contingency tables and the more general Cramér's V for nxm contingency tables indicates how correlated the categories are. Benchmark values for phi are 0.1, 0.3, and 0.5 for small, medium, and large effect sizes respectively. Cramér's V uses the benchmark values of phi divided by the square root of the degrees of freedom in the contingency table.

[iv] The Cramér's V statistic for this 3x4 contingency table is V=0.0912, which is near the 0.0707 benchmark for small effect sizes.

[v] The chi-squared test for independence on the 2x2 contingency table comparing the proportion of students using proportion-based reasoning based on whether the magnetic field was zero at the point-of-interest gives a p-value of 1.19e-12. This allows one to reject the null hypothesis that whether the magnetic field was zero at the point-of-interest does not affect the proportion of students using proportion-based reasoning.

[vi] The phi statistic for this 2x2 contingency table is 0.342, which is near the 0.3 benchmark for medium effect sizes.

[vii] The chi-squared test for independence on the 2x2 contingency table comparing the proportion of students using source-based reasoning based on whether the *Magnetostatic Fields* tutorial was required gives a p-value of 2.43e-12. This allows one to reject the null hypothesis that the *Magnetostatic Fields* tutorial does not affect the proportion of students using source-based reasoning.

[viii] The phi statistic for this 2x2 contingency table is 0.286, which is near the 0.3 benchmark for medium effect sizes.

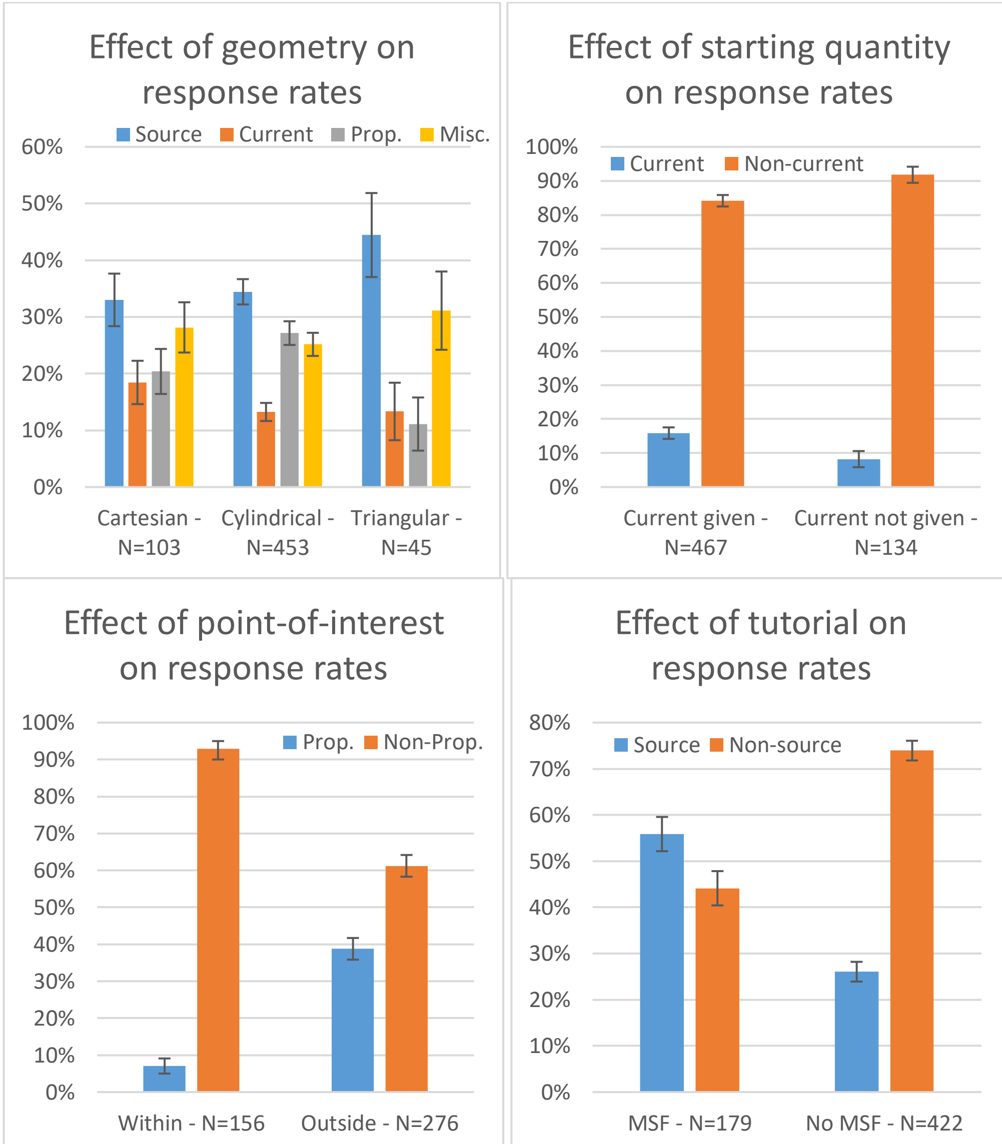

Figure 4-15: Comparison of student response rates on the magnetic vector potential task for each category.

There are individual features with some of the exams that could have contributed to the differences in response rates. The Winter 2016 exam occurred later in the quarter as a second exam. The timing placed it after students learn about the auxiliary field, which is another application of the source-based relationship. This could account for the much higher rate[i] of source-based reasoning. In

[i] +91% difference from expected of the average of all six exams, using the percent difference formula $\%diff = \frac{actual-expected}{expected}$.

comparison to the Summer 2018 exam[i], the proportion of students using source-based reasoning was significantly[ii] larger in Winter 2016, but the effect size is medium-small[iii], which is less than the point-of-interest and requirement of the *Magnetostatic Fields* tutorial[iv]. In Winter 2019, the question was presented as a comparison of magnitude of the vector potential between two points instead of evaluating the direction at a single point. Although the correct answer compared a point where $\vec{A}$ was zero to a point where it was non-zero, the comparison format elicited other types of reasoning, which helps account for the higher rate[v] of miscellaneous reasoning. In Winter 2020, the question first asked students to determine the direction of $\vec{A}$ where $\vec{B}$ was not zero (data not shown). Then, the question asked students to determine the direction of $\vec{A}$ outside where $\vec{B}$ was zero, which is compared above. Although conceptually the two questions should be answered in the same way, only 40% of students used the same reasoning type between the two parts. This helps support the idea that asking about $\vec{A}$ at a point where $\vec{B}$ is zero greatly increases[vi] the proportion of students using the proportionality misconception.

e. Reflections on curriculum development – *Magnetostatic Fields* and *Magnetic Vector Potential* tutorials

My research suggests that treating the $\vec{B} \rightarrow \vec{A}$ relationship solely by an analogy to the $\vec{J} \rightarrow \vec{B}$ relationship is ineffective compared to generalizing curl. Exam analysis shows that as students spend more time on curl applications in tutorial, their reasoning tends to shift toward the source-based approach.

[i] The most similar comparison since they both asked for a point inside and the tutorial was required, which by previous analysis were the most important factors.

[ii] The chi-squared test for independence on the 2x2 contingency table comparing the two exams to proportion of students using a source-based approach gives a p-value of 0.010. This allows one to reject the null hypothesis that the proportion of students using a source-based approach is independent of the differences between the exams.

[iii] The phi statistic for this 2x2 contingency table is 0.206, which falls between the 0.1 and 0.3 benchmarks for small and medium effect sizes, respectively.

[iv] The 2x2 contingency table comparing the two exams to proportion of students using a source-based approach gives a p-value of 0.010. The associated Phi value is 0.206, which is a medium-small effect size based on the guideline that 0.1 is small, 0.3 is medium, and 0.5 is large.

[v] +56% difference from expected.

[vi] +100% difference from expected.

This suggests that the *Magnetostatic Fields* tutorial is a major contributor to helping students build a coherent source-field model.

The largest distractor from a correct application of the $\vec{B} \to \vec{A}$ relationship is the notion that it describes a proportional relationship. This is most strongly elicited at locations where $\vec{B} = 0$, but $\vec{A}$ is non-zero. Currently, the *Magnetostatic Fields* tutorial elicits this misconception with a mock-student discussion focused more on the derivative nature of the relationship. However, even if students recognize that the derivative being zero does not imply that the field is zero, some students may still intuitively believe that the field could be zero for other reasons. An improvement to both the *Magnetostatic Fields* and *Magnetic Vector Potential* tutorials may be to focus more strongly on the spatial aspect of the source-field relationship with its link to the right-hand rule as applied to the Stokes' Theorem integrals, as this may bolster students' intuition about the direction of the field compared to its curl.

2. Applying the model to fields in matter

In the upper-division electromagnetism sequence, students are introduced to the displacement field $\vec{D}$ and auxiliary field $\vec{H}$ when dealing with polarizable or magnetizable materials, respectively. These *unfamiliar fields* provide an alternative pathway to determining the electric and magnetic fields, which are the *familiar fields* in the sense that students have worked with them before in introductory physics. Below I elaborate on fields in matter in the magnetostatic context, although each relationship has an electrostatic analogue.

a. Models for fields in matter

The equations describing the relationships involving the unfamiliar fields fall into three main categories: vector derivatives, vector addition, and linear approximations. Vector derivatives include divergence, curl, associated Green's Theorem integrals, and boundary conditions. Vector addition shows the relationship between the familiar field, unfamiliar field, and the dipole density as a sum or difference. Linear approximations include proportionality relationships between two of the three vector fields using a material property like the permeability μ, or the magnetic susceptibility χ_M. Consequently, most student responses that are articulate enough may also be categorized in this way.

i. Vector-derivative model for fields in matter

Using the source-field interpretation of the vector derivatives, one way to define the unfamiliar fields is by enumerating their divergence and curl. In statics[i], the derivatives for the auxiliary field are $\vec{\nabla} \cdot \vec{H} = -\vec{\nabla} \cdot \vec{M}$ and $\vec{\nabla} \times \vec{H} = \vec{J}_{free}$, whereas the derivatives for the magnetic field are $\vec{\nabla} \cdot \vec{B} = 0$ and $\vec{\nabla} \times \vec{B} = \mu_o \vec{J}_{net}$. The key difference is that the two fields are induced by the magnetization in different ways: the magnetization's divergence contributes to the auxiliary field as magnetic north and south poles[ii], whereas the magnetization's curl contributes to the magnetic field as free current.

In application, this allows one to compare two different source-field relationships and take the approach that is mathematically easier based on the magnetization, especially if either divergence or curl of the magnetization is not present. Then, one can take a vector addition approach to determine the other field. Another application of the differing vector derivatives is that they lead to complementary boundary conditions for the two fields. Assuming there is no free current at the boundary between two media, $\vec{\nabla} \cdot \vec{B} = 0$ requires that the perpendicular component of $\vec{B}$ is equal across the boundary, while $\vec{\nabla} \times \vec{H} = 0$ requires that the parallel component of $\vec{H}$ are continuous.

ii. Vector-addition model of fields in matter

The unfamiliar fields, familiar fields, and dipole densities are related by a vector addition equation: $\vec{H}_{net} = \frac{\vec{B}_{net}}{\mu_o} - \vec{M}$. One can also decompose the external and induced parts of the net fields to

[i] Including the time-dependent contributions complicates the model and is typically not introduced at the undergraduate level.

[ii] $\vec{\nabla} \cdot \vec{M}$ identifies the north and south poles of a magnet, which act as charge-like sources for the auxiliary field, such that the auxiliary field goes away from the north and toward the south, both inside and outside of the magnet. There is no formal name for the electrostatic analogue $\vec{\nabla} \times \vec{P}$, but it acts as a current-like source for the displacement field.

derive two other relationships: $\vec{H}_{ind} = \frac{\vec{B}_{ind}}{\mu_o} - \vec{M}$ which relates the fields due to the material, and $\vec{H}_{ext} = \frac{\vec{B}_{ext}}{\mu_o}$ which relates the fields due to external sources. The ideal application of these relationships is after one has determined one of the two fields from the sources of the material, such that the other field can be determined by a simple vector addition of two known quantities.

iii. Linear-approximation model for fields in matter

With diamagnetic and paramagnet material, the relationship between the net unfamiliar field, net familiar field, and the dipole density can be simplified to proportionality relationships. For responsive magnetic materials, the relationships are: $\mu\vec{H}_{net} = \vec{B}_{net}$ and $\vec{M} = \chi_M\vec{H}_{net}$. These relationships are simplifications of the vector-addition relationships because the materials they describe are modeled to respond to net fields with predictable dipole densities along the same axis of the field. However, these relationships do not describe non-responsive materials like permanent magnets. They also cannot be decomposed to show the induced fields, as the dipoles are modeled to respond to both the induced and the external fields collectively.

b. Tasks for determining fields in matter

A standard task that elicits student understanding of the fields in matter is to ask students to determine the direction of an unfamiliar field at various locations[i] near polarized or magnetic material. Only tasks that ask about the field inside of the material were analyzed in depth, as that is where the familiar and unfamiliar fields differ.

[i]Only tasks that ask about the field inside of the material were analyzed in depth, in order to make the comparison between tasks fair and consistent.

i. Polarized hoop task

On the third exam of Phys321 in Autumn 2015 <App – 143>, students were given an isolated hoop with fixed azimuthal polarization (Figure 4-16). The task is to determine the direction of the electric and displacement fields at a point within the hoop.

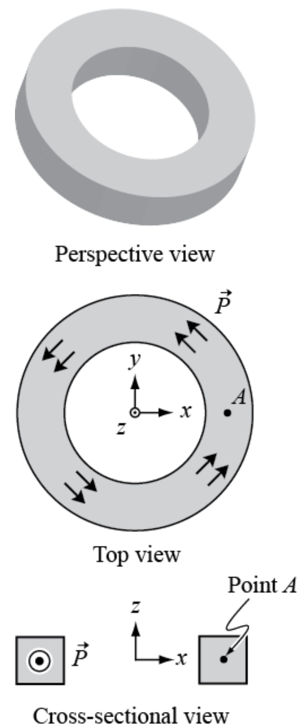

Figure 4-16: Context for the polarized hoop task.

The intended approach for this task is to use the vector-derivative model. The azimuthal polarization has no divergence, but it has curl. Since $\vec{\nabla} \cdot \vec{P} = -\rho_{bound}$, there are no bound charges to source any electric field with, but $\vec{\nabla} \times \vec{P} = \vec{\nabla} \times \vec{D}$ implies that the curl will source a similar displacement field in the same azimuthal direction as the polarization. Thus, the electric field at point *A* is zero, while the displacement field at point *A* points upwards.

ii. Bar-magnet task

The context for the bar-magnet task is a ferromagnetic rectangular prism with fixed magnetization, as shown in Figure 4-17. The task is to determine the direction of the auxiliary field at the center of the magnet. This task was given on a pretest to the *Auxiliary Field* in Phys322 <App – 67>.

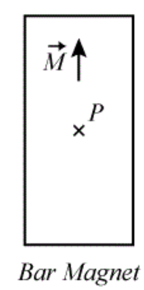

Figure 4-17: Context for the bar-magnet task.

In the pretest, students were given the context and first asked which of the following approaches they would prefer to take to determine the direction of the auxiliary field at the center: vector addition, vector derivative, or linear approximation. Regardless of their preference, they were asked questions about each approach in turn, with the prompt clearly stating that their answers from the various approaches may be inconsistent. For the vector-addition approach, they were first asked to determine the direction of $\vec{B}$, and then determine the direction $\vec{H}$ based on $\vec{H} = \frac{\vec{B}}{\mu_o} - \vec{M}$. For the vector derivative approach, students were asked to determine where the auxiliary field had divergence and curl, and then determine the direction of

$\vec{H}$. For the linear approximation approach, students were asked to determine the direction of $\vec{B}$, the sign of μ, and then determine the direction $\vec{H}$ based on $\vec{H} = \frac{\vec{B}}{\mu}$. Most questions had "cannot be determined" as an answer choice.

Since the structure of the pretest was so different than the exams, the results were filtered to report only on the direction auxiliary field using the students' preferred method. This mimics the exam environment because students typically only provide one line of reasoning to explain their answer. Had this been an exam question, the intended approach for this task would be to use the vector-derivative model. One evaluates divergence conceptually by noting that the magnetization starts at the bottom (south pole, where $\vec{\nabla} \cdot \vec{M}$ is positive) and ends at the top (north pole, where $\vec{\nabla} \cdot \vec{M}$ is negative), and relates that to the divergence of the auxiliary field by $\vec{\nabla} \cdot \vec{H} = -\vec{\nabla} \cdot \vec{M}$. Then, one uses the divergence-based source-field relationship to determine the direction of the auxiliary field by noting that the field that is produced by the top (north pole, where $\vec{\nabla} \cdot \vec{H}$ is positive) points outward and the field produced by the bottom (south pole, where $\vec{\nabla} \cdot \vec{H}$ is negative) points inward. These contributions both point downward at the center of the magnet, while there is no contribution to the induced auxiliary field from curl.

iii. Cube-in-field task

The context for the cube-in-field task is a responsive material placed in a uniform external field, *e.g.*, a diamagnetic cube in the presence of a uniform external magnetic field, as shown in Figure 4-18. The task has two variants: (1) compare the magnitude of the net and external unfamiliar fields which was given on the third exam of Phys321 in Spring 2019 <App – 144>, and (2) determine the direction of the induced unfamiliar field at the center of the cube which was given on the second exam of Phys322 in Winter 2020 <App – 145>.

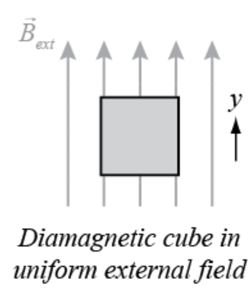

Figure 4-18: Example context for cube-in-field task.

The intended approach for this task is essentially the same as with the bar-magnet task after one determines the direction of the dipole density based on the property of the material. The comparison variant adds a layer of superposition to the reasoning. If both the induced field and the external field point in the same direction, then the magnitude of the net field will be greater than the magnitude of the external field alone.

iv. Horseshoe-magnet task

The context for the horseshoe-magnet task is an object bent in the shape of a C with fixed azimuthal magnetization (Figure 4-19). The task is to determine the direction of the auxiliary field at point *P*. This was given on the second exam of Phys322 in Winter 2017 <App – 146> and Summer 2019 <App – 147>.

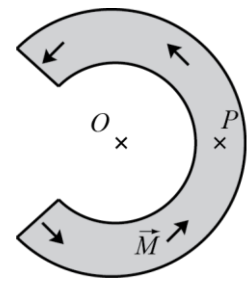

Figure 4-19: Context for the horseshoe-magnet task.

The intended approach for this task is essentially the same as the bar-magnet task, although the geometry adds a layer of complication. In quick summary, one uses the magnetization to identify the north and south poles, and then uses the poles to determine the auxiliary field, since the source of the induced auxiliary field is divergence-based with $\vec{\nabla} \cdot \vec{H} = -\vec{\nabla} \cdot \vec{M}$. The north pole at the top end of the magnet produces an auxiliary field downward and rightward at point *P*, while the south pole at the bottom produces an auxiliary field downward and leftward at point *P*. By superposition and symmetry, the horizontal components cancel so the induced auxiliary field points downward at point *P*.

c. Common student responses for determining fields in matter

Students typically justify their response using equations and/or interpretations from the vector-derivative model, vector-addition model, and linear-approximation model.

i. Common student responses with the vector-derivative model

Most correct student responses that use the vector-derivative approach are not as thorough as the intended approaches with the vector-derivative model described above, but they convey the idea that regions of the material act as sources for the unfamiliar field. There are a few responses that quote $\vec{\nabla} \cdot \vec{H} = -\vec{\nabla} \cdot \vec{M}$ and directly conclude that the induced auxiliary field is in the opposite direction as the magnetization. The lack of a description about sources suggests that students may be interpreting the equation as a linear relationship between the unfamiliar field and the dipole density. Nonetheless, these responses use the relevant vector derivative to determine the direction of the unfamiliar field inside the material productively.

The most common misconception that is elicited with the unfamiliar fields is the isolation of one vector derivative from the other, as the unfamiliar fields are the one of the first instances of physical vector fields that students encounter with both types of sources. For example, many students arrive at the conclusion that the auxiliary field is zero when there is no free current, neglecting the statement that the magnetization and auxiliary field are related through divergence. This leads to an incorrect conclusion that the auxiliary field only describes an external or applied field. This line of reasoning is likely a result of the overgeneralization of cases where bound current complicate determining the magnetic field while the magnetization has zero or negligible divergence, such that concluding that the induced auxiliary field is zero or negligible is a useful strategy for determining the magnetic field.

Additionally, some students do not incorporate the spatial aspect of the source-field relationship, which is a misconception that is present with the magnetic vector potential as well. They recognize that both divergence and curl define the behavior of the field, but they imply that if both are zero at the point-of-interest, the field is zero as well.

ii. Common student responses with the vector-addition model

Within the vector-addition model, many students misapply the vector-addition relationships by making assumptions about the relative magnitude of the quantities they are adding, either explicitly or implicitly. For example, if $\vec{M}$ points downwards, some students argue that the $\vec{H}_{ind}$ points upwards as well, because $\vec{H} = \frac{\vec{B}}{\mu_o} - \vec{M}$ shows a subtractive term. Others argue that the $\vec{H}_{ind}$ points downwards, since the $\vec{B}_{ind}$ from the bound current points downward. Others note that because $\vec{B}_{ind}$ and $\vec{M}$ point in opposite directions, they will or must cancel such that $\vec{H}_{ind}$ is zero. Consequently, students tend to apply the vector-addition relationships as a reasonableness argument[i] for their intuition, circumventing a proper proof by the vector-derivative model.

Additional complications are elicited when students are asked to compare the magnitude of a net field to the corresponding external field. With a proper interpretation of the superposition of effects from sources, this type of question essentially asks students to compare the direction of the external fields and induced fields as they superimpose $\vec{\psi}_{net} = \vec{\psi}_{ext} + \vec{\psi}_{ind}$. However, some students misrepresent the vector-addition relationships by conflating net and external fields. For example, in their interpretation of $\vec{H} = \frac{\vec{B}}{\mu_o} - \vec{M}$, they claim that $\vec{H}$ increases with the presence of $\vec{M}$ if in the opposite direction of $\vec{B}_{ext}$ since $\vec{H}_{ext} = \frac{\vec{B}_{ext}}{\mu_o}$. This suggests that some students may be implying that $\vec{H}_{net} = \frac{\vec{B}_{ext}}{\mu_o} - \vec{M}$, but could also be a variant of the compensation reasoning as they failing to note or recognize that $\vec{B}_{net}$ may be different than $\vec{B}_{ext}$ with the presence of $\vec{M}$. A few students misinterpret the superposition of effects as separation of regions. For instance, they may take the work "external" to mean the net field outside of the object rather than the part of the field caused by sources external to the objects, so they end up answering a different question.

[i] This type of compensation reasoning was also present with determining divergence and curl from mathematical or visual modes of reasoning.

iii. Common student responses with the linear-approximation model

Lastly for the linear-approximation model, many students overapply the relationships beyond the limitation of the model, as they incorrectly attempt to describe the fields of fixed polarization or magnetization. The relationship between the induced fields and these materials cannot be reduced to simple proportionality relationships, as the linear-approximation model relies on the material to be responsive to an external field.

Other students show misconceptions similar to those present in the vector-addition model. For instance, some students claim that $\vec{H}$ decreases because $\vec{B}$ decreases since they are proportional by $\vec{H}_{net} = \frac{\vec{B}_{net}}{\mu}$. Conversely, other students claim that $\vec{H}$ increases in the material because $\mu < \mu_o$ for diamagnetic material. In this way, students may be building improper intuition on these proportionality relationships and justifying their intuition with faulty reasonableness arguments. This could also be interpreted as conflation between induced, external, and net fields with the proportionality relationship. The external proportionality relationship $\vec{H}_{ext} = \frac{\vec{B}_{ext}}{\mu_o}$ uses μ_o as it does not incorporate effects of the material, while there is no proportionality relationship between induced fields inside of the material. Students who are unable to distinguish between the subdivisions of the field may be applying a proportionality relationship incorrectly, like $\vec{H}_{net} = \frac{\vec{B}_{ext}}{\mu}$.

d. Analysis of student responses for determining fields in matter

Table 4-8: Breakdown of student responses on fields-in-matter tasks. For version, v1, v2, and v3 correspond to the preliminary, sources-in-isolation, and dipole-sources versions of the tutorials, respectively. For the columns showing the approach taken, the first percentage represents the accuracy of that approach, while the second in (X% of N) represents the proportion of the sample size that took the given approach.

Version	Timing	N	Overall correct	Vector-derivative approach	Vector-addition approach	Linear-approx. approach	Misc or blank	% VDA of correct[i]
Polarized hoop task – Electric field								
v1	Phys321 Exam 3 15(4Aut)	88	10%	60% (17% of N)		0% (38% of N)	0% (45% of N)	100%
Polarized hoop task – Displacement field								
v1	Phys321 Exam 3 15(4Aut)	87	60%	21% (16% of N)	74% (49% of N)	62% (30% of N)	25% (5% of N)	6%
Bar-magnet task								
v2	Phys322 Pretest	244	27%	30% (32% of N)	25% (54% of N)	24% (14% of N)		37%
Cube-in-field task								
v2	Phys321 Exam 3 19(2Spr)	96	34%	8% (25% of N)	59% (28% of N)	33% (22% of N)	33% (25% of N)	6%
v2	Phys322 Exam 2 18(1Win)	154	47%	45% (21% of N)	69% (40% of N)	24% (25% of N)	32% (14% of N)	21%
v3	Phys322 Exam 2 20(1Win)	170	52%	73% (41% of N)	47% (38% of N)	9% (6% of N)	28% (15% of N)	57%
Horseshoe-magnet task								
v2	Phys322 Exam 2 17(1Win)	126	33%	31% (40% of N)	53% (34% of N)	5% (17% of N)	20% (8% of N)	38%
v3	Phys322 Exam 2 19(3Sum)	33	36%	55% (33% of N)	40% (30% of N)	0% (18% of N)	33% (18% of N)	50%

The data from the fields-in-matter tasks are shown in Table 4-8. Figure 4-20 through Figure 4-23 show graphical representations of this data when relevant. The two most important statistics are the proportion of student responses that used a vector-derivative approach and the accuracy within that approach, since the intended approach for these tasks is to use a vector-derivative model. Overall accuracy is also important, although for these tasks it seems to measure intuition, since there is a high proportion of responses that rely on reasonableness arguments and/or assumptions made in the vector-

[i] Proportion of correct responses that used a vector-derivative approach, *i.e.*, $\frac{\%cofVDA * \%VDAofN}{\%overallcorrect}$.

addition and linear-approximation approaches to justify the answer. In the last column of Table 4-8, the proportion of correct responses that used a vector-derivative approach is an interesting statistic which likely correlates with sophistication of reasoning, as higher proportions suggest that more students with the correct intuition can justify their answer with the vector-derivative model instead of relying on reasonableness arguments.

There are a couple trends that are evident from this set of data. Overall, the vector-addition approach tends to be the most common approach taken by students. This is likely because students tend to interpret the vector-addition relationships of the displacement and auxiliary fields as the definition[i] of the fields. The accuracy of the vector-derivative approach is the lowest on the displacement field questions, which indicates a high proportion of students who claim no free charge means no displacement field. This is likely due to the timing of the context, as the electrostatic-fields-in-matter context occurs before magnetostatics where the physical concepts of curl are developed.

Other comparisons are more assessment oriented, so they will be discussed in the following section which elaborates on differences between the tutorial versions.

e. Development of the *Displacement Field* and *Auxiliary Field* tutorials

The versions of the *Displacement Field* and *Auxiliary Field* tutorials are divided into three groups. The preliminary versions span from Autumn 2014 to Autumn 2015 and represent the initial adaptation of the CU-Boulder tutorials. These tutorials were rewritten in Winter 2016 to focus on the presence or absence of the vector derivatives and how that affects the fields, which I label as the "sources-in-isolation" version. They were rewritten again in in Summer 2019 to focus on distinguishing between curl-based and divergence-based dipole patterns, which I label as the "dipole-sources" version.

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[i] The equations $\vec{D} = \epsilon_o \vec{E} + \vec{P}$ and $\vec{H} = \frac{\vec{B}}{\mu_o} - \vec{M}$ match the Symbolic Form for the definition, $= \cdots$.

i. Preliminary versions of the *Displacement Field* and *Auxiliary Field* tutorials

Like many of the other tutorials in the Phys 321-322 sequence, the preliminary *Displacement Field* and *Auxiliary Field* tutorials are adaptations of the CU-Boulder[2] tutorials for use at UW. However, CU-Boulder did not have tutorials dedicated to fields in matter. Rather their tutorial *Spherical Linear Dielectric* focused on distinguishing between free and bound charges and properties of the electric field, but it only mentions the displacement field once as part of a derivation. Similarly, their tutorial *Magnets* discusses bound current as a model, but does not mention the auxiliary field at all.

The first version of the *Displacement Field* in Autumn 2014 <App – 100> began with a review of the proceeding tutorial, *Polarization,* using the context of a dielectric slab within a parallel plate capacitor. This was intended to help develop the idea that the displacement field ignores contribution from the bound charge. The second part investigated the fields of an object with fixed polarization, starting with a student statement claiming that the displacement field is zero if there are no free charges. This was intended to elicit ideas about the displacement field forming closed field lines, although curl was never mentioned. The third part investigated the electric and displacement fields of a half-filled parallel plate capacitor, a similar context to the half-filled spherical capacitor done in CU-Boulder's *Spherical Linear Dielectric*. This was intended to highlight the differences between the electric and displacement fields.

Only the first part of the preliminary *Auxiliary Field* tutorial <App – 104> focused on the auxiliary field. This part started by exploring bound current and its effect on the magnetic field. Then, it introduced the vector-addition relationship $\vec{H} = \frac{\vec{B}}{\mu_o} - \vec{M}$ and asked students to derive and interpret the curl of the auxiliary field. Later parts focused on paramagnetic and diamagnetic properties and did not focus on the auxiliary field, although they referenced the linear-approximation relationship $\vec{M} = \chi_M \vec{H}$.

ii. Assessment of the preliminary versions

Informal observation in the classroom suggested that student understanding of curl and its implication with Stokes' Theorem in the *Displacement Field* tutorial was lacking, which was consistent with CU-Boulder's reflections[2] that "students often did not know that [Stokes'] loops should integrate to zero." This drove development of the preliminary version to include the derivation of $\vec{\nabla} \times \vec{D} = \vec{\nabla} \times \vec{P}$ and its effect on $\vec{D}$ across boundaries. The updated Autumn 2015 version compared the two different half-filled capacitors (one horizontally split versus one vertically split) to highlight the difference in how polarization affects the two fields.

For formal assessment, the polarized hoop task was developed and given in the third midterm of Autumn 2015. The breakdown of student responses on this task is shown in Figure 4-20. The question was written to test if students could recognize that the electrostatic field could not form closed loops because $\vec{\nabla} \times \vec{E}_{stat} = 0$.

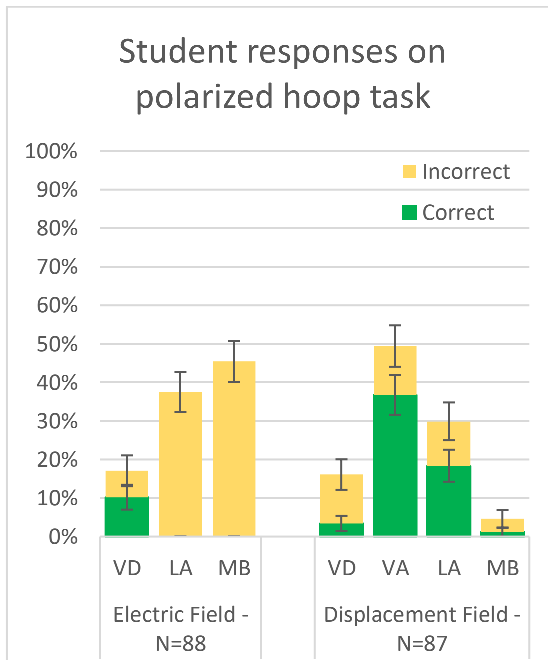

Figure 4-20: Breakdown of student responses on polarized hoop task. VD, VA, LA, and MB stand for vector-derivative approach, vector-addition approach, linear-approximation approach, and miscellaneous/blank, respectively.

However, only 10% of the students correctly identified that the electric field must be zero, all of which used a vector-derivative approach. The remaining 90% of the class gave a non-zero, azimuthal direction for the electric field. The low accuracy suggests that the task is very counterintuitive, but also highlights major shortcomings in student understanding of curl.

The high proportion of miscellaneous responses for the electric field is due to many students using heuristics about the relative directions of the electric field[i] and polarization, which are not characteristic of student reasoning for displacement and auxiliary fields in similar tasks. The heuristic could be related to the vector-derivative approach because bound charges are positively related to the $\vec{\nabla} \cdot \vec{E}$ but negatively related to $\vec{\nabla} \cdot \vec{P}$, but it could also be an interpretation of the linear-approximation model. The high prevalence of the use of heuristics suggests that student understanding of the underlying mechanism for a comparison (*i.e.,* vector derivatives or approximation models) may be lacking.

For the displacement field, 60% of the class identified the correct direction, but only 5% used a vector-addition approach. The 15% of students that used the vector-derivative approach were only 20% accurate, as 70% of students using the vector-derivative approach stated that the displacement field is zero because there are no free charges. Students who used a vector-addition approach were 75% accurate. This high accuracy is attributed to a productive use of the relationship $\vec{D} = \epsilon_o \vec{E} + \vec{P}$ in the correct case where $\vec{E} = 0$, but also if students incorrectly determined that the electric field is parallel with the polarization.

iii. Sources-in-isolation version of the *Displacement Field* and *Auxiliary Field* tutorials

Both informal observation in the classroom and exam data suggested that the preliminary versions of the *Displacement Field* tutorial were not very effective. Despite the gradual shift in focus toward curl and Stokes' integrals, students were not understanding the role and impact of curl and were implying that electric field lines can form a complete loop in electrostatics. The high rate of student responses that depended on heuristics between $\vec{E}$ and $\vec{P}$ or inappropriately applied the linear-approximation model suggested that students did not understand the source-field relationship enough to be able to apply it in the context of fields in matter, even for the electric field. Thus, the intent of the first

[i] Some explicitly state induced, some explicitly state net, and some leave it unlabeled.

major rewrite of the *Displacement Field* tutorial <App – 108> was to concentrate on developing the concept that vector derivatives describe the sources of the field. The preliminary *Auxiliary Field* tutorial was divided into two tutorials. The *Magnetization* tutorial focused on dipoles and magnetic properties, while the new *Auxiliary Field* tutorial <App – 113> mirrored the *Displacement Field* tutorial, reinforcing the concept of vector derivatives as sources.

The Spring 2016 version of the *Displacement Field* tutorial started with the vector-addition relationship $\vec{D} = \epsilon_o\vec{E} + \vec{P}$. Students were asked to derive and interpret the vector derivatives of $\vec{D}$ and identify where the polarization had divergence and curl on a finite polarized slab. The core part of the tutorial had students investigate two cases: a radially polarized sphere, and the polarized hoop task. These cases represent the two extremes of sources from the polarization, where the sphere only has divergence that sources electric field while the hoop only has curl that sources displacement field. The supplement covers the idea that the external displacement field created by the divergence-based free charges can be superimposed with the induced displacement field created by the curl of the polarization, which essentially describes Helmholtz' Theorem.

The *Auxiliary Field* tutorials from Summer 2016 on were conceptually identical to the *Displacement Field* tutorials, but with the magnetostatic analog. Minor modifications were made to both tutorials in parallel through Spring 2019 to address clarity in language and provide better scaffolding. In particular, students needed help concluding that a vector field is zero if both its vector derivatives are zero using the source interpretation, as they tended to reach the conclusion that a vector field can be uniform based on a mathematical interpretation of no derivatives means no change.

iv. Assessment of the sources-in-isolation version

With the given data, the sources-in-isolation version can be assessed in two ways. There are posttests for both the preliminary and the sources-in-isolation versions of the *Displacement Field* tutorial, so one can do a post-post comparison of the two exams. There are also data from three different instants throughout the course sequence, so one can do a longitudinal comparison between the posttest of the

Displacement Field tutorial, the pretest of the *Auxiliary Field* tutorial, and the posttest of the *Auxiliary Field* tutorial.

i. Post-post assessment of the *Displacement Field* tutorial

The third exam in Spring 2019 gave students the cube-in-field task, with the context of a dielectric cube between parallel plate capacitors. This was a comparison variant of the task, where students were asked to compare the magnitude of the net electric and displacement fields at the center of the cube to the magnitude of the respective external field. The explanation for both fields were combined into a single prompt.

Figure 4-21 shows the breakdown of student responses on this task. 25% of the students used a vector-derivative approach, of which 5% answered correctly and 75% claimed that the induced displacement field was zero. The rate of miscellaneous/blank explanations was higher than expected, likely because of the formatting of the question. Several students only provided an explicit explanation for the electric field, even though the explanation prompt asked them to “explain <their> answers for the previous two questions.”

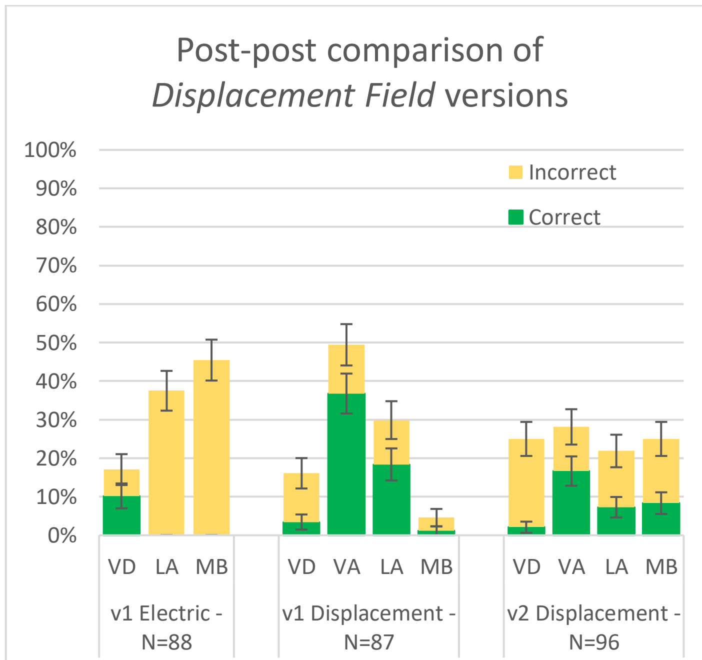

Figure 4-21: Breakdown of student responses on polarized hoop task in 15(4Aut) for the preliminary version (v1) and cube-in-field task in 19(2Spr) for the sources-in-isolation version (v2). See Figure 4-20 caption for details.

It is not a perfect comparison between the displacement field question of the two tasks because the context is very different, but the statistics for the vector-derivative approach are still insightful. It is unclear if there is a significant[i] difference between the two questions for the relative proportion of students taking a vector-derivative approach, but there is no significant[ii] difference in accuracy for students who used that approach. Both questions also only had 5% of correct responses that used a vector-derivative approach. This suggests that the sources-in-isolation version of the *Displacement Field* tutorial may have helped in drawing student attention toward the vector-derivative model, but it was not effective at helping students apply the model correctly. The similarity in low statistics on both displacement field questions could be because this topic comes before magnetostatics, in that students may not have had enough practice with curl to be able to fully understand the vector-derivative model as it applies to the displacement field.

[i] The chi-squared test for independence on the two 2x2 contingency tables comparing proportion of students taking a vector-derivative approach between the two exams gives a p-value of 0.138 without filtering miscellaneous and blank responses, but a p-value of 0.0175 when responses are filtered. It is unclear if one is able to reject the null hypothesis that the proportion of students taking the vector-derivative approach is independent of differences between the context of the two exams.

[ii] The Fisher-exact test for independence on the 2x2 contingency table comparing accuracy of students who used a vector-derivative approach between the two exams gives a p-value of 0.337. One is unable to reject the null hypothesis that student accuracy with the vector-derivative approach is independent of differences between the contexts of two exams.

ii. Longitudinal study of the fields-in-matter task with the sources-in-isolation version

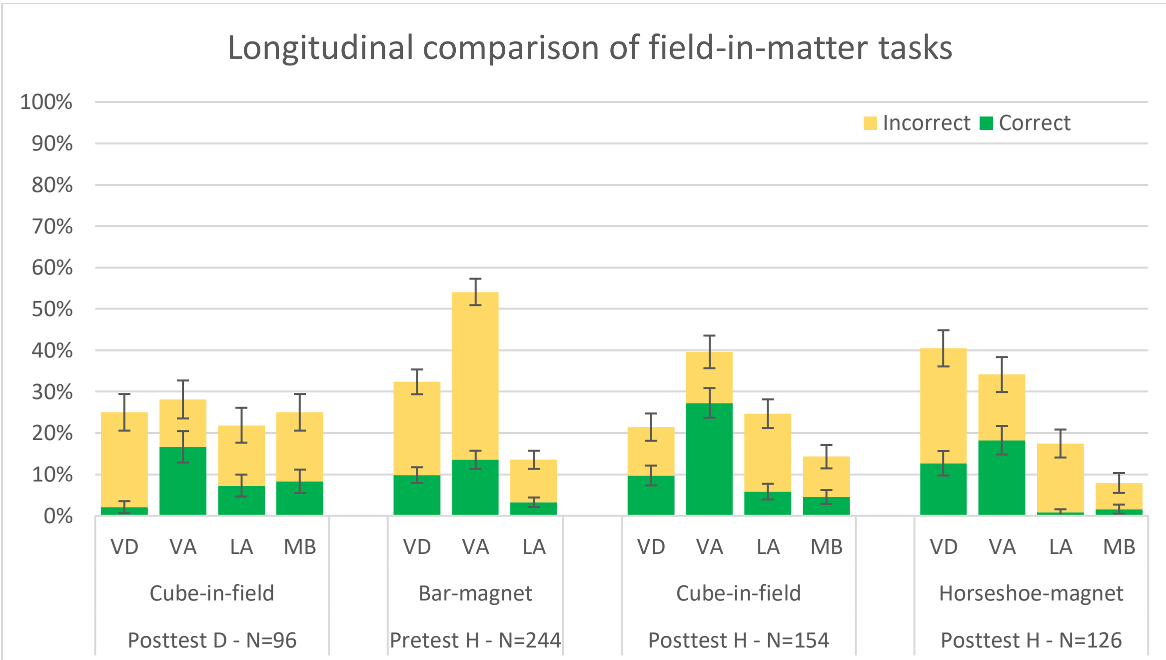

Figure 4-22: Breakdown of student responses for tasks with the sources-in-isolation version of the fields-in-matter tutorials. See Figure 4-20 caption for details.

Figure 4-22 shows a comparison of the breakdown of student responses for the longitudinal study of the fields-in-matter tasks with the sources-in-isolation version of the *Displacement Field* and *Auxiliary Field* tutorials. The key statistics-of-interest are: (1) the overall accuracy, (2) the proportion of responses using a vector-derivative approach, (3) the accuracy with that approach, and (4) the proportion of correct responses that use a vector-derivative approach.

In comparing student responses between the *Displacement Field* post-test in Spring 2019 and the *Auxiliary Field* pretest responses (1st two graphs of Figure 4-22), there is a significant[i] difference in the accuracy of the vector-derivative approach and the percent of correct answers that use the vector-

[i] The Fisher exact test for independence on the two 2x2 contingency tables comparing a statistic-of-interest between the two tasks gives p-values of 0.0328 for the accuracy of the vector-derivative approach and 0.00127 for the rate of vector-derivative approaches in correct answers. This allows one to reject the null hypothesis that these statistics are independent of differences in contexts for the two tasks.

derivative approach, but no significant[i] difference in the overall accuracy or the rate of responses using a vector-derivative approach. This suggests that instruction prior to the *Auxiliary Field* tutorial in Phys322 is not influencing intuition about fields in matter or attracting students to use a vector-derivative approach, but it is helping students develop proper reasoning with the vector-derivative approach. This supports the previous conclusion that students in Phys321 have not had sufficient practice with curl to fully understand the vector-derivative model as it applies to the displacement field.

In comparing student responses between the *Auxiliary Field* pretest to the cube-in-field question on the Winter 2018 exam (2nd and 3rd graphs of Figure 4-22), there is no significant[ii] difference in the accuracy of the vector-derivative approach, but the posttest significantly[iii] outperforms the pretest on overall accuracy. This suggests that the *Auxiliary Field* tutorial helps improve student intuition about the auxiliary field but not necessarily with a proper development of the vector-derivative model. On the other hand, the pretest significantly[iv] outperforms the posttest on both the proportion of all responses and correct responses that use a vector-derivative approach. These differences likely stem from some key differences between the question presentation rather than the *Auxiliary Field* tutorial. In particular, the posttest inadvertently[v] primed students to answer more mathematically such that the vector-addition and linear-approximation approaches were more appealing.

[i] The chi-squared test for independence on the two 2x2 contingency tables comparing a statistic-of-interest between the two tasks gives a p-value of 0.183 on the rate of the vector-derivative approach and 0.156 for the overall accuracy. One is unable to reject the null hypothesis that these statistics are independent of differences in contexts for the two tasks.

[ii] The chi-squared test for independence on the 2x2 contingency table comparing accuracy of the vector-derivative approach between the two tasks gives a p-value of 0.127. One is unable to reject the null hypothesis that the accuracy of the vector-derivative approach is independent of differences in contexts for the two tasks.

[iii] The chi-squared test for independence on the 2x2 contingency table comparing overall accuracy between the two tasks gives a p-value of 2.24e-5. This allows one to reject the null hypothesis that overall accuracy is independent of differences in contexts for the two tasks.

[iv] The chi-squared test for independence on the two 2x2 contingency tables comparing a statistic-of-interest between the two tasks gives a p-value of 0.0180 on the proportion of responses using a vector-derivative approach and 0.0330 for the proportion of correct responses using a vector-derivative approach. This allows one to reject the null hypothesis that these statistics are independent of differences in contexts for the two tasks.

[v] In the prompt of the exam question, students were notified that "$|\chi_M|$ is not small, such that $|\chi_M| \sim 1$." This was intended to remind student to not use approximations, but this backfired as many students interpreted this to mean "Use $|\chi_M| = 1$" as they attempted to calculate $\vec{H}$ mathematically.

In comparing student responses between the *Auxiliary Field* pretest to the horseshoe-magnet question on the Winter 2017 exam (2nd and 4th graphs in Figure 4-22), there are no significant[i] differences in any of the statistics-of-interest. The context between the bar magnet and horseshoe magnet are extremely similar and essentially differ only by geometry. This suggests that the sources-in-isolation version of the *Auxiliary Field* tutorial is not successful at impacting the way students understand the vector-derivative model or apply it to the auxiliary field.

Overall, the longitudinal study of the sources-in-isolation version of the fields-in-matter tutorials suggest that student improvement on the fields-in-matter tasks is largely attributed to the instruction between the *Displacement Field* posttest and the *Auxiliary Field* pretest, rather than these two tutorials. This suggests that more can be done with these tutorials to impact student understanding of the vector-derivative model.

v. Dipole-sources version of the *Displacement Field* and *Auxiliary Field* tutorials

The assessment to the sources-in-isolation versions of the tutorials shows improvement over the preliminary version of *Displacement Field* but shows stagnation of student performance in the second quarter with *Auxiliary Field*. The sources-in-isolation approach may be helping students learn about the importance of vector derivatives by exposing students to their use, but the lack of overall improvement with the auxiliary field suggests that approach is limited and can be improved.

There were a few limitations with the sources-in-isolation version. In terms of logical flow, the tutorial attempted to teach students to first determine the “easier” field based on the vector derivatives, and then determine the other field using the vector-addition relationship. However, the tutorial only used

[i] The chi-squared test for independence on the four 2x2 contingency tables comparing a statistic-of-interest between the two tasks gives a p-values of 0.122 and 0.903 on the proportion of all responses and correct responses using a vector-derivative approach, respectively, 0.178 for overall accuracy, and 0.905 on the accuracy of responses using a vector-derivative approach. One is unable to reject the null hypothesis that these statistics are independent of differences in contexts for the two tasks.

cases where the material either had no divergence or no curl, relying on the “no sources mean no field” argument to apply the source-based interpretation of the vector derivatives. Thus, one limitation was that students may learning that the vector derivatives describe sources, but they may not be learning how to generally apply non-zero divergence and curl to determine the field. The tutorial also used familiar geometries[i] as contexts. Although this may have helped reinforce model-building for the vector derivatives by reusing familiar contexts, this represented a limitation as students may not be learning to recognize or apply vector derivatives in less familiar contexts. This insight suggested that another rewrite to address some of the gaps in the sources-in-isolation version was necessary.

The intent of the second major rewrite of the *Displacement Field* and *Auxiliary Field* tutorials was to focus on how the material induces both the familiar and unfamiliar field, which is modeled by the vector derivatives. Since the materials are typically modeled to have no monopole term, the new tutorials focus on dipole fields, emphasizing the difference between a divergence-based dipole pattern and a curl-based dipole pattern. In this way, students would be able to use moderately familiar contexts to make meaning of how $\vec{\nabla} \times \vec{D} = \vec{\nabla} \times \vec{P}$ and $\vec{\nabla} \cdot \vec{H} = -\vec{\nabla} \cdot \vec{M}$ generate the unfamiliar fields.

The Summer 2019 version of the tutorial <App – 118> started with an exercise where students determine the electric field at various points around a charge dipole via superposition, and then sketch field lines. Then, they were asked to do the same thing for the magnetic field around a current dipole. Next, they were asked to analyze patterns between the two fields and consider how any heuristics they used to determine the field were related to Maxwell’s Equations.

The second part of the tutorial presented the vector-addition relationship in the context of a paramagnet in a uniform external field and a mock student discussion attempting to determine the induced auxiliary field direction from the induced version of the vector-addition relationship. The mock-student discussion included misconceptions about proportionality and assumptions about relative magnitudes.

[i] The radial vector field for divergence-only, and the circumferential vector field for curl-only.

The core part of the tutorial focused on the vector-derivative model for the auxiliary field. Students were first asked to derive the divergence and curl of the auxiliary field from the vector-addition relationship. Then, students were presented with five different vector field diagrams which included various dipole patterns, and they were asked to label the surfaces in the diagrams with divergence and/or curl. Lastly, students were asked to interpret the sources of the induced magnetic field and auxiliary field as bound current and N/S poles respectively, and then determine which of the fields given best represented each induced field.

Later versions <App – 122,127> of the two tutorials include an additional part in which students revisited the vector-addition relationship once they had correctly identified the induced fields. Students are asked to compare the vector derivatives of the familiar field and dipole density, intending to lead them to the conclusion that the magnitude of the dipole density is never less than the magnitude of the familiar field within the material, because the two vector derivatives of the dipole density contribute constructively. This is intended to allow students to correctly justify the direction of the unfamiliar field using the vector-addition approach with a heuristic based on the vector-derivative model.

vi. Assessment of the dipole-sources version

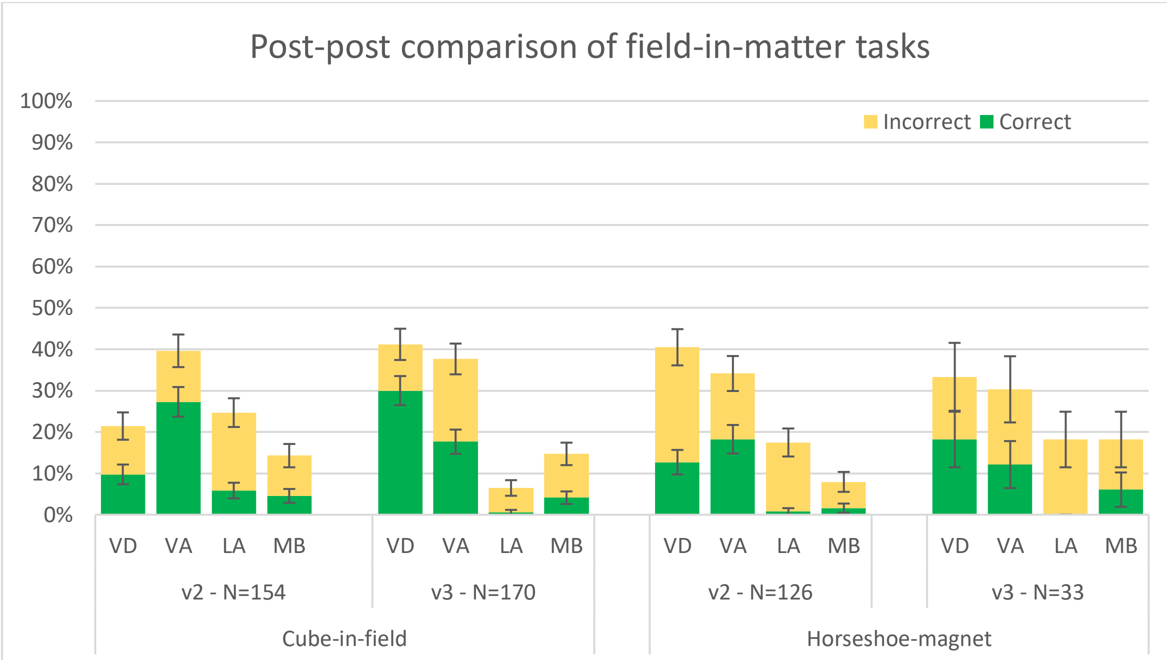

Figure 4-23: Post-post comparison of the breakdown of student responses on field-in-matter tasks between the sources-in-isolation version (v2) and the dipole-sources version (v3). See Figure 4-20 caption for details.

Figure 4-23 shows a comparison of posttests of the *Auxiliary Field* tutorial between the sources-in-isolation version and the dipole-sources version. The key statistics-of-interest are: (1) the overall accuracy, (2) the proportion of responses using a vector-derivative approach, (3) the accuracy with that approach, and (4) the proportion of correct responses that use a vector-derivative approach.

For the comparison of the cube-in-field task between the Winter 2018 and Winter 2020 exams (1st two graphs of Figure 4-23), there is no significant[i] difference in overall accuracy, but there are significant differences in the statistics relating to the vector-derivative approach. This suggests that the dipole-sources version of the tutorials is better than the sources-in-isolation version at teaching students the

[i] The chi-squared test for independence on the four 2x2 contingency tables comparing a statistic-of-interest between the two exams gives p-values of 0.373 for overall accuracy, 6.83×10^{-3} for accuracy with a vector-derivative approach, and 1.38×10^{-4} and 2.17×10^{-6} for the proportion of overall responses and correct responses that use a vector-derivative approach, respectively. One is unable to reject the null hypothesis that overall accuracy is independent of the version of the tutorial, but this allows one to reject the null hypothesis that the other statistics are independent of the version of the tutorial.

vector-derivative model for the fields-in-matter, although some of the difference could be attributed to a difference in the presentation of the task.

For the comparison of the horseshoe-magnet task between the Winter 2017 and Summer 2019 exams (last two graphs of Figure 4-23), there are no significant[i] differences in any statistic, but this is likely attributed to the small number of students in the Summer 2019 class lowering the power of the statistical test.

D. Summary and reflections on vector derivatives

The tasks used in the investigation of student understanding of vector-derivative model of sources in electrodynamics was split into four categories: (1) The interpretation of sources in differential relationships, (2) The determination of vector derivatives when sources were given, (3) The application of the vector-derivative model of sources to the magnetic vector potential, and (4) The application of the vector-derivative model of sources to fields in matter.

Data from the first category of tasks suggest that students have a strong tendency to interpret mathematical representations of vector derivatives from an operational perspective, which is inconsistent with the causal perspective that vector derivatives describe sources. On the other hand, students have a much stronger tendency to interpret the visual representation of curl from a causal perspective. The inconsistency of perspectives likely represents a hurdle to teach students to properly interpret the vector-derivative model of sources.

Data from the second category of tasks suggest that students rely on a diverse range of resources from physical, mathematical, and visual modes of reasoning to conceptually determine vector derivatives. Unfortunately, many students tend to develop misconceptions based on activation of a subset of

[i] The chi-squared test for independence on the three 2x2 contingency tables comparing a statistic-of-interest between the two exams gives p-values of 0.744 for overall accuracy, 0.454 for accuracy with a vector-derivative approach, and 0.454 and 0.459 for the proportion of overall responses and correct responses that use a vector-derivative approach, respectively. The Fisher exact test for independence on the 2x2 contingency table comparing the accuracy with a vector-derivative response gives a p-value of 0.176. One is unable to reject the null hypothesis that these statistics are independent of the version of the tutorial.

resources. This likely impedes those students from developing a coherent understanding of vector derivatives and the physical interpretation that vector derivatives can represent sources.

Data from the third category of tasks suggest that students tend to rely on one of three approaches to determine the magnetic vector potential from the magnetic field: a source-based approach going directly from $\vec{B} \rightarrow \vec{A}$ which demonstrates generalization of curl as a source, a current-based approach going from $\vec{B} \rightarrow \vec{J} \rightarrow \vec{A}$ which circumvents the generalization of curl as a source, and a proportional-based approach which implies the misconception that the magnetic field and magnetic vector potential are directly proportional to each other. The data suggest that additional tutorial instruction on the generalization of curl can increase the proportion of students that use the source-based approach, and the proportionality misconception is highly elicited in regions where the magnetic field is zero.

Data from the fourth category of tasks suggest that students tend to rely on one of three approaches to determine the unfamiliar displacement and auxiliary fields in matter: a vector-derivative approach, a vector-addition approach, and a linear approximation approach. A common misconception with a vector-derivative approach is that the unfamiliar field only depends on the free version of the source such that the material is unable to induce the unfamiliar field, while students using a vector-addition or linear approximation approach tend to rely on compensation reasoning to justify their answer. The prevalence of alternate approaches suggests that many students have difficulty understanding and applying the vector-derivative model of sources in unfamiliar contexts.

Insight gained from analysis of these tasks have guided the development of tutorials in Phys321 and Phys322 to help students better develop a coherent model of vector derivatives as sources. The current version of the *Divergence as a Source* tutorial in Phys321 presents divergence within the electrostatic context where it is easier for students to physically interpret, and it emphasizes that divergence describes the source of flux for a vector field. The current version of the *Magnetostatic Fields* and *Magnetic Vector Potential* tutorials in Phys322 attempt to help students transfer knowledge about familiar current-magnetic field relationship by showing how tools for the familiar context can be generalized to abstract curl relationships. The current version of the *Displacement Field* and *Auxiliary Field* tutorials attempt to help students distinguish between divergence-based and curl-based fields and learn to model how the material can source both the familiar electromagnetic fields and the unfamiliar fields with different vector derivatives.

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Chapter 5 Overall Impact and Reflections

Chapters 2 through 4 in this dissertation presented in-depth research based on specific tasks that I developed to probe student difficulties and the effect of curricula related to individual aspects of the source-field relationship. Each chapter concludes with a summary of the relevant research as related to student understanding of integration (Chapter 2), symmetry arguments (Chapter 3), and vector derivatives (Chapter 4). This chapter uses a more holistic lens by investigating how the entire course sequence has affected students' conceptual development of the source-field relationship through an assessment tool developed elsewhere.

I requested the use of the Colorado UppeR-division ElectrodyNamics Test[1] (CURrENT) from CU-Boulder as an externally developed assessment instrument to investigate the overall impact of the UW upper-division electromagnetism curricula on student understanding. The use of this tool provides an independent measure of the impact of the tutorials since tutorial development was not driven by the development of the instrument and vice versa. Additionally, the instrument has specific rubric guidelines to remove grader bias. In the following section, I discuss the implementation of the CURrENT at UW, overall results, and data analysis from select items that were particularly relevant to the aspects of the source-field relationship discussed in this dissertation.

A. Investigation of overall student conceptual development of the source-field relationship

The CURrENT developed by CU-Boulder seemed like a good assessment instrument for assessing the overall impact of *Tutorials in Physics: Electrodynamics*[2] on student conceptual development. Although *Tutorials in Physics: Electrodynamics* originated as an adaptation of CU-Boulder tutorials[3] for use at UW, we have modified and rewritten most of the tutorials to fit the UW physics department's course structure and student population, while developing them independently based on our research. On the other hand, CU-Boulder developed the CURrENT as an instrument to assess upper-division electromagnetism instruction more generally. Data from the CURrENT thus allow for us to compare the performance of UW students to statistics reported about other course structures and may help validate my research through tasks designed by other colleagues based on faculty-consensus learning goals.

1. Description of the CURrENT and administration at UW

CU-Boulder first created a free-response version of the CURrENT and later developed a multiple-choice version (MC_CURrENT)[4]. Preliminary results suggest that differences in student performance between the two versions were not significantly different, so the multiple-choice version was chosen here to facilitate administration and analysis.

There are pre-instructional (pretest) and post-instructional (posttest) versions[i] of the MC_CURrENT. The pretest is a 22-item instrument that is divided into 3 contexts: (1) Maxwell's equations, (2) Solenoid, and (3) Proof and Stokes' Theorem. The posttest is a 47-item instrument with 3 additional contexts: (4) Current carrying wire, (5) Charging capacitor, and (6) Electromagnetic waves. At UW, the MC_CURrENT pretest was administered once in Autumn 2019 Phys321 and replaced the first tutorial pretest of the course; the MC_CURrENT posttest was administered once in Winter 2020 Phys322 and replaced the last tutorial pretest of the course. Unlike the tutorial pretests which are typically given after relevant lecture instruction, the CURrENT pretest was administered prior to almost all course instruction of relevant concepts.

[i] The supporting documents are not provided in the appendix as they represent intellectual property of CU-Boulder, but one may request access to the instruments from their Physics Education Group. However, some of the items are discussed of the instrument are discussed with enough detail to show the relevance to my research without explicitly sharing the items.

2. Comparison of overall results of the CURrENT

One of the major benefits of using an assessment instrument is that one can compare results to published data. CU-Boulder provided rubric algorithms for both the pre- and post-test versions of the MC_CURrENT, which assigns points to each item and grades each student response, such that the data is processed objectively. However, only the results of the posttest from other universities have been published, so here I only present overall UW posttest results on the CURrENT. Additionally, the posttest contains additional items outside of the scope[i] of the pretest, so a pre-post comparison of overall rubric points is not meaningful. Instead, pretest results are discussed later as they relate to pre-post comparisons of select items related to aspects of the source-field relationship.

Table 5-1 shows a comparison of results between UW and data published in Ryan et. al.[1] UW students were shown to slightly underperform compared to published data on contexts (1) Maxwell's equations and (6) Electromagnetic waves, and perform well on context (5) Charging capacitor. The slight underperformance on context (1) may be partly attributed to a different conceptual focus of Gauss' Law in the tutorials than the concept tested for in the CURrENT, which is discussed in further detail in part 3a. below. The slight underperformance on context (6) is most likely because many concepts about electromagnetic waves that are tested for in the CURrENT are outside of the scope[ii] of Phys321 and Phys322. The relatively high performance on context (5) is likely due coverage of a very similar context in the *Energy Flow in Circuits* tutorial in Phys322, which is based on transformed CU-

Table 5-1: Comparison of CURrENT posttest scores by context. Scores from non-UW courses are estimated based on a graph published in Ryan et. al.[1]

Course	N	(1)	(2)	(3)	(4)	(5)	(6)	Overall
Traditional CU	22	64%	30%	51%	32%	41%	40%	44%
Transformed CU	47	83%	64%	76%	70%	82%	49%	71%
Transformed Other	43	74%	50%	66%	34%	57%	66%	58%
UW	125	61%	48%	55%	33%	71%	36%	51%

[i] Even within the first three contexts

[ii] The CURrENT is meant to test for concepts covered in the entire upper-division course sequence, and much of electromagnetic waves is covered in Phys323 at UW. However, Phys323 is an elective and has a smaller population of students compared to Phys322, so the CURrENT was chosen to be administered at the end of Phys322.

Boulder material with some adaptation. Overall, the UW scores fell between traditional CU-Boulder and transformed non-CU-Boulder scores.

3. Analysis of select items of the CURrENT at UW

Only a subset of the items on the CURrENT instrument are closely related to concepts discussed in this dissertation. A couple items in context (1) are related to symmetry and Gauss' Law, and a few items in contexts (2) through (4) are related to vector derivatives. No items are directly related to charge-based integrals, although a few items pertain to infinitesimal spatial elements.

For contexts (1) through (3), the relevant items on the pretest were near-identical to those on the posttest. Although a pre-post comparison of rubric points here would be valid, I have chosen to compare raw response rates instead to show in more detail how the development of the tutorials has affected student understanding of select concepts.

a. Use of surface with Gauss' Law

A pair of items on the CURrENT probe student understanding of using surfaces for applying Gauss' Law, which is somewhat related to ideas discussed in Chapter 3. The context for these items shows students an open surface with a few unlabeled vectors drawn. The first item asks if the surface can be used together with Gauss' Law in integral form. The second item asks students to choose supporting line(s) of reasoning from 5 choices. Although this task primarily probes for the understanding that Gauss' Law requires a closed surface, one of the supporting reasoning choices references the need for symmetry, while another references the relative angle between the electric field and area vector. These are items 4 and 5 on the pretest and the identical items 8 and 9 on the posttest in context (1) on Maxwell's equations.

Since the *Gauss' Law* and *Ampere's Law* tutorials focus on the use of symmetry and the simplification of the dot product within the integrals, students who have used the tutorials may be more likely to choose the symmetry and relative angle distractors in supporting their answer on the posttest than on the pretest.

Table 5-2: Breakdown of UW student responses on Gauss' Law items of CURrENT. These were items 4-5 on the pretest and 8-9 on the posttest. Percentages for supporting reasoning are shown as percentage of row total.

Version	Use of surface	Supporting reasoning contains:		
		Open surface	Symmetry	Relative angle
Pretest N=111	Correct – 58%	92% (of C)	5%	23%
	Incorrect – 41%	7% (of I)	39%	13%
Posttest N=125	Correct – 48%	87%	2%	33%
	Incorrect – 52%	6%	82%	14%

Table 5-2 shows a detailed breakdown of student responses, split by test version and accuracy on whether the open surface can be used with Gauss' Law. There is no significant[i] difference in accuracy between the two versions. There are also no significant[ii] differences in the proportion of students who answered correctly that chose supporting reasoning related to the open surface or relative angle between the electric field and area vectors. However, there is a significant[iii] difference in the proportion of students who answered incorrectly and chose supporting reasoning related to symmetry, an increase from 40% on the pretest to 80% on the posttest.

The data show that UW student responses due to instruction in Phys321 and Phys322 primarily change in the justification of the incorrect answer, with little change to the accuracy of the question, the proportion of students recognizing the correct reasoning, and the proportion of students choosing a distractor related to the dot product. This suggests that students have a much higher tendency to associate the need for symmetry with the ability to use Gauss' Law, likely due to the focus on symmetry in *Gauss' Law* and *Ampere's Law* tutorials.

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[i] The chi-squared test for independence on the 2x2 contingency table comparing accuracy between the pretest and posttest gives a p-value of 0.118. One is unable to reject the null hypothesis that the accuracy of students on this question is independent of instruction in Phys321 and Phys322.

[ii] The Fisher exact test for independence on the 2x2 contingency table comparing proportion of students with correct answers who supported their answer with reasoning related to the open surface between the pretest and posttest gives a p-value of 0.386. Likewise, the chi-squared test for independence on the 2x2 contingency table comparing the proportion of students with correct answers who supported their answer with reasoning related to the relative angle of the electric field and area vectors between the pretest and posttest gives a p-value of 0.221. One is unable to reject the null hypotheses that the student choice of supporting reasoning for correct answers on this question is independent of instruction in Phys321 and Phys322.

[iii] The chi-squared test for independence on the 2x2 contingency table comparing the proportion of students with an incorrect answer that supported their answer with reasoning related to symmetry between the pretest and posttest gives a p-value of 4.55×10^{-6}. This allows one to reject the null hypothesis that the proportion of students who support an incorrect answer on this question based on symmetry is independent of instruction in Phys321 and Phys322.

b. Determining field from curl

A set of questions on the CURrENT ask students to choose a graphical representation that best matches the field(s) of a solenoid from 6 choices, which involve the skill of determining the field from a curl-based source. This was the focus of much of the research in Chapter 4. These are item 10 on the pretest and items 14 and 17 on the posttest in context (2) Solenoid. On the pretest, item 10 asks students to choose the graph that best represents the magnetic field from a steady current carrying solenoid. On the posttest, item 14 asks students to choose the graph that best represents the magnetic field from a time-dependent current carrying solenoid at one instant in time, and item 17 asks the same for the electric field.

Table 5-3 shows the breakdown of student choices of the graphs depicting the magnetic field, organized by the functional dependence of the field within and outside the solenoid. There is a significant[i] difference in accuracy, an increase from 20% on the pretest to 35% on the posttest. For the functional dependence of the field for each region, there is no significant[ii] difference inside of the solenoid, but there is a significant[iii] difference outside, an increase from 20% to 50%.

Table 5-3: Breakdown of UW student responses for Magnetic-Field-of-Solenoid items on MC_CURrENT. These were item 10 on the pretest and 14 on the posttest.

Pretest N=108				
	Uniform inside	Linear inside	Misc. inside	Row total
Zero outside	18%	5%		22%
Decay outside	43%	10%	22%	75%
Column total	60%	15%	22%	
Posttest N=124				
	Uniform inside	Linear inside	Misc. inside	Row total
Zero outside	36%	14%		50%
Decay outside	20%	25%	4%	49%
Column total	56%	39%	4%	

[i] The chi-squared test for independence on the 2x2 contingency table comparing accuracy between pretest and posttest gives a p-value of 1.11×10^{-3}. This allows one to reject the null hypotheses that student accuracy on the magnetic field of a solenoid is independent on instruction in Phys321 and Phys322.

[ii] The chi-squared test for independence on the 2x2 contingency table comparing accuracy inside of the solenoid between pretest and posttest gives a p-value of 0.519. One is unable to reject the null hypotheses that student accuracy about the magnetic field inside the solenoid is independent on instruction in Phys321 and Phys322.

[iii] The chi-squared test for independence on the 2x2 contingency table comparing accuracy outside of the solenoid between pretest and posttest gives a p-value of 4.47×10^{-7}. This allows one to reject the null hypotheses that student accuracy on the magnetic field outside the solenoid is independent on instruction in Phys321 and Phys322.

The data suggest that instruction in Phys321 and Phys322 is somewhat effective at teaching students about the magnetic field of the solenoid, but primarily for the region outside. Many students intuitively think that the magnitude of the field always decays with distance from the source. Although this is typically true and serves as a useful heuristic for conceptually applying superposition, the geometry of the infinite solenoid and infinite sheets breaks this intuition. Thus, the increase in performance likely corresponds to students gaining familiarity with the application of symmetry and Ampere's Law in these contexts. For the spatial dependence of the magnetic field inside of the solenoid, students have likely encountered solenoids in introductory electromagnetism, where they may have learned that the field within a solenoid is approximately uniform and non-zero. This may explain why there is no difference between the pretest and posttest in upper-division electromagnetism.

One nuance with this task is that it does not prompt for explanation, which means that students may be using rote recall rather than deriving or justifying the result. One of the distracting answer choices shows a graph that has a linear increase of the field inside the solenoid and a decay outside, which is a behavior that students routinely see[i] with cylindrical geometry in the course sequence. There is a significant[ii] difference on the proportion of students who choose this distractor, from 10% on the pretest to 30% on the posttest. This suggests that many students may simply be recalling familiar graphs for this task.

Table 5-4 shows the distribution of student responses in terms of accuracy for selecting the graphical representation of the magnetic and electric field of a time-dependent solenoid on the posttest. 35% of students

Table 5-4: Breakdown of UW student responses on Fields-of-Solenoid items on posttest MC_CURrENT. These were items 14 and 17 on the posttest.

N = 125	Correct E	Incorrect E	Row total
Correct B	14%	22%	36%
Incorrect B	21%	43%	64%
Column total	34%	66%	

[i] *e.g.,* the magnetic field of a current-carrying wire with uniform volume current density.

[ii] The chi-squared test for independence on the 2x2 contingency table comparing the proportion of students selecting the choice of linear inside and decaying outside between pretest and posttest gives a p-value of 3.80×10^{-3}. This allows one to reject the null hypotheses that the proportion of students choosing this answer choice is independent on instruction in Phys321 and Phys322.

correctly determined the spatial dependence of the electric field, but only 15% of all students correctly determined both fields. Strangely, accuracy on the two problems is statistically independent[i], even though logically speaking correctly determining the electric field should be dependent on correctly determining the magnetic field first. Additionally, the use of rote recall seems to be independent[ii] for the two fields.

The data suggest that students are may not be consistently applying reasoning between the current-to-magnetic-field relationship and the magnetic-to-electric-field relationship. However, in looking at student responses for the two questions in finer detail, there were 10% of students who correctly applied the source-field relationship starting with an incorrect magnetic field to find the corresponding electric field, meaning 25% of students seem to have correctly applied the source-field relationship to determine the electric field. On the other hand, about 35% of students selected two graphs where one represented a simple derivative of the other[iii]. This suggests that these students may recognize that the induced electric field from a time-dependent magnetic field can be related by derivatives via $\vec{\nabla} \times \vec{E} = -\frac{d\vec{B}}{dt}$, but are unable to physically interpret the mathematics or properly evaluate the spatial integral in the appropriate coordinate system. In 10% of responses, students selected two identical graphs, which may represent the misconception that the source and field are directly proportional to each other without considering the spatial relationship.

c. Converting curl to integral form

A set of questions on the CURrENT ask students to identify the steps necessary to derive the integral form of $\vec{X} = \vec{\nabla} \times \vec{Y}$ with Stokes' Theorem. These are items 19-22 on the pretest and the identical items 26-29 on the posttest in context (3) Proof and Stokes' Theorem. The first two items of each version

[i] The chi-squared test for independence on the 2x2 contingency table comparing accuracy of the two fields gives a p-value of 0.551. One is unable to reject the null hypotheses that correctly determining the electric field is independent of correctly determining the magnetic field.

[ii] The chi-squared test for independence on the 2x2 contingency table comparing the selection of familiar graphs between the two fields gives a p-value of 0.451. One is unable to reject the null hypotheses that the use of rote recall of functional behavior for the electric field is independent of the use of rote recall for the magnetic field.

[iii] *e.g.*, magnetic field showed linear inside and decaying outside, while the electric field showed uniform inside and decaying outside.

ask students what rank[i] and type[ii] of integral is applied to the curl. The third item asks students which integration variable[iii] is applied. Finally, the fourth question asks students which side of the equation Stokes' Theorem is applied to.

Since students practice deriving the integral form of curl in the *Magnetic Vector Potential* tutorial, students may do better on this task after instruction.

Table 5-5: Breakdown of UW student responses on Integral-Form-of-Curl-Derivation items on MC_CURrENT. These were items 19-22 on the pretest and 26-29 on the posttest

Version	Integral rank	Integral type	Integration variable	Side for Stokes' Theorem	Integral rank and type	Rank-variable consistency	Overall correct
Pretest (N=105)	45%	48%	47%	62%	27%	70%	15%
Posttest (N=124)	50%	48%	64%	68%	28%	76%	19%

Table 5-5 shows the accuracy of students on each of the four items, as well as the combination of the integral rank and type, consistency between rank and variable in terms of dimensionality, and overall accuracy on all four items. There is a significant[iv] difference is the accuracy of the integration variable, from 45% on the pretest to 65% on the posttest; however, there are no significant[v] differences in the other six criteria.

The data suggest that the tutorials have little impact on student ability to derive the integral form of curl starting with the differential form. However, informal observation in the classroom indicates that most of the students seemed to make sense of the derivation of the integral form of curl by starting with Stokes' Theorem and replacing the curl of the field with the source vector, rather than the direct integration of the curl relationship followed by conversion with Stokes' Theorem. As both are valid

[i] *i.e.*, single, double, or triple integral
[ii] *i.e.*, open or closed integral
[iii] *e.g.*, $d\vec{l}$
[iv] The chi-squared test for independence on the 2x2 contingency table comparing student accuracy on the integration variable between the pretest and posttest gives a p-value of 9.65×10^{-3}. One is able to reject the null hypothesis that student accuracy on the integration variable is independent of instruction in Phys321 and Phys322.
[v] The chi-squared test for independence on the six 2x2 contingency tables comparing percentages between the pretest and posttest gives p-values of 0.429, 0.908, 0.356, 0.792, 0.286, and 0.414 for integral rank, type, side of Stokes' Theorem, rank-type combined, rank-variable consistency, and overall accuracy. One is unable to reject the null hypotheses that each criterion in independent of instruction in Phys321 and Phys322.

approaches for showing the conversion between the differential and integral forms of curl, the lack of improvement on this context in the CURrENT may not necessarily correlate to the impact instruction may have made to student understanding of the two forms of curl. Instead, the general lack of improvement could be due to students lacking mathematical versatility with the approach.

The one impact that is detectable from the data is the improvement on student identification of the integration variable. This could indicate that student understanding of the type of surface used with Stokes' Theorem has improved with instruction, or that student proficiency at interpreting infinitesimal spatial elements have improved, as related to some of the research discussed in Chapter 2.

d. Conceptually determining divergence

A set of questions on the CURrENT asks students to consider a segment of a long wire of uniform conductivity but steadily decreasing radius, and tests ideas about divergence. These are items 32 and 33 of the posttest in context (4) Current carrying wire. The diagram represents the current density of the wire with a field-line diagram. Item 32 asks student whether the divergence of the current density within the wire is zero or non-zero. Item 33 asks students to choose line(s) of reasoning that support their answer from 7 choices.

In Chapter 4, I discussed how resources involved with conceptually determining divergence and curl can be categorized into physical, mathematical, and visual reasoning. When the same scheme is applied to the choices for the supporting reasoning in item 33, three choices are categorized as physical, two are categorized as mathematical, and two are categorized as visual. The correct answer for this question is that the divergence of the current density is zero everywhere, with reasoning that involves a combination of choices that are categorized as physical and visual.

Table 5-6: Breakdown of UW student responses on Divergence-of-Current-Density items of posttest MC_CURrENT. These were items 32-33 on the posttest. P, M, and V stand for the selection of one or more supporting reasoning choices that are categorized as physical, mathematical, and visual, respectively. Percentages for supporting reasoning are shown as percentage of the row total.

N=125	Supporting reasoning contains:						
Accuracy	P only	M only	V only	PM	PV	VM	PMV
Correct – 42%	15% (of C)	13%	13%	8%	27%	6%	17%
Incorrect – 58%	1% (of I)	11%	41%	5%	4%	25%	12%

Table 5-6 shows the type(s) supporting reasoning chosen by students. There is a large disparity in answer selection for four of the combinations of supporting reasoning types: P only, V only, PV, and VM. 85% of students who chose supporting reasoning choices that are categorized as physical only or physical and visual correctly determined zero divergence for the current density. On the other hand, 85% of students who chose supporting reasoning choices that are categorized as visual only or visual and mathematical incorrectly determined non-zero divergence.

The data suggest that students who seem to rely on physical reasoning or in combination with visual reasoning are proficient at conceptually determining divergence, while students who seem to rely on visual reasoning or in combination with mathematical reasoning tend to do poorly. This is consistent with my findings about student reasoning for conceptually determining vector derivatives, as source-based interpretations of vector derivatives tend to be highly accurate while visual and mathematical misconceptions about vector derivatives cause student difficulties.

4. Summary of CURrENT results at UW

The overall results of the posttest version of the CURrENT suggest that the performance of UW students is similar to that of traditional CU and non-CU-Boulder transformed course structures. However, many of the items on the CURrENT are not directly related to the concepts covered in the tutorials. A more detailed pre-post analysis of UW results on select items of the CURrENT suggest that the tutorials have impacted student understanding in certain ways. They include: (1) students are more likely to associate Gauss' Law with symmetry; (2) students are more likely to identify that the magnetic field of an infinite solenoid is zero outside, which may be associated with an increased proficiency with using curl; and (3) students are more likely to correctly describe the infinitesimal spatial element in

Stokes' Theorem, which may be associated with an increased proficiency with interpreting infinitesimal quantities in general. In addition, the resource-based approach of analyzing CURrENT responses regarding the divergence of current suggests that students who use only physical or a blend of physical and visual resources do much better at conceptually determining divergence, which is consistent with my findings on similar tasks.

B. Reflections and future research

The context of my research was in the upper-division electromagnetism course sequence at UW, and most of my research about the student understanding of the source-field relationship has fueled the development of *Tutorials in Physics: Electrodynamics*. The set of tutorials is becoming more stable as compared to the first couple years when they were first adapted from the CU-Boulder tutorials, and modifications between quarters have become less frequent and less drastic. However, the curriculum development process is live and continuous, and additional research would improve our understanding on student conceptual development of the source-field relationship and help improve curricula to address student difficulties. Future research on student understanding of the source-field relationship before and after the upper-division level could provide useful insight to help optimize curriculum and instruction.

Some of the student difficulties observed at the upper-division level may stem from initial conceptualization developed in earlier courses. Basic charge-based integral formulation is a skill developed in calculus-based introductory physics, so one could continue to investigate student understanding of integrals and infinitesimal quantities at that level. Basic use of symmetry in the application of Gauss' and Ampère's Law is present in introductory electromagnetism as well, so one could also investigate to what extent introductory physics students can identify symmetry in systems, make symmetry arguments to draw conclusions about vector fields, and how often they conflate vector and field properties. Although vector derivatives are outside of the scope of introductory physics, one could investigate how students first conceptualize divergence and curl in vector calculus or similar mathematical preparatory courses. Research at earlier levels would help provide more insight as to how specific student difficulties materialize, such that we could optimize strategies to address them before or at the upper-division level.

Research beyond the upper-division level could help assess to what extent the upper-division curricula has been impactful and where to focus for improvement. Interactions with graduate teaching assistants during preparatory meetings for the tutorials informally suggest that our graduate TAs tend to be comfortable with concepts about integrals and symmetry, but many seem to struggle with consistently applying the physical concept of vector derivatives. This seems to be consistent with the conceptual development observed with the upper-division students, where the performance of tasks related to integrals and symmetry seem easy to improve upon, while students constantly struggle with conceptualizing vector derivatives and their application. Formal research on these topics with physics graduate students and faculty would provide statistics for expert-like thinking, which could serve as a benchmark for student improvement at the upper-division level.

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[1] Q. Ryan, C. Astolfi, C. Baily, and S. Pollock, Validation of a Conceptual Assessment Tool in E&M II, presented at the Physics Education Research Conference 2014, Minneapolis, MN, 2014, <https://doi.org/10.1119/perc.2014.pr.054>

[2] L. C. McDermott, P. S. Shaffer, and the Physics Education Group at the University of Washington, *Tutorials in Physics: Electrodynamics*, Preliminary edition (2021).

[3] S. Pollock, S. Chasteen, E. Kinney, M. Dubson, and R. Pepper, SEI: Junior E&M I Course Materials, (Science Education Initiative at the University of Colorado, Boulder, 2007), <https://physicscourses.colorado.edu/EducationIssues/Electrostatics/>.

[4] Q. Ryan, C. Baily, and S. Pollock, Multiple-Response Assessment for Upper-division Electrodynamics, presented at the Physics Education Research Conference 2016, Sacramento, CA, 2016, <https://doi.org/10.1119/perc.2016.pr.066>

Table of Contents

A. Guide to the appendix

This appendix is divided into three sections B-D, which correspond to supporting documents for Chapters 2-4 of the dissertation, respectively. Each section is organized by document type, starting with pretests, then tutorials, then tutorial homework if any, and then exam questions. Within each document type, they are organized by the order in which they appear in the dissertation. When an exam question is used in multiple chapters, it appears in the appendix in each corresponding section.

Table A-1:List of relevant tutorial topics by topic and associated three-letter codes

Chapter 2	Chapter 3	Chapter 4
COU - Coulomb's Law INT - Integrals with Charge MPE – Multipole Expansion	GSL – Gauss' Law AML – Ampère's Law	DEL – Vector derivatives VDS – Divergence as a source DPF – Displacement Field MSF – Magnetostatic Fields MVP – Magnetic Vector Potential AUX – Auxiliary Field

Pretests labels begin with a three-letter code (XXX)…, which corresponds to a tutorial topic as shown in Table A-1. The rest of the label is based on the version code used by the Physics Education Group to organize pretests for online administration.

Tutorial and homework labels begin with a three-letter code, followed by TUT or HW, followed by a quarter code ###. The first two numbers correspond to the year of administration, and the last number corresponds to the quarter, with 1-4 corresponding to Winter, Spring, Summer, and Autumn respectively. Phys321 is administered at Spring and Autumn, while Phys322 is administered in Winter and Summer.

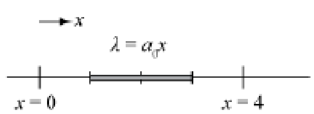
Exam-question labels begin with a quarter code, then E# corresponding to which exam in the quarter the question was administered, and then a series of three-letter codes within parentheses corresponding to tutorial topics that the question is relevant to. Some three-letter codes appear in the exam titles but are not used in this dissertation, since the scope of the exam questions is wider than that of the dissertation.

B. Supporting documents for Chapter 2

1. Pretests

a. (INT)U1a

Part I: A system consists of a line charge of varying linear change density $\lambda(x)=a_0x$ located between x=1 and x=3 as shown.



Question 1.
Determine the net charge of this system.

- 2 a<sub>0
- 3 a<sub>0
- 4 a<sub>0
- 6 a<sub>0
- 8 a<sub>0
- None of the above.

Question 2.
Explain your reasoning for the previous question.

Question 3.
The magnitude of the dipole moment for point changes in one dimension is $\Sigma(qx)$. For example, the dipole moment of 2C of charge 3m away from the origin is 6C*m. Determine the dipole moment of this system.

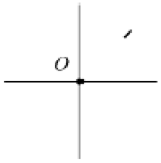
- 1/3 a<sub>0
- 2 a<sub>0
- 6 a<sub>0
- 8 a<sub>0
- (8+2/3) a<sub>0
- 9 a<sub>0
- 13 a<sub>0
- None of the above.

Question 4.
Explain your reasoning for the previous question.

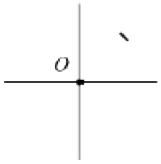
Question 5.
In five words or less, describe what it means to integrate.

Part II: In terms of two-dimensional coordinates x, y, r, and/or θ (theta), how would you describe the following infinitesimal quantities?
Example: "The length of a small step in the x-direction" is "dx".

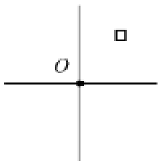
Question 6.
The length of a small step away from the origin.



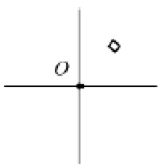
Question 7.
The length of a small step around the origin.



Question 8.
The infinitesimal area in Cartesian coordinates.



Question 9.
The infinitesimal area in polar coordinates.



b. (INT)U1b, additional questions to (INT)U1a

Question 1.
In your own words, describe what "linear charge density" means.

Question 2.
In your own words, describe what *"dx"* physically means in the context of a charge integral.

2. Tutorials

a. COU TUT 144

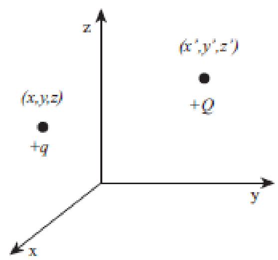
COULOMB'S LAW

ED
1

I. The separation vector in multiple coordinate systems

Coulomb's Law describes the effect that a point charge (often called the "source" charge in textbooks) has on another charge (often called the "test" charge). You should be familiar with Coulomb's law from your introductory physics classes, but in this tutorial we will look at Coulomb's law in a more formal way.

A. Shown at right is a source charge $+Q$ at a point (x', y', z'), and a test charge $+q$ at a point (x, y, z).



1. On the graph, draw the vectors $\vec{r}$ and $\vec{r}'$.

2. Your textbook defines a third vector, the separation vector $\vec{\mathscr{r}}$. In terms of the locations of the source and test charge, which direction should the separation vector point? Explain.

3. Draw this vector on the graph, and express it in terms of the vectors $\vec{r}$ and $\vec{r}'$.

4. What are the Cartesian components of $\vec{\mathscr{r}}$? Express them in terms of the Cartesian components of $\vec{r}$ and $\vec{r}'$.

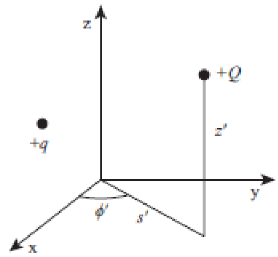
5. What are the Cartesian components of $\hat{\mathscr{r}}$, the unit vector in the direction of $\vec{\mathscr{r}}$?
(*Hint:* You can make any vector into a unit vector by normalizing it to have length 1.)

6. Write an expression for the force the source charge exerts on the test charge, including the direction in which the force acts.

ED 2 *Coulomb's Law*

B. The same two charges from part A are shown at right, in cylindrical coordinates this time.

1. How can you write the coordinates of the source charge (x',y',z') in terms of the cylindrical coordinates s', ϕ', and z'?



2. Using your answer to part 1, write the Cartesian components of $\vec{r}$ in cylindrical coordinates.

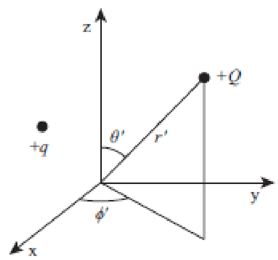
$r_x =$

$r_y =$

$r_z =$

C. The charges are now put into spherical coordinates as shown.

1. How can you write the coordinates of the source charge (x',y',z') in terms of the spherical coordinates r', θ', and ϕ'?



2. Using your answer to part 1, write the Cartesian components of $\vec{r}$ in cylindrical coordinates.

$r_x =$

$r_y =$

$r_z =$

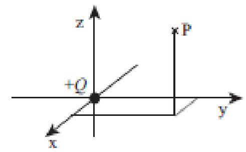
➭ Check your answers with a tutorial instructor before continuing.

II. The separation vector and electrostatic potential

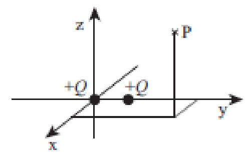
Recall from freshman physics that the potential at a test point P due to a point charge $+Q$ is given by $V = kQ/r$.

A. Shown at right is a charge $+Q$ at the origin, and a test point P at an arbitrary point.

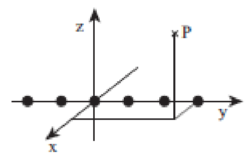
What are the vectors $\vec{r}$ and $\vec{r}'$ in this case? Use this to write an expression for the potential at point P.



B. A second $+Q$ charge is added, located at $(0,y_2,0)$. What is the potential at point P due to both charges?

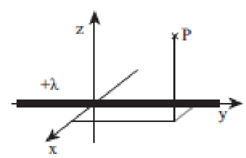


C. Many more charges are added, forming a string of N identical point charges along the y-axis, with y_N the y-coordinate of the Nth charge. Write an expression for the potential at point P due to the string of charges.



D. The string of charges is replaced by an infinite line of charge on the y-axis with uniform linear charge density $+\lambda$.

1. What is the potential at point P due to this line? (*Hint:* A good way to check to see if you've converted from a sum to an integral correctly is to verify that both expressions have the same units.)



2. In the context of this integral, what does the vector $\vec{r}$ correspond to?

What does the vector $\vec{r}'$ correspond to?

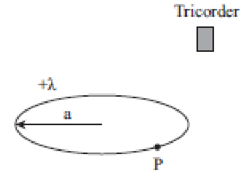
ED 4 *Coulomb's Law*

III. Charged bike tires

A hundred years in the future, a bicyclist named Thomas gets lost in the Cascades east of Seattle and gets a flat tire. Thomas pulls off the tire and consults his Tricorder to find out if there are nearby life forms that could help him. However, the flat tire has somehow become uniformly positively charged. The Tricorder complains that the electric field from the tire is very annoying.

Your goal is to calculate the electric field produced by the charged tire (a ring with linear charge density $+\lambda$).

A. The origin is at the center of the tire, and the Tricorder is off to the side as shown. An arbitrary point P is shown on the tire.



1. Label the diagram with points (x, y, z) and (x', y', z').

2. Draw and label the three vectors $\vec{r}$, $\vec{r}'$, and $\vec{\mathscr{r}}$.

3. Given the shape of the charge distribution, what choice of coordinates would be most convenient?

4. Using these coordinates, write an expression for the electric field at the Tricorder from a small piece of charge dq at point P. Be explicit – you should not leave your answer in terms of $\vec{\mathscr{r}}$, you should substitute it for the coordinates you know.

➪ Check your expression with a tutorial instructor before continuing.

B. Use this answer to write a full integral expression for the electric field at the Tricorder. Simplify your expression as much as possible.

C. Thomas notices that if he puts his Tricorder on the z-axis (straight above the center of the ring) it complains much less about the electric field.

1. What are the coordinates of the Tricorder (in the coordinate system you chose) at this position?

2. Plug these coordinates into your expression from part B, and evaluate the integral.

   (*Hint:* Most of the terms in your integral are either constant, zero, or integrate to zero. It's not a hard integral.)

3. Check the limits of your expression both for very large z, and for $z = 0$. Do these answers make sense? Sketch the E field along the z axis, including both positive and negative values of z.

4. Where should the Tricorder be placed to avoid interference from the electric field?

b. INT TUT 154

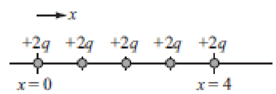
INTEGRALS WITH CHARGE

ED 1

I. Net charge

A. For an arbitrary set of point charges q_i, how do you calculate the net charge of the system? Write a mathematical expression for the net charge.

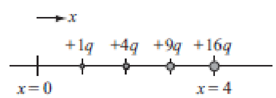
B. Consider the arrangement of charges at right.



1. What is the net charge of this arrangement?

2. Is it possible to multiply two quantities to find the net charge in this case? Explain.

C. Consider a different arrangement of charges at right.

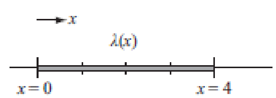


1. What is the net charge of this arrangement?

2. Is it possible to multiply two quantities to find the net charge in this case? Explain.

II. Linear charge

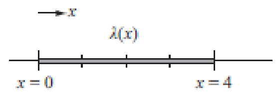
A. Consider the constant charge density at right, where $\lambda(x) = a_0$.



1. How could you use the method for net charge in I.A above to find the net charge?

2. Why is it possible to multiply λ and L to find the net charge of this system? Explain.

ED 2 *Integrals with Charge*



B. Suppose the charge density is now variable, where $\lambda(x) = b_o x^2$.

1. Consider the following student dialogue concerning the net charge of the system:

   Student 1: "I think we can still calculate the net charge by taking $b_o x^2$ and multiplying it by the length."

   Student 2: "I think we can calculate the net charge by plugging L in for x."

   Student 3: "I think we should break the line up and add up the charges."

   With which student do you agree? Explain.

2. The variable charge density is analogous to the variable charges in part I.C. Based on your answer, are you able to multiply when the charge density is variable?

Student 3 describes the process of *integration*, or summing many infinitesimal quantities.

3. To find the net charge of this system, what is the infinitesimal quantity? (*Hint:* How would you express that in terms of $\lambda(x)$?) Be sure to include a factor to show that the integrand is small!

4. Write an integral expression for the net charge of this system. You do not need to evaluate the integral.

C. How would your integral of net charge simplify if $\lambda(x)$ were constant instead of variable?

When is it possible to multiply λ by a length to find the charge of a segment? Consider both large segments and very small segments.

➪ Check your answers with a tutorial instructor before continuing.

III. Curvilinear coordinates

A. An arc with radius s spans an angle of $\pi/2$ and has a constant charge density of $\lambda(\phi) = a_o$.

1. What is the length of this arc?

2. What is the net charge of this arc?

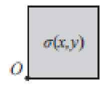
B. Suppose the charge density is now variable, such that $\lambda(\phi) = b_o \sin(\phi)$.

1. What infinitesimal length should be multiplied by $\lambda(\phi)$ to find an infinitesimal charge? Check units.

2. Write an integral expression for the net charge of this system.

IV. Surface charge

A. A 2m by 2m square with the bottom left corner at the origin has variable surface charge density $\sigma(x,y)$.



1. What small shapes would you break the charge distribution into to find the net charge? Explain.

2. What are the dimensions of your small shape? What is the area?

3. Write an integral expression for the net charge of this system.

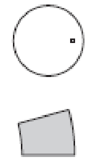
B. Consider the student statement about the circular charge distribution at right:

"To find the net charge, I can break the circle into many square boxes of area dx*dy and then integrate."

Do you agree with this student? If so, what would be the bounds of the integral? If not, explain why not.

ED 4 *Integrals with Charge*

C. Consider a tiny, slightly curved chunk (enlarged and exaggerated at right) within a circle. The sides are arcs from concentric circles, and the top and bottom are radial lines coming from the center of the circles.


Enlarged chunk

1. What is the width of this shape? The height? The area?

2. Could this shape be used to integrate the area of the circle? If so, what would be the bounds of the integral?

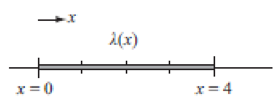
V. Spatially weighted integrals

In electrostatics, you will also integrate charge weighted by position. The simplest example is the dipole moment, which in point-charge form looks like $\vec{p} = \sum q_i \vec{r}_i$, where $\vec{r}$ is measured from the origin to the charge.

A. A $+q$ charge is located on the x-axis at $x = L$, and a $+2q$ charge is located on the x-axis at $x = 3L$. What is the dipole moment of the system? Be sure to include a unit vector!

Suppose the $+2q$ charge was moved to $x = -L$. What would be the new dipole moment?

B. Consider again the variable charge density $\lambda(x) = b_o x^2$.



1. What expression should take the place of q_i?

2. Based on the definition of dipole moment, do you integrate the expression in 1 first or multiply it with a position vector first?

3. Write an integral expression for the dipole moment of this system. Put your expression in terms of b_o, x, dx, and $\hat{x}$.

VI. Supplement: Vector integrals

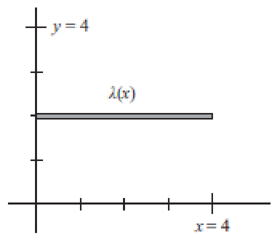
A. Suppose you have two vectors, $\vec{a} = 3\hat{x} + 2\hat{y} - 2\hat{z}$ and $\vec{b} = 2\hat{x} - 3\hat{y} - \hat{z}$.

1. Find the sum of the two vectors.

2. How many separate addition operations were required? Why?

3. Suppose you needed to integrate infinitesmial vector quantities. How many separate integrals would you need to construct? Explain.

B. Consider again the variable charge density $\lambda(x) = b_o x^2$, except this time the charge is shifted 2 units in the y-direction.



The dipole moment is given by $\vec{p} = \sum q_i \vec{r}_i$.

1. What is $\vec{r}$ in terms of Cartesian coordinates?

2. Consider a small segment of the charge density near coordinates (x,y).

   a. What is the net charge of the small segment? Answer in terms of b_o, x, y, dx, and/or dy.

   b. What is the x-component of the dipole moment of the small segment? Answer in terms of b_o, x, y, dx, and/or dy.

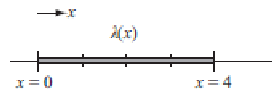
   c. What is the y-component of the dipole moment of the small segment? Answer in terms of b_o, x, y, dx, and/or dy.

3. Write integral expression(s) for the dipole moment of this system.

c. INT TUT 174 – modified second page of INT TUT 154

Integrals with Charge

B. Suppose the charge density is now variable, where $\lambda(x) = bx^2$.



1. Consider the following student dialogue concerning the net charge of the system:

   Student 1: "I think we can still calculate the net charge by taking bx^2 and multiplying it by the length."

   Student 2: "I think we can calculate the net charge by plugging L in for x."

   Student 3: "I think we should break the line up and add up the charges."

   With which student(s) do you agree? Explain.

2. The variable charge density is analogous to the variable charges in part I.C. Based on your answer, are you able to multiply when the charge density is variable?

Student 3 describes the process of *integration*, or summing many infinitesimal quantities.

3. Consider the infinitesimal quantities "dx" and "dq."

   a. What does each quantity mean? Consider both the literal meaning and a physical interpretation in your description.

   b. How are these quantities related to linear charge density λ?

4. Write an integral expression for the net charge of this system in terms of b, x, and/or dx. You do not need to evaluate the integral.

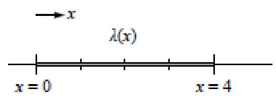
C. Based on your answers above, when is it possible to multiply λ by some length to find the charge of a segment? Consider both large segments and very small segments.

➪ Check your answers with a tutorial instructor before continuing.

d. INT TUT 184 – modified second page of INT TUT 154

Integrals with Charge

B. Suppose the charge density is now variable, where $\lambda(x) = bx^2$.



1. Consider the following student dialogue concerning the net charge of the system:

   Student 1: "I think we can still calculate the net charge by taking bx^2 and multiplying it by the length."

   Student 2: "I think we can calculate the net charge by plugging L in for x."

   Student 3: "I think we should break the line up and add up the charges."

   With which student(s), if any, do you agree? Explain your reasoning.

2. The variable charge density is analogous to the variable charges in part I.C. Based on your answer, are you able to multiply without addition when the charge density varies?

Student 3 describes the process of *integration*, or summing many infinitesimal quantities.

3. Consider the infinitesimal quantity *"dx."*

   a. What does this quantity mean? Consider the mathematical definition, physical interpretation, units, and dimensionality of this quantity.

   b. How would you combine dx with λ to describe a charge? Support your answer with dimensional analysis.

4. Write an integral expression for the net charge of this system in terms of b, x, and/or dx. You do not need to evaluate the integral.

C. Based on your answers above, when is it possible to multiply λ by some length to find the charge of a segment? Consider both large segments and very small segments.

{ Check your answers with a tutorial instructor before continuing.

3. Homework

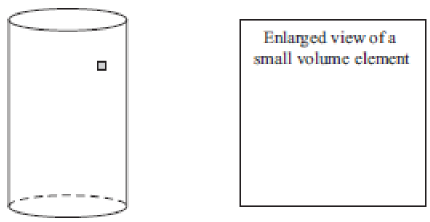
a. INT HW 164

INTEGRALS WITH CHARGE   Name:________________________ Section: ES HW–1

1. In tutorial, you investigated *dl* and *dA* elements in Cartesian and polar coordinates. We will now extend that to volume elements in cylindrical and spherical coordinates.

   a. A large cylinder is shown below.

      i. In the space provided, sketch an enlarged small volume element.
      Clearly label the width, depth, and height. Clearly indicate which side is curved.



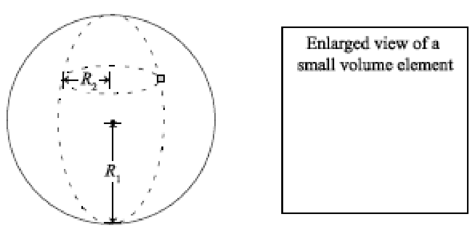
      ii. In cylindrical coordinates s, □ and z:
      What is the width of the small chunk?   What is the depth?

      What is the height?   What is the volume?

      Double check that each expression above has the correct dimensions of length.

   b. A large sphere is shown below, with two circles sketched.

      i. In the space provided, sketch an enlarged small volume element.
      Clearly label the width, depth, and height. Clearly indicate which sides are curved.



      ii. In terms of spherical coordinates r, θ, and □ determine R_1 and R_2.

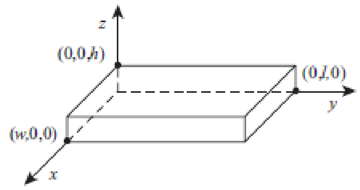
      iii. What is the width of the small chunk?   What is the depth?

      What is the height?   What is the volume?

      Double check that each expression above has the correct dimensions of length.

ES
HW–2 *Integrals with charge*

2. A rectangular slab of charge has dimensions of w, l, and h with one corner on the origin as shown at right. The volume charge density is given by $\varrho(x,y,z)$.



   a. Write an integral expression for the net charge of the slab, complete with bounds.

   b. Suppose the volume charge density does not vary with z.

      i. How does this simplify your integral above?

      ii. Can an area charge density $\sigma(x,y)$ be used to describe the charge distribution of the slab? If so, what is $\sigma(x,y)$ in terms of $\varrho(x,y,z)$, w, l, and/or h? If not, explain why not.

      iii. Can a linear charge density $\lambda(y)$ be used to describe the charge distribution of the slab? If so, what is $\lambda(y)$ in terms of $\varrho(x,y,z)$, w, l, and/or h? If not, explain why not.

   c. Suppose the volume charge density does not vary with x nor z.

      i. How does this simplify your integral expression from part a?

      ii. Can a linear charge density $\lambda(y)$ be used to describe the charge distribution of the slab? If so, what is $\lambda(y)$ in terms of $\varrho(x,y,z)$, w, l, and/or h? If not, explain why not.

   d. Based on your answers from parts a-c, express the infinitesimal charge quantity dq in three different ways.

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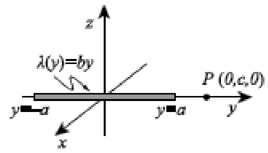
Integrals with charge Name ______________________ ES HW–3

3. Suppose we have a line charge along the y-axis from $y = -a$ to $y = a$ with a variable linear charge density of $\lambda(y) = by$, where a and b are real constants. Point P is a distance c away from the origin on the y-axis, with $c > a$.



Recall from introductory physics that the electric potential at a point relative to infinity is $V = kq/d$ for a point charge, where d is the distance between the point charge and where the electric potential is measured, and k is a constant.

a. Describe in words how you could use superposition to find the electric potential at point P.

b. On the diagram, draw and label a small segment along the line charge on the diagram. Do not use the origin. How much charge is contained in that segment?

c. Does d from the charge to point P depend on the position of the charge?

What expression should replace d in the definition of the potential above?

d. Write an integral expression for the electric potential at point P due to the line charge, relative to infinity.

e. What substitution(s) would change if instead point P was located at (l,m,n)? Explain.

f. At what location(s) (other than infinity) should the potential due to this line charge be zero? If there are no such locations, state so explicitly. Explain.

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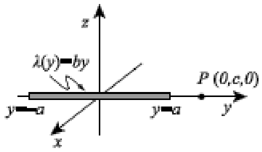
Physics Education Group, University of Washington

Preliminary Edition, 2016

ES
HW–4 *Integrals with charge*

4. Challenge question (not graded, based on the tutorial supplement in *Integrals with charge*).

The same line charge from question 3 is reproduced at right. This time, you will be asked to investigate the electric field, which is given by $\vec{E} = (kq/d^2)\hat{n}$ for a point charge, where $\hat{n}$ is the unit vector from the point charge to where the electric field is measured.



a. In what direction is the net electric field at point *P?* Explain.

b. Describe in words how you could use superposition to find the net electric field at point *P*.

c. How many non-trivial integrals would you need to evaluate to find the net electric field at point *P?* Explain.

d. Suppose point *P* were located at an arbitrary point (*l*,*m*,*n*). How would your answer to part c change? Explain.

e. At what location(s) (other than infinity) should the net electric field due to this line charge be zero? If there are no such locations, state so explicitly. Explain.

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b. INT HW 174 – modified second page of INT HW 164

ES HW–2 *Integrals with Charge*

2. Consider the verbal definition of linear charge density λ, which is "charge per unit length."
   a. Suppose there were a segment of length L_o that were uniformly charged with net charge Q_o. Determine an expression for λ.
   b. Suppose the segment were non-uniformly charged, but still had a length L_o and net charge Q_o.
      i. Why does your expression in part a. not describe λ at the center of the segment? Explain.
      ii. Describe an alternate method that would determine λ at the center of the segment. What length would you measure? What charge would you use?
   c. Based on your answers above, write a general expression for the linear charge density that would always work.
   d. Interpret the statement "charge per unit length" word by word. What sort of measurement or mathematical operation does each word refer to?

      Charge:
      Per:
      Unit:
      Length:

3. Textbooks often write integrals involving charges (or the effect of charges) as $\int f(\vec{r}, \vec{r}')\rho d\tau'$, where ρ is charge per unit volume.
   a. In this context, what are the units of $d\tau'$?
   b. Interpret the quantity $\rho d\tau'$.
      i. What are the units? Is it finite or infinitesimal? What does it describe?
      ii. Suppose you wanted to use the integral, but your charge distribution was a linear or surface charge. What similar quantities would replace $\rho d\tau'$?

Tutorials in Introductory Physics: Electrodynamics
Physics Education Group, University of Washington

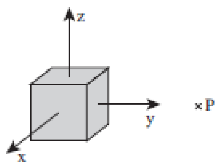
Preliminary Edition, 2018

4. Exam questions

a. 144E1 (COU, GSL)

IV. [25 points total] Tutorial question.

A cube with charge density $\rho(x,y,z) = bz$ and side length a fills the space from $x,y,z = -a/2$ to $x,y,z = +a/2$ (it's centered on the origin). A test point P is located at $y = 7$m.



A. [3 pts.] How much charge is in the cube? Explain.

B. [2 pts.] Qualitatively sketch the electric field lines of this charge distribution in the space below.

C. [4 pts.] Could you use Gauss's law to find the electric field at point P? If so, draw the Gaussian surface you would use on the diagram, and explain how you would find $\overline{E}(P)$. If not, explain why not.

D. Suppose you instead wanted to use Coulomb's law to find the electric field at P.

i. [2 pts.] What is the separation vector between a point (x',y',z') in the cube and point P?

ii. [5 pts.] Write down an expression for $\overline{E}(P)$.

(*Note:* You don't have to evaluate the integral, but you should be as explicit as possible.)

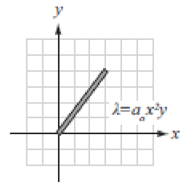
E. [4 pts.] Suppose you found $\overline{E}$ everywhere in space using one of the methods above. Where is the divergence of this electric field zero? Where is it non-zero? Explain.

Physics 321A, Fall 2014 Exam 1, page 8 ED-UWA321A144T-E1(COU, GSL).doc

b. 164E1 (INT, GSL)

IV. [25 points total] Tutorial question. This page contains two independent parts, A and B.

A. A slanted line charge whose linear charge density is given by $\lambda(x, y) = a_o x^2 y$ is shown at right. The endpoints of the line charge are at the origin and $(x, y) = (3, 4)$.



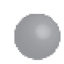
i. (2 pts) Consider the following *incorrect* student statement about determining the net charge of the system:

"The line charge varies in both x and y, so the net charge is $\int_0^3 \int_0^4 a_o x^2 y \, dx \, dy$."

Identify the flaw(s) in the student's reasoning.

ii. (8 pts) Determine the net charge of the system. You may leave your final answer in the form $\int_a^b f(u) du$ or $\int_a^b \int_c^d f(u, v) \, du dv$. Show your work, and explain your reasoning for each step and substitution.

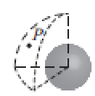
B. A uniformly charged sphere with volume charge density ρ and radius R_1 is centered on the origin.



Uniformly charged sphere

i. (5 pts) On which spherical coordinate(s) r, ϕ, and/or θ can the electric field *not depend on?* Justify your answer by using symmetry arguments.

ii. (10 pts) Consider the Gaussian surface at right. The wedge consists of a curved surface of radius R_2 concentric with the sphere, and three flat surfaces that lie in the *xy*-, *yz*- and *xz*-planes.



Gaussian surface perspective view

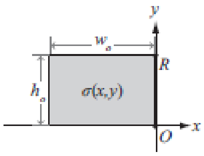
Can this Gaussian surface be used to determine the magnitude of the electric field at point P (also a distance R_2 from the center of the sphere) with Gauss' Law? Explain your reasoning.

Physics 321A, Fall 2016 Exam 1, page 5 ED-UWA321A164T-E1(INT,GSL).docx

c. 164E3 (INT, MPE)

IV. [25 points total] Tutorial question.

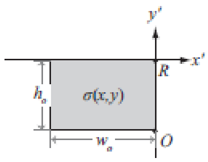
A thin rectangular sheet is located on the *xy*-plane (shown at right) with non-uniform surface charge density $\sigma(x, y) = a_o x^2$ where a_o is a positive constant. The dimensions of the sheet are width w_o along the *x*-axis and height h_o along the *y*-axis, with the origin located at point *O* at the bottom right corner.



a. [5 pts] Is the net charge of this charge distribution *positive, negative,* or *zero?* Explain your reasoning in words.

b. [10 pts] Recall that the point-charge form of the dipole moment is given by $\vec{p} = \sum Q_i \vec{r}_i$. Determine the integral expression for the *x*-**component** of the dipole moment for this charge distribution, in terms of given variables. Show your work for each substitution that you make.

c. Consider the act of choosing a new origin located at point *R* at the top right corner of the sheet without changing the charge distribution on the sheet.



i. [5 pts] Is the *x'*-component of the new dipole moment *greater than, less than,* or *equal to* the *x*-component of the old dipole moment? Explain your reasoning.

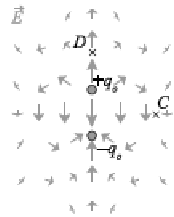
ii. [5 pts] Is the *y'*-component of the new dipole moment *greater than, less than,* or *equal to* the *y*-component of the old dipole moment? Explain your reasoning.

Physics 321A, Fall 2016 Exam 3, page 5 ED-UWA321A164T-E3(INT,MPE).docx

d. 174E1 (INT, DEL)

IV. [25 points total] This page contains two independent parts, A and B.

A. The electric field of a dipole is shown in the vector field diagram at right. The dipole consists of a single positive point charge above and a single negative point charge below.

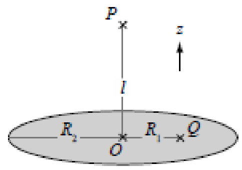


Vector field diagram
(not a field line diagram)

i. [5 pts] At point *D*, is the divergence of the electric field *positive*, *negative*, or *zero?* Explain your reasoning.

ii. [5 pts] At point *C*, in what direction is the curl of the electric field? If the curl of the electric field is zero, state so explicitly. Explain your reasoning.

B. A circular disk of radius R_2 centered at the origin and oriented in the *xy*-plane is charged with a surface charge density of $\sigma(s,\phi,z) = a_o s$. Point *P* is located a distance *l* above the origin along the *z*-axis, at coordinates (0, 0, *l*). Point *Q* is located on the disk a distance R_1 from the origin, at coordinates (R_1, 0, 0).



Perspective view

i. Consider a small region of the disk around point *Q*.

[8 pts] How much charge is located in that small region? Use an appropriate coordinate system and variables provided above. Show your work and/or explain your reasoning.

[3 pts] What is the distance from the coordinate (R_1, 0, 0) to point *P?* Show your work.

The electric potential at a point relative to infinity is $V = kq/d$ for a point charge, where *d* is the distance between the point charge and where the electric potential is measured, and *k* is a constant.

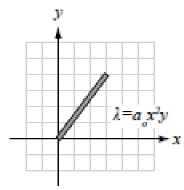
ii. [4 pts] Based on your answers above, write an integral expression for the electric potential (not electric field) at point *P* due to the charged disk, relative to infinity. You do not need to evaluate the integral.

Physics 321A, Autumn 2017 Exam 1, page 5 ED-UWA321A174T-E1(INT,DEL).docx

e. 174E3 (INT, MPE, DPF)

IV. [25 points total] Tutorial question. This page contains two independent parts, A and B.

A. A slanted line charge whose linear charge density is given by $\lambda(x,y) = a_0 x^2 y$ is shown at right. The endpoints of the line charge are at the origin and $(x, y) = (3, 4)$.



i. [6 pts] Consider the following *incorrect* student statement about determining the net charge of the system:

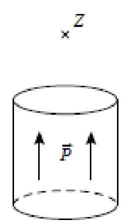
"The line charge varies in both x and y, so the monopole moment is $\int_0^3 \int_0^4 a_0 x^2 y \, dx \, dy$."

Identify the flaw(s) in the student's reasoning.

ii. [9 pts] Based on your answer above, determine an integral expression for the **x-component** of the **dipole moment,** p_x, for the charge distribution above. You do not need to evaluate the integral expression, but it should be in a form that a mathematical program can evaluate.

Be sure to explain your reasoning and/or show your work.

B. A cylinder with fixed (frozen-in) uniform polarization $\vec{P}$ is shown at right. Point Z is located directly above the cylinder, along the central axis. There are no free charges or an electric field from any other sources.



[10 pts] In what direction is the displacement field at point Z? Sketch an arrow in the box below. Explain your reasoning.

Displacement field at point Z

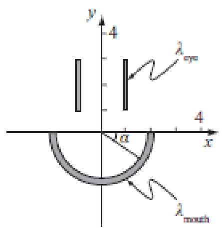
×

Physics 321A, Autumn 2017 Exam 3, page 5 ED-UWA321A174T-E3(INT,MPE,DPF).docx

f. 182E1 (INT, DEL)

IV. [25 points total] Tutorial question. For the following context, assume units (m for length, C for charge).

A. Consider the charge distribution at right. The two straight rods each have variable linear charge density λ_{eye}, whichare located at $x = 1$ and -1 spanning from $y = 1$ to 3. The curved rod makes a circular arc of radius 2 from the origin, has variable linear charge density λ_{mouth}. Take α as the angle clockwise from the x-axis.



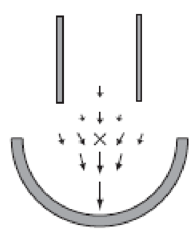
i. [5 pts] What is the net charge on one straight rod? Show your work.

ii. [5 pts] What is the net charge of the curved rod? Show your work.

iii. [10 pts] Recall that the point-charge form of the electric potential is $V = kq/d$, with the reference point at infinity. Using this, **derive an expression** in terms of k, x, y, α, λ, and/or numerical constants **for the potential at the origin** relative to infinity. You will be graded on the set-up and substitution of q and d, so you do not need to evaluate any mathematical operations in your expression.

B. [5 pts] The diagram at right shows a vector field diagram estimating the electric field near the origin (marked as ×).

Is the divergence of the electric field at the origin *positive, negative,* or *zero?* If the divergence cannot be determined, state so explicitly. Explain your reasoning.

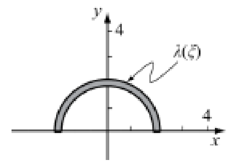


Physics 321A, Spring 2018 Exam 1, page 5 ED-UWA321A182T-E1(INT,DEL).docx

g. 192E1 (INT, GSL)

IV. [25 points total] Tutorial question. This page contains two independent parts, A and B.

A. A curved line charge is shown at right. The charge density is variable and unspecified, but can be represented by the function $\lambda(\xi)$ where ξ is some parameter based on the position. For this question, you may ignore units.

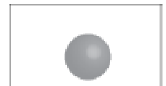


Consider the following *incorrect* student statement about determining the net charge of the object:

"Charge can be found by multiplying density with the dimensions. Thus the net charge of this object is $\lambda(\xi)$*4*2, because it is 4 wide by 2 high.

[10 pts] Identify the flaws in this student's approach. We will be grading for three distinct ideas.

B. A uniformly charged sphere with volume charge density ρ and radius R_1 is centered on the origin.



Uniformly charged sphere

i. [5 pts] On which spherical coordinate(s) r, ϕ, and/or θ can the electric field *not depend on?* Justify your answer by using symmetry arguments.

ii. [10 pts] Consider the Gaussian surface at right. The wedge consists of a curved surface of radius R_2 concentric with the sphere, and three flat surfaces that lie in the xy-, yz- and xz-planes.

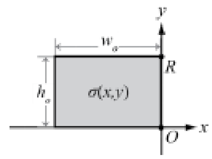
Gaussian surface perspective view

Can this Gaussian surface be used to determine the magnitude of the electric field at point P (also a distance R_2 from the center of the sphere) with Gauss' Law? Explain your reasoning.

h. 194E3 (INT, MPE)

V. [22 points total] Tutorial question.

A thin rectangular sheet is located on the *xy*-plane (shown at right) with non-uniform surface charge density $\sigma(x, y) = a_o x^2$ where a_o is a positive constant. The dimensions of the sheet are width w_o along the *x*-axis and height h_o along the *y*-axis, with the origin located at point *O* at the bottom right corner.



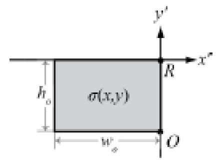
a. [4 pts] Is the net charge of this charge distribution *positive, negative,* or *zero?* Explain your reasoning **in words**.

b. Recall that the point-charge form of the dipole moment is given by $\vec{p} = \sum Q_i \vec{r}_i$.

i. [5 pts] Determine an integral expression for the *x*-**component** of the dipole moment for this charge distribution, in terms of given variables. You do not need to evaluate the integral, but it should be explicit enough for a computer to obtain a value.

ii. [3 pts] From your expression above, identify which factor(s) best represent Q_i from $\vec{p} = \sum Q_i \vec{r}_i$.

c. Consider the act of choosing a new origin located at point *R* at the top right corner of the sheet without changing the charge distribution on the sheet, with $\sigma'(x, y) = a_o x^2$.



i. [5 pts] Is the *x'*-component of the new dipole moment *greater than, less than,* or *equal to* the *x*-component of the old dipole moment? Explain your reasoning.

ii. [5 pts] Is the *y'*-component of the new dipole moment *greater than, less than,* or *equal to* the *y*-component of the old dipole moment? Explain your reasoning.

Physics 321A, Autumn 2019 Exam 3, page 6 ED-UWA321A194T-EF(INT,MPE).docx

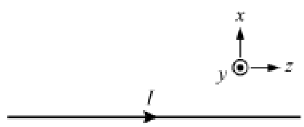
C.Supporting documents for Chapter 3

1. Pretests

a. (AML)U1c

Part I.

A uniform, infinite line current is shown below.



Question 1.
Which of the following *continuous* symmetries does this system have? Check all that apply.

- ☐ Translational symmetry along the x-axis.
- ☐ Translational symmetry along the y-axis.
- ☐ Translational symmetry along the z-axis.
- ☐ Rotational symmetry around the x-axis.
- ☐ Rotational symmetry around the y-axis.
- ☐ Rotational symmetry around the z-axis.

For the following question, base your answer only on symmetry arguments. That is, assume that you do not know the magnetic field of a line current and forgot Biot-Savart Law and Ampère's Law.

Question 2.
Consider the following true/false statements about the dependence of the magnitude of the magnetic field on cylindrical coordinates *based only on symmetry arguments*. Check all of the true statements.

- ☐ The magnitude of the magnetic field cannot depend on <i>s</i>.
- ☐ The magnitude of the magnetic field cannot depend on <i>ϕ</i>.
- ☐ The magnitude of the magnetic field cannot depend on <i>z</i>.

Question 3.
Explain your answer to the previous question.

Question 4.
You are given that the magnetic field of this current distribution cannot have a *z*-component. Which of the following explanations can be used to prove the statement above? Check all of the valid explanations.

- ☐ The Biot-Savart Law shows that every small current segment makes a magnetic field contribution perpendicular to itself.
- ☐ The curl of the magnetic field is proportional to current, and a <i>z</i>-component does not contribute to the line integral of any Ampèrian loop that encloses the current.
- ☐ A magnetic field with a <i>z</i>-component with this symmetry is caused by current distributions at infinity, which we can assume are not there.
- ☐ The divergence of the magnetic field is zero, so a Gaussian cylinder must have zero flux on both end caps.

Part II.

Consider the magnetic field specified by: $\vec{B} = c(x\text{-}y)\hat{z}$, where c is a constant.

Question 5.
For which coordinate(s) does this magnetic field depend on? Check all that apply.

☐ Depends on x.
☐ Depends on y.
☐ Depends on z.
☐ Does not depend on any coordinate.

Question 6.
Which component(s) does this magnetic field have? Check all that apply.

☐ Has an x-component.
☐ Has a y-component.
☐ Has a z-component.
☐ Does not have any component.

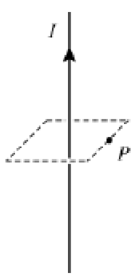
Question 7.
Along which axis (or axes) does this magnetic field have continuous translational symmetry? Check all that apply.

☐ Along the x-axis.
☐ Along the y-axis.
☐ Along the z-axis.
☐ Does not have continuous tranlational symmetry.

Question 8.
Explain your answer to the previous question.

Part III.

A closed square Ampèrian loop encloses a uniform, infinite line current as shown.



Question 9.
If you knew the linear current density but did not know the magnetic field, could you use this Ampèrian loop to determine the magnetic field at point *P*?

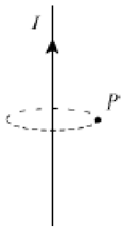
○ Yes.
○ No.

Question 10.
If you knew the magnetic field everywhere in space but did not know the linear current density, could you use this Ampèrian loop to determine the linear current density?

○ Yes.
○ No.

Question 11.
Explain your answer to the previous two questions.

The Ampèrian loop is replaced with a circular Ampèrian loop instead, as shown.



Question 12.
If you knew the linear current density but did not know the magnetic field, could you use this Ampèrian loop to determine magnetic field at point *P*?

○ Yes.
○ No.

Question 13.
If you knew the magnetic field everywhere in space but did not know the linear current density, could you use this Ampèrian loop to determine the linear current density?

○ Yes.
○ No.

Question 14.
Explain your answer to the previous two questions.

2. Tutorials

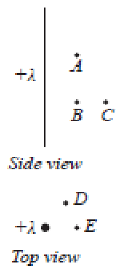
a. GSL TUT 152

GAUSS'S LAW

ED 1

I. Symmetry and electric fields

Several points are shown next to an infinite straight wire with a linear charge density $+\lambda$. Points *A* and *B* have different *z*-values, points *B* and *C* have different *s*-values, and points *D* and *E* have different ϕ-values.



A. For the following questions, *do not use Coulomb's or Gauss's Law*. Use only symmetry arguments to answer each question.

1. What types of symmetry does this charge distribution have? (*e.g.* translational, rotational, inversional)

2. Is the magnitude of the electric field at point *A* *greater than*, *less than*, or *equal to* that at point *B*? Explain.

3. Is the magnitude of the electric field at point *D* *greater than*, *less than*, or *equal to* that at point *E*? Explain.

4. Do symmetry arguments alone allow you to compare the magnitude of the electric field at point *B* to that at point *C*? Explain why or why not.

B. Consider the electric field at point *A*. State whether symmetry forbids the electric field to point in the following directions. Explain *without using Coulomb's or Gauss's Law*.

$\hat{s}$ -direction

$\hat{\varphi}$ -direction

$\hat{z}$ -direction

C. Summarize your conclusion about the electric field of the line charge thus far:

On what variable(s) can the electric field depend?

In what direction(s) can the electric field point?

➪ Check your answers to this page with a tutorial instructor before continuing.

ED 2 *Gauss's law*

II. Choosing a Gaussian surface

Gauss's Law in differential form states that the divergence of electric field is proportional to the volume charge density: $\vec{\nabla}\cdot\vec{E} = \rho / \varepsilon_o$. In integral form it states that the electric flux through a Gaussian surface is proportional to the net charge enclosed in the surface:

$$\oiint \vec{E}\cdot d\vec{A} = \iiint \rho\, dV / \varepsilon_o = Q_{encl} / \varepsilon_o .$$

Note: A Gaussian surface is an imaginary surface that fully encloses a volume, such that the area vector $d\vec{A}$ points outwards by convention.

A. A student proposes the Gaussian surface shown at right:

"I think the electric field points away from the wire. I can use a cube as my Gaussian surface because the top and bottom surfaces have no flux, and side surfaces have electric field in the same direction as the area. This makes finding the total flux easy."

The student is incorrect. Identify the flaw(s) in the student's reasoning. Explain.

B. What is a more appropriate shape for the Gaussian surface in this situation? Explain based on the properties of the charge distribution and your chosen shape.

C. On the whiteboard, sketch the charge distribution, your Gaussian surface, and the electric field at multiple points on the Gaussian surface. Label the dimensions of your Gaussian surface.

1. How much charge is enclosed in your Gaussian surface? Express your answer in terms of λ and the dimensions of your Gaussian surface.

2. Consider the flux integral $\oiint \vec{E} \cdot d\vec{A}$.
   a. For your Gaussian surface, is the dot product $\vec{E} \cdot d\vec{A}$ easy to determine for all points on the surface? Explain.
   b. For the sections of the surface where the dot product is non-zero, is the electric field on each section constant in magnitude? If not, you may need to try a different surface. If so, how does this simplify the flux integral?

3. Using Gauss's Law, find the electric field of the line charge at the surface of your Gaussian surface. Express your answer in terms of the dimensions of your Gaussian surface.

4. Suppose you changed the size of your Gaussian surface. How does the magnitude of the electric field depend on each dimension of your Gaussian surface? Explain.

D. The symmetric properties of the line charge allows us to choose an appropriate Gaussian surface such that Gauss's Law can solve for the electric field. For the following charge distributions, what shape (if any) would be an appropriate Gaussian surface? Explain.

Uniformly charged infinite sheet

Uniformly charged spherical shell

Uniformly charged cube

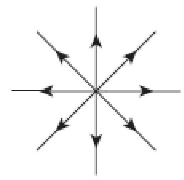
➪ Check your answers to this page with a tutorial instructor before continuing.

III. Supplement: Divergence

The *divergence* is a vector derivative that acts on a vector field $\vec{V}$, returning a scalar field ϕ, written as $\vec{\nabla} \bullet \vec{V} = \phi$.

A. We can conceptually think about the divergence by thinking about a tank of water. If water is being added to the tank at a point then the divergence is positive there. If water is flowing out of the tank through a drain then the divergence is negative there.

Points of positive divergence are called *sources*, and points of negative divergence are *sinks*.



1. The field of a single source at the origin is shown at right. Where does this field have positive divergence? Explain.

2. Does this field have negative or zero divergence anywhere?

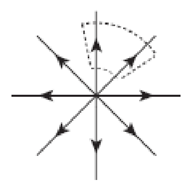
B. The divergence theorem can be used to express a volume integral of the divergence of a vector field inside a region of space in terms of the net flux of the field through that region:

$$\iiint \vec{\nabla} \cdot \vec{V} \, d\tau = \oint\!\!\oint \vec{V} \cdot d\vec{A}$$

1. Based on the equation above, if the net flux through the region is positive, what do you know about the divergence inside?

   If the net flux is zero, does this tell you the field has no divergence? Think carefully.

2. Consider the dashed region of the field from part B shown. What's the flux through this region? Does this agree with your answers to part B above?



C. In what physical systems could you use the divergence? List as many as you can think of.

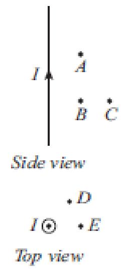
b. AML TUT 153

AMPERE'S LAW

ED 1

I. Symmetry and magnetic fields

Several points are shown next to a long straight wire carrying a current I. Points A and B have different z-values, points B and C have different s-values, and points D and E have different ϕ-values.



A. For the following questions, *do not use Ampère's law*. Use only symmetry arguments to answer each question.

1. What types of symmetry does this current distribution have? (e.g. translational, rotational, reflectional, inversional)

2. Is the magnitude of the magnetic field at point A *greater than, less than*, or *equal to* that at point B? Explain.

3. Is the magnitude of the magnetic field at point D *greater than, less than*, or *equal to* that at point E? Explain.

4. Do symmetry arguments alone allow you to compare the magnitude of the magnetic field at point B to that at point C? Explain why or why not.

B. In addition to symmetry, we know that Gauss's Law for magnetism states that $\vec{\nabla} \cdot \vec{B} = 0$ and Biot-Savart Law states that $d\vec{B} = \frac{\mu_o}{4\pi} \frac{Id\vec{l} \times (\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3}$.

State whether these laws forbid the magnetic field to point in the following directions:

$\hat{z}$ -direction $\hat{\varphi}$ -direction $\hat{s}$ -direction

C. Summarize your conclusion about the magnetic field of the line charge thus far:

On what coordinate(s) does the magnitude of the magnetic field depend?

In what direction(s) can the magnetic field point?

➯ Check your answers to this page with a tutorial instructor before continuing.

ED 2 *Ampère's law*

II. Choosing an Ampèrian loop

Ampère's law in differential form relates the curl of the magnetic field to the current density: $\vec{\nabla}\times\vec{B}=\mu_0\vec{J}$. In integral form it states that the line integral along an Ampèrian loop is proportional to the current flux through or net current encircled by the loop:

$$\oint \vec{B}\cdot d\vec{l}=\mu_o\iint \vec{J}\cdot d\vec{A}=\mu_o I_{enc}$$

Note: An Ampèrian loop is an imaginary contour that fully encircles an area, such that the area vector $d\vec{A}$ and the path vector $d\vec{l}$ are related by the right hand rule.

A. Consider again the magnetic field of a long straight wire that you analyzed in part I.

1. Consider the following student dialogue about applying Ampère's Law:

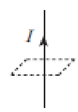
Student 1: "Ampère's law says that the line integral of the magnetic field around a loop is $\mu_o I$, but it doesn't give any information about the magnetic field directly. I can't solve that integral if I don't know the magnetic field of the wire."

Student 2: "If we know the how the magnetic field behaves, there may be certain Ampèrian loops where the magnitude of the magnetic field is constant, so it comes out of the integral. Then we can solve for the magnitude."

Student 3: "But magnetic field is a vector, so we have to consider the direction as well."

Do you agree with student 3's response? If so, how should the directions of the path and the magnetic field compare?

2. A student proposes the Ampèrian loop shown at right:



"I think the magnetic field points around the wire. I can use a square as my Ampèrian loop because the path vector at the center of each side has the same direction as the magnetic field. This makes finding the total line integral easy."

The student is incorrect. Identify the flaw(s) in the student's reasoning. Explain.

3. What is a more appropriate shape for the Ampèrian loop? Explain based on the properties of the magnetic field found in part I.

On the whiteboard, sketch the current distribution, your Ampèrian loop, and the magnetic field at multiple points along the Ampèrian loop. Label the dimensions of your Ampèrian loop.

4. Consider the line integral $\oint B \cdot dl$.

   a. For your Ampèrian loop, is the dot product $\vec{B} \cdot d\vec{l}$ easy to evaluate for all points along the path? Explain.

   b. For sections of the path where the dot product is non-zero, is the magnetic field on each section constant in magnitude? If not, you may need to try a different loop. If so, how does this simplify the line integral?

5. Suppose you doubled the perimeter of your Ampèrian loop. How would this affect the magnitude of the magnetic field at your Ampèrian loop? Explain.

6. Using Ampère's Law, find the magnitude of the magnetic field of the wire. Express your answer in terms of the dimensions of your Ampèrian loop.

B. Generalize your procedure for an arbitrary current distribution with the following questions:

1. What pieces of information must you know about the magnetic field before applying Ampère's Law?

2. How should the Ampèrian loop be chosen such that $\oint \vec{B} \cdot d\vec{l}$ can be simplified?

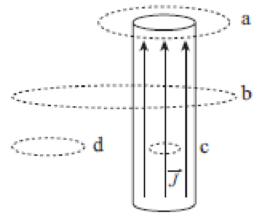
   - Which direction(s) should $d\vec{l}$ point relative to $\vec{B}$?

   - How should $|\vec{B}|$ behave along each segment of the Ampèrian loop?

➭ Check your answers to this page with a tutorial instructor before continuing.

ED 4 *Ampère's law*

C. Several Ampèrian loops are shown around a long cylindrical wire. For each loop below, explain whether or not it could give you information about the magnetic field.



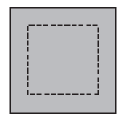
Loop a (centered on the wire):

Loop b (off-centered):

Loop c (centered on the wire):

Loop d (outside of the wire):

D. A square wire carries a uniform current density into the page. A square Ampèrian loop is shown centered inside of the wire.



Uniform current into the page

1. Two students discuss the use of this Ampèrian loop to solve for the magnetic field inside the wire:

   Student 1: "I don't think this will work. We don't have enough information about the magnetic field to do a path integral."

   Student 2: "The shape of the loop is the same as the shape of the wire. By symmetry the magnetic field is constant along this loop."

   With which student, if either, do you agree?

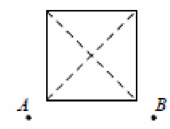
2. Does the symmetry of this current distribution show that $\vec{B}\cdot d\vec{l}$ is constant along the loop? Base your explanation on techniques explored in part I.

3. Is it possible to use Ampère's Law to find the magnetic field of a square wire?

c. GSL TUT 192

I. Symmetry and electric fields

A. Two observers A and B are looking at a square pyramid in Egypt, with locations shown on the map at right. Both observers are the same distance away from the center of the pyramid.



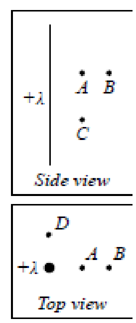
1. Suppose the two observers looked directly toward the center of the pyramid and pointed their right arm horizontally to their right. Sketch a vector showing the orientation of each observer's right arm.

2. Suppose both observers take a picture of their respective corners and compare. With no other references (*e.g.*, background, lighting, compasses, etc.) how would their pictures of the pyramid compare? Explain.

3. Mark any additional point(s) where an observer would see the pyramid in a similar orientation as observer A. Sketch a vector showing the orientation of each observer's right arm.

The exercise above shows how symmetry can tell us which points behave similarly. If a source has certain types of symmetry, the effects from that source will have the same types of symmetry.

B. Several points are shown next to an infinite straight wire with a linear charge density $+\lambda$. Points A and B have different s-values, points A and C have different z-values, and points A and D have different ϕ-values.



1. For the following questions, *do not use Coulomb's or Gauss' Law*. Use only symmetry arguments to answer each question.

   a. What type(s) of rotation can you do to the charge distribution without changing its apparent location or orientation? For each rotation, is it continuous or discrete?

   b. What type(s) of translation can you do to the charge distribution without changing its apparent location or orientation? For each translation, is it continuous or discrete?

   c. Does symmetry require that the magnitude of the electric field at point A is equal to that at points C and/or D? If so, state which symmetry requires it for each point.

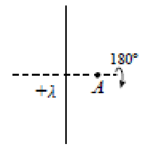
Tutorials in Introductory Physics: Electrodynamics
Physics Education Group, University of Washington

Preliminary Edition, 2019

d. Does symmetry require that the magnitude of the electric field at point *A* is equal to that at point *B*? Explain why or why not.

2. Consider rotating the line charge 180° about the dashed line.

   Use this rotation in a symmetry argument to prove that the electric field at point *A* has no $\hat{z}$ or $\hat{\phi}$ components, and explain your reasoning.



3. Consider the statement "the electric field is independent of z".

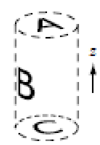
   a. Is this primarily a description of the *vector nature* or *field nature* of the electric field? Explain your reasoning.

   b. Is *continuous* or *discrete* symmetry best suited for proving the electric field is independent of a variable?

★ Check your answers to section I with a tutorial instructor before continuing.

II. Infinitesimal area vectors

An imaginary cylindrical surface is shown at right. The cylinder has length *L* and radius *R*.



A. For each surface, sketch the shape of an infinitesimal area *dA* used to integrate the partial surface, emphasizing any sides that are curved.

   • Surface A: • Surface B: • Surface C:

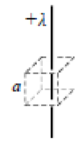
B. Write an expression for each infinitesimal area vector $d\vec{A}$ in terms of relevant quantities and infinitesimals (*i.e.* *dz*, *L*, *R*, etc.). Be sure to include a unit vector to define an outwards-pointing vector.

III. Choosing a Gaussian surface

Gauss' law in integral form states that the electric flux through an imaginary, closed surface is proportional to the net charge enclosed within the surface:

$$\oint_S \vec{E} \cdot d\vec{A} = \int_V \frac{\rho}{\epsilon_0} dV = \frac{Q_{enc}}{\epsilon_0}$$

A. A student proposes to use the Gaussian surface shown at right:



"I can use a cube as my Gaussian surface because the top and bottom surfaces have no flux, and side surfaces have electric field in the same direction as the area vector. This makes finding the electric field easy."

The student is incorrect. Identify the flaw(s) in the student's reasoning.

B. What is a more appropriate shape for the Gaussian surface in this situation? Explain based on the properties of the charge distribution and your chosen shape.

C. On a whiteboard or a large piece of paper, sketch the charge distribution, your Gaussian surface, and the electric field at multiple points on the Gaussian surface. Label the dimensions of your Gaussian surface.

1. How much charge is enclosed in your Gaussian surface? Express your answer in terms of λ and the dimensions of your Gaussian surface.

2. Consider the flux integral $\oint_S \vec{E} \cdot d\vec{A}$.

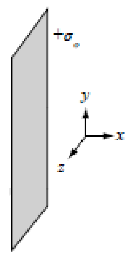
   a. For your Gaussian surface, is the dot product $\vec{E} \cdot d\vec{A}$ easy to determine for all points on the surface? Explain your reasoning.

   b. For the portion(s) of the surface where the dot product is non-zero, is the electric field constant in magnitude? If not, you may need to try a different surface. If so, how does this simplify the flux integral?

3. Using Gauss' law, find the electric field of the line charge at the surface of your Gaussian surface. Express your answer in terms of the dimensions of your Gaussian surface.

4. Suppose you changed the size of your Gaussian surface. How does the magnitude of the electric field depend on each dimension of your Gaussian surface, *i.e.* the length or diameter? Explain your reasoning.

D. An infinite sheet with a uniform surface charge density $+\sigma_o$ is shown at right.



1. What symmetries does this charged system have? For each symmetry, state the type of symmetry and whether it is continuous or discrete.

2. Using symmetry arguments alone, determine which direction(s) the electric field *cannot* point. You may wish to use a similar rotation as you used in section I.

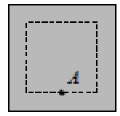
3. Using symmetry arguments alone, determine the coordinate(s) that the electric field *cannot* depend on. (*Hint:* This charge distribution only possesses 2 degrees of symmetry, so you can eliminate at most 2 coordinates.)

4. Based on your answers above, what type of Gaussian surface could make $\oint_S \vec{E} \cdot d\vec{A}$ easy to simplify in this case? Explain your reasoning.

★ Check your answers to this page with a tutorial instructor.

IV. Supplement: Discrete symmetry

A. A cube of uniform charge density is shown at right. A cubical Gaussian surface is centered inside the cube.



Cross-section of a uniformly charged cube

1. Two students discuss the use of this Gaussian surface to solve for the electric field inside the cube:

Student 1: "I don't think this will work. We don't have enough information about the electric field to simplify a surface integral."

Student 2: "The shape of the surface is the same as the shape of the cube. By symmetry the electric field is constant on the surface."

With which student, if either, do you agree?

2. Revisit the Egyptian pyramid in section I. For how many points would an observer see the same view of the pyramid as observer *A?*

Similarly, where does the electric field behave similarly to point *A* on the diagram above? Explain your reasoning.

3. Does the symmetry of this charge distribution prove that $\vec{E} \cdot d\vec{A}$ is constant for every $d\vec{A}$ on the surface?

4. Based on your answer above, would Gauss' law be useful in solving for the electric field of a uniformly charged cube?

B. The only difference between the cube above and the line charge in section III is that the cube has discrete rotational symmetry while the line charge has continuous rotational symmetry.

1. Which type of symmetry is necessary to use Gauss' law to find the electric field: *discrete, continuous, both,* or *neither?*

2. For what other shape(s) of charge distribution is Gauss' law useful in solving for the electric field?

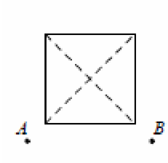
d. AML TUT 193

AMPERE'S LAW

ED
1

I. Symmetry and electric fields

A. Two observers A and B are looking at a square pyramid in Egypt, with locations shown on the map at right. Both observers are the same distance away from the center of the pyramid.



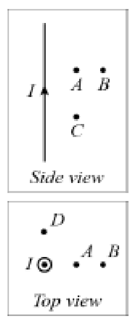
1. Suppose the two observers looked directly toward the center of the pyramid and pointed their right arm horizontally to their right. Sketch a vector showing the orientation of each observer's right arm.

2. Suppose both observers take a picture of their respective corners and compare. With no other references (*e.g.*, background, lighting, compasses, etc.) how would their pictures of the pyramid compare? Explain.

3. Mark any additional point(s) where an observer would see the pyramid in a similar orientation as observer A. Sketch a vector showing the orientation of each observer's right arm.

The exercise above shows how symmetry can tell us which points behave similarly. If a source has particular types of symmetry, the effects from that source will have the same types of symmetry.

B. Several points are shown next to an infinite straight wire carrying a current I. Points A and B have different s-values, points A and C have different z-values, and points A and D have different ϕ-values.



1. For the following questions, *do not use Ampère's law*. Use only symmetry arguments to answer each question.

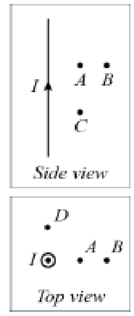
   a. What type(s) of rotation can you do to the current distribution without changing its apparent position or direction? For each rotation, is it continuous or discrete?

   b. What type(s) of translation can you do to the current distribution without changing its apparent position or direction? For each translation, is it continuous or discrete?

   c. Does symmetry require that the magnitude of the magnetic field at point A is equal to that at point C? If so, what symmetry?

ED 2 *Ampère's law*

d. Does symmetry require that the magnitude of the magnetic field at point *A* is equal to that at point *D?* If so, what symmetry?



e. Does symmetry require that the magnitude of the magnetic field at point *A* is equal to that at point *B?* If so, what symmetry?

2. In addition to symmetry, we know that Gauss's Law for magnetism states that $\vec{\nabla}\cdot\vec{B} = 0$ and the Biot-Savart Law states that $d\vec{B} = \frac{\mu_0}{4\pi}\frac{Id\vec{l}\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$.

   a. In interpreting divergence as an "in-ness" or "out-ness" around a location, which of the following component(s) does $\vec{\nabla}\cdot\vec{B} = 0$ require to be zero: B_s, B_z, and/or B_ϕ?

   b. Which of the following component(s) of the magnetic field does the cross-product in Biot-Savart Law require to be zero: B_s, B_z, and/or B_ϕ? Explain your reasoning.

   c. Based on your answers above, which component(s) of $\vec{B}$ can be non-zero?

3. Consider the statement "the magnetic field is independent of z".

   a. Is this primarily a description of the *vector nature* or *field nature* of the magnetic field? Explain your reasoning.

   b. Is *continuous* or *discrete* symmetry best suited for proving the magnetic field is independent of a variable?

★ Check your answers to this page with a tutorial instructor before continuing.

 Physics 322

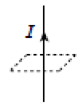
II. Choosing an Ampèrian loop

Ampère's law in integral form states that the line integral along an Ampèrian loop is proportional to the current flux through the loop or net current encircled by the loop:

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_o \int_S \vec{J} \cdot d\vec{A} = \mu_o I_{enc}$$

Note: An Ampèrian loop is an imaginary contour that fully encircles an area, such that the area vector $d\vec{A}$ and the path vector $d\vec{l}$ are related by the right hand rule.

A. A student proposes to use the Ampèrian loop shown at right:



"I think the magnetic field points around the wire. I can use a square as my Ampèrian loop because the path vector at the center of each side has the same direction as the magnetic field. This makes finding the total line integral easy."

The student is incorrect. Identify the flaw(s) in the student's reasoning. Explain.

B. What is a more appropriate shape for the Ampèrian loop? Explain based on the properties of the magnetic field found in part I.B.

C. On the whiteboard or a large piece of paper, sketch the current distribution, your Ampèrian loop, and the magnetic field at multiple points along the Ampèrian loop. Label the dimension(s) of your Ampèrian loop.

1. Consider the line integral $\oint_C \vec{B} \cdot d\vec{l}$.

   a. For your Ampèrian loop, is the dot product $\vec{B} \cdot d\vec{l}$ easy to evaluate for all points along the path? Explain your reasoning.

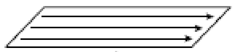
   b. For sections of the path where the dot product is non-zero, is the magnetic field on each section constant in magnitude? If not, you may need to try a different loop. If so, how does this simplify the line integral?

ED 4 *Ampère's law*

2. Using Ampère's Law, find the magnitude of the magnetic field at a point along your Ampèrian loop. Express your answer in terms of the dimension(s) of your Ampèrian loop.

3. Suppose you doubled the perimeter of your Ampèrian loop. How would this affect the magnitude of the magnetic field at a point along your Ampèrian loop? Explain.

D. An infinite sheet with a uniform surface current density $K_o\hat{x}$ is shown at right.



1. What symmetries does this current distribution have? For each symmetry, state the type of symmetry and whether it is continuous or discrete.

2. Using similar arguments to I.B.2, determine which direction(s) the magnetic field *cannot* point with symmetry, $\nabla \cdot \vec{B} = 0$, and Biot-Savart without integrating. Explain.

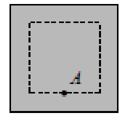
3. Using symmetry arguments alone, determine the coordinate(s) that the magnetic field *cannot* depend on. (*Hint:* This current distribution only possesses 2 degrees of continuous symmetry, so you can eliminate at most 2 coordinates.) Explain.

4. Based on your answers above, what type of Amperian loop could make $\oint_C \vec{B} \cdot d\vec{l}$ easy to simplify in this case? (*Hint:* For what relative directions is the dot product easy to evaluate?) Explain your reasoning.

Check your answers to this page with a tutorial instructor before continuing.

 Physics 322

III. Supplement: Discrete symmetry



Uniform current into the page

A. A square wire carries a uniform current density into the page. A square Ampèrian loop is shown centered inside of the wire.

1. Two students discuss the use of this Ampèrian loop to solve for the magnetic field inside the wire:

   Student 1: "I don't think this will work. We don't have enough information about the magnetic field to do a path integral."

   Student 2: "The shape of the loop is the same as the shape of the wire. By symmetry the magnetic field is constant along this loop."

   With which student, if either, do you agree?

2. Revisit the Egyptian pyramid in section I.A. For how many points would an observer see the same view of the pyramid as observer *A?*

   Similarly, where does the magnetic field behave similarly to point *A* on the diagram above? Explain your reasoning.

3. Does the symmetry of this current distribution prove that $\vec{B} \cdot d\vec{l}$ is constant for every $d\vec{l}$ along the loop?

4. Based on your answer above, would Ampère's Law be useful in solving for the magnetic field of a square wire?

B. The only difference between the square wire above and the thin wire in section II is that the square wire has discrete rotational symmetry while the thin wire has continuous rotational symmetry.

1. Which type of symmetry is necessary to use Ampère's Law to find the magnetic field: *discrete, continuous, both,* or *neither?*

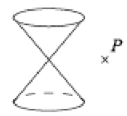
2. For what other shape(s) and/or configuration(s) of infinite wires is Ampère's Law useful in solving for the magnetic field?

3. Exam questions

a. 174E2 (GSL, POT)

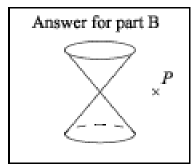
IV. [25 points total] Tutorial question.

An hourglass-shaped object (shown at right) consists of an upright cone and an upside-down (but otherwise identical) cone joined at the tip. Both cones have a uniform volume charge distribution ρ. Point P is located directly to the right of the object, in the same horizontal plane as the center point and in the plane of the paper.

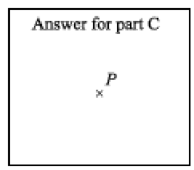


A. [6 pts] What type(s) of symmetry does this object have? Be as thorough as possible in your description of each symmetry that you identify.

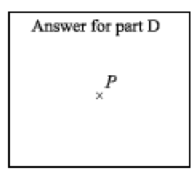
B. [6 pts] Suppose you measured the magnitude of the electric field at point P to be 2 N/C. Where else in space *must* the electric field also be 2 N/C? Sketch your answer on the diagram at right, and explain your reasoning.



C. [7 pts] In what direction(s) could the electric field point at point P? Sketch all possible vectors in the box at right. If the electric field must be zero at point P, state so explicitly. Explain your reasoning.



D. [6 pts] Based on your answer in part C, in what direction from point P could you take a small step to *not* change the value of the *electric potential?* Sketch one possible direction in the box at right. If no direction in possible, state so explicitly. Explain your reasoning.

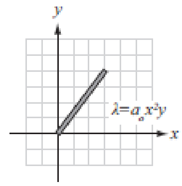


Physics 321A, Autumn 2017 Exam 2, page 5 ED-UWA321A174T-E2(GSL,POT).docx

b. 164E1 (INT, GSL)

IV. [25 points total] Tutorial question. This page contains two independent parts, A and B.

A. A slanted line charge whose linear charge density is given by $\lambda(x, y) = a_o x^2 y$ is shown at right. The endpoints of the line charge are at the origin and $(x, y) = (3, 4)$.



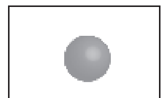
i. (2 pts) Consider the following *incorrect* student statement about determining the net charge of the system:

"The line charge varies in both x and y, so the net charge is $\int_0^3 \int_0^4 a_0 x^2 y \, dx \, dy$."

Identify the flaw(s) in the student's reasoning.

ii. (8 pts) Determine the net charge of the system. You may leave your final answer in the form $\int_a^b f(u)du$ or $\int_a^b \int_c^d f(u,v)\,du dv$. Show your work, and explain your reasoning for each step and substitution.

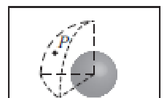
B. A uniformly charged sphere with volume charge density ρ and radius R_1 is centered on the origin.



Uniformly charged sphere

i. (5 pts) On which spherical coordinate(s) r, ϕ, and/or θ can the electric field *not depend on?* Justify your answer by using symmetry arguments.

ii. (10 pts) Consider the Gaussian surface at right. The wedge consists of a curved surface of radius R_2 concentric with the sphere, and three flat surfaces that lie in the xy-, yz- and xz-planes.



Gaussian surface perspective view

Can this Gaussian surface be used to determine the magnitude of the electric field at point P (also a distance R_2 from the center of the sphere) with Gauss' Law? Explain your reasoning.

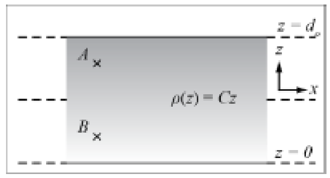
Physics 321A, Fall 2016 Exam 1, page 5 ED-UWA321A164T-E1(INT,GSL).docx

c. 184E1 (GSL, VDS)

IV. [25 points total] Tutorial question.

A charged slab with variable charge density $\rho(z) = Cz$ is shown at right, where C is a positive constant. The slab is infinite in the xy-plane, and extends from $z = 0$ to $z = d_o$. Assume that the electric field is zero at $z < 0$.

A. [6 pts] What type(s) of symmetry does this slab have (*i.e.* continuous rotational symmetry about the x-axis)? Explain how you know.



Non-uniform charged slab

B. [6 pts] Describe a Gaussian surface for which the flux integral will be easy to simplify. You may use sketches or diagrams in your answer. Explain why you chose this surface.

C. [8 pts] How does electric field within the slab depend on z (*i.e.* constant with z, proportional to $1/z$, z^2, etc.)? Explain your reasoning.

How does the electric field within the slab depend on x (*i.e.* constant with x, proportional to $1/x$, x^2, etc.)? Explain your reasoning.

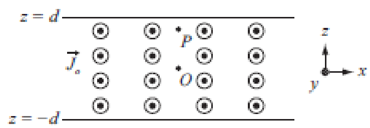
D. [5 pts] Is the divergence of the electric field at point A *greater than*, *less than*, or *equal to* the divergence of the electric field at point B? Explain your reasoning.

Physics 321A, Autumn 2018 Exam 1, page 6 ED-UWA321A184T-E1(GSL,VDS).docx

d. 161E1 (AML)

IV. [25 points total] Tutorial question.

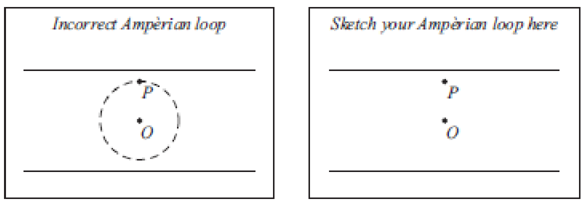
An infinite slab with uniform current density J_o directed out of the page ($-\hat{y}$) fills the space from $z = -d$ to $z = d$. The magnetic field at the origin (Point O) is zero. There are no other sources in this system.



Infinite slab of current

A. [7 pts.] What component(s) of the magnetic field due to this slab at point P **must be zero**? For each component you identified, explain your reasoning.

B. [10 pts.] A student wants to solve for the magnitude of the magnetic field at point P, but *incorrectly* chooses the Ampèrian loop shown below. Sketch a usable loop in the space provided.



- Explain why the student's loop cannot be used to calculate the magnitude of the magnetic field at point P.

- Explain why your loop can be used to calculate the magnitude of the magnetic field at point P.

C. [8 pts] Consider the magnitude of the magnetic field for points *inside* the slab.

How does the magnitude of the magnetic field depend on y (*i.e.* constant with y, proportional to $1/y$, y^2, etc.)? Explain.

How does the magnitude of the magnetic field depend on z (*i.e.* constant with z, proportional to $1/z$, z^2, etc.)? Explain.

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D.Supporting documents for Chapter 4

1. Pretests

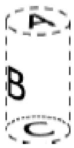
a. (MSF)U1a

Print view of '(MSF)U1a'

Print this page

Part I.

Consider a cylindrical Gaussian surface that completely encloses a volume of space. The Gaussian surface is split into three sections: Surface A, Surface B, and Surface C. (Note: The normal vector of the surface points outward.)



Question 1.
Suppose a vector field has positive divergence everywhere **within** the volume, but unknown divergence outside of the surface. Is the flux of the vector field *positive, negative, or zero* for Surface A and the entire Gaussian surface? If there is not enough information, choose the appropriate response.

Rows
Surface A
Entire Gaussian surface
- ○ Positive
- ○ Negative
- ○ Zero
- ○ Not enough information

Question 2.
Explain your reasoning for the question above.

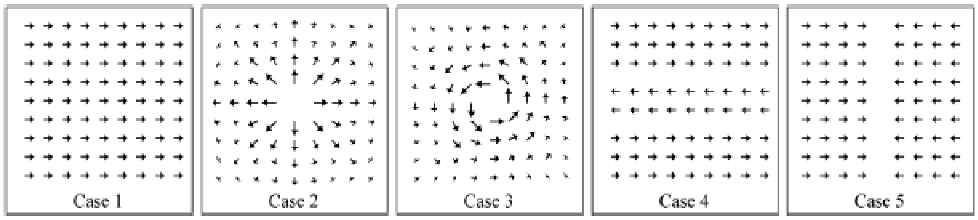
Question 3.
Suppose the net flux of a vector field through the entire Gaussian surface is zero. Which of the following statements **must** be true? Check all that apply.

- ☐ The flux of the vector field through Surface A must be zero.
- ☐ The vector field within the volume of space must be constant.
- ☐ The divergence of the vector field within the volume of space must be zero everywhere.
- ☐ The net divergence of the vector field within the volume of space must sum to zero.
- ☐ The curl of the vector field within the volume of space must be zero everywhere.
- ☐ The curl of the vector field within the volume of space must average to the zero vector.
- ☐ None of the above.

Question 4.
Explain your reasoning for the question above.

Part II.

Shown below are five vector fields, Case 1-5.



Question 5.
For which of the following cases is the **divergence** of the vector field **zero everywhere** within the shown region? Check all that apply.

- ☐ Case 1
- ☐ Case 2
- ☐ Case 3
- ☐ Case 4
- ☐ Case 5
- ☐ None of the above.

Question 6.
Explain your reasoning for the question above.

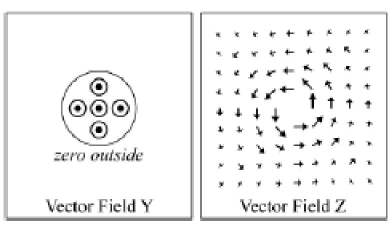
Question 7.
For which of the following cases is the **curl** of the vector field **zero everywhere** within the shown region? Check all that apply.

- ☐ Case 1
- ☐ Case 2
- ☐ Case 3
- ☐ Case 4
- ☐ Case 5
- ☐ None of the above.

Question 8.
Explain your reasoning for the question above.

Part III.

The result of a curl operator acting on one vector field is another vector field. The mathematical description is given by: $\vec{\nabla} \times \vec{W} = \vec{X}$. A visual example is shown below.



Question 9.

In physics, one vector field is considered the 'source' for the other. Which pair of vector fields are considered the 'sources'?

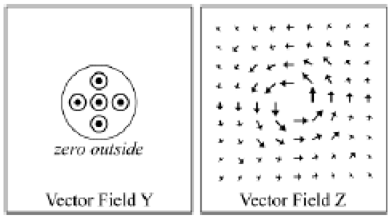
- ○ Vectors W and Y
- ○ Vectors W and Z
- ○ Vectors X and Y
- ○ Vectors X and Z

Question 10.
Explain your reasoning for the question above.

b. (MVP)U1b_NoMSF

Part I.

Below are visual representations of two unknown vector fields, $\vec{Y}$ and $\vec{Z}$. One could argue that these two vector fields represent a source-effect relationship, where one vector field is the source of the other.



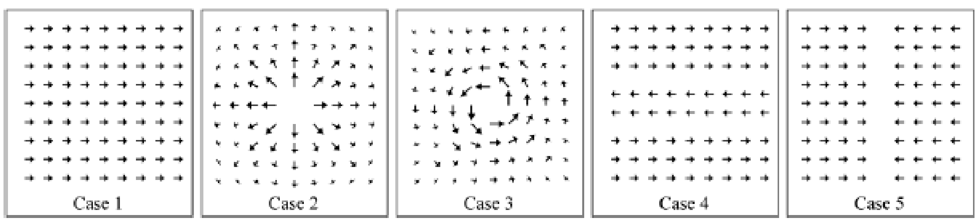
Question 1.
Which of the two vector fields is best considered the "source"?

- ○ Vector Y
- ○ Vector Z

Question 2.
Explain your reasoning for the question above.

Part II.

Shown below are five vector field diagrams (Cases 1-5) that represent the magnetic vector field, **A**.



Question 3.
For which of the following cases is the vector potential in the Coulomb's Gauge (i.e., where $\vec{\nabla} \cdot \vec{A} = 0$)?

- ☐ Case 1
- ☐ Case 2
- ☐ Case 3
- ☐ Case 4
- ☐ Case 5
- ☐ None of the above.

Question 4.
Explain your reasoning for the question above.

Question 5.
For which of the following cases is the there no magnetic field within the shown region? Check all that apply.

- ☐ Case 1
- ☐ Case 2
- ☐ Case 3
- ☐ Case 4
- ☐ Case 5
- ☐ None of the above.

Question 6.
Explain your reasoning for the question above.

Part III.

The curl relationship can also be interpreted as a source-effect relationship. The mathematical description is given by: $\vec{\nabla} \times \vec{W} = \vec{X}$, where W and X are both unknown vector fields.

Question 7.
Which of the two vector fields is best considered the "source"?

- ○ Vector W
- ○ Vector X

Question 8.
Explain your reasoning for the question above.

c. (DEL)U1b_Research2

Question 1.
Please select your 321 section below:

	Logic destinations
○ 321 AA	----------------------------...
○ 321 AB	Don't skip (default)
○ 321 AC	----------------------------...
○ 321 AD	Don't skip (default)
○ 321 AE	----------------------------...
○ 321 AF	Don't skip (default)
○ 321 AG	----------------------------...
○ Other	Don't skip (default)
No response	----------------------------...

Part I: Consider the time-derivative operator: *d/dt*. Mathematically, the relationship between velocity $\vec{v}$ and acceleration $\vec{a}$ can be written as $\frac{d\vec{v}}{dt} = \vec{a}$, where $\vec{v}$ and $\vec{a}$ are functions of *t*.

Question 2.
In physics, we can often interpret equations with a cause-effect or source-effect relationship. In the mathematical equation above, which of the following is best interpreted as the source or cause in this relationship?

○ velocity, <i>v</i>
○ acceleration, <i>a</i>
○ None: This relationship has no source or cause.

Question 3.
Explain your answer to the previous question.

Logic destination
Question 6: Now consider Newton's 2nd L...

Part I: Consider the derivative operator: *d/du*.

Mathematically, a differential relationship can be written as $\frac{df(u)}{du} = g(u)$, where *f*(*u*) and *g*(*u*) are functions of *u*.

Question 4.
In physics, we can often interpret equations with a cause-effect or source-effect relationship. In the mathematical equation above, which of the following can be best interpreted as the source or cause in this relationship?

○ <i>f</i>(<i>u</i>)
○ <i>g</i>(<i>u</i>)
○ None: Derivative relationships never have sources or causes.

Question 5.
Explain your answer to the previous question.

Question 6.

Now consider Newton's 2nd Law relating momentum $\vec{p}$ and net external force $\vec{F}_{net}$ in the equation: $\frac{d\vec{p}}{dt} = \vec{F}_{net}$.

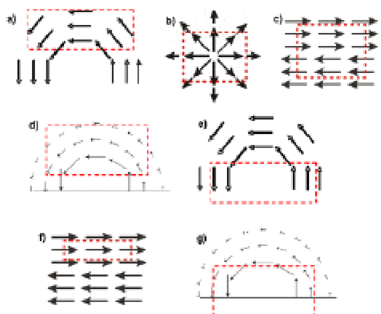
Which of the following best represents the source or cause of this relationship?

- Momentum, <i>p</i>
- Force, <i>F</i>
- None: This relationship has no source or cause.

Question 7.
If you were able to redo the previous page, would you change your answer? Choose the answer choice that best fits why.

- No, because I am unsure of my answer on this page.
- No, because this page is unrelated to the previous page.
- No, because I am confident in my answers to the previous page.
- No, because I answered this question based on the previous page.
- No, because I guessed consistently on the previous page.
- Yes, because my reasoning here contradicts my answer to the previous page.
- Yes, because I did not understand what the previous page was asking and guessed.
- Other:

Part II: Consider the following vector field diagrams. For clarification, diagrams a) and e) have radially-independent azimuthal field.



Question 8.
For each of the vector fields above, determine whether the dashed region contains divergence, *curl*, *both*, or *neither*.

Rows
Diagram a
Diagram b
Diagram c
Diagram d
Diagram e
Diagram f

Diagram g

- ○ Divergence only.
- ○ Curl only.
- ○ Both.
- ○ Neither.

Question 9.
Explain your answer to the previous question for diagram c).

Question 10.
Which of the diagrams above (a-d) could represent the following vector fields? Check all that apply.

Rows

Electric field of a line charge (any orientation)

Electric field of a sheet (any orientation)

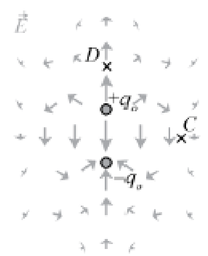
Magnetic field of a line current (any orientation)

Magnetic field of a sheet current (any orientation)

- ☐ a)
- ☐ b)
- ☐ c)
- ☐ d)
- ☐ None of the above.

Question 11.
Explain your answer to the previous question for "Electric field of a line charge."

A vector field diagram of the electric field due to two point charges (a dipole) is shown below.



Vector field diagram
(not a field line diagram)

Question 12.
Is the divergence of the electric field at point *D positive*, *negative*, or *zero?*

- ○ Positive.
- ○ Negative.
- ○ Zero.
- ○ Cannot be determined.

Question 13.
Explain your answer to the previous question.

Question 14.
Is the curl of the electric field at point *C positive*, *negative*, or *zero?*

- ○ Positive.
- ○ Negative.
- ○ Zero.
- ○ Cannot be determined.

Question 15.
Explain your answer to the previous question.

d. (VDS)U1a_ForceResearch

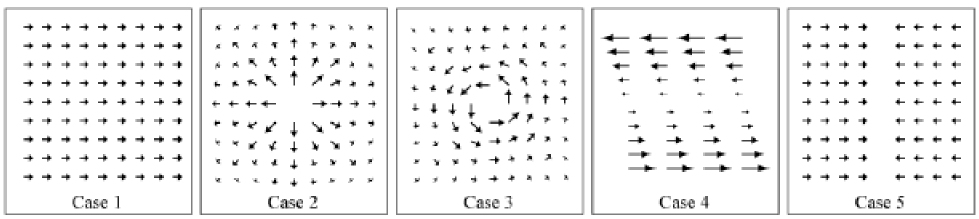
Part I: Consider Newton's second law relating momentum $\vec{p}$ and the net external force $\vec{F}_{net}$ in the equation: $\frac{d\vec{p}}{dt} = \vec{F}_{net}$.

Question 1.
Which of the following best represents the source or cause in this relationship?

- ○ Momentum, <i>p</i>
- ○ Force, <i>F</i>
- ○ Time-derivative operator, <i>d/dt</i>
- ○ None: This relationship has no source or cause.

Question 2.
Explain your answer to the previous question.

Part II: Consider the following vector field diagrams.



Question 3.
For each of the vector fields above, determine whether the picture contains *divergence*, *curl*, *both*, or *neither*.

Rows
Case 1
Case 2
Case 3
Case 4
Case 5

- ○ Divergence only.
- ○ Curl only.
- ○ Both.
- ○ Neither.

Question 4.
Explain your answer to the previous question for Case 4.

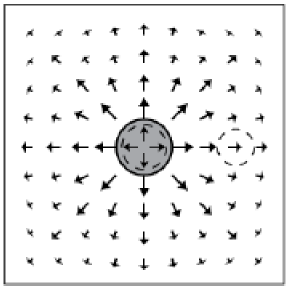
Question 5.
If the vector field diagram were describing the electric field, which of the following cases contain charge? Check all that apply. If none of the cases apply, leave this question blank.

- ☐ Case 1
- ☐ Case 2
- ☐ Case 3
- ☐ Case 4

☐ Case 5

Question 6.
Explain your answer to the previous question.

Part III. The vector-field diagram of the electric field of a uniformly charged solid sphere is shown below. The diagram is drawn as a cross-sectional plane through the center of the sphere.



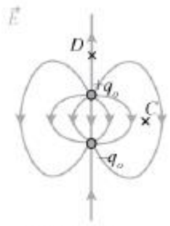
The dark area represents the uniformly charged sphere. Two dashed surfaces are shown surrounding two regions, with one inside the sphere and one to the right.

Question 7.
For each region, determine whether the net divergence of the electric field within the region is positive, negative, or zero.

Rows
Inside region
Outside region (on the right)
- ◯ Positive.
- ◯ Negative.
- ◯ Zero.
- ◯ Cannot be determined.

Question 8.
Explain your answer to the previous question.

Part IV: The field-line diagram of the electric field of a dipole is shown below. The dipole consists of two point charges (labeled $+q_0$ and $-q_0$). Two locations in empty space are marked as *C* and *D*.



Field line diagram

Question 9.
Is the divergence of the electric field at point *D positive, negative,* or *zero?*

- ◯ Positive.
- ◯ Negative.
- ◯ Zero.
- ◯ Cannot be determined.

Question 10.
Explain your answer to the previous question.

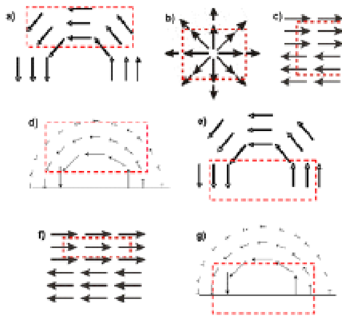
Question 11.
Is the curl of the electric field at point *C into the page (clockwise), out of the page (counterclockwise), downward,* or *zero?*

- ◯ Into the page (clockwise).
- ◯ Out of the page (counter-clockwise).
- ◯ Downward.
- ◯ Zero.
- ◯ Cannot be determined.

Question 12.
Explain your answer to the previous question.

e. (MVP)U1a_Research

Part I: In each of the diagrams below, the vector potential **A** is shown by a vector field.



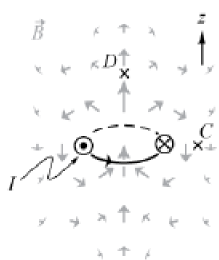
Question 1.
For which of the cases above must there be a nonzero B field inside the dashed region? Choose all that apply.

- ☐ A.
- ☐ B.
- ☐ C.
- ☐ D.
- ☐ E.
- ☐ F.
- ☐ G.
- ☐ None of them have magnetic field inside the dashed region.

Question 2.
Briefly explain how you determined whether there was magnetic field in that region.

Part II:

A vector field diagram of the magnetic field due to a loop of current (a dipole) is shown below.



Vector field diagram
(not a field line diagram)

Question 3.
Is the divergence of the magnetic field at point *D positive, negative,* or *zero?*

- ○ Positive.
- ○ Negative.
- ○ Zero.
- ○ Cannot be determined.

Question 4.
Explain your reasoning for the previous question.

Question 5.
In what direction is the curl of the magnetic field at point *C?*

- ○ Up.
- ○ Down.
- ○ In / clockwise.
- ○ Out / counterclockwise.
- ○ Zero.
- ○ Cannot be determined.

Question 6.
Explain your reasoning for the previous question.

Question 7.

Part III: The following statements about V, the electric potential, and A, the magnetic vector potential, may be true or false. Please select ALL that are true.

- ☐ V and A are physical quantities that can be directly and absolutely measured.
- ☐ There can be a non-zero A-field throughout a region with zero B-field.
- ☐ There can be a non-zero V-field throughout a region with zero E-field.
- ☐ V and A are both related to potential energy.

Question 8.
Explain your reasoning for the previous question.

Question 9.
How are V and **E** related? Please select ALL that apply.

- ☐ The divergence of V tells you E.
- ☐ The gradient of V tells you E.
- ☐ The curl of V tells you E.
- ☐ None of these.

Question 10.
How are **A** and **B** related? Please select ALL that apply.

- ☐ The divergence of A tells you B.
- ☐ The gradient of A tells you B.
- ☐ The curl of A tells you B.
- ☐ None of these.

Question 11.
Explain your reasoning for the previous two questions.

Question 12.

Part IV: Suppose you had a long needle-like material with a strong B-field running up it, but with B=0 outside the needle, i.e. $\vec{B} = C\delta^2(s)\hat{z}$, where s is the cylindrical radial coordinate and C is a constant.

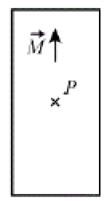
What would the vector potential look like outside the needle (far from the ends of the needle)? Please choose one.

- ○ Pointing in z-hat, and depending on s.
- ○ Pointing in z-hat, independent of s.
- ○ Pointing in s-hat, and depending on s.
- ○ Pointing in s-hat, independent of s.
- ○ Pointing in phi-hat, and depending on s.
- ○ Pointing in phi-hat, independent of s.
- ○ A superposition of some of the above answers.
- ○ Zero.

Question 13.
Explain your reasoning for the previous question.

f. (AUX)U1b

Part I. Consider a bar magnet with uniform magnetization pointing upward. Point *P* is located at the center of the bar magnet.



Bar Magnet

Question 1.
At what location would the north pole of this magnet be?

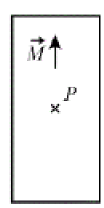
- ○ At the top.
- ○ At the right.
- ○ At the left.
- ○ At the bottom.
- ○ At the center.
- ○ Cannot be determined.

Question 2.
With what method would you prefer to use to determine the direction of the auxiliary field at point *P*?

- ○ Definition: <i>H = B/μ_o - M</i>
- ○ Derivatives: Divergence and Curl of <i>H</i>
- ○ Approximation: <i>H = B/μ</i>

Question 3.
Explain your reasoning to the previous question.

Part II. Attempt to determine the direction of the auxiliary field at point *P* by using $\boldsymbol{H} = \boldsymbol{B}/\mu_o - \boldsymbol{M}$. Your answer(s) may be inconsistent with other parts.



Bar Magnet

Question 4.
In what direction is the magnetic field at point *P*?

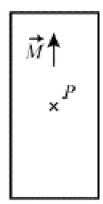
- ○ Upward.
- ○ Downward.
- ○ Zero.
- ○ Cannot be determined.

Question 5.
Based on $\boldsymbol{H} = \boldsymbol{B}/\mu_0 - \boldsymbol{M}$, in what direction is the auxiliary field at the center of the slab?

- ○ Upward.
- ○ Downward.
- ○ Zero.
- ○ Cannot be determined.

Question 6.
Explain your reasoning for the previous question.

Part III. Attempt to determine the direction of the auxiliary field at point *P* by using the vector derivatives of ***H***. Your answer(s) may be inconsistent with other parts.



Bar Magnet

Question 7.
At what location(s) is the divergence of the auxiliary field non-zero? Check all that apply.

- ☐ Top and bottom of the magnet.
- ☐ Sides of the magnet.
- ☐ Inside of the magnet.
- ☐ Outside of the magnet.
- ☐ None of the above.

Question 8.
At what location(s) is the curl of the auxiliary field non-zero? Check all that apply.

- ☐ Top and bottom of the magnet.
- ☐ Sides of the magnet.
- ☐ Inside of the magnet.

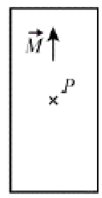
☐ Outside of the magnet.
☐ None of the above.

Question 9.
Based on the derivatives of ***H***, in what direction is the auxiliary field at point *P*?

○ Upward.
○ Downward.
○ Zero.
○ Cannot be determined.

Question 10.
Explain your reasoning for the previous question.

Part IV. Attempt to determine the direction of the auxiliary field at point *P* by using the approximation ***H*** = ***B***/*μ*. Your answer(s) may be inconsistent with other parts.



Bar Magnet

Question 11.
In what direction is the magnetic field at point *P*?

○ Upward.
○ Downward.
○ Zero.
○ Cannot be determined.

Question 12.
What is the sign of *μ* for this material?

○ Positive.
○ Negative.
○ Zero.
○ Cannot be determined.
○ Approximation cannot be used.

Question 13.
Explain your reasoning for the previous question.

Question 14.
Based on ***H*** = ***B***/*μ*, in what direction is the auxiliary field at point *P*?

○ Upward.
○ Downward.
○ Zero.
○ Cannot be determined.

2. Tutorials

a. DEL TUT 162

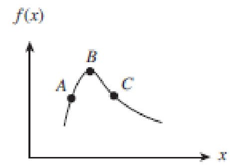
VECTOR DERIVATIVES

ED
1

I. Partial derivatives

A. Consider the one-dimensional scalar function $f(x)$ at right.

1. What information does the derivative, evaluated at each point, tell you about the behavior of the function at that point?



Suppose $f(x)$ represents the height of a hill. Consider releasing a small ball from rest at each of the marked points in turn.

2. What direction, if any, would the ball roll from each point?

3. Rank the magnitude of acceleration of the ball at each point, from greatest to least.

4. Use your answers to write an expression relating the acceleration of the ball at each point to the derivative of the function at each point. Include a unit vector in your expression.

B. Suppose the hill was two-dimensional instead, such that the height of the hill was described by a two-dimensional scalar function $f(x,y)$.

1. How could you determine the x-component of the acceleration of the ball? Would you use a total derivative or a partial derivative? Explain.

2. How could you determine the y-component of the acceleration of the ball?

The *del operator* is defined as a vector sum of partial derivatives: $\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \cdots$.

C. Based on your answers above, write a general expression for the acceleration of a ball on a hill in terms of the del operator.

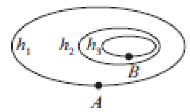
ED 2 *Vector derivatives*

II. Gradient

When the del operator acts on a scalar function, it is called the *gradient*. As you saw in the previous section, the gradient is related to the generalized slope of the function: the gradient points in the direction of *greatest increase* in the value of the function.

A. A two-dimensional function can be represented as a surface in three dimensions, where the height of the surface is the value of the function.

A contour graph of a hill is shown at right. Each closed loop corresponds to a set of points where the height is constant, with $h_1 < h_2 < h_3$. The height difference between h_1 and h_2 is the same as that between h_2 and h_3.



1. On the diagram, indicate the direction of the gradient at point *A*.

2. How does the magnitude of the gradient at point *B* compare to that at point *A*? Explain.

3. Suppose you were standing at point *A*, and took a small step in some direction. Relative to the gradient, state which direction(s) would lead to the following changes in height.

   • Move to a higher height? • Move to a lower height?

   • Stay at the same height?

4. What do your answers above imply about the direction of the gradient compared to lines of constant value?

   Use your answer to check the vectors you drew on the diagram above, and correct your drawing if necessary.

B. How does the gradient of a three-dimensional function $g(x,y,z)$ compare to the gradient of a two-dimensional function $f(x,y)$?

   What would the analogue of lines of constant value be in three dimensions?

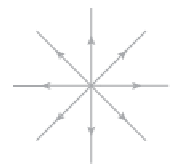
➪ Check your answers to this section with a tutorial instructor before continuing.

III. Divergence

One way to conceptually understand a vector field is to interpret the field as describing a flow of water. The field lines represent the motion of the water, and the magnitude of the field represents the speed of the water.

The dot product between the del operator and a vector field is called the *divergence* of the field: $\vec{\nabla} \cdot \vec{V}(x,y)$. In the water analogy, the divergence measures the rate of change of water at a point. If the divergence is positive at a point, that point is called a *source*, and if the divergence is negative the point is called a *sink*. If the divergence is zero, water flows through the point without gain or loss.

A. Consider the field of a single source at the origin.



1. Does this field have points of positive divergence? If so, where? Explain.

2. Does this field have points of zero divergence? If so, where? Explain.

B. Three students discuss their answers to part A above.

Student 1: "Water is added at the origin so the divergence there is positive. It flows straight away after that, so the divergence is zero everywhere else.

Student 2: "I disagree. The field lines get farther apart everywhere, so the divergence should be positive everywhere."

Student 3: "Since the field lines are getting farther apart, the magnitude of the field is decreasing everywhere. Thus the divergence is positive at the origin, but negative everywhere else."

With which student, if any, do you agree? Explain.

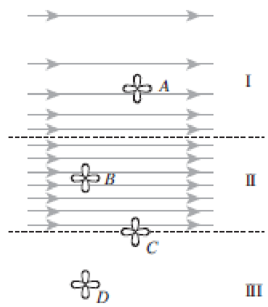
Check that your answer is consistent with the definition of divergence.

ED 4 *Vector derivatives*

IV. Curl

The del operator can also act on a vector field through the cross product, which is called the *curl:* $\vec{\nabla}\times\vec{V}(x,y)$. In the water analogy, the curl measures the net torque around a point. We can understand the curl by placing a small imaginary pinwheel in the flow of water. If the pinwheel rotates there is a nonzero curl at that point.

A. A spatially varying vector field is shown at right. In region I the field increases in magnitude towards the bottom of the page. In region II the field is uniform, and in region III the field is zero. The dashed lines show the edge of each region.



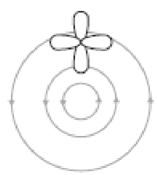
1. Four pinwheels are placed in the field as shown. Which of the pinwheels would rotate? Explain.

The direction of the curl is the direction of the net torque on the pinwheel, given by the right hand rule.

2. At each point *A-D*, which direction does the curl of the vector field point? If the curl is zero at any point, state so explicitly.

3. Considering the entire field, where in space is there curl? Explain.

B. A large pinwheel is placed in a circular vector field are shown at right.



Three students discuss the curl of the field at the pinwheel:

Student 1: "The field rotates counter-clockwise everywhere, so at the pinwheel the curl is out of the page."

Student 2: "I agree. The field pushes the upper three arms of the pinwheel counter-clockwise, but only the lower arm clockwise. The net rotation is counter-clockwise."

Student 3: "I disagree. The field is stronger on the lower arm since the field lines are closer together. The pinwheel spins clockwise, so the curl is into the page."

It is observed that the pinwheel does not rotate, so none of the students is correct. Explain the error(s) made by each student.

➪ Check your answers to this section with a tutorial instructor before continuing.

V. Supplement: Mixed derivatives

In electrodynamics, second-order derivatives are also important. In this section you will explore one mixed derivative, and in the homework you will investigate the others.

A. Consider the divergence of the curl of a vector field, $\vec{\nabla} \cdot [\vec{\nabla} \times \vec{V}]$.

1. First, sketch a simple example of a vector field with a single point of non-zero divergence.

2. Suppose your sketch above describes the curl of $\vec{V}$, *i.e.* it is a sketch of the vector field $[\vec{\nabla} \times \vec{V}]$.

   a. At the point where there is non-zero divergence, is the curl of $\vec{V}$ well-defined? Explain. (*Hint:* At this point, does $\vec{\nabla} \times \vec{V}$ have a single defined direction, or could it point in many different directions?)

   b. Given your answer above and the definition of the curl, is $\vec{V}$ well-defined? Explain.

B. In general, is it possible for the curl of a vector field to have divergence?

C. Consider a physical vector field which has no divergence anywhere in space: $\vec{\nabla} \cdot \vec{C} = 0$.

1. Could we write $\vec{C}$ as the curl of another vector field, *i.e.* $\vec{C} = \vec{\nabla} \times \vec{D}$? Explain.

2. Is $\vec{D}$ unique? (*Hint:* Could you add a constant to $\vec{D}$ without changing $\vec{C}$?)

   Given your answer above, is $\vec{D}$ physically observable? Explain why or why not.

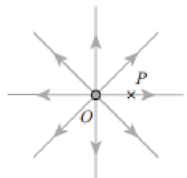
b. DEL TUT 174 – modified divergence section of DEL TUT 162

ES
9

III. Divergence

The dot product between the del operator and a vector field is called the *divergence* of the field: $\vec{\nabla} \cdot \vec{\psi}(\vec{r})$. One way to conceptualize the divergence is to think of it as a measure of the creation or destruction of the vector field, in a way that water flows *from a faucet* of positive divergence and *toward a sink* of negative divergence. In electromagnetism, the most important instance of the divergence is Gauss' Law: $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_o}$.

Consider the electric field of a single positive charge at the origin. Point *P* is located to the right of the origin.



A. At point *P*, *predict* whether the divergence of the electric field is *positive*, *negative*, or *zero*. Explain your prediction.

B. Mathematically, the divergence can be expanded as: $\vec{\nabla} \cdot \vec{\psi} = \frac{\partial \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial y} + \frac{\partial \psi_z}{\partial z}$.

1. Consider a point to the right of point *P* and a point to the left of point *P*.

   a. Is the numerical value of the *x*-component of the electric field to the right point *P* *greater than*, *less than*, or *equal to* that to the left of point *P?*

   b. Is the derivative $\frac{\partial E_x}{\partial x}$ *positive*, *negative*, or *zero* at point *P?*

2. Consider a point above point *P* and a point below point *P*.

   a. Is the numerical value of the *y*-component of the electric field above point *P* *greater than*, *less than*, or *equal to* that below point *P?*

   b. Is the derivative $\frac{\partial E_y}{\partial y}$ *positive*, *negative*, or *zero* at point *P?*

3. Based on your answers above, is $\vec{\nabla} \cdot \vec{E}$ at point *P* *positive*, *negative*, or *zero?* If there is not enough information, state so explicitly.

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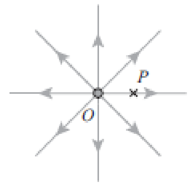
C. Visually, the divergence of a field is related to how the field is being produced or destroyed, in a sense that water flows *from a faucet* of positive divergence, and *toward a sink* of negative divergence.



1. To make a water-flow pattern like this:
   - Where must water be added?
   - Where does water flow freely?
   - Where does water drain?

2. Colloquially, *to diverge* means to separate from another route.
   a. At the origin, does the colloquial definition match the technical definition of divergence?
   b. At point *P*, does the colloquial definition match the technical definition of divergence?

D. Physically, Gauss' Law relates the divergence of the electric field to charge: $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_o}$.

1. Consider the right-hand side of Gauss' Law. Is the charge density at point *P* *positive*, *negative*, or *zero?*
2. What does that imply about the left-hand side of Gauss' Law?
3. Based on your answers above, what is the divergence of the electric field at point *P?*

A central theme in this course sequence is that the divergence of a vector field acts as a source to the vector field.

E. Recall another relationship that shows how a charge acts as a source to the electric field. How would you apply this relationship to determine the electric field from a system of charges?

➪ Check your answers to this section with a tutorial instructor before continuing.

a. VDS TUT 192

DIVERGENCE AS A SOURCE

Divergence as a source ED 13

I. Comparing Coulomb's Law and Gauss' Law

The generalized Coulomb's Law for electric fields is often given by:
$\vec{E} = \frac{1}{4\pi\varepsilon_o}\int \frac{\rho(r')d\tau'}{|\vec{r}-\vec{r'}|^2}\widehat{r-r'}$, and Gauss' Law in integral form is often given by: $\oint_S \vec{E}\cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_o}$.

A. Consider the case of a single point charge of charge $+q_o$ at the origin.

1. How does $\vec{E} = \frac{1}{4\pi\varepsilon_o}\int \frac{\rho(r')d\tau'}{|\vec{r}-\vec{r'}|^2}\widehat{r-r'}$ simplify when used to evaluate the electric field a distance d_o away from the point charge?

   What principle is the integral sign in Coulomb's Law expressing?

2. How does $\oint_S \vec{E}\cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_o}$ simplify when used to evaluate the electric field a distance d_o away from the point charge?

   What principle is necessary to simplify the integral sign in Gauss' Law?

3. Note that Coulomb's Law and Gauss' Law both have quantities related to charge, electric field, and spatial distances. What general relationship are they both describing?

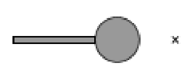
B. Consider two cases shown at right. Case A is a uniformly charged sphere, and Case B is a uniformly charged curved rod.

Case A: sphere Case B: curved rod

1. Which law above would you use to determine the electric field at a point near the sphere? Explain briefly.

2. Which law above would you use to determine the electric field at a point near the rod? Explain briefly.

3. Consider the charge distribution at right, consisting of a finite thin rod attached to a uniformly charged sphere. How would you determine the electric field at the marked point?

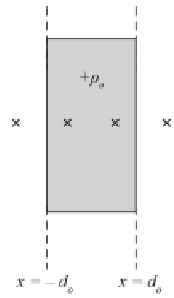


★ Check your answers with a tutorial instructor before continuing.

ED 14 *Divergence as a source*

II. Gauss' Law and Divergence

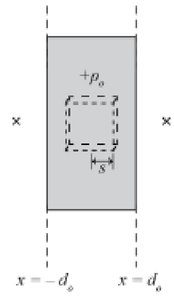
A. An isolated charged slab with uniform volume charge density $+\rho_o$ is infinite in the *yz*-plane and spans from $x = -d_o$ to $x = d_o$, as shown at upper right. The magnitude of the electric field is zero at the origin.



One can show that the magnitude of the electric field increases linearly from the center of the slab to the surface and is constant outside: such that $|\vec{E}| = C|x|$ for $-d_o < x < d_o$ and $|\vec{E}| = Cd_o$ for $|x| > d_o$, where C is a constant.

1. Based on the symmetry of the system, sketch arrows showing the direction of the electric field at each of the four points at right.

Consider a Gaussian cube with side length $2s$ centered on the origin, where $s < d_o$ as shown below right.



2. What is the net electric flux through the cube? Consider which faces contribute to flux and the location those faces as you evaluate $\oint_S \vec{E} \cdot d\vec{A}$.

3. How much charge is enclosed in the Gaussian cube?

4. Determine the constant C using Gauss' Law. *Hint:* would you expect C to be dependent on x and/or s?

The divergence theorem states that a closed surface integral of a field's flux is equal to the total divergence within the volume: $\oint \vec{\psi} \cdot d\vec{A} = \int(\vec{\nabla} \cdot \vec{\psi})dV$.

B. Based on this expression, one interpretation of the divergence is a flux density: net flux per unit volume.

1. Determine an expression for the electric flux density within the Gaussian cube above.

2. Based on your answer above, what other density is related to the divergence of the electric field?

3. *Predict* the divergence of the electric field of the slab at its center, where $x = 0$.

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4. Two students discuss the divergence of the electric field at the center of the slab.

   Student 1: "I think the divergence of the electric field at the center is zero. The electric field at that spot has zero magnitude."

   Student 2: "I agree. If you look closely with a small enough Gaussian surface, the field appears uniform. The net electric flux will be zero at that limit."

   Both students are *incorrect*. Explain the flaws in their reasoning.

C. Mathematically, the divergence of a field is considered a *vector derivative*, as the del operator $\vec{\nabla}$ is a derivative with respect to a multi-dimensional position.

1. Consider Gauss' Law in differential form, given by $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_o}$:

   a. What physical quantities are being related in this equation?

   b. Compare the physical quantities above to the quantities present in $\vec{E} = \frac{1}{4\pi\varepsilon_o}\int \frac{\rho(r')d\tau'}{|\vec{r}-\vec{r}'|^2}\widehat{r-r'}$ and $\oint_S \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_o}$. What can you infer about the nature of these three equations?

The divergence of a field is in Cartesian coordinates given by: $\vec{\nabla} \cdot \vec{\psi} = \frac{\partial \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial y} + \frac{\partial \psi_z}{\partial z}$.

2. Evaluate the divergence for the slab inside and outside the slab, whose electric field is given by: $\vec{E}_{\text{in}} = Cx\,\hat{x}$ for $-d_o < x < d_o$ (inside), and $\vec{E}_{\text{out}} = \pm C d_o \hat{x}$ for $|x| > d_o$.

   Check that your result in consistent with Gauss' Law in differential form, given your value of C in question A.4.

D. Physically, we can interpret divergence of a vector field as the "*source*" that causes *changes in flux*. Regions of positive divergence add a net outward flux, while regions of negative divergence add a net inward flux.

1. What previous interpretation of divergence leads to the idea of *source?*

2. What previous interpretation of divergence leads to the idea of *change?*

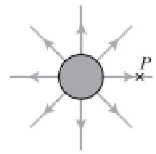
3. What previous interpretation of divergence leads to the idea of *flux?*

★ Check your answers with a tutorial instructor before continuing.

ED 16 *Divergence as a source*

III. Conceptualizing divergence

A charged sphere with uniform volume charge density and total charge $+Q_o$ is placed at the origin. The electric field outside of the sphere is given by $\vec{E} = \frac{Q_o}{4\pi\varepsilon_o r^2}\hat{r}$. The electric field lines are shown on the diagram at right. Point P is located to the right of the sphere.



A. Three students discuss the divergence of the electric field at point P:

Student 1: "The field looks like it is spreading out at point P, so the divergence of the electric field at P is positive."

Student 2: "Point P is not where the charge is, so the divergence there is zero."

Student 3: "The magnitude of electric field around point P gets weaker in the direction of the electric field, so the divergence at point P is negative."

With which student do you agree, if any?

B. Which student above is basing his or her explanation on a 'source-based' interpretation of divergence? Show how that students' answer is consistent with Gauss' Law (in either form).

C. Griffiths states that a geometric interpretation for divergence is "a measure of how much the [vector field] spreads out from the point in question."

1. Which student above is using a similar interpretation of divergence?

2. What key word(s) is that student missing from Griffiths' interpretation? Explain why those words lead to a different answer.

D. Mathematically, the divergence can be expanded as: $\vec{\nabla} \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$.

1. Which student above is using a similar interpretation of divergence?

2. What key pieces is that student missing from the mathematical definition? Explain why those pieces lead to a different answer.

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IV. Supplement: Conceptualizing curl

Although curl is a large topic of focus in junior-level magnetostatics, one can conceptualize curl with introductory-level electromagnetism and by drawing on its similarities with divergence.

A. Compare and contrast the following properties of the electric and magnetic fields with the following questions, based on what you have learned in introductory physics:

1. What are the magnetic equivalents to Coulomb's Law and Gauss' Law?

2. What physical entity(s) act as a source to the electric field? To the magnetic field?

3. How is the direction of the electric or magnetic field oriented around each source above?

B. Stokes' Theorem is given by: $\oint_C \vec{\psi} \cdot d\vec{l} = \int_A (\vec{\nabla} \times \vec{\psi}) \cdot d\vec{A}$, which relates curl to a line integral around a closed loop.

1. With the Divergence Theorem: $\oint_S \vec{\psi} \cdot d\vec{A} = \int_V \vec{\nabla} \cdot \vec{\psi}\, dV$, we could interpret divergence as the net flux per unit volume.

   Unfortunately, $\oint_C \vec{\psi} \cdot d\vec{l}$ does not have a name like flux does, but it can be likened to a "torque." What type of density could curl be interpreted as?

2. Griffiths describes curl as the pinwheel around which a vector field rotates. How is this description related to Stokes' Theorem?

C. Ampere's Law in differential form is given by: $\vec{\nabla} \times \vec{B} = \mu_o \vec{J}$.

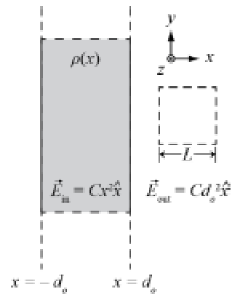
   In your own words, describe how this mathematical representation describes the relationship between magnetic field and current.

b. VDS TUT 202 – modified first three pages of VDS TUT 192

DIVERGENCE AS A SOURCE *Divergence as a source* ED 13

I. Gauss' Law and Divergence

A charged slab with *non-uniform* volume charge density $\rho(x)$ is infinite in the *yz*-plane and spans from $x = -d_o$ to $x = d_o$, as shown at right. The electric field is measured to be $\vec{E}_{in} = Cx^2\hat{x}$ for $-d_o < x < d_o$ and $\vec{E}_{out} = Cd_o^2\hat{x}$ for $|x| > d_o$, where C is a constant.



Cross-sectional view

The task is to determine the volume charge density, $\rho(x)$.

For your convenience, Gauss' Law is given by $\oint \vec{E} \cdot d\vec{A} = \frac{\int \rho dV}{\varepsilon_o}$.

A. Consider the following student approach for determining volume charge density:

> "Gauss' Law relates the net electric flux to the total charge enclosed. I can make a Gaussian cube, evaluate the net electric flux, and then divide by the volume of the Gaussian cube to get the volume charge density."

1. Consider a Gaussian cube of side length L completely outside of the slab on its right as shown, oriented such that the faces of the cube lie parallel to the *xy*-, *xz*-, and *yz*-planes.

   a. Determine the sign and absolute value of the electric flux through each face of the Gaussian cube.

Top:	Left:	Front:
Bottom:	Right:	Back:

   b. What is the net flux through the entire Gaussian cube?

   c. Does dividing the net flux by the volume of the cube (and multiplying by ε_o) give an accurate description of the charge density outside of the slab in this case?

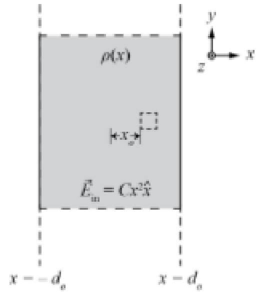
2. In general, the net flux may not be zero. What must be true about the volume density in order for this approach be accurate (*i.e.*, when is $\int \rho dV = \rho V$)? Explain your reasoning.

3. If we consider large volumes inside of the slab, $\int \rho dV \neq \rho V$, so this approach would give a poor estimate for $\rho(x)$. What would you do to make the estimate as accurate as possible? Explain your reasoning.

★ Check your answers with a tutorial instructor before continuing.

ED 14 *Divergence as a source*

4. Now consider an infinitesimal Gaussian cube of dimensions *dx*, *dy*, and *dz*, positioned such that its left face is at an *x*-coordinate of x_o, as shown at right.



Cross-sectional view

a. What is the *x*-coordinate of the right face of the Gaussian cube?

b. Determine the sign and absolute value of the electric flux through each face of the infinitesimal Gaussian surface:

Top:	Bottom:
Left:	Right:
Front:	Back

c. Determine an expression for the net electric flux through the Gaussian cube. You should expect one pair of terms to cancel, with two terms remaining.

d. Using your expression above, solve $\Phi_{net} = \frac{\rho dV}{\varepsilon_o}$ for an exact expression for ρ and simplify.

e. If you haven't already, apply the limit that *dx*, *dy*, and *dz* approach zero, and then generalize x_o with the coordinate *x*.

The process that you have just explored is called the *divergence* of a vector field, which describes the source or sink responsible for a net flux per unit volume of the vector field. Mathematically, it is written as $\vec{\nabla} \cdot \vec{\psi}$, where the del operator $\vec{\nabla}$ denotes a vector sum of spatial derivatives acting on the vector field $\vec{\psi}$.

B. The divergence theorem is given by: $\oint \vec{\psi} \cdot d\vec{A} = \int (\vec{\nabla} \cdot \vec{\psi}) dV$. In your own words, explain how the divergence theorem is consistent with the definition that the divergence describes a source of flux per unit volume.

★ Check your answers with a tutorial instructor before continuing.

C. Gauss' Law in differential form is given by $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_o}$.

1. In Cartesian coordinates, we can expand the equation above as $\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\rho}{\varepsilon_o}$.

   a. Evaluate the sum of derivatives to determine the charge density of the slab. For your convenience, $\vec{E}_{\text{in}} = Cx^2\hat{x}$ within the slab.

   b. Compare your answer to question A.4.d on the previous page. Resolve any inconsistencies if necessary.

2. Compare and contrast $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_o}$ to Coulomb's Law, given by $\vec{E} = \frac{1}{4\pi\varepsilon_o}\int \frac{\rho dV}{|\vec{r}-\vec{r}\prime|^3}(\vec{r} - \vec{r}')$ and Gauss' Law in integral form, given by $\oint \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_o}$.

   a. What physical quantities are present in all three equations?

   b. Which of the equations above contain spatial variables?

   c. Does this suggest that these three equations are *distinct relationships* or part of *one general relationship?*

   d. For each equation, describe a scenario in which it would be better suited to solving a problem than the other two equations.

   - $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_o}$

   - $\vec{E} = \frac{1}{4\pi\varepsilon_o}\int \frac{\rho dV}{|\vec{r}-\vec{r}\prime|^3}(\vec{r} - \vec{r}')$

   - $\oint \vec{E} \cdot d\vec{A} = \frac{\int \rho dV}{\varepsilon_o}$

★ Check your answers with a tutorial instructor before continuing.

c. MVP TUT 151

MAGNETIC VECTOR POTENTIAL

ED 1

I. Magnetic vector potential

The two Maxwell equations that describe static magnetic fields are $\nabla \bullet \overline{B} = 0$ and $\nabla \times \overline{B} = \mu_0 \overline{J}$.

A. Recall from the *Vector derivatives* tutorial that the divergence of the curl of a vector field is always zero.

1. Use the fact that the divergence of $\overline{B}$ is zero to write it in terms of another vector field. This new field is called the *vector potential* $\overline{A}$.

2. Use Stokes' theorem to convert your answer above into an integral form. Express your answer in terms of magnetic flux

3. Suppose you wanted to find the vector potential at some point in space. Describe how you would use the integral form above to find $\overline{A}$.

   (*Hint:* How does the integral form of your expression above compare to the integral form of Ampere's law?)

B. Suppose you added a constant field to the vector potential, which is equivalent to moving the reference point of $\overline{A}$.

1. Does this change the magnetic field anywhere in space? Explain why or why not.

2. Can a non-constant field be added to the vector potential without changing the magnetic field? If so, what type of non-constant field is allowed? If not, explain why not.

This freedom to adjust the vector potential is called *gauge invariance*. As you've seen in lecture and your textbook, this allows the vector potential to be chosen such that its divergence is zero.

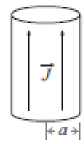
3. What is the divergence and curl of $\overline{A}$? How do they compare to those of $\overline{B}$? Explain.

➭ Check your answers to this page with a tutorial instructor before continuing.

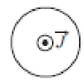
ED 2 ***Magnetic vector potential***

The fact that the equations describing the vector potential are completely analogous to those describing the magnetic field allows us to draw an analogy between the behavior of the magnetic field around currents and the behavior of the vector potential around magnetic fields.

C. A segment of an infinite wire with current uniformly distributed through it is shown at right.



Side view

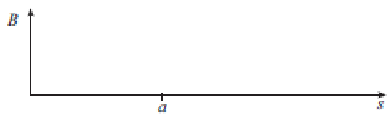


Top view

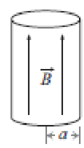
1. On the top view diagram, indicate the direction of $\vec{B}$ inside and outside the wire.

2. Use Ampere's law to find $\vec{B}$ both inside and outside the wire. What Amperian loop should you use?

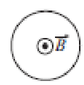
3. In the plot below, sketch a qualitative graph of the magnitude of the magnetic field.



D. The magnetic field of a solenoid is shown at right: it is uniform inside the solenoid, and zero outside. Answer the following questions by analogy to the situation above, without doing any calculations.



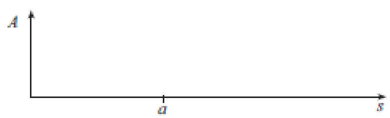
Side view



Top view

1. Which direction will the vector potential point inside and outside the wire? Explain.

2. In the plot below, sketch a qualitative graph of the magnitude of the vector potential inside and outside the wire.

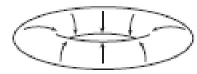


3. Use the integral expression for $\vec{A}$ from the previous page to check your answer.

➩ Check your answers to this page with a tutorial instructor before continuing.

II. Applications of the vector potential

A. A toroidal current distribution is shown at right. Current flows up the outside of the toroid and down the inside, as indicated by the arrows on the diagram.



Perspective view

1. Which direction does the magnetic field point inside the toroid? What direction does it point outside the toroid? Sketch your answers on the cross-section diagram.

Cross-section

2. To use the analogy you developed in the previous page to find the vector potential, what current distribution should you use?

3. Describe or draw the magnetic field that such a current distribution would produce.

Given your answer above, sketch the vector potential outside the toroid on the diagram at right.

Cross-section

4. Consider the following student statement:

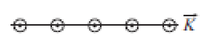
"The vector potential points in the same direction as the current density. Current flows in circles around the toroid, so the vector potential also makes circles around the toroid."

Do you agree with this student's statement? Explain why or why not.

A sheet of current directed out of the page is shown at right.

Perspective view

B. On the diagram, indicate the direction of the magnetic field above and below the sheet.



Cross-section

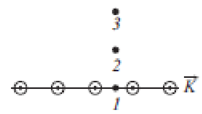
1. Use Ampere's law to find the magnitude of the magnetic field above and below the sheet. Is the field constant, or does it depend on distance from the sheet?

ED 4 *Magnetic vector potential*

2. In electrostatics, we often chose the reference point of the electric potential at infinity, because the electric field was usually zero there.

   Is infinity a good reference point for the magnetic vector potential in this case? If not, where would you put the reference point for the vector potential? Explain.

C. Three points near the current sheet are shown at right.



1. Given your choice of reference point above, what direction is the vector potential at point *1*?

2. The magnetic field is the curl of $\bar{A}$. What direction is the curl of $\bar{A}$ above the sheet?

3. Suppose we put a small pinwheel between point *1* and point *2*. Which direction will the pinwheel spin?

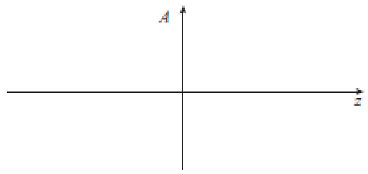
4. What direction is $\bar{A}$ at point *2*? Explain.

5. Suppose we put a small pinwheel between point *2* and point *3*. Which direction will the pinwheel spin?

6. What direction is $\bar{A}$ at point *2*? Explain.

7. How does the magnitude of $\bar{A}$ at point *3* compare to that at point *2*? Explain.

D. In the plot below, sketch a qualitative graph of the vector potential as a function of vertical distance from the sheet. Make sure your plot is consistent with your answer to question B.2 on the previous page.



Is your plot consistent with your answer to question B.1 at the end of the third page? Explain why or why not.

Does the vector potential point in the same direction as the current density? Is this surprising? Explain.

d. MSF TUT 171

MAGNETOSTATIC FIELDS

ED
1

I. Review of vector derivatives

A. Consider the divergence of a vector field, $\vec{\nabla} \cdot \vec{\psi}$.

1. What information does the divergence operator tell you about the behavior of a vector field?

2. In the space below, draw an example of a vector field with:

• positive divergence	• negative divergence	• zero divergence

B. Consider the curl of a vector field, $\vec{\nabla} \times \vec{\psi}$.

1. What information does the curl operator tell you about the behavior of a vector field?

2. In the space below, draw an example of a vector field with:

• curl into the page	• curl out of the page	• zero curl

C. In the space below, draw examples of three fields with the *same* curl but *different* divergence.

• curl into the page	• curl out of the page	• zero curl

Can you determine the behavior of a field if you only know the divergence *or* the curl? Do you need to know both?

 Physics 322

ED 2 *Magnetostatic fields*

II. Vector derivatives as sources

Magnetostatics involves several different vector fields, all which have curl but no divergence.

For each field $\vec{\psi}$, the curl is given by a source field $\vec{S}$: $\vec{\nabla}\times\vec{\psi} = \vec{S}$ and $\vec{\nabla}\cdot\vec{\psi} = 0$.

A. An example of a magnetostatic field is the magnetic field around a line current.

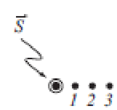
1. In the space below, draw the magnetic field around a long, straight wire.

2. Does $\vec{\psi}$ correspond to the magnetic field $\vec{B}$ or the current density $\vec{J}$? Which does $\vec{S}$ correspond to?

3. Given your answer above, where in space does $\vec{B}$ have curl? Explain how your sketch above shows this.

4. Where in space does $\vec{B}$ have divergence? Explain how your sketch above shows this.

B. Consider a field $\vec{\psi}$ with constant curl $\vec{S}$ on a single line, and zero elsewhere.



Cross-section of a line source

1. Consider the following *incorrect* statement:

   "Both the divergence and curl of $\vec{\psi}$ are zero at all three points, so the field is zero there as well."

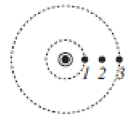
   Why is this statement incorrect? Explain your reasoning.

   At points *1–3*, in what direction is $\vec{\psi}$? Sketch your answers on the diagram above.

2. How can you use the fact that the curl is non-zero on the line to determine the direction of $\vec{F}$ at point *1*? How can you use the fact that the curl is zero at point *2* to determine the direction of $\vec{F}$ at point *3*? Explain your reasoning.

➪ Discuss your responses with a tutorial instructor before continuing.

Stokes' theorem states that the area integral of the curl over a surface is equal to the line integral around the boundary of that surface: $\iint \vec{\nabla}\times\vec{\psi}\cdot d\vec{A} = \oint \vec{\psi}\cdot d\vec{l}$.



C. Two circular Stokesian loops are shown, through points *1* and *3*, centered on the line source from the previous page.

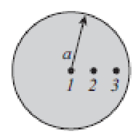
1. Is the magnitude of the area integral on the left-hand side of Stokes' theorem for the smaller loop *greater than*, *less than*, or *equal to* that for the larger loop? Explain your reasoning.

2. The two loops have different perimeters. How does this affect the right-hand integral in Stokes' theorem?

3. Given your answers above, does $\vec{\psi}$ have the same magnitude at each of the points?

   How does the magnitude of $\vec{\psi}$ depend on the distance from the line source? Does your answer depend on the magnitude or direction of $\vec{S}$? Explain your reasoning.

D. Consider a cylindrical source field pointing out of the page. The magnitude of $\vec{S}$ is constant for $s < a$, and zero for $s > a$. Point *1* is at $s = 0$.



Cylindrical volume with uniform $\vec{S}$ out of the page

1. At point *1*, the magnitude of $\vec{\psi}$ is zero. Why does this make sense, given the symmetric distribution of $\vec{S}$?

2. Consider the following student statement:

   "At point *1*, $\vec{S}$ is non-zero, so $\vec{\psi}$ has curl. However, $\vec{\psi}$ is zero at point *1*. How can a field have non-zero curl when the field is zero? That seems like a contradiction."

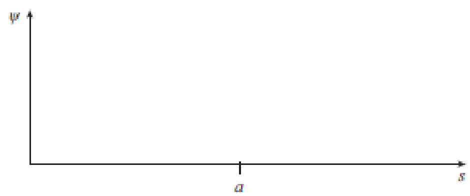
   How could you help this student resolve their confusion?

3. At points *2* and *3*, what direction does $\vec{\psi}$ point?

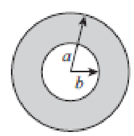
For $s < a$, is the magnitude of $\vec{\psi}$ at larger values of s *greater than*, *less than*, or *equal to* that at smaller values of s? Explain your reasoning. (*Hint:* Consider the same circular loops in part II.C. How does the magnitude of the area integral depend on the radius?)

4. For $s > a$, is the magnitude of $\vec{\psi}$ at larger values of s *greater than*, *less than*, or *equal to* that at smaller values of s? Explain your reasoning.

5. Sketch a graph of the magnitude of $\vec{\psi}$ as a function of s, both inside and outside the cylinder. Label any values on your axes.

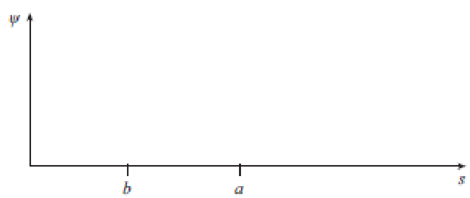


E. Suppose the cylindrical source field were hollowed out in the center, as shown.



1. Would this change affect the behavior of $\vec{\psi}$ in any region? Explain your reasoning.

2. Sketch a graph of the magnitude of $\vec{\psi}$ as a function of s in this case.



III. Supplement: Determining ψ from S

A. The vector field is governed by two spatial derivatives: $\vec{\nabla}\times\vec{\psi}(\vec{r}) = \vec{S}(\vec{r})$ and $\vec{\nabla}\cdot\vec{\psi}(\vec{r}) = 0$. Consider an analogy with a simpler derivative: $\frac{d}{dt}\vec{r}(t) = \vec{a}(t)$.

1. Physically speaking, is velocity or acceleration the source? *I.e.* does acceleration cause velocity, or does velocity cause acceleration?

2. Suppose you knew the acceleration of a particle, and used it to find the velocity. Would your answer be unique, or would there be an undetermined constant?

   Suppose instead you knew the velocity of the particle, and used it to find the acceleration. Would your answer be unique, or would there be an undetermined constant?

3. Given your answers above, what technique could you use to find the 'field' vector given the 'source' vector?

4. Is it easier to find the 'field' vector given the 'source' vector, or vice versa? Explain.

Because the 'field' vector is only specified in terms of derivative operators, solving for the 'field' from the 'source' will result in the vector equivalent of a constant of integration. However, we can often determine this constant by using other information such as symmetry.

B. Suppose the 'source' vector has a set of continuous translational or rotational symmetries.

1. If the 'source' distribution has one translational and one rotational symmetry (*i.e.*, a line), what can you conclude about the 'field' vector?

   Could the magnetic field of a line current have an arbitrary constant in addition to the field behavior determined by the curl?

2. If the 'source' distribution has two translational symmetries (*i.e.*, a sheet), what can you conclude about the 'field' vector?

   Could the magnetic field of a sheet current have an arbitrary constant in addition to the field behavior determined by the curl?

e. MVP TUT 171

MAGNETIC VECTOR POTENTIAL

ED 13

I. Review of magnetostatic fields

A. Recall from the tutorial *Magnetostatic fields* that a magnetostatic field has no divergence, and has a curl given by a source field $\vec{S}$: $\vec{\nabla} \cdot \vec{F} = 0$ and $\vec{\nabla} \times \vec{F} = \vec{S}$.

1. Stokes' theorem is given by $\iint \vec{\nabla} \times \vec{F} \cdot d\vec{a} = \oint \vec{F} \cdot d\vec{l}$. Use this to convert the curl of $\vec{F}$ into an integral equation.

2. What name does physics use to describe the type of integral with the source vector?

3. To solve this integral for $\vec{F}$ at some point, we must simplify the line integral.
   - Which direction(s) should $d\vec{l}$ point relative to $\vec{F}$ at each point on the loop?
   - How should $|\vec{F}|$ behave along each segment of the loop?

B. Consider the magnetic field created by a current distribution $\vec{J}(\vec{r})$.

1. Write the differential form of Ampère's law.

2. Use Stokes' theorem to convert your expression into a line integral. What does the flux of $\vec{J}$ mean?

C. One important result from vector calculus is that the divergence of the curl of a vector field is zero: $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0$.

Use this result with the divergence of $\vec{B}$ to write $\vec{B}$ in terms of a new vector field $\vec{A}$.

This new field is called the *magnetic vector potential*.

 Physics 322

II. Magnetic vector potential

A. The vector potential is another magnetostatic field. As you saw on the previous page, $\vec{A}$ is sourced by $\vec{B}$. In the supplement you will show that $\vec{A}$ can be chosen to have no divergence.

1. Write equations for the divergence and curl for both $\vec{A}$ and $\vec{B}$. Each equation should only involve a single derivative.

   - $\vec{\nabla} \cdot \vec{B}$
   - $\vec{\nabla} \cdot \vec{A}$
   - $\vec{\nabla} \times \vec{B}$
   - $\vec{\nabla} \times \vec{A}$

2. Show the integral form of both the curl of $\vec{A}$ and the curl of $\vec{B}$.

3. In terms of your answers above, what are the similarities between $\vec{A}$ and $\vec{B}$? Are there any noticeable differences?

4. Describe a method for solving for $\vec{A}$ if $\vec{B}$ is highly symmetric. (*Hint:* Consider your answers to question I.A.3 on the previous page.)

B. A section of the magnetic field of an infinite solenoid is shown at right. The field has uniform magnitude inside the cylindrical region, and is zero outside.

1. What loop would you use to find $\vec{A}$ in this case? Explain.

2. Imagine varying the radius of your loop, and consider how this would affect the area and line integrals in your expression above. Answer the following questions using qualitative reasoning, *without evaluating any integrals*.

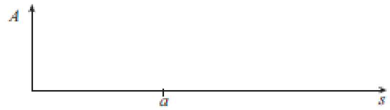
   a. How does the magnitude of $\vec{A}$ depend on the radius, *inside* the cylinder?

   b. How does the magnitude of $\vec{A}$ depend on the radius, *outside* the cylinder?

➪ Discuss your answers with a tutorial instructor before continuing.

 Physics 322

3. Sketch a qualitative graph of the magnitude of the vector potential inside and outside the solenoid.



4. Now use the line-integral expression to find mathematical expressions for $\vec{A}$, both inside and outside the solenoid.

   Do your expressions match your qualitative graph above? If not, resolve any inconsistencies.

5. Consider the following student dialogue:

   Student 1: "Both the vector potential and the magnetic field are magnetostatic fields, the only difference is what sources the curl for each."

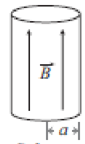
   Student 2: "That makes sense. If the cylindrical source was a current density instead, the graph for $\vec{B}$ would look the same as the graph above."

   Do you agree with student 2's conclusion? Explain why or why not.

C. Use your answers thus far to describe a general procedure to determine the vector potential, given a symmetric current distribution. Be as specific as possible.

III. Vector potential and current

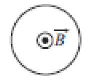
A. The solenoid from section II is shown again at right.



Side view

1. Where in space does current exist? (*Hint:* The curl of $\vec{B}$ is the current density, so there is current only where a pinwheel would spin.)

   Which direction does it point? Draw your answer on the diagram.



Top view

2. Which direction does the vector potential point? How does it compare to the direction of the current?

B. Recall from the tutorial *Ampere's law* that we need a symmetric current distribution in order to use Stokes' theorem to solve for the magnetic field.

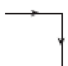
1. What type of symmetry does the current distribution need to have in order to use Stokes' theorem?

2. If the current has an appropriate set of symmetries, what can be said about the symmetries of the magnetic field?

   What can be said about the symmetries of the vector potential?

3. To find the vector potential using Stokes' theorem, what must be true about the current distribution? Explain.

C. Consider the following student dialogue about the bent wire shown at right:



Student 1: "The vector potential is always in the same direction as the current."

Student 2: "That was true for the solenoid, but the solenoid is highly symmetric. This system doesn't have any continuous symmetries, so we can't conclude that the vector potential is in the same direction as the current."

With which student do you agree? Explain.

➩ Check your answers to this page with a tutorial instructor before continuing.

IV. Supplement: Gauge choice

The requirements for a magnetostatic fields are that their curl is given by a source field, and that their divergence is zero. We have seen that $\vec{A}$ satisfies the first requirement, and in this supplement you will investigate the divergence of $\vec{A}$.

A. Write the expression relating $\vec{A}$ and $\vec{B}$.

Can you add a uniform field to $\vec{A}$ without changing $\vec{B}$? What about a non-uniform field?

Another important vector calculus identity is $\vec{\nabla}\times(\vec{\nabla}\psi) = 0$. In light of this, what is the *most general* field you can add to $\vec{A}$ without changing $\vec{B}$? Explain.

The ability to add to the potential without changing the physical field is called a *gauge transform*.

B. Suppose the vector potential for some system had a uniform divergence: $\vec{\nabla}\cdot\vec{A} = 1$ tesla.

1. Use a gauge transform $\vec{A}' = \vec{A} + \vec{\nabla}\psi$ to find a new vector potential that has no divergence, $\vec{\nabla}\cdot\vec{A}' = 0$ (this is called the *Coulomb gauge*).

   What differential equation does ψ have to satisfy? (*Hint:* The divergence of the gradient is the Laplacian operator.)

2. Is this differential equation solvable? If so, what is a possible solution? If not, explain why not.

C. Generalize your results above to answer the following questions.

1. Can you use a gauge transform to remove the divergence of the vector potential? Is this transformation unique, or are there multiple transforms that would work?

2. In section II we assumed that $\vec{A}$ was a magnetostatic field. Is this always true, or is it only true for one choice of gauge?

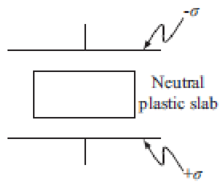
f. DPF TUT 144

DISPLACEMENT FIELD

ED
1

I. Displacement field

In the *Polarization* tutorial you saw that a slab of plastic inserted between the plates of a capacitor becomes polarized, and a bound charge appears on the top and bottom surfaces of the plastic.



A. On the diagram, sketch the induced charge distribution on the plastic.

B. Assume the polarization density inside the plastic is $\overline{P}$, pointing toward the top of the page.

1. What is the surface charge density on the top of the plastic slab? On the bottom of the slab? Explain.

   Given your answer above, can you write $\overline{P}$ in terms of σ_{bound} on the surface of the plastic?

2. What is the magnitude of the *induced* electric field of the plastic? Which direction does this induced field point? (Recall that the field of a sheet of charge is $\sigma/2\varepsilon_0$.)

3. What is the net electric field inside the plastic?

The *displacement field* $\overline{D}$ is defined in terms of the net electric field and the polarization density:

$$\overline{D} = \varepsilon_0 \overline{E}_{\text{net}} + \overline{P}\,.$$

C. What is the displacement field in this case?

If you haven't already, write your answer in terms of σ_{bound} and σ.

Given your answer above, does the displacement field depend on the free charge, the bound charge, or both?

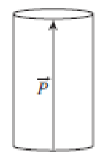
ED 2 *Displacement field*

Gauss's law in differential form states that $\nabla \bullet \overline{E} = \frac{\rho}{\varepsilon_0}$. There is a similar relation involving the polarization density: $\nabla \bullet \overline{P} = -\rho_{bound}$.

D. What would you expect the analogous expression for $\overline{D}$ to be? Express your answer in terms of ρ_{bound} and ρ_{free}.

II. Electric and displacement fields

A. A cylinder with uniform polarization pointing upwards is shown at right. This is an example of an *electret* – the electrical equivalent of a magnet.



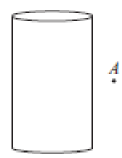
1. Consider the following student statement:

   "The displacement field is created by free charges, and there are no free charges in this system. Thus the displacement field is zero everywhere."

   Do you agree with this student's statement? Explain why or why not.

2. On the diagram above, indicate the location of any bound charges. Also sketch the electric field everywhere in space due to these bound charges.

3. At point *A*, which direction does $\overline{D}$ point? (*Hint:* What is $\overline{P}$ there?)



4. Is the divergence of $\overline{D}$ nonzero at any point? What does this imply about the shape of the displacement field lines?

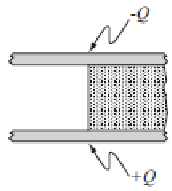
5. Sketch the displacement field on the diagram.

6. Do your answers above agree with your answer to question 1? If not, resolve any inconsistencies.

7. Compare your sketch to the sketch of $\overline{E}$ above. When are these two fields the same? When are they different?

➩ Check your answers to this page with a tutorial instructor before continuing.

III. Applications of the displacement field



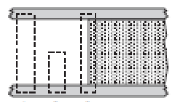
A portion of a half-filled capacitor is shown at right: the space between the right half of the plates is filled with a dielectric slab, and the space between the left half of the plates is empty.

A charge of $-Q$ is placed on the top plate, and $+Q$ on the bottom plate. The area of the plates is A. Assume the plates are large enough that you can ignore fringing fields around the edges.

A. *Predict* where there would be bound charges, and how the free charge on each plate will be distributed. Sketch your prediction on the diagram above.

   Also predict how the magnitude of the electric field in the left half would compare to that in the right half.

B. Three dashed loops are shown on the diagram at right.



   1. In general, what is the line integral of a static electric field around a closed loop?

   2. Consider the line integral of the electric field around loop 1. Both the upper and lower edges of loop 1 are inside the conducting plates.

      What does the line integral around loop 1 imply about how the *vertical* component of the electric field on the left side of the loop compares to that on the right?

   3. Now consider loop 2. What does the line integral around this loop tell you about the *horizontal* component of the electric field along the upper side? Explain.

   4. Suppose the magnitude of the electric field in the dielectric is E_d, and in free space is E_s. Considering the line integral around loop 3, is E_d *greater than*, *less than*, or *equal to* E_s? Explain.

      Were your predictions about the electric field in each region correct? If not, resolve any inconsistencies.

ED 4 ***Displacement field***

C. In the *Polarization* homework, you investigated linear dielectrics – materials whose polarization is proportional to the net electric field. The polarization in a linear dielectric is given by $\vec{P} = \varepsilon_0 \chi_E \vec{E}_{net}$, where χ_E is the *electric susceptibility* of the material.

1. Assuming the dielectric is linear, what is the magnitude of $\vec{P}$ in each region?

2. What is the magnitude of $\vec{D}$ in each region? Which direction does it point?

3. What is the *free* surface charge density on each half of the lower plate? (*Hint:* Consider a small Gaussian box, with one face inside the conducting plate where $D = 0$, and use the Gauss's law for D that you found at the top of page 2.)

4. What is the *bound* surface charge density on each half of the lower plate?

5. Use your answers above to find the *total* charge density on each half of the lower plate. Is your answer consistent with the fact that the electric field in each region is the same?

6. Given the σ_{free} you found in part 3, how much free charge is there on the left half of the bottom plate? The area of the entire plate is A.

   How much free charge is there on the right half of the bottom plate?

7. Given that the total charge on the bottom plate is $+Q$, use your answers above to find E_d.

D. You can use limiting cases to check whether your answer is correct. If the dielectric is a perfect insulator then $\chi_E = 0$, and if it is a perfect conductor then $\chi_E = \infty$.

   What is E_d in each case? Do your answers make sense?

g. AUX TUT 151

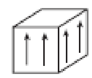
AUXILIARY FIELD

ED
1

I. Magnetization

Earlier in the course you've seen that magnetic fields are produced by currents. Currents that we directly control are called *free currents*. However, magnetic fields can arise even when there are no free currents in the system.

A. A small metal cube is placed in a uniform magnetic field (not shown). The cube acquires a uniform magnetization $\vec{M}$ pointing toward the top of the page.



Cube with uniform magnetization

1. The *bound current* in a material is given by the curl of the magnetization: $\vec{J}_{\text{bound}} = \nabla \times \vec{M}$.

   Why is the term 'bound current' appropriate for this current? Explain.

2. Where in space does bound current exist, and where does it point? Sketch it on the diagram above.

3. Consider the induced magnetic field produced by this bound current. Inside the cube, which direction does it point? What about outside the cube? Explain.

The *auxiliary field* $\vec{H}$ is defined in terms of the net magnetic field and the magnetization:

$$\vec{H} = \frac{\vec{B}_{\text{net}}}{\mu_0} - \vec{M}.$$

B. Outside the cube, what direction is $\vec{M}$? What direction is $\vec{H}$? Explain.

Ampere's law states that $\nabla \times \vec{B}_{\text{net}} = \mu_0 \vec{J}_{\text{net}}$. What is the curl of $\vec{H}$?

Does the auxiliary field depend on the free current, the bound current, or both? Explain.

➩ Check your answers to this page with a tutorial instructor before continuing.

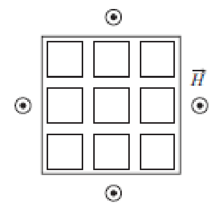
ED 2 *Auxiliary field*

II. Paramagnetism

Recall from your quantum mechanics course or from freshman physics that electrons have spin, which is an inherent magnetic dipole moment that can be thought of as a small current loop.

A. Suppose you placed an electron in a magnetic field pointing toward the top of the page, and allowed it to come to equilibrium. Which direction would the electron's spin point? Sketch the orientation of the current loop and indicate the direction current is flowing around it.

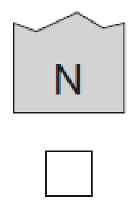
B. Several current loops are shown in cross-section at right. Each small square represents the current loop of one electron. An external $\vec{H}$ is directed out of the page.



1. On the diagram, indicate the direction of current for each loop.
2. Is there a net bound current inside the material? Is there a net bound current at the edge?
3. Is this the same as a free current? Are electrons actually flowing around the material? Explain.

When electronic spins in a material align with the auxiliary field, the material is called *paramagnetic*. The number of spins that align with the field is proportional to the field strength, so $\vec{M} = \chi_M \vec{H}$, where $\chi_M > 0$.

C. A paramagnetic cube is placed near the end of a large magnet as shown.



1. What direction does the magnetization inside the cube point?
2. Will the cube be *attracted toward* or *repelled from* the magnet? Explain.
3. Consider a point inside the cube. Is the magnetic field at that point *greater than, less than*, or *equal to* what it was before the cube was placed there? Explain.

4. Consider a point outside and to the right of the cube. Is the magnetic field at that point *greater than*, *less than*, or *equal to* what it was before the cube was placed there? Explain.

D. Electrons also obey Pauli's exclusion principle; no two electrons can have exactly the same state. Thus there can only be two electrons in any given atomic orbital, with their spins in the opposite direction.

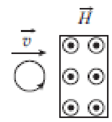
Aluminum is atomic number 13, and silicon is atomic number 14. Would you expect that either or both would be paramagnetic? Explain why or why not for each element.

➭ Check your answers to this section with a tutorial instructor before continuing.

III. Diamagnetism

Electrons not only have spin, they also revolve around the nucleus. Thus we can think of the electron's orbit as a small current loop.

A. Suppose an atom with a counterclockwise orbital current was moved into a region of constant magnetic field pointing out of the page as shown.



1. During this process does the magnetic flux through this loop change?

2. Lenz's law states that the induced current in a loop creates a magnetic field that opposes the change in flux.

   What direction does the induced current around this loop point?

   Will this induced current *increase* or *decrease* the orbital current?

3. Does this *increase* or *decrease* the magnetic field of the current loop?

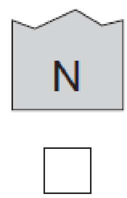
   Does this change in internal magnetic field *add to* or *oppose* the external magnetic field? Explain.

4. Suppose the orbital current was clockwise instead. Would the induced current *increase* or *decrease* the orbital current? What effect does this have on the internal magnetic field?

5. Generalize your answers above. When an object is placed in an external magnetic field, does the change in orbital magnetic field of each electron *increase* or *decrease* the external field?

When the magnetic moments of the electrons in a material change in the presence of an external $\overline{H}$, the material is called *diamagnetic*. The change in dipole moments is proportional to the field strength, so $\overline{M} = \chi_M \overline{H}$, where $\chi_M < 0$.

B. A diamagnetic cube is placed near the end of a large magnet as shown.



1. What direction does the magnetization inside the cube point?

2. Will the cube be *attracted toward* or *repelled from* the magnet? Explain.

3. Consider a point inside the cube. Is the magnetic field at that point *greater than*, *less than*, or *equal to* what it was before the cube was placed there? Explain.

4. Consider a point outside and to the right of the cube. Is the magnetic field at that point *greater than*, *less than*, or *equal to* what it was before the cube was placed there? Explain.

C. Diamagnetism is a shift in magnetic moments, while paramagnetism is a realignment of magnetic moments. Which would you expect would be a bigger effect?

What atomic numbers could you observe diamagnetism in? Explain.

h. DPF TUT 162

DISPLACEMENT FIELD

ED 1

I. Sources of the displacement field

The *displacement field* $\vec{D}$ is defined in terms of the net electric field and the polarization density:

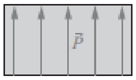
$$\vec{D} = \epsilon_o \vec{E}_{\text{net}} + \vec{P}$$

A. Gauss' law in differential form states that $\vec{\nabla} \cdot \vec{E} = \frac{\rho_{\text{net}}}{\epsilon_o} = \frac{\rho_{\text{free}} + \rho_{\text{bound}}}{\epsilon_o}$. There is a similar relation involving the polarization density: $\vec{\nabla} \cdot \vec{P} = -\rho_{\text{bound}}$.

1. What is $\vec{\nabla} \cdot \vec{D}$? Express your answer in terms of ρ_{bound} and/or ρ_{free}.

2. Do displacement field lines start and end on *free charges*, *bound charges*, *both*, or *neither?*

B. A plastic slab has the uniform polarization shown. Recall that we can visualize the curl at a point by placing a small pinwheel at the point. If the pinwheel spins, there is curl at that point.



1. Where does $\vec{P}$ have curl? Consider all faces of the plastic slab as well as the region inside the slab.

2. What is $\vec{\nabla} \times \vec{D}$? Show your work.

3. In static systems, electric field lines cannot form closed loops. Is this also true for the displacement field?

4. What must be true of $\vec{P}$ for $\vec{D}$ to behave like the electric field?

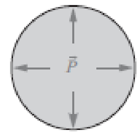
   In what direction must the polarization point, relative to the surface of a dielectric, in order for this to be true? Explain.

➪ Check your answers to this section with a tutorial instructor before continuing.

ED 2 *Displacement field*

II. Divergence of the displacement field

The sphere at right has a fixed, radial polarization $\vec{P} = C\,\hat{r}$ as shown, but has no net charge.



Radially polarized sphere

A. *Predict* the direction of each field below. Explain your reasoning.

- $\vec{E}$ inside the sphere
- $\vec{D}$ inside the sphere
- $\vec{E}$ outside the sphere
- $\vec{D}$ outside the sphere

B. Consider the sources of the displacement field.

1. Does $\vec{D}$ have non-zero divergence at any point? If so, where?

2. Does $\vec{D}$ have non-zero curl at any point? If so, where?

3. Given your answers above, what direction does $\vec{D}$ point outside the sphere? If $\vec{D}$ is zero outside the sphere, state so explicitly. Explain your reasoning.

4. What direction does $\vec{D}$ point inside the sphere? If $\vec{D}$ is zero inside the sphere, state so explicitly. Explain your reasoning.

5. Were your predictions about $\vec{D}$ correct?

C. Recall that the electric field is related to the displacement field by $\vec{D} = \epsilon_o \vec{E}_{\text{net}} + \vec{P}$.

1. Given your knowledge of $\vec{D}$ you developed above, what direction does $\vec{E}$ point outside the sphere? If $\vec{E}$ is zero outside the sphere, state so explicitly. (*Hint:* What is $\vec{P}$ outside?)

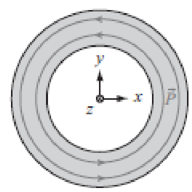
2. What direction does $\vec{E}$ point inside the sphere? If $\vec{E}$ is zero inside the sphere, state so explicitly.

3. Were your predictions about $\vec{E}$ correct?

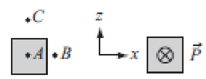
D. If a system has a divergence of $\vec{P}$ but no curl, is it easier to find $\vec{E}$ or $\vec{D}$ first? Explain.

III. Curl of the displacement field

A thick hoop with a square cross-section is shown at right. The hoop has a fixed, azimuthal polarization $\vec{P} = C\hat{\phi}$.



Top view



Cross-sectional view

A. *Predict* the direction of each field below. Explain your reasoning.

- $\vec{E}$ at point A
- $\vec{E}$ at point B
- $\vec{E}$ at point C
- $\vec{D}$ at point A
- $\vec{D}$ at point B
- $\vec{D}$ at point C

B. Consider the following statement about $\vec{D}$ of the hoop:

"The divergence of the displacement field is the free charge. There aren't any free charges on the hoop, so the displacement field is zero everywhere."

1. Do you agree with this statement? If not, identify the flaw(s) in the reasoning.

2. Does $\vec{P}$ have non-zero curl at any point on the hoop? Consider both inside the hoop as well as at the surfaces of the hoop.

3. Does $\vec{D}$ have non-zero curl at any point? If so, where?

ED 4 *Displacement field*

4. Given your answers above, can you easily determine the direction of the displacement field at points *A–C?* Explain why or why not.

C. Consider the sources of the electric field.

1. Bound charges can be found from $\vec{\nabla} \cdot \vec{P} = -\rho_{\text{bound}}$. Are there bound charges anywhere on this hoop?

2. Does $\vec{E}$ have non-zero divergence at any point? If so, where?

3. Does $\vec{E}$ have non-zero curl at any point? If so, where?

4. Given your answers above, determine the direction of the electric field at points *A–C*. If the magnitude of the electric field is zero at any point, state so explicitly.

5. Were your predictions about $\vec{E}$ correct?

6. Use the definition of the displacement field and your answers above to determine the magnitude and direction of the displacement field at points *A–C*. If the magnitude of the displacement field is zero at any point, state so explicitly.

D. If a system has a curl of $\vec{P}$ but no divergence, is it easier to find $\vec{E}$ or $\vec{D}$ first? Explain.

IV. Supplement: Curl and divergence of the displacement field

A. In systems containing dielectrics, we often consider any field caused by the dielectric an induced field, and any other field as an external field.

1. Would a displacement field sourced by free charges be considered an *external field* or an *induced field?*

   Would this field be described by *curl only*, by *divergence only*, or by *both?*

2. Would a displacement field sourced by polarization be described by *curl only*, by *divergence only*, or by *both?*

The net displacement field can be thought of as a sum of these two fields: $\vec{D}_{\text{net}} = \vec{D}_{\text{ext}} + \vec{D}_{\text{ind}}$.

3. Is this definition of $\vec{D}_{\text{net}}$ consistent with $\vec{\nabla} \cdot \vec{D} = \rho_{\text{free}}$? (*Hint:* Does one term have no divergence?)

4. Is this definition of $\vec{D}_{\text{net}}$ consistent with $\vec{\nabla} \times \vec{D} = \vec{\nabla} \times \vec{P}$? Explain.

B. Recall from the tutorial *Gauss' law* or from lecture that the divergence of a vector field can be related to the flux through a Gaussian surface.

1. Is a symmetric system necessary to use a Gaussian surface to determine a vector field? If so, does the system have to possess *continuous* or *discrete* symmetries? Explain.

2. A similar relation exists between the curl of a vector field and the line integral around a closed loop. Is symmetry also required to use a line integral to determine a vector field?

C. Based on your answers above, describe a general procedure for determining the displacement field for an arbitrary symmetric system.

i. AUX TUT 161

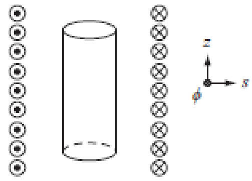
AUXILIARY FIELD

ED 1

I. Magnetization and magnetic field

The magnitude of the magnetic field caused by a solenoid is $\mu_0 K$ inside of the solenoid.

A. A long cylindrical paramagnetic is placed inside of an infinite solenoid with total surface current $\vec{K} = K_{\text{free}}\hat{\phi}$.



Long paramagnet in an infinite solenoid

1. What is the external magnetic field within the solenoid caused by this current? Specify its direction in terms of a unit vector.

2. Consider the magnetization of the paramagnet.

   a. In what direction is the magnetization of the paramagnet? Is it constant?

   Recall that the bound current of magnetization is $\vec{J}_{\text{bound}} = \vec{\nabla} \times \vec{M}$ and $\vec{K}_{\text{bound}} = \vec{M} \times \hat{n}$.

   b. Does this paramagnet have a *volume bound current*, *surface bound current*, or *both?* In what direction does it point?

   c. Suppose the magnitude of the bound current is J_{bound} and/or K_{bound}. What is the magnitude of the magnetization? (*Hint:* $\hat{n}$ is unitless.)

3. Express the net magnetic field in terms of K_{free}, J_{bound}, and/or K_{bound}, and a unit vector.

The *auxiliary field* $\vec{H}$ is defined in terms of the net magnetic field and the magnetization:

$$\vec{H} = \frac{\vec{B}_{\text{net}}}{\mu_0} - \vec{M}.$$

B. What is $\vec{H}$ inside of the paramagnet in this case? Express your answer in terms of K_{free}, J_{bound}, and/or K_{bound}, and a unit vector.

➪ Check your answers to this page with a tutorial instructor before continuing.

ED 2 *Auxiliary field*

II. Divergence and curl of the auxiliary field

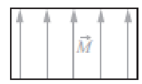
Ampère's law in differential form states that $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}_{net} = \mu_0(\vec{J}_{free} + \vec{J}_{bound})$. There is a similar relation involving the magnetization density: $\vec{\nabla} \times \vec{M} = \vec{J}_{bound}$.

The definition of the auxiliary field is reproduced here for convenience: $\vec{H} = \vec{B}/\mu_0 - \vec{M}$.

A. What is $\vec{\nabla} \times \vec{H}$? Express your answer in terms of $\vec{J}_{free}$ and/or $\vec{J}_{bound}$.

Given your answer above, do field lines for $\vec{H}$ "curl around" free currents, bound currents, or both?

B. A bar magnet has a uniform magnetization shown at right. Recall that we can visualize the divergence as the source or sink to the field.



1. Where does $\vec{M}$ have divergence? Be sure to think about all faces of the magnet in addition to the region inside.

2. What is $\vec{\nabla} \cdot \vec{H}$? Show your work.

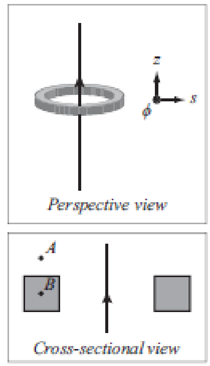
3. Given your answer above, can field lines for $\vec{H}$ to start and end, or do they have to form closed loops like the magnetic field?

4. What must be true of $\vec{M}$ for $\vec{H}$ to behave like a magnetostatic field?

   In what direction must the magnetization point relative to every surface in order for this to be true? Explain.

III. Applications of the auxiliary field

A. An infinite current-carrying wire with current passes through the center of a paramagnetic loop with a square cross-section. Point *A* is above the loop and point *B* is within the loop.



1. In what direction do you expect the magnetic field to point at points *A* and *B?*

   In what direction is the magnetization of the paramagnetic loop at point *A?*

2. Consider the surfaces of the loop.

   a. Does this magnetization have *curl*, *divergence*, or both? Explain.

   b. Sketch the direction of the bound currents on the cross-sectional view above.

3. Consider the following student discussion about the fields at points *A* and *B:*

   Student 1: "I think the auxiliary field is the same at points *A* and *B*. It only depends on the free current so we can ignore the paramagnet."

   Student 2: "I think the magnetic field is the same at points *A* and *B*. The wire has translational symmetry along the z-axis, so the magnetic field should, too."

   With which student(s) do you agree?

   What should you add to the student's explanation to make it more complete?

4. Consider two separate Amperian loops that could be used to solve for the corresponding fields at points *A* and *B*.

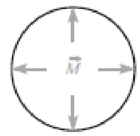
   a. Do the loops enclose the same amount of free current? What does this imply about the auxiliary field?

   b. Do the loops enclose the same amount of net current? What does this imply about the magnetic field?

➪ Check your answers to this page with a tutorial instructor before continuing.

ED 4 *Auxiliary field*

B. A ferromagnetic sphere has a fixed magnetization pointing radially outward.



Ferromagnetic sphere with radial magnetization

1. Does this system have *translational symmetry*, *rotational symmetry*, *both*, or *neither?* For each symmetry (if any), is it discrete or continuous?

   Is it possible for any relevant vector field to have a component in a non-radial direction? Explain.

2. Consider a point outside of the sphere.

   a. Is it possible for the magnetic field to have a component in the radial direction? Explain.

   b. What does this imply about $\vec{B}$ and $\vec{H}$ outside of the sphere?

3. Consider the following student discussion about the fields inside of the sphere:

   Student 1: "There is no free current in this system, so the auxiliary field is zero inside the sphere."

   Student 2: "There is no free or bound current, so the magnetic field must be zero."

   Student 3: "The sphere has magnetization, so the auxiliary field and the magnetic field cannot both be zero."

   Two of the students above are correct. Which student is incorrect? Explain.

4. Consider the surface of the magnetized sphere.

   a. Is the magnetization *perpendicular* or *parallel* to the surface of the sphere?

   b. Does the magnetization at this surface have *curl* or *divergence?*

   c. Does $\vec{H}$ for this system behave like a magnetostatic field? Explain.

IV. Supplement: Electrostatic and magnetostatic fields

A. Examine the time-independent Maxwell's equations for $\vec{E}$ and $\vec{B}$.

1. What physical entity acts as the source for the electrostatic field? Is it related to the curl or the divergence?

2. What physical entity acts as the source for the magnetostatic field? Is it related to the curl or the divergence?

3. Consider the curl and the divergence of $\vec{H}$. The auxiliary field of a simple bar magnet can be thought of as having "magnetic charge." Why is this term appropriate? What physical quantity acts as $\rho_{magnetic}$?

B. In systems containing magnetic materials, we often consider any field caused by the magnetic material an induced field, and any other field an external field.

1. Would an auxiliary field sourced by the free currents in a system be considered an *external field* or an *induced field?* Is this field similar to the *magnetostatic field* or the *electrostatic field?* Explain.

2. Would an auxiliary field sourced by the "magnetic charge" of a system be considered an *external field* or an *induced field?* Is this field similar to the *magnetostatic field* or the *electrostatic field?* Explain.

The "net" auxiliary field can be thought of as a sum of these two fields: $\vec{H}_{net} = \vec{H}_{ext} + \vec{H}_{ind}$.

3. Is this definition of $\vec{H}_{net}$ consistent with $\vec{\nabla} \times \vec{H}_{net} = \vec{J}_{free}$? (*Hint:* Does one term have no curl?) Explain.

4. Is this definition of $\vec{H}_{net}$ consistent with $\vec{\nabla} \cdot \vec{H}_{net} = -\vec{\nabla} \cdot \vec{M}$? Explain.

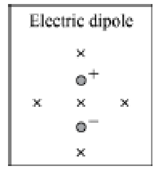
j. AUX TUT 193

AUXILIARY FIELD

ED
25

I. Dipole sources

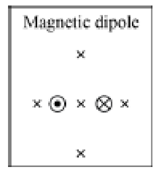
A. Consider an electric dipole, consisting of one positive and one negative charge of equal strength as shown at right. Five points are marked.



1. At each of the five points, use the principle of superposition to sketch an arrow indicating the direction of the electric field. Explain your reasoning.

2. On the diagram at right, sketch a field-line diagram for the electric field. Draw at least five field lines.

B. Now consider a magnetic dipole, consisting of two wires (or a wire loop) carrying the same amount of current, one into the page and one out of the page as shown at right. Five points are marked.



1. At each of the five points, use the principle of superposition to sketch an arrow indicating the direction of the magnetic field. Explain your reasoning.

2. On the diagram at right, sketch a field-line diagram for the magnetic field. Draw at least five field lines.

C. Both objects have dipole moments pointing upward. Compare and contrast the two dipole fields that you have drawn with the following questions:

1. How or where is the behavior of the two fields similar? How or where is the behavior different?

2. Consider any rules-of-thumb you may have used in determining the field behavior around each source. How are they related to Maxwell's Equations?

 Physics 322

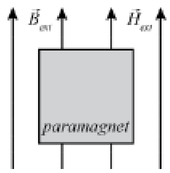
ED 26 *Auxiliary field*

II. Defining the auxiliary field

The *auxiliary field* $\vec{H}$ is defined in terms of the net magnetic field and the magnetization:

$$\vec{H}_{\text{net}} = \frac{\vec{B}_{\text{net}}}{\mu_o} - \vec{M}$$

A. A paramagnetic cube is held in place within a uniform upward external field with magnitude B_o for the magnetic field and magnitude H_o for the auxiliary field.



1. In what direction is the magnetization at the center of the cube?

2. In what direction is the *induced* magnetic field due to the paramagnetic cube at the center? Briefly explain your reasoning.

3. Several students discuss the direction of the *induced* auxiliary field based on the equation above:

Student 1: "The net auxiliary field is related to the negative of the magnetization. Thus the induced auxiliary field must point opposite to the magnetization."

Student 2: "The net auxiliary field is proportional to the net magnetic field. Thus the induced auxiliary field must point in the same direction as the induced magnetic field."

Student 3: "The magnetic field is additive while the magnetization is subtractive. Therefore, the cube's effects must cancel when looking at the auxiliary field, so there is no induced auxiliary field."

With which student(s), if any, do you agree? Explain your reasoning.

B. The curl and divergence offer additional information beyond the vector equation, as they *define the sources* of the vector field.

1. Derive an expression for $\vec{\nabla} \times \vec{H}$. Express your answer in terms of $\vec{J}_{\text{bound}}$ and/or $\vec{J}_{\text{free}}$.

2. Derive an expression for $\vec{\nabla} \cdot \vec{H}$. Express your answer in terms of vector derivatives of $\vec{B}$ and/or $\vec{M}$.

 Physics 322

III. Identifying and using divergence and curl

A. Acquire a set of transparencies and labels from a TA.

Each transparency shows a field-line diagram of an unknown vector field. Each diagram contains a dashed box for comparison. Each label represents strips of divergence and/or curl that could be present on the side of the dashed boxes.

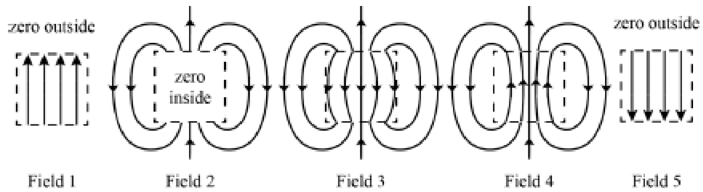
B. Fill in the following blanks with simple descriptions of how the field behaves around places of divergence and curl:

"The vector field tends to go __________ places of positive divergence, __________ places of negative divergence, __________ places of curl into-the-page, and __________ places of curl out-of-the-page.

Word bank: clockwise around, counterclockwise around, into, out of

C. For each transparency, attach the corresponding label(s) which show where the diagram has divergence and/or curl.

Below is a copy of the diagrams for your notes.



D. Let Field 1 represent $\vec{M}$ for a magnetized cube.

1. Which field best represents $\vec{B}$ for this cube? Explain your reasoning based on the divergence and/or curl.

2. Which field best represents $\vec{H}$ for this cube? Explain your reasoning based on the divergence and/or curl.

E. Compare your diagrams in part I.A and I.B to the fields you selected above. Resolve any inconsistencies.

★ Check your answers to this section with a tutorial instructor before continuing.

ED 28 *Auxiliary field*

IV. Supplement: Electrostatic and magnetostatic fields

A. Examine the time-independent Maxwell's equations for $\vec{E}$ and $\vec{B}$.

1. What physical entity acts as the source for the electrostatic field? Is it related to the curl or the divergence?

2. What physical entity acts as the source for the magnetostatic field? Is it related to the curl or the divergence?

3. Consider the curl and the divergence of $\vec{H}$. The auxiliary field of a simple bar magnet can be thought of as having "magnetic charge." Why is this term appropriate? What part of the bar magnet acts as ρ_{magnetic}?

B. In systems containing magnetic materials, we often consider any field caused by the magnetic material an induced field, and any other field an external field.

1. Would an auxiliary field sourced by the free currents in a system be considered an *external field* or an *induced field?* Is this field similar to the *magnetostatic field* or the *electrostatic field?* Explain.

2. Would an auxiliary field sourced by the "magnetic charge" of a system be considered an *external field* or an *induced field?* Is this field similar to the *magnetostatic field* or the *electrostatic field?* Explain.

The "net" auxiliary field can be thought of as a sum of these two fields: $\vec{H}_{\text{net}} = \vec{H}_{\text{ext}} + \vec{H}_{\text{ind}}$.

3. Is this definition of $\vec{H}_{\text{net}}$ consistent with $\vec{\nabla} \times \vec{H}_{\text{net}} = \vec{J}_{\text{free}}$? (*Hint:* Does one term have no curl?) Explain.

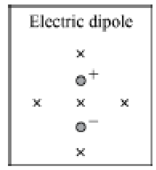
4. Is this definition of $\vec{H}_{\text{net}}$ consistent with $\vec{\nabla} \cdot \vec{H}_{\text{net}} = -\vec{\nabla} \cdot \vec{M}$? Explain.

 Physics 322

k. DPF TUT 194

I. Dipole sources

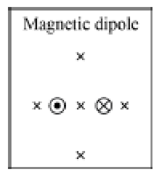
A. Consider an electric dipole, consisting of one positive and one negative charge of equal strength as shown at right. Five points are marked.



1. At each of the five points, use the principle of superposition to sketch an arrow indicating the direction of the electric field. Explain your reasoning.

2. On the diagram at right, sketch a field-line diagram for the electric field. Draw at least five field lines.

B. Now consider a magnetic dipole, consisting of two wires (or a wire loop) carrying the same amount of current, one into the page and one out of the page as shown at right. Five points are marked.



1. At each of the five points, use the principle of superposition to sketch an arrow indicating the direction of the magnetic field. Explain your reasoning.

2. On the diagram at right, sketch a field-line diagram for the magnetic field. Draw at least five field lines.

C. Both objects have dipole moments pointing upward. Compare and contrast the two dipole fields that you have drawn with the following questions:

1. How or where is the behavior of the two fields similar? How or where is the behavior different?

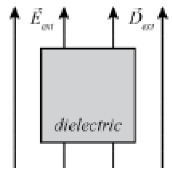
2. Consider any rules-of-thumb you may have used in determining the field behavior around each source. How are they related to Maxwell's Equations?

II. Sources of the displacement field

The *displacement field* $\vec{D}$ can be defined in terms of the net electric field and the polarization:

$$\vec{D} = \epsilon_o \vec{E}_{\text{net}} + \vec{P}$$

A. A dielectric cube is held in place within a uniform upward external field with magnitude E_{ext} for the electric field and magnitude D_{ext} for the displacement field.



1. In what direction is the polarization at the center of the cube?

2. In what direction is the *induced* electric field due to the dielectric cube at the center? Briefly explain your reasoning.

3. Several students discuss the direction of the *induced* displacement field based on the induced version of the equation above, $\vec{D}_{\text{ind}} = \epsilon_o \vec{E}_{\text{ind}} + \vec{P}$:

Student 1: "The induced displacement field is proportional to the polarization. Thus the induced displacement field must point in the same direction as the polarization."

Student 2: "The induced displacement field is proportional to the induced electric field. Thus the induced displacement field must point in the same direction as the induced electric field."

Student 3: "The induced electric field and the polarization point in opposite directions. Therefore, they cancel, so there is no induced displacement field."

With which student(s), if any, do you agree? Explain your reasoning.

B. The curl and divergence offer additional information beyond the vector equation, as they *define the sources* of the vector field.

1. Derive an expression for $\vec{\nabla} \cdot \vec{D}$. Express your answer in terms of ρ_{bound} and/or ρ_{free}.

2. Derive an expression for $\vec{\nabla} \times \vec{D}$. Express your answer in terms of vector derivatives of $\vec{E}$ and/or $\vec{P}$.

III. Identifying and using divergence and curl

A. Acquire a set of transparencies and labels from a TA.

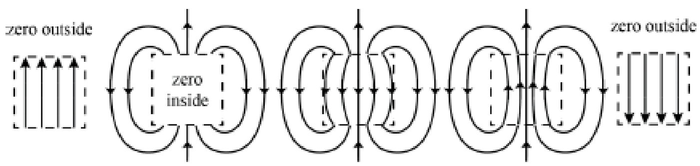
Each transparency shows a field-line diagram of an unknown vector field. Each diagram contains a dashed box for comparison. Each label represents strips of divergence and/or curl that could be present on the side of the dashed boxes.

B. Fill in the following blanks with simple descriptions of how the field behaves around places of divergence and curl:

"The vector field tends to go __________ places of positive divergence, __________ places of negative divergence, __________ places of curl into-the-page, and __________ places of curl out-of-the-page.

Word bank: clockwise around, counterclockwise around, into, out of

C. For each transparency, attach the corresponding label(s) which show where the diagram has divergence and/or curl. Below is a copy of the diagrams for your notes.



D. Let Field 1 represent $\vec{P}$ for a dielectric cube. Conventionally, we label the divergence of the polarization as *bound charges*. There is no formal name for the curl of the polarization, but it may be thought of as a *"current-like dipole"* source.

1. Consider the induced electric field:

   a. Is it sourced by the *bound charges, current-like dipole, both,* or *neither?*

   b. Which of the fields above best represents the induced electric field for this cube?

2. Now consider the induced displacement field:

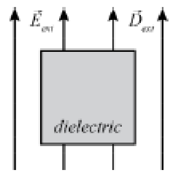
   a. Is it sourced by the *bound currents, current-like dipole, both,* or *neither?*

   b. Which of the fields above best represents the induced displacement field for this cube?

E. Compare your diagrams in parts I.A and I.B to the fields you selected above. Resolve any inconsistencies.

★ Check your answers to this section with a tutorial instructor before continuing.

IV. Comparing polarization and electric field



Consider again the equation $\vec{D} = \epsilon_o \vec{E} + \vec{P}$ and the dielectric cube as shown at right.

A. In section II, you found that it is insufficient to use this equation alone to determine the direction of the induced displacement field. What additional assumption must be made in order to determine the induced displacement field direction?

B. In section III you investigated how the regions of divergence and curl can determine the electric and displacement fields directly. Treating vector derivatives as sources for the polarization can also compare the electric field and the polarization.

1. Consider the top and bottom faces of the dielectric. Do they act as sources for the *electric field, polarization, both,* or *neither?*

2. Now consider the right and left faces of the dielectric. Do they act as sources for the *electric field, polarization, both,* or *neither?*

3. For the vector field where both sets of faces act sources:

   a. Does the top and bottom faces contribute an *upward, downward,* or *zero* field at the center of the cube? Briefly explain your reasoning.

   b. Does the right and left faces contribute and *upward, downward,* or *zero* field at the center of the cube? Briefly explain your reasoning.

   c. When you superimpose the two effects, do they *add, subtract,* or *neither?*

4. At the center of the cube, is the magnitude of $\epsilon_o \vec{E}_{\text{ind}}$ of the induced magnetic field *greater than, less than,* or *equal to* the magnitude of the polarization?

C. Using the relative magnitudes of $\epsilon_o \vec{E}_{\text{ind}}$ and polarization, determine the direction of the induced displacement field at the center of the cube. Compare your answer to the diagram in section III.

ED 53

V. Superposition of sources for the displacement field

In systems containing dielectrics, we often consider any field caused by the dielectric as an induced field, and any other field as an external field.

A. Consider the electrostatic field.

1. Is the electrostatic field sourced by *free charges, bound charges,* or *both?*

2. Suppose you wanted to determine the net electric field by superposition. What would be the most convenient way to subdivide net electric field?

   Write out the divergence and curl for each portion of the net electric field.

B. Now consider the displacement field.

1. Would a displacement field sourced only by free charges be considered an *external field* or an *induced field?*

   Would this field be sourced by *curl only,* by *divergence only,* or by *both?*

2. Would the induced displacement field due to the polarization be sourced by *curl only,* by *divergence only,* or by *both?*

The net displacement field can be thought of as a sum of these two fields: $\vec{D}_{net} = \vec{D}_{ext} + \vec{D}_{ind}$.

3. Is this definition of $\vec{D}_{net}$ consistent with $\vec{\nabla} \cdot \vec{D} = \rho_{free}$? (*Hint:* Does one term have no divergence?)

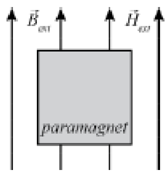
4. Is this definition of $\vec{D}_{net}$ consistent with $\vec{\nabla} \times \vec{D} = \vec{\nabla} \times \vec{P}$? Explain.

l. AUX TUT 201 – Additional page to AUX TUT 193

ED
28 *Auxiliary field*

IV. Comparing magnetization and magnetic field

Consider again the equation $\vec{H} = \frac{\vec{B}}{\mu_o} - \vec{M}$ and the uniformly magnetized cube as shown at right.



A. In section II, you found that it is insufficient to use this equation alone to determine the direction of the induced auxiliary field. What additional assumption must be made in order to determine the induced auxiliary field direction?

B. In section III you investigated how the regions of divergence and curl can determine the magnetic and auxiliary fields directly. Treating vector derivatives as sources for the magnetization can also compare the magnetic field and the magnetization.

1. Consider the top and bottom faces of the magnet. Do they act as sources for the *magnetic field, magnetization, both,* or *neither?*

2. Now consider the right and left faces of the magnet. Do they act as sources for the *magnetic field, magnetization, both,* or *neither?*

3. For the vector field where both sets of faces act as sources, do the effects from the divergence and curl sources *constructively interfere, destructively interfere,* or *neither* at the center of the cube? Explain your reasoning.

4. At the center of the cube, is the magnitude of $\frac{\vec{B}_{ind}}{\mu_o}$ of the induced magnetic field *greater than, less than,* or *equal to* the magnitude of the magnetization?

C. Using the relative magnitudes of $\frac{\vec{B}_{ind}}{\mu_o}$ and magnetization, determine the direction of the induced auxiliary field at the center of the cube. Compare your answer to the diagram in section III.

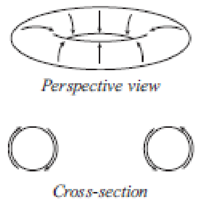
 Physics 322

3. Homework

a. MVP HW 161

MAGNETIC VECTOR POTENTIAL Name ____________________ ED HW–1

1. A toroidal current distribution is shown at right. Current flows up the outside of the toroid and down the inside, as indicated by the arrows on the diagram.



   a. Which direction does the magnetic field point inside the toroid? What direction does it point outside the toroid? Explain, and sketch your answers on the cross-section diagram.

   (*Hint:* A toroid is simply a solenoid that has been bent into a circle. What does the magnetic field of a solenoid look like?)

   b. What does the vector potential of the toroid look like? Sketch it below.

   c. Does the vector potential exactly mimic the current distribution, or is it different? If not, how is it different? Explain.

2. The current in a solenoid of radius a is a cylindrical surface current density: $\vec{J} = K\delta(s-a)\hat{\phi}$.

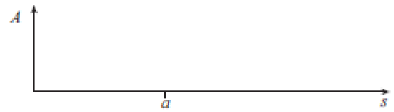
   a. What magnetic field is produced by this current density? Where is the magnetic field zero, uniform, and/or non-uniform? Explain.

Consider an analogous situation where the magnetic field is also only present on a cylindrical surface: $\vec{B} = B_0 a\,\delta(s-a)\hat{\phi}$.

   b. What vector potential is associated with this magnetic field? Explain.

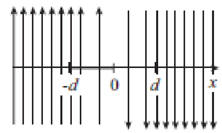
ED HW–2 ***Magnetic vector potential***

c. Sketch a graph of the magnitude of $\vec{A}$ as a function of s.

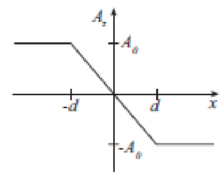


d. What current distribution would produce this magnetic field? (*Hint:* What field does a thin cylindrical shell of current produce?)

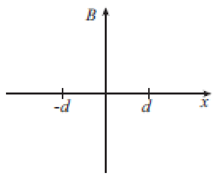
3. The diagram and graph at right show the z-component of the vector potential for some current distribution; the x and y components of the vector potential are zero everywhere in space.



a. Given this vector potential, where in space is there a *non-zero* magnetic field? Explain how you determined your answer.



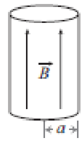
b. Sketch a quantitatively correct plot of the magnitude of the magnetic field as a function of x. Label the vertical axis appropriately, and explain your reasoning.



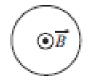
c. Where in space is there a *non-zero* current density? Explain how you determined your answer.

Magnetic vector potential Name ____________________ ED HW–3

4. A portion of an infinite solenoid is shown at right.



Side view



Top view

a. What loop should you use with Stokes' theorem to get information about the z-component of the vector potential? Explain, and draw it on the diagram.

b. Consider three loops: one entirely inside the solenoid, one partly inside the solenoid and partly outside, and a third entirely outside the solenoid.

What do these loops tell you about the magnitude of the z-component in each region? Explain.

c. Could the magnitude of the z-component be *zero* everywhere in space?

Could the magnitude of the z-component be *uniformly non-zero* everywhere in space?

d. Which of the above cases could be sourced by this magnetic field? Explain.

5. Challenge question (not graded, based on the tutorial supplement in *Magnetic vector potential*).

In electrostatics, we could use a gauge transform to choose the zero point of potential. Use the following questions as a guide to decide if you can do the same thing for the solenoid above.

a. Using Stokes' theorem with a circular loop, where is the magnitude of $\vec{A}$ equal to zero?

b. Suppose we wanted to use a gauge transform to make $\vec{A}'$ zero at the edge of the solenoid. What field would we have to add (or subtract) to the original $\vec{A}$ to do this?

c. Recall that gauge transforms must leave $\vec{B}$ unchanged. Does the added field you found above obey this rule? Explain why or why not.

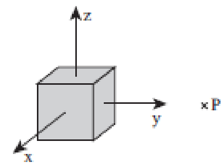
4. Exam questions

a. 144E1 (COU, GSL)

Name ______________________________ **Student ID** ____________ **Score**________
last *first*

IV. [25 points total] Tutorial question.

A cube with charge density $\rho(x,y,z) = bz$ and side length a fills the space from $x,y,z = -a/2$ to $x,y,z = +a/2$ (it's centered on the origin). A test point P is located at $y = 7$m.



A. [3 pts.] How much charge is in the cube? Explain.

B. [2 pts.] Qualitatively sketch the electric field lines of this charge distribution in the space below.

C. [4 pts.] Could you use Gauss's law to find the electric field at point P? If so, draw the Gaussian surface you would use on the diagram, and explain how you would find $\overline{E}(P)$. If not, explain why not.

D. Suppose you instead wanted to use Coulomb's law to find the electric field at P.

i. [2 pts.] What is the separation vector between a point (x',y',z') in the cube and point P?

ii. [5 pts.] Write down an expression for $\overline{E}(P)$.

(*Note:* You don't have to evaluate the integral, but you should be as explicit as possible.)

E. [4 pts.] Suppose you found $\overline{E}$ everywhere in space using one of the methods above. Where is the divergence of this electric field zero? Where is it non-zero? Explain.

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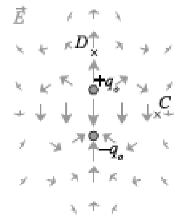
b. 174E1 (INT, DEL)

Name ________________________________ **Student ID** ____________ **Score**______
last *first*

IV. [25 points total] This page contains two independent parts, A and B.

A. The electric field of a dipole is shown in the vector field diagram at right. The dipole consists of a single positive point charge above and a single negative point charge below.

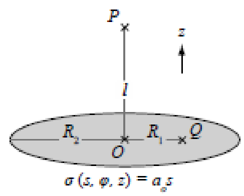


Vector field diagram
(not a field line diagram)

i. [5 pts] At point *D*, is the divergence of the electric field *positive, negative,* or *zero?* Explain your reasoning.

ii. [5 pts] At point *C*, in what direction is the curl of the electric field? If the curl of the electric field is zero, state so explicitly. Explain your reasoning.

B. A circular disk of radius R_2 centered at the origin and oriented in the *xy*-plane is charged with a surface charge density of $\sigma(s,\phi,z) = a_0 s$. Point *P* is located a distance *l* above the origin along the *z*-axis, at coordinates (0, 0, *l*). Point *Q* is located on the disk a distance R_1 from the origin, at coordinates (R_1, 0, 0).



Perspective view

i. Consider a small region of the disk around point *Q*.

[8 pts] How much charge is located in that small region? Use an appropriate coordinate system and variables provided above. Show your work and/or explain your reasoning.

[3 pts] What is the distance from the coordinate (R_1, 0, 0) to point *P*? Show your work.

The electric potential at a point relative to infinity is $V = kq/d$ for a point charge, where *d* is the distance between the point charge and where the electric potential is measured, and *k* is a constant.

ii. [4 pts] Based on your answers above, write an integral expression for the electric potential (not electric field) at point *P* due to the charged disk, relative to infinity. You do not need to evaluate the integral.

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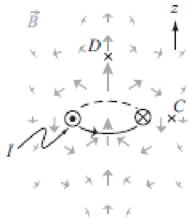
c. 171E1 (MSF, AML)

Name ________________________________ **Student ID** ____________ **Score**_______
last *first*

IV. [25 points total] Tutorial question.

The magnetic field of a single loop of current in the *xy*-plane is centered on the origin such that the *z*-axis points upwards, as shown in the vector field diagram at right.



Vector field diagram
(not a field line diagram)

A. [5 pts] At point *D*, is the divergence of the magnetic field *positive*, *negative*, or *zero?* Explain your reasoning.

B. [6 pts] At point *C*, in what direction is the curl of the magnetic field? If the curl of the magnetic field is zero, state so explicitly. Explain your reasoning.

C. [6 pts] What type(s) of symmetry does the current distribution have? Specify each symmetry as continuous or discrete, translational or rotational, and around/along which axis.

For each symmetry that you have found, what does this tell about the behavior of the magnetic field?

D. [8 pts] Can the integral form of Ampère's Law, $\oint \vec{B} \cdot d\vec{l} = \mu_o I_{\text{enc}}$, be used to determine the magnetic field at the center of the loop? Explain why or why not. (It may be helpful to draw an Ampèrian loop as an example and argue based on that.)

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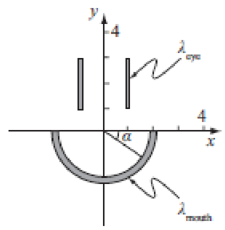
d. 182E1 (INT, DEL)

Name ______________________________ **Student ID** ____________ **Score**_______
last *first*

IV. [25 points total] Tutorial question. For the following context, assume units (m for length, C for charge).

A. Consider the charge distribution at right. The two straight rods each have variable linear charge density λ_{eye}, whichare located at $x = 1$ and -1 spanning from $y = 1$ to 3. The curved rod makes a circular arc of radius 2 from the origin, has variable linear charge density λ_{mouth}. Take α as the angle clockwise from the x-axis.



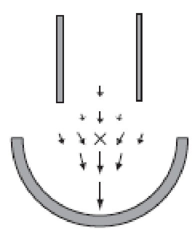
i. [5 pts] What is the net charge on one straight rod? Show your work.

ii. [5 pts] What is the net charge of the curved rod? Show your work.

iii. [10 pts] Recall that the point-charge form of the electric potential is $V = kq/d$, with the reference point at infinity. Using this, **derive an expression** in terms of k, x, y, α, λ, and/or numerical constants **for the potential at the origin** relative to infinity. You will be graded on the set-up and substitution of q and d, so you do not need to evaluate any mathematical operations in your expression.

B. [5 pts] The diagram at right shows a vector field diagram estimating the electric field near the origin (marked as ×).

Is the divergence of the electric field at the origin *positive, negative,* or *zero?* If the divergence cannot be determined, state so explicitly. Explain your reasoning.



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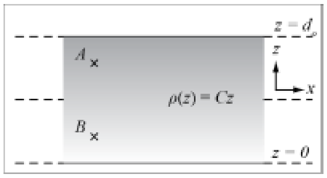
e. 184E1 (GSL, VDS)

Name ______________________________ **Student ID** ____________ **Score**______
last *first*

IV. [25 points total] Tutorial question.

A charged slab with variable charge density $\rho(z) = Cz$ is shown at right, where C is a positive constant. The slab is infinite in the xy-plane, and extends from $z = 0$ to $z = d_o$. Assume that the electric field is zero at $z < 0$.



Non-uniform charged slab

A. [6 pts] What type(s) of symmetry does this slab have (*i.e.* continuous rotational symmetry about the x-axis)? Explain how you know.

B. [6 pts] Describe a Gaussian surface for which the flux integral will be easy to simplify. You may use sketches or diagrams in your answer. Explain why you chose this surface.

C. [8 pts] How does electric field within the slab depend on z (*i.e.* constant with z, proportional to $1/z$, z^2, etc.)? Explain your reasoning.

How does the electric field within the slab depend on x (*i.e.* constant with x, proportional to $1/x$, x^2, etc.)? Explain your reasoning.

D. [5 pts] Is the divergence of the electric field at point A *greater than, less than,* or *equal to* the divergence of the electric field at point B? Explain your reasoning.

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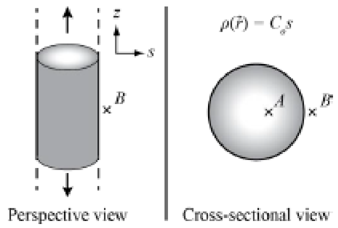
f. 194E1 (GSL, VDS)

Name ________________________________ **Student ID** ____________ **Score**______
last *first*

IV. [20 points total] Tutorial question.

A charged rod with non-uniform charge density $\rho(\vec{r}) = C_o s$ (while inside the rod) is shown at right, where C_o is a positive constant. There is no charge outside of the rod. The rod is infinitely long.

Point *A* is located near the center of the rod, whereas point *B* is located just outside the surface.



A. Suppose you were to evaluate the magnitude of the electric field at point *B* using Gauss' Law in integral form.

i. [3 pts] Describe and/or sketch a Gaussian surface for which the electric flux integral will be easy to simplify. If there is no Gaussian surface possible, state so explicitly.

ii. [5 pts] Explain *in words* why the flux integral will be (or cannot be) easily simplified. Any math must be paired with a verbal interpretation of each step.

B. [4 pts] How does electric field within the rod depend on *z* (*e.g.*, constant with *z*, proportional to $1/z$, z^2, etc.)? Explain your reasoning.

C. [8 pts] Is the divergence of the electric field at point *A greater than, less than,* or *equal to* the divergence of the electric field at point *B*? If the divergence at either point is zero, state so explicitly. Explain your reasoning.

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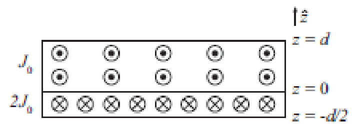
g. 151E1 (DEL, AML, MVP)

Name ______________________________ **Student ID** ____________ **Score**________
last *first*

IV. [25 points total] Tutorial question.

An infinite slab with uniform current density J_0 directed out of the page fills the space from $z = 0$ to $z = d$. A slab with twice the current density but half the thickness has current into the page, filling the space from $z = 0$ to $z = -d/2$.



Portion of two infinite slabs of current

A. [6 pts.] What direction(s) can the magnetic field point for $z > d$, above both slabs? Use symmetry arguments in your explanation.

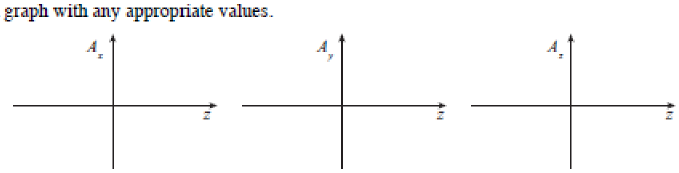
What is the magnitude of the magnetic field for $z > d$? Explain.

B. [8 pts.] On the diagram, sketch an Amperian loop that would allow you to find the magnitude of the magnetic field at $z = 0$.

Use Ampere's law to find the magnitude and direction of the magnetic field everywhere in space, and sketch a graph of the magnetic field as a function of z.

C. [6 pts.] Set the reference point of the vector potential at $z = 0$. What direction does $\bar{A}$ point in each of the following regions: above both slabs, inside the upper slab, inside the lower slab, and below both slabs? Explain your reasoning in each case.

D. [5 pts.] On the plots below, sketch a qualitatively correct graph of each component of $\bar{A}$ vs z. Label each graph with any appropriate values.



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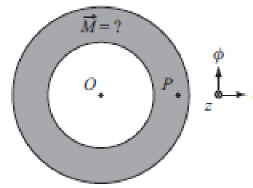
h. 161E2 (MVP, MAG, AUX)

Name ___________________________________ **Student ID** ______________ **Score**________
last *first*

IV. [25 points total] An infinite cylindrical shell has a fixed magnetization in an unknown direction. The magnetization has continuous rotational and translational symmetry. A cross-section through the origin is shown at right.



Cross-section of symmetric infinite magnetic shell

Parts A and B are independent and describe two separate systems.

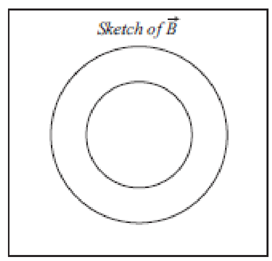
A. Suppose the bound current $\vec{J}_{\text{bound}}$ and $\vec{K}_{\text{bound}}$ is zero everywhere.

[5 pts] In what direction(s) can the magnetization point at point *P?* If any direction is possible, state so explicitly. Explain.

B. Suppose instead the auxiliary field $\vec{H}$ is zero everywhere in space.

i. [6 pts] In what direction(s) can the magnetization point at point *P?* If any direction is possible, state so explicitly. Explain.

ii. [7 pts] In the space provided, sketch a symmetrical magnetic field that is consistent with your answer above. If the magnetic field is zero anywhere, state so explicitly. Explain.



iii. [7 pts] For the magnetic field you sketched above, in what direction is the magnetic vector potential $\vec{A}$ at the origin in the Coulomb gauge? You may assume that the magnitude of the magnetic vector potential is (approximately) equal to zero at a very large radius. Explain.

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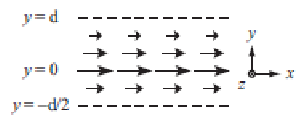
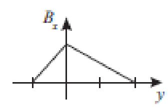
i. 163E1 (MSF, AML, MVP)

Name ______________________________________ **Student ID** _______________ **Score**________
last *first*

IV. [20 points total] Tutorial question.

Consider the following magnetic field, shown at right. The magnetic field is zero everywhere for $y < -d/2$ and $y > d$. The magnetic field from $-d/2 < y < 0$ is given by $\vec{B} = (2Cy + Cd)\,\hat{x}$ for all x and z, and from $0 < y < d$ is given by $\vec{B} = (-Cy + Cd)\,\hat{x}$ for all x and z.





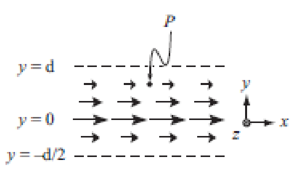
A. [6 pts.] For what region(s) and/or surface(s) of space is the current density non-zero? For each region or surface that you identify, determine the direction of current. Explain your reasoning.

B. [7 pts.] Set the reference point of the vector potential at $y = 0$. In what direction does $\vec{A}$ point in each of the following regions? If the vector potential is zero in the region, state so explicitly.

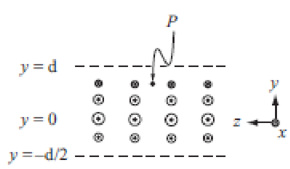
- $y < -d/2$:
- $0 < y < d$:
- $-d/2 < y < 0$:
- $y > d$:

Explain your reasoning for $y > d$.

C. [7 pts.] Again, set the reference point of the vector potential at $y = 0$. On *one* of the diagrams below, sketch the Stokesian loop necessary to determine $\vec{A}$ at point P. You do not need to solve for $|\vec{A}_P|$. Explain why your loop allows you to determine $|\vec{A}_P|$.



Original view



Rotated view

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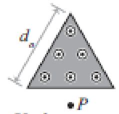
j. 183E1 (AML, MVP)

Name ______________________________ **Student ID** ____________ **Score**______
last *first*

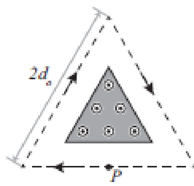
IV. [25 points total] Tutorial question.

A long wire coming out of the page carries uniform volume current density. The total current carried by the wire is I_o, coming out of the page. The cross-section of the wire makes an equilateral triangle with side lengths d_o. Consider point *P*, located a short distance below the center of the wire as shown at right.



A. [7 pts] In what direction is the magnetic field at point *P?* If the magnetic field at point *P* is zero, state so explicitly. Explain your answer.

B. [10 pts] Consider the clockwise triangular Amperian loop shown at right. The length of each side is $2d_o$. Would it be possible to determine the magnitude of the magnetic field at point *P* using Ampere's Law, $\oint \vec{B} \cdot d\vec{l} = \mu_o I_{enc}$, using this loop? Explain why or why not.



C. [8 pts] In what direction is the magnetic vector potential, $\vec{A}$, at point *P?* Assume Coulomb Gauge and that the magnitude approaches zero infinitely far away from the wire. If the magnetic vector potential is zero at point *P*, state so explicitly. Explain your reasoning.

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k. 191E1 (AML, MVP)

Name ______________________________ **Student ID** ____________ **Score**______
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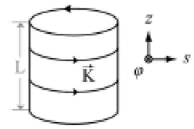
IV. [25 pts] Solenoids! This question consists of two independent parts: A and B.

A. A **finite** solenoid is shown at right.

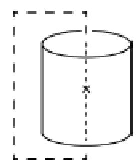


i. [3 pts] What symmetry(s) does the finite solenoid have? Limit your response to only rotations and translations.

ii. [4 pts] Could the magnetic field of this solenoid *anywhere in space* have a component in $\hat{\phi}$?
Explain why or why not.

iii. [4 pts] Could the magnetic field of this solenoid depend on z? Explain why or why not.

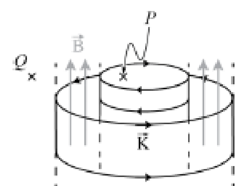
iv. [6 pts] Consider the loop shown at right. Can $\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$ be used to determine the magnitude of the magnetic field at the center of the solenoid? Explain why or why not.



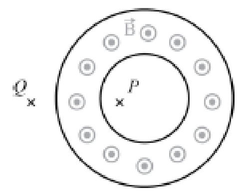
B. An **infinite** double-solenoid (shown at right) creates a uniform axial magnetic field between the inner and outer solenoids, and zero elsewhere.

Let $|\vec{A}_P|$ represent the magnitude of the magnetic vector potential at point P, which is located inside of the inner solenoid, and likewise for $|\vec{A}_Q|$ outside the outer solenoid. You should assume the Coulomb's Gauge and the reference point at $s \to \infty$.

[8 pts] Is $|\vec{A}_P|$ *greater than, less than,* or *equal to* $|\vec{A}_Q|$? If there is not enough information, state so explicitly. Explain your reasoning.



perspective view



cross-sectional view

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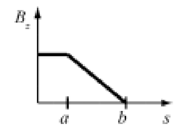
l. 201E1 (MVP)

Name __________________________________ **Student ID** _____________ **Score**_______
last *first*

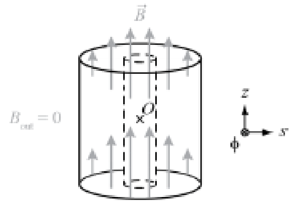
IV. [25 points total] Tutorial question.



An unknown current distribution produces the following piecewise magnetic field. In cylindrical coordinates:

$\vec{B}_{\text{in}} = B_o\hat{z}$ when $s < a$ (inside region),
$\vec{B}_{\text{mid}} = B_o(\frac{b-s}{b-a})\,\hat{z}$ for $a < s < b$ (middle region), and
$\vec{B}_{\text{out}} = 0$ when $s > b$ (outside region).

Graphical and visual representations of the magnetic field is shown at right, with point O as the origin. Cylinders of radii a and b are shown with dashed and solid lines, respectively.



Perspective view

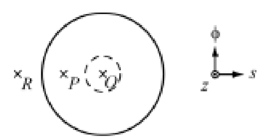
A. Qualitatively describe the current distribution that produces this magnetic field (no explanation necessary).

i. [2 pts] Where is the current non-zero?

ii. [2 pts] Is it a *volume current density*, a *surface current density*, and/or a *line current?*

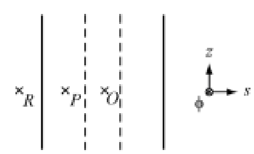
iii. [1 pt] In what direction does the current point?

B. Consider the magnetic vector potential $\vec{A}$ associated with this current distribution. Points P and R are located in the middle and outside regions respectively, as shown at right. Take the magnetic vector potential to be zero at the origin, and $\vec{\nabla} \cdot \vec{A} = 0$.



Top view

i. [5 pts] In what direction is $\vec{A}_P$, the magnetic vector potential at point P? If $\vec{A}_P$ is zero, state so explicitly. Explain.



Side view

ii. [10 pts] On one of the two diagrams at right, sketch a closed loop for which it would be easy to determine $|\vec{A}_P|$ from the magnetic field. Explain.

iii. [5 pts] In what direction is $\vec{A}_R$? If $\vec{A}_R$ is zero, state so explicitly. Explain.

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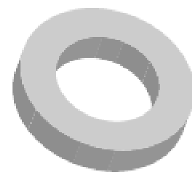
m. 154E3 (POL, DPF)

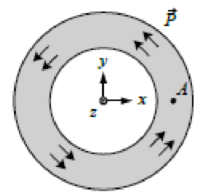
Name ______________________________ **Student ID** ____________ **Score**______
last *first*

IV. [25 points total] Tutorial question.

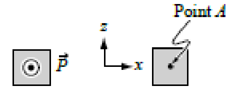
An isolated, thick hoop with a square cross-section has a uniform polarization azimuthally (around the hoop). The polarization is fixed (not caused by external sources) and is given by $\vec{P} = c_o\hat{\phi}$ within the bulk of the hoop. The inner radius of the hoop is a_o, the outer radius is b_o, and the height is $b_o - a_o$. The bound volume charge density within the bulk of the hoop is zero.



Perspective view



Top view



Cross-sectional view

a. [10 pts] At which of the following locations is the curl of $\vec{P}$ non-zero? Circle all that apply, and give the direction of the curl (*i.e.* $+\hat{\phi}$, $-\hat{x}$, etc).

Location: Direction:

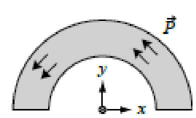
- White region $s < a_o$ along the $z = 0$ plane:
- Inner surface of the hoop at $s = a_o$:
- Gray region $a_o < s < b_o$ along the $z = 0$ plane:
- Outer surface of the hoop at $s = b_o$:
- White region $s > b_o$ along the $z = 0$ plane:
- Top surface of the hoop:
- Bottom surface of the hoop:

b. [10 pts] Point *A* is located at the center of the square cross-section along the *x*-axis as shown.

i. In what direction is the electric field at point *A*? If it is zero, state so explicitly. Explain your reasoning.

ii. In what direction is the displacement field at point *A*? If it is zero, state so explicitly. Explain your reasoning.

c. [5 pts] Suppose the hoop was cut in half along the $y = 0$ plane, as shown at right. In what *Cartesian* direction is the net dipole moment of the top half of the hoop? Explain your reasoning.



Top view

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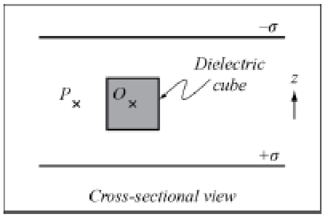
n. 192E3 (MPE, POL, DPF)

Name ______________________________ **Student ID** ____________ **Score** ______
last *first*

IV. [25 points total] Tutorial question.

A dielectric cube is placed between two charged sheets as shown at right. The sheets produce a uniform, upward electric field $\vec{E}_{\text{sheet}} = E_{\text{sheet}}\hat{z}$ and a uniform, upward displacement field $\vec{D}_{\text{sheet}} = D_{\text{sheet}}\hat{z}$.



You may use the approximation that the dielectric cube responds uniformly to the field of the sheets (*i.e.*, the polarization is uniform within the cube).

You may also assume that the magnitude of the fields from the dielectric cube is always less than that of the sheets, such that the net fields always have an upward component.

A. [5 pts] In what direction is the dipole moment of the cube? If the dipole moment is zero, state so explicitly. Explain your reasoning.

B. [10 pts] Consider point *O* which is located inside of the dielectric cube.

i. Is the magnitude of the net electric field $|\vec{E}_{\text{net},O}|$ *greater than*, *less than*, or *equal to* E_{sheet}?

ii. Is the magnitude of the net displacement field $|\vec{D}_{\text{net},O}|$ *greater than*, *less than*, or *equal to* D_{sheet}?

iii. Explain your answers for the previous two questions.

C. [10 pts] Consider point *P* which is located to the left of the dielectric cube.

i. Is the magnitude of the net electric field $|\vec{E}_{\text{net},P}|$ *greater than*, *less than*, or *equal to* E_{sheet}?

ii. Is the magnitude of the net displacement field $|\vec{D}_{\text{net},P}|$ *greater than*, *less than*, or *equal to* D_{sheet}?

iii. Explain your answers for the previous two questions. You may leave this part blank if you would like to apply the same reasoning as in part Biii.

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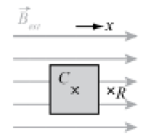
o. 201E2 (MAG, AUX)

Name ______________________________ **Student ID** ____________ **Score**______
last *first*

IV. [25 points total] Tutorial question.

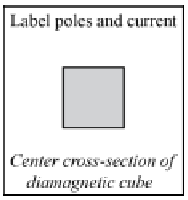
Consider a **diamagnetic** cube placed in a uniform external magnetic field pointing toward the right of the page, in $+\hat{x}$ as shown at right. Point *C* is located at the center of the cube, and point *R* is located to the right of the cube. You may assume that the magnetization in the material is uniform.



Diamagnetic cube in uniform external field

A. [2 pts.] In what direction is magnetization at point *C*? You do not need to explain. You may draw an arrow.

B. [8 pts.] In the box below, label poles of the cube with "N" for north and "S" for south, and any bound current with "⊗" for in and "⊙" for out. Explain how you determined the location(s) each source.



C. [15 pts] The following three questions can be answered independently of the other, or in any order. You may use one part to help answer the other parts, but you should avoid circular logic.

i. In the box below, draw an arrow indicating the direction of the **induced** magnetic field at point *R*. If the field is zero, state so explicitly. Explain. (*N.b.* "induced" means due to the cube.)

Direction of $\vec{B}_{ind}$ at point *R*

ii. In the box below, draw an arrow indicating the direction of the **induced** magnetic field at point *C*. If the field is zero, state so explicitly. Explain.

Direction of $\vec{B}_{ind}$ at point *C*

iii. In the box below, draw an arrow indicating the direction of the **induced** auxiliary field at point *C*. If the field is zero, state so explicitly. Explain.

Direction of $\vec{H}_{ind}$ at point *C*

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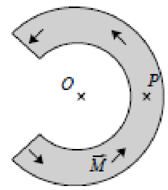
p. 171E2 (MAG, AUX)

Name ________________________ **Student ID** ____________ **Score**______
last *first*

IV. [25 points total] Tutorial question.

Consider the ferromagnetic horseshoe magnet at right. The magnetization is fixed and points around the origin *O* as shown. There is no free current.



A. [10 pts] How does this magnetization contribute to the magnetic field $\vec{B}$, if at all? Support your reasoning with relationships, and support any equation with an interpretation.

How does this magnetization contribute to the auxiliary field $\vec{H}$, if at all? Support your reasoning with relationships, and support any equation with an interpretation.

B. [8 pts] Suppose you need to determine the magnitude of $\vec{B}$ at the origin. *Qualitatively describe* the steps that you could take (*i.e.,* the relationships that you would use) to solve for $|\vec{B}|$ at the origin. If $\vec{B}$ is zero at the origin, state so explicitly. Any equations must be supported with an interpretation.

C. [7 pts] In the box at right, draw an arrow indicating the direction of $\vec{H}$ at point *P* in the magnet. If $\vec{H}$ is zero or the direction cannot be determined, state so explicitly. Explain your reasoning.

Direction of $\vec{H}$ at point *P*

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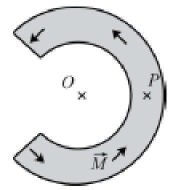
q. 193E2 (MAG, AUX)

Name ______________________________ **Student ID** ____________ **Score**______
last *first*

IV. [X points total] Tutorial question.

Consider the ferromagnetic horseshoe magnet at right. The magnetization is fixed and points around the origin *O* as shown. Point *P* is located within the magnet, in the middle along the axis of symmetry. There is no free current.



A. [Xi pts] Where does the magnetization have divergence? Specify the location(s) with positive and negative divergence. Briefly explain your reasoning.

B. [Xii pts] Suppose you need to determine the magnitude of $\vec{H}$ at point *O*, the origin. *Qualitatively describe* the steps that you could take (*i.e.*, the relationships that you would use) to solve for $|\vec{H}|$ at the origin. Any equations must be supported with an interpretation in words. You do not need to write any integrals.

C. [Xiii pts] In the box at right, draw an arrow indicating the direction of $\vec{H}$ at point *P* within the magnet. If $\vec{H}$ is zero or the direction cannot be determined, state so explicitly. Explain your reasoning.

Direction of $\vec{H}$ at point *P*

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